Math 607 - Real Variables I
Definitions, Theorems and Propositions

Kari Eifler

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Contents

The propositions and theorems marked with a ⋆ indicate that the proof was important and relatively short, and thus should be learned for exams. Definitions are shown in **blue** while theorems are shown in **purple**.

1 Set Theory

We skip the first day of class, since it likely won't show up on any exam.

1 (relation) A relation from X to Y is a subset R of $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

If $(x, y) \in R$ we write xRy

2 (countable) A is countable provided $A = \emptyset$ or there exists a function $f : \mathbb{N} \to A$ which is onto.

Remark: If A is finite, then A is countable. Here, we call A finite provided $A = \emptyset$ or there exists some $n \in \mathbb{N}$ and a function $f : \{1, 2, \ldots, n\} \to A$ which is surjective.

Lemma: If A is countable, then there exists a function $f : A \to \mathbb{N}$ which is injective.

Theorem: Suppose A is a non-empty, countable set. Then either A is finite (and there exists a unique $n \in \mathbb{N}$ and some function $f : \{1, \ldots, n\} \to A$ which is a bijection) or there exists a bijection between $\mathbb{N}$ and A.

3 (partial/linear order) A partial order on X is a relation $\leq$ on $X \times X$ having the properties:

1. $x \leq x$ for all $x \in X$ (reflexive)
2. $x \leq y$ and $y \leq x$ implies $x = y$
3. $x \leq y$ and $y \leq z$ imples $x \leq z$ (transitivity)

A partial order is called a linear order if

4. $\forall x, y \in X$ either $x \leq y$ or $y \leq x$

ex. on $\mathbb{R}^2$, we say $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. This is a partial order (not a linear order).

ex. on $\mathbb{R}^2$, we say $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 < y_1$ or if $x_1 = y_1$ and $x_2 \leq y_2$ (called the dictionary order). This is a linear order.

4 (well-ordered) We say B is well-ordered if for all $A \subseteq B$ and $A \neq \emptyset$, then A has a smallest element.

Theorem: $\mathbb{N}$ is well-ordered.

If $(X, \leq)$ is a partially ordered set, then $\leq$ is called a well-order if every non-empty subset of X has a $\leq$ smallest element.

ex. The set $\{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} \mid n \in \mathbb{N}\}$ is well-ordered under the natural order.

Properties: if $(X, \leq)$ is a well-ordered set then

- every subset of W is well-ordered by $\leq$
- if x is not the largest element of W then $x^+ = \min\{y \in W : y > x\}$ is defined (called the immediate successor of x)
- an immediate predessessor does not necessarily exist

5 (initial segment) If $(X, \leq)$ is a well ordered set, for $x \in X$, we define $I(x) = \{y \in W \mid y < x\}$ to be the initial segment at x. Then $x = \min(W \backslash I(x))$.

6 (Principle of Transfinite Induction) Suppose $(W, \leq)$ is well ordered and $A \subseteq \mathbb{N}$. Assume for every $x \in W$, if $I(x) \subseteq A$ then $x \in A$.

Then $A = W$.

Yo dummkopf, make sure you look at examples of this

7 (Theorem) Suppose $(W, \leq)$ is well ordered set and $A \subseteq W$ is some subset. Then $\cup_{x \in A} I(x)$ is equal to W or is an initial segment.

8 (order isomorphic) An order isomorphism between two well ordered sets $(W_1, \leq_1)$ and $(W_2, \leq_2)$ is some $f : W_1 \to W_2$ which is one-to-one and surjective and $f(x) \leq f(y)$ if and only if $x \leq y$.

ex. the even natural numbers are order isomorphic to $\mathbb{N}$.

Theorem: Suppose $(W_1, \leq_1)$ and $(W_2, \leq_2)$ are well ordered. Then one of the following holds:

1. W_1 is order isomorphic to W_2
2. W_1 is order isomorphic to an initial segment of W_2
3. W_2 is order isomorphic to an initial segment of W_1

Proof: We do this by defining $f : W_1 \to W_2 \cup \{\infty\}$ via

$$f(x) = \begin{cases} \min(W_2 \backslash f[I(x)] & \text{if the set is non-empty} \\ \infty & \text{if the set is empty} \end{cases}$$

We may confirm that if $f(x) \in W_2$ then $I_2(f(x)) = f(I_1(x))$.

Corollary: If A and B are two sets, then either

1. there exists some $f : A \to B$ one-to-one
2. there exists some $g : B \to A$ one-to-one

Proposition: Suppose $(W, \leq)$ is well ordered. Then if $X \subseteq W$, then either X is order isomorphic to W or to an initial segment of W. Moreover, W is not order isomorphic to an initial segment of W.

9 (transfinite recursion) You can define a function f from a well ordered set W by uniquely specifying the value $f(x)$ in terms of $f|_{I(x)}$. This is the definition of transfinite recursion.

10 (Well-ordering Theorem) Given any set A, there is a well-order on A

11 (Axiom of Choice) If $\{X_t \mid t \in I\}$ is a family of non-empty sets then $\Pi_{t \in I} X_t \neq 0$ where $\Pi_{t\in I} X_t = \{f : I \to \cup_{t \in I} X_t \mid \forall t \in I, f(A) \in X_t\}$.

2 Cardinality

12 (cardinality, $\leq$, $\geq$) We say $\text{card}(A) \leq \text{card}(B)$ if $\exists$ injective function $f : A \to B$.

We say $\text{card}(B) \geq \text{card}(A)$ if $A = \emptyset$ or if there exists a surjective function $g : B \to A$.

We say $\text{card}(A) = \text{card}(B)$ if there exists a bijection $f : A \to B$.

Theorem: $\text{card}(A) \leq \text{card}(B) \Leftrightarrow \text{card}(B) \geq \text{card}(A)$

Corollary: If A is infinite, $\text{card}(\mathbb{N}) \leq \text{card}(A)$

Notation: Write $|A| = \text{card}(A)$.

