CATapult Courseware

Module 4
Quantitative Ability

Published by IMS Learning Resources Pvt. Ltd. in the Year 2020

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QA-4.1 CYCLICITY, BASE SYSTEM AND LOGARITHMS

THEORY

Cyclicity

When a number is raised to some power, the unit's place digit of the result follows a particular pattern. This pattern is given by the rule of cyclicity.

The rule of cyclicity for numbers which end with digits 0-9 can be devised by the following table:

Number 'a' ending with	a^{4n-3}	a^{4n-2}	a^{4n-1}	a^{4n}
0	0	0	0	0
1	1	1	1	1
2	2	4	8	6
3	3	9	7	1
4	4	6	4	6
5	5	5	5	5
8	8	4	2	6
9	9	1	9	1

Where 'a' and 'n' are natural numbers.

Concept Builder 1

Find the last digit of the following:

1. 449^{219}
2. 387^{287}
3. $1^{781} + 2^{781} + \ldots\ldots\ldots + 9^{781}$
4. 302^{25}
5. $14^{13} \times 13^{14}$

Answer Key

1. 9 2. 3 3. 5
4. 2 5. 6

SOLVED EXAMPLES

Q : What will be the last digit of the number 34^{41}

A : Since 4 is in the unit digits we follow the cyclicity rule of 4

34^1 will give a units digit of 4

34^2 will give a units digit of 6

34^3 will give a units digit of 4

34^4 will give a units digit of 6

Hence the pattern repeats after 2 steps

Every odd power of 34 will give a units digit of 6

$\therefore$ 34^{41} will give a units digit of 4

Q : What will be the last digit of the sum of $2077^{19} + 1077^{21}$?

A : Since both numbers end with 7 they follow the cyclicity rule for 7

For both numbers,

Power of 1 will give a units digit of 7

Power of 2 will give a units digit of 9

Power of 3 will give a units digit of 3

Power of 4 will give a units digit of 1

Power of 5 will give a units digit of 7

Hence the pattern repeats after 4 steps

2077^{19} will give a units digit of 3

1077^{21} will give a units digit of 7

So the sum of these 2 numbers will give a units digit (of 3 + 7 = 10 i.e.,) 0

Base System

The Base system is an extension of the system of digits.

Every base corresponds to the number of digits in a particular base system.

We use the decimal system i.e. Base system 10, which has ten digits or symbols i.e. 0, 1, 2....9.

Similarly, the binary system (Base 2) has 2 digits i.e. 0 and 1.

Octal system (Base 8) has eight digits i.e. 0 to 7.

But there can be bases beyond the decimal system as well where we are required to show numbers 10, 11, 12 ...... and so on as digits. So we use symbols i.e. A, B, C .... and so on to represent the integral values beyond the number 9.

For example

$10 \rightarrow$ A

$11 \rightarrow$ B

$12 \rightarrow$ C..... And so on.

Therefore hexadecimal system will have 16 digits i.e. 0, 1, 2,...........9, A B, C ...F. where A is equal to 10, B is equal to 11 and so on, till the last digit 15 which is represented by F.

Base System conversion

1. Conversion from Decimal to any other base system 'X':

Any decimal number can be converted into a number with base 'X', by dividing the number by 'X', and then successively dividing the quotients by 'X'. The remainders written in reverse order gives the equivalent number in Base 'X'.

Example

Convert 17 to Binary system.

$$\begin{array}{r|ll} 2 & 17 & \\ \hline & 8 & 1 \\ & 4 & 0 \\ & 2 & 0 \\ & 1 & 0 \\ & 0 & 1 \end{array}$$

$\therefore (17)_{10} = (10001)_2$

Convert $(510)_{10}$ to Octal system.

$$\begin{array}{r|ll} 8 & 510 & \\ \hline & 63 & 6 \\ & 7 & 7 \\ & 0 & 7 \end{array}$$

$\therefore (510)_{10} = (776)_8$

2. Conversion from any other base 'X' to the decimal system:

Any number with a base of 'X' can be converted into equivalent decimal number by adding the product of the digits of the number from the right to the left with increasing powers of X.

Example

$(101001)_2 = [1(2)^5 + 0(2)^4 + 1(2)^3 + 0(2)^2 + 0(2)^1 + 1(2)^0]_{10}$

$= [32 + 0 + 8 + 0 + 0 + 1]_{10} = (41)_{10}$

$(4312)_8 = [4(8)^3 + 3(8)^2 + 1(8)^1 + 2(8)^0]_{10} = [2048 + 192 + 8 + 2] = (2250)_{10}$

3. Conversion from any base 'X' to base 'Y':

A number with base 'X' can be converted to an equivalent decimal number (shown in point 2 above) and then from decimal form can be converted into a number with base 'Y' (shown in point 1 above)

Example

$(17)_9 \rightarrow (\ \ )_8$

As mentioned above $(17)_9$ should be converted to a decimal number and then to a number with base 8

$(17)_9 \rightarrow (\ \ )_{10}$

$= [1 \times 9^1 + 7 \times 9^0]$

$= (16)_{10}$

$(16)_{10} \rightarrow (\ \ )_8$

$$\begin{array}{r|rr} 8 & 16 & \\ \hline 8 & 2 & 0 \\ \hline & 0 & 2 \end{array} \qquad \therefore\ (16)_{10} = (20)_8$$

$\therefore\ (17)_9 \rightarrow (20)_8$

Place value and scientific notation

The number 345 can be written as 300 + 40 + 5 or as $(3 \times 10^2) + (4 \times 10^1) + (5 \times 10^0)$. The digit 3 stands for 3 times 100; the digit 4 stands for 4 times 10; and the digit 5 stands for 5 time 1. We say these digits have the following **place values:**

3 is in the Hundred's place
4 is in the Ten's place
5 is in the Unit's place

Every digit in a decimal number has a place value. If there are digits before 345 in the number, the digits to the left of 345 are at the **thousands** place, the **ten-thousands** place, the **lakh** place, the **ten-lakh** place and so on.

Digits to the right of the decimal point also have place values. For example $0.56 = 0.5 + 0.06 = (5 \times 10^{-1}) + (6 \times 10^{-2})$; the digit 5 is in the **tenths** place, and the digit 6 is in the **hundredths** place. If there are digits beyond 6 in the number 0.56 then the place immediately to the right of 6 are the **thousandth** place, the **ten-thousandth** place and so on.

Sometimes, using the concept of place value can let you write a very big or small number in a simpler mathematical form.

For example: $3{,}40{,}00{,}000 = 3.4 \times 10^7$ or even $0.0000000054 = 5.4 \times 10^{-9}$

Because such numbers often occur in scientific calculations, writing a very large or small number as a product of a power of 10 and a number between 1 and 10 is called **scientific notation.**

SOLVED EXAMPLES

Q : On converting $(113254)_6$ to base 10, what is the number obtained?

A : $(113254)^6 = 1 \times 6^5 + 1 \times 6^4 + 3 \times 6^3 + 2 \times 6^2 + 5 \times 6^1 + 4 \times 60$

$= 7776 + 1296 + 648 + 72 + 30 + 4 = (9826)_{10}$.

Q : Convert $(148963)_{10}$ to base 12.

A :

12	148963	
	12413	7
	1034	5
	86	2
	7	2
	0	7

$\therefore (148963)_{10} = (72257)_{12}$

Concept Builder 2

Solve the following:

1. What will be value of the decimal number 313 in base 8?
2. When a binary number 1001 is converted to decimal system, what will be its value?

Answer Key

1. 471 2. 9

Logarithms

Definition of logarithm

Logarithm

For a positive number n, the logarithm of n is the power to which some number b must be raised to give n. b is called the base of the logarithm.

If $b^x = n$ then $\log_b n = x$.

$b^x = n$ is called the exponential form and $\log_b n = x$ is called the logarithmic form.

Example

$10^2 = 100 \Rightarrow \log_{10}100 = 2$

$2^3 = 8 \Rightarrow \log_2 8 = 3$

Laws of logarithm

(i) $\log_b 1 = 0$

(ii) $\log_a a = 1$

(iii) $\log_a b \times \log_b a = 1$

(iv) $\log_b(m \times n) = \log_b m + \log_b n$

(v) $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$

(vi) $\log_b (m^n) = n \log_b m$

(vii) $\log_b m = \frac{\log_a m}{\log_a b} = \log_a m \times \log_b a$

(viii) $\log_{b^\beta}(a^\alpha) = \frac{\alpha}{\beta}\log_b a$

(ix) $b^{\log_b n} = n$

Logarithms to the base 10 are called common logarithms and to the base ‘e’ are called Natural logarithms. Logarithms of 0 and negative numbers are not defined.

Concept Builder 3

1. Is $\text{Log } 2^2 = (\text{Log } 2)^2$
2. If $\text{Log}_a 32 = 5$ then find 'a'.
3. If $\text{Log}_p A = \text{Log}_q A$, where $p, q, A \in N$, then is $p = q$?
4. If Log 56 − Log 4 = Log x, then find x.
5. $\text{Log}_3 9^2 = ?$
6. If f(x) = [Log x], then find f(3456.7)
7. Which is greater: $A = \text{Log}_{12} 222$ or $B = \text{Log}_{24} 444$
8. $\text{Log}_{0.125} 2 = ?$
9. If N is a 13 digit number then [Log N] = ?
   Note: [ ] indicates 'Greatest Integer Function'

Answer Key

1. No
2. 2
3. Not necessarily
4. 14
5. 4
6. 3
7. A
8. $-\frac{1}{3}$
9. 12

SOLVED EXAMPLES

Q : Find the value of $\log_8 128$?

A : Let $\log_8 128 = x$

$8^x = 128$

$(2^3)^x = 2^7$

$\therefore 3x = 7$

$\therefore x = \frac{7}{3}$ $\quad\quad$ $\therefore \log_8 128 = \frac{7}{3}$

Q : If $\log_{10} 3 = 0.4771$ and $\log_{10} 7 = 0.8451$, find the value of $\log_{10} 2\frac{1}{3}$.

A : $\log_{10} 2\frac{1}{3} = \log_{10} \frac{7}{3} = \log_{10}(7) - \log_{10}(3) = 0.8451 - 0.4771 = 0.3680$

Q : If $\log_{10} x + \log_{10} 5 = 2$ find the value of x.

A : $\log_{10} x + \log_{10} 5 = 2$

$\Rightarrow \log_{10}(x \times 5) = 2 \Rightarrow 10^2 = 5x \Rightarrow 5x = 100 \quad \therefore\ x = 20$

Q : Solve $\log_{\sqrt{3}}\left(\log_{\sqrt{3}} x\right) = 2$.

A : $\log_{\sqrt{3}}\left(\log_{\sqrt{3}} x\right) = 2$

$\left(\sqrt{3}\right)^2 = \log_{\sqrt{3}} x$ ... (as $\log_a b = x \Rightarrow a^x = b$)

$3 = \log_{\sqrt{3}} x$

$\left(\sqrt{3}\right)^3 = x \quad \therefore\ x = 3\sqrt{3}$

Q : Find $\log_{3\sqrt{2}} 5832$.

A : $\left(3\sqrt{2}\right)^x = 5832$

$= 8 \times 729 = 8 \times 9^3 = 2^3 \times 3^6 = \left(\sqrt{2}\right)^6 \times 3^6 = \left(3\sqrt{2}\right)^6 \quad \therefore\ x = 6$

Q : Simplify: $\log \dfrac{3 \times \sqrt{32}}{40 \times \sqrt[3]{18}}$.

A : $\log\left(3 \times 32^{\frac{1}{2}}\right) - \log\left(40 \times 18^{\frac{1}{3}}\right)$

$= \log 3 + \dfrac{1}{2} \log 32 - \log 40 - \dfrac{1}{3} \log 18$

$= \log 3 + \dfrac{1}{2} \log 2^5 - \log(5 \times 2^3) - \dfrac{1}{3} \log(2 \times 3^2)$

$= \log 3 + \dfrac{5}{2} \log 2 - \log 5 - 3 \log 2 - \dfrac{1}{3} \log 2 - \dfrac{2}{3} \log 3$

$= \left(1 - \dfrac{2}{3}\right) \log 3 + \left(\dfrac{5}{2} - 3 - \dfrac{1}{3}\right) \log 2 - \log 5$

$= \dfrac{1}{3} \log 3 - \dfrac{5}{6} \log 2 - \log 5$

CLASS EXERCISE

Teaser

Imagine you are at a school that has 100 lockers for 100 students, all shut. The first student goes along the row and opens every locker. The second student then goes along and shuts every other locker beginning with locker number 2. The third student changes the state of every third locker beginning with locker number 3. (If the locker is open the student shuts it, and if the locker is closed the student opens it.) The fourth student changes the state of every fourth locker beginning with number 4. This goes on till the hundredth student who toggles the state of the last locker.

How many lockers will be open at the end?

1. Find the last digit of
   a) 653×247
   b) $23 \times 24 \times 25 \times 26 \times 27$
   c) 12^{12}
   d) 123^{23}
   e) 1234^{34}
   f) 12345^{45}
   g) $(67^8)^9$

2. Find the last two digits of
   a) 123×321
   b) 7^{777}
   c) 2007^{2007}

The last digit of the product is the product of the last digits of the individual numbers

For any number, the last digit repeats in a cycle of (maximum) 4 steps as shown below:

If x ends in	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
then x^2 ends in	0	1	4	9	6	5	6	9	4	1
then x^3 ends in	0	1	8	7	4	5	6	3	2	9
then x^4 ends in	0	1	6	1	6	5	6	1	6	1
then x^5 ends in	0	1	2	3	4	5	6	7	8	9

For a number ending in 0, 1, 5 or 6: The last digit of any power will remain the same

For a number ending in 4 or 9: The last digit will depend only on whether the power is odd or even – $(xx4)^{odd}$ ends in 4, $(xx4)^{even}$ ends in 6, $(xx9)^{odd}$ ends in 9 and $(xx9)^{even}$ ends in 1.

For a number ending in 2, 3, 7, 8: Find the remainder of the power when divided by 4. For remainder 1, 2, 3 or 0, we raise the last digit to 1, 2, 3 or 4 and note the resultant last digit.

Base Systems

3. Convert the number $(24)_{10}$ to the base 5

4. Convert the number $(44)_5$ to the base 10

5. Given the number 55 (in the decimal system), convert it into:
   a) Base 9
   b) Octal
   c) Base 5
   d) Binary
   e) Base 11
   f) Hexadecimal

6. Convert the number $(122)_x$ to base 10, where x equals:
   a) 3
   b) 5
   c) 7
   d) 9
   e) 11
   f) 16

7. Convert $(250)_{10}$ to the hexadecimal system

8. Add the numbers $(125)_6$ and $(232)_6$

To convert a number in base n to base 10:
Multiply each digit (starting from the last) by a corresponding power of the base (starting from n^0)
For example: Convert $(2134)_6$ to base 10 = $4 \times 6^0 + 3 \times 6^1 + 1 \times 6^2 + 2 \times 6^3 = 4 + 18 + 36 + 432 = (490)_{10}$

To convert a number in base 10 to a base n:
Divide the number by n and note the quotient and remainder. Take the quotient, divide it by n again, and repeat till the quotient becomes zero. Write the successive remainders in reverse order. For example:
Convert $(92)_{10}$ to base 8: 91/8 (q = 11, r = 4); 11/8 (q = 1, r = 3), 1/8 (q = 0, r = 1) $(138)_8$

A number appears larger when converted to a smaller base (and vice-versa)

A number in a base n can have n possible digits, ranging from 0 to (n–1)

9. *Find the value of $(1443)_5 + (3112)_5$

10. *Convert $(12C)_{16}$ to base 10

11. *Convert $(1011010101)_2$ to base 8 and base 16

Logarithms

12. Find the value of:

a) $\log_5 25$ b) $\log_5 \frac{1}{25}$ c) $\log_{1/5} 25$

d) $\log_5 \sqrt{5}$ e) $\log_{\sqrt{5}} 5$ f) $\log_5 \sqrt[3]{25}$

g) $\log_5 1$ h) $\log_5 (125 \times 625)$ i) $\log_5 5\sqrt{5} + \log_5 5\sqrt{5}$

f) $\log_5 125^5$ g) $\log_{125} 625$

13. If $\log 5 \approx 0.7$ find:

a) $\log 25$ b) $\log \frac{1}{5}$ c) $\log 0.5$ d) $\log 0.2$

14. If $\log_y x + \log_{2\sqrt{y}} x + \log_{3\sqrt{y}} x + \log_{4\sqrt{y}} x + \log_{5\sqrt{y}} x = 60$, then what is the relation between x and y?

