

Universal Coefficient Theorem Moore-Mayer-Vietoris Sequence for Homology of Ample Groupoids

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March 10, 2026

Topics

What are we going to talk about?

Ample groupoid homology.

Homology via the Moore chain complex of ample groupoids.

Universal coefficient theorem.

A universal coefficient theorem for discrete abelian groups.

Moore–Mayer–Vietoris sequence.

A Mayer–Vietoris type sequence for clopen saturated covers.

Ample Groupoid Homology

Ample groupoids

What do we investigate?

A topological groupoid consists of a space $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$, a unit space $\mathcal{G}_0 \subseteq \mathcal{G}$, and maps

$$u: \mathcal{G}_0 \rightarrow \mathcal{G}, \quad s, r: \mathcal{G} \rightarrow \mathcal{G}_0, \quad (-)^{-1}: \mathcal{G} \rightarrow \mathcal{G}, \quad m: \mathcal{G}_2 \rightarrow \mathcal{G}.$$

$$s(g) = g^{-1} \cdot g, \quad r(g) = g \cdot g^{-1}, \quad u(x) = 1_x.$$

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The space of composable pairs is

$$\mathcal{G}_2 \doteq \mathcal{G}_s \times_r \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = r(h)\}, \quad m(g, h) = g \cdot h.$$

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\mathcal{G} has a basis of compact open bisections $U \subseteq \mathcal{G}$, so $r|_U$ and $s|_U$ are homeomorphisms onto compact open subsets of \mathcal{G}_0 .

Standing Hypotheses

We investigate ample groupoids.

We consider $C_c(\mathcal{G}_n, A)$.

A is a topological abelian group.

\mathcal{G} is an ample groupoid.

\mathcal{G}_n is the space of n -multiplicable arrows.

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\mathcal{G} ample: compact open bisections form a basis.

$C_c(\mathcal{G}, \mathbb{Z})$ is generated by χ_K for compact open sets K .

The Nerve

On what do we compute homology?

$\mathcal{G}_\bullet := (\mathcal{G}_n, (d_i)_{i=0}^n, (s_j)_{j=0}^n)_{n \geq 0}$ is a simplicial space .

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$$d_i(g) := \begin{cases} (g_2, \dots, g_n), & i = 0, \\ (g_1, \dots, g_i \cdot g_{i+1}, \dots, g_n), & 1 \leq i \leq n-1, \\ (g_1, \dots, g_{n-1}), & i = n. \end{cases}$$

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Degeneracy maps. $s_j: \mathcal{G}_n \rightarrow \mathcal{G}_{n+1}$, $n \geq 0$:

$$s_j(g) := \begin{cases} u(x), & n = 0, x \in \mathcal{G}_0, \\ (u(r(g_1)), g_1, \dots, g_n), & n \geq 1, j = 0, \\ (g_1, \dots, g_j, u(r(g_{j+1})), g_{j+1}, \dots, g_n), & n \geq 2, 1 \leq j \leq n-1, \\ (g_1, \dots, g_n, u(s(g_n))), & n \geq 1, j = n. \end{cases}$$

$$\mathcal{G}_n := \begin{cases} \mathcal{G}_0, & n = 0, \\ \{g \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for } 1 \leq i < n\}, & n \geq 1. \end{cases}$$

Moore Chains and Boundary

Compactly supported chains on the nerve.

Chains. $C_c(\mathcal{G}_n, A)$ denotes the abelian group of continuous maps $f: \mathcal{G}_n \rightarrow A$ with compact support . If A is discrete , then every $f \in C_c(\mathcal{G}_n, A)$ is locally constant .

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Boundary. Since \mathcal{G} is étale, each face map $d_i: \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}$ is a local homeomorphism, hence the pushforward is well-defined:

$$(d_i)_*: C_c(\mathcal{G}_n, A) \rightarrow C_c(\mathcal{G}_{n-1}, A), \quad (d_i)_* f(y) := \sum_{x \in d_i^{-1}(\{y\}) \cap \text{supp}(f)} f(x),$$
$$\partial_n := \sum_{i=0}^n (-1)^i (d_i)_*: C_c(\mathcal{G}_n, A) \rightarrow C_c(\mathcal{G}_{n-1}, A).$$

Moore Chains and Boundary

If A is discrete, then $C_c(\mathcal{G}_n, A)$ is a free A -module.

Assume A is a discrete abelian group and \mathcal{G} is ample.

If $f \in C_c(\mathcal{G}_n, A)$, then for each $a \in A$ the fibre $f^{-1}(\{a\}) \subseteq \mathcal{G}_n$ is open. Since A is discrete, $f^{-1}(\{a\})$ is also closed. Thus each $f^{-1}(\{a\})$ is clopen.

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Why is f locally constant?

Fix $g \in \mathcal{G}_n$. Since A is discrete, the singleton $\{f(g)\} \subseteq A$ is open. Therefore

$$U_g := f^{-1}(\{f(g)\})$$

is an open neighborhood of g in \mathcal{G}_n . If $h \in U_g$, then

$$h \in f^{-1}(\{f(g)\}) \Leftrightarrow f(h) \in \{f(g)\} \Leftrightarrow f(h) = f(g).$$

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\mathcal{G} is ample \Rightarrow d_i local homeomorphism \Rightarrow $d_i^{-1}(\{y\})$ discrete.

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If $f \in C_c(\mathcal{G}_n, A)$, then for each $y \in \mathcal{G}_{n-1}$ the set $d_i^{-1}(\{y\}) \cap \text{supp}(f)$ is finite, as a discrete subset of the compact space $\text{supp}(f)$. Hence

$$(d_i)_* f(y) := \sum_{x \in d_i^{-1}(\{y\}) \cap \text{supp}(f)} f(x) \quad \text{is well-defined.}$$

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$(d_i)_* f \in C_c(\mathcal{G}_{n-1}, A)$: it has compact support since

$$\text{supp}((d_i)_* f) \subseteq d_i(\text{supp}(f)),$$

and it is locally constant on \mathcal{G}_{n-1} when A is discrete, by local triviality of d_i over $\text{supp}(f)$.

Moore Chains and Boundary

The identity $\partial^2 = 0$.

