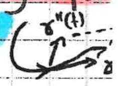
(IV) Curvature

→ quantity which describes shape of curve in given point, measure of the rate at which curve is turning in poi[nt]

∘ D.4.1: $\gamma: I \to \mathbb{R}^2$ regular param curve, the real number $\kappa(t) = \dfrac{\det[\gamma'(t)\ \gamma''(t)]}{\|\gamma'(t)\|^3}$ called curvature of γ at t



↳ idea: turning at t is described by relative position of tangent vec $\gamma'(t)$ (speed) and its derivative $\gamma''(t)$ (accelaration, rate of change of speed) → parallelogram of forces

⇒ position described by $\det$ of $[\gamma'(t)\ \gamma''(t)]$ which measures the area of the parallelogon they span

$\det[abc] = (a \times b) \cdot c$ $\begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ⇒ $a \times b = \left(0, 0, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right)$ ⇒ $|\det[abc]| = \left\| \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\| = \|a \times b\|$ → area of parallelogon

59 Example: (1) straight line with arbitrary parametrization ⇒ $\gamma'(t), \gamma''(t)$ same direction, lin. dependant ⇒ $\det[\ldots] = 0 \nearrow \kappa$

(2) circle of radius r with $\gamma(t) = (r\cos t, r\sin t)$ → $\gamma'(t) = (-r\sin t, r\cos t)$, $\gamma''(t) = (-r\cos(t), -r\sin(t))$

⇒ $\kappa(t) = \frac{r^2}{r^3} = \frac{1}{r}$ → for clockwise parametrization $\gamma(t) = (r\cos t, -r\sin t)$ we have $\kappa(t) = -\frac{1}{r}$ → const curve

(3) ellipse with $a > b$, $\gamma(t) = (a\cos t, b\sin t)$ we have $\gamma'(t) = (-a\sin t, b\cos t)$, $\gamma''(t) = (-a\cos(t), -b\sin t)$

⇒ $\kappa(t) = \dfrac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$ → κ attains max value $\frac{a}{b^2}$ when $\sin t = 0$ (minimal denominator) and minimal value $\frac{b}{a^2}$ when $\cos t = 0$ (denominator max)

(4) graph of smooth fct $\gamma(t) = (t, h(t))$, $\gamma'(t) = (1, h'(t))$, $\gamma''(t) = (0, h''(t))$ ⇒ $\kappa(t) = \dfrac{h''(t)}{(1+h'(t)^2)^{3/2}}$

60 ↳ if $h'(t) = 0$ ⇒ $\kappa(t) = h''(t)$

∘ T.4.1: curvature of a plane curve is unchanged under direction-preserving reparametrization and multiplied by -1 under direction-reversing reparametrization

Proof: $\beta(u) = \gamma(\Phi(u))$, $e = \pm$ sign of Φ' ⇒ $\beta'(u) = \Phi'(u)\gamma'(\Phi(u))$, $\beta''(u) = \Phi''(u)\gamma'(\Phi(u)) + \Phi'(u)^2\gamma''(\Phi(u))$

⇒ det linear used for β'': $\det[\beta'(u)\beta''(u)] = \Phi'(u)^3 \det[\gamma'(\Phi(u))\gamma''(\Phi(u))]$, $\|\beta'(u)\| = |\Phi'(u)| \cdot \|\gamma'(\Phi(u))\|$

⇒ $\kappa(\Phi(u)) \cdot e$ is curvature of β at u if insert into formula of curvature ∎ $\wedge 3$ crucial

Proof is executed by simple computation $\|\beta'(u)\| = |\Phi'(u)| \cdot \|\gamma'(\Phi(u))\|$.

