

CATapult Courseware

**Module 1**  
**Quantitative Ability**

Published by IMS Learning Resources Pvt. Ltd. in the Year 2020

Registered Office: 6th Floor, NCL Building, 'E' Block, Near Bandra Family Court,  
Bandra Kurla Complex (BKC), Bandra (E), Mumbai - 400051  
Tel.: +91 22 66170000  
Toll Free: 1800-1234-467  
CIN : U80220MH1999PTC121823  
E-mail : support@imsindia.com Website: www.imsindia.com

**Copyright © IMS Learning Resources Pvt. Ltd.**

All copyrights to this material vests with IMS Learning Resources Pvt. Ltd. No part of this material either in part or as a whole shall be copied, reprinted, reproduced, sold, distributed or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, or stored in any retrieval system of any nature without the permission of IMS Learning Resources Pvt. Ltd., and any such violation would entail initiation of suitable legal proceedings.

The views of and opinions expressed in this book are not necessarily those of the publishers. While every effort has been made to ensure that all facts are stated correctly, the publishers regret their inability to accept responsibility for any inadvertent errors or inaccuracies. Readers are advised in their own interest to reconfirm facts before acting upon them.

The publishers shall endeavour, wherever possible to remedy all errors of commission and omission which are brought to their attention in subsequent editions.

This book is sold subject to the condition that it shall not, but way of trader or otherwise, be lent, resold, hired out, or otherwise circulated without the publisher's prior written consent in any form of binding or cover other than that in which it is published and without a similar condition including this condition being imposed on the subsequent purchaser and without limiting the rights under copyright reserved above.



## QA-Intro | INTRODUCTION TO QA-DI

## CLASS EXERCISE

### Teaser

Step 1: Take a 3-digit number

Step 2: Write the same number again next to it to make it a 6-digit number (for example if you chose 365, the 6-digit number would be 365365)

Step 3: Divide this 6-digit number by 7

Step 4: Divide the resultant number (from step 3) by 11.

Step 5: Divide the resultant number (from step 4) by 13.

What do you get?

## Quantitative Ability

- 1) Add plus or minus signs in between the numbers below such that the result is 100

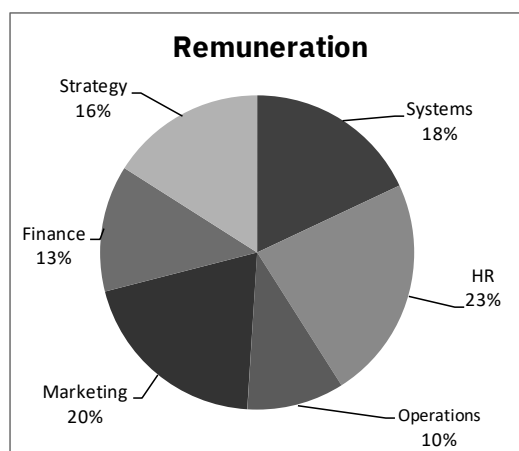
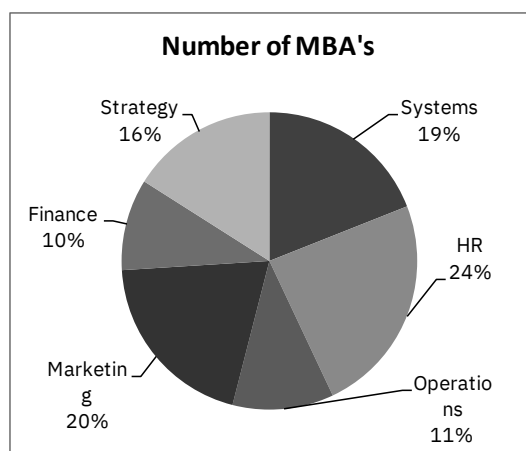
**1 2 3 4 5 6 7 8 9**

(For example, a possible solution is  $12 + 3 - 4 + 5 + 67 + 8 + 9 = 100$ . Can you find two more?)

- 2) If you write numbers from 1 to 100, how many digits will you have written?  
 1) 101                      2) 189                      3) 192                      4) 300
- 3) If a sum of Rs 200 is to be paid using notes of denomination Rs 2, Rs 5, Rs 10 and Rs 25 only, using at least one note of each kind, then which of the following could be the number of notes used?  
 1) 11                      2) 28                      3) 84                      4) 99
- 4) What is the number of digits in the product  $(100 - 1)(100 - 2)(100 - 3) \dots (100 - 199)(100 - 200)$ ?  
 1) 1    2) 400  
 3) Cannot be determined                      4) None of the above
- 5) If 4 hens lay 4 eggs in 4 days, in how many days will 8 hens lay 8 eggs?  
 1) 2                      2) 4                      3) 8                      4) 16

## Data Interpretation - Calculative

The charts below show the results of a survey conducted on 21,853 MBAs who passed out from the top 50 B-schools in 2007. The first chart shows the percentage breakup of the total respondents currently employed in a particular specialisation. The second shows the percentage breakup of the total current remuneration (Rs 2622 million per month) by specialisation.



- 1) Approximately how many MBAs who graduated in 2007 are currently working in Systems?  
 1) 3675                      2) 3989                      3) 4152                      4) 4391

- 2) If the above data is representative, which field of specialisation offers the highest average salary?  
1) HR                      2) Finance                      3) Systems                      4) Marketing
- 3) Approximately how much more did Systems MBAs earn per month than Finance MBAs, in total?  
1) 156 Million                      2) 141 Million                      3) 118 Million                      4) 131 Million
- 4) How much was the average annual salary of a Finance MBA in 2007?  
1) 18.72 lakhs                      2) 2.31 lakhs                      3) 1.56 lakhs                      4) 15.6 lakhs
- 5) Approximately how much more did a typical Marketing MBA earn than a Strategy MBA?  
1) < 2000                      2) 2000-9000                      3) 9000-20000                      4) Cannot be determined

### Data Interpretation – Reasoning based

- 1) Three people are discussing their ages. They discover that the sum of their age is exactly 100. Also, all three of them have a different age. If the youngest among them is 32 years old, how old is the eldest? (Assume that the age of each of the three people is an integer value)  
1) 33 years                      2) 34 years                      3) 35 years                      4) 36 years
- 2) Five friends A, B, C, D and E played a game of dice. Each of them rolled 3 fair dice (bearing numbers from 1 to 6), and added the three numbers to calculate their scores. The person with the highest total won the game. It was additionally observed that:
  - Each person had three different numbers on his three dice.
  - No two people achieved the same total score.
  - D, the person who achieved the lowest score rolled a 6.
  - The total score of all the five friends was 55.
 How much is the score of A, if A won the game?  
1) 12                      2) 13                      3) 14                      4) 15

### Answer questions 3 – 5 based on the following information:

Four people A, B, C and D are playing a game. Each of them starts with a different number of points. At the end of each round, one winner is declared. Each of the other three players gives exactly half his points to the winner. After three rounds, the scores for A, B, C and D respectively are 42, 12, 4 and 24.

- 3) Who won the first round?  
1) A                      2) B                      3) C                      4) D
- 4) Who won the second round?  
1) A                      2) B                      3) C                      4) D
- 5) How many points did B have at the start of the game?  
1) 14                      2) 12                      3) 24                      4) 96

**Data Sufficiency**

***In the following questions, mark:***

***(1) if the Question can be answered from one of the Statements alone but not from the other alone.***

***(2) if the Question can be answered from Statement I alone as well as from Statement II alone.***

***(3) if the Question can be answered from Statements I and II together but not from either alone.***

***(4) if the Question cannot be answered even from Statements I and II together.***

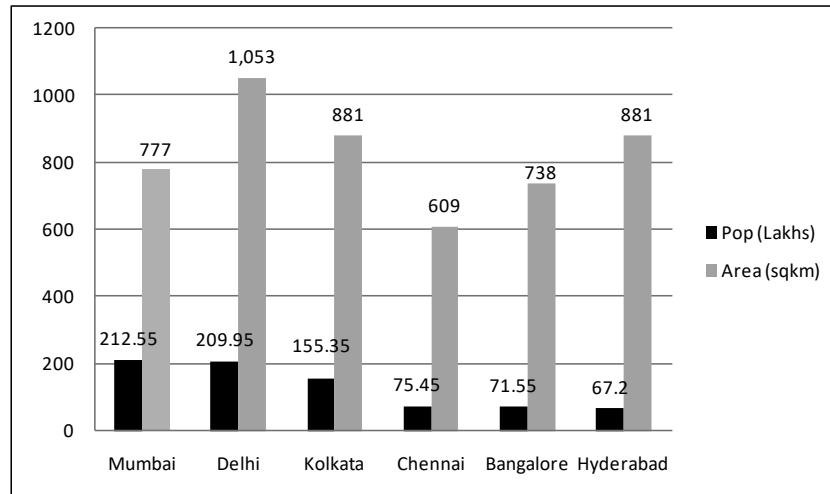
- 1) Is Jack taller than Jill?
  - I) Jack is taller than Tom, who is taller than Jill
  - II) Jill is shorter than Bartholomew, who is shorter than Tom
- 2) Who won the elections in Ramgarh in 2010?
  - I) 65% of the population of Ramgarh did not vote for Jai
  - II) 49% of the population of Ramgarh did not vote for Veeru
- 3) How much money does Peter have?
  - I) If the amount of money Peter has is tripled, it will be less than \$100
  - II) If the amount of money Peter has is doubled, it will be more than \$65
- 4) There are some cats and some pigeons in a room. How many cats and how many pigeons are there?
  - I) The cats and pigeons have a total of 5 heads
  - II) The cats and pigeons have a total of 14 legs
- 5) In a certain year, is 31st Dec of that year a Sunday?
  - I) 31st January of the next year will be a Wednesday
  - II) 31st December of the previous year was a Saturday

### Challengers

- 1) Let the sum of the digits of a number X be represented by Sum(X). For example Sum(256) = 2 + 5 + 6 = 13. Suppose A and B are two 3-digit numbers, what could be the maximum value of Sum(A) – Sum (B)?  
 1) 24                                  2) 26                                  3) 27                                  4) 29
- 2) If it takes 1 second to cut 1 m of rope, how long will it take to cut a 200 m long rope into 1 m pieces?  
 1) 3 min 20 sec                  2) 3 min 19 sec                  3) 3 min 30 sec                  4) None of these
- 3) A ladder hangs over the side of a ship anchored in a port. The bottom rung touches the water. The distance between rungs is 20 cm and the length of the ladder is 180 cm. The tide is rising at the rate of 15 cm each hour. When will the water reach the seventh rung from the top?  
 1) 9 hours                              2) 10 hours                              3) 12 hours                              4) None of the above

**Directions to questions 4 and 5: Refer to the bar graph below and answer the questions that follow:**

The following graph gives data collected by Rahul from the internet for a class project on the Areas and Populations of some major metropolitan cities of India



**Note:** Population density for a given city (in people per square km) is defined as  $\frac{\text{Population}}{\text{Area}}$

- 4) Which city has the lowest population density among the given cities?  
 1) Delhi                                  2) Hyderabad                                  3) Chennai                                  4) Bangalore
- 5) What is the population density of Delhi, in people per square km?  
 1) 0.20132                                  2) 21231.7                                  3) 19938.2                                  4) 20013.2
- 6) Three friends P, Q and R are playing a series of games. Each game had only one winner. P lost 3 games, Q lost 4 games and R lost 5 games. How many games did they play in all?  
 1) 8    2) 9    3) 6    4) 12
- 7) There are 123 apples in an orchard. The farmer plucks 61 of them. How many apples does he have?  
 1) 60    2) 61    3) 62    4) 63



## PRACTICE EXERCISE

**Directions to questions 1 to 5: Choose the correct alternative.**

- 1) A basket contains 5 apples. Can you divide them among 5 kids so that each one has an apple and one apple stays in the basket?
- 2) What mathematical symbol can be put between 5 and 7 to get a value bigger than 5 and smaller than 7?
- 3) If 5 bakers make 5 cakes in 5 days, how many cakes will 10 bakers make in 10 days?  
1) 5                                      2) 10                                      3) 15                                      4) 20
- 4) Mary's mum has four children. One child is called April, another May and a third June. What is the name of the fourth child?  
1) March                                      2) July  
3) Cannot be determined                                      4) None of the above
- 5) A three-digit number 'n' (less than 500) is taken. A six-digit number is formed by writing the number 'n' as the first three digits and the number '2n' as the last three. Which of the following numbers is not necessarily a divisor of the number so obtained?  
1) 6                                      2) 37                                      3) 167                                      4) 501

**Directions to questions 6 to 10: Refer to the data below and answer the questions that follow:**

A multi-round election was conducted in which there were 5 candidates P, Q, R, S, T.

- In the first round, 60 votes were cast, and on counting it was observed that all the candidates got a different number of votes, and the results (in descending order) were T, P, R, S, Q. The last candidate, Q, who had scored 10 votes, was eliminated.
  - In the second round, again 60 votes were cast. T scored the same number of votes as he had in the 1st round, and as a result was eliminated while P topped the round.
  - In the third and final round, again featuring 60 votes, S managed to get 6 more votes than she did in the second round and as a result was declared the winner. P finished in second position.
- 6) How many votes did T get in the first round?  
1) 12                                      2) 14  
3) 15                                      4) Cannot be determined
  - 7) How many votes did P get in the second round?  
1) 15                                      2) 16  
3) 17                                      4) Cannot be determined
  - 8) How many votes did S get in the second round?  
1) 15                                      2) 16  
3) 17                                      4) Cannot be determined
  - 9) How many votes did R get in the third round?  
1) 17                                      2) 18  
3) 19                                      4) Cannot be determined
  - 10) How many votes, in total, did S score across all the three rounds?  
1) 50                                      2) 49                                      3) 48                                      4) 47



## QA-1.1 | PROPERTIES OF NUMBERS



### Integers

An **Integer** is any number in the set {...,-3, -2, -1, 0, 1, 2, 3.....}.Integers comprise **natural numbers** (1,2,3.....),their negatives (-1,-2,-3.....) and the number zero (0).

**Natural numbers** are counting numbers starting with '1'. The set of natural numbers is usually denoted by N i.e. {1,2,3.....}

**Whole numbers** comprise natural numbers and the number zero. The set of whole numbers is usually denoted by W i.e. {0,1,2,3.....}

### Fractions

**Fractions:** A fraction is a number in the form  $\frac{x}{y}$  (where x and y are both integers and  $y \neq 0$ ). In the above number form, 'x' is called the numerator of the fraction and 'y' is called the denominator of the fraction

Fractions can be classified as:

**Proper Fractions:** All fractional values where the numerator is less than the denominator.

e.g.  $\frac{3}{7}, \frac{4}{9}$ .

**Improper Fractions:** All fractional values where the numerator is greater than or equal to the denominator. e.g.  $\frac{13}{9}, \frac{15}{8}$ .

An improper fraction can be expressed as a **mixed fraction**.

#### Example

$$\frac{17}{4} = \frac{16+1}{4} = \frac{16}{4} + \frac{1}{4} = 4 + \frac{1}{4} = 4\frac{1}{4}$$

Similarly, a mixed fraction can be expressed as an improper fraction.

#### Example

$$3\frac{2}{5} = 3 + \frac{2}{5} = \frac{15+2}{5} = \frac{17}{5}$$

### Decimal Fractions

A fraction in which the denominator can be expressed as a power of 10 is called a decimal fraction.

#### Example

$$\frac{6}{10}, \frac{7}{100}, \frac{8}{1000} \dots \text{etc.}$$

### Rational numbers

Integers and fractions together constitute **rational numbers**. Rational numbers are either terminating or recurring. e.g. 3, -5,  $\frac{2}{7}$ , etc

### Terminating Numbers

$$\frac{1}{2} = 0.5 \quad \frac{1}{4} = 0.25 \quad \frac{1}{8} = 0.125$$

All the above values terminate after a particular point. All such numbers are terminating numbers.

### Recurring Numbers

$$1.333... = 1.\overline{3}, 2.565656... = 2.\overline{56}$$

$$\frac{1}{7} = 0.14285714857..... = 0.\overline{142857} \quad \frac{1}{14} = 0.0714285714285..... = 0.\overline{0714285}$$

Above values are non-terminating and repetition of digit/s starts after a particular point. Every recurring number can be expressed as a regular fraction.

### Example

i.  $0.\overline{3} = 0.3333 \dots$

Suppose  $x = 0.\overline{3} = 0.3333 \dots$  (I)

$\therefore 10x = 3.3333 \dots$  (II)

(II)-(I) gives—

$$9x = 3$$

$$\therefore x = 0.\overline{3} = \frac{3}{9}$$

ii.  $0.\overline{23} = 0.232323 \dots$

Suppose  $x = 0.\overline{23} = 0.23232323 \dots$  (I)

$\therefore 100x = 23.232323 \dots$  (II)

(II)-(I) gives—

$$99x = 23$$

$$\therefore x = 0.\overline{23} = \frac{23}{99}$$

### Irrational numbers

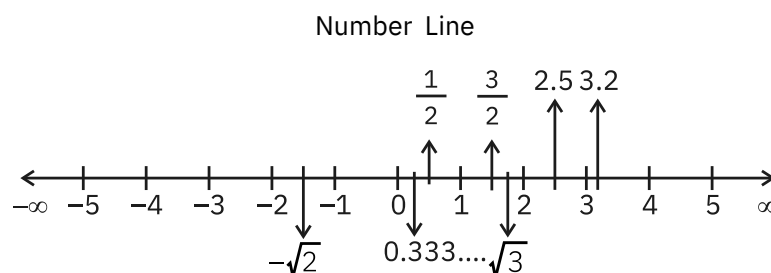
**Irrational numbers** are non-terminating and non-recurring.

e.g.  $\sqrt{2} = 1.414213562373095.....$  ,  $\sqrt{5} = 2.236067977499.....$

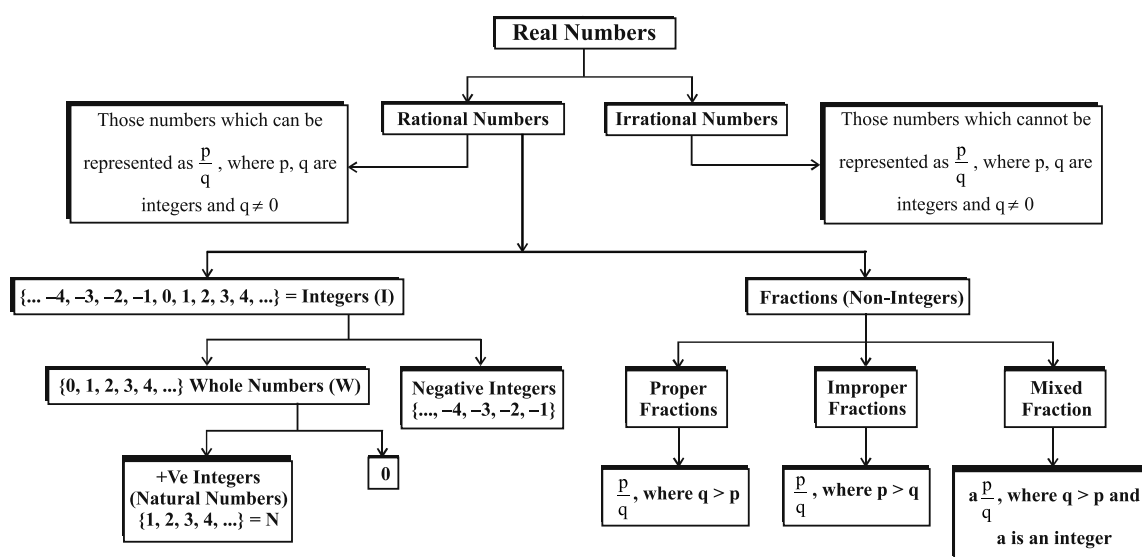
### Real numbers

All rational and irrational numbers constitute **Real numbers**. Real numbers can be represented on a number line.

The numbers can be represented on the number line as follows:



Summarizing what we have learnt so far:



### Complex numbers

It was found that the system of real numbers was inadequate, for it does not contain square roots of negative numbers. Imaginary numbers were introduced to overcome this inadequacy.

A number of the form  $bi$ , where  $b$  is a real number,  $b \neq 0$  and  $i = \sqrt{-1}$  is called an **imaginary number**.

### Example

$$5i, -2i, \sqrt{3}i$$

A number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$  is called a complex number.

### Example

$$\sqrt{5} + i\sqrt{2}, 1 + i^3\sqrt{2}$$

For the complex number  $a + bi$ ,  $a$  is called the real part and  $b$  the imaginary part. The set of complex numbers is denoted by  $C$ .

### About Complex Numbers

- I. If  $a = 0$  and  $b \neq 0$  then the number becomes a imaginary number.
- II. Any real number is a complex number since any real number  $x$  can be written as  $x+i(0)$
- III.  $i = \sqrt{-1}$ ;  $i^2 = -1$ ;  $i^3 = -i$ ;  $i^4 = 1$

### Conjugate of a complex number

The conjugate of a complex number is obtained by changing the sign of the imaginary part. If  $(a + bi)$  is a complex number, then  $(a - bi)$  is its complex conjugate and vice versa.

### Example

$2 + 3i$  is the conjugate of  $2 - 3i$

The product of a complex number and its conjugate will be a real number.

$$(2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13.$$

To simplify  $\frac{1}{a+bi}$ , we can multiply it by the conjugate of  $(a + bi)$  i.e.  $a - bi$ .

$$\text{It gives, } \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}.$$

### Concept Builder 1

**Directions for questions 1 to 7: Mark the following statements as True/False:**

1. Every complex number is a real number.
2. Every irrational number is a complex number.
3. Every real number can be represented on a number line.
4. All rational numbers are recurring.
5. Every rational number is an integer.
6. Every integer is a whole number.
7. Every complex number can be represented on a number line.
8. Solve: The value of the decimal 0.454545 will be .....

### Answer Key

- |          |          |         |          |          |          |          |                    |
|----------|----------|---------|----------|----------|----------|----------|--------------------|
| 1. False | 2. False | 3. True | 4. False | 5. False | 6. False | 7. False | 8. $\frac{45}{99}$ |
|----------|----------|---------|----------|----------|----------|----------|--------------------|

## Properties of Integers

### Consecutive integers

Integers that occur in a sequence and where the difference between two successive integers is 1, are **consecutive integers**.

Consecutive integers can be expressed in the form:

$n, n + 1, n + 2, n + 3, \dots$ , where 'n' is an integer.

### Even numbers

All natural numbers that are divisible by 2 are called **even numbers**. These numbers can be represented in the form ' $2n$ ' (where 'n' is a natural number). e.g. 2, 4, 14, 18 etc

**Consecutive even numbers** can be expressed in the form  $2n, 2n+2, 2n+4, 2n+6, \dots$ , where 'n' is a natural number.

### Odd numbers

All natural numbers that are not divisible by 2 are called **odd numbers**. These numbers can be represented in the form ' $2n+1$ ' (where 'n' is a whole number) or ' $2n-1$ ' (where 'n' is a natural number). e.g. 1, 3, 5, 7, 9, 15 etc

**Consecutive odd numbers** can be expressed in the form  $2n+1, 2n+3, 2n+5, 2n+7, \dots$ , where 'n' is a whole number

### Factors and Multiples

An integer 'n' is called a **factor or a divisor** of another integer 'm' if 'n' divides 'm' exactly (i.e. the remainder is zero) and in such a case 'm' is called a **multiple** of 'n'.

e.g. 1, 2, 3, 4, 6 and 12 are the factors (or divisors) of 12 and 12, 24, 36, 48, ..... are its multiples.

### Prime numbers

A natural number greater than 1, that has exactly two factors, i.e., 1 and the number itself, is called a **prime number**. e.g. 5, 7, 11, 13 etc

**Note:** 2 is the only even prime number.

Any prime number greater than 3 can be expressed as  $6k+1$  or  $6k-1$  (where 'k' is a positive integer). However all the numbers in that form are not necessarily prime.

e.g.  $5 = 6 \times 1 - 1$ ;  $7 = 6 \times 1 + 1$ . However  $25 = 6 \times 4 + 1$  (is not prime)

### Composite numbers

Any number that has more than two factors (i.e. other than itself and the number 1) is called a **composite number**. e.g. 4, 6, 8 etc

**Note:** 1 is neither prime nor composite

### HCF (Highest Common Factor)

HCF is the largest integer that perfectly divides two or more given integers. It is also known as the GCD (Greatest Common Divisor).

#### Example

Find the HCF of 12 and 18

The factors or divisors of 12 are: 1, 2, 3, 4, **6**, 12

The factors or divisors of 18 are: 1, 2, 3, **6**, 9 and 18

The highest factor common to both 12 and 18 is 6,  $\therefore$  the HCF of 12 and 18 is 6.

### LCM (Lowest Common Multiple)

**LCM** is the smallest integer that is perfectly divisible by two or more given integers.

#### Example

Find the LCM of 12 and 18

The multiples of 12 are: 12, 24, **36**, 48, 60.....

The multiples of 18 are: 18, **36**, 54, 72 .....

The least multiple that is common to both numbers is 36,  $\therefore$  the LCM of 18 and 12 is 36.

### Co-Prime or Relatively prime numbers

If two numbers have no factor common other than the number 1, then the numbers are **co-prime** to each other. e.g. 2 and 3 are co-prime to each other. 15 and 28 are co-prime to each other.

$$15 = 3 \times 5 \text{ and}$$

$$28 = 2 \times 2 \times 7$$

There is no common factor.

LCM of two prime (or co-prime) numbers is the product of the two numbers and the HCF of two prime (or co-prime) numbers is 1.

If a & b are prime (or co-prime) numbers,

$$\text{LCM (a, b)} = a \times b \text{ and } \text{HCF (a, b)} = 1.$$

### Absolute value of an integer

**Absolute value** is the distance of the integer from zero on the number line. We use modulus (| |) sign to indicate absolute value.

$$|n| = n, \text{ where 'n' is an integer.}$$

#### Example

$$|3| = 3; |-5| = 5$$

### Factorial

A factorial is a number obtained by multiplying all the positive integers less than or equal to a given positive integer. The factorial of a given integer 'n' is usually written as  $n!$  or  $\underline{n}$ .

$$n! = 1 \times 2 \times 3 \times 4 \times \dots \times n$$

### Example

$$1! = 1$$

$$2! = 1 \times 2 = 2$$

$$3! = 1 \times 2 \times 3 = 6$$

$$4! = 1 \times 2 \times 3 \times 4 = 24$$

### About Factorials

1. By convention,  $(\text{zero})! = \text{unity i.e., } 0! = 1$
2. Factorial is defined only for whole numbers.
3.  $n! = n \times (n - 1)!$

## Basic Arithmetic Operations

### Addition

Addition is the process of finding a single number or decimal fraction or fraction, which is equal to two or more given quantities taken together.

### Subtraction

Subtraction is the method of finding what quantity is left when a smaller number/decimal/ fraction is taken from greater one. The greater number/decimal/fraction is called the **minuend**, the smaller, the **subtrahend** and the number left is called the **remainder**. The difference between any two numbers is unaltered if the same number be added to each.

### Multiplication

Multiplication is the short process of finding the sum of a given number of repetitions of the same number. The number to be repeated or multiplied is called the **multiplicand** and the number, which indicates how often the multiplicand is to be repeated is called the **multiplier**. The sum of the repetitions is called the **product**.

#### Example

$17 \times 2 = 34$ . 17 is the multiplicand; 2 is the multiplier and 34 is the product.

The multiplier and the multiplicand may also be interchanged.

#### Example

$2 \times 3 = 6$  also  $3 \times 2 = 6$ .

#### Example

$3 \times 0 = 0$

If one factor is zero, the product is zero.

#### Example

$100 \times 3000$  multiply 1 and 3

$\Rightarrow 1 \times 3 = 3$  add 5 zeros to 3 i.e. 300000 i.e.,  $100 \times 3000 = 300000$ .

#### Example

$5 \times 5 = 5^2$ ;  $3 \times 3 \times 3 \times 3 = 3^4$



### Power of a number

The product obtained by multiplying together several factors all equal to the same number is called the **power of that number**.

#### Example

$$3 \times 3 \times 3 = 3^3, 4 \times 4 \times 4 \times 4 \times 4 = 4^5.$$

### Division

Division is the method of finding how often a given number (called the **divisor**) is contained in another given number (called the **dividend**). The number expressing this is called the **quotient** and the excess of the dividend over the product of the divisor and quotient is called the **remainder**.

#### Example

$64 \div 5$ , 64 is the dividend, 5 is the divisor, 12 the quotient and 4 the remainder.

$$64 = (5 \times 12) + 4$$

**Note:** Dividend = (Quotient  $\times$  Divisor) + Remainder

If the Remainder = 0, then the Divisor is a factor of the dividend, the Quotient is a factor of the dividend and the dividend is the multiple of both the divisor and the quotient.

To divide a number by 10, 100, 1000, etc. cut off one, two, three, etc. digits respectively from the right of the number, the digits cut off represent the remainder, and the remaining figure is the quotient.

#### Example

$1563 \div 100 \Rightarrow$  the quotient is 15 and the remainder is 63.

### Basic Arithmetic Operations on Even and Odd Numbers

1. Sum of two odd numbers is always even.

#### Example

$$1 + 3 = 4, 3 + 7 = 10.$$

2. Sum of two even numbers is always even.

#### Example

$$2 + 4 = 6, 4 + 6 = 10.$$

3. Sum of odd and even number is always odd.

#### Example

$$1 + 2 = 3, 3 + 4 = 7.$$

4. Product of two even numbers is even.

**Example**

$$2 \times 4 = 8, \quad 4 \times 6 = 24.$$

5. Product of two odd numbers is odd.

**Example**

$$1 \times 3 = 3, \quad 3 \times 5 = 15, \quad 3 \times 7 = 21.$$

6. Product of odd and even numbers is even.

**Example**

$$2 \times 3 = 6, \quad 4 \times 9 = 36.$$

7. The base number determines whether the power of the number is even or odd.

$$x^n = x \times x \times \dots \times x \quad n \text{ number of times}$$

**Example**

- (i) Any integer power of an even number is even.

$$\text{e.g., } 2^5 = 32, \quad 2^6 = 64$$

- (ii) Any integer power of an odd number is odd.

$$\text{e.g., } 3^4 = 81, \quad 3^5 = 243.$$

### Basic Arithmetic Operations on Fractions

1. Fractions having the same denominator are called like fractions and they can be easily added.

**Example**

$$\frac{4}{7} + \frac{6}{7} = \frac{10}{7}$$

2. **Reciprocal** of a number means one divided by that number.

**Example**

$$\text{reciprocal of } \frac{3}{2} \text{ is } \frac{2}{3}$$

3. Division by a fraction means multiplication by its reciprocal.

**Example**

$$\frac{3}{5} \div \frac{5}{11} = \frac{3}{5} \times \frac{11}{5} = \frac{33}{25}$$

4. Calculation of fractions using LCM

If the fractions have different denominators, they must first be reduced to like fractions, having the same denominator. This can be done by taking the LCM of the various denominators as the common denominator.

### Example

$$\frac{3}{8} + \frac{9}{10} + \frac{7}{12} = \frac{45}{120} + \frac{108}{120} + \frac{70}{120} = \frac{223}{120} \text{ (LCM of 8, 10 \& 12 = 120)}$$

In adding mixed fractions, the integral and fractional parts should be collected separately.

### Example

$$\begin{aligned} 5\frac{95}{100} + \frac{65}{52} &= 5\frac{19}{20} + \frac{5}{4} = 5\frac{19}{20} + 1\frac{1}{4} \\ &= 5 + 1 + \frac{19}{20} + \frac{1}{4} = 6 + \frac{24}{20} = 6 + 1\frac{4}{20} = 7\frac{1}{5} \end{aligned}$$

### Rules of Signs

The sign of the final value in case of a multiplication or a division of two or more numbers depends on the sign of the numbers involved. By convention we use the following rules:

1. (-ve number)  $\times$  (-ve number) = +ve number
2. (-ve number)  $\times$  (+ve number) = -ve number
3. (-ve number)  $\div$  (-ve number) = +ve number
4. (-ve number)  $\div$  (+ve number) = -ve number
5. (+ve number)  $\div$  (-ve number) = -ve number

### Perfect Number

If the sum of the divisors of N excluding N itself is equal to N, then N is called a perfect number.

#### Example

6, 28, 496, 8128, 33550336.

$6 = 1 + 2 + 3$ , where 1, 2 and 3 are the divisors of 6

$28 = 1 + 2 + 4 + 7 + 14$

$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$

The sum of the reciprocals of the divisors of a perfect number including that of its own is always = 2

#### Example

For the perfect number 28,  $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$

### VBODMAS Rule

When we have to perform a series of mathematical operations, there is a rule regarding the order in which we should perform these operations. This rule is **VBODMAS** rule. Here,

- V** – means Vinculum (bar) eg.  $3 \times \overline{2+5}$
- B** – means Bracket, {}, [], ()
- O** – means Of (i.e. multiplication)
- D** – means Division ( $\div$ ,  $/$ )
- M** – means Multiplication ( $\times$ )
- A** – means Addition (+)
- S** – means Subtraction ( $\div$ )

It gives the order in which the various operations are to be performed.

#### Note:

- Vinculum can be used to give priority to an operation over other operations.  
e.g.  $3 \times \overline{2+5} = 3 \times 7 = 21$   
Here, the part of the expression having a bar over it is solved first.
- If there are brackets inside brackets, the innermost bracket is solved first.  
 $[3 \times \{15 \div (8 - 3)\}]$   
 $= [3 \times \{15 \div 5\}] \dots\dots ( )$   
 $= [3 \times 3] \dots\dots \{ \}$   
 $= 9 \dots\dots [ ]$
- Of means multiplication  
eg.  $\frac{1}{3}$  of 27 =  $\frac{1}{3} \times 27 = 9$
- If divisions and multiplications are the consecutive operations, then perform the operations in the order of appearance.  
eg.  $15 \times 4 \div 12 = 60 \div 12 = 5$
- If additions and subtractions are the consecutive operations, then perform the operations in the order of appearance.  
eg.  $100 - 47 + 32 = 53 + 32 = 85$

#### Example

$$\begin{aligned}
 \text{(i)} \quad & \frac{1}{4} \text{ of } \frac{4}{3} \{ (2 \times 3) + 4 \times \overline{4+1} \} + \frac{1}{3} \\
 &= \frac{1}{4} \text{ of } \frac{4}{3} \{ (2 \times 3) + 4 \times 5 \} + \frac{1}{3} \quad \text{(Vinculum)} \\
 &= \frac{1}{4} \text{ of } \frac{4}{3} \{ 6 + 26 \} + \frac{1}{3} \quad \text{(Inner bracket)} \\
 &= \frac{1}{4} \text{ of } \frac{4}{3} \times 26 + \frac{1}{3} \quad \text{(Outer bracket)} \\
 &= \frac{1}{3} \times 26 + \frac{1}{3} \quad \text{(Of)} \\
 &= \frac{26}{3} + \frac{1}{3} = \frac{27}{3} = 9
 \end{aligned}$$

$$\text{(ii)} \quad \frac{(1+3) \times 2 \div 4}{2 \times 2 - 2 \div 2 + 2} = \frac{4 \times \frac{2}{4}}{2 \times 2 - 1 + 2} = \frac{2}{4 - 1 + 2} = \frac{2}{4 + 1} = \frac{2}{5}$$

### Symbol Based BODMAS

Addition, Subtraction, Multiplication and Division are the fundamental operators in Mathematics denoted by symbols  $+$ ,  $-$ ,  $\times$  and  $\div$  respectively.

These operators need not always be represented by the same symbol in different streams. e.g., multiplication in pure mathematics is denoted by ' $\times$ ' whereas in C computer language, the symbol '\*' denotes multiplication. The statement less than or equal to can be denoted by  $\leq$  or  $<=$ . The symbol '||' which usually indicates 'parallel' in geometry, also represents the logical operator 'or' in C language. The symbol '%' which means percentage in Arithmetic also implies the integer value when an integer 'a' is divided by another 'b' and is written in the form  $a \% b$ .

In symbol-based problems a symbol or an operator is given and the operation which could be performed using that symbol is also defined.

The objective of these type of problem is to test the ability to perform mathematical and logical operation, not going by the conventional symbols but by the new symbols defined. To solve these kinds of questions one has to replace the new symbols by the real ones and arrive at the answer.

While solving mathematical expression after replacement of the symbols by the original mathematical symbol proceed according to the rule of BODMAS.

#### Example

If ' $+$ ' mean ' $\times$ ' and ' $\times$ ' means ' $+$ ', then find the value of  $7 \times 8 + 5$

Substituting the actual symbols.

$$7 + 8 \times 5 = 7 + 40 = 47$$

### Concept Builder 2

Solve the following according to the instructions as per the question:

- Is the expression  $7^{13} + 6^{12} + 1^{11} + 4^7$  even/odd?
- Find the value of  $\frac{72540}{5} + \frac{72540}{25} + \frac{72540}{125}$ .
- Find the value of the expression  $1 + 4 \div 3 \times 9$  of  $(6 - 2)$ .
- If  $x \# y = 2\left(\frac{x+y}{x-y}\right)$ , then find the value of  $\frac{(4\#3)}{7}$ .
- Find the value of  $\frac{|133 - 9| + [5 - 211] + |8 - 9|}{|110 - 9| + [19 - 101] + 8}$ .

#### Answer Key

1. Even      2. 17989.92      3. 49      4. 2      5. -3

## SOLVED EXAMPLES

**Q :** Express  $\frac{316}{17}$  as a mixed fraction.

**A :** Here we divide the numerator by the denominator. The quotient gives the number of units in the whole number, and the remainder (or number of parts over) gives the numerator of the proper fraction. The denominator of the proper fraction is the same as that of the original improper fraction. The intermediate step of the written work may be dispensed with after a little practice.

$$\frac{316}{17} = 18\frac{10}{17}$$

$$\begin{array}{r} 17 \overline{) 316} \phantom{(18)} \\ \underline{17} \phantom{0} \\ 146 \\ \underline{136} \\ 10 \end{array}$$

**Q :** Express  $7\frac{3}{11}$  as an improper fraction.

$$\mathbf{A : } 7\frac{3}{11} = \frac{7 \times 11 + 3}{11} = \frac{77 + 3}{11} = \frac{80}{11}$$

Here we reverse the process. Since each unit contains 11 elevenths, 7 units = 77 elevenths. To this product we add 3 to obtain the numerator of the improper fraction.

**Q :** Reduce  $\frac{609}{1595}$  to its lowest terms.

$$\mathbf{A : } \frac{609}{1595} = \frac{3 \times 203}{5 \times 319} = \frac{3 \times 7 \times 29}{5 \times 11 \times 29} = \frac{21}{55}$$

Here, no common factor is at once evident, but we can easily recognize the prime factors 3, 7, 5 and 11. Thus, revealing the common factor 29.

**Q :** To find whether 467 is prime or not.

**A :** We find by trial that 467 is not divisible by any of the primes 2, 3, 5, 7 ..... 19, 23. But on dividing by 23 we get a quotient less than 23, and we need not go further. For, if 467 contained a prime factor beyond 23, the quotient obtained on dividing 467 by that prime factor would be less than 23, and must have been revealed as a factor by the former trials. Hence, 467 has no prime factor greater or less than 23; that is, it is prime number. In general, to find whether a number is prime or not, we need to check whether it is divisible by any of the primes up to the square root of that number.

The square of a composite number must contain the square of every factor of that number.

e.g.,  $6 = 2 \times 3 : 6^2 = 2 \times 3 \times 2 \times 3 = 2^2 \times 3^2$

$56 = 2^3 \times 7 : 56^2 = 2^3 \times 7 \times 2^3 \times 7 = 2^6 \times 7^2$

It will be noticed that each prime factor of the number is repeated an even number of times in the square of the number. Conversely, when a square number has been expressed in prime factors, its square root can be written down at once by simply halving the index of the power of each prime factor.

**Q :** If  $x$  is an even number and  $y$  is an odd number, then which of the following statements is false?

- (i)  $(x + x^x)(y + y^y)$  is even.
- (ii)  $(x + y) + (xy + y^x) + (x^y + x^x)$  is odd.
- (iii)  $x + y^x$  is odd.
- (iv)  $(x + y) + (xy + y) + x^y + y^x$  is odd.

**A :** (i)  $x + x^x = \text{even} + \text{even} = \text{even}; y + y^y = \text{odd} + \text{odd} = \text{even}.$

$(x + x^x)(y + y^y)$  is even. Hence, true.

(ii)  $(x + y) + (xy + y^x) + (x^y + x^x)$   
 $= \text{odd} + (\text{even} + \text{odd}) + (\text{even} + \text{even})$   
 $= \text{odd} + \text{odd} + \text{even} = \text{even}.$  Hence, false.

(iii)  $x + y^x = \text{even} + \text{odd} = \text{odd}.$  Hence, true.

(iv)  $(x + y) + (xy + y) + x^y + y^x = \text{odd} + (\text{even} + \text{odd}) + \text{even} + \text{odd}$   
 $= \text{odd} + \text{odd} + \text{even} + \text{odd} = \text{even} + \text{odd} = \text{odd}.$  Hence, true.

**Q :** Let  $\overline{a + bi}$  denote the complex conjugate of  $a + bi$ . If  $a > 0$ ,  $b > 0$  and  $a < b$ , then determine the relation between the following two expressions.

(i)  $(\overline{a + bi})^2 + 2(abi + b^2)$

(ii)  $(\overline{a + bi})^2 - 2abi$

**A :** Consider (i):

$$(\overline{a + bi})^2 + 2(abi + b^2) = (a - bi)^2 + 2(abi + b^2) \dots \{\text{Complex conjugate of } a + bi \text{ is } a - bi\}$$

$$\Rightarrow (a^2 - 2abi + (bi)^2) + 2abi + 2b^2 = a^2 - 2abi - b^2 + 2abi + 2b^2$$

$$\dots \{\because i^2 = -1\} \Rightarrow a^2 + b^2$$

$$\Rightarrow \text{As } a > 0, b > 0 \therefore a^2 > 0 \text{ and } b^2 > 0 \Rightarrow a^2 + b^2 > 0$$

Consider (ii):

$$\begin{aligned} \overline{(a-bi)}^2 - 2abi &= (a+bi)^2 - 2abi \dots \{\text{Complex conjugate of } (a-bi) \text{ is } (a+bi)\} \\ \Rightarrow a^2 + 2abi + (bi)^2 - 2abi &= a^2 + 2abi - b^2 - 2abi \dots \{ \because i^2 = -1 \} \Rightarrow a^2 - b^2 \\ \Rightarrow \text{As } a > 0, b > 0 \text{ and } a < b &\Rightarrow a^2 < b^2 \Rightarrow a^2 - b^2 < 0. \\ \text{Thus, (i) is greater than (ii).} \end{aligned}$$

**Q :** Express  $0.0032\overline{45}$  as a fraction.

$$\mathbf{A : } 0.0032\overline{45} = \frac{32.\overline{45}}{10000} = \frac{32}{10000} + \frac{0.\overline{45}}{10000}$$

$$\text{Now, let } 0.\overline{45} = x \therefore 100x = 45.\overline{45}$$

$$\Rightarrow 100x = 45 + x \Rightarrow 99x = 45 \Rightarrow x = \frac{45}{99}$$

$$\therefore 0.0032\overline{45} = \frac{32 + \frac{45}{99}}{10000} = \frac{3213}{990000}$$

**Directions for next 3 examples: Read the data below and answer the questions that follow.**

$$A \alpha B = \frac{A+B}{2}; A \beta B = \frac{A-B}{2}; A \gamma B = \frac{A+B}{A-B} \text{ and } A \theta B = \frac{A-B}{A+B}.$$

**Q :** Find the value of  $(10 \alpha 8) \gamma (7 \beta 5)$

$$\mathbf{A : } (10 \alpha 8) \gamma (7 \beta 5) = \left( \frac{10+8}{2} \right) \gamma \left( \frac{7-5}{2} \right) = 9 \gamma 1 = \frac{9+1}{9-1} = \frac{10}{8} = \frac{5}{4}$$

**Q :** Find the value of  $[1 \beta (4 \gamma 5)] \theta (8 \alpha 2)$

$$\begin{aligned} \mathbf{A : } [1 \beta (4 \gamma 5)] \theta (8 \alpha 2) &= \left[ 1 \beta \left( \frac{4+5}{2} \right) \right] \theta \left( \frac{8+2}{2} \right) = \left[ \frac{1-(-9)}{2} \right] \theta (5) \\ &= \left( \frac{1+9}{2} \right) \theta (5) = 5 \theta 5 = \frac{5-5}{5+5} = \frac{0}{10} = 0 \end{aligned}$$

**Q :** Which of the following is true?

$$(i) A \alpha B = -(A \beta B) \quad (ii) A \gamma B = \frac{A \alpha B}{A \beta B} \quad (iii) A \theta B = \frac{A \beta B}{A \alpha B}$$

$$\mathbf{A : } (i) \text{ LHS} = A \alpha B = \frac{A+B}{2}$$



$$\text{RHS} = -(A \beta B) = -\left(\frac{A-B}{2}\right) = \frac{-A+B}{2}$$

$\therefore$  (i) is not true.

$$\text{(ii) LHS} = A \gamma B = \frac{A+B}{A-B}$$

$$\text{RHS} = \frac{A \alpha B}{A \beta B} = \frac{\frac{A+B}{2}}{\frac{A-B}{2}} = \frac{A+B}{A-B}$$

$\therefore$  (ii) is true.

$$\text{(iii) LHS} = A \theta B = \frac{A-B}{A+B}$$

$$\text{RHS} = \frac{A \beta B}{A \alpha B} = \frac{\frac{A-B}{2}}{\frac{A+B}{2}} = \frac{A-B}{A+B}$$

$\therefore$  (iii) is true.

**Directions for next 3 examples: Read the data below and answer the questions that follow.**

$$\begin{aligned} \lambda(a, b) &= (a - b)^2 && \text{if } a \geq b \\ &= a + b && \text{if } b > a \\ \theta(a, b) &= \frac{a+b}{2} && \text{if } a + b \geq 0 \\ &= -(a + b) && \text{if } a + b < 0 \end{aligned}$$

**Q :** Find  $\theta(\lambda(1, 3), \theta(1, 3))$

$$\mathbf{A :} \theta\left(1+3, \frac{1+3}{2}\right) = \theta(4, 2) = \frac{4+2}{2} = 3$$

**Q :** Find the value of  $\frac{[\lambda(-(-3+2), (-2+3))]}{(-(-2+1))}$ .

$$\mathbf{A :} \frac{\lambda(-(-3+2), (-2+3))}{(-(-2+1))} = \frac{\lambda(1, 1)}{1} = \frac{(1-1)^2}{1} = 0$$

**Q :**  $\theta[0, \lambda(0, \theta(0, 1))]$

**A :**  $\theta\left(0, \lambda\left(0, \left(\frac{0+1}{2}\right)\right)\right) = \theta\left(0, \lambda\left(0, \frac{1}{2}\right)\right) = \theta\left(0, \frac{1}{2}\right) = \frac{0+\frac{1}{2}}{2} = \frac{1}{4}$

**Q :** Simplify  $7-1\frac{3}{4}-4\frac{7}{8}$

**A :** Given expression =  $(7 - 1 - 4) - \frac{3}{4} - \frac{7}{8} = 2 - \left(\frac{3}{4} + \frac{7}{8}\right) = 2 - \frac{6+7}{8}$   
 $= 2 - \frac{13}{8} = \frac{16-13}{8} = \frac{3}{8}$

**Alternatively,**

The expression =  $\left(2 - \frac{3}{4}\right) - \frac{7}{8} = 1\frac{1}{4} - \frac{7}{8} = \frac{10}{8} - \frac{7}{8} = \frac{3}{8}$

**Q :** Simplify (i)  $\frac{\frac{23}{6} + \frac{3}{2}}{\frac{17}{3} - \frac{5}{4}}$  (ii)  $\frac{3\frac{3}{4} - \frac{11}{12}}{\frac{7}{15} \times 1\frac{7}{8} \text{ of } \frac{8}{21}}$

**A :** (i) First method:  $\frac{\frac{23}{6} + \frac{3}{2}}{\frac{17}{3} - \frac{5}{4}} = \frac{32}{6} \times \frac{12}{53} = \frac{64}{53} = 1\frac{11}{53}$

Second method: We multiply the numerator and the denominator by 12, which is the L.C.M. of 6, 2, 3 and 4.

Thus the given fraction =  $\frac{46+18}{68-15} = \frac{64}{53} = 1\frac{11}{53}$

(ii) Numerator =  $\frac{15}{4} - \frac{11}{12} = \frac{45-11}{12} = \frac{34}{12} = \frac{17}{6}$

Denominator =  $\frac{7}{15} \times \frac{15}{8} \times \frac{8}{21} = \frac{1}{3}$

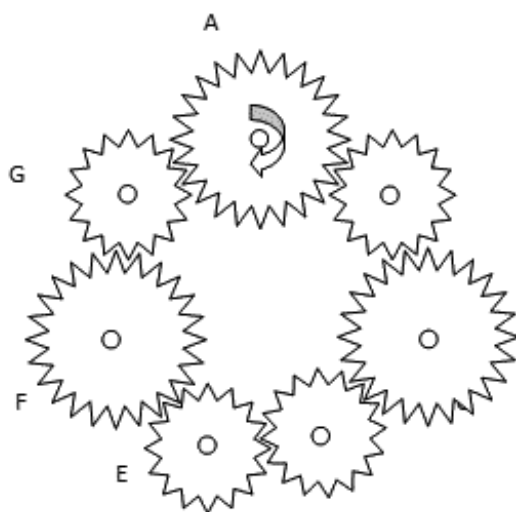
$\therefore$  the whole fraction =  $\frac{17}{6} \div \frac{1}{3} = \frac{17}{6} \times 3 = 8\frac{1}{2}$



## CLASS EXERCISE

### Teaser

There is a system of 7 gears A, B, C, D, E, F and G as shown in the figure. When gear A is turned clockwise, in which direction will gear E move?