15. If $\log 2 \approx 0.3$, find the number of digits in 2^{34}

Challengers

1. Find the last digit of

a) 9^{9^9}

1) 1 2) 3 3) 7 4) 9

b) $808^{9!}$

1) 2 2) 4 3) 6 4) 8

2. Find the value of $\log 0.2 + \log 0.4 + \log 0.8 + \log 0.5 + \log 0.25 + \log 0.125$

1) -3 2) -4 3) -5 4) -6

3. Find the sum of $\log_{\sqrt{3}} 3 + \log_{\sqrt{3}} 9 + \log_{\sqrt{3}} 27$ ...up to n terms

1) 2n 2) $\frac{n(n+1)}{2}$ 3) 3n 4) $n^2 + n$

4. A car was brought in for servicing when its odometer read 2500 km. During the servicing, it was found that the odometer had a fault because of which it always skipped the digits 4 and 6. How much distance had the car actually travelled?

1) 1250 km 2) 1344 km 3) 2000 km 4) 1280 km

5. If $\log 5 \approx 0.7$, find the number of leading zeroes in 25^{-12}

1) 17 2) 16 3) 8 4) 15

PRACTICE EXERCISE-1

DIRECTIONS for questions 1 and 2: Solve as directed.

1. Find the last digit of
   a) 98^{98}
   b) 169^{961}
   c) 73^{37}
   d) $2019^{2017^{2015^{2013}}}$

2. Find the equivalent in base 10 of
   a) The first 4-digit number in base 3
   b) The last 3-digit number in base 8
   c) the number 121 in base 7
   d) the largest 3-digit number with distinct digits in base 5
   e) the smallest 4-digit number with distinct digits in base 6

DIRECTIONS for questions 3 to 14: Choose the correct alternative.

3. $(4654)_7$ in base 10 when divided by $(14)_{10}$ gives a remainder of:

   1) 0 2) 11 3) 7 4) 3

4. Addition of HCF and LCM of two numbers $(3565)_{10}$ and $(1773)_{10}$ is carried out. The result of the addition in base 5 is written as:

   1) 3104230441 2) 3014230442 3) 302430410 4) 3104203443

5. Suppose, $\log_3 x = \log_{12} y = a$, where x, y are positive numbers. If G is the geometric mean of x and y, and $\log_6 G$ is equal to:

   1) $\sqrt{a}$ 2) $2a$ 3) $\frac{a}{2}$ 4) a

   (Past CAT question)

6. The value of $\log_{0.008}\sqrt{5} + \log_{\sqrt{3}} 81 - 7$ is equal to

   1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) $\frac{5}{6}$ 4) $\frac{7}{6}$

   (Past CAT question)

7. If x is a real number such that $\log_3 5 = \log_5(2 + x)$, then which of the following is true?

   1) $0 < x < 3$ 2) $23 < x < 30$ 3) $x > 30$ 4) $3 < x < 23$

   (Past CAT question)

8. What is the value of

$\log_{a_2} a_1^2 \times \log_{a_3} a_2^2 \times \log_{a_4} a_3^2 \times .... \times \log_{a_n} (a_{n-1})^2 \times \log_{10} a_n$?

1) 1 2) $2^n \log_{10} a_1$ 3) $2^{n-1} \log_{10} a_1$ 4) $2^{n-3} \log_{10} a_1$

9. If $\log_{81} 32 = \frac{5}{4} \log_{11} 2 \times \log_a b$, then find the value of b given $3 \leq a \leq 11$.

1) $11 \leq b < 121$ 2) $11 \leq b < 1331$ 3) $3 \leq b \leq 11$ 4) $121 \leq b < 1331$

10. If a, b and c are three positive numbers such that $\log\left(\frac{a}{b}\right)$, $\log\left(\frac{b}{a}\right)$ and $\log\left(\frac{bc}{a^2}\right)$ are in A.P., then which of the following is true?

1) $2b = a + c$ 2) $c = \frac{ab}{a+b}$ 3) $b^2 = ac$ 4) $ab = c^2$

11. How many roots does the equation $x^2 = \log_{10} x$ have?

1) 0 2) 1 3) 2 4) More than 2

12. If $\log_y x = (a \cdot \log_z y) = (b \cdot \log_x z) = ab$, then which of the following pairs of values for (a, b) is not possible?

1) $(-2, \frac{1}{2})$ 2) (1, 1) 3) (0.4, 2.5) 4) $(\pi, \frac{1}{\pi})$ 5) (2, 2)

(Past CAT question)

13. In a number system the product of 44 and 11 is 1034. The number 3111 of this system, when converted to the decimal number system, becomes ___________.

1) 406 2) 1086 3) 213 4) 691

(Past CAT question)

14. Two numbers, $(297)_B$ and $(792)_B$, belong to base B number system. If the first number is a factor of the second number then the value of B is:

1) 11 2) 12 3) 15 4) 17 5) 19

(Past XAT question)

DIRECTIONS for question 15: Solve as directed.

15. The average of three numbers, $\log_2 a$, $\log_4 a^2$ and $\log_8 a^3$, is 3. What is the value of a?

DIRECTIONS for questions 16 to 22: Choose the correct alternative.

16. Which of the following is equivalent to $\frac{\log_4 7 + \log_3 7}{\log_4 7 \times \log_3 7}$?

1) $\log_{12} 7$ 2) $\log_7 4$ 3) $\log_7 3$ 4) $\log_7 12$

17. If $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \ldots + \frac{1}{\log_9 x} = \frac{1}{\log_2 y} + \frac{1}{\log_3 y} + \ldots + \frac{1}{\log_8 y} = 2$, find $\frac{x}{y}$.
(Given x, y > 1)

1) 3 2) 1 3) $\frac{1}{3}$ 4) $\frac{9}{8}$

18. If log2 = 0.3010, then the number of digits in 5^{30} is

1) 19 2) 20 3) 21 4) 22

19. If $\log_{(x-a)}(8x^3 - 36x^2 + 54x - 27) = 3 + \log_{(x-a)}8$ then find the value of 'a' if it is a real number.

1) 3 2) 2 3) $\frac{3}{2}$ 4) $\frac{2}{3}$

20. If $x \geq y$ and $y > 1$, then the value of the expression $\log_x\left(\frac{x}{y}\right) + \log_y\left(\frac{y}{x}\right)$ can never be

1) −1 2) −0.5 3) 0 4) 1

(Past CAT question)

21. Let $u = (\log_2 x)^2 - 6\log_2 x + 12$ where x is a real number. Then the equation $x^u = 256$, has

1) no solution for x 2) exactly one solution for x
3) exactly two distinct solutions for x 4) exactly three distinct solutions for x

(Past CAT question)

22. If $\log x - \log\sqrt{x} + \log\sqrt[3]{x} - \log\sqrt[4]{x} = \frac{7}{24}$, which of the following is true about the value of $\log\left(\frac{x^3 - x^2 + x - 1}{x - 1}\right)$? (All logarithms are to the base 10.)

1) It is between 0.25 and 0.5. 2) It is between 0.5 and 1.
3) It is between 1 and 1.5. 4) It is between 1.5 and 2.

PRACTICE EXERCISE-2

DIRECTIONS for questions 1 to 20: Choose the correct alternative.

1. The rightmost non-zero digit of the number 20^{2520} is:
   1) 2 2) 4 3) 6 4) 8

2. Find the digit in 469th place from right of 90^{468}.
   1) 1 2) 3 3) 9 4) 0

3. When we convert $(4760)_{10}$ to base 12, the number obtained is:
   1) 2908 2) 2818 3) 3012 4) 2753

4. $(A)_B \equiv$ A is an integer, B is the base system. $\frac{(3)_8 \times (16)_8 \times (5)_{10}}{(3)_4 \times (101)_2} = (A)_{10}$. What is the value of A?
   1) 14 2) 16 3) 12 4) None of these

5. What are the last two digits of 7^{2008}?
   1) 21 2) 61 3) 01 4) 41 5) 81

   (Past CAT question)

6. What is the digit in the unit's place of 2^{51}?
   1) 2 2) 8 3) 1 4) 4

   (Past CAT question)

7. If $\log_{12}18 = \frac{x + 2y}{2x + y}$, find $\frac{x}{y}$.
   1) $\log_3 2$ 2) $\log_2 3$ 3) $\log_2 4$ 4) $\log_2 9$

8. If $\log_{12}81 = p$, then $3\left(\frac{4 - p}{4 + p}\right)$ is equal to:
   1) $\log_2 8$ 2) $\log_6 16$ 3) $\log_6 8$ 4) $\log_4 16$

   (Past CAT question)

9. If x is a positive quantity such that $2^x = 3^{\log_5 2}$, then x is equal to
   1) $1 + \log_3 \frac{5}{3}$ 2) $\log_5 9$ 3) $\log_5 8$ 4) $1 + \log_5 \frac{3}{5}$

   (Past CAT question)

10. If $\log_2(5 + \log_3 a) = 3$ and $\log_5(4a + 12 + \log_2 b) = 3$, then a + b is equal to:

1) 40 2) 67 3) 59 4) 32

(Past CAT question)

11. If $p^3 = q^4 = r^5 = s^6$, then the value of $\log_s(pqr)$ is equal to

1) $\frac{47}{10}$ 2) $\frac{16}{5}$ 3) $\frac{24}{5}$ 4) 1

(Past CAT question)

12. $\frac{1}{\log_2 100} - \frac{1}{\log_4 100} + \frac{1}{\log_5 100} - \frac{1}{\log_{10} 100} + \frac{1}{\log_{20} 100} - \frac{1}{\log_{25} 100} + \frac{1}{\log_{50} 100}$?

1) 0 2) $\frac{1}{2}$ 3) 10 4) −4

(Past CAT question)

13. When the curves $y = \log_{10}x$ and $y = x^{-1}$ are drawn in the x-y plane, how many times do they intersect for values $x \geq 1$?

1) Never 2) Once 3) Twice 4) More than twice.

(Past CAT question)

14. If $\log_3 2$, $\log_3(2^x - 5)$, $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, then the value of x is equal to:

1) 5 2) 4 3) 2 4) 3

(Past CAT question)

15. The micro manometer in a certain factory can measure the pressure inside the gas chamber from 1 unit to 999999 units. Lately this instrument has not been working properly. The problem with the instrument is that it always skips the digit 5 and moves directly from 4 to 6. What is the actual pressure inside the gas chamber if the micro manometer displays 0030l6?

1) 2201 2) 2202 3) 2600
4) 2960 5) None of the above options

(Past XAT question)

16. If log A, log B and log C are in arithmetic progression, what can be said about B, AC, and ABC?

1) They are in arithmetic progression.
2) They are in geometric progression.
3) They are in harmonic progression.
4) Nothing can be conclusively said about A, B and C

17. If log 2 = 0.3010 and log 3 = 0.4771, find the number of digits in $(324)^5$.

1) 15 2) 14 3) 13 4) 12

18. For how many three-digit numbers 'abc', $(abc)_8 - (abc)_6 = (90)_{10}$, where $(n)_6$, $(n)_8$ and $(n)_{10}$ represent number 'n' in base 6, 8 and 10 respectively?

1) 4 2) 6 3) 8 4) More than 8

19. Arindham was given a 2-digit base 4 number. He had to convert the number to base 10 and find the square of the number. He mistook the base of the number to be 5. Which of the following cannot be a valid difference between the actual and the wrong answer in base 10?

1) 93 2) 40 3) 44 4) 79

(Past CAT question)

20. A positive whole number M less than 100 is represented in base 2 notation, base 3 notation, and base 5 notation. It is found that in all three cases the last digit is 1, while in exactly two out of the three cases the leading digit is 1. Then M equals

1) 31 2) 63 3) 75 4) 91

(Past CAT question)

DIRECTIONS for question 21: Solve as directed.

21. If x & y are 2-digit positive integers, then how many solutions does $(1 + \log_2 x)(\log_2 x - 6) = y$ have?

QA-4.2 | SET THEORY

THEORY

Introduction

A set is a well-defined collection of objects. The objects, which belong to the set, are called elements of the set. The elements of a set may be objects, persons, letters, numbers etc. A set of pupils weighing 30 kg or over, is a well defined set whereas a set of heavy pupils, is not well defined.

A set is denoted by a capital letter and the elements of a set are denoted by small letters. The Greek letter epsilon '$\in$' is used to denote that an element 'belongs to' a set. The symbol '$\notin$' denotes 'does not belong to'.

Methods of Representation

1. Tabulation form (Roster form)

A set is described by listing all its elements enclosed in curly brackets. The elements are seperated by commas and each element is written only once.

Example

P is a set of all the letters in the word 'perpendicular'.
$P \equiv \{p, e, r, n, d, i, c, u, l, a\}$
A is a set of all even numbers less than 20.
$A \equiv \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

2. Set Builder form

In this form, a set is defined by specifying the property, which determines the elements of the set uniquely.

Example

$P \equiv \{p \mid p$ is a letter in the word 'perpendicular'$\}$
$A \equiv \{a : a$ is an even number less than $20\}$
The curly braces stand for 'the set of', the lower case italic letter represents any element of the given set, the stroke or colon stands for 'such that'. Set P is read as 'P is a set of all p such that p is a letter in the word perpendicular'.

Classification of Sets

1. Finite Set

A set is finite if it consists of definite number of different elements.

Example

V is a set of all vowels in the English language. V {a, e, i, o, u}

2. Infinite Set

A set, which is not finite i.e., the elements of which are too innumerable to count, is called an infinite set.

Example

P ≡ {p | p is a point on a line between the distinct points A and B}
N ≡ {n | n is a natural number}

3. Empty Set or Null Set

The set, which contains no elements at all, is called an empty set or null set. The empty set is written as { } or ϕ

Example

A {a | a is an even prime number greater than five} i.e., A ≡ { }

4. Singleton Set

A set containing only one element is a singleton set.

Example

A ≡ {b | b is an integer lying between 5 and 7}
A ≡ {6}

5. Equal Sets

Two sets are said to be equal if they contain exactly the same elements.

Example

A {x | x is a letter in the word ‘area’} i.e., A ≡ {a, r, e}
B {y | y is a letter in the word ‘ear’} i.e., B ≡ {a, r, e}
Here, A and B are equal sets.

6. Subsets

A set ‘A’ is a subset of set ‘B’ if and only if every element of A is an element of B. The symbol $\subseteq$ is used to denote ‘subset of’. If A is a subset of B (denoted by $A \subseteq B$), then B is called a superset of A (denoted by $B \supseteq A$)

Example

A ≡ {a | a ∈ N and a is a multiple of 5}
i.e., A ≡ {5, 10, 15, 20, ...}
B ≡ {b | b ∈ N and b is a multiple of 10}
B ≡ {10, 20, ...} B ⊆ A

If a set B is a subset of A and B is not equal to A, then B is a proper subset of A. The symbol ⊂ is used to denote proper subset.

About Subsets

(i) Every set is a subset of itself.

(ii) The empty set is a subset of every set; $\phi \subset A$, $\phi \subseteq \phi$.

(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(iv) A set having n elements has 2^n subsets, including the set itself and the null set and $2^n - 1$ number of proper subsets i.e., excluding the set itself.

7. Power set

The collection of all the subsets of a set is known as the 'Power Set' of that set.

Example

If A = {1, 2}, then Power Set of A or P(A) = { {}, {1}, {2}, {1, 2} }

8. Universal Set

Consider the sets A, B, C and D. Any set of which A, B, C and D are subsets is called the universal set. The universal set is usually denoted by U.

Example

(i) The universal set is the set of Real numbers, while considering the set of Natural numbers, Whole numbers, Integers and Rational numbers.

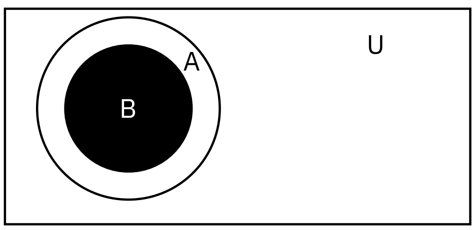
(ii) The set of alphabets is the universal set from which the letters of any word may be chosen to form a set.