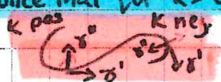
→ for a curve with unit speed (not a serious limitation due to 3.3, 4.1) expression for curvature simpler, $\gamma: I \to \mathbb{R}^2$ unit speed curve, $\hat{\gamma}'(s) = \begin{pmatrix} -\gamma_2' \\ \gamma_1' \end{pmatrix}$ normal vector of $\gamma'(s) = \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix}$ ⇒ $\hat{\gamma}'$ unit vector perpendicular to $\gamma'(s)$ and pointing to left

∘ T.4.2: For a curve with unit speed $\gamma'' = \kappa\hat{\gamma}'$ ⇒ $\kappa = \pm\|\gamma''\|$ where sign is + (curve turns in pos. direction/counter clockwise) if γ'' and $\hat{\gamma}'$ same direction and − if they have opposite directions (curve turns in negative/clockwise direction)

Proof: $n = 2$ and $F(t) = \gamma'(t)$ used for following lemma implies that $\gamma''(s)$ scalar multiple of $\hat{\gamma}'(s)$ and that

61 scalar given by: $\hat{\gamma}'(s) \cdot \gamma''(s) = \det[\gamma'(s)\gamma''(s)] = \kappa(s)$ ∎

Notice that for $\kappa > 0$, γ'' and $\hat{\gamma}'$ have same direction and curve turns left, $\kappa < 0$ opposite dir + turns right



∘ L.4.2: $F(t) \in \mathbb{R}^n$ smooth fct with $\|F(t)\| = 1\ \forall t \in \mathbb{R}$ ⇒ $F(t) \cdot F'(t) = 0\ \forall t \in I$ and for $n = 2$: $F'(t) = (\hat{F}(t) \cdot F'(t))\hat{F}(t)\ \forall t \in I$ → $\|F(t)\| = F(t) \cdot F(t) = 1$ ⇒ derivative $(f \cdot g)' = f'g + f\ldots$

unit vec ↳ scalar depending on t when inner product is computed

T.4.2

↳ suggests a way to determine a plane unit speed curve from its curvature (up to translations + d... primitives:

⇒ curvature fct $\kappa(s) = \gamma'' : \hat{\gamma}'$, $\gamma(s) = (x(t), y(t))$, $x'' = -\kappa y'$, $y'' = \kappa x'$

→ integrate both sides twice and combine the two, to get formulas for x and y → curve with unit speed can be determined from curvature fct. in theory up to constant (need one $\gamma(s_0)$)

∘ C.4.2: A regular param. curve is part of line ⇔ $\kappa \equiv 0$

62 $\kappa(s) = 0\ \forall s \in I$ ⇒ $\gamma''(s) = 0 \overset{\text{integr.}}{\Rightarrow} \gamma(s) = p + sq$ where p, q constant vectors

→ any unit vector $w \in \mathbb{R}^2$ can be written as $w = (\cos\theta, \sin\theta)$, θ determined up to integral multiples of 2π

$\hat{\wedge} = \times$ ⇒ $w = \gamma'(t)/\|\gamma'(t)\|$ of a regular plane curve ⇒ θ tangent angle and $\theta(t)$ its fct.

Example: (1) $\gamma(t) = (r\cos t, r\sin t)$ ⇒ $\gamma'(t)/\|\gamma'(t)\| = (-\sin t, \cos t) = (\cos(t + \frac{\pi}{2}), \sin(t + \frac{\pi}{2}))$

(2) $\gamma(t)=(t,t^2) \Rightarrow \gamma'(t)=(1,2t)$, $\|\gamma'(t)\|=\sqrt{1+4t^2}$ $\left[\Rightarrow \frac{\gamma'(t)}{\|\gamma'(t)\|}=\left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right)\right] \Rightarrow \theta(t)=\tan^{-1}(2t)$

→ first coordinate pos $\frac{\sin}{\cos}$ $\Rightarrow \theta(t)=\tan^{-1}(2t)$

(3) → because of ambiguity in choice of θ ($+k2\pi$) not obvious that $\theta(t)$ can be ~~assumed~~ choosen to be smooth ⇒ 4.3 possi[illegible]

○ 4.3: $\omega(t)$ unit vector in $\mathbb{R}^2$ depending smoothly on a parameter t in $I\subset\mathbb{R}$ open $\Rightarrow \exists\theta: I\to\mathbb{R}$ smooth such that $\omega(t)=(\cos\theta(t),\sin\theta(t))\ \forall t\in I$ 63

○ T.4.3: $\theta(s)$ smooth tangent angle for plane curve (possible to choose so that it's smooth) with unit speed $\Rightarrow \kappa(s)=\theta'(s)$

→ $\|\gamma'(t)\|=1$, $\gamma'(t)\overset{4.2}{=}(\cos\theta(t),\sin\theta(t)) \Rightarrow \gamma''(t)=(-\theta'(t)\sin(\theta(t)),\theta'(t)\cos\theta(t)) \Rightarrow \det[\gamma'(t)\ \gamma''(t)]=\theta'(t)$

