Universeller Koeffizientensatz
Moore-Mayer-Vietoris Sequenz
für die Homologie von einfachen Gruppoiden

Luciano Melodia M.A., B.Sc., B.A.
Bewerbung an der Friedrich-Alexander Universität Erlangen-Nürnberg
Lehrstuhl für theoretische Informatik

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Themen
Worum wird es heute gehen?

Homologie einfacher Gruppoide.
Homologie über den Moore-Kettenkomplex von einfachen Gruppoiden.

Universeller Koeffzientensatz.
Ein universeller Koeffizientensatz für diskrete abelsche Gruppen.

Moore–Mayer–Vietoris sequence.
Eine Sequenz vom Mayer-Vietoris Typ für abgeschlossen-offene saturierte Überdeckungen.

Homologie einfacher Gruppoide

Standing Hypotheses

We investigate ample groupoids.

We consider $C_c(\, \mathcal{G}_n \,, A\,)$.

A is a topological abelian group.

$\mathcal{G}$ is an ample groupoid.

$\mathcal{G}_n$ is the space of n-multiplicable arrows.

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Why is this important?

$\mathcal{G}$ étale: structure maps in the nerve, such as face maps d_i and degeneracies s_j, are local homeomorphisms, so pushforwards $(d_i)_*$ are defined by finite fibre sums on compact support.

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$\mathcal{G}$ ample: compact open bisections form a basis.

$C_c(\mathcal{G}, \mathbb{Z})$ is generated by χ_K for compact open sets K.

The Nerve

On what do we compute homology?

$\mathcal{G}_\bullet := (\mathcal{G}_n, (d_i)_{i=0}^n, (s_j)_{j=0}^n)_{n \geq 0}$ is a simplicial space .

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Face maps. $d_i \colon \mathcal{G}_n \to \mathcal{G}_{n-1}$, $n = 1$: $d_0 = s$, $d_1 = r$. For $n \geq 2$:

$$
d_i(\mathbf{g}) \coloneqq \begin{cases} (g_2, .., g_n), & i = 0, \\ (g_1, .., g_i \cdot g_{i+1}, .., g_n), & 1 \leq i \leq n-1, \\ (g_1, .., g_{n-1}), & i = n. \end{cases}
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$\boxed{\text{Degeneracy maps.}}$ $s_j \colon \mathcal{G}_n \to \mathcal{G}_{n+1}$, $n \geq 0$:

$$s_j(\mathbf{g}) := \begin{cases} u(x), & n = 0, \ x \in \mathcal{G}_0, \\ (u(r(g_1)), g_1, .., g_n), & n \geq 1, \ j = 0, \\ (g_1, .., g_j, u(r(g_{j+1})), g_{j+1}, .., g_n), & n \geq 2, \ 1 \leq j \leq n-1, \\ (g_1, .., g_n, u(s(g_n))), & n \geq 1, \ j = n. \end{cases}$$

$$\mathcal{G}_n := \begin{cases} \mathcal{G}_0, & n = 0, \\ \{\mathbf{g} \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for } 1 \leq i < n\}, & n \geq 1. \end{cases}$$

Moore Chains and Boundary

Compactly supported chains on the nerve.

Chains. $C_c(\mathcal{G}_n, A)$ denotes the abelian group of continuous maps $f\colon \mathcal{G}_n \to A$ with compact support. If A is discrete, then every $f \in C_c(\mathcal{G}_n, A)$ is locally constant.

