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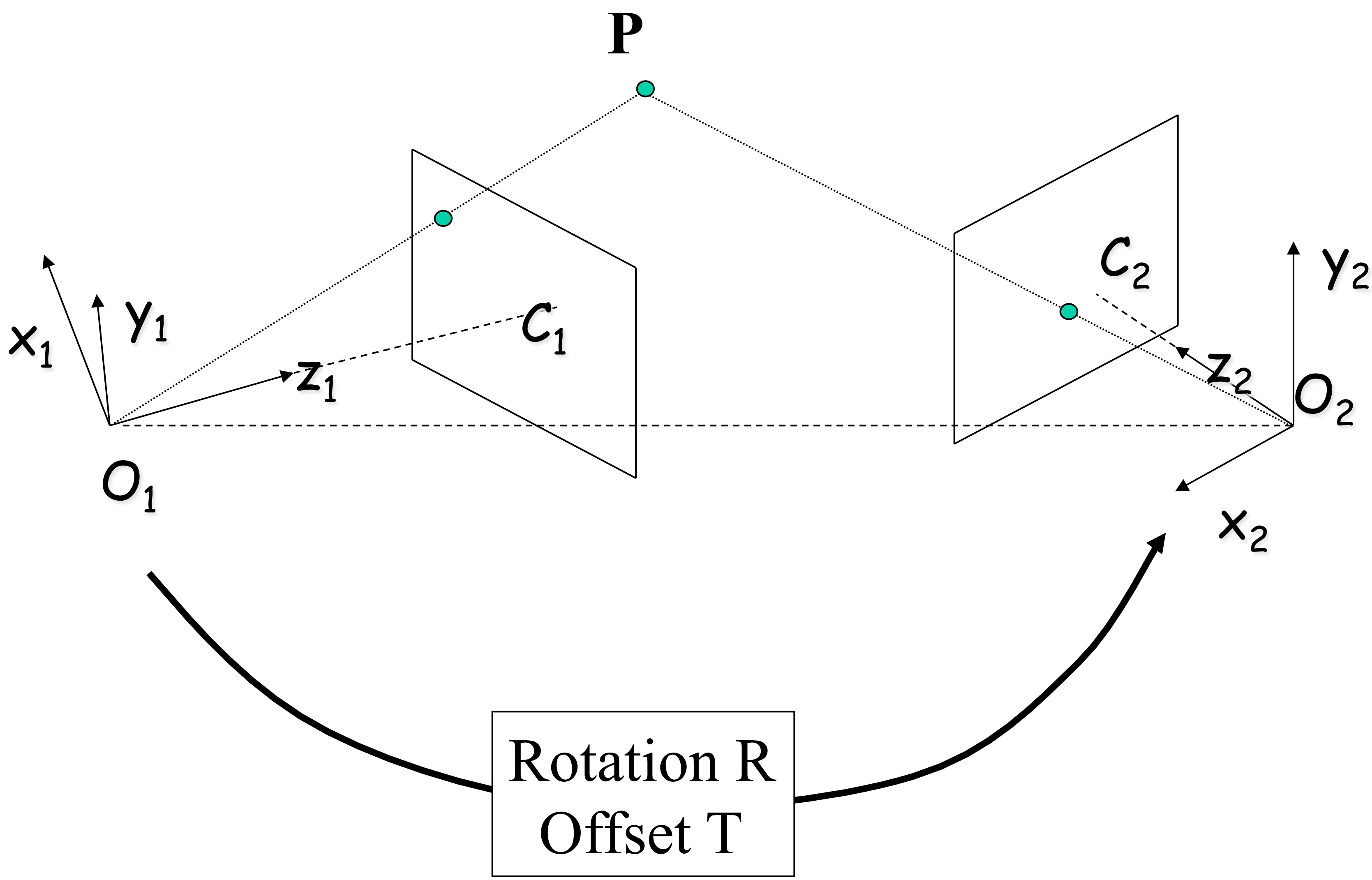
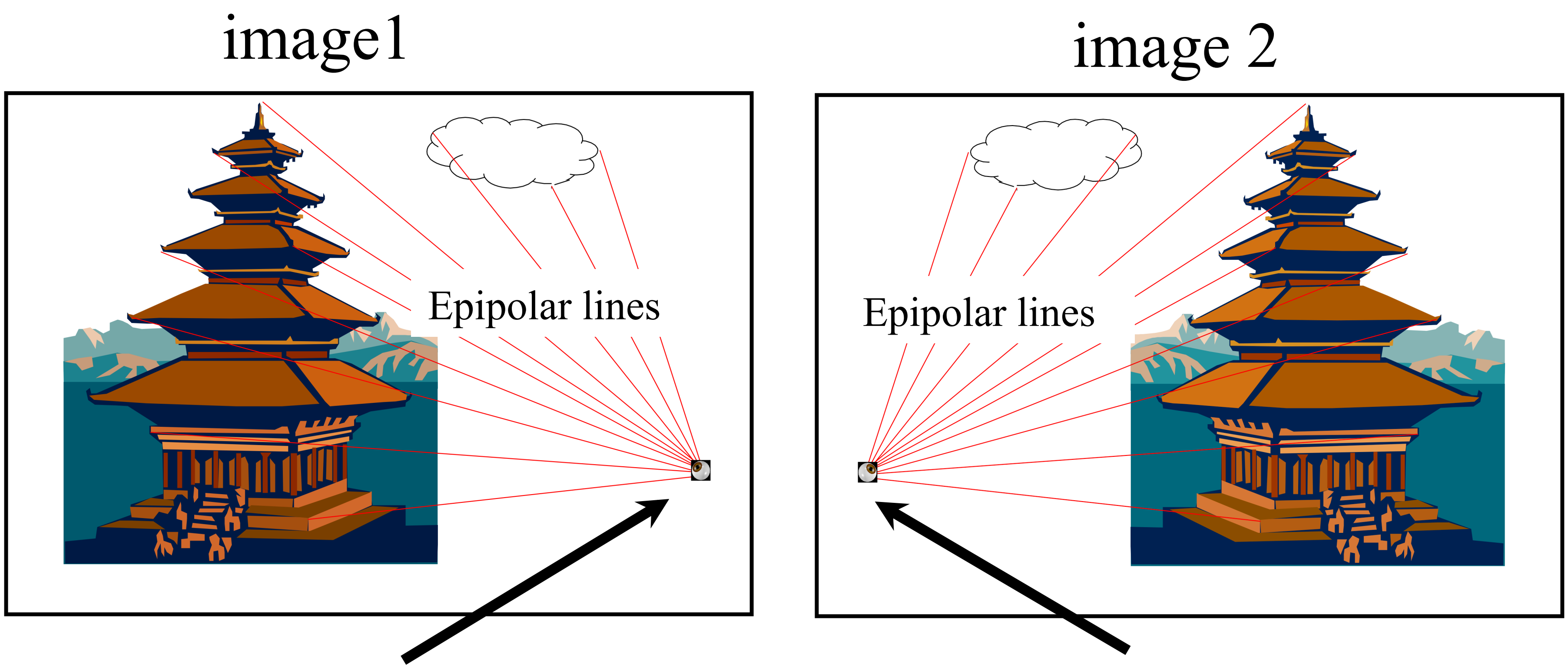
Generalized Stereo:
Essential and Fundamental Matrices

