V. The second fundamental form

→ extend notion of curvature to surfaces: will not be possible to described it by a number. The description is based on concept of normal curvature, which associates a numbers to each of the infinite many unit tangent vectors at a given point p.

$t(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|}$ unit tangent vector, $t = \binom{t_1}{t_2}$, $\hat{t} = \binom{-t_2}{t_1}$

5.1 Shape operator (Weingarten map)

→ for plane curve with unit speed ($t' = \kappa \hat{t}$) the κ is given by the rate of change of dir. of the tangent line angle t.
for surfaces we will look at the rate of change of the tangent space at p. it will show that it's more convenient to use the negative of the derivative

→ Let $\sigma: U \to \mathbb{R}^3$ a regular surface: position of $T_p\sigma$ in $\mathbb{R}^3$ completely determined by unit normal vec $N = \frac{\sigma_u' \times \sigma_v'}{\|\sigma_u' \times \sigma_v'\|}$

77 ↳ for rate of change, both $-N_u'$ and $-N_v'$ have to be considered ($N_u', N_v' \in T_p\sigma$ as N a unit vec)

∘ **D.5.1:** Let $p = (u_0, v_0) \in U$. The linear map $W = W_p : T_p\sigma \to T_p\sigma$, $W(a\sigma_u' + b\sigma_v') = -aN_u' - bN_v'$ $\forall a, b \in \mathbb{R}$ is called the shape operator of σ at p. ($\sigma_u', \sigma_v', N_u', N_v'$ all evaluated in p) $W(\sigma_u') = -N_u'$, $W(\sigma_v') = -N_v'$

Examples: (1) $\sigma(u,v) = p + uq_1 + vq_2$ with q_1, q_2 linearly independent $\Rightarrow N = \frac{q_1 \times q_2}{\|q_1 \times q_2\|}$ constant and W the zero operator

(2) unit sphere with standard spherical coordinates: $N(u,v) = -\sigma(u,v)$ $\Rightarrow N_u' = -\sigma_u', N_v' = -\sigma_v'$ $\Rightarrow W = id_{T_p\sigma}$ identity

(3) $\gamma: I \to \mathbb{R}^2$ plane line with unit speed. Since $(\hat{t})' = \hat{t'}$ $\Rightarrow (\hat{t})' = \hat{t'} = \kappa\hat{\hat{t}} = -\kappa t$, ($\hat{\hat{t}} = -t$)

78 N and $\hat{t}$ analogues of each other $\Rightarrow W(a\gamma'(t_0)) = -a(\hat{t})'(t_0) = a\kappa t$ → W a higher dim. version of κ

surface normal along curve $-N(\mu(t)) \cdot \mu'(t)$ 4.3 $\Rightarrow m'(t) = aN_u' + bN_v'$ for $a\sigma_u' + b\sigma_v' = \gamma'(t) = t(t)$

∘ **L.5.1:** $\gamma = \sigma \circ \mu$ with $\mu(t_0) = p$ $\Rightarrow W(\gamma'(t_0)) = -m'(t_0)$ where $m = N \circ \mu$ the surface normal along the curve

↳ W more directly related to a geometric prop of σ than N_u', N_v'

↳ W associates to a tan vec. the derivative of $-N$ along any curve that has the tan. vec as its tangent in that point

∘ **T.5.1:** The shape operator W is unchanged under dir preserving reparametrization and changes to $-W$ for reversing

Proof: $\tau = \sigma \circ \Phi$ reparam with diffeo $\Phi: V \to U$. Claim that $W^\tau = \pm W^\sigma \overset{2.4}{\Rightarrow} \forall \omega \in T_p\sigma \, \exists t_0: \omega = \gamma'(t_0)$ for some curve curve $\gamma = \sigma \circ \mu$

$\Rightarrow W^\sigma(\gamma'(t_0)) = -(N^\sigma \circ \mu)'(t_0)$, $\gamma = \sigma \circ \mu = \tau \circ \Phi^{-1} \circ \mu : I \to V$ plane curve, $\tau_s' \times \tau_t' = \det(D\Phi)\, \sigma_u' \times \sigma_v'$

$\Rightarrow N^\tau = \pm N^\sigma \circ \Phi$ with $+ \Leftrightarrow \det(D\Phi) > 0$, $N^\tau = \frac{\tau_s' \times \tau_t'}{\|\tau_s' \times \tau_t'\|}$ $\Rightarrow W^\tau(\gamma'(t_0)) = -(N^\tau \circ \mu)'(t_0) = -(\pm N^\sigma \circ \Phi \circ \Phi^{-1})'(t_0)$
$= \pm(-N^\sigma \circ \mu)'(t_0) = \pm W^\sigma(\gamma'(t_0))$ ∎

→ nochmal ansehen

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5.2 Second fundamental form

→ introduce another object, closely related to W, to relate W with κ_n (minus prod.)

