22.01.2023

(III.) First fundamental form

$\to$ introduce notion to treat metric questions (on curves/surfaces: length of curve, area of subset of surface)
↳ notion of distance along curve closely associated with length of a vector in Euclidean space $\mathbb{R}^2, \mathbb{R}^3$, can be used on tangent vectors to define distances (infinitesimal distances) of very close points, 'global' distance between two points along curve then obtained by integration over these local distances

○ Basic definitions: $\gamma: I \to \mathbb{R}^n$ smooth curve $\Rightarrow$ speed of γ at $t \in I$ is $\|\gamma'(t)\|_{\mathbb{R}^n}$ (physical interpretation: velocity, motion of particle in n-space $\to \gamma'(t)$ velocity vec. for particle at time t)
– distance along γ between $\gamma(t_1)$ and $\gamma(t_2)$: $\int_{t_1}^{t_2} \|\gamma'(t)\|\,dt$ $\to$ Idea: vector from $\gamma(t)$ to $\gamma(t+\Delta t)$ approximately $\gamma'(t)\Delta t$ (first lin Approx of γ) $\to$ distance roughly $\|\gamma'(t)\|\Delta t$ $\to$ Adding up all distances, taking limit $\Delta t \to 0$ $\to$ formula $\Rightarrow$ no rigorous proof

39 ○ 3.1: Let $\gamma: I \to \mathbb{R}^n$ smooth curve $\Rightarrow$ arc length of γ from $t_1 \in I$ to $t_2 \in I$ is $\int_{t_1}^{t_2} \|\gamma'(t)\|\,dt$
↳ arc-length fct for γ a primitive of the speed fct, $t \mapsto \|\gamma'(t)\|$, that is, a differentiable fct $L: I \to \mathbb{R}$ with $\ell'(t) = \|\gamma'(t)\|$. Arc length from t_1 to t_2 then $L(t_2) - L(t_1)$
$\Rightarrow$ Notice: do not require $t_1 \leq t_2$, if $t_2 < t_1$, arc length negative, also L not unique but constant factor in difference irrelevant

$P_1, P_2 \in \mathbb{R}^n$ $t \mapsto P_1 + t(P_2 - P_1)$ $\to$ line segment length $\|P_2 - P_1\|$

arc-length invariant up to ±1

Example: $\gamma(t) = p + tq$ a straight line $(q \neq 0)$ $\Rightarrow$ arc length from $p + t_1 q$ to $p + t_2 q$ ist $\int_{t_1}^{t_2} \|q\|\,dt = \|q\|(t_2 - t_1) \overset{\text{if } t_1 < t_2}{=} \|\gamma(t_2) - \gamma(t_1)\|$ otherwise – necessary
circle: (2) $\gamma(t) = (r\cos t, r\sin t)$ $\Rightarrow \|\gamma'(t)\| = r$ $\forall t \in I$ $\to$ speed of γ constant $\Rightarrow \int_0^t \|\gamma'(t)\| = tr$
helix: (3) $\gamma(t) = (\lambda t, r\cos(\omega t), r\sin(\omega t))$ $\Rightarrow \|\gamma'(t)\| = \sqrt{\lambda^2 + \omega^2 r^2}$ constant arclength: constant times t for $\int_0^t$
$\to$ reasonable geometric notions invariant under reparametrization: arc length as well; as long as they preserve orient.
○ T.3.1: $\gamma: I \to \mathbb{R}^n$ param curve, $\beta = \gamma \circ \Phi: J \to \mathbb{R}^n$ a reparametrization.
Let $u_1, u_2 \in J$ and $t_i = \Phi(u_i)$ for $i \in \{1, 2\}$. If $\det \Phi > 0$ (preserve o.) $\Rightarrow \int_{t_1}^{t_2} \|\gamma'(t)\|\,dt = \int_{u_1}^{u_2} \|\beta'(t)\|\,dt$
40 and if $\det \Phi < 0$ $\to$ same absolute value, opposite signs
Proof: chain rule $\int_{u_1}^{u_2} \|\beta'(u)\|\,du = \int_{u_1}^{u_2} \|\gamma'(\Phi(u))\| \cdot |\Phi'(u)|\,du$, then substitution $dt = \ldots$ for $t = \Phi(u)$

