Lista sem gabarito

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1 Questão 1

(a)

A matriz A é similar a matriz B se, e somente se, existe uma matriz inversível tal que:

$$A = P \cdot B \cdot P^{-1}$$

Pelo enunciado, sabemos que:

$$A = Q \cdot R \tag{1}$$

$$B = R \cdot Q \tag{2}$$

Utilizando a equação (2):

$$B = R \cdot Q$$

Como Q é ortogonal, então $Q^{-1} = Q^T$, assim Q^{-1} existe:

$$B \cdot Q^{-1} = R \cdot Q \cdot Q^{-1}$$

$$B \cdot Q^{-1} = R \tag{3}$$

Substituindo (3) em (1):

$$A = Q \cdot B \cdot Q^{-1}$$

Dessa forma, A e B são similares.

(b)

O método utilizado para a decomposição QR foi o de Gram-Schmidt.
Sabendo que a matriz A composta por $R \cdot Q$, onde A, Q e R são da forma:

$$A = \begin{pmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_4 \\ | & | & | & | \\ | & | & | & | \end{pmatrix}$$

$$Q = \begin{pmatrix} | & | & | & | \\ q_1 & q_2 & q_3 & q_4 \\ | & | & | & | \\ | & | & | & | \end{pmatrix}$$

$$R = \begin{pmatrix} q_1^T \cdot a_1 & q_1^T \cdot a_2 & q_1^T \cdot a_3 & q_1^T \cdot a_4 \\ 0 & q_2^T \cdot a_2 & q_2^T \cdot a_3 & q_2^T \cdot a_4 \\ 0 & 0 & q_3^T \cdot a_3 & q_3^T \cdot a_4 \\ 0 & 0 & 0 & q_4^T \cdot a_4 \end{pmatrix}$$

Considerando que cada coluna da matriz A é um vetor com nome a_1, a_2, a_3, a_4 e cada coluna da matriz Q serão os 4 vetores ortonormais encontrados em cada iteração (q_1, q_2, q_3, q_4) de modo que na primeira iteração teremos os dados fornecidos pelo professor:

$$a_1 = \begin{pmatrix} 0.7140299 \\ 0.6918437 \\ 0.6748934 \\ 0.7884872 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} 0.6918437 \\ 1.2492256 \\ 1.0307285 \\ 0.6275049 \end{pmatrix}$$

$$a_3 = \begin{pmatrix} 0.6748934 \\ 1.0307285 \\ 1.0893471 \\ 0.5567456 \end{pmatrix}$$

$$a_4 = \begin{pmatrix} 0.7884872 \\ 0.6275049 \\ 0.5567456 \\ 0.9550583 \end{pmatrix}$$

Sabendo que a norma de um vetor $u \in R^n$ é o número real $\|u\|_2 = \sqrt{<u, u>}$. Então, segue-se o seguinte algoritmo:

1. $q_1 = \frac{a_1}{\|a_1\|_2}$

2. $q_2' = \left(I - q_1 \cdot q_1^T\right) \cdot a_2 = a_2 - q_1 \cdot q_1^T \cdot a_2$

3. $q_2 = \frac{q_2'}{\|q_2'\|_2}$

4. $q_3' = \left(I - q_1 \cdot q_1^T - q_2 \cdot q_2^T\right) \cdot a_3 = a_3 - q_1 \cdot q_1^T \cdot a_3 - q_2 \cdot q_2^T \cdot a_3$

5. $q_3 = \frac{q_3'}{\|q_3'\|_2}$

6. $q_4' = \left(I - q_1 \cdot q_1^T - q_2 \cdot q_2^T - q_3 \cdot q_3^T\right) \cdot a_4 = a_4 - q_1 \cdot q_1^T \cdot a_4 - q_2 \cdot q_2^T \cdot a_4 - q_3 \cdot q_3^T \cdot a_4$

