Universal Coefficient Theorem
Moore-Mayer-Vietoris Sequence
for Homology of Ample Groupoids

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Topics

What are we going to talk about?

Ample groupoid homology.

Homology via the Moore chain complex of ample groupoids.

Universal coefficient theorem.

A universal coefficient theorem for discrete abelian groups.

Moore–Mayer–Vietoris sequence.

A Mayer–Vietoris type sequence for clopen saturated covers.

Ample Groupoid Homology

Ample groupoids

What do we investigate?

A topological groupoid consists of a space $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$, a unit space $\mathcal{G}_0 \subseteq \mathcal{G}$, and maps

$$u\colon \mathcal{G}_0 \to \mathcal{G}, \qquad s, r\colon \mathcal{G} \to \mathcal{G}_0, \qquad (-)^{-1}\colon \mathcal{G} \to \mathcal{G}, \qquad m\colon \mathcal{G}_2 \to \mathcal{G}.$$

$$s(g) = g^{-1} \cdot g, \qquad r(g) = g \cdot g^{-1}, \qquad u(x) = 1_x.$$

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The space of composable pairs is

$$\mathcal{G}_2 \coloneqq \mathcal{G} \,{}_s\!\times_r \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = r(h)\}, \qquad m(g, h) = g \cdot h.$$

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$\mathcal{G}$ has a basis of compact open bisections $U \subseteq \mathcal{G}$, so $r|_U$ and $s|_U$ are homeomorphisms onto compact open subsets of $\mathcal{G}_0$.

Standing Hypotheses

We investigate ample groupoids.

We consider $C_c(\mathcal{G}_n, A)$.

A is a topological abelian group.

$\mathcal{G}$ is an ample groupoid.

$\mathcal{G}_n$ is the space of n-multiplicable arrows.

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$\mathcal{G}$ ample: compact open bisections form a basis.

$C_c(\mathcal{G}, \mathbb{Z})$ is generated by χ_K for compact open sets K.

The Nerve

On what do we compute homology?

$\mathcal{G}_\bullet \coloneqq (\mathcal{G}_n, (d_i)_{i=0}^n, (s_j)_{j=0}^n)_{n\geq 0}$ is a simplicial space.

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Face maps. $d_i \colon \mathcal{G}_n \to \mathcal{G}_{n-1}$, $n = 1$: $d_0 = s$, $d_1 = r$. For $n \geq 2$:

$$d_i(\mathbf{g}) \coloneqq \begin{cases} (g_2, .., g_n), & i = 0, \\ (g_1, .., g_i \cdot g_{i+1}, .., g_n), & 1 \leq i \leq n-1, \\ (g_1, .., g_{n-1}), & i = n. \end{cases}$$

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Degeneracy maps. $s_j \colon \mathcal{G}_n \to \mathcal{G}_{n+1}$, $n \geq 0$:

$$s_j(\mathbf{g}) := \begin{cases} u(x), & n = 0,\ x \in \mathcal{G}_0, \\ (u(r(g_1)), g_1, .., g_n), & n \geq 1,\ j = 0, \\ (g_1, .., g_j,\ u(r(g_{j+1})),\ g_{j+1}, .., g_n), & n \geq 2,\ 1 \leq j \leq n-1, \\ (g_1, .., g_n,\ u(s(g_n))), & n \geq 1,\ j = n. \end{cases}$$

$$\mathcal{G}_n := \begin{cases} \mathcal{G}_0, & n = 0, \\ \{\mathbf{g} \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for } 1 \leq i < n\}, & n \geq 1. \end{cases}$$

Moore Chains and Boundary

Compactly supported chains on the nerve.

Chains. $C_c(\mathcal{G}_n, A)$ denotes the abelian group of continuous maps $f\colon \mathcal{G}_n \to A$ with compact support. If A is discrete, then every $f \in C_c(\mathcal{G}_n, A)$ is locally constant.

