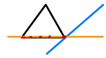
# LECTURE 6

**DEFINITION: SUPPORTING HYPERPLANE:** $P \subseteq \mathbb{R}^n$ POLYHEDRA, $c \in \mathbb{R}^n, t \in \mathbb{R}$.

EXAMPLE



$h = \{x \in \mathbb{R}^n \mid c^T x = t\}$ IS CALLED A SUPPORTING POLYHEDRA, IF $h \cap P \neq \emptyset$ AND $c^T x \leq t \quad \forall x \in P$

**DEFINITION: FACE:** $P \subseteq \mathbb{R}^n$ POLYHEDRON, $h$ SUPPORTING HYPERPLANE. THEN $F = P \cap h$ IS A FACE OF P. ALSO $P, \emptyset$ ARE FACES

FACES WHICH ARE NOT P NOR $\emptyset$ ARE CALLED PROPER FACES

- FACE OF dim: dim 0: VERTEX    dim 1: EDGE    dim p-1: FACETS

IF DELETE X FROM P, YOU CANNOT EXPRESS IT WITH OTHER POINTS.

**THEOREM:** LET $P \subseteq \mathbb{R}^n$ BE A POLYTOPE. V IS THE SET OF ITS VERTICES. $V_{EXT} = \{x \in P \mid x \notin \text{conv}(P \setminus \{x\})\}$

THEN $V = V_{EXT}$ AND $P = \text{conv}(V)$

**PROOF:**

$\forall x \in V_0 : \text{conv}(V_0 \setminus \{x\}) \neq P$

- $V_0$ BE A MINIMAL SET S.T. $P = \text{conv}(V_0)$. → SET OBTAINED REMOVING BY INDUCTION UNNECESSARY VERTICES
- WE WILL PROVE: $V \subseteq V_{EXT} \subseteq V_0 \subseteq V$
- $V \subseteq V_{EXT}$: CHOOSE $z \in V$. V, BEING THE SET OF VERTICES, HAS A SUPPORTING HYPERPLANE WHO DETERMINES z.

i.e. $c^T z = t$ FOR SOME $c \in \mathbb{R}^n, t \in \mathbb{R}$ S.T. $c^T x < t \quad \forall x \in P \setminus \{z\}$. z IS THUS A VERTEX. LET'S PROVE IT

CONVEXITY PRESERVES INEQUALITIES

CONSIDER $y \in \text{conv}(P \setminus \{z\}) \Rightarrow c^T y < t \Rightarrow z \notin \text{conv}(P \setminus \{z\}) \Rightarrow z \in V_{EXT}$

$V_0 \subseteq P \setminus \{z\}$

- $V_{EXT} \subseteq V_0$: IF $z \notin V_0 \Rightarrow z \in P = \text{conv}(V_0) \subseteq \text{conv}(P \setminus \{z\}) \Rightarrow z \notin V_{EXT}$ BY DEFINITION OF $V_{EXT}$
- $V_0 \subseteq V$: LET $z \in V_0$; $D := \text{conv}(V_0 \setminus \{z\})$. $V_0$ WAS MINIMAL, THUS THE POLYTOPE WILL BECOME SMALLER.

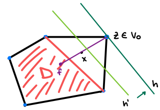
D AND $\{z\}$ ARE CLOSED, DISJOINT: USE SEPARATION THEOREM $\Rightarrow \exists c \in \mathbb{R}^n, t \in \mathbb{R}$ S.T. $c^T z > r$ AND $c^T x < r \quad \forall x \in D$. (GIVING $h'$)

CHOOSE $t := c^T z$, $h = \{x \mid c^T x = t\}$. $c^T x < r < t \quad \forall x \in D$, SINCE WE MOVED HYPERPLANE FURTHER FROM D, INEQUALITY STAYS THE SAME.

ALSO, $c^T x = 0$. $\Rightarrow \forall x \in \text{conv}(V_0) \setminus \{z\} = P \setminus \{z\}$: $c^T x < t$. IDEA: SINCE x IS IN THE MIDDLE OF SOME POINT IN D, CALL IT f, AND z,

AND SINCE $c^T z = t$, AND $c^T f < t \Rightarrow$ X WILL SATISFY THE INEQUALITY SHARPLY, i.e. $c^T x < t$. * RMK

PICTURE




FORMALLY $\Rightarrow$ h IS SUPPORTING HYPERPLANE OF P, AND $h \cap P = \{z\}$. $\Rightarrow \underline{z \in V}$ ■

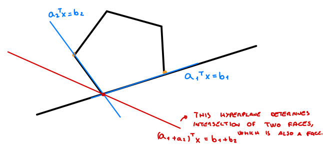
* **REMARK:** $x = \sum_{v_i \neq V_0} \lambda_i v_i \Rightarrow v_i \in V_0 \setminus \{z\}$ S.T. $\lambda_i > 0$. $\forall v_i \in V_0$: $c^T v_i \leq t \Rightarrow \sum \lambda_i v_i \leq t$.

