Quantitative Aptitude

P & C, Probability, Series, Functions, Binomial Expansion, Trigonometry & Co-ordinate Geometry

Number of Questions : 40 **CEX–0512/18**

Directions for questions 1 to 8: Answer the questions on the basis of the following information.

In a class of 12 students, 7 are boys and 5 are girls. The class has 4 sessions each day, one each of arithmetic, algebra, geometry and probability. These classes are to be held one after the other in 4 distinct time slots and can be in any sequence (unless otherwise stated in a question). Further there are 2 teachers available and they can teach any topic (unless otherwise stated in a question). For all the following questions a 'session' refers to a particular combination of topic and a teacher at a particular time slot.

1. In how many distinct ways can the sessions be scheduled for two consecutive days?
   (a) 384 (b) 576 (c) 9,216
   (d) 1,47,456 (e) 192

2. In how many distinct ways can the sessions be scheduled for a particular day if one teacher could teach only arithmetic and algebra whereas the other teacher could teach only geometry and probability?
   (a) 24 (b) 48 (c) 384
   (d) 576 (e) 192

3. In how many distinct ways can the sessions be scheduled for a particular day if one faculty could not teach arithmetic and the other faculty could not teach probability?
   (a) 24 (b) 96 (c) 384
   (d) 576 (e) 192

4. In how many distinct ways can the sessions be scheduled for two consecutive days if both the teachers have to have equal number of sessions?
   (a) 576 (b) 9,216 (c) 40,320
   (d) 73,728 (e) 384

5. If the sequence of class has to be arithmetic, algebra, geometry and probability in this particular order itself, then the number of distinct ways in which the sessions can be planned for an entire week of 6 working days is
   (a) 24^2 (b) 6^{24} (c) 24^6
   (d) 2^{24} (e) 2^{20}

6. The entire class of 12 students is divided equally into two different divisions and then the sessions are scheduled for these two divisions with classes being held simultaneously in two classrooms. In how many ways can the sessions be planned for these two divisions for a particular day?
   (a) 576 (b) 576×2^8 (c) 576×8^2
   (d) 9216 (e) 576×8

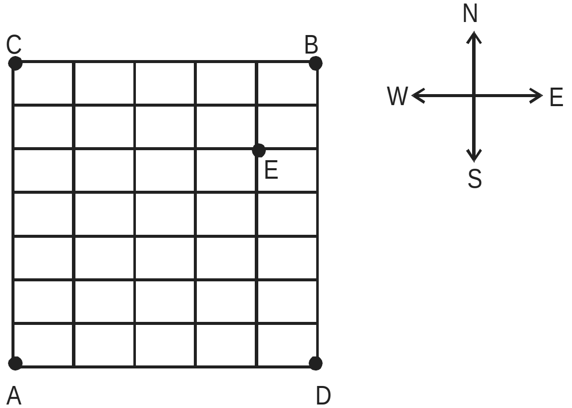
7. In question 6, in how many ways can the class be divided into two equal divisions?
   (a) ${}^{12}C_6 \times {}^{6}C_6$
   (b) $12 \times 11 \times 10 \times 9 \times 8 \times 7$
   (c) $\dfrac{12!}{(6!)^2}$
   (d) $\dfrac{12!}{(2!)^6}$ (e) $\dfrac{{}^{12}C_6}{2!}$

8. The seating arrangement in the class consists of 6 desks with two students sitting on each desk. The 6 desks are arranged in two columns with 3 desks in each column. In how many distinct ways can the 12 students sit in this particular arrangement for a particular session?

(a) ${}^{12}C_2$ (b) $\frac{12!}{(2!)^6}$ (c) $\frac{12!}{(2!)^6}6!$

(d) $\frac{12!}{2!10!}$ (e) 12!

Directions for questions 9 to 13: Answer the questions on the basis of the following information. Chandigarh has six roads in North-South direction and eight roads in East-West direction as shown. The roads in the North-South direction are perpendicular to the roads in the East-West direction. Any adjacent pair of parallel roads are 0.5 km apart.



9. How many distinct shortest routes are possible if one has to travel from A to B?

(a) 2 (b) $\frac{14!}{6!\,8!}$

(c) 2^{14} (d) 792

(e) None of these

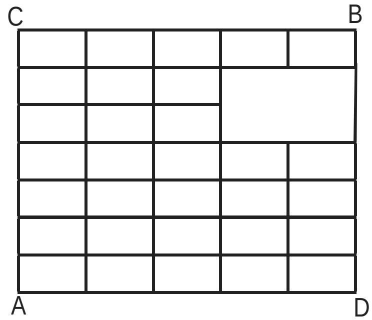
10. How many distinct shortest routes are possible if one has to travel from A to B via E?

