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EXACT SOLU (OBLIG 1 oppg. 1.2.3)

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$$

Skal vise at dette tilfredsstiller bølgeligningen:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Løsn.:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-i\omega e^{i(k_x x + k_y y - \omega t)}\right)$$

$$= -\omega^2 e^{i(k_x x + k_y y - \omega t)}$$

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$

$$= \frac{\partial}{\partial x}\left(ik_x e^{i(k_x x + k_y y - \omega t)}\right)$$

$$+ \frac{\partial}{\partial y}\left(ik_y e^{i(k_x x + k_y y - \omega t)}\right)$$

$$= -k_x^2 e^{i(k_x x + k_y y - \omega t)} - k_y^2 e^{i(k_x x + k_y y - \omega t)}$$

$\Rightarrow$

$$-\omega^2 e^{i(k_x x + k_y y - \omega t)} = c^2(-k_x^2 - k_y^2) e^{i(k_x x + k_y y - \omega t)}$$

$$-\omega^2 = c^2(-k_x^2 - k_y^2)$$

$$\omega^2 = c^2(k_x^2 + k_y^2)$$

som er dispersjonsrelasjonen, så da er u en løsning av bølgelikningen.

DISPERSION COEFF. (OBLIG 1, oppg. 1.2.4)

La $m_x = m_y$ s.a. $k_x = k_y = k$

Diskret løsning:

$$u_{ij}^n = e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \quad , \hat{\imath} = \sqrt{-1}$$

der $\tilde{\omega}$ er numersk dispersjonskoeff., dvs. num. approx av ω.

Diskretisert bølgelikn.:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2} \right)$$

Setter inn løsn. i likn. for å finne $\tilde{\omega}$:

V.S.:

$$\begin{aligned} u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} \\ &\quad - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \\ &\quad + e^{\hat{\imath}(kh(i+j) - \tilde{\omega}(n-1)\Delta t)} \\ &= e^{\hat{\imath} kh(i+j)} \left[e^{\hat{\imath}(-\tilde{\omega} n \Delta t - \tilde{\omega}\Delta t)} - 2e^{\hat{\imath}(-\tilde{\omega} n \Delta t)} + e^{\hat{\imath}(-\tilde{\omega} n \Delta t + \tilde{\omega}\Delta t)} \right] \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{\hat{\imath}(-\tilde{\omega}\Delta t)} - 2 + e^{\hat{\imath}\tilde{\omega}\Delta t} \right] \end{aligned}$$

H.S. 1:

$$\begin{aligned} u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n &= e^{\hat{\imath}(kh(i+1+j) - \tilde{\omega} n \Delta t)} - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t} + e^{\hat{\imath}(kh(i-1+j) - \tilde{\omega} n \Delta t} \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{\hat{\imath} kh} - 2 + e^{\hat{\imath}(-kh)} \right] \end{aligned}$$

H.S. 2:

$$\begin{aligned} u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n &= e^{\hat{\imath}(kh(i+j+1) - \tilde{\omega} n \Delta t)} - 2e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} + e^{\hat{\imath}(kh(i+j-1) - \tilde{\omega} n \Delta t)} \\ &= e^{\hat{\imath}(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{\hat{\imath} kh} - 2 + e^{\hat{\imath}(-kh)} \right] \end{aligned}$$

Sett sammen:

$$u_{ij}^{n+1} - 2u_{ij}^{n} + u_{ij}^{n-1} = \overbrace{\frac{c^2 \Delta t^2}{h^2}}^{\bar{c}^2} \left(u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n} \right)$$

$$e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{i(-\tilde{\omega}\Delta t)} - 2 + e^{i\tilde{\omega}\Delta t} \right]$$

$$= \bar{c}^2 \Big(e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{ikh} - 2 + e^{i(-kh)} \right]$$

$$+ e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{ikh} - 2 + e^{i(-kh)} \right] \Big)$$

$$e^{i(-\tilde{\omega}\Delta t)} - 2 + e^{i\tilde{\omega}\Delta t} = \bar{c}^2 \left(e^{ikh} - 2 + e^{i(-kh)} + e^{ikh} - 2 + e^{i(-kh)} \right)$$

$$e^{i\tilde{\omega}\Delta t} + e^{i(-\tilde{\omega}\Delta t)} - 2 = \bar{c}^2 \left(2e^{ikh} + 2e^{i(-kh)} - 4 \right)$$

$\boxed{\bar{c} = \frac{1}{\sqrt{2}}} \rightarrow$

$$2\, \frac{e^{i\tilde{\omega}\Delta t} + e^{i(-\tilde{\omega}\Delta t)}}{2} - 2 = 2\, \frac{e^{ikh} + e^{i(-kh)}}{2} - 4 \cdot \frac{1}{2} \quad \Big| :2$$

$$\cos(\tilde{\omega}\Delta t) - 1 = \cos(kh) - 1$$

$$\cos(\tilde{\omega}\Delta t) = \cos(kh)$$

Da må

$$\tilde{\omega}\Delta t = kh$$

$$\tilde{\omega} = \frac{h}{\Delta t} k = \frac{\sqrt{2}}{\sqrt{2}} \frac{h}{\Delta t} k = \sqrt{2}\, \bar{c} \frac{h}{\Delta t} k = \sqrt{2}\, \frac{c \cancel{\Delta t}}{\cancel{h}} \frac{\cancel{h}}{\cancel{\Delta t}} k = \sqrt{2}\, ck \quad \text{Nesten!}$$

Hvis $\tilde{\omega}\Delta t = \frac{1}{\sqrt{2}} kh \Rightarrow \tilde{\omega} = ck \Rightarrow \tilde{\omega} = \omega$