Curvature Theorems

Def: $\gamma: I \to \mathbb{R}^2$ regular param. curve, then $\kappa(t) = \frac{\det(\gamma'(t)\ \gamma''(t))}{\|\gamma'(t)\|^3}$ is called curvature of γ at t.

Theorem: Curvature of a plane curve is unchanged under direction-preserving reparametrization and multiplied by -1 under direction-reversing reparametrization.

Theorem: For a curve with unit speed $\gamma'' = \kappa \hat{\gamma}'$ and $\kappa = \pm\|\gamma''\|$ where $+$ if γ'' and $\hat{\gamma}'$ have the same direction (the curve turns counterclockwise) or $-$ if γ'' and $\hat{\gamma}'$ have opposite direction (the curve turns clockwise).

Corollary: A regular param. curve is part of a line iff $\kappa \equiv 0$.
For a curve with unit speed the curvature is rate of change of tangent angle.

Def: $\gamma: I \subseteq \mathbb{R} \to \mathbb{R}^3$ space curve, regular. Then $\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} \geq 0$ is the curvature of γ at t.

For a unit speed curve $\kappa(s) = \|\gamma''(t)\|$.
The κ is unchanged under reparametrization.

Def.: $\gamma: I \to \mathbb{R}^3$ regular curve with curvature $\kappa(t): t \in I$ with $\kappa(t) \neq 0$
then $\tau(t) = \frac{\det(\gamma'(t)\ \gamma''(t)\ \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$ is called torsion.
If the curve is contained in a fixed plane then $\tau \equiv 0$.
Torsion is unchanged under reparametrization.

Theorem: For a curve in $\mathbb{R}^3$ with unit speed and non-zero curvature we have $b' = -\tau n$, $\tau = \pm\|b'\|$.

Corollary: A regular space curve is contained in a fixed plane iff $\tau = 0$ everywhere.

Frenet formulas: For curve with unit speed and non-zero curvature:
1.) $t' = \kappa n$
2.) $n' = -\kappa t + \tau b$ $\qquad b(s) = t(s) \times n(s)$
3.) $b' = -\tau n$ $\qquad u(s) = m(s) \times t(s)$
Possible in principle to determine curve from torsion, curvature up to integration constants.

Def: $N = \frac{\sigma'_u \times \sigma'_v}{\|\sigma'_u \times \sigma'_v\|}$ is called the unit normal vector of σ.

Theorem: $\kappa_g(t) = \frac{\det(\gamma'(t)\ \gamma''(t)\ m(t))}{\|\gamma'(t)\|^3}$ the geodesic curvature and $\kappa_n(t) = \frac{\gamma''(t)\, m(t)}{\|\gamma'(t)\|^2}$ the normal curvature.
For a unit speed curve they are $\kappa_g(s) = \gamma''(s) \cdot u(s)$ and $\kappa_n(s) = \gamma''(s) \cdot m(s)$
where $u(s) = m(s) \times t(s)$ and $t(s) = \gamma'(s)$.

$(t(s), n(s), b(s))$ moving frame of Frenet $\qquad$ osculating plane
$(t(s), u(s), m(s))$ moving frame of Darboux
$\kappa_g(t)$ is unchanged for direction preserving reparametrization of γ and $\cdot(-1)$ for direction reversing.
$\kappa_n(t)$ is unchanged for direction preserving reparametrization of γ and $\cdot(-1)$ for direction reversing.

Corollary: $\kappa^2 = \kappa_g(t)^2 + \kappa_n(t)^2$

Theorem: Given a point $p = (u_0, v_0) \in U$ and $w_0 \in T_p\sigma \setminus \{0\}$, all curves $\gamma = \sigma \circ \mu$ with $\mu(t_0) = p$ and $\gamma'(t_0) = w_0$ all have the same $\kappa_n(t_0)$.

Def: A geodesic on a surface is a regular curve on the surface whose $\kappa_g \equiv 0$.

Theorem: $\gamma = \sigma \circ \mu$ a regular curve on σ: γ geodesic with constant speed iff $\gamma''(t)$ normal to $T_{\mu(t)}\sigma$ for all $t \in I$.
$\Rightarrow$ A geodesic has constant speed if there is no acceleration in the tangent direction. The only acceleration is that which is necessary to keep the object on the surface and it is normal to the surface.

First fundamental form Theorems

Def.: Let $\gamma: I \to \mathbb{R}$ smooth curve, arc-length of γ from $t_1 \in I$ to $t_2 \in I$ is $\int_{t_1}^{t_2} \|\gamma'(t)\|\, dt$.
The arc-length function for γ is a primitive of the speed function $t \mapsto \|\gamma'(t)\|$.
Differentiable function $l: I \to \mathbb{R}$ with $l'(t) = \|\gamma'(t)\|$. Arc-length then $l(t_2) - l(t_1)$.

Theorem: $\gamma: I \to \mathbb{R}^n$ param. curve, $\beta = \gamma \circ \phi: J \to \mathbb{R}^n$ a reparametrization. Let $u_1, u_2 \in J$ and $t_i = \phi(u_i)$ for $i \in \{1, 2\}$. If $\det(\phi) > 0 \Rightarrow \int_{u_1}^{u_2} \|\beta'(t)\|\, dt = \int_{t_1}^{t_2} \|\gamma'(t)\|\, dt$, else opposite signs.

Theorem: $\gamma: I \to \mathbb{R}^N$ param. curve $\Rightarrow$ if $t_1 < t_2$ in I and let L arc length of γ from t_1 to t_2 $\Rightarrow L \geq \|\gamma(t_2) - \gamma(t_1)\|$.