⇒ for curve with unit speed, curvature is rate of change of tangent angle → for circle with $r=1$ unit speed $\theta(t)$

in $\mathbb{R}^2$ only two possible direction left (+) right (−) indicated by $\kappa>0$, $\kappa<0$

4.4 Curvature for space curves

denominator for reparam. later; factor for unit speed; area of spanned parallelogram → similar motivation

tells us how much curve turns but not the direction → makes sense, in $\mathbb{R}^3$ infinite directions for vector

○ D.4.4: $\gamma: I\to\mathbb{R}^3$ space curve, regular $\Rightarrow \kappa(t)=\frac{\|\gamma'(t)\times\gamma''(t)\|}{\|\gamma'(t)\|^3}\geq 0$ the curvature of γ at t. For unit speed curve $\kappa(s)=\|\gamma''(t)\|$ → derived from $\|a\times b\|^2=\|a\|^2\|b\|^2-(a\cdot b)^2$, $=0$

Notice: ○ $\mathbb{R}^3$: κ unchanged under reparametrization (direction irrelev. here; pres., reversing) and describes for unit speed the rate of change of direction of the curve

○ C.4.2. valid for space curves with similiar proofs 64

○ can apply this def to plane curve, viewed as space curve in xy-plane $\gamma(t)=(x(t),y(t),0)$

$\Rightarrow \gamma'\times\gamma''=(0,0,\det\begin{pmatrix}x' & x''\\ y' & y''\end{pmatrix}) \Rightarrow \|\gamma'\times\gamma''\|=|\det[\gamma'\ \gamma'']|$ → def for space curves more primitive

○ T.4.4: curvature of space curve unchanged under reparametrization → similar: $\beta'(u)\times\beta''(u)=\phi'(u)^3\gamma'(\phi(u))\times\gamma''(\phi(u))$

Example: helix $\gamma(t)=(\lambda t, r\cos\omega t, r\sin\omega t)$ has constant curvature $\kappa\equiv\frac{r\omega^2}{r^2\omega^2+\lambda^2}$ → geometric interpret.: same shape everywhere 65

4.5 Torsion:

→ describes twisting of the curve, for plane curve regarded as space curve in $\mathbb{R}^3$ torsion is 0

○ D.4.5: $\gamma: I\to\mathbb{R}^3$ regular curve with curvature $\kappa(t): t\in I$ with $\kappa(t)\neq 0 \Rightarrow \tau(t)=\frac{\det[\gamma'(t)\ \gamma''(t)\ \gamma'''(t)]}{\|\gamma'(t)\times\gamma''(t)\|^2}$ called torsion. Notice: denom. $\|\gamma'(t)\times\gamma''(t)\|^2=\kappa(t)^2\|\gamma'(t)\|^6$

denominator always for reparam, motivation later

(0) curve contained in fixed plane ω: $\gamma'(t),\gamma''(t),\gamma'''(t)$ parallel to plane $\Rightarrow \tau=0$

Example: (1) helix has torsion $\tau=\frac{\lambda\omega}{r^2\omega^2+\lambda^2}$, again obtain a constant which is reasonable, just as state above

○ T.4.5: torsion is for space curves unchanged under a reparametrization

→ apply chain rule for all three derivatives of γ, compute determinant, vec product and reduce (kürzen) the fraction 66

Notice that torsion also unchanged when direction is reversed. Sign of the torsion allows us to separate spaces curves with non-zero curvature a. torsion into 'right' and 'left'. E.g. helix for which $\lambda\omega>0$ (same sign) called right helix and for opposite signs → left helix, first conventional screw.