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Boundary. Since $\mathcal{G}$ is étale, each face map $d_i\colon \mathcal{G}_n \to \mathcal{G}_{n-1}$ is a local homeomorphism, hence the pushforward is well-defined:

$$(d_i)_* : C_c(\mathcal{G}_n, A) \to C_c(\mathcal{G}_{n-1}, A), \qquad (d_i)_* f(y) := \sum_{x \in d_i^{-1}(\{y\}) \cap \mathrm{supp}(f)} f(x),$$

$$\partial_n := \sum_{i=0}^{n} (-1)^i (d_i)_* \colon C_c(\mathcal{G}_n, A) \to C_c(\mathcal{G}_{n-1}, A).$$

Moore Chains and Boundary

If A is discrete, then $C_c(\mathcal{G}_n, A)$ is a free A-module.

Assume A is a discrete abelian group and $\mathcal{G}$ is ample.

If $f \in C_c(\mathcal{G}_n, A)$, then for each $a \in A$ the fibre $f^{-1}(\{a\}) \subseteq \mathcal{G}_n$ is open. Since A is discrete, $f^{-1}(\{a\})$ is also closed. Thus each $f^{-1}(\{a\})$ is clopen.

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$\mathrm{supp}(f)$ is compact $\Rightarrow \mathrm{im}(f) \subseteq A$ is compact, hence finite.

Why is f locally constant?

Fix $g \in \mathcal{G}_n$. Since A is discrete, the singleton $\{f(g)\} \subseteq A$ is open. Therefore

$$U_g \coloneqq f^{-1}\big(\{f(g)\}\big)$$

is an open neighborhood of g in $\mathcal{G}_n$. If $h \in U_g$, then

$$h \in f^{-1}\big(\{f(g)\}\big) \Leftrightarrow f(h) \in \{f(g)\} \Leftrightarrow f(h) = f(g).$$

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How is this a chain complex?

$\mathcal{G}$ is ample $\Rightarrow$ d_i local homeomorphism $\Rightarrow$ $d_i^{-1}(\{y\})$ discrete.

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If $f \in C_c(\mathcal{G}_n, A)$, then for each $y \in \mathcal{G}_{n-1}$ the set $d_i^{-1}(\{y\}) \cap \operatorname{supp}(f)$ is finite, as a discrete subset of the compact space $\operatorname{supp}(f)$. Hence

$$(d_i)_* f(y) \coloneqq \sum_{x \in d_i^{-1}(\{y\}) \cap \operatorname{supp}(f)} f(x) \quad \text{is well-defined.}$$

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$$(d_i)_* f(y) := \sum_{x \in d_i^{-1}(\{y\}) \cap \operatorname{supp}(f)} f(x) \quad \text{is well-defined.}$$

$(d_i)_* f \in C_c(\mathcal{G}_{n-1}, A)$: it has compact support since

$$\operatorname{supp}\big((d_i)_* f\big) \subseteq d_i\big(\operatorname{supp}(f)\big),$$

and it is locally constant on $\mathcal{G}_{n-1}$ when A is discrete, by local triviality of d_i over $\operatorname{supp}(f)$.

Moore Chains and Boundary

The identity $\partial^2 = 0$.

$\partial_{n-1}\partial_n = 0:$ By definition,

$$\partial_{n-1}\partial_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}(d_i)_* \circ (d_j)_*.$$

For $i < j$, the simplicial identities give $d_i d_j = d_{j-1} d_i$, hence by functoriality of pushforward,

$$(d_i)_* \circ (d_j)_* = (d_i d_j)_* = (d_{j-1} d_i)_* = (d_{j-1})_* \circ (d_i)_*.$$

Therefore the terms cancel in pairs and

$$\partial_{n-1}\partial_n = \sum_{0\le i<j\le n}(-1)^{i+j}\Big((d_i)_*(d_j)_* - (d_{j-1})_*(d_i)_*\Big) = 0.$$

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A discrete abelian group.
For all $n \geq 0$ there is a natural short exact sequence in $\mathbf{Ab}$:

$$0 \to H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big) \to 0.$$

The sequence splits, though not canonically:

$$H_n(\mathcal{G}; A) \cong \big(H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A\big) \oplus \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big).$$

Universal Coefficient Theorem

If A is discrete, then $C_c(\mathcal{G}_n, A)$ is a free A-module.