~~Theorem: $\gamma: I \to \mathbb{R}^n$ a param. curve $\Rightarrow$ if $t_1 < t_2$ in I~~

Theorem: A regular param. curve γ allows a direction-preserving reparam. with unit speed (parametrized by arc-length).

Def.: $\sigma: U \to \mathbb{R}^3$ param. surface: Three functions on U, associated with σ:
1.) $E(p) = \|\sigma'_u(p)\|^2$
2.) $F(p) = \sigma'_u(p) \bullet \sigma'_v(p)$
3.) $G(p) = \|\sigma'_v(p)\|^2$
By Cauchy-Schwarz inequality we get: $F(p)^2 \leq E(p)G(p)$
with strict inequality iff σ is regular at p.

Def: σ regular at p: map $I_p: T_p\sigma \to \mathbb{R}$, $w \mapsto \|w\|^2 = Ea^2 + Fab + Gb^2$ is called the first fundamental form of σ in p and E, F, G the component functions.
I_p quadratic form on $T_p\sigma$ $\forall p \in U$.

Theorem: The arc-length of a param. curve $\gamma(t) = \sigma(u(t), v(t))$ on σ is given with respect to coordinates $(u(t), v(t))$ as follows: $\int_{t_1}^{t_2} (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2}\, dt$ where E, F, G are evaluated in $(u(t), v(t))$.

Def.: $D = [a, b] \times [c, d]$: rectangle with area $A(D) = (b-a)(d-c)$.
If $f: D \to \mathbb{R}^2$ continuous, integral over D is $\int_D f\, dA = \int_a^b \int_c^d f(u,v)\, dv\, du = \int_c^d \int_a^b f(u,v)\, du\, dv$
Fubini-Theorem, both integrals are finite, interchanging is possible.

Def.: Block-set: finite union of closed rectangles

Def.: Partition of the block-set is refinement such that blocks overlap only at the boundary
then we have
$\circ\ |\int_K f\, dA| \leq A(K) \sup_{p \in K} |f(p)|$ $\qquad \circ\ \int_{K_1 \cup K_2} f\, dA = \int_{K_1} f\, dA + \int_{K_2} f\, dA$

Def: Set $D \subset \mathbb{R}^2$ is called elementary domain if it is closed and bounded (compact) and if its boundary ∂D is a finite union of (the trace of) smooth curves defined on closed intervals.

Def: Area of D is defined by $A(D) = \sup_{K \subseteq D} A(K)$ then $\int_D f\, dA = \sup_{K \subseteq D} \int_K f\, dA$ $\leftarrow f(p) \geq 0$

Suprema are finite, since D is bounded, it is contained in a sufficiently large square with side-length N
$\int_D f\, dA \leq A(K) \sup_{p \in K} f(p) \leq A(D) \sup_{p \in D} f(p)$

Remove assumption that $f(p) \geq 0$ splitting

$$f_+(x) = \max\{0, f(x)\}, \quad f_-(x) = \max\{0, -f(x)\}$$

$$\int_D f\, dA = \int_D f_+\, dA - \int_D f_-\, dA$$

$$\int_D f + g\, dA = \int_D f\, dA + \int_D g\, dA$$

$$\int_D c f\, dA = c \int_D f\, dA$$

$$\left|\int_D f\, dA\right| \leq \int_D |f|\, dA$$

$$\int_{D_1 \cup D_2} f\, dA = \int_{D_1} f\, dA + \int_{D_2} f\, dA$$

Lemma: $\gamma: [a, b] \to \mathbb{R}^2$ smooth, the trace $\gamma([a, b])$ is a null set.
Thus boundaries of elementary domains are null-sets.

Theorem: $U \subseteq \mathbb{R}^2$ open set, $f: U \to [0, \infty[$ be a continuous function with $D \subset U$ an elementary domain.
$\int_D f\, dA = \inf_{D \subseteq K} \int_K f\, dA$ where K is a block-set.

Lemma: $\gamma:[a,b]\to\mathbb{R}^2$ smooth. The trace $\gamma([a,b])$ is a null set.
Thus, boundaries of elementary domains are null sets.

Theorem: $U\subset\mathbb{R}^2$ open set, $f:U\to[0,\infty[$ be a continuous function with $D\subseteq U$ an elementary domain
$\int_D f\,dA = \inf_{D\subseteq K\subseteq U}\int_K f\,dA$ where K are Nah-sets

Lemma: let $\phi,\psi:[a,b]\to\mathbb{R}$ smooth fct. with $\phi(u)<\psi(u)$ $\forall u\in(a,b)$.
The set $D=\{(u,v)\mid a\le u\le b,\ \phi(u)\le v\le\psi(u)\}$ is an elementary domain.

Theorem: $A(D)=\int_a^b[\psi(u)-\phi(u)]\,du$ and the plane integral of a continuous fct f over D is
$\int_D f\,dA=\int_a^b\int_{\phi(u)}^{\psi(u)} f(u,v)\,dv\,du$.

Theorem: $D\subseteq\mathbb{R}^2$ closed and bounded and contained in W. If D is an elementary domain, then so its image $\phi(D)\subseteq U$.
Moreover $\int_{\phi(D)} f\,dA=\int_D(f\circ\phi)\cdot|\det(D\phi)|\,dA$ for $f:U\to\mathbb{R}$ is continuous

Def.: The area of the surface $\sigma:U\to\mathbb{R}^3$ over $D\subseteq\mathbb{R}^2$ elementary domain with $D\subseteq U$ is
$A(\sigma,D)=\int_D\|\sigma_u'\times\sigma_v'\|\,dA$.