4.6 Osculating plane and binormal vector:

→ now we will explain geometric significance of torsion. Let $\gamma: I\to\mathbb{R}^3$ regular curve with non-zero curvature $\kappa(t) \Rightarrow \gamma'(t),\gamma''(t)\in\mathbb{R}^3$ linearly independent span osculating plane through $\gamma(t)$, kisses curve in $\gamma(t)$ because of Taylor approx. of order two: $\gamma(t+\Delta t)\simeq\gamma(t)+\Delta t\,\gamma'(t)+\frac{1}{2}(\Delta t)^2\gamma''(t)$

⇒ right hand side belongs to osculat. plane $\forall\Delta t\in\mathbb{R}^+$

→ we will show that torsion describes rate of change of osculating plane:

○ can be easily shown (chain rule, different scaler, different values of t) that osculating plane is unchanged under reparametrization, just like the torsion ⇒ Assume unit speed for given curve

→ notation: $t(s)=\frac{\gamma'(s)}{\|\gamma'(s)\|}$ ~~$=\gamma'(s)$~~ unit tangent vec., keep assumption $\kappa(t)\neq 0$: $n(s)=\frac{\gamma''(s)}{\|\gamma''(s)\|}$ the principal normal ortho to $t(s)$ due to 4.2, $b(s)=t(s)\times n(s)$ the binormal of the curve, normal to oscul. plane

4.2 unit speed $t(s)\perp n(s)$

⇒ rate of change of osculating plane described by size of $b'(s)$ (4.6: exactly what torsion measures) 67

b unit vector $\|a\times b\|^2=\|a\|^2\|b\|^2-(a\cdot b)^2$, $=0$

unit tangent $t(s)$
principal normal $n(s)$
binormal $t(s)\times n(s)$

size of derivative measures torsion

○ **T.4.6:** for curve in $\mathbb{R}^3$ with unit speed and non-zero curvature we have $b' = -\tau n$, $\tau = \pm \|b'\|$

→ Proof: show b' proportional to n by showing $(b' \perp t) \wedge (b' \perp b)$ just like n:

4.2: $b' \perp b$, $b \cdot t = 0 \Rightarrow b' \cdot t + b \cdot t' = 0 \Rightarrow (b' \perp t) \Leftrightarrow (b \perp t')$ T.4.2: $t' = \kappa n$

$\Rightarrow (b \perp t')$ follows from $(b \perp n) \Rightarrow b' = cn$, claim that it is $-\tau$

$\Rightarrow$ since $\gamma'' = \kappa n \Rightarrow \gamma''' = \kappa' n + \kappa n' \Rightarrow \det[\gamma' \gamma'' \gamma'''] = (\gamma' \times \gamma'') \cdot \gamma''' = (t \times \kappa n) \cdot (\kappa' n + \kappa n') =$

$= \kappa^2 (t \times n) \cdot n' + \kappa\kappa' \underbrace{(t \times \kappa n) \cdot n}$ $= \kappa^2 (t \times n) \cdot n'$ → vector ortho to n $\Rightarrow \tau = (t \times n) \cdot n' = b \cdot n'$

$\Rightarrow$ from $b \cdot n = 0 \Rightarrow b \cdot n' = -b' \cdot n = -cn \cdot n = -c \cdot \|n\| = -c \Rightarrow c = -\tau \Rightarrow b' = -\tau n$ ∎

○ **C.4.6:** A regular space curve with $\kappa \neq 0$ is contained in fixed plane $\Leftrightarrow \tau = 0$ everywhere

→ Proof: „$\Leftarrow$" $b'(s) = 0 \Rightarrow b$ constant vector. Since $t(s) \perp b \Rightarrow \gamma'(s) \cdot b = 0 = (\gamma \cdot b)'(s) \Rightarrow \gamma \cdot b$ is constant c

$\Rightarrow \gamma(s) \in \{\xi \in \mathbb{R}^3 \mid \xi \cdot b = c\}\ \forall s$

„$\Rightarrow$" example (0) from 4.5.1

4.7 Frenet formulas:

→ $(t(s), n(s), b(s))$ form a positively ordered (det > 0) orthonormal basis for $\mathbb{R}^3$ depending on s, called moving frame of Frenet for curve. We have seen $t' = \kappa n$, $b' = -\tau n$ $\Rightarrow$ interest of determining n' 68

○ **T.4.7:** for curve with unit speed and non-zero curvature: 1) $t' = \kappa n$ 2) $n' = -\kappa t + \tau b$ 3) $b' = -\tau n$

→ Proof: 4.2: $n' \cdot n = 0$, T.4.6: $b \cdot n' = \tau$

since (t, n, b) orthonormal basis $n' = \underbrace{(n' \cdot t)}_{-n \cdot t'} t + (n' \cdot n) n + \underbrace{(n' \cdot b)}_{= -b' \cdot n = \tau \cdot \|n\| = \tau} b = \tau b + (-n \cdot t') t = \tau b - \kappa t$ (ortho-basis; $= -n \cdot \kappa n = -\kappa$)

→ formulas of frenet: since t, n, b have three coordinates $\Rightarrow$ essentially a system of 9 first order differential equations in three coordinates $\Rightarrow$ by solving system (at least in principal) possible to determine curve from torsion, curvature up to integration constants.