Let $f \in C_c(\mathcal{G}_n, A)$. Since A is discrete and $\mathrm{supp}(f)$ is compact, $\mathrm{im}(f) = \{a_1, .., a_m\}$ is finite. Set $K_i := f^{-1}(\{a_i\})$ for $1 \leq i \leq m$.

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Each K_i is clopen in $\mathcal{G}_n$, the sets K_i are pairwise disjoint, and $\mathrm{supp}(f) = \bigsqcup_{i=1}^{m} K_i$.

$$f = \sum_{i=1}^{m} a_i \chi_{K_i} \qquad \text{in } C_c(\mathcal{G}_n, A).$$

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Therefore the canonical map

$$\bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A \longrightarrow C_c(\mathcal{G}_n, A), \qquad (a_K)_{K \in \mathcal{K}(\mathcal{G}_n)} \longmapsto \sum_{K \in \mathcal{K}(\mathcal{G}_n)} a_K \chi_K,$$

is surjective. It is injective: if $\sum_{i=1}^m a_i \chi_{K_i} = 0$ with K_i pairwise disjoint compact open, then evaluating at any $g \in K_i$ gives $a_i = 0$. Thus

$$C_c(\mathcal{G}_n, A) \cong \bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A, \qquad \text{free as an } A\text{-module}.$$

Proof of the UCT

Step 1: Chain-level identification.

Let $f \in C_c(\mathcal{G}_n, A)$ and write $\mathrm{im}(f) = \{a_1, .., a_m\}$. Set $K_i \coloneqq f^{-1}(\{a_i\})$. Then $\mathrm{supp}(f) = \bigsqcup_{i=1}^{m} K_i$ with K_i clopen and $f|_{K_i} \equiv a_i$.

Extension by 0: $\chi_{K_i} \in C_c(\mathcal{G}_n, \mathbb{Z})$ and

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Define the canonical $\mathbb{Z}$-bilinear map

$$\Phi_{\mathcal{G}_n} \colon C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow C_c(\mathcal{G}_n, A), \qquad \xi \otimes a \longmapsto a \cdot \xi.$$

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$$C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_n, A) \qquad \text{for discrete } A.$$

Proof of the UCT

Step 2: Compatibility with the boundary.

For each face map $d_i \colon \mathcal{G}_n \to \mathcal{G}_{n-1}$, the pushforward $(d_i)_*$ is $\mathbb{Z}$-linear and satisfies

$$(d_i)_*(\xi \cdot a) = \big((d_i)_*\xi\big) \cdot a \qquad \text{for } \xi \in C_c(\mathcal{G}_n, \mathbb{Z}),\ a \in A.$$

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$$\Phi_{\mathcal{G}_{n-1}} \circ (\partial_n \otimes \mathrm{id}_A) = \partial_n \circ \Phi_{\mathcal{G}_n}.$$

Therefore $\Phi_{\mathcal{G}_\bullet}$ is an isomorphism of chain complexes

$$C_c(\mathcal{G}_\bullet, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_\bullet, A).$$

Proof of the UCT

Step 3: Apply the classical algebraic UCT.

The Moore complex $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ is a chain complex of free abelian groups. Applying the classical algebraic UCT to $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ and transporting across the chain isomorphism from Steps 1–2 yields, for all $n \geq 0$, a short exact sequence

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This sequence is natural in $\mathcal{G}$ and in discrete A. In general, for non-discrete topological abelian groups A, Moore homology need not satisfy such a UCT.

Non-discrete Coefficients: What Fails

The general result.

