Algebraic and Topological Persistence

by

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Abstract

Dedication

I would like to dedicate this work to the people closest to me, without whom I would not have been able to write these pages, without whom I would not have been able to embark on this trip, without whom I would not have had the desire to continue on this journey.

My path led me to study German with a minor in Italian, to major in Information Science with a minor in Media Informatics and finally to a Master's degree in Information Science in Regensburg. After a three-year doctorate in the field of Computer Science in Erlangen, I finally decided not to continue with my dissertation but to devote myself to studying theoretical mathematics. I was asked whether I could make up my mind and whether I was finally satisfied with what I was doing. To be honest, I was always interested in the structure of, well, basically elements. That just translated into language or computer programs. I'm glad I was able to discover that for myself.

I would like to thank my mother Beata, who was always skeptical about these decisions, but always supported them once they were made.

I would like to thank my late father Domenico, who always encouraged me to think more about yesterday and less about tomorrow.

I would like to thank my sister Dominique, who never missed an opportunity to ask me questions about physics whenever she could.

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I would also like to thank all those whom I cannot mention by name. If you feel touched by reading these lines, then I'm sure they affect you in one way or another.

Perhaps I have learned a lesson or two from you.

Declaration

I hereby declare that I

- alone wrote the submitted Bachelor's thesis without illicit or improper assistance.
- did not use any materials other than those listed in the bibliography and that all passages taken from these sources in part or in full have been marked as citations and their sources cited individually in the thesis. Citations include the version (edition and year of publication), the volume and page numbers of the cited work.
- have not submitted this Bachelor's thesis to another institution and that it has never been used for other purposes or to fulfill other requirements, in part or in full.

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My special thanks go to my supervisor, Prof. Dr. Kang Li. Prof. Li always motivated me with his willingness to discuss when I had already invested too much time in reading the scientific papers and lacked a clear view in the forest of theorems. Often also when there were tricky passages in the research literature whose evidence first had to be deciphered. Prof. Li also took the time to reduce the sprawling content to an appropriate level for a bachelor's thesis. I would like to thank him for his excellent supervision.

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We would like to emphasize that a collection of points $X = \{x_0, x_1, \ldots, x_d\}$ in $\mathbb{R}^n$ is considered to be affinely independent if these points do not lie within any affine subspace of dimension lower than d.

Definition 2.2.1 *(d-Simplex) [1, §2.1]* Given a set $X = \{x_0, x_1, \ldots, x_d\} \subset \mathbb{R}^n$ consisting of $d + 1$ affinely independent points, the d-dimensional simplex $\sigma^{(d)}$, also known as a d-simplex, is defined as the set of all convex combinations of these points.

$$\sigma^{(d)} := \left\{ \sum_{i=0}^{d} \lambda_i x_i \mid \sum_{i=0}^{d} \lambda_i = 1,\ \lambda_i \geq 0 \right\}. \tag{2.1}$$

As a convention, the empty set $\emptyset$ is included as a face, representing the simplex formed by the empty subset of vertices. A 0-simplex represents a single point, a 1-simplex represents a line segment connecting two points, a 2-simplex represents a triangle, and a 3-simplex represents a tetrahedron. It is worth mentioning that the d-simplex is homeomorphic to the d-dimensional disk D^d.

Theorem 2.2.1 *The d-simplex $\sigma^{(d)}$ is homeomorphic to the d-dimensional disk D^d.*

Proof: Define the standard d-simplex $\sigma^{(d)}$ as

$$\sigma^{(d)} = \left\{ (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} x_i = 1, x_i \geq 0 \right\}$$

and the d-dimensional disk D^d as

$$D^d = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i^2 \leq 1 \right\}.$$

We construct a homeomorphism $f : \sigma^{(d)} \to D^d$ by

$$f(x_1, \ldots, x_{d+1}) = (\sqrt{x_1}, \ldots, \sqrt{x_d}),$$

where $x_{d+1} = 1 - \sum_{i=1}^{d} x_i$. This map is well-defined since $\sum_{i=1}^{d} (\sqrt{x_i})^2 = \sum_{i=1}^{d} x_i \leq 1$.