4.8 Curvature of curves on a surface

→ $\gamma: I \to \mathbb{R}^3$ with $\gamma = \sigma \circ \mu$ where $\mu: I \to U$ plane curve. Assume γ regular and σ regular at $\mu(t)\ \forall t \in I$

we denote $N = \dfrac{\sigma_u' \times \sigma_v'}{\|\sigma_u' \times \sigma_v'\|}$, the unit normal vec of σ and put $m(t) = N(\mu(t))$ the unit normal vec of σ along γ

○ 4.8: $\kappa_g(t) = \dfrac{\det[\gamma'(t)\ \gamma''(t)\ m(t)]}{\|\gamma'(t)\|^3}$ the geodesic curvature and $\kappa_n(t) = \dfrac{\gamma''(t) \cdot m(t)}{\|\gamma'(t)\|^2}$ the normal curvature of γ at t with respect to σ. → 2x change of columns in $(a \times b) \cdot c = \det[abc]$ 69

For a unit speed curve they are $\kappa_g(s) = \gamma''(s) \cdot u(s)$ and $\kappa_n(s) = \gamma''(s) \cdot m(s)$, where $u(s) = m(s) \times t(s)$ with $t(s) = \gamma'(s)$. The vec $u(s)$ is called tangent normal of γ → lies in tangent space

○ Note: $(t(s), u(s), m(s))$ form a pos. ordered orthonormal base of $\mathbb{R}^3$ ('moving frame of $\boxed{\text{Darboux}}$').

→ first two vec. span the tangent space $T_{\mu(s)}\sigma$

→ since for unit speed $\gamma'(s) \perp \gamma''(s)$ it follows that decomposition of $\gamma''(s)$ according to the orthonormal basis reads $\gamma''(s) = \kappa_g(s) u(s) + \kappa_n(s) m(s)$  $\gamma'' = \kappa_g u + \kappa_n m$. Since $\kappa_g(s) u(s) \in T_{\mu(s)}\sigma$ and $\kappa_n(s) m(s) \perp T_{\mu(s)}\sigma$ this explains the reasoning behind the names.

○ **T.4.8:** κ_g is unchanged under dir. preserving reparam of γ and $\cdot(-1)$ for direction reparam. of γ.

Both κ_g and κ_n unchanged for dir preserving reparam of σ and $\cdot(-1)$ both for reversing

Proof: statements of reparam of γ follow from proof of T.4.1 and the ones for σ are straightforward since σ only present in $m(t)$ 70

○ **C.4.8:** The formula: $\kappa^2 = \kappa_g^2 + \kappa_n^2$ follows from $\kappa(t) = \|\gamma''(t)\|$ for unit speed and the decomp. of γ'' from above with theorem of Pythagoras.

Examples: (1) plane curve regarded as space curve $\gamma(t) = (x(t), y(t), 0)$ in the xy-plane $\Rightarrow m = (0,0,1)$ and κ_g the curvature of the plane curve and $\kappa_n = 0$

(2) compute κ_g and κ_n for circle on a sphere with radius 1. Such curve is called a great circle if the plane that passes through it contains the origin, otherwise it called a small circle.

Due to a possible spatial rotation around the center of the sphere, we can assume that line is horizontal → fixed lattitude

(...) $m(t) = -\sigma(u, t)$, 4.1.2: $\kappa = \dfrac{1}{\cos u}$, $\kappa_g(t) = -\tan u$, $\kappa_n(t) = 1$ 71

$\Rightarrow \tan^2 u + 1 = \dfrac{1}{\cos^2 u}$ verifies C4.8

4.9 Interpretation of normal curvature

→ curve on a surface has to follow the shape of the surface ⇒ forced to some amount of curvature. The interpret. of κ_n is, that it is exactly the part of κ (C.4.8) which curve is forced to have by being on σ:

∘ T.4.9: Given a point $p=(u_0,v_0)\in U$ and $\omega_0 \in T_p\sigma \setminus \{0\}$. All curves $\gamma = \sigma \circ \mu$ with $\mu(t_0)=p$ and $\gamma'(t_0)=\omega_0$ all have the same $\kappa_n(t_0)$

↳ κ_n a lower bound to κ due to C.4.8

∘ L.4.9: $\gamma = \sigma \circ \mu$ a parametrized curve, $m(t) = N(\mu(t))$ for $t \in I$, $p = \mu(t)$ and $(a,b) = \mu'(t)$ for a given $t \in I$:
$\gamma'(t) = a\sigma_u'(p) + b\sigma_v'(p)$ $\qquad$ $m'(t) = a N_u'(p) + b N_v'(p)$

Proof: both by chain rule

72 Proof of theorem: $\gamma'(t)\cdot m(t) = 0 \Rightarrow \gamma''(t)\cdot m(t) + \gamma'(t)\cdot m'(t) = 0 \Rightarrow \kappa_n = \frac{\gamma''(t)\cdot m(t)}{\|\gamma'(t)\|^2} = -\frac{\gamma'(t)\cdot m'(t)}{\|\gamma'(t)\|^2} \overset{L.4.9}{=} -\frac{\omega_0 \cdot m'(t)}{\|\underbrace{\gamma'(t)}_{\omega_0}\|^2}$

→ ω_0 and $m'(t)$ due to L.4.9 the same for all curves ⇒ $\kappa_n(t)$ the same ∎

Examples: (1) $\kappa_n = 1$ at all points for all curves on the unit sphere since ω_0 direction the same for a unique great circle with $\kappa_n = 1$

→ κ_n more a property of the surface rather than of the curve γ

∘ D.4.9: The normal curvature of σ with direction ω_0 is the one of all curves (any) with $\gamma = \sigma \circ \mu$, $\mu(t_0) = p$, $\mu'(t_0) = \omega_0$

↳ follows from 4.8 that κ_n is unchanged for dir preserving dir and -1, sign changes for reversing

4.10 Geodesics

∘ D.4.10: A geodesic on a surface is a regular curve on the surface whose $\kappa_g \equiv 0$

↳ curve which in each point is as straight as possible, in the sense that it has the least possible curvature of a curve in that point with that direction

↳ property of being a geodesic is unchanged under (both reversing and preserving) reparam. of γ (T.4.8)

Examples: (1) geodesics on xy planes are the straight lines contained in it

73 (2) great circles on S^2 are geodesics and small circles are not (even the only geodesics on S^2):
$\gamma(s)$ unit speed curve on $S^2 \Rightarrow m(s) = -\gamma(s)$: $\kappa_g = 0, \kappa_n = 1 \Rightarrow \gamma''(s) = -\gamma(s) \Rightarrow \tau \equiv 0 \overset{C.4.6}{\Rightarrow}$ curve contained in fixed plane ⇒ $\gamma''(s)$ parallel to plane ⇒ great circle

∘ T.4.10: $\gamma = \sigma \circ \mu$ a regular curve on σ: $\quad$ γ geodesic with constant speed ⇔ $\gamma''(t)$ normal to $T_{\mu(t)}\sigma$ $\forall t \in I$

Proof: „⇒" γ a geodesic, $\gamma''(s) = \kappa_g(s)\,u(s) + \kappa_n(s)\,m(s) \Rightarrow \gamma''(s) = \kappa_n(s)\,m(s) \perp \gamma'(t)$

unit speed reparam $t = cs$ for constant speed only changes second derivative by scalar mult.

„⇐" $\gamma''(t)\cdot\gamma'(t) = 0 \Rightarrow \frac{d}{dt}\|\gamma'(t)\|^2 = 0 \Rightarrow$ constant speed, after reparam to unit speed, $\gamma''(s) = \kappa_n(s)m(s) + \kappa_g(s)u(s)$
and $\gamma''(t) \perp T_{\mu(t)}\sigma \overset{u(s)\neq 0,\text{ orthon. basis}}{\Rightarrow} \kappa_g(s) \equiv 0$ ∎

⇒ follows: a geodesic has constant speed if there is no acceleration in the tangent direction. The only acceleration is that which is necessary to keep the object on the surface and it is normal to the surface.