### Classification of Numbers

1. Classify the following into the appropriate category (Natural, Whole, Integer, Rational, Real, Complex)

- |                  |                       |                      |             |
|------------------|-----------------------|----------------------|-------------|
| a) $\sqrt[3]{8}$ | b) $\sqrt{-23}$       | c) $\frac{2}{13}$    | d) $3\pi$   |
| e) 12.712368     | f) $27.4\overline{3}$ | g) $5\frac{3}{7}$    | h) $2 + 3i$ |
| i) $-28/4$       | j) 0                  | k) $4i - 5\sqrt{13}$ | l) 0.0011   |

2. Find the value of the following:

- |          |          |             |                     |
|----------|----------|-------------|---------------------|
| a) $i^2$ | b) $i^3$ | c) $i^{82}$ | d) $(2+3i)(2 - 3i)$ |
|----------|----------|-------------|---------------------|

3. Convert as directed:

- |   |   |
|---|---|
| a) $5\frac{3}{7}$ into an improper fraction | b) 4.333..... into an improper fraction |
| c) $\frac{12}{7}$ into a mixed fraction     | d) 0.686868..... into a proper fraction |

4. State True / False / Cannot be Determined

- Even + Odd + Odd = Even
- Even x Odd x Even x Odd = Even
- (Even + Odd) x (Even + Odd) = Even
- Positive + Positive + Negative = Positive
- Positive x Positive x Negative x Negative = Positive
- (Positive x Positive) + (Negative x Negative) - Negative = Positive
- All fractions can be written as mixed fractions
- All recurring decimals can be written in a rational format

**N (Natural Numbers)** = Normal counting numbers 1, 2, 3...

**W (Whole Numbers)** = N + Zero

**I (Integers)** = W + Negative Numbers

**Q (Rational Numbers)** = I + Fractions

**R (Real Numbers)** = Q + Irrational Numbers

**C (Complex Numbers)** = R + Imaginary Numbers

5. \*Classify the following:

- |                     |                          |                     |                           |
|---------------------|--------------------------|---------------------|---------------------------|
| a) $\frac{125}{99}$ | b) $\frac{1.3165}{17.4}$ | c) $\frac{54}{2.7}$ | d) $\frac{\sqrt{31}}{12}$ |
| e) $\sqrt{6}$       | f) $\sqrt{-6}$           | g) $-\sqrt{-6}$     | h) $-\sqrt{6}$            |

6. Insert  $>$ ,  $=$  or  $<$  signs in between the following pairs of fractions:

a)  $\frac{11}{14}$    $\frac{13}{16}$

b)  $\frac{11}{8}$    $\frac{13}{10}$

c)  $\frac{14}{19}$    $\frac{7}{9}$

d)  $\frac{3}{14}$    $\frac{5}{23}$

e)  $\frac{217}{229}$    $\frac{219}{231}$

f)  $\frac{271}{263}$    $\frac{461}{473}$

7. Find the value of the following:

a)  $7!$

b)  $3! + 4! + 5!$

c)  $0! \times 1! \times 2! \times 3!$

d)  $(3 \times 6!) + (4 \times 6!)$

8. Identify whether the following numbers are Prime or not:

a) 71

b) 63

c) 117

d) 299

9. Show that for any number  $N$ , it is true that  $N \times N! = (N + 1)! - N!$

**To check if a number  $N$  is prime, one can check if  $N$  is divisible by any prime number less than or equal to  $\sqrt{N}$**

10. \*Arrange the following in ascending order:  $\frac{11}{9}, \frac{11}{8}, \frac{7}{11}, \frac{12}{7}, \frac{6}{13}, \frac{13}{8}, \frac{11}{12}$

11. \*Check which of the following numbers are prime: 101, 199, 201, 301, 499, 501

### The VBODMAS Rule

12. What is the value of the following expressions?

a)  $17.3 + 8.2 \times (2 - 1) + 4.5$

b)  $12.4 \div (62 \div (2.5 \times 8)) - (7 + \overline{4 - 7})$

c)  $(0.003 \times 5 + 0.005 \times 7) + 0.05$

d)  $(\overline{7.5 \times 0.4}) \div 6 + 2.1 \times 2$

13. Which of the following expressions yields a different result from the other three?

a)  $30 + (\overline{11 - 4.6 \text{ of } 20}) - 10$

b)  $(200 - 300 \div \overline{3.4 + 2.6}) - 2.5 \times 0.8$

c)  $80 + 40 + 20 + 10 - 5 + 4 - 3 + 2 - 1$

d)  $\sqrt{9} \times 7^2 + \overline{17 - 4^2}$

14. Match the following expressions with their approximate values:

	Expression		Value
1	$\sqrt{80.88} + 7 \div 4.1 - (-0.9) - (2.99)^2$	a	1.0
2	$117 \div (31 + 4 \times 2) \div 2$	b	3.0
3	$0.25 \times 1.99 + 1.99 + 16.39 \div 8.19$	c	1.4
4	$7.17 - \sqrt{35.78} + 1 + 0.8$	d	1.8
5	$(2.1699 + 3.8298) \times (0.465 - 0.161)$	e	1.5

**The order of priority for basic mathematical operations is VBODMAS, where:**

**V** (Vinculum or bar) } these need to be solved before anything else  
**B** (Bracket) }  
**O** (Of) - indicates a multiplication of higher priority  
**D** (Division) } if only division and multiplication remain, we go left to right  
**M** (Multiplication) }  
**A** (Addition) } if only addition and subtraction remain, we go left to right  
**S** (Subtraction) }

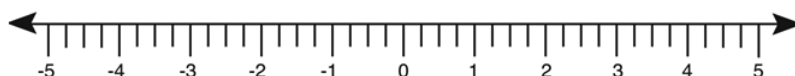
(A good mnemonic for remembering this order: **VBODMAS = Very Bored Of Doing Maths At School!**)

15. \*Compute the following:

- i)  $3.53 + 1.6 \times 2 - 4.41 \div 2.1$   
 ii)  $10 \times (9 + 8 \text{ of } \overline{7+6} - 5 \times (4 \times 3 \div 2 - 1))$

### The Real Number Line

Let us define: **|x|** as the absolute value of x (or the modulus of x)  
**[x]** as the greatest integer less than or equal to x (or the rounded-down integer value of x)  
**L(x)** as the least integer greater than or equal to x (or the rounded-up integer value of x)



16. Plot the following points on the given Number Line

- a) 3.14                      b)  $\frac{4}{6}$                       c)  $L(\sqrt{2})$                       d)  $|-5|$   
 e)  $\sqrt{16}$                       f)  $-3.2$                       g)  $0.\overline{33}$                       h)  $[3.14]$   
 i)  $[-3.14]$

17. Find the value of the following expressions:

- a)  $|3.2| + |-3.2| + [3.2] + [-3.2]$       b)  $[-4.7] \times L(3.2) \times |-2| \times L(-1.735)$   
c)  $|[-4.6]| + [| - 4.6|]$       d)  $|3.2 - 4.8|$

18. State True/False/Cannot Be Determined:

- a) There are 20 integer values of  $x$  such that  $|x| \leq 10$   
b) If  $|x + y| \leq 10$  then  $|xy| < 100$

**In terms of a number line, we can define the following functions for a real number  $x$**   
 **$[x]$**  : Greatest Integer Less Than or Equal to  $x$  = the first number to the left of  $x$  on the number line. Also known as rounded-down value.  
 **$L(x)$**  : Least Integer Greater Than or Equal to  $x$  = the first number to the right of  $x$  on the number line. Also known as rounded-up value.  
 **$|x|$**  : Absolute Value of  $x$  = the distance from 0 to  $x$  on the number line. Also known as modulus.  
**Note that:**  
 $[x] \leq x \leq L(x)$  (equal only if  $x$  is an integer)  
 $|x| \leq x$  (equal only when  $x$  is non-negative)

19. \* State True/False/Cannot Be Determined:

- a) If  $|xy| < 10$  then  $|x + y| < 100$       b) When  $x < |y|$ , then  $|y| > |x|$

### Challengers

- Let  $n! = 1 \times 2 \times 3 \times 4 \times \dots \times n$  for integer  $n \geq 1$ . What could be the value of  $1! + (2 \times 2!) + (3 \times 3!) + (4 \times 4!) + \dots (10 \times 10!)$ ?  
 1)  $11 \times 11!$       2)  $11!$       3)  $11! - 1$       4)  $11! + 1$
- The number of people writing a test is a prime number less than 200. The ratio of the people who pass to those who fail, could be:  
 1)  $82 : 83$       2)  $75 : 78$       3)  $76 : 81$       4)  $91 : 78$
- The post office has stamps of Rs 1, Rs 3 and Rs 7. Which of the following totals cannot be achieved by buying exactly 100 stamps?  
 1) Rs 298      2) Rs 600      3) Rs 473      4) Rs 440
- Let  $p$ ,  $q$  and  $r$  be distinct integers, that are odd and positive. Which of the following must be true?  
 1)  $(p + q)(q + r)(r - p)$  is negative      2)  $(p + q)(q + r)(r + p)$  is even  
 3)  $(q - p)(q - r)(p - r)$  is positive      4)  $(p + q)(q + r)(r + p)$  is odd
- Which of the following statements is true?  
 1)  $[x] - L(x) = -1$       2)  $[x] + |x| \geq 0$   
 3)  $|x + y| > |x - y|$       4)  $L(x) + |x| \geq 0$



## PRACTICE EXERCISE

**DIRECTIONS for question 1: Choose the correct alternative.**

1.  $SM(X_1, X_2, X_3, \dots, X_n)$  means  $X_1 + X_2 + X_3 + \dots + X_n$

$AVG(X_1, X_2, X_3, \dots, X_n)$  means  $\frac{(X_1 + X_2 + X_3 + \dots + X_n)}{n}$

$DN(X_1, X_2, X_3, \dots, X_n)$  means  $\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \dots + \frac{1}{X_n}$ .

Find the value of  $AVG(DN(1, 1), SM(1, 2), DN(2, 2), SM(2, 3))$

- 1)  $\frac{17}{4}$                       2)  $\frac{11}{4}$                       3)  $\frac{7}{17}$                       4)  $\frac{4}{11}$

**DIRECTIONS for questions 2 to 5: Refer to the data below and answer the questions that follow.**

For two numbers 'a' and 'b',

$P(a, b) = a - b$ ;  $Q(a, b) = a \times b$ ;

$R(a, b) = a \div b$ ;  $S(a, b) = a + b$ .

Given  $a = 1$ ,  $b = 2$ ,  $c = 3$ .

2. The value of  $P(a, Q(b, R(b, c)))$  is:

- 1)  $> 0$                       2)  $< 0$                       3) 1                      4) 0

3. Solve:  $P(Q(R(c, a), S(a, b)), S(Q(a, c), P(c, a)))$ .

- 1) 2                      2) -4                      3) 4                      4) 3

4. Solve:  $R(Q(S(x, y), P(x, y)), P(Q(3, S(x, y)), S(x, y)))$ .

- 1) 0                      2)  $xy$                       3)  $\frac{x+y}{2}$                       4)  $\frac{x-y}{2}$

5. Which value is the greatest?

I.  $S(P(x, y), P(x, y))$                       II.  $Q(S(x, y), R(x, y))$                       III.  $P(S(x, y), Q(x, y))$

- 1) I                      2) II  
3) III                      4) Cannot be determined

**DIRECTIONS for questions 6 to 15: Choose the correct alternative.**

6. Mr. X forgot his telephone number. All he remembers is that the 1<sup>st</sup> digit of the number is an even prime number and the rest of the telephone number is square of a particular number. Now this number (which is to be squared) comprises of the first four natural numbers in ascending order. What is the telephone number of Mr. X?

- 1) 31522756                      2) 21567432                      3) 21730400                      4) 21522756



7. Perform mathematical operations in the following order of priority:  
(1) Addition (2) Subtraction  
(3) Division (4) Multiplication  
other operations and constraints being kept normal.  
What is the value of  $0.\overline{35} + 0.\overline{63} \times 100 \div 0.\overline{4} - 0.\overline{34}$  ?  
1) 990                      2)  $999.\overline{9}$                       3)  $989.\overline{89}$                       4) 999
8. Anil wants to divide Rs.100 into a number of bags so that one can ask for any amount between Rs.1 and Rs.100, he can give the proper amount by giving certain number of these bags without taking out the amount from them. What is the minimum number of bags he will require if each bag has whole number of rupees?  
1) 5                      2) 6                      3) 7                      4) 8
9.  $A = 0.a_1a_1a_1\dots$  and  $B = 0.a_2a_2a_2\dots$ , where ' $a_1$ ' and ' $a_2$ ' are multiples of 3 and also, ' $a_1$ ' and ' $a_2$ ' are distinct integers from 1 to 8. Find the value of  $A + B$ .  
1)  $\frac{a_1 + a_2}{2}$                       2)  $\frac{a_1 + a_2}{99}$                       3) 1                      4) Cannot be determined
10. Sachin writes all the numbers from 1 to 1000 on a paper in order. Find the 2883<sup>rd</sup> digit written by him.  
1) 0                      2) 9                      3) 7                      4) 8
11. 72 hens cost Rs.     96.7   . Then what does each hen cost, where two digits in place of '   ' are not visible or are written in illegible hand?  
1) Rs. 3.23                      2) Rs. 5.11                      3) Rs. 5.51                      4) Rs. 7.22  
**(Past CAT question)**
12. Manisha has to multiply 2 numbers together, but she uses 35 instead of 53 in this product, and finds that the product has decreased by 540. What is this increased product?  
1) 1050                      2) 540                      3) 1440                      4) 1590  
**(Past CAT question)**
13. Let S be the set of prime numbers greater than or equal to 2 and less than 100. Multiply all elements of S. With how many consecutive zeros will be product end?  
1) 1                      2) 4                      3) 5                      4) 10  
**(Past CAT question)**
14. The alphabets a, b, c represent integers forming a two digit number 'ab' and a three digit number 'ccb'. Both are defined under the usual decimal number system. If  $(ab)^2 = ccb$  and  $ccb > 300$ , then the value of b is:  
1) 1                      2) 0                      3) 5                      4) 6  
**(Past CAT question)**

15.  $x$ ,  $y$  and  $z$  are three positive integers such that  $x > y > z$ . Which of the following is closest to the product  $xyz$ ?

- 1)  $(x - 1)yz$                       2)  $x(y - 1)z$                       3)  $xy(z - 1)$                       4)  $x(y + 1)z$

(Past CAT question)

**Direction for questions 16 and 17: Each question is followed by two statements, I and II. Mark the answer as:**

**Choose (1), if the question can be answered by one of the statements alone but not by the other.**

**Choose (2), if the question can be answered by using either statement alone.**

**Choose (3), if the question can be answered by using both the statements together, but cannot be answered by using either statement alone.**

**Choose (4), if the question cannot be answered even by using both the statements together.**

16.  $x$ ,  $y$  and  $z$  are natural numbers. Is the difference between  $z$  and  $x$  even ?

- I.  $xyz$  is odd.  
II.  $xy + yz + zx$  is even.

(Past CAT question)

17. Is  $x + y - z + t$  even ?

- I.  $x + y + t$  is even.  
II.  $t$  and  $z$  are odd.

(Past CAT question)

**DIRECTIONS for questions 18 to 20: Choose the correct alternative.**

18. Let  $x < 0.50$ ,  $0 < y < 1$ ,  $z > 1$ . Given a set of numbers, the middle number, when they are arranged in ascending order, is called the median. So the median of the numbers  $x$ ,  $y$ , and  $z$  would be:

- 1) less than one                      2) between 0 and 1  
3) greater than 1                      4) cannot say

(Past CAT question)

19. Given odd positive integers  $x$ ,  $y$  and  $z$  which of the following is not necessarily true?

- 1)  $x^2y^2z^2$  is odd                      2)  $3(x^2 + y^3)z^2$  is even.  
3)  $5x + y + z^4$  is odd                      4)  $z^2(x^4 + y^4)/2$  is even

(Past CAT question)

20. Let  $x < 0$ ,  $0 < y < 1$ ,  $z > 1$ . Which of the following may be false?

- 1)  $(x^2 - z^2)$  has to be positive.                      2)  $yz$  has to be positive.  
3)  $xy$  can never be zero.                      4)  $(y^2 - z^2)$  is always negative.

(Past CAT question)

## QA-1.2 | DIVISIBILITY AND GCD-LCM



### Divisibility

When a number 'x' is divided by another number 'y' and the remainder is zero, then the number 'x' is said to be divisible by 'y'. To check divisibility of numbers, we have certain divisibility tests as mentioned below:

#### Divisibility tests for numbers

1. A number is divisible by 2, when its units digit is divisible by 2 or is 0.

##### Example

179 is not divisible by 2 as the units digit is odd.

2. A number is divisible by 3, when the sum of its digits is divisible by 3.

##### Example

123 (Sum of digits is 6) is divisible by 3.

3. A number is divisible by 4, when the number formed by the last two right hand digits is divisible by 4, or if the last two right hand digits are 0's.

##### Example

1900 is divisible by 4 (as the last two digits are 00),

1902 is not divisible by 4 (as the last two right hand digits i.e., 02, are not divisible by 4).

4. A number is divisible by 5, when its unit's digit is divisible by 5 or is 0.
5. A number is divisible by 6, when it is divisible by 2 as well as by 3.

##### Example

123 is divisible by 3 but not by 2, therefore it is not divisible by 6.

12312 is divisible by 2 as well as by 3, therefore it is divisible by 6.

#### 6. Divisibility test of 7

##### Method 1

If the digits a, b, c & d of a four-digit number abcd are such that ' $2b + 3c + d - a$ ' is divisible by 7, then so is the original number.

##### Example

1981

Using the above algebraic expression, 1981 can be expressed as:

$2 \times 9 + 3 \times 8 + 1 - 1 = 42$  which is divisible by 7. Hence, 1981 is divisible by 7.

**Method 2**

A number is divisible by 7 if the sum of the products of the digits of a number taken from left to right and digits 1, -2, -3, -1, 2, 3,... in succession is divisible by 7 or is 0.

**Example**

392

The required sum =  $3 \times 1 - 9 \times 2 - 3 \times 2 = -21$ , which is divisible by 7.

Hence, 392 is divisible by 7.

**Method 3**

An integer 'I' is divisible by 7, if the difference of the number of its thousands and the remainder of its division by thousand is divisible by 7.

**Example**

439187

Difference =  $439 - 187 = 252$  which is divisible by 7.

Hence, 439187 is divisible by 7.

**Method 4**

Any number is divisible by 7, if the number of tens added to five times the units digit is divisible by 7.

**Example**

308

Number of tens = 30

The required sum =  $30 + 5 \times 8 = 70$  which is divisible by 7.

Hence, 308 is divisible by 7.

**Method 5**

Any number is divisible by 7, if the number of tens added to (-2) times the units digit is divisible by 7.

**Example**

6727

Number of tens = 672

(-2) times the units digit = -14

$672 - 14 = 658$

Number of tens in 658 = 65

(-2) times the units digit = -16

$65 - 16 = 49$  which is divisible by 7.

Hence, 6727 is divisible by 7.

**Method 6**

Any number of the form  $\overline{abcabc}$  is divisible by 7. eg. 345345, 284284, 596596 are divisible by 7.

7. A number is divisible by 8 when the number formed by the last three right hand digits is divisible by 8, or when the last three digits are 0's.

**Example**

546728 is divisible by 8 as 728 is divisible by 8

11100329642 is not divisible by 8 as 642 is not divisible by 8.

8. A number is divisible by 9, when the sum of its digits is divisible by 9.

**Example**

729729 (sum of digits 36) is divisible by 9

1111----- (81 times) is divisible by 9 (sum of the digits is 81).

9. A number is divisible by 10, when its unit's digit is 0.
10. A number is divisible by 11, when the difference between the sum of the digits in the odd places and the sum of the digits in the even places is 0 or a multiple of 11.

**Note:** When any number with an even number of digits is added to its reverse, the sum is always a multiple of 11.

11. A number is divisible by 12, when it is divisible by 3 and 4 both.

**12. Divisibility test of 13**

**Method 1**

Any four digit number  $abcd$  is divisible by 13 if ' $a + 4b + 3c - d$ ' is divisible by 13.

**Example**

9373

$9 + 4 \times 3 + 3 \times 7 - 3 = 39$  which is divisible by 13.

Hence, 9373 is divisible by 13.

**Method 2**

A number is divisible by 13 if the sum of the product of the digits of the number from left or right with 1, 4, 3, -1, -4, -3, 1, 4, 3, ... successively is divisible by 13 or is 0.

**Example**

195

The sum of the products  $\Rightarrow 1 \times 1 + 9 \times 4 + 5 \times 3 = 52$ , which is divisible by 13.

Hence, the number 193 is divisible by 13.

**Method 3**

An integer  $I$  is divisible by 13, if the difference of the number of its thousands and the remainder of its division by thousand is divisible by 13.

### Example

160485

Number of its thousands = 160

Remainder of its division by 1000 = 485

$160 - 485 = 325$  which is divisible by 13.

Hence, 160485 is divisible by 13.

### Method 4

Any number is divisible by 13, if the number of tens added to four times the units digit is divisible by 13.

### Example

6058

Number of tens = 605

4 times units digit = 32

$605 + 32 = 637$  which is divisible by 13.

Hence, 6058 is divisible by 13.

Any number of the form  $\overline{abcabc}$  is divisible by 13.

eg. 238238, 521521, 896896 are divisibly by 13.

13. A number is divisible by 15, when it is divisible by 3 and 5 both.

### Example

225 is divisible by 3 as well as 5  $\therefore$  it is divisible by 15

1225 is divisible by 5 but not by 3  $\therefore$  it is not divisible by 15

**Note:** When a number is divisible by two or more co-prime numbers, it is also divisible by their products.

14. **Divisibility test of 17**

### Method 1

A number is divisible by 17, if the number of tens added to 12 times the units digit is divisible by 17.

### Example

153

Number of tens = 15

The required sum =  $15 + 12 \times 3 = 51$ , which is divisible by 17

Hence, 153 is divisible by 17.

### Method 2

A number is divisible by 17, if the number of tens added to  $(-5)$  times the units digit is divisible by 17.

### Example

In the same case as above, the required sum =  $15 + (-5) \times 3 = 0$  is divisible by 17.

Hence, 153 is divisible by 17.

**15. Divisibility test of 19**

Any number is divisible by 19, if the number of tens added to twice the units digit is divisible by 19.

**Example**

228

Number of tens = 22

The required sum =  $22 + 2 \times 8 = 38$ , which is divisible by 19

Hence, 228 is divisible by 19.

16. A number is divisible by 25, when the number formed by the last two right hand digits is divisible by 25 or if the last two right hand digit are zero.

**17. Divisibility test of 29**

Any number is divisible by 29, if the number of tens added to thrice the units digit is divisible by 29.

**Example**

348

Number of tens = 34

The required sum =  $34 + 3 \times 8 = 58$ , which is divisible by 29

Hence, 348 is divisible by 29.

18. A number is divisible by 125, when the number formed by the last three right hand digits is divisible by 125 or the last three right hand digits are zero.

**Note:** The results obtained by the divisibility tests are also useful in determining the value of the remainder in a division operation

**Example**

34689

sum of digits =  $3 + 4 + 6 + 8 + 9 = 30$

Remainder of  $\left(\frac{30}{9}\right) = 3$ . Hence, remainder of  $\left(\frac{34689}{9}\right) = 3$ .

**Concept Builder 1**

**Find the value of 'x' if:**

1.  $59x$  is divisible by 6.
2.  $204853x9$  is divisible by 9.
3.  $x557x55$  is divisible by 11.
4.  $932168932168x$  is divisible by 12.

**Answer Key**

1. 4    2. 5    3. 9    4. 8

### Factorisation

To express a number  $N$  (where  $N$  is a natural number) as a product of its prime factors is called **factorisation**.

The number of divisors of a given number  $N$  (including 1 and the number itself) where

$$N = a^m \cdot b^n \cdot c^p$$

where  $a, b, c$  are prime numbers, is  $(1 + m)(1 + n)(1 + p)$

The sum of the divisors is:  $\frac{a^{m+1} - 1}{a - 1} \cdot \frac{b^{n+1} - 1}{b - 1} \cdot \frac{c^{p+1} - 1}{c - 1}$

### Example

$$70 = 2^1 \times 5^1 \times 7^1$$

$$\text{Number of divisors} = (1 + 1)(1 + 1)(1 + 1) = 2 \times 2 \times 2 = 8$$

i.e., 1, 2, 5, 7, 10, 14, 35, 70

$$\text{Sum of divisors} = 1 + 2 + 5 + 7 + 10 + 14 + 35 + 70$$

$$= \frac{2^2 - 1}{2 - 1} \times \frac{5^2 - 1}{5 - 1} \times \frac{7^2 - 1}{7 - 1} = 144$$

### About prime factors

- The number of ways in which a number  $N$  can be expressed as the product of two factors which are relatively prime to each other is  $2^{m-1}$  where  $m$  is the number of different prime factors of  $N$ .

### Example

$$N = 540 = 4 \times 27 \times 5 = 2^2 \times 3^3 \times 5$$

(i.e., number of different prime factors = 3 =  $m$ )

$$\therefore \text{Number of ways} = 2^{3-1} = 4$$

$\therefore$  540 can be expressed as product of two factors which are relatively prime to each other in 4 ways.

$$\text{i.e., } 540 = 20 \times 27 = 108 \times 5 = 4 \times 135 = 540 \times 1$$

- Can you think of the smallest number  $N$  such that it can be expressed as a product of two factors which are relatively prime in 8 ways?

For  $N$  to be expressed as product of two relatively prime factors in 8 ways

$N = a^l \times b^m \times c^n \times d^p$ , where  $a, b, c$  and  $d$  are distinct prime factors and  $l, m, n$  and  $p$  are natural numbers.

For  $N$  to be the least,  $l = m = n = p = 1$  and  $a, b, c$  and  $d$  should be the least prime numbers  $\therefore N = 2 \times 3 \times 5 \times 7 = 210$ .



### SOLVED EXAMPLES

**Q :** If  $N = 2^a \times 3^b \times 5^c \times 10^d$ , then find

1. Total number of factors of  $N$
2. Total number of odd factors of  $N$
3. Total number of even factors of  $N$

**A :**  $N = 2^a \times 3^b \times 5^c \times 10^d$

Factorising  $N$  into prime factors we get,

$$N = 2^{a+d} \times 3^b \times 5^{c+d} \quad (\because 10^d = 2^d \times 5^d)$$

1. The total number of factors of  $N = (a+d+1)(b+1)(c+d+1)$
2. To calculate the total number of odd factors of  $N$  we consider only the product of those factors that will yield odd values i.e. 3 and 5 and the product of their powers.

$$\text{Total number of odd factors of } N = \text{Total number of factors of } 3^b \times 5^{c+d}$$

$$\Rightarrow (b+1)(c+d+1)$$

3. The total number of even factors of  $N$  can be written as

$$\text{Total number of factors of } N - \text{Total number of odd factors of } N$$

$$\Rightarrow (a+d+1)(b+1)(c+d+1) - (b+1)(c+d+1)$$

$$\Rightarrow (b+1)(c+d+1)[(a+d+1) - 1]$$

$$\Rightarrow (b+1)(c+d+1)(a+d)$$

**Q :** If  $N = 72$ , then find the number of pairs  $(a, b)$ , such that  $a$  and  $b$  are the factors of  $N$  and  $a \times b = N$  having the value of ' $a$ ' less than ' $b$ '.

**A :**  $N = 72$

The factors of 72 are:

1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72

If we form pairs  $(a, b)$ , then the lowest factor when multiplied by the highest factor will give 72, the 2<sup>nd</sup> lowest factor when multiplied by the second highest factor will give 72 and so on.

The total number of such pairs can be given as:

$[1 \times 72], [2 \times 36], [3 \times 24], [4 \times 18], [6 \times 12], [8 \times 9]$  i.e. a total of 6 pairs

**Note:** To find the number of such pairs such that  $a \times b = N$

1. For numbers with odd number of factors

$$\text{Total number of pairs where } a \times b = N = \frac{(\text{Total number of factors} + 1)}{2}$$

2. For numbers with even number of factors

$$\text{Total number of pairs where } a \times b = N = \frac{(\text{Total number of factors})}{2}$$

**Q :** If  $N = 72$ , then find the number of pairs  $(a,b)$  for  $N$  such that  $a$  and  $b$  are the factors of  $N$  and are co-prime to each other.

**A :** The numbers that have no common factor other than 1 are co-prime to one another

Total number of factors of  $N = (3+1) \times (2+1) = 4 \times 3 = 12$

1. All factors will be co-prime with 1.

2. Powers of 2 will be co-prime with powers of 3.

To find the number of pairs as in (1) above,

With 1 we have 11 pairs of factors that are co-prime with each other

To find the number of pairs as in (2) above,

Powers of 2 (that are factors of 72) =  $2^1$ ,  $2^2$  and  $2^3$  i.e. a total of 3 factors

Powers of 3 (that are factors of 72) =  $3^1$  and  $3^2$  i.e. a total of 2 factors

So the number of possible pairs =  $3 \times 2 = 6$  i.e.,  $[2, 3]$ ,  $[2, 9]$ ,  $[4, 3]$ ,  $[4, 9]$ ,  $[8, 3]$ ,  $[8, 9]$

In all, the total number of co-prime pairs will be  $11 + 6 = 17$

## HCF & LCM of integers

### Highest Common Factor or (GCD)

Highest Common Factor (HCF) is the largest integer that perfectly divides two or more given numbers. It is also known as GCD (Greatest Common Divisor).

#### Example

The HCF of 12 and 18 is 6.

### Methods to find the HCF of the given numbers

#### 1] By method of factorisation

Express the given number as the product of its prime factors. Now take the product of the least power of the common factors, which is the HCF.

#### Example

Find HCF of 136, 144, 168

$$136 = 2 \times 2 \times 2 \times 17 = 2^3 \times 17$$

$$144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 = 2^4 \times 3^2$$

$$168 = 2 \times 2 \times 2 \times 3 \times 7 = 2^3 \times 3 \times 7$$

$$\text{HCF} = 2^3 = 8$$

## 2] By division method

Suppose two numbers are given. Divide the greater number by the lesser one; divide the lesser by the remainder; divide the first remainder by the new remainder, and so on till there is no remainder. The last divisor is the required H.C.F.

### Example

Find the H.C.F. of 12 and 15.

$$\begin{array}{r} 12 \overline{) 15} \quad (1 \\ \underline{-12} \\ 3 \overline{) 12} \quad (4 \\ \underline{-12} \\ 00 \end{array} \quad \therefore \text{HCF} = 3$$

In case more than two numbers are given then choose any two of them and find their HCF.

The HCF of these two and the third gives the HCF of three numbers and so on.

**Note:** If 'Z' is HCF of 'X' and 'Y', then Z is also a factor of  $aX + bY$  where a, b are integers.

## Least Common Multiple

Least common multiple (LCM) of two or more given numbers is the least number which is exactly divisible by each of the given numbers.

### Methods to find LCM of the given numbers

#### 1] By method of factorisation

Resolve each one of the given numbers into prime factors, then their LCM is the product of highest powers of all the factors, that occur in these numbers.

### Example

Find the LCM of 136, 144 and 168

$$136 = 2^3 \times 17$$

$$144 = 2^4 \times 3^2$$

$$168 = 2^3 \times 3 \times 7$$

$$\text{LCM} = 2^4 \times 3^2 \times 17 \times 7 = 17136$$

#### 2] Using the formula: Product of numbers = HCF $\times$ LCM

If two numbers are given, find their HCF. Their LCM is given as

$$\text{LCM} = \frac{\text{Product of two numbers}}{\text{HCF}}$$

### Example

Find the LCM of 136 and 144.

$$\text{LCM} = \frac{136 \times 144}{8} = 2448$$

**Note:** If three numbers are given, then find the LCM of any two of the numbers. Then, the LCM of the **LCM of the two numbers** and the third number gives the LCM of the three numbers.

### 3] Short cut method using factorisation

Find LCM of 18, 28, 108, 105

2	18,	28,	108,	105
2	9,	14,	54,	105
3	9,	7,	27,	105
3	3,	7,	9,	35
3	1,	7,	3,	35
5	1,	7,	1,	35
7	1,	7,	1,	7
	1,	1,	1,	1

$$\text{LCM} = 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 7 = 3780$$

Alternatively,

We can strike out 18 at once, since any multiple of 108 must contain 18 as a factor.

2	28,	108,	105
2	14,	54,	105
3	7,	27,	105
7	7,	9,	35
	1,	9,	5

$$\text{LCM} = 2 \times 2 \times 3 \times 7 \times 9 \times 5 = 3780$$

**Note:** For any given numbers, the HCF is necessarily a factor of their L.C.M.

HCF is the product of the least powers of the common factors.

LCM is the product of the highest power of all the factors.

### HCF and LCM of fractions

$$1. \quad \text{HCF of Fractions} = \frac{\text{HCF of Numerators}}{\text{LCM of Denominators}}$$

$$2. \quad \text{LCM of Fractions} = \frac{\text{LCM of Numerators}}{\text{HCF of Denominators}}$$

#### Example

Find HCF and LCM of  $\frac{4}{5}$ ,  $\frac{8}{35}$ ,  $\frac{2}{15}$ ,  $\frac{6}{25}$

$$\therefore \text{HCF} = \frac{\text{HCF of } 4, 8, 2, 6}{\text{LCM of } 5, 35, 15, 25} = \frac{2}{5 \times 7 \times 3 \times 5} = \frac{2}{525}$$

$$\text{LCM} = \frac{\text{LCM of } 2, 8, 4, 6}{\text{HCF of } 5, 35, 15, 25} = \frac{24}{5}$$

**Note:** While finding the HCF or LCM of fractions by the above method, the fraction should be represented in their reduced form.

### Comparison of Fractions

Arrange the fractions  $\frac{1}{3}$ ,  $\frac{2}{4}$ ,  $\frac{5}{6}$ ,  $\frac{7}{8}$  in descending order.

Take LCM of all denominators.

$\therefore$  LCM of 3, 4, 6, 8 = 24

$$\therefore \frac{1}{3} = \frac{8}{24}, \frac{2}{4} = \frac{12}{24}, \frac{5}{6} = \frac{20}{24}, \frac{7}{8} = \frac{21}{24}$$

$\therefore$  Descending order  $\frac{21}{24}$ ,  $\frac{20}{24}$ ,  $\frac{12}{24}$ ,  $\frac{8}{24}$  or  $\frac{7}{8} > \frac{5}{6} > \frac{2}{4} > \frac{1}{3}$

Arrange the following fractions in ascending order.

$$\frac{13}{15}, \frac{11}{13}, \frac{8}{10}, \frac{9}{11}, \frac{5}{7}$$

**Rule:** For fraction less than 1, if the difference between numerator and denominator of each of the given fractions remains the same, then the one with larger numerator is larger.

$\therefore$  In above example difference between the numerator and denominator of all these fractions is same and equal to 2, the one with larger numerator is larger. Proceeding in similar way:

$$\text{Descending order: } \frac{13}{15} > \frac{11}{13} > \frac{9}{11} > \frac{8}{10} > \frac{5}{7}$$

$$\text{Ascending order: } \frac{5}{7} < \frac{8}{10} < \frac{9}{11} < \frac{11}{13} < \frac{13}{15}$$

### H.C.F and L.C.M. of Decimal numbers

In the given numbers, make the same number of places of decimals by annexing zeros in some numbers if necessary. Consider these numbers without decimal point, find HCF or LCM as the case may be. Now, in the result mark off as many decimal places as there are in each of the numbers.

### Some Applications of LCM and HCF of Numbers

1. LCM of 2 numbers  $\times$  HCF of 2 numbers = Product of 2 numbers

The above relation is true only for 2 numbers.

#### Example

The LCM of 2 natural numbers is 216 and their product is 15552. Find the HCF of the two numbers.

As per definition LCM of 2 numbers  $\times$  HCF of 2 numbers = Product of 2 numbers

$$216 \times \text{HCF} = 15552 \quad \therefore \text{HCF} = 72$$

**Note :** LCM should always be a multiple of HCF or, HCF is always a factor of LCM.  
This relation is even true for more than 2 numbers.

2. If the HCF and sum of 2 numbers is known, then sometimes we can find the value of the two numbers.

### Example

- i. The HCF of 2 natural numbers is 24, and the sum of the two numbers is 144. Find the 2 numbers.  
HCF of 2 numbers = 24, Both numbers have to be multiple of 24. (Both a and b have to be co-prime)  
The numbers can be represented as 24a and 24b.  
Now,  $\therefore$  sum of both numbers is 144  
 $\Rightarrow 24a + 24b = 144 = 24(a + b) = 144 \Rightarrow (a + b) = 6$   
The values of [a, b] can be [1, 5], [2, 4], [3, 3] (Not necessarily in that order)  
The only co-prime values of [a, b] are [1, 5]. The 2 numbers are 24 and 120.
- ii. The HCF of 2 distinct natural numbers is 39 and the product of the two numbers is 15210. How many such pairs of numbers are possible?  
The 2 numbers are 39a and 39b  
According to the given data,  
 $39a \times 39b = 15210 = a \times b = 10$   
Now the possible values of [a, b] are [1,10] and [2,5]  
Since both the above pairs are co-prime, we have 2 possible pairs of numbers

3. If the LCM of 'x' consecutive natural numbers is known, then we can find the LCM of 'x + n' consecutive natural numbers.

### Example

The LCM of the 1<sup>st</sup> 100 natural numbers is k. Find the LCM of the first 105 natural numbers.  
Given, the LCM of 100 natural numbers = k.

To find the LCM of the first 105 natural numbers, factorize the first 5 numbers after 100.

101  $\rightarrow$  Prime

102  $\rightarrow 2^1 \times 3^1 \times 17^1$

103  $\rightarrow$  Prime

104  $\rightarrow 2^3 \times 13^1$

105  $\rightarrow 3^1 \times 5^1 \times 7^1$

As the LCM of 100 numbers is 'k', the powers of 2, 3, 5, 7 and 17 are already taken into consideration. The only number that will affect the LCM of the numbers will be prime numbers between 100 and 105 i.e. 101 and 103

So, the LCM of the first 105 natural numbers will be  $101 \times 103 \times k = 10403k$ .

Concept Builder 2

Solve the following:

- Find the number of factors of 8820.
- Find the number of non-prime factors of  $2^{10} \times 7^3 \times 5^9$ .
- Find the HCF of  $\frac{21}{25}, \frac{7}{30}, \frac{28}{15}$ .
- Find the LCM of 0.18, 0.72 and 0.324.
- Find the HCF of the numbers 'a' and 'b' where:  
 $a = 4^2 \times 5^3 \times 7^2$        $b = 2^3 \times 5 \times 7$
- The product of two numbers is 2430 and their HCF is 9. What will be their LCM?
- The LCM of 24 and 'x' is 72 and their HCF is 12 then, find the value of 'x'.
- The sum of two numbers is 528 and their HCF is 33. Find the total number of such pairs of numbers possible.
- What is the maximum length of the scale that can be used to measure 168m, 350m, 343m and 441m exactly?
- A time-bomb beeps after every 60 seconds. Another time-bomb beeps after every 66 seconds. They beeped together at 6pm. What will be the time at which they will again beep together?
- Five bells begin to toll together. They toll at intervals of 2, 3, 4, 6 and 10 seconds respectively. How many times will they toll together in half an hour excluding the time when they toll together at the start?

Answer Key

11. 30				
6. 270	7. 36	8. 4	9. 7m	10. 6:11pm
1. 54	2. 437	3. $\frac{150}{7}$	4. 6.48	5. 280

## Remainders

A natural number 'm' when divided by a natural number 'n' will yield the remainder that is a whole number and less than n i.e., 0, 1, 2, ..... n - 1

## Application of Divisibility and GCD-LCM in Remainder based questions

The questions based on remainders are classified as below:

### 1. Divisibility based

#### Example

A number when divided by 81 leaves a remainder of 22. What is the remainder when the same number is divided by 9?

According to the given data,

The number N can be expressed as  $\Rightarrow 81x + 22$

where 'x' is the quotient and 22 is the remainder

To find the remainder when the number 'N' is divided by 9, divide the following term:

$$\frac{(81x + 22)}{9}$$

$$\frac{81x}{9} = 9x \text{ and leaves a remainder of } 0$$

$$\frac{22}{9} \Rightarrow \text{gives a quotient of } 2 \text{ and leaves a remainder of } 4$$

$\therefore$  The remainder of the number N when divided by 9 leaves a remainder of  $0 + 4 = 4$

### 2. LCM based

Questions based on LCM can be classified as below

**Case 1:** To find the smallest number (say x), which when divided by two or more divisors, leaves the same remainder in each case.

In general,  $x = \text{Remainder} + \text{LCM of divisors} \times k$  [where, k is a positive integer]

#### Example

Find the smallest 3 digit number which when divided by 3, 4 and 5 leaves a remainder of 2 in each case

Remainder = 2

LCM of divisors = LCM (3, 4 and 5) = 60

Let the smallest 3 digit number be x

$x = \text{Remainder} + \text{LCM of divisors} \times k$

To get the smallest 3 digit number we have to multiply the LCM by 2 i.e.,  $k = 2$

$\therefore x = 2 + 60 \times 2 = 2 + 120 = 122$



**Case 2:** To find the smallest number (say  $x$ ), which when divided by two or more divisors, leaves a different remainder in each case, such that there is a common difference between the divisor and the corresponding remainder.

In general,  $x = \text{LCM of divisors} \times k$  (where  $k$  is a positive integer) – Common difference

### Example

A number when divided by 7, 11 and 13, leaves remainders 4, 8 and 10 respectively. Find the smallest such possible number.

According to given data, the number  $N$  can be expressed as,

$$N = 7a + 4 \text{ (Divisor} = 7; \text{Remainder} = 4)$$

$$N = 11b + 8 \text{ (Divisor} = 11; \text{Remainder} = 8)$$

$$N = 13c + 10 \text{ (Divisor} = 13; \text{Remainder} = 10)$$

The common difference between the divisor and the corresponding remainder is 3

$$\text{Smallest number} = \text{LCM} [7, 11, 13] - 3 = 1001 - 3 \Rightarrow 998$$

**Case 3:** To find a number which when divided by two divisors, say 'a' and 'b' leave different remainders (say  $k_1$  and  $k_2$  respectively) such that the difference between the divisor and the corresponding remainder, in both cases is different.

The smallest number 'N' which satisfies the above conditions can be expressed as :

$$N = ax + k_1 \text{ or } N = by + k_2$$

Equating the two equations we get:

$$ax + k_1 = by + k_2 \Rightarrow x = \frac{(by + k_2 - k_1)}{a}$$

Take the lowest possible integral value of 'y' in the above equation such that  $x$  is also an integral value.

In general,

$$\text{Required number} = N + \text{LCM of divisors} \times k \text{ (where 'k' is a whole number)}$$

### Example

A number when divided by 5 leaves a remainder of 1 and when divided by 6 leaves a remainder of 5. Find the largest such 3 digit number.

According to given data, the number  $N$  can be expressed as,

$$N \Rightarrow 5x + 1 \text{ or } N = 6y + 5$$

Equating the two equations we get,

$$5x + 1 = 6y + 5 \Rightarrow x = \frac{(6y + 5 - 1)}{5} \Rightarrow x = \frac{(6y + 4)}{5}$$

Putting  $y = 1$ , we get  $x = 2$

So the smallest number is  $\Rightarrow 5 \times 2 + 1 = 11$  (Also,  $6 \times 1 + 5 = 11$ )

Largest 3 digit number satisfying the above condition =  $N + \text{LCM of divisors} \times k$

$\Rightarrow 11 + 30k$  (LCM of 5 and 6 is 30)

Now  $11 + 30k < 1000$

$\Rightarrow 30k < 989$

For  $k = 32$ ,  $30k = 960$

For  $k = 33$ ,  $30k = 990$

$\Rightarrow$  So  $k = 32$  Required number =  $11 + 30 \times 32 = 971$

### 3. HCF based

Questions based on HCF can be classified as below

**Case 1:** The largest number which divides the numbers  $a$ ,  $b$  and  $c$  giving remainders of  $d$ ,  $e$  and  $f$  respectively will be the HCF of the three numbers  $(a - d)$ ,  $(b - e)$  and  $(c - f)$ .

#### Example

Find the largest number that leaves a remainder of 2, 5 and 9 when it divides 142, 265 and 389 respectively.

Largest number = HCF  $[(142-2), (265-5), (389-9)]$

= HCF  $[140, 260, 380] = 20$

**Case 2:** The largest number which divides the numbers  $a$ ,  $b$  and  $c$ , such that the remainders are the same for all the numbers will be the HCF  $[|a-b|, |a-c|]$  or HCF  $[|a-b|, |b-c|]$  or HCF  $[|b-c|, |a-c|]$ .

#### Example

Find the largest number that divides 566, 746 and 1286, leaving the same remainder in each case.

Largest number = HCF  $[|566-746|, |746 - 1286|]$

= HCF  $[180, 540] = 180$

### Concept Builder 3

1. A number  $N$  when divided by 120 gives a remainder of 76. What would be the remainder obtained on dividing the same number by 8?
2. What is the least number that should be subtracted from 371298, if the number has to be a multiple of 4?
3. Find the largest number that divides 52, 152 and 227 to leave the same remainder in each case.
4. Find the five digit smallest number exactly divisible by 16, 32, 48 and 54.

#### Answer Key

1. 4      2. 2      3. 25      4. 10368

### SOLVED EXAMPLES

**Q :** Factorize 7020.

**A :** The number is obviously divisible by 10; the first quotient satisfies the test for 9, and the next quotient that for 6. The quotient 13 is prime.

$$\begin{array}{r|l}
 10 & 7020 \\
 \hline
 9 & 702 \\
 \hline
 6 & 78 \\
 \hline
 & 13
 \end{array}$$

$$\begin{aligned}
 \text{Thus, } 7020 &= 10 \times 9 \times 6 \times 13 = 2 \times 5 \times 3 \times 3 \times 2 \times 3 \times 13 \\
 &= 2^2 \times 3^3 \times 5 \times 13
 \end{aligned}$$

When divisors are not evident, the prime numbers should be tried in succession.

**Q :** Find the H.C.F. of 126, 396, and 1080.

**A :** Expressing the numbers in their prime factors

$$126 = 2 \times 3^2 \times 7$$

$$396 = 2^2 \times 3^2 \times 11$$

$$1080 = 2^3 \times 3^3 \times 5$$

The highest power of 2, which will divide 2,  $2^2$ , and  $2^3$ , is 2.

The highest power of 3, which will divide  $3^2$  and  $3^3$ , is  $3^2$ .

There are no other common factors.

Thus, the H.C.F. is  $2 \times 3^2$  or 18.

**Q :** Find the H.C.F. of 55224 and 122012.

$$\begin{array}{rcl}
 \text{A : } 55224 & ) & 122012 \quad ( 2 \\
 & \underline{110448} & \\
 & 11564 & ) 55224 \quad ( 4 \\
 & & \underline{46256} \\
 & & 8968 & ) 11564 \quad ( 1 \\
 & & & \underline{8968} \\
 & & & 2596 & ) 8968 \quad ( 3 \\
 & & & & \underline{7788} \\
 & & & & 1180 & ) 2596 \quad ( 2 \\
 & & & & & \underline{2360} \\
 & & & & & 236 & ) 1180 \quad ( \\
 & & & & & & \underline{1180} \\
 & & & & & & 0
 \end{array}$$

$$\text{H.C.F} = 236$$

If the two given numbers are large (as above) then the above method can be tedious and time-consuming. In such cases we can find the HCF as follows:

$$55224 = 4 \times 2 \times 9 \times 767$$

$$122012 = 4 \times 11 \times 2773$$

Note here that the common factor above is '4' --- (i)

Now, find the H.C.F. of 767 and 2773 as follows:

$$\begin{array}{r}
 767 \overline{) 2773} \quad (3 \\
 \underline{2301} \\
 472 \overline{) 767} \quad (1 \\
 \underline{472} \\
 295 \overline{) 472} \quad (1 \\
 \underline{295} \\
 177 \overline{) 295} \quad (1 \\
 \underline{177} \\
 118 \overline{) 177} \quad (1 \\
 \underline{118} \\
 59 \overline{) 118} \quad (2 \\
 \underline{118} \\
 0
 \end{array}$$

Thus, the H.C.F. of 767 and 2773 = 59

Therefore, H.C.F. of 552244 and 122012 =  $4 \times 59 = 236$  (from i)

When there are more than 2 numbers, not easily put into factors, we find the H.C.F. of two; then the H.C.F. of the result and a third number, and so on.

The final H.C.F. is the required one.

**Q :** What is the greatest length which can be used to measure exactly the following lengths: 20 ft., 13 ft. 9 inches, 17 ft. 6 inches, 21 ft. 3 inches?

**A :** We must express these lengths in the same denomination and find their greatest common divisor. Expressed in inches we have 240 inches, 165 inches, 210 inches, 255 inches, and G.C.D. of these is 15 inches, or 1 ft. 3 inches.

**Q :** Find the L.C.M. of 299, 221 and 759.

**A :** The H.C.F. of 299 and 221 is 13. By division we find that  $299 = 13 \times 23$ , and  $221 = 13 \times 17$ . Since all these factors must be factors of the L.C.M. we make trial of the factors 17 and 23 applied to the number 759. Thus we find:

$$759 = 23 \times 3 \times 11$$

$$\text{Thus L.C.M. required} = 13 \times 23 \times 17 \times 3 \times 11 = 167739$$

**Q :** Arrange the fractions  $\frac{2}{15}, \frac{3}{10}, \frac{5}{21}$  in the ascending order of their respective magnitudes.

**A :**

5	15,	10,	21
3	3,	2,	21
	1,	2,	7

$$\text{L.C.M. of 15, 10, 21} = 5 \times 3 \times 2 \times 7 = 210$$

$$\frac{2}{15} = \frac{2 \times 14}{15 \times 14} = \frac{28}{210}, \dots\dots\dots (1)$$

$$\frac{3}{10} = \frac{3 \times 21}{10 \times 21} = \frac{63}{210}, \dots\dots\dots (2)$$

$$\frac{5}{21} = \frac{5 \times 10}{21 \times 10} = \frac{50}{210}, \dots\dots\dots (3)$$

The factors 14, 21, and 10 are obtained by dividing 210 by the denominators of the given fractions in turn. By comparing the numerators we see that  $\frac{28}{210}$  is the least and  $\frac{63}{210}$  is the greatest of the given fractions. Thus  $\frac{2}{15}, \frac{5}{21}, \frac{3}{10}$  are the original fractions when arranged in an ascending order.

**Q :** Which of the following statement/s is/are definitely false?

- I. LCM of a set of numbers is necessarily greater than the largest number in that set.
- II. HCF of a set of numbers can be equal to the smallest number in that set.
- III. The HCF of a, b and c is H. a, b and c are therefore multiples of H.
- IV. The LCM of a, b and c is L. a, b and c are therefore factors of L.

**A :** I. a] Take 4, 6 and 12 as the given set of numbers

LCM = 12 (the largest number)

b] Take 2, 4 and 7 as the given set of numbers

LCM = 28 (greater than the largest number)

**∴ LCM of a set of numbers would be greater than or equal to the largest number in that set.**

It would be equal to the largest number when the largest number is a multiple of all the other numbers in that set. e.g. In the first case,  
 $12 = 4 \times 3$  and  $12 = 6 \times 2$   
 12 is a multiple of 4 and 6  
 ∴ 12 is the LCM.

II. a] Take 2, 6 and 12 as the given set of numbers.

HCF = 2 (the smallest number)

b] Take 4, 6 and 12 as the given set of numbers.

HCF = 2 (smaller than the smallest number)

**∴ HCF of a given set of numbers would be smaller than or equal to the smallest number in that set.**

HCF would be equal to the smallest number when the smallest number is a factor of all the other numbers in that set.  
 e.g., In the first case i.e. 2, 6 and 12  
 $(6 = 2 \times 3 \text{ and } 12 = 2 \times 6)$

2 is a factor of both 6 and 12.

