

LECTURE 2

• ZERO - SUM GAMES

• CONSIDER A WAR BETWEEN B AND E, CONTENDING 3 MOUNTAIN PASSES

EACH B AND E HAVE 5 SOLDIERS.

• DECISION BY COMMANDERS \rightarrow HOW TO DIVIDE THE 5 - SOLDIERS. (k, l, m) ; $k \leq l \leq m$; $k + l + m = 5$
THE GROUPS OF SOLDIERS DIVIDES THEMSELVES RANDOMLY BETWEEN THE 3 MOUNTAIN PASSES

• GIVEN A MOUNTAIN PASS, WHO SENDS MORE SOLDIERS CAPTURES THE PASS. IF # OF SOLDIERS IS THE SAME, YOU HAVE A DRAW.

• OBJECTIVE IS MAXIMIZING THE PROBABILITIES OF WINNING. THIS PROBLEM IS CALLED ZERO-SUM GAME SINCE PROFIT OF ONE IS LOSS OF OTHER.

EXAMPLE: B $(0, 0, 5)$; E $(0, 0, 5)$ \rightarrow TWO CASES: (a) BOTH GO IN SAME SPOT $(\frac{1}{3}) \rightarrow$ DRAW \rightarrow OUTCOME: 0

(b) THEY GO IN DIFFERENT SPOT $(\frac{2}{3}) \rightarrow$ 1 PASS EACH \rightarrow OUTCOME: 0

THE VARIOUS CASES ARE SUMMARIZED IN THE TABLE BELOW:

	$(0, 0, 5)$	$(0, 1, 4)$	$(0, 2, 3)$	$(1, 1, 3)$	$(1, 2, 2)$
$(0, 0, 5)$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	-1
$(0, 1, 4)$	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$(0, 2, 3)$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$
$(1, 1, 3)$	1	$\frac{1}{3}$	0	0	$-\frac{1}{3}$
$(1, 2, 2)$	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0

$(0, 2, 3)$ IS UNIQUE

DEFINITION: NASH EQUILIBRIUM: PAIR OF STRATEGY WHICH ARE BOTH RESPONSE TO ONE - ANOTHER (PURE - NASH IF CHOICE IS UNIQUE)

• LET'S TAKE ANOTHER EXAMPLE - ROCK, PAPER, SCISSOR

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

THERE IS NO SINGLE ACTION THAT COULD BE BEST RESPONSE TO ANOTHER

A: $X = (x_1, x_2, x_3)$, $x_1, x_2, x_3 \geq 0$; $x_1 + x_2 + x_3 = 1 \rightarrow$ MIXED STRATEGY

FACT: EVERY ZERO-SUM GAME HAS A MIXED - NASH EQUILIBRIUM

NEW PROBLEM: ROCK-PAPER-SCISSOR BY SANTA-CLAUS AND EASTER BUNNY

	R	P
R	0	-1
P	1	0
S	-1	1

LET'S FIX SC STRATEGIES : $x = (x_1, x_2, x_3)$

$$\min_y \sum_{i=1,2} x_i M_{i,j} y_i ; \quad E \left[\text{OUTCOME}, \text{GIVEN } x \right] \rightarrow \exists j \text{ s.t. } \text{OUTCOME}(x, j) \leq E \left[\text{OUTCOME}(x, j) \right]$$

• CHOICE FOR SANTA-CLAUS : ADD VARIABLE:

$$x = \arg \max \min \{ x_1 - x_3, -x_1 + x_3 \}$$

MAX x_0

S.T.

$$x_0 \leq x_1 - x_3$$

$$x_0 \leq -x_1 + x_3$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

OPTIMAL $\left(\overset{x_0}{\frac{1}{3}}, \overset{\text{ROCK}}{0}, \overset{\text{PAPER}}{\frac{2}{3}}, \overset{\text{SCISSOR}}{\frac{1}{3}} \right)$

• CHOICE FOR EASTER BUNNY:

$$y = \arg \min \max \{ -y_2, y_1, -y_1 + y_2 \} : \text{ADD VARIABLE}$$

MIN y_0

S.T.

$$y_0 \geq -y_2$$

$$y_0 \geq y_1$$

$$y_0 \geq -y_1 + y_2$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \geq 0$$

OPTIMAL $\left(\overset{y_0}{\frac{1}{3}}, \underset{\text{ROCK}}{\frac{1}{3}}, \underset{\text{PAPER}}{\frac{2}{3}} \right)$

• GEOMETRY PART

• **LINEAR SPACE**: $S \subseteq \mathbb{R}^d$ S.T. CLOSED TO MULTIPLICATION BY SCALAR AND ADDITION OF VECTORS. LINES/PLANES PASSING THROUGH ORIGIN.