$\to$ geometric interpret of following theo is that the linear curve from P_1 to P_2, $t \mapsto P_1 + t(P_2 - P_1)$ is shortest from P_1 to P_2 but not unique in this respect: reparametrization
○ T.3.2: $\gamma: I \to \mathbb{R}^n$ a param. curve $\Rightarrow$ if $t_1 < t_2$ in I and let L, arclength of γ from t_1 to t_2 $\Rightarrow L \geq \|\gamma(t_2) - \gamma(t_1)\|$
Proof: consider fct $\varphi(t) = \gamma(t) \overset{\text{inner product}}{\cdot} (\gamma(t_2) - \gamma(t_1))$ $\Rightarrow \varphi(t_2) - \varphi(t_1) = \|w\|^2 = \int_{t_1}^{t_2} \varphi'(t)\,dt$
41 $\Rightarrow \varphi'(t) = \gamma'(t) \cdot w$ $\Rightarrow$ CSU: $\varphi'(t) \leq |\varphi'(t)| \leq \|\gamma'(t)\| \cdot \|w\|$ $\Rightarrow \|w\|^2 \leq \int_{t_1}^{t_2} \|\gamma'(t)\| \cdot \|w\|\,dt = L \cdot \|w\|$ ■

3.3 Unit speed parametrization

$\to$ param curve has unit speed if $\|\gamma'(t)\| = 1$ $\forall t \in I$ $\to$ common practice to replace symbol for var by s
$\Rightarrow$ determination of arc length simple with unit speed as $\int_{s_1}^{s_2} \|\gamma'(t)\| = s_2 - s_1$

we say parametrized by arc-length

○ 3.3T.: A regular param curve γ allows a direction-preserving reparam. with unit speed ('parametrized by arc length')
Proof: $\ell(t)$ an arbitrary arc-length fct (not a unique primitive but always from 0 to t). $\ell'(t) = t \mapsto \|\gamma'(t)\|$ is smooth as γ is smooth and as regular never 0 $\Rightarrow \|\gamma'(t)\| > 0$, ℓ smooth
$\overset{2.5}{\Rightarrow} \ell$ bijective onto its image, $\Phi = \ell^{-1}$ smooth $\Rightarrow \Phi'(s) = {\ell^{-1}}'(s) \overset{2.5}{=} \frac{1}{\ell'(t)} = \frac{1}{\|\gamma'(t)\|} > 0$ where $t = \Phi(s) \Leftrightarrow s = \ell(t)$
$\Rightarrow (\gamma \circ \Phi)'(s) = \gamma'(\Phi(s))\,\Phi'(s) = \gamma'(t) / \|\gamma'(t)\|$ where $t = \Phi(s)$ $\Rightarrow \gamma \circ \Phi$ unit speed ■

Example: (1) curve with constant speed $c \neq 0$: $\ell(t) = ct$ $\Rightarrow \Phi(s) = \ell^{-1}(s) = \frac{s}{c}$ $\to$ unit speed param. achieved by inserting $t = \frac{s}{c}$ into expression γ
↳ e.g. for $\gamma(t) = (\cos t \cdot r, r\sin t)$ $\Rightarrow \ell(t) = rt$ $\Rightarrow \beta(s) = \gamma(\frac{s}{r}) = (r\cos\frac{s}{r}, r\sin\frac{s}{r})$ $\to$ circle
42 ↳ helix: $\beta(s) = (\lambda\frac{s}{c}, r\cos(\omega\frac{s}{c}), r\sin(\omega\frac{s}{c}))$ where $c = \sqrt{\lambda^2 + r^2\omega^2}$ $\to$ helix constant speed

First Fundamental form:

inner product with itself gets $u \cdot \| \cdot \|$ — inner product — where $p \in U$

○ Def: $\sigma: U \to \mathbb{R}^3$ param. surface: Three fct on U, associated with σ: $E(p) = \|\sigma_u'(p)\|^2$, $F(p) = \sigma_u'(p) \cdot \sigma_v'(p)$, $G(p) = \|\sigma_v'(p)\|^2$
↳ three fcts together analogue for surfaces of speed $\|\gamma'(t)\|$ of curve (or rather $\|\sigma'(t)\|^2$) — wenn is angle $\pm 1 \Rightarrow \theta = 0, \theta =$
⇒ Cauchy-Schwartz-inequality: $F(p)^2 \leq E(p)G(p)$ with strict inequ. ⇔ σ regular at p ⇔ σ_u', σ_v' lin indep.