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Boundary. Since $\mathcal{G}$ is étale, each face map $d_i: \mathcal{G}_n \to \mathcal{G}_{n-1}$ is a local homeomorphism, hence the pushforward is well-defined:

$$(d_i)_* : C_c(\mathcal{G}_n, A) \to C_c(\mathcal{G}_{n-1}, A), \qquad (d_i)_* f(y) := \sum_{x \in d_i^{-1}(y) \cap \operatorname{supp}(f)} f(x),$$

$$\partial_n := \sum_{i=0}^{n} (-1)^i (d_i)_* : C_c(\mathcal{G}_n, A) \to C_c(\mathcal{G}_{n-1}, A).$$

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For all $n \geq 0$ there is a natural short exact sequence in **Ab**:

$$0 \to H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big) \to 0.$$

The sequence splits, though not canonically:

$$H_n(\mathcal{G}; A) \cong \big(H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A\big) \oplus \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big).$$

Universal Coefficient Theorem

If A is discrete, then $C_c(\mathcal{G}_n, A)$ is a free A-module.

Let $f \in C_c(\mathcal{G}_n, A)$. Since A is discrete and $\mathrm{supp}(f)$ is compact, $\mathrm{im}(f) = \{a_1, .., a_m\}$ is finite. Set $K_i := f^{-1}(\{a_i\})$ for $1 \leq i \leq m$.

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Each K_i is clopen in $\mathcal{G}_n$, the sets K_i are pairwise disjoint, and $\mathrm{supp}(f) = \bigsqcup_{i=1}^m K_i$.

$$f = \sum_{i=1}^m a_i \, \chi_{K_i} \qquad \text{in } C_c(\mathcal{G}_n, A).$$

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Therefore the canonical map

$$\bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A \longrightarrow C_c(\mathcal{G}_n, A), \qquad (a_K)_{K \in \mathcal{K}(\mathcal{G}_n)} \longmapsto \sum_{K \in \mathcal{K}(\mathcal{G}_n)} a_K \chi_K,$$

is surjective. It is injective: if $\sum_{i=1}^m a_i \chi_{K_i} = 0$ with K_i pairwise disjoint compact open, then evaluating at any $g \in K_i$ gives $a_i = 0$. Thus

$$C_c(\mathcal{G}_n, A) \cong \bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A, \qquad \boxed{\text{free as an } A\text{-module}}.$$

Non-discrete Coefficients: What Fails

The general result.

For X locally compact , totally disconnected , Hausdorff with a

basis of compact open sets and an abelian group A, consider the canonical map

$$\Phi_X \colon C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(X, A), \qquad \chi_U \otimes a \mapsto a\,\chi_U.$$

Then

$$\mathrm{im}(\Phi_X) \subseteq \{\xi \in C_c(X, A) \mid \xi(X) \text{ is finite}\}.$$

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In particular, Φ_X may fail to be surjective for non-discrete A . Moreover,

$$\Phi_X \text{ surjective} \Leftrightarrow \forall\, \xi \in C_c(X, A) : \ \xi(X) \text{ finite} \Leftrightarrow \Phi_X \text{ is an isomorphism}.$$

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If A is discrete, then Φ_X is an isomorphism. The converse can fail.

Moore-Mayer-Vietoris Sequence

Mayer-Vietoris vs. Moore-Mayer-Vietoris

From saturated covers to homology.

Mayer–Vietoris: **Moore–Mayer–Vietoris:**

$X = U_1 \cup U_2$,
$U_1, U_2 \subseteq X$ open.

Good cover:
U_1, U_2, $U_1 \cap U_2$ contractible.

In general: any nonempty finite
intersection is contractible.

Compute $H_\bullet(X)$ from :
$H_\bullet(U_1)$, $H_\bullet(U_2)$, $H_\bullet(U_1 \cap U_2)$.

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$\mathcal{G}|_U \coloneqq \{g \in \mathcal{G} \mid s(g), r(g) \in U\}$,
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Compute $H_\bullet(\mathcal{G})$ from:
$H_\bullet(\mathcal{G}|_{U_1}), \ H_\bullet(\mathcal{G}|_{U_2}), \ H_\bullet(\mathcal{G}|_{U_1 \cap U_2}).$

Reductions and Moore–Mayer–Vietoris

For $U \subseteq \mathcal{G}_0$ define the reduction

$$\mathcal{G}|_U \coloneqq \{g \in \mathcal{G} \mid s(g), r(g) \in U\}, \qquad (\mathcal{G}|_U)_0 = U,$$

with structure maps the restrictions of $u, m, s, r, -^{-1}$ to $\mathcal{G}|_U$. For $i \in \{1, 2\}$ and $U_{12} \coloneqq U_1 \cap U_2$ we write $\mathcal{G}|_{U_i}$ and $\mathcal{G}|_{U_{12}}$.

