

# **Universal Coefficient Theorem Moore-Mayer-Vietoris Sequence for Homology of Ample Groupoids**

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## Setting

We investigate ample groupoids.

We consider  $C_c(\mathcal{G}_n, A)$ .

$A$  is a topological abelian group.

$\mathcal{G}$  is an ample groupoid.

$\mathcal{G}_n$  is the space of  $n$ -multiplicable arrows.

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Why is this important?

$\mathcal{G}$  étale: structure maps in the nerve, such as face maps  $d_i$  and degeneracies  $s_j$ , are local homeomorphisms, so pushforwards  $(d_i)_*$  are defined by finite fibre sums on compact support.

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$\mathcal{G}$  ample: compact open bisections form a basis.

$C_c(\mathcal{G}, \mathbb{Z})$  is generated by  $\chi_K$  for compact open sets  $K$ .

# The Nerve

On what do we compute homology?

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Face maps.  $d_i: \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}$ ,  $n = 1$ :  $d_0 = s$ ,  $d_1 = r$ . For  $n \geq 2$ :

$$d_i(g) := \begin{cases} (g_2, \dots, g_n), & i = 0, \\ (g_1, \dots, g_i \cdot g_{i+1}, \dots, g_n), & 1 \leq i \leq n-1, \\ (g_1, \dots, g_{n-1}), & i = n. \end{cases}$$

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Degeneracy maps.  $s_j: \mathcal{G}_n \rightarrow \mathcal{G}_{n+1}$ ,  $n \geq 0$ :

$$s_j(g) := \begin{cases} u(x), & n = 0, x \in \mathcal{G}_0, \\ (u(r(g_1)), g_1, \dots, g_n), & n \geq 1, j = 0, \\ (g_1, \dots, g_j, u(r(g_{j+1})), g_{j+1}, \dots, g_n), & n \geq 2, 1 \leq j \leq n-1, \\ (g_1, \dots, g_n, u(s(g_n))), & n \geq 1, j = n. \end{cases}$$

$$\mathcal{G}_n := \begin{cases} \mathcal{G}_0, & n = 0, \\ \{g \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for } 1 \leq i < n\}, & n \geq 1. \end{cases}$$

# Moore Chains and Boundary

Compactly supported chains on the nerve.

Chains.  $C_c(\mathcal{G}_n, A)$  denotes the abelian group of continuous maps  $f: \mathcal{G}_n \rightarrow A$  with compact support . If  $A$  is discrete , then every  $f \in C_c(\mathcal{G}_n, A)$  is locally constant .

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**Boundary.** Since  $\mathcal{G}$  is étale, each face map  $d_i: \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}$  is a local homeomorphism, hence the pushforward is well-defined:

$$(d_i)_*: C_c(\mathcal{G}_n, A) \rightarrow C_c(\mathcal{G}_{n-1}, A), \quad (d_i)_* f(y) := \sum_{x \in d_i^{-1}(y) \cap \text{supp}(f)} f(x),$$
$$\partial_n := \sum_{i=0}^n (-1)^i (d_i)_*: C_c(\mathcal{G}_n, A) \rightarrow C_c(\mathcal{G}_{n-1}, A).$$

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For all  $n \geq 0$  there is a natural short exact sequence in **Ab**:

$$0 \rightarrow H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \rightarrow 0.$$

The sequence splits, though not canonically:

$$H_n(\mathcal{G}; A) \cong (H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A) \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A).$$

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If  $A$  is discrete, then  $C_c(\mathcal{G}_n, A)$  is a free  $A$ -module.

Let  $f \in C_c(\mathcal{G}_n, A)$ . Since  $A$  is discrete and  $\text{supp}(f)$  is compact,  $\text{im}(f) = \{a_1, \dots, a_m\}$  is finite. Set  $K_i := f^{-1}(\{a_i\})$  for  $1 \leq i \leq m$ .