$\partial_{n-1}\partial_n = 0$: By definition,

$$\partial_{n-1}\partial_n = \sum_{i=0}^{n-1} \sum_{j=0}^n (-1)^{i+j} (d_i)_* \circ (d_j)_*.$$

For $i < j$, the simplicial identities give $d_i d_j = d_{j-1} d_i$, hence by functoriality of pushforward,

$$(d_i)_* \circ (d_j)_* = (d_i d_j)_* = (d_{j-1} d_i)_* = (d_{j-1})_* \circ (d_i)_*.$$

Therefore the terms cancel in pairs and

$$\partial_{n-1}\partial_n = \sum_{0 \leq i < j \leq n} (-1)^{i+j} \left((d_i)_*(d_j)_* - (d_{j-1})_*(d_i)_* \right) = 0.$$

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For all $n \geq 0$ there is a natural short exact sequence in **Ab**:

$$0 \rightarrow H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \rightarrow 0.$$

The sequence splits, though not canonically:

$$H_n(\mathcal{G}; A) \cong (H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A) \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A).$$

Universal Coefficient Theorem

If A is discrete, then $C_c(\mathcal{G}_n, A)$ is a free A -module.

Let $f \in C_c(\mathcal{G}_n, A)$. Since A is discrete and $\text{supp}(f)$ is compact, $\text{im}(f) = \{a_1, \dots, a_m\}$ is finite. Set $K_i := f^{-1}(\{a_i\})$ for $1 \leq i \leq m$.

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Each K_i is clopen in \mathcal{G}_n , the sets K_i are pairwise disjoint, and $\text{supp}(f) = \bigsqcup_{i=1}^m K_i$.

$$f = \sum_{i=1}^m a_i \chi_{K_i} \quad \text{in } C_c(\mathcal{G}_n, A).$$

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Therefore the canonical map

$$\bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A \longrightarrow C_c(\mathcal{G}_n, A), \quad (a_K)_{K \in \mathcal{K}(\mathcal{G}_n)} \longmapsto \sum_{K \in \mathcal{K}(\mathcal{G}_n)} a_K \chi_K,$$

is surjective. It is injective: if $\sum_{i=1}^m a_i \chi_{K_i} = 0$ with K_i pairwise disjoint compact open, then evaluating at any $g \in K_i$ gives $a_i = 0$. Thus

$$C_c(\mathcal{G}_n, A) \cong \bigoplus_{K \in \mathcal{K}(\mathcal{G}_n)} A, \quad \text{free as an } A\text{-module}.$$

Proof of the UCT

Step 1: Chain-level identification.

Let $f \in C_c(\mathcal{G}_n, A)$ and write $\text{im}(f) = \{a_1, \dots, a_m\}$. Set $K_i := f^{-1}(\{a_i\})$. Then $\text{supp}(f) = \bigsqcup_{i=1}^m K_i$ with K_i clopen and $f|_{K_i} \equiv a_i$.

Extension by 0: $\chi_{K_i} \in C_c(\mathcal{G}_n, \mathbb{Z})$ and

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Define the canonical \mathbb{Z} -bilinear map

$$\Phi_{\mathcal{G}_n}: C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow C_c(\mathcal{G}_n, A), \quad \xi \otimes a \longmapsto a \cdot \xi.$$

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$\Phi_{\mathcal{G}_n}$ is surjective, and injective since $C_c(\mathcal{G}_n, \mathbb{Z})$ is free on $\{\chi_K \mid K \in \mathcal{K}(\mathcal{G}_n)\}$.

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$$C_c(\mathcal{G}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_n, A) \quad \text{for discrete } A.$$

Proof of the UCT

Step 2: Compatibility with the boundary.

For each face map $d_i: \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}$, the pushforward $(d_i)_*$ is \mathbb{Z} -linear and satisfies

$$(d_i)_*(\xi \cdot a) = ((d_i)_*\xi) \cdot a \quad \text{for } \xi \in C_c(\mathcal{G}_n, \mathbb{Z}), a \in A.$$

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Hence $\Phi_{\mathcal{G}_\bullet}$ intertwines the Moore boundary:

$$\Phi_{\mathcal{G}_{n-1}} \circ (\partial_n \otimes \text{id}_A) = \partial_n \circ \Phi_{\mathcal{G}_n}.$$

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Therefore $\Phi_{\mathcal{G}_\bullet}$ is an isomorphism of chain complexes

$$C_c(\mathcal{G}_\bullet, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong C_c(\mathcal{G}_\bullet, A).$$

Proof of the UCT

Step 3: Apply the classical algebraic UCT.

The Moore complex $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ is a chain complex of free abelian groups. Applying the classical algebraic UCT to $C_c(\mathcal{G}_\bullet, \mathbb{Z})$ and transporting across the chain isomorphism from Steps 1–2 yields, for all $n \geq 0$, a short exact sequence

$$0 \rightarrow H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \rightarrow 0.$$

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This sequence is natural in \mathcal{G} and in discrete A . In general, for non-discrete topological abelian groups A , Moore homology need not satisfy such a UCT.

Non-discrete Coefficients: What Fails

The general result.

For X locally compact, totally disconnected, Hausdorff with a basis of compact open sets and an abelian group A , consider the canonical map

$$\Phi_X: C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a\chi_u.$$

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In particular, Φ_X may fail to be surjective for non-discrete A . Moreover,

$$\Phi_X \text{ surjective} \Leftrightarrow \forall \xi \in C_c(X, A) : \xi(X) \text{ finite} \Leftrightarrow \Phi_X \text{ is an isomorphism.}$$

Non-discrete Coefficients: What Fails

The general result.

For X locally compact, totally disconnected, Hausdorff with a basis of compact open sets and an abelian group A , consider the canonical map

$$\Phi_X: C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a\chi_u.$$

Then

$$\text{im}(\Phi_X) \subseteq \{\xi \in C_c(X, A) \mid \xi(X) \text{ is finite}\}.$$

In particular, Φ_X may fail to be surjective for non-discrete A . Moreover,

$$\Phi_X \text{ surjective} \Leftrightarrow \forall \xi \in C_c(X, A) : \xi(X) \text{ finite} \Leftrightarrow \Phi_X \text{ is an isomorphism.}$$

If A is discrete, then Φ_X is an isomorphism. The converse can fail.

Non-discrete Coefficients: What Fails

Failure of the tensor comparison.