Reductions and Moore–Mayer–Vietoris

Long Natural Moore-Mayer-Vietoris Sequence for Homology.

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Moore–Mayer–Vietoris long exact homology sequence:

$$\cdots \longleftarrow H_{n-1}(\mathcal{G}|_{U_1}; A) \oplus H_{n-1}(\mathcal{G}|_{U_2}; A) \xleftarrow{H_{n-1}(\alpha_\bullet)} H_{n-1}(\mathcal{G}|_{U_1 \cap U_2}; A) \longleftarrow$$

$$\overbrace{\qquad \partial_n \qquad}$$

$$\longleftarrow H_n(\mathcal{G}; A) \xleftarrow{H_n(\beta_\bullet)} H_n(\mathcal{G}|_{U_1}; A) \oplus H_n(\mathcal{G}|_{U_2}; A) \xleftarrow{H_n(\alpha_\bullet)} H_n(\mathcal{G}|_{U_1 \cap U_2}; A) \longleftarrow$$

$$\overbrace{\qquad \partial_{n+1} \qquad}$$

$$\longleftarrow H_{n+1}(\mathcal{G}; A) \xleftarrow{H_{n+1}(\beta_\bullet)} H_{n+1}(\mathcal{G}|_{U_1}; A) \oplus H_{n+1}(\mathcal{G}|_{U_2}; A) \longleftarrow \cdots$$

Takeaways

What you get and how to use it.

Setting: $\mathcal{G}$ ample étale, A a discrete abelian group,

Moore chains $C_c(\mathcal{G}_n, A)$ with boundary $\partial = \sum_{i=0}^{n} (-1)^i (d_i)_*$.

Two structural tools:

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UCT for discrete coefficients:

$$0 \to H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \to 0.$$

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Moore–Mayer–Vietoris

for a clopen saturated cover $U_1 \cup U_2 = \mathcal{G}_0$.

There is a natural long exact sequence relating

$$H_{\bullet}(\mathcal{G}; A), H_{\bullet}(\mathcal{G}|_{U_1}; A), H_{\bullet}(\mathcal{G}|_{U_2}; A), H_{\bullet}(\mathcal{G}|_{U_{12}}; A).$$

Takeaways
What you get and how to use it.

For non-discrete A, the canonical comparison

$$\Phi_X : \ C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(X, A), \qquad \chi_U \otimes a \mapsto a\,\chi_U,$$

need not be surjective. Tensor-level reduction in UCT can fail.

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How to use in practice:

Choose a clopen saturated cover.
$U_1, U_2 \subseteq \mathcal{G}_0$ so that reductions $\mathcal{G}|_{U_1}$, $\mathcal{G}|_{U_2}$, $\mathcal{G}|_{U_{12}}$ are computable.
Compute integral homology.
$H_\bullet(\mathcal{G}|_{U_i})$, $H_\bullet(\mathcal{G}|_{U_{12}})$, then glue to $H_\bullet(\mathcal{G})$ via MMV.

Thank you.

Homology of SFT Groupoids

Example: Diaconu-Renault Groupoid

Computing Homology with Moore–Mayer–Vietoris + UCT.

$A \in \mathrm{Mat}(N \times N, \mathbb{N}_0)$ with no zero row and no zero column.
$(E_A^{(1)}, E_A^{(0)}, s_{E_A^{(1)}}, r_{E_A^{(1)}})$ a finite directed graph whose adjacency matrix is A.

