CENG 371 - Scientific Computing
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Homework 3

Kargı, Bora
e2380566@ceng.metu.edu.tr

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1.1

File is submitted through ODTUCLASS.

1.2

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1.3

Largest eigenvalue of the matrix A can be found by using the power method whereas to find the smallest eigenvalue, we can use shifted inverse method by setting $\alpha = 0$. The results of the findings are give in the Figure 1. It can be seen that the largest eigenvector is $v_1 = [0.288675, -0.5, 0.57735, -0.50.288675]$ with an eigenvalue of $\lambda_1 3.732$ where as smallest eigenvector is $v_n = [0.288675, 0.5, 0.57735, 0.50.288675]$ with an eigenvalue of $\lambda_n = 0.268$.

```
>> power_method
Total runtime : 3.474944
The 1'th largest eigenvalue is 3.732051
The corresponding eigenvector is: [0.288675 -0.5 0.57735 -0.5 0.288675 ]
```

```
>> inverse_power_method
Total runtime : 2.830400
The eigenvalue closes to 0.000000 is 0.267949
The corresponding eigenvector is: [0.288675 0.5 0.57735 0.5 0.288675 ]
```

Figure 1: Finding the largest and smallest eigenvalues and their corresponding eigenvectors by using power and inverse power methods.

1.4

Largest eigenvalue found using power method is given in the Figure 2. To check, lets solve it by pen and paper as instructed. It is easy to show that in first iteration,

$$Bx = \mathbf{0}$$

So in theory, our algorithm shouldn't work. However Figure 2 shows that it still works. The reason can be explained with rounding errors. This algorithm is an example of algorithms that works thanks to rounding errors.

```
>> power_method
Total runtime : 2.714396
The 1'th largest eigenvalue is 1.342686
The corresponding eigenvector is: [-0.407031 -0.0287614 0.912961 ]

 Total error = 0.000000
```

Figure 2: Finding the largest eigenvalue using the power method.Error shows the difference between Av and λv to check whether the vector we have found is an eigenvector with the true eigenvalue.

2

2.1

The basic idea of the given method is finding the next largest eigenvalue by deleting the largest one.We can rewrite :

$$A = Q\Lambda Q^T$$

where $Q = [\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||}..\frac{v_n}{||v_n||}]$ are the normalized eigenvectors of the matrix A and $\Lambda = [\lambda_1, \lambda_2, \lambda_3..\lambda_n]$ is the diagonal matrix that has eigenvalue in its columns. This matrix multiplication leads to:

$$A = \sum_{i=1}^{n} \lambda_i \frac{v_i v_i^T}{v_i^T v_i}$$

This decomposition is called spectral decomposition of a matrix A. Using these facts, if we subtract $\frac{\lambda_i v_i v_i^T}{v_i^T v_i}$, we would get :

$$A = \sum_{i=2}^{n} \lambda_i \frac{v_i v_i^T}{v_i^T v_i}$$

which means we make the eigenvalue with the largest magnitude equal to the 0 - so if we use power method, we would get second largest eigenvalue. If we keep apply this technique, we can get the k'th largest eigenvalue iteratively.

2.2

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2.3

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2.4

Let us compare the performance difference between two methods on can_{229} matrix of the University of Florida Sparse Matrix Collection. The result of the comparisons are given in the figure belows. It can be seen that methods converges to same eigenvalues, however we have used early stopping so that we can compare convergence speeds. From the results shown in the Figure 3, Figure 4 and Figure 5, we can assume that :

- With small k values (number of eigenvalue we are trying to find), power k method converges faster than the subspace iteration method.
- As we increase the k value, subspace iteration method starts to perform better than the power k method.

Note that even if we compute the eigenvalues, there are still differences between them. The reason is we have used same iteration number (10^4) epsilon value (10^{-14}) in early stopping in two kind of structure - matrix and vector. So this might be misleading. For this reason, now let us compare the raw speeds - without early stopping - of the algorithms to make our assumption we have made above stronger.

