Algorithms and Data Structures 2
CS 1501

Fall 2022

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(Slides are adapted from Dr. Ramirez's and Dr. Farnan's CS1501 slides.)

Announcements

- Upcoming Deadlines
  - Homework 6: this Friday @ 11:59 pm
  - Lab 5: next Monday @ 11:59 pm
  - Assignment 1 Late Deadline Wednesday Oct 12^{th} @ 11:59 pm
- Autograder issues and debugging hints
- If you think you lost points in a lab assignment because of the autograder or because of a simple mistake
  - please reach out to Grader TA over Piazza
- **Student Support Hours** of the teaching team are posted on the Syllabus page

Previous lecture

- Huffman Compression
  - How to compute character frequencies
- Run-length Encoding
- LZW
  - compression and expansion algorithms

This Lecture

- LZW
  - implementation concerns
- Shannon’s Entropy
- Comparing LZW vs Huffman
- Burrows-Wheeler Compression Algorithm

LZW implementation concerns: codebook

- How to represent/store during:
  - Compression
  - Expansion
- Considerations:
  - What operations are needed?
  - How many of these operations are going to be performed?
- Discuss

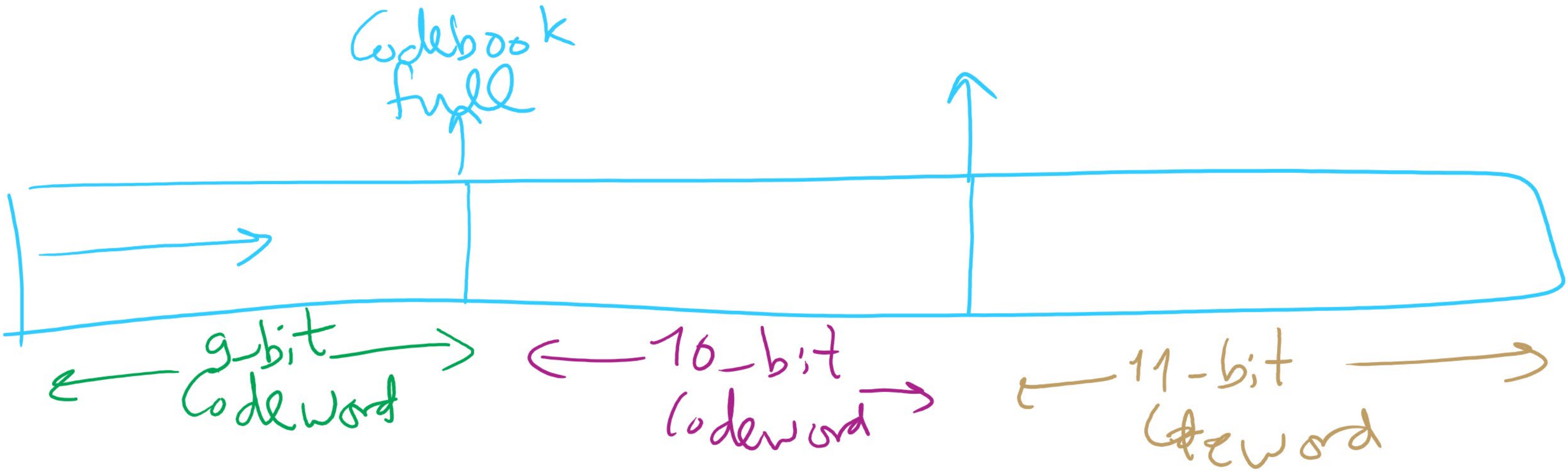
Further implementation issues: codeword size

- How long should codewords be?
  - Use fewer bits:
    - Gives better compression earlier on
    - But, leaves fewer codewords available, which will hamper compression later on
  - Use more bits:
    - Delays actual compression until longer patterns are found due to large codeword size
    - More codewords available means that greater compression gains can be made later on in the process

Variable width codewords

- This sounds eerily like variable length codewords…
  - Exactly what we set out to avoid!
- Here, we're talking about a different technique
- Example:
  - Start out using 9 bit codewords
  - When codeword 512 is inserted into the codebook, switch to outputting/grabbing 10 bit codewords
  - When codeword 1024 is inserted into the codebook, switch to outputting/grabbing 11 bit codewords…
  - Etc.

Adaptive Codeword Size



Even further implementation issues: codebook size

- What happens when we run out of codewords?
  - Only 2^n possible codewords for n bit codes
  - Even using variable width codewords, they can't grow arbitrarily large…
- Two primary options:
  - Stop adding new keywords, use the codebook as it stands
    - Maintains long already established patterns
    - But if the file changes, it will not be compressed as effectively
  - Throw out the codebook and start over from single characters
    - Allows new patterns to be compressed
    - Until new patterns are built up, though, compression will be minimal

Can we reason about how much a file can be compressed?

- Yes! Using Shannon Entropy

Information theory in a single slide...

- Founded by Claude Shannon in his paper "A Mathematical Theory of Communication"
- *Entropy* is a key measure in information theory
  - Slightly different from thermodynamic entropy
  - A measure of the unpredictability of information content
  - Example: which is more unpredicatble?
    - a character that occurs with a probabillity of 0.5 or
    - a character that occurs with probability 0.25
    - which should have more entropy?

Entropy

- Entropy equation: $H(c) = -1 * \log_2 Pr(c)$
  - Pr(c) is the probability of character c
- Examples:
  - $Pr(c1) = 0.5 \rightarrow H(c1) = -1 * \log_2(0.5) = -1*-1 = 1$ bit
  - $Pr(c2) = 0.25 \rightarrow H(c2) = -1*\log_2(0.25) = -1*-2 = 2$ bits
  - $Pr(c3) = 1/2^{100} \rightarrow H(c3) = -1*\log_2(2^{-100}) = -1*-100 = 100$ bits

Implications on Lossless Compression

- On average, a lossless compression scheme cannot compress a message to have more than 1 bit of entropy per bit of compressed message
- By losslessly compressing data, we represent the same information in less space
  - entropy of 8 bits of compressed data >
    entropy of 8 bits of uncompressed data

Entropy of a file

- The average number of bits required to store a character in that file
- So, it is the average entropy of all unique characters in the file
- H(file) = $\text{sum}_{\text{each unique character } c}\ H(c)*Pr(c)$
- How can we determine the probability of each character in the file?
    - if depends only on file contents
        - *Pr(c)* = *f(c)* / file size
    - However, may also depend on receiver and sender contexts and their world knowledge

Entropy applied to language:

- the average number of bits required to store a letter of the language
- Entropy of a language * length of message = amount of information contained in that message
- Uncompressed, English has between 0.6 and 1.3 bits of entropy per letter

The showdown you've all been waiting for...

HUFFMAN vs LZW

- In general, LZW will give better compression
  - Also better for compressing archived directories of files
    - Why?
      - Very long patterns can be built up, leading to better compression
      - Different files don't "hurt" each other as they did in Huffman
        - Remember our thoughts on using static tries?

So lossless compression apps use LZW?

- Well, gifs can use it
  - And pdfs
- Most dedicated compression applications use other algorithms:
  - DEFLATE (combination of LZ77 and Huffman)
    - Used by PKZIP and gzip
  - Burrows-Wheeler transforms
    - Used by bzip2
  - LZMA
    - Used by 7-zip
  - brotli
    - Introduced by Google in Sept. 2015
    - Based around a " ... combination of a modern variant of the LZ77 algorithm, Huffman coding[,] and 2nd order context modeling ... "

Is there a univeral compression algorithm?

- Nope!
- No algorithm can compress every bitstream
  - Assume we have such an algorithm
  - We can use to compress its own output!
  - And we could keep compressing its output until our compressed file is 0 bits!
    - Clearly this can't work
- Proofs in Proposition S of Section 5.5 of the text

Is finding the best algorithm for a given file possible?

- Nope!
- This problem is undecidable
- Example:
  - A Fibonacci sequence of one billion numbers can be compressed by a program to generate Fibonacci numbers

A final note on compression evaluation

- "Weissman scores" are a made-up metric for Silicon Valley (TV)

Burrows-Wheeler Data Compression Algorithm

- Best compression algorithm (in terms of compression ratio) for text
- The basis for UNIX's bzip2 tool

Adapted from: https://www.cs.princeton.edu/courses/archive/spr03/cos226/assignments/burrows.html

BWT: Compression Algorithm

- Three steps
  - Cluster same letters as close to each other as possible
    - Burrows-Wheeler Transform
  - Move-To-Front Encoding
    - Convert output of previous step into an integer file with large frequency differences
  - Huffman Compression
    - Compress the file of integers using Huffman

BWT: Expansion Algorithm

- Apply the inverse of compression steps in reverse order
  - Huffman decoding
  - Move-To-Front decoding
  - Inverse Burrows-Wheeler Transform

Move-To-Front Encoding

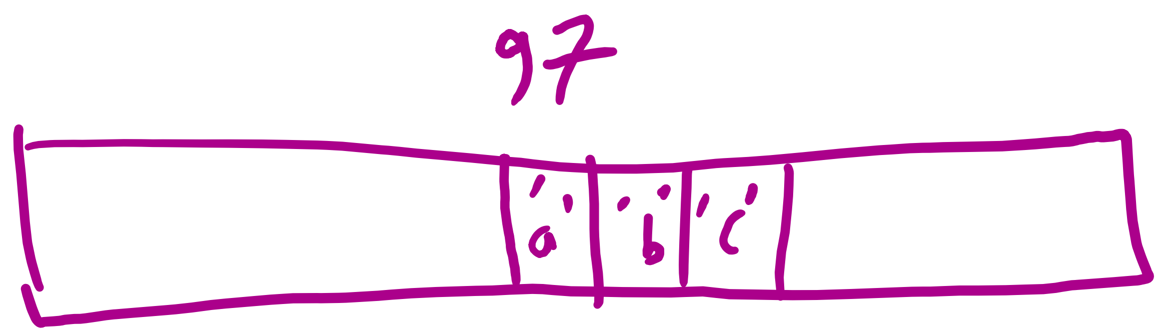
- Initialize an ordered list of the 256 ASCII characters
  - extended ASCII character *i* appears *i*th in the list
- For each character c from input
  - output the index in the list where c appears
  - move c to the front of the list

