

Simplicial Homology

An introduction to simplicial homology theory

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Abstract

This paper explores the foundational concepts of simplicial structures that form the basis of simplicial homology theory. It also introduces singular homology as a means to establish the equivalence of homology groups for homeomorphic topological spaces. The paper concludes by providing a proof of the equivalence between simplicial and singular homology groups.

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1 Simplicial Complexes

We would like to emphasize that a collection of points $X = \{x_0, x_1, \dots, x_d\}$ in \mathbb{R}^n is considered to be *affinely independent* if these points do not lie within any affine subspace of dimension lower than d .

Definition 1.1. *Given a set $X = \{x_0, x_1, \dots, x_d\} \subset \mathbb{R}^n$ consisting of $d + 1$ affinely independent points, the d -dimensional simplex σ , also known as a*

d -**simplex**, is defined as the set of all convex combinations of these points.

$$\sigma := \left\{ \sum_{i=0}^d \lambda_i x_i \mid \sum_{i=0}^d \lambda_i = 1, \lambda_i \geq 0 \right\}. \quad (1)$$

As a convention, the empty set \emptyset is included as a face, representing the simplex formed by the empty subset of vertices. A 0-simplex represents a single point, a 1-simplex represents a line segment connecting two points, a 2-simplex represents a triangle, and a 3-simplex represents a tetrahedron. It is worth mentioning that the d -simplex is homeomorphic to the d -dimensional disk D^d .

Furthermore, it is worth noting that σ represents the convex hull of the points X , which can be defined as the smallest convex subset of \mathbb{R}^n that contains all the points x_0, x_1, \dots, x_d . The *faces* of the simplex σ with vertex set X are simplices formed by subsets of X . An d -*face* of a simplex refers to a subset of the vertices of the simplex with a cardinality of $d+1$. The faces of an d -simplex with a dimension less than d are known as its *proper faces*. Two simplices are considered to be *properly situated* if their intersection is either empty or a face of both simplices. By identifying simplices along entire faces, we can construct the resulting *simplicial complexes*.

Definition 1.2. A **simplicial complex** K is a finite collection of simplices that satisfies the following properties:

1. For every simplex σ in K and every face τ of σ , it follows that τ is also in K .
2. If σ and τ are both simplices in K , then they are properly situated.

The *dimension* of K is defined as the highest dimension among its simplices. For a simplicial complex K in \mathbb{R}^n , its *underlying space* $|K|$ is the union of all the simplices in K . The topology of K is determined by the topology induced on $|K|$ by the standard topology in \mathbb{R}^n . It is important to note that when the vertex set is known, a simplicial complex in \mathbb{R}^n can be fully characterized by listing its simplices. As a result, we can describe it purely in terms of combinatorics using *abstract simplicial complexes*.

Definition 1.3. Consider a finite set $V = \{v_1, \dots, v_n\}$. An **abstract simplicial complex** \tilde{K} with vertex set V is a collection of finite subsets of V that satisfies the following two conditions:

1. All elements of V are included in \tilde{K} .
2. If σ is a subset of \tilde{K} and τ is a subset of σ , then τ is also a subset of \tilde{K} .

The abstract simplicial complex \tilde{K} associated with a simplicial complex K is commonly referred to as its *vertex scheme*. Conversely, if an abstract complex \tilde{K} serves as the vertex scheme for a complex K in \mathbb{R}^n , then K is known as a *geometric realization* of \tilde{K} .

Lemma 1.4. *Every finite abstract simplicial complex \tilde{K} can be realized geometrically in a Euclidean space.*

Proof. Let v_1, v_2, \dots, v_n denote the vertex set of \tilde{K} , where n represents the number of vertices in \tilde{K} . Consider $\sigma \subset \mathbb{R}^n$, the simplex formed by the span of e_1, e_2, \dots, e_n , where e_i represents the i th unit vector. In this context, K refers to the subcomplex of σ such that $[e_{i_0}, \dots, e_{i_d}]$ is a d -simplex of K if and only if $[v_{i_0}, \dots, v_{i_k}]$ is a simplex of \tilde{K} . \square

Note: All realizations of an abstract simplicial complex are homeomorphic to each other. The specific realization mentioned above is referred to as the *natural realization*. Furthermore, it has been proven that any finite abstract simplicial complex of dimension n can be realized as a simplicial complex in \mathbb{R}^{2d+1} .

