Simplicial Homology

An introduction to simplicial homology theory

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Abstract

This paper explores the foundational concepts of simplicial structures that form the basis of simplicial homology theory. It also introduces singular homology as a means to establish the equivalence of homology groups for homeomorphic topological spaces. The paper concludes by providing a proof of the equivalence between simplicial and singular homology groups.

Contents

1 Simplicial Complexes

We would like to emphasize that a collection of points $X = \{x_0, x_1, \dots, x_d\}$ in $\mathbb{R}^n$ is considered to be *affinely independent* if these points do not lie within any affine subspace of dimension lower than d.

Definition. *Given a set $X = \{x_0, x_1, \dots, x_d\} \subset \mathbb{R}^n$ consisting of $d+1$* ***affinely independent*** *points, the d-dimensional simplex σ, also known as a* d-**simplex**, *is defined as the set of all convex combinations of these points.*

$$\sigma := \left\{ \sum_{i=0}^{d} \lambda_i x_i \mid \sum_{i=0}^{d} \lambda_i = 1,\ \lambda_i \geq 0 \right\}. \tag{1}$$

As a convention, the empty set $\emptyset$ is included as a face, representing the simplex formed by the empty subset of vertices. A 0-simplex represents a single point, a 1-simplex represents a line segment connecting two points, a 2-simplex represents a triangle, and a 3-simplex represents a tetrahedron. It is worth mentioning that the d-simplex is homeomorphic to the d-dimensional disk D^d.

Furthermore, it is worth noting that σ represents the convex hull of the points X, which can be defined as the smallest convex subset of $\mathbb{R}^n$ that contains all the points $x_0, x_1, \ldots, x_d$. The *faces* of the simplex σ with vertex set X are simplices formed by subsets of X. An *d-face* of a simplex refers to a subset of the vertices of the simplex with a cardinality of $d+1$. The faces of an d-simplex with a dimension less than d are known as its *proper faces.* Two simplices are considered to be *properly situated* if their intersection is either empty or a face of both simplices. By identifying simplices along entire faces, we can construct the resulting *simplicial complexes.*

Definition. *A* ***simplicial complex*** *K is a finite collection of simplices that satisfies the following properties:*

1. *For every simplex σ in K and every face τ of σ, it follows that τ is also in K.*
2. *If σ and τ are both simplices in K, then they are properly situated.*

The *dimension* of K is defined as the highest dimension among its simplices. For a simplicial complex K in $\mathbb{R}^n$, its *underlying space* $|K|$ is the union of all the simplices in K. The topology of K is determined by the topology induced on $|K|$ by the standard topology in $\mathbb{R}^n$. It is important to note that when the vertex set is known, a simplicial complex in $\mathbb{R}^n$ can be fully characterized by listing its simplices. As a result, we can describe it purely in terms of combinatorics using *abstract simplicial complexes.*

Definition. *Consider a finite set $V = \{v_1, \ldots, v_n\}$. An* ***abstract simplicial complex*** *$\tilde{K}$ with vertex set V is a collection of finite subsets of V that satisfies the following two conditions:*

1. *All elements of V are included in $\tilde{K}$.*
2. *If σ is a subset of $\tilde{K}$ and τ is a subset of σ, then τ is also a subset of $\tilde{K}$.*

The abstract simplicial complex $\tilde{K}$ associated with a simplicial complex K is commonly referred to as its *vertex scheme.* Conversely, if an abstract complex $\tilde{K}$ serves as the vertex scheme for a complex K in $\mathbb{R}^n$, then K is known as a *geometric realization* of $\tilde{K}$.

Lemma. *Every finite abstract simplicial complex $\tilde{K}$ can be realized geometrically in a Euclidean space.*

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertex set of $\tilde{K}$, where n represents the number of vertices in $\tilde{K}$. Consider $\sigma \subset \mathbb{R}^n$, the simplex formed by the span of $e_1, e_2, \ldots, e_n$, where e_i represents the ith unit vector. In this context, K refers to the subcomplex of σ such that $[e_{i_0}, \ldots, e_{i_d}]$ is a d-simplex of K if and only if $[v_{i_0}, \ldots, v_{i_k}]$ is a simplex of $\tilde{K}$. □

Note: All realizations of an abstract simplicial complex are homeomorphic to each other. The specific realization mentioned above is referred to as the *natural realization*. Furthermore, it has been proven that any finite abstract simplicial complex of dimension n can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$.

2 Homology Groups

Given a set V representing the vertices of a simplex σ, we can establish an *orientation* for the simplex by selecting a specific ordering for the vertices. If the vertex ordering differs from our chosen order by an odd permutation, it is considered *reversed*, while even permutations are said to *preserve* the orientation. Consequently, any simplex can have only two possible orientations. Moreover, the orientation of a d-simplex induces an orientation on its $(d-1)$-faces. To be more precise, if $\sigma^{(d)} := (v_0, v_1, \ldots, v_d)$ represents an oriented d-simplex, then the orientation of the $(d-1)$-face τ of $\sigma^{(d)}$ with the vertex set $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is given by $\tau_i = (-1)^i(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$.

Definition. *Given a set $\{\sigma_1^{(d)}, \ldots, \sigma_k^{(d)}\}$ of arbitrarily oriented d-simplices of a complex K and an abelian group G, we define a d-**chain** c with coefficients $g_i \in G$ as a formal sum.*

$$c := g_1\sigma_1^{(n)} + g_2\sigma_2^{(n)} + \ldots + g_k\sigma_k^{(n)} = \sum_{i=1}^{k} g_i\sigma_i^{(n)}. \tag{2}$$

Note: Henceforth we will assume that $G = (\mathbb{Z}, +)$.

