Simplicial Homology

An introduction to simplicial homology theory

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Abstract

This paper explores the foundational concepts of simplicial structures that form the basis of simplicial homology theory. It also introduces singular homology as a means to establish the equivalence of homology groups for homeomorphic topological spaces. The paper concludes by providing a proof of the equivalence between simplicial and singular homology groups.

Contents

1 Simplicial Complexes

We would like to emphasize that a collection of points $X = \{x_0, x_1, \ldots, x_d\}$ in $\mathbb{R}^n$ is considered to be *affinely independent* if these points do not lie within any affine subspace of dimension lower than d.

Definition 1.1. *Given a set $X = \{x_0, x_1, \ldots, x_d\} \subset \mathbb{R}^n$ consisting of $d + 1$* ***affinely independent*** *points, the d-dimensional simplex σ, also known as a*

d**-simplex**, *is defined as the set of all convex combinations of these points.*

$$\sigma := \left\{ \sum_{i=0}^{d} \lambda_i x_i \mid \sum_{i=0}^{d} \lambda_i = 1,\ \lambda_i \geq 0 \right\}. \tag{1}$$

As a convention, the empty set $\emptyset$ is included as a face, representing the simplex formed by the empty subset of vertices. A 0-simplex represents a single point, a 1-simplex represents a line segment connecting two points, a 2-simplex represents a triangle, and a 3-simplex represents a tetrahedron. It is worth mentioning that the d-simplex is homeomorphic to the d-dimensional disk D^d.

Furthermore, it is worth noting that σ represents the convex hull of the points X, which can be defined as the smallest convex subset of $\mathbb{R}^n$ that contains all the points $x_0, x_1, \ldots, x_d$. The *faces* of the simplex σ with vertex set X are simplices formed by subsets of X. An *d-face* of a simplex refers to a subset of the vertices of the simplex with a cardinality of $d+1$. The faces of an d-simplex with a dimension less than d are known as its *proper faces.* Two simplices are considered to be *properly situated* if their intersection is either empty or a face of both simplices. By identifying simplices along entire faces, we can construct the resulting *simplicial complexes.*

Definition 1.2. *A **simplicial complex** K is a finite collection of simplices that satisfies the following properties:*

1. *For every simplex σ in K and every face τ of σ, it follows that τ is also in K.*

2. *If σ and τ are both simplices in K, then they are properly situated.*

The *dimension* of K is defined as the highest dimension among its simplices. For a simplicial complex K in $\mathbb{R}^n$, its *underlying space* $|K|$ is the union of all the simplices in K. The topology of K is determined by the topology induced on $|K|$ by the standard topology in $\mathbb{R}^n$. It is important to note that when the vertex set is known, a simplicial complex in $\mathbb{R}^n$ can be fully characterized by listing its simplices. As a result, we can describe it purely in terms of combinatorics using *abstract simplicial complexes.*

Definition 1.3. *Consider a finite set $V = \{v_1, \ldots, v_n\}$. An **abstract simplicial complex** $\tilde{K}$ with vertex set V is a collection of finite subsets of V that satisfies the following two conditions:*

1. *All elements of V are included in $\tilde{K}$.*

2. *If σ is a subset of $\tilde{K}$ and τ is a subset of σ, then τ is also a subset of $\tilde{K}$.*

The abstract simplicial complex $\tilde{K}$ associated with a simplicial complex K is commonly referred to as its *vertex scheme.* Conversely, if an abstract complex $\tilde{K}$ serves as the vertex scheme for a complex K in $\mathbb{R}^n$, then K is known as a *geometric realization* of $\tilde{K}$.

Lemma 1.4. *Every finite abstract simplicial complex $\tilde{K}$ can be realized geometrically in a Euclidean space.*

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertex set of $\tilde{K}$, where n represents the number of vertices in $\tilde{K}$. Consider $\sigma \subset \mathbb{R}^n$, the simplex formed by the span of $e_1, e_2, \ldots, e_n$, where e_i represents the ith unit vector. In this context, K refers to the subcomplex of σ such that $[e_{i_0}, \ldots, e_{i_d}]$ is a d-simplex of K if and only if $[v_{i_0}, \ldots, v_{i_k}]$ is a simplex of $\tilde{K}$. □

Note: All realizations of an abstract simplicial complex are homeomorphic to each other. The specific realization mentioned above is referred to as the *natural realization.* Furthermore, it has been proven that any finite abstract simplicial complex of dimension n can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$.