∘ **D.5.2:** The map $w \in T_p\sigma \mapsto II_p(w) = w \cdot W(w) \in \mathbb{R}$ is called the second fundamental form of σ in p, which does not change under reparametrizations, except by a sign if dir reversing (T

∘ **T.5.2:** The normal curvature in direction w_0 is $\kappa_n = \frac{II_p(w_0)}{\|v_0\|^2}$ (recall κ_n the same for all curves through p with $\gamma'(t_0) = w_0$.

Proof: was proofen in 4.5 that $\kappa_n = -\gamma'(t_0) \cdot m'(t_0) / \|\gamma'(t_0)\|^2$ with $\gamma'(t_0) = w_0$ for a γ with $\gamma = \sigma \circ \mu$ and $\gamma'(t_0) = w_0$

5.3 Coordinate expressions for second fundamental form

$M = N \cdot \sigma_{uv}'' = \frac{\det[\sigma_u', \sigma_v', \sigma_{uv}'']}{\|\sigma_u' \times \sigma_v'\|}$

$N = N \cdot \sigma_{vv}'' = \frac{\det[\sigma_u', \sigma_v', \sigma_{vv}'']}{\|\sigma_u' \times \sigma_v'\|}$

→ give an explicit expression by which II can be computed for a given parametrization.

∘ **T.5.3:** $II_p(a\sigma_u' + b\sigma_v') = La^2 + 2Mab + Nb^2$ with respect to basis (σ_u', σ_v') with $L := N \cdot \sigma_{uu}'' = \frac{\det[\sigma_u', \sigma_v', \sigma_{uu}'']}{\|\sigma_u' \times \sigma_v'\|}$,

80 (where all terms are eval in given $p \in U$) $M = N \cdot \sigma_{uv}''$ and $N = N \cdot \sigma_{vv}''$

$\Rightarrow \begin{pmatrix} L & M \\ M & N \end{pmatrix} \Rightarrow II_p(a\sigma_u' + b\sigma_v') = \binom{a}{b}^t \begin{pmatrix} L & M \\ M & N \end{pmatrix} \binom{a}{b}$ → second fundamental is a quadratic form

81 Example: $\sigma(u,v) = (r\cos u \cos v, r\cos u \sin v, r\sin u)$, $r > 0$ $\Rightarrow N(u,v) = -\sigma(u,v)$ $\Rightarrow L = r, M = 0, N = r\cos^2 u$ $\Rightarrow II(a\sigma_u' + b\sigma_v') = r(a^2 + b^2\cos^2 u)$

∘ **T.5.4:** The matrix for the shape operator W with respect to (σ_u', σ_v') is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

Proof: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, write $W(\sigma_u') = h\sigma_u' + j\sigma_v'$, $W(\sigma_v') = i\sigma_u' + k\sigma_v'$ $\Rightarrow$ mat for. W_p of form $\begin{pmatrix} h & i \\ j & k \end{pmatrix}$

$\overset{\text{calc}}{\Rightarrow} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} h & i \\ j & k \end{pmatrix} \Leftrightarrow \begin{pmatrix} h & i \\ j & k \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ ∎

82 → ⚠ W_p in general not symmetric (product of symm mat not necessarily symmetric)

5.5 Diagonalization of the second fundamental form

Eigenvalue

- **D5.5:** An eigenvec. for W_p is called a principal vector in $T_p\sigma$ and the corresponding principal curvature at p.
  - ↳ if $w \in T_p\sigma$ is principal vec. with corresponding principal curvature λ $\Rightarrow \kappa_n = II_p(w) = \lambda$
  - ↳ T5.1: principal vectors unchanged under reparam., while corresponding principal curvatures have opposite for direction reversing reparam.
  - ↳ $W: T_p\sigma \to T_p\sigma$ symmetric with respect to dot product: $w_1 \cdot W(w_2) = W(w_1) \cdot w_2 \quad \forall w_1, w_2 \in T_p\sigma$