BUT $\forall v_i \in V_0 \setminus \{z\}$: $c^T v_i < t \Rightarrow \sum \lambda_i v_i < t$

**THEOREM:** IF P IS POLYHEDRON: $V = V_{EXT}$ (ANYTHING ABOUT conv(V), BY UNBOUNDEDNESS)

**THEOREM:** LET E, F BE FACES OF P. THEN $E \cap F$ IS ALSO A FACE OF P.

PICTURE




**PROOF:**

- $x \in E \cap F$: $a_1^T x = b_1$, $a_2^T x = b_2$. $\Rightarrow (a_1 + a_2)^T x = b_1 + b_2 \Rightarrow x \in h \cap P$.
- IF $x \in P \setminus \{E \cap F\}$: $a_1^T x \leq b_1$, $a_2^T x \leq b_2$ BUT AT LEAST ONE IS STRICT (OTHERWISE $x \in E \cap F$)

$\Rightarrow (a_1 + a_2)^T x < b_1 + b_2$ ■

• $P = \{x \mid Ax \le b\}$ , $F = P \cap h$ , $c^T x = v$ $\Rightarrow$ $F = \{x \mid Ax \le b$ AND $c^T x = v\}$

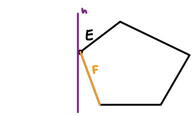
$\Rightarrow$ THE FACE OF A POLYHEDRON IS A POLYHEDRON

**THEOREM:** LET F BE A FACE OF P : THEN $E \subseteq F$ IS A FACE OF P IFF E IS A FACE OF F

**PROOF:**

"$\Rightarrow$" : h SUPPORTING HYPERPLANE S.T. $E = P \cap h$. $\Rightarrow$ $E \subseteq F \cap h \overset{*}{\subseteq} P \cap h = E$

→ PICTURE : ⇒




$F \cap h = E$ $\Rightarrow$ WHAT SHOWED *

"$\Leftarrow$" : • LET $F = \{x \in P \mid c^T x = t\}$ , $E = \{x \in P \mid d^T x = v\}$. THOSE ARE HYPERPLANE WHO DETERMINES THEIR FACES. I.E. $c^T x < t$ $\forall x \in P \setminus F$ , $d^T x < v$ $\forall x \in F \setminus E$

• FOR $\lambda > 0$, DEFINE $h_\lambda = \{x \mid (\lambda c + d)^T x = \lambda t + v\}$

FOR SOME LARGE $\lambda$ , h WILL BECOME MORE AND MORE AS h'.

NOW SHOW THERE IS A SUITABLE $\lambda$. $V :=$ SET OF VERTICES OF P, $c^T x < t$ $\forall x \in V \setminus F$

AND $(\lambda c + d)^T \overset{*}{\le} t + v$ $\forall x \in V \cap F$. CLARIFYING. $\forall x \in F$: $c^T x \le t$ AND $d^T x \le v$

→ $\exists$ $\lambda > 0$ S.T. $(\lambda c + d)^T < \lambda t + v$ $\forall x \in V \setminus F$ . IT CAN BE $d^T x > v$ FOR SOME $x \in V \setminus F$.

SO $\forall x \in V \setminus F$ $\exists$ $\lambda_x$ S.T. $d^T x - v < \lambda (t - c^T x)$. $\lambda$ ROTATES h TO h'.

CHOOSE $\lambda := \max \{\lambda_x \mid x \in V \setminus F\}$. $\Rightarrow$ $d^T x - v < \lambda (t - c^T x)$ $\forall x \in V \setminus F$

• $(\lambda c + d)^T x \le \lambda t + v$ $\forall x \in P$ AND $(\lambda c + d)^T x = \lambda t + v$ $\forall x \in E$

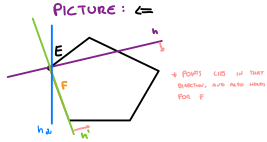
$\Rightarrow$ $h_\lambda$ IS A SUPP. HYPERPLANE OF P

• PROVE $E = P \cap h_\lambda$ • IF $x \in F \setminus E$ $\Rightarrow$ $d^T x < v$ $\Rightarrow$ $x \notin h_\lambda$

• IF $x \in P \setminus F$ $\Rightarrow$ x HAS POSITIVE COEFFICIENT AT $v \in V \setminus F$ IN CONVEX OF V

$\Rightarrow$ $x \notin h_\lambda$ ■

PICTURE : ⇐




→ RECTIFY IN SOME WAY IT STILL REMAINS BOTH IN THE INTERSECTION BUT ALSO AS A SUPPORT FOR P

$$\frac{5 \cdot 4}{2} = 10$$

$$\frac{1}{5}$$

$$10 \cdot \left(\frac{1}{5}\right)^3 \cdot \left(\frac{4}{5}\right)^2$$

$$\binom{3}{2} \cdot \left(\frac{1}{2}\right)^3$$

$$\frac{3!}{2!} \qquad 3 \cdot \frac{1}{8} \qquad ccc \quad \left(\frac{1}{2}\right)^3$$

$$\frac{\overline{X} - z_{0.95} \cdot \sigma}{\sqrt{n}} \qquad \overline{X}$$

$$\frac{\overline{X} - 0{,}17 \cdot \sigma}{\sqrt{n}}$$

$$\frac{250 - 0{,}17 \cdot 15}{\sqrt{36}}$$

$$1 - \left(\frac{3}{4}\right)^n = 0.1$$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

$$V(X) + V(Y) - 2\,Cov(X,Y)$$

$$1 - P(X=0) - P(X=1)$$

$1 -$

$$\frac{9!}{2!\,7!} \qquad \frac{6}{36} \qquad \binom{4}{2} \qquad \frac{4!}{2!\,2!} \qquad 6$$

$$\binom{4}{3} \qquad \frac{4!}{3!}$$

$$\frac{4!}{4!}$$

$$\frac{6}{36} + \frac{4}{36} + \frac{1}{36}$$