Review: General Stereo



Review: General Stereo



Epipole : location of cam2 as seen by cam1.

Epipole : location of cam1 as seen by cam2.

Corresponding points lie on conjugate **epipolar lines**.

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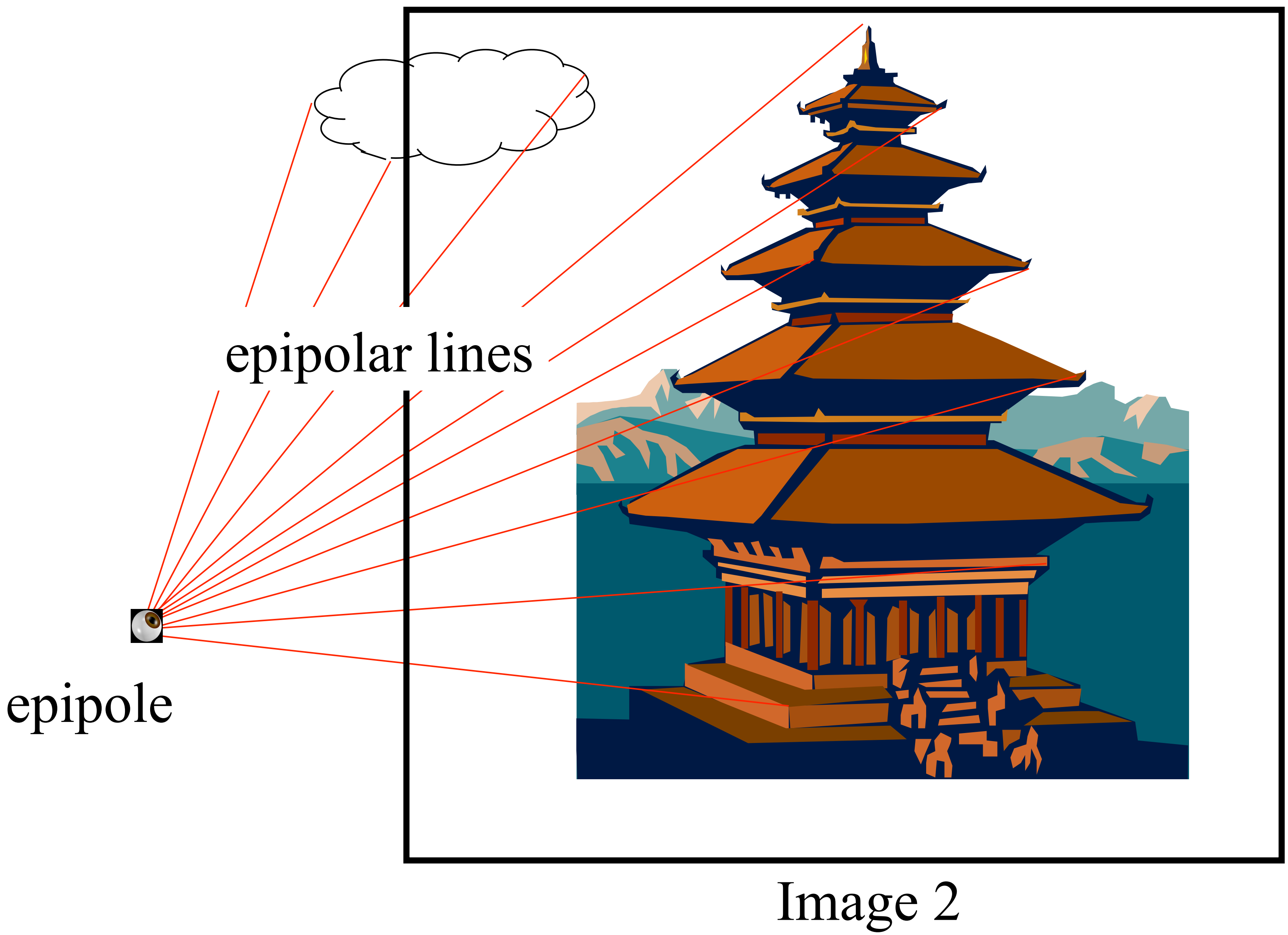
Note: Epipole Does not Have to be in the Image