$\|\sigma_u'\times\sigma_v'\|^2=(EG-F^2)^{\frac{1}{2}}$, $\|a\times b\|=\|a^2\|\|b^2\|-(a\cdot b)^2$

Theorem: The surface area is invariant under reparametrization $\tau=\sigma\circ\phi:W\to\mathbb{R}^3$ where $\phi:W\to U$ is a diffeomorphism!

Tangents Theorems

Def.: $\gamma:I\to\mathbb{R}^n$ a param. smooth curve with $t_0\in I$.
The curve is called regular in t_0 if $\gamma'(t_0)\neq0$. Otherwise it is called singular.

Def.: $\gamma'(t_0)$ tangent vector (velocity vector)
$\mu(t_0)=\gamma(t_0)+t\gamma'(t_0)$ (tangent line)

Theorem: Assume γ regular at t_0 and let $v=\frac{\gamma'(t_0)}{\|\gamma'(t_0)\|}$ be the unit vector in direction of tangent vector,

then $v=\lim_{t\to t_0^+}\frac{\gamma(t)-\gamma(t_0)}{\|\gamma(t)-\gamma(t_0)\|}=\lim_{t\to t_0^-}\frac{\gamma(t)-\gamma(t_0)}{\|\gamma(t)-\gamma(t_0)\|}$ $\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ tangent normal vector

Def.: The tangent line of a level set is $\frac{\partial f}{\partial x}(p)(x-x_0)+\frac{\partial f}{\partial y}(p)(y-y_0)=0$ (Gradient is normal vector to tangent line)
Tangent line depends on the level set C through fct. f but independent of chosen param. γ.

Def.: A param. surface σ is called regular at $p=(u_0,v_0)$ if the partial derivatives σ_u' and σ_v' evaluated at p are linearly independent, otherwise singular.

Def.: The linear subspace of $\mathbb{R}^3$ spanned by $\sigma_u'(p)$ and $\sigma_v'(p)$ is called the tangent space of σ at p, denoted $T_p\sigma$.

Def.: The tangent plane of σ at (u_0,v_0) is $\frac{\partial f}{\partial x}(p)(x-x_0)+\frac{\partial f}{\partial y}(p)(y-y_0)+\frac{\partial f}{\partial z}(p)(z-z_0)=0$.

Def.: A param. curve on a param. surface $\sigma:U\to\mathbb{R}^3$ is a param. curve $\gamma=\sigma\circ\mu:I\to\mathbb{R}^3$ where $\mu:I\to U$ is a param. plane curve.

Lemma: Let $\gamma=\sigma\circ\mu$ be a param. curve on σ, then $\gamma'(t)=u'(t)\sigma_u'(\mu(t))+v'(t)\sigma_v'(\mu(t))$.

Theorem: Let $p\in U$. The tangent space $T_p\sigma$ is equal to the set of tangent vectors $\gamma'(t_0)$ of all param. curves $\gamma=\sigma\circ\mu$ on σ through p, with $t_0\in I$ and $\mu(t_0)=p$.

Def.: $\gamma:I\to\mathbb{R}^n$ a param. curve and $\phi:J\to I$ a smooth bijective map with smooth inverse.
The curve $\beta=\gamma\circ\phi:J\to\mathbb{R}^n$ is called a reparametrization of γ.

Theorem: Let $J\subseteq\mathbb{R}$ be an open interval and $\phi:J\to\mathbb{R}$ a smooth map. Let $I=\phi(J)$, following are equivalent:

1.) $\begin{cases}1)\ \phi'(u)\neq0\ \ \forall u\in J.\\ 2)\ \phi:J\to I \text{ bijective, } I \text{ open, } \phi^{-1}\text{ smooth}\end{cases}$

2.) then $(\phi^{-1})'(t)=\frac{1}{\phi'(\phi^{-1}(t))}$ $\forall t\in I$.

Def.: Let $U,W\subseteq\mathbb{R}^n$ be open sets. A map $\phi:W\to U$ which is smooth, bijective and has a smooth inverse is called a diffeomorphism.

Def.: $\sigma:U\to\mathbb{R}^3$ a param. surface, $\phi:W\to U$ a diffeomorphism ($U,W\subseteq\mathbb{R}^2$ open sets)
$\tau:\sigma\circ\phi:W\to\mathbb{R}^3$ is called a reparametrization of σ.

Theorem: $\tau=\sigma\circ\phi$ reparametrization of surface σ
$\Rightarrow$ tangent space identical: $T_p\tau=T_{\phi(q)}\sigma$, $q\in W$
(if τ is regular at q $\Leftrightarrow$ σ is regular at $\phi(q)$)
Tangent space is invariant under reparametrization!

Def.: $N(p)=\frac{\sigma_u'\times\sigma_v'}{\|\sigma_u'\times\sigma_v'\|}$ unit normal at p.

Level sets are graphs, away from critical points.
Graphs $y=h(x)$ are level sets $y-h(x)=0$
Graphs are regular param. curves.
Regular param. curves can be param. as graphs of smooth fcts. in Nbh. of its points.

Example: $\gamma(t)=(\cos(t),\sin(t))$ with $t\in\mathbb{R}$
$\gamma'(t)=(-\sin(t),\cos(t))$ with $t\in]0,\pi[$, $x'(t)\neq0$
$\to\gamma(\phi(u))=(u,\sin(\cos^{-1}(u)))=(u,\sqrt{1-u^2}),\ u\in J=]-1,1[$

Def.: A regular param. curve γ is locally injective, if there exists a nbh. around each $t_0\in I$ such that the restriction of γ to this nbh is injective.