- Example:

a b b b a a b b b b a c c a b b a a a b c

97

- 'a' is 97 in ASCII



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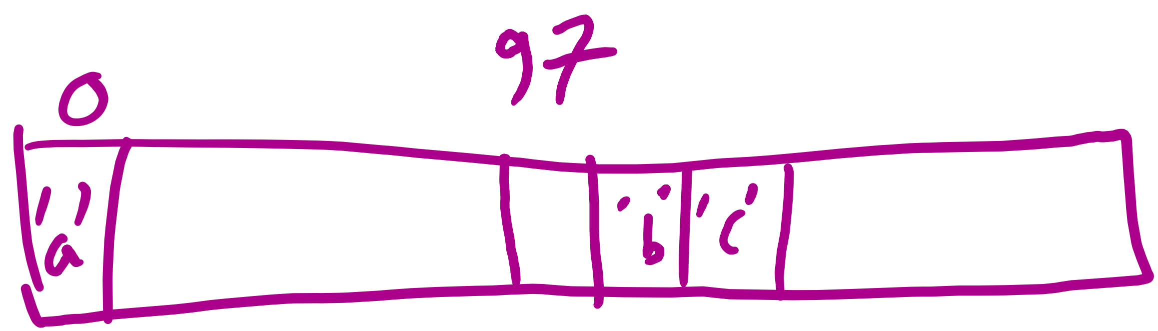

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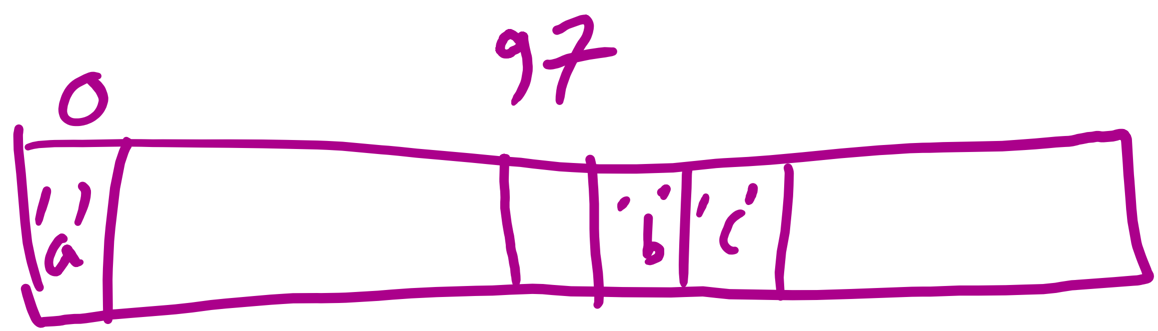

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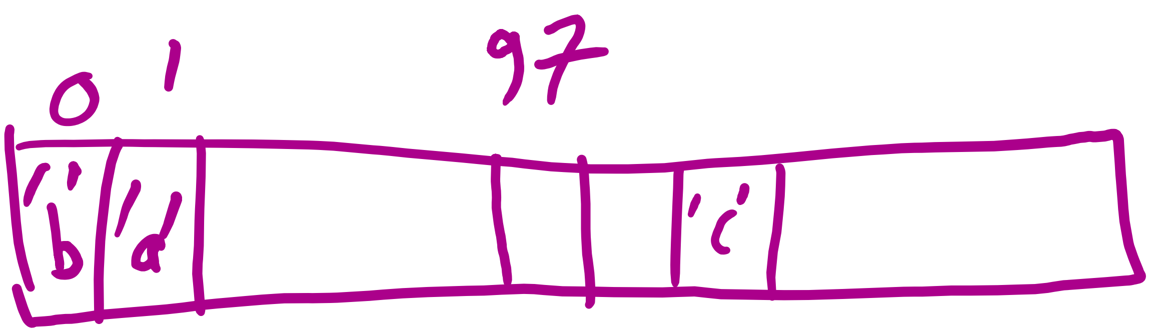

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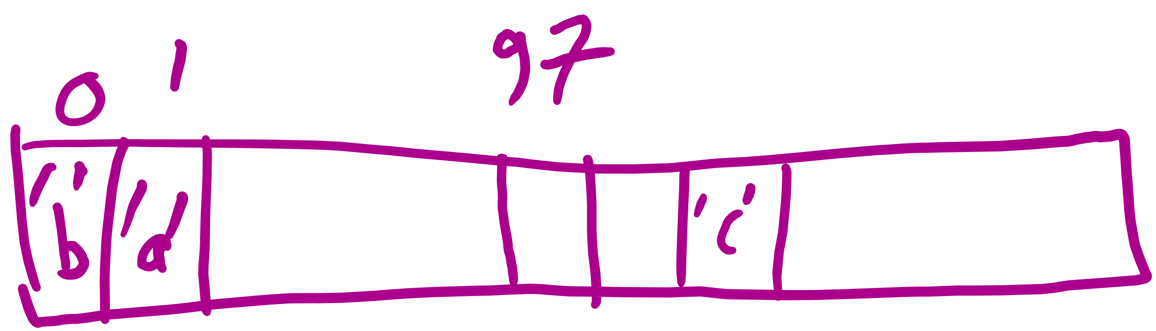
97 98 0



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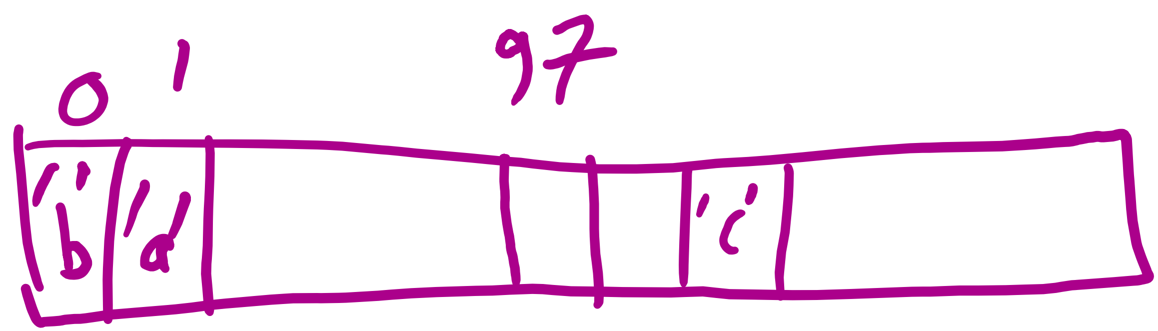

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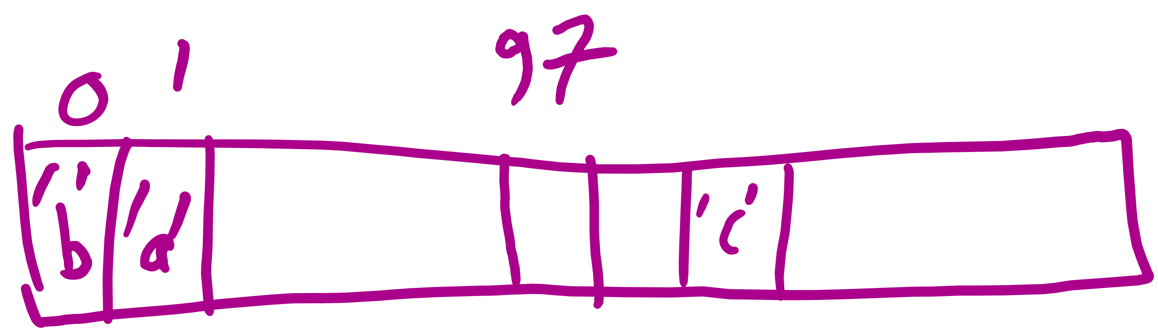

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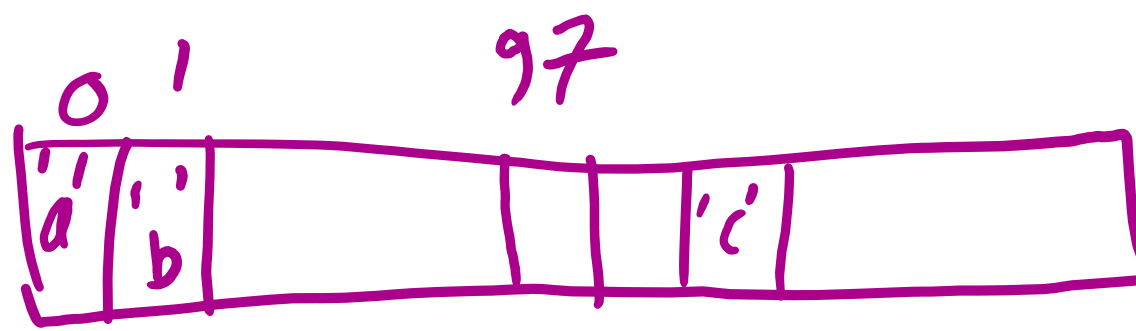