13 (Cantor-Schrïder-Bernstein) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A)$ then $\text{card}(A) = \text{card}(B)$.

14 (card($\mathcal{P}(A)$)) For any set A, $\text{card}(A) < \text{card}(\mathcal{P}(A))$ where $\mathcal{P}(A)$ is the power set of A.

15 (cardinal arithmetic) We write $\text{card}(A) + \text{card}(B) = \text{card}(C)$ if $\text{card}(C) = \text{card}(A \sqcup B)$ where $\sqcup$ is the disjoint union.

That is, for A', B' with $A' \cap B' = \emptyset$ and $\text{card}(A) = \text{card}(A')$, $\text{card}(B) = \text{card}(B')$ then define $\text{card}(A) + \text{card}(B) = \text{card}(A' \cup B')$.

Theorem: If A is infinite, then $|A| + |A| = |A|$. If A is infinite and B is arbitrary, then $|A| + |B| = \max\{|A|, |B|\}$.

16 (Zorn's Lemma) Assume $(X, \leq)$ is a partially ordered set. Assume every limiting order subset (i.e. chain) of X has an upper bound. Then X has a maximal element.

17 (Hausdorff Maximal Principle) Let $(X, \leq)$ be a partially ordered set. Then there exists a maximal chain in X

i.e. if $Y \subseteq X$ such that $(Y, \leq)$ is linearly ordered and if $Z \subseteq X$ with Z linearly ordered and $Z \supseteq Y$ then $Z = Y$.

18 (important equivalence) TFAE:

1. well-ordering theorem
2. Axiom of Choice
3. Zorn's Lemma
4. Hausdorff Maximal Principle

Note: these proofs are obnoxiously long. Probably not worth looking into. Unless you'd rather be studying than hanging out by the pool.

19 ($A^{|B|}$) We let $2^{|A|} = \text{card}\{f : A \to \{0,1\}\} = \text{card}(\mathcal{P}(A))$.

More generally, $A^{|B|} = \text{card}\{f : B \to A\}$

Theorem: $2^{|\mathbb{N}|} = |\mathbb{N}|^{|\mathbb{N}|} = |\mathbb{R}|$

3 Measures

20 (algebra) We define $\mathcal{A} \subseteq \mathcal{P}(X)$ to be an algebra if

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies $A^C = X \backslash A \in \mathcal{A}$
3. $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$.

A σ-algebra is an algebra $\mathcal{A}$ such that if $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}$ then $\cup_n A_n \in \mathcal{A}$.

ex. finite unions of the form $(a, b] \cap \mathbb{R}$ for $-\infty \leq a < b \leq \infty$ is an algebra (but not a σ-algebra)

ex. If X is an infinite set then $\mathcal{A} = \{A \subseteq X \mid A \text{ is finite or } A^c \text{ is finite}\}$ is an algebra (but not a σ-algebra)

21 (measure) If $\mathcal{A}$ is an algebra and $\mu : \mathcal{A} \to [0, \infty]$ satisfies

1. $\mu(\emptyset) = 0$
2. If $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$

then μ is called a finitely additive measure on $\mathcal{A}$ (for obvious reasons).

If $\mu : \mathcal{A} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$ and is countably additive (i.e. for pairswise disjoint $\{A_n\}$ then $\mu(\cup A_n) = \sum \mu(A_n)$) then we call μ a premeasure.

A measure is a premeasure on a σ-algebra. That is, if $\mathcal{A}$ is a σ-algebra, then $\mu : \mathcal{A} \to [0, \infty]$ is a measure if

1. $\mu(\emptyset) = 0$
2. If $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_n A_n \in \mathcal{A}$ then $\mu(\cup A_n) = \sum_n \mu(A_n)$

ex. We may define the counting measure to be

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

ex. the dirac measure are those where we fix $x_0 \in X$ and for $A \subseteq X$ set

$$\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

We call a premeasure (i.e. countably additive function on an algebra)

- finite if $\mu(X) < \infty$
- a probability if $\mu(X) = 1$
- σ-finite if there exists $(A_n)_{n \in \mathbb{N}} \subseteq A$ such that $X = \cup A_n$ and $\mu(A_n) < \infty$ for all n
- simi-finite if for all $A \subseteq X$ such that $\mu(A) \neq 0$, there exists some $B \subseteq A$ with $0 < \mu(B) < \infty$

22 (Disjointification lemma) Given an algebra $\mathcal{A}$ and $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$, there exists $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ such that

1. for all n, $B_n \subseteq A_n$
2. (B_n) is pairwise disjoint
3. for all N, $\cup_{n=1}^N A_n = \cup_{n=1}^N B_n$

23 ($\mathcal{A}(\mathcal{E}), \mathcal{M}(\mathcal{E})$) Let $\mathcal{E}$ be the collection of all subsets of X. Let

$\mathcal{A}(\mathcal{E}) = \cap\{\mathcal{A} \mid \mathcal{E} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{A}$ is an algebra$\}$

$\mathcal{M}(\mathcal{E}) = \cap\{\mathcal{M} \mid \mathcal{E} \subseteq \mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{M}$ is a σ-algebra$\}$

Proposition: $\mathcal{A}(\mathcal{E})$ is an algebra and $\mathcal{M}(\mathcal{E})$ is a σ-algebra.

Let (X, d) be a metric space. We set

$\mathcal{B}_X$ to be the Borel subsets of X to be $\mathcal{M}(\{\text{open subsets of } X\})$.