$\therefore$  HCF = 2

III and IV are both true as explained in detail for I and II above.

$\therefore$  I is false, rest are true.

**Q :** A number P4571203R is divisible by 18. Which of the following values can P and R take?

(i) 1, 2

(ii) 2, 3

(iii) 6, 8

(iv) 3, 3

**A :** The number is divisible by 18. i.e., it has to be divisible by 2 and 9.

[ $\because$  2 and 9 are relatively prime.]

$\therefore$  R can take on any values 0, 2, 4, 6, 8

Now,  $P + 4 + 5 + 7 + 1 + 2 + 0 + 3 + R = P + R + 22$

i.e.,  $P + R$  could be 5, 14. [as the sum can't exceed 18 since P and R are each less than 10].

So, from the choices, P and R can take the values 6, 8.

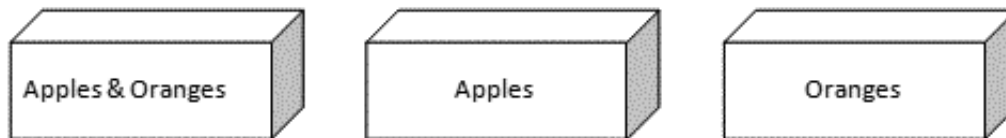


## CLASS EXERCISE

### Teaser

There are three boxes of fruit. The first box contains just Apples, second contain just Oranges and the third box contains a mixture of both. Each box is labelled – one says “Apples”, the second says “Oranges”, and the last says “Apples & Oranges.” But it is known that none of the boxes are labelled correctly.

The task is to label each of the boxes correctly. However you are only allowed to pick one box and exactly one piece of fruit from that box will be shown to you. What will be your strategy?



### Divisibility tests, Factorization, GCD-LCM

1. Write in prime-factorised form:
  - i) 400
  - ii) 888
  - iii) 30030
2. Is 198765432 divisible by 24?
3. Can 990924 be completely divided by 66?
4. Find GCD and LCM of:
  - i) 140 and 240
  - ii) 400 and 375
  - iii) 48, 72, 64
  - iv) 16 and 25
  - v) 28 and 308
  - vi)  $n$  and  $n + 1$
  - vii)  $\frac{5}{6}$  and  $1\frac{7}{8}$
  - viii)  $\frac{6}{9}$ ,  $\frac{4}{12}$  and  $\frac{3}{18}$
5. I have 132 oranges, 84 apples and 144 guavas. If each type of fruit can be distributed equally amongst all the students in my class, what could be the strength of my class? What is the maximum possible strength?

GCD of a set of numbers  $\leq$  each of the numbers  $\leq$  LCM of the set

GCD is a factor of each number in the set, and each number is a factor of the LCM

The LCM of a set of fractions  $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3} \dots$  is given by  $\frac{\text{LCM}(N_1, N_2, N_3, \dots)}{\text{GCD}(D_1, D_2, D_3, \dots)}$

The HCF of a set of fractions  $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3} \dots$  is given by  $\frac{\text{GCD}(N_1, N_2, N_3, \dots)}{\text{LCM}(D_1, D_2, D_3, \dots)}$

6. \* Which of the following are divisible by 7, 9 and 11?
  - i) 1386
  - ii) 594
  - iii) 3465
  - iv) 7623
7. \* In how many ways can 120 be written as a product of 2 even factors?

### Applications of Factorisation, GCD and LCM

8. Find the least number which leaves remainder 1 when divided by 18 or 24 or 30.
9. Find the least number which leaves remainder 1 when divided by 2, 2 when divided by 3, and 3 when divided by 4.
10. Find the least number which leaves remainder 1 when divided by 2, 3, 4, 5 or 6 but is completely divisible by 7.
11. Find the largest 3-digit number which leaves remainder 2 when divided by 14 or 35.
12. Find the greatest number which leaves remainders 10, 8 and 6 when it divides 70, 80 and 90 respectively.



13. Two numbers have  $\text{GCD} = 6$  and  $\text{LCM} = 1320$ . If one of the numbers is 120, find the other number.
14. Find two numbers which have  $\text{GCD} = 6$  and  $\text{LCM} = 90$ .
15. Find two 2-digit numbers which have  $\text{GCD} = 6$  and  $\text{LCM} = 432$ .

Note that for two numbers  $a$  and  $b$ :

- $a = \text{GCD} \times \text{its uncommon factors (UCFa)}$
- $b = \text{GCD} \times \text{its uncommon factors (UCFb)}$
- UCFa and UCFb are relatively prime
- $\text{LCM} = \text{GCD} \times (\text{UCFa}) \times (\text{UCFb})$
- $\text{LCM} \times \text{GCD} = a \times b$
- $a$  and  $b$  are said to be relatively prime (or co-prime) if their  $\text{GCD}(a, b) = 1$

16. \*Find two 2-digit numbers which have  $\text{GCD} = 6$  and product = 2880.
17. \*Instead of multiplying a number by 12, I divided it by 12 and got the answer 12. What is the correct answer?
18. \*When a number is divided by 15, the quotient is an odd number and the remainder is 1. What will be the remainder when divided by 30?
19. \*Arjun takes 6 minutes while Parth takes  $7\frac{1}{2}$  minutes to take a walk around Simmons Circle. If they started together at 10:00 am from the same spot in the same direction around the circle, at what time will they both be at the starting point again?
20. \*In a class when only Arjun was absent, the students could be divided into groups of 6 students each. But when both Arjun and Parth were absent, the rest could be divided into groups of 7 each. What is the minimum possible number of students in the class?
21. State true or false:
  - i. If a number is divisible by both 4 and 6 then it must be divisible by 24
  - ii. If a number is divisible by both 3 and 8 then it must be divisible by 24
  - iii. If a number is divisible by 5 then its square must be divisible by 25
  - iv. If a number is divisible by both 40 and 60 then it must be divisible by 120
22. Find the smallest number ( $>1$ ) and the largest number ( $<n$ ) that is relatively prime to  $n$ , where  $n$  equals:
 

i. 21	ii. 420	iii. 90
-------	---------	---------
23. Find the smallest and largest composite numbers ( $<n$ ) that are relatively prime to  $n$ , where  $n$  equals:
 

i. 42	ii. 60	iii. 420
-------	--------	----------

24. When a number is divided by 45 the remainder is 6. Find the remainder when the same number is divided by:  
i. 15                      ii. 5                      iii. 3
25. A three digit number is divisible by 7. What could be the possible remainders when it is divided by  
i. 21                      ii. 49                      iii. 3
26. \*When a number is divided by 120 the remainder is 48. Find the remainder when the same number is divided by:  
i. 15                      ii. 12                      iii. 9                      iv. 6
27. \*A number leaves a remainder 2 when divided by 9. Which of the following could not be the remainder when it is divided by 45?  
1) 20                      2) 30                      3) 29                      4) 38

### Challengers

1. N is the least number such that when divided by 180 or 144 the remainder is 7 but when divided by 7 the remainder is 1. Then which of these is true:  
1)  $0 < N < 1000$                       2)  $1000 < N < 2000$   
3)  $2000 < N < 4000$                       4)  $N > 4000$
2. Can there be a perfect square whose digits consist of exactly 4 ones, 4 twos and 4 zeroes in any order?
3. How many numbers from 1 to 1000 are divisible by:  
1) 3                      2) 5  
3) Both 3 and 5                      4) 3 but not by 5  
5) 15 but not by 30
4. How many numbers between 400 and 800 are divisible by both 88 and 72?  
1) 1                      2) 2                      3) 4                      4) 5
5. What is the remainder when  $3^3 \times 4^4 \times 5^5$  is divided by 6000?  
1) 0                      2) 15                      3) 60                      4) 100



## PRACTICE EXERCISE - 1

**DIRECTIONS for questions 1 and 2: Solve as directed.**

1. The sum of two numbers is 1560 and their HCF is 65. Find the total number of such pairs of numbers possible.
2. What is the maximum weight that can be used to measure 259 kg, 333 kg, 481 kg and 555 kg respectively.

**DIRECTIONS for questions 3 to 10: Choose the correct alternative.**

3. 'a' and 'b' are prime numbers. Which of the following is true?
  - I.  $a^2$  has three positive integer factors.
  - II.  $ab$  has four positive integer factors.
  - III.  $a^3$  has four positive integer factors.
  - 1) I and II only
  - 2) II and III only
  - 3) All of these
  - 4) None of these
4. The number of common terms in the two sequences 17, 21, 25, ... , 417 and 16, 21, 26, ... , 466 is.
  - 1) 78
  - 2) 19
  - 3) 20
  - 4) 77
5. Let  $k$  denote the number of factors of a number  $n$ . How many numbers ( $n$ ) less than 20 satisfy the condition  $k \geq \frac{n}{2}$ ?
  - 1) 7
  - 2) 5
  - 3) 4
  - 4) 8
6. Using only 2,5,10,25 and 50 paise coins, what will be the minimum number of coins required to pay exactly 36 paise, 77 paise, and Rs.2.03 to three different persons?
  - 1) 21
  - 2) 22
  - 3) 18
  - 4) 19
7. In a lighthouse, three lights flash after every 120 sec, 72 sec and 135 sec. If they all flash simultaneously at 8:00:00 according to a watch which loses 2 sec in every minute, when will they flash again simultaneously according to this watch?
  - 1) 8 : 18 : 12
  - 2) 8 : 17 : 36
  - 3) 8 : 18 : 36
  - 4) 8 : 17 : 24
8. If the numbers from 1 to 1000 which are exactly divisible by 3 are written in an ascending order, which would be the 100<sup>th</sup> digit from the left?
  - 1) 1
  - 2) 3
  - 3) 8
  - 4) None of these

9. If  $198a$  is a prime number, find the value of  $a$ .
- 1) 1                      2) 3                      3) 7                      4) 9
10. What is the minimum number of square marble tiles required to tile a floor of length 2 metres 56 cm and width 3 metres 36 cm?
- 1) 256                      2) 336                      3) 5376                      4) 748

**DIRECTIONS for questions 11 and 12: Refer to the data below and answer the questions that follow.**

Three thieves robbed a diamond shop independently but distributed their diamonds according to the rule that the person or persons with maximum diamonds should give diamonds equal to the H.C.F. of their individual number of diamonds to the person or persons with the lowest number of diamonds. This process is repeated till all have equal diamonds.

11. If they have 39, 52, 26 diamonds, how many times should the process be repeated?
- 1) 1                      2) 3                      3) 4                      4) Not possible
12. If they have 51, 85, 102 diamonds, how many times should the process be repeated?
- 1) 3                      2) 4                      3) 5                      4) Not possible

**DIRECTIONS for questions 13 to 20: Choose the correct alternative.**

13. Three containers having  $4\frac{1}{2}$ ,  $3\frac{1}{4}$  and  $6\frac{1}{3}$  litres of three different cold drinks are to be served equally to all guests in a party such that each one gets maximum. How many maximum number of guests could be entertained?
- 1) 50                      2) 150                      3) 169                      4) 196
14. Kant says, "I have selected two numbers and their HCF is the product of the first two odd prime numbers". Disha says, "I have selected two numbers and their LCM is the product of the first four prime numbers". If both had selected the same numbers, what would the numbers be given that none of the numbers was 15?
- 1) 14, 225                      2) 30, 105                      3) 42, 75                      4) 35, 90
15. A teacher bought a certain number of gifts to distribute equally among her ten students. But on the day of distribution four students were absent. Still the teacher was able to distribute the gifts equally. What is the least number of gifts that the teacher must have bought?
- 1) 60                      2) 30                      3) 45                      4) 15

16.  $x$  and  $y$  are natural numbers such that  $(x > y)$  and the LCM of  $x$  and  $y$  differs from the LCM of  $y$  and  $x - y$  by 18. Then,  $y$  could be equal to:
- 1) 16                      2) 18                      3) 27                      4) 4
17. Find the least number which when divided by 6, 15, 17 leaves a remainder 1, but when divided by 7 leaves no remainder.
- 1) 211                      2) 511                      3) 1022                      4) 86
- (Past CAT question)**
18. Let  $k$  be a positive integer such that  $(k + 4)$  is divisible by 7. Then the smallest positive integer  $n$ , greater than 2, such that  $k + 2n$  is divisible by 7 equals \_\_\_\_\_.
- 1) 9                      2) 7                      3) 5                      4) 3
- (Past CAT question)**
19. How many numbers ( $n$ ) are there between 1 and 200 such that  $\frac{n}{2}$ ,  $\frac{n}{3}$  and  $\frac{2n+1}{5}$  are all composite natural numbers?
- 1) 0                      2) 1                      3) 2                      4) 3
20. The HCF and LCM of two numbers is given. It is possible to find out the two numbers uniquely if
- I. either the sum or the difference between the two numbers is known.  
 II. HCF of the two numbers = LCM of the two numbers.  
 III.  $\frac{\text{LCM}}{\text{HCF}}$  = prime number.
- 1) I and II only              2) II only                      3) II and III only              4) I, II and III



## PRACTICE EXERCISE - 2

**DIRECTIONS for questions 1 to 18: Choose the correct alternative.**

1. The remainder obtained when a prime number greater than 6 is divided by 6 is  
 1) 1 or 3                      2) 1 or 5                      3) 3 or 5                      4) 4 or 5  
**(Past CAT question)**
2. Three bells chime at an interval of 18 min, 24 min and 32 min. At a certain time they begin to chime together. What length of time will elapse before they chime together again?  
 1) 2 hr and 24 min                      2) 4 hr and 48 min  
 3) 1 hr and 36 min                      4) 5 hr  
**(Past CAT question)**
3. After the division of a number successively by 3, 4 and 7, the remainders obtained are 2, 1 and 4 respectively. What will be the remainder if 84 divides the same number?  
 1) 80                      2) 76                      3) 41                      4) 53  
**(Past CAT question)**
4. There are 2 lights, one red and the other green. The red one flashes 3 times every minute and the green one flashes 5 times every two minutes. If the lights start flashing together, then the total number of times both will have flashed together in an hour is:  
 1) 30                      2) 24                      3) 48                      4) None of these  
**(Past CAT question)**
5. The integers 34,041 and 32,506 when divided by a three-digit integer  $n$  leave the same remainder. What is  $n$ ?  
 1) 289                      2) 367                      3) 453                      4) 307  
**(Past CAT question)**
6. A is the set of positive integers which when divided by 2, 3, 4, 5 and 6, leave the remainders 1, 2, 3, 4 and 5 respectively. How many integers between 0 and 100 belong to set A?  
 1) 0                      2) 1                      3) 2                      4) None of these  
**(Past CAT question)**
7. Each of the numbers  $x_1, x_2, \dots, x_n$ ,  $n \geq 4$ , is equal to 1 or  $-1$ . Suppose,  
 $x_1 x_2 x_3 x_4 + x_2 x_3 x_4 x_5 + x_3 x_4 x_5 x_6 + \dots + x_{n-3} x_{n-2} x_{n-1} x_n + x_{n-2} x_{n-1} x_n x_1 + x_{n-1} x_n x_1 x_2 + x_n x_1 x_2 x_3 = 0$ , then  
 1)  $n$  is even                      2)  $n$  is odd  
 3)  $n$  is an odd multiple of 3                      4)  $n$  is prime  
**(Past CAT question)**

8. Three wheels can complete 60, 36 and 24 revolutions per minute. There is a red spot on each wheel that touches the ground at time zero. After how much time, all these spots will simultaneously touch the ground again?
- 1)  $\frac{5}{2}$  seconds      2)  $\frac{5}{3}$  seconds      3) 6 seconds      4) 7.5 seconds  
(Past CAT question)
9. A certain number, when divided by 899, leaves a remainder 63. Find the remainder when the same number is divided by 29.
- 1) 5      2) 4      3) 1      4) Cannot be determined  
(Past CAT question)
10. Number of students who have opted for subjects A, B and C are 60, 84 and 108 respectively. The examination is to be conducted for these students such that only the students of the same subject are allowed in one room. Also the number of students in each room must be same. What is the minimum number of rooms that should be arranged to meet all these conditions?
- 1) 28      2) 60      3) 12      4) 21  
(Past CAT question)
11. A number is formed by writing first 54 natural numbers in front of each other as 12345678910111213... Find the remainder when this number is divided by 8.
- 1) 1      2) 7      3) 2      4) 0  
(Past CAT question)
12. If  $n$  is an integer, how many values of  $n$  will give an integral value of  $\frac{(16n^2 + 7n + 6)}{n}$  ?
- 1) 2      2) 3      3) 4      4) None of these  
(Past CAT question)
13.  $P$  and  $Q$  are two positive integers such that  $PQ = 64$ . Which of the following cannot be the value of  $P + Q$ ?
- 1) 20      2) 65      3) 16      4) 35  
(Past CAT question)
14. If  $m$  and  $n$  are integers divisible by 5, which of the following is not necessarily true?
- 1)  $m - n$  is divisible by 5      2)  $m^2 - n^2$  is divisible by 25  
3)  $m + n$  is divisible by 10      4) None of these  
(Past CAT question)

15. If  $n$  is any odd number greater than 1, then  $n(n^2 - 1)$  is
- 1) divisible by 96 always
  - 2) divisible by 48 always
  - 3) divisible by 24 always
  - 4) None of these
- (Past CAT question)**

16. If a number 774958A96B is to be divisible by 8 and 9, the values of A and B, respectively, can be \_\_\_\_\_.
- 1) 7, 8
  - 2) 8, 0
  - 3) 5, 8
  - 4) None of these
- (Past CAT question)**

17. For the product  $n(n + 1)(2n + 1)$ ,  $n \in \mathbb{N}$ , which one of the following is not necessarily true?
- 1) It is even
  - 2) Divisible by 3
  - 3) Divisible by the sum of the square of first  $n$  natural numbers
  - 4) Never divisible by 237
- (Past CAT question)**

18. Which is the least number that must be subtracted from 1856, so that the remainder when divided by 7, 12, and 16 is 4.
- 1) 137
  - 2) 1361
  - 3) 140
  - 4) 172
- (Past CAT question)**

**Direction for questions 19 and 20: Each of these questions is followed by two statements, I and II. Mark the answer as:**

1. if the question can be answered with the help of statement I alone.
2. if the question can be answered with the help of statement II, alone.
3. if both statement I and statement II are needed to answer the question.
4. if the question cannot be answered even with the help of both the statements.

19. 4 students were added to a dance class. Will the teacher be able to divide them into groups of 8?
- I. If 12 students were added she would be able to divide them in groups of 8.
  - II. Currently no. of students is not divisible by 8.
- (Past CAT question)**

20. What is the number  $x$ ?
- I. The LCM of  $x$  and 18 is 36.
  - II. The HCF of  $x$  and 18 is 2.
- (Past CAT question)**



## QA-1.3 | INDICES AND SURDS



### Square numbers

#### Square of a number

If a number is multiplied by itself, the product so obtained is called the square of that number.

##### Example

Square of 7 =  $7 \times 7 = 49$ .

i.e.,  $7^2 = 7 \times 7 = 49$ .

#### Perfect square

The square of a natural number is called a perfect square.

#### Some properties of perfect squares

- (i) A square cannot end with an odd number of zeros.
- (ii) A square number cannot end with 2, 3, 7 or 8.
- (iii) The square of an odd number is odd.
- (iv) The square of an even number is even.
- (v) Every square number is a multiple of 3, or exceeds a multiple of 3 by unity.
- (vi) Every square number is a multiple of 4 or exceeds a multiple of 4 by unity.
- (vii) If a square number ends in 9, the preceding digit is even.

#### Shortcut methods of Calculation of Powers of a Number

##### Square of a number ending in 5

Multiply the number of tens by the next higher integer and annex 25 to the right of the product.

##### Example

$105^2 : 10 \times 11 = 110$ . Hence, square is 11025.

##### Square of a number ending in 25

Multiply the number of hundreds by a number consisting of the number of hundreds with a 5 to its right. Thereafter, annex 625 to the right of the product.

##### Example

$625^2 : 6 \times 65 = 390$ . Hence, square of 625 is 390625.

$1125^2 : 11 \times 115 = 1265$ . Hence, square of 1125 is 1265625.

### Squares of $1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$ etc

Multiply the integral portion by the next higher integer and add  $\frac{1}{4}$ ,

#### Example

$$\left(6\frac{1}{2}\right)^2 = 6 \times 7 + \frac{1}{4} = 42\frac{1}{4}.$$

### Squares, cubes and higher powers of a number consisting wholly of 9s.

- (i) The square is found by writing down  $(n - 1)$  nines followed by 8 and then  $(n - 1)$  zeros followed by 1 (where  $n$  is the total number of 9s in the number).

#### Example

$$999^2 = 998001$$

- (ii) The cube is found by writing down  $(n - 1)$  nines followed by 7, then  $(n - 1)$  zeros followed by 2 and finally  $n$  nines.

#### Example

$$999^3 = 997002999$$

- (iii) The fourth power is found by writing down  $(n - 1)$  nines followed by 6, then  $(n - 1)$  zeros followed by 5, then  $(n - 1)$  nines followed by 6, and lastly  $(n - 1)$  zeros followed by 1.

#### Example

$$999^4 = 996005996001$$

### Cube of a Number

A cube of number is a number that is the product of three numbers which are the same. In other words, if a number is multiplied by itself and then by itself again, the product obtained is called the cube of that number.

$$\text{eg., } 7^3 = 7 \times 7 \times 7 = 7 \times 7^2 = 343$$

### Armstrong Numbers

Observe the following numbers. The cubes of the digits that form them, add up to the number itself. These numbers are called Armstrong Numbers.

$$153 = 1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$$

$$\begin{aligned} 370 &= 3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370 \\ 371 &= 3^3 + 7^3 + 1^3 = 27 + 343 + 1 = 371 \\ 407 &= 4^3 + 0^3 + 7^3 = 64 + 0 + 343 = 407 \end{aligned}$$

## Methods to Calculate Square roots & Cube roots

### Square root

The square root of a given number is a number whose square is equal to the given number.

Note: Any positive real number has two square roots, one positive and one negative. For example, square roots of 9 are  $-3$  and  $+3$  since  $(-3)^2 = (+3)^2 = 9$ . Any non negative real number (say  $x$ ) has a unique non negative square root (say  $r$ ) called the principal square root. For example, the principal square root of 9 is  $+3$ . In common usage, unless otherwise specified, the square root is generally taken as the principal square root.

- (i) **To find the square root of a number by prime factorization.**

$$\sqrt{1334025} = \sqrt{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 11 \times 11} = 3 \times 5 \times 7 \times 11 = 1155$$

- (ii) **To find the square root of a number by division method:**

The given number is separated in pairs from right to left e.g., 2119936 will be grouped as 2, 11, 99, 36 (as it has an odd number of digits) and 105625 will be grouped as 10, 56, 25 (as it has an even number of digits)

Step 1: The integer whose square is less than or equal to the first group is (on the extreme left) written on the top and left of the number.

Step 2: The square of the integer is subtracted from the first group.

Step 3: The next group is brought down and written with the remainder and forms the new dividend.

Step 4: Add the quotient to the previous quotient and put it in the divisor.

e.g., (i)	e.g., (ii)
$\begin{array}{r rrrr} & 1 & & & \\ 1 & 2, & 11, & 99, & 36, \\ 1 & 1 & & & \\ \hline 2 & 1, & 11 & & \end{array}$	$\begin{array}{r rr} & 3 & \\ 3 & 10, & 56, & 25 \\ 3 & 9 & & \\ \hline 6 & 1, & 56 & \end{array}$

Step 5: The new divisor is obtained by writing a suitable digit say  $A$  to the right of the integer from step 4 and this number is multiplied by  $A$ . The suitable digit is also the quotient.

For example, if  $A = 3$  in example (i),  $23 \times 3 = 69$  which is well within 111 so try  $A = 4$ ;  $24 \times 4 = 96$ ; with  $A = 5$ ;  $25 \times 5 = 125$  is out of range of 111. Likewise, we put  $A = 2$  in example (ii).

e.g., (i)					e.g., (ii)				
	1	4				3	2		
1	2,	11,	99,	36,	3	10,	56,	25	
1	1				3	9			
24	1,	11			62	1,	56		
		96				1	24		
		15					32		

Step 6: Repeat step 3 to 5

e.g., (i)						e.g., (ii)					
	1	4	5	6			3	2	5		
1	2,	11,	99,	36,		3	10,	56,	25		
1						3	9				
24		111				62	1,	56			
4		96				2	1	24			
285		15	99			645		32	25		
5		14	25			5		32	25		
2906		1	74	36		650			0		
6		1	74	36							
2912				0							

$$\therefore \sqrt{2119936} = 1456$$

$$\sqrt{105625} = 325$$

(iii) **To find the square root of a positive decimal fraction**

If the given number is in the form of a decimal fraction, the digits to the left of the decimal point and those to the right of the decimal point are grouped independently. In the integral part, groups of two digits each are formed, starting from the decimal point and moving to the left.

In the fractional part, starting from the decimal point the groups are formed towards the right.

e.g., (i)						e.g., (ii)					
	23.25						107.5				
2	540.5625					1	11556.25				
2	4					1	1				
43	140					20	15				
3	129					0	0				
462	1156					207	1556				
2	924					7	1449				
4645	23225					2145	10725				
5	23225					5	10725				
4650	0					2150	0				

$1^2 = 1$  and  $9^2 = 81$ . So the square root of a one digit or a 2 digit number is always a one digit number.  $10^2 = 100$  and  $99^2 = 9801$ . So the square root of a three digit number or a four digit number is always a two digit number.

Thus, the number of digits in the square root of a 'n' digit number is

(i)  $\frac{n+1}{2}$ , where n is odd.

(ii)  $\frac{n}{2}$ , where n is even.

(iv) **Square root of fraction**

The square root of a fraction, if the numerator and the denominator are both perfect squares, is found by taking the square root of the numerator and denominator separately. In the case of a fraction whose denominator is not a perfect square, we may convert the fraction into a decimal and then find the square root of the decimal as discussed in the previous point.

**(Note:** In case we need to find the square root of a mixed fraction, it must first be expressed as an improper fraction.)

**Cube Root**

The cube root of a number is a value that when used in a multiplication three times gives that number.

Example:  $3 \times 3 \times 3 = 27$ , so the cube root of 27 is 3.

Also  $-5 \times -5 \times -5 = -125$ , so the cube root of  $-125$  is  $-5$ .

**To find the cube root by prime factorisation:**

**Example**

(i)  $\sqrt[3]{15625} = \sqrt[3]{5 \times 5 \times 5 \times 5 \times 5 \times 5} = 5 \times 5 = 25$

(ii)  $\sqrt[3]{2744} = \sqrt[3]{2 \times 2 \times 2 \times 7 \times 7 \times 7} = 2 \times 7 = 14$

## SOLVED EXAMPLES

**Q :** Which of the following is/are true?

- I. If a number has  $n$  digits, then its square will have  $2n$  or  $2n - 1$  digits.
- II. A square number can always be written in the form  $3n$  or  $3n + 1$  where  $n$  is a whole number.
- III. A square number can always be written in the form  $4n$  or  $4n + 1$  where  $n$  is a whole number.

**A :** I. Squares of numbers from 1 to 9 have 1 or 2 digits. Square of numbers from 10 to 99 have 3 or 4 digits. Generalising, if a number has  $n$  digits its square will have  $2n$  or  $2n - 1$  digits. Thus, I is true.

II. Observe the squares of numbers from 0 to 10 i.e., 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Each of these can be written in the form  $3n$  or  $3n + 1$  where  $n$  is a whole number. Thus, II is true.

III. Again observe squares of numbers from 0 to 10, i.e., 0, 1, 4, 9, 16, 25... Each of these can be written in the form  $4n$  or  $4n + 1$ , where  $n$  is a whole number. Thus, III is true.

**Q :** Which of the following is/are true?

- 1] The square of a square cannot end in 2, 3, 4, 7, 8 or 9.
- 2] The square of a cube cannot end in 2, 3, 7 or 8.
- 3] The cube of a square cannot end in 2, 3, 7 or 8.

**A :** Consider numbers from 0 to 9. Their squares are 0, 1, 4, 9, 16, 25, 36, 49, 64, 81.

So a square of a square will always end in 0, 1, 5 and 6. The cubes of numbers from 0 to 9 are 0, 1, 8, 27, 64, 125, 216, 343, 512, 729. So a cube can end in any number from 0 to 9.

- 1] Square will end in 0, 1, 4, 5, 6 and 9 and the square of a square will end only in 0, 1, 5 or 6. Thus, [1] is true.
- 2] A cube can end in any number from 0 to 9. So the square of a cube can end in 0, 1, 4, 5, 6 and 9. Thus, [2] is true.
- 3] A square will end in 0, 1, 4, 5, 6 and 9. So the cube of square will end in 0, 1, 4, 5, 6 or 9. Thus, [3] is true.

**Q :** Find the first four digits of the square root of 5.23

**A :**

	2.2 8 6
2	5.230000 ...
2	4
42	123
2	84
448	3900
8	3584
4566	31600
6	27396
4572	4204

$$\sqrt{5.23} \approx 2.286$$

After bringing down the last significant figure, two ciphers are brought down at each stage, and appended to the remainder at that stage. Hence no new figure in the root can produce a 'divisor' which will bring the process to an end.

**Q :** A rectangular garden, the length of which is four times its breadth, has an area of 2 acres 361 sq. yards. Find its length and breadth to the nearest yard.

**A :** Let the length and breadth be denoted by  $4x$  and  $x$  yards respectively.

$$\therefore \text{Area} = 4x \times x = 4x^2 \text{ sq. yards.}$$

$$\text{Now, 1 acre} = 4840 \text{ sq. yards.}$$

$$\therefore 2 \text{ acres } 361 \text{ sq. yards.}$$

$$= 2 \times 4840 + 361 = 10041 \text{ sq. yards.}$$

$$\therefore 4x^2 = 10041 \quad \therefore x^2 = 2510.25$$

	50.1
5	2510.25
5	25
100	10
0	0
1001	1025
1	1001
1002	24

$$\therefore x = 50.1 \text{ yards} \Rightarrow 4x = 200.4 \text{ yards}$$

Thus, the required lengths are 50 yards, and 200 yards.

**Q :** Find the cube root of the number 3652264.

**A :**

8	3652264
7	456533
7	65219
7	9317
	1331

$$3652264 = 8 \times 7^3 \times 1331 = 2^3 \times 7^3 \times 11^3$$

$$\therefore \text{Cube root of } 3652264 = 2 \times 7 \times 11 = 154$$



## Indices

By the term 'a<sup>m</sup>' we mean a × a × ... m times i.e. the product of m numbers each equal to 'a' is represented by a<sup>m</sup>, where 'a' is called the **base** and 'm' is called the **power**.  $\sqrt[n]{a}$  or  $a^{\frac{1}{n}}$  denotes the nth root of a.

## Laws of Indices

(i)  $a^m \times a^n = a^{m+n}$

(ii)  $a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$

(iii)  $(a^m)^n = a^{mn}$

(iv)  $a^m \div a^n = a^{m-n}$

(v)  $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ , where a and p are real numbers and q ≠ 0.

(vi)  $a^{\frac{1}{n}} = \sqrt[n]{a}$ , where a and n are real numbers and n ≠ 0.

(vii)  $a^{-n} = \frac{1}{a^n}$

(viii)  $a^0 = 1$  (where a ≠ 0)

(ix)  $(a \times b)^m = a^m \times b^m$  and  $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

(x) If  $a^m = a^n$  and a ≠ -1, 0, 1 then m = n.

## SOLVED EXAMPLES

**Q :** Simplify  $16^{\frac{5}{4}}$

**A :**  $16^{\frac{5}{4}} = \sqrt[4]{16^5} = \sqrt[4]{(24)^5} = 2^{\frac{20}{4}} = 2^5 = 32.$

**Q :** Simplify  $27^{\frac{-4}{3}}$

**A :**  $27^{\frac{-4}{3}} = \frac{1}{27^{\frac{4}{3}}} = \frac{1}{(3^3)^{\frac{4}{3}}} = \frac{1}{3^4} = \frac{1}{81}$

**Q :** Find the value of  $2^3 \times 2^6$

**A :**  $2^3 \times 2^6 = 2^{3+6} = 2^9 = 512$

**Q :** Simplify  $\left(27^{\frac{-2}{3}}\right)^{\frac{1}{2}}$

**A :**  $\left(27^{\frac{-2}{3}}\right)^{\frac{1}{2}} = 27^{\frac{-1}{3}} = 3^{-1} = \frac{1}{3}$

*Alternatively,*

$$\frac{1}{(27)^{\frac{1}{3}}} = \frac{1}{(3^3)^{\frac{1}{3}}} = \frac{1}{3}$$

**Q :** Simplify  $(2^3)^2$

**A :**  $(2^3)^2 = 2^6 = 64$

**Q :** Solve:  $2^x + 5 = 2^x + 3 + 6$

**A :**  $2^x \times 2^5 = 2^x \times 2^3 + 6$

$$2^x (2^5 - 2^3) = 6$$

$$2^x (32 - 8) = 6$$

$$2^x = \frac{6}{24} = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$$

$$\therefore x = -2$$

**Q :** Simplify:  $-\sqrt{\frac{36t^{12}}{25m^6}}$

**A :**  $-\sqrt{\frac{36t^{12}}{25m^6}} = \frac{-6t^6}{5m^3}$

### Concept Builder 1

Solve the following indices:

1.  $\left(\frac{1}{125}\right)^{\frac{-2}{3}} \div \left(\frac{1}{8}\right)^{\frac{-4}{3}}$

2.  $\left(\frac{1}{6561}\right)^{\frac{-1}{2}}$

3.  $\left(\frac{27}{125}\right)^{\frac{2}{3}} \times \left(\frac{64}{27}\right)^{\frac{1}{3}} \div \left(\frac{25}{49}\right)^{\frac{1}{2}}$

4.  $\left(\frac{729}{512}\right)^{\frac{2}{3}}$

5.  $\left(\frac{2401}{625}\right)^{\frac{-3}{2}}$

**Answer Key**

3.  $\frac{84}{125}$

4.  $\frac{64}{81}$

5.  $\frac{76}{56}$

1.  $\frac{25}{16}$

2.  $\frac{81}{16}$

## Surds

Any root of a rational number, which cannot be exactly found is called a surd (an irrational number)

### Example

$$\sqrt{2}, \sqrt[3]{4}, 2 + \sqrt{2}$$

For a surd  $\sqrt[n]{a}$ ,  $a$  is called the radicand and  $n$  is called the order of the surd.  $\sqrt{\phantom{x}}$  is the radical sign.

For  $\sqrt[n]{a}$  to be a surd

- (i) the radicand  $a$  should be a positive rational number.
- (ii)  $n$  should be a natural number.
- (iii)  $\sqrt[n]{a}$  is an irrational number.

## Laws of Surds

- (i)  $(\sqrt[n]{a})^n = a$
- (ii)  $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ . Also  $(\sqrt[n]{a} \cdot \sqrt[n]{b})^n = a \cdot b$
- (iii)  $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ . Also  $\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \frac{a}{b}$
- (iv)  $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$
- (v)  $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$

## Types of Surds

### Pure Surd

A surd which consists wholly of an irrational number is called a pure surd. A pure surd has 1 as the only rational factor. e.g.  $\sqrt{8}, \sqrt[4]{5}$

### Mixed Surd

If a surd has one factor which is a rational number other than one and the other factor is an irrational number then the surd is called a mixed surd.

### Example

$$2\sqrt{5}, \frac{5}{9}\sqrt{75}, -2\sqrt[3]{29}$$

Every mixed surd can be expressed as a pure surd. However, a pure surd cannot always be converted into a mixed one.

### Example

$$2\sqrt[3]{2} = \sqrt[3]{2 \times 2^3} = \sqrt[3]{16}$$

### Similar surds

By similar surds, we mean surds which have the same surd in their simplest form.

### Example

$\sqrt{72}$  and  $\sqrt{200}$  are similar as  $\sqrt{72} = 6\sqrt{2}$  and  $\sqrt{200} = 10\sqrt{2}$  have the same surd  $\sqrt{2}$  in their simplest form.

### Rationalisation

If two surds be such that their product is rational, each of them is said to be rationalised when multiplied by the other and either of them is said to be a rationalising factor of the other.

For example, if  $\sqrt{2}$  is multiplied by  $\sqrt{2}$ , it will give 2, which is a rational number. So  $\sqrt{2}$  is a rationalizing factor of  $\sqrt{2}$ .

Similarly,  $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ .

So  $\sqrt{2}$  is a rationalizing factor of  $\sqrt{8}$ .

### Binomial Quadratic Surd

A binomial quadratic surd is a binomial in which the surds present are of order 2. Such surds have rationalisation factor as the conjugates.

### Example

The conjugate of  $\sqrt{3} + \sqrt{2}$  is  $\sqrt{3} - \sqrt{2}$  as  $(\sqrt{3} + \sqrt{2}) \times (\sqrt{3} - \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$ , which is a rational number.

Hence,  $\sqrt{3} - \sqrt{2}$  which is the conjugate of  $\sqrt{3} + \sqrt{2}$  is the rationalizing factor of  $\sqrt{3} + \sqrt{2}$ .

Further,  $\sqrt{3} + \sqrt{2}$  is also the rationalizing factor of  $\sqrt{3} - \sqrt{2}$ .

**Note:** If  $a + \sqrt{b} = x + \sqrt{y}$ , where  $a$  and  $x$  are rational and  $\sqrt{b}$  and  $\sqrt{y}$  are irrational, then  $a = x$  and  $b = y$ .

### Properties of numbers between 0 and 1

If  $x$  is a number between 0 and 1 (both numbers excluded) then:

$$\dots \sqrt[3]{x} > \sqrt{x} > x > x^2 > x^3 \dots$$

#### Example

For  $x = \frac{1}{64}$

According to the above property,

$$\begin{array}{ccccccccc} \frac{1}{4} & > & \frac{1}{8} & > & \frac{1}{64} & > & \frac{1}{(64)^2} & > & \frac{1}{(64)^3} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \sqrt[3]{x} & & \sqrt{x} & & x & & x^2 & & x^3 \end{array}$$

#### Some Important Formulae

$$(a + b)^2 = a^2 + b^2 + 2ab$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

### SOLVED EXAMPLES

**Q :** Convert  $5\sqrt{3}$  into a pure surd.

**A :**  $5\sqrt{3} = (5^2)^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = (5^2 \times 3)^{\frac{1}{2}} = (25 \times 3)^{\frac{1}{2}} = \sqrt{75}$

**Q :** Convert  $\sqrt{32}$  into a mixed surd.

**A :**  $\sqrt{32} = \sqrt{16 \times 2} = (4^2 \times 2)^{\frac{1}{2}} = 4\sqrt{2}$

**Q :** Convert  $\sqrt[3]{40}$  into a mixed surd.

**A :**  $\sqrt[3]{40} = \sqrt[3]{8 \times 5} = 2\sqrt[3]{5}$

**Q :** Find the value of (i)  $\sqrt{147} + \sqrt{27}$  (ii)  $\sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40}$

**A :** (i)  $\sqrt{147} + \sqrt{27} = \sqrt{49 \times 3} + \sqrt{9 \times 3} = 7\sqrt{3} + 3\sqrt{3} = 10\sqrt{3}$

$$\begin{aligned} \text{(ii)} \quad \sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40} &= \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt[3]{8 \times 5} \\ &= 5\sqrt[3]{5} - 3\sqrt[3]{5} + 2\sqrt[3]{5} = 4\sqrt[3]{5} \end{aligned}$$

**Q :** Which is greater  $\sqrt[3]{9}$  or  $\sqrt[4]{20}$ ?

$$\mathbf{A :} \quad \sqrt[3]{9} = 9^{\frac{1}{3}} = 9^{\frac{4}{12}} = \sqrt[12]{9^4} = \sqrt[12]{6561}$$

$$\sqrt[4]{20} = 20^{\frac{1}{4}} = 20^{\frac{3}{12}} = \sqrt[12]{20^3} = \sqrt[12]{8000}$$

(Since L.C.M. of 3 and 4 is 12, we convert both the surds to order 12)

$$\text{Since } 6561 < 8000, \sqrt[3]{9} < \sqrt[4]{20}$$

**Q :** Expand  $(\sqrt{5} + 2)^2$

$$\begin{aligned} \mathbf{A :} \quad (\sqrt{5} + 2)^2 &= (\sqrt{5})^2 + (\sqrt{2})^2 + 2 \times \sqrt{5} \times 2 \\ &= 5 + 4 + 4\sqrt{5} \\ &= 9 + 4\sqrt{5} \end{aligned}$$

**Q :** Expand  $(\sqrt{5} - 2)^2$

$$\begin{aligned} \mathbf{A :} \quad (\sqrt{5} - 2)^2 &= (\sqrt{5})^2 + (-\sqrt{2})^2 - 2 \times \sqrt{5} \times 2 \\ &= 5 + 4 - 4\sqrt{5} \\ &= 9 - 4\sqrt{5} \end{aligned}$$

**Q :** Find the positive square root of  $7 + 4\sqrt{3}$

$$\begin{aligned} \mathbf{A :} \quad 7 + 4\sqrt{3} &= 4 + 3 + 4\sqrt{3} \\ &= (2)^2 + (\sqrt{3})^2 + 2 \times 2 \times \sqrt{3} \\ &= (2 + \sqrt{3})^2 \end{aligned}$$

Now, square root of  $7 + 4\sqrt{3} = \pm (2 + \sqrt{3})$

However, positive square root of  $7 + 4\sqrt{3} = + (2 + \sqrt{3})$

since  $-(2 + \sqrt{3}) = -2 - \sqrt{3}$  is negative

**Q :** Find the positive square root of  $8 - 2\sqrt{15}$

**A :**  $8 - 2\sqrt{15} = 5 + 3 - 2\sqrt{15}$

$$= (\sqrt{5})^2 + (\sqrt{3})^2 - 2 \times \sqrt{5} \times \sqrt{3}$$

$$= (\sqrt{5} - \sqrt{3})^2$$

Square root of  $8 - 2\sqrt{15} = \pm (\sqrt{5} - \sqrt{3})$

However, positive square root of  $8 - 2\sqrt{15}$  will be  $+(\sqrt{5} - \sqrt{3})$

as  $-(\sqrt{5} - \sqrt{3}) = \sqrt{3} - \sqrt{5}$ , which is negative.

**Q :** Rationalise denominator of  $\frac{4}{2 - \sqrt{3}}$

**A :** Multiply the numerator and denominator by  $2 + \sqrt{3}$

$$\frac{4}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{4(2 + \sqrt{3})}{4 - 3} = \frac{4(2 + \sqrt{3})}{1} = 4(2 + \sqrt{3})$$

## Concept Builder 2

Express/solve the following surds to their simplest form:

1.  $3\sqrt{2} \times \sqrt[3]{5}$

2.  $\sqrt[6]{2187}$

3.  $\sqrt[3]{-216} \times \sqrt[4]{4096}$

4.  $\sqrt[3]{13824} - 4$

5.  $\sqrt{192} + \sqrt{147}$

**Answer Key**

1.  $3\sqrt[6]{200}$  2.  $3\sqrt[3]{3}$  3. -48 4.  $20\sqrt{3}$  5. 15





## CLASS EXERCISE

### Teaser

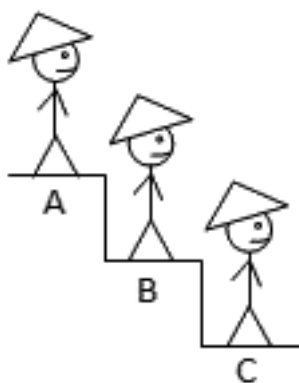
This is a variant of the famous prisoners and caps puzzle:

Three prisoners were arrested for a crime. When they begged for release, the warden made them an offer:

“I have five caps”, he said “two black and three white. The three of you will stand on these stairs facing downwards (see figure below), with C in front and A at the back. You will then be blindfolded and I shall place one cap on each one's head. The blindfolds will then be removed so that each of you can see the caps of the prisoners in front but not your own. Each of you will then be asked the colour of the cap on your own head. If you are able to answer correctly you will be released but if the answer is wrong you will be shot. However you have a choice not to answer and continue in prison. You cannot communicate with each other. But you can listen to each other's answers.

The warden started with A, who had the advantage of seeing the others. A replied that he could not answer. Then the warden asked B who also replied that he was not able to answer. The warden then asked C who thought for a while and answered. He was released promptly.

What was C's answer and how was he able to deduce it?



### Squares, Square-roots, Cubes, Cube-roots

a)  $(-2)^3 \times (-3)^2 =$

b)  $\frac{\sqrt{136}}{\sqrt{1.36}} =$

c) Find  $\sqrt[3]{126 \times 12 \times 49}$

d) Find the square-roots of

- 576

- 5184

- 2809

- 0.000441

- 42

- 146.41

e) By what least number should 77077 be multiplied to get a perfect square?

f) \*By what least number should 1323 be multiplied to get a perfect cube?

g) \*Find the value of a if  $\frac{\alpha}{21} \times \frac{\alpha}{189} = 1$

h) \*  $\sqrt{\frac{\sqrt{40.96}}{10}}$

i) \*Find c where  $\sqrt{1 + \frac{c}{121}} = \frac{12}{11}$

j) \*  $\frac{(0.7)^3}{0.0049} + \frac{(0.5)}{(0.05)^2}$

## Indices

Find the value of the given expression:

- |                   |                   |                     |
|-------------------|-------------------|---------------------|
| a) $2^8 =$        | b) $\sqrt{2^8}$   | c) $2^6 \times 2^4$ |
| d) $2^6 \div 2^4$ | e) $2^4 \div 2^6$ | f) $(2^3)^2$        |

What's the meaning of:

- |             |                      |                       |
|-------------|----------------------|-----------------------|
| a) $4^{-2}$ | b) $4^{\frac{1}{2}}$ | c) $4^{-\frac{1}{2}}$ |
|-------------|----------------------|-----------------------|

Simplify:

- |  |   |
|--|---|
| a) $19^{20} \times 19^{19}$              | g) Find m where $9^9 = 3^m$   |
| b) $19^{20} \div 19^5$                   | h) Find p and q where $90^{90} = 9^9 \times 10^{10} \times 9^p \times 10^q$ |
| c) $19^{20} \times 19^{10} \div 19^{15}$ | i) Find t where $3^2 \times 9^2 = 3^t$                                      |
| d) $(19^4)^5 \times (19^2)^{10}$         | j) Find k where $38^4 = 19^4 \times 4^k$                                    |
| e) $19^{23} \div [19^2]^3$               |   |
| f) $19^{20} \div 19^{-10}$               |   |

Solve:

- |  |  |
|--|--|
| a) $\left(\frac{1}{3}\right)^{20} \div \left(\frac{1}{9}\right)^4 = 3^k$ . Find k. | k) *Find z where $3^6 \times 8^z = 36^3$   |
| b) Find k where $(3^{42} \times 9^{12})^{1/3} = 3^k$                               | l) *Find n where $49\sqrt{7} = 7^n$  |
| c) $(0.4^4 \div 0.2^8)^{\frac{1}{2}}$  | m) *Find A and B where $3^{12} \times 6^8 = 3^A \times 4^B$                                  |
| d) Find m where $0.0012 = 1.2 \times 10^m$   | n) *Find k if $\sqrt{343^{\frac{2k}{3}}} = \sqrt[3]{7^2}$                                    |
| e) Find n where $4 = 32^n$   | o) * $25^{2.5} =$  |
| f) Find p where $4^{60} = 8^p$   | p) *Find a and b where $0.0025 = \left(\frac{2}{5}\right)^a \times 10^b$                     |
| g) If $x^{\frac{1}{2}} = 8$ then find $x^{\frac{2}{3}}$                            | q) *Find mn where $144 = \frac{2^m}{3^n}$  |
| h) Find q where $3^{44} \times 9^{20} = 27^q$                                      | r) * $\frac{(62.5)^3 \times (0.005)^5}{(1.25)^2 \times (2.5)^4}$                             |
| i) Find k where $\sqrt{9^3} \div \sqrt[3]{9^2} = 3^k$                              | s) * $\sqrt[3]{\sqrt{\frac{9^3}{(0.9)^2}}} \times \sqrt[3]{\sqrt{\frac{6^4 \times 4}{100}}}$ |
| j) Find the value of $(0.008)^{-\frac{2}{3}}$                                      |  |

### Surds

1. Convert to mixed surd:

a)  $\sqrt{180}$                       b)  $\frac{\sqrt{216} \times \sqrt{75}}{\sqrt{45}}$                       c)  $\sqrt[3]{250}$

2. Convert to pure surd:

a)  $2\sqrt{5}$                       b)  $3\sqrt[3]{3}$

3. Rationalize the denominator

a)  $\frac{1}{2\sqrt{2}}$                       b)  $\frac{8\sqrt{2}}{\sqrt{3}}$                       c)  $\frac{\sqrt{10}}{\sqrt{15}}$

d)  $\frac{4}{\sqrt{7}-\sqrt{5}}$                       e)  $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

4. Expand:  $(2 + \sqrt{3})^2$

5. Simplify:

a)  $\frac{\sqrt{675}-\sqrt{12}-\sqrt{3}}{2\sqrt{27}}$                       b)  $\frac{\sqrt{75}+\sqrt{27}}{\sqrt{75}-\sqrt{27}}$                       c)  $\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} + \frac{4\sqrt{3}}{\sqrt{2}-\sqrt{6}}$

6. Find the positive square root of:

a)  $3 + 2\sqrt{2}$                       b)  $5 + \sqrt{24}$                       c)  $7 - \sqrt{24}$

d)  $*7 + 4\sqrt{3}$                       e)  $*7 - \sqrt{48}$                       f)  $*14 + 4\sqrt{6}$

7. Find  $\sqrt{x} + \frac{1}{\sqrt{x}}$  where  $x = 3 + \sqrt{8}$ .

8. If  $y = 13 - 2\sqrt{6}$ , find the value of  $y^2 - 26y + 140$

9. Compare

a)  $2^{30}$  and  $3^{20}$                       b)  $\left(\frac{1}{4}\right)^{50}$  and  $\left(\frac{1}{5}\right)^{40}$

c)  $^2\sqrt{30}$  and  $^3\sqrt{40}$                       d)  $\frac{2}{3}^{\frac{3}{2}}$  and  $\frac{3}{2}^{\frac{2}{3}}$

e)  $40^{\frac{2}{3}}$  and  $30^{\frac{3}{2}}$                       f)  $(0.8)^2$  and  $(0.8)^3$

### Challengers

1. Find the smallest out of the following:
 

1) $0.12 \times (0.1)^{-4}$	2) $1.2 \times (0.01)^{-3}$
3) $0.012 \times (0.001)^{-2}$	4) All are equal
  
2. Find  $m + n$  where  $\left(\frac{2}{5}\right)^5 \times \frac{2^{-5}}{10^5} = 2^m \times 5^n$ 

1) 10	2) 15	3) - 10	4) - 15
-------	-------	---------	---------
  
3. Which of the following best describes the value of  $\frac{5^{\frac{1}{4}} \times 125^{0.25}}{\sqrt[4]{25} \times 5\sqrt{5}}$ ?
 