- **T5.5:** There exists for each $p \in U$ an orthonormalbasis (w_1, w_2) for $T_p\sigma$ consisting of principal vectors with corresponding principal curvatures $\kappa_1, \kappa_2 \in \mathbb{R}$. With respect to this basis:
  $II_p(aw_1 + bw_2) = \kappa_1 a^2 + \kappa_2 b^2 \ \forall a, b \in \mathbb{R}$

↳ linear map, $\kappa_1 \perp \kappa_2$

Proof: first one follows from C.D.1 in Appendix, other formula follows from evaluation of $w \cdot W(w)$ with $w = aw_1 + bw_2$

- **C5.5.1:** Let w_1, w_2 and κ_1, κ_2 as above, $\theta \in \mathbb{R}$, $w_\theta = \cos\theta w_1 + \sin\theta w_2 \Rightarrow \kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ 83
  in particular $\kappa_n \in [\kappa_1, \kappa_2]$
  ↳ $= II_p(w_\theta) = \kappa_n$ then 5.5

- **C5.5.2:** The principal curvatures κ_1, κ_2 are the roots κ of equation: $\det\left(\underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{:D} - \kappa\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$
  The corresponding principal vectors are $a\sigma_u' + b\sigma_v'$ where (a,b) is non-zero and solves $D\binom{a}{b} = \kappa_i \binom{a}{b}$

Example: cylinder $\sigma(u,v) = (\cos v, \sin v, u) \Rightarrow E = G = 1, F = 0, N = (-\cos v, -\sin v, 0) \overset{!}{=} \sigma_{vv}''$, $\sigma_{uu}'' = \sigma_{uv}'' = 0$, $L = M = 0, N = 1$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \kappa_1 = 0, \kappa_2 = 1$ → normal curvature in dir σ_u' (vertical) is zero and (horizontal) σ_v'-dir is 1 84

5.6 graph of a quadratic form

t transport, transposition

→ A quadratic form on $\mathbb{R}^2$ is a function $q: \mathbb{R}^2 \to \mathbb{R}$ of the form $q(x,y) = ax^2 + 2bxy + cy^2$ for some $a, b, c \in \mathbb{R}$
$q(x,y) = (x,y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\binom{x}{y}$

→ Recall from linear algebra that every symmetric matrix A is orthogonally diagonalizable, C is an orthogonal matrix such that $D = C^{-1}AC$ is a diagonal mat. with real entries and the columns of C $(C^{-1} = C^t)(\det C \in \{\pm$ form an orthonormal basis of eigenvectors. Coordinates of $w = \binom{x}{y} \in \mathbb{R}^2$ with respect to the basis given by the columns of C are denoted $\binom{x'}{y'} \Rightarrow w = C\binom{x'}{y'}$, $w' = \binom{x'}{y'} \Rightarrow w = Cw' \Rightarrow q(w) = w^t A w =$
$= (Cw')^t A (Cw') = w'^t C^t A C w' = w'^t D w'$ if λ_1, λ_2 be th eigenval. in the diagonal $\Rightarrow q(w) = \lambda_1 x'^2 + \lambda_2 y'^2$
↳ change of variables results in simplification of the expression for q (xy term dissapears)

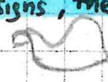
vectors still an orthonormal basis

→ Notice $\det C = \pm 1$ and by changing the sign of one column (if necessary) we can arrange $\det C = 1$
$\Rightarrow C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$ → basis vecs of C are obtained from the standard basis vec e_i excactly by this rotation, x' and y' new coordinates of w with respect to this new basis (rotated 85
$w = \binom{x'}{y'} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{C^T}\binom{x}{y}$

counter clockwise, C^T clockwise

- **T5.6:** Let $q(w) = w^t A w$ a quadratic form on $\mathbb{R}^2$ with a symm. 2x2 matrix A. There exists a rotation of $\mathbb{R}^2$ such that in the rotated $x'y'$-coord. $q(w) = \lambda_1 x'^2 + \lambda_2 y'^2$ where λ_1, λ_2 are eigenvalues of A

→ in these rotated coordinates we can easily describe the graph of q: vertical crosssections of graph, obtained by taking intersection with one of the two vertical coordinate planes ($x'z$-plane and $y'z$-plane).
Therefore surface is called paraboloid.
$\Rightarrow$ shape of the horizontal cross sections depends on the eigenvalues λ_1 and λ_2. If they are both positive or both negative, then horizontal cross section of graph is an ellipse and graph called elliptic paraboloid. If $\lambda_1 \neq 0 \neq \lambda_2$ and different signs, then graph is called a hyperbolic paraboloid because every horizontal crossection is a hyperbola. 86



form of a saddle

both pos

both neg.