$G_\delta(x) = \{$ countable intersections of open sets $\}$

$F_\sigma(x) = \{$ countable union of closed sets $\}$

Proposition: Let $\mathcal{B}_\mathbb{R} = \mathcal{M}(\text{open subsets of } \mathbb{R})$ is also equal to

- $\mathcal{M}(\text{open intervals})$
- $\mathcal{M}(\text{bounded open intervals})$
- $\mathcal{M}(\text{closed bounded sets})$
- $\mathcal{M}(\text{bounded intervals } [a, b))$
- $\mathcal{M}(\text{bounded intervals } (a, b])$
- $\mathcal{M}\{(a, \infty) \mid a \in \mathbb{R}\}$
- $\mathcal{M}\{(-\infty, a) \mid a \in \mathbb{R}\}$
- $\mathcal{M}\{[a, \infty) \mid a \in \mathbb{R}\}$
- $\mathcal{M}\{(-\infty, a] \mid a \in \mathbb{R}\}$

24 (elementary family) $\mathcal{E} \subset \mathcal{P}(X)$ is called an elementary family provided

1. $\emptyset \in \mathcal{E}$
2. If $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$
3. If $E \in \mathcal{E}$ then E^C is the finite disjoint union of sets in $\mathcal{E}$

Proposition: Let $\mathcal{E}$ be an elementary family. Then $\mathcal{A}(\mathcal{E}) = \{$ all finite disjoint unions of sets in $\mathcal{E}\}$.

4 Measure Theory

25 (measurable space) If $\mathcal{M}$ is a σ-algebra of subsets then $(X, \mathcal{M})$ is called a measurable space. If $(X, \mathcal{M})$ is a measurable space and μ is a measure on $\mathcal{M}$ then $(X, \mathcal{M}, \mu)$ is called a measure space.

26 (continuous from below/above) We say μ is continuous from below if $E_1 \subseteq E_2 \subseteq \ldots$ and $E = \cup E_n$ are all in $\mathcal{A}$ then $\mu(E) = \lim \mu(E_n)$.

Similarly, μ is continuous from above if for $E_1 \supseteq E_2 \supseteq \ldots$ and $E = \cap E_n$ and $\mu(E_1) < \infty$ then $\mu(E) = \lim \mu(E_n)$.

We say μ is continuous from above at 0 if for $E_1 \supseteq E_2 \supseteq \ldots$ and $\emptyset = \cap E_n$ and $\mu(E_1) < \infty$ then $0 = \lim \mu(E_n)$.

Proposition: If $\mu(X) < \infty$ then

μ is countably additive $\Leftrightarrow$ μ is continuous from below

$\Rightarrow \mu$ is continuous from above

$\Rightarrow \mu$ is continuous from above at 0.

27 (complete measure space) A measure space $(X, \mathcal{M}, \mu)$ is called complete provided whenever $A \subseteq X$ and there exists some $E \in \mathcal{M}$ such that $A \subseteq E$ and $\mu(E) = 0$, then $A \in \mathcal{M}$.

28 (outer measure) Suppose X is a set. An outer measure is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ which satisfies

1. $\mu^*(\emptyset) = 0$
2. If $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$
3. $\mu^*(\cup E_n) \le \sum \mu^*(E_n)$

Proposition: Fix $\mathcal{E} \subseteq \mathcal{P}(X)$ with $\emptyset, X \in \mathcal{E}$. For $\rho : \mathcal{E} \to (0, \infty]$ with $\rho(\emptyset) = 0$, we may define for $E \subseteq X$

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \mid E \subseteq \cup E_n,\ E_n \in \mathcal{E} \right\}$$

Then μ^* is an outer measure, and is called the outer measure induced by ρ.

Given an outer measure μ^* on X, we say a subset A of X is called μ^*-measurable provided A splits every subset of X in an additive way. That is, for every $E \subseteq X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C)$

29 (Caratheodory) Suppose μ^* is an outer measure on X and set $\mathcal{M} = \mathcal{M}_{\mu^*} =$ all μ^*-measurable subsets of X. Then $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

30 (Proposition) Say $\mathcal{A}$ is an algebra of subsets of X and μ_0 is a premeasure on $\mathcal{A}$, μ^* the outer measure generated by μ_0. Then $\mu^*|_{\mathcal{A}} = \mu_0$, $\mathcal{A} \subseteq \mathcal{M}_{\mu^*}$, and $(X, \mathcal{M}_{\mu^*}, \mu = \mu^*|_{\mathcal{M}_{\mu^*}})$ is a complete measure.

31 (J1.1) Suppose $\mathcal{E}$ is an elementary family, $\rho : \mathcal{E} \to [0, \infty]$ satisfies $\rho(\emptyset) = 0$. Then if ρ is finitely additive on $\mathcal{E}$ then there exists a unique finitely additive ρ_1 on $\mathcal{A}(\mathcal{E})$ such that $\rho = \rho_1|_{\mathcal{E}}$.

Moreover, if ρ is countably additive on $\mathcal{E}$ then ρ_1 is a premeasure on $\mathcal{A}(\mathcal{E})$.

32 (μ_F) Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically increasing, right continuous and let $\mathcal{E} = \{(a, b] \cap \mathbb{R} \mid -\infty \le a \le b \le \infty\}$, so $\mathcal{E}$ is an elementary family and $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$. Define ρ on $\mathcal{E}$ by $\rho((a, b] \cap \mathbb{R}) := F(b) - F(a)$. Then

1. $\rho(\emptyset) = 0$

2. ρ is monotone
3. ρ is countably additive

By our extension theorem, ρ extends to a (unique) measure μ (we will call it μ_F) on $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$.

Denote by $\overline{\mu_F}$ the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ on the μ_F measurable set. We usually denote by $\mathcal{M}_{\mu_F}$, and we usually write μ_F for $\overline{\mu_R}$. μ_F on $\mathcal{M}_{\mu_F}$ is called the Lebesgue-Stielites measure generated by F.

If $F(x) = x$, then $m = \mu_F$ is the Lebesgue measure restricted to $\mathcal{B}_{\mathbb{R}}$. Lebesgue measure itself is the completion $\overline{m}$.