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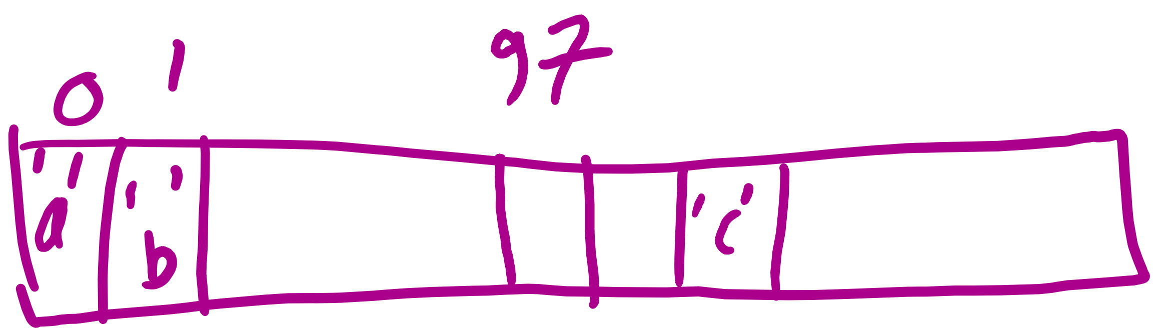

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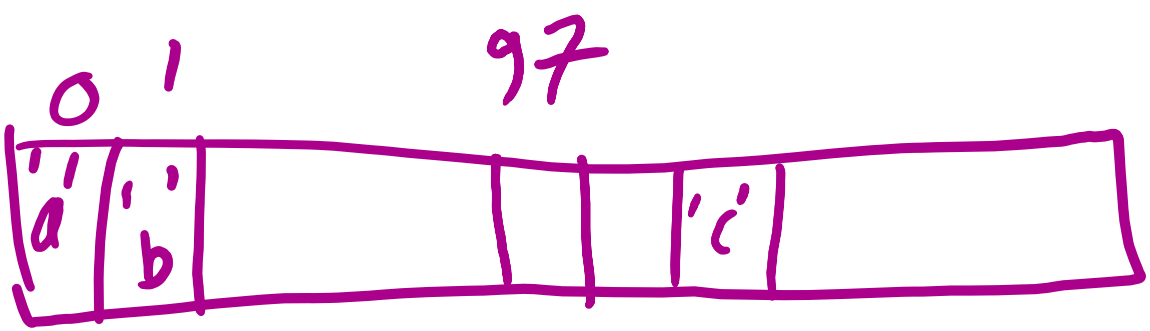

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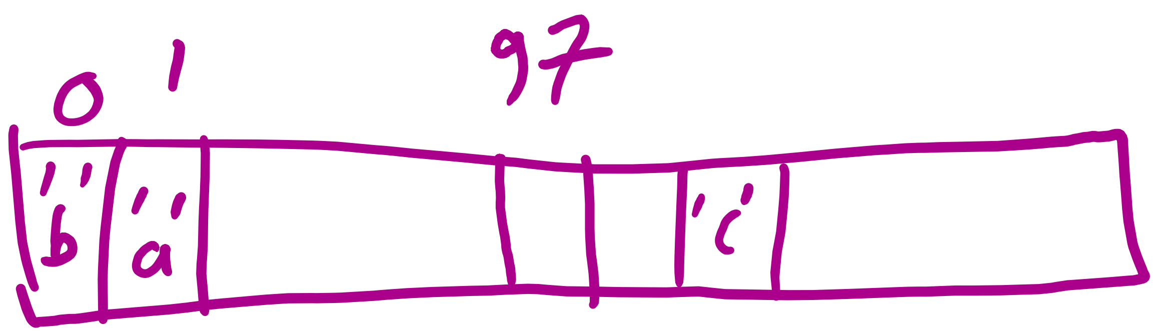

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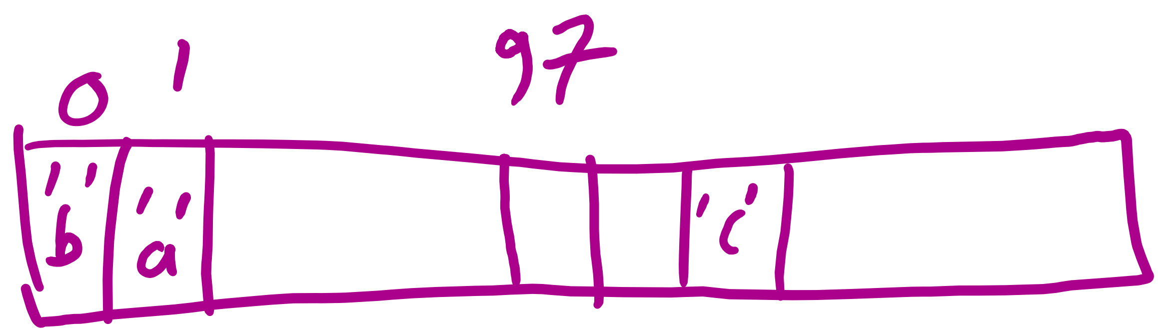

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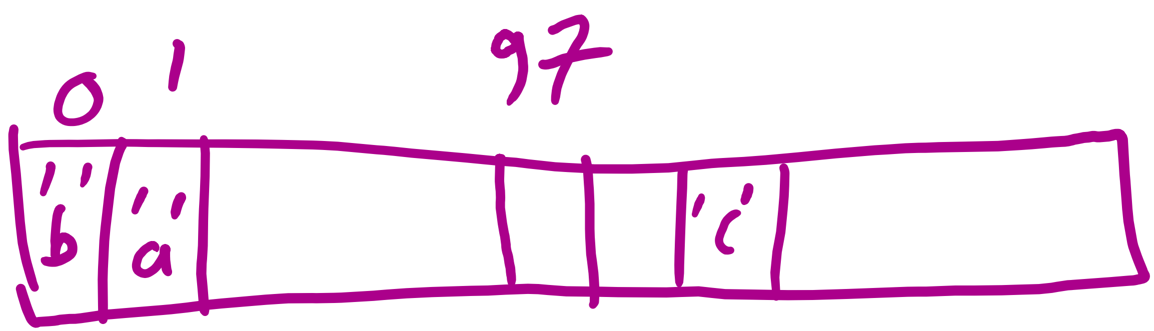

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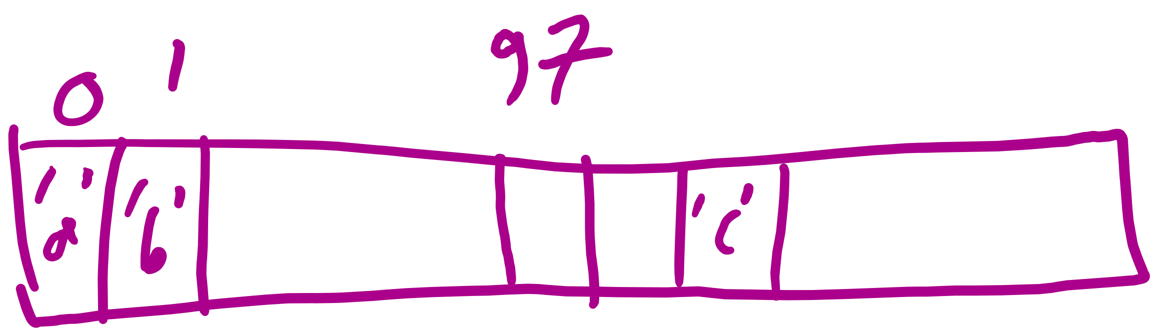

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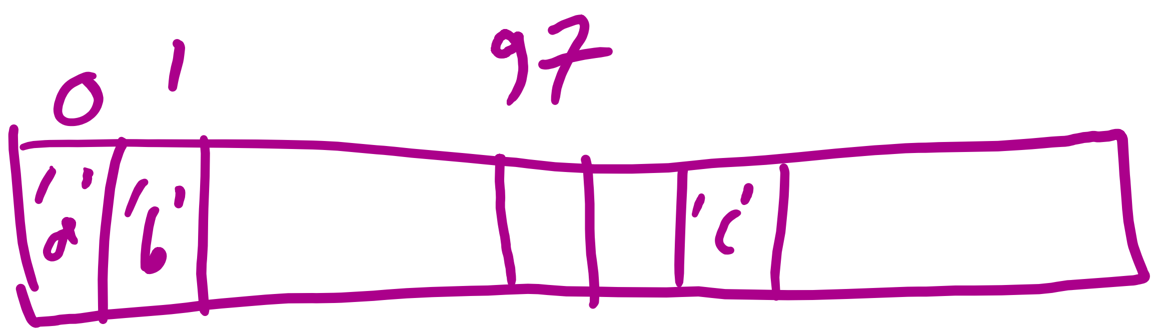
a b b b a a b b b b a **c** c a b b a a a b c

97 98 0 0 1 0 1 0 0 0 1 99



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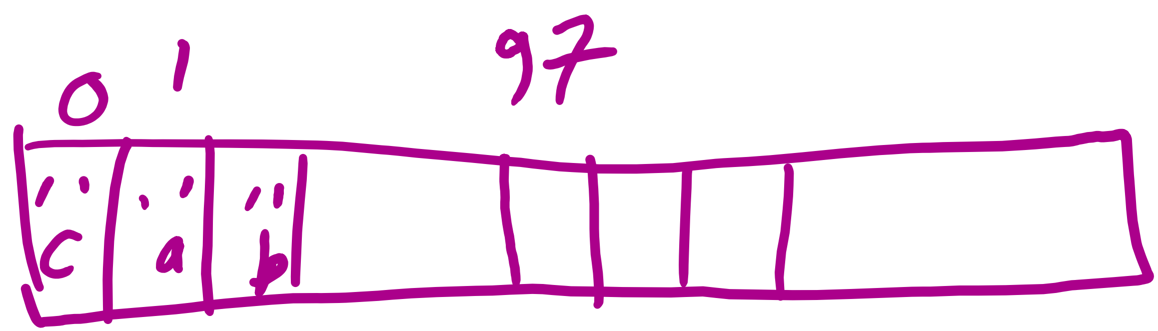

