

We have the distribution function

$$\phi(s) = \phi_0 \begin{cases} s^{-p_1}, & s \leq s_{\text{break}} \\ s_{\text{break}}^{p_2-p_1} \times s^{-p_2}, & s > s_{\text{break}} \end{cases},$$

Neglecting the factor of  $\phi_0$ , which is cancelled by the numerator, the integral in the denominator of Eq 9 is

$$I = \int_{s_{\min}}^{s_{\text{break}}} \frac{s^{-p_1}}{1 + (s/d_i\theta_0)^{-\beta}} ds + s_{\text{break}}^{p_2-p_1} \int_{s_{\text{break}}}^{s_{\max}} \frac{s^{-p_2}}{1 + (s/d_i\theta_0)^{-\beta}} ds$$

Let's sub in both integrals  $x = \frac{s}{d_i\theta_0}$  and  $dx = \frac{ds}{d_i\theta_0}$ , so we get

$$I = (d_i\theta_0)^{1-p_1} \int_{s_{\min}/d_i\theta_0}^{s_{\text{break}}/d_i\theta_0} \frac{x^{-p_1}}{1 + x^{-\beta}} dx + s_{\text{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} \int_{s_{\text{break}}/d_i\theta_0}^{s_{\max}/d_i\theta_0} \frac{x^{-p_2}}{1 + x^{-\beta}} dx$$

or in terms of  $x$ ,

$$\begin{aligned} I &= (d_i\theta_0)^{1-p_1} \int_{x_{\min}}^{x_{\text{break}}} \frac{x^{-p_1}}{1 + x^{-\beta}} dx + s_{\text{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} \int_{x_{\text{break}}}^{x_{\max}} \frac{x^{-p_2}}{1 + x^{-\beta}} dx \\ &= (d_i\theta_0)^{1-p_1} I_1 + s_{\text{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} I_2. \end{aligned}$$

If we introduce the variables  $\gamma_1 = 1 + \beta - p_1$  and  $\gamma_2 = 1 + \beta - p_2$  for convenience, we find [say, with Mathematica] that

$$\begin{aligned} I_1 &= \frac{1}{\gamma_1} \left[ x_{\text{break}}^{\gamma_1} F_2^1 \left( 1, \frac{\gamma_1}{\beta}, 1 + \frac{\gamma_1}{\beta}, -x_{\text{break}}^{\beta} \right) - x_{\min}^{\gamma_1} F_2^1 \left( 1, \frac{\gamma_1}{\beta}, 1 + \frac{\gamma_1}{\beta}, -x_{\min}^{\beta} \right) \right] \\ I_2 &= \frac{1}{\gamma_2} \left[ x_{\max}^{\gamma_2} F_2^1 \left( 1, \frac{\gamma_2}{\beta}, 1 + \frac{\gamma_2}{\beta}, -x_{\max}^{\beta} \right) - x_{\text{break}}^{\gamma_2} F_2^1 \left( 1, \frac{\gamma_2}{\beta}, 1 + \frac{\gamma_2}{\beta}, -x_{\text{break}}^{\beta} \right) \right], \end{aligned}$$

where  $F_2^1$  is the Gaussian hypergeometric function.

This is implemented in the function `analytic_integrand_weights_double_power_law()`