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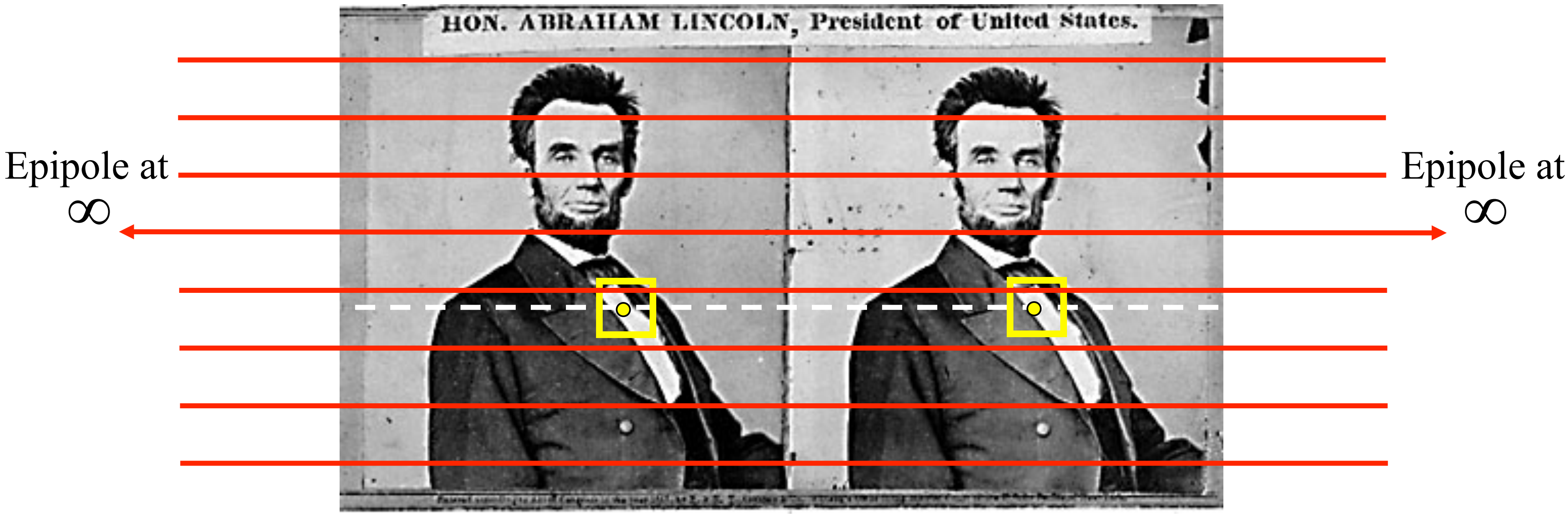
Simple Stereo



For simple stereo, epipolar lines are parallel and the epipoles are thus "at infinity."

Note, in homogenous coordinates, (x y 0) is a point at infinity and it can be manipulated like any other point.

Preview for Today: Essential/Fundamental Matrices

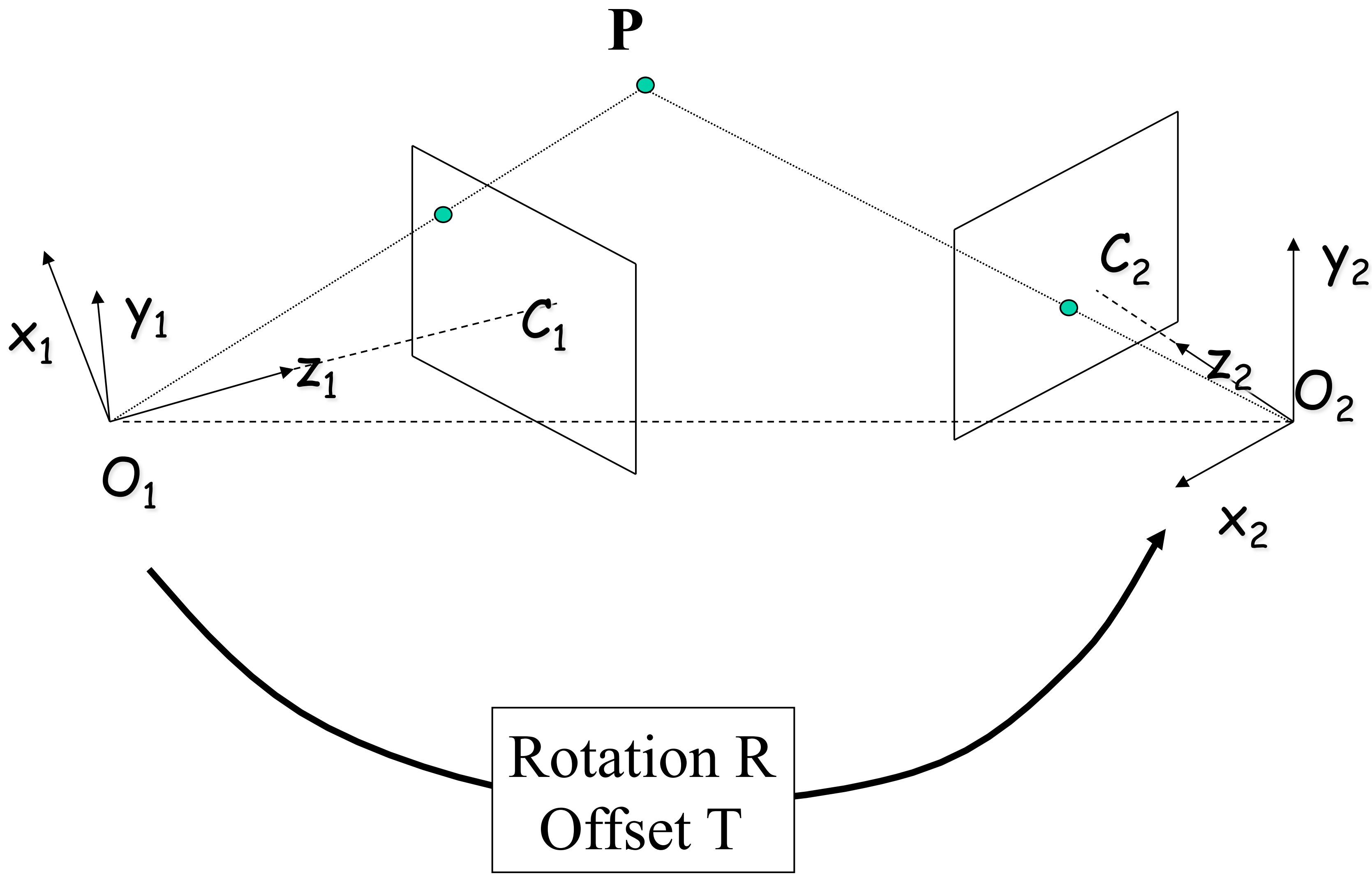
The essential and fundamental matrices are 3x3 matrices that “encode” the epipolar geometry of two views.

