# On the Throughput and Optimal Packet Length of an Uncoded ARQ System over Slow Rayleigh Fading Channels

ShaoYang Liu, XiaoHe Wu, Yong Xi, and JiBo Wei

***Abstract*—This letter presents a simplified expression for the average packet error rate (PER) for an uncoded packet transmission system with large packet length over slow Rayleigh fading channels, by assuming that the bit error rate (BER) has an exponential form. Then the throughput performance of an uncoded packet transmission system with automatic repeat request (ARQ) is investigated, and the optimal packet length is derived to maximize the throughput. The theoretical analysis is validated by simulations.**

***Index Terms*—Packet error rate (PER), automatic repeat request(ARQ), Rayleigh fading, optimal packet length.**

## I. Introduction

In modern wireless communication system, user information is generally transmitted in packets, combined with automatic repeat request (ARQ) schemes, and the packet error rate (PER) is then an important quality of service metric for wireless system, which largely determines the user throughput performance.

Although with a long history research, to evaluate the PER performance over fading channels still constitutes an important and tough work [1], considering the various modulation and coding schemes. In this context, usually bounding techniques are simplifying methods for the PER evaluation, e.g., in [2] Jenson's inequality is applied to bound the PER, and in [3] Chernoff upper bound is employed.

In [4] it is found that for a conventional packet transmission system over slow Rayleigh fading channels, the average PER of such system can be generally upper bounded by $1 - exp(-w_0/\overline{\gamma})$, where $w_0$ defined by an integral expression,corresponds exactly to the inverse of the coding gain.

Although being widely employed, the performance of various ARQ schemes over fading channels are usually evaluated by the information-theoretic bounds [5], which hinders the further investigation for user throughput, considering the practical modulation schemes and packet lengths. In [3] the authors obtained the throughput bounds for typical ARQ schemes considering the practical modulations, however the PER is complicatedly derived as the sum of serials, which obstructs the further optimization for packet length.

In this letter, based on the prior work in [4], we first obtain a simplified PER expression for an uncoded packet transmission system with large packet length, over slow Rayleigh fading channels. Then we investigate the throughput performance for an uncoded packet ARQ system, and derive the optimal packet length to maximize the throughput.

Manuscript received March 26, 2012. The associate editor coordinating the review of this letter and approving it for publication was A. Panagopoulos.

The authors are with the School of Electronic Science and Engineering, National University of Defense Technology, 410073, Changsha, Hunan, P.R. China (e-mail: {xiyong8260, wjbhw}@nudt.edu.cn).

Digital Object Identifier 10.1109/LCOMM.2012.060812.120626

## II. The Average PER Analysis

Let $f(\gamma)$ be the function relating PER to the instantaneous SNR under additive white Gaussian noise (AWGN), and let $p(\gamma)$ be the probability density function (PDF) of the received SNR. Then the average PER, denoted as $P_f(\overline{\gamma})$, can be computed by integrating $f(\gamma)$ over $p(\gamma)$ as:

$$P_f(\overline{\gamma}) = \int_0^\infty f(\gamma)p(\gamma)d\gamma, \tag{1}$$

where $\overline{\gamma}$ is the average SNR.

For Rayleigh fading, $\gamma$ is exponentially distributed with $p(\gamma) = \exp(-\gamma/\overline{\gamma})/\overline{\gamma}$, then (1) is given by:

$$P_f(\overline{\gamma}) = \frac{1}{\overline{\gamma}} \int_0^\infty f(\gamma) exp(-\gamma/\overline{\gamma})\, d\gamma. \tag{2}$$

In [4] it is concluded that the average PER of a system over a slow Rayleigh fading channel is upper bounded by

$$P_f(\overline{\gamma}) \leq 1 - exp(-w_0/\overline{\gamma}), \tag{3}$$

where $w_0$ is defined by

$$w_0 = \int_0^\infty f(\gamma)d\gamma, \tag{4}$$

in the high SNR region, the asymptotic average PER is given by:

$$P_f(\overline{\gamma}) \approx 1 - exp(-w_0/\overline{\gamma}). \tag{5}$$

It is found that when the packet length increases, (5) also becomes tighter [4].

Usually it is difficult to obtain an explicit closed-form expression for $w_0$ for a given modulation and coding scheme, which obstructs further investigation. However by making some simplifying assumptions, proposition 1 can be obtained.

