LECTURE 3

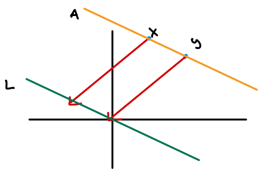
LINEAR COMBINATION : $\sum \lambda_i x_i \quad \lambda_i \in \mathbb{R}$

LINEAR DEPENDENCE : $\sum \lambda_i x_i = 0$ FOR SOME $\lambda_i \in \mathbb{R}$

AFFINE COMBINATION : $\sum \lambda_i x_i \; , \; \sum \lambda_i = 1$

AFFINE DEPENDENCE : $\sum \lambda_i x_i = 0$ FOR SOME $\sum \lambda_i = 0$ (· SAME AS L. DEP. OF $(x_1 - x_n) \dots (x_{n-1} - x_n)$)

EX: AFFINE COMBINATION



$t x + (1-t) y = y + t(x-y) \quad t \in \mathbb{R}$

DEFINITION AFFINE DIMENSION: $A = a + L$, $\dim(A) := \dim(L)$. TAKE $X \subseteq \mathbb{R}^d$. $\dim(X) := \dim(\text{aff. hull}(X))$?

- OBSERVE THAT $\dim(X) = \max k$, SO THAT X CONTAINS $k+1$ INDEPENDENT VECTORS.

- AFFINE SPACES

LINE: 1-dim AFFINE SPACE , PLANE: 2-dim AFFINE SPACE , HYPER-PLANE IN $\mathbb{R}^d$: $(d-1)$-dim AFFINE SPACE

- HYPERPLANE H $\quad H = \{ x \in \mathbb{R}^d \mid a^T x = b \}$, $a \in \mathbb{R}^d \setminus \{0\}$, $b \in \mathbb{R}$

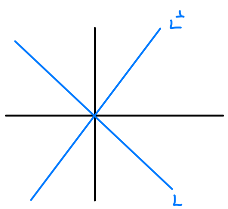
- * TAKE K-dim AFFINE SPACES, $k \le d-1$, ARE INTERSECTION OF HYPERPLANES. $k = d-1$, $A = a + L$. DEFINE $a^T(x,y) = b$

THERE IS ONLY ONE $\perp$ DIRECTION TO PLANE L. CALL IT a, WHICH IS ALSO UNIQUE. $A = y + L$ NOW

NOW $\forall x \in L$: $a^T x = 0$. $\forall x \in A$: $a^T(x-y) = 0 \iff a^T(x) = a^T(y) = b$

- COME BACK TO WHAT WE HAVE SEEN BEFORE. HOW TO EXPRESS $A = y + L$ AS INTERSECTION OF HYPERPLANES *

GRAPHICALLY:



$\forall x \in L^\perp , \; x' \in L : \quad x^T \cdot x' = 0$

ORTHOGONAL BASIS OF $\mathbb{R}^d$

SINCE $\underbrace{b_1 \dots b_k}_{\text{BASIS OF } L} , \underbrace{b_{k+1} \dots b_d}_{\text{BASIS OF } L^\perp}$

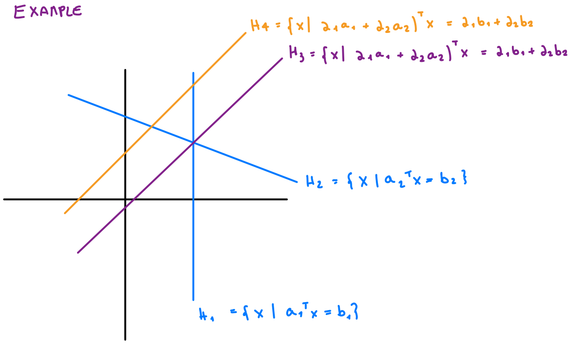
- TAKE $a_1 \dots a_{d-k}$ BASIS OF $L^\perp$