The inverse $g : D^d \to \sigma^{(d)}$ is given by

$$g(y_1, \ldots, y_d) = (y_1^2, \ldots, y_d^2, 1 - \sum_{i=1}^{d} y_i^2),$$

ensuring that g is well-defined because $\sum_{i=1}^{d} y_i^2 \leq 1$ implies $1 - \sum_{i=1}^{d} y_i^2 \geq 0$.

Both f and g are continuous and inverses of each other as shown by $f(g(y)) = y$ for all $y \in D^d$ and $g(f(x)) = x$ for all $x \in \sigma^{(d)}$. □

Furthermore, it is worth noting that $\sigma^{(d)}$ represents the convex hull of the points X, which can be defined as the smallest convex subset of $\mathbb{R}^n$ that contains all the points $x_0, x_1, \ldots, x_d$. The faces of the simplex $\sigma^{(d)}$ with vertex set X are simplices formed by subsets of X. An d-face of a simplex refers to a subset of the vertices of the simplex with a cardinality of $d + 1$. The faces of a d-simplex with a dimension less than d are known as its proper faces. Two simplices are considered to be properly situated if their intersection is either empty or a face of both simplices. By identifying simplices along entire faces, we can construct the resulting simplicial complexes.

Definition 2.2.2 *(Simplicial Complex) [1, §2.2]* A simplicial complex K is a finite collection of simplices that satisfies the following properties:

1. For every simplex $\sigma^{(d)}$ in K and every face $\tau^{(k)}$ with $k < d$ of $\sigma^{(d)}$, it follows that $\tau^{(k)}$ is also in K.
2. If $\sigma^{(d)}$ and $\tau^{(k)}$ are both simplices in K, then they are properly situated.

The dimension of K is defined as the highest dimension among its simplices. For a simplicial complex K in $\mathbb{R}^n$, its underlying space $|K|$ is the union of all the simplices in K. The topology of K is determined by the topology induced on $|K|$ by the standard topology in $\mathbb{R}^n$. It is important to note that when the vertex set is known, a simplicial complex in $\mathbb{R}^n$ can be fully characterized by listing its simplices. As a result, we can describe it purely in terms of combinatorics using abstract simplicial complexes.

Definition 2.2.3 *(Abstract Simplicial Complex) [1, §2.3]* Consider a finite set $V = \{v_1, \ldots, v_n\}$. An abstract simplicial complex $\tilde{K}$ with vertex set V is a collection of finite subsets of V that satisfies the following two conditions:

1. All elements of V are included in $\tilde{K}$.
2. If $\sigma^{(d)}$ is a subset of $\tilde{K}$ and $\tau^{(k)}$ is a subset of $\sigma^{(d)}$, then $\tau^{(k)}$ is also a subset of $\tilde{K}$.

The abstract simplicial complex $\tilde{K}$ associated with a simplicial complex K is commonly referred to as its vertex scheme. Conversely, if an abstract complex $\tilde{K}$ serves as the vertex scheme for a complex K in $\mathbb{R}^n$, then K is known as a geometric realization of $\tilde{K}$.

Lemma 2.2.2 *Every finite abstract simplicial complex $\tilde{K}$ can be realized geometrically in an Euclidean space.*

Proof: Let $\{v_1, v_2, \ldots, v_n\}$ denote the vertex set of $\tilde{K}$, where n represents the number of vertices in $\tilde{K}$. Consider $\sigma^{(n-1)} \subset \mathbb{R}^n$, the simplex formed by the span of $\{e_1, e_2, \ldots, e_n\}$, where e_i represents the ith unit vector. In this context, K refers to the subcomplex of $\sigma^{(d)}$ such that $[e_{i_0}, \ldots, e_{i_d}]$ is a d-simplex of K if and only if $[v_{i_0}, \ldots, v_{i_d}]$ is a simplex of $\tilde{K}$. □

Remark 2.2.4 *All realizations of an abstract simplicial complex are homeomorphic to each other. The specific realization mentioned above is referred to as the natural realization.*

Furthermore, it has been proven that any finite abstract simplicial complex of dimension d can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$.

