Lecture 7

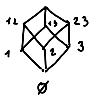
① Polytope is a convex hull of its vertices

② Intersection of faces is a face

③ Face of a face of P is a face of P

Ex:

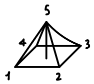
Face lattice $\{1,2,3\}$



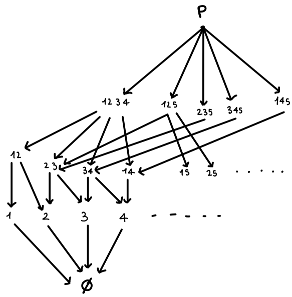
$a \cap b$ in this case
$\parallel$
$a \wedge b$ = largest point smaller than both of them . Ex $12 \wedge 23 = 1$

$a \vee b = a \cup b$

Ex



$\Rightarrow$



• Observations

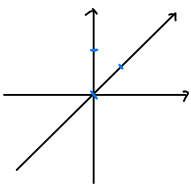
. The meet is given by property ②

. For any two faces there is a unique face which contains both of them and is the smallest one $(a \vee b)$

Every face is convex hull of its vertices

• Every face is an intersection of all facet which contain it

• Each face of dimension(P) −2 , is an intersection of exactly 2 facets



Definition: Min. face: does not contain any other proper face

· Min. faces are always affine spaces

· Minimal faces in a polyhedron have same dimension

· Every facet is a convex hull of its minimal faces.

· Every min. face is intersection of all facets containing it

- $P = \{x \in \mathbb{R}^n \mid A'x = b' \text{ AND } A''x \leq b''\}$. m' EQUALITIES , m'' INEQUALITIES

DEFINITION : MINIMAL DESCRIPTION: SUCH DESCRIPTION OF P IS MINIMAL IF :

- OMISSION OF ANY CONSTRAINT RESULTS IN CHANGE OF P
- WE CANNOT CONVERT ANY INEQ. INTO EQUALITY WITHOUT CHANGING P

THEOREM: IF $P \neq \emptyset$, THEN $\exists\ z \in P$ S.T. $A''z < b''$

PROOF:

$\forall_{i=1..m''}: \exists\ z^{(i)} \in P$, S.T. $A''_i z^{(i)} < b''_i$. THIS IS TRUE BY MINIMALITY OF DESCRIPTION, SINCE OTHERWISE CHANGING IT TO EQUALITY WOULD CHANGE ANYTHING FROM THE POLYHEDRON, VIOLATING THE 2nd PROPERTY.

- CHOOSE $z := \sum_i^{m''} \frac{1}{m''} z^{(i)} \in P$. THIS IS A CONVEX COMBINATION OF POINTS BELONGING TO THE POLYHEDRON.
- $z \in P$ AND $A''z < b$ ■

THEOREM: IF THERE IS $z \in P$, S.T. $A''z < b''$, THEN: • $\dim(P) = n - \text{rank}(A)$ • z DOES NOT BELONG TO ANY PROPER FACE OF P

PROOF:

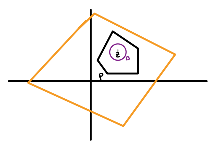
- $\overset{\text{AFF. SPACE}}{A} = \{x \mid A'x = b'\}$; THUS $P \subseteq A \implies \dim(P) \leq \dim(A) = n - \text{rank}(A')$
- CHOOSE $\varepsilon > 0$ S.T. $B = \{x \in A \mid \|x - z\| < \varepsilon\} \subseteq P$
- $B \subseteq P = \dim(P) \geq \dim(B)$
- $\dim(B) = \dim(A)$? $A = z + L$, $S :=$ BASIS OF L, WITH VECTORS OF LENGTH 1

$\forall s \in S: z + \varepsilon s \in B$, SINCE $\varepsilon s \in L \implies z + \varepsilon s \in A$. BUT $z + \varepsilon s$ ARE POINTS IN THE BALL , THUS IT IS TRUE.

- z TOGHETHER WITH $\{z + \varepsilon s \mid s \in S\}$ ARE AFFINELY INDEPENDENT

$\implies \dim(B) = \dim(L) = \dim(A)$ ■ (1st. PART)

PICTURE



- TAKE $h = \{x \mid a^T x = b\} \supseteq A \implies h \cap P = P$ (SINCE A CONTAINS P)

- ALL POSSIBLE HYPERPLANES ARE THE ONES DEFINED BY THE EQUATIONS THAT DEFINE THE AFFINE SPACE WHERE P BELONGS TO AFFINE SPACE IN THE PLANE IS THE ORANGE PICTURE.

$P \cap A$ IS ALWAYS THE WHOLE POLYHEDRON

- CONSIDER A PLANE $\perp$ TO A, PASSING THROUGH z.
- TAKE $h = \{x \mid a^T x = b\} \not\supseteq A$; $z \in h$: $a^T z = b$.