Venn Diagrams

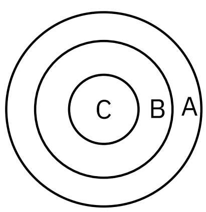
A set can be represented by closed figures like circles, triangles, rectangles etc. The point in the interior of the figure represents the elements of the set. Such a representation is called a Venn diagram.

Example

(i) $B \subset A$ can be represented as



(ii)



A ≡ Pupils in Mumbai, B ≡ Pupils in a Mumbai school, C ≡ Pupils in std. VI of that school.

Operations on sets

1. Union of Sets

The union of two sets A and B is a set which consists of all the elements of A and all the elements of B. The symbol '$\cup$' denotes union.

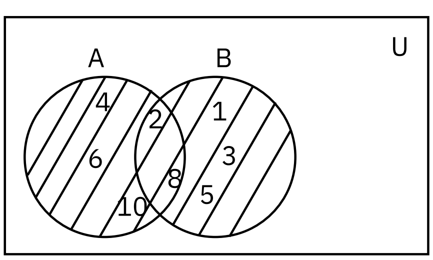
$A \cup B \equiv \{x \mid x \in A \text{ or } x \in B\}$

Some important points

(i) $A \cup B = B \cup A$

(ii) $A \cup \phi = A$

Example



$A \equiv \{2, 4, 6, 8, 10\}$

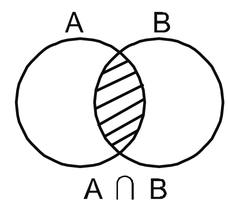
B ≡ {1, 2, 3, 5, 8}

A ∪ B ≡ {1, 2, 3, 4, 5, 6, 8, 10}

2. Intersection of Sets

The intersection of two sets A and B is a set which consists of all the common elements of A and B. The symbol '∩' denotes intersection.

A B {x | x ∈ A and x ∈ B}



Example

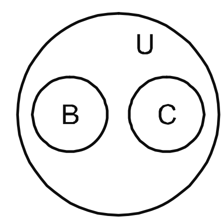
A ≡ {2, 4, 6, 8, 10}

B ≡ {1, 2, 3, 5, 8}

A ∩ B ≡ {2, 8}

Disjoint sets

If the sets A and B are such that they have no common elements i.e., A ∩ B = ϕ then set A and set B are called disjoint or mutually exclusive sets.



Example

(i) A ≡ {2, 4, 6, 8}
B ≡ {1, 3, 5, 7, 9}

(ii) U ≡ Pupils in a school
A ≡ Boys in the school
B ≡ Girls in the school

3. Complement of a set

Let U be the universal set and the set A ⊆ U. Complement of set A with respect to the universal set U is the set of all those elements of U which are not the elements of A. It is denoted by A′ or A^c

A′ = {x | x U and x A}

Example

If U ≡ {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and

A ≡ {3, 4, 5, 6, 7} then

A' ≡ {1, 2, 8, 9, 10}

Some important points

(i) Complement of the universal set is the null set and vice versa.

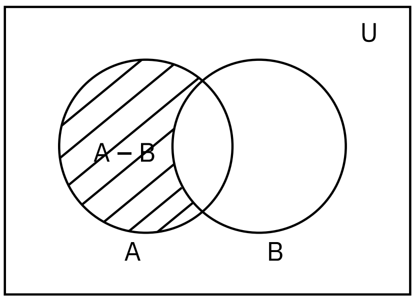
(ii) (A′)′ = A

(iii) If A ⊆ B then B' ⊆ A'

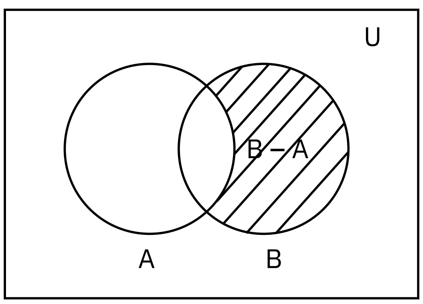
(iv) A ∪ A' = U (i.e., Universal Set)

4. Difference of Sets

If A and B are two sets, then the set of all elements which belong to A but do not belong to B is called the difference of sets A and B and is denoted by A − B. The set of all elements which belong to B but do not belong to A is called the difference of sets B and A and is denoted by B − A. $\therefore$ $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.



$B - A = \{x \mid x \in B \text{ and } x \notin A\}$.



Some important points

(i) $A - B \neq B - A$

(ii) If $A \subset B$ then $A - B = \phi$

(iii) $A - B \subseteq A$

(iv) The sets $A - B$, $A \cap B$ and $B - A$ are mutually disjoint.

Important Properties & Formulae

If A, B and C are any three sets, then

1. Distributive property of union and intersection

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2. Cardinal Number

The number of elements in a set is often referred to as the cardinal number of the set.
The cardinal number of a set is written as n(A).

A $\{2, 4, 6, 8, 10\}$; $n(A) = 5$

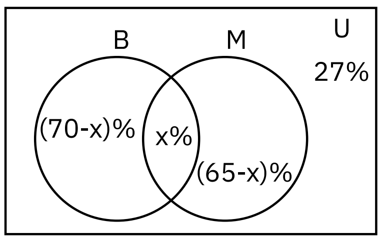
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Example

Q : In an examination, 70% of the candidates passed in English, 65% in Mathematics, 27% failed in both the subjects and 248 passed in both the subjects. Find the total number of candidates.

A :



Let the sets E and M represent students who passed in English and Mathematics respectively.

$n(E \cup M) = (100 - 27)\% = 73\%$

$n(E \cup M) = n(E) + n(M) - n(E \cap M)$

$73\% = 70\% + 65\% - x\%$

$x\% = 62\%$

Now, $62\% \equiv 248$

$\therefore$ Total number of candidates $= \dfrac{248 \times 100}{62} = 400$

3. De-Morgan's Laws

There are two identities involving union, intersection and complement.

(i) $(A \cup B)' = A' \cap B'$

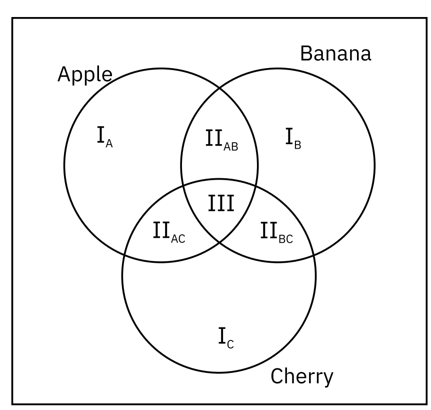
(ii) $(A \cap B)' = A' \cup B'$

Approach to Venn Diagrams Questions with 3 Sets

The following illustration shows you how to approach a venn diagram question with 3 sets

Example

Q : A survey was conducted on 100 children about their liking for 3 fruits i.e. Apple, Banana & Cherry. It was found that 20 children did not like any of these fruits. There were 30 children who liked Apple, 30 liked Banana and 40 liked Cherry. Also, the number of children who liked all the three fruits were 5. Find the number of children who liked exactly two fruits.



A : The given data can be represented through a Venn diagram, as shown below:

I_A = Number of Children who liked only Apple

I_B = Number of Children who liked only Banana

I_C = Number of Children who liked only Cherry

II_{AB} = Number of Children who liked Apple & Banana only

II_{AC} = Number of Children who liked Apple & Cherry only

II_{BC} = Number of Children who liked Banana & Cherry only

III = Number of Children who liked all the three fruits

Where,

U = Total Number of Children Surveyed = 100

A = Number of Children who liked Apple ($I_A + II_{AB} + II_{AC} + III$) = 30 ... (i)

B = Number of Children who liked Banana ($I_B + II_{AB} + II_{BC} + III$) = 30... (ii)

C = Number of Children who liked Cherry ($I_C + II_{AC} + II_{BC} + III$) = 40 ... (iii)

Let,

Y = Number of Children who did not like any of these fruits = 20

X = Number of Children who like at least one of the fruits = U − Y = 80

S = Sum of all the three Sets = A + B + C = 30 + 30 + 40 = 100

Then,

Number of children who liked exactly one fruit (either Apple or Banana or Cherry)

$= I = I_A + I_B + I_C$

Number of children who liked exactly two fruits (Apple-Banana or Apple-Cherry or Banana-Cherry)

$= II = II_{AB} + II_{AC} + II_{BC}$

X = I + II + III

Looking at (i), (ii) & (iii) and rearranging the terms we get,

$S = I_A + I_B + I_C + II_{AB} + II_{AC} + II_{AB} + II_{BC} + II_{AC} + II_{BC} + III + III + III$

$\Rightarrow$ I + 2 II + 3 III

So, S − X = (I + 2 II + 3 III) − (I + II + III)

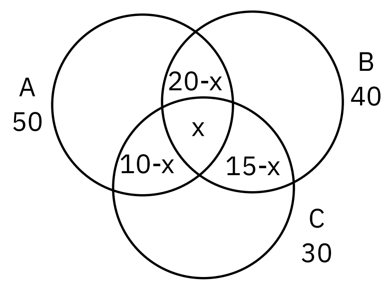
S − X = II + 2 III

Putting the values in above result, we get: 100 − 80 = II + 2 (5)

So, Number of children who liked exactly two fruits = II = 10

Q : In a survey of 100 students, it was found that 50 used the college library, 40 had their own library and 30 borrowed books. Of these 20 used both the college library and their own, 15 used their own library and borrowed books and 10 used the college library and borrowed books. How many students used all the three sources of books?

A :



A: Students using the college library

B: Students using their own library.

C: Students who borrow books.

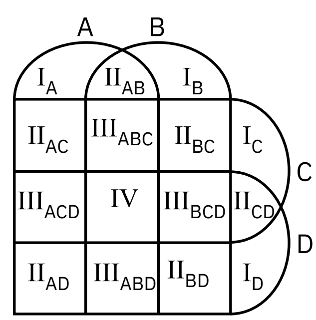
Suppose x students use all the three sources

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$\therefore\ 100 = 50 + 40 + 30 - 20 - 15 - 10 + x \qquad \therefore\ x = 25$

Approach to Venn Diagrams Questions with 4 Sets

Given below is a representation that can be used to solve a Venn diagram question consisting of 4 sets. i.e., A, B, C and D



For the given four sets,

$I = I_A + I_B + I_C + I_D$

$II = II_{AB} + II_{AC} + II_{AD} + II_{BC} + II_{BD} + II_{CD}$

$III = III_{ABC} + III_{BCD} + III_{ACD} + III_{ABD}$

$X = I + II + III + IV$

$S = I + 2\ II + 3\ III + 4\ IV$

So, $S - X = II + 2\ III + 3\ IV$

Concept Builder

1. Is set A = {0} a null set?
2. Sets ‘A’ & ‘B’ are defined as follows: A = {1, 2, 3}   B = {a, b, c}. Are these two Equal Sets?
3. Set ‘A’ is formed by all the natural numbers less than 100 which are multiples of 7 and another set ‘B’ is formed by all the natural numbers less than 200 which are multiples of 17. Are these two Disjoint Sets?
4. Find the number of Subsets of the set P = {3, 4, 5, 6}
5. In the above question, Find the number of Proper Subsets of set ‘P’
6. How many elements will be there in the Union of the following two sets: A = {a | a < 10, aN} B = {b | 6 < b < 20, b $\in$ N } ?
7. In the above question, what is ‘A-B’
8. In a Company of 50 employees, 20 like Tea and 20 like Coffee. Can we find out the number of employees who do not like any of these drinks?
9. In a class of 100 students, if 30 like English and 40 like History, then how many students at most, do not like any of these subjects?
10. In the above question, what is the minimum number of students who like at least one of these subjects?

Answers

1. No
2. No
3. Yes
4. 16
5. 15
6. 19
7. A-B = {1, 2, 3, 4, 5, 6}
8. No
9. 60
10. 40

SOLVED EXAMPLES

Q : A $\{x \mid 3x^2 - 7x - 6 = 0\}$

B $\{x \mid 6x^2 - 5x - 6 = 0\}$

Find $A \cap B$

A : $3x^2 - 7x - 6 = 0$

$\therefore\ 3x^2 - 9x + 2x - 6 = 0\ \therefore\ 3x(x - 3) + 2(x - 3) = 0$

$\therefore\ (3x + 2)\ (x - 3) = 0$

$\therefore\ x = -\frac{2}{3}$ or $x = 3$ $\therefore\ A \equiv \left\{-\frac{2}{3}, 3\right\}$

$6x^2 - 5x - 6 = 0$

$6x^2 - 9x + 4x - 6 = 0 \quad \therefore\ 3x(2x - 3) + 2(2x - 3) = 0$

$\therefore\ (3x + 2)\ (2x - 3) = 0$

$\therefore\ x = -\frac{2}{3}$ or $x = \frac{3}{2}$ $\quad \therefore\ B \equiv \left\{-\frac{2}{3}, \frac{3}{2}\right\} = \therefore\ A \cap B = \left\{-\frac{2}{3}\right\}$

Q : If N is a set of natural numbers and W is a set of whole numbers, then which of the following are true?

(i) $W \subseteq N$ (ii) $N \cup \{\ \} = W$

A : $N = \{1, 2, 3, 4, ...\}$ $W = \{0, 1, 2, 3, 4, ...\}$

$\therefore\ N \subseteq W$. Hence, (i) is not true

{ } represents null set (empty set)

$N\ \{\ \} \neq W$ $\therefore$ (ii) is also not true.

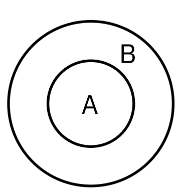
Q : Two finite sets have x and y number of elements. The total number of subsets of the first set is four times the total number of subsets of the second set. Find the value of x – y.

A : Number of subsets of the two sets are 2^x and 2^y respectively.

$\frac{2^x}{2^y} = 4 \Rightarrow 2^{x-y} = 4 \Rightarrow 2^{x-y} = 22 \qquad \therefore\ x - y = 2$

Q : Set A has 4 elements and set B has 7 elements. What can be the minimum number of elements in $A \cup B$?

A :

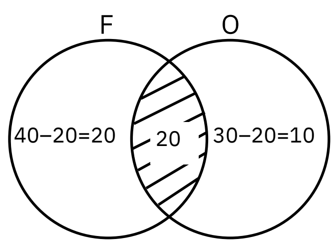


$A \cup B$ will have minimum number of elements, if set A is a subset of B.

$\therefore\ n\ (A \cup B) = n(B) = 7$

Q : A firm has 40 workers working in the factory premises, 30 working in its office and 20 working in both the factory premises and the office premises. How many workers are there in the firm, if all the workers work either in factory or in office? How many are working in (i) the factory alone (ii) office alone?

A :



$n(F) = 40$ $n(O) = 30$ $n(F \cap O) = 20$

The number of workers working in the factory only

$= n(F) - n(F \cap O) = 40 - 20 = 20.$

The number of workers working in the office only

$= n(O) - n(F \cap O) = 30 - 20 = 10$

Total number of workers $= 20 + 20 + 10 = 50$

Q : In a school, there are 250 students. 140 study Physics and 180 study Biology. How many students study both the subjects?

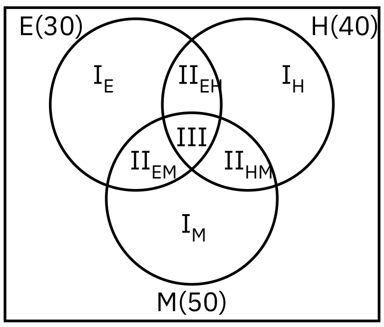
A : The answer of the given question cannot be determined because we don't know whether every student studies at least one of these subjects or not. There may be a case where certain students don't study either Physics or Biology.