If one eigenvalue is zero but the other isn't, then graph is called a parabolic cylinder (cylinder where the cross section is a parabola instead of a circle). Finally if $\lambda_1 = \lambda_2 = 0 \Rightarrow$ graph is a plane.

In the rotated coordinates we obtain a graph $\sigma(u,v) = (u, v, \lambda_1 u^2 + \lambda_2 v^2)$. A simple calculation shows that at $(u,v) = (0,0)$ we have $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 2\lambda_1 & 0 \\ 0 & 2\lambda_2 \end{pmatrix}$ in rotated coordinates

$\Rightarrow$ rotation of coordinates has excactly the effect that shape operator is diagonalized. The principal curvatures are $2\lambda_1$ and $2\lambda_2$ and the principal vectors are along the two horizontal axes for rotated basis.

<u>Example</u>: $q(x,y) = x^2 + xy + y^2$ corresponds to $\begin{pmatrix} 1 & 0{,}5 \\ 0{,}5 & 1 \end{pmatrix}$ $\Rightarrow D = C^T A C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$ where $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ 87

$\rightarrow$ quadratic form becomes $\frac{1}{2}x'^2 + \frac{3}{2}y'^2$ in rotated coordinates, graph an eliptic paraboloid. Its axes are rotated from the x and y axes by angle θ, determined ~~by~~ from $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{-1}{\sqrt{2}}$ that is clockwise by 45 degrees

5.7 type of a surface at a point

$\rightarrow$ principal curvatures and vectors can be explained geometrically:
can assume that $\sigma(p)$ is the origin and the xy-plane is $T_p\sigma$ (can always be arranged by translation followed by a suitable rotation of $\mathbb{R}^3$). Can be shown that transformation does not alter κ_1 and κ_2. Furthermore, it follows from 2.11 and its proof that σ allows orientation preserving reparam. as smooth graph over xy-plane (principal curvatures unchanged by such a reparam.). We therefore assume $\sigma(u,v) = (u, v, h(u,v))$ with h smooth.

Since $\sigma(p) = (0,0,0) \Rightarrow p = (0,0)$, $h(0,0) = 0$, $\sigma'_u = (1, 0, h'_u)$, $\sigma'_v = (0, 1, h'_v)$ and as $T_p\sigma$ is xy-plane $\Rightarrow h'_u = h'_v = 0$ in (0

$\Rightarrow E = G = 1$, $F = 0$ in p.

$N = (0,0,1)$ and $\sigma''_{uu} = (0,0,h''_{uu})$, $\sigma''_{vv} = (0,0,h''_{vv})$, $\sigma''_{uv} = (0,0,h''_{uv})$ $\overset{5.3}{\Rightarrow} L = h''_{uu}(0,0)$, $M = h''_{uv}(0,0)$, $h''_{vv}(0,0)$

$\Rightarrow$ Taylor expansion to order two of σ: $\sigma(u,v) \approx \sigma(0,0) + u\sigma'_u(0,0) + v\sigma'_v(0,0) + \frac{1}{2}(u^2\sigma''_{uu}(0,0) + 2uv\sigma''_{uv}(0,0) + v^2\sigma''_{vv}(0,0))$

$= (u, v, \frac{1}{2}II_p(u\sigma'_u + v\sigma'_v))$ $\Rightarrow \sigma$ approxed near p by $\frac{1}{2}II_p$ (its graph)

$\rightarrow$ we can now read off the shape of σ from this shape of the graph

$\Rightarrow$ conclusion: after suitable rotation of xy-plane which brings the principal vectors in the direction of the axes, surface will appear as in 5.6 (depending on signatures of κ_1, κ_2) 88

- D.5.7: the type of σ at $p \in U$ is defined as follows. It's called elliptic point if principal curvatures κ_1, κ_2 are both non-zero with same sign, a hyperbolic point if non-zero (both) with opposite signs. If one is zero but other one is not $\rightarrow$ point is called parabolic and if $\kappa_1 = \kappa_2 = 0$ planar

  $\rightarrow$ type of a point unchanged under reparam as principal curvatures are either unchanged or both change signs.