$\implies$ THERE IS A BALL B AROUND z. IF THERE AN HYPERPLANE PASSING THROUGH z, AND WHICH NOT CONTAIN THE WHOLE AFFINE SPACE, IT NEEDS TO CUT THE BALL IN TWO PARTS.

THAT'S WHY THE POLYTOPE P CONTAINS POINTS ON BOTH SIDES OF THIS HYPERPLANE. SINCE B LIES IN THE POLYHEDRON, THEN THIS HYPERPLANE CANNOT BE A SUPPORTING HYPERPLANE, SINCE IT WOULD NEED TO HAVE THE WHOLE POLYHEDRON ON THE SAME SIDE. $\implies$ z CANNOT BE A PROPER FACE, SINCE EVERY HYPERPLANE WHICH CONTAINS IT, CUT THE POLYHEDRON IN HALF, SO IT CAN BE A SUPPORTING ONE $\implies$ z CANNOT LIE IN A SUPPORTING HYPERPLANE

THEOREM: $P \subseteq \mathbb{R}^n$ BE A POLYHEDRON.

- EVERY FACE IS A FACE OF SOME OF THE FACETS P
- THERE IS ONE-TO-ONE CORRESPONDANCE BETWEEN FACETS OF P AND THE INEQUALITIES IN A MIN. DESCRP OF P

COROLLARY: IF $\dim(P) = n$, THEN ITS MINIMAL DESCRIPTION IS UNIQUE UP TO MULTIPLICATION OF THE INEQUALITIES.

PROOF:

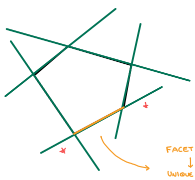
- NOTE THAT IF POLYTOPE HAS FULL DIMENSION, ITS MINIMAL DESCRIPTION DOES NOT CONTAIN ANY EQUALITY, SINCE EACH EQUALITY DROPS DIMENSION BY 1
- FROM THM, $\underset{\text{FOR EACH FACET, UNIQUE INEQ.}}{F_K} \longrightarrow A''_K x \le b''_K$ IN MINIMAL DESCRIPTION. LET $h_K = \{x \mid A''_K x = b''_K\}$ IS A SUPPORTING HYPERPLANE.

$F_K = h_K \cap P$. THIS IS THE CORRESPONDANCE COMING FROM THE THEOREM.

- F_K IS A FACET, $\dim(P) = n \Rightarrow \dim|F_K| = n-1$. THUS F_K CONTAINS n AFFINELY INDEPENDENT POINTS

$\Rightarrow$ SO THERE IS ONLY A SINGLE HYPERPLANE PASSING THROUGH F_K, SINCE ITS dim IS SO LARGE.

PICTURE • $\{x \mid A'x = b', Ax'' \le b'', A''_K = b''_K\}$



* IF YOU CHANGE ONE OF THOSE SIDES, THE POLYTOPE WILL BE DIFFERENT

FACET, $\dim(n-1)$
↓
UNIQUE AFF. SPACE, THE AFFINE HULL OF THIS SPACE, WHICH CONTAINS IT. HYPERPLANE IS UNIQUE AND THE DESCRIPTION IS UNIQUE

COROLLARY: EVERY PROPER FACE OF P IS AN INTERSECTION OF SOME OF ITS FACETS

PROOF:

- E PROPER FACE, $k = \dim(E)$, $d = \dim(P)$
- $E \subseteq F$, F IS A FACET OF P
- $P' = P = \{x \mid A'x = b, A''x \le b''\}$; $\mathbb{F} = \emptyset$

FOR $i = 1 \ldots k$: $F :=$ FACE OF P', S.T. $E \subseteq P'$. WE ALWAYS FIND F S.T. $E \subseteq F$

$F = \{x \mid \underbrace{A'x = b', A''x \le b''}_{\text{MIN. DESCRIPTION OF } P'}, \underbrace{A''_K x = b''_K}_{\text{CONSTRAINT CORRESPONDING TO } F}\}$

$G =$ FACET OF P CORRESPONDING $A''_K x = b''_K$

P' STARTS BEING P, BUT IT BECOMES SOMETHING SMALLER, BUT EVEN IF P' IS SMALLER, • IS FACET (BY THM OF ONE-TO-ONE APPLIED TO P', BUT THEN, A FACET ALSO CORRESPONDING TO •

ADD G TO $\mathbb{F}$, $P := F$.

- $\dim(P') = k$, $E = \bigcap_{F \in \mathbb{F}} F$, SINCE $P' = \{x \mid Ax = b', A''x \le b'', A''_K x = b''_K\}$