VII Geodesics

- Property of being geodesic is intrinsic!
- $|\kappa_g(t)|$ is an intrinsic property of a curve on a surface. It is invariant under reparametrization. } intrinsic invariant $|\kappa_g(t)|$

Th. 7.1 Let $\gamma = \sigma \circ \mu$ be a regular parametrized curve on σ. The geodesic curvature $\kappa_g(t)$ satisfies $\kappa_g = \|\gamma'\|^{-3} \det(g_{ij})^{1/2} ((u_1)' \Lambda_2 - (u_2)' \Lambda_1)$ where g_{ij} is the first fundamental form of σ at $\mu(t)$, u_1, u_2 are the coordinates of $\mu(t)$ and $(u_1)', (u_2)'$ are their derivatives with respect to t, and where Λ_i denotes the function

$$\Lambda_i(t) = (u_i)''(t) + \sum_{j,k=1}^{2} \Gamma^i_{jk}(\mu(t))(u_j)'(t)(u_k)'(t)$$

for $i = 1, 2$, in terms of the Christoffel symbols Γ^i_{jk}.

Geodesic curvature can be expressed through the Christoffel-symbols

Rem: Λ_i can be determined from E, F, G, thus also κ_g. Hence, $|\kappa_g|$ is intrinsic invariant.

Cor. 7.1: Let $\gamma(s) = \sigma(u_1(s), u_2(s))$ be a regular parametrized smooth curve on σ. Then γ is a geodesic and has constant speed if and only if the coordinate functions u_1 and u_2 satisfy the following system of second order differential equations:

$$(u_i)'' + \sum_{j,k=1}^{2} \Gamma^i_{jk}(u_j)'(u_k)' = 0 \qquad i = 1, 2. \qquad \text{(geodesic equation)}$$

with coefficients Γ^i_{jk} evaluated at $\mu(t)$.

Rem.: Solving geodesic equation determines geodesic on a surface.

7.2 Existence of geodesics

Th. 7.2 Through every point of a regular param. surface passes a unique geodesic curve in each direction. More precisely, let $p \in U$ and $w \in T_p\sigma \setminus \{0\}$ be given. There exists a geodesic curve $\gamma = \sigma \circ \mu : I \to \mathbb{R}^3$ on σ with

$$p = \mu(t_0) \text{ and } w = \gamma'(t_0) \quad \text{for some } t_0 \in I.$$

Moreover, if two unit speed geodesics defined on intervals I, J both satisfy (3) for some common $t_0 \in I \cap J$, then they agree on $I \cap J$.

7.3. Geodesic coordinates

Def. 7.3. Let $\gamma : J \to \mathbb{R}^3$ be a unit speed curve. A regular param. surface $\sigma : U \to \mathbb{R}^3$ is called a geodesic coordinate system transversal to γ if $U = I \times J$ for some interval I and:

(i) there exists $u_0 \in I$ such that $\gamma(v) = \sigma(u_0, v)$ for all v, and this curve is a geodesic on σ;

(ii) all the coordinate curves $I \ni u \mapsto \sigma(u, v)$ are unit speed geodesics on σ, which intersect orthogonally with γ (that is, the tangent vector $\sigma_u(u_0, v)$ is orthogonal to $\gamma'(v) = \sigma_v(u_0, v)$ for all $v \in J$).

Ex. - Standard coordinates in $\mathbb{R}^n$ are geodesic.
- Spherical coordinates on the unit sphere.

Thm. 7.3 Let $\sigma : U \to \mathbb{R}^3$ be a regular param. surface, and let a point $p \in U$ and a unit speed geodesic $\gamma = \sigma \circ \mu$ on σ be given with $\mu(0) = p$. There exists an open rectangle $W = I \times J$ around $(0, 0)$ in $\mathbb{R}^2$ and a diffeomorphism ϕ of W onto an open neighborhood $U' \subset U$ of p such that $\phi(0,0) = p$ and such that the reparametrization $\tau(s, t) = \sigma(\phi(s, t))$ of $\sigma|_{U'}$ is a geodesic coordinate system transversal to $\gamma|_J$.

The first fundamental form of a geodesic coordinate system

Let $\sigma : U \to \mathbb{R}^3$ be a regular surface.

Thm. 7.4: The surface σ is a geodesic coordinate system transversal to γ if and only if the following condition hold. The coefficients of the first fundamental form satisfy

$$E(u,v) = 1, \quad F(u,v) = 0$$

for all $(u,v) \in U$ and

$$G(u_0,v) = 1, \quad G'_u(u_0,v) = 0 \quad \text{for all } v \in I.$$

Rem.: $u \mapsto \sigma(u,v)$ is geodesic iff $E(u,v) = 1$ and $E'_v(u,v) - 2F'_u(u,v) = 0$.

Lemma 7.4: Let $\sigma : U \to \mathbb{R}^3$ be a regular param. surface. The coordinate curve $u \mapsto \sigma(u,v_0)$ is a unit speed geodesic if and only if $E = 1$ and $E'_v - 2F'_u = 0$ in all points of the curve.

7.5. Interpretation of the Gauss-Theorem

Thm. 7.5 Let $\sigma : U \to \mathbb{R}^3$ be a geodesic coordinate system around $p = (0,0) \in U$. The Gauss curvature K of σ in p is given by

$$K = -\frac{3}{2} \lim_{\varepsilon \to 0} \varepsilon^{-4} \left(A(\sigma, D_\varepsilon) - A(D_\varepsilon) \right).$$

Gauss curvature measures the difference between area of a small square about p and the corresponding area of a plane square.
Area is intrinsic property, definition of square (geodesics and right angles) is intrinsic.
In elliptic point $\sigma(D_\varepsilon) < D_\varepsilon$
In hyperbolic point $\sigma(D_\varepsilon) > D_\varepsilon$. (for ε sufficiently small).