Theorem 2.2.3 *Any finite abstract simplicial complex of dimension d can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$.*

Proof: Let K be a finite simplicial complex of dimension d. We construct an injective geometric realization $f : \tilde{K} \to \mathbb{R}^{2d+1}$ where $\tilde{K}$ denotes the abstract simplicial complex associated with K.

Define an injective map $\tilde{f} : V(K) \to \mathbb{R}^{2d+1}$ for the vertex set $V(K)$ of K. Since $V(K)$ is finite and $\mathbb{R}^{2d+1}$ has sufficient dimensionality, injectivity is guaranteed.

Adjust $\tilde{f}$ if necessary to ensure the images of the vertices of each simplex $\sigma \in K$ are affinely independent. This can be achieved by slight perturbations within $\mathbb{R}^{2d+1}$, leveraging the ample dimensionality to avoid overlaps.

We extend $\tilde{f}$ to a map $f : \tilde{K} \to \mathbb{R}^{2d+1}$ by defining it on each simplex $\sigma = [v_0, \ldots, v_k]$ through the unique affine map $v_i \mapsto \tilde{f}(v_i)$.

The injective and affine properties of $\tilde{f}$ on vertices guarantee that f is injective over each simplex and preserves the simplicial structure, meaning $f(|\sigma| \cap |\tau|) = f(|\sigma|) \cap f(|\tau|)$ for any simplices $\sigma, \tau \in K$.

Thus, f realizes K as a geometric simplicial complex in $\mathbb{R}^{2d+1}$. □

2.3 Simplicial Homology Groups

Given a set V representing the vertices of a d-simplex $\sigma^{(d)}$, we can establish an orientation for the simplex by selecting a specific ordering for the vertices. If the vertex ordering differs from our chosen order by an odd permutation, it is considered reversed, while even permutations are said to preserve the orientation. Consequently, any simplex can have only two possible orientations. Moreover, the orientation of a d-simplex induces an orientation on its $(d-1)$-faces. To be more precise, if $\sigma^{(d)} := (v_0, v_1, \ldots, v_d)$ represents an oriented d-simplex, then the orientation of the $(d-1)$-face $\tau^{(d-1)}$ of $\sigma^{(d)}$ with the vertex set $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is given by $\tau_i^{(d-1)} = (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$.

Definition 2.3.1 *(d-Chain) [7, §2.3]* Given a set $\{\sigma_1^{(d)}, \ldots, \sigma_k^{(d)}\}$ of arbitrarily oriented d-simplices of a complex K and an abelian group G, we define a d-chain c with coefficients $g_i \in G$ as a formal sum.

$$c := g_1\sigma_1^{(d)} + g_2\sigma_2^{(d)} + \ldots + g_k\sigma_k^{(d)} = \sum_{i=1}^{k} g_i\sigma_i^{(d)}. \tag{2.2}$$

Henceforth we will assume that $G = (\mathbb{Z}, +)$.

Lemma 2.3.1 *The set of simplicial d-chains C_d^Δ is an abelian group $(C_d^\Delta, +)$.*

Proof: The identity element of the group is represented by the empty chain $\sum_{i\in\emptyset} g_i\sigma_i^d = e_{C_d^\Delta} = e_G = 0$.

The sum of two chains is defined as $c + c' = \sum_{i=1}^k g_i\sigma_i^{(d)} + \sum_{j=1}^l g'_j\sigma_j^{(d)} = \sum_{i=1}^k (g_i + g'_i)\sigma_i^{(d)} + \sum_{j=k+1}^l g'_j\sigma_j^{(d)}$ if $k \leq l$ and $c + c' = \sum_{i=1}^k g_i\sigma_i^{(d)} + \sum_{j=1}^l g'_j\sigma_j^{(d)} = \sum_{i=1}^l (g_i + g'_i)\sigma_i^{(d)} + \sum_{j=l+1}^k g_j\sigma_j^{(d)}$ if $k > l$, thus, we can conclude that $c + c' \in C_d^\Delta$.

The associativity of the group operation in C_d^Δ follows directly from the associativity of the group operation in G.

