(VI) Teorema Egregium

Which geometric quantities can be determined solely by computation involving the arc-length? Such quantities are called intrinsic. The Gaussian curvature is in fact intrinsic.

Recall: If $U \subset \mathbb{R}^n$ is m-dimensional with $m \leq n$ and $L: U \to U$ a linear map, the determinant of L is the determinant of the $m \times m$-Matrix that represents L in some basis for U.

6.1 The Gaussian curvature (or total curvature) $K(p)$ of σ at p is the determinant of the map W, that is

Determinant is a group homomorphism

$$K(p) = \det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right) = \frac{LN - M^2}{EG - F^2}.$$

Note: det does not depend on change of basis, as $\det(UAU^{-1}) = \det(U)\det(A)\det(U)^{-1}$.

There exist surfaces with different shape, that have the same curvature.

For the plane the shape operator $W = 0$. For unit sphere $W = id$.

Example (1) $\sigma(u,v) = p + uq_1 + vq_2$ is the plane with $K = 0$. For the unit sphere $W = id$, thus $K = 1 \Rightarrow$ The Gaussian curvature of a sphere of radius r is $K = \frac{1}{r^2}$.

(2) Consider the cylinder $\sigma(u,v) = (\cos(v), \sin(v), u)$. The Gauss curvature in the point $\sigma(u,v)$ can be computed using that $E = G = 1$, $F = 0$ and $L = M = 0$, $N = 1$. Thus $K = 0$.

Eigenvalues of $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ principal curvatures

Product of principal curvatures

6.1. The Gauss curvature of σ at p is the product $K(p) = \kappa_1 \kappa_2$. ⚠
In particular, σ is elliptic at p if and only if $K(p) > 0$, it is hyperbolic at p if and only if $K(p) < 0$, and it is parabolic or planar at p if and only if $K(p) = 0$.

Diagonalized shape operator

Proof: With respect to a basis of eigenvectors, the matrix of W is diagonal with κ_1, κ_2 in the diagonal. The determinant is then the product of these entries ■

Note: Gaussian curvature does not allow distinction between parabolic and planar point.

The coefficients $E = \|\sigma_u'\|^2$, $G = \|\sigma_v'\|^2$ of the first fundamental form can be determined by measuring the arc lengths of the curves $t \mapsto \sigma(t,v)$, $t \mapsto \sigma(u,t)$ to which σ_u' and σ_v' are tangent vectors. By measuring arc lengths along $t \mapsto \sigma(t,t)$, whose tangent vector is $\sigma'(t,t) = (\sigma_u'(t), \sigma_v'(t)) \cdot (1,1) = \sigma_u' + \sigma_v'$ we can determine $\|\sigma_u' + \sigma_v'\|$ and since $\|\sigma_u' + \sigma_v'\|^2 = E + G + 2F$ we can determine F as well.

$\Rightarrow$ ⚠ Any quantity that can be expressed by E, F and G can also be expressed by the length of curves. The property of being expressible by arc-length is equivalent to the property of being expressible by the first fundamental form.

6.2 A quantity or property of a parametrized surface σ, which can be expressed purely in terms of the coefficient functions E, F and G of the first fundamental form for σ, is called intrinsic. If in addition it is invariant under reparametrization of σ, it is called intrinsic invariant.

Ex. (1) The arc-length $\int_{t_0}^{t} \|\gamma'(t)\| \, dt$ is an intrinsic invariant.

(2) E, F, G are intrinsic but not invariant, as they change by reparam.

(3) The coordinates in $\mathbb{R}^3$ of $\sigma(u,v)$ are not intrinsic since they can't be determined from E, F and G alone. Notice that a translation changes coordinates but not E, F, G.

(4) The coefficients L, M, N of the second fundamental form are not intrinsic. Plane and cylinder have $E = G = 1$ and $F = 0$ but different second fundamental form.

(5) The shape operator W and the principal curvatures κ_1 and κ_2 are invariant under reparametrization (up to $\pm$) but they are not intrinsic.

6.3 Christoffel symbols

Change of notation: $\sigma'_1 = \sigma'_u$, $\sigma'_2 = \sigma'_v$
$\sigma''_{11} = \sigma''_{uu}$, $\sigma''_{12} = \sigma''_{uv}$, etc.