Inverse function theorem: Let $F:U\to\mathbb{R}^n$, $U\subset\mathbb{R}^n$ open and $q\in U$ with $\det(DF(p))\neq0$.
Then there exists an open set $W\subseteq U$ containing q and an open set $V\subseteq\mathbb{R}^n$ containing $F(q)$ such that $V=F(W)$ and the restriction $F|_W:W\to V$ is a diffeomorphism of W onto V.

Content:

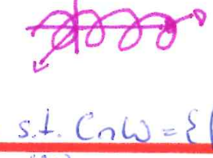
Parametrized Curves and Surfaces Theorems on extra sheet.

Presentations:

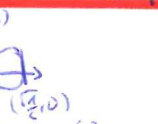
Param. Curves and Surfaces ...

Continuous map, $I\subseteq\mathbb{R}$ open interval, not necessarily proper

- $\gamma:I\to\mathbb{R}^n,\mathbb{R}^2,\mathbb{R}^3$
  $\gamma(t)=(t,0)$, $\tilde\gamma(t)=(e^t,0)$, $\gamma(t)\neq\tilde\gamma(t)$, $\mathrm{im}(\gamma(t))=\mathrm{im}(\tilde\gamma(t))$ on $]0,\infty[$. image is not equal to curve distinguish orbit and curve parametrization must not be injective curve is differentiable in C^∞ iff $f_1,\dots,f_n$ in C^∞
  - $t\mapsto p+qt$, $q\neq0$
  - Ellipse: $C=\{(x,y)\mid\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a,b>0\}$, $\gamma(t)=(a\cos(t),b\sin(t))$
  - ~~Cylinder~~ Hyperbola: $C=\{(x,y)\mid\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ a,b>0,\ x>0\}$, $\gamma(t)=(a\cosh(t),b\sinh(t))$
- $\sigma:U\to\mathbb{R}^3$, $U\subseteq\mathbb{R}^2$ critical point $\nabla f(p)=0$ What are critical points? How do they look like? σ smooth if $\sigma_u',\sigma_v'\in C^\infty$ differentiable up to all orders.
  - $\sigma(u,v)=p+uq_1+vq_2$
  - $\sigma(u,v)=(\cos(u)\cos(v),\cos(u)\sin(v),\sin(u))$
  - $r>0$, $\sigma(u,v)=(r\cos(v),r\sin(v),u)$, $S=\{(x,y,z)\mid x^2+y^2=r^2\}$ apart from critical points level sets can be parametrized as curves.
- $h:A\to B$, $(x,h(x))\in A\times B$ always as parametrized curve $t\mapsto(t,h(t))$
  - $\lambda=1$: $h:\mathbb{R}\to\mathbb{R}^2$, $h(t)=(r\cos(\omega t),r\sin(t))$ parametrized by $\gamma(t)=(\lambda t,r\cos(\omega t),r\sin(\omega t))$
  - $h(u,v)=\sqrt{1-u^2-v^2}$ on $U=\{(u,v)\mid u^2+v^2<1\}$ is a half sphere.



- Replace by $\mathbb{R}^{n-m}$ Replace by $\mathbb{R}^n$ with $m<n$
  $f:\Omega\to\mathbb{R}$, $\Omega\subseteq\mathbb{R}^2$, $C=\{(x,y)\in\Omega\mid f(x,y)=c\}$.
  $p=(x_0,y_0)\in C$, $\frac{\partial f}{\partial y}(p)\neq0$, then $\exists I,J$ $x_0\in I$, $y_0\in J$, $h:I\to J$ s.t. $C\cap W=\{(x,h(x))\mid x\in I\}$.
  consider Df or consider $A=\frac{\partial f_i}{\partial y_j}$ and $\det(A)\neq0$
  - ⚠ $\frac{\partial f}{\partial y}\neq0$ not necessary e.g. $f(x,y)=y^2$, $c=0$



  - $f(x,y)=x^4-x^2+y^2$, $c=0$, $\partial_x f(x,y)=\partial_y f(x,y)=0$, $\gamma(t)=(\cos(t),\cos(t)\sin(t))$
    Parametrized by as smooth function, but not as ...
- $f:\Omega\to\mathbb{R}^m$ ... $\|\partial f_i\|\neq0$

First fundamental form

Treat metric questions on curves on surfaces.
Distance along curve assoc. with distance in Euclidean space.

- $\int_{t_1}^{t_2} \|\gamma'(t)\|\,dt$, $\gamma(t)$ to $\gamma(t+\Delta t) \approx \|\gamma'(t)\|\,\Delta t$ — Arc-length invariant under reparametrization.
  $\ell: t \longmapsto \|\gamma'(t)\|$, $\ell'(t) = \|\gamma'(t)\|$, $\ell(t_2) - \ell(t_1)$ — regular param. curve allows reparametrization by arc-length unit speed, s.t. $\|\gamma'(t)\| = 1$.
- $\|\gamma'(t)\| = 1 \quad \forall t \in I \Rightarrow \int_{t_1}^{t_2} \|\gamma'(t)\|\,dt = t_2 - t_1$

functions good for computing length of tangent vectors

- $E(p) = \|\sigma'_u(p)\|^2$, $F(p)^2 \leq E(p)G(p)$
  $G(p) = \|\sigma'_v(p)\|^2$ — component functions of first fundamental form
  $F(p) = \sigma'_u \cdot \sigma'_v$

Block-set is finite union of closed rectangles. Partition of the block-set allows rectangles to only overlap at their boundaries.