(a) 378 (b) $\frac{11!}{6!\,5!}\times\frac{3!}{2!}$

(c) 4 (d) 192

(e) None of these

11. If the network of roads is as shown in the following figure, then find how many distinct shortest routes are possible if one has to travel from A to B.

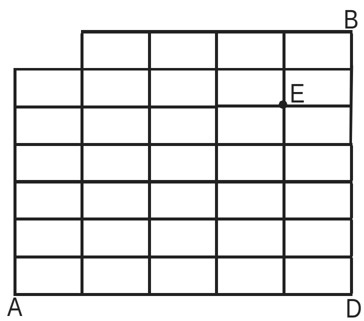


(a) 128 (b) 414

(c) 210 (d) 288

(e) None of these

12. If the network of roads is as shown in the figure given below, then find how many distinct shortest routes are possible if one has to travel from A to B.



(a) 791 (b) 1

(c) 188 (d) 792

(e) None of these

13. The number of distinct shortest route if one has to travel from A to D and then to C and then to B is

(a) $\frac{14!}{6!\,8!}$ (b) 2

(c) 192 (d) 792

(e) None of these

14. The probability that an event A occurs in one trial of an experiment is 0.4. Three independent trials of experiment are performed. Find the probability that event A occurs at least once.

(a) 0.936 (b) 0.784
(c) 0.964 (d) 0.6
(e) None of these

15. If P(A) represents probability of event A, then for two events A and B, $P(A \cap B)$ is

I. not less than P(A) + P(B) – 1
II. not greater than P(A) + P(B)
III. equal to $P(A) + P(B) - P(A \cup B)$
IV. equal to $P(A) + P(B) + P(A \cup B)$

Which among the following is correct?
(a) Only I and III are true
(b) Only III and IV are true
(c) Only I, II and III are true
(d) Only II and III are correct.
(e) All of the above are true

16. A mapping is selected at random from the set of all the mappings of the set A = {1, 2, ..., n} into itself. Find the probability that the mapping selected is an injection.

(a) $\frac{1}{n^n}$ (b) $\frac{1}{n!}$ (c) $\frac{(n-1)!}{n^{n-1}}$

(d) $\frac{n!}{n^{n-1}}$ (e) n!

17. Find the sum of infinite terms of the series S, where S = $(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + ...$

(a) $\frac{3+2\sqrt{2}}{\sqrt{2}}$ (b) $\frac{3-2\sqrt{2}}{\sqrt{2}}$

(c) $\frac{2\sqrt{2}-3}{\sqrt{2}}$ (d) $\frac{3+2\sqrt{2}}{2\sqrt{2}}$

(e) None of these

18. Find the sum of first 10 terms of the series S, where S =

$$\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+ \ldots$$

$(x \neq 0, 1)$

(a) $\left(\frac{x^{20}-1}{x^2-1}\right)\left(\frac{x^{22}+1}{x^{20}}\right)+20$

(b) $\left(\frac{x^{18}-1}{x^2-1}\right)\left(\frac{x^{11}+1}{x^{9}}\right)+20$

(c) $\left(\frac{x^{18}-1}{x^2-1}\right)\left(\frac{x^{11}-1}{x^{9}}\right)+20$

(d) $\left(\frac{x^{18}+1}{x^2-1}\right)\left(\frac{x^{11}+1}{x^{9}}\right)+20$

(e) None of these

19. If a, b and c are in HP, then find the value of $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$.

(a) $\frac{2}{bc}+\frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$

(c) $\frac{3}{b^2}+\frac{2}{ab}$ (d) $\frac{3}{ab}+\frac{2}{b^2}$

(e) None of these

20. Let f(x + 2) + f(5x + 6) = 2x – 1 for all real x. Find the value of f(1).

(a) –2 (b) –1 (c) $\frac{-5}{2}$

(d) $\frac{-3}{2}$ (e) $\frac{3}{2}$

21. Find the value of P, where P

$= \underbrace{(666...6)}_{n\,digits}^2 + \underbrace{(888...8)}_{n\,digits}$

(a) $\frac{4}{9}(10^n + 1)$ (b) $\frac{4}{9}(10^{2n} - 1)$

(c) $\frac{4}{9}(10^n - 1)$ (d) $\frac{2}{9}(10^n - 1)$

(e) None of these

22. For all integers x and y, f(x, y) is defined as below: f(0, y) = 2y + 1 and f(x + 2, 0) = f(x + 1, 1). If f(x + 2, y + 1) = f(x, f(x + 1, y)), find the value of f(2, 1).