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Each  $K_i$  is clopen in  $\mathcal{G}_n$ , the sets  $K_i$  are pairwise disjoint, and  $\text{supp}(f) = \bigsqcup_{i=1}^m K_i$ .

$$f = \sum_{i=1}^m a_i \chi_{K_i} \quad \text{in } C_c(\mathcal{G}_n, A).$$

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Therefore the canonical map

$$\bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A \longrightarrow C_c(\mathcal{G}_n, A), \quad (a_K)_{K \in \mathcal{K}(\mathcal{G}_n)} \longmapsto \sum_{K \in \mathcal{K}(\mathcal{G}_n)} a_K \chi_K,$$

is surjective. It is injective: if  $\sum_{i=1}^m a_i \chi_{K_i} = 0$  with  $K_i$  pairwise disjoint compact open, then evaluating at any  $g \in K_i$  gives  $a_i = 0$ . Thus

$$C_c(\mathcal{G}_n, A) \cong \bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A, \quad \text{free as an } A\text{-module}.$$

# Moore-Mayer-Vietoris Sequence

# Mayer-Vietoris vs. Moore-Mayer-Vietoris

From saturated covers to homology.

## Mayer–Vietoris:

$X = U_1 \cup U_2$ ,  
 $U_1, U_2 \subseteq X$  open.

### Good cover:

$U_1, U_2, U_1 \cap U_2$  contractible.

In general: any nonempty finite intersection is contractible.

## Moore–Mayer–Vietoris:

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### Compute $H_\bullet(\mathcal{G})$ from:

$$H_\bullet(\mathcal{G}|_{U_1}), H_\bullet(\mathcal{G}|_{U_2}), H_\bullet(\mathcal{G}|_{U_1 \cap U_2}).$$

## Reductions and Moore–Mayer–Vietoris

Long Natural Moore–Mayer–Vietoris Sequence for Homology.

For  $U \subseteq G_0$  define the reduction

$$G|_U := \{g \in G \mid s(g), r(g) \in U\}, \quad (G|_U)_0 = U,$$

with structure maps the restrictions of  $u, m, s, r, -^{-1}$  to  $G|_U$ . For  $i \in \{1, 2\}$  and  $U_{12} := U_1 \cap U_2$  we write  $G|_{U_i}$  and  $G|_{U_{12}}$ .

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### Moore–Mayer–Vietoris long exact homology sequence:

$$\begin{array}{ccccccc} \cdots & \longleftarrow & H_{n-1}(G|_{U_1}; A) \oplus H_{n-1}(G|_{U_2}; A) & \xleftarrow{H_{n-1}(\alpha_\bullet)} & H_{n-1}(G|_{U_1 \cap U_2}; A) & \longleftarrow & \cdots \\ & & \overbrace{\hspace{10em}}^{\partial_n} & & & & \\ H_n(G; A) & \xleftarrow{H_n(\beta_\bullet)} & H_n(G|_{U_1}; A) \oplus H_n(G|_{U_2}; A) & \xleftarrow{H_n(\alpha_\bullet)} & H_n(G|_{U_1 \cap U_2}; A) & \longleftarrow & \cdots \\ & & \overbrace{\hspace{10em}}^{\partial_{n+1}} & & & & \\ H_{n+1}(G; A) & \xleftarrow{H_{n+1}(\beta_\bullet)} & H_{n+1}(G|_{U_1}; A) \oplus H_{n+1}(G|_{U_2}; A) & \longleftarrow & \cdots & & \end{array}$$

## Takeaways

What you get.

Setting:  $\mathcal{G}$  ample étale,  $A$  a discrete abelian group,

Moore chains  $C_c(\mathcal{G}_n, A)$  with boundary  $\partial = \sum_{i=0}^n (-1)^i (d_i)_*$ .

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Two structural tools:

UCT for discrete coefficients:

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Moore–Mayer–Vietoris

for a clopen saturated cover  $U_1 \cup U_2 = \mathcal{G}_0$ .

