Double power law

We have the distribution function

$$\phi\left(s\right) = \phi_0 \begin{cases} s^{-p_1}, & s \leq s_{\mathrm{break}} \\ s_{\mathrm{break}}^{p_2-p_1} \times s^{-p_2}, & s > s_{\mathrm{break}} \end{cases},$$

Neglecting the factor of ϕ_0, which is cancelled by the numerator, the integral in the denominator of Eq 9 is

$$I = \int_{s_{\min}}^{s_{\mathrm{break}}} \frac{s^{-p_1}}{1 + (s/d_i\theta_0)^{-\beta}} \, \mathrm{d}s + s_{\mathrm{break}}^{p_2-p_1} \int_{s_{\mathrm{break}}}^{s_{\max}} \frac{s^{-p_2}}{1 + (s/d_i\theta_0)^{-\beta}} \, \mathrm{d}s$$

Let's sub in both integrals $x = \frac{s}{d_i\theta_0}$ and $dx = \frac{ds}{d_i\theta_0}$, so we get

$$I = (d_i\theta_0)^{1-p_1} \int_{s_{\min}/d_i\theta_0}^{s_{\mathrm{break}}/d_i\theta_0} \frac{x^{-p_1}}{1 + x^{-\beta}} \, \mathrm{d}x + s_{\mathrm{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} \int_{s_{\mathrm{break}}/d_i\theta_0}^{s_{\max}/d_i\theta_0} \frac{x^{-p_2}}{1 + x^{-\beta}} \, \mathrm{d}x$$

or in terms of x,

$$I = (d_i\theta_0)^{1-p_1} \int_{x_{\min}}^{x_{\mathrm{break}}} \frac{x^{-p_1}}{1 + x^{-\beta}} \, \mathrm{d}x + s_{\mathrm{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} \int_{x_{\mathrm{break}}}^{x_{\max}} \frac{x^{-p_2}}{1 + x^{-\beta}} \, \mathrm{d}x$$

$$= (d_i\theta_0)^{1-p_1} I_1 + s_{\mathrm{break}}^{p_2-p_1} (d_i\theta_0)^{1-p_2} I_2.$$

If we introduce the variables $\gamma_1 = 1 + \beta - p_1$ and $\gamma_2 = 1 + \beta - p_2$ for convenience, we find [say, with Mathematica] that

$$I_1 = \frac{1}{\gamma_1} \left[x_{\mathrm{break}}^{\gamma_1} F_2^1\left(1, \frac{\gamma_1}{\beta}, 1 + \frac{\gamma_1}{\beta}, -x_{\mathrm{break}}^{\beta}\right) - x_{\min}^{\gamma_1} F_2^1\left(1, \frac{\gamma_1}{\beta}, 1 + \frac{\gamma_1}{\beta}, -x_{\min}^{\beta}\right) \right]$$

$$I_2 = \frac{1}{\gamma_2} \left[x_{\max}^{\gamma_2} F_2^1\left(1, \frac{\gamma_2}{\beta}, 1 + \frac{\gamma_2}{\beta}, -x_{\max}^{\beta}\right) - x_{\mathrm{break}}^{\gamma_2} F_2^1\left(1, \frac{\gamma_2}{\beta}, 1 + \frac{\gamma_2}{\beta}, -x_{\mathrm{break}}^{\beta}\right) \right],$$

where F_2^1 is the Gaussian hypergeometric function.

This is implemented in the function analytic_integrand_weights_double_power_law().

Single power law

In this case, the distribution function is just

$$\phi\left(s\right) = \phi_0 s^{-p}.$$

We can again ignore the factor of ϕ_0. The integral in the denominator of Eq 9 is

$$I = \int_{s_{\min}}^{s_{\max}} \frac{s^{-p}}{1 + (s/d_i\theta_0)^{-\beta}} \, \mathrm{d}s.$$

Let's sub $x = \frac{s}{d_i\theta_0}$ and $\mathrm{d}x = \frac{\mathrm{d}s}{d_i\theta_0}$, so we get

$$I = (d_i\theta_0)^{1-p} \int_{x_{\min}}^{x_{\max}} \frac{x^{-p}}{1 + x^{-\beta}} \, \mathrm{d}x = (d_i\theta_0)^{1-p} I_1$$

Where $x_{\max} = s_{\max}/(d_i\theta_0)$, and the same for $x_{\min}$. If we introduce $\gamma = 1 + \beta - p$, then we find

$$I_1 = \frac{1}{\gamma} \left[x_{\max}^{\gamma} F_2^1\left(1, \frac{\gamma}{\beta}, 1 + \frac{\gamma}{\beta}, -x_{\max}^{\beta}\right) - x_{\min}^{\gamma} F_2^1\left(1, \frac{\gamma}{\beta}, 1 + \frac{\gamma}{\beta}, -x_{\min}^{\beta}\right) \right].$$

This is implemented in the function analytic_integrand_weights_single_power_law().

1

Einasto-like profile

In this case, the distribution function is

$$\phi(s) = \phi_0 e^{-(s/s_0)^\alpha}.$$

We can ignore the normalization factor of ϕ_0, for the same reasons as above. The integral in the denominator of Eq 9 is

$$I = \int_{s_{\min}}^{s_{\max}} \frac{e^{-(s/s_0)^\alpha}}{1 + (s/d_i\theta_0)^{-\beta}} \, \mathrm{d}s.$$

Let's sub $x = \frac{s}{d_i\theta_0}$ and $\mathrm{d}x = \frac{\mathrm{d}s}{d_i\theta_0}$ and $x_0 = s_0/d_i\theta_0$, so we get

$$I = d_i\theta_0 \int_{x_{\min}}^{x_{\max}} \frac{e^{-(x/x_0)^\alpha}}{1 + x^{-\beta}} \, \mathrm{d}x = d_i\theta_0 I_1$$

Where $x_{\max} = s_{\max}/(d_i\theta_0)$, and the same for $x_{\min}$. Unfortunately, I_1 is not analytic, but we can calculate it numerically.