(a) 5 (b) 6 (c) 7
(d) 8 (e) 4

23. Let,

$S_n = \frac{1}{1^3} + \frac{1+2}{1^3+2^3} + ... + \frac{1+2+...+n}{1^3+2^3+...+n^3}$;

n = 1, 2, 3, ... Find the value of S_{35}.

(a) $\frac{8}{9}$ (b) $\frac{9}{10}$ (c) $\frac{25}{18}$

(d) $\frac{19}{20}$ (e) $\frac{35}{18}$

24. The number of common terms of the two sequences 17, 21, 25, ..., 417 and 16, 21, 26,..., 466.

(a) 21 (b) 29 (c) 25
(d) 91 (e) 20

Directions for questions 25 and 26:
Let f(g(x)) = 2 – 3x and g(f(x)) = 1 + 3x for all real x.

25. Find the value of $f\left(\frac{-1}{2}\right)$.

(a) $\frac{1}{2}$ (b) $\frac{-3}{2}$ (c) $\frac{-1}{2}$

(d) $\frac{3}{2}$ (e) 0

26. Find the value of $2g\left(\frac{1}{2}\right) + f\left(\frac{-1}{2}\right)$.

(a) $\frac{-1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$

(d) $\frac{-3}{2}$ (e) 0

27. If the coefficient of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in AP, then the value of n can be

(a) 7 only (b) 14 only (c) 7 or 14
(d) 10 (e) None of these

28. Find the term which is independent of x in the expansion of $(x - 2x^{-1})^{2n}$.

(a) $(n + 1)^{th}$ (b) n^{th} (c) $(n - 1)^{th}$

(d) $\left(\frac{n}{2}\right)^{th}$ (e) None of these

29. Find the coefficient of x^{18} in $(ax^4 - bx)^9$.

(a) $84a^2 b^7$ (b) $36a^5 b^4$ (c) $72a^3 b^6$
(d) $84a^3b^6$ (e) $84a^5 b^3$

30. If in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$, x^{-17} occurs in r^{th} term, then find the value of r.

(a) r = 10 (b) r = 11 (c) r = 12
(d) r = 13 (e) r = 9

31. If in the expansion of $(1+x)^m (1-x)^n$, the coefficients of x and x^2 are 3 and –6 respectively, then find the value of m

(a) 6 (b) 9 (c) 12
(d) 24 (e) 18

32. Find the sum of the rational terms in the expansion of $\left(\sqrt{2}+3^{\frac{1}{5}}\right)^{10}$.

(a) 32 (b) 41
(c) 48 (d) 24
(e) None of these

33. If $y = \dfrac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta}$, then which of the following is true?

(a) $\frac{1}{3} \le y \le 3$ (b) $y \notin \left[\frac{1}{3}, 3\right]$
(c) $-3 < y < -\frac{1}{3}$ (d) $\frac{1}{3} \le y < 3$
(e) None of these

34. One solution of equation

$$\frac{1-\sin x+\sin^2 x \ldots+(-1)^n \sin^n x+\ldots}{1+\sin x+\sin^2 x \ldots+\sin^n x+\ldots} = \frac{1-\cos 2x}{1+\cos 2x}$$

is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$
(d) 0 (e) $\frac{\pi}{8}$

35. A line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle keeping the origin fixed, the same line L has intercepts p and q. Which of the following is true?

(a) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

(b) $\frac{1}{a^2}+\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{q^2}$

(c) $a^2 + p^2 = b^2 + q^2$

(d) $\frac{1}{a^2}-\frac{1}{b^2}=\frac{1}{p^2}-\frac{1}{q^2}$

(e) it depends on angle of rotation

36. Find the number of tangents which can be drawn from the point (–1, 2) to the circle $x^2 + y^2 + 2x - 4y + 4 = 0$.

(a) 1 (b) 2 (c) 3
(d) 0 (e) 4

37. A triangle PQR is inscribed in the circle $x^2 + y^2 = 25$ with vertices P, Q and R on the circle. If Q and R have coordinates (3, 4) and (–4, 3) respectively, then find the measure of $\angle QPR$.

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$
(d) $\frac{\pi}{6}$ (e) 0

38. If circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other, then which of the following is true?

(a) $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$

(b) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

(c) $\frac{1}{a}+\frac{1}{b}=c^2$

(d) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$

(e) $\frac{1}{a}+\frac{1}{b}=c$

39. If $3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$, then find the value of x.

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) 1 (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$
(e) None of these

40. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45°. If the angle of elevation of the top of the complete pillar at the same point is to be 60°, then the height of the incomplete pillar is to be increased by

(a) $50\sqrt{2}$ m
(b) 100 m
(c) $100\left(\sqrt{3}-1\right)$ m
(d) $100\left(\sqrt{3}+1\right)$m
(e) $100\sqrt{3}$m

Centriod of a triangle:
Centriod of a triangle formed by the points
$A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be G, then

$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

E.g. Find the centroid of a triangle whose vertices are A(-1, 2), B(3, 2) and C(-1, 5).