For X locally compact, totally disconnected, Hausdorff with a basis of compact open sets and an abelian group A, consider the canonical map

$$\Phi_X \colon C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(X, A), \qquad \chi_U \otimes a \mapsto a\,\chi_U.$$

Then

$$\operatorname{im}(\Phi_X) \subseteq \{\xi \in C_c(X, A) \mid \xi(X) \text{ is finite}\}.$$

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$$\Phi_X \colon C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(X, A), \qquad \chi_U \otimes a \mapsto a\,\chi_U.$$

Then

$$\mathrm{im}(\Phi_X) \subseteq \{\xi \in C_c(X, A) \mid \xi(X) \text{ is finite}\}.$$

In particular, Φ_X may fail to be surjective for non-discrete A. Moreover,

$$\Phi_X \text{ surjective} \Leftrightarrow \forall\, \xi \in C_c(X, A) : \ \xi(X) \text{ finite} \Leftrightarrow \Phi_X \text{ is an isomorphism.}$$

Non-discrete Coefficients: What Fails

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If A is discrete, then Φ_X is an isomorphism. The converse can fail.

Non-discrete Coefficients: What Fails

Failure of the tensor comparison.

If A is non-discrete and $0 \in A$ is not isolated, then surjectivity of Φ_X can fail even for compact, totally disconnected spaces with a basis of clopen subsets. Set

$$
\begin{aligned}
X &\coloneqq \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \;\middle|\; a_n \in \{0,2\} \right\} \subset [0,1], \\
A &\coloneqq (\mathbb{R}, \mathcal{O}_{\mathrm{std}}), \\
\xi &\colon X \to A, \quad x \mapsto x.
\end{aligned}
$$

Then X is compact, Hausdorff, totally disconnected, and has a basis of clopen subsets. Hence $\xi \in C_c(X, A)$ and $\xi(X) = X$ is infinite. Therefore $\xi \notin \mathrm{im}(\Phi_X)$, so Φ_X is not surjective.

Non-discrete Coefficients: What Fails

Isomorphism without discreteness.

$$A \coloneqq (\mathbb{R}, \mathcal{O}_{\mathrm{std}}), \qquad (\{\bullet\}, \mathcal{O}_{\{\bullet\}} \coloneqq \{\varnothing, \{\bullet\}\}).$$

Then $\{\bullet\}$ is locally compact, totally disconnected, Hausdorff and compact open.

$$\begin{aligned} C_c(\{\bullet\}, \mathbb{Z}) &\cong \mathbb{Z}, \\ C_c(\{\bullet\}, A) &\cong A, \\ C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A &\cong \mathbb{Z} \otimes_{\mathbb{Z}} A \cong A. \end{aligned}$$

Under these identifications the canonical map

$$\Phi_{\{\bullet\}} : \; C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(\{\bullet\}, A), \qquad \chi_{\{\bullet\}} \otimes a \mapsto a \cdot \chi_{\{\bullet\}},$$

is the standard isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} A \to A$, $1 \otimes a \mapsto a$.

Moore-Mayer-Vietoris Sequence

Mayer-Vietoris vs. Moore-Mayer-Vietoris

From saturated covers to homology.

Mayer–Vietoris:

$X = U_1 \cup U_2$,

$U_1, U_2 \subseteq X$ open.

Open cover:

$U_1,\ U_2,\ U_1 \cap U_2$.

Compute $H_\bullet(X)$ from:

$H_\bullet(U_1)$, $H_\bullet(U_2)$, $H_\bullet(U_1 \cap U_2)$.

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$U_1, U_2 \subseteq \mathcal{G}_0$ clopen.

Saturated cover:

$\forall\, x \in U :\ s(g) = x \Rightarrow r(g) \in U$.

$\mathcal{G}|_U \coloneqq \{g \in \mathcal{G} \mid s(g), r(g) \in U\}$,

$\mathcal{G}|_U \hookrightarrow \mathcal{G}$ open.

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Compute $H_\bullet(\mathcal{G})$ from:

$H_\bullet(\mathcal{G}|_{U_1})$, $H_\bullet(\mathcal{G}|_{U_2})$, $H_\bullet(\mathcal{G}|_{U_1 \cap U_2})$.