7. $q_4 = \frac{q_4'}{\|q_4'\|_2}$

8. $R = \begin{pmatrix} q_1^T \cdot a_1 & q_1^T \cdot a_2 & q_1^T \cdot a_3 & q_1^T \cdot a_4 \\ 0 & q_2^T \cdot a_2 & q_2^T \cdot a_3 & q_2^T \cdot a_4 \\ 0 & 0 & q_3^T \cdot a_3 & q_3^T \cdot a_4 \\ 0 & 0 & 0 & q_4^T \cdot a_4 \end{pmatrix}$

9. $Q = \begin{pmatrix} | & | & | & | \\ q_1 & q_2 & q_3 & q_4 \\ | & | & | & | \\ | & | & | & | \end{pmatrix}$

10. $A = R \cdot Q$

Na próxima iteração, as matrizes-colunas de A (a_1, a_2, a_3, a_4) mudarão, de forma que cada matriz-coluna de A será igual a cada matriz-coluna de $R \cdot Q$.

Fez-se várias vezes as iterações até que na quinta iteração, os valores na diagonal principal da matriz $R \cdot Q$ ficarão iguais aos da quarta iteração.

1° Iteração:

```
a1  =   [[0.7140299]
  [0.6918437]
  [0.6748934]
  [0.7884872]]
a2  =   [[0.6918437]
  [1.2492256]
  [1.0307285]
  [0.6275049]]
a3  =   [[0.6748934]
  [1.0307285]
  [1.0893471]
  [0.5567456]]
a4  =   [[0.7884872]
  [0.6275049]
  [0.5567456]
  [0.9550583]]
q1  =   [[0.49680381]
  [0.48136722]
  [0.46957363]
  [0.54860931]]
q2' =   [[-0.18914089]
  [ 0.3956148 ]
  [ 0.19803132]
  [-0.34534662]]
q2  =   [[-0.31935716]
  [ 0.66798046]
  [ 0.33436832]
  [-0.58310456]]
q3' =   [[ 0.01965057]
  [-0.10515473]
  [ 0.14390639]
  [-0.0487033 ]]
q3  =   [[ 0.10575722]
  [-0.5659313 ]
  [ 0.77448851]
  [-0.26211586]]
q4' =   [[-1.71294927e-03]
  [-9.10805482e-05]
  [ 5.57797235e-04]
  [ 1.15367355e-03]]
q4  =   [[-0.80000774]
  [-0.04253783]
  [ 0.26051099]
  [ 0.53880625]]
```

```
45  R =  [[ 1.43724722  1.7733048    1.64841302   1.47917066]
46   [ 0.           0.59225504   0.51257672  -0.20338878]
47   [ 0.           0.           0.1858083   -0.0908793 ]
48   [ 0.           0.           0.           0.00214117]]
49  Q =  [[ 0.49680381 -0.31935716   0.10575722  -0.80000774]
50   [ 0.48136722   0.66798046  -0.5659313   -0.04253783]
51   [ 0.46957363   0.33436832   0.77448851   0.26051099]
52   [ 0.54860931  -0.58310456  -0.26211586   0.53880625]]
53  A = RQ =  [[ 3.15317879e+00  4.14203696e-01   3.73934457e-02   1.17466352e-03]
54   [ 4.14203696e-01   6.85601138e-01   1.15120544e-01  -1.24852359e-03]
55   [ 3.73934457e-02   1.15120544e-01   1.67727299e-01  -5.61233532e-04]
56   [ 1.17466352e-03  -1.24852359e-03  -5.61233533e-04   1.15367355e-03]]
```