2 Homology Groups

Given a set V representing the vertices of a simplex σ, we can establish an *orientation* for the simplex by selecting a specific ordering for the vertices. If the vertex ordering differs from our chosen order by an odd permutation, it is considered *reversed*, while even permutations are said to *preserve* the orientation. Consequently, any simplex can have only two possible orientations. Moreover, the orientation of a d-simplex induces an orientation on its $(d-1)$-faces. To be more precise, if $\sigma^{(d)} := (v_0, v_1, \ldots, v_d)$ represents an oriented d-simplex, then the orientation of the $(d-1)$-face τ of $\sigma^{(d)}$ with the vertex set $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is given by $\tau_i = (-1)^i(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$.

Definition 2.1. *Given a set $\{\sigma_1^{(d)}, \ldots, \sigma_k^{(d)}\}$ of arbitrarily oriented d-simplices of a complex K and an abelian group G, we define a d-**chain** c with coefficients $g_i \in G$ as a formal sum.*

$$c := g_1\sigma_1^{(d)} + g_2\sigma_2^{(d)} + \ldots + g_k\sigma_k^{(d)} = \sum_{i=1}^{k} g_i\sigma_i^{(d)}. \tag{2}$$

Note: Henceforth we will assume that $G = (\mathbb{Z}, +)$.

Lemma 2.2. *The set of d-chains L_d is an abelian group $(L_d, +)$.*

Proof. The identity element of the group is represented by the empty chain $\sum_{i=1}^{k} e_G\sigma_i^d = e_G$. The sum of two chains is defined as $c + c' = \sum_{i=1}^{k} g_i\sigma_i^{(d)} + \sum_{j=1}^{l} g'_j\sigma_j^{(d)} = \sum_{i=1}^{k}(g_i+g'_i)\sigma_i^{(d)} + \sum_{j=k}^{l} g'_j\sigma_j^{(d)}$ if $k \leq l$ and $c+c' = \sum_{i=1}^{k} g_i\sigma_i^{(d)} + \sum_{j=1}^{l} g'_j\sigma_j^{(d)} = \sum_{i=1}^{l}(g_i + g'_i)\sigma_i^{(d)} + \sum_{j=l}^{k} g_j\sigma_j^{(d)}$ if $k > l$, thus, we can conclude that $c + c' \in L_d$. The associativity of the group operation in L_d follows directly from the associativity of the group operation in G. The inverse element is defined as $e_{L_d} = c + (-c) = \sum_{i=1}^{k} g_i\sigma_i^{(d)} - \sum_{i=1}^{k}(-g_i)\sigma_i^{(d)} = \sum_{i=1}^{k}(g_i - g_i)\sigma_i^{(d)}$ with $c, -c \in L_d$. □

Definition 2.3. *Let $\sigma^{(d)}$ be an oriented d-simplex in a complex K. The* ***boundary*** *of $\sigma^{(d)}$ is defined as the $(d-1)$-chain of K with coefficients in the abelian group $G = \mathbb{Z}$, given by*

$$\partial(\sigma^{(d)}) = \sigma_0^{(d-1)} + \sigma_1^{(d-1)} + \ldots + \sigma_d^{(d-1)} = \sum_{i=1}^{d} \sigma_i^{(d-1)} \tag{3}$$

where $\sigma_i^{(d-1)}$ is an $(d-1)$-face of $\sigma^{(d)}$. If $d = 0$, we define $\partial(\sigma^{(0)}) = e_G = 0$.

Since $\sigma^{(d)}$ is an oriented simplex, the $\sigma_i^{(d-1)}$-faces also have associated orientations. We can extend the definition of the boundary linearly to all elements of L_d.