- $T_p\sigma = \operatorname{Span}_{\mathbb{R}}\{\sigma'_u, \sigma'_v\}$, $w \in T_p\sigma : w = a\sigma'_u + b\sigma'_v$
  $\|w\|^2 = (a\sigma'_u + b\sigma'_v)(a\sigma'_u + b\sigma'_v) = Ea^2 + 2Fab + Gb^2$

  $I_p : T_p\sigma \to \mathbb{R}$, $w \longmapsto \|w\|^2 \Rightarrow I_p(w) = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ — First fundamental form.

Vertauschbar mit Fubini

- $\gamma(t) = \sigma(u(t), v(t))$, $\boxed{\int_{t_1}^{t_2} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt}$ arc-length of param. curve
  $\gamma'(t) = u'\sigma'_u + v'\sigma'_v$, linear combination of tangent vectors
- 2. $A(K)$, $\int_K f\,dA$, $\left|\int_K f\,dA\right| \leq A(K) \sup_{p \in K} |f(p)|$ — Zuerst: 1. $A(D) = (b-a)(d-c)$, $\int_D f\,dA = \int_a^b \int_c^d f(u,v)\,dv\,du$

  $$\int_{K_1 \cup K_2} f\,dA = \int_{K_1} f\,dA + \int_{K_2} f\,dA$$

Gebiet; Block-sets are der finite elementary domains. Elementary domain is closed and bounded thus (compact) and boundary ∂D is finite union of trace of smooth curves on closed intervals.

- $D \subseteq \mathbb{R}^2$ elementary domain
  $f : D \to \mathbb{R}$, $f(p) \geq 0 \ \forall p \in D$: $\int_D f\,dA = \sup_{K \subset D} \int_K f\,dA$

$\gamma([a,b]) \to \mathbb{R}^2$ be smooth. The trace $\gamma([a,b])$ is a null-set.

- $A(\sigma, D) = \int_D \|\sigma'_u \times \sigma'_v\|\,dA$
  $\boxed{= \int_D (EG - F^2)^{\frac{1}{2}}\,dA}$ — often $A(\sigma(D))$, but depends on both, σ and D. Not legitimate

Tangents

- $\gamma(t) = (t^3, t^2)$  ? — Does not correspond to intuition of being smooth
  $\gamma(t)$ regular $\Leftrightarrow \gamma'(t) \neq 0$
- σ is regular $\Leftrightarrow \sigma'_u \times \sigma'_v \neq 0$ — Regular, if σ'_u and σ'_v are linearly independent or $\dim_{\mathbb{R}} T_p\sigma = \dim_{\mathbb{R}} \sigma$
  - $\sigma(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ — singular at poles $(0,0,\pm 1)$, depends on parametrization
    $\sigma'_u \times \sigma'_v = -\cos(u)\,\sigma(u,v) = 0 \Leftrightarrow u \in \frac{\pi}{2} + 2\pi k$
  - $\sigma'_u(u,v) = (1, 0, \frac{\partial h}{\partial u})$, $\sigma'_v(u,v) = (0, 1, \frac{\partial h}{\partial v})$ regular, σ'_u and σ'_v are always linearly independent.
- $\gamma = \sigma \circ \mu : I \to \mathbb{R}^3$, $\mu : I \to U \subseteq \mathbb{R}^2$ — $\gamma'(t) = u'(t)\sigma'_u(\mu(t)) + v'(t)\sigma'_v(\mu(t))$
- $T_p\sigma = \gamma'(t_0)$ of all $\gamma = \sigma \circ \mu$ on σ through p — $\phi : J \to I$ bijective, smooth, smooth inverse $\Rightarrow$ diffeomorphism

Inverse function theorem in one variable:

- $J \subseteq \mathbb{R}$, $\phi : J \to \mathbb{R}$, $I = \phi(J)$, $\phi'(u) \neq 0 \ \forall u \in J$, $\phi : J \to I$ bijective, ϕ^{-1} smooth and I open — nicht notwendig für den großen Zusammenhang
  $\Rightarrow (\phi^{-1})'(t) = \dfrac{1}{\phi'(\phi^{-1}(t))} \quad \forall t \in I$, I open.
  zu Main Theorem 1: $\phi' > 0$ preserves dir, $\phi' < 0$ reverses dir
- $\tau = \sigma \circ \phi$ reparametrization, ϕ diffeomorphism
  $\Rightarrow T_p\tau = T_{\phi(p)}\sigma$, $p \in W$ — This goes to the sphere

$\boxed{\text{Main Theorem 1:}}$

$\gamma : I \to \mathbb{R}^2$, regular
$\exists I' : t_0 \in I' \subseteq I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, $\phi : \cancel{J \to I'}$ $I' \to J$ open bijective
with ϕ^{-1} smooth, s.t. $\gamma(\phi(u)) = (u, h(u))$

Ex.: $\gamma(t) = (\cos(t), \sin(t))$ ~~$t \in \mathbb{R}$~~ $t \in (0, \pi)$
$\gamma'(t) = (-\sin(t), \cos(t))$
$\cos(t)$ bijective, $I' = (0, \pi) \to (-1, 1)$
$\cos^{-1} : J \to I'$ glatt
$\gamma(\phi(u)) = (u, \sin(\cos^{-1}(u))) = (u, \sqrt{1-u^2})$

- $N(p) = \dfrac{\sigma'_u(p) \times \sigma'_v(p)}{\|\sigma'_u(p) \times \sigma'_v(p)\|}$, for sphere $N(u,v) = -\sigma(u,v)$
  - $\det(D(\phi)) > 0$: prev. orient. — $\phi' > 0$ preserves
    $\det(D(\phi)) < 0$: rev. orient. — $\phi' < 0$ reverse
- level sets = Graphs (except critical points)
  Graphs $y = h(x)$ are level-sets $y - h(x) = 0$
  Graphs are reg. param. curves.
  Reg. param. curves can be param. as graphs of smooth fcts. in nbh. of its points.

