# Interesting Integral

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## 1 Problem Intro

$$\boxed{I = \int \sqrt{x^3\sqrt{x^4\sqrt{x^5\sqrt{\ldots}}}}\ dx} = \int x^{\frac{3}{2}} x^{\frac{4}{2^2}} x^{\frac{5}{2^3}} \ldots dx = \int \prod_{i=1}^{\infty} x^{\frac{2+i}{2^i}} dx$$

$$\text{Property: } \prod_n x^{b_n} = x^{\sum_n b_n}, \text{Let } S = \sum_{i=1}^{\infty} \frac{2+i}{2^i}$$

$$\Rightarrow \quad I = \int x^S dx = \frac{x^{S+1}}{S+1} + C, \quad C \in \mathbb{R}$$

$$S = \sum_{i=1}^{\infty} \left(\frac{2}{2^i} + \frac{i}{2^i}\right) = \overbrace{\sum_{i=1}^{\infty} \frac{2}{2^i}}^{\text{Geometric Series}} + \sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{1}{1-\frac{1}{2}} + \sum_{i=1}^{\infty} \frac{i}{2^i}$$

$$= 2 + \underbrace{\left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\right)}_{=Q} \Rightarrow S = 2 + Q$$

## 2 Sum Simplification

| Simplification Step | Q |
|---|---|
| $n=1$ | $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$ |
| $n=2$ | $\frac{4}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \cdots$ |
| $n=3$ | $\frac{11}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \frac{6}{2^6} + \cdots$ |
| $\vdots$ | let $a_n$ be the numerator of the first term in the $n^{th}$ step |
| $n=\infty$ | $\lim_{n\to\infty} \frac{a_n}{2^n}$* |

*Note this is allowed because $\lim_{n\to\infty} \frac{n}{2^n} = 0$

## 3 Solving the Recurrence Relation

Clearly, $a_1 = 1$, and we are given:

$$\frac{a_{n+1}}{2^{n+1}} = \frac{a_n}{2^n} + \frac{n+1}{2^{n+1}}$$

Multiplying both sides by $2^{n+1}$:

$$a_{n+1} = 2a_n + (n+1) \quad \text{(1st order linear recurrence relation)}$$

We rewrite this as:

$$a_{n+1} - 2a_n = n + 1$$

Let the general solution be:

$$a_n = h_n + p_n$$

## 4 Combining Homogeneous and Particular Solutions

### Homogeneous Solution

Solve the homogeneous part:

$$h_{n+1} - 2h_n = 0 \Rightarrow h_{n+1} = 2h_n$$

This is a geometric sequence:

$$h_n = c_1 \cdot 2^n, \quad c_1 \in \mathbb{R}$$

### Particular Solution

Guess a particular solution of the form $p_n = An + B$. Substitute into the recurrence:

$$p_{n+1} - 2p_n = n + 1$$

$$A(n+1) + B - 2(An + B) = n + 1$$

$$An + A + B - 2An - 2B = n + 1$$

$$-An + A - B = n + 1$$

Matching coefficients:

$$\begin{cases} -A = 1 \\ A - B = 1 \end{cases} \rightarrow \begin{cases} A = -1 \\ -1 - B = 1 \end{cases} \rightarrow \begin{cases} A = -1 \\ B = -2 \end{cases} \Rightarrow p_n = -n - 2$$

### General Solution

Combine homogeneous and particular solutions:

$$a_n = c_1 \cdot 2^n - n - 2$$

Use the initial condition $a_1 = 1$ to solve for $c_1$:

$$1 = c_1 \cdot 2^1 - 1 - 2 \Rightarrow 1 = 2c_1 - 3 \Rightarrow c_1 = 2$$

Thus,

$$a_n = 2 \cdot 2^n - n - 2 = 2^{n+1} - n - 2$$

# 5 Limit Evaluation

Evaluate:

$$Q = \lim_{n\to\infty} \frac{a_n}{2^n} = \lim_{n\to\infty} \left( \cancelto{2}{\frac{2^{n+1}}{2^n}} - \cancelto{0}{\frac{n}{2^n}} - \cancelto{0}{\frac{2}{2^n}} \right)$$

$$\implies Q = 2$$

# 6 Integral Expression

If $S = 2 + Q = 2 + 2 = 4$, then the integral becomes:

$$I = \frac{x^{4+1}}{4+1} + C$$

# 7 Final Answer

$$\boxed{I = \frac{x^5}{5} + C}$$