2 Homology Groups

Given a set V representing the vertices of a simplex σ , we can establish an *orientation* for the simplex by selecting a specific ordering for the vertices. If the vertex ordering differs from our chosen order by an odd permutation, it is considered *reversed*, while even permutations are said to *preserve* the orientation. Consequently, any simplex can have only two possible orientations. Moreover, the orientation of a d -simplex induces an orientation on its $(d-1)$ -faces. To be more precise, if $\sigma^{(d)} := (v_0, v_1, \dots, v_d)$ represents an oriented d -simplex, then the orientation of the $(d-1)$ -face τ of $\sigma^{(d)}$ with the vertex set $\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_d\}$ is given by $\tau_i = (-1)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_d)$.

Definition 2.1. *Given a set $\{\sigma_1^{(d)}, \dots, \sigma_k^{(d)}\}$ of arbitrarily oriented d -simplices of a complex K and an abelian group G , we define a d -**chain** c with coefficients $g_i \in G$ as a formal sum.*

$$c := g_1 \sigma_1^{(n)} + g_2 \sigma_2^{(n)} + \dots + g_k \sigma_k^{(n)} = \sum_{i=1}^k g_i \sigma_i^{(n)}. \quad (2)$$

Note: Henceforth we will assume that $G = (\mathbb{Z}, +)$.

Lemma 2.2. *The set of d -chains L_d is an abelian group $(L_d, +)$.*

Proof. The identity element of the group is represented by the empty chain $\sum_{i=1}^k e_G \sigma_i^{(d)} = e_G$. The sum of two chains is defined as $c + c' = \sum i = 1^k (g_i + g'_i) \sigma_i^{(d)}$, thus, we can conclude that $c + c' \in L_d$. The associativity of the group operation in L_d follows directly from the associativity of the group operation in G . The inverse element is defined as $eL_d = c + (-c) = \sum_{i=1}^k g_i \sigma_i^{(d)} - \sum_{i=1}^k (-g_i) \sigma_i^{(d)} = \sum_{i=1}^k (g_i - g_i) \sigma_i^{(d)}$ with $c, -c \in L_d$. \square

Definition 2.3. Let $\sigma^{(d)}$ be an oriented d -simplex in a complex K . The **boundary** of $\sigma^{(d)}$ is defined as the $(d-1)$ -chain of K with coefficients in the abelian group $G = \mathbb{Z}$, given by

$$\partial(\sigma^{(d)}) = \sigma_0^{(d-1)} + \sigma_1^{(d-1)} + \dots + \sigma_d^{(d-1)} = \sum_{i=1}^d \sigma_i^{(d-1)} \quad (3)$$

where $\sigma_i^{(d-1)}$ is an $(d-1)$ -face of $\sigma^{(d)}$. If $d = 0$, we define $\partial(\sigma^{(0)}) = e_G = 0$.

Since $\sigma^{(d)}$ is an oriented simplex, the $\sigma_i^{(d-1)}$ -faces also have associated orientations. We can extend the definition of the boundary linearly to all elements of L_d .

Lemma 2.4. The **boundary operator** is a group homomorphism $\partial : L_d \rightarrow L_{d-1}$.

Proof. We define the boundary operator for a d -chain $c = \sum_{i=1}^k g_i \sigma_i^{(d)}$ as follows: $\partial(c) = \sum_{i=1}^k g_i \partial(\sigma_i^{(d)}) = \sum_{i=1}^k g_i \sum_{j=1}^d \sigma_j^{(d-1)} = \sum_{i=1}^k \sum_{j=1}^d g_i \sigma_j^{(d-1)} \in L_{d-1}$, where $\sigma_i^{(d)}$ are the d -simplices of K . We can compute this by

$$\partial(c + c') = \partial\left(\sum_{i=1}^k g_i \sigma_i^{(d)} + \sum_{j=1}^l g'_j \sigma_j^{(d)}\right) \quad (4)$$

