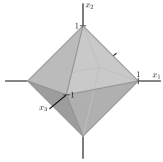
LECTURE 5

- CROSS POLYTOPE : $\{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. YOU NEED 2^n CONSTRAINTS: $\forall \sigma \in \{\pm 1\}^n : \sigma^T x \leq 1$



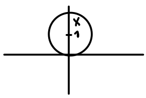
DEFINITION: CONVEX CONE : $\emptyset \neq C \in \mathbb{R}^n$ IS CALLED CONVEX CONE IF $\lambda x + \mu y \in C$ $\forall x, y \in C$, $\lambda, \mu \geq 0$

DEFINITION OF CONE GENERATED BY X : CONE(X): $X \subseteq \mathbb{R}^n$, CONE GENERATED BY X IS THE SMALLEST CONE CONTAINING X

- $\text{CONE}(X) = \{\sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N}_+, x_i \in X, \lambda_i \geq 0\}$

LEMMA: EVERY FINITELY GENERATED CONE IS CLOSED

EXAMPLE



ALL HALFSPACE + ORIGIN IS PART OF CONE GENERATED BY X. NO X-AXIS SINCE O IS INCLUDE

THEOREM: FARKAS LEMMA: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ EXACTLY ONE OF THE FOLLOWING STATEMENTS IS TRUE

① $\exists x \in \mathbb{R}^n$ S.T $Ax = b$, $x \geq 0$ (SUCH LP HAS FEASIBLE SOLUTION) ② $\exists y \in \mathbb{R}^m$ S.T. $y^T A \geq 0^T$ AND $y^T b < 0$ (CERTIFICATE LP DOESN'T HAVE SOLUTION)

2nd POINT: $\underbrace{(y^T A)}_{\geq 0} x = y^T(Ax) = y^T(b) < 0$. NO WAY LAST LHS < 0 IF $x \geq 0$. $\Rightarrow$ NO SOLUTION TO LP IN THAT CASE

THEOREM: FARKAS LEMMA (EQUIVALENT) : $a_1 .. a_n$, $b \in \mathbb{R}^m$. EXACTLY ONE OF THE FOLLOWING IS TRUE

① $b \in \text{CONE}(\{a_1 \dots a_n\}) = C$ THIS CAN BE SEEN AS b THAT CAN BE WRITTEN AS NON-NEGATIVE COMBINATION OF $a_1 \dots a_n$.

THIS COMBINATION IS GIVEN BY $A \cdot x$

② THERE IS A HYPERPLANE SEPARATING b FROM C. $h = \{x \in \mathbb{R}^m \mid y^T x = 0\}$. $y^T b < 0$ AND $y^T a_i \geq 0$ $\forall i = 1 \dots n$

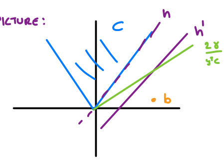
PROOF

- ① AND ② CANNOT HAPPEN AT SAME TIME.
  - IF $b \in C \Rightarrow b = \sum \lambda_i a_i$, $\lambda_i \geq 0$. $\forall i$ $y^T a_i \geq 0 \Rightarrow y^T b \geq 0$. NON-NEGATIVITY HOLDS ALSO FOR b IF IT BELONGS TO CONE
- ① IS FALSE $\Rightarrow$ ② IS TRUE

$b \notin C$. APPLY SEPARATION THEOREM TO $\{b\}$ AND C. $\Rightarrow$ BY SEP. $\exists$ $h = \{x \in \mathbb{R}^m \mid y^T x = \gamma\}$. $y^T b < \gamma$, $y^T c > \gamma$ $\forall c \in C$.

OBSERVE $0 \in C \Rightarrow y^T \cdot 0 > \gamma \Rightarrow \gamma < 0$

PICTURE:



CLAIM: $h = \{x \in \mathbb{R}^m \mid y^T x = 0\}$ SATISFIES ② . YOU NEED TO MOVE A BIT THE HYPERPLANE

THIS HAPPENS IF $y^T b < 0$ AND $y^T a_i \geq 0$ $\forall i = 1 \dots n$. SHOW FIRST $y^T b < 0$

- WE KNOW $y^T b < \gamma$. BUT $\gamma < 0$ $\Rightarrow y^T b < 0$
- NOW $y^T a_i \geq 0$ $\forall i = 1 \dots n$. ASSUME BY CONTRADICTION $\exists c \in C$ S.T. $y^T c < 0$. TAKE $y^T \underbrace{\left(\frac{2 \cdot \gamma}{y^T c}\right) \cdot c}_{\in C} = 2\gamma < \gamma$

$\Rightarrow y^T c < \gamma$, BY SEPARATION THM. IS FALSE

THEOREM (MINKOWSKI – WEYL): $P \subseteq \mathbb{R}^n$ / POLYTOPES CAN BE REPRESENTED BOTH BY $Ax \leq b$ OR $\mathrm{conv}(V)$

- P IS A POLYTOPE IFF $\exists$ A FINITE SET $V \subseteq \mathbb{R}^n$ S.T. $P = \mathrm{conv}(V)$.
- GIVEN $V \subseteq \mathbb{R}^n$ $\max\{c^T x \mid x \in \mathrm{conv}(V)\}$

  1. FIND $P = \mathrm{conv}(V)$ SPECIFIED BY CONSTRAINTS. $P = \{x \mid Ax \leq b\}$ $\Rightarrow$ USE L.PROGRAMMING

     CROSS POLYTOPE: $P = \{x \mid \sum |x_i| \leq 1\}$. $P = \mathrm{conv}(\{\pm a_1 \dots a_n\})$ $O(n^2)$ $\Rightarrow$ $2 \cdot n$ VERTICES OF n COORDINATES.

     $P = \{x \mid \sigma^T x \leq 0,\ \sigma \in \{\pm 1\}^n\}$ IT GIVES 2^n CONSTRAINT $\rightarrow$ YOU CANNOT EVEN WRITE THOSE SINCE INPUT IS n^2

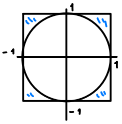
  2. $\max\{c^T x \mid x \in V\}$.

     $x \in P = \mathrm{conv}(V)$: $x = \sum \lambda_i v_i$ $\Rightarrow$ $\exists i$ S.T. $c^T x_i \geq c^T x$. $O(|V|)$ EVALUATION OF OBJECTIVE.

- GIVEN $P = \{x \mid Ax \leq b\}$; TRY ALL VERTICES: FIND V, S.T. $P = \mathrm{conv}(V)$ $\Rightarrow$ CUBE: $[-1, 1]^n$. $|\{-1, +1\}^n| = 2^n$ $\rightarrow$ USE LP DIRECTLY

EXAMPLES

$\mathrm{VOL}([0,1]^n) = 1$; $\mathrm{VOL}([-1,1]) = 2^n$. $\mathrm{BALL}(0,1) = \{x \mid \|x\|_2 \leq 1\}$



$\mathrm{VOL}(\mathrm{BALL}_n) \approx \left(\frac{\pi e}{2n}\right)^{n/2}$. BALL IS DECREASING VOLUME, LOST INSIDE CUBE

- MASS ACCUMULATES IN CORNERS