The inverse element is defined by $e_{C_d^\Delta} = c + (-c) = \sum_{i=1}^k g_i\sigma_i^{(d)} + \sum_{i=1}^k (-g_i)\sigma_i^{(d)} = \sum_{i=1}^k (g_i - g_i)\sigma_i^{(d)}$ with $c, -c \in C_d^\Delta$. □

Definition 2.3.2 *(Boundary) [6, p.106]* Let $\sigma^{(d)}$ be an oriented d-simplex in a complex K. The boundary of $\sigma^{(d)}$ is defined as the simplicial $(d-1)$-chain of K with coefficients in the abelian group $g_i \in G$, given by

$$\partial(\sigma^{(d)}) = g_0\sigma_0^{(d-1)} + g_1\sigma_1^{(d-1)} + \ldots + g_d\sigma_d^{(d-1)} = \sum_{i=1}^{d} g_i\sigma_i^{(d-1)} \tag{2.3}$$

where $\sigma_i^{(d-1)}$ is an $(d-1)$-face of $\sigma^{(d)}$. If $d = 0$, we define $\partial(\sigma^{(0)}) = e_G = 0$.

In the following, we will set $G = \mathbb{Z}$. Since $\sigma^{(d)}$ is an oriented simplex, the $\sigma_i^{(d-1)}$-faces also have associated orientations. We can extend the definition of the boundary linearly to elements of C_d^{Δ}.

Lemma 2.3.2 *The* ***boundary operator*** *is a group homomorphism*

$$\partial : C_d^{\Delta} \to C_{d-1}^{\Delta}.$$

Proof: We define the boundary operator for a d-chain $c = \sum_{i=1}^{k} g_i\sigma_i^{(d)}$ as follows: $\partial(c) = \sum_{i=1}^{k} g_i\ \partial(\sigma_i^{(d)}) = \sum_{i=1}^{k} g_i \sum_{j=1}^{d} \sigma_j^{(d-1)} = \sum_{i=1}^{k}\sum_{j=1}^{d} g_i\sigma_j^{(d-1)}$ which is an element of C_{d-1}^{Δ}, where $\sigma_i^{(d)}$ are the d-simplices of K.

We can compute this by

$$\partial(c + c') = \partial(\sum_{i=1}^{k} g_i\sigma_i^{(d)} + \sum_{j=1}^{l} g_j'\sigma_j^{(d)}) \tag{2.4}$$

$$= \sum_{i=1}^{k} g_i\partial(\sigma_i^{(d)}) + \sum_{j=1}^{l} g_j'\partial(\sigma_j^{(d)}) \tag{2.5}$$

$$= \partial(c) + \partial(c'). \tag{2.6}$$

□

Example 2.3.3 *Let's consider the* 2*-simplex* $\sigma^{(2)}$ *with vertices* v_0*,* v_1*, and* v_2*. The* 1*-faces of this simplex are* $e_0 = (v_1, v_2)$ *connecting* v_1 *and* v_2*,* $e_1 = (v_2, v_0)$ *connecting* v_2 *and* v_0*, and* $e_2 = (v_0, v_1)$ *connecting* v_0 *and* v_1*. Now, let's proceed with the computation.*

$$\partial(\partial(\sigma^{(2)})) = \partial(e_0 + e_1 + e_2) \tag{2.7}$$

$$= \partial(e_0) + \partial(e_1) + \partial(e_2) \tag{2.8}$$

$$= \partial(v_1, v_2) + \partial(v_2, v_0) + \partial(v_0, v_1) \tag{2.9}$$

$$= [(v_2) - (v_1)] + [(v_0) - (v_2)] + [(v_1) - (v_0)]. \tag{2.10}$$

We observe that C_0^Δ is an abelian group and that oppositely oriented simplices cancel each other out, resulting in $\partial(\partial(\sigma^{(2)})) = 0$. This property can be generalized to higher dimensions through induction. Therefore, since ∂ is a linear operator and the chain c is a sum of d-simplices, we can conclude that $\partial^2(c) = 0$ for any d-chain c in C_d^Δ. Consequently, the boundary of the boundary is zero. Moreover, if the boundary of a simplex is zero, it is referred to as a cycle. By this definition, we can deduce that the boundary of any simplex is a cycle.