33 (regularity properties of Lebesgue-Stiltjes measure) If $A \in \mathcal{M}_\mu$, then

1. $\mu(A) = \inf\{\mu(\mathcal{O}) \mid A \subseteq \mathcal{O}, \mathcal{O} : \text{open}\}$
2. $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K : \text{compact}\}$

34 (Theorem) For $E \subseteq \mathbb{R}$, TFAE:

1. $E \in \mathcal{M}_\mu$
2. there exists a $V \supseteq E$, V is a G_δ-set, $\mu(V \backslash E) = 0$
3. there exists a $H \subseteq E$, H is a F_σ-set, $\mu(E \backslash H) = 0$

5 Cantor Sets

35 (cantor set) A metric space X is called a Cantor set provided $|X| > 2$, X is compact, X has no isolated points (ie. for all $x \in X$, $\{x\}$ is not open) and X is totally disconnected (ie. no subset of X having 2 or more points is connected).

$X \subseteq \mathbb{R}$ is totally disconnected if it does not contain any non-degenerate intervals.

ex. the regular Cantor middle thirds

ex. $X = \{0,1\}^{\mathbb{N}}$

Theorem: Any two Cantor sets are homeomorphic.

36 (Fat Cantor sets) Suppose $\mu = \mu_F$ with F increasing and continuous. Then for all $E \in \mathcal{M}_\mu$,

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ is a Cantor set}\}$$

So

$$\mu([0,1]) = \sup\{\mu(K) \mid K \subseteq [0,1]\ K \text{ is totally disconnected, compact, no isolated points}\}$$

Thus, for every $\epsilon > 0$, there exists some Cantor set $K \subseteq [0,1]$ with $\mu(K) > 1-\epsilon$. We call these fat cantor sets.

6 Measurable-ness

37 (non-measurable set) There exists $A \subseteq [0,1]$ that is universally non-measurable in the following sense: if F is monotonically increasing and continuous and $F(1) - F(0) > 0$ then A is not μ_F measurable.

38 (J1.17) Assume $A \subseteq [0,1]$ satisfies $\forall C \subseteq [0,1]$ closed and uncountable, $A \cap C \neq \emptyset \neq A^C \cap C$. If F is monotonically increasing and continuous, $\mu = \mu_F$ then

1. $\mu^*(A) = \mu([0,1])$
2. $E \in \mathcal{M}_\mu$, $E \subseteq [0,1] \Rightarrow \mu^*(E \cap A) = \mu(E)$
3. $E \in \mathcal{M}_\mu$, $E \subseteq [0,1], \mu(E) > 0$ then $E \cap A \notin \mathcal{M}_\mu$

In particular, if $F(1) > F(0)$ then A is not μ-measurable.

39 (measurable function) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, where $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras. Then $f : X \to Y$ is called measurable provided for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.

For $(X, \mathcal{M})$ a measurable space, a function $f : X \to \mathbb{R}$ is called measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}_\mathbb{R}$. Let $f' : X \to \overline{\mathbb{R}}$. Then f' is measurable when we consider $\mathcal{M}(\mathcal{B}_\mathbb{R} \cup \{\pm\infty\})$. Then we require $f'^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}_\mathbb{R}$, $f'^{-1}(\infty) \in \mathcal{M}$ and $f'^{-1}(-\infty) \in \mathcal{M}$.

A function $f : X \to \mathbb{C}$ is measurable if and only if $\Re(f)$ and $\Im(f)$ are measurable.

Proposition: Assume $\mathcal{E} \subseteq \mathcal{P}(X)$ such that $\mathcal{M}(\mathcal{E}) = \mathcal{N}$. If $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$ then f is measurable.

Proposition: Let $(X, \mathcal{M}), (Y, \mathcal{N}), (Z, \mathcal{P})$ be measurable spaces, $f : X \to Y$, $g : X \to Z$ and define $h : X \to Y \times Z$ by $h(x) = (f(x), g(x))$.

Then h is $\mathcal{M} - \mathcal{N} \otimes \mathcal{P}$ measurable if and only if f and g are measurable (where $\mathcal{N} \otimes \mathcal{P} = \mathcal{M}(\{A \times B \mid A \in \mathcal{N}, B \in \mathcal{P}\})$.

Proposition: For a measurable space $(X, \mathcal{M})$ and $f, g : X \to \mathbb{R}$, $a \in \mathbb{R}$ then $a \cdot f$, $f + g$, and $f \cdot g$ are all measurable.

Moreover, if $f, g : X \to \overline{\mathbb{R}}$ then $f \wedge g$ and $f \vee g$ are measurable, as is $|f|$.

If we define $0 \cdot \infty = 0 = \infty \cdot 0$, $\infty - \infty = 0$ then so is $f + g$ and $f \cdot g$ for $f, g : X \to \overline{\mathbb{R}}$.

Propositon: If (f_j) is a sequence of $\overline{\mathbb{R}}$ measurable functions on $(X, \mathcal{M})$, then $g_1 = \sup f_j$, $g_2 = \inf f_j$, $g_3 = \limsup f_j$, and $g_4 = \liminf f_j$ are all measurable functions.

40 (simple function) A function $f : X \to \mathbb{R}$ is simple if f is measurable and $f[X]$ is finite.

Any simple function may be written as $f = \sum_{j=1}^{n} a_j \chi_{E_j}$ for measurable E_j.

41 (Important approximation theorem) Let $(X, \mathcal{M})$ be a measurable space and

1. $f : X \to [0, \infty]$ a measurable function. Then there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ of simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ and $\phi_n \to f$ pointwise.
2. Moreover, if f is bounded, then ϕ_n converges to f uniformly

   ie. $\sup_{x \in X} |f(x) - \phi_n(x)| \to 0$.
3. $f : X \to \mathbb{R}$ a measurable function. Then there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ of simple functions to $\mathbb{R}$ such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$ and $\phi_n \to f$ pointwise.

42 ($f = g$ a.e.) Suppose $(X, \mathcal{M}, \mu)$ is a measure space. We say $f = g$ μ-a.e. if there exists some $E \in \mathcal{M}$ with $\mu(E) = 0$ such that $\{x \mid f(x) \neq g(x)\} \subseteq E$.