Lemma 2.4. *The* ***boundary operator*** *is a group homomorphism $\partial : L_d \to L_{d-1}$.*

Proof. We define the boundary operator for a d-chain $c = \sum_{i=1}^{k} g_i \sigma_i^{(d)}$ as follows: $\partial(c) = \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) = \sum_{i=1}^{k} g_i \sum_{j=1}^{d} \sigma_j^{(d-1)} = \sum_{i=1}^{k} \sum_{j=1}^{d} g_i \sigma_j^{(d-1)} \in L_{d-1}$, where $\sigma_i^{(d)}$ are the d-simplices of K. We can compute this by

$$\begin{aligned}
\partial(c + c') &= \partial(\sum_{i=1}^{k} g_i \sigma_i^{(d)} + \sum_{j=1}^{l} g'_j \sigma_j^{(d)}) && (4)\\
&= \partial\left(\sum_{i=1}^{k} g_i \sigma_i^{(d)}\right) + \partial\left(\sum_{j=1}^{l} g'_j \sigma_j^{(d)}\right) && (5)\\
&= \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) + \sum_{j=1}^{l} g'_j \partial(\sigma_j^{(d)}) && (6)\\
&= \partial(c) + \partial(c'). && (7)
\end{aligned}$$

□

Example 2.5. *Let's consider the 2-simplex $\sigma^{(2)}$ with vertices v_0, v_1, and v_2. The 1-faces of this simplex are $e_0 = (v_1, v_2)$ connecting v_1 and v_2, $e_1 = (v_2, v_0)$ connecting v_2 and v_0, and $e_2 = (v_0, v_1)$ connecting v_0 and v_1. Now, let's proceed with the computation.*

$$\begin{aligned}
\partial(\partial(\sigma^{(2)})) &= \partial(e_1 + e_2 + e_3) && (8)\\
&= \partial e_1 + \partial e_2 + \partial e_3 && (9)\\
&= \partial(v_0, v_1) + \partial(v_1, v_2) + \partial(v_2, v_0) && (10)\\
&= [(v_1) - (v_0)] + [(v_2) - (v_1)] + [(v_0) - (v_2)]. && (11)
\end{aligned}$$

We observe that L_0 is an abelian group and that oppositely oriented simplices cancel each other out, resulting in $\partial(\partial(\sigma^{(2)})) = 0$. This property can be generalized to higher dimensions through induction. Therefore, since ∂ is a linear *operator and the chain c is a sum of d-simplices, we can conclude that* $\partial^2(c) = 0$ *for any d-chain c in* L_d*. Consequently, the boundary of the boundary is zero. Moreover, if the boundary of a simplex is zero, it is referred to as a* cycle*. By this definition, we can deduce that the boundary of any simplex is a cycle.*

Definition 2.6. *A d-chain is referred to as a* ***cycle*** *if its boundary is equal to zero. We denote the set of d-cycles of a complex K over the group* $\mathbb{Z}$ *as* Z_d*, the* ***cycle group****. It is important to note that* Z_d *is a subgroup of* L_d *and can also be expressed as* $Z_d = \ker(\partial)$*.*

Definition 2.7. *A d-cycle o of a k-complex K is said to be* ***homologous to zero*** *if it can be expressed as the boundary of an* $(d + 1)$*-chain in K, where* $d = 0, 1, \ldots, k - 1$*. In other words, a cycle is considered a boundary if it can be "filled in" by a higher-dimensional chain. This equivalence relation is denoted as* $x \sim 0$*, and the subgroup of* Z_d *consisting of boundaries is referred to as the* ***boundary group*** B_d*. It is worth noting that* B_d *is equal to the image of the boundary operator* ∂*.*

Since B_d is a subgroup of Z_d and Z_d is an abelian group, every subgroup of Z_d is normal. Therefore, we can construct the quotient group $H_d = Z_d/B_d$.

Definition 2.8. *The group* H_d *represents the d-dimensional* ***homology group*** *of the complex K over* $\mathbb{Z}$*. It can be expressed as the quotient group* $\ker(\partial)/im(\partial)$*.*

Next, we want to examine the structure of this homology group by shedding light on its connection to the connected components of a simplicial complex. We will find that the homology groups of the connected components of the complex, which in turn form a complex themselves, yield the direct sum of the homology group of the entire complex.

Definition 2.9. *A* ***subcomplex*** *is defined as a subset S of the simplices belonging to a complex K, where S itself forms a complex.*

The collection of all simplices in a complex K with dimensions less than or equal to d is referred to as the d-skeleton of K. By definition, the d-skeleton forms a subcomplex.

