LECTURE 2

• ZERO - SUM GAMES

• CONSIDER A WAR BETWEEN B AND E, CONTENDING 3 MOUNTAIN PASSES

EACH B AND E HAVE 5 SOLDIERS.

• DECISION BY COMMANDERS → HOW TO DIVIDE THE 5-SOLDIERS. (k, l, m) ; $k \leq l \leq m$; $k+l+m=5$

THE GROUPS OF SOLDIERS DIVIDES THEMSELVES RANDOMLY BETWEEN THE 3 MOUNTAIN PASSES

• GIVEN A MOUNTAIN PASS, WHO SENDS MORE SOLDIERS CAPTURES THE PASS. IF # OF SOLDIERS IS THE SAME, YOU HAVE A DRAW.

• OBJECTIVE IS MAXIMIZING THE PROBABILITY OF WINNING. THIS PROBLEM IS CALLED ZERO-SUM GAME SINCE PROFIT OF ONE IS LOSS OF OTHER.

EXAMPLE: $B(0,0,5)$; $E(0,0,5)$ → TWO CASES: ⓐ BOTH GO IN SAME SPOT $\left(\frac{1}{3}\right)$ → DRAW → OUTCOME: 0

ⓑ THEY GO IN DIFFERENT SPOT $\left(\frac{2}{3}\right)$ → 1 PASS EACH → OUTCOME: 0

THE VARIOUS CASES ARE SUMMARIZED IN THE TABLE BELOW:

	$(0,0,5)$	$(0,1,4)$	$(0,2,3)$	$(1,1,3)$	$(1,2,2)$
$(0,0,5)$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	-1
$(0,1,4)$	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$(0,2,3)$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$
$(1,1,3)$	1	$\frac{1}{3}$	0	0	$-\frac{1}{3}$
$(1,2,2)$	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0

$(0,2,3)$ IS UNIQUE

DEFINITION: NASH EQUILIBRIUM: PAIR OF STRATEGY WHICH ARE BOTH RESPONSE TO ONE-ANOTHER (PURE-NASH IF CHOICE IS UNIQUE)

• LET'S TAKE ANOTHER EXAMPLE - ROCK, PAPER, SCISSOR

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

THERE IS NO SINGLE ACTION THAT COULD BE BEST RESPONSE TO ANOTHER

A: $X = (x_1, x_2, x_3)$, $x_1, x_2, x_3 \geq 0$; $x_1 + x_2 + x_3 = 1$ → MIXED STRATEGY

FACT: EVERY ZERO-SUM GAME HAS A MIXED-NASH EQUILIBRIUM

NEW PROBLEM: ROCK-PAPER-SCISSOR BY SANTA-CLAUS AND EASTER BUNNY

	R	P
R	0	-1
P	1	0
S	-1	1

LET'S FIX SC STRATEGY : $x = (x_1, x_2, x_3)$

$\min_{y} \sum_{i,j} x_i M_{ij} y_j$; E_y [OUTCOME, GIVEN x] $\rightarrow \exists y$ S.T. OUTCOME$(x,y) \leq E$[OUTCOME(x,y)]

- CHOICE FOR SANTA-CLAUS : ADD VARIABLE:

$$x = \arg\max \min \{x_2 - x_3 ; -x_1 + x_3\}$$

$$\max \ x_0$$

S.T.

$$x_0 \leq x_2 - x_3$$
$$x_0 \leq x_3 - x_1$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

OPTIMAL $\left(\overset{x_0}{\frac{1}{3}}, \overset{\text{ROCK}}{0}, \overset{\text{PAPER}}{\frac{2}{3}}, \overset{\text{SCISSOR}}{\frac{1}{3}}\right)$

- CHOICE FOR EASTER BUNNY:

$Y = \arg\min \max \{-y_2, y_1, -y_1 + y_2\}$: ADD VARIABLE

$$\min \ y_0$$

S.T.

$$y_0 \geq -y_2$$
$$y_0 \geq y_1$$
$$y_0 \geq -y_1 + y_2$$
$$y_1 + y_2 = 1$$
$$y_1, y_2 \geq 0$$

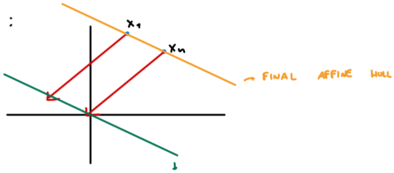
OPTIMAL $\left(\overset{y_0}{\frac{1}{3}}, \underset{\text{ROCK}}{\frac{1}{3}}, \underset{\text{PAPER}}{\frac{2}{3}}\right)$