Reductions and Moore–Mayer–Vietoris

Long Natural Moore-Mayer-Vietoris Sequence for Homology.

For $U \subseteq \mathcal{G}_0$ define the reduction

$$\mathcal{G}|_U \coloneqq \{g \in \mathcal{G} \mid s(g), r(g) \in U\}, \qquad (\mathcal{G}|_U)_0 = U,$$

with structure maps the restrictions of $u, m, s, r, -^{-1}$ to $\mathcal{G}|_U$. For $i \in \{1, 2\}$ and $U_{12} \coloneqq U_1 \cap U_2$ we write $\mathcal{G}|_{U_i}$ and $\mathcal{G}|_{U_{12}}$.

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Moore–Mayer–Vietoris long exact homology sequence:

$$\begin{tikzcd}
\cdots & H_{n-1}(\mathcal{G}|_{U_1};A) \oplus H_{n-1}(\mathcal{G}|_{U_2};A) \arrow[l] & H_{n-1}(\mathcal{G}|_{U_1\cap U_2};A) \arrow[l, "H_{n-1}(\alpha_\bullet)"'] \\
H_n(\mathcal{G};A) \arrow[urr, "\partial_n"] & H_n(\mathcal{G}|_{U_1};A) \oplus H_n(\mathcal{G}|_{U_2};A) \arrow[l, "H_n(\beta_\bullet)"'] & H_n(\mathcal{G}|_{U_1\cap U_2};A) \arrow[l, "H_n(\alpha_\bullet)"'] \\
H_{n+1}(\mathcal{G};A) \arrow[urr, "\partial_{n+1}"] & H_{n+1}(\mathcal{G}|_{U_1};A) \oplus H_{n+1}(\mathcal{G}|_{U_2};A) \arrow[l, "H_{n+1}(\beta_\bullet)"'] & \cdots \arrow[l]
\end{tikzcd}$$

Proof of Moore–Mayer–Vietoris

Proof idea for $H_n(\alpha_\bullet)$.

$(\iota_i)_n\colon (\mathcal{G}|_{U_{12}})_n \hookrightarrow (\mathcal{G}|_{U_i})_n$ is an open embedding, hence a local homeomorphism. Therefore the functorial pushforward on Moore chains

$$((\iota_i)_n)_*\colon C_c\big((\mathcal{G}|_{U_{12}})_n, A\big) \to C_c\big((\mathcal{G}|_{U_i})_n, A\big)$$

is given by a finite fibre sum. Since $(\iota_i)_n$ is injective, it is extensible by zero:

$$((\iota_i)_n)_* f(x) \coloneqq \begin{cases} f(x), & x \in (\mathcal{G}|_{U_{12}})_n, \\ 0, & x \in (\mathcal{G}|_{U_i})_n \setminus (\mathcal{G}|_{U_{12}})_n. \end{cases}$$

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Define the chain map

$$\begin{aligned} \alpha_n\colon C_c\big((\mathcal{G}|_{U_{12}})_n, A\big) &\to C_c\big((\mathcal{G}|_{U_1})_n, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_n, A\big), \\ f &\mapsto \big(((\iota_1)_n)_* f, -((\iota_2)_n)_* f\big). \end{aligned}$$

Compatibility: $\partial\,((\iota_i)_n)_* = ((\iota_i)_{n-1})_*\,\partial, \partial\,\alpha_n = \alpha_{n-1}\,\partial$. Hence $\alpha_\bullet$ induces $H_n(\alpha_\bullet)$.

Moore–Mayer–Vietoris

Proof idea for $H_n(\beta_\bullet)$.