Q : 100 residents of Valentine Tower speak at least one of the three languages i.e. English, Hindi and Marathi. Number of residents who speak English, Hindi and Marathi are 30, 40 and 50 respectively. If the number of residents who speak only 'English & Hindi' are 5, then find:

I. Maximum number of residents who speak all the three languages

II. Minimum number of residents who speak all the three languages

A : Using the 'Approach to Venn Diagram questions' the given data can be represented as follows:



U = X = 100

Here,

E = (Residents speaking English) = 30

H = (Residents speaking Hindi) = 40

M = (Residents speaking Marathi) = 50

X = (Residents speaking at least one of the three languages) = 100

S = (Sum of all the three sets) = 30 + 40 + 50 = 120

II_{EH} = (Residents speaking English & Hindi only) = 5

S − X = II + 2 III = 20 .... (i)

<u>To find Maximum Value of 'III'</u>

if a + b = Constant., then 'a' will be maximum when 'b' is minimum.
So, take minimum value of 'II', which is 5 (as we already know that 5 residents speak 'English & Hindi' only)
Putting II = 5 in Eq. (i), we get III = 7.5, which is not possible as Number of residents speaking all three languages has to be an Integer value.
So, take II = 6, then III = 7 which will be the required maximum value
Ans. 7

<u>To find Minimum Value of 'III'</u>

Take III = 0 in Eq. (ii). So, II = 20 which is not contradicting with any of the given data. So, Minimum value of III = 0.

Q : Kingfisher Airlines has a fleet size of 120 planes which may travel to any of these destinations i.e. Ahmedabad, Bengaluru, Chennai & Delhi. It is known that 40 planes travel to Ahmedabad, 20 to Bengaluru, 50 to Chennai & 35 to Delhi. If 20 planes travel to exactly 2 destinations, 8 to exactly three destinations and 3 to exactly four destinations then find out the number of planes that do not travel to any of these destinations?

A : Using the 'Approach to Venn Diagram questions' the given data can be represented as follows:

S – X = II + 2 III + 3 IV .... (i)

Where,

U = 120

X = ? (Number of planes that travel to at least one of the destinations)

S = A + B + C + D = 145

II = 20

III = 8

IV = 3

Putting these value in Eq. (i),

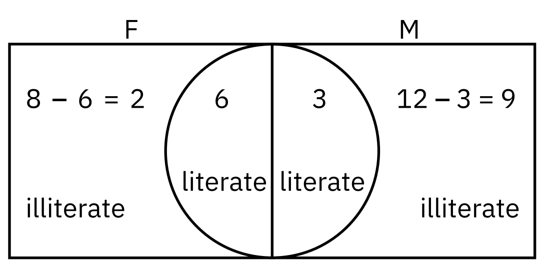
X = 100

So, the number of planes that do not travel to any of the above destinations

= U – X = 20

Q : In a group of 20 adults there are 8 females, 9 literate and 6 female literate. Find the number of male illiterates in the group.

A :



Number of males = 20 – 8 = 12

Number of male literate = 9 – 6 = 3

$\therefore$ Number of male illiterate = 12 – 3 = 9

CLASS EXERCISE

Teaser

During a vacation, a student noticed that when it rained in the morning, the evenings were clear and when it rained in the evenings, the mornings were clear. In all it rained on 9 days. He also noticed that in all six mornings and seven evenings were clear. How long was his vacation?

Sets and Counting

Consider the sets defined below and answer t he questions that follow:
Universal set U = {Positive real numbers less than 300}
A = $\{x \in N|\ x$ is divisible by 15}
B = {2, 4, 6, 8 ... 198, 200}
C = The set of all natural numbers divisible by 5
D = $\{x \in R|\ x < 100\}$
E = {3, 6, 9, 12...}

1. Which of the sets described above are finite sets?

2. Find the following sets (describe them explicitly in any of the above forms):

(a) $A \cap B$

(b) $A \cap D$

(c) $A \cup C$

(d) $C \cap E \cap A'$

(e) *$(A \cap D) \setminus (B \cap C)$

3. Find the number of elements in the sets A\B and B\A. Are these two sets equal? Are they equivalent?

4. Find:

(a) $n(B \cap C \setminus A)$

(b) $|B \cap C \cap A^c|$

(c) $|B \cup C|$

(d) $|E \cup B \cup C|$

(e) * $n(A \cap E \cap D^c)$

Counting Rules:

For any sets A, B, C we have:
1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Let U be the universal set containing sets A and B. Then
$(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Questions pertaining to two sets:

5. In a class of 50 students, 30 students play cricket, 29 students play hockey and 19 students play both the games.

(a) How many students play at least 1 of the two games? (A U B)

(b) How many students play exactly 1 of the two games?

(c) How many students play neither of the two games? (A U B)'

(d) How many students play only Cricket?

(e) How many students play only Hockey?

Questions pertaining to three sets:

6. A survey was conducted among 80 people to find out their genre preferences – Romance (R), Comedy (C) or Horror (H). It was found that 54 people liked Romance, 42 people liked Comedy and 38 people like Horror. 30 like both Romance and Comedy, 22 like Comedy and Horror while 26 like Romance and Horror. 12 people like all the three.

(a) How many people like at least 1 of the three genres? (A U B U C)

(b) How many people like at least 2 of the 3 genres?

(c) How many people like exactly 1 of the three genres?

(d) How many people like none of the 3 genres? (A U B U C)'

(e) How many people like only Romance?

(f) How many people like only Comedy?

(g) How many people like only Horror?

(h) How many like Romance and Comedy but not Horror?

(i) How many like Horror and Comedy but not Romance?

(j) How many like Romance and Horror but not Comedy?

7. Amongst 100 people who were surveyed 90 like Amitabh, 77 like Abhishek, 75 like Hrithik and 70 like Shahrukh.

(a) Minimum number of people who like all 4.

(b) Maximum number of people who like all 4.

8. In a class of 70 students, if 60 like coffee and 40 like tea, then how many like both tea and coffee, if it is known that each student likes at least one of the two drinks?

9. In a class of 70 students, if 60 like coffee and 40 like tea, then how students at the most like both tea and coffee? How students at the least like both tea and coffee?

10. In a class of 70 students the number of students who like tea exceeds the number of students who like coffee by 10 and is same as the number of students who don't like tea. How many students like coffee?

11. In a class, 10% students like tea but not coffee, 20% like coffee but not milk and 30% like milk but not tea. 12 students like all the three drinks and there is no one who likes none of the drinks. How many students are there in the class?

12. In a class, 40% of the boys drink tea but 40% of the tea drinkers are girls. If every student drinks only one of the two drinks, then

(a) The number of boys who drink coffee is _____ % of the number of girls who drink tea.

(b) The coffee drinkers are at least ____% of the tea drinkers.

*13. In a class of 70 students, 20 students do not like coffee. 10 like both tea and coke. 40 like exactly two drinks. There is no one who likes neither tea nor coffee. What is the ratio of students who like only coffee to those who like only tea and coke?

14. P, Q and R go jogging in Central Park. They all start at 7 a.m. from the same point and in the same direction on a circular track. P takes 4 minutes to complete a round, while Q takes 5 minutes and R takes 6 minutes. All three keep running till 9 a.m.

(a) How many times will all three of them meet at the starting point (excluding the start)?

(b) How many times will P and R meet at the start without Q also being there?

(c) How many times will exactly one of the three runners be at the starting point?

(d) Consider a camera which takes a picture of the starting point every 1 minute starting from 7 a.m. From 7 a.m. to 9 a.m., how many times will none of P, Q, or R be there in the picture?

Challengers

1. The Power Set of a set is defined as the number of distinct subsets of that set.

   Let A = {x < 100: x is a prime number}. Let Ap denote the power set of A.

(a) Is {2, 5, 13} a subset of Ap?

(b) Find |Ap|

(c) Find the number of sets in Ap that have exactly 3 elements

(d) * How many non empty sets are elements of the power set of Ap?

2. Planning to introduce some new items in the menu, Pizza Hut conducts a survey on its existing customers to find out their preferences.

   It was observed that out of those surveyed 96 people liked Pasta, 71 liked Chocolate Mousse and 75 liked Fruit Beer. 12 people liked all the three, while 22 people did not like any of the three. Out of those who liked Chocolate Mousse, 40 did not like any other item and out of those who liked Fruit Beer, 43 did not like any other item.

(a) What is the maximum number of people who like Chocolate Mousse and Fruit Beer but not Pasta?

1) 16 2) 17 3) 18 4) 19

(b) Which of the following could be the number of people who like only Pasta?

1) 43 2) 54 3) 65 4) 76

(c) What is the minimum number of people who like Fruit Beer and Pasta but not Chocolate Mousse?

1) 0 2) 1 3) 2 4) 3

(d) If the number of people who like Chocolate Mousse and Fruit Beer is less than the number of people who like Chocolate Mousse and Pasta, then what is the maximum number of people who like only Pasta?

1) 63 2) 54 3) 65 4) 76

(e) What is the minimum number of people covered by the survey?

1) 206 2) 225 3) 196 4) 201

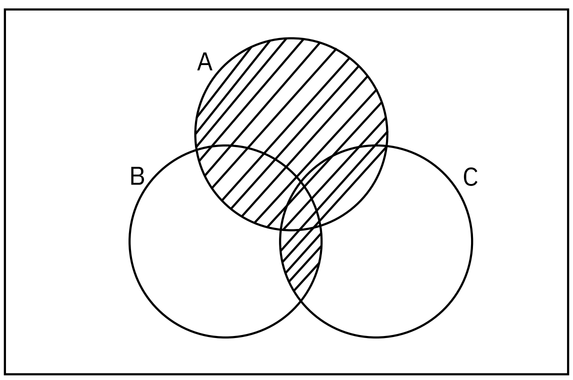
PRACTICE EXERCISE-1

DIRECTIONS for questions 1 to 5: Choose the correct alternative.

1. If A and B are two sets such that neither A is a subset of B nor B is a subset of A then which of the following statement is true:

   1) $A \cup B \subset A$
   2) $A \subset A \cap B$
   3) $A \subset A \cup B$
   4) $A \cup B \subset B$

2.



The shaded region is represented by:

   1) $(A \cup B) \cup (A \cup C)$
   2) $(A \cap B) \cup (A \cap C)$
   3) $(A \cup B) \cap (A \cup C)$
   4) None of these

3. Set S ≡ {x | x is a natural number}. Which of the following is a proper subset of S?

   1) A ≡ {x | absolute value of x is a prime integer}
   2) B ≡ {x | x^2 is a positive integer}
   3) C ≡ {x | x is a square root of an integer}
   4) None of these

4. Which of the following is an empty set?

   1) A ≡ {x | x∈I, 2x = 0}
   2) B ≡ {x | x∈N, x^3 is prime}
   3) C {x | x∈R, x^3 + 1 = 0}
   4) D ≡ {x | x∈N, x is even, x is prime}

5. A = Set of all quadrilaterals.
   B = Set of parallelograms.
   C = Set of squares.
   D = Set of rectangles.
   Then which of the following is/are true?
   I. B is a subset of A and C is a subset of D.
   II. $A \cup B = A$ and $C \cup D = D$.
   III. $A \cap C = C$ and $B \cap D = D$
   1) I only
   2) II only
   3) I and II only
   4) I, II and III

DIRECTIONS for questions 6 and 7: Refer to the data given below and answer the questions that follow.

A class of 30 students comprises of boys who can play at least one of the three games Cricket, Hockey and Football. 3 boys play only cricket, 3 boys play only Hockey and 2 play only football. 4 boys could play all three games, while 11 could play Football and Cricket, and 10 boys could play Football and Hockey.

6. How many boys played Cricket and Hockey but not Football?
   1) 1 2) 2 3) 3 4) 5

7. How many boys can play at least two games?
   1) 16 2) 18 3) 10 4) 22

DIRECTIONS for questions 8 to 13: Refer to the data below and answer the questions that follow.

In ABC Ltd., there are 600 employees working across 3 departments viz., HR, Design and Marketing. An employee can work in either one or two or all the three departments. The number of employees working in exactly two departments is 120. The employee strength of HR, Design and Marketing is 300, 350 and 100 respectively.

8. The number of employees working in all the three departments is:
   1) 10 2) 5 3) 15 4) 30

9. The number of employees working in only one department is
   1) 465
   2) 600
   3) 135
   4) Cannot be determined

10. The number of employees working in at least two departments is:
    1) 130
    2) 135
    3) 270
    4) None of these

11. If there are 250 employees such that some of them are only in HR and the rest are only in Marketing, then find the number of employees working only in Design.

    1) 100 2) 400
    3) 215 4) Cannot be determined

12. Find the number of employees working only in HR. (Use information from previous questions if necessary).

    1) 38 2) 250
    3) 265 4) Cannot be determined

13. Find the number of employees working in both Design and Marketing. (Use information from previous questions if necessary).

    1) 20 2) 25
    3) 48 4) Cannot be determined

DIRECTIONS for question 14: Choose the correct alternative.

14. According to a survey conducted in a locality, all the persons are employed in either manufacturing, teaching or IT sectors. There are 100 persons working in manufacturing sector, 60 working in teaching sector and one person working only in IT sector. Totally 10 persons work in both IT as well as teaching sector and 6 persons work in teaching sector as well as in manufacturing sector. Also 20 persons are employed in at least 2 sectors types. Find the population of the locality surveyed.

    1) 200 2) 148 3) 155 4) 199

DIRECTIONS for questions 15 and 16: Refer to the data given below and answer the questions that follow.

There are 35 employees in an office. 49 diaries, each having either a pink cover a blue cover, or a red cover, are distributed in a way that employees get at least one diary. The number of employees who got only one diary each with either red or blue or pink cover are 21, of which, less than 5 employees got only one diary each with a blue cover, more than 4 employees got only one diary each with a red cover and more than 8 employees got only one diary each with pink cover. The number of employees who got both blue and red covered diaries but not a pink covered one is one more than those who got only red covered diary. Employees who got blue and pink covered diaries but not a red covered one are 4 less than the employees who got only pink covered diary. Nobody got all the three diaries. The number of employees getting only red covered diary is not less than the number of those who got both blue and pink covered diaries but not a red covered one. Nobody got both red and pink covered diaries.

15. What is the minimum possible number of red covered diaries?

    1) 15 2) 17 3) 18 4) 19

16. If the total number of diaries is 52, find the minimum number of pink covered diaries that would be required.

    1) 14 2) 15 3) 16 4) 19

DIRECTIONS for questions 17 and 18: Choose the correct alternative.

17. In a school, in all 60 students study at least one of the three foreign languages, namely German, Spanish and French. Total 30 students study German, 20 students study Spanish while 40 students study French. What is the maximum number of students who study all the three languages?

    1) 5 2) 10 3) 15 4) 20

18. Let T be the set of integers {3, 11, 19, 27, ... 451, 459, 467} and S be the a subset of T such that the sum of no two elements of S is 470. The maximum possible number of elements in S is:

    1) 32 2) 28 3) 29 4) 30

(Past CAT question)

DIRECTIONS for questions 19 and 20: Solve as directed.

19. In a class of 60 students, the number of students who don't play football or hockey is 12, the number of students who don't play cricket or hockey is 15 and the number of students who don't play football or cricket is 18. If it is known that every student plays at least one of the three sports, what can be the maximum number of students who play exactly two sports?

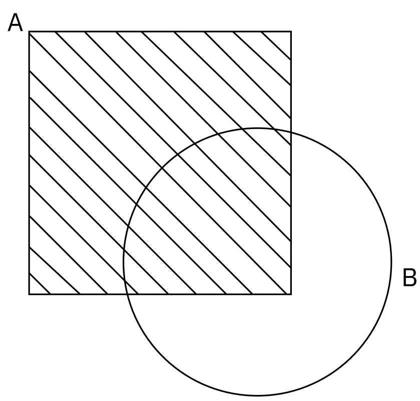
20. In a class of 73 students, 30 students like Physics and 37 students like Mathematics. The number of students who like both Physics and Chemistry (but not Mathematics), both Chemistry and Mathematics (but not Physics), and both Physics and Mathematics (but not Chemistry) are 10, 12 and 0 respectively. If the number of students who like only Physics or only Mathematics is 45, what is the number of students who like only Chemistry? (It is known that each student likes at least one subject.)

PRACTICE EXERCISE-2

DIRECTIONS for questions 1 to 5: Choose the correct alternative.