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- Example:

a b b b a a b b b b a c c **c a b b a a a b c**

97 98 0 0 1 0 1 0 0 0 1 99 0 1 2 0 1 0 0 1 2

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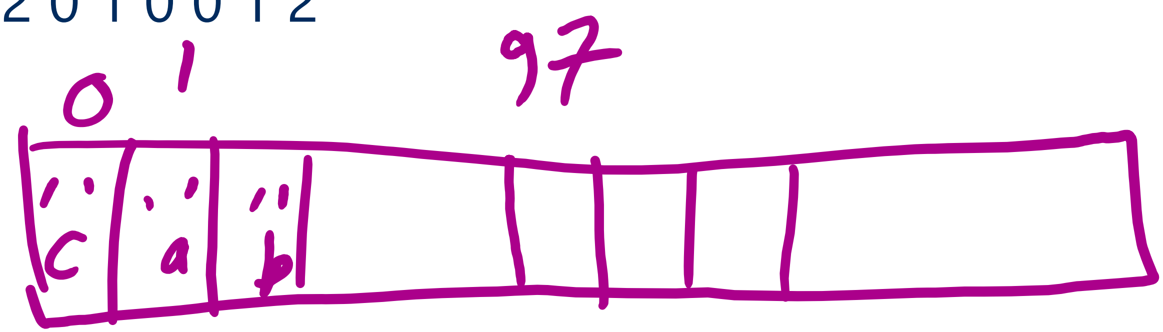


Move-To-Front Encoding

In the output of MTF Encoding, smaller integers have higher frequencies than larger integers

Move-To-Front Decoding

- Initialize an ordered list of 256 characters
  - same as encoding
- For each integer i (i is between 0 and 255)
  - print the ith character in the list
  - move that character to the front of the list

Burrows-Wheeler Transform

- Rearranges the characters in the input
  - lots of clusters with repeated characters
  - still possible to recover the original input
- Intuition: Consider **hen** in English text
  - most of the time the letter preceding it is t or w
  - group all such preceding letters together (mostly t's and some w’s)

Burrows-Wheeler Transform

- For each block of length N
  - generate **N strings** by cycling the characters of the block one step at a time
  - sort the strings
  - output is the last column in the sorted table and the index of the original block in the sorted array

Burrows-Wheeler Transform

- Example: Let’s transform “ABRACADABRA”

- N = 11

- Cyclic Versions of the string:
  - ABRACADABRA
  - BRACADABRAA
  - RACADABRAAB
  - ACADABRAABR
  - CADABRAABRA
  - ADABRAABRAC
  - DABRAABRACA
  - ABRAABRACAD
  - BRAABRACADA
  - RAABRACADAB
  - AABRACADABR

- After Sorting
  - AABRACADABR
  - ABRAABRACAD
  - ABRACADABRA
  - ACADABRAABR
  - ADABRAABRAC
  - BRAABRACADA
  - BRACADABRAA
  - CADABRAABRA
  - DABRAABRACA
  - RAABRACADAB
  - RACADABRAAB

2

RDARCAAAABB

Downsides of Burrows-Wheeler Algorithm

- Have to process blocks of input file
  - Compare to LZW, which processes the input one character at time
- The larger the block size, the better the compression
  - But, the longer the sorting time

Repetitive Minimum Problem

- Input:
  - a (large) dynamic set of data items
- Output:
  - repeatedly find a minimum item
- You are implementing an algorithm that repeats this problem
  - examples of such an algorithm?
    - Selection sort and Huffman tree construction
- What we cover today applies to the repetitive maximum problem as well

Let’s create an ADT!

- The Priority Queue ADT

- Let’s generalize min and max to highest **priority**
- Primary operations of the PQ:
  - Insert
  - Find item with highest priority
    - e.g., findMin() or findMax()
  - Remove an item with highest priority
    - e.g., removeMin() or removeMax()
- We mentioned priority queues in building Huffman tries
- How do we implement these operations?
  - Simplest approach: arrays

Unsorted array PQ

- Insert:
  - Add new item to the end of the array
  - $\Theta(1)$
- Find:
  - Search for the highest priority item (e.g., min or max)
  - $\Theta(n)$
- Remove:
  - Search for the highest priority item and delete
  - $\Theta(n)$

Sorted array PQ

- Insert:
  - Add new item in appropriate sorted order
  - $\Theta(n)$
- Find:
  - Return the item at the end of the array
  - $\Theta(1)$
- Remove:
  - Return and delete the item at the end of the array
  - $\Theta(1)$

So what other options do we have?

- What about a balanced binary search tree?
  - Insert
    - $\Theta(\lg n)$
  - Find
    - $\Theta(\lg n)$
  - Remove
    - $\Theta(\lg n)$
- OK, all operations are $\Theta(\lg n)$
  - No constant time operations

Which implementation should we choose?

- Depends on the application
- We can compare the *amortized runtime* of each implementation
- Given a set of operations performed by the application:

$$\text{Amortized runtime} = \frac{\text{Total runtime of a sequence of operations}}{\#\text{operations}}$$

Example: Huffman Trie Construction

- K-1 iterations
  - K is the # unique characters in the file to be compressed
- Each iteration:
  - 2 removeMin calls
  - 1 insert call
- Unsorted Array: Total time Huffman Trie Construction =(K-1)*[2 * K + 1 * 1] = $O(K^2)$
- Sorted Array: Total time Huffman Trie Construction =(K-1)*[2 * 1 + 1 * K] = $O(K^2)$
- Balanced BST: Total time Huffman Trie Construction =(K-1)*[2 * log K + 1 * log K] = O(K log K)

Is a BST overkill?

- Our find and remove operations only need the highest priority item, not to find/remove *any* item
    - Can we take advantage of this to improve our runtime?
        - Yes!

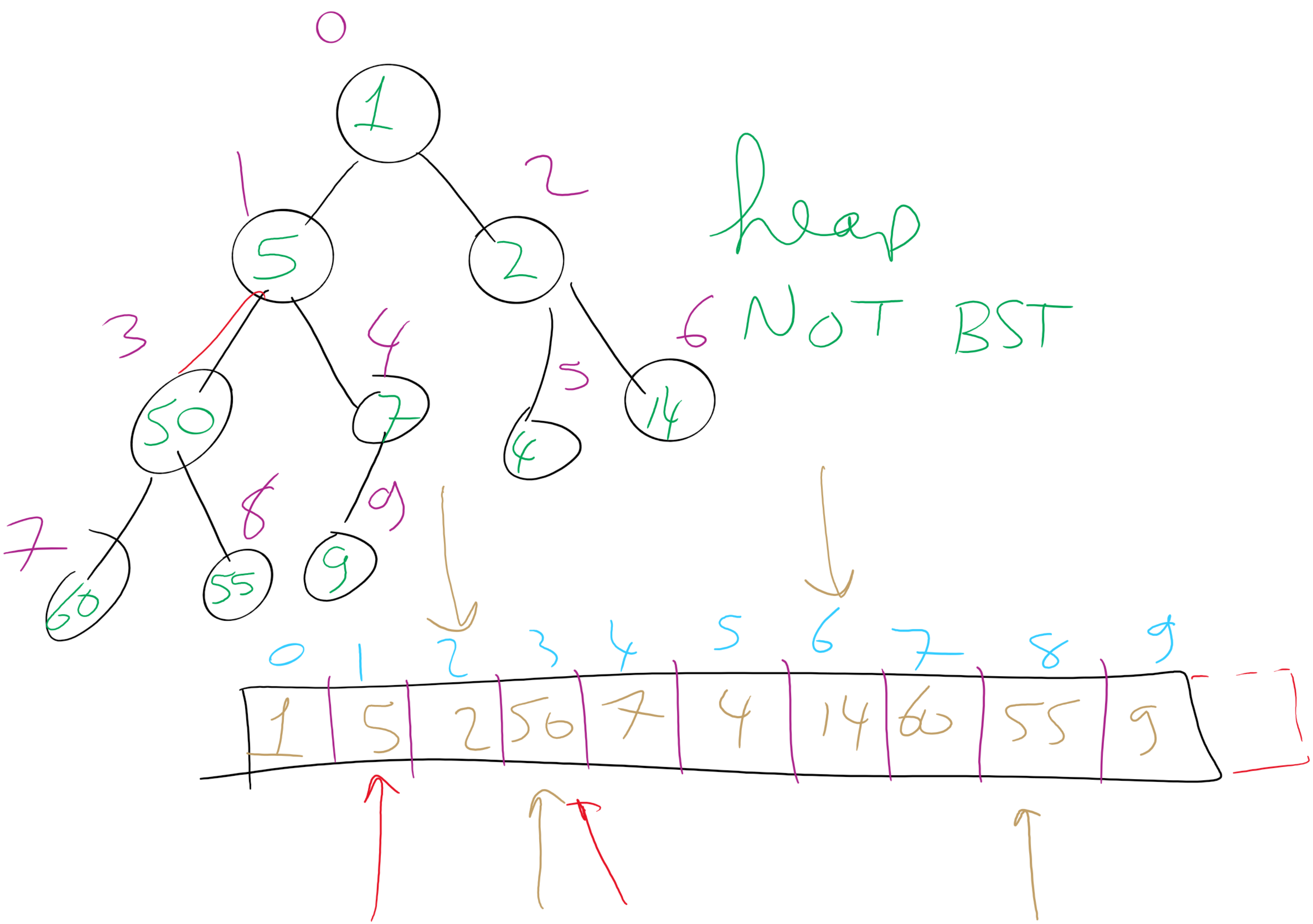
The heap

- A heap is complete binary tree such that for each node T in the tree:
  - T.item is of a higher priority than T.right_child.item
  - T.item is of a higher priority than T.left_child.item