↳ fcts E, F, G useful for computing lengths of tangent vectors. Assume σ regular at p, $(\sigma_u'(p), \sigma_v'(p))$ then basis for $T_p\sigma$ ⇒ $w \in T_p\sigma$, $w = a\sigma_u' + b\sigma_v'$ ⇒ $\|w\|^2 = (a\sigma_u' + b\sigma_v') \cdot (a\sigma_u' + b\sigma_v') = Ea^2 + 2Fab + Gb^2$

○ D.3.4: σ regular at p: map $I_p: T_p\sigma \to \mathbb{R}$, $w \mapsto \|w\|^2 = E(p)a^2 + 2F(p)ab + G(p)b^2$ (→ associates to tangent vec the square of its length) is called first fundamental form of σ in p, E, F, G component fcts
↳ matrix form: $I_p(w) = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ $\left[\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (D\sigma)^T \cdot D\sigma = \begin{pmatrix} \sigma_u' \\ \sigma_v' \end{pmatrix} (\sigma_u', \sigma_v') \right]$ 43
$e, f, g \in \mathbb{R}$
↳ I_p quadratic form on $T_p\sigma$ $\forall p \in U$ (quad. f: vector space V with base (v_1, v_2), $Q: V \to \mathbb{R}$, $w = av_1 + bv_2 \mapsto ea^2 + 2fab + gb^2$)

Example: (1) $\sigma(u,v) = p + uq_1 + vq_2$, $q_1, q_2 \in \mathbb{R}^3$ assumed to be lin. independant ⇒ $\sigma_u' = q_1$, $\sigma_v' = q_2$ ⇒ $E = \|q_1\|^2$, $F = q_1 \cdot q_2$, $G = \|q_2\|^2$
↳ if q_1, q_2 orthonormal (norm 1, inner prod. 0) pair ⇒ $E = G = 1$, $F = 0$ — component f constant
(2) $\sigma(u,v) = (r\cos v, r\sin v, u)$ (cylinder) → $\sigma_u' = (0,0,1)$, $\sigma_v' = (-r\sin v, r\cos v, 0)$ ⇒ $E = 1$, $F = 0$, $G = r^2$
(3) unit sphere: (spherical coordinates), 2.3.1 ⇒ $E = 1$, $G = \cos^2 u$ (not constant), $F = 0$

smooth fcts, vector $\in U$

○ T.3.4: The arc length of param curve $\gamma(t) = \sigma(u(t), v(t))$ on σ (2.4: $\gamma' = u'\sigma_u' + v'\sigma_v'$) is given with respect to coordinates $(u(t), v(t))$ as follows: $\int_{t_1}^{t_2} (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2}\, dt$
where E, F, G are evaluated in $(u(t), v(t))$
and derivatives u', v' are eval. in t

Proof: $\|\gamma'(t)\|^2 = I_p(\gamma'(t)) = Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2$ and rest is def. of arc length ... 44

Example: (1) circle $\gamma(t) = \sigma(u, t)$ on unit sphere with fixed lattitude u ⇒ $u' = 0$, $v'(t) = 1$ → with values of E, F, G of ex. above we obtain: total length of γ: $\int_0^{2\pi} (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2} dt = \int_0^{2\pi} \cos u \, dt = 2\pi \cos u$