Let $\kappa_i\colon \mathcal{G}|_{U_i} \hookrightarrow \mathcal{G}$ be the inclusion of reductions.
$(\kappa_i)_n\colon (\mathcal{G}|_{U_i})_n \hookrightarrow \mathcal{G}_n$ is an open embedding, hence a local homeomorphism. Therefore the pushforward on Moore chains extends by zero:

$$((\kappa_i)_n)_* g(x) \coloneqq \begin{cases} g(x), & x \in (\mathcal{G}|_{U_i})_n, \\ 0, & x \in \mathcal{G}_n \setminus (\mathcal{G}|_{U_i})_n. \end{cases}$$

Define

$$\begin{aligned} \beta_n\colon C_c\big((\mathcal{G}|_{U_1})_n, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_n, A\big) &\to C_c(\mathcal{G}_n, A), \\ (g_1, g_2) &\mapsto ((\kappa_1)_n)_* g_1 + ((\kappa_2)_n)_* g_2. \end{aligned}$$

Compatibility: $\partial\,((\kappa_i)_n)_* = ((\kappa_i)_{n-1})_*\,\partial, \partial\,\beta_n = \beta_{n-1}\,\partial$. Hence $\beta_\bullet$ induces $H_n(\beta_\bullet)$.

Moore–Mayer–Vietoris

Proof idea for ∂_n.

Assume a SES of Moore chain complexes

$$0 \to C_c\big((\mathcal{G}|_{U_{12}})_\bullet, A\big) \xrightarrow{\alpha_\bullet} C_c\big((\mathcal{G}|_{U_1})_\bullet, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_\bullet, A\big) \xrightarrow{\beta_\bullet} C_c\big(\mathcal{G}_\bullet, A\big) \to 0.$$

Here ∂ denotes the Moore boundary.

Let $[c] \in H_n(\mathcal{G}; A)$ with $\partial c = 0$ and choose b with $\beta_n(b) = c$.

Then $\beta_{n-1}(\partial b) = \partial(\beta_n(b)) = \partial c = 0$, hence

$$\partial b \in \ker(\beta_{n-1}) = \operatorname{im}(\alpha_{n-1}).$$

Choose $a \in C_c\big((\mathcal{G}|_{U_{12}})_{n-1}, A\big)$ with $\alpha_{n-1}(a) = \partial b$ and define

$$\partial_n([c]) := [a] \in H_{n-1}(\mathcal{G}|_{U_{12}}; A).$$

Standard homological algebra: ∂_n is well-defined, independent of choices, and yields exactness at $H_n(\mathcal{G}; A)$.

Takeaways

What you get and how to use it.

Setting: $\mathcal{G}$ ample étale, A a discrete abelian group,
Moore chains $C_c(\mathcal{G}_n, A)$ with boundary $\partial = \sum_{i=0}^{n}(-1)^i(d_i)_*$.

Two structural tools:

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Two structural tools:

UCT for discrete coefficients:

$$0 \to H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big) \to 0.$$

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Moore–Mayer–Vietoris

for a clopen saturated cover $U_1 \cup U_2 = \mathcal{G}_0$.
There is a natural long exact sequence relating

$$H_\bullet(\mathcal{G}; A), H_\bullet(\mathcal{G}|_{U_1}; A), H_\bullet(\mathcal{G}|_{U_2}; A), H_\bullet(\mathcal{G}|_{U_{12}}; A).$$

Takeaways

What you get and how to use it.

Why discreteness matters:

For non-discrete A, the canonical comparison

$$\Phi_X : \; C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(X, A), \qquad \chi_U \otimes a \mapsto a\,\chi_U,$$

need not be surjective. Tensor-level reduction in UCT can fail.

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What you get and how to use it.

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For non-discrete A, the canonical comparison

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need not be surjective. Tensor-level reduction in UCT can fail.

How to use in practice:

Choose a clopen saturated cover.
$U_1, U_2 \subseteq \mathcal{G}_0$ so that reductions $\mathcal{G}|_{U_1}$, $\mathcal{G}|_{U_2}$, $\mathcal{G}|_{U_{12}}$ are computable.
Compute integral homology.
$H_\bullet(\mathcal{G}|_{U_i})$, $H_\bullet(\mathcal{G}|_{U_{12}})$, then glue to $H_\bullet(\mathcal{G})$ via MMV.