- It does not matter how T.left_child.item relates to T.right_child.item
  - This is a relaxation of the approach needed by a BST

The *heap property*

Heap Example



Heap PQ runtimes

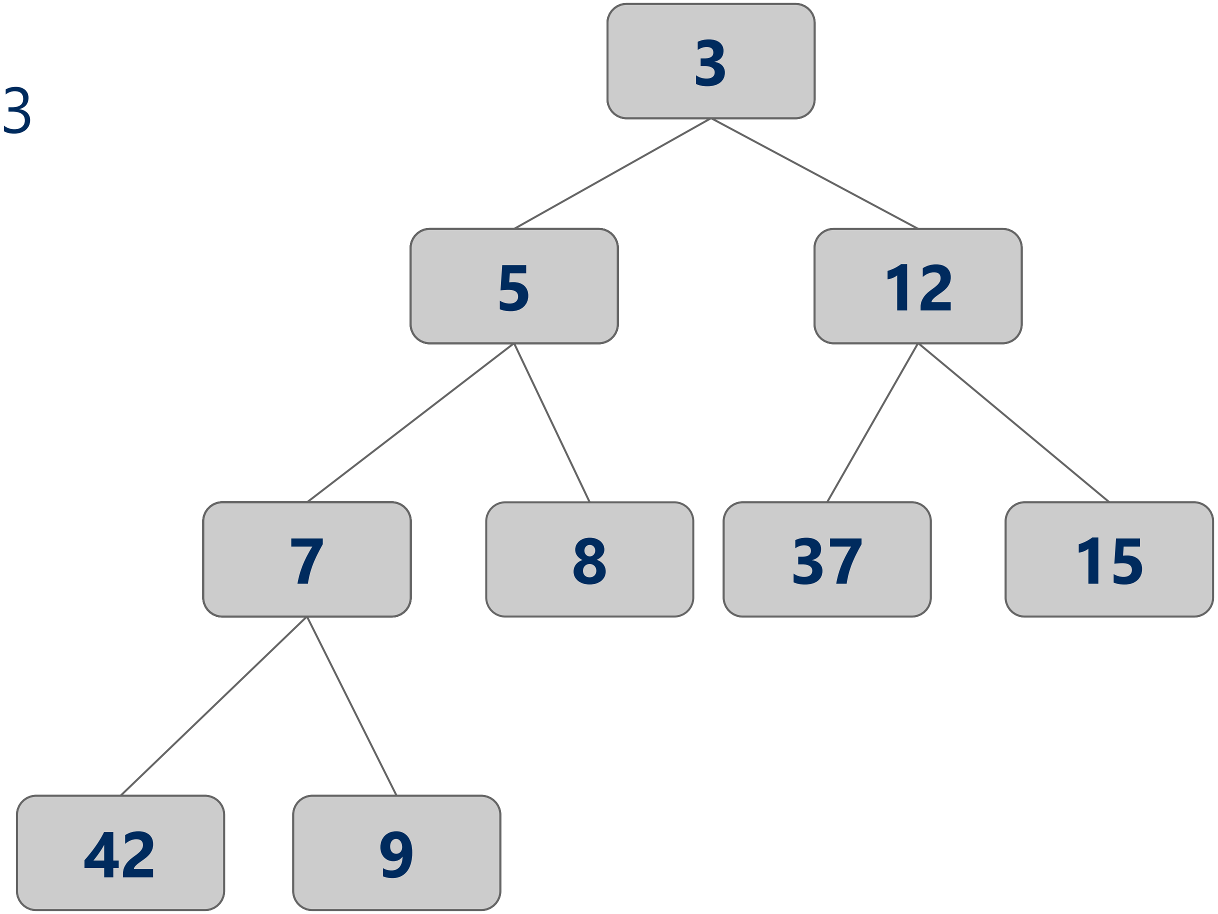
- Find is easy
  - Simply the root of the tree
    - $\Theta(1)$
- Remove and insert are not quite so trivial
  - The tree is modified and the heap property must be maintained

Heap insert

- Add a new node at the next available leaf
- Push the new node up the tree until it is supporting the heap property

Min heap insert

Insert:
7, 42, 37, 5, 8, 15, 12, 9, 3



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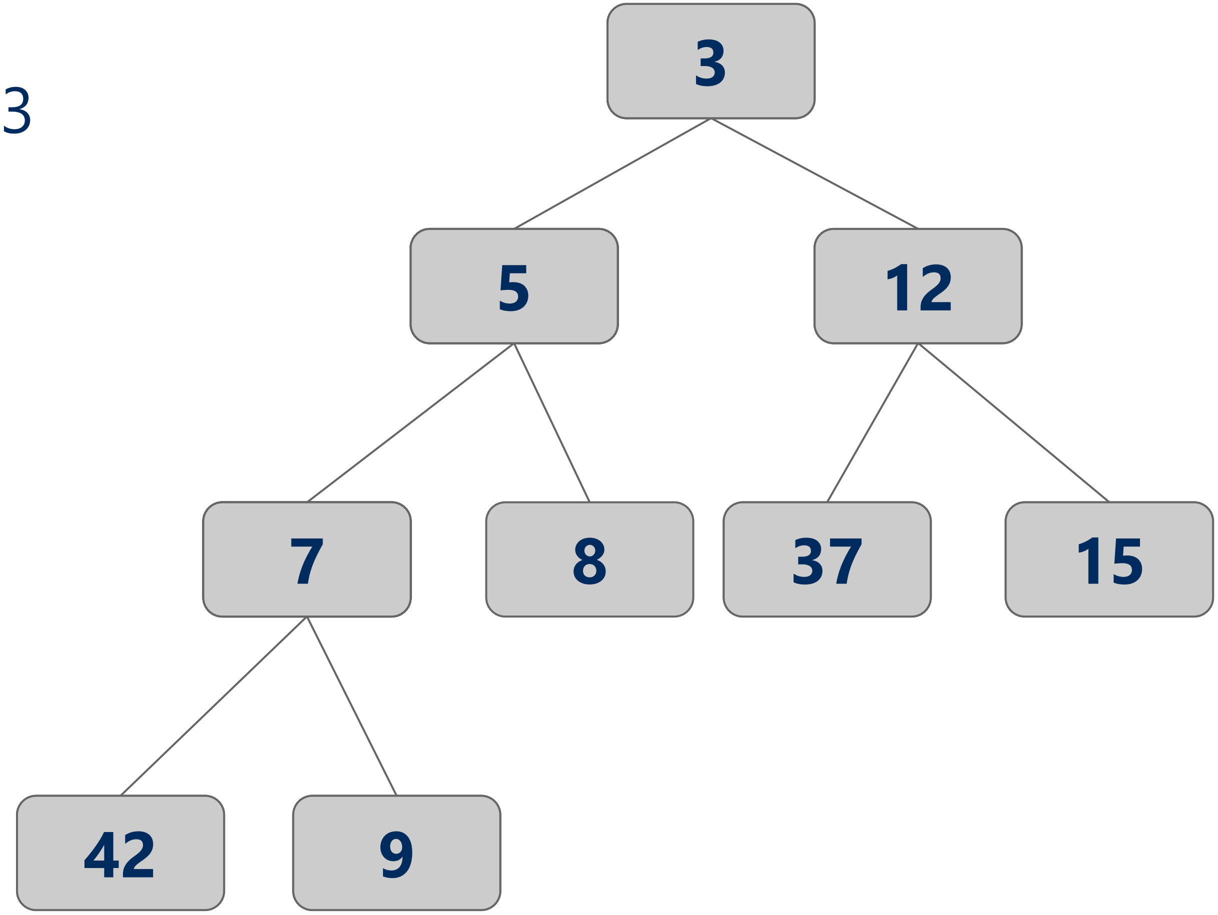
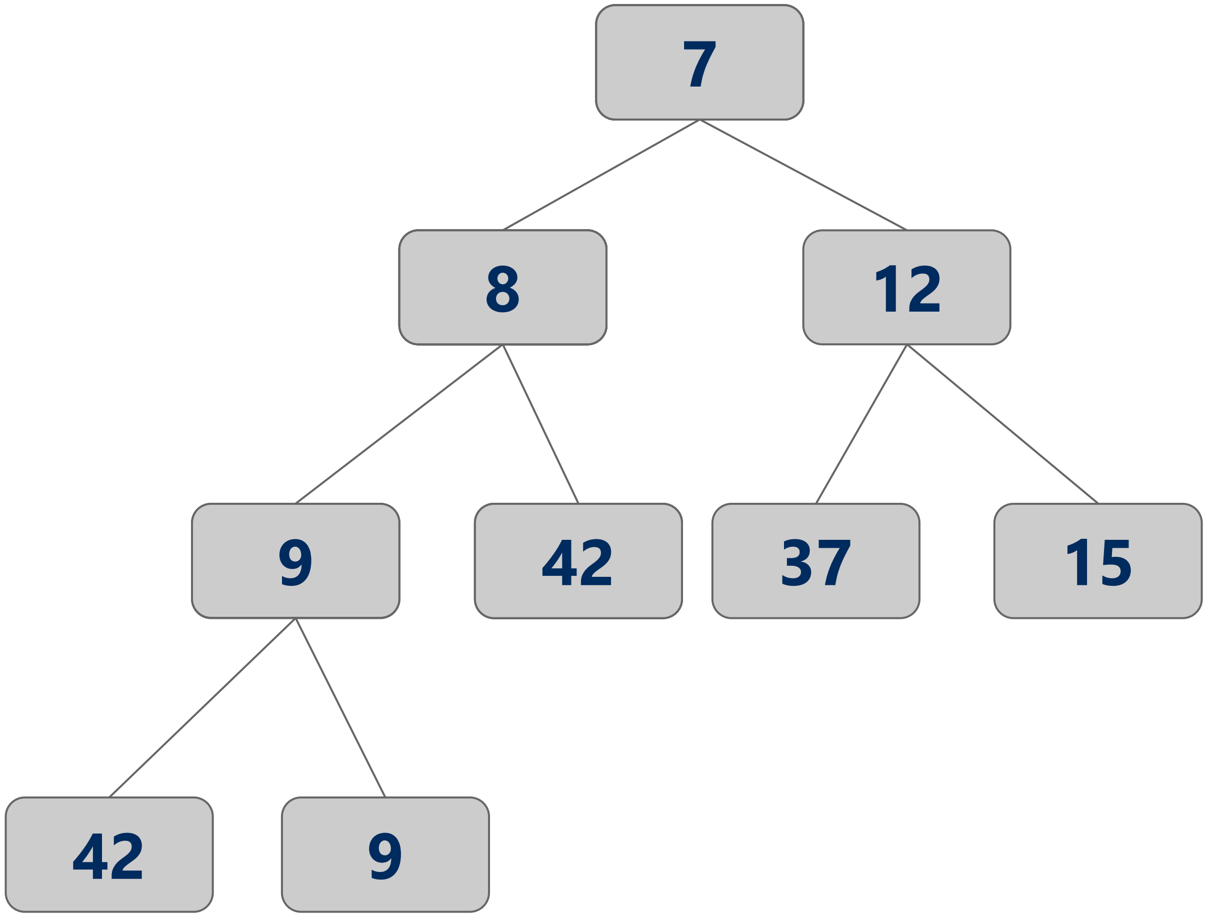