Motivation: Given a point in one image, multiplying by the essential/fundamental matrix will tell us which epipolar line to search along in the second view.

Furthermore: Finding the left and right nullspaces of E/F tells us where the epipoles are.

Deriving Essential Matrix

On blackboard (or see previous lecture's slides)



Essential Matrix

$$P_r^T E P_l = 0$$

$$E = RS$$

Where S= $\begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$

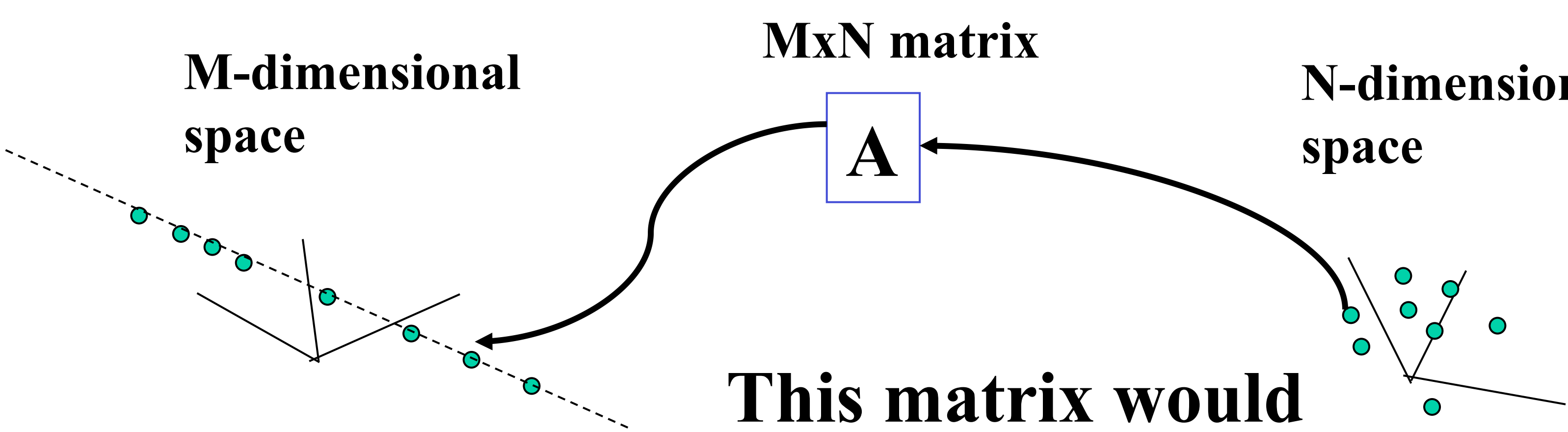
- has rank 2 (because S does)
- depends only on the EXTRINSIC Parameters (R & T)

Review: Rank of a Matrix

What is rank of a matrix?

Number of columns (rows) that are linearly independent.

If matrix A is treated as a linear mapping, it is the <u>intrinsic</u> dimension of the space that is mapped into.