(a) $\left(\frac{1}{3},3\right)$ (b) (2, 3)
(c) (0, 2) (d) $\left(\frac{1}{3},2\right)$

Sol. a Centroid $G \equiv \left(\frac{-1+3-1}{3}, \frac{2+2+5}{3}\right) = \left(\frac{1}{3}, 3\right)$.

Incenter of a triangle:
Incenter of a triangle formed by the points
L $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be I, then

$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Where a is the length of the side opposite to A, b is the length of the side opposite to B, and c is the length of the side opposite to C.

E.g. Find the incenter of a triangle whose vertices are A(–1, 2), B(3, 2) and C(–1, 5).
(a) (1, 2) (b) (0, 3)
(c) (0, 2) (d) (1, 3)

Sol. b $AB = \sqrt{(-1-3)^2+(2-2)^2} = \sqrt{16+0} = 4 = c$

$BC = \sqrt{(3-(-1))^2+(2-5)^2} = \sqrt{16+9} = 5 = a$

$CA = \sqrt{(-1-(-1))^2+(5-2)^2} = \sqrt{0+9} = 3 = b$

Then, incenter I

$$\equiv \left(\frac{5\times -1+3\times 3+4\times -1}{5+3+4}, \frac{5\times 2+3\times 2+4\times 5}{5+3+4}\right) = (0, 3).$$

Excenter of a triangle:
Excenter of a triangle formed by the points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ opposite to vertex A be I_A. Then,

$$I_A \equiv \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

Similarly, excenters of a triangle formed by the points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ opposite to vertex B and C be I_B and I_C, then

$$I_B \equiv \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right) \text{ and}$$

$$I_C \equiv \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

E.g. Find the excenter opposite to vertex A of a triangle whose vertices are A(–1, 2), B(3, 2) and C(–1, 5).
(a) (3, 3) (b) (5, 3)
(c) (3, 8) (d) (5, 8)

Sol. d a = 5, b = 3 and c = 4
Excenter

$$I_A \equiv \left(\frac{-5\times -1+3\times 3+4\times -1}{-5+3+4}, \frac{-5\times 2+3\times 2+4\times 5}{-5+3+4}\right)$$

= (5, 8).

Answers and Explanations

1	d	**2**	a	**3**	b	**4**	c	**5**	d	**6**	d	**7**	e	**8**	e	**9**	d	**10**	a
11	b	**12**	a	**13**	d	**14**	b	**15**	c	**16**	c	**17**	a	**18**	a	**19**	b	**20**	d
21	b	**22**	c	**23**	e	**24**	e	**25**	a	**26**	a	**27**	c	**28**	a	**29**	d	**30**	c
31	c	**32**	b	**33**	a	**34**	a	**35**	a	**36**	d	**37**	c	**38**	d	**39**	b	**40**	c

1. d Lets find the number of distinct ways of scheduling sessions for a particular day. Consider each time slot as a particular box where a particular topic has to be assigned (topics cannot be repeated) and also a particular faculty is to be assigned (faculty can be repeated). The number of ways topics can be sequenced is 4! = 24. For each of these topics and time slot, there are two faculty available. Thus the number of distinct ways of scheduling the session for one day is $24 \times 2^4 = 384$. Thus for two days it will be 384 × 384 = 1,47,456.

2. a The number of ways in which the topics can be sequenced is 4! = 24. Now we have just one faculty for the sessions as only one faculty can teach the topics. Thus total number of ways of scheduling the session is 24.

3. b The number of ways the topics can be sequenced is 4! = 24. For the session on algebra and geometry, we have option of either of the two faculty, but for arithmetic and probability, only one faculty is available. Thus total number of distinct ways of scheduling the session is $24 \times 2^2 = 96$.

4. c The topics can be sequenced in 4! ways for a day and hence in 24 × 24 = 576 ways for two days. Now there are 8 sessions over the two days that have to be divided equally among the two faculty. This can be done in ${}^8C_4$ ways, i.e. in 70 ways. Thus the total number of schedules = 576 × 70 = 40,320

5. d Here we have 24 sessions (ordering of topics already decided) across 6 days which have to be assigned to one of the two faculty such that any faculty can get any number of classes. This can be done in 2^{24} ways.

6. d For one particular division, the topics can be sequenced in 24 ways. Similarly, for the other division they can be sequenced in 24 ways. Thus the total number of ways of sequencing the topics for both the division are 24 × 24 = 576. Now for a particular time slot, there are two ways of assigning the two faculty. Thus for 4 time slots, there are $2^4 = 16$ ways of assigning the sessions faculty. Thus total number of ways of scheduling = 576 × 16 = 9216.