○ First fundamental form can be used to determine angle between (non-zero) tangent vectors, say between $w = a\sigma_u'(p) + b\sigma_v'(p)$, $\tilde{w} = \tilde{a}\sigma_u'(p) + \tilde{b}\sigma_v'(p)$ in $T_p\sigma$ ⇒ $\theta \in [0, \pi]$, $\cos\theta = \frac{w \cdot \tilde{w}}{\|w\| \cdot \|\tilde{w}\|}$
⇒ $\cos\theta = \frac{Ea\tilde{a} + F(a\tilde{b} + b\tilde{a}) + Gb\tilde{b}}{(Ea^2 + 2Fab + Gb^2)^{1/2}(E\tilde{a}^2 + 2F\tilde{a}\tilde{b} + G\tilde{b}^2)^{1/2}}$ — angle between σ_u' and σ_v' $= \cos\theta = \frac{F}{\sqrt{EG}}$
↳ parametrized surface $\sigma(u,v)$ called orthogonal, if $F(p) = 0$ $\forall p \in U$ ⇔ $\sigma_u'(p)$ and $\sigma_v'(p)$ always perpendicular ⇔ $\theta = \frac{\pi}{2}$ $\forall p \in U$ 45

(2) $\gamma(t)$ curve on unit sphere → $\gamma(t) = \sigma(u(t), v(t))$ → determine angle between tan vect $\gamma'(t)$ an direction (North) of meridians → meridians characterized by fixed longitude v → tan vect. of meridian has direction σ_u' ⇒ $(a = u'(t), b = v'(t)) \wedge (\tilde{a} = 1, \tilde{b} = 0)$ ⇒ values of E, F, G above ⇒ $\cos\theta = \frac{u'}{(u'^2 + \cos^2 u (v')^2)^{1/2}}$

Areas and plane curves:

→ consider $D \subseteq \mathbb{R}^2$. If $D = [a,b] \times [c,d]$ → rectangle with area $A(D) = (b-a)(d-c)$. If $f: D \to \mathbb{R}$ continuous we define integral of f over D by $\int_D f\, dA = \int_a^b \int_c^d f(u,v)\, dv\, du = \int_c^d \int_a^b f(u,v)\, du\, dv$
(→ inner integral depends continu on u)
↳ Fubini-theorem: both integrals finite ⇒ can change order of integration → can always change in our course

K is a block-set

→ if D not a rectangle → more diff to define its area and integrals over it. By a block set we understand a finite union of closed rectangles → by decomposing rectangles further, we can achieve that they only overlap on boundaries, such a decomposition then called partition of the block set (not unique). 46
⇒ independant of choice of partition: $A(K)$ is sum of rectangles, $\int_K f\, dA$ sum of integrals over them — area of K is integral of constant fct 1 over

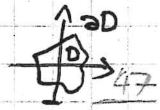
○ $\left| \int_K f\, dA \right| \leq A(K) \sup_{p \in K} |f(p)|$

○ $\int_{K_1 \cup K_2} f\, dA = \int_{K_1} f\, dA + \int_{K_2} f\, dA$ if the two block sets only overlap on their boundaries

∘ smooth curves on closed integrals: $\gamma : [a,b] \to \mathbb{R}^2$, $-\infty < a < b < \infty$ $\to \gamma$ smooth on (a,b) and has a continuous extension as well as all its derivatives to $[a,b]$ (that is, they have limits from left and right at the endpoints)

Gebiet

∘ 3.5.1: set $D \subset \mathbb{R}^2$ called an elementary domain if it is closed and bounded (compact) and if its boundary ∂D is a finite union of (the trace of) smooth curves def. on closed intervals 47



↳ block-sets are elementary domains since their boundaries are unions of line segments

∘ 3.5.2: $D \subset \mathbb{R}^2$ an elementary domain. The area of D def by: $A(D) = \sup_{K \subset D} A(K)$ where K are block-sets

↳ continuous fct $f : D \to \mathbb{R}$, $f(p) \geq 0$ $\forall p \in D$ def by: $\int_D f \, dA = \sup_{K \subset D} \int_K f \, dA$

↳ supremums are finite: since D is bounded, it is contained in a sufficiently large square with side-length N