Proposition: Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Then

1. f is measurable and $f = g$ μ-a.e. then g is measurable
2. (f_n) measurable and $g = \sup f_n$ μ-a.e. then g is measurable
3. (f_n) measurable and $f_n \to f$ μ-a.e. then f is measurable

7 Integration

43 (integral) For a simple, positive function ϕ, we have the canonical representation $a_1 < a_2 < \cdots < a_n$, $E_i = [f = a_i]$, so $f = \sum_{i=1}^{n} a_i \chi_{E_i}$. We define the integral to be

$$\int_X f d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$$

Proposition: For $\phi, \psi \in \mathrm{Simp}(X)^+$, then

1. $\int \phi d\mu \geq 0$
2. $a \geq 0$ then $\int a\phi d\mu = a \int \phi d\mu$

3. $\int(\phi+\psi)d\mu = \int \phi d\mu + \int \psi d\mu$
4. If $\phi \leq \psi$ then $\int \phi d\mu \leq \int \psi d\mu$
5. Define $\nu : \mathcal{M} \to (0,\infty]$ by $\nu(E) := \int_E \phi d\mu = \int \chi_E \phi d\mu$. Then ν is a measure on $\mathcal{M}$
6. $\phi = 0$ μ-a.e. $\Leftrightarrow \int \phi d\mu = 0$

For an arbitrary $f \in L^+$, define

$$\int f d\mu = \sup\left\{\int \phi d\mu \mid 0 \leq \phi \leq f, \phi \in \mathrm{Simp}(X)^+\right\}$$

All remarks of the above proposition hold true for all functions in L^+.

44 (monotone convergence) If $0 \leq f_1 \leq f_2 \leq \ldots$ with $f_n \in L^+$ and $f = \lim_n f_n$ pointwise, then $\int f_n d\mu \to \int f d\mu$.

45 (Fatou's Lemma) For $f_n \in L^+$ then

$$\int \liminf f_n \leq \liminf \int f_n$$

46 (Dominated Convergence Theorem, v1) If $0 \leq f_n \leq g$ are all measurable and $f_n \to_X f$, $\int g < \infty$ then $\int f_n \to \int f$.

47 (Dini's Theorem) For $f_n \in \mathcal{C}([0,1])$, $f_1 \geq f_2 \geq \ldots$, $f_n \to_{[0,1]} 0$ then f_n converges to 0 uniformly on $[0,1]$.

48 (μ-integrable) We say $f : X \to \overline{\mathbb{R}}$ is μ-integrable provided f is measurable and $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. Define $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$.

Let $L^1(\mu)$ be the set of all μ-integrable real-valued functions.

Proposition: $L^1(\mu)$ is a vector space and a lattice and $\int \cdot d\mu$ is a positive linear functional on $L^1(\mu)$.

We say a function $f : X \to \mathbb{C}$ is μ-integrable if both $\Re(f)$ and $\Im(f)$ are μ-integrable. Define $\int f = \int \Re(f) + i \int \Im(f)$.

Proposition: If $f \in L^1_{\mathbb{C}}(\mu)$ then $|\int f| \leq \int |f|$.

49 (Generalized Dominated Convergence Theorem) Let $g, g_n \in L^+$ be measurable, $|f_n| \leq g_n$ μ-a.e., $f_n \to f$ and $g_n \to g$ μ-a.e. with $\int g_n \to \int g < \infty$.

Then $\int f_n \to \int f$. Moreover, $\int |f - f_n| \to 0$

50 (norm on $L^1(\mu)$) For a measure space $(X, \mathcal{M}, \mu)$, the norm on $L^1(\mu)$ is defined to be

$\|f\|_1 = \int |f| d\mu$.

Properties:

- $\|f\|_1 \geq 0$ and $\|f\|_1 = 0$ if and only if $f = 0$ μ-a.e.
- $\|af\|_1 = |a| \|f\|_1$
- $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$

Proposition: If we assume $\mu = \mu_F$ for some increasing, right continuous F on $\mathbb{R}$ then

1. the integrable simple functions are dense in $L^1(\mu)$
2. the linear combinations of χ_I (for I bounded and open interval) is dense in $L^1(\mu)$
3. the continuous functions with compact support are dense in $L^1(\mu)$

Theorem: $L^1(\mu)$ is complete

51 (oscillation) Suppose $f : X \to Y$ where (X, ρ_X) and (Y, ρ_Y) are metric spaces. For $\epsilon > 0$, define

$$\omega(f, x)(\epsilon) = \sup\{\rho_Y(f(z), f(y)) \mid y, z \in B_\epsilon(x)\}$$

If $0 \leq \epsilon_1 < \epsilon_2$ then $\omega(f, x)(\epsilon_1) \leq \omega(f, x)(\epsilon_2)$. Define the oscillation of f at x to be

$$\omega(f, x) = \lim_{\epsilon} \omega(f, x)(\epsilon)$$

$\omega(f, x) > \delta \Leftrightarrow$ for every open $\mathcal{O}$ with $x \in \mathcal{O}$, there exists $y, z \in \mathcal{O}$ such that $\rho_Y(f(y), f(z)) > \delta$.

Set $D_\delta(f) = \{x \in X \mid \omega(f, x) \geq \delta\}$.

Lemma: For any function f, $D_\delta(f)$ is a closed set.

Set $D(f) = \cup_{n \in \mathbb{N}} D_{1/n}(f)$ so $D(f) \in F_\sigma$.

52 (Lebesgue-Stieltzes vs Riemann) If $\int_a^b g(t) dF(t)$ exists, then so does $\int_{[a,b]} g d\mu_F$ and the integrals are the same.

53 ($D(g)$ gives existence of integral) If F is monotonically increasing and continuous, g bounded on $[a, b]$ then if $\mu_F(D(g)) = 0$ then the Riemann-Stieltjes Integral $\int_a^b g(t) dF(t)$ exists.

8 Convergence

54 (converges in measure) If f_n, f are $\mathbb{C}$-valued measurable functions on the measure space $(X, \mathcal{M}, \mu)$ then we say $f_n \to f$ (μ) (ie. f_n converges to f in measure) provided $\forall \epsilon > 0$, $\lim_n \mu[|f - f_n| > \epsilon] = 0$

Let $A_n(\epsilon) = \sup_{k \geq n} \mu[|f - f_k| > \epsilon]$ so $f_n \to f$ (μ) if and only if for every $\epsilon > 0$, $\lim_n A_n(\epsilon) = 0$.