1. Which of the following is not an empty set?
   1) A ≡ {x/x ⊂ I, x^2 is not positive}
   2) B ≡ {x/x ⊂ N, 2x + 1 is even}
   3) C ≡ {x/x ⊂ N, x is odd and x^2 is even}
   4) D ≡ {x/x ⊂ R, $x^2 + 1 = 0$}

2. Which of the following correctly represents the shaded region, if the universal set is equal to A U B?



   1) [B∩A] ∪[(B∪A) – B]
   2) A
   3) [B∩A] + [B'∩A]
   4) All of these

3. The intersection of which of the following two sets is a null set?
   A ≡ {x | x∈I, x is even}
   B ≡ {x | x is prime}
   C ≡ {x | x∈N, x^2 is odd}
   1) A and C
   2) B and C
   3) A and B
   4) None of these

4. Which of the following is/are true?

   I. A set of all natural numbers is a well-defined set.
   II. A null set contains an element '0'.
   III. S1 = {a, b, c} and S2 = {a, b, c}. S2 is a subset of S1.
   IV. S1 = {a, b, c, d, e} and S2 = {a, b, c, m}. S2 is an improper subset of S1.

   1) I only
   2) I and III only
   3) II and IV only
   4) I, II and IV only

5. The market research of a certain breakfast cereal firm interviewing 100 people, found that on a certain morning for breakfast 72 people had cereals, 39 people had an egg, 75 people had toast; 32 people had cereal and egg, 53 people had cereal and toast, 26 people had toast and egg. Of these 21 had cereal, toast and egg. How many of those interviewed had neither cereal, toast nor egg?

   1) 6
   2) 4
   3) 5
   4) 8

DIRECTIONS for questions 6 and 7: Refer to the data given below and answer the questions that follow.

In a certain residential colony 120 persons are obese; also 50 persons, of whom 20 are obese, suffer from heart disease. Further there are 15 high BP patients, 10 of whom are obese. Of the heart disease patients, 6 also have high BP. 5 persons suffer from all the three ailments.

6. How many persons suffer only from High BP.?

   1) 1
   2) 9
   3) 5
   4) 4

7. How many obese persons have neither heart disease nor high BP?

   1) 105
   2) 95
   3) 80
   4) Data insufficient

DIRECTIONS for questions 8 and 9: Choose the correct alternative.

8. The term lucid dreaming refers to dreaming while knowing that you are dreaming. The survey conducted by Mr.Backmore and Mr.Gren in city X estimated that 40% of illiterate population never had lucid dream, 80% of literate population had at least one lucid dream. Difference in the number of people who are literate and illiterate is 40% of the total population. Find the number of people who had at least one lucid dream as a percentage of total population of city X, if there are more illiterate people in city X compared to literate people who had at least one lucid dream.

   1) 34%
   2) 74%
   3) 66%
   4) Cannot be determined

9. In a derby, the same number of people bet money on each of the horses Chetak, Toofan and Tej. The number of people who bet on both Toofan and Chetak is half the number of those who bet on both Toofan and Tej and is $\frac{1}{3}^{rd}$ the number of people who bet on Chetak and Tej. 2000 people bet on both Chetak and Toofan but not on Tej, and 10000 people bet on both Toofan and Tej but not on Chetak. If the number of people who bet only on Tej is 20% of the number of people who bet only on Chetak, find the number of people who bet on Tej.

1) 18000　　2) 34000　　3) 36000　　4) 4000

DIRECTIONS for questions 10 to 12: Refer to the data given below and answer the questions that follow.

Natraj Dance Academy conducts courses for 4 styles of dances-Tango, Salsa, Jive and Hip-hop. The number of students learning Tango and Salsa are 26 each, those learning Jive are 36 and those learning Hip-hop are 22. Students who want to learn 3 dance styles can opt for Tango, Salsa and Jive. There are 4 such students. 8 students are learning Tango and Salsa both. 10 are learning both Salsa and Jive while 12 are learning both Tango and Jive. Students learning Hip-hop can opt for Jive but not for Tango and Salsa.

10. If there are 8 students who are learning Jive and Hip-hop both, what was the total number students who were admitted in the Dance Academy?

1) 74　　2) 76
3) 75　　4) Cannot be determined

11. The ratio of students learning only Tango to those learning only Salsa is:

1) 5 : 6　　2) 6 : 5　　3) 2 : 3　　4) 2 : 5

12. How many students like Jive but not Tango and Salsa?

1) 18　　2) 12　　3) 10　　4) 8

DIRECTIONS for questions 13 to 18: Choose the correct alternative.

13. The number of positive integers not greater than 100, which are not divisible by 2, 3 or 5 is:

1) 26　　2) 18
3) 31　　4) None of these

14. In a class, 30% of the students like QA and 25% like DI. If 60% don't like either of the two sections, which of the following is true?

    1) 10% of the students like only QA.
    2) 10% of the students like both the sections.
    3) 15% of the students like only DI.
    4) 25% of the students like exactly one section.

15. In a class, Parvez correctly found out that 40 students like Math and 31 students like Science. However, by mistake he recorded the number of students who like both the subjects as the number of students who like only Science. Similarly, by mistake, he recorded the number of students who like only Science as the number of students who like both the subjects. If, according to Parvez, the ratio of the number of students who like exactly one subject to the number of students who like both the subjects is 35:18, what is the actual ratio of the number of students who like exactly one subject to the number of students who like both the subjects? (Note that each student in the class likes at least one subject)

    1) 22 : 13
    2) 37 : 13
    3) 40 : 31
    4) 45 : 13

16. Consider the set S = { 2, 3, 4, ..., 2n+1}, where n is a positive integer larger than 2007. Define X as the average of the odd integers in S and Y as the average of the even integers in S. What is the value of X–Y?

    1) $\frac{1}{2}n$
    2) $\frac{n+1}{2n}$
    3) 2008
    4) 1

(Past CAT question)

17. $n(P \cup Q \cup R) = n(P) + n(Q) + n(R)$. Which of the following will be true?

    1) $(P \cap Q) = \phi$
    2) $(R \cap Q) \cup (P \cap R) = \phi$
    3) $(P \cap Q \cap R) = \phi$
    4) All of these

18. If $A \equiv \{x \mid 6x^2 + x - 15 = 0\}$,
    $B \equiv \{x \mid 2x^2 - 5x + 3 = 0\}$, and
    $C \equiv \{x \mid 2x^2 + x - 3 = 0\}$; find $A \cap B \cap C$.

    1) $\{1\}$
    2) $\left\{\frac{3}{2}\right\}$
    3) $\left\{\frac{3}{2}, \frac{-5}{3}, 1\right\}$
    4) None of these

DIRECTIONS for questions 19 and 20: Solve as directed.

19. In a class, students like either apple or mango or both or none. 62.5% of the students who like apple like no other fruit. The number of students who do not like any fruit is 30% of those who like at least one fruit and 50% of those who like mango. 18 students like only mango. What is the total number of students in the class?

20. In a foreign language training academy, three languages, namely German, Spanish and French, are taught. There are 100 students in the academy and each of them studies at least one language. The number of students who study German, Spanish and French are 30, 40 and 50 respectively. What can be the maximum number of students who study all the three languages?

QA-4.3 | BASIC P&C - I

THEORY

Counting Methods

There are several powerful methods for counting objects and sets of objects without actually having to list down the elements of the sets. All these methods are based on the basic counting principles.

There are two such principles.
1. Multiplication principle of counting
2. Addition principle of counting

Multiplication Principle of Counting

Consider the following example:

Alice wants to buy one of the three Music Systems A, B and C available in a shop. How many choices does she have? Obviously 3, as she can buy either the Music System A, or the Music System B or the Music System C.

Similarly, if the shop has 2 Ovens X and Y for sale and Alice wants to buy one of the ovens, she has 2 choices.

Now, if Alice wants to buy one of the Music Systems and also one of the Ovens, in how many ways can she do that?

The choices available to Alice are

AX	AY
BX	BY
CX	CY

We find that corresponding to each of the 3 ways Alice can buy a Music System she can buy an Oven in two ways. Therefore, the total number of ways in which she can buy one of the Music Systems **and** one of the Ovens is $3 \times 2 = 6$.

If an event can happen in m number of ways and if corresponding to each of the m ways, a second event can happen in n number of ways, both the events together can happen in $m \times n = mn$ number of ways.

Let us take another example. If a coin is flipped 5 times, how many different results are possible? Every flipping of the coin can result in two different outcomes, either head (H) or tail (T). Thus, 5 tosses together can result in $2 \times 2 \times 2 \times 2 \times 2$ or 2^5 different outcomes, such as the following. HHHHH, HHHHT, HHHTH, HHHTT… and
THHHH, THHHT, THHTH, THHTT…

Example

Ronny wears a shirt, a pair of trousers and a pair of shoes to the office. If he has 5 shirts, 3 pairs of trousers and 2 pairs of shoes, how many combinations does Ronny have for office wear?

Ronny can choose a shirt in 5 ways, a pair of trousers in 3 ways and a pair of shoes in 2 ways. Therefore, he can choose a shirt, a pair of trousers and a pair of shoes in $5 \times 3 \times 2 = 30$ ways. In general, if an operation can be performed in n_1 ways, and for each of these a second operation can be performed in n_2 ways, and for each of the latter a third operation can be performed in n_3 ways, ..., and for each of the latter a k^{th} operation can be performed in n_k ways, then the entire sequence of k operations can be performed in $n_1 \times n_2 \times n_3 .... \times n_k$ ways.

Addition Principle of Counting

Now, if the shop had 3 different Music Systems and two different Ovens just as in the earlier case, but Alice was to buy **either** a Music System **or** an Oven (not both), in how many ways could she do that?

The choices available to Alice would be

A B C X Y

Alice can buy any one of the 3 Music Systems in 3 ways or buy any of the two Ovens in 2 ways, and therefore, she could buy a Music System or an Oven in $3 + 2 = 5$ ways.

If an event can happen in m number of ways and if a second event can happen in n number of ways, (subject to the condition that both the events cannot happen together) then the first event or the second event can happen in (m + n) number of ways.

Example

A coach has three teams A, B and C under him. There are 5, 10 and 12 players in Teams A, B and C respectively. If the coach has to select one player from Team A or Team C, in how many ways can he do that?

The coach can choose a player from Team A in 5 ways or he can choose a player from Team C in 12 ways. Therefore, he has $5 + 12 = 17$ choices in all.

Factorial

An operation that will often be used in the basic counting principle is the factorial. If m is an integer greater than 1, then m factorial, denoted by the symbol m!, is defined as the product of all the integers from 1 to m.

Thus,

$2! = 1 \times 2 = 2$

$3! = 1 \times 2 \times 3 = 6$

$4! = 1 \times 2 \times 3 \times 4 = 24$ and so on.

By definition, $1! = 1$ and $0! = 1$

Example

Find the value of $\frac{7! \times 5}{14} - (5 \times 5!)$

$\frac{7! \times 5}{14} - (5 \times 5!) = \frac{7 \times 6! \times 5}{7 \times 2} - (5 \times 5!)$

$= \frac{6! \times 5}{2} - (5 \times 5!) = \frac{6 \times 5! \times 5}{2} - (5 \times 5!)$

$= (3 \times 5! \times 5) - (5 \times 5!)$

$= 15 \times 5! - 5 \times 5! = 10 \times 5! = 10 \times 120 = 1200$

Note: Factorial is not defined for fractions or negative integers.

Comparison of Permutations & Combinations

In how many ways can you arrange three persons A, B and C in a row? In 6 ways: namely ABC, ACB, BCA, BAC, CAB, and CBA. This is what is called a permutation.

A **permutation** is an ordered arrangement of some or all the elements of a set of objects.

Order is important in a permutation. Therefore, permutations with the same objects in different orders are considered distinct arrangements.

You have seen that three persons A, B and C can be arranged in a row in 6 different ways. But, how many three member teams can be formed out of three persons A, B and C? Only one, because ABC, ACB, BCA, BAC, CAB, or CBA do not make any difference. The team containing A, B and C and the team containing A, C and B are one and the same. Here the order (even the order in which they are picked up) does not matter. This is what is called a combination.

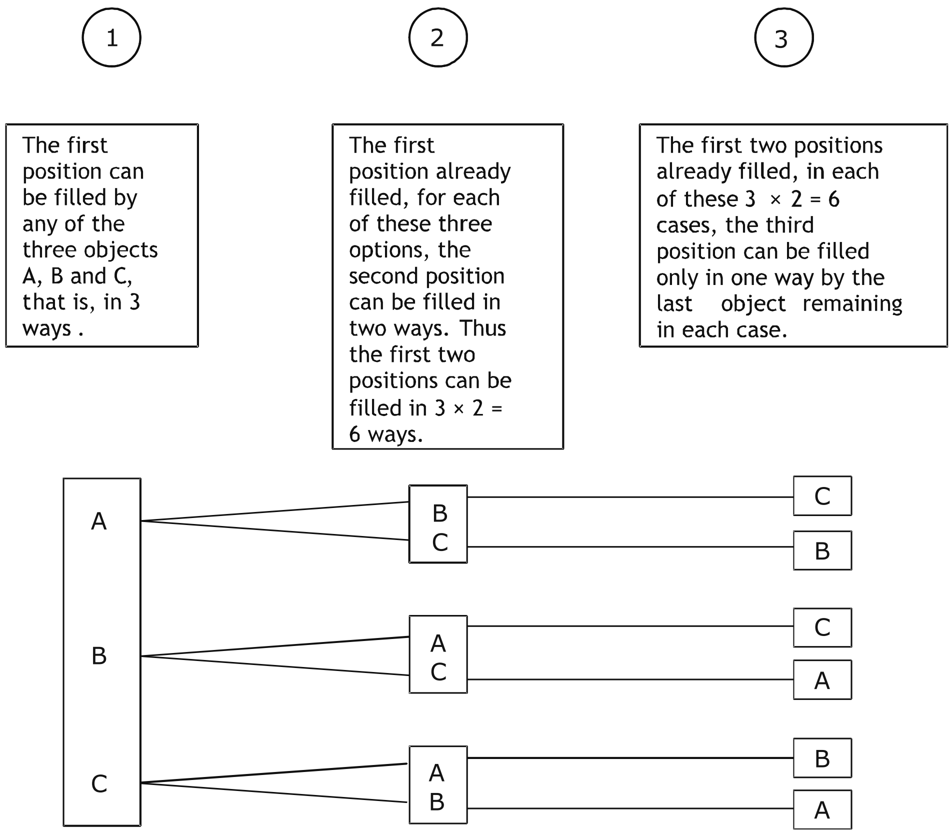
Combination refers to **choosing** a certain specified number of elements from a set of objects. Combination is, therefore, **selecting** a specified number of elements or **forming a group** or **team** of a specified number of elements from a set of objects. Therefore, order does not matter.

Permutation and Combination — distinguished

The difference between Permutation and combination may be summarized as below.
Permutation refers to arrangements so that order is important and different arrangements are treated as distinct and counted as different permutations.
Combination refers to selection/grouping/team-formation where order does not matter and groups of same elements in different orders are treated as one and the same, and hence counted as one. In case of permutation, ABC and ACB are counted as two; whereas, in case of combination, ABC and ACB are treated as one.

Theory of Permutations

To understand 'Theory of Permutations' we consider the illustration given below
Let us examine in how many ways we can arrange three objects A, B and C in a row.
Let us indicate the positions by circles 1, 2 and 3.



Thus, we find that the first, second, and third positions can be filled in 3 ways, 2 ways and 1 way respectively, and the three positions together can be filled in $3 \times 2 \times 1 = 6$ ways.

As we already know this is 3!.

Therfore, **n distinct objects taken all at a time can be arranged in n! ways.**

Example

In how many ways can you arrange the letters of the word GLAD?
This is a case of arranging 4 distinct objects (in this case, letters) taking all at a time. Therefore, the number of arrangements possible is $4! = 24$.

Different Permutation cases

1. Permutation of n distinct objects taken r at a time

Look at the example below:

If you have 5 objects and 3 of those are to be arranged in a row, in how many ways can you do that?

Just as in the previous case, here you have three positions to fill. Also, as one position is filled, the number of options available for the next position get reduced by one. Thus, the first position can be filled in 5 ways, the second position can be filled in 4 ways, and the third and last position can be filled in 3 ways.

Hence, there are 5 × 4 × 3 = 60 possible arrangements.