7. e 12 students can be divided into two equal groups in

$$\frac{{}^{12}C_6 \times {}^6C_6}{2!} = 3{,}32{,}640.$$

8. e A googly ! Just consider it as 12 distinct places and 12 distinct students, so the number, of distinct ways of sitting is 12!.

9. d The shortest route will be to continuously move northwards or eastwards. It will not be only A-D-B and A-C-B. Even a zig-zag path of moving north and eastwards means travelling 12 roads segments. Any shortest path will involve moving 5 horizontal and 7 vertical road segments. The total number of ways will be the number of ways of arranging hhhhhvvvvvvv.

This can be done in $\frac{12!}{5!\,7!}$ ways. (hhvvvhvvhvhv means first going two roads segments to east, next three roads segments to north and so on and will be one of the shortest routes).

$$\frac{12!}{5!\,7!} = \frac{8 \times 9 \times 10 \times 11 \times 12}{2 \times 3 \times 4 \times 5} = 792.$$

10. a From A to E one has to move four horizontal road segments and five vertical road segments and the number of distinct shortest routes will be $\frac{9!}{4!\,5!} = 126$ ways. From E to B, the number of shortest routes will be $\frac{3!}{2!\,1!} = 3$.

Thus, total number of distinct shortest routes = 126 × 3 = 378.

11. b The required number of routes
= Total number of routes – Number of routes passing through E
= 792 – 378 = 414.

12. a There will be only one shortest route from A to B passing through C.
Thus, the number of shortest routes from A to B if C did not exist = 792 – 1 = 791.

13. d From A to D there is Just one shortest path.
From D to C, there will be 792 distinct shortest routes (Same as A to B)
From C to B again there is just one shortest path.
Thus, total number of distinct shortest routes = 792.

14. b Here p = 0.4, q = 0.6 and n = 3.
$\therefore$ The required probability = P(A occurring at least once)
$^3C_1(0.4) \times (0.6)^2 + {}^3C_2\,(0.4)^2 \times (0.6) + {}^3C_3\,(0.4)^3$

$$= \left(3 \times \frac{4}{10} \times \frac{36}{100} + 3 \times \frac{16}{100} \times \frac{6}{10} + \frac{64}{1000}\right)$$

$$= \frac{784}{1000} = 0.784.$$

Short cut:
P(A occurring at least once) = 1 – P (A not occurring in any trial) = 1 – 0.6 × 0.6 × 0.6 = 0.784.

15. c (i) For arbitrary events A, B
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ... (i)
Since the probability of an event is less than or equal to 1.
$\therefore P(A \cup B) \le 1.$
$\therefore P(A) + P(B) - P(A \cap B) \le 1$ [From (i)] or
$P(A \cap B) \ge P(A) + P(B) - 1.$
$\therefore$ Statement (I) is true.
(ii) Since the probability lies between 0 and 1, $1 \ge P(A \cup B) \ge 0.$
$\Rightarrow P(A) + P(B) - P(A \cap B) \ge 0$
$\Rightarrow P(A) + P(B) \ge P(A \cap B)$
$\therefore$ Statement (II) is true.
(iii) As $P(A \cup B) = P(A) + P(B) - P(A \cap B).$
$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B).$
$\therefore$ Statement (III) is also true. But statement (IV) is false.

16. c The total number of functions from A to itself is n^n and the total number of bijections from A to itself is n!. (Since A is a finite set, therefore every injective mapping from A to itself is bijective also.)

$$\therefore \text{The required probability} = \frac{n!}{n^n} = \frac{(n-1)!}{n^{n-1}}.$$

17. a $S = (\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \ldots\infty$

Here $a = (\sqrt{2} + 1)$, and the common ratio $r = \dfrac{1}{\sqrt{2}+1}$.

$$\therefore S = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-\dfrac{1}{\sqrt{2}+1}} = \frac{3+2\sqrt{2}}{\sqrt{2}}.$$

18. a

$$S = \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \ldots + \left(x^{10} + \frac{1}{x^{10}}\right)^2$$

$S = (x^2 + x^4 + x^6 + \ldots + x^{20}) +$

$$\left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ \ldots\ + \frac{1}{x^{20}}\right) + 20$$

$$= x^2\,\frac{(x^{20}-1)}{(x^2-1)} + \frac{1}{x^2}\,\frac{\left(1 - \dfrac{1}{x^{20}}\right)}{\left(1 - \dfrac{1}{x^2}\right)} + 20$$

$$= \left(\frac{x^{20}-1}{x^2-1}\right)\left(\frac{x^{22}+1}{x^{20}}\right) + 20.$$

19. b If a, b and c are in HP, then $\dfrac{2}{b} = \dfrac{1}{a} + \dfrac{1}{c}$.

$$\therefore \left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right) = \left(\frac{3}{b} - \frac{2}{a}\right)\left(\frac{1}{b}\right) = \frac{3}{b^2} - \frac{2}{ab}$$

Hence (c) is incorrect.
Again,

$$\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right) = \left(\frac{2}{c} - \frac{1}{b}\right)\left(\frac{1}{b}\right) = \frac{2}{bc} - \frac{1}{b^2}$$

Hence (a) is incorrect.