The fundamental form then reads:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

that is

$$g_{ij} = \sigma'_i \cdot \sigma'_j, \quad b_{ij} = \sigma''_{ij} \cdot N$$

6.3 The expression $\sigma''_{ij} \cdot \sigma'_k$ is intrinsic. It can be determined from the coefficients of the first fundamental form by means of the following formulas:

$$\sigma''_{ij} \cdot \sigma'_k = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u_j} + \frac{\partial g_{jk}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_k}\right), \quad (i,j,k = 1,2).$$

6.3 The Christoffel symbols associated with σ are the functions $\Gamma^k_{ij} : U \to \mathbb{R}$ defined for $i,j,k = 1,2$ by $\begin{pmatrix} \Gamma^1_{ij} \\ \Gamma^2_{ij} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \sigma''_{ij} \cdot \sigma'_1 \\ \sigma''_{ij} \cdot \sigma'_2 \end{pmatrix}$

Rem.: At any given point p all the three vectors σ'_u, σ'_v, N constitute a basis for $\mathbb{R}^3$, which can be seen as analogous to the the moving frame (t,n,b) of a curve, although not orthonormal. (Moving frame see Section 4.7).

6.3 Let coefficients Γ^k_{ij} for $i,j,k = 1,2$ be defined as above. Then $\sigma''_{ij} = \Gamma^1_{ij}\sigma'_1 + \Gamma^2_{ij}\sigma'_2 + b_{ij}N$.

Rem.: The Christoffel symbols can be determined from E, F, G. Thus, they are intrinsic. But they are not intrinsic invariants, because they change under reparametrization.

6.3 The Christoffel symbols Γ^k_{ij} are intrinsic. They can be expressed by a formula which involves only the coefficients of the first fundamental form and their (first order) derivatives with respect to u and v.

Rem.: Let the inverse matrix of g_{ij} be denoted by g^{ij}.

It follows that: $\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\right)$ (Formula of Gauss)

Consider the case when we have an orthogonal parametrization, thus $F = 0$, then:

$$\sigma''_{11} \cdot \sigma'_1 = \tfrac{1}{2}E'_u, \quad \sigma''_{12} \cdot \sigma'_1 = 1/2\, E'_v, \quad \sigma''_{22} \cdot \sigma'_1 = -1/2\, G'_u$$
$$\sigma''_{11} \cdot \sigma'_2 = -1/2\, E'_v, \quad \sigma''_{12} \cdot \sigma'_2 = 1/2\, G'_u, \quad \sigma''_{22} \cdot \sigma'_2 = 1/2\, G'_v \quad \text{and}$$

$$\Gamma^1_{11} = \frac{1}{2E}E'_u, \quad \Gamma^1_{12} = \Gamma^1_{21} = 1/2E\, E'_v, \quad \Gamma^1_{22} = -1/2E\, G'_u,$$
$$\Gamma^2_{11} = -1/2G\, E'_v, \quad \Gamma^2_{12} = \Gamma^2_{21} = 1/2G\, G'_u, \quad \Gamma^2_{22} = 1/2G\, G'_v.$$

Ex. 6.3.1: (1) Christoffel symbols for a plane $\sigma(u,v) = p + ue_1 + ve_2$ are all zero, since σ''_{ij} vanish or $E = G = 1$ and $F = 0$.
(2) Consider $\sigma(u,v) = (u\cos(v), u\sin(v), 0)$. $\Rightarrow \sigma'_u = (\cos(v), \sin(v), 0)$ and $\sigma'_v = (-u\sin(v), u\cos(v), 0)$
Hence $E = 1$, $F = 0$ and $G = u^2$. Then $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = \Gamma^2_{22} = 0$ and $\Gamma^1_{22} = -u$ and $\Gamma^2_{12} = 1/u$.
In particular, they differ from those in (1), though this is a plane in polar coordinates.

6.4 The remarkable theorem of Gauss

6.4. (Teorema egregium) The Gauss curvature K is intrinsic invariant, that is, there exists a general formula expressing K by means of the component functions E, F and G of the first fundamental form.

If $F=0$ the Gauss curvature becomes

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right).$$

6.5 Isometries

Isometry is a distance preserving map.