• GEOMETRY PART

- **LINEAR SPACE:** $S \subseteq \mathbb{R}^d$ S.T. CLOSED TO MULTIPLICATION BY SCALAR AND ADDITION OF VECTORS. LINES/PLANES PASSING THROUGH ORIGIN.
- **LINEAR INDEPENDENCE** $\{x_1 \dots x_k\}$ LIN. DEPENDENT IF $\exists\ a_1 \dots a_k$ S.T. $\sum a_i x_i = 0$ WITH AT LEAST ONE $a_i \neq 0$.
- **AFFINE SPACE:** $S = L + x$, $L \in \mathbb{R}^d$ LINEAR SPACE. EX: LINE SHIFTED, THAT DOES NOT PASS TO THE ORIGIN
- **AFFINE HULL:** $X = \{x_1 \dots x_n\}$; MOVE THIS SET TO TOUCH ORIGIN: $x_1 - x_n, \dots, x_{n-1} - x_n, 0$. TAKE NOW L. COMB OF THIS

$\rightarrow \beta(x_1 - x_n) + \dots + \beta_n(x_{n-1} - x_n) + x_n$ ↗ TO SHIFT IT BACK

GRAPHICALLY:



LINEAR HULL: LINE THAT REPRESENTS L. COMBINATION OF TWO POINTS IN NEW POSITION → YOU HAVE TO SHIFT IT BACK

- $\beta_1(x_1 - x_n) + \dots + \beta_n(x_{n-1} - x_n) + x_n = \beta_1 x_1 + \dots + \beta_n x_n + (1 - \beta_1 - \dots - \beta_{n-1})\, x_n$

DEFINITION (AFFINE COMB): $X = \{x_1 \dots x_n\}$. $\sum_{i}^{n} a_i x_i$; S.T. $\sum a_i = 1$, $a_1 \dots a_n \in \mathbb{R}$

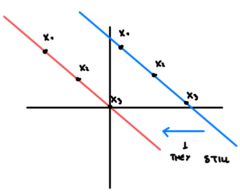
DEFINITION (AFFINE HULL): GIVEN X $\quad$ AFF. HULL$(X) = \bigcap_{A \in \mathcal{A}} A$; $\mathcal{A} = \{A \mid \text{AFFINE SPACE}, X \subseteq A\}$

LEMMA: AFF. HULL$(X) = \{x \mid x = \sum_{i}^{n} a_i x_i ,\ \sum a_i = 1\}$

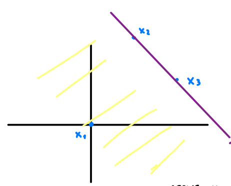
DEFINITION: X IS AFFINE DEPENDENT IF $\exists\ a_1 \dots a_n$ S.T. $\sum a_i = 0$ AND $\sum a_i x_i = 0$, WITH AT LEAST ONE $a_i \neq 0$

- SAYING $(x_1 - x_n), \dots, (x_{n-1} - x_n)$ ARE LINEARLY DEPENDENT IS EQUIVALENT TO WHAT SAID ABOVE.

EXAMPLE:



↓ THEY STILL REMAIN DEPENDENT.



JUST x_2, x_3 → ADD x_1 TO COMBINATION

AFFINE HULL WITH x_1

↓

"ZERO" POINTS IN SET OF POINTS DO NOT MAKE THE SET BEING DEPENDENT

- SAYING $(x_1 - x_n), \dots, (x_{n-1} - x_n)$ ARE LINEARLY DEPENDENT IS EQUIVALENT TO WHAT SAID ABOVE.

SINCE IT MEANS

$\beta_1(x_1 - x_n) + \dots + \beta_{n-1}(x_{n-1} - x_n) = 0 \iff \beta_1 x_1 + \dots + \beta_{n-1} x_{n-1} - (\beta_1 + \dots + \beta_{n-1})\, x_n$ → SEE COEFFICIENTS SUM UP TO ZERO

EXAMPLE: TAKE $X = \{x_1 \dots x_n\} \subseteq \mathbb{R}^d$. THERE CAN BE AT MOST $d+1$ AFF. IND. VECTORS

TAKE MATRIX A WITH i-TH COLUMN $(x_i - x_n)$ $\quad A = \begin{array}{c} n-1 \text{ COL} \\ \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} \end{array}$ d ROWS $\quad X$ IS AFF. INDEP $\iff \det(A) \neq 0$