The **infinite path space** is given by **Sims 2021, 2.5**:
$$E_A^\infty = \left\{ (e_n)_{n \geq 1} \in (E_A^1)^{\mathbb{N}} \; \middle| \; r_{E_A^{(1)}}(e_n) = s_{E_A^{(1)}}(e_{n+1}) \text{ for all } n \geq 1 \right\} \text{ with}$$
$\sigma \colon E_A^\infty \to E_A^\infty, (e_0, e_1, e_2, ..) \mapsto (e_1, e_2, e_3, ..).$
$(\mathcal{G}_A)_0 = E_A^\infty.$
$(\mathcal{G}_A)_1 = \{(x, n, y) \in E_A^\infty \times \mathbb{Z} \times E_A^\infty \mid \exists\, k, \ell \in \mathbb{N}_0 : \; n = k - \ell, \; \sigma^k(x) = \sigma^\ell(y)\}.$
$s(x, n, y) = y, \; r(x, n, y) = x, \; 1_x = (x, 0, x),$
$(x, n, y)^{-1} = (y, -n, x), \; (x, n, y) \cdot (y, m, z) = (x, n + m, z) \text{ if } s(x, n, y) = r(y, m, z).$

Example: Diaconu-Renault Groupoid

Homology of SFT-Groupoids is well known.

$1 - A^\mathsf{T}$ acts on $\mathbb{Z}^N$ and we have by Matui 2012, 4.14:

$$H_0(\mathcal{G}_A) \cong \operatorname{coker}(1 - A^\mathsf{T}),$$

$$H_1(\mathcal{G}_A) \cong \ker(1 - A^\mathsf{T}),$$

$$H_n(\mathcal{G}_A) = 0 \text{ for } n \geq 2.$$

Consider now:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad C = (3).$$

We compute the integral homology of $\mathcal{G}_A$, $\mathcal{G}_B$, and $\mathcal{G}_C$.

Example: Diaconu-Renault Groupoid

Computing Homology for $\mathcal{G}_A$.

For A we have

$$\mathbf{1} - A^{\mathsf{T}} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \det(\mathbf{1} - A^{\mathsf{T}}) = -2.$$

Hence $\mathbf{1} - A^{\mathsf{T}}$ has full rank over $\mathbb{Z}$ and $\ker(\mathbf{1} - A^{\mathsf{T}}) = 0$.

Moreover, the Smith normal form is

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{C_2 \leftarrow C_2 - C_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

so $\operatorname{coker}(\mathbf{1} - A^{\mathsf{T}}) \cong \mathbb{Z}/2\mathbb{Z}$.

We get $H_0(\mathcal{G}_A) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_A) = 0$, $H_n(\mathcal{G}_A) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid
Computing Homology for $\mathcal{G}_B$.

For B we have

$$\mathbf{1} - B^{\mathsf{T}} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$(\mathbf{1} - B^{\mathsf{T}})(x, y)^{\mathsf{T}} = 0 \Leftrightarrow -x - y = 0$, hence $\boxed{\ker(\mathbf{1} - B^{\mathsf{T}}) \cong \mathbb{Z}}$.

The image is generated by $(1, 1)$, which is primitive in $\mathbb{Z}^2$, so

$\boxed{\operatorname{coker}(\mathbf{1} - B^{\mathsf{T}}) \cong \mathbb{Z}^2 / \langle (1, 1) \rangle_{\mathbb{Z}} \cong \mathbb{Z}}$.

Thus, we have for homology $\boxed{H_0(\mathcal{G}_B) \cong \mathbb{Z}}$, $\boxed{H_1(\mathcal{G}_B) \cong \mathbb{Z}}$, $\boxed{H_n(\mathcal{G}_B) = 0}$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

Computing Homology for $\mathcal{G}_C$.

For C we have $\mathbf{1} - C^{\mathsf{T}} = -2$, so
$\ker(\mathbf{1} - C^{\mathsf{T}}) = 0$ and $\operatorname{coker}(\mathbf{1} - C^{\mathsf{T}}) \cong \mathbb{Z}/2\mathbb{Z}$.