It is worth observing $5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$

This can be generalized as follows:

Taken r at a time, n distinct objects ($r \leq n$) can be arranged in ^nP_r or (n, r) ways, where $^nP_r = \frac{n!}{(n-r)!}$.

Example

How many 3 digit numbers can be formed using the letters 1, 2, 3, 4 and 5 without repeating any of the digits?

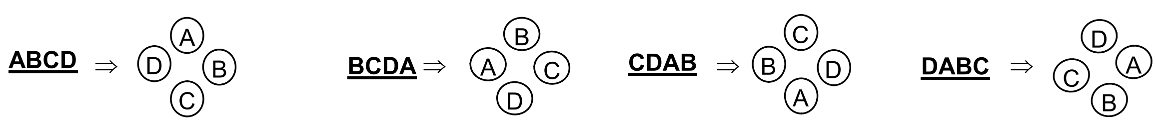
This is a case of arranging 5 distinct objects (here the digits being the objects) taken 3 at a time.

Therefore, the number of arrangements possible is $^5P_3 = \frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$.

2. Circular Permutations

When four objects A, B, C and D are arranged in a row the total number of different arrangements possible is 4!, i.e., 24. These include, among others, the four arrangements ABCD, BCDA, CDAB and DABC, which are considered as four distinct permutations. In case of linear arrangements the beginning, the end as well as the intervening relative positions of the objects distinguish one arrangement from another.

But if the objects A, B, C and D are to be arranged in a circle, the arrangements will not have any beginning (left-end) or end (right-end), as they have in the case of linear arrangements. In circular arrangements, it is only the relative positions of the objects that distinguish one arrangement from another.



Consequently, in the above circular arrangements, as the relative positions of the objects are unchanged, the four permutations ABCD, BCDA, CDAB, and DABC are not distinct or different from each other. They are one and the same arrangement, viewed from different positions, and hence are to be counted as a single arrangement.

Thus the number of circular arrangements of 4 distinct objects taken all at a time is $\frac{4!}{4}$ or (4 − 1)! as every 4 linear arrangements will be counted as 1 circular arrangement.

The foregoing can be generalized as follows.

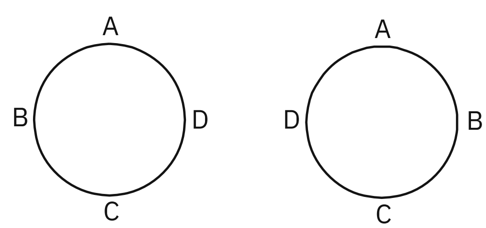
The number of permutations of n distinct objects that are arranged in a circle is given by $\frac{n!}{n}$ or (n − 1)!

This can also be thought of as keeping the position of one out of n objects fixed and arranging the remaining (n – 1) objects so that the relative positions undergo change. This can be done in (n – 1)! ways.

Clockwise and Anti-clockwise Permutations

There are two types of circular permutations depending on whether the clockwise and anticlockwise arrangements are distinguishable.

(i) When n number of people have to be seated around a circular table, the clockwise and anti-clockwise arrangements are distinct arrangements.



In the clockwise arrangement B is to the left of A, and D is to the right of A. While in the anti-clockwise arrangement D is to the left of A and B to the right.

Hence, these are 2 distinct arrangements.

The total arrangements (n – 1)! include such pairs of clockwise and anti-clockwise arrangements.

(ii) When n different beads are to be arranged on a necklace then the clockwise and anti-clockwise arrangements are not distinguishable as the necklace can be turned over.

If the clockwise arrangement is turned over you get the anticlockwise arrangement. The two views pertain to the same arrangement. Hence, the clockwise and anti-clockwise views are not distinguishable. In other words, the two arrangements are one and the same arrangement seen from two sides of the plane of the necklace.

Therefore, the number of ways in which n different beads can be arranged in a necklace is $\frac{1}{2} \times (n - 1)!$.

Rules of Permutation

1. $^nP_n = {}^nP_{n-1} = n!$
2. $^nP_1 = n$
3. $^nP_0 = 1$

Concept Builder 1

1. Find the value of $^9P_2$?
2. Is $^{35}P_{34} = {}^{35}P_1$?
3. Which is greater: $^{10}P_5$ or $^8P_7$?
4. In how many ways can 6 people be seated on 6 chairs for a photograph?
5. A = number of ways of arranging 3 out of 5 people on 3 chairs & B = number of ways of arranging 3 people on 5 chairs. Is A equal to B?
6. How many three-letter arrangements can be formed using the letters A, B, C, D, E without repeating any of the letter?
7. What are the total number of possible outcomes, if two dice, are rolled together?
8. How many different necklaces can be prepared using 10 different coloured pearls?

Answer Key

1. 72
2. No
3. $^8P_7$
4. 6!
5. Yes
6. 60
7. 36
8. $\frac{9!}{2}$

Theory of Combinations

Consider the following example:

There are 5 distinct objects and you are required to arrange the objects taking three at a time. The question is, in how many ways can you do that?

The task will involve selecting three objects and arranging every such set of three objects in different ways.

The number of ways three objects can be selected from a set of 5 distinct objects is usually denoted by ${}^5C_3$.

Now, every set of 3 objects can be arranged in 3! ways. Therefore, the number of ways 3 objects can be selected and arranged is ${}^5C_3 \times 3!$.

As you already know, the number of ways 5 objects can be arranged in a row taken 3 at a time is given by ${}^5P_3$ or $\frac{5!}{(5-3)!}$.

Therefore,

${}^5C_3 \times 3! = {}^5P_3$ or $\frac{5!}{(5-3)!}$.

Hence, ${}^5C_3 = \frac{{}^5P_3}{3!} = \frac{5!}{(5-3)!3!}$

This can be generalized as below.

Number of Combinations of 'n' different things, taken 'r' at a time is given by ${}^nC_r = \frac{n!}{r!(n-r)!}$.

This means that the number of different groups of **r** objects (or things or persons) that can be formed out of **n** distinct objects (or things or persons) is $\frac{n!}{r!(n-r)!}$.

Note that ${}^nC_r$ is the number of r-element subsets of the superset containing n elements.
You can look at the above also from the following point of view.
Consider the case of arranging the letters A, B, C, D, and E taking 3 letters at a time. You know that it can be done in ${}^5P_3 = 60$ ways. This set of 60 arrangements is comprised of 10 sub-sets, each sub-set constituted of 6 ordered arrangements of 3 letters, such as the following.

- ABC, ACB, BCA, BAC, CAB, and CBA having only the letters A, B, and C.
- ACD, ADC, CAD, CDA, DAC, and DCA having only the letters A, C, and D, and so on.

Each of the above subsets has 3! or 6 three-letter arrangements, which are the six elements in the respective subset.

However, when you consider forming groups taking 3 letters at a time, every such subset (containing 6 ordered arrangements of 3 letters) has to be counted as one group — as the 3 letters are the same and while forming a group, order does not matter. Thus, the number of such groups is $\frac{{}^5P_3}{3!}$ $= \frac{60}{6} = 10$

Therefore, ${}^5C_3 = \frac{{}^5P_3}{3!}$

In general terms, when you have n distinct objects, every r! arrangements out of the ${}^nP_r$ arrangements is counted as one combination of n objects taken r at a time, so that ${}^nC_r = \frac{{}^nP_r}{r!}$.

(i) There are 6 girls and 5 boys in a class. In how many ways can you form a team of 3 girls and 2 boys?
To form a team of 3 girls and 2 boys, we have to select 3 girls out of 6 girls **and** select 2 boys out of 5 boys. The first task can be done in ${}^6C_3$ ways and the second task can be done in ${}^5C_2$ ways. Therefore, the number of ways both can be done together is ${}^6C_3 \times {}^5C_2 = 20 \times 10 = 200$ ways.

(ii) Here is a simple variation of the previous problem.
There are 6 girls and 5 boys in a class. In how many ways can you form a team of 3 girls or 2 boys?
To form a team of 3 girls and 2 boys, we have to select 3 girls out of 6 girls **or** select 2 boys out of 5 boys. The first task can be done in ${}^6C_3$ ways and the second task can be done in ${}^5C_2$ ways. Therefore, the number of ways either the first or the second task can be done is ${}^6C_3 + {}^5C_2 = 20 + 10 = 30$ ways.

Rules of Combinations

1. ${}^nC_r = {}^nC_{n-r}$

   The physical implication of this relationship can be understood in the following manner. When there are n objects, every time we form a group taking r objects, (n – r) objects are left out; in other words, a group of (n – r) objects is formed correspondingly. Thus, the number of groups formed by taking r objects at a time, is the same as the number of groups formed by taking (n – r) objects at a time.

   ${}^nC_r = {}^nC_{n-r}$ is the mathematical expression for the same.
2. ${}^nC_0 = {}^nC_n = 1$
3. ${}^nP_r = r!\ {}^nC_r$
4. ${}^nP_1 = {}^nC_{n-1} = n$
5. ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n$

   or, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$

Example

Suppose a child is presented with a pencil, a pen and an eraser, and told to choose some of these 3 gifts; he may choose nothing or only a pencil or only an eraser or only a pen or a pen and a pencil or an eraser and a pencil, or an eraser and a pen, or all three; thus he has $8 = 2^3$ choices. In general, the total number of combinations of 'n' distinct things taken some or all at a time = ${}^nC_0 + {}^nC_1 + {}^nC_3 + \ldots.\ {}^nC_n = 2^n$.

Note that ${}^nC_0$ (=1) is the number of groups with zero objects. If such a group is not admissible the relationship becomes ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$.

The number of different groups as above can also be arrived at in the following manner. For every object you have two options — either to include it or to exclude it, which results in formation of all the different groups possible. Thus, for n objects, the total number of groups possible is $2 \times 2 \times 2\ldots$ (n times) $= 2^n$. This includes the option where all the objects are excluded, which gives rise to the group with no objects.

6. ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$

   $({}^{n-1}C_{r-1})$ is the number of combinations of n distinct objects taken r at a time when one particular object is always included. $({}^{n-1}C_{r})$ is the number of combinations of n distinct objects taken r at a time when one particular object is never included. Thus, these two terms together equal ${}^{n}C_{r}$.

7. ${}^{n+1}C_{r+1} = {}^{n}C_{r+1} + {}^{n-1}C_{r} + {}^{n-1}C_{r-1}$

Concept Builder 2

1. What will you call the group of letters used in your name? A Permutation or a Combination?
2. Find the value of the expression: 7! – (14 × 6! – 5! × 42)
3. What is the value of $^{11}C_2$?
4. If $^nC_2 = {}^nC_{12}$, then n = ?
5. Find the value of $^{1111}C_{1110}$.
6. In how many ways can you select two balls out of available five identical green balls?
7. How many different selections are possible for the given four different fruits?
8. How many different selections are possible for the given four identical fruits?
9. Find the number of ways of forming a committee of 4 members from a group of 5 Women and 5 Men.

Answer Key

1. Permutation	6. 1
2. 0	7. 16
3. 55	8. 5
4. 14	9. $_{10}C_4$
5. 1111	

SOLVED EXAMPLES

Q : Find $^6P_4$ and $^{25}C_{22}$.

A : $^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$

$^{25}C_{22} = \frac{25!}{(25-22)!22!} = \frac{25!}{3!22!} = \frac{25 \times 24 \times 23}{3 \times 2} = 2300$

Q : How many three digit numbers with distinct digits can be formed using 1, 2, 3, 4?

A : Number of three digit numbers that can be formed
= Number of permutations of 4 things taken 3 at a time
= $^4P_3 = 4 \times 3 \times 2 = 24$

Q : How many three digit even numbers are there?

A : For the number to be even, the last (unit place) digit of the number should be any one of 0, 2, 4, 6, 8. So, the total number of ways to select the last digit = 5 ways
The hundred's place digit of the number can be any one of 1 to 9, i.e. 9 ways
And the ten's place digit of the number can be any one of 0 to 9, i.e. 10 ways
So, total three digit numbers = $9 \times 10 \times 5 = 450$

Q : Find the number of ways of inviting at least one executive out of five executives to a conference.

A : Here, we have two options for each executive - either he is invited or he is not invited So, the total number of ways of inviting these 5 executives = 2^5. But in these selections, there would be a case where all executives are not invited, which we need to exclude.
So, the total number of ways = $2^5 - 1 = 31$

Q : In how many ways can the numbers 1 to 15 be assigned to the 25 squares in a 5×5 grid, leaving 10 squares empty?

A : 15 permutations of 25 squares
$= {}^{25}P_{15} = \frac{25!}{10!} = 11 \times 12 \times 13 \times ..... \times 25$
Alternatively,
There are 25 squares.
One of 1, 2, —— 15 is placed in these 25 squares in 25 ways.
After placing 1st number, 2nd can be placed in the remaining 24 squares in 24 ways.
3rd number in 23 squares in 23 ways.
Similarly, 15th number in 11 squares in 11 ways.
$\therefore$ Number of ways = $11 \times 12 \times ..... \times 23 \times 24 \times 25$

CLASS EXERCISE

Teaser

A standard pack of cards has 52 cards, with four suits of 13 cards each. Hearts and Diamonds are “red suits” while Spades and Clubs are “black suits”. These cards are emptied into a box.

a) How many cards (minimum) must one pick from the box to ensure two cards of the same colour?

b) How many cards (minimum) must one pick from the box to ensure two cards of different colours?

c) How many cards (minimum) must one pick from the box to ensure two cards of the same suit?

d) How many cards (minimum) must one pick from the box to ensure two cards of different suits?

e) How many cards (minimum) must one pick from the box to ensure five cards of the same suit?

The Fundamental Principle of Counting

a) Arun has to go for a job interview tomorrow. He has 4 formal shirts and 3 formal trousers. In how many ways can he choose his outfit for the interview?

b) Anita has to go to a party. She decides to wear either a saree or a dress. If Anita has 4 sarees and 6 dresses, in how many ways can she choose her outfit?

c) Anuja is planning to go for a party. She plans to wear either a saree or a skirt and top, along with high-heeled party shoes. She has 4 sarees, 3 skirts, 7 tops, and 12 pairs of high-heeled shoes. In how many ways can she choose her outfit?

d) There are five children A, B, C, D and E in a class. In how many ways can two different students be awarded the 1st and 2nd ranks in the class?

e) In the above question, in how many ways can the ranks be awarded if B cannot be awarded 1st rank?

The Fundamental Principle of Counting
If an action A can be performed in "m" ways, and for each of those "m" cases, another action B can be performed in "n" ways, then the number of ways of performing both A and B is m x n. Similarly, the number of ways of performing either A or B (but not both!) is m + n.

f) How many 4-digit numbers exist?

g) How many 4-digit odd numbers exist?

h) How many 4-digit odd numbers exist with all digits distinct?

i) How many 4-digit even numbers exist with all digits distinct?

j) A test contains 10 multiple choice questions, each having 5 options (a, b, c, d and e). In how many ways can Ameya mark the answers, if he attempts all the questions?

k) *There are five children A, B, C, D and E taking part in two competitions. In how many ways can the first position in the two competitions be awarded?

l) *How many 4-digit numbers exist with all digits odd?

m) *A test contains 10 multiple choice questions, each having 5 options (a, b, c, d and e). In how many ways can Ameya mark the answers, if he need not attempt all the questions?

Selection and Arrangement of Distinct Objects: Basic Permutations and Combinations

In how many ways can:

a) 8 people be arranged in 8 seats in a straight line?

b) 5 people be arranged in 8 seats in a straight line?

c) 5 out of 8 people be seated in 5 seats in a straight line?

d) 5 people out of 8 be selected to form a team?

e) a team of 2 boys and 3 girls be selected from a group of 3 boys and 5 girls?

f) a person attempt 10 True/False questions and get exactly 6 correct?

g) 8 people be seated in a circle?

h) 8 people be seated round a square table (with 2 seats on each side)?

i) 6 precious stones of different colours be arranged to form a circular necklace?