There is a natural long exact sequence relating

$$H_{\bullet}(\mathcal{G}; A), H_{\bullet}(\mathcal{G}|_{U_1}; A), H_{\bullet}(\mathcal{G}|_{U_2}; A), H_{\bullet}(\mathcal{G}|_{U_{12}}; A).$$

## Takeaways

What you get.

Why discreteness matters:

For non-discrete  $A$ , the canonical comparison

$$\Phi_X : C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a \chi_u,$$

need not be surjective. Tensor-level reduction in UCT can fail.

**Thank you.**

# **Homology of SFT Groupoids**

## Example: Diaconu–Renault Groupoid

Computing Homology with Moore–Mayer–Vietoris + UCT.

$A \in \text{Mat}(N \times N, \mathbb{N}_0)$  with no zero row and no zero column.

$(E_A^{(1)}, E_A^{(0)}, s_{E_A^{(1)}}, r_{E_A^{(1)}})$  a finite directed graph whose adjacency matrix is  $A$ .

The infinite path space is given by Sims 2021, 2.5:

$E_A^\infty = \left\{ (e_n)_{n \geq 1} \in (E_A^1)^\mathbb{N} \mid r_{E_A^{(1)}}(e_n) = s_{E_A^{(1)}}(e_{n+1}) \text{ for all } n \geq 1 \right\}$  with

$\sigma: E_A^\infty \rightarrow E_A^\infty, (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, e_3, \dots)$ .

$(\mathcal{G}_A)_0 = E_A^\infty$ .

$(\mathcal{G}_A)_1 = \{(x, n, y) \in E_A^\infty \times \mathbb{Z} \times E_A^\infty \mid \exists k, \ell \in \mathbb{N}_0 : n = k - \ell, \sigma^k(x) = \sigma^\ell(y)\}$ .

$s(x, n, y) = y, r(x, n, y) = x, 1_x = (x, 0, x),$

$(x, n, y)^{-1} = (y, -n, x), (x, n, y) \cdot (y, m, z) = (x, n+m, z) \text{ if } s(x, n, y) = r(y, m, z).$

## Example: Diaconu-Renault Groupoid

Homology of SFT-Groupoids is well known.

$\mathbf{1} - A^T$  acts on  $\mathbb{Z}^N$  and we have by Matui 2012, 4.14:

$$H_0(\mathcal{G}_A) \cong \text{coker}(\mathbf{1} - A^T),$$

$$H_1(\mathcal{G}_A) \cong \text{ker}(\mathbf{1} - A^T),$$

$$H_n(\mathcal{G}_A) = 0 \text{ for } n \geq 2.$$

Consider now:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = (3).$$

We compute the integral homology of  $\mathcal{G}_A$ ,  $\mathcal{G}_B$ , and  $\mathcal{G}_C$ .

## Example: Diaconu-Renault Groupoid

Computing Homology for  $\mathcal{G}_A$ .

For  $A$  we have

$$\mathbf{1} - A^T = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \det(\mathbf{1} - A^T) = -2.$$

Hence  $\mathbf{1} - A^T$  has full rank over  $\mathbb{Z}$  and  $\ker(\mathbf{1} - A^T) = 0$ .

Moreover, the Smith normal form is

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{C_2 \leftarrow C_2 - C_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

so  $\text{coker}(\mathbf{1} - A^T) \cong \mathbb{Z}/2\mathbb{Z}$ .

We get  $H_0(\mathcal{G}_A) \cong \mathbb{Z}/2\mathbb{Z}$ ,  $H_1(\mathcal{G}_A) = 0$ ,  $H_n(\mathcal{G}_A) = 0$  for  $n \geq 2$ .

## Example: Diaconu-Renault Groupoid

Computing Homology for  $\mathcal{G}_B$ .