Hence $H_0(\mathcal{G}_C) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_C) = 0$, $H_n(\mathcal{G}_C) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid
The disjoint union groupoid.

We have $\mathcal{G} = \mathcal{G}_A \sqcup \mathcal{G}_B \sqcup \mathcal{G}_C$, the disjoint union groupoid.
The nerve decomposes levelwise to $\mathcal{G}_n = (\mathcal{G}_A)_n \sqcup (\mathcal{G}_B)_n \sqcup (\mathcal{G}_C)_n$.
The Moore chain complex splits as a direct sum, thus

$$H_n(\mathcal{G}) \cong H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \text{ for } n \geq 0.$$

In particular

$$H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad H_1(\mathcal{G}) \cong \mathbb{Z}, \quad H_n(\mathcal{G}) = 0 \text{ for } n \geq 2.$$

Define $U_1 := (\mathcal{G}_A)_0 \sqcup (\mathcal{G}_B)_0, U_2 := (\mathcal{G}_B)_0 \sqcup (\mathcal{G}_C)_0$.

Example: Diaconu-Renault Groupoid
The reduction groupoids.

The reductions are

$$\mathcal{G}|_{U_1} = \mathcal{G}_A \sqcup \mathcal{G}_B, \quad \mathcal{G}|_{U_2} = \mathcal{G}_B \sqcup \mathcal{G}_C, \quad \mathcal{G}|_{U_1 \cap U_2} = \mathcal{G}_B.$$

This yields the long exact sequence

$$\cdots \to H_n(\mathcal{G}_B) \xrightarrow{\alpha_n} H_n(\mathcal{G}_A \sqcup \mathcal{G}_B) \oplus H_n(\mathcal{G}_B \sqcup \mathcal{G}_C) \xrightarrow{\beta_n}$$

$$\xrightarrow{\beta_n} H_n(\mathcal{G}) \xrightarrow{\partial_n} H_{n-1}(\mathcal{G}_B) \to \cdots.$$

Example: Diaconu-Renault Groupoid
Explicit formulas for α_n, β_n, ∂_n

$$\alpha_n : H_n(\mathcal{G}_B) \to H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$
$$[b] \mapsto ([0], [b], [-b], [0]),$$
$$\beta_n : H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \to H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$
$$([a], [b_1], [b_2], [c]) \mapsto ([a], [b_1 + b_2], [c]).$$

∂_n vanishes in this example by exactness $\partial_n = 0 \colon H_n(\mathcal{G}) \to H_{n-1}(\mathcal{G}_B)$, since β_n is surjective and $\ker(\beta_n) = \{([0], [b], [-b], [0]) \mid b \in H_n(\mathcal{G}_B)\} = \operatorname{im}(\alpha_n)$.

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Finite coefficients via UCT

Final homology groups for $\mathbb{Z}/p\mathbb{Z}$.

Fix a prime p. Assume $H_2(\mathcal{G}) = 0, H_1(\mathcal{G}) \cong \mathbb{Z}, H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

Vanishing in higher degrees: $H_n(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 2$.

Degree 0:

$$H_0(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong H_0(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

Degree 1 via UCT:

$$0 \to H_1(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}\big(H_0(\mathcal{G}), \mathbb{Z}/p\mathbb{Z}\big) \to 0,$$

hence

$$H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

Proof of the UCT

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Let $f \in C_c(\mathcal{G}_n, A)$ and write $\mathrm{im}(f) = \{a_1, .., a_m\}$. Set $K_i := f^{-1}(\{a_i\})$. Then $\mathrm{supp}(f) = \bigsqcup_{i=1}^m K_i$ with K_i clopen and $f|_{K_i} \equiv a_i$.

