Simplicial Homology

An introduction to simplicial homology theory

Luciano Melodia

September 26, 2023

Abstract

This paper explores the foundational concepts of simplicial structures that form the basis of simplicial homology theory. It also introduces singular homology as a means to establish the equivalence of homology groups for homeomorphic topological spaces. The paper concludes by providing a proof of the equivalence between simplicial and singular homology groups.

Contents

1 Simplicial Complexes

We would like to emphasize that a collection of points $X = \{x_0, x_1, \dots, x_d\}$ in $\mathbb{R}^n$ is considered to be *affinely independent* if these points do not lie within any affine subspace of dimension lower than d.

Definition. *Given a set $X = \{x_0, x_1, \dots, x_d\} \subset \mathbb{R}^n$ consisting of $d+1$ **affinely independent** points, the d-dimensional simplex σ, also known as a d-**simplex**,* *is defined as the set of all convex combinations of these points.*

$$\sigma := \left\{ \sum_{i=0}^{d} \lambda_i x_i \mid \sum_{i=0}^{d} \lambda_i = 1,\ \lambda_i \geq 0 \right\}. \tag{1}$$

As a convention, the empty set $\emptyset$ is included as a face, representing the simplex formed by the empty subset of vertices. A 0-simplex represents a single point, a 1-simplex represents a line segment connecting two points, a 2-simplex represents a triangle, and a 3-simplex represents a tetrahedron. It is worth mentioning that the d-simplex is homeomorphic to the d-dimensional disk D^d.

Furthermore, it is worth noting that σ represents the convex hull of the points X, which can be defined as the smallest convex subset of $\mathbb{R}^n$ that contains all the points $x_0, x_1, \ldots, x_d$. The *faces* of the simplex σ with vertex set X are simplices formed by subsets of X. An *d-face* of a simplex refers to a subset of the vertices of the simplex with a cardinality of $d+1$. The faces of an d-simplex with a dimension less than d are known as its *proper faces.* Two simplices are considered to be *properly situated* if their intersection is either empty or a face of both simplices. By identifying simplices along entire faces, we can construct the resulting *simplicial complexes.*

Definition. *A **simplicial complex** K is a finite collection of simplices that satisfies the following properties:*

1. *For every simplex σ in K and every face τ of σ, it follows that τ is also in K.*
2. *If σ and τ are both simplices in K, then they are properly situated.*

The *dimension* of K is defined as the highest dimension among its simplices. For a simplicial complex K in $\mathbb{R}^n$, its *underlying space* $|K|$ is the union of all the simplices in K. The topology of K is determined by the topology induced on $|K|$ by the standard topology in $\mathbb{R}^n$. It is important to note that when the vertex set is known, a simplicial complex in $\mathbb{R}^n$ can be fully characterized by listing its simplices. As a result, we can describe it purely in terms of combinatorics using *abstract simplicial complexes.*

Definition. *Consider a finite set $V = \{v_1, \ldots, v_n\}$. An **abstract simplicial complex** $\tilde{K}$ with vertex set V is a collection of finite subsets of V that satisfies the following two conditions:*

1. *All elements of V are included in $\tilde{K}$.*
2. *If σ is a subset of $\tilde{K}$ and τ is a subset of σ, then τ is also a subset of $\tilde{K}$.*

The abstract simplicial complex $\tilde{K}$ associated with a simplicial complex K is commonly referred to as its *vertex scheme.* Conversely, if an abstract complex $\tilde{K}$ serves as the vertex scheme for a complex K in $\mathbb{R}^n$, then K is known as a *geometric realization* of $\tilde{K}$.

Lemma. *Every finite abstract simplicial complex $\tilde{K}$ can be realized geometrically in a Euclidean space.*

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertex set of $\tilde{K}$, where n represents the number of vertices in $\tilde{K}$. Consider $\sigma \subset \mathbb{R}^n$, the simplex formed by the span of $e_1, e_2, \ldots, e_n$, where e_i represents the ith unit vector. In this context, K refers to the subcomplex of σ such that $[e_{i_0}, \ldots, e_{i_d}]$ is a d-simplex of K if and only if $[v_{i_0}, \ldots, v_{i_k}]$ is a simplex of $\tilde{K}$. □

Note: All realizations of an abstract simplicial complex are homeomorphic to each other. The specific realization mentioned above is referred to as the *natural realization*. Furthermore, it has been proven that any finite abstract simplicial complex of dimension n can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$.