For  $B$  we have

$$\mathbf{1} - B^T = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$$(\mathbf{1} - B^T)(x, y)^T = 0 \Leftrightarrow -x - y = 0, \text{ hence } \ker(\mathbf{1} - B^T) \cong \mathbb{Z}.$$

The image is generated by  $(1, 1)$ , which is primitive in  $\mathbb{Z}^2$ , so

$$\text{coker}(\mathbf{1} - B^T) \cong \mathbb{Z}^2 / \langle (1, 1) \rangle_{\mathbb{Z}} \cong \mathbb{Z}.$$

Thus, we have for homology  $H_0(\mathcal{G}_B) \cong \mathbb{Z}$ ,  $H_1(\mathcal{G}_B) \cong \mathbb{Z}$ ,  $H_n(\mathcal{G}_B) = 0$  for  $n \geq 2$ .

## Example: Diaconu-Renault Groupoid

Computing Homology for  $\mathcal{G}_C$ .

For  $C$  we have  $\mathbf{1} - C^T = -2$ , so

$$\ker(\mathbf{1} - C^T) = 0 \quad \text{and} \quad \text{coker}(\mathbf{1} - C^T) \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence  $H_0(\mathcal{G}_C) \cong \mathbb{Z}/2\mathbb{Z}$ ,  $H_1(\mathcal{G}_C) = 0$ ,  $H_n(\mathcal{G}_C) = 0$  for  $n \geq 2$ .

## Example: Diaconu-Renault Groupoid

The disjoint union groupoid.

We have  $\mathcal{G} = \mathcal{G}_A \sqcup \mathcal{G}_B \sqcup \mathcal{G}_C$ , the disjoint union groupoid.

The nerve decomposes levelwise to  $\mathcal{G}_n = (\mathcal{G}_A)_n \sqcup (\mathcal{G}_B)_n \sqcup (\mathcal{G}_C)_n$ .

The Moore chain complex splits as a direct sum, thus

$$H_n(\mathcal{G}) \cong H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \text{ for } n \geq 0.$$

In particular

$$H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad H_1(\mathcal{G}) \cong \mathbb{Z}, \quad H_n(\mathcal{G}) = 0 \text{ for } n \geq 2.$$

Define  $U_1 := (\mathcal{G}_A)_0 \sqcup (\mathcal{G}_B)_0$ ,  $U_2 := (\mathcal{G}_B)_0 \sqcup (\mathcal{G}_C)_0$ .

## Example: Diaconu-Renault Groupoid

The reduction groupoids.

The reductions are

$$\mathcal{G}|_{U_1} = \mathcal{G}_A \sqcup \mathcal{G}_B, \quad \mathcal{G}|_{U_2} = \mathcal{G}_B \sqcup \mathcal{G}_C, \quad \mathcal{G}|_{U_1 \cap U_2} = \mathcal{G}_B.$$

This yields the long exact sequence

$$\cdots \rightarrow H_n(\mathcal{G}_B) \xrightarrow{\alpha_n} H_n(\mathcal{G}_A \sqcup \mathcal{G}_B) \oplus H_n(\mathcal{G}_B \sqcup \mathcal{G}_C) \xrightarrow{\beta_n} \\ \xrightarrow{\beta_n} H_n(\mathcal{G}) \xrightarrow{\partial_n} H_{n-1}(\mathcal{G}_B) \rightarrow \cdots .$$

## Example: Diaconu-Renault Groupoid

Explicit formulas for  $\alpha_n$ ,  $\beta_n$ ,  $\delta_n$

$$\alpha_n : H_n(\mathcal{G}_B) \rightarrow H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$

$$[b] \mapsto ([0], [b], [-b], [0]),$$

$$\beta_n : H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \rightarrow H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$

$$([a], [b_1], [b_2], [c]) \mapsto ([a], [b_1 + b_2], [c]).$$

$\delta_n$  vanishes in this example by exactness  $\delta_n = 0 : H_n(\mathcal{G}) \rightarrow H_{n-1}(\mathcal{G}_B)$ , since  $\beta_n$  is surjective and  $\ker(\beta_n) = \{([0], [b], [-b], [0]) \mid b \in H_n(\mathcal{G}_B)\} = \text{im}(\alpha_n)$ .