We say f_n is Cauchy in measure provided for every $\epsilon > 0$, $\mu[|f_n - f_m| > \epsilon] \to 0$.

Letting $B_n(\epsilon) = \sup_{k,m \geq n} \mu[|f_k - f_m| > \epsilon]$ then (f_n) is Cauchy in measure $\Leftrightarrow$ $\lim_n B_n(\epsilon) = 0$.

Theorem: In a finite measure space, $f_n \to f$ in measure $\Leftrightarrow$ every subsequence of f_n has a futher subsequence that converges to f a.e.

Proposition: Suppose $\|f_n - f\|_1 \to 0$. Then $f_n \to f(\mu)$.

Theorem: Let (f_n) be Cauchy in measure. Then there exists a measurable f and a subsequence (f_{n_k}) such that

1. $f_{n_k} \to f$ a.e.
2. $f_n \to f$ (μ)

55 (almost uniformly) We say $f_n \to f$ almost uniformly provided $\forall \epsilon > 0$, $\exists E$ such taht $\mu(E^C) < \epsilon$ and f_n converges to f uniformly on E.

Theorem: Suppose $f_n \to f$ almost uniformly. Then $f_n \to f$ in measure and $f_n \to f$ a.e.

Note: converse is not true

56 (Egoroff's Theorem) Suppose $f_n \to f$ a.e. and $\mu(D) < \infty$. Then $\chi_D f_n \to \chi_D f$ almost uniformly.

57 (metric on $\mathcal{L}_{\mathbb{C}}(\mathcal{M})$) Let $(X, \mathcal{M}, \mu)$ be a finite metric space (ie. $\mu(X) < \infty$). Define d on $\mathcal{L}_{\mathbb{C}}(\mathcal{M})$ via $d(f, g) = \int |f - g| \wedge 1 d\mu = d(f - g, 0)$.

Then $d(f, g) = 0$ if and only if $f = g$ μ-a.e.

Then d is a metric on $\mathcal{L}_{\mathbb{C}}(\mathcal{M})$.

58 (types of convergence)

(A) $f_n \rightrightarrows f$ (uniform)

i.e. $\|f_n - f\|_{\sup} \to 0$

(B) $f_n \to f$ pointwise

i.e. $f_n(x) \to f(x)$ for all x

(C) $f_n \to f$ a.e.

i.e. $\mu(\{x \mid f_n(x) \nrightarrow f(x)\}) = 0$ (this is not a topological mode of convergence)

(D) $f_n \to f$ (μ) (in measure)

i.e. $\forall \epsilon > 0$, $\lim_n \mu[|f - f_n| > \epsilon] = 0$

(E) $L^1(\mu)$ convergence

i.e. $\|f_n - f\|_1 \to 0$

(F) $f_n \to f$ almost uniformly

i.e. $\forall \epsilon > 0$, $\exists E$ such that $\mu(E^C) < \epsilon$ and $f_n \rightrightarrows_E f$

The following diagram shows the implications where blue arrows mean on any measure space and gray arrows mean it only holds on finite measure spaces.

$$\begin{tikzcd}
(A) \arrow[r] \arrow[dd] \arrow[dr] & (B) \arrow[r] & (C) \arrow[dl, bend right] \\
& (F) \arrow[ur, bend right] \arrow[dd] & \\
(E) \arrow[dr] & & \\
& (D) &
\end{tikzcd}$$

(F) $\nrightarrow$ (E) and (D) $\to$ (C) for a subsequence.

(C) or (D) + (dominated or monotonicity) $\to$ (E)

$f_n \to f$ in L^1 $\Leftrightarrow$ every subsequence of f_n has a further subsequence which converges to f in L^1.

9 Product measures / spaces / stuff

59 (product of measure spaces) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Then $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ is a measure space where $\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{M} \times \mathcal{N})$. For $A \times B \in \mathcal{M} \times \mathcal{N}$ we set $(\mu \times \nu)(A \times B) := \mu(A)\nu(B)$.

$\mu \times \nu$ is a premeasure on $\mathcal{M} \times \mathcal{N}$ and so we may apply Caratheodory to get a measure space.

Theorem Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and $E \in \mathcal{M} \otimes \mathcal{N}$. Set

$$E_x = \{y \in Y \mid (x, y) \in E\} \qquad E^y = \{x \in X \mid (x, y) \in E\}$$

Then

1. $\forall x \in X$, $E_x \in \mathcal{N}$

2. Define $g_E : X \to [0,\infty]$ by $x \mapsto \nu(E_x)$. Then g_E is measurable.
3. $\int \chi_E d(\mu\times\nu) = (\mu\times\nu)(E) = \int g_E(x)d\mu(x) = \int \nu(E_x)d\mu(x) = \int(\int \chi_E(x,y)d\nu(y))d\mu(x)$.

By symmetry, the same holds for y. Note: this is Tonelli's theorem with $f = \chi_E$.

60 (Tonelli) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and $f : X \times Y \to [0,\infty]$ be a measurable function. Then

1. Define $f_x : Y \to [0,\infty]$ by $y \mapsto f(x,y)$. Then f_x is measurable for all $x \in X$
2. $x \mapsto \int f(x,y)d\nu(y)$ is a measurable function on X
3. $\int f d(\mu \times \nu) = \int(\int f(x,y)d\nu(y))d\mu(x)$

61 (Fubini) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $f \in L^1(\mu \times \nu)$. Then

1. for μ-a.e. $x \in X$, $f(x,\cdot) \in L^1(\nu)$
2. $x \mapsto \int_Y f(x,y)d\nu(y) \in L^1(\mu)$
3. $\int f d\mu \times \nu = \int(\int f(x,y)d\nu(y))d\mu(x)$

If f is measurable on $X \times Y$ then $|f|$ is measurable on $X \times Y$.