$\boxed{\text{Inverse function theorem:}}$

$F : U \to \mathbb{R}^m$, smooth, $U \subset \mathbb{R}^n$ and $q \in U$ with $\det(F(q)) \neq 0$.
$\Rightarrow \exists W \subset U$ open: $q \in W$ and $V \subset \mathbb{R}^n$ open: with $F(q) \in V$ s.t. $V = F(W)$ and $F|_W : W \to V$ is diffeo.

Curvature

parallelogram spanned of vectors, measure the area of the parallelo. span spanned by $\gamma'(t), \gamma''(t)$ — Smooth and reg. param.

Content:
- Curvature plane and space
- Torsion
- Frenet formulas
- Geodesic and normal curvature

- $\kappa(t) = \dfrac{\det(\gamma'(t)\ \gamma''(t))}{\|\gamma'(t)\|^2}$, $\gamma(t) : I \to \mathbb{R}^2$ — for plane curves
  - $\gamma(t) = p + tq \Rightarrow \gamma'(t), \gamma''(t)$ lin. dep. $\Rightarrow \kappa(t) = 0$ — curvature on a line is zero
  - $\gamma(t) = (r\cos(t), r\sin(t))$, $\kappa(t) = \frac{1}{r}$ for clockwise parametrization, $-\frac{1}{r}$ for counter clockwise
  - unchanged u. dir. prev. rep.
- $\gamma''(s) = \kappa(s)\hat{\gamma}'(s) \Rightarrow \kappa(s) = \pm\|\gamma''(s)\|$ + counter clock − clock — unit speed curve

smooth and reg. param.

- $\kappa(t) = \dfrac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$, $\gamma(t) : I \to \mathbb{R}^3$ — for space curves

How much does the curve fail to lie on a plane.
Torsion describes twisting of curve, is zero when curve lies in a plane — if in fixed plane determinant is zero

- $\gamma : I \to \mathbb{R}^3$, $\kappa(t) \neq 0$, $\tau(t) = \dfrac{\det(\gamma'(t)\ \gamma''(t)\ \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2} = \dfrac{\det(\gamma'(t)\ \gamma''(t)\ \gamma'''(t))}{\kappa(t)^2\|\gamma'(t)\|^6}$
- $\gamma'(t), \gamma''(t) \in \mathbb{R}^3$ lin. indp. osculating plane $\operatorname{span}_{\mathbb{R}}(\gamma'(t), \gamma''(t))$
  kisses curve in t, Taylor approx.: $\gamma(t+\Delta t) \approx \gamma(t) + \Delta t\,\gamma'(t) + \frac{1}{2}(\Delta t)^2\gamma''(t)$
- $t(s) = \dfrac{\gamma'(s)}{\|\gamma'(s)\|}$, $n(s) = \dfrac{\gamma''(s)}{\|\gamma''(s)\|}$, $b(s) = t(s) \times n(s)$
- $b'(s) = -\tau(s)n(s)$, $\tau = \pm\|b'(s)\|$
  $(t(s), n(s), b(s))$ Frenet
  1.) $t'(s) = \kappa(s)n(s)$
  2.) $n'(s) = -\kappa(s)t(s) + \tau(s)b(s)$
  3.) $b'(s) = -\tau(s)n(s)$

Formulas of Frenet: since fcts. t, n, b have three coordinates it is essentially a system of 9 first order differential equations in three coordinates. By solving system one can in principle determine the curve from curvature and torsion.

Curve on a surface has its own curvature and the "given" curvature by the space surrounding.

- $\boxed{\kappa_g(t) = \dfrac{\det(\gamma'(t)\ \gamma''(t)\ m(t) = N(\mu(t)))}{\|\gamma'(t)\|^3}, \quad \kappa_n(t) = \dfrac{\gamma''(t) \cdot m(t)}{\|\gamma'(t)\|^2}}$
- $\kappa_g(s) = \gamma''(s) \cdot u(s)$, $\kappa_n(s) = \gamma''(s) \cdot m(s)$, $u(s) = m(s) \times t(s) = \gamma'(s)$
  $(t(s), u(s), m(s))$ Darboux — for unit speed $\gamma'(s) \perp \gamma''(s)$ $\quad \gamma''(s) = \underbrace{\kappa_g(s)u(s)}_{\in T_{\mu(s)}\sigma} + \underbrace{\kappa_n(s)m(s)}_{\perp T_{\mu(s)}\sigma}$
- unchanged u. dir. prev. rep.
- $\kappa^2 = \kappa_g^2 + \kappa_n^2$ — $\gamma'' = \kappa_g u(s) + \kappa_n m(s)$, $n(s) = \pm\|\gamma''(s)\|$
- Reg. curve on surface s.t. $\kappa_g \equiv 0$ geodesic.

curve in each point as straight as possible
great circles on S^2 are examples
$\Rightarrow$ geodesic has constant speed if there is no acceleration in the tangent direction. The only acceleration is that one, that is necessary to keep the object on the surface and it is normal to the surface.