*Proposition 1:* Consider an uncoded packet transmission with length $n$, if the BER function for the AWGN channels can be expressed as $b(\gamma) = ae^{-c\gamma}$, where $0 < a \leq 1$ and $c > 0$, the inverse coding gain over a slow Rayleigh fading channel is given by :

$$w_0(n) = \frac{1}{c}\left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{(1-a)^k}{k}\right), \tag{6}$$

when packet length $n >> 1$, $w_0(n)$ is asymptotically approximated by

$$w_0(n) \approx \frac{1}{c}\left(ln(n) + \gamma_e + ln(a)\right), \tag{7}$$

where $\gamma_e \approx 0.577$ is the Euler constant [6, eq. (6.1.3)].

*Proof:* The PER of an uncoded packet transmission with length $n$ for the AWGN channels can be given by:

$$f(\gamma) = 1 - (1 - b(\gamma))^n. \tag{8}$$

Substituting (8) into (3), it follows that:

$$\begin{aligned} w_0(n) &= \int_0^\infty (1 - (1 - b(\gamma))^n)\, d\gamma \\ &= \frac{1}{c}\int_0^a \left((1 - (1-y)^n)\frac{1}{y}\right) dy \quad (y \doteq ae^{-c\gamma}) \\ &= \frac{1}{c}\int_{1-a}^1 \left(\frac{1}{1-x}(1 - x^n)\right) dx \quad (x \doteq 1 - y) \\ &= \frac{1}{c}\left(\sum_{k=1}^n 1/k - \sum_{k=1}^n (1-a)^k/k\right). \end{aligned} \tag{9}$$

In (6), when $n >> 1$, the first item in the bracket can be approximated [6, eq. (6.1.3)] by :

$$\sum_{k=1}^n 1/k \approx ln(n) + \gamma_e, \tag{10}$$

and the second item is approximated by:

$$\sum_{k=1}^n \frac{(1-a)^k}{k} \approx \sum_{k=1}^\infty \frac{(1-a)^k}{k} = -ln(a). \tag{11}$$

Combining (6) (10) (11),when $n >> 1$, we arrive at (7). ■

In (7), letting $k_1 = 1/c$, $k_2 = \frac{1}{c}(\gamma_e + ln(a))$, then (7) can be rewritten as

$$w_0(n) \approx k_1 ln(n) + k_2. \tag{12}$$

Combining (5) and (12), the average PER for an uncoded packet transmission with large packet length over a slow Rayleigh fading channel is then approximated by:

$$\begin{aligned} P_f(\overline{\gamma}, n) &\approx 1 - exp\left(-\frac{k_1 ln(n) + k_2}{\overline{\gamma}}\right) \\ &= 1 - exp\left(-\frac{k_2}{\overline{\gamma}}\right) n^{\left(-\frac{k_1}{\overline{\gamma}}\right)}. \end{aligned} \tag{13}$$

Approximation (7) yields a quite good accuracy. For example, for FSK non-coherent demodulation, where $a = 1/2$,$c = 1/2$ [7], when $n = 32$ bits, (7) is less than (6) with 0.5% deviation, and when $n = 1024$ bits, the deviation is less than 0.01%. Therefore the accuracy of (13) is mainly determined by (5). As shown in [4], the accuracy of (5) improves with the increase of the SNR or the packet length.

To draw proposition 1, we have assumed that the BER has a form of $b(\gamma) = ae^{-c\gamma}$, which is true for FSK and DPSK non-coherent demodulations, with the BER forms of $\frac{1}{2}e^{-\frac{x}{2}}$ and $\frac{1}{2}e^{-x}$ respectively [7]. For many modulation schemes, e.g., M-ASK, M-PAM, MSK, M-PSK, the BER has the form $aQ(\sqrt{c\gamma})$ [7], where $Q(.)$ denotes the Gaussian Q_function. For $Q(.)$, there is a well known exponential upper bound $Q(x) < \frac{1}{2}exp\left(-x^2/2\right)$. Recently, the author in [8] also obtained a lower and approximate bound $Q(x) \approx \frac{1}{4}exp\left(-\frac{2}{\pi}x^2\right)$, which infers that (13) generally holds for broad modulation schemes. We also derived a tighter lower bound $Q(x) \approx 0.15\exp\left(-1.0636x^2\right)$ for the PER evaluation with large packet length. Due to the limit of this letter, we just omit the deduction details.