Now $\left(\dfrac{1}{b} + \dfrac{1}{c} - \dfrac{1}{a}\right)\left(\dfrac{1}{c} + \dfrac{1}{a} - \dfrac{1}{b}\right)$

$$= \left(\frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right)\right)$$

$$= \frac{1}{4}\left(\frac{1}{a} + \frac{1}{c}\right)^2 + \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = \frac{1}{4}\left(\frac{3}{c^2} + \frac{2}{ca} - \frac{1}{a^2}\right).$$

20. d Given, f(x + 2) + f(5x + 6) = 2x – 1
At, $x + 2 = 5x + 6 \Rightarrow x = -1$

$$\Rightarrow f(1) + f(1) = -3 \Rightarrow f(1) = \frac{-3}{2}.$$

21. b

$$\underbrace{(666\ldots6)}_{n\text{ digits}} = 6 + 60 + 600 + 6000 + \ldots + \underbrace{600000\ldots0}_{n\text{ digits}}$$

$$= 6 + 6.10 + 6.10^2 + \ldots + 6.10^{n-1} = \frac{2}{3}(10^n - 1).$$

$\therefore (666\ldots6)^2 = 4/9\ (10^n - 1)^2.$

Similarly, $(888\ldots8) = \dfrac{8}{9}(10^n - 1).$

$$\therefore \underbrace{(66\ldots6)}_{n\text{ digits}}^2 + \underbrace{(888\ldots8)}_{n\text{ digits}} = \frac{4}{9}(10^n - 1)^2 + \frac{8}{9}(10^n - 1)$$

$$= \frac{4}{9}(10^{2n} - 1).$$

22. c $f(x + 2, y + 1) = f(x, f(x + 1, y))$
At, $x = y = 0 \Rightarrow f(2, 1) = f(0, f(1, 0))$
Now, $f(x + 2, 0) = f(x + 1, 1)$
At $x = -1 \Rightarrow f(1, 0) = f(0, 1)$
$\therefore f(2, 1) = f(0, f(0, 1))$
Now, $f(0, y) = 2y + 1$
$\therefore f(0, 1) = 3$
$\therefore f(2, 1) = f(0, 3) = 2 \times 3 + 1 = 7.$

23. e We have

$$S_n = \frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + ... + \frac{1+2+3+...+n}{1^3+2^3+3^3+...+n^3}$$

or $S_n = a_1 + a_2 + a_3 + ... + a_n$ (say), where

$$a_n = \frac{1+2+...+n}{1^3+2^3+...+n^3} = \frac{\frac{n(n+1)}{2}}{\left(\frac{n(n+1)}{2}\right)^2} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\therefore S_n = 2\left(\frac{1}{1} - \frac{1}{2}\right) + 2\left(\frac{1}{2} - \frac{1}{3}\right) +$$

$$2\left(\frac{1}{3} - \frac{1}{4}\right) + ... + 2\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2\left(1 - \frac{1}{n+1}\right) = 2 - \frac{2}{n+1} < 2.$$

$$\therefore S_{35} = 2 - \frac{2}{36} = \frac{35}{18}.$$

24. e Let m^{th} term of the first sequence be equal to the n^{th} term of the second sequence. Then,
$17 + 4(m - 1) = 16 + 5(n - 1) \Rightarrow 4m + 13 = 5n + 11$
$\Rightarrow 4m + 2 = 5n$
$\Rightarrow n = m - \frac{1}{5}(m - 2)$. Since n is an integer, $m - 2$ must be a multiple of 5, i.e. m must be of the form $5k + 2$, with $k \geq 0$. Then $n = 4k + 2$. The first sequence has 101 terms and the second has 91 terms.
$\therefore 0 \leq 5k + 2 < 101$ and $0 \leq 4k + 2 \leq 91 \Rightarrow 0 \leq k \leq 19$.
Hence, the given sequences have 20 common terms.

25. a $f(g(x)) = 2 - 3x$ and $g(f(x)) = 1 + 3x$
$\therefore f[g(f(x))] = 2 - 3f(x)$
or, $f(1 + 3x) = 2 - 3f(x)$.