Thank you.

Homology of SFT Groupoids

Example: Diaconu-Renault Groupoid

Computing Homology with Moore–Mayer–Vietoris + UCT.

$A \in \mathrm{Mat}(N \times N, \mathbb{N}_0)$ with no zero row and no zero column.
$(E_A^{(1)}, E_A^{(0)}, s_{E_A^{(1)}}, r_{E_A^{(1)}})$ a finite directed graph whose adjacency matrix is A.

The infinite path space is given by Sims 2021, 2.5:
$E_A^\infty = \left\{ (e_n)_{n\geq 1} \in (E_A^1)^{\mathbb{N}} \;\middle|\; r_{E_A^{(1)}}(e_n) = s_{E_A^{(1)}}(e_{n+1}) \text{ for all } n \geq 1 \right\}$ with
$\sigma\colon E_A^\infty \to E_A^\infty, (e_0, e_1, e_2, ..) \mapsto (e_1, e_2, e_3, ..)$.
$(\mathcal{G}_A)_0 = E_A^\infty$.
$(\mathcal{G}_A)_1 = \{(x, n, y) \in E_A^\infty \times \mathbb{Z} \times E_A^\infty \mid \exists\, k, \ell \in \mathbb{N}_0 :\; n = k - \ell,\; \sigma^k(x) = \sigma^\ell(y)\}$.
$s(x, n, y) = y,\; r(x, n, y) = x,\; 1_x = (x, 0, x)$,
$(x, n, y)^{-1} = (y, -n, x),\; (x, n, y) \cdot (y, m, z) = (x, n + m, z)$ if $s(x, n, y) = r(y, m, z)$.

Example: Diaconu-Renault Groupoid

Homology of SFT-Groupoids is well known.

$\mathbf{1} - A^{\mathsf{T}}$ acts on $\mathbb{Z}^N$ and we have by Matui 2012, 4.14:

$$H_0(\mathcal{G}_A) \cong \mathrm{coker}(\mathbf{1} - A^{\mathsf{T}}),$$
$$H_1(\mathcal{G}_A) \cong \ker(\mathbf{1} - A^{\mathsf{T}}),$$
$$H_n(\mathcal{G}_A) = 0 \quad \text{for } n \geq 2.$$

Consider now:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad C = (3).$$

We compute the integral homology of $\mathcal{G}_A$, $\mathcal{G}_B$, and $\mathcal{G}_C$.

Example: Diaconu-Renault Groupoid

Computing Homology for $\mathcal{G}_A$.

For A we have

$$\mathbf{1} - A^{\mathsf{T}} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \det(\mathbf{1} - A^{\mathsf{T}}) = -2.$$

Hence $\mathbf{1} - A^{\mathsf{T}}$ has full rank over $\mathbb{Z}$ and $\ker(\mathbf{1} - A^{\mathsf{T}}) = 0$.
Moreover, the Smith normal form is

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{C_2 \leftarrow C_2 - C_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

so $\operatorname{coker}(\mathbf{1} - A^{\mathsf{T}}) \cong \mathbb{Z}/2\mathbb{Z}$.

We get $H_0(\mathcal{G}_A) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_A) = 0$, $H_n(\mathcal{G}_A) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

Computing Homology for $\mathcal{G}_B$.

For B we have

$$\mathbf{1} - B^{\mathsf{T}} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$(\mathbf{1} - B^{\mathsf{T}})(x, y)^{\mathsf{T}} = 0 \Leftrightarrow -x - y = 0$, hence $\ker(\mathbf{1} - B^{\mathsf{T}}) \cong \mathbb{Z}$.
The image is generated by $(1, 1)$, which is primitive in $\mathbb{Z}^2$, so $\operatorname{coker}(\mathbf{1} - B^{\mathsf{T}}) \cong \mathbb{Z}^2/\langle(1, 1)\rangle_{\mathbb{Z}} \cong \mathbb{Z}$.