Lemma. *The set of d-chains L_d is an abelian group $(L_d, +)$.*

Proof. The identity element of the group is represented by the empty chain $\sum_{i=1}^{k} e_G\sigma^d i = e_G$. The sum of two chains is defined as $c + c' = \sum i = 1^k(g_i + g'i)\sigma_i^{(d)}$, thus, we can conclude that $c + c' \in L_d$. The associativity of the group operation in L_d follows directly from the associativity of the group operation in G. The inverse element is defined as $eL_d = c + (-c) = \sum_{i=1}^{k} g_i\sigma_i^{(d)} - \sum_{i=1}^{k}(-g_i)\sigma_i^{(d)} = \sum_{i=1}^{k}(g_i - g_i)\sigma_i^{(d)}$ with $c, -c \in L_d$. □

Definition. *Let $\sigma^{(d)}$ be an oriented d-simplex in a complex K. The* ***boundary*** *of $\sigma^{(d)}$ is defined as the $(d-1)$-chain of K with coefficients in the abelian group $G = \mathbb{Z}$, given by*

$$\partial(\sigma^{(d)}) = \sigma_0^{(d-1)} + \sigma_1^{(d-1)} + \ldots + \sigma_d^{(d-1)} = \sum_{i=1}^{d} \sigma_i^{(d-1)} \tag{3}$$

where $\sigma_i^{(d-1)}$ is an $(d-1)$-face of $\sigma^{(d)}$. If $d = 0$, we define $\partial(\sigma^{(0)}) = e_G = 0$.

Since $\sigma^{(d)}$ is an oriented simplex, the $\sigma_i^{(d-1)}$-faces also have associated orientations. We can extend the definition of the boundary linearly to all elements of L_d.

Lemma. *The* ***boundary operator*** *is a group homomorphism $\partial : L_d \to L_{d-1}$.*

Proof. We define the boundary operator for a d-chain $c = \sum_{i=1}^{k} g_i \sigma_i^{(d)}$ as follows: $\partial(c) = \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) = \sum_{i=1}^{k} g_i \sum_{j=1}^{d} \sigma_j^{(d-1)} = \sum_{i=1}^{k} \sum_{j=1}^{d} g_i \sigma_j^{(d-1)} \in L_{d-1}$, where $\sigma_i^{(d)}$ are the d-simplices of K. We can compute this by

$$\begin{aligned}
\partial(c + c') &= \partial(\sum_{i=1}^{k} g_i \sigma_i^{(d)} + \sum_{j=1}^{l} g'_j \sigma_j^{(d)}) && (4)\\
&= \partial\left(\sum_{i=1}^{k} g_i \sigma_i^{(d)}\right) + \partial\left(\sum_{j=1}^{l} g'_j \sigma_j^{(d)}\right) && (5)\\
&= \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) + \sum_{j=1}^{l} g'_j \partial(\sigma_j^{(d)}) && (6)\\
&= \partial(c) + \partial(c'). && (7)
\end{aligned}$$

□

Example. *Let's consider the 2-simplex $\sigma^{(2)}$ with vertices v_0, v_1, and v_2. The 1-faces of this simplex are $e_0 = (v_1, v_2)$ connecting v_1 and v_2, $e_1 = (v_2, v_0)$ connecting v_2 and v_0, and $e_2 = (v_0, v_1)$ connecting v_0 and v_1. Now, let's proceed with the computation.*

$$\begin{aligned}
\partial(\partial(\sigma^{(2)})) &= \partial(e_1 + e_2 + e_3) && (8)\\
&= \partial e_1 + \partial e_2 + \partial e_3 && (9)\\
&= \partial(v_0, v_1) + \partial(v_1, v_2) + \partial(v_2, v_0) && (10)\\
&= [(v_1) - (v_0)] + [(v_2) - (v_1)] + [(v_0) - (v_2)]. && (11)
\end{aligned}$$

We observe that L_0 is an abelian group and that oppositely oriented simplices cancel each other out, resulting in $\partial(\partial(\sigma^{(2)})) = 0$. This property can be generalized to higher dimensions through induction. Therefore, since ∂ is a linear operator and the chain c is a sum of d-simplices, we can conclude that $\partial^2(c) = 0$

for any d-chain c in L_d. Consequently, the boundary of the boundary is zero. Moreover, if the boundary of a simplex is zero, it is referred to as a cycle. *By this definition, we can deduce that the boundary of any simplex is a cycle.*

Definition. *A d-chain is referred to as a* ***cycle*** *if its boundary is equal to zero. We denote the set of d-cycles of a complex K over the group $\mathbb{Z}$ as Z_d, the* ***cycle group****. It is important to note that Z_d is a subgroup of L_d and can also be expressed as $Z_d = \ker(\partial)$.*

Definition. *A d-cycle o of a k-complex K is said to be* ***homologous to zero*** *if it can be expressed as the boundary of an $(d+1)$-chain in K, where $d = 0, 1, \ldots, k-1$. In other words, a cycle is considered a boundary if it can be "filled in" by a higher-dimensional chain. This equivalence relation is denoted as $x \sim 0$, and the subgroup of Z_d consisting of boundaries is referred to as the* ***boundary group*** *B_d. It is worth noting that B_d is equal to the image of the boundary operator ∂.*

Since B_d is a subgroup of Z_d and Z_d is an abelian group, every subgroup of Z_d is normal. Therefore, we can construct the quotient group $H_d = Z_d/B_d$.

Definition. *The group H_d represents the d-dimensional* ***homology group*** *of the complex K over $\mathbb{Z}$. It can be expressed as the quotient group* $\ker(\partial)/im(\partial)$.

3 Singular Homology

4 Chain Complexes

5 Exact Sequences

6 Relative Homology Groups

7 The Equivalence of H_k^{Δ} and H_k

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