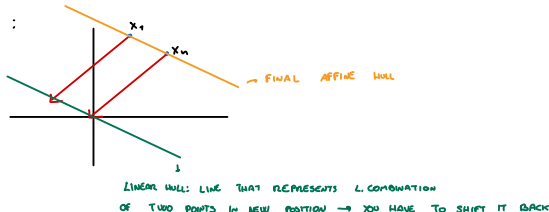
• **LINEAR INDEPENDENCE** $\{x_1 \dots x_n\}$ LIN. DEPENDENT IF $\exists \beta_1 \dots \beta_n$ S.T. $\sum \beta_i x_i = 0$ WITH AT LEAST ONE $\beta_i \neq 0$.

• **AFFINE SPACE**: $S = L + x$, $L \in \mathbb{R}^d$ LINEAR SPACE. EX: LINE SHIFTED, THAT DOES NOT PASS TO THE ORIGIN

• **AFFINE HULL**: $X = \{x_1 \dots x_n\}$; NONE THIS SET TO TOUCH ORIGIN: $x_1 - x_n, \dots, x_{n-1} - x_n, 0$. TAKE NOW L. COMB OF THU

$$\rightarrow \beta(x_1 - x_n) + \dots + \beta_{n-1}(x_{n-1} - x_n) + x_n \xrightarrow{\text{TO SHIFT IT BACK}}$$

GRAPHICALLY:



$$\beta_1(x_1 - x_n) + \dots + \beta_{n-1}(x_{n-1} - x_n) + x_n = \beta_1 x_1 + \dots + \beta_{n-1} x_{n-1} + (1 - \beta_1 - \dots - \beta_{n-1}) x_n$$

DEFINITION (AFFINE COMB): $X = \{x_1 \dots x_n\}$. $\sum \beta_i x_i$; S.T. $\sum \beta_i = 1$, $\beta_1 \dots \beta_n \in \mathbb{R}$

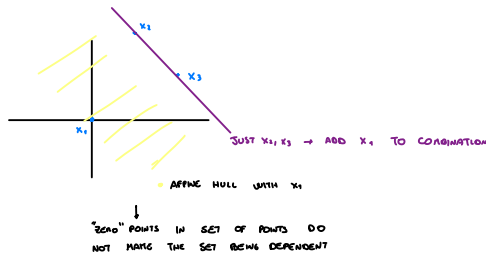
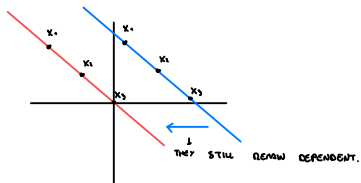
DEFINITION (AFFINE HULL): GIVEN X $\text{AFF. HULL}(X) = \bigcap_{A \in \mathcal{A}} A$; $A = \{A \mid \text{AFFINE SPACE}, X \subseteq A\}$

LEMMA: $\text{AFF. HULL}(X) = \{x \mid x = \sum \beta_i x_i, \sum \beta_i = 1\}$

DEFINITION: X IS AFFINE DEPENDENT IF $\exists \beta_1 \dots \beta_n$ S.T. $\sum \beta_i = 0$ AND $\sum \beta_i x_i = 0$, WITH AT LEAST ONE $\beta_i \neq 0$

• SAYING $(x_1 - x_n) \dots, (x_{n-1} - x_n)$ ARE LINEARLY DEPENDENT IS EQUIVALENT TO WHAT SAID ABOVE.

EXAMPLE:



• SAYING $(x_1 - x_n) \dots, (x_{n-1} - x_n)$ ARE LINEARLY DEPENDENT IS EQUIVALENT TO WHAT SAID ABOVE.

↳ SINCE IT MEANS

$$\beta_1(x_1 - x_n) + \dots + \beta_{n-1}(x_{n-1} - x_n) = 0 \Leftrightarrow \beta_1 x_1 + \dots + \beta_{n-1} x_{n-1} - (\beta_1 + \dots + \beta_{n-1}) x_n \rightarrow \text{SEE COEFFICIENTS SUM UP TO ZERO}$$

EXAMPLE: TAKE $X = \{x_1 \dots x_n\} \subseteq \mathbb{R}^d$. THERE CAN BE AT MOST $d+1$ AFF. IND. VECTORS

TAKE MATRIX A WITH i -TH COLUMN $(x_i - x_n)$ $A = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$ d ROWS X IS AFF. INDEP $\Leftrightarrow \det(A) \neq 0$