If A is non-discrete and $0 \in A$ is not isolated, then surjectivity of Φ_X can fail even for compact, totally disconnected spaces with a basis of clopen subsets. Set

$$X := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n \in \{0, 2\} \right\} \subset [0, 1],$$

$$A := (\mathbb{R}, \mathcal{O}_{\text{std}}),$$

$$\xi: X \rightarrow A, \quad x \mapsto x.$$

Then X is compact, Hausdorff, totally disconnected, and has a basis of clopen subsets. Hence $\xi \in C_c(X, A)$ and $\xi(X) = X$ is infinite. Therefore $\xi \notin \text{im}(\Phi_X)$, so Φ_X is not surjective.

Non-discrete Coefficients: What Fails

Isomorphism without discreteness.

$$A := (\mathbb{R}, \mathcal{O}_{\text{std}}), \quad (\{\bullet\}, \mathcal{O}_{\{\bullet\}} := \{\emptyset, \{\bullet\}\}).$$

Then $\{\bullet\}$ is locally compact, totally disconnected, Hausdorff and compact open.

$$C_c(\{\bullet\}, \mathbb{Z}) \cong \mathbb{Z},$$

$$C_c(\{\bullet\}, A) \cong A,$$

$$C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \cong \mathbb{Z} \otimes_{\mathbb{Z}} A \cong A.$$

Under these identifications the canonical map

$$\Phi_{\{\bullet\}} : C_c(\{\bullet\}, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(\{\bullet\}, A), \quad \chi_{\{\bullet\}} \otimes a \mapsto a \cdot \chi_{\{\bullet\}},$$

is the standard isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} A \rightarrow A$, $1 \otimes a \mapsto a$.

Moore-Mayer-Vietoris Sequence

Mayer-Vietoris vs. Moore-Mayer-Vietoris

From saturated covers to homology.

Mayer–Vietoris:

$X = U_1 \cup U_2,$
 $U_1, U_2 \subseteq X$ open.

Moore–Mayer–Vietoris:

Open cover:
 $U_1, U_2, U_1 \cap U_2.$

Compute $H_\bullet(X)$ from:

$H_\bullet(U_1), H_\bullet(U_2), H_\bullet(U_1 \cap U_2).$

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$X = U_1 \cup U_2,$
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$\mathcal{G}_0 = U_1 \cup U_2,$
 $U_1, U_2 \subseteq \mathcal{G}_0$ clopen.

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 $U_1, U_2 \subseteq \mathcal{G}_0$ clopen.

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$\forall x \in U : s(g) = x \Rightarrow r(g) \in U.$

$\mathcal{G}|_U := \{g \in \mathcal{G} \mid s(g), r(g) \in U\},$
 $\mathcal{G}|_U \hookrightarrow \mathcal{G}$ open.

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 $\mathcal{G}|_U \hookrightarrow \mathcal{G}$ open.

Compute $H_\bullet(\mathcal{G})$ from:

$H_\bullet(\mathcal{G}|_{U_1}), H_\bullet(\mathcal{G}|_{U_2}), H_\bullet(\mathcal{G}|_{U_1 \cap U_2}).$

Reductions and Moore–Mayer–Vietoris

Long Natural Moore–Mayer–Vietoris Sequence for Homology.

For $U \subseteq G_0$ define the reduction

$$G|_U := \{g \in G \mid s(g), r(g) \in U\}, \quad (G|_U)_0 = U,$$

with structure maps the restrictions of $u, m, s, r, -^{-1}$ to $G|_U$. For $i \in \{1, 2\}$ and $U_{12} := U_1 \cap U_2$ we write $G|_{U_i}$ and $G|_{U_{12}}$.

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$$\mathcal{G}|_U := \{g \in \mathcal{G} \mid s(g), r(g) \in U\}, \quad (\mathcal{G}|_U)_0 = U,$$

with structure maps the restrictions of $u, m, s, r, -^{-1}$ to $\mathcal{G}|_U$. For $i \in \{1, 2\}$ and $U_{12} := U_1 \cap U_2$ we write $\mathcal{G}|_{U_i}$ and $\mathcal{G}|_{U_{12}}$.

Moore–Mayer–Vietoris long exact homology sequence:

$$\cdots \longleftarrow H_{n-1}(\mathcal{G}|_{U_1}; A) \oplus H_{n-1}(\mathcal{G}|_{U_2}; A) \xleftarrow{H_{n-1}(\alpha_\bullet)} H_{n-1}(\mathcal{G}|_{U_1 \cap U_2}; A) \longleftarrow$$

∂_n

$$H_n(\mathcal{G}; A) \xleftarrow{H_n(\beta_\bullet)} H_n(\mathcal{G}|_{U_1}; A) \oplus H_n(\mathcal{G}|_{U_2}; A) \xleftarrow{H_n(\alpha_\bullet)} H_n(\mathcal{G}|_{U_1 \cap U_2}; A) \longleftarrow$$

∂_{n+1}

$$H_{n+1}(\mathcal{G}; A) \xleftarrow{H_{n+1}(\beta_\bullet)} H_{n+1}(\mathcal{G}|_{U_1}; A) \oplus H_{n+1}(\mathcal{G}|_{U_2}; A) \longleftarrow \cdots$$

Proof of Moore–Mayer–Vietoris

Proof idea for $H_n(\alpha_\bullet)$.

$(\iota_i)_n : (\mathcal{G}|_{U_{12}})_n \hookrightarrow (\mathcal{G}|_{U_i})_n$ is an open embedding, hence a local homeomorphism.
Therefore the functorial pushforward on Moore chains

$$((\iota_i)_n)_* : C_c((\mathcal{G}|_{U_{12}})_n, A) \rightarrow C_c((\mathcal{G}|_{U_i})_n, A)$$

is given by a finite fibre sum. Since $(\iota_i)_n$ is injective, it is extensible by zero:

$$((\iota_i)_n)_* f(x) := \begin{cases} f(x), & x \in (\mathcal{G}|_{U_{12}})_n, \\ 0, & x \in (\mathcal{G}|_{U_i})_n \setminus (\mathcal{G}|_{U_{12}})_n. \end{cases}$$

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Define the chain map

$$\begin{aligned} \alpha_n: C_c((\mathcal{G}|_{U_{12}})_n, A) &\rightarrow C_c((\mathcal{G}|_{U_1})_n, A) \oplus C_c((\mathcal{G}|_{U_2})_n, A), \\ f &\mapsto ((\iota_1)_n)_* f, -((\iota_2)_n)_* f. \end{aligned}$$

Compatibility: $\partial ((\iota_i)_n)_* = ((\iota_i)_{n-1})_* \partial$, $\partial \alpha_n = \alpha_{n-1} \partial$. Hence α_\bullet induces $H_n(\alpha_\bullet)$.