2° Iteração:

```
1   a1 =  [[3.15317879e+00]
2    [4.14203696e-01]
3    [3.73934457e-02]
4    [1.17466352e-03]]
5   a2 =  [[ 0.4142037 ]
6    [ 0.68560114]
7    [ 0.11512054]
8    [-0.00124852]]
9   a3 =  [[ 0.03739345]
10   [ 0.11512054]
11   [ 0.1677273 ]
12   [-0.00056123]]
13  a4 =  [[ 0.00117466]
14   [-0.00124852]
15   [-0.00056123]
16   [ 0.00115367]]
17  q1 =  [[9.91413667e-01]
18   [1.30232769e-01]
19   [1.17571427e-02]
20   [3.69334423e-04]]
21  q2' =  [[-0.08278005]
22   [ 0.62031702]
23   [ 0.10922683]
24   [-0.00143367]]
25  q2 =  [[-0.1303052 ]
26   [ 0.97644952]
27   [ 0.17193545]
28   [-0.00225675]]
29  q3' =  [[ 0.00159135]
30   [-0.02508144]
31   [ 0.14364405]
32   [-0.00027342]]
33  q3 =  [[ 0.01091262]
34   [-0.17199536]
35   [ 0.98503553]
36   [-0.00187499]]
37  q4' =  [[-7.35338576e-07]
38   [ 2.10683582e-06]
```

```
39   [ 2.56381985e-06]
40   [ 1.14937149e-03]]
41  q4 =  [[-6.39771708e-04]
42   [ 1.83302494e-03]
43   [ 2.23061792e-03]
44   [ 9.99995628e-01]]
45  R =  [[ 3.18048751e+00  5.01287967e-01  5.40366269e-02  9.95806374e-04]
46   [ 0.00000000e+00  6.35278124e-01  1.36376375e-01 -1.47128452e-03]
47   [ 0.00000000e+00  0.00000000e+00  1.45826262e-01 -3.27439177e-04]
48   [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.14937651e-03]]
49  Q =  [[ 9.91413667e-01 -1.30305205e-01  1.09126206e-02 -6.39771708e-04]
50   [ 1.30232769e-01  9.76449518e-01 -1.71995361e-01  1.83302494e-03]
51   [ 1.17571427e-02  1.71935450e-01  9.85035530e-01  2.23061792e-03]
52   [ 3.69334423e-04 -2.25675407e-03 -1.87499273e-03  9.99995628e-01]]
53  A = RQ =  [[ 3.21909859e+00  8.43368823e-02  1.71437924e-03  4.24504242e-07]
54   [ 8.43368823e-02  6.43768272e-01  2.50734429e-02 -2.59386022e-06]
55   [ 1.71437924e-03  2.50734429e-02  1.43644663e-01 -2.15507257e-06]
56   [ 4.24504312e-07 -2.59386012e-06 -2.15507261e-06  1.14937149e-03]]
```

3° Iteração:

```
1   a1 =  [[3.21909859e+00]
2    [8.43368823e-02]
3    [1.71437924e-03]
4    [4.24504312e-07]]
5   a2 =  [[ 8.43368823e-02]
6    [ 6.43768272e-01]
7    [ 2.50734429e-02]
8    [-2.59386012e-06]]
9   a3 =  [[ 1.71437924e-03]
10   [ 2.50734429e-02]
11   [ 1.43644663e-01]
12   [-2.15507261e-06]]
13  a4 =  [[ 4.24504242e-07]
14   [-2.59386022e-06]
15   [-2.15507257e-06]
16   [ 1.14937149e-03]]
17  q1 =  [[9.99656843e-01]
18   [2.61899221e-02]
19   [5.32382246e-04]
20   [1.31825301e-07]]
21  q2' =  [[-1.68099277e-02]
22   [ 6.41118335e-01]
23   [ 2.50195756e-02]
24   [-2.60719841e-06]]
25  q2 =  [[-2.61907655e-02]
26   [ 9.98896621e-01]
27   [ 3.89818356e-02]
28   [-4.06215442e-06]]
29  q3' =  [[ 6.97307379e-05]
30   [-5.55728747e-03]
31   [ 1.42450500e-01]
32   [-2.03109159e-06]]
```

```
33  q3 =  [[ 4.89136416e-04]
34   [-3.89824023e-02]
35   [ 9.99239777e-01]
36   [-1.42473878e-05]]
37  q4' =  [[-2.65737072e-10]
38   [ 4.02144661e-09]
39   [ 1.65450142e-08]
40   [ 1.14937145e-03]]
41  q4 =  [[-2.31202082e-07]
42   [ 3.49882244e-06]
43   [ 1.43948366e-05]
44   [ 1.00000000e+00]]
45  R =  [[ 3.22020363e+00   1.01181531e-01   2.44693632e-03   3.55429767e-07]
46   [ 0.00000000e+00   6.41826513e-01   3.06004091e-02  -2.69079391e-06]
47   [ 0.00000000e+00   0.00000000e+00   1.42558877e-01  -2.06848723e-06]
48   [ 0.00000000e+00   0.00000000e+00   0.00000000e+00   1.14937145e-03]]
49  Q =  [[ 9.99656843e-01  -2.61907655e-02   4.89136416e-04  -2.31202082e-07]
50   [ 2.61899221e-02   9.98896621e-01  -3.89824023e-02   3.49882244e-06]
51   [ 5.32382246e-04   3.89818356e-02   9.99239777e-01   1.43948366e-05]
52   [ 1.31825301e-07  -4.06215442e-06  -1.42473878e-05   1.00000000e+00]]
53  A = RQ =  [[ 3.22174983e+00   1.68256775e-02   7.58958146e-05   1.51443706e-10]
54   [ 1.68256775e-02   6.42311196e-01   5.55720670e-03  -4.66901591e-09]
55   [ 7.58958146e-05   5.55720670e-03   1.42450500e-01  -1.63754932e-08]
56   [ 1.51516237e-10  -4.66892431e-09  -1.63755407e-08   1.14937145e-03]]
```