Heap remove

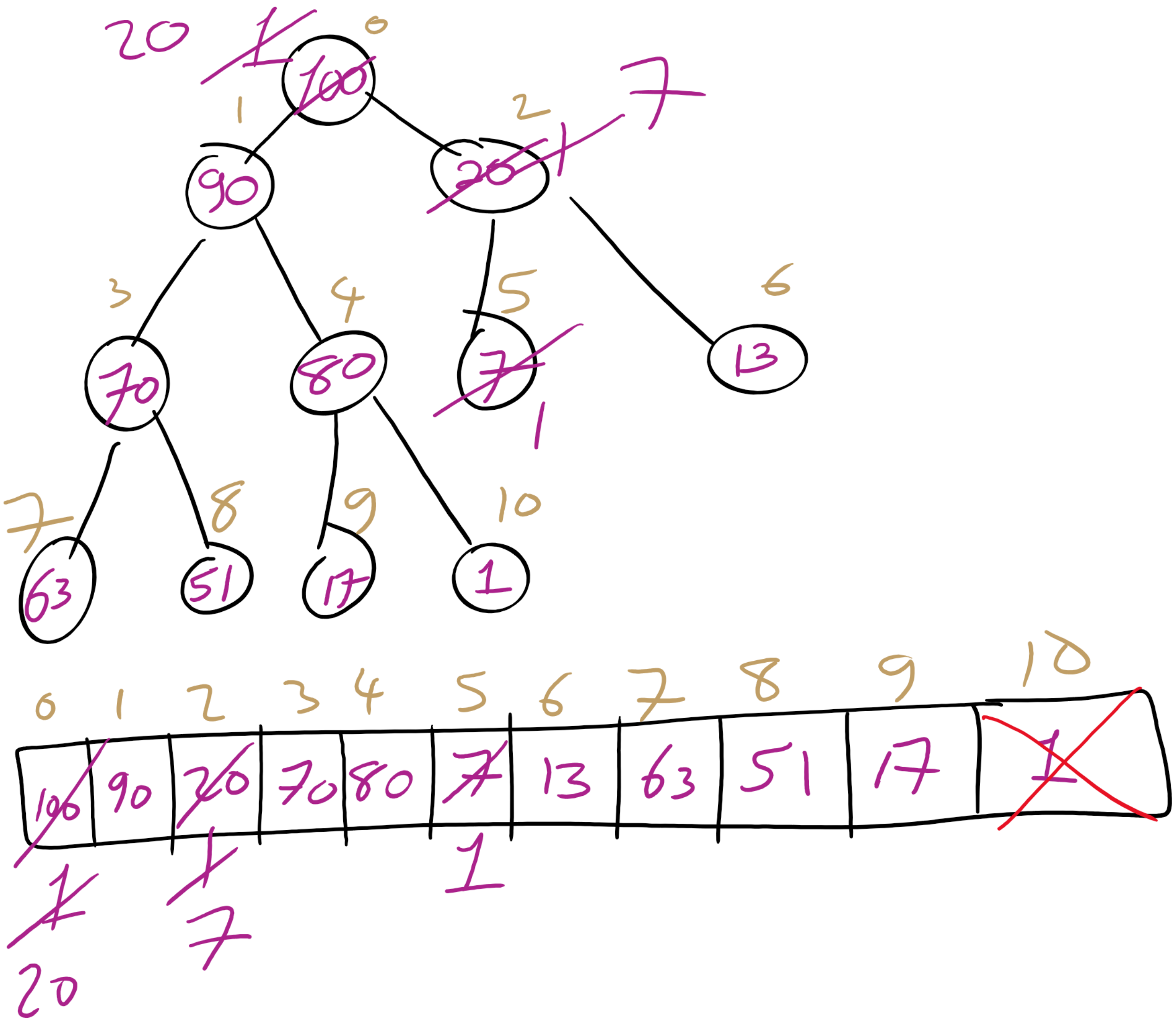
- Tricky to delete root…
  - So let's simply overwrite the root with the item from the last leaf and delete the last leaf
    - But then the root is violating the heap property…
      - So we push the root down the tree until it is supporting the heap property

Min heap removal

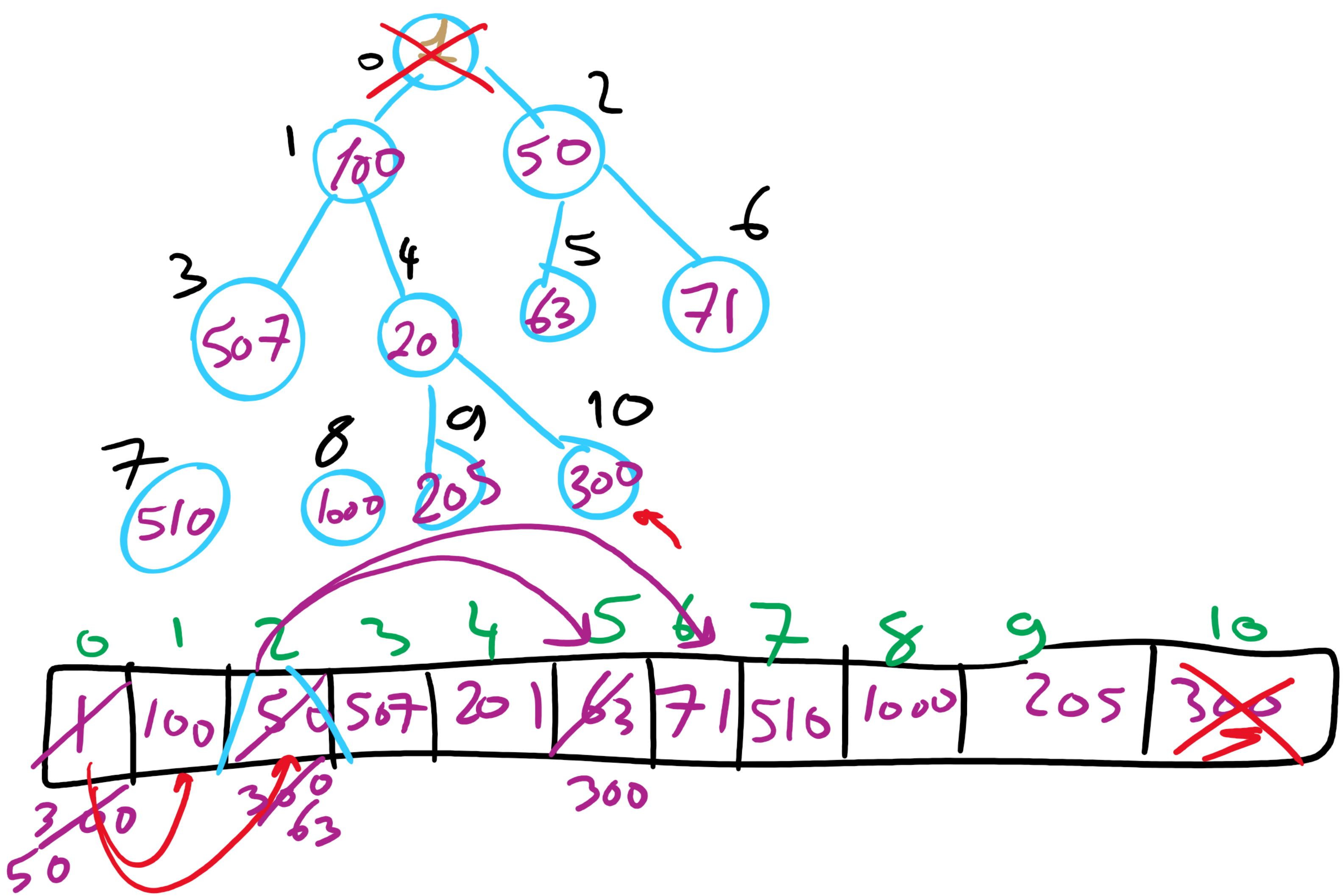
NO!