$\Rightarrow \int_D f A \leq A(K) \sup_{p \in K} f(p) \leq A(D) \sup_{p \in D} f(p)$ which is finite since f continuous (max as compact domain)

split function into positive and negative part

$\Rightarrow$ remove assumption $f \geq 0$ $\Rightarrow f : U \to \mathbb{R}$ continuous, $f_+(x) = \max\{0, f(x)\}$, $f_-(x) = \max\{0, -f(x)\}$ so that $f_+, f_- \geq 0$ and $f = f_+ - f_-$ $\Rightarrow \int_D f \, dA = \int_D f_+ \, dA - \int_D f_- \, dA$

∘ $\int_D f + g \, dA = \int_D f \, dA + \int_D g \, dA$ ∘ $\int_D cf \, dA = c \int_D f \, dA$ ∘ $\left| \int_D f \, dA \right| \leq \int |f| \, dA$

∘ $\int_{D_1 \cup D_2} f \, dA = \int_{D_1} f \, dA + \int_{D_2} f \, dA$ where D_1, D_2 only intersect with their boundaries 48

Null sets:

$\to$ theorem below shows that we don't 'miss' parts of D by forming supremum over blocksets $K \overset{!}{\subset} D$

$\to$ say that a closed bounded set is a null set if $\forall \varepsilon > 0$ $\exists K$ block set of area $< \varepsilon$ such that $D \subset K$

∘ L 3.6: $\gamma : [a,b] \to \mathbb{R}^2$ smooth, the trace $\gamma([a,b])$ is a null set.

Proof: using the continuous arc length fct $\to$ divide γ in N pieces of equal length $\frac{N}{\ell}$, ℓ total length of curve (finite as $[a,b]$ + continuous!)

$\to$ cover trace with squares of side-length $\frac{N}{\ell}$ $\to$ union of areas $N(\ell/N)^2 = \ell^2/N \leq \varepsilon$ for sufficiently large N

$\to$ follows that boundaries of elementary domains are null sets:

∘ T. 3.6: $U \subset \mathbb{R}^2$ open set, $f : U \to [0, \infty[$ be a continuous fct with $D \subset U$ an elementary domain

$\Rightarrow \int_D f \, dA = \inf_{D \subset K \subset U} \int_K f \, dA$ where K are block-sets $\to$ didn't proof it 49

↳ can also approximate from the outside

Double integrals:

$\to$ Let $\Phi, \Psi : [a,b] \to \mathbb{R}$ smooth fcts with $\Phi(u) < \Psi(u)$ $\forall u \in (a,b)$. The set $D = \{(u,v) \mid a \leq u \leq b, \Phi(u) \leq v \leq \Psi(u)\}$ of points between graphs is an elementary domain 50

∘ 3.7 T: set D from above has area $A(D) = \int_a^b [\Psi(u) - \Phi(u)] \, du$ and the plane integral of a continuous fct f over D is $\int_D f \, dA = \int_a^b \int_{\Phi(u)}^{\Psi(u)} f(u,v) \, dv \, du$

↳ more complicated sets are treated by means of a disjoint division into subsets of this form (possibly u and v interchanged)

Example: triangle $D = \{(u,v) \mid 0 \leq u, 0 \leq v, 2u + v \leq 2\}$ $\to v = 2 - 2u$, $\Psi(u) = 2 - 2u$, $\Phi(u) = 0$ $\Rightarrow A(D) = \int_0^1 \Psi(u) = 1$ 51

$\to f(u,v) = v \Rightarrow \int_D v \, dA = \int_0^1 \int_0^{2-2u} v \, dv \, du = \int_0^1 \frac{1}{2}(2 - 2u)^2 \, du = \frac{2}{3}$

$\to D$ can also be regarded as set $0 \leq v \leq 2$, $0 \leq u \leq 1 - \frac{1}{2}v$ $\Rightarrow A(D) = \int_0^2 (1 - \frac{1}{2}v) \, dv = 1$

$\int_D v \, dA = \int_0^2 \int_0^{1 - \frac{1}{2}v} v \, du \, dv = \int_0^2 v(1 - \frac{1}{2}v) \, dv = \frac{2}{3}$

Transformation of integrals

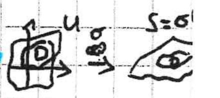
$\to$ generalization of the formula for substitution of variables in ordinary integrals.