# Finite coefficients via UCT

Final homology groups for  $\mathbb{Z}/p\mathbb{Z}$ .

Fix a prime  $p$ . Assume  $H_2(\mathcal{G}) = 0$ ,  $H_1(\mathcal{G}) \cong \mathbb{Z}$ ,  $H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ .

Vanishing in higher degrees:  $H_n(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) = 0$  for all  $n \geq 2$ .

Degree 0:

$$H_0(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong H_0(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

Degree 1 via UCT:

$$0 \rightarrow H_1(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \rightarrow H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_0(\mathcal{G}), \mathbb{Z}/p\mathbb{Z}) \rightarrow 0,$$

hence

$$H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

# **Proof of the UCT**

## Proof of the UCT

Step 1: Chain-level identification.

Let  $f \in C_c(\mathcal{G}_n, A)$  and write  $\text{im}(f) = \{a_1, \dots, a_m\}$ . Set  $K_i := f^{-1}(\{a_i\})$ . Then  $\text{supp}(f) = \bigsqcup_{i=1}^m K_i$  with  $K_i$  clopen and  $f|_{K_i} \equiv a_i$ .

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$$C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_n, A) \quad \text{for discrete } A.$$

## Proof of the UCT

Step 2: Compatibility with the boundary.

For each face map  $d_i: \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}$ , the pushforward  $(d_i)_*$  is  $\mathbb{Z}$ -linear and satisfies

$$(d_i)_*(\xi \cdot a) = ((d_i)_*\xi) \cdot a \quad \text{for } \xi \in C_c(\mathcal{G}_n, \mathbb{Z}), a \in A.$$

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Therefore  $\Phi_{\mathcal{G}_\bullet}$  is an isomorphism of chain complexes

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## Proof of the UCT

Step 3: Apply the classical algebraic UCT.

The Moore complex  $C_c(\mathcal{G}_\bullet, \mathbb{Z})$  is a chain complex of free abelian groups. Applying the classical algebraic UCT to  $C_c(\mathcal{G}_\bullet, \mathbb{Z})$  and transporting across the chain isomorphism from Steps 1–2 yields, for all  $n \geq 0$ , a short exact sequence

$$0 \rightarrow H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \rightarrow 0.$$

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This sequence is natural in  $\mathcal{G}$  and in discrete  $A$ . In general, for non-discrete topological abelian groups  $A$ , Moore homology need not satisfy such a UCT.

# **Failure of Isomorphism**

## Non-discrete Coefficients: What Fails

The general result.

For  $X$  locally compact, totally disconnected, Hausdorff with a basis of compact open sets and an abelian group  $A$ , consider the canonical map

$$\Phi_X: C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a\chi_u.$$

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If  $A$  is discrete, then  $\Phi_X$  is an isomorphism. The converse can fail.

## Non-discrete Coefficients: What Fails

Failure of the tensor comparison.

If  $A$  is non-discrete and  $0 \in A$  is not isolated, then surjectivity of  $\Phi_X$  can fail even for compact, totally disconnected spaces with a basis of clopen subsets. Set

$$X := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n \in \{0, 2\} \right\} \subset [0, 1],$$

$$A := (\mathbb{R}, \mathcal{O}_{\text{std}}),$$

$$\xi: X \rightarrow A, \quad x \mapsto x.$$

Then  $X$  is compact, Hausdorff, totally disconnected, and has a basis of clopen subsets. Hence  $\xi \in C_c(X, A)$  and  $\xi(X) = X$  is infinite. Therefore  $\xi \notin \text{im}(\Phi_X)$ , so  $\Phi_X$  is not surjective.