1) $5^2$	2) $\frac{1}{5}$	3) $\sqrt{5}$	4) $\frac{1}{\sqrt{5}}$
----------	------------------	---------------	-------------------------
  
4. Find the value of  $m$  where  $8^{2m-4} = 4^{m+6}$ 

1) 3	2) 4	3) 5	4) 6
------	------	------	------
  
5. Which of these is the smallest:
 

1) $2^{-0.2}$	2) $3^{-0.3}$	3) $4^{-0.4}$	4) $5^{-0.5}$
---------------	---------------	---------------	---------------
  
6.  $\frac{2^{22} + 2^{23} + 2^{24}}{2^{23} + 2^{24} + 2^{25}} = ?$ 

1) $\frac{1}{2}$	2) $\frac{1}{8}$	3) $\frac{1}{6}$	4) None of these
------------------	------------------	------------------	------------------



## PRACTICE EXERCISE

**DIRECTIONS for questions 1 to 5: Solve as directed.**

1. Solve:

a)  $(10^3)^4 \times (10^4)^5 \div (10^5)^6$

b)  $\frac{2^5 \times 2^{12}}{2^6}$

c)  $2^2 \times 2^2 \times 2^2 \times 2^2 \div 20^2$

d)  $\left(\frac{27}{64}\right)^{\frac{1}{3}}$

e)  $\sqrt{2^{81} \div 2^9}$

f)  $\sqrt{2^8 \times 4^6 \div 8^2}$

g) Find x where  $16 = 8^x$

h) Find t where  $0.00081 = (0.3)^4 \times 10^t$

2. Solve:

a)  $\frac{81^2 \times 27^2}{9^3 \times 3^5}$

b) Find n where  $2^8 \times 2^6 \times 2^4 = 4^n$

c)  $(64^{-\frac{1}{5}})^{\frac{5}{2}}$

d)  $4^5 \times 5^4 \times 25^2$

e)  $2^4 \times 3^6 \div 6^3$

f)  $\frac{64^3 \times 27^{-4} \times 8^{-5}}{9^{-1} \times 16^2 \times 81^{-3}}$

g)  $\frac{27^4}{81^3}$

h)  $0.25^{0.5}$

3. Simplify:

a)  $\sqrt{8} - \sqrt[3]{54} + \sqrt{54} - \sqrt[3]{16} + \sqrt{50}$

b)  $\frac{1}{\sqrt{27}} + \frac{\sqrt{3}}{27}$

c)  $\frac{1}{\sqrt[3]{81}} + \frac{2}{\sqrt[3]{24}}$

d)  $\frac{\sqrt[3]{2}}{\sqrt{3}} = \sqrt[6]{?}$

e)  $\frac{\sqrt[3]{40}}{\sqrt[12]{64}} \times \sqrt[6]{8} \times \sqrt[3]{25}$

f)  $\sqrt[3]{2\sqrt{4}} = \sqrt[12]{x}$

g)  $\frac{1}{\sqrt{2} + \sqrt{7}} + \frac{\sqrt{2} - \sqrt{7}}{25}$

4. Find the positive square root of:

a)  $10 + 4\sqrt{6}$

b)  $11 + 4\sqrt{6}$

c)  $25 + 4\sqrt{6}$

5. Which is greater:  $(1.2)^4$  or  $(1.25)^3$

**DIRECTIONS for questions 6 to 20: Choose the correct alternative.**

6. If  $x = 2$  and  $y$  is a positive integer, for which of the following values of  $y$ ,  $x^{88} < y^{11}$ ?

1) 258

2) 69

3) 35

4) 18

7. A man plants his orchard with 5625 trees and arranges them so that there are as many rows as there are trees in a row. How many rows are there?

1) 125                      2) 25                      3) 75                      4) None of these

8. Find the value of  $\sqrt{14 \frac{21}{44} \times 2 \frac{7}{55} \times \frac{9}{20}}$

1)  $\sqrt{4 \frac{19}{200}}$                       2)  $3 \frac{159}{220}$                       3)  $\sqrt{5 \frac{4}{163}}$                       4)  $6 \frac{10}{135}$

9. Which of the following is the greatest?

1)  $(3)^{-\frac{1}{2}}$                       2)  $(-2)^{-\frac{1}{3}}$                       3)  $\left(\frac{3}{2}\right)^{-\frac{2}{3}}$                       4)  $(2)^{-\frac{3}{2}}$

10.  $\frac{2^5 \times 16^x - 64 \times 2^{4x-2}}{8 \times 4^{2x}} - \frac{2^x \times 4}{2^{x+1} - 2^x} = ?$

1) 2                      2) -2                      3) 1                      4) 0

11. If  $a = 2^{56} \times 3^{36} \times 11^{44}$ ,  $b = 2^{54} \times 3^{37} \times 11^{45}$ ,  
 $c = 2^{58} \times 3^{38} \times 11^{44}$ ,  $d = 2^{57} \times 3^{36} \times 11^{45}$ ,  
then the order of a, b, c and d from largest to smallest is:

1) d, a, c, b                      2) c, d, b, a                      3) c, d, a, b                      4) d, c, a, b

12. Which of the following can be reduced to an integral value?

1)  $\sqrt{27} + \sqrt{147}$                       2)  $\sqrt[3]{-8} + \sqrt{36}$                       3)  $\sqrt[3]{8} + \sqrt{-36}$                       4) Both (2) and (3)

13. When reduced,  $\frac{8^{\frac{2}{3}} + \left(\frac{1}{4}\right)^{-\frac{3}{2}} + \left(\frac{1}{27}\right)^{-\frac{4}{3}}}{(144)^{\frac{1}{2}}}$  becomes a/an:

1) Proper fraction                      2) Improper fraction  
3) Surd                      4) Complex number

14. a is a natural number greater than one and it is not a perfect square. [n] is the largest integer lesser than or equal to n. b is the largest perfect square less than or equal to a.

Then  $\frac{[\sqrt{a}]^{\sqrt{b}}}{\sqrt{b}} = ?$

1)  $[\sqrt{b}]^{\sqrt{b}+1}$                       2)  $[\sqrt{a}]^{2b}$                       3)  $[\sqrt{b}]^{\sqrt{b}-1}$                       4)  $[\sqrt{b}]^{\sqrt{b}+a-1}$

15. What is the value of 'n' if  $\frac{64 \times 4^{3n} + 4 \times 4^{3n+3}}{16 \times 4^{2n} - 16^n} = \frac{256}{3}$ ?

- 1)  $\frac{1}{2}$                       2) 1                      3)  $\frac{3}{2}$                       4) 2

(Past CAT question)

16. Which among  $2^{\frac{1}{2}}$ ,  $3^{\frac{1}{3}}$ ,  $4^{\frac{1}{4}}$ ,  $6^{\frac{1}{6}}$  and  $12^{\frac{1}{12}}$  is the largest?

- 1)  $2^{\frac{1}{2}}$                       2)  $3^{\frac{1}{3}}$                       3)  $4^{\frac{1}{4}}$                       4)  $6^{\frac{1}{6}}$

(Past CAT question)

17. A calculator has a key which replaces the displayed entry with its square and another key which replaces the resultant displayed entry with its reciprocal. Let "y" be the final result, if one starts with an entry say  $x$  ( $x \neq 0$ ) and alternately squares and reciprocates "n" times each (by beginning with pressing the key that displays the square value of the displayed entry). Assuming the case is completely accurate (ex: no round off errors/overflow errors) then  $y = ?$

- 1)  $x^{((-2)^n)}$                       2)  $x^{2n}$                       3)  $x^{(-2)^n}$                       4)  $x^{((-1)^{n2n})}$

18.  $n^3$  is odd. Which of the following statement(s) is/are true?  
I.  $n$  is odd.                      II.  $n^2$  is odd.                      III.  $n^2$  is even.

- 1) I only                      2) II only                      3) I and II only                      4) I and III only

(Past CAT question)

19. If  $5^a = 26$  and  $125^b = 676$  then

- 1)  $3a = 2b$                       2)  $a = 2b$                       3)  $2a = 3b$                       4)  $b = 2a$

(Past CAT question)

**DIRECTIONS for question 20: The question is followed by two statements, A and B. Answer each question using the following instructions.**

**Choose (1), if the question can be answered by one of the statements alone but not by the other.**

**Choose (2), if the question can be answered by using either statement alone.**

**Choose (3), if the question can be answered by using both the statements together, but cannot be answered by using either statement alone.**

**Choose (4), if the question cannot be answered even by using both the statements together.**

20. Is  $a^{44} < b^{11}$ , given that  $a = 2$  and  $b$  is an integer?

- A.  $b$  is even                      B.  $b$  is greater than 16

(Past CAT question)



## QA-1.4 | LINEAR AND QUADRATIC EQUATIONS



### Basic concepts of Algebra

An algebraic expression is obtained by combining numbers and letters together with arithmetic operators.

#### Example

$$x^2 + 3xy + y^2.$$

### Algebraic term

When one or more of +, – signs appear, the expression is separated into parts, each called a term.

#### Example

In  $x^2 + 3xy + y^2$ ; the terms are  $x^2$ ,  $3xy$  and  $y^2$ .

In a term  $7x^2y$ , 7 is called the numerical coefficient,  $x^2y$  is called the **variable factor**.

**Like terms** are those in which the powers of the variables are the same.

#### Example

$2ab^2$  and  $3ab^2$  are like terms.

If an expression contains only one term, then it is called a **monomial**.

#### Example

$3xy$ ,  $4xz$ ,  $8xyz$ , 36.

If an expression contains two terms, then it is called a **binomial**.

#### Example

$l + 3m$ ,  $p + 2r$

If an expression contains three terms, then it is called a **trinomial**.

#### Example

$l + 3m + 2n$ ,  $p + q - 2r$

If an expression contains two or more terms, then it is called a **polynomial**.

**Example**

$$3x + 5y + 4z, 4a - 5b + 3c - 2d$$

An algebraic expression of the form  $a + bx + cx^2 + dx^3 + \dots$  is called a **polynomial in one variable x**.

**Example**

$$4 + 3x + 8x^2 + 9x^3 + 5x^4$$

The **degree of a polynomial** is the greatest of the exponents (indices) of its various terms.

**Example**

$$-6x + 7x^3 + 4x^2 \text{ is a polynomial of degree 3.}$$

## Simple Equations

### Framing a simple equation

An equation is a statement in which two algebraic expressions are equal. If the equation is of the first order, (i.e. highest power of the variable is 1) then it is called a **linear equation** or a **simple equation**.

A **linear equation in one variable** is represented by  $ax + b = 0$ , where  $x$  is the variable (the value of which is not known), and 'a' and 'b' are real numbers ( $a \neq 0$ )

**Example**

$$(i) 2x - 13 = 7; \quad (ii) \frac{3x+7}{5} = 11; \quad (iii) 2x + 3y = 4$$

While solving a simple equation with one variable, simplify the two sides separately by removing the brackets, if any. Transpose all the terms containing the unknown variable to the left-hand side and the remaining terms to the right-hand side. For this add/subtract the same number to/from both sides of the equation or/and multiply/divide both sides of the equation by the same non-zero number as required. A linear equation in one variable always has a unique solution.

**Example**

$$\begin{aligned} (i) \quad & 3x + 5(2 - x) = -16 \\ & 3x + 10 - 5x = -16 \\ & -2x = -26 \quad \therefore x = 13 \\ & \therefore 13 \text{ is the solution of the given equation.} \end{aligned}$$

(ii)  $\frac{x-6}{5} + \frac{x-4}{3} = 8 - \frac{x-2}{7}$

Multiply both sides by the LCM of 5, 3, and 7, i.e. 105.

$$21(x - 6) + 35(x - 4) = 105 \times 8 - 15(x - 2)$$

$$21x + 35x + 15x = 840 + 30 + 126 + 140$$

$$71x = 1136 \quad \therefore x = 16 \quad \therefore 16 \text{ is the solution of the given equation.}$$

**Note:** Arithmetic operations like addition, subtraction, multiplication or division of terms on both sides of the equations of / by the same non zero number does not change the solution of the given equation.

### Concept Builder 1

1.  $2x - 4 = 6x - 22$
2.  $(3x + 4)(2x - 5) - 122 = (6x - 1)(x + 10)$
3.  $\frac{5x-7}{3} = \frac{x+9}{4}$
4.  $(x - 2)(2x - 5) - (2x^2 - 8x + 11) = 0$
5. Find  $x$ , if  $\frac{x-8}{4} + \frac{x+5}{5} = 6 + \frac{x+2}{11}$
6. If  $8a + 3 = 27$ , then the value of  $3a + 6$  is:

### Answer Key

$$\begin{array}{llll} 1. & x = 4.5 & 2. & x = -2 \\ 3. & x = \frac{17}{55} & 4. & x = -1 \\ 5. & 20 & 6. & 15 \end{array}$$

### Translation of Verbal Statements to a Simple Equation

Verbal statements can be expressed as algebraic equations, that help us solve a mathematical problem. In these problems, the unknown quantity is denoted by a variable 'x'. Use the information about the unknown variable (i.e., x), given by the statement in the question and form an equation. Then solve the equation to obtain a solution.

Some of the common type of translations are illustrated below.

1. Consider 3 consecutive natural numbers. If 'x' is taken as the lowest number, then 3 consecutive natural numbers (including x) are 'x', 'x + 1' and 'x + 2'.
2. Consider 3 consecutive odd numbers. If 'x' is taken as the lowest odd number, then 3 consecutive odd numbers (including x) are x, x + 2, and x + 4.

**Note:** Similarly, 3 consecutive even numbers can also be expressed as 'x', 'x + 2' and 'x + 4', If x is taken as the lowest even number.

3. If Deepa's present age is 'x' years, then:  
Deepa's age 3 years hence will be 'x + 3' years.  
Deepa's age four years ago was 'x - 4' years.  
Deepa's mother's present age, which is twice that of Deepa's age, is '2x' years.
4. A 3 digit number 'xyz' where x is the hundreds' digit, y is the tens' digit and z is the unit digit can be expressed as '100x + 10y + z'.  
The number formed by reversing the digits is 100z + 10y + x.
5. A person has a certain amount of money only in 10 paise, 25 paise and 50 paise coins. He has 'x' coins of 10 paise. The number of 25 paise coins is 2 less than the number of 10 paise coins. The number of 50 paise coins is 3 more than the number of 10 paise coins.  
Total Amount with the person (in paise) = 10x + 25(x - 2) + 50(x + 3).

### Concept Builder 2

1. Arun is twice as old as Amal now and the sum of their ages 5 years hence will be 100. What is the present age of Arun
2. The sum of three consecutive natural numbers is equal to 60. Find the product of the lowest and highest number.
3. The sum of three consecutive even numbers is 42. What is the highest number?
4. When a number 'x' is multiplied by 6, the result exceeds the number by 40. What is the value of x?

### Answer Key

1. 60 years      2. 399      3. 16      4. 8

### SOLVED EXAMPLES

**Q :** Solve:  $\left(x - \frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 = x + 2$

**A :**  $\left(x - \frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 = x + 2$

$$\therefore \left(x^2 - x - \frac{1}{4}\right) - \left(x^2 - 3x + \frac{9}{4}\right) = x + 2$$

$$\therefore 2x - 2 = x + 2 \quad \therefore x = 4$$

**Q :** Solve:  $\frac{2x-3}{2.5} = \frac{3x-4}{12.5} + 0.262$ .

**A :**  $\frac{2x-3}{2.5} = \frac{3x-4}{12.5} + 0.262$

$$\frac{2(2x-3)}{5} = \frac{2(3x-4)}{25} + 0.262$$

Multiply both sides by 25,

$$20x - 30 = 6x - 8 + 6.55$$

$$14x = 28.55 \quad \therefore x \approx 2.04$$

**Q :** Solve:  $\frac{2x-1}{x+3} + \frac{1-2x}{x-3} = \frac{4-3x}{x^2-9}$ .

**A :**  $\frac{2x-1}{x+3} + \frac{1-2x}{x-3} = \frac{4-3x}{x^2-9}$

Multiply both sides by LCM of  $x + 3$  and  $x - 3$ , i.e.,  $x^2 - 9 = (x - 3)(x + 3)$ .

$$\frac{(2x-1)}{(x+3)} \times (x-3)(x+3) + \frac{(1-2x)}{(x-3)} \times (x-3)(x+3)$$

$$= \frac{4-3x}{(x+3)(x-3)} \times (x-3)(x+3)$$

$$\therefore (2x-1)(x-3) + (1-2x)(x+3) = 4-3x$$

$$\therefore 2x^2 - 7x + 3 + 3 - 5x - 2x^2 = 4 - 3x$$

$$\therefore -7x - 5x + 3x = 4 - 3 - 3$$

$$\therefore -9x = -2 \quad \therefore x = \frac{2}{9}$$

**Q :** Find a number such that the difference between 8 times the number and 3 times the number is 55.

**A :** Let the number be  $x$ .

$$\therefore 8x - 3x = 55 \quad \therefore x = 11 \quad \therefore \text{The number is 11.}$$

**Q :** A father is now 3 times as old as his son. 5 years ago, he was 4 times as old as his son. Find their present ages.

**A :** If ' $x$ ' is the son's present age, then  $3x$  is the father's present age.

$$5 \text{ years ago, son's age} = x - 5.$$

$$5 \text{ years ago, father's age} = 3x - 5.$$

$$\therefore 3x - 5 = 4(x - 5) \quad \therefore 3x - 5 = 4x - 20 \quad \therefore x = 15 \text{ and } 3x = 45$$

Hence, son's present age is 15 years and father's present age is 45 years.

**Q :** Anand has Re.1 and Bindu has 10 paise. How much must Anand give to Bindu in order that he may have just 4 times as much as Bindu?

**A :** Let ' $x$ ' be the number of paise that Anand gives to Bindu.

Then Anand will have  $(100 - x)$  paise, and Bindu will have  $(10 + x)$  paise. But, according to the question, Anand now has 4 times as much as Bindu.

$$\therefore 100 - x = 4(10 + x) \quad \therefore x = 12$$

Thus, Anand must give 12 paise to Bindu.

**Q :** Shiva has a certain number of 10 paise coins, 3 times as many 25 paise coins as 10 paise coins, and 5 more 50 paise coins than 25 paise coins. If the total value is Rs.120, then how many 10 paise coins are there?

**A :** Let the number of 10 paise coins be ' $x$ '.

$$\therefore \text{Number of 25 paise coins} = 3x$$

$$\& \text{ Number of 50 paise coins} = 3x + 5$$

$$\therefore 10x + 25(3x) + 50(3x + 5) = 12000$$

$$10x + 75x + 150x + 250 = 12000$$

$$\therefore 235x = 11750 \quad \therefore x = 50$$

Thus, there are 50 ten paise coins.

## Simultaneous Equations

### Linear equation in 2 variables

If  $a, b, c$  are real numbers where  $a \neq 0, b \neq 0$ ; then  $ax + by = c$  is a **linear equation in two variables** of degree 1. The values of ' $x$ ' and ' $y$ ', for which both sides of the equation assume the same value are called the **solution** of the equation.

#### Example

$$2x + 4y = 24.$$

The value  $x = 6$  and  $y = 3$  satisfies the above equation.

Therefore, the pair  $(6, 3)$  is a solution of the equation.

$(4, 4)$  and  $(8, 2)$  are also solutions of the above equation

Therefore, for a linear equation with two variables, there exists an infinite number of solutions; it does not have a unique solution.

**Note:** In a solution set  $(x, y)$  the value of ' $x$ ' is written first, followed by the value of ' $y$ '.

### Simultaneous equations

A set of 2 (or more) linear equations in two (or more) variables, which are satisfied simultaneously (i.e., at the same time) by the same unique solution, are called **simultaneous equations**.

For example,  $x + y = 5$  and  $x - y = 3$  are only satisfied by the pair,  $x = 4$  and  $y = 1$ , only.

### Methods of Solving Simultaneous Equations

#### 1. By Elimination Method

To solve simultaneous equations, it is necessary to eliminate one of the two variables and obtain an equation, which has only one variable. For this, the coefficients of the variable to be eliminated are first made equal in both the equations and then the two equations are either added or subtracted.

Types of questions based on simultaneous questions that can be solved by the Elimination method are discussed below.

**Case 1:** The coefficient of one of the variables is the same in both the equations

At times the equations will have a variable with the same coefficient. In such cases, the two equations are either added or subtracted to eliminate one variable.

#### Example

Solve  $x + y = 8$  and  $x - y = 4$ .

$$x + y = 8 \quad \dots (i)$$

$$x - y = 4 \quad \dots (ii)$$

The variable  $y$  has different signs, but the same coefficient 1 in the first and second equation.

$\therefore$  If we add the two equations, the variable ' $y$ ' will be eliminated.

Adding equations (i) and (ii),  $2x = 12$ .

$$\therefore x = 6$$

Now, substitute this value of ' $x$ ' in any one of the equations (i) or (ii).

Substituting the value of ' $x$ ' in equation (i),  $6 + y = 8 \therefore y = 2$

$\therefore$  The solution set is  $(6, 2)$ .

**Case 2: The coefficient of both the variables is different in the 2 equations.**

In such cases we either multiply or divide these equations with real numbers other than 0 to obtain new equations equivalent to the original ones, so that one of the variables can be easily eliminated.

**Example**

Solve  $4x - 3y = 5$  and  $3x - 2y = 4$ .

Here, the coefficients of the variable  $x$  are 4 and 3 and that of  $y$  is  $-3$  and  $-2$ . Hence, neither of the variables can be eliminated by simple addition or subtraction.

$$4x - 3y = 5 \quad \dots (i)$$

$$3x - 2y = 4 \quad \dots (ii)$$

If we multiply the first equation by 3 and the second equation by 4, then the coefficient of the variable  $x$  will be 12 in both the equations and can easily be eliminated on subtraction.

*Alternatively,*

If we multiply the first equation by 2 and the second equation by 3, then the coefficient of the variable  $y$  in both equations will be  $-6$ , which can easily be eliminated by subtracting the equations.

Multiplying equation (i) by 3, and (ii) by 4.

$$12x - 9y = 15 \quad \dots (iii)$$

$$12x - 8y = 16 \quad \dots (iv)$$

Subtracting equation (iv) from (iii),

$$12x - 9y = 15$$

$$12x - 8y = 16$$

$$\begin{array}{r} - \quad + \quad - \\ \hline \end{array}$$

$$-y = -1 \quad \therefore y = 1$$

Substituting this value of  $y$  in equation (i),  $4x - 3 \times 1 = 5$

$$4x = 8 \quad \therefore x = 2$$

Hence, the solution set is  $(2, 1)$ .

**2. By Substitution Method**

Simultaneous equations can also be solved by expressing one of the variables in an equation in terms of the other and then substituting this value of the variable in the second equation.

**Example**

Solve  $5x + 2y = 24$  and  $3x - y = 10$ .

$$5x + 2y = 24 \quad \dots (i)$$

$$3x - y = 10 \quad \dots (ii)$$

From (ii),  $y = 3x - 10$

Substituting this value of ' $y$ ' in equation (i),

$$5x + 2(3x - 10) = 24$$

$$5x - 20 + 6x = 24$$

$$11x = 44 \quad \therefore x = 4$$

Substituting the value of ' $x$ ' in equation (i),

$$5 \times 4 + 2y = 24$$

$$2y = 4 \quad \therefore y = 2 \quad \therefore \text{The solution set is } (4, 2).$$



### 3. By Graphical Method

Before discussing this method, let us look at the graphical representation of a linear equation in 2 variables. A Linear equation in 2 variables will always be a straight line.

Consider the equation  $2x + 4y = 24$ .

The value  $x = 6$  and  $y = 3$  satisfies the above equation.

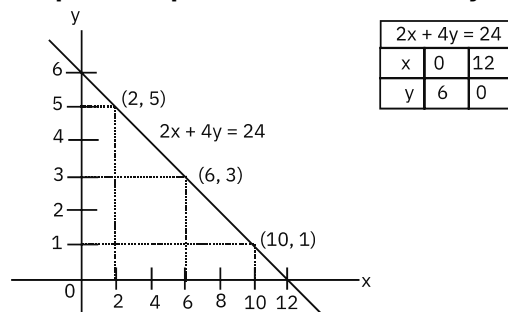
Therefore, the pair  $(6, 3)$  is a solution of the equation.

In the same equation if we take  $x = 4$ , then we get  $2 \times 4 + 4 \times y = 24$ .

$\therefore 4y = 16 \quad \therefore y = 4$

So,  $(4, 4)$  is also a solution of the equation. Thus, infinite number of solutions can be found for this equation.

#### Graphical representation of $2x + 4y = 24$



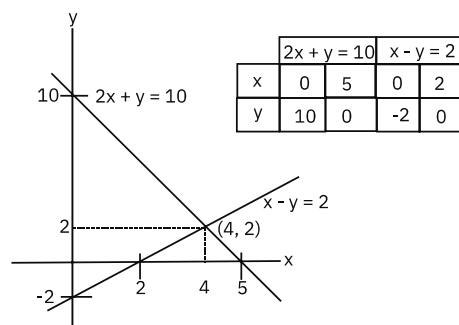
The points  $(10, 1)$ ,  $(6, 3)$ ,  $(2, 5)$  all are different solutions of the equation  $2x + 4y = 24$

In the graphical method, we first plot the lines represented by both the equations.

The co-ordinates of the point of intersection of the two lines, represented by linear equations, will give the solution of the simultaneous equations.

#### Example

Solve  $2x + y = 10$  and  $x - y = 2$  using the graphical method



Hence, the solution is  $(4, 2)$

### Special cases in Simultaneous Equations

Suppose  $ax + by = c$  and  $dx + ey = f$  are two equations involving variable  $x$  and  $y$  then

**Case (i)** If  $\frac{a}{d} \neq \frac{b}{e}$  then the two equations will give a unique solution for  $x$  and  $y$ .

#### Example

Solve  $a + 3b = 9$  and  $3a + 2b = 13$

$$a + 3b = 9 \quad \dots (i)$$

$$3a + 2b = 13 \quad \dots (ii)$$

(i)  $\times 3$  – (ii) gives

$$\begin{array}{r} 3a + 9b = 27 \\ - \quad 3a + 2b = 13 \\ \hline \end{array}$$

$$7b = 14$$

$$\therefore b = 2, a + 3 \times 2 = 9$$

$$\therefore a = 9 - 6 = 3$$

Hence the solution set is  $(3, 2)$

**Case (ii)** If  $\frac{a}{d} = \frac{b}{e} \neq \frac{c}{f}$  then we get no solution for the given equation.

#### Example

The cost of 3 pens and 5 pencils is Rs. 37 and the cost of 9 pens and 15 pencils is Rs. 91. Find the cost of 5 pens.

Let the cost of each pen be Rs. 'x' and the cost of each pencil be Rs. 'y'.

$$3x + 5y = 37 \quad \dots (i)$$

$$9x + 15y = 91 \quad \dots (ii)$$

$$\frac{3}{9} = \frac{5}{15} \neq \frac{37}{91}$$

The two equations can't be solved to get a solution.  $\therefore$  The solution does not exist.

**Note:** If we plot two lines for the two equations we will get two parallel lines which will never intersect. Hence, no solution is possible.

**Case (iii)**  $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$ , then we get infinite solutions for the given equations.

#### Example

The cost of 7 horses and 4 camels is Rs. 6300 and the cost of 28 horses and 16 camels is Rs. 25200. Find the cost of 2 horses.

Let the cost of each horse be Rs. 'h' and the cost of each camel be Rs. 'c'.

$$7h + 4c = 6300 \quad \dots (i)$$

$$28h + 16c = 25,200 \quad \dots (ii)$$

$$\frac{7}{28} = \frac{4}{16} = \frac{6300}{25200} = \frac{1}{4}$$

The two equations will have infinite solutions.

**Note:** If we plot two lines for the equations, the two line will coincide i.e., they will intersect each other at infinite points. Hence, infinite solutions are possible

We can summarize the above results as follows.

Suppose  $ax + by = c$  and  $dx + ey = f$  are two equations involving two variables  $x$  and  $y$ . Then

- (i) If  $\frac{a}{d} \neq \frac{b}{e}$  then the two equations will give a unique solution for  $x$  and  $y$
- (ii) If  $\frac{a}{d} = \frac{b}{e} \neq \frac{c}{f}$  then we get no solution for the given equations
- (iii) If  $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$  then we get infinite solutions for the given equations

### Important algebraic formulae

1.  $(a + b)(a - b) = a^2 - b^2$
2.  $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
3.  $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
4.  $(a + b)^3 = a^3 + 3ab(a + b) + b^3$
5.  $(a - b)^3 = a^3 - 3ab(a - b) - b^3$
6.  $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 - ab + b^2)$
7.  $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + ab + b^2)$
8.  $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$   
 If  $a + b + c = 0$  then  $a^3 + b^3 + c^3 = 3abc$

## Concept Builder 2

**Solve:**

1.  $4x + y = 9, 3x = 6$
2.  $3x + 8y = 43$  and  $3x - 7y = 13$ .
3.  $2x + 5y = 22, x + 2y = 9$
4.  $5x + 7t = 8, 7x - 4t = 25$
5.  $2x + y = 18; 5x - 2y = 27$ .

**Directions for questions 6 to 8: How many solutions will each of the following simultaneous equations have.**

6.  $7x - 5y = 13, 21x - 15y = 39$
7.  $-3x + 4y = 41, 45x - 60y = 32$
8.  $x + 5 = 3y, 4x + y = 2$
9. The sum of the ages of Haseeb and Ronak is 24 and the difference between their ages is 6. Find their ages given that Ronak is older than Haseeb
10. In a shop, 6 apples and 5 mangoes cost Rs.60 while 8 apples and 9 mangoes cost Rs. 80. What is the cost of a mango and an apple?
11. The sum of two numbers is 10 and the difference between them is 2. Find the two numbers.
12. Divya's mother's present age is 4 times her own age. Five years ago, sum of their ages was 30 years. Find their present ages.
13. Find the value of  $\frac{x}{y}$ , where  $\frac{3x}{2} - \frac{2y}{3} = 5$  and  $\frac{5x}{6} + \frac{7y}{12} = \frac{17}{2}$ .
14. If  $x + y = 10$  and  $\frac{1}{x} + \frac{1}{y} = \frac{5}{12}$ , then the value of  $x - y$  is:

**Answer Key**

- |     |   |    |                     |
|-----|---|----|---------------------|
| 9.  | Haseeb = 9 years, Ronak = 15 years                    | 1. | $x = 2, y = 1$      |
| 10. | 10  | 2. | $x = 9, y = 2$      |
| 11. | 6 and 4   | 3. | $x = 1$ and $y = 4$ |
| 12. | Divya's age = 8 years and her mother's age = 32 years | 4. | $x = 3, t = -1$     |
| 13. | 1   | 5. | $x = 7, y = 4$      |
| 14. | 2 or -2   | 6. | Infinite            |
|     |   | 7. | No solution         |
|     |   | 8. | 1                   |

### SOLVED EXAMPLES

**Q :** Solve  $41m - 47n = 35$ ;  $47m - 41n = 53$ .

**A :**  $41m - 47n = 35$  ..... (i)  
 $47m - 41n = 53$  ..... (ii)  
 Adding (i) and (ii),  
 $88m - 88n = 88$  ..... (iii)  
 Dividing (iii) by 88,  
 $m - n = 1$  ..... (iv)  
 Subtracting (ii) from (i),  $-6m - 6n = -18$  ..... (v)  
 Dividing (v) by 6,  $-m - n = -3$  ..... (vi)  
 Adding (iv) and (vi),  
 $-2n = -2 \quad \therefore n = 1$   
 Substituting the value of 'n' in equation (iv),  
 $m - 1 = 1 \quad \therefore m = 2$   
 Hence, the solution set is (2, 1).

**Q :** Solve  $\frac{x}{3} + \frac{y}{4} = -\frac{1}{2}$ ; and  $\frac{x}{3} - \frac{y}{4} = -\frac{3}{2}$ .

**A :**  $\frac{x}{3} + \frac{y}{4} = -\frac{1}{2}$  ..... (i)                       $\frac{x}{3} - \frac{y}{4} = -\frac{3}{2}$  ..... (ii)  
 Adding (i) and (ii),  
 $\frac{2x}{3} = -\frac{4}{2}; \quad \therefore x = -3$   
 Substituting the value of 'x' in equation (i),  
 $-1 + \frac{y}{4} = -\frac{1}{2} \quad \therefore \frac{y}{4} = \frac{1}{2}; \quad \therefore y = 2$   
 Hence, the solution set is (-3, 2).

**Q :** Solve  $4(x - y) = -20$ ;  $\frac{5x}{8} + \frac{7y}{18} = 6$ .

**A :**  $4(x - y) = -20$   
 $\therefore x - y = -5$  ..... (i)  
 $\therefore \frac{5x}{8} + \frac{7y}{18} = 6$  ..... (ii)  
 LCM of 8 and 18 is 72, multiplying equation (ii) by 72,  
 $45x + 28y = 432$  ..... (iii)  
 Multiplying equation (i) by 28,  
 $28x - 28y = -140$  ..... (iv)  
 Adding equations (iii) and (iv),  
 $73x = 292 \quad \therefore x = 4$

Substituting the value of  $x$  in equation (i),  
 $4 - y = -5 \therefore y = 9$   
 Hence, the solution set is  $(4, 9)$ .

**Q :** A two-digit number is equal to thrice the sum of its digits. The difference between the number obtained by interchanging the number and the original number is 45. Find the number.

**A :** Let the digit in the unit's place be ' $x$ ' and the digit in the ten's place be ' $y$ '.

The original number =  $10y + x$ .

Number obtained by interchanging the digits =  $10x + y$ .

Now,  $10y + x = 3(x + y)$ .

$2x - 7y = 0$  .... (i)

Also,  $(10x + y) - (10y + x) = 45$ .

$\therefore 10x + y - 10y - x = 45 \therefore 9x - 9y = 45$

$\therefore x - y = 5$  .... (ii)

Multiplying (ii) by 2,

$2x - 2y = 10$  .... (iii)

Subtracting (iii) from (i),

$-5y = -10 \therefore y = 2$

Substituting the value of  $y$  in equation (ii),

$x - 2 = 5 \therefore x = 7 \therefore$  The number is  $10 \times 2 + 7 = 27$ .

**Q :** When 1 is added to both the numerator and the denominator of a certain fraction, it becomes  $\frac{1}{2}$  and when 1 is added to the numerator and 1 is subtracted from the denominator, the fraction becomes equal to 1. Find the fraction.

**A :** Let the fraction be  $\frac{x}{y}$ .

$\therefore \frac{x+1}{y+1} = \frac{1}{2} \therefore 2(x+1) = y+1$

$\therefore 2x - y = -1$  .... (i)

$\therefore \frac{x+1}{y-1} = 1 \therefore x+1 = y-1$

$\therefore x - y = -2$  .... (ii)

Subtracting (ii) from (i),  $x = 1$  & Substituting the value of ' $x$ ' in equation (i),

$2 - y = -1 \therefore y = 3 \Rightarrow$  Hence, the fraction is  $\frac{1}{3}$ .

**Q :** There is a certain number consisting of 3 digits which is equal to 13 times the sum of the digits and if 495 is added to the number, the digits will be reversed. Also the sum of the extreme digits is greater than the middle digit by 2. Find the number.

**A :** Let the hundred's place digit and unit's place digit be ' $x$ ' and ' $y$ ' respectively.

$\therefore$  The ten's place digit will be  $x + y - 2$ .

The original number =  $100x + 10(x + y - 2) + y$ .

Now,  $100x + 10(x + y - 2) + y = 13(x + x + y - 2 + y)$ .

$$\therefore 100x + 10x + 10y - 20 + y = 26x + 26y - 26$$

$$\therefore 84x - 15y = -6 \quad \dots (i)$$

The number obtained by interchanging the digits =  $100y + 10(x + y - 2) + x$ .

$$\therefore 100y + 10(x + y - 2) + x = 495 + 100x + 10(x + y - 2) + y$$

$$\therefore 100y + x = 495 + 100x + y$$

$$99y - 99x = 495$$

$$-x + y = 5 \quad \dots (ii)$$

$$\text{Multiplying (ii) by 15, we get, } -15x + 15y = 75 \quad \dots (iii)$$

Adding (i) and (iii),  $69x = 69$ .

$$\therefore x = 1$$

Substituting the value of 'x' in equation (ii), we get

$$-1 + y = 5 \quad \therefore y = 6$$

The ten's place digit is  $6 + 1 - 2 = 5$ .  $\therefore$  The number is 156.

**Q :** A few tickets of a show are sold at Rs.10 per ticket and the other tickets at Rs.8 per ticket. In all 105 tickets were sold. If the amount collected on a day was Rs.922, find the number of tickets sold at Rs.10.

**A :** Let 'x' tickets be sold at Rs.10 and 'y' tickets be sold at Rs.8.

$$x + y = 105 \quad \dots (i)$$

$$10x + 8y = 922 \quad \dots (ii)$$

Multiplying equation (i) by 10,

$$10x + 10y = 1050 \quad \dots (iii)$$

Subtracting equation (ii) from (iii),

$$2y = 128 \quad \therefore y = 64$$

Substituting the value of 'y' in equation (i),

$$64 + x = 105 \quad \therefore x = 41$$

Hence, 41 tickets were sold at Rs.10.

**Q :** Find the value of  $\frac{8.73 \times 8.73 \times 8.73 + 4.27 \times 4.27 \times 4.27}{8.73 \times 8.73 - 8.73 \times 4.27 + 4.27 \times 4.27}$ .

$$\begin{aligned} \mathbf{A :} & \quad \frac{8.73^3 + 4.27^3}{8.73^2 - 8.73 \times 4.27 + 4.27^2} \\ &= \frac{(8.73 + 4.27)(8.73^2 - 8.73 \times 4.27 + 4.27^2)}{8.73^2 - 8.73 \times 4.27 + 4.27^2} \quad [\text{As } a^3 + b^3 = (a + b)(a^2 - ab + b^2)] \\ &= 8.73 + 4.27 = 13 \end{aligned}$$

## Quadratic Polynomial

A polynomial of degree 2 is called a **quadratic polynomial**. The general form of a quadratic polynomial is  $ax^2 + bx + c$ ; where  $a$ ,  $b$  and  $c$  are real numbers,  $a \neq 0$  and  $x$  is a variable.

### Example

$$x^2 - 6x + 4; 3x^2 + \sqrt{2}x$$

## Quadratic Equation

An equation  $p(x) = 0$ , where  $p(x)$  is a quadratic polynomial, is called a **quadratic equation**. The general form of a quadratic equation is  $ax^2 + bx + c = 0$  where  $a$ ,  $b$  and  $c$  are real numbers,  $a \neq 0$  and  $x$  is a variable.

### Example

$$x^2 - 2x + 2 = 0$$

## Roots of a quadratic equation

The roots of a quadratic equation are the values of  $x$ , which satisfy the given equation. A quadratic equation cannot have more than two distinct roots. Discussed below are methods of finding the roots of a quadratic equation.

### 1. By method of factorisation by splitting the middle term.

Consider the equation  $ax^2 + bx + c = 0$

Factorise the term ' $ac$ ' such that the sum of the two factors is equal to  $b$ .

### Example

$$x^2 - 5x + 6 = 0$$

$$a = 1, b = -5 \text{ and } c = 6; a \times c = 1 \times 6 = 6$$

The factors of 6 whose sum is equal to  $-5$  is  $-3$  and  $-2$ .

$-5x$  can be replaced with  $-3x$  and  $-2x$ .

$$x^2 - 3x - 2x + 6 = 0$$

$$x(x - 3) - 2(x - 3) = 0$$

$$(x - 2)(x - 3) = 0$$

In general, if the product of  $p$  and  $q$  is zero, i.e.,  $pq = 0$ , then either  $p = 0$  or  $q = 0$ .

$$\therefore x - 2 = 0 \text{ or } x - 3 = 0$$

$$\therefore x = 2 \text{ or } x = 3$$

$\therefore$  The roots of the quadratic equation  $x^2 - 5x + 6 = 0$  are 3 and 2.

### 2. By method of completing the square.

Consider the equation  $ax^2 + bx + c = 0$

At times, it is not easy to factorize the product ' $ac$ ' such that the sum of the two factors is  $b$ .

The roots can also be found by the method of completing the square.



**Example**

Consider  $3x^2 + 7x + 1 = 0$

In this method, the equation is first divided by 'a' and the term  $\frac{c}{a}$  is taken on the right hand side.

$$x^2 + \frac{7}{3}x = -\frac{1}{3} \dots \text{(dividing by 3 and taking the constant term on the right hand side)}$$

To make the left hand side i.e.,  $x^2 + \frac{b}{a}x$  a perfect square, add  $\left(\frac{1}{2} \times \text{coefficient of } x\right)^2$  to both left hand side and right hand side.

Add  $\left(\frac{1}{2} \times \text{coefficient of } x\right)^2 = \left(\frac{1}{2} \times \frac{7}{3}\right)^2 = \left(\frac{7}{6}\right)^2$  on both the sides.

$$\therefore x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{1}{3} + \left(\frac{7}{6}\right)^2$$

$$\therefore \left(x + \frac{7}{6}\right)^2 = \frac{37}{36}$$

$$\therefore x + \frac{7}{6} = \pm \sqrt{\frac{37}{36}} = \pm \frac{\sqrt{37}}{6} \quad \text{i.e., } x = \frac{-7 + \sqrt{37}}{6} \text{ or } x = \frac{-7 - \sqrt{37}}{6}$$

$$\therefore \text{The two roots of } 3x^2 + 7x + 1 = 0 \text{ are } \frac{-7 + \sqrt{37}}{6} \text{ and } \frac{-7 - \sqrt{37}}{6}.$$

**3. By using formula.**

By completing the square on x for the general form of the quadratic equation  $ax^2 + bx + c = 0$ , a general formula for the roots is established.

The two roots of the quadratic equation are given by  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

### Example

$$x^2 - 8x + 10 = 0$$

$$a = 1, b = -8, c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times 10}}{2 \times 1} = \frac{8 \pm \sqrt{64 - 40}}{2} = \frac{8 \pm \sqrt{24}}{2}$$

$$= \frac{8 \pm 2\sqrt{6}}{2} = \frac{2(4 \pm \sqrt{6})}{2} = 4 \pm \sqrt{6}; \quad \therefore x = 4 + \sqrt{6} \text{ or } x = 4 - \sqrt{6}$$

$\therefore$  The two roots of  $x^2 - 8x + 10$  are  $4 + \sqrt{6}$  and  $(4 - \sqrt{6})$ .

### Note:

The factorization method can be used only if the last term is easily factorable, whereas the method of completing the square and the formula for the roots can be used to solve any quadratic equation

### Concept Builder 3

**Directions for questions 1 to 6: Find the roots of the following quadratic equations.**

1.  $x^2 + 2x - 63 = 0$
2.  $12x^2 + x - 6 = 0$
3.  $2x^2 + 3x - 12 = 0$
4.  $2x^2 + x - 528 = 0$
5.  $(x + 5)(x - 5) = 39$
6.  $2x^2 - 5x + 3 = 0$

### Answer Key

1.  $-9$  and  $7$
2.  $-\frac{3}{2}$  and  $\frac{4}{3}$
3.  $x = \frac{-3 + \sqrt{105}}{2}$  or  $\frac{-3 - \sqrt{105}}{2}$
4.  $x = 16$  and  $-\frac{33}{2}$
5.  $x = 7$  and  $8$
6.  $x = 1$  and  $\frac{2}{3}$



## CLASS EXERCISE

### Teaser

Sati and Savitri visited three temples to offer flowers. At the first temple, Sati offered two-third of her flowers, plus one extra flower. At the second temple also she offered two-third of her flowers, and one extra flower. After repeating the process at the third temple, she was left with one flower. Savitri started with the same number of flowers as Sati, but offered 20 flowers at each temple. How many flowers did Savitri have left after visiting the third temple?

## Algebraic Formulae

### *Some useful formulae*

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

### Algebra: Basics

- Classify the following on the basis of order, number of variables, and number of terms:
  - $3x^3 + 5y^2 + 6xy + 2x - 7y + 4$
  - $(8 - x)^3$
  - $x^4 + y^5 + 3x^2y + 7y - 4x^3y^3 + 2xyz - 3yx^2 - 4$
- Simplify
  - $3x^3 + 4xy^2 + y^3 - 2y^2x + 3x(x^2 + y^2)$
  - $(5x + 4y)(4y - 5x)$
  - $(3p - 4q) + (5p + 2q) - (8p - 2q)$
- Solve:
  - If 12 is added to three times a number, and the result is divided by 5, the original number is obtained. Find the number.
  - If a number is multiplied by 10, the resultant number is the square of the original number. Find the number.
- Find the value of:
  - $(1.25)^2 + (0.75)(2.5) + (0.75)^2$
  - $999^2$
  - $(74.75)^2 - (25.25)^2$
- Simplify:  $\frac{1 + p + \frac{p^2}{4}}{1 - \frac{p^2}{4}}$
- Which of these is/are factors of  $81e^4 - 1$ ?
  - $27e^3 - 1$
  - $9e^2 + 1$
  - $3e^2 + 1$
  - $9 - e^2$
  - $e + \frac{1}{3}$
- What should be added to  $4x^2 - 16x$  to make it a perfect square?
- Find the value of  $(1003 \times 997) - (1005 \times 995)$ .
- Find the value of  $\sqrt{(2+1)(2^2+1)(2^4+1)(2^8+1)+1}$

### Simple Linear Equations

10. Rajni is seven times as old as his son. After three years, he will be five times his son's age. What is Rajni's present age?
11. Amitabh is 30 years older than Abhishek. 4 years back, Amitabh was thrice the age of Abhishek. How old is Amitabh?
12. Madhubala was asked her age. She replied "in three years, my age will be one and a half times what it was six years ago. If she is speaking the truth, how old is Madhubala?"
13. Raj is multiplying two numbers. By mistake, he read one of the numbers as 52 instead of 25 and as a result his answer increased by 486, What was the other number?
14. Amir went to the market to buy a dozen apples, but found himself 4 rupees short. Instead, he bought ten apples and returned with 10 rupees left over. How much is the cost of an apple?
15. Madhuri takes a 2-digit number. She then reverses it and subtracts the result from the original number, getting 54 as her answer. How many such numbers are possible?
16. Sridevi takes a 2-digit number and subtracts it from the number obtained by reversing the original number. She also gets the answer 54. How many such numbers are possible?
17. Prabhudeva took a number, and reversed it. He then divided it by 24 and cubed the result. To his surprise, he got back his original number. What number did he start with?
18. There is a bowl of grapes lying on the table. Rishi walks in and eats half the grapes plus 3 extra grapes. Then Shashi walks in and eats half the remaining grapes plus 3 extra grapes. Finally, Shammi walks in and eats half the remaining grapes plus 3 extra grapes. If the bowl is now empty, how many grapes were there to begin with?
19. \* Nargis takes a 3-digit number  $N$  and reverses it. She then subtracts the original number from the new number obtained, getting an answer of 594. How many possible values of  $N$  could exist?
20. \* Find a three digit number which, when reversed, becomes equal to 17 times the square of its cube root.
21. \* Anil's age is 11 times his grandson's age. After 3 years, he will be 8 times his grandson's age. Find Anil's age.
22. \* Ravindra has three times as much money as Aravinda does. If Ravindra gives a quarter of his money to Aravinda, then Aravinda will have Rs 80 less than Ravindra. How much money does Aravinda start with?
23. \* A vendor is selling eggs in the market. He sells his first customer half his eggs and half an egg, the second customer half the remaining eggs and half an egg, the third customer half the remaining eggs and half an egg, and the fourth and last customer half the remaining eggs and half an egg. No eggs were broken. If he is left with no eggs, how many eggs did the vendor start with?

24. \* When a two digit number is reversed and added to the original number, the result is divisible by 13. How many such numbers are possible?
25. \* Instead of multiplying a number by 1.23, Ranjan multiplied it by 1.32. As a result, the value reduced by 4.95. What was the original number?

### Simultaneous Equations

26. Find unique solutions, if they exist, to the following pairs of equations:
- |                    |                     |                    |
|--------------------|---------------------|--------------------|
| a) $3x + 4y = 11$  | b) $7p - 2q = 15$   | c) $4m + 6n = 24$  |
| $5x + 2y = 9$      | $2p + 4q = 18$      | $6m + 9n = 36$     |
| d) $6c - 12d = 36$ | e) $35p + 21r = 91$ | f) $17z - 4x = 48$ |
| $8d - 4c = 4$      | $34p + 102r = 170$  | $17x = 85$         |
27. 12 cashews and 13 pears cost Rs 43.50, while 8 cashews and 17 pears cost Rs 41.50. How much will 6 cashews and 9 pears cost?
28. 1 dozen pencils and 8 erasers cost Rs 28, while 4 pencils and 16 erasers cost Rs 27. How much will 2 erasers and 3 pencils cost?
29. As part of his summer internship in sales, Dev is observing the purchasing habits of customers at a cutlery shop. The first customer buys 3 forks, 7 knives and 4 spoons, paying Rs 235. The second buys 3 knives, 2 spoons and 11 forks, paying Rs 250. The third customer picks up 7 forks, 2 spoons and 4 knives for Rs 210. If Dev wants to buy 2 forks at the same shop, how much will it cost him?

Suppose  $ax + by = c$  and  $dx + ey = f$  are two equations involving two variables  $x$  and  $y$ . Then

- If  $\frac{a}{d} \neq \frac{b}{e}$  then the two equations will give a unique solution for  $x$  and  $y$
- If  $\frac{a}{d} = \frac{b}{e} \neq \frac{c}{f}$  then we get no solution for the given equations
- If  $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$  then we get infinite solutions for the given equations

30. \* 12 apples and 17 bananas cost Rs 87.25, while 16 apples and 25 bananas cost Rs 119.25. How much will 2 bananas and an apple cost?
31. \* With Rs 500, it is possible to buy 3 pizzas and 1 garlic bread, or 2 pizzas and 4 garlic breads. How many times the cost of a garlic bread is the cost of a pizza?
32. \* If 16 skirts and 12 blouses cost Rs 6800, while 15 blouses and 20 skirts cost Rs 8500, what will be the cost of 9 blouses and a dozen skirts?