The second fundamental form

symmetric with respect to dot product
Eigenvalues are principal curvatures
Eigenvectors are principal vectors

- $W_p : T_p\sigma \to T_p\sigma$, $W(a\sigma'_u + b\sigma'_v) = -aN'_u - bN'_v$
  $\gamma = \sigma \circ \mu$ with $\mu(t_0) = p \Rightarrow W(\gamma'(t_0)) = -m'(t_0)$ — unchanged under direction preserving reparametrization ~~and~~ and sign reversed under direction reversing reparametrization
  - unchanged u. dir. prev. rep.
- $II_p : T_p\sigma \to T_p\sigma$, $w \longmapsto w \cdot W(w)$
  - unchanged u. dir. prev. rep.
  $\kappa_n = \dfrac{II_p(w)}{\|w\|^2}$

Content:
- definition of shape operator
- second fundamental form
- normal curvature by second fundamental form
- Weingarten Eigenvalues
- Second fundamental form as linear combinations of its eigenvectors
- Shape of surface by principal curvature

- $II_p(a\sigma'_u + b\sigma'_v) = La^2 + 2Mab + Nb^2$

  $M = N\sigma''_{uv} = \dfrac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{uv})}{\|\sigma'_u \times \sigma'_v\|}$

  $N = N \cdot \sigma''_{vv} = \dfrac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{vv})}{\|\sigma'_u \times \sigma'_v\|}$ $\qquad II_p(a\sigma'_u + b\sigma'_v) = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

  $L = N \cdot \sigma''_{uu} = \dfrac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{uu})}{\|\sigma'_u \times \sigma'_v\|}$

Matrix in general not symmetric, as product of symmetric matrices does not need to be symmetric itself.

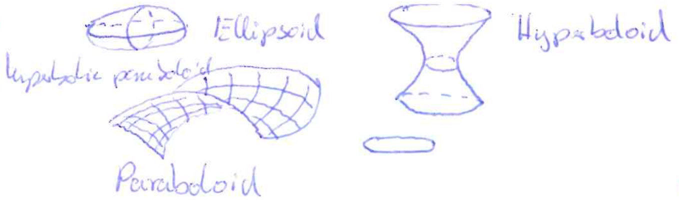
- $W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

orthonormal basis (w_1, w_2) for $T_p\sigma$ consisting of principal vectors with corresponding principal curvatures.

- $II_p(aw_1 + bw_2) = \kappa_1 a^2 + \kappa_2 b^2$
- $\det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} - \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$, solve $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \kappa \begin{pmatrix} a \\ b \end{pmatrix}$.

elliptic paraboloid

- elliptic point: $\kappa_1, \kappa_2 \neq 0$, $\mathrm{sgn}(\kappa_1) = \mathrm{sgn}(\kappa_2)$
- hyperbolic point: $\kappa_1, \kappa_2 \neq 0$, $\mathrm{sgn}(\kappa_1) \neq \mathrm{sgn}(\kappa_2)$
- parabolic point: $\kappa_1 = \kappa_2 = 0$ (planar)
- parabolic cylinder: $\kappa_1 = 0, \kappa_2 \neq 0$.



Geodesics ••

Content:
- Geodesic intrinsic invariant
- Geodesic curvature in terms of E, F, G
- Geodesic in any direction
- unit speed geodesics
- Geodesic coordinates
- Interpretation Gaussian curvature

- Property is intrinsic, geodesic is reg. param curve with $\kappa_g \equiv 0$. $|\kappa_g(t)|$ is intrinsic invariant

- $\kappa_g = \|\gamma'\|^{-3} \det(g_{ij})^{1/2} \left((u_1)'\lambda_2 - (u_2)'\lambda_1\right)$
  $\lambda_i(t) = (u_i)''(t) + \sum_{j,k=1}^{2} \Gamma^i_{jk}(\mu(t))(u_j)'(t)(u_k)'(t)$
  can be determined from E, F, G as Γ^k_{ij} are intrinsic. geodesic curvature expressible through Christoffel symbols

- Solving geodesic equations determines geodesic on surface. Geodesic with constant speed iff $(u_i)'' + \sum_{j,k=1}^{2} \Gamma^i_{jk}(u_j)'(u_k)' = 0$.
  Solving geodesic equations determines geodesic on a surface.

- Through every point unique geodesic in each direction.
  $p \in U$, $w \in T_p\sigma \setminus \{0\}$, $\exists\, \gamma := \sigma \circ \mu : I \to \mathbb{R}^2$, $p = \mu(t_0)$, $w = \gamma'(t_0)$, γ geodesic
  if they have unit speed and satisfy these conditions, they agree on $I \cap J$.

Examples:
- Standard coordinates.
- Spherical coordinates.

- $\gamma : J \to \mathbb{R}^3$, $\|\gamma'(t)\| = 1$, $\sigma : U \to \mathbb{R}^3$, $U = I \times J$, (geodesic coordinate system transversal to γ)
  1. $\exists u_0 \in I$: $\gamma(v) = \sigma(u_0, v)$ $\forall v$ and γ is geodesic.
  2. $I \ni u \mapsto \sigma(u, v)$ are unit speed geodesics on σ, $\sigma'_u(u_0, v) \perp \gamma'(v) = \sigma'_v(u_0, v)$
- σ is geodesic coordinate system transversal to γ iff
  $E(u,v) = 1$, $F(u,v) = 0$ for all $(u,v) \in U$ and
  $G(u_0, v) = 1$, $G'_u(u_0, v) = 0$ for all $v \in J$.
  $\iff$ $E(u,v) = 1$ and $E'_v(u,v) - 2F'_u(u,v) = 0$, then $u \mapsto \sigma(u,v)$ is geodesic.

- $\sigma : U \to \mathbb{R}^3$ geodesic coordinate system around $p = (0,0) \in U$. Then $K = -\frac{3}{2} \lim_{\varepsilon \to 0} \varepsilon^{-4}\left(A(\sigma, D_\varepsilon) - A(D_\varepsilon)\right)$
  Gauss curvature measures the difference between area of a small square about p and the corresponding area of a plane square. If $\sigma(D_\varepsilon) < D_\varepsilon$, point is elliptic, if $\sigma(D_\varepsilon) > D_\varepsilon$ point is hyperbolic for sufficiently small ε.