Definition 2.10. *A complex K is considered* ***connected*** *if it cannot be expressed as the disjoint union of two or more non-empty subcomplexes. A geometric complex is* ***path-connected*** *if there exists a path consisting of* 1*-simplices connecting any vertex to any other vertex.*

Lemma 2.11. *Path-connectedness* $\Longleftrightarrow$ *connectedness.*

Proof. "$\Longrightarrow$": Let us assume that K is not connected. In this case, we can choose two separate subcomplexes, namely L and M, which do not share any common elements, but when combined, they form the entire complex $L \cap M = K$. Now, let's suppose that there exists a path between a vertex l_0 in L and a vertex m_0 in M. However, if we consider the last vertex l_i in this path that belongs to L, we observe that the 1-simplex connecting l_i to the next vertex in the path cannot be a part of either L or M. If it were, then L and M would have a nonempty intersection, which contradicts our initial assumption that K is not connected.

"$\Longleftarrow$": Now, let's consider the other direction. Suppose there are two points, namely l_0 and m_0, in K that do not have a path connecting them. In this case, we can define L as the path-connected subcomplex of K that contains l_0, and M as the path-connected subcomplex that contains m_0. If there exists a vertex v_0 in the intersection of L and M (i.e., $v_0 \in L \cap M \neq \emptyset$), then we can find a path from l_0 to v_0 and another path from v_0 to m_0. By concatenating these paths, we obtain a path from l_0 to m_0, which contradicts our initial assumption that there is no path between l_0 and m_0. Therefore, we conclude that L and M must have an empty intersection ($L \cap M = \emptyset$), indicating that K is not connected. □

Theorem 3. *Let $K_1, \ldots, K_p$ be the collection of all connected components of a complex K. Furthermore, let H_{d_i} represent the dth homology group of K_i, and H_d denote the dth homology group of K. In this context, we can establish that H_d is isomorphic to the direct sum $H_{d_1} \oplus \cdots \oplus H_{d_p}$.*

Proof. Let L_d represent the group of d-chains of K, and K_i denote the ith component of K. We can define L_{d_i} as the group of d-chains of K_i. It is evident that L_{d_i} is a subgroup of L_d. Furthermore, we observe that L_d can be expressed as the direct sum of $L_{d_1}, \ldots, L_{d_p}$:

$$L_d = L_{d_1} \oplus \cdots \oplus L_{d_p}. \tag{12}$$

Our goal is to demonstrate that a similar decomposition can be applied to the groups B_d and Z_d. By considering B_{d_i} as the image of ∂ restricted to the subgroup L_{d_i}, we can represent the group B_d as the direct sum of these restrictions:

$$B_d = B_{d_1} \oplus \cdots \oplus B_{d_p}. \tag{13}$$

Thus, for any element $c \in L_{d+1}$, which can be represented as:

$$c = c_1 + \cdots + c_p, \quad \partial(c) = \partial c_1 + \cdots + \partial c_p \in B_d, \tag{14}$$

where $c_i \in L_{d+1_i}$. Let us define Z_{d_i} as the intersection of the kernel of ∂ and L_{d_i}. It follows that Z_d can be expressed as the direct sum of $Z_{d_1}, \ldots, Z_{d_p}$:

$$Z_d = Z_{d_1} \oplus \cdots \oplus Z_{d_p}. \tag{15}$$

To verify this, we observe that for an element $c \in L_d$ to belong to Z_d, we require $\partial(c) = 0$. However, we can express $\partial(c)$ as $\partial(c_1) + \cdots + \partial(c_p)$. Therefore, for $\partial(c) = 0$ to hold, it implies that $\partial(c_i) = 0$, indicating that $c_i \in Z_{d_i}$. Since both Z_d and B_d can be decomposed componentwise, we can conclude that:

$$Z_d/B_d = Z_{d_1}/B_{d_1} \oplus \cdots \oplus Z_{d_p}/B_{d_p}, \tag{16}$$

and consequently:

$$H_d = H_{d_1} \oplus \cdots \oplus H_{d_p}. \tag{17}$$

□

Definition 3.1. *The* ***index*** *of a chain* $c = \sum_{i=1}^{k} g_i \sigma_i^{(n)}$ *is defined as the sum of the coefficients* $I(c) = \sum_{i=1}^{k} g_i$.