### Quadratic Polynomials

33. Factorize the following quadratic polynomials and find their roots:

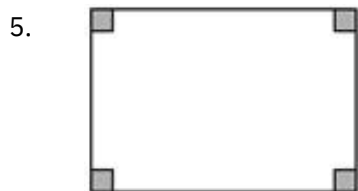
- |                     |                     |                      |
|---------------------|---------------------|----------------------|
| a) $x^2 - 5x - 6$   | b) $6x^2 - x - 2$   | c) $x^2 - 3$         |
| d) $x^2 + 4$        | e) $x^2 + 4x - 6$   | f) $x^2 + 4x + 4$    |
| g) $* x^2 + 6x + 5$ | h) $* x^2 - 6x + 5$ | i) $* 2x^2 - 3x - 9$ |
| j) $* 4x^2 - 1$     |                     |                      |

### Challengers

**Answer Questions 1 – 3 based on the following information:**

A telephone service provider charges a price of P for a new connection, and allows 30 free calls within that. On every extra call beyond the 30th one, the charge is Rs 5. Raja has 2 connections, on one of which he makes Q calls, and on the other, R calls, (where both Q and R are greater than or equal to 30) resulting in 2 bills of Rs 500 and Rs 400 respectively. Had he made all the calls on a single connection instead, his bill would have come to just Rs 800.

- What is the value of P?  
 1) Rs 150                      2) Rs 200                      3) Rs 250                      4) Rs 300
- What is the value of Q?  
 1) 30                              2) 50                              3) 60                              4) 80
- What is the value of R?  
 1) 30                              2) 50                              3) 60                              4) 80
- Chunky is offered a contract with a startup for two years. The agreed payment is a payment of Rs. 10 lakhs and company stock worth Rs X. After 16 months, he leaves the project; the company pays him a pro-rated remuneration Rs 5.2 lakhs and company stock worth Rs X. How much is the value of X?  
 1) Rs 4.4 lakh                  2) Rs 4.8 lakh                  3) Rs 5 lakh                      4) Rs 5.2 lakh



A rectangle of sides 20 cm and 25 cm is shown alongside. Squares were cut out at each of its corners to make a box of volume 462 cu cm. Find the area of the portion that was cut out.



## PRACTICE EXERCISE

**DIRECTIONS for questions 1 to 20: Choose the correct alternative.**

- From each of the two given numbers, half the smaller number is subtracted. Of the resulting numbers the larger one is three times as large as the smaller. What is the ratio of the two numbers?  
 1) 2 : 1                      2) 3 : 1                      3) 3 : 2                      4) None
- Let  $x$ ,  $y$  and  $z$  be distinct positive integers satisfying  $x < y < z$  and  $x + y + z = k$ . What is the smallest value of  $k$  that does not determine  $x$ ,  $y$ ,  $z$  uniquely?  
 1) 9                              2) 6                              3) 7                              4) 8
- Three distinct numbers are given. The difference between the largest and the smallest is added to the third number. One third of resultant number is less than the average of the original three numbers by 14. Find the smallest number (among the original three numbers).  
 1) 42                              2) 12                              3) 21                              4) Cannot be determined
- In a class of 100 students,  $\frac{1}{3}$ rd of the boys failed in Maths while  $\frac{2}{5}$ th of the girls failed in English. But nobody failed in both subjects. If the total number of boys who passed in Maths and girls who passed in English is 64, how many boys are there in the class?  
 1) 40                              2) 50                              3) 60                              4) Cannot be determined
- Ashish, an intelligent boy got a lot of chocolates on his birthday which he did not want to share with his sister. When his sister came and asked him, "How many chocolates do you have?", after pausing for a moment he replied, "If I divide the total number of chocolates into two unequal numbers, then 27 times the difference between the two numbers equals the difference between the square of the two numbers." The sister looked puzzled, but finally found the number. What is the total number of chocolates Ashish got?  
 1) 15                              2) 35                              3) 27                              4) 30
- The number of votes not cast for the Praja Party in the National General Election increased by 25% over those not cast for it in the previous Assembly Polls, and the Praja Party lost by a majority twice as large as that by which it had won the Assembly Polls. If a total 2,60,000 people voted each time. How many voted for the Praja Party in the Assembly Elections.  
 1) 1,10,000                      2) 1,50,000                      3) 1,40,000                      4) 1,20,000
- In an election for the President, 261 valid votes are cast, for the 5 contestants. Find the least number of votes a candidate requires to receive to win the election.  
 1) 53                              2) 54                              3) 257                              4) 72



8. If  $x$  is defined as  $x = \frac{(108.2^2 + 104^2 - 208 \times 108.2)(108.2^2 + 104^2 + 104 \times 108.2)}{(108.2^3 - 104^3)}$ , what is the value of  $\left[\frac{x+0.8}{x-3.2}\right]^2$ ?  
 1) 24.98                      2) 25                      3) 25.01                      4) 25.06
9. A tiled floor of a room has dimensions  $m \times m$  sq.m. The dimensions of the tile used are  $n \times n$  sq.m. All tiles used are green tiles except diagonal tiles which are red. After some years some green tiles are replaced by red tiles to form an alternate red and green tile pattern. How many green tiles are removed? ( $m$ ,  $n$  and the total number of tiles are odd).  
 1)  $\frac{m^2 - 4mn + 2n^2}{2n^2}$       2)  $\frac{(m - 2n)^2 - n^2}{2n^2}$       3)  $\frac{m^2 - 4mn - n^2}{2n^2}$       4)  $\frac{m^2 - 4mn - 2n^2}{2n^2}$
10. If  $x + y + z = 0$ , find  $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$ .  
 1) 0                      2) 1                      3) 2                      4) 3
11. If  $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$ , then  $(x + y + z)^3 =$ ?  
 1)  $27xyz$                       2) 1                      3)  $xyz$                       4)  $9yxz$
12. The sum of two integers is 10 and the sum of their reciprocals is  $\frac{5}{12}$ . Then the larger of these integers is  
 1) 2                      2) 4                      3) 6                      4) 8
13. Three consecutive positive even numbers are such that thrice the first number exceeds double the third by 2, then the third number is  
 1) 10                      2) 14                      3) 16                      4) 12
14. Two positive integers differ by 4 and sum of their reciprocals is  $\frac{10}{21}$ . Then one of the numbers is  
 1) 3                      2) 1                      3) 5                      4) 21
15. What is the value of  $k$  for which the following system of equations has no solution:  $2x - 8y = 3$  and  $kx + 4y = 10$ ?  
 1) -2                      2) 1                      3) 2                      4) -1
16. Three friends, returning from a movie, stopped to eat at a restaurant. After dinner, they paid their bill and noticed a bowl of mints at the front counter. Sita took  $\frac{1}{3}$  of the mints, but returned four because she had a momentary pang of guilt. Fatima then took  $\frac{1}{4}$  of what was left but returned three for similar reasons. Eswari then took half of the remainder but threw two back into the bowl. The bowl had only 17 mints left when the raid was over. How many mints were originally in the bowl?  
 1) 38                      2) 31                      3) 41                      4) None of these

17. S and D are two numbers. D is a two digit number such that the sum of squares of the individual digits is equal to S. The difference between S and D is 13. What is the value of D?
- 1) 24                      2) 54                      3) 34                      4) 45
18. Three consecutive integers are such that their product is three times the middle number multiplied by 40. Find the absolute value of the largest number.
- 1) 12                      2) 11                      3) 10                      4) (1) or (3)
19. Three oranges, five bananas and two apples cost Rs.40. However four oranges, ten bananas and four apples cost Rs.70. Find the cost of ten oranges.
- 1) Rs.25    2) Rs.50  
3) Rs.30    4) Cannot be determined
20. Two numbers are such that one is 'x' less than 43 and the other is 'x' greater than 34. If product of both the numbers is 1482, find the smaller number.
- 1) 39    2) 38  
3) 19    4) Unique value cannot be determined

## QA-1.5 | PERCENTAGES, PROFIT & LOSS, INTEREST



### THEORY

#### Percentages

The term per cent means for every hundred. A fraction whose denominator is 100 is called a percentage and the numerator of the fraction is called the rate per cent. It is denoted by the symbol %.

#### Example

In a college X, 240 students passed out of 300 students and in college Y, 300 students passed out of 400 students. Can we say that the performance of college Y is better than that of X because  $300 > 240$ . Not really, because we want to compare the performance of the two colleges, we shall have to compare  $\frac{240}{300}$  with  $\frac{300}{400}$  instead of comparing 240 with 300. Can we now say which is the better of the two. The two fractions can be compared when their denominators are the same. Let us write both the fractions with 100 as their denominator.

$$\text{College X: } \frac{240}{300} = \frac{240}{300} \times \frac{100}{100} = \frac{80}{100} = 80 \text{ percent (or 80\%)}$$

$$\text{College Y: } \frac{300}{400} = \frac{300}{400} \times \frac{100}{100} = \frac{75}{100} = 75 \text{ percent (or 75\%)}$$

Now it is obvious that college X has a better performance than college Y. Thus, we see that percentages is a useful tool for comparison of fractions where the denominators are unequal.

#### Applications involving percentage

To find the % equivalent of a fraction or decimal

Express the fraction with the denominator 100 and the numerator is the required answer.

#### Example

$$\frac{11}{24} = \frac{\frac{11}{24} \times 100}{100} = \frac{45\frac{5}{6}}{100} = 45\frac{5}{6} \%$$

The percent equivalent of a decimal can be obtained by moving the decimal point two places to the right and adding the percent sign.

#### Example

$$0.356 = 0.356 \times 100\% = 35.6\%$$

To find the fraction or decimal equivalent of 'a' %:

Divide 'a' by 100.

**Example**

$$21\frac{7}{8}\% = \frac{21\frac{7}{8}}{100} = \frac{175}{800} = \frac{7}{32}$$

The decimal equivalent of a percent is obtained by moving the decimal point two places to the left and discarding the percent sign.

**Example**

$$46\% = \frac{46}{100} \text{ or } 0.46$$

To increase a number by a given %:

Multiply the number by the factor  $\frac{100 + \text{Rate}}{100}$

**Example**

Increase 20 by 15%

$$20 \times \left( \frac{100 + 15}{100} \right) = 20 \times \frac{115}{100} = 23 \quad \text{OR} \quad 20 \times 1.15 = 23$$

To decrease a number by a given %:

Multiply the number by the factor  $\frac{100 - \text{Rate}}{100}$

**Example**

Decrease 30 by 20%

$$30 \times \left( \frac{100 - 20}{100} \right) = 30 \times \frac{80}{100} = 24 \quad \text{OR} \quad 30 \times 0.8 = 24$$

To find the % increase of a number:

$$\% \text{ increase} = \frac{\text{Total increase}}{\text{Initial value}} \times 100 = \frac{\text{Final value} - \text{Initial value}}{\text{Initial value}} \times 100$$

**Example**

The population of a village in 1980 was 3000 and in 1990 was 3200. Find the % increase.

$$\% \text{ increase} = \frac{3200 - 3000}{3000} \times 100 = \frac{200}{3000} \times 100 \approx 6.67\%$$

To find the % decrease of a number:

$$\% \text{ decrease} = \frac{\text{Total decrease}}{\text{Initial value}} \times 100 = \frac{\text{Initial value} - \text{Final value}}{\text{Initial value}} \times 100$$

### Example

The cost of a bike last year was Rs.19000. Its cost this year is Rs.17000. Find the % decrease in its cost.

$$\% \text{ decrease} = \frac{19000 - 17000}{19000} \times 100 = \frac{2000}{19000} \times 100 \approx 10.5\%$$

### Absolute percentage change

Absolute percentage change is concerned only with the actual quantity of percentage change between two terms and not the positive or negative value of the percentage change.

$$\text{Absolute percentage change} = \frac{|\text{New value} - \text{Original value}|}{\text{Original value}} \times 100$$

### Example

The sales of a company are Rs.2500 crores in the year 2002 and is Rs.2200 crores in the year 2003. Find the absolute percentage change in the sales of the company from 2002 to 2003.

$$\text{Absolute percentage change} = \frac{|2200 - 2500|}{2500} \times 100 = \frac{|-300|}{2500} \times 100 = \frac{300}{25} = 12\%$$

Given below are some typical cases of percentage based problems

1. If the price of a commodity increases by  $r\%$ , then reduction in consumption, so as not to increase the expenditure is  $\left(\frac{r}{100 + r} \times 100\right)\%$
2. If the price of a commodity decreases by  $r\%$ , then the increase in consumption so as not to decrease the expenditure is  $\left(\frac{r}{100 - r} \times 100\right)\%$
3. If A's income is  $r\%$  more than that of B, then B's income is less than that of A by  $\left(\frac{r}{100 + r} \times 100\right)\%$ .
4. If A's income is  $r\%$  less than that of B, then B's income is more than that of A by  $\left(\frac{r}{100 - r} \times 100\right)\%$ .

### SOLVED EXAMPLES

**Q :** The total population of a country is 294000, out of which 150000 are males. Out of every 100 males, 98 can read and write, but only 53% of the total population can do so. Find the percentage of women who can read and write.

**A :** Number of men who can read and write =  $150000 \times \frac{98}{100} = 147000$

Number of men and women who can read and write =  $294000 \times \frac{53}{100} = 155820$

Number of women who can read and write =  $155820 - 147000 = 8820$

Thus, out of 144000 women 8820 can read and write.

$\therefore$  Required percentage =  $\frac{8820 \times 100}{144000} \% = 6.125\%$

**Q :** If the price of coal be raised by 20%, then find by how much a householder must reduce his consumption of this commodity so as not to increase his expenditure?

**A :** Reduction in consumption =  $\left( \frac{20}{100 + 20} \times 100 \right) \% = \left( \frac{20}{120} \times 100 \right) \% \approx 16.67\%$

**Q :** A man lost  $12\frac{1}{2}\%$  of his money and after spending 70% of the remainder, has Rs.210 left.

How much did the man have initially?

**A :** 30% of the money he did not lose is represented by 210

$\therefore$  Total money he did not lose =  $210 \times \frac{100}{30} = \text{Rs.}700$

87.5% of his total money is equal to Rs.700

$\therefore$  Total money he had =  $700 \times \frac{100}{87.5} = \text{Rs.}800$

**Q :** A's income is 10% more than B's. How much less is B's income than that of A's ?

**A :** B's income is less than A's by  $\left( \frac{10}{100 + 10} \times 100 \right) \% = \left( \frac{1000}{110} \right) \% = 9.09\%$

**Q :** In the annual budget of a certain school, the annual income was estimated at Rs.250 lakhs and annual expenditure was estimated at Rs.210 lakhs. Actually, that year the annual income increased by 5 percent than estimated income and expenditure increased by 10 percent than estimated. Find the difference between the actual income and actual expenditure.

**A :** Actual income =  $250 \times \left( \frac{100 + 5}{100} \right) = 250 \times 1.05 = \text{Rs.}262.5 \text{ lakh}$

Actual expenditure =  $210 \times \left( \frac{100 + 10}{100} \right) = 210 \times 1.1 = \text{Rs.}231 \text{ lakh}$

$\therefore$  Required difference =  $262.5 - 231 = \text{Rs.}31.5 \text{ lakh}$

**Q :** A merchant purchased calculators worth Rs.16800, mobile phones worth Rs.28900 and lighters worth Rs.6800. The duty that he had to pay was 5% on calculators, 8% on mobile phone and 3% on lighters. How much duty did he pay?

**A :** Duty on calculator =  $16800 \times 0.05 = \text{Rs.}840$

Duty on mobile phones =  $28900 \times 0.08 = \text{Rs.}2312$

Duty on lighters =  $6800 \times 0.03 = \text{Rs.}204$

$\therefore$  Total Duty =  $840 + 2312 + 204 = \text{Rs.}3356$

**Q :** 75% of a number when added to 75 is equal to that number. What is the number?

**A :** Let the required number be 'x'.

$\therefore 0.75x + 75 = x \Rightarrow 0.25x = 75 \Rightarrow x = \frac{75}{0.25} = 300$

**Q :** In a class, 65% of students are boys. On a particular day 80% of girl students were present. What was the fraction of boys who were present that day if the total number of students present that day was 70%?

**A :** Let the class have 100 students.

	Girls	Boys	Total
Students	35	65	100
Students present	28	42	70

$\therefore$  Required fraction =  $\frac{42}{65}$

### Concept Builder 1

1. A basket contains red, blue and yellow coloured balls. If 50% of the total number of balls are red, 20% of the remaining are blue, how many balls are yellow in colour given that the total number of balls in the basket is 250?
2. What will be 70% of a number whose 150% is 90?
3. A man earns Rs.7500. He spends 60% of his income. If his income increases by 20% and his expenditure increases by 10%, find the increase in his savings.
4. Om has some cards out of which he gave 32% of total cards to Ram and 16% of total cards to Sita. If 50% of Sita's cards was taken back by Om and he is left with 75 cards, then the total number of cards is:
5. A person buys a watch worth Rs.750, a mobile worth Rs.3600 and a television worth Rs.10500 and pays a duty of 4%, 7%, 9% respectively. Find the total duty paid by him.

### Answer Key

1. 100. 2. 42. 3. Rs.1050 4. 125. 5. Rs.1227



### Theory of Simple & Compound Interest

The concept of Simple & Compound Interest is based on Percentages.

When a sum of money is lent by A to B, A is called the lender (creditor), B the borrower (debtor). The sum lent is called the principal (P).

Interest (I) is the extra money paid by the borrower to the lender for the use of the money for a specified time.

The time for which the money is borrowed is called period (N).

The interest paid per 100 rupees in a year is called rate per cent per annum (R).

The sum of interest and principal is called the Amount (A)

$$A = P + I$$

### Simple Interest (S.I.)

When interest is paid as it falls due, it is called simple interest i.e., throughout the loan period,

interest is charged on the original sum (principal) borrowed.  $S.I. = \frac{PNR}{100}$

#### Example

Find the interest to be paid on Rs.1500 at 10% per annum for a period of 2 years.

P = 1500, R = 10%, N = 2 years

$$S.I. = \frac{PNR}{100} = \frac{1500 \times 2 \times 10}{100} = \text{Rs.}300$$

### Compound Interest (C.I.)

Money is said to be lent at compound interest when at the end of a year or other fixed period, the interest that has become due is not paid to the lender, but is added to the sum lent, and the amount thus obtained becomes the principal for the next year or period. The process is repeated until the amount for the last period has been found. The difference between the final amount and the original principal is the compound interest (C.I.).

In compound interest, the interest is calculated on the accrued interest also.

This difference between S.I. & C.I. will be clear from the following illustration:

On a sum of Rs.1000 at a rate of 10% per annum, S.I. & C.I. will accrue for 4 years as follows.

<u>Year</u>	<u>Simple Interest</u>	<u>Compound Interest</u>
1	Rs100	Rs100
2	Rs100	Rs110
3	Rs100	Rs121
4	Rs100	Rs133.1

In case of Compound Interest, after the first year, the previous year's interest is added to the original sum of Rs1000 to calculate the interest.

For C.I., Amount =  $P \left[ 1 + \frac{R}{100} \right]^N$

Compound Interest = Amount – Principal

### Some Important Rules and Formulae in C.I.

1. The time period after which the interest is added each time to form a new principal is called conversion period. It may be annually, semi annually or quarterly.
2. In case, interest is paid semi annually (half yearly) N is the number of half years and R is the rate percent per half year i.e., Number of years  $\times 2$  = Number of half years. Rate percent per annum  $\div 2$  = Rate percent per half year.
3. In case interest is paid quarterly, N is the number of quarters and R is the rate percent per quarter.

Number of years  $\times 4$  = Number of quarters

Rate percent per year  $\div 4$  = Rate percent per quarter.

4. When rates are different for different years, say  $R_1, R_2, R_3$  percent for first, second and third

year respectively then Amount =  $\left( 1 + \frac{R_1}{100} \right) \left( 1 + \frac{R_2}{100} \right) \left( 1 + \frac{R_3}{100} \right)$

5. In case the time is a fraction of a year, say  $y \frac{x}{z}$  years, then Amount =  $P \left( 1 + \frac{R}{100} \right)^y \times \left( 1 + \frac{\frac{x}{z} \times R}{100} \right)$

### Difference between S.I. & C.I.

1. When interest is payable only on the principal sum of money at the end of every year, the total interest paid at the end of the year is the simple interest.
2. When interest is charged on the principal and also on the interest accrued in the previous period, the total interest payable at the end of the period is termed as compound interest.
3. In two years, the difference between CI and SI is nothing but the interest on the first year's interest taken as principal. In the example with S.I. we solved earlier, 1st year's interest was Rs.150. If we are to calculate compound interest,

$$\text{Amount} = 1500 \left( 1 + \frac{10}{100} \right)^2 = 1500 = 1500 \times (1.1)^2 = \text{Rs.1815}$$

$$\therefore \text{CI} = 1815 - 1500 = 315$$

$\therefore$  Difference between CI and SI was Rs.15 which was the interest on Rs.150.

### Population formula & Depreciation formula

The original population of a town is  $P$  and the annual increase is  $R\%$ , then the population of the town in  $n$  years is  $P\left(1 + \frac{R}{100}\right)^n$  and if the annual decrease is  $R\%$ , then the population of the town in  $n$  years is given by a change of sign in the formula i.e.,  $P\left(1 - \frac{R}{100}\right)^n$ . The second formula mentioned alongside is also called as the Depreciation formula.

#### Example

If the annual increase in the population of a town is  $4\%$  and the present population is  $15625$  what will be the population in  $3$  years?

Required population =  $15625 (1.04)^3 = 17576$

### Hire Purchase

In a hire purchase plan, a customer can make use of the goods while paying for them. The amount paid at the time of purchase is called the down payment. The remainder is paid in equal installments and each is the monthly installment. The difference between the total amount to be paid and the cash price is called the installment charge.

$$\text{Monthly Installment} = \frac{\text{Amount to be paid} - \text{Down payment}}{\text{Number of installments}}$$

#### Example

If a transistor is available at Rs.400 cash or Rs.100 down payment and Rs.70 per month for 5 months, find

- |   |                                    |
|---|------------------------------------|
| <p>(i) Total amount paid for it</p> <p>Amount paid = <math>100 + 70 \times 5 = \text{Rs.}450</math></p> <p>Installment charge = <math>450 - 400 = \text{Rs.}50</math></p> | <p>(ii) The installment charge</p> |
|---|------------------------------------|

### SOLVED EXAMPLES

**Q :** If the annual increase in the population of a town is 4% and the present population be 17576, what was it three years ago?

**A :** Population 3 years ago  $\times \left(\frac{104}{100}\right)^3$  = present population

$$\therefore \text{Population 3 years ago} \times \frac{26 \times 26 \times 26}{25 \times 25 \times 25} = 17576$$

$$\therefore \text{Population 3 years ago} = 17576 \times \frac{25 \times 25 \times 25}{26 \times 26 \times 26} = 15625$$

**Q :** A lent Rs.600 to B for 2 years, and Rs.150 to C for 4 years and received altogether from both Rs.90 as interest. Find the rate of interest, simple interest being calculated (assume that money was lent at the same rate of interest to B and C).

**A :** Rs.600 for 2 years = Rs.1200 for 1 year and Rs.150 for 4 years = Rs.600 for 1 year

$\therefore$  Total principal = Rs.1800 for 1 year

$$\text{Interest} = \text{Rs.90}; \quad \therefore \text{Rate} = \frac{90 \times 100}{1800 \times 1} = 5\%$$

**Q :** The simple interest on a sum of money is  $\frac{1}{9}$  th of the principal, and the number of years is equal to the rate percent per annum. Find the rate percent.

**A :** If the principal = P, Time = n years, Rate percent = n, then  $\frac{P \times n \times n}{100} = \frac{P}{9}$

$$\therefore n^2 = \frac{100}{9} \quad \therefore n = \frac{10}{3} = 3\frac{1}{3}; \quad \therefore \text{Rate percent is } 3\frac{1}{3}.$$

**Q :** If Rs.5600 amount to Rs.6678 in  $3\frac{1}{2}$  years, what will Rs.9400 amount to in  $5\frac{1}{4}$  years at the same rate percent per annum simple interest?

**A :** We first find the rate percent as follows:

$$\text{Interest on Rs.5600} = \text{Rs.6678} - \text{Rs.5600} = \text{Rs.1078}$$

$$\therefore \text{Rate \%} = \frac{100 \times 1078}{5600 \times 3\frac{1}{2}} = \frac{2 \times 100 \times 1078}{5600 \times 7} = 5\frac{1}{2}$$

$$\text{Interest on Rs.9400} = \text{Rs.} \frac{9400 \times 21 \times 11}{100 \times 4 \times 2} = \frac{10857}{4} = \text{Rs.}2714.25$$

$$\therefore \text{The required amount} = \text{Rs.}9400 + \text{Rs.}2714.25 = \text{Rs.}12114.25$$

**Q :** A man derives his income from the investment of Rs.4150 at a certain rate of simple interest and Rs.3500 at 1% higher than the rate of interest on Rs.4150. His whole income for 4 years is Rs.1211. Find the rates of interest.

**A :** Income for 1 year = Rs.  $\frac{1211}{4}$

Since the rate of interest for Rs.3500 is 1% higher, therefore, if we subtract 1% on Rs.3500 from Rs.  $\frac{1211}{4}$ , the remainder will be one year's interest on (Rs.4150 + Rs.3500) at the lower rate of interest. Interest on (Rs.4150 + Rs.3500) = Rs.  $\frac{1211}{4}$  - 1% on Rs.3500

$$\therefore \text{Interest on Rs.7650} = \text{Rs.} \frac{1211}{4} - \text{Rs.}35 = \text{Rs.} \frac{1071}{4}$$

$$\therefore \text{Interest on Rs.100} = \text{Rs.} \frac{1071}{4} \times \frac{100}{7650} = \text{Rs.}3\frac{1}{2}$$

$$\therefore \text{The lower rate of interest is } 3\frac{1}{2}\% \text{ and the higher rate of interest is } 4\frac{1}{2}\%.$$

**Q :** A property decreases in value every year at the rate of  $6\frac{1}{4}\%$  on its value at the beginning of that year. If its value at the end of 3 years be Rs.21093.75, what was its worth at the beginning of these three years?

**A :** Here,  $P \left(1 - \frac{6\frac{1}{4}}{100}\right)^3 = 21093\frac{3}{4}$

$$P \left(\frac{15}{16}\right)^3 = \frac{84375}{4}; P = \frac{84375 \times 16 \times 16 \times 16}{4 \times 15 \times 15 \times 15} = 25600$$

$\therefore$  Original value of the property is Rs.25600.

**Q :** On what sum will the difference between the simple and compound interests for 3 years at 5 percent per annum amount to Rs.12.20?

**A :** Compound Interest on Re.1 for 3 years at 5 percent

$$= 1 \left\{ \left( 1 + \frac{5}{100} \right)^3 - 1 \right\} = \frac{1261}{8000}$$

$$\text{Simple Interest on Re.1 for 3 years at 5 percent} = \frac{15}{100}$$

$$\therefore \text{Difference} = \frac{1261}{8000} - \frac{15}{100} = \frac{61}{8000}$$

But the given difference is Rs.12.20 or Rs.  $\frac{61}{5}$

$$\therefore \text{Sum} = \frac{\frac{61}{5}}{\frac{61}{8000}} = \frac{8000}{5} = \text{Rs.1600.}$$

**Q :** If the compound interest on a certain sum for 2 years at 3 percent be Rs.101.50, what would be the simple interest for the same time period and at the same rate of interest?

$$\text{A : Compound Interest on Re.1} = \left( \frac{103 \times 103}{100 \times 100} - 1 \right) = \frac{609}{10000}$$

$$\text{Simple Interest on Re.1} = \frac{6}{100}$$

$$\therefore \frac{\text{Simple Interest}}{\text{Compound Interest}} = \frac{6}{100} \times \frac{10000}{609} = \frac{200}{203}$$

$$\therefore \text{Simple Interest} = \frac{200}{203} \text{ of Rs.101.50} = \frac{200}{203} \times \frac{203}{2} = \text{Rs.100}$$

*Alternatively, first find the principal and then the simple interest.*

**Q :** The compound interest on a certain sum for 2 years is Rs.40.80 and the simple interest is Rs.40. Find the sum and the rate percent.

**A :** A little reflection will show that the difference between the simple and compound interests for 2 years is the interest on the first year's interest. Find a year's simple interest on Rs.20.

$$\text{Compound Interest} - \text{Simple Interest} = \text{Rs.40.80} - \text{Rs.40} = \text{Rs.0.80}$$

$$\text{Interest on Rs.20 for 1 year} = \text{Rs.0.80}$$

$$\text{Interest on Rs.100 for 1 year} = \frac{80 \times 100}{100 \times 20} = \text{Rs.4}$$

The rate is 4 p.c.p.a.

$$\text{Now the principal P is given by } P = \frac{100 \times I}{nr} = \frac{100 \times 40}{2 \times 4} = \text{Rs.500}$$

Hence, the required principal is Rs.500.

**Q :** Find the present worth of Rs.481.25 due  $2\frac{1}{2}$  years hence, reckoning simple interest at 4 percent per annum. What is the discount?

**A :** Interest on Rs.100 for  $2\frac{1}{2}$  years at 4 percent = Rs.10

$$\therefore \text{Amount of Rs.100} = \text{Rs.100} + \text{Rs.10} = \text{Rs.110}$$

$$\therefore \text{Present worth of Rs.110} = \text{Rs.100}$$

$$\therefore \text{Present worth of Rs.481.25} = \frac{100}{110} \times 481.25 = \text{Rs.437.50}$$

$$\text{Discount} = \text{Rs.481.25} - \text{Rs.437.50} = \text{Rs.43.75}$$

The discount may also be found directly thus: The discount on Rs.110 = Rs.10

$$\text{The discount on Rs.481.25} = \frac{10}{110} \times 481.25 = \text{Rs.43.75}$$

*Alternatively,*

By applying the formula : present worth =  $\frac{100A}{100 + RT}$ , we get

$$\text{Present worth} = \frac{100 \times 481.25}{100 + 4 \times 2\frac{1}{2}} = \frac{48125}{110} = \text{Rs.437.50}$$

$$\text{Also, total discount} = \frac{A \times R \times T}{100 + RT}$$

$$\therefore \text{Total discount} = \frac{481.25 \times 4 \times 2\frac{1}{2}}{100 + 4 \times 2\frac{1}{2}} = \frac{481.25 \times 10}{110} = \text{Rs.43.75}$$

## Concept Builder 2

- Find the compound interest on Rs.4400 at the rate of interest of 10% p.a. with interest reckoned half-yearly for a year.
- At the rate of  $8\frac{1}{2}\%$  p.a. simple interest, a sum of Rs.4800 will earn how much interest in 2 years and 3 months?
- A sum of money becomes  $\frac{5}{4}$  times of itself in 4 years at a certain rate of simple interest. The rate per annum is:
- At what rate of percent compound interest does a sum of money become four times itself in 2 years?
- The difference between the simple interest on Rs.1500 for 4 years at some rate of interest and Rs.3000 for 2 years at some other rate of interest is Rs.360. Then the difference between the rate of interest is:
- A vessel contains some bacteria; the rate of reproduction of bacteria is 10% per day. After 3 days, the vessel was full of bacteria having 1331 bacteria in it. Find the increase in the number of bacteria.
- Ram borrowed Rs.5000 out of which he borrowed 25% at the rate of 6% S.I. and the remaining at the rate of 10% S.I. After 3 years, it will amount to:
- The difference between the C.I. and S.I. on a certain sum is Rs.1.25 at 5% rate of interest for 2 years. Find the sum.
- In how many years, will a man get Rs.87 as simple interest on Rs.725 at 4% per annum?
- A certain sum amounts to Rs.5150 in 3 years and Rs.5350 in 7 years at simple interest. Find the rate of interest.
- A sum of Rs.800 amounts to Rs.920 in 3 years at simple interest. If the interest rate is increased by 3%, how much would it amount to?
- What is the principal amount which earns Rs.252 as compound interest for two years at 10% per annum?
- If B borrowed Rs.2460 at the rate of 6% per annum simple interest, what is the amount that he will pay to clear the debt after 5 years?
- Simple interest on a certain sum is  $\frac{9}{25}$  times of the sum. Find the rate percent and time, if both are equal.
- If Rs.7500 amounts to Rs.8112 in 2 years compounded annually, find the rate of interest per annum.
- Find the amount of Rs.4000 for 3 years compounded annually, the rate of interest being 5% for the first 2 years and 10% for the next one year.

### Answer Key

- |    |        |    |         |     |         |     |               |
|----|--------|----|---------|-----|---------|-----|---------------|
| 1. | 451    | 4. | 100%.   | 9.  | 3 years | 13. | Rs.3198       |
| 2. | Rs.918 | 5. | 6.      | 10. | 1%      | 14. | R = 6%, N = 6 |
| 3. | 1      | 6. | 331     | 11. | Rs.992  | 15. | 4%.           |
|    |        | 7. | 6350    | 12. | Rs.1200 | 16. | Rs.4851       |
|    |        | 8. | Rs.500. |     |         |     |               |



## Profit & Loss

### Basic terminology

#### Cost Price (CP)

The price for which an article is bought is called its cost price.

#### Selling Price (SP)

The price at which an article is sold is called its selling price.

#### Profit (Gain)

The difference between the selling price and cost price (i.e., when selling price is greater than cost price) is called the profit.

#### Loss

The difference between the cost price and the selling price (i.e., when cost price is greater than the selling price) is called the loss.

Profit and loss are generally represented as a percent of the cost price, unless otherwise stated.

#### Overhead charges

If an individual has to spend some money on transportation etc., then this extra expenditure is called overhead charges.

#### Marked price (MP)

The price on the label is called the marked price or list price.

#### Discount

The reduction made on the 'marked price' of an article is called discount. When no discount is given, 'selling price' is the same as 'marked price'.

#### Markup

Markup is the difference between the cost of an article and its marked price. A markup is added on to the cost price or purchase price of the producer/trader of an article in order to create a profit..

### Important formulae in Profit & Loss

1. Profit = SP – CP                      ... (SP > CP)
2. Loss = CP – SP                      ... (CP > SP)
3. % Profit =  $\frac{\text{Profit}}{\text{CP}} \times 100 = \frac{\text{SP} - \text{CP}}{\text{CP}} \times 100$
4. % Loss =  $\frac{\text{Loss}}{\text{CP}} \times 100 = \frac{\text{CP} - \text{SP}}{\text{CP}} \times 100$
5. Profit =  $\frac{\text{Profit}\% \times \text{CP}}{100}$

$$6. \text{ Loss} = \frac{\text{Loss\%} \times \text{CP}}{100}$$

$$7. \text{ SP} = \frac{100 + \text{Profit\%}}{100} \times \text{CP}$$

$$8. \text{ SP} = \frac{100 - \text{Loss\%}}{100} \times \text{CP}$$

$$9. \text{ CP} = \text{SP} \times \left( \frac{100}{100 + \text{Profit\%}} \right)$$

$$10. \text{ CP} = \text{SP} \times \left( \frac{100}{100 - \text{Loss\%}} \right)$$

$$11. \text{ C.P} + \text{Mark up} = \text{M.P OR C.P} + \% \text{ Mark up on C.P} = \text{M.P}$$

$$12. \text{ Discount \%} = \frac{\text{Discount}}{\text{Marked Price}} \times 100$$

### 13. Relation Between S.P and M.P

Whenever you buy an article at a discounted price, the following formula applies:

Selling price = Marked price – Discount

Where,

‘selling price’ is the price you actually pay for an article

‘marked price’ is the price of an article without a discount,

‘discount’ is the reduction on the marked price of the article.

### Example

A bag that normally costs Rs.2000 is being sold for only Rs.1200. Calculate the discount as a percentage of the marked price.

Given: marked price = Rs2000, selling price = Rs.1200

selling price = marked price – discount

so discount = marked price – selling price

= Rs.2000 – Rs.1200

= Rs.800

Expressing the discount as a percentage of the marked price:

$$\text{Discount \%} = \frac{\text{Discount}}{\text{Marked Price}} \times 100$$

$$\text{Discount \%} = \frac{800}{2000} \times 100 = 40\%$$

**Note:** If the article is sold without discount then M.P = S.P

### Cases related to Profit & Loss

1. If two items are sold, each at Rs.x, one at a gain of p% and the other at a loss of p%, there

is an overall loss given by  $\frac{p^2}{100}\%$ . The absolute value of the loss is given by  $\frac{2p^2x}{100^2 - p^2}$ .

This can be proved as follows:

Suppose SP of each item = Rs.x.

$$\text{CP of one item} = \frac{100}{100 + p} \times x$$

$$\text{CP of other item} = \frac{100}{100 - p} \times x$$

$$\text{Total CP} = \frac{100x}{100 + p} + \frac{100x}{100 - p} = \frac{20000x}{(100)^2 - p^2}$$

$$\text{Total SP} = 2x$$

$$\text{Loss} = \frac{20000x}{10000 - p^2} - 2x = \frac{2xp^2}{(100)^2 - p^2} = \frac{2p^2 \times \text{SP}}{(100)^2 - p^2}$$

$$\text{Loss \%} = \frac{2p^2 \times \text{SP}}{20000\text{SP}} \times 100 = \frac{p^2}{100}$$

2. If CP of two items is same and % Loss and % Gain on the two items are equal, then net loss or net profit is zero.

3. Buy x get y free i.e., if x + y articles are sold at cost price of x articles, then the percentage

$$\text{discount} = \frac{y}{x + y} \times 100.$$

4. By using false weight, if a substance is sold at cost price, the overall gain % is given by

$$\frac{100 + \text{Gain\%}}{100} = \frac{\text{True Scale or Weight}}{\text{False Scale or Weight}}.$$

5. In case of successive discounts a% and b%, the effective discount is  $\left(a + b - \frac{ab}{100}\right)\%$ .

### SOLVED EXAMPLES

**Q** : A boy buys eggs at 9 for Rs.16, and sells them at 11 for Rs.20. What is his gain or loss per-cent?

**A** : To avoid fractions, let the number of eggs bought and sold be the LCM of 11 and 9, i.e.99.

$$\text{CP of 99 eggs} = \frac{99 \times 16}{9} = \text{Rs.176};$$

$$\text{SP of 99 eggs} = \frac{99 \times 20}{11} = \text{Rs.180}$$

$$\text{Profit on Rs.176} = 4$$

$$\% \text{ Profit} = \frac{4 \times 100}{176} = 2\frac{3}{11}\% = 2.272\%$$

**Q** : A dishonest dealer professes to sell his goods at cost price, but he uses a weight of 960 gm for the kg weight. Find his gain percent.

**A** : Supposing the goods cost the dealer Re.1 for the kg., he sells for Re.1 goods which cost him Rs.0.96.

$$\text{Gain on Rs.0.96} = 0.04; \% \text{ Gain} = \frac{0.04 \times 100}{0.96} = 4\frac{1}{6}\% = 4.16\%$$

*Alternatively,*

$$\text{By using the formula: } \frac{100 + \text{Gain}\%}{100} = \frac{1000}{960}$$

$$\therefore \text{Gain}\% = 4.16\%$$

**Q** : A sells a bicycle to B at a profit of 20% and B sells to C at a profit of 25%. If C pays Rs.225 for the bicycle, what did A pay for it?

**A** : Suppose A pays Rs.100, then B pays Rs.120 and C pays  $\frac{120 \times 125}{100} = 150$

Thus, if C pays Rs.150, A pays Rs.100.

$$\text{If C pays Rs.225, A pays } \frac{100 \times 225}{150} = \text{Rs.150.}$$

**Q** : A trader allows a discount of 5% for cash payment. How much percent above the cost price must he mark his goods to make a profit of 10%?

**A** : let the CP be Rs.100. For 10% profit, SP should be Rs.110. After 5% discount, 95% of the MP = Rs.110.

$MP = \frac{100}{95} \times 110 = \text{Rs.}115 \frac{15}{19} = \text{Rs.}115.79$ . Thus, the goods must be marked 15.79% above cost price to make a profit of 10%.

**Q :** 1 kg of tea and 4 kg of sugar cost Rs.6.28. But if the cost of sugar rose by 50% and that of tea by 10%, the same quantity of tea and sugar would cost Rs.7.42. Find the prices per kg of tea and sugar.

**A :** If the price of both tea and sugar were to rise by 50%, the cost of 1 kg of tea and 4 kg of sugar would be  $\text{Rs.}6.28 \times \frac{150}{100}$  or Rs.9.42.

But the cost of tea rises by only 10%

$\therefore$  40% of the cost of 1 kg of tea =  $\text{Rs.}9.42 - \text{Rs.}7.42 = \text{Rs.}2$

$\therefore$  Cost of 1 kg of tea =  $\text{Rs.}2 \times \frac{100}{40} = \text{Rs.}5$

$\therefore$  Cost of 4 kg of sugar =  $\text{Rs.}6.28 - \text{Rs.}5 = \text{Rs.}1.28$

$\therefore$  Cost of 1 kg of sugar = Rs.0.32 or 32 paise.

*Alternatively,*

Let the price of 1 kg of tea be Rs.t and that of 1 kg of sugar be Rs.s

$\therefore t + 4s = 6.28 \quad \dots (i)$

$1.1t + 4 \times 1.5s = 7.42$

$\therefore 1.1t + 6s = 7.42 \quad \dots (ii)$

Multiply (i) by 1.1

$\therefore 1.1t + 4.4s = 6.908 \quad \dots (iii)$

Subtract (iii) from (ii)

$1.6s = 0.512$

$\therefore s = \frac{0.512}{1.6} = 0.32 \quad \therefore t + 4 \times 0.32 = 6.28 \quad \therefore t = 6.28 - 1.28 = 5$

$\therefore$  The price of 1 kg of tea is Rs.5 and one kg of sugar is 32 paise.

**Q :** I bought 8 kg. of sugar for Rs. 88 and was obliged to sell it at a loss of as much money as I received for 2 kg. At what price per kg did I sell the sugar?

**A :** CP of 8 kg – SP of 8 kg = SP of 2 kg

SP of 8 kg + SP of 2 kg = Rs.88

$\therefore$  SP of 10 kg = Rs.88  $\therefore$  SP of 1 kg =  $\frac{88}{10} = \text{Rs.}8.8$

**Q :** A woman buys chickoos at Rs.25 a dozen and an equal number at Rs.30 a score. She sells them at Rs.36 a score and thus makes a profit of Rs.10. How many chickoos did she buy?

**A :** The LCM of 12 and 20 = 60.

Suppose she buys 60 chickoos of each kind.

$$\text{Cost of 60 chickoos at Rs.25 a dozen} = \frac{60}{12} \times \text{Rs.25} = \text{Rs.125}$$

$$\text{Cost of 60 chickoos at Rs.30 a score} = \frac{60}{20} \times \text{Rs.30} = \text{Rs.90}$$

$$\therefore \text{Total cost of 120 chickoos} = \text{Rs.125} + \text{Rs.90} = \text{Rs.215.}$$

$$\text{Selling price of 120 chickoos at Rs.36 a score} = \text{Rs.216.}$$

$$\therefore \text{Profit on 120 chickoos} = \text{Rs.216} - \text{Rs.215} = \text{Rs.1}$$

But the actual profit is Rs.10.

Hence, the proportion of Rs.1 : Rs.10 :: 120 chickoos : the required number.

$$\therefore \text{Required number} = 120 \times 10 = 1200.$$

**Q :** A man purchases a certain number of oranges at 25 a rupee and the same number at 20 a rupee. He mixes them together and sells them at 45 for two rupees. How much percent does he gain or lose in the transaction?

**A :** Suppose the man buys 900 (i.e., LCM of 25, 20 and 45) oranges of each kind.

$$\text{Cost of 900 oranges of the first kind} = \text{Rs.36}$$

$$\text{Cost of 900 oranges of the second kind} = \text{Rs.45}$$

$$\therefore \text{Total cost of } 900 \times 2 \text{ oranges} = \text{Rs.36} + \text{Rs.45} = \text{Rs.81}$$

$$\text{Selling price of } 900 \times 2 \text{ oranges} = \text{Rs.} \frac{1800 \times 2}{45} = \text{Rs.80}$$

$$\therefore \text{Loss on Rs.81} = \text{Rs.81} - \text{Rs.80} = \text{Rs.1}$$

$$\therefore \text{Loss on Rs.100} = \frac{1}{81} \times 100 = \text{Rs.} \frac{100}{81} \therefore \text{Loss} = 1\frac{19}{81}\% \approx 1.23\%$$

*Alternatively,*

$$\text{Cost of 45 oranges at 25 a rupee} = \text{Rs.1.8}$$

$$\text{Cost of 45 oranges at 20 a rupee} = \text{Rs.2.25}$$

$$\text{Cost price of 90 oranges} = \text{Rs.1.8} + 2.25 = \text{Rs.4.05}$$

$$\text{Selling price of 90 oranges at 45 for two rupees} = \text{Rs.4}$$

$$\therefore \text{Loss} = 4.05 - 4 = \text{Rs.0.05}$$

$$\text{Percentage loss} = \frac{0.05}{4.05} \times 100 = 1.23\%$$

**Q** : The cost price of 10 articles is equal to the selling price of 9 articles. Find the profit percent.

**A** : Let the cost price of 1 article be Re.1

∴ Cost of 10 articles is Rs.10

∴ Selling price of 10 articles =  $\frac{10 \times 10}{9} = \text{Rs.} \frac{100}{9}$

∴ Gain on Rs.10 =  $\text{Rs.} \frac{100}{9} - \text{Rs.} 10 = \text{Rs.} \frac{10}{9}$

∴ Gain on Rs.100 =  $\text{Rs.} \frac{100}{9} = \text{Rs.} 11 \frac{1}{9}$  ∴ Profit per cent is  $11 \frac{1}{9}\%$ .

**Q** : If oranges are bought at the rate of 25 for a rupee, how many must be sold for a rupee so as to gain 25%?

**A** : SP of 25 oranges =  $1 \times \frac{125}{100} = \text{Rs.} 1.25$

Number of oranges to be sold for Rs.1.25 = 25

Number of oranges to be sold for Re.1 =  $\frac{25}{1.25} = 20$

**Q** : A trader allows two successive discounts of 20% and 15%. If he gets Rs.136 for an article, find its marked price.

**A** : Effective discount =  $\left(a + b - \frac{ab}{100}\right)\% = \left(20 + 15 - \frac{20 \times 15}{100}\right)\% = 32\%$

∴ If MP is 100, then SP =  $100 - 32 = 68$

∴ If SP = 136, MP =  $\frac{136 \times 100}{68} = \text{Rs.} 200$

*Alternatively,*

$136 = \frac{80}{100} \times \frac{85}{100} \times \text{MP}$  ∴ MP = Rs.200

**Q** : Two goods are sold at the same price one at a loss of 15% and other at a profit of 15%. Find the overall profit/loss.

**A** : We know that when goods are sold at the same S.P. there is always a loss incurred.

∴ Overall loss =  $\frac{p^2}{100} = \frac{15^2}{100} = 2.25\%$

### Concept Builder 3

- If the profit made by a trader is one-fifth of the SP, find the actual profit percentage.
- If 5 quintals of rice are sold for Rs.720, there is a profit of 20%. Find the cost price of 5 quintals rice.
- A man buys 2 watches each costing Rs.x. He sells one watch at a gain of  $9\frac{1}{3}\%$  and the other at a loss of  $9\frac{1}{3}\%$ . Find his net loss or gain percent.
- The cost price of 12 apples is equal to the SP of 16 apples. What is the loss or gain percent?
- A dealer sold two watches for Rs.400. He sells one at a profit of 20% and the other at a loss of 20%. If the SP of each is the same, find his total loss or gain percent.
- x sells a bicycle to y at a profit of 25% and y sells it to z at a profit of 20%. If z pays Rs.156, how much did x pay for it?
- A milkman buys milk at Rs.10 per litre and adds one-third of water, then he sells the mixture at Rs.11 per litre. The gain is:
- A vendor bought bananas at 6 for Rs.10 and sold them at 4 for Rs.6. What is his gain or loss percent?
- The price of a DVD is marked at Rs.3500 and 5% discount is allowed on it. The selling price of DVD is:
- A dishonest dealer uses a scale of 90 cm instead of 1 metre scale and claims to sell at cost price. His profit is:
- An article was sold for Rs.y after giving a discount of x%. Then its listed price is:
- A man purchases two televisions for Rs.1000 each. He sells one television at 10% gain and the second one at 10% loss. Find his gain or loss percent on the whole transaction.
- By selling an article for Rs.300, a trader made a profit of 25%. Now, if the trader gives a successive discount of 15% and 10% on the existing selling price, then how much profit/loss does he earn?
- An article's listed price is Rs.200. After allowing a discount of 20% on it, its selling price is:
- After receiving 3 successive discounts of 10%, 20% and 25% on MP of furniture, Maya purchased the furniture for Rs.9720. Determine the MP of the furniture.

### Answer Key

1.	25%.	8.	10% loss	12.	no loss, no profit
2.	Rs.600.	9.	Rs.3325.	13.	4.375% loss
3.	0%.	14.	Rs.160	15.	Rs.18000.
4.	25% loss	10.	$11\frac{1}{9}\%$		
5.	4% loss	11.	$\frac{100y}{100 - x}$		
6.	Rs.104				
7.	46.66%				



## CLASS EXERCISE

### Teaser

A farmer and his son used to sell 30 oranges each at the local market daily. The farmer used to sell at the rate of 2 oranges for a rupee while his son, being poor at bargaining, used to sell at the rate of 3 oranges for a rupee. They used to return with Rs.25 daily.

One day the farmer was sick and asked his son to go to the market alone. They decided that since they used to sell 2 for a rupee and 3 for a rupee respectively, the effective rate should be 5 oranges for 2 rupees. So the son went and sold all the 60 oranges at the rate of 5 oranges for 2 rupees, making Rs.24 in all!

How did the son end up making a loss of 1 rupee?



## Percentage

1. Fill in the blanks:
  - a) 5% of 20 is \_\_\_\_\_.
  - b) \_\_\_\_\_ % of 20 is 5.
  - c) 5% of \_\_\_\_\_ is 20.
  - d) 120% of 200 is 200% of \_\_\_\_\_.
  - e) \* 25% less than 300 is \_\_\_\_\_ % more than 200.
  - f) \* \_\_\_\_\_ % of 4% of 5 equals 4% of 5% of 6.
  - g) \*  $33\frac{1}{3}$  % of 300 =  $3\frac{1}{3}$  % of \_\_\_\_\_.
2. Fill in the blanks (Units of measurement):
  - a) 60 paise = \_\_\_\_\_ % of Rs.10
  - b) 5 cm = \_\_\_\_\_ % of 10 km
  - c) \* 5% of 10 kg = 2% of \_\_\_\_\_ gm
  - d) \* 45 min = \_\_\_\_\_ % of a day
3. Half of one third of three-fifths of M = Two thirds of \_\_\_\_\_ % of M.
4. 45% of the students in a class are girls and the number of boys exceeds the number of girls by 30. How many students are there in the class?
5. The price of my Chevrolet Spark depreciates by 10% every year. If I bought it for 3.6 lakhs in the year 2011, by what percent would its price have fallen in 2015?
6. The prices at McDonalds rose by 15% in the period 2005 – 2008 and by 20% in the period 2008 – 2011. If the price of a McChicken burger is Rs.69 now, what was its price in 2005?
7. I spend Rs.3300 per month at a nearby mall. However, this month the prices of all the items were hiked by 10%. What percent reduction in my consumption will help me keep my budget constant?
8. If Sudesh spends 25% more than Ganesh, then Ganesh spends \_\_\_\_\_ % less than Sudesh.
9. \* In a test, the total of the marks scored by Sampat and Ganpat was 125% more than Sampat's marks. Then the difference in their marks will be \_\_\_\_\_ % of Ganpat's marks.
10. \* Anand's salary is 10% less than Swanand's and 10% more than Paramanand's. By what percent is Swanand's salary more than Paramanand's?
11. \* A bottle of my favourite drink contains 12% alcohol. On my doctors advice I added 20ml water to the bottle to reduce the concentration to 10%. What was the quantity of drink in my bottle initially?