At $x = \frac{-1}{2}, f\left(\frac{-1}{2}\right) = 2 - 3f\left(\frac{-1}{2}\right)$

$\Rightarrow 4f\left(\frac{-1}{2}\right) = 2 \Rightarrow f\left(\frac{-1}{2}\right) = \frac{1}{2}.$

26. a $f(g(x)) = 2 - 3x$ and $g(f(x)) = 1 + 3x$
$g[f(g(x))] = 1 + 3g(x)$
$\Rightarrow g(2 - 3x) = 1 + 3g(x)$

At $x = \frac{1}{2}$

$\Rightarrow g\left(\frac{1}{2}\right) = 1 + 3g\left(\frac{1}{2}\right) \Rightarrow g\left(\frac{1}{2}\right) = \frac{-1}{2}$

$\therefore 2g\left(\frac{1}{2}\right) + f\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right) + \frac{1}{2} = \frac{-1}{2}.$

27. c Coefficient of $T_5 = {}^nC_4$, $T_6 = {}^nC_5$ and $T_7 = {}^nC_6$.
According to the condition, $2\,{}^nC_5 = {}^nC_4 + {}^nC_6$.

$$\Rightarrow 2\left[\frac{n!}{(n-5)!\ 5!}\right] = \left[\frac{n!}{(n-4)!\ 4!} + \frac{n!}{(n-6)!\ 6!}\right]$$

$$\Rightarrow 2\left[\frac{1}{(n-5)\ 5}\right] = \left[\frac{1}{(n-4)(n-5)} + \frac{1}{6 \times 5}\right]$$

After solving, we get n = 7 or 14.

28. a $T_{r+1} = {}^{2n}C_r\ x^{2n-r}\left(-2x^{-1}\right)^r$

$= {}^{2n}C_r\ x^{2n-r-r}(-2)^r$

$= 2n - 2r = 0 \Rightarrow r = n$
i.e. T_{n+1} is independent of x.

29. d $T_{r+1} = {}^9C_r\ (ax^4)^{9-r}\ (-bx)^r$
$\Rightarrow 36 - 4r + r = 18 \Rightarrow r = 6$
$\therefore$ Coefficient $= {}^9C_6\ (a^3)\ (-b)^6 = 84a^3b^6$.

30. c $\left(x^4 - \frac{1}{x^3}\right)^{15}$

$T_r = {}^{15}C_{r-1}\left(x^4\right)^{15-r+1}\left(\frac{-1}{x^3}\right)^{r-1}$

$= {}^{15}C_{r-1}(-1)^{r-1}x^{64-4r-3r+3} = {}^{15}C_{r-1}(-1)^{r-1}x^{67-7r}$

x^{-17} occurs in rth term

$\Rightarrow 67 - 7r = -17 \Rightarrow r = \frac{67+17}{7} = \frac{84}{7} = 12.$

$\therefore r = 12.$

31. c $(1+x)^m = {}^mC_0 + {}^mC_1x + {}^mC_2x^2 + ...$

$= 1 + mx + \frac{m(m-1)}{2}.x^2 + ...$

And

$(1-x)^n = {}^nC_0 + {}^nC_1(-x) + {}^nC_2(-x)^2 + ...$

$= 1 - nx + \frac{n(n-1)}{2}x^2 ...$

$$(1+x)^m(1-x)^n = \left(1 + mx + \frac{m(m-1)}{2}x^2 + ...\right)$$

$$\times\left(1 - nx + \frac{n(n-1)}{2}x^2 ...\right)$$

$\therefore$ m – n = 3 $\Rightarrow$ n = m – 3

$$\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn=-6$$

$\Rightarrow$ m(m – 1) + n(n – 1) – 2mn = –12

$\therefore$ m(m – 1) + (m – 3)(m – 4) – 2m(m – 3) = –12

$\Rightarrow m^2-m+m^2-7m+12-2m^2+6m=-12$

$\Rightarrow$ 2m = 24 $\Rightarrow$ m = 12.

32. b $\left(\sqrt{2}+3^{\frac{1}{5}}\right)^{10}=\left(2^{\frac{1}{2}}+3^{\frac{1}{5}}\right)^{10}$

$T_{r+1}={}^{10}C_r\,2^{\frac{10-r}{2}}.3^{\frac{r}{5}}$ be a rational term

$\Rightarrow$ r is a multiple of 5 and as $0\le r\le 10$

r = 0, 5 or 10

$\frac{10-r}{2}$ is also an integer $\Rightarrow$ $r\ne 5$

$\Rightarrow$ r = 0 or 10

Rational terms in its expansion are first and last, all other terms contain irrational parts.