6.5 Let $\sigma: U \to \mathbb{R}^3$ and $\rho: V \to \mathbb{R}^3$ be parametrized surfaces. Then σ and ρ are said to be <u>isometric</u> if a diffeomorphism $q: U \to V$ exists such that σ and $\rho \circ q$ have identical first fundamental forms. That is,

$$E_\sigma = E_{\rho\circ q}, \quad F_\sigma = F_{\rho \circ q}, \quad G_\sigma = G_{\rho\circ q}.$$

In this case q is said to <u>induce an isometry</u> from σ to ρ.

<u>Rem.</u>: If q induces an isometry from σ to ρ, then q^{-1} induces an isometry from ρ to σ. If $\tau: W \to \mathbb{R}^3$ is given, together with a diffeomorphism $\phi: V \to W$ inducing an isometry from ρ to τ, then $\phi \circ q$ induces an isometry from σ to τ.

$\Psi(\sigma(p)) = \rho(q(p))$, $(p \in U)$ and $\Psi: \sigma(U) \to \rho(V)$
If σ is injective, such a map exists and is unique. We call Ψ a <u>lift</u>.
When q induces an isometry a lift is said to be a <u>bending</u> from $\sigma(U)$ to $\rho(V)$.

Ex. 6.5.2. Let $\rho(u,v), \sigma(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$ with $(u,v) \in \mathbb{R}^2$, with Domain $U = \{(u,v) \mid -\frac{\pi}{2} < u < \frac{\pi}{2}, v \in \mathbb{R}\}$. Let $\alpha \in \mathbb{R}$ be given. Then the map $U \to V$ defined by $q(u,v) = (u, v+\alpha)$ induces an isometry from ~~ρ to~~ σ to ρ.
This follows from the fact that E, F, G are independent of v.
The lift is a rotation of the sphere around the z-axis, by the angle α.

6.5 Let $\gamma = \sigma \circ \mu: I \to \mathbb{R}^3$ be a param. curve on σ and q induces an isometry from σ to ρ. Let $\delta = \rho \circ q_1: I \to \mathbb{R}^3$ be defined as a param. curve on ρ. When q induces an isometry the arc lengths of γ and δ are equal. That is, let $t_1, t_2 \in I$ then the arc length of γ from t_1 to t_2 is equal to the arc length of δ from t_1 to t_2.

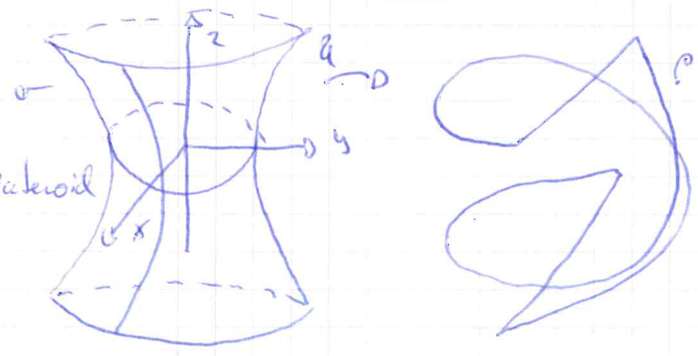
6.5 Assume that $q: U \to V$ induces an isometry from σ to ρ. Then the Gauss curvature of σ in p is equal to the Gauss curvature of ρ in $q(p)$, for all $p \in U$.

<u>Rem.</u> <u>The Gauss curvature is invariant under isometries</u>.

Ex. 6.5.4 The sphere has Gauss curvature different from zero in all points, thus no portion of the sphere can be mapped isometrically into a plane. Such a map would be called <u>ideal map</u>.

Ex. 6.5.5 $\sigma(u,v) = (a\cosh(u)\cos(v), a\cosh(u)\sin(v), au)$ for $(u,v) \in U = \mathbb{R}^2$ where $a > 0$. This is called <u>catenoid</u>.

σ is only injective for $\{(u,v) \mid -\pi < v < \pi\}$



$\rho(s,t) = (s\cos(t), s\sin(t), at)$. This surface is the helicoid. $q(u,v) = (a\sinh(u), v)$ induces an isometry. $E = G = a^2\cosh^2(u)$ and $F = 0$, in both cases.