## Profit and Loss

We earn Profit = SP – CP if SP > CP and we incur Loss = CP – SP if CP > SP.

$$\text{Percentage profit / loss} = \frac{\text{profit/loss}}{\text{cost price}} \times 100$$

1. Answer the following questions on the basis of the data given below:  
Ankit has purchased a blackberry from Kittu for Rs.24000.
  - a) If he sells it to Tina for Rs.27000 then what percentage profit will he earn?
  - b) If he sells it to Tania at 10% loss, what is the cost price for Tania?
  - c) If Kittu earlier purchased it for Rs.20000 what percentage profit did she earn?
  - d) If Kittu incurred a loss of 20% in the deal, at what price had Kittu purchased it?
  - e) \* If Ankit wants to sell it making a profit of one-fifths of the selling price, at what price should he sell it?
  
2. Answer the following questions on the basis of the data given below:  
VJ Sales purchases plasma TVs in bulk at Rs. 60000 each. However the marked price on a plasma TV at the VJ Sales showroom is Rs. 63000.
  - a) What percentage profit does VJ Sales make on plasma TVs if it is sold at the marked price?
  - b) If VJ Sales offers a discount of 5% on each TV, what percentage profit/loss will they make?
  - c) If VJ Sales raises the marked price by 20% and then offers a discount of 20% on each TV, what will be the percentage change in their selling price?
  - d) If VJ Sales wants to give a discount of 10% on plasma TVs and yet earn 11% profit, what should be new marked price?
  
3. Answer the following questions:
  - a) Anish bought 30 watches at wholesale rate. He sold 20 of them at 10% profit and the rest at 20% profit. What percentage profit did he make on the whole?
  - b) Shashi offers one watch free on the purchase of every 4 watches. What is the effective percentage discount offered by him?
  - c) Ashish sold two watches to Ishan for Rs. 891 each. He made a profit of 10% on one watch and a loss of 10% on the other. How much profit or loss did he make on the whole?
  - d) Nisha sold 10 watches to Asha making a profit equal to the selling price of 2 watches. How much percent profit did she make in the deal?
  - e) Asha sold 10 watches to Isha making a profit equal to the cost price of 2 watches. What was her profit percentage in the deal?

- f) Isha sold some watches to Amisha such that the selling price of 12 watches was equal to the cost price of 10 watches. How much percentage profit or loss did she make in the deal?

**\* Faulty balance and weights:**

4. A shopkeeper professes to sell his goods at cost price. However:
- a) He uses a faulty balance that balances 960 gm of tomatoes with a 1 kg weight. What percentage profit or loss will he incur?
  - b) He uses a faulty weight of 960 gm in place of a 1 kg weight. What percentage profit or loss will he incur?
  - c) He uses a faulty balance that has one arm 4% longer than the other. What percentage profit can he earn?

### Simple & Compound Interest

Let P = Principal invested, R = Rate of interest (pcpa), N = Number of years,

I = Interest and A = Amount. Then,

At Simple Interest:

$$I = \frac{P \times N \times R}{100} \quad \text{and} \quad A = P + I$$

At Compound interest:

$$A = \left(1 + \frac{R}{100}\right)^N \quad \text{and} \quad I = A - P$$

*For one year, SI is always equal to CI.*

*In case of CI, the interest obtained after one year gets re-invested and hence added to the principal*

1. After how many years will an amount of Rs.800 amount to Rs.1000 at simple interest at the rate of 5 pcpa?
2. After how many years will an amount get doubled at the rate of 5 pcpa at simple interest?
3. \* At what rate will an amount get tripled after 25 years at simple interest?
4. \* How much amount should I invest to receive an amount of Rs.100 as simple interest every month at the rate of 5pcpa?
5. Find the principal invested at the rate of 5% pa simple interest if an amount of Rs.1210 was obtained at the end of 2 years?
6. Find the difference between the simple and compound interest obtained on Rs.20000 invested at the rate of 10% pa after:
  - a) 1 year
  - b) 2 years
  - c) \* 2 years compounded half-yearly
7. \* Find the amount invested if the difference between the simple and compound interest obtained at the rate of 4% pa after 2 years is Rs.160.

## Challengers

### \*\* SET 1

1. I bought a book for Rs.60. Then I sold it off to a friend for Rs.70. But after a while I felt sorry that I sold it and bought it back for Rs.80. How much loss did I incur?
2. Matthew sold a watch to Joanne at Rs.20 profit. After a couple of days, Joanne sold the same watch back to Matthew at Rs.20 profit. Was it an even deal for Matthew?
3. Grandpa is writing a book. Every morning he starts writing vigorously and fills a lot of pages. But post-lunch he goes through all that he's written that far (right from day one) and deletes one-fifth of it. He doesn't write anything more that day. At the end of the day the content of his book is still 20% more than that at the end of the previous day. How much percent new content does Grandpa write in the morning?
4. A student wrote 3 hours 3 min as 3.3 hours. By what percent of an hour did he go wrong?
5. A can of juice was 80% full. 80% of the contents were emptied into a glass and 84ml juice was added to the can. Then the can became full to the brim. What is the capacity of the can?
6. Rushabh sold some CDs to Rushant at the rate of four CDs for the cost price of three. After a few days Rushant sold the same CDs back to Rushabh at the rate of 3 CDs for the price he had paid for four. Find the percentage loss incurred by Rushabh in the transaction.  
 1) 25%                      2) 56.25%                      3) 77.7 %                      4) 43.75

### SET 2

1. In a class 40% of the students are girls. If 50% of the boys and 60% of the girls pass the final exam, find the passing percentage of the class.
2. An agent offers a commission of 5% on the sale value if the sale to a single customer is upto Rs. 1000. Further, a commission of 10% on the sale value is offered if the sale to a single customer exceeds Rs. 1000. A salesman sold goods worth Rs.1500 to each of five customers and goods worth Rs.800 to each of four others. How much percent commission will he get on the whole?
3. Cotton Cottage displays the following offers on different types of kurtis:
  - o Type FL:                      25% discount
  - o Type HL:                      Buy 3 get 1 free
  - o Type SS:                      Buy 1 get 50% off on 2nd
  - o Type FS:                      Buy 5 get 2 free
 On which type is the percentage discount the greatest?
4. In a test, Calvin secured 40% marks and failed by 60 marks while Susie secured 60% marks and passed by 40 marks. What is the passing percentage?
5. In Norway, when the price of eggs dropped by 20%, 20 more eggs could be obtained for 20 Kroner. What was the price of one egg earlier?



## PRACTICE EXERCISE - 1

**DIRECTIONS for questions 1 to 10: Fill in the blanks.**

1. Bill sold a marble showpiece worth \$1000 to Ted at a profit of 10%. Ted then sold it to Benny at a loss of 10%. If Benny had bought the piece at the same price directly from Bill, then Bill would have incurred \_\_\_\_\_ (% profit/ % loss/ no profit no loss).
2. If Ethel lends \$250 to Sam at the rate of \_\_\_\_\_ pcpa simple interest, then she will receive a sum of \$300 after 5 years.
3. If the price of onions drops by 25%, then we can increase our consumption by \_\_\_\_\_% without increasing our expenditure.
4. If the selling price of 16 pens is same as the cost price of 20 pens then the \_\_\_\_\_ (profit/loss) percentage is\_\_\_\_\_.
5. A supermarket offers a flat 10% discount on the sale of hosiery items and yet earns a profit of 17% on their sale. This is possible because it marks up the price of the items by \_\_\_\_\_%
6. \_\_\_\_\_ percent of an hour is a minute.
7. A shopkeeper sold 40% of his goods at 60% profit and the rest at 40% loss. On the whole he must have incurred \_\_\_\_\_ (% profit/ % loss/ no profit no loss).
8. By selling a T-shirt for Rs.750, Pierre Cardin earns a profit of 25%. If the price is reduced by 4%, the profit will drop to \_\_\_\_\_%
9. A vendor professes to sell vegetables at cost price. However he uses an 800 gm weight in place of a 1kg weight. Then he earns a profit of \_\_\_\_\_ %
10. Ryan purchased a cabinet worth Rs.3000. However he did not have enough money for a cash down payment so he opted for an option of Rs.600 down payment followed by 3 monthly instalments of Rs.840 each. Then he was charged simple interest at the rate of \_\_\_\_\_ pcpa.

**DIRECTIONS for questions 11 to 20: Choose the correct alternative.**

11. A certain piece of property having depreciated by Rs.2355, is now worth Rs.3925. What was its original value? By what per cent has it depreciated?
 

1) Rs.6280, 37.5%	2) Rs.6280, 30%
3) Rs.1570, 37.5%	4) None of these

12. A woman has a certain number of mangoes, of which 13% are bad. She gives 75% of the remainder to charity, and then has 261 left. How many did she have initially?  
1) 1300                      2) 1250                      3) 1200                      4) 610
13. A bookseller gets a discount of  $12\frac{1}{2}\%$  on a certain book, which is priced at Rs.8 per copy from the wholesaler. If he purchases 15 copies of the book, what is the total discount that he gets?  
1) Rs.17                      2) Rs.18                      3) Rs.14                      4) Rs.15
14. The population of two parts of a town i.e., the upper and the lower parts of a town were equal. After the former had fallen 20 per cent and the latter has risen by 15 per cent, the total number of inhabitants was 39390. What was the population of each part at first?  
1) 32825                      2) 20200                      3) 21435                      4) 76810
15. If in 4 years Rs.2520 amounts to Rs.3024, in what time at the same rate will Rs.520 amount to Rs.611 at simple interest?  
1) 3 yrs.                      2) 2.5 yrs.                      3) 3.5 yrs.                      4) 4 yrs.
16. If the price of electricity is decreased by 20% and Varun decides to spend 10% less on electricity. By how much percent does his consumption get affected?  
1) 12.5% decrease                      2) 10.25% decrease  
3) 11.75% increase                      4) 12.5% increase
17. Rakesh has applied for a loan of Rs.300000. He has two options, the first is a fixed rate of compound interest of 6% for three years and the other is a floating rate of 7% for the first year, 6% for the second year and 5% for the third year. If he repays the loan after three years, then which option is beneficial to Rakesh and by how much?  
1) Fixed Rate, Rs.318                      2) Floating Rate, Rs.318  
3) Floating Rate, Rs.31.8                      4) Fixed Rate, Rs.31.8
18. A person borrows two equal sums at the same time at 5 and 4 percent respectively and finds that if he repays the former sum with simple interest on a certain date 6 months before the latter on which simple interest is also charged, he will have to pay in each case the same amount, viz., Rs.1100. Find the sum of money borrowed.  
1) Rs.850                      2) Rs.1000                      3) Rs.995                      4) Rs.990
19. By selling 5 dozen mangoes for Rs.156 it was found that  $\frac{3}{10}$ th of the outlay was gained. What should the retail price per mango be in order to gain 60%?  
1) Rs.4                      2) Rs.2                      3) Rs.3.2                      4) Rs.4.2
20. A trader allows a discount of 5 percent to his customers. What price should he mark on an article, the cost price of which is Rs.800 so as to make a clear profit of 25 percent on his outlay?  
1) Rs.1000                      2) Rs.1053                      3) Rs.1200                      4) Rs.1123





## PRACTICE EXERCISE - 2

**DIRECTIONS for questions 1 to 16: Choose the correct alternative**

1. Find the sum of money, which at  $3\frac{3}{4}$  percent for 3 years and 4 months will yield the same simple interest as Rs.160 at  $2\frac{1}{2}$  % for 2 years and 6 months.  
 1) Rs.95                      2) Rs.62                      3) Rs.75                      4) Rs.80
2. I bought a scooter at a certain price. Its value depreciates every year at the rate of 10%. If its value at the end of 3 years is Rs.21600, approximately what was its cost when I purchased it?  
 1) Rs.29630                      2) Rs.30000                      3) Rs.25000                      4) None of these
3. A sells a camera to B at a profit of 20% and B sells it to C at a profit of 10%. C sells it to D at a loss of Rs.120. If the difference between A's C.P. and D's C.P. is Rs.200, what is the cost of the camera to A?  
 1) Rs.500                      2) Rs.1000                      3) Rs.1500                      4) None of these
4. Mark up percent is the percentage increase in C.P. such that the C.P. (after the increase) becomes equal to the marked price. A man makes a profit of 4.15% after giving a discount of 3.84% on his marked price. Find his mark up percent.  
 1) 8.04%                      2) 8.43%                      3) 8.2%                      4) 8.33%
5. In a party, 55 persons were present (men and women). 40% of the women wore earrings (consider 2 ear rings per woman) and the remaining 60% wore goggles. Among the men, 50% wore goggles. Total number of goggles in the party were 1.5 times the total number of earrings. Find the total number of earrings in the party assuming that only women wear earrings at the party.  
 1) 10                      2) 16                      3) 20                      4) 24
6. 40% of the students in a school are boys. In a particular school examination, attended by all the students, only 20% of the girls passed, though 40% of all the students had passed. Find the percentage of boys who passed the test.  
 1) 30                      2) 50                      3) 70                      4) 28
7. At a particular examination, Seema got 16 more marks than Meena, who got 10% points less than Beena. Also Beena scored 2% points more than Teena. If Teena had a score of 500 out of 600, what percentage of marks did Seema get? (All percentage are expressed as marks scored as a percentage of total marks in the examinations)  
 1) 85                      2) 80                      3) 82                      4) 78

8. In a town in France, women wear dresses having either 1 or 2 or 3 colours. The number of women having at least 2 colours on their dress is 70% of the female population. If the difference between the number of women wearing 2 and 3 coloured dresses respectively is 1400 and 1200 women wear 1 coloured dresses, find the total female population of the town.
- 1) 4000                      2) 3500                      3) 3200  
4) 2400                      5) Data Insufficient
9. The marks obtained by a student in English, Maths, Science and Hindi in standard X are as follows (Maximum marks per subject = 100):
- The marks obtained in Maths are 1.5 times the marks obtained in English.
  - He got a total of 64% in these four subjects.
  - He got the maximum and minimum marks in Science and Hindi respectively, with a difference of 48 marks between them.
  - An addition of 50% of the marks obtained in English to the final score gives an overall percentage of 70%.
- What would be his percentage if only Maths and Science Marks are counted?
- 1) 80                      2) 82                      3) 84                      4) 86
10. A purchased a car from a showroom at Rs.5 lakh. He sold to B at a loss of 15% who sold to C at a loss of 10% who sold it to D at a loss of 18% who sold it to E at a profit of 11%. Find E's C.P.
- 1) Rs.348151.5              2) Rs.347052.6              3) Rs.362315.1              4) Rs.352802.4
11. A shopkeeper buys certain goods. He sells one-third of them at 20% profit and three-eighth of the remaining at 20% loss. What should be the mark-up on the remaining goods so that after giving a discount of 20% on them while selling, the shopkeeper gets a profit of 10% on overall transaction?
- 1) 50%                      2) 40%                      3) 25%                      4) 20%
12. A merchant bought some watches at Rs.6000 per dozen and an equal number at Rs.15000 per 25 pieces. He sold them at Rs.17000 per 25 pieces thus making a total profit of Rs.39000. How many watches did he buy?
- 1) 300                      2) 30                      3) 450                      4) 50
13. A manufacturer marks his goods in advance at 80 percent more than the cost price, but he allows 15 articles to the dozen and also 10 percent discount for cash. What rate of profit on his outlay does he obtain from a customer who pays cash?
- 1) 18.2%                      2) 25%  
3) 29.6%                      4) Cannot be determined

14. Equal sum of money is invested for three years in two different schemes.  
Scheme I pays by compound interest at rate of 10% p.a for three years.  
Scheme II pays by simple interest at rate of 10% p.a for the first two years and rate of interest changes for 3rd year.  
If the returns on the investments are equal in both the schemes, find the rate at which the interest is paid on the sum of money invested in scheme II for the third year.  
1) 13.1%                      2) 12.1%                      3) 13%                      4) Cannot be determined.
15. A careless shopkeeper sells his goods at the Cost Price, uses a weight of 1.2 kg, instead of using a weight of 1 kg to sell an item. What is his overall loss percentage?  
1)  $66\frac{2}{3}\%$                       2)  $16\frac{2}{3}\%$                       3) 83.33%                      4) None of these
16. If the price of sugar increases by 50%, by how much should a family reduce the consumption of sugar so that the increase in the expenditure on sugar is 20%?  
1) 25%                      2) 20%                      3) 30%                      4) 40%

**Directions for Questions 17 and 18: Solve as directed.**

17. One merchant correctly calculates his percentage profit on the cost price; another wrongly calculates it on the selling price. Find the difference in actual profits if both claim to make  $17\frac{1}{2}\%$  profit on goods sold at Rs.3760.
18. If a certain principal amounts to Rs. 6272 in 2 years and to Rs. 7024.64 in 3 years at the same rate of compound interest, calculate the principal amount (in Rs.).

**Directions for Questions 19 and 20: Choose the correct alternative.**

19. If the cost price of 5 cricket bats is equal to the selling price of 4 cricket bats, what is the profit/loss percent?  
1) Loss of 20%                      2) Loss of 25%                      3) Profit of 20%                      4) Profit of 25%
20. A seller sells two diamonds of different sizes at different selling prices. The selling price of the larger diamond is more than the selling price of the smaller diamond. The seller makes 20% profit on selling the smaller diamond and 10% loss on selling the larger diamond. He makes an overall loss of  $1\frac{9}{11}\%$  on selling the two diamonds. What is the ratio of the selling price of the larger diamond to the selling price of the smaller diamond?  
1) 3:2  
2) 2:1  
3) 4:3  
4) More information is needed to answer this question



## PRACTICE EXERCISE - 3

**DIRECTIONS for questions 1 and 2: Solve as directed.**

1. If the ratio of the compound interest (compounded annually) to the simple interest earned on a principal at the same rate of interest per annum for 3 years is 331:300, what is the rate of interest per annum?
2. The price of a house depreciates by 12.5% per annum. When the price was Rs. 80 lakh, Rosesh invested a sum of Rs. 40 lakh at 12.5% per annum simple interest. What is the minimum number of years (integer value) that it will take for him to be able to buy the house using his investment?

**Directions for Questions 3 to 13: Choose the correct alternative.**

3. Sher Singh invested a sum of Rs. P on a deposit scheme for a year. At the end of the year, the interest obtained was Rs. 105. For the next year, he invested the sum Rs. P, the interest obtained in the first year and additional Rs. 45 in another deposit scheme such that by the end of the second year, his investment grew to Rs. 1765.50. If the rate of interest per annum in both the schemes is the same, which of the following can be the rate of interest?  
1) 4%                      2) 5%                      3) 5.5%                      4) 7%
4. After selling a certain quantity of fruits, Mohanlal realized that he was selling at a loss of 15%. He then sold the remaining fruits that he had at 20% profit such that he ended up with 0 net profit and 0 net loss. If he sold 40 kg of fruits on loss, how many kgs of fruits did he sell on profit?  
1) 10 kg                      2) 20 kg                      3) 30 kg                      4) 40 kg
5. Find the approximate value of x if the Selling Price of a commodity after three successive discounts of 10% each on its marked price is equal to its Selling Price after two successive discounts of x% each on the marked price.  
1) 15                      2) 14.62                      3) 15.38                      4) 13.80
6. The rate of inflation was 1000%. What will be the cost of an article, which costs 6 units of currency now, two years from now?  
1) 666                      2) 660                      3) 720                      4) 726  
**(Past CAT question)**
7. If equal number of people are born on each day, find the approximate percentage of the people whose birthday will fall on 29th February. (If we are to consider people born in the 20th century and assume no deaths.)  
1) 0.374%                      2) 0.5732%                      3) 0.0684%                      4) 0.124%  
**(Past CAT question)**

8. A student took five papers in an examination, where the full marks were the same for each paper. His marks in these papers were in the proportion of 6 : 7 : 8 : 9 : 10. In all the papers together, the candidate obtained 60% of the total marks. Then the number of papers in which he got more than 50% marks is:

1) 2                      2) 3                      3) 4                      4) 5

(Past CAT question)

9. A college has raised 75% of the amount it needs for a new building by receiving an average donation of Rs. 600 from the people already solicited. The people already solicited represent 60% of the people whom the college will ask for donations. If the college is to raise exactly the amount needed for the new building, what should the average donation from the remaining people to be solicited be?

1) Rs. 300                      2) Rs. 250                      3) Rs. 400                      4) Rs. 500

(Past CAT question)

10. Fresh grapes contain 90% water by weight while dried grapes contain 20% water by weight. What is the weight of dry grapes available from 20 kg of fresh grapes?

1) 2 kg                      2) 2.4 kg                      3) 2.5 kg                      4) None of these

(Past CAT question)

11. The owner of an art shop conducts his business in the following manner: Every once in a while he raises his prices by X%, then a while later he reduces all the new prices by X%. After one such up-down cycle, the price of a painting decreased by Rs. 441. After a second up-down cycle the painting was sold for Rs. 1,944.81. What was the original price of the painting?

1) 2756.25                      2) 2256.25                      3) 2500                      4) 2000

(Past CAT question)

12. 40% of the employees of an organization are men. Of these 75% earn more than Rs. 25,000 per year. If 45% of the total employees of the company earn more than Rs. 25,000 per year, then what fraction of the women earn more than Rs. 25,000 per year?

1)  $\frac{2}{11}$                       2)  $\frac{1}{4}$                       3)  $\frac{1}{3}$                       4)  $\frac{3}{4}$

(Past CAT question)

13. If a seller gives a discount of 15% on retail price, she still makes a profit of 2%. Which of the following ensures that she makes a profit of 20%?

1) Give a discount of 5% on retail price.  
2) Give a discount of 2% on retail price.  
3) Increase the retail price by 2%.  
4) Sell at retail price.

(Past CAT question)

**Directions for Questions 14 and 15: Refer to the data below and answer the questions that follow.**

A company purchases components A and B from Germany and USA, respectively. A and B form 30% and 50% of the total production cost. The current gain is 20%. Due to change in the international scenario, the cost of the German Mark increased by 30% and that of the US Dollar increased by 22%. Due to market conditions the selling price cannot be increased beyond 10%. Then,

**(Past CAT question)**

14. What is the maximum current gain possible?  
 1) 10%                      2) 12.5%                      3) 0%                      4) 7.5%
15. If the US Dollar becomes cheaper by 12% over its original cost and the cost of the German Mark increases by 20%, and the selling price is not altered, what will be the gain?  
 1) 10%                      2) 20%                      3) 15%                      4) 7.5%

**Directions for Questions 16 to 19: Choose the correct alternative.**

16. A man earns  $x\%$  on the first 2000 rupees and  $y\%$  on the rest of his income, If he earns Rs.700 from Rs.4000 and Rs.900 from an income of Rs.5000, find  $x$  :  
 1) 20                      2) 15                      3) 25                      4) None of these.

**(Past CAT question)**

17. Instead of a metre scale, a cloth merchant uses a 120 cm scale while buying, but uses an 80 cm scale while selling the same cloth. If he offers a discount of 20% on cash payment, what is his overall percentage of profit?  
 1) 20%                      2) 25%                      3) 40%                      4) 15%

**(Past CAT question)**

18. A person who has a certain amount with him goes to the market. He can buy 50 oranges or 40 mangoes. He retains 10% of the amount for taxi fares and buys 20 mangoes and with the balance he purchases oranges. The number of oranges he can purchase is \_\_\_\_\_.  
 1) 36                      2) 40                      3) 15                      4) 20

**(Past CAT question)**

19.  $\frac{2}{5}$  of the voters promised to vote for P and the rest promised to vote for Q. Of these, on the last day, 15% of the voters went back on their promise to vote for P and 25% of the voters went back on their promise to vote for Q, and P lost by 2 votes. Then the total number of voters is \_\_\_\_\_.  
 1) 100                      2) 110                      3) 90                      4) 95

**(Past CAT question)**

**DIRECTIONS for question 20: Solve as directed.**

20. Arun's present age in years is 40% of Barun's. In another few years, Arun's age will be half of Barun's. By what percentage will Barun's age increase during this period?

**(Past CAT question)**

## QA-1.6 | RATIO, PROPORTION, AND VARIATION



### THEORY

#### Ratio

The ratio of two quantities of the same kind is the fraction that one quantity is of the other. In other words, to say how many times a given number is in comparison to another number, or to say what part a given number is in comparison to another, is to state the ratio that holds between them.

The ratio of two terms 'a' and 'b' is denoted by  $a : b$  (read as 'a is to b') and is measured by  $\frac{a}{b}$  or  $a \div b$ . The numerator 'a' is called the 'antecedent' and the denominator 'b' is called the 'consequent'.

#### While comparing two quantities in terms of ratio:

- (i) the two quantities must be of the same kind.
- (ii) the units of measurement of the two quantities must be the same.

#### Example

if  $a = \text{Rs.}2$  and  $b = 50$  paise, then  $a : b = 200 : 50 = 4 : 1$ .

- (iii) the ratio is a pure number, i.e., without any unit of measurement as it denotes how many times a given quantity is in comparison to another.
- (iv) the ratio would stay unaltered even if both the antecedent and the consequent are multiplied or divided by the same non-zero number.

#### Example

$$a : b = \frac{a}{b} = \frac{a \times m}{b \times m} = \frac{a / m}{b / m}.$$

#### Compounding of ratios

1. If two different ratios (say  $a : b$  and  $c : d$ ) are expressed in different units, then, if we require to combine these two ratios, we do so through **compounding**.

Here, the compound of  $a : b$  and  $c : d$  is  $\frac{a \times c}{b \times d}$ , i.e.,  $ac : bd$ .

#### Example

To complete a certain job, 1 man works for 4 hours a day for 5 days; 1 boy works for 6 hours a day for 10 days. We want to know how many boys are required to do the same job that 1 man can do?

To know this, we need to combine two different units, hours and days.

The ratio of days worked is  $5 : 10$ , the ratio of hours worked is  $4 : 6$ .

The compound ratio (day-hours) is  $\frac{5 \times 4}{10 \times 6} = 20 : 60 = 1 : 3$

So, 1 man can do the job that requires 3 boys.

2. The **duplicate** ratio of  $a : b$  is  $a^2 : b^2$ .
3. The **triplicate** ratio of  $a : b$  is  $a^3 : b^3$ .
4. The **subduplicate** ratio of  $a : b$  is  $\sqrt{a} : \sqrt{b}$ .
5. The **subtriplicate** ratio of  $a : b$  is  $\sqrt[3]{a} : \sqrt[3]{b}$ .

### Order of ratios

To determine which of the two given ratios  $a : b$  and  $c : d$  is greater or smaller, we compare  $a \times d$  and  $b \times c$ . For  $b > 0$  and  $d > 0$ :

1. If  $a \times d > b \times c$ , then  $\frac{a}{b} > \frac{c}{d}$ .
2. If  $a \times d < b \times c$ , then  $\frac{a}{b} < \frac{c}{d}$ .
3. If  $a \times d = b \times c$ , then  $\frac{a}{b} = \frac{c}{d}$ .

### Ratio of Equality, Greater Inequality or Lesser Inequality

1. A ratio of equality is unaltered, a ratio of greater inequality is diminished and a ratio of lesser inequality is increased if the same positive quantity is added to both its terms.

Let  $\frac{a}{b}$  be the given ratio and  $x$  be a positive quantity and  $x > 0$ .

- (i) If  $\frac{a}{b} = 1$ , then  $\frac{a+x}{b+x} = \frac{a}{b} = 1$ .
- (ii) If  $\frac{a}{b} > 1$ , then  $\frac{a+x}{b+x} < \frac{a}{b}$ .
- (iii) If  $\frac{a}{b} < 1$ , then  $\frac{a+x}{b+x} > \frac{a}{b}$ .

2. A ratio of equality is unaltered, a ratio of greater inequality is increased and a ratio of lesser inequality is diminished if some positive quantity not greater than the smaller term be subtracted from each of its terms.

Let  $\frac{a}{b}$  be the given ratio,  $x$  be a positive quantity and  $x < b$ .

- (i) If  $\frac{a}{b} = 1$ , then  $\frac{a-x}{b-x} = \frac{a}{b} = 1$ .
- (ii) If  $\frac{a}{b} > 1$ , then  $\frac{a-x}{b-x} > \frac{a}{b}$ .
- (iii) If  $\frac{a}{b} < 1$ , then  $\frac{a-x}{b-x} < \frac{a}{b}$ .

### Properties of ratios

1. If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{b}{a} = \frac{d}{c}$ , i.e., the inverse ratios of two equal ratios are equal. The property is called **invertendo**.



2. If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a}{c} = \frac{b}{d}$ , i.e., the ratios of antecedents and consequents of two equal ratios are equal. This property is called **alternendo**.

3. If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a+b}{b} = \frac{c+d}{d}$ , i.e., adding 1 to both sides.

This property is called **componendo**.

4. If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a-b}{b} = \frac{c-d}{d}$ , i.e., subtracting 1 from both sides.

This property is called **dividendo**.

5. If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ , i.e., dividing the results of componendo by dividendo.

This property is called **componendo-dividendo**.

#### 6. Equal Ratio

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}; \text{ each ratio} = \frac{\text{Sum of Numerators}}{\text{Sum of Denominators}} = \frac{a+c+e\dots}{b+d+f\dots}$$

The principle can also be applied after multiplying the Numerator and Denominator of any fraction by the same number.

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots, \text{ then each ratio is equal to } \left( \frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots} \right)^{\frac{1}{n}}, \text{ where } p, q, r, n..$$

may have any values except that they must not all be zeros.

### SOLVED EXAMPLES

**Q :** What is the least integer which when subtracted from both the numerator and denominator of  $\frac{60}{70}$  will give a ratio equal to  $\frac{16}{21}$ ?

**A :** Let  $x$  be the required integer.

$$\therefore \frac{60-x}{70-x} = \frac{16}{21}$$

$$\therefore 1260 - 21x = 1120 - 16x$$

$$5x = 140$$

$$x = 28$$

**Q :** If  $\frac{x}{y} = \frac{4}{5}$ , find the value of  $\frac{3x+4y}{4x+3y}$ .

**A :** 
$$\frac{3x+4y}{4x+3y} = \frac{\frac{3x}{y} + 4}{\frac{4x}{y} + 3} = \frac{3 \times \frac{4}{5} + 4}{4 \times \frac{4}{5} + 3} = \frac{32}{31}$$

**Q :** Find the greatest of three natural numbers which are in the ratio 3 : 2 : 5, if the sum of their squares is 1862.

**A :** If the numbers are 3x, 2x and 5x, then  $9x^2 + 4x^2 + 25x^2 = 1862$ .

$$\therefore x^2 = 49 \quad \therefore x = 7$$

The greatest number = 5x = 35.

**Q :** The monthly salaries of two persons are in the ratio 3 : 5. If each receives an increase of Rs.20 in the salary, the ratio is altered to 13 : 21. Find their respective salaries.

**A :** Let the salaries be 3x and 5x.

$$\frac{3x+20}{5x+20} = \frac{13}{21}$$

$$\therefore 21(3x+20) = 13(5x+20)$$

$$\therefore 63x + 420 = 65x + 260 \quad \therefore x = 80$$

$\therefore$  The required salaries are Rs.240 and Rs.400.

**Q :** Divide Rs.2430 among A, B and C so that if their shares are diminished by Rs.5, Rs.10, Rs.15 respectively, the remainders are in the ratio 3 : 4 : 5.

**A :** Divide  $2430 - (5 + 10 + 15) = 2400$  in the ratio 3 : 4 : 5.

$$\therefore \text{A's share} = \frac{3}{12} \times 2400 + 5 = \text{Rs.605};$$

$$\therefore \text{B's share} = \frac{4}{12} \times 2400 + 10 = \text{Rs.810};$$

$$\therefore \text{C's share} = \frac{5}{12} \times 2400 + 15 = \text{Rs.1015}.$$

**Q :** What is the least integer which when added to both integers of the ratio 5 : 9 will make a ratio greater than 7 : 10?

**A :** Let the two integers be  $5k$  &  $9k$ .

Let the least integer to be added =  $x$

$$\therefore \frac{5k+x}{9k+x} > \frac{7}{10} \quad \therefore 50k + 10x > 63k + 7x$$

$$\therefore 3x > 13k \quad \therefore x > \frac{13}{3}k.$$

$x$  will be the minimum if  $k$  is minimum.

$$\therefore x > \frac{13}{3} \times 1 \quad \therefore x > \frac{13}{3}.$$

The least integer greater than  $\frac{13}{3}$  is 5.

**Q :** The prime cost of an article was 3 times the value of materials used. The cost of raw materials increases in the ratio of 3 : 7 and that of productive wages in the ratio 4 : 9.

Find the present prime cost of an article which could formerly be made for Rs.18.

**A :** Prime cost =  $3x$ , if value of material =  $x$ .

$$3x = 18 \quad \therefore \text{Value of material} = x = 6.$$

$$\therefore \text{Productive wages} = 18 - 6 = 12$$

$$\text{Present productive wages} = 12 \times \frac{9}{4} = 27$$

$$\text{Present material cost} = 6 \times \frac{7}{3} = 14$$

$$\therefore \text{Present prime cost} = 14 + 27 = \text{Rs.41}.$$

**Q :** If  $\frac{a}{b+c-a} = \frac{b}{c+a-b} = \frac{c}{a+b-c}$ , find the value of each ratio.

$$\mathbf{A :} \text{ Each ratio} = \frac{\text{Sum of Numerators}}{\text{Sum of Denominators}} = \frac{a+b+c}{a+b+c} = 1.$$

**Q :** In an alloy of zinc, tin and lead, the quantity of zinc is  $\frac{3}{4}$ th that of tin and the quantity of tin is  $\frac{4}{5}$ th that of lead. How much of each metal will there be in 12 kg. of the alloy?

**A :** The alloy contains zinc, tin and lead in the ratio 3 : 4 : 5. If zinc is  $3x$ , tin is  $4x$  and lead is  $5x$ .

$$\therefore 12x = 12$$

$$\therefore x = 1$$

$$\therefore \text{zinc} = 3\text{kg.}, \text{tin} = 4 \text{ kg. and lead} = 5 \text{ kg.}$$

**Q :** If  $\frac{x^2 + 3x - 5}{3x - 5} = \frac{x^2 + 2x - 7}{2x - 7}$ , find  $x$ .

**A :**  $\frac{(x^2 + 3x - 5) - (3x - 5)}{3x - 5} = \frac{(x^2 + 2x - 7) - (2x - 7)}{2x - 7}$  ... (by dividendo)

$$\frac{x^2}{3x - 5} = \frac{x^2}{2x - 7}$$

If  $x = 0$ , then LHS = RHS

If  $x \neq 0$ , then  $3x - 5 = 2x - 7 \therefore x = -2 \Rightarrow x = 0$  or  $x = -2$ .

**Q :** Find the value of  $\frac{x+a}{x-a} + \frac{x+b}{x-b}$ , if  $x = \frac{2ab}{a+b}$ .

**A :**  $x = \frac{2ab}{a+b}; \frac{x}{a} = \frac{2ab}{a(a+b)}; \frac{x+a}{x-a} = \frac{3b+a}{b-a}$  ... (componendo-dividendo)

$$\text{Similarly, } \frac{x}{b} = \frac{2a}{a+b}; \frac{x+b}{x-b} = \frac{3a+b}{a-b}$$

$$\therefore \frac{x+a}{x-a} + \frac{x+b}{x-b} = \frac{3b+a}{b-a} + \frac{3a+b}{a-b} = -\frac{(3b+a)}{a-b} + \frac{3a+b}{a-b} = \frac{2a-2b}{a-b} = 2$$

### Concept Builder 1

- The sides of a triangle are in the ratio  $\frac{1}{3} : \frac{1}{4} : \frac{1}{5}$  and its perimeter is 94 cms. What is the length of the smallest side?
- If  $\frac{a^3 + b^3}{a^3 - b^3} = \frac{76}{49}$ , find the value of  $a : b$ .
- In a ratio, which is equal to  $5 : 6$ , if the consequent is 42, then the antecedent is:
- The salaries of A, B and C are in the ratio  $1 : 2 : 3$ . If the increments of 20%, 10% and 5% are allowed, respectively in their salaries, then what will be the new ratio of their salaries?
- The average age of three girls is 24 years and their ages are in the ratio  $3 : 4 : 5$ . The age of the oldest girl is:

### Answer Key

1. 24 cms. 2.  $\frac{3}{5}$  3. 35 4. 24 : 44 : 63. 5. 30

### Proportion

When the ratio of two terms is equal to the ratio of two other terms, then these four terms are said to be in **proportion**, i.e., if  $a : b = c : d$ , then  $a, b, c$  and  $d$  are in proportion. It is also represented as  **$a : b :: c : d$**

$a, b, c$  and  $d$  are called the first, second, third and fourth proportionals respectively. The terms  $a$  and  $d$  are called the **extremes**, while  $b$  and  $c$  are called the **means**.

If  $a : b = c : d$ , then  $ad = bc$ , i.e., the product of extremes is equal to the product of the means.

### Mean Proportion

When  $\frac{a}{b} = \frac{b}{c}$ , then  $a, b$  and  $c$  are said to be in continued proportion and  $b$  is called the **mean proportional** between  $a$  and  $c$ .

Also,  $b^2 = a \times c$ , i.e.,  $b = \sqrt{ac}$

$b$  is also called the geometric mean of  $a$  and  $c$ . The concept of geometric mean will be covered in detail in the next chapter of this module.

### Third Proportional

If  $a : b = b : c$  then  $c$  is said to be the third proportional to  $a$  and  $b$ .

If 3 quantities are proportional, then the ratio of the 1<sup>st</sup> to the 3<sup>rd</sup> is the duplicate ratio of the 1<sup>st</sup> to the 2<sup>nd</sup>.

i.e. For  $a : b :: b : c \Rightarrow a : c = a^2 : b^2$

### Continued Proportion

If  $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \dots$ , then  $a, b, c, d, e, \dots$  are said to be in continued proportion.

The concept of continued proportion is useful in the following situation.

1. If four quantities  $a, b, c$  and  $d$  are such that  $a : b, b : c$  and  $c : d$  are known then

we can find  $a : d$  as  $\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$

2. If  $a, b, c$  and  $d$  are four quantities and  $a : b, b : c, c : d$  are known, then we can find  $a : b : c : d$ .

#### Example

If  $a : b = 2 : 3; b : c = 4 : 5, c : d = 7 : 9$ , find  $a : b : c : d$ .

$a : b = 8 : 12, b : c = 12 : 15$  (LCM of 3 and 4 is 12)

$\therefore a : b : c = 8 : 12 : 15$

$a : b : c = 56 : 84 : 105, c : d = 105 : 135$  (LCM of 7 and 15 is 105)

$\therefore a : b : c : d = 56 : 84 : 105 : 135$

## SOLVED EXAMPLES

**Q :** Divide Rs.581 among A, B and C such that four times A's share is equal to 5 times B's share which is equal to seven times C's share.

**A :** 4 times A's share = 5 times B's share = 7 times C's share

$$\frac{\text{A's share}}{35} = \frac{\text{B's share}}{28} = \frac{\text{C's share}}{20} \dots (\text{dividing by LCM of 4, 5 and 7, i.e., 140})$$

$$\therefore A : B : C = 35 : 28 : 20$$

$$\therefore \text{Share of A} = \frac{35}{35 + 28 + 20} \times 581 = \text{Rs.245}$$

$$\text{Share of B} = \frac{28}{83} \times 581 = \text{Rs.196}$$

$$\text{Share of C} = \frac{20}{83} \times 581 = \text{Rs.140}$$

**Q :** A sum of Rs.3115 is divided among A, B and C, so that if Rs.25, Rs.28 and Rs.52 be diminished from their respective shares, the remainders will be in the ratio 8 : 15 : 20. Find the share of each.

**A :** (A's share - 25) : (B's share - 28) : (C's share - 52) = 8 : 15 : 20

$$\therefore \frac{\text{A's share} - 25}{8} = \frac{\text{B's share} - 28}{15} = \frac{\text{C's share} - 52}{20} = k \text{ (say)}$$

$$\therefore \text{A's share} - 25 = 8k \quad \therefore \text{A's share} = 8k + 25$$

$$\text{Similarly, B's share} = 15k + 28 \text{ and C's share} = 20k + 52$$

$$\therefore 8k + 25 + 15k + 28 + 20k + 52 = 3115 \quad \therefore k = 70$$

$$\therefore \text{A gets Rs.585, B gets Rs.1078 and C gets Rs.1452.}$$

**Q :** A, B and C play cricket. A's runs are to B's runs and B's runs are to C's runs as 3 : 2 each. Altogether, they score 342 runs. How many runs does each make?

**A :** A : B = 3 : 2 = 9 : 6; B : C = 3 : 2 = 6 : 4 ... (making B equal)

$$\therefore A : B : C = 9 : 6 : 4 \quad \therefore \text{Runs made by A} = \frac{9}{9 + 6 + 4} \times 342 = 162$$

$$B = \frac{6}{19} \times 342 = 108$$

$$C = \frac{4}{19} \times 342 = 72$$

**Q :** If  $ab$ ,  $bc$ ,  $x$  and  $c^2$  are in proportion, find  $x$ .

**A :**  $\frac{ab}{bc} = \frac{x}{c^2} \quad \therefore x = ac.$

**Q :** The sum of two numbers is 10 and their geometric mean is 4. Find the numbers.

**A :** If one number is  $x$ , the other number is  $10 - x$ .

$$(GM)^2 = x(10 - x); \quad 4^2 = 10x - x^2$$

$$\therefore x^2 - 10x + 16 = 0$$

$$\therefore (x - 8)(x - 2) = 0 \Rightarrow x = 8 \text{ or } x = 2$$

$\therefore$  The numbers are 8 and 2.

### Concept Builder 2

- Rs.252 is divided among A, B and C such that for every Rs.2 that A receives, B receives Rs.5 and for every Rs.3 that B receives, C receives Rs.7. Find A's share.
- The third proportional to  $(x^2 - y^2)$  and  $(x - y)$  is:
- If  $A : B = 4 : 5$  and  $B : C = 4 : 5$ , then  $A : C$  is:
- A sum of Rs.350 is divided as Rs.A, Rs.B, Rs.C and Rs.D among four people such that  $12A = 15B = 20C = 30D$ . What is the value of  $(A + C) - (B + D)$ ?
- The ratio of the incomes of A and B is  $5 : 4$  and the ratio of their expenditures is  $3 : 2$ . If at the end of the year, each saves Rs.1600, then the income of A is:
- An amount of Rs.735 was divided between A, B and C. If each of them had received Rs.25 less, then their shares would have been in the ratio of  $1 : 3 : 2$ . The money received by C was:

### Answer Key

6. Rs. 245

5. Rs.4000

4. Rs. 50

3.  $\frac{25}{16}$

2.  $\frac{x - y}{x + y}$

1. Rs.27.

## Variation

### Direct Proportion (Variation)

If two quantities 'a' and 'b' are related, such that any increase or decrease in 'b' produces a proportionate increase or decrease in 'a' or vice-versa, then the two quantities are said to be in direct proportion.

In other words, the ratio of 'a' and 'b' is a constant.

The statement 'y' varies as 'x' is symbolically written as  $y \propto x$ .

If  $y \propto x$ , then  $y = kx$ , where  $k$  is a non-zero constant called the constant of variation.

$y = kx$  is called the equation of direct variation.

### Inverse Proportion (Variation)

If two quantities 'a' and 'b' are related, such that any increase in 'a' would lead to a decrease in 'b' or any decrease in 'a' would lead to an increase in 'b', then the two quantities 'a' and 'b' are said to be in inverse proportion.

The statement 'y' varies inversely as 'x' is symbolically written as  $y \propto \frac{1}{x}$ .

If  $y \propto \frac{1}{x}$ , then  $y = \frac{k}{x}$ , i.e.,  $xy = k$ , where  $k$ , a non-zero constant, is the constant of variation.

$xy = k$  is called the equation of indirect variation.

### Mixed Variation

1. In this type we have both a direct and inverse relationship between the variables involved. If a variable  $x$  is dependent on two variables  $y$  and  $z$  and varies inversely as  $y$  (when  $z$  is constant) and directly as  $z$  (when  $y$  is constant), the relationships can be written as

$$x \propto \frac{1}{y} \text{ and } x \propto z$$

We can combine the two as

$$x \propto \frac{z}{y} \text{ or } x = \frac{kz}{y}, \text{ where } k \text{ is a non-zero constant of proportionality.}$$

### Example

If the square of  $x$  varies as the cube of  $y$ , and  $x = 2$ , when  $y = 3$ , find the equation between  $x$  and  $y$ .

$$x^2 \propto y^3; x^2 = ky^3$$

$$4 = 27k; \therefore \frac{4}{27} = k; \therefore x^2 = \frac{4}{27}y^3; \therefore 27x^2 = 4y^3$$



2. Another type of mixed variation is when the variable  $x$  can be expressed as a sum of a constant  $c$  and a variable  $y$  (to which it is directly proportional) i.e.,  
 $x = c + ky$

### Example

The expenses of a boarding house are partly fixed and partly variable as the number of boarders. The charge is Rs.70 per head when there are 25 boarders and Rs.60 per head when there are 50 boarders. Find the charge per head when there are 100 boarders.

Expenses = Constant + Variable

$\therefore E = m + kb$ , where  $m$  and  $k$  are constant.

$\therefore 1750 = m + 25k; 3000 = m + 50k$

Subtracting the first equation from the second,

$\therefore 25k = 1250; k = 50, m = 500 \quad \therefore E = 500 + 50b$

$\therefore$  Expense when there are 100 boarders =  $E = 500 + 50 \times 100 = \text{Rs.}5500$

$\therefore$  Charge per head = Rs.55.

### Important results of proportion

- If  $A \propto B$  and  $B \propto C$ , then  $A \propto C$
- If  $A \propto C$  and  $B \propto C$ , then  $A \pm B \propto C$  and  $\sqrt{AB} \propto C$
- If  $A \propto BC$ , then  $B \propto \frac{A}{C}$  and  $C \propto \frac{A}{B}$
- If  $A \propto B$  and  $C \propto D$ , then  $AC \propto BD$
- If  $A \propto B$ , then  $A^n \propto B^n$
- If  $A \propto B$ , then  $AP \propto BP$ , where  $P$  is any quantity, variable or constant.

### Applications of Ratios & Proportion in Numbers

- Given the sum of two unknown variables, the product of the two variables say  $(a$  and  $b)$  is highest when they are equal.

#### Example

Given  $a + b = 10$

Possible values of  $(a, b)$  are  $(0, 10) (1, 9) (2, 8) \dots (5, 5) \dots (8, 2) \dots (10, 0)$

$0 \times 10 = 0, 1 \times 9 = 9, 2 \times 8 = 16, 5 \times 5 = 25$

$\therefore$  the product of the two numbers is highest when they are equal.

- Given that two variables are inversely proportional to each other i.e., the product of the two variables is constant, the sum of the two variables will be least when they are equal.

**Example**

Given  $a \times b = 100$

Possible values of  $(a, b)$  are  $(1, 100)$   $(2, 50)$   $(4, 25)$  .....  $(10, 10)$  .....  $(20, 5)$  .....  $(100, 1)$

$1 + 100 = 101$ ,  $2 + 50 = 52$ ,  $4 + 25 = 29$ ,  $10 + 10 = 20$

$\therefore$  the sum of the two numbers is least when they are equal.

3.  $a^x \times b^y \times c^z$  is maximum when  $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$  provided  $a + b + c$  is constant.

**Example**

Find the maximum value  $a^3 \times b^2$ , where  $a + b = 10$

Now, the product will be maximum when  $\frac{a}{3} = \frac{b}{2} \Rightarrow 2a = 3b \Rightarrow 2a - 3b = 0$  ..... (i)

Also given  $a + b = 10$  ..... (ii)

Solving (i) and (ii) we get  $a = 6$  and  $b = 4$

Maximum value of  $a^3 \times b^2 = 6^3 \times 4^2 = 3456$

**SOLVED EXAMPLES**

**Q :** The cost of a diamond varies as the square of its weight. A diamond weighing 10 decigrams costs Rs.1600. Find the loss incurred when it breaks into two pieces whose weights are in the ratio 2:3.

**A :** If  $C$  is the cost and  $W$  is the weight,  $C \propto W^2$ ;  $\therefore C = KW^2$

When  $C = 1600$ ,  $W = 10$ ;  $K = \frac{C}{W^2} = \frac{1600}{100} = 16$ ;  $\therefore C = 16W^2$

The weights of the two pieces (in decigrams) are:  $\frac{2}{5} \times 10$  and  $\frac{3}{5} \times 10$ , i.e., 4 and 6.

$\therefore$  Cost of the two pieces are  $16 \times 4^2$  and  $16 \times 6^2$ ,

$\therefore$  Total cost =  $256 + 576 = \text{Rs.}832$

$\therefore$  Loss incurred =  $1600 - 832 = \text{Rs.}768$

**Q :** If  $z$  varies as  $px + y$  and if  $z = 3$ , when  $x = 1$  and  $y = 2$ , and  $z = 5$ , when  $x = 2$  and  $y = 3$ , find  $p$ .

**A :**  $z = k(px + y)$ , where  $k$  is a constant. From the two conditions given,

$\therefore 3 = k(p + 2)$  and  $5 = k(2p + 3)$

$\therefore \frac{3}{5} = \frac{k(p+2)}{k(2p+3)}$ ;

$\therefore 3(2p + 3) = 5(p + 2) \therefore 6p + 9 = 5p + 10 \therefore p = 1$

**Q :** If  $x$  varies directly as  $y$  and inversely as  $z$ , and if  $x = a$  when  $y = b$  and  $z = c$ , find  $x$  when  $y = b^2$  and  $z = c^2$ .

**A :**  $x \propto \frac{y}{z}$ ;  $\therefore x = \frac{ky}{z}$ , where  $k$  is a constant.

$$\therefore a = k \times \frac{b}{c}; \therefore k = \frac{ac}{b}$$

$$\text{When } y = b^2 \text{ and } z = c^2, x = \frac{ky}{z} = \frac{ac}{b} \times \frac{b^2}{c^2} = \frac{ab}{c}$$

### Concept Builder 3

1. If the cost of a diamond varies directly with the square of its weight and if the weight of the diamond is 5 grams, then the cost is Rs.1500. What will be the cost of 12 grams diamond?
2. The electricity bill of a certain establishment is partly fixed and partly varies as the number of units of electricity consumed. When in a certain month 480 units are consumed, the bill is Rs.1600. In another month when 550 units are consumed, the bill is Rs.1810. In another month when 600 units are consumed, the bill for that month would be:

**Answer Key**

1. Rs.8640
2. 1960

## Partnership

**Partnership** is an association of two or more persons who invest their money in order to carry on a certain business. A partner who manages the business is called the working partner and the one who simply invests the money is called the Sleeping Partner or the Dormant Partner. Partnership is based on the concept of Ratio and Proportion.