$\therefore$ Sum of rational terms

$$={}^{10}C_0\left(2^{\frac{1}{2}}\right)^{10}+{}^{10}C_{10}\left(3^{\frac{1}{5}}\right)^{10}=2^5+3^2=32+9=41.$$

33. a We have

$$\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}=y\Rightarrow\frac{1+x^2-x}{1+x^2+x}=y,$$

where $\tan\theta=x$

$\Rightarrow x^2(y-1)+x(y+1)+y-1=0$

Since $\tan\theta = x$ is real, therefore Discriminant ≥ 0

$\Rightarrow (y+1)^2-4(y-1)^2\ge 0\Rightarrow -3y^2+10y-3\ge 0$

$\Rightarrow 3y^2-10y+3\le 0\Rightarrow y\in\left[\frac{1}{3},3\right]$

34. a $\dfrac{\frac{1}{1-(-\sin x)}}{\frac{1}{1-\sin x}}=\dfrac{1-\cos 2x}{1+\cos 2x}\Rightarrow\dfrac{1-\sin x}{1+\sin x}=\dfrac{2\sin^2 x}{2\cos^2 x}$

$=\dfrac{2\sin^2 x}{2-2\sin^2 x}$, provided |sin x| < 1

$\Rightarrow 2(1-\sin x)^2(1+\sin x)=2\sin^2 x(1+\sin x)$, i.e.

$\Rightarrow (1-\sin x)^2=\sin^2 x$ as $1+\sin x\ne 0$

$\therefore \sin x=\frac{1}{2}\Rightarrow x=\frac{\pi}{6}.$

35. a Rotation of axes does not affect the perpendicular distance of the line from origin. If the equations of lines in two cases are $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x'}{p}+\frac{y'}{q}=1$, then

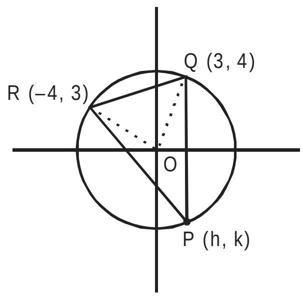
$$\left|\frac{0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|=\left|\frac{0+0-1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}\right| \text{ or } \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}.$$

36. d $S = x^2 + y^2 + 2x - 4y + 4$

$S|_{x=-1,\,y=2} = 1 + 4 - 2 - 8 + 4 = -1$

$S|_{x=-1,\,y=2} < 0$, therefore the point (–1, 2) lies inside the circle, therefore no tangents can be drawn.

37. c



Centre O(0, 0)

Slope of RO = $-\frac{3}{4}=m_1$

Slope of QO = $\frac{4}{3}=m_2$. Since $m_1m_2 = -1$, $\angle ROQ = \pi/2$

and hence $\angle QPR = \pi/4$, as angle at the centre is double angle at the circumference.

Alternate solution:

$$\frac{\sin(\angle QPR)}{QR}=\frac{1}{2R}$$

$$\Rightarrow \sin(\angle QPR)=\frac{QR}{2R}=\frac{\sqrt{7^2+1^2}}{10}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \angle QPR=\frac{\pi}{4}$$

38. d C_1 (–a, 0); C_2 (0, –b); $R_1 = \sqrt{a^2 - c}$; $R_2 = \sqrt{b^2 - c}$;

$C_1C_2 = \sqrt{a^2 + b^2}$, since, they touch each other, therefore

$\sqrt{a^2 - c} + \sqrt{b^2 - c} = \sqrt{a^2 + b^2}$ (assume external touching)

$\Rightarrow a^2b^2 - b^2c - a^2c = 0$ or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

Note that condition for internal touching also gives the same result.

39. b $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1+x^2} = \frac{\pi}{3}$

Putting $x = \tan\theta$, $3\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

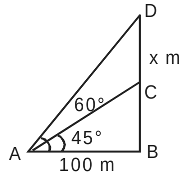
$-4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + 2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{\pi}{3}$

$\Rightarrow 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$

$\Rightarrow 3\ (2\theta) - 4\ (2\theta) + 2\ (2\theta) = \frac{\pi}{3} \Rightarrow 6\theta - 8\theta + 4\theta = \frac{\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{6} \Rightarrow \tan^{-1} x = \frac{\pi}{6} \Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

40. c



Let BC be the incomplete and BD be the complete pillar. In ΔABC and ΔABD, we have

$\tan 45^\circ = \frac{BC}{AB}$ and $\tan 60^\circ = \frac{BD}{AB}$

$\Rightarrow BC = 100$ m and $BD = 100\sqrt{3}$ m

$\Rightarrow BC + CD = 100\sqrt{3}$

$\Rightarrow 100 + x = 100\sqrt{3} \Rightarrow x = 100\ (\sqrt{3} - 1)$ m.