Number of ways of:	
i. Arranging r distinct objects in r distinct places (without repetition)	$= {}^{r}P_{r} = r!$
ii. Arranging r out of n distinct objects in r distinct places (without repetition)	$= {}^{n}P_{r} = \dfrac{n!}{(n-r)!}$
iii. Selecting r out of n distinct objects and placing them in r identical places	$= {}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
iv. Arranging n people in a circle	$= (n-1)!$
v. Arranging n beads/stones in a circle	$= \dfrac{(n-1)!}{2}$

In how many ways can:

j) *5 people out of 8 be selected to form a team, given that one person, Abdul, is always selected?

k) *5 people out of 8 be arranged in 5 seats, given that one person, Abdul, is one of the five?

l) *2 boys and 3 girls achieve the first 5 ranks in a class of 3 boys and 5 girls (no two getting the same rank)

m) *8 people be seated around a circular table having 12 seats?

n) *8 people be seated around a square table with 3 seats on each side?

o) *8 people be seated around an equilateral triangular table with 4 seats on each side?

p) *In how many ways can 8 spherical beads of different colours be arranged to form a straight chain?

q) There are 4 men (A, B, C, D) and 6 women (Z, Y, X, W, V, U) in an expedition team.

In how many way can we

i) select a subgroup of 3 people?
ii) select a subgroup of 7 people?
iii) select a subgroup of 7 people, such that X is selected?
iv) select a subgroup of 7 people, such that X is not selected?
v) select a subgroup of 2 men and 4 women?
vi) select a subgroup of 2 men and 4 women, such that B and Y are selected
vii) select and arrange a subgroup of 6 people?
viii) arrange a subgroup of 2 men and 4 women, such that B and Y are selected

r) Find the value of $\dfrac{{}^{50}C_4}{{}^{50}C_{48}}$

s) If ${}^{100}C_{63}$ = a, then find the value of ${}^{100}C_{36}$ in terms of 'a'

The following results can be useful when dealing with combinations:

${}^{n}C_r = {}^{n}C_{n-r}$ (Specially, ${}^{n}C_0 = {}^{n}C_n = 1$ and ${}^{n}C_1 = {}^{n}C_{n-1} - n$)

${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$

${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

Distribution of Distinct Objects into Distinct Groups

In how many ways can:

a) 8 people be sent to 5 classrooms?

b) 5 distinct coins be put into 8 pockets?

c) distinct words of 5 letters each be made using the English Alphabet?

d) distinct words, each of 5 distinct letters, be made using the English Alphabet?

e) distinct words, each of up to 5 distinct letters, be made using the English Alphabet?

In general, the number of ways of sending 'r' distinct objects to 'n' groups (with repetition) is n^r Another way of visualising it would be $\text{groups}^{\text{objects}}$

Challengers

1. A combination lock has 3 digits and each digit can take values from 0 to 9. How many trials would be needed to determine the correct combination?

   1) 729 2) 900 3) 999 4) 1000

2. There are six friends writing the TAC test for admission to nine MIIs. In how many ways could their final results be achieved, if it is known that each student can get into at most one MII?

   1) ${}^{9}P_{6}$ 2) 6^9 3) 9^6 4) 1 million

3. N people attend a dinner party and sit round a circular table. Each person knows only the people sitting immediately next to him and has to be introduced to everyone else. If the total number of pairs of people introduced to each other is 20, then what is the value of N?

   1) 6 2) 8 3) 10 4) 12

4. What is the approximate ratio of the number of five-digit numbers that can be made by using 0,1, 2, 3, 4, 5, 6 and 7 where none of digits is repeated to that where digits can be repeated?

   1) 1 : 4 2) 1 : 5 3) 1 : 3 4) 1 : 6

5. Let n be the total number of different 5-digit numbers with all distinct digits, formed using 2, 3, 4, 5 and 6 and divisible by 4. What is the value of n?

   1) 44 2) 32 3) 36 4) None of these

PRACTICE EXERCISE

DIRECTIONS for questions 1 to 18: Choose the correct alternative.

1. How many signals consisting of at least one flag can be made by hoisting 5 flags of different colours one above the other when any number of them may be hoisted at a time?

   1) 5 2) 240 3) 326 4) 325

2. Renny hosted a party to which he invited 20 persons. In how many ways can he and the guests be seated at a circular table? In how many ways of these will two particular persons be seated on either side of the host?

   1) 20!, 2 × 18! 2) 21!, 2 × 19! 3) 21!, 2 × 18! 4) 20!, 2 × 19!

3. Given 5 different green dyes, 4 different blue dyes, 3 different red dyes, how many combinations of dyes can be chosen taking atleast 1 green and 1 blue dye?

   1) 4096 2) 3255 3) 3720 4) None of these

4. In how many ways can three teams having 2, 2 and 3 members respectively be formed out of 7 players?

   1) 35 2) 210 3) 105 4) None of these

5. In how many ways can three cards be drawn one after the other at random from a regular pack of 52 cards such that all three belong to different suits?

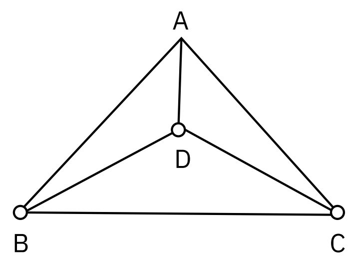
   1) 6×13^3 2) 12×13^3 3) 24×13^3 4) 36×13^3

6. There are six boxes numbered 1, 2, 3, 4, 5 and 6. Each box is to be filled up either with a white ball or a black ball in such a manner that at least one box contains a black ball and all the boxes containing black balls are consecutively numbered. The total number of ways in which this can be done equals _____.

   1) 15 2) 21 3) 63 4) 64

(Past CAT question)

7. Four cities are connected by a road network as shown in the figure. In how many ways can you go from a particular city and come back to it without travelling on the same road more than once?



1) 8 2) 12 3) 16 4) 20

(Past CAT question)

8. In an Antakshari competition, three participants are vying for the prize. In all, four judges cast their votes in the competition such that each judge votes in favour of only one participant. In how many ways can the four judges cast their votes?

1) 4 2) 14 3) 15 4) 81

9. A man has nine friends, four boys and five girls. In how many ways can he invite them, if there have to be exactly three girls among the invitees?

1) 320 2) 160 3) 80 4) 200

(Past CAT question)

10. How many triangles can be formed by joining 12 points, 7 of which are collinear?

1) 220 2) 35 3) 10 4) 185

11. In how many ways can the eight directors, the Vice-chairman and the Chairman of a firm be seated at a round table, if the Chairman has to sit between the Vice-chairman and a director?

1) $9! \times 2$ 2) $2 \times 8!$ 3) $2 \times 7!$ 4) None of these

(Past CAT question)

12. Ravi and his wife has put a lock which has a number code with three digits each digit ranging from 1 to 5. A thief tried to open the lock. There was a mechanical lock along with a digital lock which can be easily opened by the thief. In minimum how many attempts the thief will definitely come to know the number code of the lock if the number on the lock is 111?

1) 60 2) 125 3) ${}^{125}P_3$ 4) None of these

13. From among six couples, a committee of five members is to be formed. If the selected committee has no couple, then in how many ways can the committee be formed?

1) 6 2) 792 3) 40 4) 192

14. A student can select one of 6 different Maths books, one of 3 different Chemistry books and one of 4 different Science books. In how many different ways can the student select a Maths book, Chemistry book and Science book?

1) 14 2) 12 3) 72 4) 74

(Past CAT question)

15. To buy a computer system, a customer can choose one of 4 monitors, one of 2 keyboards, one of 4 computers and one of 3 printers. Determine the number of possible systems that a customer can choose from.

1) 96 2) 98 3) 98.5 4) 94

(Past CAT question)

16. Ten points are marked on a straight line and eleven points are marked on another straight line. How many triangles can be constructed with vertices from among the above points?

1) 495 2) 550 3) 1045 4) 2475

(Past CAT question)

17. For a scholarship, at most n candidates out of 2n + 1 can be selected. If the number of different ways of selection of at least one candidate is 63, the maximum number of candidates that can be selected for the scholarship is:

1) 3 2) 4 3) 2 4) 5

(Past CAT question)

18. From 3 mangoes, 4 apples and 2 oranges, how many selections of fruits can be made, taking at least one of each kind? (Assume that each fruit in a particular type is distinct from the other)

1) 256 2) 512 3) 315 4) 255

DIRECTIONS for questions 19 and 20: Solve as directed.

19. 5 doctors and 5 lawyers came together for a conference. During the conference, their coats had been sent to the laundry. When the laundryman returned the coats, it was found that each doctor received a lawyer's coat and each lawyer received a doctor's coat. In how many ways could this be done, considering that each coat was unique to the respective owner?

20. There are 12 points in a plane such that there are two distinct sets of 6 points that are collinear. No point in any set is collinear with the points of the other set. How many distinct lines can be drawn that pass through at least two of these 12 points?

QA-4.4 | BASIC P&C - II

THEORY

In the previous class, we studied Fundamental Principle of Counting and basic concepts in Permutations and Combinations. In this class, we will study the concepts in P & C in greater depth.

Different Permutation Cases

1. Permutation of n distinct objects taken r at a time, if repetition is allowed

Now, consider arranging numbers such that you are allowed to repeat the numbers.

How many three digit numbers can be formed using the digits 1, 2, 3, 4 and 5?

Here the task is to fill up three positions using the said five digits. Also, since no restriction is imposed, a digit can be used more than once.

Therefore, the first position can be filled in 5 ways. Since repetition is allowed, the second position can also be filled in 5 ways, and the third and last position can also be filled in 5 ways.

Thus the three positions can be filled up in $5 \times 5 \times 5 = 5^3$ ways.

On generalization, we get:

If repetition is allowed, for n distinct things taken r at a time, the number of arrangements possible is n^r.

Example

Four students Alex, Bob, Claire and David are participating in three competitions – Recitation, Story Telling and a Test of Reasoning. There is one prize at stake in each competition. In how many ways can the three prizes be won by the four students?

The Recitation prize may be won by any of the four students, so that there are 4 ways that the prize can go. Similarly, the other two prizes can also go in 4 ways each. Therefore, the three prizes can go in $4 \times 4 \times 4 = 4^3 = 64$ ways.

Note that it would have been a very laborious way to solve this problem if we considered the number of ways the first student could get one or more prizes and then the number of ways the next student could get one or more prizes, as the events would become dependent.

2. Permutation of n objects of which a few are identical

If there are n objects of which p objects are identical and of one kind, q identical objects of another kind and r identical objects of a third kind, and so on, then the number of permutations of all n objects taken all at a time is $\dfrac{n!}{p!q!r!}$.

Example

In how many ways can you arrange the letters of the word INTERNATIONAL?

There are 13 letters in all, including 2 I's, 3 N's, 2T's, and 2A's. There is one each of 4 other letters E, R, O, and L, which are all distinct.

Therefore, the number of ways the letters can be arranged is $\frac{13!}{2!3!2!2!}$.

Rules of Permutation

1. Suppose, 3 out of 5 persons A, B, C, D, E are to be seated on 3 chairs, for a photograph, such that A is always photographed. Then in actuality, we just have to arrange 2 of the 4 persons B, C, D and E on 2 chairs after having selected any 1 of the 3 chairs (in 3 ways) for A.

   $\therefore$ The number of different arrangements (photos) = $3 \times {}^4P_2 = 3 \times {}^{5-1}P_{3-1}$.

   $\therefore$ **The total number of arrangements of n things taken r at a time, in which a particular thing always occurs =** $r\left({}^{n-1}P_{r-1}\right)$

2. Next, suppose on the other hand, 3 out of 5 persons A, B, C, D, E are to be seated on 3 chairs for a photograph, such that A is never photographed. Then, in essence, we have to arrange 3 of the remaining 4 persons B, C, D and E on 3 chairs.

   The number obviously is ${}^4P_3 = {}^{5-1}P_3$.

   $\therefore$ **The total number of permutations of n distinct things taken r at a time in which a particular thing never occurs =** ${}^{n-1}P_r$

3. As the above two cases are mutually exclusive and exhaustive, we can conclude that the total number of permutations of n distinct things taken r at a time, i.e., ${}^nP_r$ will be equal to the sum of permutations from the above two cases.

   i.e. ${}^nP_r = {}^{n-1}P_r + r\left({}^{n-1}P_{r-1}\right)$

Rules of Combinations

1. Suppose we are to select 2 out of 3 letters A, B, C such that A is always selected; then effectively we only have to select just one of the two letters B and C (as one letter, A, has already been selected).

   $\therefore$ The required number of ways = ${}^{2}C_{1} = {}^{3-1}C_{2-1}$.

   In general, **the number of combinations of 'n' things taken r at a time in which 'p' particular things will always occur = ${}^{n-p}C_{r-p}$.**

2. On the other hand, if in the above example, A is never to be selected, then our job remains to select 2 out of 2 letters (B and C).

   The required number of ways = ${}^{2}C_{2} = {}^{3-1}C_{2}$.

   In general, **the number of combinations of n things taken r at a time in which p particular things never occur = ${}^{n-p}C_{r}$.**

3. **Different Possible Selections for Given 'n' identical objects**

   Example

   How many different selections are possible for the given three identical balls?

   We can select either 0 or 1 or 2 or all 3 balls at a time

   But, the number of ways of selecting 0, 1, 2 or 3 identical balls are the same, i.e. 1 each.

   So, the total number of different selections possible = 1 + 1 + 1 + 1 = 4

   In general, the number of different selections possible for n identical objects = (n + 1)

   Using the formula ${}^{n}C_{r}$ is applicable when 'n' distinct objects are available and a selection of 'r' objects is to be made out of them.

SOLVED EXAMPLES

Q : In how many ways can 4 cards be selected from a pack of 52 cards so as to include at least one diamond card.

A : Four cards can be selected from 52 cards in ${}^{52}C_4$ ways.

If none of these selected cards is a diamond card, they can be selected from the 39 cards in ${}^{39}C_4$ ways.

$\therefore$ The remaining selection will have at least one diamond card.

$\therefore$ Four cards with at least one diamond can be selected in ${}^{52}C_4 - {}^{39}C_4$.

Note: In a pack of cards there are 52 cards of four suits with 13 cards each. Two of the suits, diamond and hearts are red in colour and the other two, clubs and spades are black in colour. The 13 cards of each suit are valued as Ace, 2 to 10, Jack, Queen and King.

Q : A committee of 2 hawkers and 3 shopkeepers is to be formed from 7 hawkers and 10 shopkeepers. Find the number of ways in which this can be done if a particular shopkeeper is included and a particular hawker is excluded.

A : A particular shopkeeper is included

$\therefore$ We have to choose 2 from remaining 9 in ${}^{9}C_2 = \dfrac{9!}{7!2!} = \dfrac{8 \times 9}{2} = 36$ ways.

1 hawker is excluded

$\therefore$ We have to choose 2 hawkers from remaining 6 in ${}^{6}C_2 = \dfrac{6!}{4!2!} = \dfrac{5 \times 6}{2} = 15$ ways

Total number of ways = $36 \times 15 = 540$.

Q : In how many ways can a team of 11 cricketers be chosen from 6 bowlers, 4 wicket keepers and 11 batsmen to give a majority of batsmen if at least 4 bowlers are to be included and there is one wicket keeper?

A : 1 wicket keeper from 4 can be selected in ${}^{4}C_1 = \dfrac{4!}{3!1!} = 4$ ways

If 4 bowlers are chosen then remaining 6 batsmen can be chosen in ${}^{11}C_6$.

$${}^{6}C_4 \,.\, {}^{11}C_6 = \frac{6!}{4!2!} \times \frac{11!}{6!5!} = \frac{5 \times 6}{2} \times \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2} = 15 \times 14 \times 33 = 6930$$

If we choose 5 bowlers then we have to choose 5 batsmen.

$\therefore$ There is no majority.

$\therefore$ Total number of ways = $4 \times 6930 = 27720$.

Q : How many eight distinct letter words can be formed from the letters of the words “COURTESY” beginning with C and ending with Y?

A : The first place will always be taken by C and last place always by Y. The remaining six places can be filled by the remaining six letters in ${}^6P_6$ ways.

$\therefore$ Total number of words = $1 \times 1 \times {}^6P_6 = 6! = 720$

Q : In how many ways can the seven letters A, B, C, D, E, F and G be arranged so that B and C are always together?

A : Considering B and C as one, the six things can be arranged in ${}^6P_6$ ways. B and C can be arranged among themselves in 2 ways. i.e., BC and CB.

Total number of arrangements = ${}^6P_6 \times 2 = 2 \times 720 = 1440$ ways.