### Simple & Compound Partnership

If the capitals of the partners are invested for the same period, the partnership is called **simple**, and if for different periods, is called **compound**.

#### About Simple Partnership

If the period of investment is the same for each partner, then the profit or loss is divided in the ratio of their investments.

1. If A and B are partners in a business, then:

$$\frac{\text{Investment of A}}{\text{Investment of B}} = \frac{\text{Profit of A}}{\text{Profit of B}} \text{ OR } \frac{\text{Investment of A}}{\text{Investment of B}} = \frac{\text{Loss of A}}{\text{Loss of B}}$$

2. If A, B and C are partners in a business, then:

$$\text{Investment of A : Investment of B : Investment of C}$$

$$= \text{Profit of A : Profit of B : Profit of C OR Loss of A : Loss of B : Loss of C}$$

#### About Compound Partnership

**Monthly Equivalent Investment** is the product of the capital invested and the period for which it is invested.

If the period of investment is different, then the profit or loss is divided in the ratio of their Monthly Equivalent Investment.

1. If A and B are partners in a business, then,

$$\frac{\text{Monthly Equivalent Investment of A}}{\text{Monthly Equivalent Investment of B}} = \frac{\text{Profit of A}}{\text{Profit of B}}$$

$$\text{i.e., } \frac{\text{Investment of A} \times \text{Period of Investment of A}}{\text{Investment of B} \times \text{Period of Investment of B}} = \frac{\text{Profit of A}}{\text{Profit of B}} \text{ OR}$$

$$\frac{\text{Monthly Equivalent Investment of A}}{\text{Monthly Equivalent Investment of B}} = \frac{\text{Loss of A}}{\text{Loss of B}}$$

$$\text{i.e., } \frac{\text{Investment of A} \times \text{Period of Investment of A}}{\text{Investment of B} \times \text{Period of Investment of B}} = \frac{\text{Loss of A}}{\text{Loss of B}}$$

2. If A, B and C are partners in a business, then,

$$\text{Monthly Equivalent Investment of A : Monthly Equivalent Investment of B :}$$

$$\text{Monthly Equivalent Investment of C} = \text{Profit of A : Profit of B : Profit of C}$$

OR

$$\text{Monthly Equivalent Investment of A : Monthly Equivalent Investment of B}$$

$$: \text{Monthly Equivalent Investment of C} = \text{Loss of A : Loss of B : Loss of C}$$

### SOLVED EXAMPLES

**Q :** Three partners A, B and C invest Rs.1600, Rs.1800 and Rs.2300 respectively in a business. How should they divide a profit of Rs.399?

**A :** Profit is to be divided in the ratio 16 : 18 : 23

$$\text{A's share of profit} = \frac{16}{16+18+23} \times 399 = \frac{16}{57} \times 399 = \text{Rs.112}$$

$$\text{B's share of profit} = \frac{18}{57} \times 399 = \text{Rs.126}$$

$$\text{C's share of profit} = \frac{23}{57} \times 399 = \text{Rs.161}$$

**Q :** A, B and C enter into a partnership. A advances Rs.1200 for 4 months, B gives Rs.1400 for 8 months and C Rs.1000 for 10 months. They gain Rs.585 altogether. Find the share of profit each.

**A :** Monthly Equivalent Investment of A =  $1200 \times 4 = \text{Rs.4800}$

Monthly Equivalent Investment of B =  $1400 \times 8 = \text{Rs.11200}$

Monthly Equivalent Investment of C =  $1000 \times 10 = \text{Rs.10000}$

Profit is divided in the ratio 48 : 112 : 100, i.e., 12 : 28 : 25

$$\text{A's share of profit is } \frac{12}{65} \times 585 = \text{Rs.108}$$

$$\text{B's share of profit is } \frac{28}{65} \times 585 = \text{Rs.252}$$

$$\text{C's share of profit is } \frac{25}{65} \times 585 = \text{Rs.225}$$

**Q :** Three persons A, B, C rent the grazing of a park for Rs.570. A puts 126 oxen in the park for 3 months, B puts in 162 oxen for 5 months and C puts in 216 oxen for 4 months. What part of the rent should each person pay?

**A :** Monthly Equivalent Rent of A =  $126 \times 3 = \text{Rs.378}$

Monthly Equivalent Rent of B =  $162 \times 5 = \text{Rs.810}$

Monthly Equivalent Rent of C =  $216 \times 4 = \text{Rs.864}$

$\therefore$  Rent is to be divided in the ratio 378 : 810 : 864, i.e., 7 : 15 : 16

$\therefore$  A would have to pay  $\frac{7}{7+15+16} = \frac{7}{38}$  of the rent.

∴ B would have to pay  $\frac{15}{38}$  of the rent.

∴ C would have to pay  $\frac{16}{38}$  i.e.,  $\frac{8}{19}$  of the rent.

**Q :** A, B and C enter into a partnership by investing 1500, 2500 and 3000 rupees respectively. A, as manager gets one-tenth of the total profit and the remaining profit is divided among the three in the ratio of their investment. If A's total share is Rs.369, find the shares of B and C.

**A :** If total profit is x, A's share is  $\frac{1}{10}x + \frac{3}{14}$  of the balance  $\frac{9}{10}x$

$$\therefore \frac{1}{10}x + \frac{27}{140}x = \text{Rs.}369; \therefore 14x + 27x = 369 \times 140$$

$$\therefore x = \frac{369 \times 140}{41} = 9 \times 140 = 1260; \therefore \text{B's share} = \frac{5}{14} \times \frac{9}{10} \times 1260 = \text{Rs.}405$$

$$\text{C's share} = \frac{6}{14} \times \frac{9}{10} \times 1260 = \text{Rs.}486$$

**Q :** Three men A, B and C subscribe Rs.4700 for a business. A subscribes Rs.700 more than B and B Rs.500 more than C. How much will each receive out of a profit of Rs.423?

**A :** If C subscribes Rs.x, then,

B subscribes Rs.(x + 500) and A subscribes Rs.(x + 1200)

$$\therefore 3x + 1700 = 4700; \therefore x = \text{Rs.}1000$$

$$\therefore \text{Ratio of profits of C, B and A} = 1000 : 1500 : 2200 \text{ i.e. } 10 : 15 : 22$$

$$\therefore \text{C's share of profit} = \frac{10}{47} \times 423 = \text{Rs.}90$$

$$\therefore \text{B's share of profit} = \frac{15}{47} \times 423 = \text{Rs.}135$$

$$\therefore \text{A's share of profit} = \frac{22}{47} \times 423 = \text{Rs.}198$$

**Q :** A and B start a business. A invests Rs.600 more than B for 4 months and B for 5 months. A's share is Rs.48 more than that of B, out of a total profit of Rs.528. Find the capital contributed by each.

$$\text{A : B's profit} = \frac{528 - 48}{2} = \text{Rs.}240;$$

$$\text{A's profit} = \text{Rs.}288$$

$$\frac{\text{A's capital} \times 4}{\text{B's capital} \times 5} = \frac{288}{240} = \frac{6}{5}; \therefore \frac{\text{A's capital}}{\text{B's capital}} = \frac{6}{5} \times \frac{5}{4} = \frac{3}{2}$$

$$\therefore \frac{\text{B's capital} + 600}{\text{B's capital}} = \frac{3}{2} \quad \therefore \text{B's capital} = \text{Rs.1200 and A's capital} = \text{Rs.1800.}$$

**Q :** A and B enter into a partnership. A puts in Rs.2000 but at the end of 3 months, withdraws Rs.500 and again at the end of 8 months withdraws Rs.300. Out of a total profit of Rs.900 at the end of the year, B's share was Rs.400. Find B's capital.

**A :** Ratio of profits = (A's Rs.2000 for 3 months) + (A's Rs.1500 for 5 months) + (A's Rs.1200 for 4 months) : (B's capital  $\times$  for 12 months) = (6000 + 7500 + 4800) :

$$12x = \frac{18300}{12x} = \frac{500}{400}; \quad \therefore \frac{1525}{x} = \frac{5}{4}; \quad \therefore x = \text{Rs.1220}$$

**Q :** Two partners invested Rs.1250 and Rs.850 respectively in a business. They decided to distribute equally 60% of the profit, and the remaining as the interest on their capital. If one receives Rs.320 more than the other, find the total profit.

**A :** If the total profit is  $100x$ , each gets  $30x$  as equal distribution.

Balance profit of  $40x$  is divided in the ratio of capital =  $1250 : 850 = 25 : 17$

$\therefore$  One partner gets  $\frac{25}{42} \times 40x$  and the other gets  $\frac{17}{42} \times 40x$

$\therefore$  This difference  $\frac{25}{42} \times 40x - \frac{17}{42} \times 40x = 320$ , i.e.,  $\frac{8}{42} \times 40x = 320$

$\therefore x = 42 \therefore$  Total profit = Rs.4200.

### Concept Builder 4

1. Sunil and Anil started a business and invested Rs.120000 and Rs.180000 respectively. After 3 months Anil left and Kapil joined by investing Rs.150000. At the end of the year, they realized a profit of Rs.148000. What was Kapil's share?
2. P, Q and R invest Rs.30000 for a business. P invests Rs.1000 more than Q and Q invests Rs.4000 more than R. Out of the total profit of Rs.18,000, R receives:
3. A, B and C started a business with their investments in the ratio 1 : 2 : 3. After 6 months, A invested the same amount as before and B and C withdrew half of their investments. The ratio of their profits at the end of the year is:
4. A, B and C enter into a partnership. They invest Rs.25000, Rs.50000 and Rs.75000, respectively. At the end of the first year, B withdraws Rs.25000 while at the end of the second year C withdraws Rs.50000. In what ratio the profits have to be shared at the end of 3 years?
5. P, Q and R hire a meadow for Rs.1560. If P puts in 12 cows for 15 days; Q puts 18 cows for 10 days and R puts 20 cows for 8 days, then the rent paid by Q is:
6. P and Q invest in a business in the ratio 5 : 3. If 10% of the total profit goes to charity and P's share is Rs.270, the total profit is:
7. Three partners A, B and C started a business. A's capital is equal to twice B's capital and B's capital is two-third times C's capital. Out of a total profit of Rs.14400 at the end of the year, C's share is:
8. Two partners X and Y start a business. Twice Y's capital is equal to thrice X's capital. Out of an annual profit of Rs.40,000 at the end of the year, X's share is:
9. If the profits earned by X, Y and Z are in the ratio of 3 : 5 : 9, respectively and the period for which each of them invested the amount in business is in the ratio of 3 : 4 : 6 for X, Y and Z, respectively, then what is the ratio of the investment of X, Y and Z?
10. Raju, Ravi and Ramu invest Rs.15000, Rs.25000 and Rs.30000, respectively in a business. After one year, Raju removed his money but Ravi and Ramu continued for one more year. If the net profit after 2 years is Rs.25200, then Raju's share in the profit is:
11. A, B and C started a business, two times of A's capital is equal to 3 times of B's capital and equal to 5 times of C's capital. Out of a profit of Rs.1550, A's share is:
12. Varun and Tarun started a business with capitals in the ratio of 3 : 4. After one year, Varun's profit is Rs.2100. Then what is the total annual profit?

### Answer Key

- |    |            |    |            |     |           |
|----|------------|----|------------|-----|-----------|
| 1. | Rs.60000   | 5. | Rs.540.    | 9.  | 4 : 5 : 6 |
| 2. | Rs.4200.   | 6. | Rs.480.    | 10. | Rs.3024.  |
| 3. | 2 : 2 : 3. | 7. | Rs.4800    | 11. | Rs. 750.  |
| 4. | 3 : 4 : 7. | 8. | Rs. 16000. | 12. | Rs.4900.  |

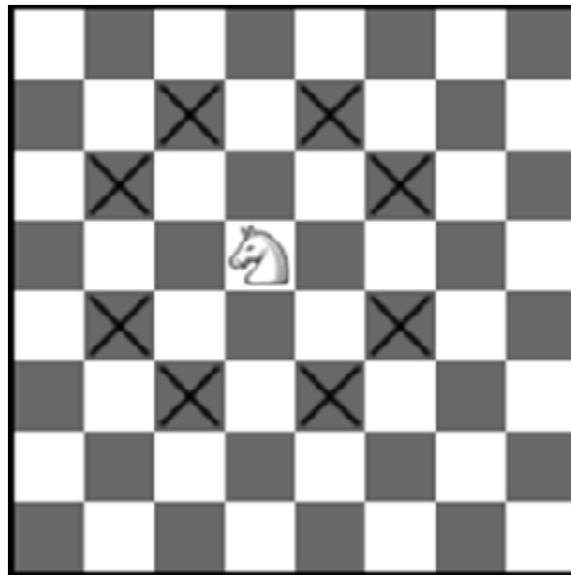




## CLASS EXERCISE

### Teaser

In a game of chess, the knight moves in a very specific manner; a knight's move consists of going 2 squares in one direction and then 1 square in a perpendicular direction (for example, as shown in the figure below, the knight in the position depicted can move to any of the squares marked with an X)



If a knight starts at the top left corner of the board, can it reach the square on the bottom right corner (using only legal moves), visiting every single square on the board exactly once?

### Ratio, Proportion and Variation

- 1) An ingot is made up of copper, zinc and tin. It contains copper and zinc in the ratio 3 : 5, while the ratio of copper and tin is 5 : 3. What is the fraction of tin in the ingot?
- 2) In his will, a man divided half his wealth among his sons Ravi and Shashi in the ratio 3 : 5 and the remaining half among his daughters Chitra and Devi in the ratio 7 : 9. If Chitra got Rs 6000 more than one of her brothers, then how much did Shashi get?
- 3) A bag contains 3 types of coins: nickels (worth 5 cents each), dimes (worth 10 cents each) and quarters (worth 25 cents each) in the ratio 5 : 3 : 2. If the total value of the dimes is 50 cents more than that of the nickels, then how many quarters does the bag contain?
- 4) The salaries of Radha and Sham were in the ratio 7 : 6 a year ago. Currently, each of them has received a raise of Rs 3000 and hence their salaries are now in the ratio 15 : 13. What is their combined salary currently?
- 5) How can Rs 5678 be divided among Prerna, Karna and Swarna such that if the share of each is reduced by Rs 26, the remaining amounts would be in the ratio 6 : 5 : 3?
- 6) 3 friends went to eat pizza. John ate  $\frac{1}{2}$  of a pizza, Mary ate  $\frac{1}{3}$  of a pizza and Sherrie ate  $\frac{1}{4}$  of a pizza. If the bill came to \$23.4, how much should each of them pay?
- 7) If  $\frac{a}{b} = \frac{c}{d}$  then prove that the following are also true:
  - a)  $\frac{2a + 5b}{2c + 5d} = \frac{4a - 7b}{4c - 7d}$
  - b)  $\frac{2a + 5c}{2b + 5d} = \frac{4a - 7c}{4b - 7d}$
  - c)  $\frac{bc}{d} - a = \sqrt{\frac{bcd}{a}} - d$
  - d)  $5a - 2b - 5c + 2d = \frac{(a - c)(5c - 2d)}{c}$
  - e)  $\frac{a^2d - b^2d}{b^2c - b^2d} = \frac{c + d}{d}$

- 8) If  $\frac{c}{d} = \frac{7}{4}$  then how much is  $\frac{3c^2 + 5d^2}{3c^2 - 5d^2}$  ?
- 9) If  $\frac{2p^3 + 3q^3}{2p^3 - 3q^3} = \frac{39}{15}$  then how much is  $\frac{p^2 + 2q^2}{p^2 - 2q^2}$  ?
- 10) If  $3m + 5n = 0$  then what is the value of  $\frac{m^3 + 5n^3}{m^3 - 5n^3}$  ?
- 11) 40 workers can dig 30 pits in 10 days, working 6 hours a day.
  - a) How many days will 20 workers take to dig 15 pits working 12 hours a day?
  - b) How many workers are required to dig 6 pits in 4 days working 5 hours a day?
- 12) I travelled from Kolhapur to Nashik in 6 hours at the speed of 77 kmph. By how much should I have increased my speed in order to reach half an hour earlier?
- 13) Rakesh, remained absent from work for 20 days in the month of April and earned Rs. 4000. How many days should he have remained absent in order to earn 20% more if he earns daily wages?
- 14) At constant temperature the pressure of a gas is inversely proportional to its volume. 1140 litres of a certain gas exerts a pressure of 750 units. If the pressure is increased by 10 units keeping the temperature constant, by how much will its volume decrease?
- 15) A fort has enough provision for its 250 soldiers for 10 weeks. If another 100 soldiers join then, how long will the provisions last?
- 16) If 12 men, working 8 hours a day, take 24 days to build a wall 120 m long, 6 m tall and 1 m thick, then how many days will 24 men working 6 hours a day will be required to build a wall 160 m long, 9 m tall and 0.5 m wide?
- 17) In a zoo, 3 lions eat as much food as 4 leopards, while 5 bears eat as much as 9 leopards. If they have enough food for 9 lions and 14 leopards for 30 days, how long will this food last if there are 12 lions, 10 bears and 5 leopards?
- 18) On Sunday morning, a bookstore has an opening stock of 480 copies of a best-selling book. Every day that week, during the day, exactly half the opening stock gets sold, Also, every day, at closing time, 16 additional copies arrive from the publisher. When the stock goes below 50, the computer system gives a warning and a new bulk order is placed. On which day will the warning be given?

If  $\frac{a}{b} = \frac{c}{d} = k$ , then we can also say that:

$$\frac{a}{c} = \frac{b}{d} \quad (\text{Alternendo})$$

$$\frac{b}{a} = \frac{d}{c} \quad (\text{Invertendo})$$

$$\frac{a+b}{b} = \frac{c+d}{d} \quad (\text{Componendo})$$

$$\frac{a-b}{b} = \frac{c-d}{d} \quad (\text{Dividendo})$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad (\text{Componendo and Dividendo})$$

$$\frac{a+c}{b+d} = k$$

$$\frac{pa+qc}{pb+qd} = k$$

$$\frac{pa^n+qc^n}{pb^n+qd^n} = k^n$$

- 19) \* Two numbers are taken such that their ratio is equal to  $7/8$ . After increasing the first by 3 and the second by 8, the ratio becomes  $3/4$ . Find the original numbers
- 20) \* The volume of a cylinder is directly proportional to its height and to the square of its radius. If the height is halved while the radius is tripled, what will be the percentage change in the volume of the cylinder?
- 21) \* If  $\frac{a}{b} = \frac{c}{d}$  then show that
- i)  $a + b + c + d = \frac{(a+b)(a+c)}{a}$       ii)  $\sqrt{\frac{(a+b)(c+d)}{bd}} = \frac{(a+b)c}{ad}$
- 22) \* In a particular volunteer job, 3 women can do the same work as 5 men or 6 children. Currently, the volunteers include 8 women, 10 men and 14 children, and the project is expected to last for 7 weeks. After two weeks, they get joined by 5 men, 8 women and 6 children. How much further time will be required?
- 23) \* The ratio of cows to goats on a farm is 3 : 4 and that of goats to hens is 5 : 6. If there are 27 more hens than cows, what is the total number of legs of all the livestock?
- 24) \* The radiation measured by a detector varies inversely as the square of its distance from the source of radiation. If at a distance of 2.4 m, the radiation from a source is measured at 375 units, how much will be the radiation recorded at a distance of 4 m?

## Partnership

- 25) Kane and Abel start a business investing Rs 7 lakh and Rs 9 lakh respectively. At the end of a year, they make a profit of Rs 3,20,000. How should they divide this amount?
- 26) Hull Inc. hired two consultants to work on a project. Brock worked for 35 days on the project, while Lauren worked for 50 days. If they were paid a total of Rs 1,19,000, how should they split the amount?
- 27) Suraj starts a company investing Rs 12 lakh. After 8 months, Chand joins him investing Rs 15 lakh. At the end of two years, the company realises a profit of Rs 13.2 lakh. How should it be apportioned?
- 28) Two visiting platoons eat at the army canteen for a while. The first one consists of thirty men who visit for 2 weeks, while the second consists of seventy men who visit for 9 days. How should the bill of Rs 20000 be divided?
- 29) On 1st January 2020, Rekha invested an amount 'a' to start a business. After some months, Nisha joined her, investing an amount 'b'. Rekha and Nisha divided up the entire profit at the end of 2020, in the ratio 6 : 5. At some point during 2021, they were joined by Neha, who invested an amount 'c'. At the end of 2021, Rekha, Nisha and Neha shared the entire profit for that year in the ratio 4 : 5 : 2. At some point during 2022, Nisha left to pursue an MBA. Simultaneously, Radhika joined the company investing an amount 'd'. At the end of 2022, Rekha, Nisha, Neha and Radhika shared the entire profit for that year in the ratio 6 : 5 : 12 : 3.
  - a) For how many months was Nisha a part of the business?
    - 1) 28 months
    - 2) 22 months
    - 3) 18 months
    - 4) Cannot be determined
  - b) If 'b' = Rs 10,000, how much is 'd' – 'a'?
    - 1) Rs. 2000
    - 2) Rs. 4000
    - 3) Rs. 6000
    - 4) Cannot be determined
  - c) Approximately on what date did Nisha leave?
    - 1) 1st July '22
    - 2) 1st August '22
    - 3) 1st September '22
    - 4) Cannot be determined
  - d) \*For how many months was Neha with the company, as of the end of 2022?
    - 1) 18 months
    - 2) 15 months
    - 3) 20 months
    - 4) Cannot be determined

- e) \*Approximately what percentage of the total profit over the three years did Nisha earn?
- |          |                         |
|----------|-------------------------|
| 1) 33.7% | 2) 32.7%                |
| 3) 31.7% | 4) Cannot be determined |
- f) \*If the remaining three continue the business for 2023, in what ratio will they share the profit?
- |              |                         |
|--------------|-------------------------|
| 1) 2 : 4 : 3 | 2) 4 : 5 : 6            |
| 3) 4 : 5 : 8 | 4) Cannot be determined |

### Challengers

- 1) If  $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b} = R$  and  $a + b + c \neq 0$ , then what can be said about  $R$ ?
- 1)  $0 \leq R < 0.33$       2)  $0.33 \leq R < 0.66$       3)  $0.66 \leq R < 1$       4)  $R \geq 1$

2)

Age	% of population
0-9	28
10-19	22
20-39	28
40-59	15
60 and above	7

The table shows the breakup of population by age in Bihar. It is known that the total number of people in Bihar aged 60 or above is 59.93 lakhs. If it is also known that, among the people in the age group 20 – 39, the ratio of illiterate to literate people is 2 : 3, then roughly how many people in this age-group (20 – 39) are illiterate?

- 1) 160 lakhs      2) 120 lakhs      3) 96 lakhs      4) 24 lakhs
- 3) When one becomes a member of the Pegasus library, one pays a certain initial cost Rs C, which allows one to issue the first 30 books for free. After that one has to pay Rs K per book issued. Harold became a member, issued 60 books and paid Rs 800. Helen also became a member, issued 50 books and paid Rs 700. If instead they had taken a single membership, how much would they have saved?
- 1) Rs 200      2) Rs 100      3) No saving      4) None of these
- 4) An engine is used to pull several railway coaches. The unloaded engine runs at a speed of 60 mph. The reduction in its speed is directly proportional to the square root of the number of coaches pulled. When 16 coaches are attached, the speed drops to 44 mph.
- a) What will be the speed when 36 coaches are attached?
- 1) 24 mph      2) 33 mph  
3) 36 mph      4) 40 mph
- b) What will be the maximum number of coaches which could be pulled by the engine such that the speed does not drop below 30 mph?
- 1) 49      2) 56  
3) 57      4) 64
- 5) An alchemist was trying to mix three elements – copper, silver and gold – according to an ancient recipe for making the “Philosopher’s Stone”. He mixed them in the ratio 5 : 7 : 9. But on a closer look, he realised that they were actually supposed to be mixed in the ratio 9 : 7 : 5. Consequently, he decided to add X grams of copper and Y grams of silver to the mixture to achieve the required ratio. What is X : Y?
- 1) 5 : 2      2) 5 : 1      3) 9 : 5      4) 2 : 1



**DIRECTIONS** for questions 1 to 18: Choose the correct alternative.

- The ratio of ages of Aman and Boman is 3 : 4. Five years hence, the ratio changes to 7 : 9. Find their current ages (in years).  
1) 15 and 20                  2) 60 and 80                  3) 3 and 4                  4) 30 and 40
- The price of a diamond varies according to the square of its weight. The price of gold is directly proportional to its weight. Three rings of equal weight, each composed of a diamond and gold are priced at a,b and c. The diamonds in them weigh 3, 4 and 5 gms respectively. What is the value of diamond of 1 gm?  
1)  $\frac{a+c+d}{2}$                   2)  $a + c - 2b$                   3)  $\frac{a+c}{2} - b$                   4) None of these
- The costs of three grades of apple, A, B and C, are Rs.5, Rs.6 and Rs.8 per kg respectively. In what proportion should they be mixed so that the price of the mixture may be Rs.7 per kg?  
1) 1 : 1 : 2    2) 1 : 1 : 3  
3) 1 : 2 : 3    4) Multiple answers are possible
- Three completely filled identical vessels, containing a soft drink, have fruit juice, soda and water in the ratio 1 : 3 : 5, 6 : 2 : 3 and 7 : 4 : 1. If all the contents of the three vessels are poured into an empty container, find the percentage of water in that container.  
1) 50                                  2) 30.4                                  3) 25.33                                  4) 28.125
- The wages of labourers in a factory increased in the ratio 22 : 25 and there was a reduction in their number in the ratio 15 : 11. Find the original wage bill if the present bill is Rs.5000.  
1) Rs.2500                          2) Rs.3000                          3) Rs.5000                          4) Rs.6000
- Annual incomes of A and B are in the ratio 4 : 3 and their expenses, as 3 : 2. If each saves Rs.600 at the end of the year, find the annual income of each.  
1) 1200, 900                      2) 1500, 1125                      3) 2400, 1800                      4) 3600, 2700
- The value of a silver coin varies directly as the square of its diameter when the thickness remains the same and directly as its thickness while the diameter is constant. Two silver coins have their diameters as 4 : 3. Find the ratio of their thickness if the value of the first is 4 times the value of the second.  
1) 4 : 3                                  2) 3 : 2                                  3) 2 : 1                                  4) 9 : 4



8. If the radius of a sphere is increased by 50%, then the ratio of the percentage increase in volume to the percentage increase in the surface area of the sphere is:  
 1) 2 : 1                      2) 10 : 19                      3) 19 : 10                      4) 1 : 2
9. A student gets an aggregate of 60% marks in five subjects in the ratio 10 : 9 : 8 : 7 : 6. If the passing marks are 50% of the maximum marks and each subject has the same maximum marks, in how many subjects did he pass the examination?  
 1) 2                      2) 3                      3) 4                      4) 5
- (Past CAT question)**
10. A student got as many marks in Physics as he got in Maths. The ratio of his marks in Biology and Chemistry was 5 : 7. The ratio of his marks in Biology and Maths is 3 : 2. If he gets an aggregate of 64% in all the four papers and each subject had equal maximum marks, in how many subjects has he scored more than 50% marks?  
 1) 1                      2) 2                      3) 3                      4) 4
11. The charges of a goldsmith is partly fixed and partly variable with the amount in grams of gold. The charge is Rs.300 for 20 grams of gold and Rs.550 for 45 grams of gold. Find the total charges for 100 grams of gold.  
 1) Rs.900                      2) Rs.1000                      3) Rs.1100                      4) Rs.9000
12. The rent of a mini bus hired for a trip is partly fixed and partly vary in direct proportion to the number of passengers. For a particular trip carrying 10 passengers, the rent turned out to be Rs.2200; for another trip with 20 passengers, the rent was Rs.3400. Find the fixed rent.  
 1) Rs.100                      2) Rs.1200                      3) Rs.1000                      4) Rs.120
13. Expenses at a boarding house are partly fixed and are partly varying on the number of boarders at the house. The average expense per boarder is Rs. 700 with 25 boarders. The average expense comes to Rs. 600 when the number of boarders is 50. What will be the average cost per boarder when the number of boarders is 100?  
 1) Rs.550                      2) Rs.560                      3) Rs.540                      4) Rs.570
- (Past CAT question)**
14. From each of the two given numbers, half the smaller number is subtracted. Of the resulting numbers the larger one is three times as large as the smaller. What is the ratio of the two numbers?  
 1) 2 : 1                      2) 3 : 1                      3) 3 : 2                      4) None
- (Past CAT question)**

15. Iqbal dealt some cards to Mushtaq and himself from a full pack of playing cards and laid the rest aside. Iqbal then said to Mushtaq. "If you give me a certain number of your cards, I will have four times as many cards as you will have.  
If I give you the same number of cards, I will have thrice as many cards as you will have". Of the given choices, which could represent the number of cards with Iqbal?
- 1) 9                      2) 31                      3) 12                      4) 35
16. The square of A varies directly as the cube of B. B varies directly as the square root of C. The value of C when A = 4 is 81. Find C when A = 3.
- 1)  $9\frac{4}{3}$                       2)  $\left(\frac{9}{4}\right)^{\frac{8}{3}}$                       3)  $(4.5)^{\frac{8}{3}}$                       4) Data Insufficient
17. I have one-rupee coins, 50-paisa coins and 25-paisa coins. The number of coins are in the ratio 2.5 : 3 : 4. If the total amount with me is Rs. 210, find the number of one-rupee coins.
- 1) 90                      2) 85                      3) 100                      4) 105  
(Past CAT question)
18. The cost of diamond varies directly as the square of its weight. Once, this diamond broke into four pieces with weights in the ratio 1 : 2 : 3 : 4. When the pieces were sold, the merchant got Rs. 70,000 less. Find the original price of the diamond.
- 1) Rs. 1.4 lakh                      2) Rs. 2 lakh                      3) Rs. 1 lakh                      4) Rs. 2.1 lakh  
(Past CAT question)

**DIRECTIONS for questions 19 and 20: Refer to the data below and answer the questions that follow.**

An airline has a certain free luggage allowance and charges for excess luggage at a fixed rate per kg. Two passengers, Raja and Praja have 60 kg of luggage between them, and are charged Rs 1200 and Rs 2400 respectively for excess luggage. Had the entire luggage belonged to one of them, the excess luggage charge would have been Rs 5400.

19. What is the weight of Praja's luggage?
- 1) 25 kg                      2) 30 kg                      3) 35 kg                      4) 40 kg  
(Past CAT question)
20. What is the free luggage allowance?
- 1) 10 kg                      2) 15 kg                      3) 20 kg                      4) 25 kg  
(Past CAT question)



## PRACTICE EXERCISE - 2

**DIRECTIONS for questions 1 to 5: Choose the correct alternative.**

1. If  $\frac{a}{b} = \frac{1}{3}$ ,  $\frac{b}{c} = 2$ ,  $\frac{c}{d} = \frac{1}{2}$ ,  $\frac{d}{e} = 3$  and  $\frac{e}{f} = \frac{1}{4}$ , then what is the value of  $\frac{abc}{def}$ ?

1)  $\frac{3}{8}$                       2)  $\frac{27}{8}$                       3)  $\frac{3}{4}$                       4)  $\frac{27}{4}$                       5)  $\frac{1}{4}$

(Past CAT question)

2. If  $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b} = r$  then  $r$  cannot take any value except

1)  $\frac{1}{2}$                       2)  $-1$                       3)  $\frac{1}{2}$  or  $-1$                       4)  $-\frac{1}{2}$  or  $-1$

(Past CAT question)

3. The funds raised during a 4-player special cricket match were Rs.20000. This amount was to be divided among the four players in the ratio of their runs scored. If the first three players scored 10, 5 and 5 runs, how many runs, at the minimum, should the fourth batsman try to score if he wants at least Rs.15000 as his share?

1) 65                      2) 55                      3) 58                      4) 60

4. Rs.79.20 is divided among 7 men, 11 women and 5 boys, so that each woman has thrice as much as each boy has and each man has as much as a woman and a boy together have. Find the share of a man.

1) Rs.1.20                      2) Rs.2.40                      3) Rs.4.80                      4) Rs.5

5. In his will, Melaram left Rs.70000 for his 3 sons Suresh, Dinesh and Hemant in the ratio 3 : 6 : 5. The three sons decided to start a business with the inheritance. Suresh put in  $\frac{2}{3}$ rd of his inheritance, while Dinesh invested  $\frac{2}{5}$ th of his inheritance. Hemant added 13 parts out of 25 from his inheritance. What was the ratio of Suresh's contribution to the total capital invested in the business ?

1) 2 : 7                      2) 1 : 3                      3) 3 : 8                      4) 12 : 35

**DIRECTIONS for questions 6 to 8: Refer to the data below and answer the questions that follow.**

Alphonso, on his death bed, keeps half his property for his wife and divide the rest equally among his three sons Ben, Carl and Dave. Some years later Ben dies leaving half his property to his widow and half to his brothers Carl and Dave together, shared equally. When Carl makes his will he keeps half his property for his widow and the rest he bequeaths to his younger brother Dave. When Dave dies some years later, he keeps half his property for his widow and the remaining for his mother. The mother now has Rs. 1,575,000.

6. What was the worth of the total property?  
 1) Rs. 30 lakh      2) Rs. 8 lakh      3) Rs. 18 lakh      4) Rs.24 lakh  
 (Past CAT question)
7. What was Carl's total share?  
 1) Rs. 4 lakh      2) Rs. 12 lakh      3) Rs. 6 lakh      4) Rs. 5 lakh  
 (Past CAT question)
8. What was the ratio of the property owned by the widows of the three sons, in the end?  
 1) 7 : 9 : 13      2) 8 ; 10 : 15      3) 5 : 7 : 9      4) 9 : 12 : 13  
 (Past CAT question)

**DIRECTIONS for questions 9 and 10: Refer to the data below and answer the questions that follow.**

The ratio of white collar to blue collar employees in a firm is 8 : 3. Also ratio of male employees to female employees is 7 : 4. It is observed that 60% of the white collar employees are males.

9. What is the ratio of female white collar employees to male blue collar employees?  
 1) 16 : 11      2) 24 : 11      3) 3 : 1      4) Data Insufficient
10. If there are 48 female blue collar employees, what is the difference between male white collar employees and female blue collar employees?  
 1) 288      2) 240      3) 220      4) Data Insufficient

**DIRECTIONS for questions 11 to 20: Choose the correct alternative.**

11. Two bags have certain number of mangoes. If half the mangoes from bag 1 are transferred to bag 2, the ratio of mangoes in bag 1 to bag 2 gets reversed. Now one-third and two-third of the mangoes from the first and the second bag are transferred to the second and first bag respectively. The ratio of mangoes in bag 1 to bag 2 now is:  
 1) 2 : 1      2) 2 : 2      3) 4 : 5      4) 5 : 4

12. Three different brands of diesel oils are mixed in the proportion of 4 : 5 : 7 by weight to obtain a volume of 1050 litres. Find the volume of the first brand of diesel oil if the densities are in the proportion 4 : 2 : 1. [Mass = Volume  $\times$  Density]
- 1) 100 litres                      2) 105 litres                      3) 250 litres                      4) 225 litres

13. According to a will, a property worth Rs.345 lakh was divided among three beneficiaries in the ratio  $\frac{1}{3} : \frac{1}{18} : \frac{1}{27}$ . Find the largest share?
- 1) 180 lakh                      2) 270 lakh                      3) 45 lakh                      4) 540 lakh

14. The table below shows the age-wise distribution of the population of Reposia. The number of people aged below 35 years is 400 million.

Age group	Percentages
Below 15 years	30.00
15 – 24	17.75
25 – 34	17.00
35 – 44	14.50
45 – 54	12.50
55 – 64	7.10
65 and above	1.15

If the ratio of females to males in the 'below 15 years' age group is 0.96, then what is the number of females (in millions) in that age group?

- 1) 82.8                      2) 90.8                      3) 80.0                      4) 88.0

(Past CAT question)

15. Three travelers are sitting around a fire, and are about to have a meal. One of them has 5 small loaves of bread, the second has 3 small loaves of bread. The third has no food, but has eight coins. He offers to pay for some bread. They agree to share the loaves equally among the three travelers, and the third traveler will pay 8 coins for his share of the eight loaves. All loaves were of the same size. The second traveler (who had three loaves) suggests that he should be paid three coins, and the first traveler be paid five coins. The first traveler says that he should get more than five coins. How much should the first traveler get?
- 1) 5                      2) 7                      3) 1                      4) None of these

(Past CAT question)

16. Three shepherds A, B and C rented a pasture for a year. A grazed 22 sheep for 4 months, B grazed 16 sheep for 8 months and C grazed 32 sheep for 6 months. If C's share of rent is Rs.600, the total rent for the year was:
- 1) Rs.3550                      2) Rs.2000                      3) Rs.1275                      4) Rs.675

17. Ram, Shyam and Ghanshyam invest Rs.2000, Rs.10000 and Rs.5000 to set up a stall. Ram gets 15% of the total profit for running the stall. If in a year, Ram gets Rs.500 less than Shyam and Ghanshyam together, then the total profit of that year is:
- 1) Rs.1000                      2) Rs.2000                      3) Rs.1500                      4) Rs.3000
18. A and B enter into a business as partners. As A runs the business, B offers him 18% of the total profit or Rs.3500, apart from his share of the profit. A prefers to keep Rs.3500. Had he accepted to keep 18% of the total profit, he would have got Rs.100 more. Find the total profit?
- 1) Rs.2000                      2) Rs.40000                      3) Rs.20000                      4) None of these
19. Ram and Seeta entered into a joint venture, investing in a ratio of 5 : 6. After 3 months, Ram withdrew  $\frac{1}{5}$ th of his money and Seeta withdrew  $\frac{1}{6}$ th of her money. Profit at the end of the year was Rs.22800. What will be Seeta's share?
- 1) Rs.12600                      2) Rs.63000                      3) Rs.11400                      4) None of these
20. A, B, C and D invested into a business, the ratio of their capitals being 7 : 12 : 14 : 10. If the ratio of their returns is 14 : 6 : 28 : 30, find the period (in years) for which D invested if B invested for 2 years.
- 1) 6                                  2) 2                                  3) 12                                  4) 3

## QA-1.7 | MEANS AND WEIGHTED AVERAGES THEORY

### Averages

A central value around which a group of values show a tendency to concentrate is called an **average**. Thus an average is a single value that is in some way indicative of a group of values.

### Measures of central tendency

#### Mode

Mode is the number that occurs most frequently in a given set of numbers.

#### Example

Find the mode of the set of numbers 2, 4, 6, 6, 5, 4, 3, 6, 2, 6, 4, 1, 4, 5, 4. The number 4 occurs maximum number of times, hence the mode is 4. In a given set of data, if two or more values occur the same number of times, then a unique mode does not exist.

#### Median

Median is the middle value of a group of numbers arranged in an ascending or descending order.

If the number of values (n) in a given set of data is odd, then the Median is the  $\left(\frac{n+1}{2}\right)$ th value.

#### Example

Find the median of the numbers 40, 41, 33, 48, 41, 39 and 38.

On arranging the values in ascending order 33, 38, 39, 40, 41, 41, 48.

Median is the  $\left(\frac{7+1}{2}\right)$ th value, the 4<sup>th</sup> value i.e., 40

If the number of values (n) in a given set of data is even, then there will be two middle values say a and b, the median is taken as  $\frac{a+b}{2}$ .

#### Example

Find the median of the numbers 16, 15, 13, 14, 23, 44.

On arranging the values in descending order 44, 23, 16, 15, 14, 13. The two middle values are 16 and 15.

$$\therefore \text{Median} = \frac{16+15}{2} = \frac{31}{2} = 15.5$$

**Empirical Relation between mean, mode and median: Mode = 3 Median – 2 Mean**

#### Arithmetic mean

The most commonly used average is the arithmetic mean (AM) or simply the average.

The arithmetic mean of n numbers  $x_1, x_2, x_3, \dots, x_n$  is denoted by  $\bar{x}$  and calculated as

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} \quad \bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

where the Greek letter  $\Sigma$  (sigma) indicates “Sum of the terms of the type”.  $i$  takes the value from 1 to  $n$ .

The arithmetic mean of  $a$  and  $b$  is  $\frac{a+b}{2}$ .

### Example

Find the arithmetic mean of 10, 15, 17, 20.

$$\bar{x} = \frac{10 + 15 + 17 + 20}{4} = 15.5$$

When large numbers are involved, an Arithmetic Mean is generally assumed to facilitate calculation. If  $a$  is the assumed Arithmetic mean, then the true Arithmetic mean  $\bar{x}$  of the group of  $n$  numbers

$x_1, x_2, x_3, \dots, x_n$  is  $\bar{x} = a + \frac{\sum_{i=1}^n d_i}{n}$  where  $d_i = x_i - a$  is the deviation from the assumed mean.

### Example

Find the Arithmetic mean of 42, 46, 48, 50, 55, 60, 70  
Let the assumed mean be 50

$$\sum_{i=1}^7 d_i = -8 - 4 - 2 + 0 + 5 + 10 + 20 = 21$$

$x_i$	$d_i$
42	$42 - 50 = -8$
46	$46 - 50 = -4$
48	$48 - 50 = -2$
50	$50 - 50 = 0$
55	$55 - 50 = 5$
60	$60 - 50 = 10$
70	$70 - 50 = 20$

$$\therefore \bar{x} = 50 + \frac{\sum_{i=1}^7 d_i}{7} = 50 + \frac{21}{7} = 50 + 3 = 53.$$

### Weighted Arithmetic Mean

If the values  $x_1, x_2, \dots, x_n$  are assigned weights  $w_1, w_2, \dots, w_n$  respectively, then

$$\text{Weighted Arithmetic Mean} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}$$

The weighted arithmetic mean denoted by  $\bar{x}_w$  is calculated as,  $\bar{x}_w = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$ .



The term 'weight' stands for the relative importance that is attached to the different values. 'w' can also be replaced by 'f' in a frequency distribution where 'f' denotes the frequency or simply the number of times a particular observation occurs.

### Example

$x_i$	$w_i$	$w_i x_i$
20	3	60
30	1	30
40	5	200
50	1	50

i)  $\sum w_i = 10 \quad \sum x_i w_i = 340 \quad \therefore \bar{x}_w = \frac{\sum_{i=1}^4 w_i x_i}{\sum_{i=1}^4 w_i} = \frac{340}{10} = 34$

- ii) The mean marks scored by 50 boys are 70 and those scored by 100 girls are 55. Find the average marks scored by this group.

$$\bar{x}_c = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2} = \frac{70 \times 50 + 100 \times 55}{150} = \frac{9000}{150} = 60$$

### Geometric mean

The geometric mean (G.M.) of  $n$  numbers  $x_1, x_2, \dots, x_n$  is the  $n$ th root of their products

i.e.,  $G.M. = \sqrt[n]{x_1 \times x_2 \times x_3 \times \dots \times x_n}$

### Example

- i) Geometric mean of 2, 3 and 4 is  $\sqrt[3]{2 \times 3 \times 4} = \sqrt[3]{24} = 2.88$ .

The geometric mean of  $a$  and  $b$  is  $\sqrt{ab}$ .

- ii) Find the Geometric mean of 9 and 4.

$$G.M. = \sqrt{9 \times 4} = 6$$

Geometric mean is generally used to calculate rate of growth.

If the geometric mean of one group of 'a' numbers is 'x' and that of another group of 'b' numbers

is 'y' then the geometric mean of the combined groups is  $(x^a y^b)^{\frac{1}{a+b}}$

### Harmonic Mean

The harmonic mean (H.M.) of  $n$  numbers  $x_1, x_2, \dots, x_n$  is calculated as  $\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$ . The harmonic

mean of  $a$  and  $b$  is  $\frac{2ab}{a+b}$ .

### Example

Find the Harmonic mean of 1, 0.5, 10.

$x_i$	$\frac{1}{x_i}$
1	1
0.5	2
10	0.1

$$\sum \frac{1}{x_i} = 3.1$$

$$HM = \frac{3}{3.1} = \frac{30}{31}.$$

This is an important result and is generally used to find the average speed when equal distances are covered at different speeds. (When different distances are covered in the same time, arithmetic mean is used to determine the average Speed).

### Example

A person travels from A to B at 6 kmph and returns at 4 kmph, find his average speed.

Average Speed = Harmonic Mean of 6 and 4

$$\frac{2 \times 6 \times 4}{6 + 4} = \frac{48}{10} = 4.8 \text{ kmph}$$

### Weighted Harmonic Mean

The Weighted Harmonic Mean of 3 numbers  $x_1$ ,  $x_2$  and  $x_3$  with weights  $w_1$ ,  $w_2$  and  $w_3$  respectively

is  $\frac{\sum_{i=1}^3 w_i}{\sum_{i=1}^3 \frac{w_i}{x_i}}$ . This is used to calculate the average speed when different distances are covered at different speeds.

### Example

A cyclist covers his first 5 km at an average speed of 10 kmph, another 3 km at 8 kmph and the last 2 km at 5 kmph. Find his average speed during the journey.

$$\text{Average Speed} = \text{Weighted HM} = \frac{5 + 3 + 2}{\frac{5}{10} + \frac{3}{8} + \frac{2}{5}} = \frac{10}{\frac{1}{2} + \frac{3}{8} + \frac{2}{5}} = \frac{400}{51} = 7.84 \text{ kmph}$$

**Note:** This concept will be again covered in Time, Speed and Distance.

### Relation between the three Means i.e., AM, GM, HM

- The Arithmetic Mean of two numbers  $a$  and  $b$ , is  $\frac{a+b}{2}$ , their Geometric Mean is  $\sqrt{ab}$  and their Harmonic Mean is  $\frac{2ab}{a+b}$ .  
(GM)<sup>2</sup> = AM  $\times$  HM

### Example

If the Arithmetic Mean of two numbers is 6.5 and their Geometric Mean is 6, what is their Harmonic Mean?

$$AM \times HM = (GM)^2 \therefore HM = \frac{6 \times 6}{6.5} = 5.54$$

2. Arithmetic Mean  $\geq$  Geometric Mean  $\geq$  Harmonic Mean

### Example

Arithmetic Mean of 2 and 4 is 3.

Geometric Mean of 2 and 4 is 2.82.

Harmonic Mean of 2 and 4 is 2.66

$$3 > 2.82 > 2.66$$

## Measures of Dispersion

Measures of dispersion gives an idea about the manner in which the values of the variables are dispersed or scattered around an average value.

### Range

It is the difference between the largest and smallest values of the observations in a distribution.

### Example

Find the range of the following distribution.

44, 25, 17, 88, 67, 82, 74

$$\text{Range} = 88 - 17 = 71$$

## Standard deviation (S.D.) & Variance (Var)

It is the positive square root of the arithmetic mean of the squares of all the deviations of the observations from their arithmetic mean. If the observations are  $x_1, x_2, x_3, x_4, \dots, x_n$  then,

$$\text{S.D.} = \sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \dots + (x_n - \bar{x})^2}{n}} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}$$

$$\text{For a frequency distribution table, } \sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

The square of the standard deviation is called the **variance**.

### Example

Find the standard deviation of the following numbers.

11, 17, 8, 25, 22, 34, 38, 12, 13, 71, 24

$$\bar{x} = \frac{275}{11} = 25 \text{ and } n = 11$$

x	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
11	-14	196
17	-8	64
8	-17	289
25	0	0
22	-3	9
34	9	81
38	13	169
12	-13	169
13	-12	144
71	46	2116
24	-1	1
		<hr/>
		$\sum (x_i - \bar{x})^2 = 3238$

$$\therefore \text{S.D.} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} = \sqrt{\frac{3238}{11}} = 17.16$$

### SOLVED EXAMPLES

**Q :** Find the Geometric Mean of 17, 10, 28, 35 and 3.

**A :** Geometric Mean of 17, 10, 28, 35 and 3 is

$$\sqrt[5]{17 \times 10 \times 28 \times 35 \times 3} = \sqrt[5]{499800} = 13.796.$$

**Q :** Find the Geometric Mean of the sequence 1, 2, 4, 8, 16 .....  $2^n$ .

**A :** The terms are  $1, 2^1, 2^2, 2^3, \dots, 2^n$ .

$$\text{Geometric Mean} = \sqrt[n+1]{1 \times 2^1 \times 2^2 \times 2^3 \times \dots \times 2^n}$$

$$= \sqrt[n+1]{2^{1+2+3+4+\dots+n}} = \sqrt[n+1]{2^{\frac{n(n+1)}{2}}} = 2^{\frac{n}{2}}$$

$$\dots \text{ as } 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

**Q :** The population of 4 towns is 35560, 30000, 27500 and 25600 respectively. What is the average population of a town?

**A :** Average =  $\frac{35560 + 30000 + 27500 + 25600}{4} = \frac{118660}{4} = 29665.$

**Q :** The average weight of 8 oarsmen in a boat is increased by 1 kg, when one of the crew, who weighs 60 kg is replaced by a new man. What is the weight of the new man?

**A :** The average weight of 8 men is increased by 1 kg. Therefore, the total weight is increased by  $8 \times 1 = 8$  kg. Hence, the weight of the new man is  $60 + 8 = 68$  kg.

**Q :** The average marks of three batches of students containing 70, 50 and 30 students respectively, are 50, 55 and 45. Find the average marks of all the 150 students.

**A :** Total marks of I batch =  $70 \times 50 = 3500$

Total marks of II batch =  $50 \times 55 = 2750$

Total marks of III batch =  $30 \times 45 = 1350$

Total marks = 7600

Hence average marks of 150 students  $\frac{7600}{150} = 50.67.$

**Q :** The following are the weekly incomes of 10 workers. Calculate the arithmetic mean income (Rs) : 115, 114, 123, 152, 105, 112, 150, 125, 120, 127.

**A :** Let 120 be the assumed Arithmetic Mean.

$$d_i = x_i - 120$$

So, deviations are: -5, -6, 3, 32, -15, -8, 30, 5, 0, 7

$$\text{Arithmetic Mean} = 120 + \frac{\sum_{i=1}^{10} d_i}{10}$$

$$\text{Arithmetic Mean} = 120 + \frac{43}{10} = 120 + 4.3 = 124.3$$

**Q :** A man employs 20 men, 15 women and 5 children and pays Rs.10 to each male, Rs.8 to each female, and Rs.4 to each child. Find the average wages per day.

**A :**

Wages ( $x_i$ )	Number of employees ( $f_i$ )	Total Wages ( $f_i w_i$ )
10	20	200
8	15	120
4	5	20

$$\sum f_i = 40 \quad \sum f_i x_i = 340$$

$$\text{Average wages per day} = \frac{340}{40} = \text{Rs.8.5 per day.}$$

**Q :** The average rainfall from Monday to Saturday is 0.3 inches, but the average for the whole week is 0.5 inches. What was the rainfall on Sunday?

