One Way $\longrightarrow$ Two way

$Y_{it} = \mu + \tau_i + \varepsilon_{it}$

$Y_{ijt} = \mu + \tau_{ij} + \varepsilon_{ijt}$

also have: $\varepsilon_{ijt} \sim N(0, \sigma^2)$

complete 2-way anova: $Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}$

$$E(Y_{ijt}) = \mu + \boxed{\alpha_i + \beta_j + (\alpha\beta)_{ij}}$$
$$= \mu + \tau_{ij}$$

eg: $E(Y_{11t} - Y_{12t}) = (\cancel{\mu + \alpha_1} + \beta_1) - (\cancel{\mu + \alpha_1} + \beta_2) = \beta_1 - \beta_2$

$E(Y_{21t} - Y_{22t}) = (\cancel{\mu + \alpha_2} + \beta_1) - (\cancel{\mu + \alpha_2} + \beta_2) = \beta_1 - \beta_2$

the difference in mean response between $j=1$ & $j=2$ (factor β) is not affected by the level of A

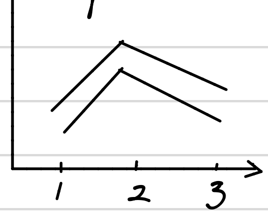
for the complete model:

$E(Y_{11t} - Y_{12t}) = (\cancel{\mu + \alpha_1} + \beta_1 + (\alpha\beta)_{11}) - (\cancel{\mu + \alpha_1} + \beta_2 + (\alpha\beta)_{12}) = \beta_1 - \beta_2 + (\alpha\beta)_{11} - (\alpha\beta)_{12}$

$E(Y_{21t} - Y_{22t}) = (\mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21}) - (\mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22}) = \beta_1 - \beta_2 + (\alpha\beta)_{21} - (\alpha\beta)_{22}$

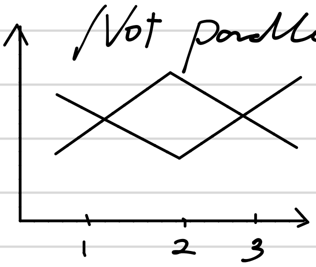
- Interaction plot:

parallel lines



No Interaction

Not parallel lines



Has Interaction.

Two-Way ANOVA:

$$\bar{Y}_{ij\cdot} = \frac{1}{r_{ij}} \sum_{t=1}^{r_{ij}} Y_{ijt}$$

$$\bar{Y}_{i\cdot\cdot} = \frac{1}{br} \sum_{j=1}^{b} \sum_{t=1}^{r} Y_{ijt}$$

$$\bar{Y}_{\cdot j\cdot} = \frac{1}{ar} \sum_{i=1}^{a} \sum_{t=1}^{r} Y_{ijt}$$

$$\bar{Y}_{\cdots} = \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{t=1}^{r} Y_{ijt}}{n}$$

$$Y_{ijt}-\bar{Y}_{...}=\underbrace{(\bar{Y}_{i..}-\bar{Y}_{...})}_{\text{Effect A}}+\underbrace{(\bar{Y}_{.j.}-\bar{Y}_{...})}_{\text{Effect B}}+\underbrace{(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})}_{\text{Effect of Interaction}}+\underbrace{(Y_{ijt}-\bar{Y}_{ij.})}_{\text{Residual}}$$

$$SS_{total}=\sum_i\sum_j\sum_t(Y_{ijt}-\bar{Y}_{...})^2 \qquad df = rab-1$$

$$SSA=\sum_{i=1}^{a} br(\bar{Y}_{i..}-\bar{Y}_{...})^2 \qquad df = a-1$$

$\hookrightarrow$ the number of variables of the other factor

$$SSB=\sum_{j=1}^{b} ar(\bar{Y}_{.j.}-\bar{Y}_{...})^2 \qquad df = b-1$$

$\hookrightarrow$ the number of variables of the other factor

$$SSAB=r\sum_i\sum_j\left(\bar{Y}_{ij.}-(\bar{Y}_{i..}+\bar{Y}_{.j.}-\bar{Y}_{...})\right)^2 \qquad df=(a-1)(b-1)$$

$$SSE=\sum_i\sum_j\sum_t(Y_{ijt}-\bar{Y}_{ij.})^2 \qquad df = ab(r-1)$$

$$\frac{SSE}{\sigma^2}\sim\chi^2_{ab(r-1)} \qquad \frac{SSA}{\sigma^2}\sim\chi^2_{a-1}$$