Moore–Mayer–Vietoris

Proof idea for $H_n(\beta_\bullet)$.

Let $\kappa_i: \mathcal{G}|_{U_i} \hookrightarrow \mathcal{G}$ be the inclusion of reductions.

$(\kappa_i)_n: (\mathcal{G}|_{U_i})_n \hookrightarrow \mathcal{G}_n$ is an open embedding, hence a local homeomorphism. Therefore the pushforward on Moore chains extends by zero:

$$((\kappa_i)_n)_* g(x) := \begin{cases} g(x), & x \in (\mathcal{G}|_{U_i})_n, \\ 0, & x \in \mathcal{G}_n \setminus (\mathcal{G}|_{U_i})_n. \end{cases}$$

Define

$$\begin{aligned} \beta_n: C_c((\mathcal{G}|_{U_1})_n, A) \oplus C_c((\mathcal{G}|_{U_2})_n, A) &\rightarrow C_c(\mathcal{G}_n, A), \\ (g_1, g_2) &\mapsto ((\kappa_1)_n)_* g_1 + ((\kappa_2)_n)_* g_2. \end{aligned}$$

Compatibility: $\partial ((\kappa_i)_n)_* = ((\kappa_i)_{n-1})_* \partial$, $\partial \beta_n = \beta_{n-1} \partial$. Hence β_\bullet induces $H_n(\beta_\bullet)$.

Moore–Mayer–Vietoris

Proof idea for ∂_n .

Assume a SES of Moore chain complexes

$$0 \rightarrow C_c((\mathcal{G}|_{U_{12}})_\bullet, A) \xrightarrow{\alpha_\bullet} C_c((\mathcal{G}|_{U_1})_\bullet, A) \oplus C_c((\mathcal{G}|_{U_2})_\bullet, A) \\ \xrightarrow{\beta_\bullet} C_c(\mathcal{G}_\bullet, A) \rightarrow 0.$$

Here ∂ denotes the Moore boundary.

Let $[c] \in H_n(\mathcal{G}; A)$ with $\partial c = 0$ and choose b with $\beta_n(b) = c$.

Then $\beta_{n-1}(\partial b) = \partial(\beta_n(b)) = \partial c = 0$, hence

$$\partial b \in \ker(\beta_{n-1}) = \text{im}(\alpha_{n-1}).$$

Choose $a \in C_c((\mathcal{G}|_{U_{12}})_{n-1}, A)$ with $\alpha_{n-1}(a) = \partial b$ and define

$$\partial_n([c]) := [a] \in H_{n-1}(\mathcal{G}|_{U_{12}}; A).$$

Standard homological algebra: ∂_n is well-defined, independent of choices, and yields exactness at $H_n(\mathcal{G}; A)$.

Takeaways

What you get and how to use it.

Setting: \mathcal{G} ample étale, A a discrete abelian group,

Moore chains $C_c(\mathcal{G}_n, A)$ with boundary $\partial = \sum_{i=0}^n (-1)^i (d_i)_*$.

Two structural tools:

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Two structural tools:

UCT for discrete coefficients:

$$0 \rightarrow H_n(\mathcal{G}) \otimes_{\mathbb{Z}} A \xrightarrow{\iota_n^{\mathcal{G}}} H_n(\mathcal{G}; A) \xrightarrow{\kappa_n^{\mathcal{G}}} \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathcal{G}), A) \rightarrow 0.$$

Takeaways

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Moore–Mayer–Vietoris

for a clopen saturated cover $U_1 \cup U_2 = \mathcal{G}_0$.

There is a natural long exact sequence relating

$$H_\bullet(\mathcal{G}; A), H_\bullet(\mathcal{G}|_{U_1}; A), H_\bullet(\mathcal{G}|_{U_2}; A), H_\bullet(\mathcal{G}|_{U_{12}}; A).$$

Takeaways

What you get and how to use it.

Why discreteness matters:

For non-discrete A , the canonical comparison

$$\Phi_X : C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a\chi_u,$$

need not be surjective. Tensor-level reduction in UCT can fail.

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What you get and how to use it.

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For non-discrete A , the canonical comparison

$$\Phi_X : C_c(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow C_c(X, A), \quad \chi_u \otimes a \mapsto a\chi_u,$$

need not be surjective. Tensor-level reduction in UCT can fail.

How to use in practice:

Choose a clopen saturated cover.

$U_1, U_2 \subseteq \mathcal{G}_0$ so that reductions $\mathcal{G}|_{U_1}$, $\mathcal{G}|_{U_2}$, $\mathcal{G}|_{U_{12}}$ are computable.

Compute integral homology.

$H_{\bullet}(\mathcal{G}|_{U_i})$, $H_{\bullet}(\mathcal{G}|_{U_{12}})$, then glue to $H_{\bullet}(\mathcal{G})$ via MMV.

Thank you.

Homology of SFT Groupoids

Example: Diaconu–Renault Groupoid

Computing Homology with Moore–Mayer–Vietoris + UCT.

$A \in \text{Mat}(N \times N, \mathbb{N}_0)$ with no zero row and no zero column.

$(E_A^{(1)}, E_A^{(0)}, s_{E_A^{(1)}}, r_{E_A^{(1)}})$ a finite directed graph whose adjacency matrix is A .

The infinite path space is given by Sims 2021, 2.5:

$E_A^\infty = \left\{ (e_n)_{n \geq 1} \in (E_A^1)^\mathbb{N} \mid r_{E_A^{(1)}}(e_n) = s_{E_A^{(1)}}(e_{n+1}) \text{ for all } n \geq 1 \right\}$ with

$\sigma: E_A^\infty \rightarrow E_A^\infty, (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, e_3, \dots)$.

$(\mathcal{G}_A)_0 = E_A^\infty$.

$(\mathcal{G}_A)_1 = \{(x, n, y) \in E_A^\infty \times \mathbb{Z} \times E_A^\infty \mid \exists k, \ell \in \mathbb{N}_0 : n = k - \ell, \sigma^k(x) = \sigma^\ell(y)\}$.

$s(x, n, y) = y, r(x, n, y) = x, 1_x = (x, 0, x),$

$(x, n, y)^{-1} = (y, -n, x), (x, n, y) \cdot (y, m, z) = (x, n+m, z) \text{ if } s(x, n, y) = r(y, m, z).$

Example: Diaconu-Renault Groupoid

Homology of SFT-Groupoids is well known.