## III. The Throughput And Optimal Packet Length Analysis For ARQ System

For wireless packet systems, to improve the transmission reliability, various ARQ schemes are widely employed. In this section, we will discuss the throughput performance for a typical stop-wait ARQ system over a slow Rayleigh fading channel with unlimited retries. By the slow fading model,it is assumed that the channel state remains static during one ARQ round, but changes independently from round to round.

For stop-wait ARQ protocol with unlimited retries, the average retransmission number for a packet to be transmitted successfully would be [5]:

$$\overline{N} = \frac{1}{1 - P_f(\overline{\gamma}, n)}. \tag{14}$$

Let the time duration for a single ARQ round be $T_f = t_b(n + N_{extra})$, where $t_b$ denotes the time for one bit to be transmitted, and $N_{extra}$ denotes the number of equivalent bit inducing by ARQ protocol overheads, typically including the time duration for acknowledgement (ACK) transmitting, the guard interval for processing and propagation delay, etc.

For total $N$ user information bits to be transmitted, with the packet length $n$, the total transmission time would be $T_{total} = \frac{NT_f}{n(1-P_f(\overline{\gamma},n))}$, and the throughput, denoted as $\eta(n)$, is then given by:

$$\eta(n) = \frac{N}{T_{total}} = \frac{n(1 - P_f(\overline{\gamma}, n))}{T_f}. \tag{15}$$

Substituting (13) into (15), the throughput performance of a packet ARQ system over slow Rayleigh fading channels is derived as:

$$\eta(n) \approx exp\left(-\frac{k_2}{\overline{\gamma}}\right)\frac{1}{t_b(n + N_{extra})} n^{\left(1-\frac{k_1}{\overline{\gamma}}\right)}. \tag{16}$$

When $1 - \frac{k_1}{\overline{\gamma}} > 0$, it is found that $\eta(n)$ is a convex function, so we can discuss the optimal packet length to maximize the throughput.

In (16), taking the derivative of $\eta(n)$ with respect to $n$ and equating to zero, we resolve the optimal packet length as:

$$n_{opt} \approx N_{extra}\frac{\overline{\gamma} - k_1}{k_1} \tag{17}$$

Formula (17) shows that the optimal packet length depends on the modulation scheme, simply linearly increases with the number of equivalent overhead bit, and also increases with the average SNR.

When $1 - \frac{k_1}{\overline{\gamma}} \leq 0$, (16) is a monotonically decreasing function, which means one should choose the packet length as short as possible, subjected to the allowed minimum packet size of a system.

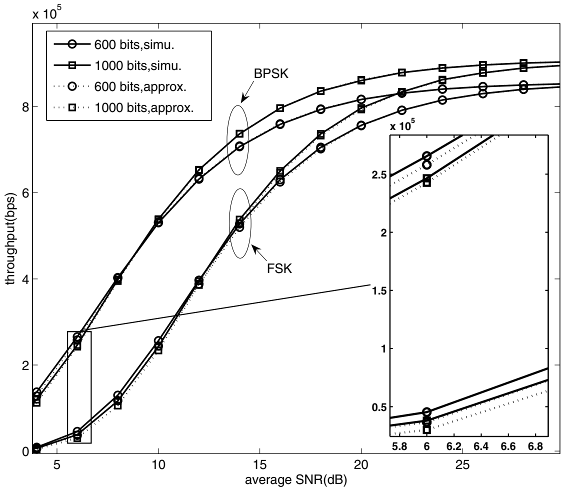


Fig. 1. throughput of a stop-wait ARQ system against average SNR.

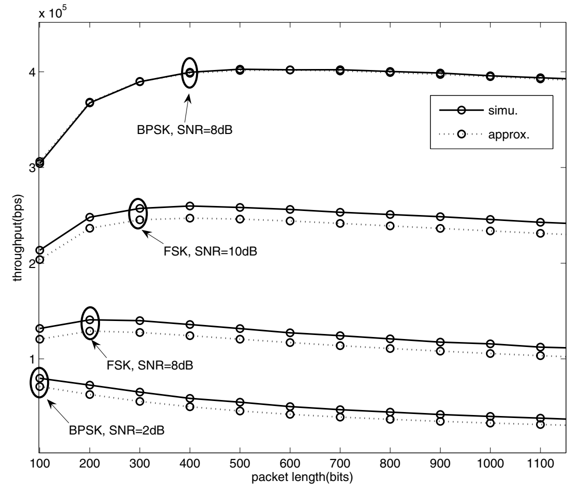


Fig. 2. throughput of a stop-wait ARQ system against packet length.