ANOVA:

	SS	df	MS	F
A	SSA	$a-1$	SSA/dfA	MSA/MSE
B	SSB	$b-1$	SSB/dfB	MSB/MSE
AB	SSAB	$(a-1)(b-1)$	SSAB/dfAB	MSAB/MSE
Error	SSE	$ab(r-1)$	SSE/dfe	
total	SSTOTAL	$abr-1$		

Hypothesis Testing:

① $H_0: (\alpha\beta)_{ij}=0$ (no interaction)

$$F=\frac{MSAB}{MSE}=\frac{SSAB/(a-1)(b-1)}{SSE/ab(r-1)}\sim F_{(a-1)(b-1),\,ab(r-1)}$$

② $H_0: \alpha_1=\alpha_2=\dots=\alpha_a$ (no effect of A on mean response)

$$F=\frac{MSA}{MSE}=\frac{SSA/(a-1)}{SSE/ab(r-1)}\sim F_{a-1,\,ab(r-1)}$$

③ $H_0: \beta_1=\beta_2=\dots=\beta_B$ (no effect of B on mean response)

$$F=\frac{MSB}{MSE}=\frac{SSB/(b-1)}{SSE/ab(r-1)}\sim F_{b-1,\,ab(r-1)}$$

④ if reject H_0 from ①
$\Rightarrow$ if significant interaction

$$H_0: \underline{\tau_{ij}} - \tau_{kl} = 0$$

where $\tau_{ij} = \alpha_i + \beta_j + (\alpha\beta)_{ij}$

⑤ if interaction not significant

$H_0: \alpha_i - \alpha_j = 0$
$H_0: \beta_i - \beta_j = 0$
} (all the pairwise comparison in all mean of factor A)

- to determine whether it's estimable?

$A=1 \quad A=2$

if $\frac{1}{b}\sum_{j=1}^{b}(\mu+\tau_{1j}) - \frac{1}{b}\sum_{j=1}^{b}(\mu+\tau_{2j})$ is estimable

$$= (\alpha_1 - \alpha_2) + \boxed{\frac{1}{b}\sum_{j=1}^{b}(\alpha\beta)_{1j} - \frac{1}{b}\sum_{j=1}^{b}(\alpha\beta)_{2j}}$$

$=0$ if estimable

- CI

$$(\bar{Y}_{ij\cdot} - \bar{Y}_{kl\cdot}) \pm t\text{-critical} \cdot \sqrt{MSE\left(\frac{1}{r}+\frac{1}{r}\right)}$$

* Bonferroni correction: $(\bar{Y}_{ij\cdot} - \bar{Y}_{kl\cdot}) \pm \left|t\left(\frac{\alpha}{2g}, ab(r-1)\right)\right| \sqrt{MSE\left(\frac{1}{r}+\frac{1}{r}\right)}$

$$T_{ij-kl} = \frac{\bar{Y}_{ij\cdot} - \bar{Y}_{kl\cdot}}{\sqrt{MSE\cdot\frac{2}{r}}} \Rightarrow |\bar{Y}_{ij\cdot} - \bar{Y}_{kl\cdot}| > t\left(1-\frac{\alpha}{2g}, ab(r-1)\right) \cdot \sqrt{MSE\cdot\frac{2}{r}}$$

* Tukey procedure: $\hat{D} \pm T\, \underbrace{s(\hat{D})}_{\sqrt{MSE\left(\frac{2}{r}\right)}}$

$$T = \frac{1}{\sqrt{2}} q(1-\alpha, ab, ab(r-1))$$

$q = \frac{\sqrt{2}\hat{D}}{s(\hat{D})}$ if $q = \frac{\sqrt{2}\hat{D}}{s(\hat{D})} > q(1-\alpha, ab, ab(r-1))$ then reject H_0

compare A= i & i'

$$\widehat{\alpha_i - \alpha_{i'}} = \left(\frac{1}{b}\sum_{j=1}^{b} \overline{Y_{ij\cdot}}\right) - \left(\frac{1}{b}\sum_{j=1}^{b} \overline{Y_{i'j\cdot}}\right) = \overline{Y_{i\cdot\cdot}} - \overline{Y_{i'\cdot\cdot}}$$

$$\widehat{\alpha_i - \alpha_{i'}} \pm \boxed{T}\,\boxed{SD(\widehat{\alpha_i - \alpha_{i'}})}$$

$$= t\left(1-\frac{\alpha}{2g}; ab(r-1)\right)\cdot\sqrt{MSE\left(\frac{1}{br}+\frac{1}{br}\right)}$$