2 Homology Groups

Given a set V representing the vertices of a simplex σ, we can establish an *orientation* for the simplex by selecting a specific ordering for the vertices. If the vertex ordering differs from our chosen order by an odd permutation, it is considered *reversed*, while even permutations are said to *preserve* the orientation. Consequently, any simplex can have only two possible orientations. Moreover, the orientation of a d-simplex induces an orientation on its $(d-1)$-faces. To be more precise, if $\sigma^{(d)} := (v_0, v_1, \ldots, v_d)$ represents an oriented d-simplex, then the orientation of the $(d-1)$-face τ of $\sigma^{(d)}$ with the vertex set $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is given by $\tau_i = (-1)^i(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)$.

Definition. *Given a set $\{\sigma_1^{(d)}, \ldots, \sigma_k^{(d)}\}$ of arbitrarily oriented d-simplices of a complex K and an abelian group G, we define a d-**chain** c with coefficients $g_i \in G$ as a formal sum.*

$$c := g_1\sigma_1^{(n)} + g_2\sigma_2^{(n)} + \ldots + g_k\sigma_k^{(n)} = \sum_{i=1}^{k} g_i\sigma_i^{(n)}. \tag{2}$$

Note: Henceforth we will assume that $G = (\mathbb{Z}, +)$.

Lemma. *The set of d-chains L_d is an abelian group $(L_d, +)$.*

Proof. The identity element of the group is represented by the empty chain $\sum_{i=1}^{k} e_G\sigma^d i = e_G$. The sum of two chains is defined as $c + c' = \sum i = 1^k(g_i + g'i)\sigma_i^{(d)}$, thus, we can conclude that $c + c' \in L_d$. The associativity of the group operation in L_d follows directly from the associativity of the group operation in G. The inverse element is defined as $eL_d = c + (-c) = \sum_{i=1}^{k} g_i\sigma_i^{(d)} - \sum_{i=1}^{k}(-g_i)\sigma_i^{(d)} = \sum_{i=1}^{k}(g_i - g_i)\sigma_i^{(d)}$ with $c, -c \in L_d$. □

Definition. *Let $\sigma^{(d)}$ be an oriented d-simplex in a complex K. The* ***boundary*** *of $\sigma^{(d)}$ is defined as the $(d-1)$-chain of K with coefficients in the abelian group $G = \mathbb{Z}$, given by*

$$\partial(\sigma^{(d)}) = \sigma_0^{(d-1)} + \sigma_1^{(d-1)} + \ldots + \sigma_d^{(d-1)} = \sum_{i=1}^{d} \sigma_i^{(d-1)} \tag{3}$$

where $\sigma_i^{(d-1)}$ is an $(d-1)$-face of $\sigma^{(d)}$. If $d = 0$, we define $\partial(\sigma^{(0)}) = e_G = 0$.

Since $\sigma^{(d)}$ is an oriented simplex, the $\sigma_i^{(d-1)}$-faces also have associated orientations. We can extend the definition of the boundary linearly to all elements of L_d.

Lemma. *The* ***boundary operator*** *is a group homomorphism $\partial : L_d \to L_{d-1}$.*

Proof. We define the boundary operator for a d-chain $c = \sum_{i=1}^{k} g_i \sigma_i^{(d)}$ as follows: $\partial(c) = \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) = \sum_{i=1}^{k} g_i \sum_{j=1}^{d} \sigma_j^{(d-1)} = \sum_{i=1}^{k} \sum_{j=1}^{d} g_i \sigma_j^{(d-1)} \in L_{d-1}$, where $\sigma_i^{(d)}$ are the d-simplices of K. We can compute this by

$$\begin{aligned}
\partial(c + c') &= \partial(\sum_{i=1}^{k} g_i \sigma_i^{(d)} + \sum_{j=1}^{l} g'_j \sigma_j^{(d)}) && (4)\\
&= \partial\left(\sum_{i=1}^{k} g_i \sigma_i^{(d)}\right) + \partial\left(\sum_{j=1}^{l} g'_j \sigma_j^{(d)}\right) && (5)\\
&= \sum_{i=1}^{k} g_i \partial(\sigma_i^{(d)}) + \sum_{j=1}^{l} g'_j \partial(\sigma_j^{(d)}) && (6)\\
&= \partial(c) + \partial(c'). && (7)
\end{aligned}$$