↳ $U, W \subset \mathbb{R}^2$ open, $\Phi : W \to U$ a diffeomorphism

∘**T.3.8:** $D \subset \mathbb{R}^2$ closed and bounded and contained in W. If D is an elementary domain, then so is its image $\Phi(D) \subset U$. Moreover $\int_{\Phi(D)} f\,dA = \int_D (f \circ \Phi) \cdot |\det(D\Phi)|\,dA$ for $f: U \to \mathbb{R}$ continuous

↳ $f=1$: $A(\Phi(D)) = \int_D |\det(D\Phi)|\,dA$ 5

∘**D.3.9:** The area of the surface $\sigma: U \to \mathbb{R}^3$ over $D \subset \mathbb{R}^2$ elementary domain with $D \subset U$ is



~~$A(\sigma(D))$~~ $\neq A(\sigma, D) = \int_D \|\sigma'_u \times \sigma'_v\|\,dA$

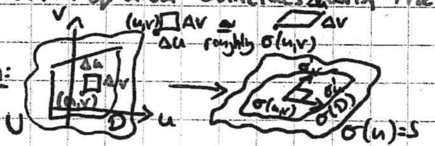
Recall that $\sigma'_u \times \sigma'_v$ is a normal vec to tangent plane $\Rightarrow \|\sigma'_u \times \sigma'_v\| = (EG - F^2)^{1/2}$ as general rule of vec. calculus: $\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a.b)^2$

↳ often denote $A(\sigma(D))$ although not legitimate as it depends on both σ and D and not just $\sigma(D)$ unless some injectivity is ok

↳ if we consider (x,y)-plane as $\sigma(u,v) = (u,v,0) \Rightarrow E = G = 1, F = 0 \Rightarrow A(\sigma, D) = \int_D 1\,dA$

⇒ new notion of area coincides with the previous one for plane sets 53

→ motivation:



area of parallelogram, spanned by $\Delta u \sigma'_u$, $\Delta v \sigma'_v$ is roughly (linear approx. of σ) the same as of $\square^{\Delta v}_{\Delta u}$

$\|\Delta u \sigma'_u \times \Delta v \sigma'_v\| = \Delta u \Delta v \|\sigma'_u \times \sigma'_v\|$

→ adding up all those areas and taking limit $(\Delta u, \Delta v) \to (0,0)$ leads to formula 3.9

∘**T.3.9:** Surface area invariant under reparametrization $\tau = \sigma \circ \Phi: W \to \mathbb{R}^3$ where $\Phi: W \to U$ diffeo ($\subset \mathbb{R}^2$)

Proof: let $E \subset U$ elementary domain $\Rightarrow D = \Phi^{-1}(E) \subset W$ elementary domain (→ preimage of compact domain under diffeo)

⇒ since $\tau = \sigma \circ \Phi \Rightarrow \tau(D) = \sigma(E)$ → appears to be tautology but it is not as area defined by both parametrization and image, not just their image. $A(\tau, D) = \int_D \|\tau'_s \times \tau'_t\|\,dA \overset{(*)}{=} A(\sigma, E) = \int_E \|\sigma'_u \times \sigma'_v\|\,dA$

(∗) 2.8: $\tau'_s(q) \times \tau'_t(q) = \det(D\Phi)(q)\,\sigma'_u(\Phi(q)) \times \sigma'_v(\Phi(q))$ → insert into $A(\tau, D)$, then use 3.8 ∎ 54

Example: compute surface area of unit sphere (param. with spherical coordinates) Let D be rectangle where $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ and $-\pi \leq v \leq \pi$ ⇒ σ injective on interior of D (overlap on a line which is not important as it's a null set) boundary

$E = 1, F = 0, G = \cos^2 u \Rightarrow (EG - F^2)^{1/2} = \cos u \Rightarrow \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \cos u\,du\,dv = 4\pi 1^2$