Definition 2.3.4 *(Cycles) [6, p.106]* A d-chain is a cycle if its boundary is equal to zero.

We denote the set of d-cycles of a complex K over the group $\mathbb{Z}$ as Z_d^Δ, the simplicial cycle group. It is important to note that Z_d^Δ is a subgroup of C_d^Δ and can also be expressed as $Z_d^\Delta = \ker(\partial_d)$.

A d-cycle of a k-complex K is said to be homologous to zero if it can be expressed as the boundary of an $(d+1)$-chain in K, where $d = 0, 1, \dots, k-1$. In other words, a cycle is considered a boundary if it can be "filled in" by a higher-dimensional chain. This equivalence relation is denoted as $c \sim 0$.

Definition 2.3.5 *(Boundary Group) [7, §2.3]* The subgroup of Z_d^Δ consisting of boundaries is referred to as the simplicial boundary group B_d^Δ.

It is worth noting that B_d^Δ is equal to the image of the boundary operator ∂_{d+1}. Since B_d^Δ is a subgroup of Z_d^Δ and Z_d^Δ is an abelian group, every subgroup of Z_d^Δ is normal. Therefore, we can construct the group quotient $H_d^\Delta = Z_d^\Delta / B_d^\Delta$.

Definition 2.3.6 *(Simplicial Homology Group) [6, §2.1]* The group H_d^Δ represents the d-dimensional simplicial homology group of the complex K over $\mathbb{Z}$. It can be expressed as the group quotient $\ker(\partial_d)/\mathrm{im}(\partial_{d+1})$.

2.4 Persistent Homology

2.5 Persistent (Co)homology

Inverse problems primarily involve inferring geometric shapes from measurements like path integrals. Classical methods such as Fourier transforms provide extensive information but struggle with nonlinearity and ill-posed conditions, requiring substantial regularization. Topology, particularly through persistent homology, offers alternative methods for deducing topological rather than geometric information. This approach is especially useful in high-dimensional, discrete sets of points, exemplified in the finite case by geological sonar to detect subterranean features based on density variations [5, §1]. Persistent homology identifies topological features represented as intervals in a barcode or persistence diagram, crucial for understanding the presence and persistence of features such as holes or voids in topological spaces. This method is statistically robust and can provide both qualitative and quantitative insights into point sets, which we suspect to lie on some compact topological object [2, 3].

In our work, we continue to examine the consequences of absolute and relative homology and cohomology groups for filtrations of cell complexes, such as the introduced simplicial complexes. In particular, we derive the theory in the context of filtered simplicial complexed upon sets of points, embedded into some metric space. While we can apply the entire theory to simplicial complexes and their (co)homology, cell complexes provide a much broader context and significantly simplify notation in many instances.

In particular, there are at least four naturally arising persistent objects that can be extracted from a filtration of any topological space. We follow the results of de Silva, Morozov, and Vejdemo-Johansson for our explanations [5, §1]. They are

$$\text{persistent} \begin{Bmatrix} \text{absolute} \\ \text{relative} \end{Bmatrix} \begin{Bmatrix} \text{homology} \\ \text{cohomology} \end{Bmatrix}.$$

In this work, we address the computation of barcodes for all four types of persistent objects. We demonstrate that both absolute and relative (co)homologies yield identical barcodes and that transitions between these states are facilitated by established duality principles. The duality between homology and cohomology is akin to the duality in vector spaces, whereas a global duality specific to persistent topology allows for a unique interchange:

$$\begin{aligned} \text{Absolute homology} &\rightleftarrows \text{relative cohomology.} \\ \text{Absolute cohomology} &\rightleftarrows \text{relative homology.} \end{aligned}$$

The main results from the literature suggest that a single calculation is sufficient to compute all four persistent objects due to the commutative nature of the global duality.

2.5.1 Filtrations of Complexes

Individual (co)homology groups are consistently defined with coefficients in a fixed field $\mathbb{K}$, organizing persistent (co)homology as a graded module over $\mathbb{K}[x]$, as can be seen in the next chapter **??**. Replacing $\mathbb{K}$ with the ring $\mathbb{Z}$ leads to substantial issues, discussed in [7, §3.1].