Heap removeMax Example



Heap removeMin Example



Heap runtimes

- Find
  - $\Theta(1)$
- Insert and remove
  - Height of a complete binary tree is lg n
  - At most, upheap and downheap operations traverse the height of the tree
  - Hence, insert and remove are $\Theta(\lg n)$

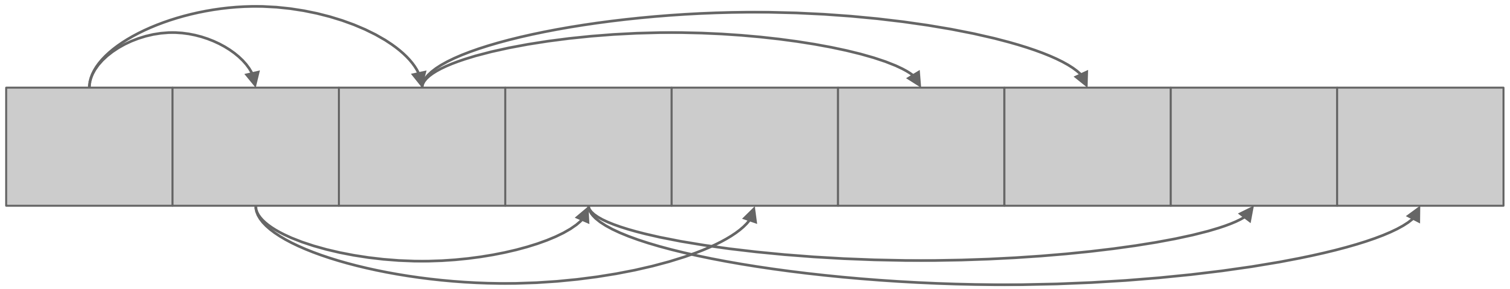
Heap implementation

- Simply implement tree nodes like for BST
  - This requires overhead for dynamic node allocation
  - Also must follow chains of parent/child relations to traverse the tree
- Note that a heap will be a complete binary tree...
  - We can easily represent a complete binary tree using an array

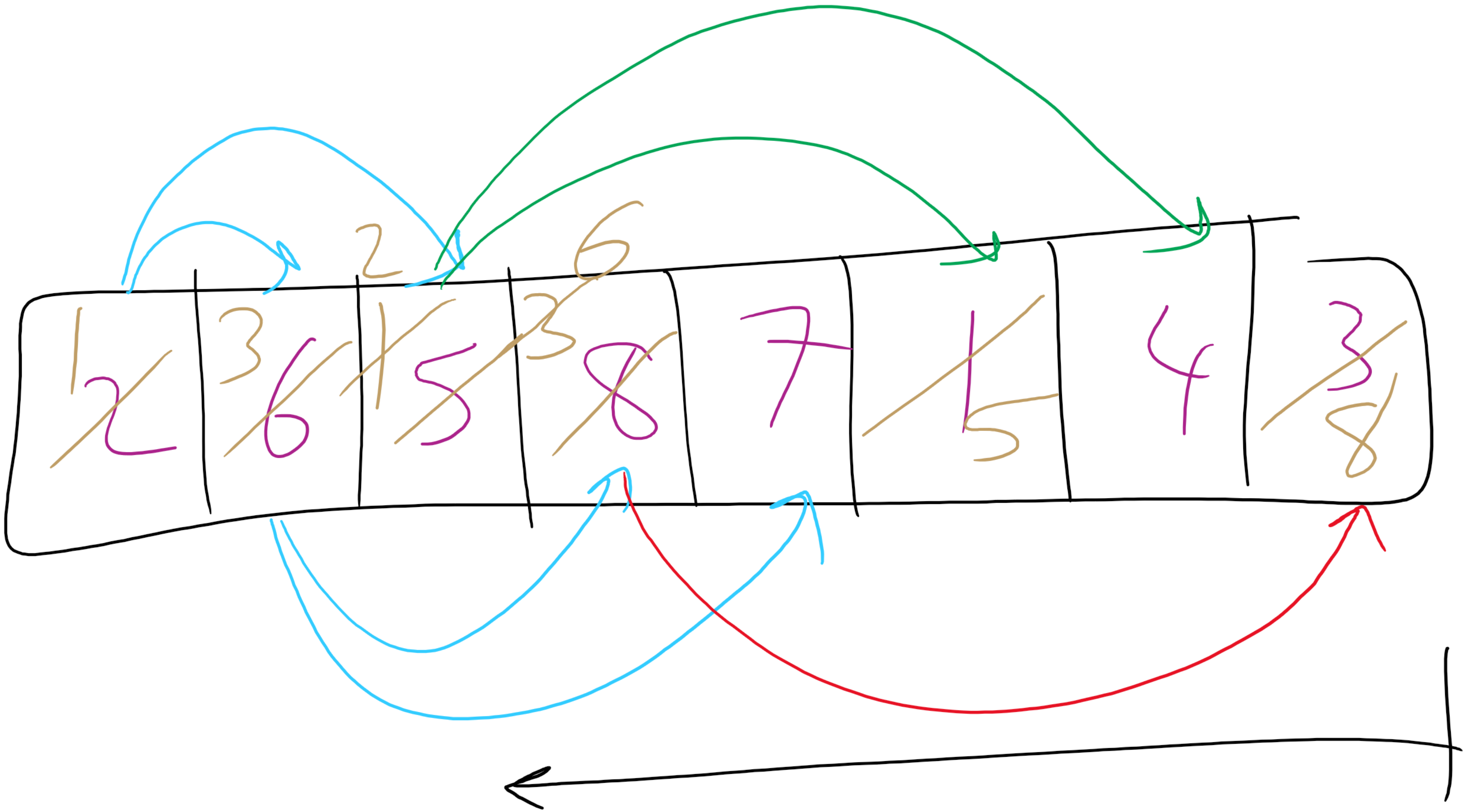
Storing a heap in an array

- Number nodes row-wise starting at 0
- Use these numbers as indices in the array
- Now, for node at index i
  - parent(i) = $\lfloor (i - 1) / 2 \rfloor$
  - left_child(i) = 2i + 1
  - right_child(i) = 2i + 2