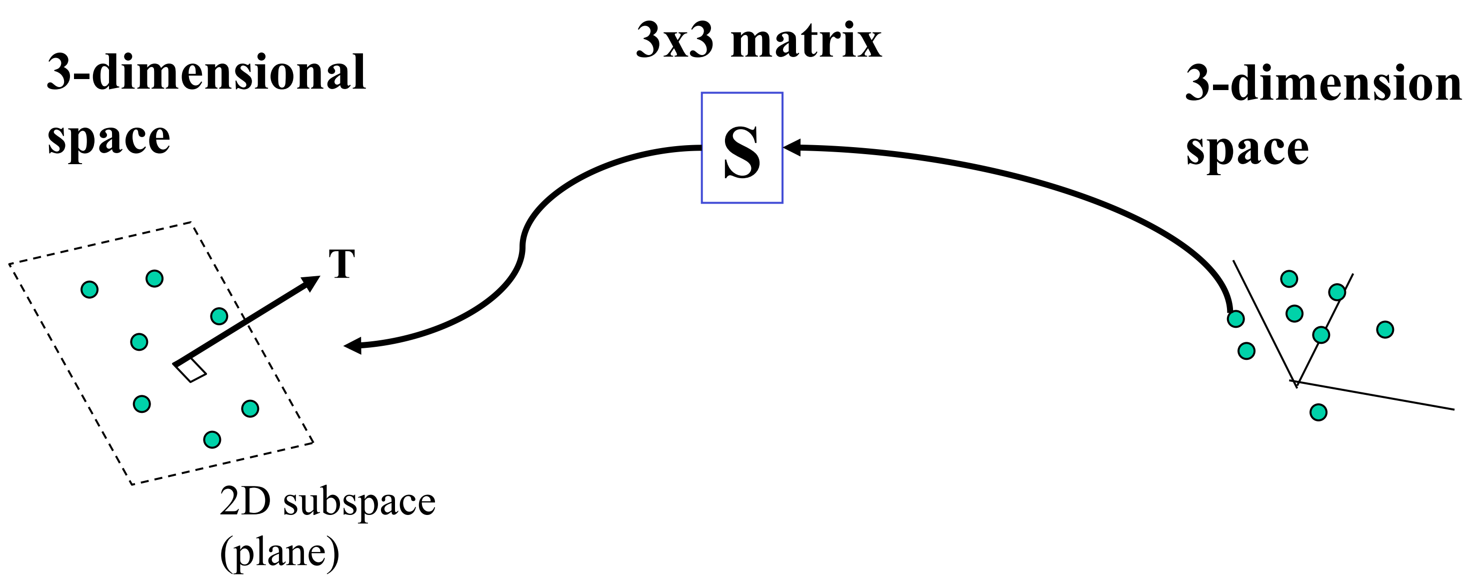


This matrix would have rank 1

Rank of S is 2

S is carefully constructed to map vector v into vector T x v (cross product of T and v)

Therefore it maps vectors into a 2D plane perpendicular to T



Nullspace(s) of S

Since 3x3 matrix S has rank 2, it has 1D nullspaces

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Right nullspace

$$[\, p \quad q \quad r \,] \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} = [\, 0 \quad 0 \quad 0 \,]$$

Left nullspace

Question: what are the left and right nullspaces of S?

Longuet-Higgins equation

$$P_r^T E P_l = 0$$

$$p_l = \frac{f_l}{Z_l} P_l \qquad p_r = \frac{f_r}{Z_r} P_r$$

$$\left(\cancel{\frac{Z_r}{f_r}} p_r\right)^T E \left(\cancel{\frac{Z_l}{f_l}} p_l\right) = 0$$

$$\boxed{p_r^T E p_l = 0}$$

Longuet-Higgins equation

$$P_r^T E P_l = 0$$

This relates viewing rays

Importance of Longuet-Higgens ...

This relates 2D film points

$$\boxed{p_r^T E p_l = 0}$$

Fundamental Matrix

The essential matrix uses film plane coordinates

To use image (pixel) coordinates we must consider the INTRINSIC camera parameters:

Pixel coord (row,col)

Affine transform matrix

Camera (film) coord

$$\bar{p_l} = M_l p_l \qquad p_l = M_l^{-1} \bar{p_l}$$

$$\bar{p_r} = M_r p_r \qquad p_r = M_r^{-1} \bar{p_r}$$

Fundamental Matrix

$$p_l = M_l^{-1} \bar{p}_l$$

$$p_r = M_r^{-1} \bar{p}_r$$

$$p_r^T E p_l = 0$$

$$(M_r^{-1} \bar{p}_r)^T E (M_l^{-1} \bar{p}_l) = 0$$

$$\bar{p}_r{}^T (M_r^{-T} E M_l^{-1}) \bar{p}_l = 0$$

$$\boxed{\bar{p}_r{}^T F \bar{p}_l = 0}$$

short version: The same equation works in pixel coordinates too!

Fundamental Matrix Properties

$$F = M_r^{-T} R S M_l^{-1}$$

- has rank 2 (because S does)
- depends on the INTRINSIC and EXTRINSIC Parameters
  (f,sx,sy,ox,oy ; R & T)

Essential / Fundamental Matrices

The essential and fundamental matrices are 3x3 matrices that "encode" the epipolar geometry of two views.

Motivation: Given a point in one image, multiplying by the essential/fundamental matrix will tell us which epipolar line to search along in the second view.