$\mathbf{1} - A^T$ acts on \mathbb{Z}^N and we have by Matui 2012, 4.14:

$$H_0(\mathcal{G}_A) \cong \text{coker}(\mathbf{1} - A^T),$$

$$H_1(\mathcal{G}_A) \cong \text{ker}(\mathbf{1} - A^T),$$

$$H_n(\mathcal{G}_A) = 0 \text{ for } n \geq 2.$$

Consider now:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = (3).$$

We compute the integral homology of \mathcal{G}_A , \mathcal{G}_B , and \mathcal{G}_C .

Example: Diaconu-Renault Groupoid

Computing Homology for \mathcal{G}_A .

For A we have

$$\mathbf{1} - A^T = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \det(\mathbf{1} - A^T) = -2.$$

Hence $\mathbf{1} - A^T$ has full rank over \mathbb{Z} and $\ker(\mathbf{1} - A^T) = 0$.

Moreover, the Smith normal form is

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{C_2 \leftarrow C_2 - C_1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

so $\text{coker}(\mathbf{1} - A^T) \cong \mathbb{Z}/2\mathbb{Z}$.

We get $H_0(\mathcal{G}_A) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_A) = 0$, $H_n(\mathcal{G}_A) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

Computing Homology for \mathcal{G}_B .

For B we have

$$\mathbf{1} - B^T = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

$$(\mathbf{1} - B^T)(x, y)^T = 0 \Leftrightarrow -x - y = 0, \text{ hence } \ker(\mathbf{1} - B^T) \cong \mathbb{Z}.$$

The image is generated by $(1, 1)$, which is primitive in \mathbb{Z}^2 , so

$$\text{coker}(\mathbf{1} - B^T) \cong \mathbb{Z}^2 / \langle (1, 1) \rangle_{\mathbb{Z}} \cong \mathbb{Z}.$$

Thus, we have for homology $H_0(\mathcal{G}_B) \cong \mathbb{Z}$, $H_1(\mathcal{G}_B) \cong \mathbb{Z}$, $H_n(\mathcal{G}_B) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

Computing Homology for \mathcal{G}_C .

For C we have $\mathbf{1} - C^T = -2$, so

$$\ker(\mathbf{1} - C^T) = 0 \quad \text{and} \quad \text{coker}(\mathbf{1} - C^T) \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence $H_0(\mathcal{G}_C) \cong \mathbb{Z}/2\mathbb{Z}$, $H_1(\mathcal{G}_C) = 0$, $H_n(\mathcal{G}_C) = 0$ for $n \geq 2$.

Example: Diaconu-Renault Groupoid

The disjoint union groupoid.

We have $\mathcal{G} = \mathcal{G}_A \sqcup \mathcal{G}_B \sqcup \mathcal{G}_C$, the disjoint union groupoid.

The nerve decomposes levelwise to $\mathcal{G}_n = (\mathcal{G}_A)_n \sqcup (\mathcal{G}_B)_n \sqcup (\mathcal{G}_C)_n$.

The Moore chain complex splits as a direct sum, thus

$$H_n(\mathcal{G}) \cong H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \text{ for } n \geq 0.$$

In particular

$$H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad H_1(\mathcal{G}) \cong \mathbb{Z}, \quad H_n(\mathcal{G}) = 0 \text{ for } n \geq 2.$$

Define $U_1 := (\mathcal{G}_A)_0 \sqcup (\mathcal{G}_B)_0$, $U_2 := (\mathcal{G}_B)_0 \sqcup (\mathcal{G}_C)_0$.

Example: Diaconu-Renault Groupoid

The reduction groupoids.

The reductions are

$$\mathcal{G}|_{U_1} = \mathcal{G}_A \sqcup \mathcal{G}_B, \quad \mathcal{G}|_{U_2} = \mathcal{G}_B \sqcup \mathcal{G}_C, \quad \mathcal{G}|_{U_1 \cap U_2} = \mathcal{G}_B.$$

This yields the long exact sequence

$$\cdots \rightarrow H_n(\mathcal{G}_B) \xrightarrow{\alpha_n} H_n(\mathcal{G}_A \sqcup \mathcal{G}_B) \oplus H_n(\mathcal{G}_B \sqcup \mathcal{G}_C) \xrightarrow{\beta_n} \\ \xrightarrow{\beta_n} H_n(\mathcal{G}) \xrightarrow{\partial_n} H_{n-1}(\mathcal{G}_B) \rightarrow \cdots .$$

Example: Diaconu-Renault Groupoid

Explicit formulas for α_n , β_n , δ_n

$$\alpha_n : H_n(\mathcal{G}_B) \rightarrow H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$

$$[b] \mapsto ([0], [b], [-b], [0]),$$

$$\beta_n : H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C) \rightarrow H_n(\mathcal{G}_A) \oplus H_n(\mathcal{G}_B) \oplus H_n(\mathcal{G}_C),$$

$$([a], [b_1], [b_2], [c]) \mapsto ([a], [b_1 + b_2], [c]).$$

δ_n vanishes in this example by exactness $\delta_n = 0 : H_n(\mathcal{G}) \rightarrow H_{n-1}(\mathcal{G}_B)$, since β_n is surjective and $\ker(\beta_n) = \{([0], [b], [-b], [0]) \mid b \in H_n(\mathcal{G}_B)\} = \text{im}(\alpha_n)$.

Finite coefficients via UCT

Final homology groups for $\mathbb{Z}/p\mathbb{Z}$.

Fix a prime p . Assume $H_2(\mathcal{G}) = 0$, $H_1(\mathcal{G}) \cong \mathbb{Z}$, $H_0(\mathcal{G}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

Vanishing in higher degrees: $H_n(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 2$.

Degree 0:

$$H_0(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong H_0(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

Degree 1 via UCT:

$$0 \rightarrow H_1(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \rightarrow H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_0(\mathcal{G}), \mathbb{Z}/p\mathbb{Z}) \rightarrow 0,$$

hence

$$H_1(\mathcal{G}; \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{for } p \text{ odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3, & \text{for } p = 2. \end{cases}$$

$$\mathrm{C}_c(\mathcal{G},\mathbb{Z})$$

$C_c(\mathcal{G}, \mathbb{Z})$

The group is generated by χ_U for $U \in \text{Bis}(\mathcal{G})$.