- Blocking (No hypothesis testing) (Randomization within each blocks)

$$Y_{it} = \mu + \tau_i + \varepsilon_{it}$$

$$= \mu + \tau_i + \underbrace{(R+\varepsilon)}_{\text{error}}$$

$$Y_{ij} = \underbrace{\mu + \tau_i}_{\text{the } i\text{th treatment}} + \underbrace{B_j}_{j\text{th block effect}} + \varepsilon_{ij} \sim N(0, \sigma^2)$$

		df
SSTOTAL	$= \sum_i \sum_j (y_{ij} - \overline{y}_{\cdot\cdot})^2$	$ab-1$
SST	$= b\sum_{i=1}^{a} (\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2$	$a-1$
SSBlock	$= a\sum_{j=1}^{b} (\overline{y}_{\cdot j} - \overline{y}_{\cdot\cdot})^2$	$b-1$
SSE	$= \sum\sum (y_{ij} - \overline{y}_{i\cdot} - \overline{y}_{\cdot j} + \overline{y}_{\cdot\cdot})^2$	$(a-1)(b-1)$

$$SST = b\sum_{i=1}^{a} (\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2$$

$$= b\sum_{i=1}^{a} (\overline{y}_{i\cdot}^2 - 2\overline{y}_{\cdot\cdot}\overline{y}_{i\cdot} + \overline{y}_{\cdot\cdot}^2)$$

$$= b\sum_{i=1}^{a} \overline{y}_{i\cdot}^2 - \underbrace{2b\,\overline{y}_{\cdot\cdot}\underbrace{\sum_{i=1}^{a}\overline{y}_{i\cdot}}_{a\overline{y}_{\cdot\cdot}}}_{2ab\overline{y}_{\cdot\cdot}^2} + ab\,\overline{y}_{\cdot\cdot}^2$$

$$= b\sum_{i=1}^{a} \overline{y}_{i\cdot}^2 - ab\,\overline{y}_{\cdot\cdot}^2$$

$$SSTOTAL = \sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij} - \bar{y}_{..})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}^2 - 2y_{ij}\bar{y}_{..} + \bar{y}_{..}^2)$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - 2\bar{y}_{..}\underbrace{\boxed{\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}}}_{ab\,\bar{y}_{..}} + ab\,\bar{y}_{..}^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - ab\,\bar{y}_{..}^2$$

$$SSBLOCK = a\sum_{j=1}^{b}(\bar{y}_{.j} - \bar{y}_{..})^2$$

$$= a\sum_{j=1}^{b}(\bar{y}_{.j}^2 - 2\bar{y}_{..}\bar{y}_{.j} + \bar{y}_{..}^2)$$

$$= a\sum_{j=1}^{b}\bar{y}_{.j}^2 - ab\,\bar{y}_{..}^2$$

$$SSE = SSTOTAL - SSA - SSB$$

- Multiple comparison of D (Tukey's Method)

$$H_0: \tau_i - \tau_{i'} = 0$$

$$\hat{\tau}_i - \hat{\tau}_{i'} = \bar{y}_{i.} - \bar{y}_{i'.}$$

$$u = \frac{1}{\sqrt{2}} \cdot q_{tukey}(1-\alpha, b, (a-1)(b-1))$$

$$se(\bar{y}_{i.} - \bar{y}_{i'.}) = \sqrt{MSE\left(\frac{1^2}{b} + \frac{(-1)^2}{b}\right)} = \sqrt{MSE \cdot \frac{2}{b}}$$

- Notes: Multipliers

① Bonferroni: $B = t(1-\frac{\alpha}{2g}; (a-1)(b-1))$

② Tukey: $T = \frac{1}{\sqrt{2}} q(1-\alpha; b, (a-1)(b-1))$

③ Scheffe: $S = \sqrt{(b-1)F(1-\alpha; b-1, (a-1)(b-1))}$

Smaller the multiplier, more efficient the procedure