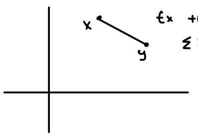
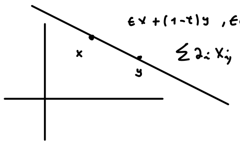
# LECTURE 4

- CONVEX COMBINATION




$tx + (1-t)y,\ t \in [0,1]$
$\sum \lambda_i = 1,\ \lambda_i \geq 0$

- AFFINE COMBINATION




$tx + (1-t)y,\ t \in \mathbb{R}$
$\sum \lambda_i x_i\quad \sum \lambda_i = 1$

- TAKE $C$ CONVEX, $\dim(C) = d$. WE WANT $C = \text{CONV}(X)$. HOW BIG $X$ DO WE NEED? SPOILER: $d+1$

**THEOREM CARATHEODORY:** LET $X \subseteq \mathbb{R}^n$, $\dim(X) = d$ (I CAN FIND $d+1$ VERTICES TO REPRESENT ANY POINT THERE)

THEN $\text{CONV}(X) = \left\{ \sum_i^{d+1} \lambda_i x_i \mid x_i \in X,\ \lambda_i \geq 0,\ \sum_i^{d+1} \lambda_i = 1 \right\}$

## PROOF:

$x \in \text{CONV}(X)$: $x = \sum_i^K \lambda_i x_i$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$. CHOOSE THIS CONVEX COMBINATION S.T. $K$ IS MINIMAL

BY CONTRADICTION, ASSUME $K > d+1$. REMEMBER $\dim(X) = d$. THUS $K > d+1 \Rightarrow x_1 \dots x_K$ ARE AFFINELY DEPENDENT

$\Rightarrow \sum_i^K \beta_i x_i = 0$ FOR SOME $\beta \neq 0$, $\sum \beta_i = 0$. THUS BOTH NEGATIVE AND POSITIVE $\beta_i$ NEEDS TO EXISTS.

- CHOOSE MAXIMAL $\gamma$, S.T. $\gamma \beta_i + \lambda_i \geq 0 \quad \forall i = 1 \dots K$.

HERE $\gamma$ IS CHOSEN FROM A SET S.T.: ① NON-EMPTY; ② BOUNDED (SINCE $\exists\ \beta_i < 0$) ③ CLOSED

- THIS ALL MEANS COMPACT + NON-EMPTY, SO THIS $\gamma$ WILL EXISTS FOR SURE.

$x = \gamma \sum_i^K \beta_i x_i + \sum_i^K \lambda_i x_i$. THIS CAN BE DONE SINCE LHS SUM UP TO ZERO (BY AFF. DEP).

$= \sum_i^{K} (\gamma \beta_i + \lambda_i) x_i$ WHICH IS A CONVEX COMBINATION. THIS SINCE $(\gamma\beta_i + \lambda_i) \geq 0$, $\sum (\gamma \beta_i + \lambda_i) = 1$

**CLAIM:** $\exists\ j \in \{1 \dots K\}$ S.T. $(\gamma \beta_i + \lambda_i) = 0$. THIS IS TRUE SINCE WE CANNOT HAVE $(\gamma \beta_i + \lambda_i) > 0$ ALWAYS, SINCE IN THAT CASE

I COULD INCREASE TO $(\gamma + \varepsilon)\beta_i + \lambda_i \geq 0$, BUT THIS IS A CONTRADICTION, SINCE I'VE CHOSEN $\gamma$ MAXIMAL.

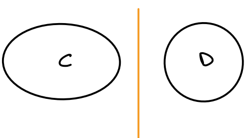
NOW, DOING A RECAP, I HAVE EXPRESSED $x$ AS A CONVEX COMBINATION BY $K$ VECTORS, AND ONE OF THEM HAS ZERO-COEFFICIENT

BY THE PREVIOUS CLAIM. THUS $x$ IS EXPRESSED BY A CONVEX COMBINATION OF $K-1$ VECTORS. CONTRADICTION SINCE WE ASSUMED

THIS IS A CONTRADICTION SINCE WE ASSUMED $K$ MINIMAL, AND USED THE FACT THAT $x_1 \dots x_K$ ARE AFF. DEPENDENT (IF $K > d+1$).