Lemma: Let \mathcal{G} be ample, then every $f \in C_c(\mathcal{G}, \mathbb{Z})$ is locally constant and has compact open support for discrete \mathbb{Z} . $C_c(\mathcal{G}, \mathbb{Z})$ is generated as abelian group by $\chi_U \in \text{Bis}(\mathcal{G})$.

Proof:

$C_c(\mathcal{G}, \mathbb{Z})$

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Proof:

Local constancy: $g \in \mathcal{G}$, $n = f(g)$, $\{n\} \in \mathbb{Z}$ open, $f^{-1}(\{n\}) \ni g$ open neighborhood, $f(f^{-1}(\{n\})) = \{n\}$ is constant.

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Compact open support: $f^{-1}(\{n\})$ clopen,

$$U := \bigcup_{n \neq 0} f^{-1}(\{n\}) = \{g \in \mathcal{G} \mid f(g) \neq 0\}, \text{supp}(f) = \overline{U} = U.$$

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Lemma: Let \mathcal{G} be ample, then every $f \in C_c(\mathcal{G}, \mathbb{Z})$ is locally constant and has compact open support for discrete \mathbb{Z} . $C_c(\mathcal{G}, \mathbb{Z})$ is generated as abelian group by $\chi_U \in \text{Bis}(\mathcal{G})$.

Proof:

Local constancy: $g \in \mathcal{G}$, $n = f(g)$, $\{n\} \in \mathbb{Z}$ open, $f^{-1}(\{n\}) \ni g$ open neighborhood, $f(f^{-1}(\{n\})) = \{n\}$ is constant.

Compact open support: $f^{-1}(\{n\})$ clopen,
 $U := \bigcup_{n \neq 0} f^{-1}(\{n\}) = \{g \in \mathcal{G} \mid f(g) \neq 0\}$, $\text{supp}(f) = \overline{U} = U$.

Generation by $\chi_U \in \text{Bis}(\mathcal{G})$: $K := \text{supp}(f)$ compact, f locally constant, $f(K) \subset \mathbb{Z}$ finite, $n_1, \dots, n_m \in \text{im}(f)$ non zero with $n_i \neq n_j$ for $i, j \in \{1, \dots, n\}$, $S_j := f^{-1}(\{n_j\})$ for $1 \leq j \leq n$.

$C_c(\mathcal{G}, \mathbb{Z})$

The group is generated by χ_U for $U \in \text{Bis}(\mathcal{G})$.

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Proof:

Claim: Then each S_j is clopen, contained in K , the sets S_1, \dots, S_m are pairwise disjoint and

$$K = \bigsqcup_{j=1}^m S_j, \quad f(g) = \sum_{j=1}^m n_j \chi_{S_j}(g). \tag{1}$$

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Fix j . S_j is compact open in \mathcal{G} , since \mathcal{G} is ample, $\text{Bis}(\mathcal{G})$ is a basis of compact open bisections, hence for every $g \in S_j$ there exists $U_g \in \text{Bis}(\mathcal{G})$ with $g \in U_g \subseteq S_j$. By compactness of S_j choose $U_{j,1}, \dots, U_{j,\ell_j} \in \text{Bis}(\mathcal{G})$ with $S_j = \bigcup_{k=1}^{\ell_j} U_{j,k}$.

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Proof:

Refine the cover: For $1 \leq k, r \leq \ell_j$ define $W_{j,k} := U_{j,k} \setminus \bigcup_{r < k} U_{j,r}$. Each $W_{j,r}$ is compact open, as difference of compact open sets. Each $W_{j,k}$ is compact open, being the difference of compact open sets. Moreover $W_{j,j} \subset U_{j,k}$, hence $W_{j,k}$ is a bisection. The sets $W_{j,1}, \dots, W_{j,\ell_j}$ are pairwise disjoint and satisfy $S_j = \bigsqcup_{k=1}^{\ell_j} W_{j,k}$, so $\chi_{S_j}(g) = \sum_{k=1}^{\ell_j} \chi_{W_{j,k}}(g)$. Therefore

$$f(g) = \sum_{j=1}^m n_j \chi_{S_j}(g) = \sum_{j=1}^m \sum_{k=1}^{\ell_j} n_j \chi_{W_{j,k}}(g). \quad (2)$$

Research Questions

Research Questions

Relations to $B\mathcal{G}$, Haar-measures, étale case.

Haar–Moore chains:

Let \mathcal{G} be a locally compact Hausdorff étale groupoid with a Haar system λ and let A be a Hausdorff topological abelian group. Construct a Haar based chain complex $C_{\bullet}^{\lambda}(\mathcal{G}; A)$ by integrable A valued functions on \mathcal{G}_{\bullet} and define face pushforwards by fiber integration. Determine minimal hypotheses guaranteeing $\partial^2 = 0$ and identify when $H_{\bullet}^{\lambda}(\mathcal{G}; A) \cong H_{\bullet}(\mathcal{G}; A)$ for discrete A .

Research Questions

Relations to $B\mathcal{G}$, Haar-measures, étale case.

Mayer–Vietoris beyond discreteness:

For an open saturated cover $\mathcal{G}_0 = U \cup V$, find conditions on λ and A such that extension by zero yields a short exact sequence of complexes

$$0 \rightarrow C_{\bullet}^{\lambda}(\mathcal{G}|_{U \cap V}; A) \rightarrow C_{\bullet}^{\lambda}(\mathcal{G}|_U; A) \oplus C_{\bullet}^{\lambda}(\mathcal{G}|_V; A) \rightarrow C_{\bullet}^{\lambda}(\mathcal{G}; A) \rightarrow 0$$

and hence a long exact Mayer–Vietoris sequence.

Research Questions

Relations to $B\mathcal{G}$, Haar-measures, étale case.

Comparison with classifying space homology:

Find necessary and sufficient hypotheses on \mathcal{G} and A giving a natural isomorphism

$$H_n(\mathcal{G}; A) \cong H_n^{\text{sing}}(B\mathcal{G}; A),$$

and determine the correct geometric realization, that restores comparison in cases where $B\mathcal{G}$ fails. In that sense, answer when there is a natural transformation from Moore–homology to $H_{\bullet}^{\text{sing}}(B\mathcal{G}; A)$.

$$H_{\bullet}(\mathcal{G};\mathbb{Z}) \neq H_{\bullet}(B\mathcal{G};\mathbb{Z})$$

$$H_{\bullet}(\mathcal{G}; \mathbb{Z}) \neq H_{\bullet}(B\mathcal{G}; \mathbb{Z})$$

Unit groupoid of the Cantor set X .