Essential Matrix -
works in film plane coordinates
(calibrated cameras)

Fundamental Matrix –
works in pixel coordinates
(uncalibrated cameras)

Background: Point on a Line in Homogeneous Coordinates

- Let l be a line in the image:

$$au + bv + c = 0$$

- Using homogeneous coordinates:

$$\tilde{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \tilde{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \boxed{\tilde{p}^T \tilde{l} = \tilde{l}^T \tilde{p} = 0}$$

Vector equation representing that point p lies on line l

Epipolar Lines

- Given point in left image, what is corresponding epipolar line in right image?

$$p_r^T E p_l = 0$$

$$\tilde{l}_r = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

p_r belongs to epipolar line in the right image defined by

$$\tilde{l}_r = E p_l$$

Epipolar Lines

- Similarly:

$$p_r^T E p_l = 0$$

$$\tilde{l}_l^{\,T} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T$$

p_l belongs to epipolar line in the left image defined by

$$\tilde{l}_l = E^T p_r$$

Example

"Left" image

"Right" image

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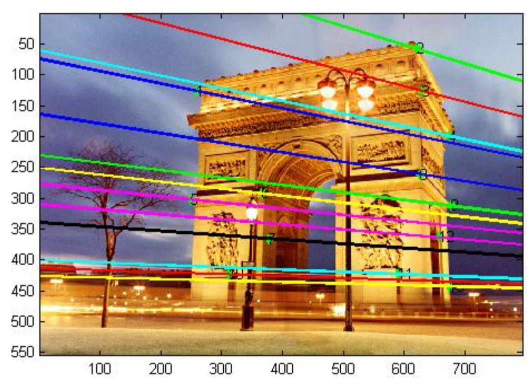
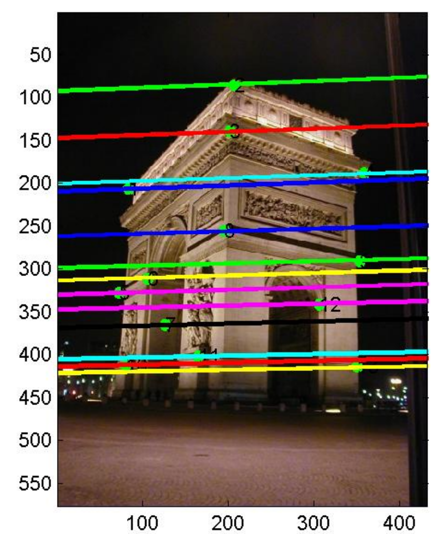
Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix}$$

We will discuss how to compute this matrix in a future lecture.





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Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \begin{pmatrix} 343.53 \\ 221.70 \\ 1.0 \end{pmatrix}$$

u

v

u = 343.5300 v = 221.7005

$$\begin{matrix} 0.0001 \\ 0.0045 \\ -1.1942 \end{matrix} \rightarrow \begin{matrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{matrix}$$

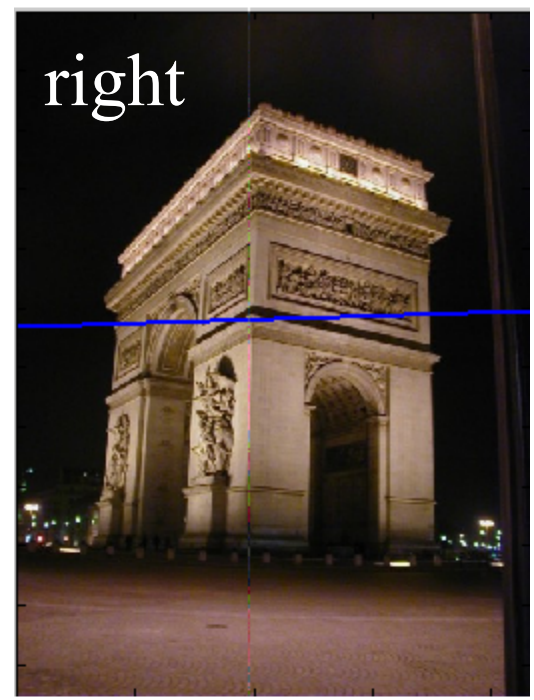
normalize so sum of squares
of first two terms is 1 (optional)

Example

$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \begin{pmatrix} 343.53 \\ 221.70 \\ 1.0 \end{pmatrix}$$

u = 343.5300 v = 221.7005

$$\begin{matrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{matrix}$$

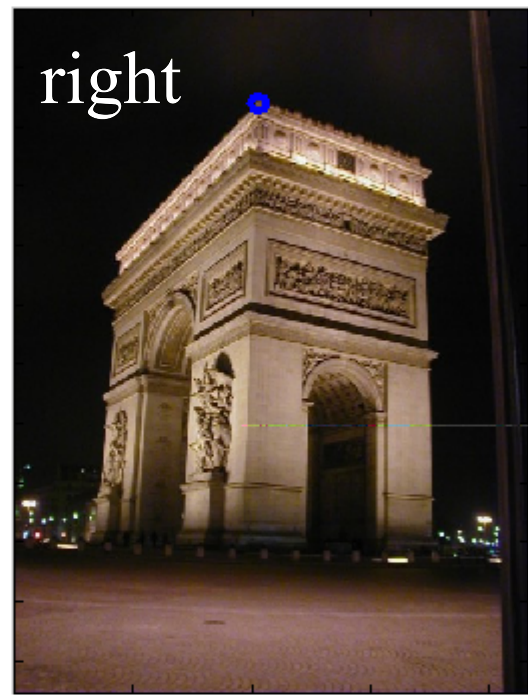


Example

$$\begin{pmatrix} 205.5526 & 80.5 & 1.0 \end{pmatrix} \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix}$$