We investigate the persistent topology of filtered topological spaces, with a primary focus on the prototypical example of a filtered cell complex. This structure is defined by a sequence $\mathbb{X}$ of cell complexes:

$$\mathbb{X} : \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_n = X_\infty,$$

where X_1 starts with a single vertex σ_1, and each subsequent complex X_i is constructed by adding a single cell to the previous complex:

$$X_i := X_{i-1} \cup \sigma_i.$$

Here, the indexing set is $\{1, 2, \ldots, n\}$. Additionally, associated real values a_i are assigned to these indices, satisfying $a_1 \leq a_2 \leq \cdots \leq a_n$. This formulation clearly delineates the stepwise enlargement of the complex, illustrating the dynamic evolution of its topology as new cells are incrementally incorporated. We will use the same running example as in [5, §2.2].

Example 2.5.1 *The chosen illustrative example involves a cellular filtration of the 2-sphere, denoted by $\mathbb{S}^2$. This process constructs a cell complex and introduces an ordering among the cells to facilitate differentiation between them. The development of the filtration can be articulated as follows [5, §2.2]:*

$$\begin{aligned} \mathbb{S}^2 :\ & \emptyset \\ & \subset S_1 = \{\sigma_1\} \\ & \subset S_2 = \{\sigma_1, \sigma_2\} \\ & \subset S_3 = \{\sigma_1, \sigma_2, \sigma_3 := (\sigma_1, \sigma_2)\} \\ & \subset S_4 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4 := (\sigma_2, \sigma_1)\} \\ & \subset S_5 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 := (\sigma_3, \sigma_4)\} \\ & \subset S_6 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 := (\sigma_4, \sigma_3)\}. \end{aligned}$$

In this sequence, the cell σ_1 symbolizes the initial point. The set S_2 incorporates two distinct points. In S_3, a path connecting σ_1 and σ_2 is introduced. S_4 augments this path with its reverse—from σ_2 to σ_1—clearly distinguishing it from σ_3. Finally, S_5 and S_6 progressively differentiate between cells that represent the upper and lower halves of the sphere, respectively.

2.5.2 Persistent Homology on Complexes

In the context of algebraic topology, applying the homology functor $\mathfrak{H}$ to a filtration of a complex $\mathbb{X}$ yields a sequence of algebraic structures:

$$\mathfrak{H}(\mathbb{X}) : \quad \mathfrak{H}(X_1) \to \mathfrak{H}(X_2) \to \cdots \to \mathfrak{H}(X_n), \tag{2.11}$$

where $\mathfrak{H}(-)$ generally represents either the k-th dimensional homology, denoted by $H_k(-; \mathbb{F})$, or the total homology, expressed as $H_\bullet(-; \mathbb{F})$. Here, this diagram characterizes a sequence of abelian groups or finite-dimensional vector spaces, interconnected through vector space homomorphisms, forming what is known as a persistence module.

Persistence modules are central to understanding how the features of a space evolve over time. They can typically be decomposed into a direct sum of interval modules (3.5). Each interval module is associated with an ordered pair of integers (p, q) where $1 \leq p \leq q \leq n$, within a finite filtration. These pairs (p, q) signify topological features that persist over an index set $I := \{p, \ldots, q\}$, where $\inf\{I\} = p$ and $\sup\{I\} = q$. Conventionally, these tuples are interpreted as half-open intervals $[a_p, a_{q+1})$, with $a_{n+1} = \infty$ being a customary notation when the sequence extends beyond the largest indexed space.

The decomposition of a persistence module into its constituent interval modules is represented in a persistence diagram (3.4), or a barcode (3.3). This barcode is a multiset of ordered tuples (p, q) or, alternatively, a multiset of half-open intervals $[a_p, a_{q+1})$. This collection is formally expressed through the forgetful functor Pers$(-)$:

$$\begin{aligned} \mathrm{Pers}(\mathfrak{H}(\mathbb{X})) &= \{(p_1, q_1), \ldots, (p_m, q_m)\} && (2.12) \\ &\cong \{[a_{p_1}, a_{q_1+1}), \ldots, [a_{p_m}, a_{q_m+1})\}. && (2.13) \end{aligned}$$

In practical applications, intervals where $a_p = a_{q+1}$ are usually omitted, as they represent ephemeral topological features.