4° Iteração:

```
1   a1 =  [[3.22174983e+00]
2    [1.68256775e-02]
3    [7.58958146e-05]
4    [1.51516237e-10]]
5   a2 =  [[ 1.68256775e-02]
6    [ 6.42311196e-01]
7    [ 5.55720670e-03]
8    [-4.66892431e-09]]
9   a3 =  [[ 7.58958146e-05]
10   [ 5.55720670e-03]
11   [ 1.42450500e-01]
12   [-1.63755407e-08]]
13  a4 =  [[ 1.51443706e-10]
14   [-4.66901591e-09]
15   [-1.63754932e-08]
16   [ 1.14937145e-03]]
17  q1 =  [[9.99986363e-01]
18   [5.22245641e-03]
19   [2.35570058e-05]
20   [4.70285337e-11]]
21  q2' =  [[-3.35406847e-03]
22   [ 6.42205806e-01]
23   [ 5.55673132e-03]
24   [-4.66987335e-09]]
25  q2 =  [[-5.22246474e-03]
26   [ 9.99948932e-01]
```

```
27   [ 8.65212910e-03]
28   [-7.27124362e-09]]
29  q3' =  [[ 3.07998696e-06]
30   [-1.23203871e-03]
31   [ 1.42391758e-01]
32   [-1.63261811e-08]]
33  q3 =   [[ 2.16295642e-05]
34   [-8.65213413e-03]
35   [ 9.99962569e-01]
36   [-1.14652493e-07]]
37  q4' =   [[-9.48482264e-14]
38   [ 7.21648718e-12]
39   [ 1.31845677e-10]
40   [ 1.14937145e-03]]
41  q4 =   [[-8.25218223e-11]
42   [ 6.27863794e-09]
43   [ 1.14711112e-07]
44   [ 1.00000000e+00]]
45  R =  [[ 3.22179377e+00   2.01800212e-02   1.08272756e-04   1.26726204e-10]
46   [ 0.00000000e+00   6.42238604e-01   6.78902666e-03  -4.81960862e-09]
47   [ 0.00000000e+00   0.00000000e+00   1.42397088e-01  -1.64662583e-08]
48   [ 0.00000000e+00   0.00000000e+00   0.00000000e+00   1.14937145e-03]]
49  Q =   [[ 9.99986363e-01  -5.22246474e-03   2.16295642e-05  -8.25218223e-11]
50   [ 5.22245641e-03   9.99948932e-01  -8.65213413e-03   6.27863794e-09]
51   [ 2.35570058e-05   8.65212910e-03   9.99962569e-01   1.14711112e-07]
52   [ 4.70285337e-11  -7.27124362e-09  -1.14652493e-07   1.00000000e+00]]
53  A = RQ =   [[ 3.22185522e+00   3.35422304e-03   3.35444903e-06  -1.89539327e-14]
54   [ 3.35422304e-03   6.42264546e-01   1.23203799e-03  -8.44816195e-12]
55   [ 3.35444903e-06   1.23203799e-03   1.42391758e-01  -1.31730023e-10]
56   [ 5.40532539e-14  -8.35735981e-12  -1.31778302e-10   1.14937145e-03]]
```