## Non-discrete Coefficients: What Fails

Isomorphism without discreteness.

$$A := (\mathbb{R}, \mathcal{O}_{\text{std}}), \quad (\{\bullet\}, \mathcal{O}_{\{\bullet\}} := \{\emptyset, \{\bullet\}\}).$$

Then  $\{\bullet\}$  is locally compact, totally disconnected, Hausdorff, and  $X$  is compact open.

$$C_c(\{\bullet\}, \mathbb{Z}) \cong \mathbb{Z},$$

$$C_c(\{\bullet\}, A) \cong A,$$

$$C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong \mathbb{Z} \otimes_{\mathbb{Z}} A \cong A.$$

Under these identifications the canonical map

$$\Phi_{\{\bullet\}} : C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(\{\bullet\}, A), \quad \chi_{\{\bullet\}} \otimes a \mapsto a \cdot \chi_{\{\bullet\}},$$

is the standard isomorphism  $\mathbb{Z} \otimes_{\mathbb{Z}} A \rightarrow A$ ,  $1 \otimes a \mapsto a$ .

# Proof of Moore–Mayer–Vietoris

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Proof idea for  $H_n(\alpha_\bullet)$ .

$(\iota_i)_n: (\mathcal{G}|_{U_{12}})_n \hookrightarrow (\mathcal{G}|_{U_i})_n$  is an open embedding, hence a local homeomorphism.  
Therefore the functorial pushforward on Moore chains

$$((\iota_i)_n)_*: C_c((\mathcal{G}|_{U_{12}})_n, A) \rightarrow C_c((\mathcal{G}|_{U_i})_n, A)$$

is given by a finite fibre sum. Since  $(\iota_i)_n$  is injective, it is extension by zero:

$$((\iota_i)_n)_* f(x) := \begin{cases} f(x), & x \in (\mathcal{G}|_{U_{12}})_n, \\ 0, & x \in (\mathcal{G}|_{U_i})_n \setminus (\mathcal{G}|_{U_{12}})_n. \end{cases}$$

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Define the chain map

$$\begin{aligned} \alpha_n: C_c((\mathcal{G}|_{U_{12}})_n, A) &\rightarrow C_c((\mathcal{G}|_{U_1})_n, A) \oplus C_c((\mathcal{G}|_{U_2})_n, A), \\ f &\mapsto ((\iota_1)_n)_* f, -((\iota_2)_n)_* f. \end{aligned}$$

Compatibility:  $\partial ((\iota_i)_n)_* = ((\iota_i)_{n-1})_* \partial$ ,  $\partial \alpha_n = \alpha_{n-1} \partial$ . Hence  $\alpha_\bullet$  induces  $H_n(\alpha_\bullet)$ .

## Proof of Moore–Mayer–Vietoris

Proof idea for  $H_n(\beta_\bullet)$ .

Let  $\kappa_i: \mathcal{G}|_{U_i} \hookrightarrow \mathcal{G}$  be the inclusion of reductions.

$(\kappa_i)_n: (\mathcal{G}|_{U_i})_n \hookrightarrow \mathcal{G}_n$  is an open embedding, hence a local homeomorphism. Therefore the pushforward on Moore chains extends by zero:

$$((\kappa_i)_n)_* g(x) := \begin{cases} g(x), & x \in (\mathcal{G}|_{U_i})_n, \\ 0, & x \in \mathcal{G}_n \setminus (\mathcal{G}|_{U_i})_n. \end{cases}$$

Define

$$\begin{aligned} \beta_n: C_c((\mathcal{G}|_{U_1})_n, A) \oplus C_c((\mathcal{G}|_{U_2})_n, A) &\rightarrow C_c(\mathcal{G}_n, A), \\ (g_1, g_2) &\mapsto ((\kappa_1)_n)_* g_1 + ((\kappa_2)_n)_* g_2. \end{aligned}$$

Compatibility:  $\partial ((\kappa_i)_n)_* = ((\kappa_i)_{n-1})_* \partial$ ,  $\partial \beta_n = \beta_{n-1} \partial$ . Hence  $\beta_\bullet$  induces  $H_n(\beta_\bullet)$ .