$$L = (0.0010 \quad -0.0030 \quad -0.4851)$$

$$\rightarrow (0.3211 \quad -0.9470 \; -151.39)$$

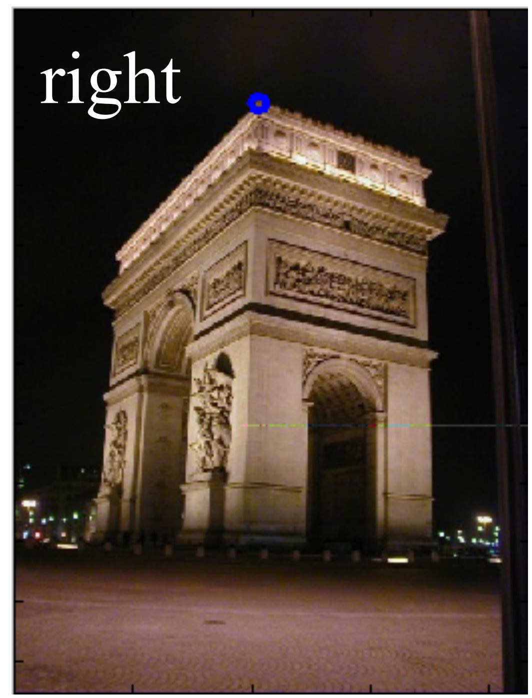


u = 205.5526 v = 80.5000

Example

$$
\begin{pmatrix} 205.5526 & 80.5 & 1.0 \end{pmatrix}
\begin{pmatrix}
-0.00310695 & -0.0025646 & 2.96584 \\
-0.028094 & -0.00771621 & 56.3813 \\
13.1905 & -29.2007 & -9999.79
\end{pmatrix}
$$

$$L = (0.3211 \quad -0.9470 \quad -151.39)$$



u = 205.5526 v = 80.5000

Finding the Epipoles

- Epipoles belong to each epipolar line:

$$e_r^T E p_l = 0 \qquad p_r^T E e_l = 0$$

- And they belong to <u>all</u> the epipolar lines, therefore:

$$e_r^T E = 0 \qquad E e_l = 0$$

Aka left and right nullspaces of E!

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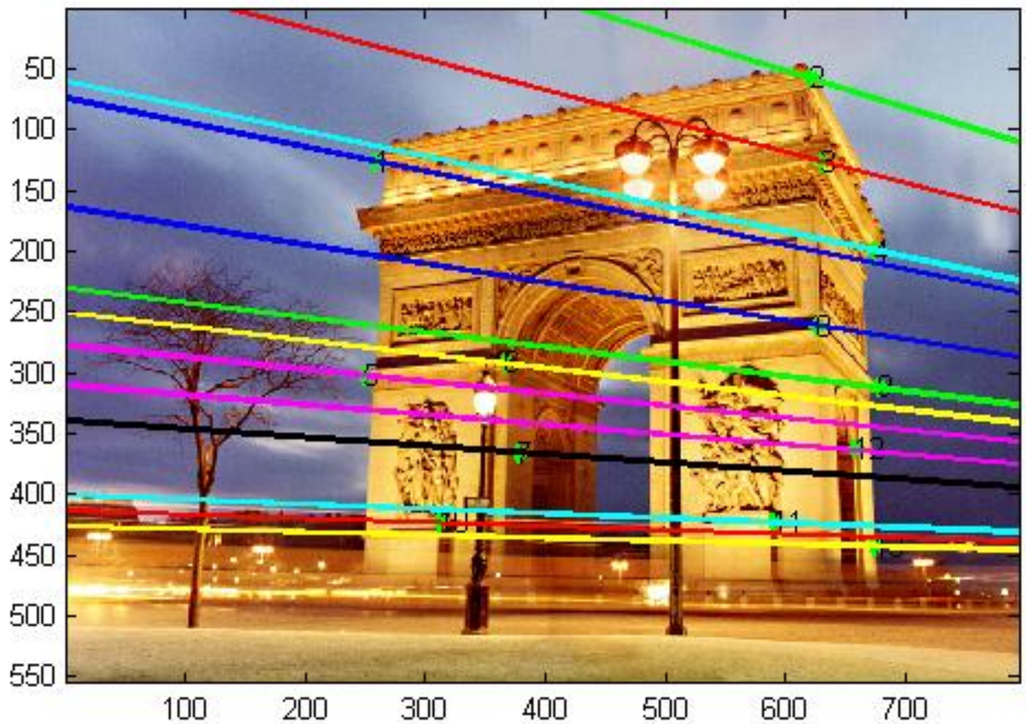
Example

where is the epipole?



$$F * e_L = 0$$

vector in the right nullspace of matrix F

However, due to noise, F may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of F' *F

Math Background

assume v is in right null-space of F so that

$$F\, v = 0$$

then $(F\, v)^T\, (F\, v) = 0$

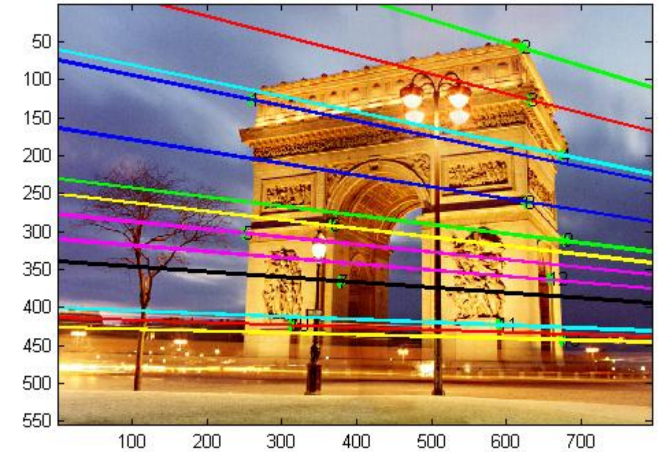
$$v^T\, F^T\, F\, v = 0$$

and $F^T\, F$ is a real, symmetric matrix. So v is an eigenvector of $F^T\, F$ with eigenvalue 0.