Example 2.5.2 *In the further elaboration of the example previously cited, which is also described in 2.5.1, we consider the topological subspaces S_1, S_3, S_5, all of which are contractible. Meanwhile, S_2, S_4, S_6 are homeomorphic to the 0-sphere, 1-sphere, and 2-sphere, respectively. This structural distinction leads to four distinct intervals in the persistence diagram of the total homology of a sphere, specifically $\mathbb{S}^2$:*

$$\begin{aligned} Pers(H_\bullet(\mathbb{S}^2)) &= \{(1,6)_0, (2,2)_0, (4,4)_1, (6,6)_2\} && (2.14) \\ &= \{(1,\infty)_0, (2,3)_0, (4,5)_1, (6,\infty)_2\}. && (2.15) \end{aligned}$$

Here, the subscript k in $(p, q)_k$ or $[a_p, a_{q+1})_k$ denotes a topological feature in the k-dimensional homology.

2.5.3 The Four Standard Persistence Modules

The standard module of persistent homology, $H_\bullet(\mathbb{X})$, illustrates how the absolute homology groups $H_\bullet(X_i)$ relate to each other as the index i changes. Similar observations can be made by considering the absolute cohomology groups $H^\bullet(X_i)$, the relative homology groups $H_\bullet(X_n, X_i)$, and the relative cohomology groups $H^\bullet(X_n, X_i)$ [5, §2.4]. This leads to the following sequences:

$$\begin{aligned}
H_\bullet(\mathbb{X}) :&\quad H_\bullet(X_1) \to \cdots \to H_\bullet(X_{n-1}) \to H_\bullet(X_n)\\
H^\bullet(\mathbb{X}) :&\quad H^\bullet(X_1) \leftarrow \cdots \leftarrow H^\bullet(X_{n-1}) \leftarrow H^\bullet(X_n)\\
H_\bullet(X_\infty, \mathbb{X}) :&\quad H_\bullet(X_n) \to H_\bullet(X_n, X_1) \to \cdots \to H_\bullet(X_n, X_{n-1})\\
H^\bullet(H_\infty, \mathbb{X}) :&\quad H^\bullet(X_n) \leftarrow H^\bullet(X_n, X_1) \leftarrow \cdots \leftarrow H^\bullet(X_n, X_{n-1}).
\end{aligned}$$

2.5.4 Barcode Isomorphisms

2.5.5 Persistent Chain Complexes

2.5.6 Cohomology of Chain Complexes

2.6 Distances and the Stability Theorem

Chapter 3

Algebraic Persistence

3.1 Persistence Modules

Definition 3.1.1 *(Persistence complex)* A **persistence complex** is an indexed family of chain complexes $\{C^i_\bullet, \partial)\}_{i \in I}$ along with chain maps $f^i : C^i_\bullet \to C^{i+1}_\bullet$.

Every finite filtration $\emptyset \subset K^0 \subset K^1 \subset \cdots \subset K^m$ generates a finite persistence complex. From each of the simplicial complexes K^i one can generate a chain complex $C^i_\bullet$ according to **??**. The chain map $f^i C^i_\bullet \to C^{i+1}_\bullet$ is an inclusion, since every n-simplex in K^i is also contained in K^{i+1} and the inclusion is naturally transferred to n-chains, since these are merely linear combinations of n-simplices.

Definition 3.1.2 *(Persistence module [4, §1.1])* A **persistence $\mathbb{R}$-module** $\mathbb{V}$ is an indexed family of vector spaces $(V_r \mid r \in \mathbb{R})$ and a doubly-indexed family of linear maps $(v^t_s : V_s \to V_t \mid s \le t)$ which satisfy the composition law $v^t_s \circ v^s_r = v^t_r$ whenever $r \le s \le t$, and where $v^t_t = \mathrm{id}_{V_t}$.