Teorema Egregium •••••

- Intrinsic (invariants).
  Explain intrinsic as all properties that can be described by the arc-length. This is equivalent to be expressible by the first fundamental form. Invariant, if invariant under reparametrization.

- $K(p) := \det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right) = \frac{LN - M^2}{EG - F^2} = \kappa_1 \kappa_2$
  plane and cylinder have same K (Gauss curvature) as $E = G = 1$, $F = 0$ and $L = M = 0$, $N = 1$.
  - σ is elliptic at p if $K(p) > 0$
  - σ is hyperbolic at p if $K(p) < 0$
  - σ is parabolic or planar if $K(p) = 0$

Christoffel symbols

$$\begin{pmatrix} \Gamma^1_{ij} \\ \Gamma^2_{ij} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} \sigma''_{ij} \cdot \sigma'_1 \\ \sigma''_{ij} \cdot \sigma'_2 \end{pmatrix}}_{\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u_j} + \frac{\partial g_{jk}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_k}\right)}$$

are intrinsic but not intrinsic invariant. They change under reparametrization.

$$K = -\frac{1}{E}\left(\frac{\partial}{\partial u}\Gamma^2_{12} - \frac{\partial}{\partial v}\Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right)$$

- $\int_{t_1}^{t_2} \|\gamma'(t)\| dt$ intrinsic invariant
  E, F, G intrinsic
  (x, y, z) not intrinsic
  L, M, N not intrinsic — plane and cylinder have different 1-Fundamental Form
  W, κ_1, κ_2 invariant under reparametrization but not intrinsic.

Content:
- intrinsic invariant
- Gauss curvature
- F=0, Curvature
- Isometries
- Ideal maps

- Teorema Egregium: Gauss curvature K is intrinsic invariant. Formula using E, F, G exists.
  σ'_u, σ'_v, N are Basis for $\mathbb{R}^3$, similar to (t, n, b) but not orthonormal.

- For $F = 0$: $K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{G'_u}{\sqrt{EG}}\right)'_u + \left(\frac{E'_v}{\sqrt{EG}}\right)'_v\right)$ Invariant under isometries.

- Isometry: $\sigma : U \to \mathbb{R}^3$, $\rho : V \to \mathbb{R}^3$, $\overset{\text{diffeo}}{\varphi} : U \to V$ s.t. $E_\sigma = E_{\rho \circ \varphi}$; $F_\sigma = F_{\rho \circ \varphi}$; $G_\sigma = G_{\rho \circ \varphi}$.
  $\Psi(\sigma(p)) = \rho(\varphi(p))$, $\Psi : \sigma(u) \to \rho(V)$, if σ injective, map unique called lift.
  φ induces an isometry, Ψ is called bending.
- $K(\sigma(p)) = K((\rho \circ \varphi)(p))$

Sphere has Gauss curvature different from zero everywhere: thus no portion of the sphere can be mapped isometrically into a plane. Such a map would be called an ideal map.

Important formulas: The second fundamental form

$t'(s) = \kappa(s)\hat{t}(s) \iff \gamma''(s) = \kappa(s)\hat{\gamma}'(s)$

$W_p : T_p\sigma \to T_p\sigma$, $w \mapsto -aN'_u - bN'_v$, $w = a\sigma'_u + b\sigma'_v$

$W(\gamma'(t)) = -(N \circ \mu)'(t)$ for $\gamma(t) = (\sigma \circ \mu)(t)$ (a space curve) plane curve on a surface

$II_p : T_p\sigma \to \mathbb{R}$, $a\sigma'_u + b\sigma'_v \mapsto (a\sigma'_u + b\sigma'_v) \cdot W(a\sigma'_u + b\sigma'_v) = w \cdot W(w) = La^2 + 2Mab + Nb^2$

$\hookrightarrow w \cdot (W(w)) = W(w) \cdot w$ Symmetric for dot product.

$\kappa_n = \frac{II_p(w_0)}{\|w_0\|^2}$

$II_p(a\sigma'_u + b\sigma'_v) = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$L := N \cdot \sigma''_{uu} = \frac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{uu})}{\|\sigma'_u \times \sigma'_v\|}$

$M := N \cdot \sigma''_{uv} = \frac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{uv})}{\|\sigma'_u \times \sigma'_v\|}$

$N := N \cdot \sigma''_{vv} = \frac{\det(\sigma'_u\ \sigma'_v\ \sigma''_{vv})}{\|\sigma'_u \times \sigma'_v\|}$

Matrix for shape operator W_p: $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ in general not symmetric, as product of two symmetric matrices does not need to be symmetric.

$II_p(aw_1 + bw_2) = \kappa_1 a^2 + \kappa_2 b^2$, (w_1, w_2) orthonormal basis $T_p\sigma$ w_1, w_2 are principal vectors (Eigenvectors of W)

Principal curvatures are roots of $\det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} - \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$.

Solve $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \kappa \begin{pmatrix} a \\ b \end{pmatrix}$ as Eigenvalue-Problem.

Graph $\sigma(u, v)$ = $(u, v, \lambda_1 u^2 + \lambda_2 v^2)$

$\kappa_1, \kappa_2 \neq 0$ and $\mathrm{sgn}(\kappa_1) = \mathrm{sgn}(\kappa_2)$: elliptic point
$\kappa_1, \kappa_2 \neq 0$ and $\mathrm{sgn}(\kappa_1) \neq \mathrm{sgn}(\kappa_2)$: hyperbolic point
$\kappa_1, \kappa_2 = 0$: planar point
$\kappa_1 = 0, \kappa_2 \neq 0$: parabolic point