similarly, if v is in left null-space of F, then

$v^T\, F = 0$ implies v is zero-eigenvector of $F\, F^T$

Example



>> [u,d] = eigs(F' * F)

$$u = \begin{array}{cc|c|} -0.0013 & 0.2586 & -0.9660 \\ 0.0029 & -0.9660 & -0.2586 \\ 1.0000 & 0.0032 & -0.0005 \end{array} \qquad d = 1.0e8* \begin{array}{ccc} -1.0000 & 0 & 0 \\ 0 & -0.0000 & 0 \\ 0 & 0 & -0.0000 \end{array}$$

eigenvector associated with smallest magnitude eigenvalue

>> uu = u(:,3)

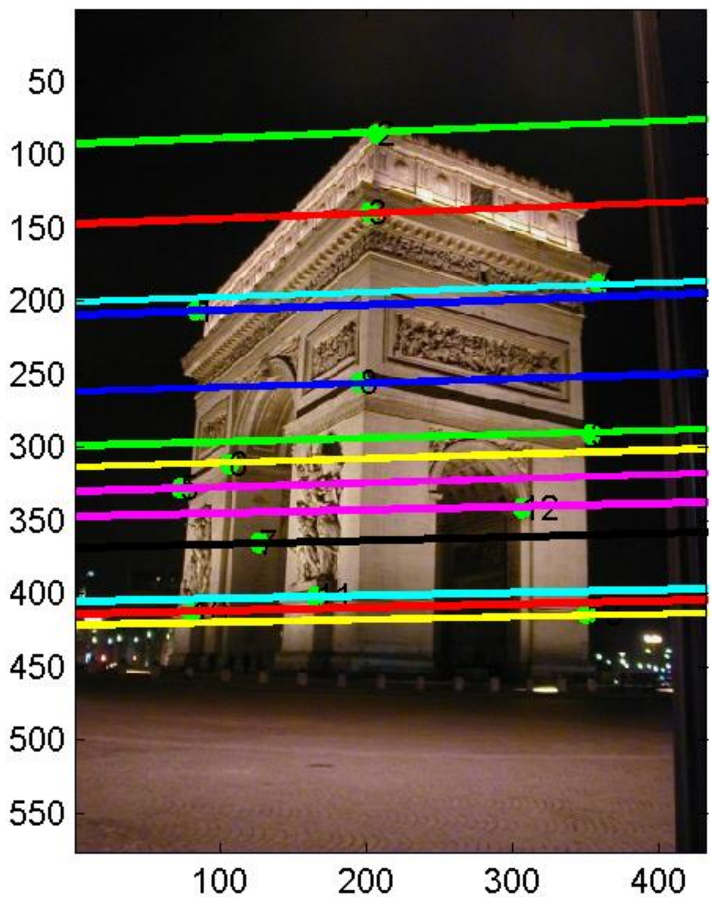
uu = (-0.9660 -0.2586 -0.0005)

>> uu / uu(3) : to get pixel coords

(1861.02 498.21 1.0)

Example

where is the epipole?



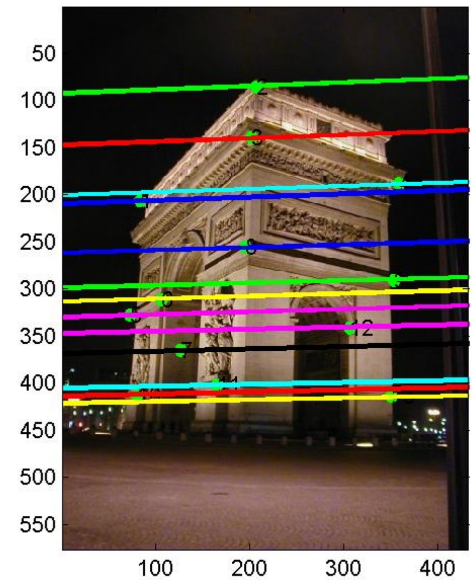
$e'_r * F = 0$

$\rightarrow F' * e_r = 0$

vector in the right nullspace of matrix F'

However, due to noise, F' may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of $F*F'$

Example



\>> [u,d] = eigs(F * F')

u =

-0.0003	-0.0618	-0.9981
-0.0056	-0.9981	0.0618
1.0000	-0.0056	0.0001

d = 1.0e8*

-1.0000	0	0
0	-0.0000	0
0	0	-0.0000

eigenvector associated with smallest magnitude eigenvalue

\>> uu = u(:,3)

uu = (-0.9981 0.0618 0.0001)

\>> uu / uu(3) : to get pixel coords

(-19021.8 1177.97 1.0)

Summary: Essential/Fundamental Matrices

The essential and fundamental matrices are 3x3 matrices that "encode" the epipolar geometry of two views.

Motivation: Given a point in one image, multiplying by the essential/fundamental matrix will tell us which epipolar line to search along in the second view.

Furthermore: Finding the left and right nullspaces of E/F tells us where the epipoles are.

E/F Matrix Summary

Longuet-Higgins equation $$p_r^T E p_l = 0$$

Epipolar lines:

$$\tilde{p_r}^T \tilde{l_r} = 0 \qquad \tilde{p_l}^T \tilde{l_l} = 0$$

$$\tilde{l_r} = E p_l \qquad \tilde{l_l} = E^T p_r$$

Epipoles:

$$e_r^T E = 0 \qquad E e_l = 0$$

E vs F: E works in film coords (calibrated cameras)
F works in pixel coords (uncalibrated cameras)