Extension by 0: $\chi_{K_i} \in C_c(\mathcal{G}_n, \mathbb{Z})$ and

$$f = \sum_{i=1}^m a_i \chi_{K_i} \qquad \text{in } C_c(\mathcal{G}_n, A).$$

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Define the canonical $\mathbb{Z}$-bilinear map

$$\Phi_{\mathcal{G}_n} \colon C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow C_c(\mathcal{G}_n, A), \qquad \xi \otimes a \longmapsto a \cdot \xi.$$

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$$C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_n, A) \qquad \text{for discrete } A.$$

Proof of the UCT

Step 2: Compatibility with the boundary.

For each face map $d_i \colon \mathcal{G}_n \to \mathcal{G}_{n-1}$, the pushforward $(d_i)_*$ is $\mathbb{Z}$-linear and satisfies

$$(d_i)_*(\xi \cdot a) = \big((d_i)_* \xi\big) \cdot a \qquad \text{for } \xi \in C_c(\mathcal{G}_n, \mathbb{Z}), \ a \in A.$$

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Therefore $\Phi_{\mathcal{G}_\bullet}$ is an isomorphism of chain complexes

$$C_c(\mathcal{G}_\bullet, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_\bullet, A).$$

Proof of the UCT
Step 3: Apply the classical algebraic UCT.

The Moore complex $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ is a chain complex of free abelian groups. Applying the classical algebraic UCT to $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ and transporting across the chain isomorphism from Steps 1–2 yields, for all $n \geq 0$, a short exact sequence

$$0 \to H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \mathrm{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(\mathcal{G}), A\big) \to 0.$$

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This sequence is natural in $\mathcal{G}$ and in discrete A. In general, for non-discrete topological abelian groups A, Moore homology need not satisfy such a UCT.

Failure of Isomorphism

Non-discrete Coefficients: What Fails

Failure of the tensor comparison.

If A is non-discrete and $0 \in A$ is not isolated, then surjectivity of Φ_X can fail even for compact, totally disconnected spaces with a basis of clopen subsets. Set

$$X := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \ \middle| \ a_n \in \{0, 2\} \right\} \subset [0, 1],$$

$$A := (\mathbb{R}, \mathcal{O}_{\mathrm{std}}),$$

$$\xi \colon X \to A, \quad x \mapsto x.$$

Then X is compact, Hausdorff, totally disconnected, and has a basis of clopen subsets. Hence $\xi \in C_c(X, A)$ and $\xi(X) = X$ is infinite. Therefore $\xi \notin \mathrm{im}(\Phi_X)$, so Φ_X is not surjective.

Non-discrete Coefficients: What Fails

Isomorphism without discreteness.

$$A := (\mathbb{R}, \mathcal{O}_{\mathrm{std}}), \qquad (\{\bullet\}, \mathcal{O}_{\{\bullet\}} := \{\varnothing, \{\bullet\}\}).$$

Then $\{\bullet\}$ is locally compact, totally disconnected, Hausdorff, and X is compact open.

$$C_c(\{\bullet\}, \mathbb{Z}) \cong \mathbb{Z},$$
$$C_c(\{\bullet\}, A) \cong A,$$
$$C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong \mathbb{Z} \otimes_{\mathbb{Z}} A \cong A.$$

Under these identifications the canonical map

$$\Phi_{\{\bullet\}}: \; C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to C_c(\{\bullet\}, A), \qquad \chi_{\{\bullet\}} \otimes a \mapsto a \cdot \chi_{\{\bullet\}},$$

is the standard isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} A \to A$, $1 \otimes a \mapsto a$.

Proof of Moore–Mayer–Vietoris

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Proof idea for $H_n(\alpha_\bullet)$.