Thus, we have for homology $H_0(\mathcal{G}_B) \cong \mathbb{Z}$, $H_1(\mathcal{G}_B) \cong \mathbb{Z}$, $H_n(\mathcal{G}_B) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

Computing Homology for $\mathcal{G}_C$.

For C we have $\mathbf{1} - C^T = -2$, so
$\ker(\mathbf{1} - C^T) = 0$ and $\operatorname{coker}(\mathbf{1} - C^T) \cong \mathbb{Z}/2\mathbb{Z}$.

Hence $H_0(\mathcal{G}_C) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_C) = 0$, $H_n(\mathcal{G}_C) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

The disjoint union groupoid.

We have $\mathcal{G} = \mathcal{G}_A \sqcup \mathcal{G}_B \sqcup \mathcal{G}_C$, the disjoint union groupoid.
The nerve decomposes levelwise to $\mathcal{G}_n = (\mathcal{G}_A)_n \sqcup (\mathcal{G}_B)_n \sqcup (\mathcal{G}_C)_n$.
The Moore chain complex splits as a direct sum, thus

$$H_n(\mathcal{G}) \cong H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \text{ for } n \geq 0.$$

In particular

$$H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad H_1(\mathcal{G}) \cong \mathbb{Z}, \quad H_n(\mathcal{G}) = 0 \text{ for } n \geq 2.$$

Define $U_1 \coloneqq (\mathcal{G}_A)_0 \sqcup (\mathcal{G}_B)_0, U_2 \coloneqq (\mathcal{G}_B)_0 \sqcup (\mathcal{G}_C)_0$.

Example: Diaconu-Renault Groupoid

The reduction groupoids.

The reductions are

$$\mathcal{G}|_{U_1} = \mathcal{G}_A \sqcup \mathcal{G}_B, \quad \mathcal{G}|_{U_2} = \mathcal{G}_B \sqcup \mathcal{G}_C, \quad \mathcal{G}|_{U_1 \cap U_2} = \mathcal{G}_B.$$

This yields the long exact sequence

$$\cdots \to H_n(\mathcal{G}_B) \xrightarrow{\alpha_n} H_n(\mathcal{G}_A \sqcup \mathcal{G}_B) \oplus H_n(\mathcal{G}_B \sqcup \mathcal{G}_C) \xrightarrow{\beta_n}$$
$$\xrightarrow{\beta_n} H_n(\mathcal{G}) \xrightarrow{\partial_n} H_{n-1}(\mathcal{G}_B) \to \cdots.$$

Example: Diaconu-Renault Groupoid

Explicit formulas for α_n, β_n, ∂_n

$$
\begin{aligned}
\alpha_n &: H_n(\mathcal{G}_B) \to H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),\\
&[b] \mapsto ([0], [b], [-b], [0]),\\
\beta_n &: H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \to H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),\\
&([a], [b_1], [b_2], [c]) \mapsto ([a], [b_1 + b_2], [c]).
\end{aligned}
$$

∂_n vanishes in this example by exactness $\partial_n = 0 \colon H_n(\mathcal{G}) \to H_{n-1}(\mathcal{G}_B)$, since β_n is surjective and $\ker(\beta_n) = \{([0], [b], [-b], [0]) \mid b \in H_n(\mathcal{G}_B)\} = \mathrm{im}(\alpha_n)$.

Finite coefficients via UCT

Final homology groups for $\mathbb{Z}/p\mathbb{Z}$.

Fix a prime p. Assume $H_2(\mathcal{G}) = 0, H_1(\mathcal{G}) \cong \mathbb{Z}, H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

Vanishing in higher degrees: $H_n(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) = 0 \quad$ for all $n \geq 2$.

Degree 0:

$$H_0(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong H_0(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

Degree 1 via UCT:

$$0 \to H_1(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}\big(H_0(\mathcal{G}), \mathbb{Z}/p\mathbb{Z}\big) \to 0,$$

hence

$$H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

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