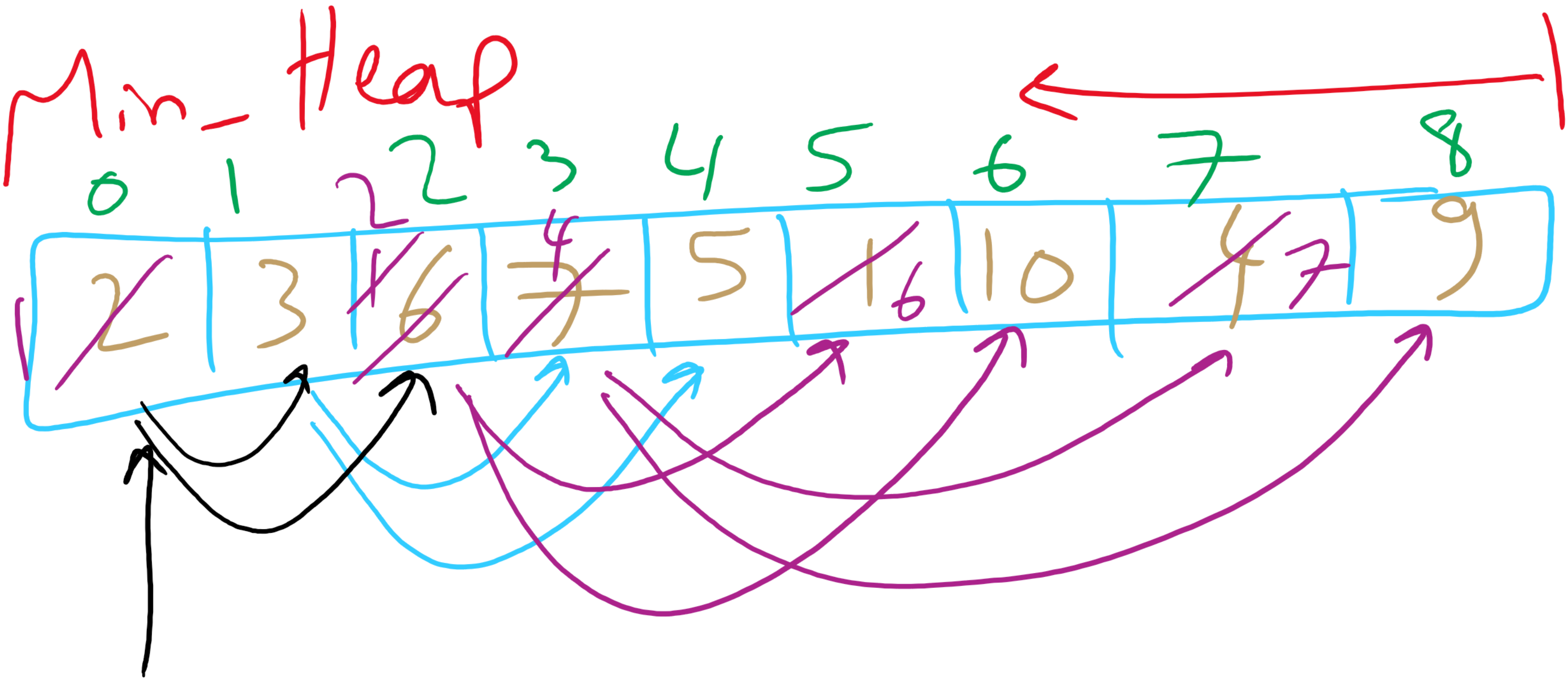
For arrays indexed from 0



Heapify Operation

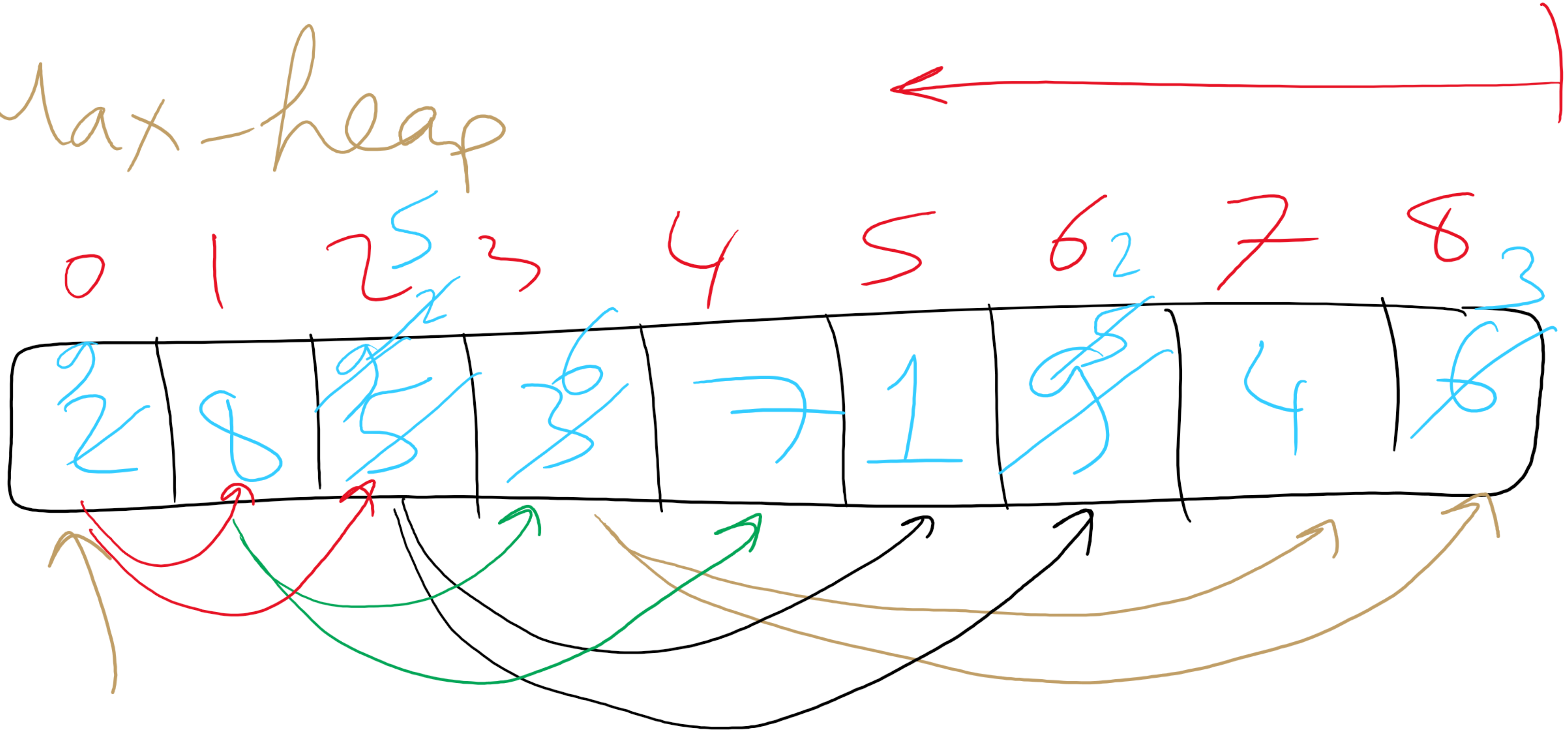


Heapify Example



Heapify Example

Max-heap



HeapSort Pseudo-code

HeapSort(Array a)

- Heapify(a) — $\Theta(n)$

ascending: MAX_HEAP

descending: MIN_HEAP

end = n − 1

for i = 0 … n − 1 (n)

- remove highest priority item by swapping with the end of the array — log n
- end--

Heap Sort

- Heapify the numbers
  - MAX heap to sort ascending
  - MIN heap to sort descending
- "Remove" the root
  - Don’t actually delete the leaf node
- Consider the heap to be from 0 .. length - 1
- Repeat

Heap sort analysis

- Runtime:
  - Worst case:
    - n log n
- In-place?
  - Yes
- Stable?
  - No

Storing Objects in PQ

- What if we want to update an Object?
  - What is the runtime to find an arbitrary item in a heap?
    - $\Theta(n)$
    - Hence, updating an item in the heap is $\Theta(n)$
  - Can we improve of this?
    - Back the PQ with something other than a heap?
    - Develop a clever workaround?

Indirection

- Maintain a second data structure that maps item IDs to each item’s current position in the heap
- This creates an *indexable* PQ

Indirection example setup

- Let's say I'm shopping for a new video card and want to build a heap to help me keep track of the lowest price available from different stores.
- Keep objects of the following type in the heap:

```
class CardPrice implements Comparable<CardPrice>{
    public String store;
    public double price;
    public CardPrice(String s, double p) { … }
    public int compareTo(CardPrice o) {
        if (price < o.price) { return -1; }
        else if (price > o.price) { return 1; }
        else { return 0; }
    }
}
```

Indirection example

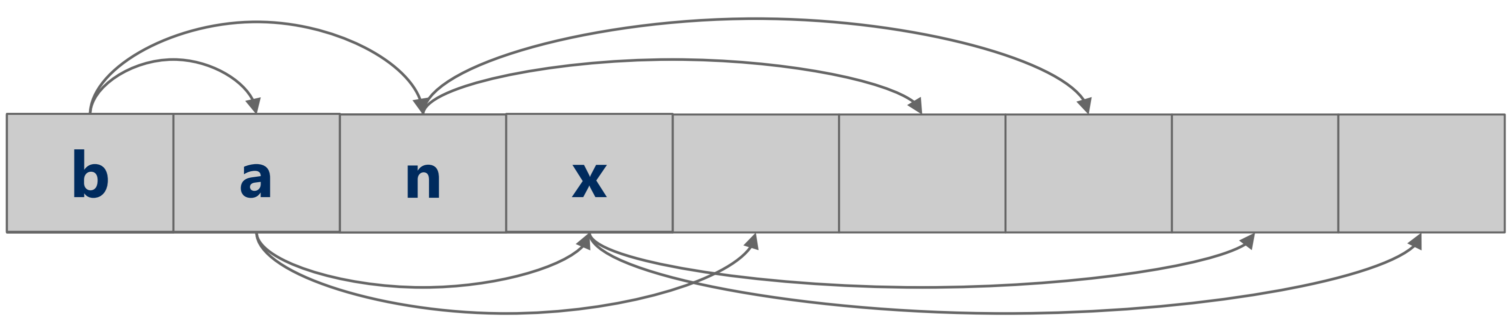
- n = new CardPrice("NE", 333.98);
- a = new CardPrice("AMZN", 339.99);
- x = new CardPrice("NCIX", 338.00);
- b = new CardPrice("BB", 349.99);

- Update price for NE: 340.00

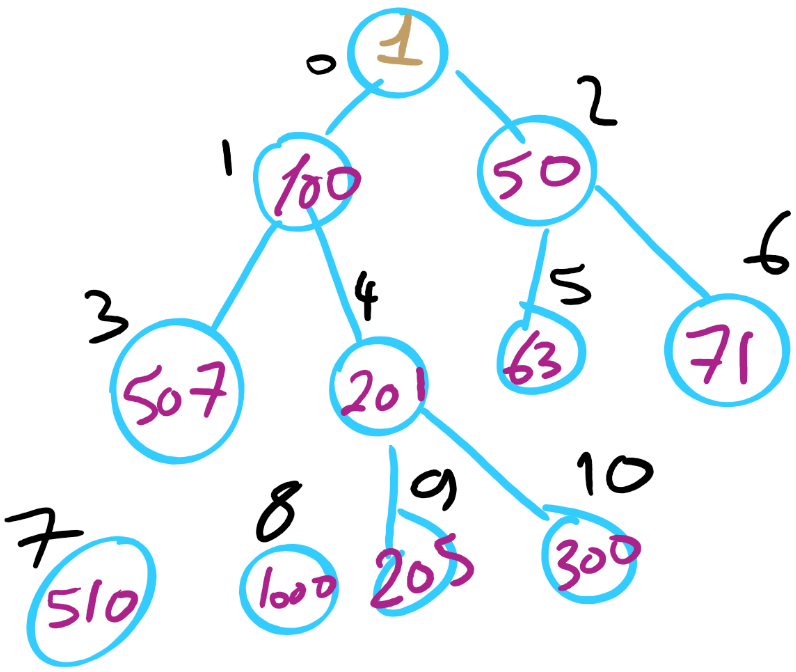
- Update price for NCIX: 345.00

- Update price for BB: 200.00

Indirection
"NE":2
"AMZN":1
"NCIX":3
"BB":0



Indexable PQ Example



Indexable PQ Example