5° Iteração:

```
1   a1 =   [[3.22185522e+00]
2    [3.35422304e-03]
3    [3.35444903e-06]
4    [5.40532539e-14]]
5   a2 =   [[ 3.35422304e-03]
6    [ 6.42264546e-01]
7    [ 1.23203799e-03]
8    [-8.35735981e-12]]
9   a3 =   [[ 3.35444903e-06]
10   [ 1.23203799e-03]
11   [ 1.42391758e-01]
12   [-1.31778302e-10]]
13  a4 =   [[-1.89539327e-14]
14   [-8.44816195e-12]
15   [-1.31730023e-10]
16   [ 1.14937145e-03]]
17  q1 =   [[9.99999458e-01]
18   [1.04108378e-03]
19   [1.04115392e-06]
20   [1.67770495e-14]]
```

```
21  q2' =   [[-6.68648487e-04]
22   [ 6.42260358e-01]
23   [ 1.23203380e-03]
24   [-8.35742730e-12]]
25  q2 =   [[-1.04108386e-03]
26   [ 9.99997618e-01]
27   [ 1.91827325e-03]
28   [-1.30124914e-11]]
29  q3' =   [[ 1.36114290e-07]
30   [-2.73141271e-04]
31   [ 1.42388871e-01]
32   [-1.31758716e-10]]
33  q3 =   [[ 9.55931730e-07]
34   [-1.91827330e-03]
35   [ 9.99998160e-01]
36   [-9.25342499e-10]]
37  q4' =   [[-3.38370023e-17]
38   [ 1.29159273e-14]
39   [ 1.06358898e-12]
40   [ 1.14937145e-03]]
41  q4 =   [[-2.94395710e-14]
42   [ 1.12373832e-11]
43   [ 9.25365758e-10]
44   [ 1.00000000e+00]]
45  R =   [[ 3.22185697e+00   4.02287371e-03   4.78535372e-06  -2.78670350e-14]
46   [ 0.00000000e+00   6.42261887e-01   1.50517786e-03  -8.71577246e-12]
47   [ 0.00000000e+00   0.00000000e+00   1.42389133e-01  -1.32777137e-10]
48   [ 0.00000000e+00   0.00000000e+00   0.00000000e+00   1.14937145e-03]]
49  Q =   [[ 9.99999458e-01  -1.04108386e-03   9.55931730e-07  -2.94395710e-14]
50   [ 1.04108378e-03   9.99997618e-01  -1.91827330e-03   1.12373832e-11]
51   [ 1.04115392e-06   1.91827325e-03   9.99998160e-01   9.25365758e-10]
52   [ 1.67770495e-14  -1.30124914e-11  -9.25342499e-10   1.00000000e+00]]
53  A = RQ =   [[ 3.22185941e+00   6.68650002e-04   1.48249003e-07  -7.30823464e-14]
54   [ 6.68650002e-04   6.42263245e-01   2.73141265e-04  -1.05589498e-13]
55   [ 1.48249004e-07   2.73141265e-04   1.42388871e-01  -1.01510922e-12]
56   [ 1.92830617e-17  -1.49561861e-14  -1.06356225e-12   1.14937145e-03]]
```

Somente na 5° Iteração os valores da diagonal principal de $R \cdot Q$ são iguais aos da 4° Iteração com aproximação de 4 casas decimais e a matriz $R \cdot Q$ se tornou uma triangular superior, uma vez que todos os valores abaixo da diagonal principal são próximos de zero em 4 casas decimais. De forma que os autovalores da matriz A são revelados, arredondando em 4 casas decimais: 3.2219, 0.6423, 0.1424, 0.0011.