Remark 3.1.3 *In general, we can define a persistence module over any partially ordered set T instead of the real numbers. This is called a **T-persistence module**.*

Example 3.1.4 *Consider the following persistence modules:*

1. *Let X be a topological space and let $f : X \to \mathbb{R}$ be a function. Consider the sublevelsets $X^t := (X, f)^t = \{x \in X \mid f(x) \le t\}$. The inclusion maps $(i^t_s : X^s \to X^t \mid s \le t)$ satisfy trivially the composition law $i^t_s \circ i^s_r = i^t_r$ whenever $r \le s \le t$ with i^t_t the identity on X^t. This information is also called sublevelset filtration of (X, f) and is denoted as $\mathbb{X}^f_{sub}$ [4, p.7].*

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_{n+2}"] & \vdots \arrow[d, "\partial_{n+2}"] & \vdots \arrow[d, "\partial_{n+2}"] & \vdots \arrow[d, "\partial_{n+2}"] & \\
\cdots \arrow[r, "f^{i-2}_{n+1}"] & C^{i-1}_{n+1} \arrow[r, "f^{i-1}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i}_{n+1} \arrow[r, "f^{i}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i+1}_{n+1} \arrow[r, "f^{i+1}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i+2}_{n+1} \arrow[r, "f^{i+2}_{n+1}"] \arrow[d, "\partial_{n+1}"] & \cdots \\
\cdots \arrow[r, "f^{i-2}_{n}"] & C^{i-1}_{n} \arrow[r, "f^{i-1}_{n}"] \arrow[d, "\partial_{n}"] & C^{i}_{n} \arrow[r, "f^{i}_{n}"] \arrow[d, "\partial_{n}"] & C^{i+1}_{n} \arrow[r, "f^{i+1}_{n}"] \arrow[d, "\partial_{n}"] & C^{i+2}_{n} \arrow[r, "f^{i+2}_{n}"] \arrow[d, "\partial_{n}"] & \cdots \\
\cdots \arrow[r, "f^{i-2}_{n-1}"] & C^{i-1}_{n-1} \arrow[r, "f^{i-1}_{n-1}"] \arrow[d, "\partial_{n-1}"] & C^{i}_{n-1} \arrow[r, "f^{i}_{n-1}"] \arrow[d, "\partial_{n-1}"] & C^{i+1}_{n-1} \arrow[r, "f^{i+1}_{n-1}"] \arrow[d, "\partial_{n-1}"] & C^{i+2}_{n-1} \arrow[r, "f^{i+2}_{n-1}"] \arrow[d, "\partial_{n-1}"] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Figure 3.1: A persistence complex, where moving to the right increases the filtration index, while moving downwards decreases the dimension.

2. *In general one obtains a persistence module by applying any functor from topological spaces to vector spaces. For example, let* $H := H_p(-;\mathbb{F})$ *be the functor of the* p*-dimensional singular homology on the category of topological spaces with coefficients in some abelian group* G*, in our case in some field* $\mathbb{F}$*. We define a persistence module* $\mathbb{V}$ *by setting* $V_t = H(X^t)$*, and* $v_s^t = H(i_s^t) : H(X^s) \to H(X^t)$*, which are the maps on homology induced by the inclusion maps [4, p.7].*
   In brief we yield $\mathbb{V} = H(\mathbb{X}^f_{sub})$*.*

3.2 Interval Decompositions

3.3 Persistence Barcodes

3.4 Persistence Diagrams

3.5 Interval-indecomposable Persistence Modules

3.6 Zigzag Persistence

Instead of a persistence module, zigzag persistences are always considered when we look at a zigzag diagram of a topological space or vector space, i.e. a sequence of spaces $V_1, \ldots, V_n$ where each adjacency pair is connected by a mapping $V_i \to V_{i+1}$ or $V_i \leftarrow V_{i+1}$. This means that the persistence modules discussed so far are also zigzag persistence modules in which the morphisms all point in one direction.

Definition 3.6.1 *(Zigzag Module) [***<empty citation>***]*

3.7 Multiparameter Persistence

Chapter 4

Categorification of Persistent Homology

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