$\forall x \in L$: $\quad a_i^T x = 0$

MOREOVER $\forall x \in A \quad a_i^T(x-y) = 0 \; \forall i$; $\; a_i^T x = a_i^T y = b$

- $H_i = \{ x \in \mathbb{R}^d \mid a_i^T x = b \} \qquad A = \bigcap_{i=1}^{d-k} H_i$

EXAMPLE



DEFINITION : HALFSPACES : $\{ x \in \mathbb{R}^d \mid a^T x \le b \}$ $a \in \mathbb{R}^d \setminus \{0\}$, $b \in \mathbb{R}$

DEFINITION : LP - STANDARD FORM . $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

$$\max \ c^T x$$
$$\text{s.t.} \quad Ax \le b \iff a_i^T \le b \quad \forall i = 1..m$$ → I WANT TO FIND INTERSECTION OF HALFSPACES

- ANY LP CAN BE TRANSFORMED TO STANDARD FORM. ALSO WE NEED WEAK INEQUALITIES

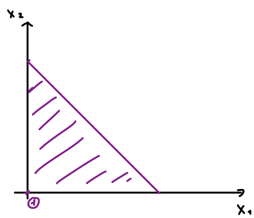
DEFINITION: LP IN EQUATIONAL FORM : $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$:

$$\max \ c^T x \quad \text{s.t.} \quad Ax = b, \ x \ge 0$$

- LP IN STANDARD FORM CAN BE TRANSFORMED TO LP IN EQ. FORM .
  - $Ax \le b \Rightarrow Ax + z = b$, $z \ge 0$, $z \in \mathbb{R}^n$.
  - $x_i \in \mathbb{R} \Rightarrow x_i^+ - x_i^-$, $x_i^+ \ge 0$, $x_i^- \ge 0$

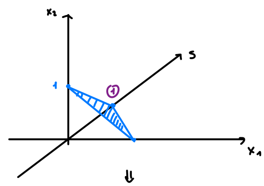
EXAMPLE

- $-x_1 \le 0$
- $-x_2 \le 0$
- $x_1 + x_2 \le 1$



EQ. FORM →

$x_1 + x_2 + s = 1$
$x_1, x_2 \ge 0$
$s \ge 0$



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THIS IS AN EQUILATER TRIANGLE, THIS CHANGED.

- THERE WILL BE CORRESPONDANCE BETWEEN FEASIBLE SOLUTION

DEFINITION: CONVEXITY : $X \subseteq \mathbb{R}^d$ IS CONVEX IF $\forall x, y \in X : tx + (1-t)y \in X$, $t \in [0,1]$

DEFINITION : CONVEX HULL $\operatorname{conv}(X) = \bigcap \{ C \text{ convex} \mid X \subseteq C \}$

DEFINITION: CONVEX COMBINATION : $x_1 .. x_k$, $\sum_i^k \lambda_i x_i$, $\sum \lambda_i = 1$, $\lambda_i \ge 0$

- SEE THAT IF $\forall x \in X : a^T x \le b \Rightarrow$ ANY CONVEX COMBINATION y OF X ALSO SATISFIES $a^T y \le b$

<u>PROOF:</u> $y = \sum_i^k \lambda_i x_i \Rightarrow a^T y = \sum_i^k \lambda_i a^T x_i \le \sum_i^k \lambda_i b = b$

↳ THIS WORKS ONLY IF $\lambda_i \ge 0$, ASSUMPTION WE HAVE IN CONVEXITY AND NOT IN AFFINITY.

- SEE $X \subseteq C$, C CONVEX $\Rightarrow$ IF Y IS CONVEX COMBINATION OF $X \Rightarrow y \in C$

- $\operatorname{CONV}(X) = \{ \sum \lambda_i x_i \mid k \in \mathbb{N}_+, x_i \in X, \lambda_i \ge 0, \sum \lambda_i = 1 \}$

<u>COROLLARY</u> : IF X IS FINITE $\Rightarrow \operatorname{CONV}(X)$ IS COMPACT