$(\iota_i)_n \colon (\mathcal{G}|_{U_{12}})_n \hookrightarrow (\mathcal{G}|_{U_i})_n$ is an open embedding, hence a local homeomorphism. Therefore the functorial pushforward on Moore chains

$$((\iota_i)_n)_* \colon C_c\big((\mathcal{G}|_{U_{12}})_n, A\big) \to C_c\big((\mathcal{G}|_{U_i})_n, A\big)$$

is given by a $\boxed{\text{finite fibre sum}}$. Since $(\iota_i)_n$ is injective, it is extension by zero:

$$((\iota_i)_n)_* f(x) \coloneqq \begin{cases} f(x), & x \in (\mathcal{G}|_{U_{12}})_n, \\ 0, & x \in (\mathcal{G}|_{U_i})_n \setminus (\mathcal{G}|_{U_{12}})_n. \end{cases}$$

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Define the chain map

$$\alpha_n \colon C_c\big((\mathcal{G}|_{U_{12}})_n, A\big) \to C_c\big((\mathcal{G}|_{U_1})_n, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_n, A\big),$$
$$f \mapsto \big(((\iota_1)_n)_* f, -((\iota_2)_n)_* f\big).$$

$\boxed{\text{Compatibility:}}$ $\partial\,((\iota_i)_n)_* = ((\iota_i)_{n-1})_*\,\partial,\, \partial\,\alpha_n = \alpha_{n-1}\,\partial$. Hence $\alpha_\bullet$ induces $H_n(\alpha_\bullet)$.

Proof of Moore–Mayer–Vietoris
Proof idea for $H_n(\beta_\bullet)$.

Let $\kappa_i \colon \mathcal{G}|_{U_i} \hookrightarrow \mathcal{G}$ be the inclusion of reductions.
$(\kappa_i)_n \colon (\mathcal{G}|_{U_i})_n \hookrightarrow \mathcal{G}_n$ is an open embedding, hence a local homeomorphism. Therefore the pushforward on Moore chains extends by zero:

$$((\kappa_i)_n)_* g(x) := \begin{cases} g(x), & x \in (\mathcal{G}|_{U_i})_n, \\ 0, & x \in \mathcal{G}_n \setminus (\mathcal{G}|_{U_i})_n. \end{cases}$$

Define

$$\beta_n \colon C_c\big((\mathcal{G}|_{U_1})_n, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_n, A\big) \to C_c(\mathcal{G}_n, A),$$
$$(g_1, g_2) \mapsto ((\kappa_1)_n)_* g_1 + ((\kappa_2)_n)_* g_2.$$

Compatibility: $\partial\,((\kappa_i)_n)_* = ((\kappa_i)_{n-1})_*\,\partial, \partial\,\beta_n = \beta_{n-1}\,\partial$. Hence $\beta_\bullet$ induces $H_n(\beta_\bullet)$.

Proof of Moore–Mayer–Vietoris

Proof idea for ∂_n.

Assume a SES of Moore chain complexes

$$0 \to C_c\big((\mathcal{G}|_{U_{12}})_\bullet, A\big) \xrightarrow{\alpha_\bullet} C_c\big((\mathcal{G}|_{U_1})_\bullet, A\big) \oplus C_c\big((\mathcal{G}|_{U_2})_\bullet, A\big)$$
$$\xrightarrow{\beta_\bullet} C_c\big(\mathcal{G}_\bullet, A\big) \to 0.$$

Here ∂ denotes the Moore boundary.

Let $[c] \in H_n(\mathcal{G}; A)$ with $\partial c = 0$ and choose b with $\boxed{\beta_n(b) = c}$.

Then $\beta_{n-1}(\partial b) = \partial(\beta_n(b)) = \partial c = 0$, hence

$$\partial b \in \ker(\beta_{n-1}) = \operatorname{im}(\alpha_{n-1}).$$

Choose $a \in C_c\big((\mathcal{G}|_{U_{12}})_{n-1}, A\big)$ with $\boxed{\alpha_{n-1}(a) = \partial b}$ and define

$$\partial_n([c]) \coloneqq [a] \in H_{n-1}(\mathcal{G}|_{U_{12}}; A).$$

Standard homological algebra: ∂_n is well-defined, independent of choices, and yields exactness at $H_n(\mathcal{G}; A)$.

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