Let X be the Cantor set and $\mathcal{G} := (X \rightrightarrows X)$ the unit groupoid. For $n \geq 1$ one has

$$\mathcal{G}_n = \{(x_1, \dots, x_n) \in X^n \mid x_1 = \dots = x_n\} = \{(x, \dots, x) \mid x \in X\}.$$

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Define inverse homeomorphisms

$$\iota_n: X \rightarrow \mathcal{G}_n, \quad \iota_n(x) = (x, \dots, x), \quad \pi_n: \mathcal{G}_n \rightarrow X, \quad \pi_n(x, \dots, x) = x,$$

so that $\pi_n \iota_n = \text{id}_X$ and $\iota_n \pi_n = \text{id}_{\mathcal{G}_n}$.

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Since X is compact and \mathbb{Z} is discrete, $C_n(\mathcal{G}; \mathbb{Z}) = C_c(\mathcal{G}_n, \mathbb{Z}) = C(\mathcal{G}_n, \mathbb{Z})$.

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$$(\iota_n)_*: C(\mathcal{G}_n, \mathbb{Z}) \rightarrow C(X, \mathbb{Z}), \quad (\pi_n)_*: C(X, \mathbb{Z}) \rightarrow C(\mathcal{G}_n, \mathbb{Z}),$$

$$(\iota_n)_*(\pi_n)_* = (\pi_n \iota_n)_* = (\text{id}_X)_* = \text{id}_{C(X, \mathbb{Z})},$$

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Unit groupoid of the Cantor set X .

For $f \in C(X, \mathbb{Z})$ and $x \in X$ one computes

$$\begin{aligned} ((\iota_{n-1})_*(d_i)_*(\pi_n)_* f)(x) &= \sum_{u \in \iota_{n-1}^{-1}(x)} ((d_i)_*(\pi_n)_* f)(u) \\ &= \sum_{\gamma \in d_i^{-1}(\iota_{n-1}(x))} ((\pi_n)_* f)(\gamma) \\ &= \sum_{y \in \pi_n^{-1}(d_i(\iota_n(x)))} f(y) = f(x), \end{aligned}$$

where we used $d_i \iota_n = \iota_{n-1}$ and $\pi_n \iota_n = \text{id}_X$. Hence

$$(\iota_{n-1})_*(d_i)_*(\pi_n)_* = \text{id}_{C(X, \mathbb{Z})}.$$

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$(\iota_{n-1})_*(d_i)_*(\pi_n)_* = \text{id}_{C(X, \mathbb{Z})}$. Using $(\pi_{n-1})_*(\iota_{n-1})_* = \text{id}_{C(\mathcal{G}_{n-1}, \mathbb{Z})}$ and

$(\iota_n)_*(\pi_n)_* = \text{id}_{C(X, \mathbb{Z})}$ this implies

$$(d_i)_* = (\pi_{n-1})_*(\iota_{n-1})_*(d_i)_*(\pi_n)_*(\iota_n)_* = (\pi_{n-1})_* \text{id}_{C(X, \mathbb{Z})} (\iota_n)_* = \text{id}_{C(\mathcal{G}_n, \mathbb{Z})}.$$

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Unit groupoid of the Cantor set X .

Therefore

$$\partial_n = \sum_{i=0}^n (-1)^i (d_i)_* = \left(\sum_{i=0}^n (-1)^i \right) id_{C(X, \mathbb{Z})} = \begin{cases} 0, & n \text{ odd}, \\ id, & n \text{ even}, \end{cases}$$

and the Moore complex is

$$\dots \xrightarrow{id} C(X, \mathbb{Z}) \xrightarrow{0} C(X, \mathbb{Z}) \xrightarrow{id} C(X, \mathbb{Z}) \xrightarrow{0} C(X, \mathbb{Z}).$$

Consequently

$$H_0(\mathcal{G}; \mathbb{Z}) = C(X, \mathbb{Z}), \quad H_n(\mathcal{G}; \mathbb{Z}) = 0 \text{ for } n \geq 1.$$

$$H_{\bullet}(\mathcal{G}; \mathbb{Z}) \neq H_{\bullet}(B\mathcal{G}; \mathbb{Z})$$

Cardinality of $H_0(\mathcal{G}; \mathbb{Z})$.

$$X = \{0, 1\}^{\mathbb{N}}, \quad [\varepsilon] = \{x \in X \mid x_1 = \varepsilon_1, \dots, x_n = \varepsilon_n\}, \quad \varepsilon \in \{0, 1\}^n.$$

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$$|C(X, \mathbb{Z})| \leq \sum_{k \in \mathbb{N}} |\mathbb{Z}^k| |\text{Clop}(X)^k| \leq \sum_{k \in \mathbb{N}} \aleph_0 \cdot \aleph_0 = \aleph_0.$$

$$|H_0(\mathcal{G}; \mathbb{Z})| \leq \aleph_0.$$

$$H_*(\mathcal{G}; \mathbb{Z}) \neq H_*(B\mathcal{G}; \mathbb{Z})$$

Cardinality of $H_0^{\text{sing}}(B\mathcal{G}; \mathbb{Z})$.

$\mathcal{G}_n = X$ for all n , $d_i = s_j = \text{id}_X$, so the realization is

$$B\mathcal{G} = |\mathcal{G}_\bullet| \cong \left(\coprod_{n \geq 0} X \times \Delta^n \right) / \sim \cong X \times |\Delta^\bullet|.$$

$|\Delta^\bullet|$ is contractible, $X \times |\Delta^\bullet| \rightarrow X$ is a homotopy equivalence, hence $B\mathcal{G} \simeq X$.

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$$\delta \text{ injective} \Rightarrow \left| H_0^{\text{sing}}(B\mathcal{G}; \mathbb{Z}) \right| = \left| \bigoplus_{x \in X} \mathbb{Z} \right| \geq |X| = 2^{\aleph_0}.$$

$\mathcal{B}\mathcal{G} \simeq X$ for the unit groupoid

We have

$$|\mathcal{G}_\bullet| = \left(\coprod_{n \geq 0} \mathcal{G}_n \times \Delta^n \right) / \sim = \left(\coprod_{n \geq 0} X \times \Delta^n \right) / \sim,$$

where \sim is generated by

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Therefore \sim never changes the X coordinate and only imposes the usual simplicial identifications on Δ^\bullet , so for

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$$\Phi: |\mathcal{G}_\bullet| \rightarrow X \times |\Delta^\bullet|, \quad \Phi([x, t]) = (x, [t]), \quad \Psi: X \times |\Delta^\bullet| \rightarrow |\mathcal{G}_\bullet|, \quad \Psi(x, [t]) = [x, t].$$

Φ and Ψ are well defined and continuous, and $\Phi\Psi = \text{id}$, $\Psi\Phi = \text{id} \Rightarrow |\mathcal{G}_\bullet| \cong X \times |\Delta^\bullet|$.

