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EXACT SOLU (OBLIG 1 oppg. 1.2.3)

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$$

Skal vise at dette tilfredsstiller bølgeligningen:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Løsn.:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-i\omega e^{i(k_x x + k_y y - \omega t)}\right)$$

$$= -\omega^2 e^{i(k_x x + k_y y - \omega t)}$$

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$

$$= \frac{\partial}{\partial x}\left(ik_x e^{i(k_x x + k_y y - \omega t)}\right)$$

$$+ \frac{\partial}{\partial y}\left(ik_y e^{i(k_x x + k_y y - \omega t)}\right)$$

$$= -k_x^2 e^{i(k_x x + k_y y - \omega t)} - k_y^2 e^{i(k_x x + k_y y - \omega t)}$$

$\Rightarrow$

$$-\omega^2 e^{i(k_x x + k_y y - \omega t)} = c^2(-k_x^2 - k_y^2)\, e^{i(k_x x + k_y y - \omega t)}$$

$$-\omega^2 = c^2(-k_x^2 - k_y^2)$$

$$\omega^2 = c^2(k_x^2 + k_y^2)$$

som er dispersjonsrelasjonen, så da er u en løsning av bølgelikningen.

DISPERSION COEFF. (OBLIG 1, oppg. 1.2.4)

La $m_x = m_y$ s.a. $k_x = k_y = k$

Diskret løsning:

$$u_{ij}^n = e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \quad , \hat{\imath} = \sqrt{-1}$$

der $\tilde{\omega}$ er numersk dispersjonskoeff., dvs. num. approx av ω.

Diskretisert bølgelikn.:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2}\right)$$

Setter inn løsn. i likn. for å finne $\tilde{\omega}$:

V.S.:

$$\begin{aligned} u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} \\ &\quad - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \\ &\quad + e^{\hat{\imath}(kh(i+j) - \tilde{\omega}(n-1)\Delta t)} \\ &= e^{\hat{\imath} kh(i+j)}\left[e^{\hat{\imath}(-\tilde{\omega} n\Delta t - \tilde{\omega}\Delta t)} - 2e^{\hat{\imath}(-\tilde{\omega} n \Delta t)} + e^{\hat{\imath}(-\tilde{\omega} n\Delta t + \tilde{\omega}\Delta t)}\right] \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n\Delta t)}\left[e^{\hat{\imath}(-\tilde{\omega}\Delta t)} - 2 + e^{\hat{\imath}\tilde{\omega}\Delta t}\right] \end{aligned}$$

H.S. 1:

$$\begin{aligned} u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n &= e^{\hat{\imath}(kh(i+1+j) - \tilde{\omega} n\Delta t)} - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t} + e^{\hat{\imath}(kh(i-1+j) - \tilde{\omega} n\Delta t} \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n\Delta t)}\left[e^{\hat{\imath} kh} - 2 + e^{\hat{\imath}(-kh)}\right] \end{aligned}$$

H.S. 2:

$$\begin{aligned} u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n &= e^{\hat{\imath}(kh(i+j+1) - \tilde{\omega} n\Delta t)} - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n\Delta t)} + e^{\hat{\imath}(kh(i+j-1) - \tilde{\omega} n\Delta t)} \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n\Delta t)}\left[e^{\hat{\imath} kh} - 2 + e^{\hat{\imath}(-kh)}\right] \end{aligned}$$

Sett sammen:

$$u_{ij}^{n+1}-2u_{ij}^{n}+u_{ij}^{n-1}=\overbrace{\frac{c^2\Delta t^2}{h^2}}^{\bar{C}^2}\left(u_{i+1j}^{n}-2u_{ij}^{n}+u_{i-1j}^{n}+u_{ij+1}^{n}-2u_{ij}^{n}+u_{ij-1}^{n}\right)$$

$$e^{\hat{\imath}(kh(i+j)-\tilde{\omega}n\Delta t)}\left[e^{\hat{\imath}(-\tilde{\omega}\Delta t)}-2+e^{\hat{\imath}\tilde{\omega}\Delta t}\right]$$

$$=\bar{C}^2\Big(e^{\hat{\imath}(kh(i+j)-\tilde{\omega}n\Delta t)}\left[e^{\hat{\imath}kh}-2+e^{\hat{\imath}(-kh)}\right]$$

$$+e^{\hat{\imath}(kh(i+j)-\tilde{\omega}n\Delta t)}\left[e^{\hat{\imath}kh}-2+e^{\hat{\imath}(-kh)}\right]\Big)$$

$$e^{\hat{\imath}(-\tilde{\omega}\Delta t)}-2+e^{\hat{\imath}\tilde{\omega}\Delta t}=\bar{C}^2\left(e^{\hat{\imath}kh}-2+e^{\hat{\imath}(-kh)}+e^{ikh}-2+e^{\hat{\imath}(-kh)}\right)$$

$$e^{\hat{\imath}\tilde{\omega}\Delta t}+e^{\hat{\imath}(-\tilde{\omega}\Delta t)}-2=\bar{C}^2\left(2e^{\hat{\imath}kh}+2e^{\hat{\imath}(-kh)}-4\right)$$

$\boxed{\bar{C}=\frac{1}{\sqrt{2}}}\rightarrow$

$$2\,\frac{e^{\hat{\imath}\tilde{\omega}\Delta t}+e^{\hat{\imath}(-\tilde{\omega}\Delta t)}}{2}-2=2\,\frac{e^{\hat{\imath}kh}+e^{\hat{\imath}(-kh)}}{2}-4\cdot\frac{1}{2}\quad\Big|:2$$

$$\cos(\tilde{\omega}\Delta t)-1=\cos(kh)-1$$

$$\cos(\tilde{\omega}\Delta t)=\cos(kh)$$

Da må

$$\tilde{\omega}\Delta t=kh$$

$$\tilde{\omega}=\frac{h}{\Delta t}k=\frac{\sqrt{2}}{\sqrt{2}}\frac{h}{\Delta t}k=\sqrt{2}\,\bar{C}\frac{h}{\Delta t}k=\sqrt{2}\,\frac{c\cancel{\Delta t}}{\cancel{h}}\frac{\cancel{h}}{\cancel{\Delta t}}k=\sqrt{2}\,ck=\omega$$

$$\boxed{\begin{aligned}\omega^2&=c^2|\vec{k}|^2\\&=c^2\sqrt{k_x^2+k_y^2}^{\,2}\\&=c^2\sqrt{2k^2}^{\,2}\\&=c^2\,2k^2\\\omega&=\sqrt{2}\,ck\end{aligned}}$$