62 (Approximation properties of m^n) We let m^n be the completion of $m \times \cdots \times m$ where m is the Lebesgue measure on $\mathbb{R}$. So $\mathcal{L}^n$ is the Lebesgue measurable sets on $\mathbb{R}^n$.

Take $E \in \mathcal{L}^n$. Then

1. $m^n(E) = \inf\{m^n(\mathcal{O}) \mid E \subseteq \mathcal{O} \text{ and } \mathcal{O} \text{ is open}\}$

   $= \sup\{m^n(K) \mid K \subseteq E \text{ and } K \text{ is compact}\}$.
2. $E = A_1 \backslash N_1$ where A_1 is G_δ and $m^n(N_1) = 0$

   $E = A_2 \cup N_2$ where A_2 is F_σ and $m^n(N_2) = 0$
3. $m^n(E) < \infty$ implies $\forall \epsilon > 0$, $\exists (R_j)_{j=1}^N$ of disjoint open rectangles such that $m^n(E \triangle (\cup R_j)) = 0$

63 (uniqueness of Haar measure on $\mathbb{R}^n$) Let ν be a measure on $\mathcal{B}_{\mathbb{R}^n}$ that is translation invariant (i.e. $\nu(E) = \nu(a + E)$ for $a \in \mathbb{R}^n$, $E \in \mathcal{B}_{\mathbb{R}^n}$) such that $\nu([0,1]^n) < \infty$.

Then ν is equal to the scalar $\nu([0,1]^n)$ times m^n.

10 Signed Measure and Differentiation Theory

64 (signed measure) Let $(X, \mathcal{M})$ be a measurable space. We call $\mu : \mathcal{M} \to \overline{\mathbb{R}}$ a signed measure provided

- $\mu(\emptyset) = 0$
- μ is valued on $[-\infty, \infty)$ or $(-\infty, \infty]$
- μ is countably additive

Then $\mu(\cup E_n) = \sum \mu(E_n)$ is absolutely convergent.

65 (positive/negative/null for ν) Let ν be a signed measure on $(X, \mathcal{M})$. Then $P \in \mathcal{M}$ is called positive for ν if for all $A \in \mathcal{M}$ with $A \subseteq P$, $\nu(A) \geq 0$. N is negative for ν provided for all $B \in \mathcal{M}$ with $B \subseteq N$, $\nu(B) \leq 0$. We say N is null for ν if N is both positive and negative.

66 (Hahn-Decomposition Theorem) Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists $P \in \mathcal{M}$ which is positive for ν and $N = P^C$ is negative for ν.

Moreover, the decomposition $X = P \cup N$ is essentially unique: if P_1 is positive for ν and $N_1 = P_1^C$ is negative for ν, then $P \triangle P_1 = N \triangle N_1$ is null for ν.

67 (mutually singular) The signed measures μ and λ are mutually singular ($\mu \perp \lambda$) if ther exists a measurable decomposition $A \cup B = X$, $A \cap B = \emptyset$ such that A is null for λ and B is null for μ.

68 (Jordan Decomposition) Take the Hahn decomposition and let $\nu^+(E) := \nu(E \cap P)$, $\nu^-(E) := -\nu(E \cap N)$ so that $\nu = \nu^+ - \nu^-$. Note that $\nu^+ \perp \nu^-$.

Note that the Jordan Decomposition is unique.

The total variation of ν is defined to be $|\nu|(E) = \nu^+(E) + \nu^-(E)$.

69 (absolutely continuous wrt) We say ν is absolutely continuous with respect to μ (written $\nu \ll \mu$) if $\mu(E) = 0$ then E is null for ν.

70 (Lebesgue Decomposition Theorem) Let μ be a measure on $(X, \mathcal{M})$ and ν a σ-finite signed measure. Then $\nu = \nu_1 + \nu_2$ where $\nu_1 \perp \mu$, $\nu_2 \ll \mu$. Moreover, this decomposition is unique.

71 (Radon-Nikodyn Theorem) If f is $\mathcal{M}$-measurable and $\max\{\int f^+ d\mu, \int f^- d\mu\} < \infty$ then we say f is extended μ-integrable, and $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$.

If $(X, \mathcal{M})$ is a measurable space, μ a σ-finite measure on $\mathcal{M}$ and ν a σ-finite signed measure on $\mathcal{M}$ with $\nu \ll \mu$, then there exists an extended μ-integrable f such that $\nu = \nu_f$ where $\nu_f(E) = \int_E f d\mu$.

Moreover, we have uniqueness. If $\nu_f = \nu_g$ then $f = g$ μ-a.e.

Theorem: If μ is a measure, and ν a finite signed measure, then $\nu \ll \mu$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\mu(E) < \delta$ implies $|\nu|(E) < \epsilon$.

Proposition: Let $(X, \mathcal{M})$ be a measurable space, ν a σ-finite signed measure, and μ, λ σ-finite measures such that $\nu \ll \mu \ll \lambda$. If $\nu(E) = \int_E f d\mu$, we write $\frac{d\nu}{d\mu}$ for f. Then

1. $g \in L^1(\nu)$ and $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and
$$\int g d\nu = \int g\frac{d\nu}{d\mu} d\mu$$
2. $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$

72 (Covering Lemma) Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$ and $\mathcal{U} = \cup\mathcal{C}$ is an open set. If $c < m(\mathcal{U})$ then there exists disjoint $B_1, \ldots, B_n \in \mathcal{C}$ such that

$$m^n(\mathcal{U}) \geq m^n\left(\cup_{j=1}^k B_j\right) = \sum_{j=1}^k m^n(B_j) > \frac{1}{3^n}c$$

73 (locally integrable) A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is called locally integrable if $\int_E |f| d\mu < \infty$ for all bounded Borel sets E.