**A :** Total rainfall from Monday to Saturday is  $6 \times 0.3 = 1.8$  inches

$$\text{Rainfall for the whole week} = 7 \times 0.5 = 3.5 \text{ inches}$$

$$\therefore \text{Rainfall on Sunday} = 3.5 - 1.8 = 1.7 \text{ inches.}$$

**Q :** Find two numbers whose Arithmetic mean is 12.5 and Geometric mean is 10.

**A :** Let the two numbers be a and b.

$$\therefore \frac{a+b}{2} = 12.5 \text{ i.e., } a + b = 25$$

$$\sqrt{ab} = 10 \text{ i.e., } ab = 100 \therefore a = 20, b = 5$$

**Q :** A plane flies along the four sides of a square ground at speeds of 100, 200, 300 and 400 kmph. Find the average speed.

$$\text{Average speed} = \frac{4}{\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400}} = \frac{4 \times 1200}{12 + 6 + 4 + 3} = 192 \text{ kmph}$$

**Note:** The concept in the above example will be again covered in the topic Time, Speed and Distance.

### Concept Builder 1

1. The weight of 5 persons are 45, 55, 50, 60 and 35. What is the average weight of a person?
2. The average of three numbers is 30. The first is twice of second and second is one-third of third number. Find the numbers.
3. The average weight of 8 men is reduced by 2 kg when one man weighing 80 kg is replaced by a new man. Find the weight of new man.
4. The average temperature of the first three days of a week is  $26.5^{\circ}\text{C}$  and that of the next three days is  $29^{\circ}\text{C}$ . If the weekly average is  $27.4^{\circ}\text{C}$ , what is the temperature of the last day?
5. If the average marks of 2 classes of 8<sup>th</sup> standard is 60.25 and the average marks of next 5 classes is 50 and average marks of one class is 31.5, then average marks of all 8<sup>th</sup> standard classes is:
6. In a shop, average weights of 3 red bags, 4 blue bags and 4 green bags are 27 kg, 32 kg and 30 kg respectively. Now there are 2 black bags, where the weight of the first bag is half of the second bag. If the average weight of all the 13 bags is 29 kg, then find the weight of the heavier black bag.
7. If arithmetic mean of 12, 10, 20.5, x, 13, 15, 10.5, 11 is 13, then find value of x.
8. The average of 5 consecutive odd numbers is 17. Find the largest number of the five.
9. The average temperature from 1<sup>st</sup> to 5<sup>th</sup> January is  $39^{\circ}\text{C}$  and from 2<sup>nd</sup> to 6<sup>th</sup> January is  $40^{\circ}\text{C}$ . The temperature recorded on 1<sup>st</sup> January is  $37^{\circ}\text{C}$ . The temperature on 6<sup>th</sup> January is:

### Answer Key

- |          |                |           |                             |           |
|----------|----------------|-----------|-----------------------------|-----------|
| 1. 49 kg | 2. 30, 15, 45. | 3. 64 kg. | 4. $25.3^{\circ}\text{C}$ . | 5. 50.25. |
| 6. 32 kg | 7. 12          | 8. 21     | 9. $42^{\circ}\text{C}$ .   |           |

## Mixtures

Mixtures is based on the concept of weighted average.

### Types of Mixtures

#### Simple Mixture

When two different ingredients are mixed together, it is known as a simple mixture.

#### Compound Mixture

When two or more simple mixtures are mixed together to form another mixture, it is known as a compound mixture.

### Alligation Rule

The alligation rule states that, “When different quantities of the same or different ingredients, of different costs (one cheap and the other dear) are mixed together to produce a mixture of a mean cost, the ratio of their quantities is inversely proportional to the difference in their cost from the mean cost.”

$$\frac{\text{Quantity of Cheap}}{\text{Quantity of Dear}} = \frac{\text{Price of Dear} - \text{Mean Price}}{\text{Mean Price} - \text{Price of Cheap}}$$

### Results based on Alligation rule

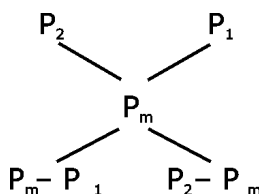
- For simple mixtures

If  $P_1$  and  $P_2$  are the prices of two quantities  $Q_1$  and  $Q_2$ , then the average price of the mixture, given by  $P_m$  is:

$$P_m = \frac{P_1 Q_1 + P_2 Q_2}{Q_1 + Q_2}$$

$$\frac{Q_1}{Q_2} = \frac{P_2 - P_m}{P_m - P_1}$$

This can be also represented as:



Required ratio of quantities of  $P_1$  and  $P_2$  is  $P_2 - P_m : P_m - P_1$

Similarly, when more than two ingredients are mixed:

$$P_m = \frac{P_1 Q_1 + P_2 Q_2 + P_3 Q_3 + \dots + P_n Q_n}{Q_1 + Q_2 + Q_3 + \dots + Q_n}$$



As can be seen the alligation rule is useful in:

- To find the mean value of a mixture when the prices of two or more ingredients, which are mixed together and the proportion in which they are mixed are given.
- To find the proportion in which the ingredients at given prices must be mixed to produce a mixture at a given price.

### Example

- A grocer mixes 28 kg of tea costing Rs.6.20 per kg with 12 kg costing Rs.8.40 per kg. Find the cost of the mixture per kg.

$$\text{Cost of the mixture per kg.} = \frac{6.20 \times 28 + 8.40 \times 12}{40} = \frac{173.60 + 100.80}{40} = \text{Rs.6.86}$$

- In what proportion must a grocer mix tea at Rs.1.02 per kg and Rs.1.44 per kg so as to make a mixture worth Rs.1.26 per kg?

$$\text{Using the alligation rule, } \frac{\text{Amount of cheap tea}}{\text{Amount of the dear tea}} = \frac{1.44 - 1.26}{1.26 - 1.02} = \frac{0.18}{0.24} = \frac{3}{4}$$

### 2. For compound mixtures

When two mixtures  $M_1$  and  $M_2$ , each containing ingredient A and B in the ratio  $a : b$  and  $x : y$ , respectively, are mixed, the proportion of the ingredients A and B i.e.,  $q_A : q_B$ , in the compound mixture is given by:

$$\frac{q_A}{q_B} = \frac{M_1 \times \left( \frac{a}{a+b} \right) + M_2 \times \left( \frac{x}{x+y} \right)}{M_1 \times \left( \frac{b}{a+b} \right) + M_2 \times \left( \frac{y}{x+y} \right)}$$

and, the quantity in which  $M_1$  and  $M_2$  is to be mixed when the quantity of A and B i.e.  $q_A$  and  $q_B$  in the compound mixture is known is given by:

$$\frac{\text{Quantity of } M_1}{\text{Quantity of } M_2} = \frac{\left( \frac{x}{x+y} \right) - \left( \frac{q_A}{q_A + q_B} \right)}{\left( \frac{q_A}{q_A + q_B} \right) - \left( \frac{a}{a+b} \right)}$$

**Example**

Vessel A contains milk and water in the ratio 4 : 5 and vessel B contains the same in the ratio 5 : 1. In what proportion should quantities be taken from A and B to form a mixture in which milk and water are in the proportion 5 : 4?

$$\begin{array}{ccc} \frac{4}{9} & & \frac{5}{6} \\ & \searrow \quad \nearrow & \\ & \frac{5}{9} & \\ & \nearrow \quad \searrow & \\ \frac{5}{18} & & \frac{1}{9} \end{array}$$

$$\text{Ratio of the quantities} = \frac{5}{18} : \frac{1}{9} = 5 : 2$$

Alternatively,

Let  $9x$  be taken from A and  $6y$  from B.

$$\text{Total milk taken from A and B} = \left(\frac{4}{9} \times 9x\right) + \left(\frac{5}{6} \times 6y\right) = 4x + 5y$$

$$\text{Total water taken} = 5x + 1y$$

$$\therefore \frac{4x + 5y}{5x + y} = \frac{5}{4}; \therefore 25x + 5y = 16x + 20y; \therefore 9x = 15y \therefore \frac{x}{y} = \frac{15}{9};$$

$$\therefore \text{Ratio of the quantities} = \frac{9x}{6y} = \frac{9}{6} \times \frac{15}{9} = \frac{5}{2} = 5 : 2$$

Alternatively,

$$\text{By using the formula, } \frac{\text{Quantity to be taken from A}}{\text{Quantity to be taken from B}} = \frac{\frac{5}{6} - \frac{5}{9}}{\frac{5}{9} - \frac{4}{9}} = \frac{5 \times 9 - 5 \times 6}{6 \times 9} \times 9 = \frac{15}{6} = \frac{5}{2}$$

3. When a mixture of three ingredients A, B and C is given, take any two ingredients such that cost of the mixture is between the costs of the two chosen ones and find the ratio. Once again, take two more ingredients and find their ratio. Then find the combined ratio. This will give an infinite number of solutions.
4. If a vessel contains 'a' litres of liquid A, and if 'b' litres be withdrawn and replaced by liquid B, then if 'b' litres of mixture be withdrawn and replaced by liquid B, and the operation repeated 'n' times in all, then:

$$\frac{\text{Liquid A left in vessel after } n^{\text{th}} \text{ operation}}{\text{Initial quantity of liquid A in vessel}} = \left(\frac{a-b}{a}\right)^n$$

$$\frac{\text{Liquid A left after } n^{\text{th}} \text{ operation}}{\text{Liquid B left after } n^{\text{th}} \text{ operation}} = \frac{\left(\frac{a-b}{a}\right)^n}{1 - \left(\frac{a-b}{a}\right)^n}$$

### Example

From a cask of wine containing 64 litres, 8 litres are drawn out and the cask is filled up with water. If the same process is repeated a second time, then a third time, how many litres of wine will be left in the cask? What will be the proportion of wine to water in the resulting mixture?

$$\frac{\text{Wine left in the cask after third operation}}{\text{Initial quantity of wine in the vessel}}$$

$$= \left( \frac{64-8}{64} \right)^3 = \left( \frac{56}{64} \right)^3 = \left( \frac{7}{8} \right)^3 = \frac{343}{512}$$

$$\therefore \text{Proportion of wine to water} = \frac{\frac{343}{512}}{1 - \frac{343}{512}} = \frac{343}{512 - 343} = \frac{343}{169}$$

### SOLVED EXAMPLES

**Q :** A tea merchant buys two kinds of tea, the price of the first kind being twice that of the second. He sells the mixture at Rs.14 per kg thereby making a profit of 40%. If the ratio of the first to the second kind of tea in the mixture is 2 : 3, find the cost price of each kind of tea.

**A :** The cost of mixture =  $\frac{14 \times 100}{140}$  = Rs.10 per kg.

Ratio in which the cheap and dear is mixed = 3 : 2

Let the price of cheap tea be Rs.x per kg. and dear tea be Rs.2x per kg.

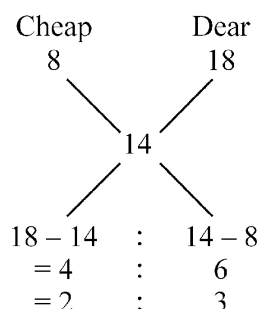
$\therefore$  By applying the alligation rule:  $\frac{3}{2} = \frac{2x - 10}{10 - x}$

$$\therefore 30 - 3x = 4x - 20; \therefore 7x = 50; \therefore x = 7\frac{1}{7}$$

$\therefore$  Cost of cheap tea = Rs.  $7\frac{1}{7}$  and cost of dear tea = Rs.  $14\frac{2}{7}$

**Q :** A merchant has 50 kg of sugar, part of which he sells at 8% gain and the rest at 18%. He gains 14% on the whole. Find the quantity sold at 18% profit.

**A :**



$$\therefore \text{Quantity sold at 18\% gain} = \frac{3}{5} \times 50 = 30 \text{ kg.}$$

*Alternatively,*

Let  $x$  be the cost per kg of the sugar.

$\therefore$  Price of sugar sold at 8% gain =  $1.08x$  per kg and, price of sugar sold at 18% gain =  $1.18x$  per kg

And the mean price at which it is sold =  $1.14x$  per kg

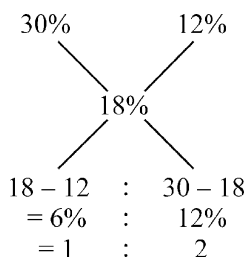
$\therefore$  By alligation,

$$\frac{\text{Quantity sold at 8\% gain}}{\text{Quantity sold at 18\% gain}} = \frac{1.18x - 1.14x}{1.14x - 1.08x} = \frac{0.04x}{0.06x} = \frac{2}{3}$$

$$\therefore \text{Quantity sold at 18\% gain} = \frac{3}{5} \times 50 = 30 \text{ kg}$$

**Q :** A butler stole wine from a butt of sherry, which contained 30% spirit and he replaced what he had stolen by wine containing 12% spirit. The butt was then of 18% strength. How much of the butt did he steal?

**A :**



By the alligation rule, we find that wine containing 30% of spirit and wine containing 12% spirit should be mixed in the ratio 1 : 2 to produce a mixture containing 18% spirit.

This means that  $\frac{1}{3}$  of the butt of sherry was left; in other words the butler drew out  $\frac{2}{3}$  of the butt.  $\therefore \frac{2}{3}$  of the butt was stolen.

**Q :** Three equal glasses are filled with a mixture of spirit and water. The proportion of spirit to water in each glass is as follows: in the first glass 2 : 3, in the second glass 3 : 4 and the third 4 : 5. The contents of the three glasses are emptied into a single vessel. What is the proportion of spirit and water in it?

**A :** Let the capacity of each glass be 1 unit.

$$\text{The quantity of spirit in the vessel} = \frac{2}{5} + \frac{3}{7} + \frac{4}{9} = \frac{401}{315}$$

$$\text{The quantity of water in the vessel} = \frac{3}{5} + \frac{4}{7} + \frac{5}{9} = \frac{544}{315}$$

$\therefore$  The required ratio = 401 : 544

**Note:** The total quantity of water in the vessel can also be found out by subtracting  $\frac{401}{315}$  from 3.

**Q :** Nine litres are drawn from a cask full of wine, and it is then filled with water. Nine litres of the mixture are drawn and the cask is again filled with water. The quantity of wine now left in the cask is to that of the water in it is 16 : 9. How much does the cask hold?

**A :** Here,  $\frac{\text{Wineleft}}{\text{Water}} = \frac{16}{9}$

$$\therefore \frac{\text{Wineleft}}{\text{Wineleft} + \text{Water}} = \frac{16}{(16 + 9)} = \frac{16}{25}$$

$$\therefore \left( \frac{\text{Initial quantity of wine} - 9}{\text{Initial quantity of wine}} \right)^2 = \frac{16}{25} = \left( \frac{4}{5} \right)^2$$

$$\therefore \frac{\text{Initial quantity of wine} - 9}{\text{Initial quantity of wine}} = \frac{4}{5}$$

Let initial quantity of wine = x litres.

$$\therefore \frac{x - 9}{x} = \frac{4}{5}$$

$$\therefore x = 45 \text{ litres.}$$

$\therefore$  The initial quantity of wine i.e., the capacity of the cask = 45 litres

*Alternatively,*

$$\frac{\text{Wine left}}{\text{Water left}} = \frac{\left( \frac{a-9}{a} \right)^2}{1 - \left( \frac{a-9}{a} \right)^2}; \therefore \frac{16}{9} = \frac{\left( 1 - \frac{9}{a} \right)^2}{1 - \left( 1 - \frac{9}{a} \right)^2}$$

$$\therefore \frac{16}{9} - \frac{16}{9} \left(1 - \frac{9}{a}\right)^2 = \left(1 - \frac{9}{a}\right)^2 \quad \therefore \frac{16}{9} = \frac{25}{9} \left(1 - \frac{9}{a}\right)^2$$

$$\therefore \left(1 - \frac{9}{a}\right) = \sqrt{\frac{16}{25}} = \frac{4}{5}; \quad \therefore \frac{9}{a} = \frac{1}{5}; \quad \therefore a = 45 \text{ litres}$$

**Q :** The cost prices of 3 kinds of sugar are Rs.5, Rs.6 and Rs.6.80, respectively. In what proportion should they be mixed so that the price of the mixture may be Rs.6.50 per bag?

**A :** Suppose the quantities of the three kinds of sugar are x, y and z respectively.

$$\therefore \frac{5x + 6y + 6.80z}{x + y + z} = 6.5$$

$$\therefore 5x + 6y + 6.80z = 6.5x + 6.5y + 6.5z$$

$$\therefore 0.3z = 1.5x + 0.5y$$

Depending on different values of x and y, we will get different values of z.

Therefore multiple answers are possible.

### Concept Builder 2

- 5 kg of rice at Rs.9 per kg and 3 kg of rice at Rs.5 per kg are mixed. Find the price of the mixture per kg.
- A grocer mixes 4 kg of tea costing Rs.6 per kg with 'x' kg of tea costing Rs.10 per kg and the cost of the mixture per kg is Rs.8. Then find 'x'.
- There are two mixtures of water and honey. Volumes of first and second mixtures are 30 litres and 40 litres respectively. The ratios of water to honey in both the mixtures are 2 : 3 and 3 : 5, respectively. Find the ratio of water to honey when the two types are mixed.
- A mixture of 150 litres of spirit and water contains 8% of water. Then the mixture contains how many litres of spirit?
- A mixture of 20 litres of milk and water contains 10% of water. How much water should be added to it to increase the percentage of water to 25%?
- The cost of type-I oil is Rs.40 per litre and type-II oil is Rs.45 per litre. If both type-I and type-II are mixed in the ratio of 3 : 2, then the price per litre of the mixture of oil is:
- Two solutions of 90% and 97% purity were mixed, resulting in 42 litres of mixture of 94% purity. How much is the quantity of the first solution in the resulting mixture?
- A vessel contains 5 litres of milk. From this, 1 litre of milk was taken out and replaced by water. This process was repeated further for two times. How much milk is now contained by the container?

### Answer Key

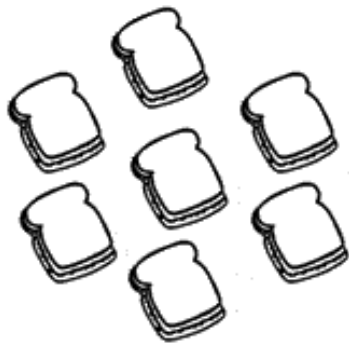
- |    |          |    |            |    |              |
|----|----------|----|------------|----|--------------|
| 1. | Rs.7.50. | 4. | 138 litres | 7. | 18 litres.   |
| 2. | 4kg      | 5. | 4 litres   | 8. | 2.56 litres. |
| 3. | 27 : 43. | 6. | Rs.42      |    |              |



## CLASS EXERCISE

### Teaser

Anthony was travelling to the city of Far Far Away. Along the way he met two other travellers, Basil and Cecil, who were headed in the same direction. When they stopped for the night, Anthony realised that he had brought no food. Basil had packed four sandwiches while Cecil had packed three. They agreed that all three of them would share the food equally, and Anthony would pay 7 silver coins as his share. How should Basil and Cecil divide the money?



## Measures of Central Tendency

Data for questions 1 and 2: The table below gives the age and height details of the ten finalists of the “Miss Latvia” contest 2019. It also gives the final score awarded by the judges to the contestants.

Sr. No.	Name	Age (Years)	Height (cm)	Final Score
1	Petra A.	19	176	636
2	Beatrise P.	23	172	633
3	Velna A.	19	178	631
4	Daina K.	18	169	630
5	Zuzanna S.	21	171	627
6	Alise A.	21	164	622
7	Regina M.	22	168	621
8	Brigita N.	20	171	621
9	Jana K.	19	172	617
10	Emilija S.	21	169	612

- 1)
  - a) What is the average age of the finalists?
  - b) What is the median age of the finalists?
  - c) What is the mode of the finalists’ ages?
- 2)
  - a) Using the method of assumed average, compute the average height of the finalists?
  - b) Using the method of assumed average, compute the average final score of the finalists?

The Mean or simple average of a set of values can be calculated by adding up all the values and dividing the result by the number of values.

The Median of a set of numbers is found by arranging them in increasing or decreasing order, and selecting the middle number.

The Mode of a set of values is the value (or values) which occurs most frequently.

- 3)\* Find the arithmetic mean, median and mode of the following values:
  - a) 156, 158, 161, 158, 171, 165, 158
  - b) 2123, 2017, 2134, 2099, 2092, 2083, 2056, 2092
  - c) 22.7, 23.1, 24.3, 22.3, 21.4, 22.7, 21.3, 22.9, 23.1, 24.6



### Different types of Means and their properties

- 4) Two men, Ralph and Sean, take up a typing job requiring 100 pages to be typed. Sean charges \$20 per hour while Ralph (being 1.5 times as fast as Sean) charges \$30 per hour. Both of them work for 6 hours. What is the average hourly wage they earn?
- 5) Two men, Ralph and Sean, take up a typing job requiring 100 pages to be typed. Each of them types 50 pages. Sean charges \$20 per hour while Ralph (being 1.5 times as fast as Sean) charges \$30 per hour. What is the average hourly wage they earn?
- 6) Apeksha is test-driving her new car. She first drives for 45 minutes at 30 kmph, and then drives a further 45 minutes at 70 kmph. What is her average speed for the entire duration?
- 7) Apeksha is test-driving her new car. She first drives for 45 km at 30 kmph, and then drives a further 45 km at 70 kmph. What is her average speed for the entire duration?
- 8) Kanika went to the market to buy onions. She bought 4 kg from a vendor at Rs 50 per kg. Then she found another vendor who offered onions at Rs 40 per kg, so she bought another 4 kg. What is the average price she paid per kg?
- 9) Kanika went to the market to buy onions. She bought 4 kg from a vendor at Rs 50 per kg. Then she found another vendor who offered onions at Rs 40 per kg, so she bought another 5 kg. What is the average price she paid per kg?
- 10) A cuboidal aquarium has dimensions  $1.5 \times 4 \times 4.5$  m. It is to be emptied into another aquarium which is a perfect cube. What is the minimum side the new aquarium could have?

Arithmetic Mean (AM) of  $a_1, a_2, a_3 \dots a_n$  is an additive mean.  $AM = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$

Geometric Mean (GM) of  $a_1, a_2, a_3 \dots a_n$  is a multiplicative mean.  $GM = \sqrt[n]{a_1 \times a_2 \times a_3 \times \dots \times a_n}$

Harmonic Mean (HM) of  $a_1, a_2, a_3 \dots a_n$  is a reciprocative mean.  $HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}}$

For any set of positive numbers it can be shown that  $AM > GM > HM$

For two numbers  $a$  and  $b$ ,  $AM = \frac{a+b}{2}$ ,  $GM = \sqrt{ab}$  and  $HM = \frac{2ab}{a+b}$ , and hence  $AM \times HM = GM^2$ .

- 11) \* A reads 60 pages an hour, while B reads 40. If both read for 6 hours, what is the average number of pages per person per hour? If each reads 400 pages, how will the answer change?
- 12) \* A invests in a scheme which gives him 28% interest. The next year, he removes the entire amount and invests it in a new scheme which gives him 100% interest. What is the average annual return he receives?

**Weighted Averages and Alligation**

- 13) In a school, the average weight of the 40 students in class VII was found to be 45 kg, while that of 30 students in class VIII was found to be 52 kg. What is the average weight of all 70 students?
- 14) Two groups of students wrote a test. The average score of 25 students of the 1st group was 120, while that of the 35 students in the second group was 126. What is the average score of both groups?
- 15) Kanika went to the market to buy onions. She bought 4 kg from a vendor at Rs 50 per kg. Then she found another vendor who offered onions at Rs 40 per kg, so she bought another 6 kg. What is the average price she paid per kg?
- 16) 9 members of the Indian squad turned up for practice yesterday. Their average height was measured to be 174 cm. The remaining 6 members turned up today and their average height was found to be 189 cm. What is the average height of the entire squad?
- 17) India played 6 matches in 2015 and the average attendance was 28000. Kenya played 4 matches with an average attendance of 18000. England played 5 matches with an average attendance of 24000. If no two of these three teams played each other, what is the average attendance of their matches?
- 18) Amol invests Rs 2 lakhs in two options A and B. At the end of a year, A gives him a return of 31% while B gives him 21%, and his total earning is Rs 56,000. How much did he invest in A?
- 19) A survey of tourists at the Taj Mahal showed the average age of 50 respondents as 34 years. It was later discovered that two of the respondents were tourist guides and not tourists, both 58 years of age. What should have been the actual average age of the tourists?
- 20) In a company, the average monthly salary of Managers is 43000, that of Executives is 32000 and that of Workers is 24000. The average monthly salary of Managers and Executives combined is 35000 and that of Executives and Workers combined is 26000. What fraction of the workforce is made up of Workers?
- 21) Three varieties of rice cost respectively 18, 25 and 28 Rs per kilo. Find a possible ratio in which the three could be mixed so that the resultant would be worth Rs 26 per kilo.
- 22) 4 litres of a mixture of milk and water in the ratio 7 : 1 is mixed with 6 litres of a mixture of milk and water in the ratio 9 : 7. What is the ratio of milk to water in the resultant mixture?
- 23) Two jars contain acid at 74% and 44% concentration respectively. If 20 litres are drawn from the 1st jar, how many litres from the 2nd jar should be added so that the concentration of the resultant is 50%?
- 24) A vessel contains 30 litres of 60% nitric acid. How much water should be added to the vessel so that the concentration of acid becomes 40%?
- 25) A decanter contains 7 parts iced tea to 3 parts water. Another decanter contains 21 parts iced tea to 4 parts water. In what ratio should the contents be mixed to get 3 parts of iced tea to 1 part of water?
- 26) A milkman purchases milk at Rs 30 per litre and added 6 litres of water. By selling the resultant mixture at Rs 30 per litre, he made a profit of 15%. How many litres of milk did he buy initially?

- 27) A scientist has a 100 litre cistern filled with pure acid. With a 20-litre jug, he removes precisely 20 litres from the cistern, replacing it with water. He repeats this process again. What is the concentration of acid left in the cistern? After how many such processes will the concentration drop below 50%?

Weighted Average: If the 1<sup>st</sup> group has  $x_1$  objects with average  $A_1$ , the 2<sup>nd</sup> group has  $x_2$  objects with average  $A_2$ , ...and the  $n^{\text{th}}$  group has  $x_n$  objects which average  $A_n$ , the overall average is given by:

$$A_w = \frac{\sum_{i=1}^n x_i A_i}{\sum_{i=1}^n x_i} = \frac{x_1 A_1 + x_2 A_2 + x_3 A_3 + \dots + x_n A_n}{x_1 + x_2 + x_3 + \dots + x_n}$$

The Principle of Alligation: When different quantities of two mixtures of different costs are mixed, the ratio of their quantities is inversely proportional to the difference in their costs from the mean cost.

If quantity  $Q_1$  of the cheaper mixture at cost  $C$  and quantity  $Q_2$  of the dearer one at cost  $D$  are mixed, then the mean cost  $X =$  and hence  $\frac{Q_1 C + Q_2 D}{Q_1 + Q_2} = \frac{Q_1}{Q_2} = \frac{D - X}{X - C}$

- 28) \* A man was taking the average age of a group of 16 children on a picnic. By mistake he included the 40-year old teacher, which increased his answer by 2. What is the actual average age of the 16 children?
- 29) \* Two ingots are melted together in a furnace. One weighs 8 kg and contains 43% iron and the other weighs 11 kg and contains 81% iron. What percentage of iron is present in the resultant large ingot?
- 30) \* Two companies (with average ages of their workforces being 31 and 39 years) merged. The average age of the resultant workforce was 36 years. Find the ratio of sizes of the original workforces of the companies.
- 31) \* A laboratory has a tank containing 500 gallons of 60% salt solution in water. On heating it, water starts evaporating. How much water should evaporate such that the resultant solution contains 75% salt?
- 32) \* A greedy worker at a honey manufactory steals 10 litres from a full vat of honey, replacing it by an equal quantity of water to hide the crime. After some time he steals another 10 litres from the same vat, replacing it again by water. The ratio of honey to water is now 49 : 32. What is the capacity of the vat?
- 33) \* In what ratio should a solution with water and alcohol in the ratio 2:3 be mixed with a solution with water and alcohol in the ratio 7:3 to get a solution with a 11:9 ratio of water and alcohol?

## Challengers

- 1) 5 friends – Ajax, Betty, Cody, Donna and Elsie – decide to contribute to a charity. Ajax contributes  $\frac{1}{6}$ th as much as all the others combined, Betty contributes  $\frac{1}{5}$ th as much as all the others combined, Cody contributes  $\frac{1}{4}$ th as much as all the others combined and Donna contributes  $\frac{1}{3}$ rd as much as all the others combined. If the total contribution of the 5 is \$ 42,000, then how much did Elsie pay?

1) \$ 10000                  2) \$ 10100                  3) \$ 11000                  4) \$ 11100
- 2) 3 solutions have acid content 40%, 60% and 80% respectively. When x, (x+1) and (x+2) ml respectively are taken from the three and mixed, the resultant is 65% acid. What can be said about x?

1)  $0.1 < x \leq 0.5$  ml    2)  $0.5 < x \leq 1.5$  ml  
3)  $1.5 < x \leq 2.5$  ml    4)  $2.5 < x \leq 3.5$  ml
- 3) Consider the set of all positive integers from 1 till k (where  $k = 2m$  and m is an integer greater than 2012). Let the average of all the odd numbers in this set be O and the average of all the even numbers in this set be E. Then what is the maximum value of |E – O|?

1) 1                          2)  $m/2$                           3)  $k - m + 1$                           4) 2011
- 4) A milkman mixes 25 litres of water with 75 litres of milk. He sells  $\frac{1}{5}$ th of the mixture and then adds water to replenish the quantity which he has sold. What is the resultant proportion of milk to water?

1) 2 : 3                          2) 1 : 3                          3) 3 : 2                          4) 1 : 4
- 5) A shopkeeper buys two varieties of olive oil, one at Rs 55 per litre and one at Rs 66 per litre. In what ratio should he mix the two so that when he sells the mixture at Rs 69 per litre, he gains 15%?

1) 5 : 6                          2) 4 : 7                          3) 6 : 5                          4) 7 : 4
- 6) A start-up has 10 employees whose average age in 2010 was 45 years. In 2013, the two eldest employees retired at the age of 60 years each. In 2014, 3 new trainee employees, with a combined age of 50 years, joined the company. What is the average age of the company's employees in 2015?

1) 37 years                          2) 40 years                          3) 38 years                          4) 39 years



## PRACTICE EXERCISE - 1

**DIRECTIONS for questions 1 to 3: Choose the correct alternative.**

1. The median and mode of 25, 26, 28, 29, 30, 21, 25, 28, 27, 25 is:  
 1) 26, 25                      2) 26.5, 25                      3) 25.5, 26                      4) 25, 25
2. Find the harmonic mean of two numbers whose arithmetic mean and geometric mean are 8 and 6 respectively.  
 1)  $\frac{32}{3}$                       2) 10                      3) 4.5                      4) 4.2
3. If the harmonic mean between two positive numbers is to their geometric mean as 12 : 13; then the numbers could be in the ratio:  
 1) 12 : 13                      2)  $\frac{1}{12} : \frac{1}{13}$                       3) 4 : 9                      4) 2 : 3

**(Past CAT question)**

**Direction for questions 4 to 6: Refer to the data below and answer the questions that follow.**

There are 60 students in a class. These students are divided into three groups A, B and C of 15, 20 and 25 students each. The groups A and C are combined to form group D.

4. What is the average weight of the students in group D?  
 1) More than the average weight of A  
 2) More than the average weight of C  
 3) Less than the average weight of C  
 4) Cannot be determined
5. If one student from group A is shifted to group B, which of the following will be true?  
 1) The average weight of both groups increases  
 2) The average weight of both the groups decreases  
 3) The average weight of the class remains the same  
 4) Cannot be determined
6. If all the students of the class have the same weight, then which of the following is false?  
 1) The average weight of all the four groups is the same  
 2) The total weight of A and C is twice the total weight of B  
 3) The average weight of D is greater than the average weight of A  
 4) The average weight of all the groups remains the same even if a number of students are shifted from one group to another

**(Past CAT question)**

**(Past CAT question)**

**(Past CAT question)**

**DIRECTIONS for questions 7 to 20: Choose the correct alternative.**

7. The average of the scores of a student in 10 papers is 80 marks. If the highest and the lowest scores are not considered, the average is 81 marks. If his highest score is 92 marks, find the lowest score.

1) 55                      2) 60                      3) 62                      4) Cannot be determined

**(Past CAT question)**

8. The average marks lost on a certain test (max. marks 20) in a class is 13 if the maximum score obtained is excluded. If the marks obtained by the highest scorer are included, the average of marks obtained rises to 8. How many students are there in the class, if it is known that the minimum of marks lost by any student is 3?

1) The given situation is not possible                      2) 9  
3) 11                      4) 10

9. There are seven consecutive integers. The average of the first 5 is  $n$ . What is the average of all seven?

1)  $n$                       2)  $n + 1$   
3)  $kn$ , here  $k$  is a function of  $n$                       4)  $n + \frac{2}{7}$

**(Past CAT question)**

10. Three natural numbers are such that the first number is the average of the other two. Also the sum of two of these numbers is the value of the largest number. Find the smallest number if the largest number is 15.

1) 5                      2) 10                      3) 7                      4) -5

11. The mean of 20 observations was found out to be 65 but later on it was found that 69 was misread as 96. Find the correct mean.

1) 63.65                      2) 12.37                      3) 69.5                      4) 65.95

12. Amol was asked to calculate the arithmetic mean of ten positive integers each of which had two digits. By mistake, he interchanged the two digits, say  $a$  and  $b$ , in one of these ten integers. As a result, his answer for the arithmetic mean was 1.8 more than what it should have been. Then  $b - a$  equals

1) 1                      2) 2                      3) 3                      4) None of these

**(Past CAT question)**

13. The average score of 60 students in an exam is 90. If the scores of the top five students are not considered, then the average of the remaining drops by 5. If the second ranker scored less than 140 and all the students got integral scores, find the minimum marks that could have been scored by the topper.

1) 161                      2) 169                      3) 157                      4) 153

14. In an examination of 7 subjects with maximum 100 marks in each, I scored 40% in all. Average marks in 3 subjects were 48 while the highest marks scored among these three subjects were 64. If passing marks in any subject were 35, then what was the minimum number of subjects in which I failed?  
 1) 1                      2) 2                      3) 0                      4) None of these
15. The (integral) marks obtained by Dennice in 4 subjects at an examination are in the ratio 5 : 6 : 7 : 8. If he has obtained 80 marks in one of the subjects, find his average marks. (The maximum marks that can be secured in any subject are equal to 100.)  
 1) 65                      2) 98                      3) 50                      4) 66.67
16. Twenty four men along with another man 'X' agree to provide a sum of money in charity. Each of the twenty four men contributes Rs.20. X agrees to pay Rs.3 more than twice the average of the 25 men. What is the whole sum contributed by the 25 men?  
 1) Rs.500                      2) Rs.525                      3) Rs.550                      4) Rs.540
17. The average age of a committee of seven trustees is the same as it was five years ago, a younger man having been substituted for one of them. How much younger was he than the trustee whose place he took?  
 1) 32 years                      2) 35 years                      3) 33 years                      4) 34 years
18. The average weight of 10 men is increased by  $1\frac{1}{2}$  kg when one of the men, who weighs 68 kg is replaced by a new man. Find the weight of the new man.  
 1) 82 kg                      2) 81 kg                      3) 80 kg                      4) 83 kg
19. Which of the following is/are true?  
 I. When the Arithmetic Mean of a set of positive values and number of values in the set are known, the Geometric Mean of the set can be found out.  
 II. When the Arithmetic Mean of a set of positive values and number of values in the set are known, we can find the sum of all the elements in the set.  
 III. When the Geometric Mean of a set of positive values and number of values in the set are known, the Arithmetic Mean of the set can be found out.  
 1) Only I                      2) Only II                      3) Only I and II                      4) All except II
20. Which of the following statement/s is/are true?  
 I. The average of a set of values will always lie between the lowest and the largest of these values.  
 II. If each of the values in a set is increased by a constant k, then the new average of the set is increased by k.  
 III. If each of the values in a set is multiplied by a constant m, then the new average will also be 'm' times the old average.  
 1) Only I and II                      2) Only II and III  
 3) Only I and III                      4) I, II and III



## PRACTICE EXERCISE - 2

**DIRECTIONS for questions 1 to 11: Choose the correct alternative.**

1. The average daily wage of 15 workers is Rs.20 and that of 20 clerks and 17 office assistants is Rs.35 and Rs.25 respectively. Find the average daily wage of these 52 employees.  
 1) Rs.29.4                      2) Rs.28.2                      3) Rs.26                      4) Rs.27.4
2. Cask A contains wine and water in the ratio 6 : 7 and cask B in the ratio 9 : 4. In what ratio must the contents of the two casks be mixed to give a mixture of wine and water in the ratio 8 : 5?  
 1) 1 : 3                      2) 2 : 3                      3) 1 : 2                      4) None of these
3. In a residential colony, there are 30 men, 25 women and 15 children. The average weight of the men is 60 kg., that of the women is 50 kg., while that of the children is 40 kg. Find (approximately) the average weight of the entire colony.  
 1) 52.14                      2) 63.64                      3) 49.22                      4) 50.00
4. A trader has 380 kg. of tea, some of which he sells at 6% profit and the rest at 13% loss. He gains 2% on the whole. How much tea was sold at 6% profit?  
 1) 80 kg.                      2) 300 kg.                      3) 325 kg.                      4) None of these
5. A man buys spirit at Rs. 60 per litre, adds water to it and then sells it at Rs. 75 per litre. What is the ratio of spirit to water if his profit in the deal is 37.5%?  
 1) 9 : 1                      2) 10 : 1                      3) 11 : 1                      4) None of these  
 (Past CAT question)
6. Two liquids A and B are in the ratio 5 : 1 in container 1 and 1 : 3 in container 2. In what ratio should the contents of the two containers be mixed so as to obtain a mixture of A and B in the ratio 1 : 1?  
 1) 2 : 3                      2) 4 : 3                      3) 3 : 2                      4) 3 : 4  
 (Past CAT question)
7. A dealer buys dry fruits at Rs. 100, Rs. 80 and Rs. 60 per kilogram. He mixes them in the ratio 3 : 4 : 5 by weight, and sells at a profit of 50%. At what price per kilogram does he sell the dry fruit?  
 1) Rs. 80                      2) Rs. 100                      3) Rs. 95                      4) None of these  
 (Past CAT question)
8. A salt trader buys two types of salt, the first one having a price thrice that of the other. He sells the mixture at Rs.18 thus making a profit of 80%. If the proportion of mixing the first and the second types of salt is 1 : 5, find the cost price of the cheaper salt.  
 1) Rs.7                      2) Rs.7.5                      3) Rs.8                      4) Rs.10



9. A  $x\%$  concentrated solution of an acid is one which contains  $x\%$  acid and  $(100 - x)\%$  water. In a chemical experiment, the contents of two bottles of an acid of concentrations  $30\%$  and  $45\%$  are mixed, the first one being of half the volume of the other. Find the concentration of the mixture.
- 1)  $35\%$                       2)  $40\%$                       3)  $42.5\%$                       4) None of these
10.  $40\%$  of the employees of an organization are men and the rest are women. Of these  $75\%$  earn more than Rs. 25,000 per year. If  $45\%$  of the total employees of the company earn more than Rs. 25,000 per year, then what fraction of the women earn more than Rs. 25,000 per year?
- 1)  $\frac{2}{11}$                       2)  $\frac{1}{4}$                       3)  $\frac{1}{3}$                       4)  $\frac{3}{4}$

(Past CAT question)

11. There are two containers: the first contains 500 ml of alcohol, while the second contains 500 ml of water. Three cups of alcohol from the first container is taken out and is mixed well in the second container. Then three cups of this mixture is taken out and is mixed in the first container. Let A denote the proportion of water in the first container and B denote the proportion of alcohol in the second container. Then
- 1)  $A > B$                       2)  $A < B$   
3)  $A = B$                       4) Cannot be determined

(Past CAT question)

**DIRECTIONS for Questions 12 and 13: Refer to the data below and answer the questions that follow.**

The following table presents the sweetness of different items relative to sucrose whose sweetness is taken to be 1.00.

Lactose	0.16
Maltose	0.32
Glucose	0.74
Sucrose	1.00
Fructose	1.70
Saccharin	675.00

12. What is the maximum amount of sucrose (to the nearest gram) that can be added to one gram of saccharin to make a mixture that will be at least 100 times as sweet as glucose?
- 1) 7                      2) 8                      3) 9                      4) 100
13. Approximately how many times sweeter than sucrose is a mixture consisting of glucose, sucrose, fructose in the ratio 1: 2: 3?
- 1) 1.3                      2) 1                      3) 0.6                      4) 2.3

(Past CAT question)

**DIRECTIONS for questions 14 to 19: Choose the correct alternative.**

14. Three Maths classes : X, Y, and Z, take an algebra test.  
The average score in class X is 83.  
The average score in class Y is 76.  
The average score in class Z is 85.  
The average score of all students in classes X and Y together is 79.  
The average score of all students in classes Y and Z together 81.  
What is the average for all the three classes ?  
1) 81                      2) 81.5                      3) 82                      4) 84.5  
**(Past CAT question)**
15. The two mixtures A and B contains alcohol and water. Three litres of mixture A and four litres of mixture B are mixed to get a mixture with 30% alcohol. Four litres of mixture A and three litres of mixture B are mixed, we get a mixture with 25% alcohol. In what ratio should we mix the two mixtures A and B to get a new mixture with 27.5% alcohol?  
1) 1 : 2                      2) 2 : 1                      3) 2 : 3                      4) None of these  
**(Past CAT question)**
16. A dishonest milkman adds 20% water to the existing quantity of milk and sells it at the same price for which he buys it. After selling a few litres of the mixture he is caught and has to pay fine of Rs.200. After paying the fine he realises that he has suffered a loss of Rs.100. How many litres of the mixture did he sell if he bought pure milk for Rs.10 per litre?  
1) 100 litres                      2) 85 litres                      3) 50 litres                      4) 60 litres
17. A bucket was full of oil. Seven litres of oil was removed and replaced with water. Again 7 litres of the mixture was removed and replaced with water. The proportion of oil left in the bucket then was  $\frac{25}{36}$  of the total mixture (in the bucket). What is the capacity of the bucket?  
1) 42 litres                      2) 45 litres                      3) 40 litres                      4) 50 litres
18. I bought 5 litres of soda and 3 litres of water. But some quantity of soda was stolen and an equal quantity of water was added, after which I mixed the two and found the ratio of soda : water as 9 : 7. Find the quantity of soda stolen.  
1) 500 ml                      2) 250 ml                      3) 575 ml                      4) None of these

19. A certain medicine has two variants, I and II. Each variant has 4 ingredients A, B, C and D. The proportions of the four ingredients in variant I are as follows:  
A is  $\frac{3}{4}$  times B. D is twice A and C is  $\frac{7}{4}$  times B. Compared to 1 unit of variant I, one unit of variant II has 10 gms more of A. The proportion of A and C in variant II is 5 : 7. The quantity of C in variant II is the same as in variant I. Find the quantity of D in one unit of variant I.
- 1) 30 gm                      2) 20 gm                      3) 14 gm                      4) 15 gm

**Direction for question 20: Each question is followed by two statements A and B. Indicate your responses based on the following directives:**

**Mark (1) if the question can be answered using A alone but not using B alone.**

**Mark (2) if the question can be answered using B alone but not using A alone.**

**Mark (3) if the question can be answered using A and B together, but not using either A or B alone.**

**Mark (4) if the question can be answered even using A and B together.**

20. The average weight of a class of 100 students is 45 kg. The class consists of two sections, I and II, each with 50 students. The average weight,  $W_I$ , of Section I is smaller than the average weight,  $W_{II}$ , of Section II. If the heaviest student, say Deepak, of Section II is moved to Section I, and the lightest student, say Poonam, of Section I is moved to Section II, then the average weights of the two sections are switched, i.e., the average weight of Section I becomes  $W_{II}$  and that of Section II becomes  $W_I$ . What is the weight of Poonam?
- A:  $W_{II} - W_I = 1.0$
- B: Moving Deepak from Section II to I (without any move from I to II) makes the average weights of the two sections equal.



## PRACTICE EXERCISE - 3

**DIRECTIONS for questions 1 to 20: Choose the correct alternative.**

1. In XYZ construction company, male labourers are paid Rs.100 per day while female labourers are paid Rs. 25 per day. For a particular project the company is planning to spend Rs.80 per day per person. How many female labourers per day does the company hire if the total number of labourers per day is 45?  
1) 4                      2) 15                      3) 12                      4) None of these
2. 400 students took SimCAT in Delhi. 60% of the boys and 80% of the girls cleared the cut off in the examination. If the total percentage of students qualifying is 65%, how many girls appeared in the examination?  
1) 100                      2) 120                      3) 150                      4) 300
3. Alloy A contains iron and copper in the ratio 2 : 3 (not necessarily in that order). Alloy B contains copper and zinc in the ratio 4 : 1 (not necessarily in that order). Alloys A and B are mixed in the ratio 3 : 1 (in that order). Which of the following cannot be the percentage of copper in this mixture?  
1) 35%                      2) 40%                      3) 50%                      4) 65%
4. When Tur dal, Masoor dal and Urad dal are mixed in the ratio 1 : 2 : 3, then the resultant cost price of the mixture is Rs. 35 per kg. When Masoor dal, Chana dal and Urad dal are mixed in the ratio 1 : 2 : 2, the resultant cost price of the mixture is Rs. 43 per kg. What will be the resultant cost price (in Rs. per kg) of a mixture of Tur dal, Masoor dal, Chana dal and Urad dal which are mixed in the ratio 1:3:2:5?  
1) Rs. 39                      2) Rs. 39.54                      3) Rs. 38.63                      4) Data insufficient
5. The average age of the professor, assistant professor and all the students in a class is 17 years. The average of the ages of the professor and the assistant professor is more than the average of the ages of all the students in the class by a value which is 2 more than the number of students in the class. What is the average age of all the students in the class?  
1) 18 years                      2) 15 years                      3) 12 years                      4) 9 years
6. Ten years ago, the average age of a newly-wed couple was 25. Five years ago, the average age of the man, his wife and their son was 21. At present, the average age of the man, his wife, their son and their daughter is 20. What is the age difference (in years) between the son and the daughter?  
1) 4                      2) 3                      3) 6                      4) 2
7. Six men A, B, C, D, E, F agree with a seventh man G to provide a sum of money among them. A, B, C, D, E, F are to subscribe Rs.10 each, and G is to pay Rs.3 more than the average of the seven. What is the whole sum to be provided?  
1) Rs.73.50                      2) Rs.74                      3) Rs.73                      4) Rs.72.50

8. The average age of a team of 15 employees is 36 years. The youngest of them is 20 years old and the eldest is 56 years old. Two of them with average age 28 years leave the team. If one of the two comes back on the condition that he will be made the team-leader then which of the following can possibly be the average age of the new team so formed?  
 1) 35 years                      2) 36 years                      3) 40 years                      4) 39 years
9. The average age of a group of 20 women (of whom the youngest is 32 years old and the eldest is 56 years old) is 49 years. If two women with ages 45 years and 50 years leave the group and three women join the group, the average remains unchanged. What is the average age of the three women who joined the group later?  
 1) 32 years                      2) 38 years                      3) 45 years                      4) 48 years
10. The ratio of men and women in an organization is 7 : 5. If 20% of the men and 35% of the women are managers, find the percentage of employees who are managers.  
 1) 21.75%                      2) 26.25%                      3) 32.25                      4) 37.5%
11. In a mixture of wheat and barley the wheat is 60%. To 400 quintals of this mixture some quantity of barley is added and then the wheat is  $53\frac{1}{3}\%$ . How many quintals of barley are added?  
 1) 25 quintals                      2) 50 quintals                      3) 30 quintals                      4) 40 quintals
12. A cask contains a mixture of two liquids A and B in the ratio 7 : 5. When 9 gallons of the mixture is drawn off and the cask is filled with liquid B, the proportion becomes A : B :: 7 : 9. How many gallons does the cask hold?  
 1) 36 gallons                      2) 48 gallons                      3) 56 gallons                      4) 60 gallons
13. Five litres of wine is drawn from a cask full of wine and it is then filled with water. Five litres of the mixture is drawn and the cask is again filled with water. The ratio of the quantity of wine now left in the cask to that of the water is 361 : 39. How much does the cask hold?  
 1) 400 litres                      2) 250 litres                      3) 100 litres                      4) 75 litres
14. A milkman mixed 4 litres of water with 10 litres of milk. He boiled one fourth of this mixture till the water content, in it, halved. Then he added it to remaining mixture. What is the proportion of milk in the entire solution?  
 1)  $\frac{7}{10}$                       2)  $\frac{13}{15}$                       3)  $\frac{20}{27}$                       4)  $\frac{5}{6}$
15. The percentage of alcohol in two wine tanks is 20 and 40 respectively and rest is grape juice. If a wine glass is filled to its half by the tank 1 and 2 in the ratio 2 : 3 and the rest is filled again by pure alcohol, then what is ratio of the grape juice to alcohol in the wine glass?  
 1) 17 : 16                      2) 16 : 33                      3) 17 : 50                      4) 17 : 33

16. Three people A, B and C weigh themselves in a particular order. First A, B, C weigh themselves individually and then AB, BC, CA and ABC together respectively. The recorded weight for the last measure is 180 kgs. The average of the 7 measures is:  

1) 320 kgs

2)  $\frac{360}{7}$  kgs

3)  $\frac{720}{7}$  kgs

4) Cannot be determined
17. A dishonest milkman mixes 1 litre of water for every 15 litres of milk. How much pure milk (in litres) should he add to 240 litres of his adulterated milk so that his profit percentage is halved? Assume that he sells milk at its cost price.  

1) 480 litres

2) 240 litres

3) 225 litres

4) 180 litres
18. A man buys milk at Rs.5 a litre and after adding water, sells it at Rs.6 a litre, thereby making a profit of  $33\frac{1}{3}\%$ . Find the proportion of water to milk in the mixture.  

1) 1 : 9

2) 1 : 10

3) 5 : 9

4) 4 : 9
19. The average of  $(n - 1)$  distinct numbers, which are from the set of  $n$  consecutive natural numbers starting from 1 is  $12\frac{9}{23}$ . What can be the value of the number, which has not been considered?  

1) 13

2) 15

3) 17

4) 20
20. A cask of wine when fully filled holds 10 litres. 2 litres of wine is removed and filled with water. Then 4 litres of the solution is replaced with water. Then, 6 and 8 litres respectively. At the end of the 4th operation, the ratio of wine to water is:  

1)  $\frac{4!}{(5)^4}$

2)  $\frac{4!}{5^4 - 4!}$

3)  $\frac{8!}{(10)^4}$

4)  $\frac{8!}{(10)^4 - 8!}$