From the Figure 6, Figure 7 and Figure 8, it is clear that for the same iteration number (10^3), subspace iteration method performs faster as the k gets larger and power k algorithm is faster for smaller k values. Lastly, in Figure 9, we have found all the eigenvectors/eigenvalues for this particular matrix for the iteration number 10^4 to show that with higher iteration numbers and k values, subspace iteration method is the winner in terms of speed.

```
>> subspace_iteration
(K = {20})
Total runtime : 0.932133
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.989592
The 10'th largest eigenvalue is 6.997727
The 11'th largest eigenvalue is 6.939202
The 12'th largest eigenvalue is 6.939193
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.150998
The 15'th largest eigenvalue is 6.123666
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.891500
The 18'th largest eigenvalue is 5.816168
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

```
>> power_k_method
(K = {20})
Total runtime : 1.720574
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998533
The 10'th largest eigenvalue is 6.988804
The 11'th largest eigenvalue is 6.939188
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.891500
The 18'th largest eigenvalue is 5.816168
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

Figure 3: Comparison of running time and eigenvalues of power k and subspace iteration methods with k = 20.

```
>> subspace_iteration
(K = {100})
Total runtime : 1.225253
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.996958
The 10'th largest eigenvalue is 6.990370
```

```
>> power_k_method
(K = {100})
Total runtime : 9.788179
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998533
The 10'th largest eigenvalue is 6.988804
```

Figure 4: Comparison of running time and eigenvalues of power k and subspace iteration methods with k = 100.

```
(K = {200})
Total runtime : 1.003945
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.996332
The 10'th largest eigenvalue is 6.990999
```

```
(K = {200})
Total runtime : 19.684587
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998533
The 10'th largest eigenvalue is 6.988804
```

Figure 5: Comparison of running time and eigenvalues of power k and subspace iteration methods with k = 200.

```
>> subspace_iteration
(K = {20})
Total runtime : 1.620830
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998060
The 10'th largest eigenvalue is 6.989129
The 11'th largest eigenvalue is 6.939289
The 12'th largest eigenvalue is 6.939237
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.891500
The 18'th largest eigenvalue is 5.816168
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

```
>> power_k_method
(K = {20})
Total runtime : 0.203406
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998525
The 10'th largest eigenvalue is 6.988813
The 11'th largest eigenvalue is 6.939188
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.816168
The 18'th largest eigenvalue is 5.891500
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

Figure 6: Comparison of running time and eigenvalues of power k and subspace iteration methods without earlystopping with k = 20.

```
>> subspace_iteration
(K = {100})
Total runtime : 1.513162
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998385
The 10'th largest eigenvalue is 6.988945
The 11'th largest eigenvalue is 6.939196
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.891500
The 18'th largest eigenvalue is 5.816168
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

```
>> power_k_method
(K = {100})
Total runtime : 1.115291
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998525
The 10'th largest eigenvalue is 6.988813
The 11'th largest eigenvalue is 6.939188
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.816168
The 18'th largest eigenvalue is 5.891500
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

Figure 7: Comparison of running time and eigenvalues of power k and subspace iteration methods without earlystopping with k = 100.

```
>> subspace_iteration
(K = {200})
Total runtime : 1.591081
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.990107
The 10'th largest eigenvalue is 6.997231
The 11'th largest eigenvalue is 6.939188
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154956
The 15'th largest eigenvalue is 6.119708
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.891500
The 18'th largest eigenvalue is 5.816168
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

```
>> power_k_method
(K = {200})
Total runtime : 2.343432
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998525
The 10'th largest eigenvalue is 6.988813
The 11'th largest eigenvalue is 6.939188
The 12'th largest eigenvalue is 6.939188
The 13'th largest eigenvalue is 6.293620
The 14'th largest eigenvalue is 6.154966
The 15'th largest eigenvalue is 6.119698
The 16'th largest eigenvalue is 5.891500
The 17'th largest eigenvalue is 5.816168
The 18'th largest eigenvalue is 5.891500
The 19'th largest eigenvalue is 5.816168
The 20'th largest eigenvalue is 5.490218
```

Figure 8: Comparison of running time and eigenvalues of power k and subspace iteration methods without earlystopping with k = 200.

```
>> subspace_iteration
(K = {229})
Total runtime : 15.212711
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998533
The 10'th largest eigenvalue is 6.988804
```

```
>> power_k_method
(K = {229})
Total runtime : 19.382032
The 1'th largest eigenvalue is 8.696485
The 2'th largest eigenvalue is 8.474378
The 3'th largest eigenvalue is 8.474378
The 4'th largest eigenvalue is 7.901888
The 5'th largest eigenvalue is 7.901888
The 6'th largest eigenvalue is 7.727336
The 7'th largest eigenvalue is 7.442476
The 8'th largest eigenvalue is 7.442476
The 9'th largest eigenvalue is 6.998533
The 10'th largest eigenvalue is 6.988804
```

Figure 9: Comparison of running time and eigenvalues of power k and subspace iteration methods without earlystopping with k = 229.