We denote this $f \in L^1_{loc}(m)$. For $f \in L^1_{loc}(m)$ and $r > 0$, define

$$\begin{aligned} A_r(f) &: \mathbb{R}^n \to \mathbb{R} \\ A_r(f)(x) &\mapsto \frac{\int_{B(r,x)} f(y)dy}{m(B(r,x))} = \frac{\int_{B(r,x)} f(y)dy}{r^n m(B(1,0))} \end{aligned}$$

Define the Hardy-Littlewood Maximal function of $f \in L^1_{loc}(m)$ to be

$$(Hf)(v) = \sup_{r>0} A_r|f(x)|$$

Lemma: Define $g : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ via $g(r, x) = A_r f(x)$. Then g is jointly continuous (hence, H is measurable).

Moreover, $[H > t] = \cup_{r\geq 0}[A_r|f| > t]$.

Theorem: $\epsilon m[Hf > \epsilon] \leq 3^n\|f\|_1$.

Theorem: Suppose $f \in L^1_{loc}(\mathbb{R}^n)$. Then $\lim_{r\to 0^+} A_r(f)(x) = f(x)$ m-a.e.

74 (Lebesgue set of f) For $f \in L^1_{loc}(\mathbb{R}^n)$, define the Lebesgue set of f to be

$$L_f = [x \mid \lim_{r\to 0^+} \frac{\int_{B(r,x)} |f(y) - f(x)| dy}{m(B(r,x))} = 0]$$

Theorem: Then $m(L_f^C) = 0$.

Comment 1:

$$\left| \frac{\int_{B(r,x)} f(y) dy}{m(B(r,x))} - f(x) \right| = \left| \frac{\int_{B(r,x)} f(y) - f(x) dy}{m(B(r,x))} \right|$$

so this theorem strengthens the last.

Lebesgue Density Theorem:

$$\frac{m(B(r,x) \cap A)}{m(B(r,x))} \to \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad a.e. \text{ as } r \to 0^+$$

75 (Lebesgue Differential Theorem) Fix $x \in \mathbb{R}^n$. We say $\{E_r\} \subseteq \mathcal{B}_{\mathbb{R}^n}$ shrinks nicely to x if

- $E_r \subseteq B(r,x) \qquad \forall r > 0$
- $\exists \alpha > 0$ such that $\forall r > 0$, $m(E_r) \geq \alpha m(B(r,x))$

Lebesgue Differential Theorem: For $f \in L^1_{loc}(\mathbb{R}^n)$, then for all $x \in L_f$ and for all $\{E_r\}$ shrinking nicely to x, we have

$$\lim_{r\to 0^+} \frac{\int_{E_r} |f(y) - f(x)| dy}{m(E_r)} = 0$$

$$f(x) = \lim_{r\to 0^+} \frac{\int_{E_r} f(y) dy}{m(E_r)}$$

76 (Theorem) **Theorem:** Assume μ is a signed measure on $\mathbb{R}^n$, $\mu \ll m$, $|\mu(E)| < \infty$ for all bounded E_r strinking nicely to x. Then for m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{\mu(E_r)}{m(E_r)} = 0$$

77 (differentiation on $\mathbb{R}$) If F is increasing on $\mathbb{R}$, then

1. $D(F)$ is countable, where $D(F)$ is the set of discontinuities of F
2. $G(x) := F(x^+) = \lim_{y \downarrow x^+} F(y)$ is right continuous and $F(x) = G(x)$ for all but countably many x
3. $F'(x)$ and $G'(x)$ exist almost everywhere and $F'(x) = G'(x)$ a.e.

78 (bounded variation) Define $T(F,P) = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$ where $P = x_0 < x_1 < \cdots < x_n$ is a partition. Let $T_F(x) = \sup T(F,P)$ which is clearly monotocially increasing.

Let $T_F(\infty) = \lim_{x\to\infty} T_F(x)$ and so we say F is of bounded variation ($F \in BV$) provided $T_F(\infty) < \infty$.

F bounded and monotone implies $F \in BV$. Moreover, if G, H are bounded and monotonically increasing, then $G - H \in BV$.

Properties:

1. $T_{cF} = |c| T_F$
2. $T_{F+G} \le T_F + T_G$
3. $T_{constant} = 0$
4. $T_{F+constant} = T_F$
5. $F \in BV$ implies $T_F(-\infty) = 0$
6. F monoton and bounded, then $F \in BV$, $T_F(x) = F(x) - F(-\infty)$
7. $F \in BV$ then $F = G - H$ where G, H are bounded increasing functions
8. $F \in BV$ implies $F(x+)$, $F(x-)$ exist
9. $F \in BV$ then $F'(x)$ exiss m-a.e. and $F'(x) = G'(x)$ on m-a.e. x where $G(x) = F(x+)$
10. $F \in BV$ and right continuous implies T_F is right continuous.

We say F is normalized bounded variation ($F \in NBV$) if $F \in BV$, F is right continuous and $F(-\infty) = 0$.

Theorem: If μ is a finite signed measure on $\mathcal{B}_{\mathbb{R}}$ define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \mu(-\infty, x]$. Then $F \in NBV$.

Conversely, if $F \in NBV$ then there exists a unique Borel measure μ_F such that for all x, $\mu_F(-\infty, x] = F(x)$.

79 (F absolutely continuous) We have $F : \mathbb{R} \to \mathbb{R}$ absolutely continuous provided $\forall \epsilon > 0$, $\exists \delta > 0$ such that $((a_i, b_i))$ disjoint open intervals such taht $\sum b_i - a_i < \delta$ then $\sum |F(b_i) - F(a_i)| < \epsilon$.

Theorem: Let $F \in NBV$. Then F is absolutely continuous if and only if $\mu_F \ll m$.

Proposition: $F \in NBV$ then

1. $F' \in L^1(m)$
2. $\mu_F \perp m$ implies $F' = 0$ m-a.e.

3. $\mu_F \ll m$ implies $F(x) = \int_0^x F'(t)dt$

If we write $\mu_F = \lambda + \nu$ where $\lambda \perp m$, $\nu \ll m$ then $F'(x) = \frac{d\nu}{dm}(x)$.

Theorem: If $F, G \in NBV$ and G is continuous, then

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a).$$