Proposition 3.2. *If* K *is a connected complex and* c *is a* 0*-chain with* $I(c) = 0$*, then the condition* $I(c) = 0$ *is equivalent to* $c \sim 0$*, where* $\sim$ *denotes homology equivalence. Furthermore, in this case, the zeroth homology group* $H_0(K, \mathbb{Z})$ *is isomorphic to the integers* $\mathbb{Z}$.

Proof. We begin by proving that $c \sim 0 \implies I(c) = 0$. Let $\sigma^{(1)} = (v_0, v_1)$ be a 1-simplex. Then, for a chain $c = \partial(g\sigma^{(1)}) = gv_1 - gv_0$, we have $c \sim 0$. It is clear that $I(c) = I(g\sigma^{(1)}) = g - g = 0$. Since $I(c + c') = I(c) + I(c')$, I is a group homomorphism. For any $c \in L_1$ of the form $\sum_{i=1}^{k} g_i \sigma_i^{(1)}$, where $\sigma_i^{(1)} = (v_i, v_{i+1})$, we have $c = \partial(c) \sim 0 \implies I(c) = I(\partial(c)) = 0$.

To prove the forward direction, $I(x) = 0 \implies c \sim 0$, we consider two vertices v and w of K. Since K is connected, there exists a path between v and w consisting of 1-simplices $\sigma_i^{(1)} = (v_i, v_{i+1})$, $i = 1, \ldots, k-1$, where $v_0 = v$ and $v_k = w$. We consider the boundary of the chain $c = \sum_{i=1}^{k} g\sigma_i^{(1)}$, given by $\partial(c) = \sum_{i=1}^{k} g\partial(\sigma_i^{(1)}) = \sum_{i=1}^{k} g[(v_{i+1}) - (v_i))] = gw - gv$. The index of the chain $c = \sum_{i=1}^{k} g_i \sigma_i^{(n)}$ is defined as $I(c) = \sum_{i=1}^{k} g_i$. Since $\partial(c)$ is a boundary, we have $c = \partial(c) \sim 0$. This implies that $(gw - gv) \sim 0$, which further implies $gw \sim gv$. Therefore, any 0-chain c in K is homologous to the chain gv. We observe that homologous chains have equal indices, i.e., $I(c) = I(gv) = g$. Thus, we have $c \sim gv \implies c \sim I(c)v$. This shows that if $I(c) = 0$, then $c \sim 0$. Hence, $I(c) = 0$ is equivalent to $c \sim 0$.

As mentioned, I is a homomorphism from $L_0 = Z_0$ to $\mathbb{Z}$. For a 0-simplex c and $g \in \mathbb{Z}$, the chain $gc \in L_0$ is a cycle with $I(gc) = g$. Therefore, $I(Z_0) = \mathbb{Z}$. Since $I(c) = 0$ is equivalent to $c \sim 0$, we have $B_0 = \ker(I)$. This implies that $H_0 = Z_0/B_0 \cong \mathbb{Z}$. □

4 Singular Homology

5 Chain Complexes

6 Exact Sequences

7 Relative Homology Groups

8 The Equivalence of H_k^{Δ} and H_k

References

[1] Boissonnat, J. D., Chazal, F., Yvinec, M. (2018). Geometric and Topological Inference (Vol. 57). Cambridge University Press.

[2] Edelsbrunner, H., Harer, J. L. (2022). Computational Topology: An Introduction. American Mathematical Society.

[3] Hatcher, A. (2005). Algebraic Topology. Cambridge University Press.

[4] Jonsson, J. (2011). Introduction to Simplicial Homology. Königliche Technische Hochschule. URL: `https://people.kth.se/~jakobj/doc/homology/homology.pdf`.

[5] Khoury, M. (2022). Lecture 6: Introduction to Simplicial Homology. Topics in Computational Topology: An Algorithmic View. Ohio State University. URL: `http://web.cse.ohio-state.edu/~wang.1016/courses/788/Lecs/lec6-marc.pdf`.

[6] Melodia, L., Lenz, R. (2021). Estimate of the Neural Network Dimension Using Algebraic Topology and Lie Theory. In Pattern Recognition. ICPR International Workshops and Challenges.

[7] Nadathur, P. (2007). An Introduction to Homology. University of Chicago. URL: `https://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Nadathur.pdf`.

[8] Pontryagin L. S. (1952): Foundations of Combinatorial Topology. Graylock Press.