Double power law

We have the distribution function

$$\phi\left(s\right)=\phi_0\begin{cases}s^{-p_1}, & s\leq s_{\mathrm{break}}\\ s_{\mathrm{break}}^{p_2-p_1}\times s^{-p_2}, & s>s_{\mathrm{break}}\end{cases},$$

Neglecting the factor of ϕ_0, which is cancelled by the numerator, the integral in the denominator of Eq 9 is

$$I=\int_{s_{\min}}^{s_{\mathrm{break}}}\frac{s^{-p_1}}{1+\left(s/d_i\theta_0\right)^{-\beta}}\mathrm{d}s+s_{\mathrm{break}}^{p_2-p_1}\int_{s_{\mathrm{break}}}^{s_{\max}}\frac{s^{-p_2}}{1+\left(s/d_i\theta_0\right)^{-\beta}}\mathrm{d}s$$

Let's sub in both integrals $x=\frac{s}{d_i\theta_0}$ and $dx=\frac{ds}{d_i\theta_0}$, so we get

$$I=\left(d_i\theta_0\right)^{1-p_1}\int_{s_{\min}/d_i\theta_0}^{s_{\mathrm{break}}/d_i\theta_0}\frac{x^{-p_1}}{1+x^{-\beta}}\mathrm{d}x+s_{\mathrm{break}}^{p_2-p_1}\left(d_i\theta_0\right)^{1-p_2}\int_{s_{\mathrm{break}}/d_i\theta_0}^{s_{\max}/d_i\theta_0}\frac{x^{-p_2}}{1+x^{-\beta}}\mathrm{d}x$$

or in terms of x,

$$\begin{aligned}I&=\left(d_i\theta_0\right)^{1-p_1}\int_{x_{\min}}^{x_{\mathrm{break}}}\frac{x^{-p_1}}{1+x^{-\beta}}\mathrm{d}x+s_{\mathrm{break}}^{p_2-p_1}\left(d_i\theta_0\right)^{1-p_2}\int_{x_{\mathrm{break}}}^{s_{\max}}\frac{x^{-p_2}}{1+x^{-\beta}}\mathrm{d}x\\&=\left(d_i\theta_0\right)^{1-p_1}I_1+s_{\mathrm{break}}^{p_2-p_1}\left(d_i\theta_0\right)^{1-p_2}I_2.\end{aligned}$$

If we introduce the variables $\gamma_1=1+\beta-p_1$ and $\gamma_2=1+\beta-p_2$ for convenience, we find [say, with Mathematica] that

$$\begin{aligned}I_1&=\frac{1}{\gamma_1}\left[x_{\mathrm{break}}^{\gamma_1}F_2^1\left(1,\frac{\gamma_1}{\beta},1+\frac{\gamma_1}{\beta},-x_{\mathrm{break}}^{\beta}\right)-x_{\min}^{\gamma_1}F_2^1\left(1,\frac{\gamma_1}{\beta},1+\frac{\gamma_1}{\beta},-x_{\min}^{\beta}\right)\right]\\I_2&=\frac{1}{\gamma_2}\left[x_{\max}^{\gamma_2}F_2^1\left(1,\frac{\gamma_2}{\beta},1+\frac{\gamma_2}{\beta},-x_{\max}^{\beta}\right)-x_{\mathrm{break}}^{\gamma_2}F_2^1\left(1,\frac{\gamma_2}{\beta},1+\frac{\gamma_2}{\beta},-x_{\mathrm{break}}^{\beta}\right)\right],\end{aligned}$$

where F_2^1 is the Gaussian hypergeometric function.

This is implemented in the function analytic_integrand_weights_double_power_law().

Single power law

In this case, the distribution function is just

$$\phi\left(s\right)=\phi_0s^{-p}.$$

We can again ignore the factor of ϕ_0. The integral in the denominator of Eq 9 is

$$I=\int_{s_{\min}}^{s_{\max}}\frac{s^{-p}}{1+\left(s/d_i\theta_0\right)^{-\beta}}\mathrm{d}s.$$

Let's sub $x=\frac{s}{d_i\theta_0}$ and $\mathrm{d}x=\frac{\mathrm{d}s}{d_i\theta_0}$, so we get

$$I=\left(d_i\theta_0\right)^{1-p}\int_{x_{\min}}^{x_{\max}}\frac{x^{-p}}{1+x^{-\beta}}\mathrm{d}x=\left(d_i\theta_0\right)^{1-p}I_1$$

Where $x_{\max}=s_{\max}/\left(d_i\theta_0\right)$, and the same for $x_{\min}$. If we introduce $\gamma=1+\beta-p$, then we find

$$I_1=\frac{1}{\gamma}\left[x_{\max}^{\gamma}F_2^1\left(1,\frac{\gamma}{\beta},1+\frac{\gamma}{\beta},-x_{\max}^{\beta}\right)-x_{\min}^{\gamma}F_2^1\left(1,\frac{\gamma}{\beta},1+\frac{\gamma}{\beta},-x_{\min}^{\beta}\right)\right].$$

This is implemented in the function analytic_integrand_weights_single_power_law().