$\Rightarrow K$ HAS TO BE EQUAL TO $d+1$ ■

## INTUITION BEHIND SEPARATION THEOREM




IF DISJOINT, $\exists$ HYPERPLANE DIVIDES THEM. MOREOVER, WE NEED BOTH CLOSED, AND ONE BOUNDED

$a^T x = b \Rightarrow a^T c < b\ \forall c \in C$
$a^T d > b\ \forall d \in D$

- THIS THEOREM IS USEFUL SINCE IT TELLS THERE IS INEQUALITY BETWEEN OUR POLYEDRA AND THAT INFEASIBLE POINT

**THEOREM: SEPARATION THEOREM**: LET $C, D \subseteq \mathbb{R}^n$, BOTH NON-EMPTY, CLOSED, DISJOINT AND $C$ IS BOUNDED

THEN, $\exists$ HYPERPLANE $\{x \mid a^T x = b\}$, S.T. $C \subseteq \{x \mid a^T x < b\}$, $D \subseteq \{x \mid a^T x > b\}$

**PROOF**:

YOU NEED TO "ADD" BOUNDEDNESS OF B TO USE THIS ARGUMENT

① FIND $c \in C$, $d \in D$ WHOSE DISTANCE IS MINIMAL.

- $(c,d) \in C \times D$, WHICH IS A COMPACT SET. WE ARE MINIMIZING THE EUCLIDIAN DISTANCE, WHICH IS A CONTINUOUS FUNCTION. THEN $(c,d)$ $\exists$.

CHOOSE $c' \in C$, $d' \in d$; $\beta := \|c', d'\|_2$. $\lambda = \mathrm{diam}(C)$. TAKE BALL $B := \{x \mid \|c' - x\|_2 \leq \lambda + \beta\}$

TAKE $D' = D \cap B$, WHICH IS COMPACT. WHENEVER $\|x-y\|_2 \leq \beta$ FOR SOME $x \in C, y \in D$ => $y \in D'$ BY TRIANG. INEQUALITY

$\|c' - y\| \leq \|c' - x\| + \|x - y\| \leq \lambda + \beta$ => $y \in D'$. THUS $C, D'$ ARE COMPACT => CHOOSE $c \in C, d \in D'$ WITH MIN. DISTANCE

=> $\forall x \in C, y \in D: \|x - y\| \geq \|c - d\|$

② CHOOSE THE HYPERPLANE

$a := d - c$, $\quad b = \frac{a^T c + a^T d}{2} = a^T\left(\frac{c+d}{2}\right)$

$$a^T d - a^T c = \overset{a = d-c}{a^T a} = \overset{\text{SINCE } d,c \text{ DISJOINT AND CLOSED}}{\|c-d\|_2 > 0}$$

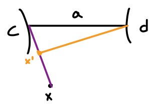
$$a^T a = \sum (c_i - d_i)^2 = (c-d)^T (c-d)$$

THUS $a^T d \neq a^T c$ => $a^T c < b < a^T d$

③ $\forall x \in C$ $\quad a^T x < b$ AND $\forall y \in D: a^T y > b$ $\qquad$ * OTHER POINT FOLLOWS IN SAME WAY.

START PROVING $a^T x \leq a^T c \leq b$. SUPPOSE BY CONTRADICTION $a^T x > a^T c$

$a^T(x - c) > 0$. $\exists$ $x' \in \mathrm{conv}(\{c, x\})$, S.T. $\|x' - d\|_2 < \|c - d\|_2$. CONTRADICTION SINCE $(c,d)$ HAD MIN. DISTANCE * ■




: GRAPHICAL SITUATION

**DEFINITION: POLYEDRON**: $P \subseteq \mathbb{R}^n$ IS A POLYHEDRON IF IT IS AN INTERSECTION OF FINITELY MANY HALFSPACES

**DEFINITION: POLYTOPE**: A POLYHEDRON $P$ IS POLYTOPE IF IT IS BOUNDED.

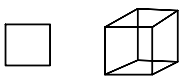
- POLYHEDRA IS CLOSED, POLYTOPE IS COMPACT. POLYHEDRA IS ALSO CONVEX SINCE HALFSPACES ARE CONVEX.

  $\max_{x \in P} c^T x$ ALWAYS EXISTS WHENEVER $P$ IS A NON-EMPTY POLYTOPE.

**EXAMPLES**:

- $[-1, 1]^n$

$n = 2$ $\qquad$ $n = 3$



YOU CAN EXPRESS THOSE AS INTERSECTION OF $2 \cdot n$ HALFSPACES: $x_1 \leq 1$, $x_1 \geq -1$ .... $x_n \leq 1$, $x_n \geq -1$

- REGULAR $n$- SIMPLEX

$n = 2$ $\qquad$ $n = 3$

CONSTRAINTS $\sum_{i}^{n+1} x_i = 1$, $x_1 \ldots x_{n+1} \geq 0$ FOR $n$-dim SIMPLEX