## IV. Simulation Results And Discussion

In this section, the simulation results are presented to validate the theoretical analysis in (16) and (17). The uncoded packets are transmitted with FSK and BPSK modulation schemes respectively over Rayleigh fading channels, with $t_b$ of 1us, or equivalently, 1Mbps data rate, and with $N_{extra}$ of 100 bits. For BPSK modulation, which has the BER form of $Q(\sqrt{2\gamma})$ [7], we employ our approximation $Q\left(x\right) \approx 0.15\exp\left(-1.0636x^2\right)$.

In Fig.1, the throughput against average SNR for two modulation schemes are presented with the packet lengths of 600 and 800 bits respectively, by both simulation and the approximation of (16). It is shown that the theoretical approximation curves fit the simulation results well when the average SNR is larger than 20dB.

For FSK with 600 bits packet length, it is found that the theoretical approximation yields a 20% deviation from the simulation results at SNR 6dB, and the deviation reduces rapidly to 0.1% at 20dB. It validates a conclusion that the accuracy of (13) improves with the increase of the average SNR. It is also observed that the accuracy of BPSK is much better than that of FSK, with the deviations of 3% and 0.04% for SNR 6dB and 20dB respectively for the same packet size. It is because that BPSK scheme has a lower BER than FSK, and thus has a steeper $f(\gamma)$ function, which leads to tighter approximation in (5). For the definition of the term 'steep', please refer to [4].

When the SNR increase large enough,the throughput curves for different modulation schemes converge to the same value, which can be inferred by (16), when $\bar{\gamma} \to \infty$, the modulation factors $k_1$ and $k_2$ could be ignored, therefore,$\eta(n) \approx \frac{n}{t_b(n+N_{extra})}$.

In Fig.2, we plot the throughput curves against packet length, by both simulation and the approximation of (16), to validate the optimal packet length analysis in (17). For FSK, two average SNRs of 8dB and 10dB are selected, while for BPSK, the SNRs are selected as 2dB and 8dB. By (17) it is calculated that the optimal packet lengths for FSK are 215 and 400 bits respectively, and 68 and 570 bits respectively for BPSK, which agree with the simulation results well. It is shown that the optimal packet length increases as the SNR increases. It is also indicated that with the same SNR, the optimal packet length of BPSK is larger than that of FSK, which can be explained by (17), since FSK has a larger $k_1$ ($k_1 = 2$) than BPSK ($k_1 \approx 0.94$).

## V. Conclusion

In this letter, we first derive the exact closed-form expression of $w_0$ for an uncoded packet system, by making exponential BER assumption, and then obtain an extremely simplified PER expression for large packet length. With the derived simplified PER expression, we further investigate the throughput performance for an uncoded packet ARQ system over slow Rayleigh fading channels, and then derive a simplified expression for the optimal packet length to maximize the throughput.

## References

[1] E. Malkamaki and H.Leib, "Type-II hybrid ARQ with convolutional codes over block fading Rayleigh channels," in *Proc. 1997 IEEE PIMRC*.
[2] M. Chiani, "Error probability for block codes over channels with block interference," *IEEE Trans. Inf. Theory*, vol. 44, no. 7, pp. 2998–3008, Nov. 1998.
[3] L. Cao and P. Kam, "On the performance of packet ARQ schemes in Rayleigh fading: the role of receiver channel state information and its accuracy," *IEEE Trans. Veh. Technol.*, vol. 60, no. 2, pp. 2998–3008, Nov. 2011.
[4] Y. Xi, A. Burr, J. B. Wei, and D. Grace, "A general upper bound to evaluate packet error rate over quasi-static fading channels," *IEEE Trans. Wireless Commun.*, vol. 10, no. 5, pp. 1373–1377, May 2011.
[5] G. Caire and D. Tuninetti, "The throughput of hybrid-ARQ protocols for the Gaussian collision channel," *IEEE Trans. Inf. Theory*, vol. 47, no. 4, pp. 1971–1988, July 2001.
[6] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th edition. Dover, 1970.
[7] J. G. Proakis, *Digital Communications*, 4th edition. McGraw-Hill, 2001.
[8] M. Wu, X. Lin, and P. Y. Kam, "New exponential lower bounds on the Gaussian Q-function via Jensen's inequality," in *Proc. 2011 IEEE VTC – Spring*.