## Proof of Moore–Mayer–Vietoris

Proof idea for  $\partial_n$ .

Assume a SES of Moore chain complexes

$$0 \rightarrow C_c((\mathcal{G}|_{U_{12}})_\bullet, A) \xrightarrow{\alpha_\bullet} C_c((\mathcal{G}|_{U_1})_\bullet, A) \oplus C_c((\mathcal{G}|_{U_2})_\bullet, A) \\ \xrightarrow{\beta_\bullet} C_c(\mathcal{G}_\bullet, A) \rightarrow 0.$$

Here  $\partial$  denotes the Moore boundary.

Let  $[c] \in H_n(\mathcal{G}; A)$  with  $\partial c = 0$  and choose  $b$  with  $\beta_n(b) = c$ .

Then  $\beta_{n-1}(\partial b) = \partial(\beta_n(b)) = \partial c = 0$ , hence

$$\partial b \in \ker(\beta_{n-1}) = \text{im}(\alpha_{n-1}).$$

Choose  $a \in C_c((\mathcal{G}|_{U_{12}})_{n-1}, A)$  with  $\alpha_{n-1}(a) = \partial b$  and define

$$\partial_n([c]) := [a] \in H_{n-1}(\mathcal{G}|_{U_{12}}; A).$$

Standard homological algebra:  $\partial_n$  is well-defined, independent of choices, and yields exactness at  $H_n(\mathcal{G}; A)$ .

## References I

- Armstrong, B., N. Brownlowe, and A. Sims (2021). *Simplicity of twisted C\*-algebras of Deaconu–Renault groupoids*. DOI: [10.48550/arXiv.2109.02583](https://doi.org/10.48550/arXiv.2109.02583). arXiv: [2109.02583](https://arxiv.org/abs/2109.02583) [math.OA].
- Crainic, M. and I. Moerdijk (1999). *A homology theory for étale groupoids*. DOI: [10.48550/arXiv.math/9905011](https://doi.org/10.48550/arXiv.math/9905011). arXiv: [math/9905011](https://arxiv.org/abs/math/9905011) [math.KT].
- Crainic, M. and I. Moerdijk (2000). “A homology theory for étale groupoids”. In: *Journal für die Reine und Angewandte Mathematik* 2000.521, pp. 25–46. DOI: [10.1515/crll.2000.029](https://doi.org/10.1515/crll.2000.029).
- Matui, H. (2012). “Homology and topological full groups of étale groupoids on totally disconnected spaces”. In: *Proceedings of the London Mathematical Society* 104.1, pp. 27–56. DOI: [10.1112/plms/pdr058](https://doi.org/10.1112/plms/pdr058).
- Matui, H. (2022). *Long exact sequences of homology groups of étale groupoids*. DOI: [10.48550/arXiv.2111.04013](https://doi.org/10.48550/arXiv.2111.04013). arXiv: [2111.04013](https://arxiv.org/abs/2111.04013) [math.DS].

## References II

- Matui, H. and T. Mori (2024). *Cup and cap products for cohomology and homology groups of ample groupoids*. DOI: [10.48550/arXiv.2411.14906](https://doi.org/10.48550/arXiv.2411.14906). arXiv: [2411.14906](https://arxiv.org/abs/2411.14906) [math.OA].
- Sims, A. (2017). *Étale groupoids and their C\*-algebras*. DOI: [10.48550/arXiv.1710.10897](https://doi.org/10.48550/arXiv.1710.10897). arXiv: [1710.10897](https://arxiv.org/abs/1710.10897) [math.OA].
- Sims, A. (2018). *Hausdorff étale groupoids and their C\*-algebras*. DOI: [10.48550/arXiv.1710.10897](https://doi.org/10.48550/arXiv.1710.10897). arXiv: [1710.10897](https://arxiv.org/abs/1710.10897) [math.OA].