A contraction of $|\Delta^\bullet|$

Fix the basepoint $*$:= $[(1)] \in |\Delta^\bullet|$.

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Define for each n the affine map

$$h_n : \Delta^n \times [0, 1] \rightarrow \Delta^{n+1}, \quad h_n((t_0, \dots, t_n), s) = ((1-s)t_0, \dots, (1-s)t_n, s).$$

Let \tilde{H} be the induced map on the disjoint union,

$$\tilde{H} : \left(\coprod_{n \geq 0} \Delta^n \right) \times [0, 1] \rightarrow \coprod_{n \geq 0} \Delta^n, \quad \tilde{H}((t, n), s) := (h_n(t, s), n + 1).$$

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$$\tilde{H} : \left(\coprod_{n \geq 0} \Delta^n \right) \times [0, 1] \rightarrow \coprod_{n \geq 0} \Delta^n, \quad \tilde{H}((t, n), s) \coloneqq (h_n(t, s), n + 1).$$

For every simplicial $\theta : \Delta^m \rightarrow \Delta^n$ define $\theta' : \Delta^{m+1} \rightarrow \Delta^{n+1}$ by

$$\theta'(u_0, \dots, u_m, u_{m+1}) = (\theta(u_0, \dots, u_m), u_{m+1}).$$

Then θ' is simplicial and for all $t \in \Delta^m$, $s \in [0, 1]$,

$$\tilde{H}(\theta(t), s) = \theta'(\tilde{H}(t, s)) \Rightarrow \tilde{H}(\theta(t), s) \approx \tilde{H}(t, s).$$

A contraction of $|\Delta^\bullet|$

Hence \tilde{H} descends to a continuous homotopy

$$H: |\Delta^\bullet| \times [0, 1] \rightarrow |\Delta^\bullet|, \quad H([t], s) := [h_n(t, s)] \text{ for } t \in \Delta^n,$$

$$H([t], 0) = [t], \quad H([t], 1) = [(0, \dots, 0, 1)].$$

Choose $\theta: \Delta^0 \rightarrow \Delta^{n+1}$ with $\theta(1) = (0, \dots, 0, 1)$. Then $\theta(1) \approx (1)$, hence

$$H([t], 1) = [(0, \dots, 0, 1)] = [(1)] = *, \quad H([t], 0) = [t],$$

so H contracts $|\Delta^\bullet|$ to $*$.

Let $p: X \times |\Delta^\bullet| \rightarrow X$ be the projection and $s: X \rightarrow X \times |\Delta^\bullet|$ be $s(x) = (x, *)$. Then $p \circ s = \text{id}_X$ and the homotopy

$$K: (X \times |\Delta^\bullet|) \times [0, 1] \rightarrow X \times |\Delta^\bullet|, \quad K((x, u), t) := (x, H(u, t))$$

satisfies $K(-, 0) = \text{id}_{X \times |\Delta^\bullet|}$ and $K(-, 1) = s \circ p$. Hence $s \circ p \simeq \text{id}_{X \times |\Delta^\bullet|}$.

Thus p is a homotopy equivalence and $\mathcal{BG} \simeq X$.

References I

- Armstrong, B., N. Brownlowe, and A. Sims (2021). *Simplicity of twisted C*-algebras of Deaconu–Renault groupoids*. DOI: [10.48550/arXiv.2109.02583](https://doi.org/10.48550/arXiv.2109.02583). arXiv: [2109.02583](https://arxiv.org/abs/2109.02583) [math.OA].
- Crainic, M. and I. Moerdijk (1999). *A homology theory for étale groupoids*. DOI: [10.48550/arXiv.math/9905011](https://doi.org/10.48550/arXiv.math/9905011). arXiv: [math/9905011](https://arxiv.org/abs/math/9905011) [math.KT].
- Crainic, M. and I. Moerdijk (2000). “A homology theory for étale groupoids”. In: *Journal für die Reine und Angewandte Mathematik* 2000.521, pp. 25–46. DOI: [10.1515/crll.2000.029](https://doi.org/10.1515/crll.2000.029).
- Matui, H. (2012). “Homology and topological full groups of étale groupoids on totally disconnected spaces”. In: *Proceedings of the London Mathematical Society* 104.1, pp. 27–56. DOI: [10.1112/plms/pdr058](https://doi.org/10.1112/plms/pdr058).
- Matui, H. (2022). *Long exact sequences of homology groups of étale groupoids*. DOI: [10.48550/arXiv.2111.04013](https://doi.org/10.48550/arXiv.2111.04013). arXiv: [2111.04013](https://arxiv.org/abs/2111.04013) [math.DS].

References II

- Matui, H. and T. Mori (2024). *Cup and cap products for cohomology and homology groups of ample groupoids*. DOI: [10.48550/arXiv.2411.14906](https://doi.org/10.48550/arXiv.2411.14906). arXiv: [2411.14906](https://arxiv.org/abs/2411.14906) [math.OA].
- Melodia, L. (2026). $H_0(C_c(\mathcal{G}_\bullet, \mathbb{Z})) \neq H_0^{\text{sing}}(BG; \mathbb{Z})$ for the Cantor unit groupoid. DOI: [10.48550/arXiv.2602.13375](https://doi.org/10.48550/arXiv.2602.13375). arXiv: [2602.13375](https://arxiv.org/abs/2602.13375) [math.AT].
- Sims, A. (2017). *Étale groupoids and their C^* -algebras*. DOI: [10.48550/arXiv.1710.10897](https://doi.org/10.48550/arXiv.1710.10897). arXiv: [1710.10897](https://arxiv.org/abs/1710.10897) [math.OA].
- Sims, A. (2018). *Hausdorff étale groupoids and their C^* -algebras*. DOI: [10.48550/arXiv.1710.10897](https://doi.org/10.48550/arXiv.1710.10897). arXiv: [1710.10897](https://arxiv.org/abs/1710.10897) [math.OA].