□

Example. *Let's consider the 2-simplex $\sigma^{(2)}$ with vertices v_0, v_1, and v_2. The 1-faces of this simplex are $e_0 = (v_1, v_2)$ connecting v_1 and v_2, $e_1 = (v_2, v_0)$ connecting v_2 and v_0, and $e_2 = (v_0, v_1)$ connecting v_0 and v_1. Now, let's proceed with the computation.*

$$\begin{aligned}
\partial(\partial(\sigma^{(2)})) &= \partial(e_1 + e_2 + e_3) && (8)\\
&= \partial e_1 + \partial e_2 + \partial e_3 && (9)\\
&= \partial(v_0, v_1) + \partial(v_1, v_2) + \partial(v_2, v_0) && (10)\\
&= [(v_1) - (v_0)] + [(v_2) - (v_1)] + [(v_0) - (v_2)]. && (11)
\end{aligned}$$

We observe that L_0 is an abelian group and that oppositely oriented simplices cancel each other out, resulting in $\partial(\partial(\sigma^{(2)})) = 0$. This property can be generalized to higher dimensions through induction. Therefore, since ∂ is a linear operator and the chain c is a sum of d-simplices, we can conclude that $\partial^2(c) = 0$

for any d-chain c in L_d. Consequently, the boundary of the boundary is zero. Moreover, if the boundary of a simplex is zero, it is referred to as a cycle. *By this definition, we can deduce that the boundary of any simplex is a cycle.*

Definition. *A d-chain is referred to as a* ***cycle*** *if its boundary is equal to zero. We denote the set of d-cycles of a complex K over the group $G = \mathbb{Z}$ as Z_d. It is important to note that Z_d is a subgroup of L_d and can also be expressed as $Z_d = \ker(\partial)$.*

Definition. *A d-cycle o of a k-complex K is said to be* ***homologous to zero*** *if it can be expressed as the boundary of an $(d+1)$-chain in K, where $d = 0, 1, \ldots, k-1$. In other words, a cycle is considered a boundary if it can be "filled in" by a higher-dimensional chain. This relationship is denoted as $x\tilde{0}$, and the subgroup of Z_d consisting of boundaries is referred to as the* ***boundary group*** *B_d. It is worth noting that B_d is equal to the image of the boundary operator ∂.*

3 Singular Homology

4 Chain Complexes

5 Exact Sequences

6 Relative Homology Groups

7 The Equivalence of H_k^{Δ} and H_k

References

[1] Boissonnat, J. D., Chazal, F., Yvinec, M. (2018). Geometric and Topological Inference (Vol. 57). Cambridge University Press.

[2] Edelsbrunner, H., Harer, J. L. (2022). Computational Topology: An Introduction. American Mathematical Society.

[3] Hatcher, A. (2005). Algebraic Topology. Cambridge University Press.

[4] Jonsson, J. (2011). Introduction to Simplicial Homology. Königliche Technische Hochschule. URL: `https://people.kth.se/~jakobj/doc/homology/homology.pdf`.

[5] Khoury, M. (2022). Lecture 6: Introduction to Simplicial Homology. Topics in Computational Topology: An Algorithmic View. Ohio State University. URL: `http://web.cse.ohio-state.edu/~wang.1016/courses/788/Lecs/lec6-marc.pdf`.

[6] Melodia, L., Lenz, R. (2021). Estimate of the Neural Network Dimension Using Algebraic Topology and Lie Theory. In Pattern Recognition. ICPR International Workshops and Challenges.

[7] Nadathur, P. (2007). An Introduction to Homology. University of Chicago. URL: `https://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Nadathur.pdf`.

[8] Pontryagin L. S. (1952): Foundations of Combinatorial Topology. Graylock Press.