$$= \partial\left(\sum_{i=1}^k g_i \sigma_i^{(d)}\right) + \partial\left(\sum_{j=1}^l g'_j \sigma_j^{(d)}\right) \quad (5)$$

$$= \sum_{i=1}^k g_i \partial(\sigma_i^{(d)}) + \sum_{j=1}^l g'_j \partial(\sigma_j^{(d)}) \quad (6)$$

$$= \partial(c) + \partial(c'). \quad (7)$$

□

Example 2.5. Let's consider the 2-simplex $\sigma^{(2)}$ with vertices v_0 , v_1 , and v_2 . The 1-faces of this simplex are $e_0 = (v_1, v_2)$ connecting v_1 and v_2 , $e_1 = (v_2, v_0)$ connecting v_2 and v_0 , and $e_2 = (v_0, v_1)$ connecting v_0 and v_1 . Now, let's proceed with the computation.

$$\partial(\partial(\sigma^{(2)})) = \partial(e_1 + e_2 + e_3) \quad (8)$$

$$= \partial e_1 + \partial e_2 + \partial e_3 \quad (9)$$

$$= \partial(v_0, v_1) + \partial(v_1, v_2) + \partial(v_2, v_0) \quad (10)$$

$$= [(v_1) - (v_0)] + [(v_2) - (v_1)] + [(v_0) - (v_2)]. \quad (11)$$

We observe that L_0 is an abelian group and that oppositely oriented simplices cancel each other out, resulting in $\partial(\partial(\sigma^{(2)})) = 0$. This property can be generalized to higher dimensions through induction. Therefore, since ∂ is a linear

operator and the chain c is a sum of d -simplices, we can conclude that $\partial^2(c) = 0$ for any d -chain c in L_d . Consequently, the boundary of the boundary is zero. Moreover, if the boundary of a simplex is zero, it is referred to as a cycle. By this definition, we can deduce that the boundary of any simplex is a cycle.

Definition 2.6. A d -chain is referred to as a **cycle** if its boundary is equal to zero. We denote the set of d -cycles of a complex K over the group \mathbb{Z} as Z_d , the **cycle group**. It is important to note that Z_d is a subgroup of L_d and can also be expressed as $Z_d = \ker(\partial)$.

Definition 2.7. A d -cycle c of a k -complex K is said to be **homologous to zero** if it can be expressed as the boundary of an $(d + 1)$ -chain in K , where $d = 0, 1, \dots, k - 1$. In other words, a cycle is considered a boundary if it can be "filled in" by a higher-dimensional chain. This equivalence relation is denoted as $x \sim 0$, and the subgroup of Z_d consisting of boundaries is referred to as the **boundary group** B_d . It is worth noting that B_d is equal to the image of the boundary operator ∂ .

Since B_d is a subgroup of Z_d and Z_d is an abelian group, every subgroup of Z_d is normal. Therefore, we can construct the quotient group $H_d = Z_d/B_d$.

Definition 2.8. The group H_d represents the d -dimensional **homology group** of the complex K over \mathbb{Z} . It can be expressed as the quotient group $\ker(\partial)/\text{im}(\partial)$.

Next, we want to examine the structure of this homology group by shedding light on its connection to the connected components of a simplicial complex. We will find that the homology groups of the connected components of the complex, which in turn form a complex themselves, yield the direct sum of the homology group of the entire complex.

Definition 2.9. A **subcomplex** is defined as a subset S of the simplices belonging to a complex K , where S itself forms a complex.

The collection of all simplices in a complex K with dimensions less than or equal to d is referred to as the d -skeleton of K . By definition, the d -skeleton forms a subcomplex.

Definition 2.10. A complex K is considered **connected** if it cannot be expressed as the disjoint union of two or more non-empty subcomplexes. A geometric complex is **path-connected** if there exists a path consisting of 1-simplices connecting any vertex to any other vertex.

Lemma 2.11. Path-connectedness \iff connectedness.

Proof.

□

3 Singular Homology

4 Chain Complexes

5 Exact Sequences

6 Relative Homology Groups

7 The Equivalence of H_k^Δ and H_k

References

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