Q : How many 7 letter words can be constructed using the 26 letters of the alphabet series if each word contains exactly 3 vowels? (repetition is allowed)

A : If there are 3 vowels, 3 places for 3 vowels can be chosen in ${}^7C_3$ ways, for each of the three places, there are 5 letters

$\therefore$ vowel positions = 5^3;

$\therefore$ remaining 4 places are occupied by 21 consonants

$\therefore$ consonant positions = 21^4.

$\therefore$ Number of words with 3 vowels is ${}^7C_3 \times 5^3 \times 21^4$

Q : In how many ways can 7 Englishmen and 7 Americans sit around a circular table such that, no 2 Americans sit together?

A : Putting 1 Englishman in a fixed position, the remaining 6 can be arranged in 6! = 720 ways.

For each such arrangement, there are 7 positions for the 7 Americans and they can be arranged in 7! ways.

Total number of arrangements = $7! \times 6! = 3628800$

Derangement

A Derangement is a permutation of the elements of a set such that none of the elements appear in their original position.

Let us consider the following example.

Three men A, B and C get together for a party; each brings a walking stick along. In the party they all get drunk. In how many different ways is it possible that they leave, none carrying his own stick?

The possibilities are:

1. A carries B's stick, B carries C's stick, C carries A's stick
2. A carries C's stick, B carries A's stick, C carries B's stick

Thus there are 2 ways in all.

Note that this is a case of derangement of 3 objects and can be calculated using the formula as below:

$$3!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right]=6\left[\frac{1}{2}-\frac{1}{6}\right]=2$$

In general, we have, the number of arrangements of n objects such that no object is at its desired or proper place, i.e., the number of derangements of n objects

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right]$$

Theory of Partitioning

This theory is useful in finding out the number of ways of distributing one or more distinct/identical items into distinct/identical groups.

Different Cases of Partitioning

1. **Dividing 'n' distinct objects into 'r' distinct groups (some groups may be empty)**

   Suppose we have three different books A, B and C and two shelves, say upper and lower shelf, then the number of different ways in which this may be done is 8, i.e.,

   (a) A, B and C in upper shelf.
   (b) A, B and C in lower shelf.
   (c) A alone in upper shelf.
   (d) B alone in upper shelf.
   (e) C alone in upper shelf.
   (f) A alone in lower shelf.
   (g) B alone in lower shelf.
   (h) C alone in lower shelf.

   Now, $8 = 2^3$

   Thus, in general, the **number of ways of dividing (partitioning) n distinct things into r distinct groups (where some groups may be empty) = r^n**.

2. Dividing 'n' distinct objects into 'r' identical groups (some groups may be empty)

Now, if we were placing the books as in the above example, not in distinct shelves, but say in identical cartons, then we either

(a) place all books in one carton (1 way, just place them in any).

(b) place 2 books in one carton and 1 in the other (3 ways depending on which book is kept alone).

Thus, we have 4 ways in all; but $4 = \frac{8}{2} = \frac{2^3}{2}$

Thus, in general, the **number of ways of dividing (partitioning) n distinct things into r identical groups (where some groups may be empty) =** $\frac{r^n}{r!}$

In the above case, groups are not numbered.

3. Dividing 'n' identical objects into 'r' distinct groups (some groups may be empty)

Let us look at the example below:

Suppose we want to distribute 5 chocolates between 4 children such that all the chocolates may go to a single child. In how many different ways can this be done?

This can be thought of as follows:

We arrange the 5 chocolates in a row, shown by C's, and then introduce 3 vertical bars between them

| C C C | C C |

The chocolates before the first vertical bar go to the first child, the chocolates between the first two vertical bars go to the second child, the chocolates between the next two vertical bars go to the third child and the remaining chocolates go to the fourth child.

Thus the problem is that of arranging 5 C's and 3 |'s in all possible ways.

The required answer, then, is $\frac{(5+3)!}{5!\,3!} = {}^{5+3}C_3 = {}^{5+4-1}C_{4-1}$.

In general, The number of ways of partitioning n non-distinct things into r distinct groups (some groups may be empty) is ${}^{n+r-1}C_{r-1}$.

4. Dividing 'n' identical objects into 'r' distinct groups (no group is empty)

Number of ways of partitioning n non-distinct things into r distinct groups (all groups non-empty) is ${}^{n-1}C_{r-1}$.

This is so because all groups being non empty, each of the r groups must contain at least one object i.e., $n \geq r$ and these are objects out of n are already placed, one in each group. Hence, we only have to partition the remaining (n – r) objects into r groups, where some (now) may not get a share of any of these (n – r) objects.

Hence, by the formula (3), the required number of ways = ${}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

SOLVED EXAMPLES

Q : Find the number of ways of inviting at least one executive out of five executives to a conference.

A : Here, we have two options for each executive - either he is invited or he is not invited So, the total number of ways of inviting these 5 executives = 2^5. But in these selections, there would be a case where all executives are not invited, which we need to exclude.

So, the total number of ways = $2^5 - 1 = 31$

Q : In how many ways can 21 identical white balls and 19 identical black balls be arranged in a row so that no 2 black balls are together?

A : First arrange 21 white balls in a row. This can be done in 1 way (since they are identical). Now there are 22 places for the 19 black balls and so the places can be filled in

$^{22}C_{19}$ ways = $\frac{22!}{3!\,19!}$ ways OR $^{22}C_3 = \frac{22 \times 21 \times 20}{2 \times 3} = 1540$

Concept Builder

1. How many 3-digit numbers are there which ends with either 5 or 7?
2. In how many ways can four parrots be put in four cages, each meant for a particular parrot, such that none of the parrot goes in its own cage?
3. Find the number of ways of selecting a team of 5 players from a group of 8 players such that a particular player is always selected?
4. How many 3-digit numbers are there which ends with either 5 or 7?

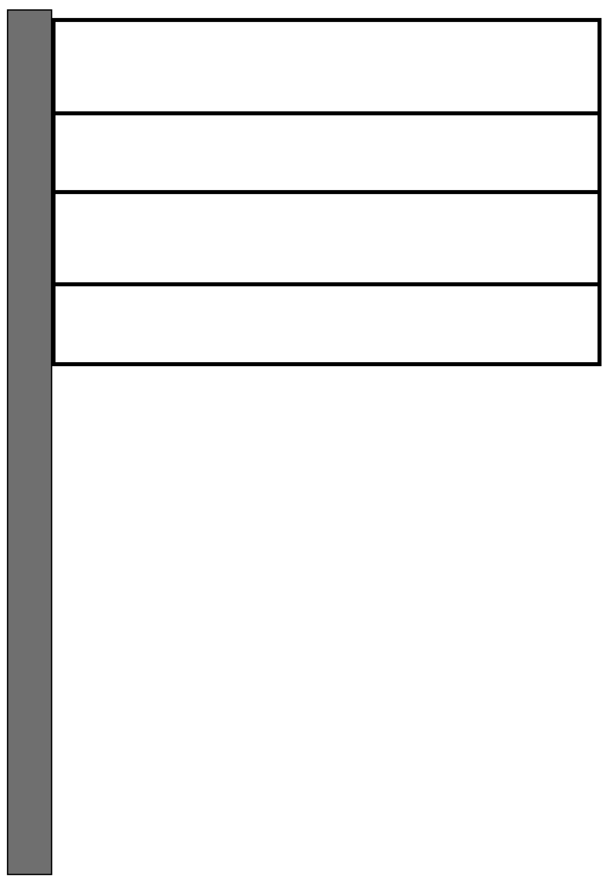
Answer Key

1. 180
2. 9
3. 35
4. 180

CLASS EXERCISE

Teaser

A new flag has to be designed. It must have 4 horizontal stripes, as shown in the figure below. Each stripe of the flag is to painted a single colour, such that no two adjacent stripes have the same colour. The available colours are Black, Red, Green and Blue. In how many different ways can such a flag be created?



Arrangement of objects with additional conditions:

1. In how many ways can 7 boys and 4 girls be arranged in a straight line such that
   a) no two girls are together?
   b) all the girls are together?
   c) two specific boys X and Y are together?
   d) all the girls are together and all the boys are together?

2. *In how many ways can 5 distinct History textbooks, 4 distinct Geography textbooks and 3 distinct Civics textbooks be arranged in a straight line such that all books of a given subject are together?

3. *In how many ways can 7 boys and 4 girls be arranged in a circle such that no two girls are together?

If we want to arrange (m + n) objects such that a certain group of m objects should be kept together, arrange them amongst themselves at the start and thereafter treat them as a single object: $m!\,(n+1)!$

Conversely, if we want to arrange (m + n) objects such that no two of a certain group of m objects should be together, arrange the other n first, and then place the m objects in the n + 1 spaces formed: $n!\ {}^{n+1}P_m$

Arrangement of objects, not all distinct:

4. In how many ways can:
   a) the letters of the word ORANGE be arranged?
   b) the letters of the word APPLE be arranged?
   c) the letters of the word PARATROOPER be arranged?
   d) 10 people be divided into groups of 5, 3 and 2?

Number of ways of:

i. Arranging n distinct objects in r distinct places (with repetition) = r^n

ii. Arranging n objects (not all distinct), out of which ‘p’ of one type are alike, ‘q’ of a second type are alike, ‘r’ of a third type are alike, and so on = $\dfrac{n!}{p!q!r!\ldots}$

   e) *How many 3 letter words starting and ending with a vowel can be made in the English language?
   f) *In how many ways can some or all of 5 distinct coins be put into 8 pockets?
   g) *How many words of 2 distinct vowels and 3 distinct consonants can be made using the English Alphabet?

h) * How many words of 5 distinct letters each, beginning and ending with a consonant, can be made using the English Alphabet?

i) * In how many ways can 6 white and 4 black marbles be arranged in a straight line?

Factors of a number

5. How many factors does the number 90 have?

6. What is the sum of the factors of the number 90?

In general, if a number N is written in the form $N = a^p \times b^q \times c^r$... where a, b, c are distinct primes, then

- the number of factors of N (including 1 and N itself) is given by $(p+1) \times (q+1) \times (r+1)$...
- the sum of all factors of N is given by $(1 + a + a^2 ... + a^p) \times (1 + b + b^2 ... + b^q) \times (1 + c + c^2 ... + c^r)$ ...

7. * What is the number of factors of 48? And of 100? Also find their sum?

Completely wrong arrangements: Derangements

8. In how many ways can 3 letters be placed in 3 addressed envelopes such that none of them go into the correct envelope?

9. In how many ways can 10 letters be placed in 10 addressed envelopes such that exactly 9 letters are in the correct envelope?

10. *6 people carrying identical briefcases meet for dinner at a restaurant. At the end of the meal, each of them picks up a briefcase. In how many ways could none of them get the right briefcase (i.e. his own)?

The standard formula for calculating D(n) (the number of ways of deranging 'n' objects) is

$$D(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{(-1)^n}{n!}\right]$$

A quicker iterative way to calculate D(n) is as follows:

D(1)			= 0
D(2)	= D(1) × 2 + 1	= 0 × 2 + 1	= 1
D(3)	= D(2) × 3 − 1	= 1 × 3 − 1	= 2
D(4)	= D(3) × 4 + 1	= 2 × 4 + 1	= 9
D(5)	= D(4) × 5 − 1	= 9 × 5 − 1	= 44
. . .	. . .	. . .	
D(n)	$= D(n - 1) \times n + (-1)^n$ in general		

Distribution of identical objects into groups

11. In how many ways can 5 apples be divided between 2 people - Sid and Manny?

12. In how many ways can 5 apples be divided among 3 people - Sid, Manny and Diego?

13. In how many ways can 15 chocolate éclairs be distributed among 5 children, given that each child receives at least 1 éclair?

14. Five friends are sitting around a table eating a bowl of grapes. There were a hundred grapes initially in the bowl. If the five friends finish off the grapes, and each of them eats at least 10 grapes, in how many ways could they have distributed the grapes among themselves?

15. Amy has 20 dimes and 4 pockets. In how many ways can she put the coins into these pockets?

16. A group of 6 hunters went to hunt foxes. They managed to bag a total of 20 foxes. In how many different ways could they have done so?

The number of ways in which n identical objects can be distributed into r distinct groups

- such that groups can be empty is ${}^{(n+r-1)}C_{(r-1)}$
- such that no group is empty is ${}^{(n-1)}C_{(r-1)}$

17. * In how many ways could 28 identical marbles be put into 8 boxes so that no box is empty?

18. * Rose Day is being celebrated in Bayport Engineering College. Between them, the 18 girls in the first year batch get 75 red roses. In how many ways could this happen, given that each girl gets at least 2 red roses?

19. * In an ice-cream eating competition, there are forty participants. Each of them eats at least 4 scoops of vanilla ice-cream and in all 250 scoops of vanilla ice-cream were consumed. In how many ways could they have eaten the ice-cream?

20. * In a test match innings, the total number of runs scored by the 11 batsmen from England was 350. Given that no batsman failed to score, how many different scorecards could be possible?

21. * Scrat has 24 acorns and 4 holes to hide them in. In how many ways can he distribute the acorns given that one particular hole, his favourite, must have at least 10 acorns?

22. In how many ways up-to 10 identical objects can be distributed among 5 people?

23. How many 4 digits number exists such that sum of digits is equal to 7?

24. How many 4 digits numbers exist such that sum of digits is up to 7?

25. *How many numbers having up-to 4 digits exist such that sum of digits is equal to 7?

26. *How many number having up-to 4 digits exist such that sum of digits is up-to 7?

Challengers

1. A group of 20 hunters went to hunt foxes. They managed to bag a total of 6 foxes. In how many different ways could they have done so?

   1) ${}^{20}C_6$ 2) ${}^{19}C_5$ 3) ${}^{25}C_5$ 4) ${}^{25}C_6$

2. How many terms does expansion of $(a + b + c + d)^{10}$ have?

   1) 286 2) 84 3) 55 4) 120

3. How many 4-digit numbers exist which are not less than 2000 but not greater than 6000 and which have all digits even or zero?

   1) 72 2) 250 3) 49 4) 251

4. There are 'p' committees in a class (where $p \geq 5$), each consisting of 'q' members (where $q \geq 6$). No two committees are allowed to have more than 1 student in common. What is the minimum and maximum number of students possible?

   1) $(p - 1)(q - 1)$, pq 2) $pq - {}^{q}C_2$, pq
   3) $pq - {}^{p}C_2$, pq 4) $pq - p$, $pq - 1$

5. There are seven tasks (1, 2, 3, 4, 5, 6, 7) which have to be done by seven people (A, B, C, D, E, F and G). Each person can do only one task. Task 1 must be done by A, B or C. Task 4 and 5 cannot be done by either F or G. In how many ways can the tasks be accomplished?

   1) 1728 2) 864 3) 5040 4) 216

6. A person has 3 books in French, 4 in Hindi and 6 in English. All the books in a given language are distinct. In how many ways can she arrange them in a line such that all the French books are together, but no two Hindi books are together.

   1) $11! \times 3! - 8! \times 3! \times 4!$ 2) $7! \times 3! \times {}^{8}P_4$
   3) $13! - 8! \times 3! \times 4!$ 4) $7! \times 3! \times {}^{10}P_4$

7. A contestant in a game show has to solve a 'Match the following' style question. She is given the names of 10 animals, and their pictures, and has to assign the correct name to each picture. She manages to assign exactly 6 of them correctly. In how many ways could she have done it?

   1) ${}^{10}C_4 \times {}^{7}P_4$ 2) ${}^{10}P_6$ 3) ${}^{10}C_6 \times 4!$ 4) ${}^{10}C_6 \times 9$

8. Find the sum of all the factors of 216 that are divisible by 12.

   1) 468 2) 432 3) 600 4) 576

9. 'Mega Mind' is a televised game show, where aspirants are required to write an entrance test to qualify to be a contestant. For last week's episode, 4 doctors, 5 engineers and 3 lawyers took the test. In how many ways could the contestants be selected, given that there was at least one contestant from each of the three given professions and that at least one engineer was not selected??

   1) 3255 2) 4096 3) 3150 4) 2520

PRACTICE EXERCISE

DIRECTIONS for questions 1 to 12: Choose the correct alternative.

1. How many numbers greater than two lakhs can be made using the digits 0, 4, 5 and 6 where only the digits 4 and 5 may be repeated and that too only once?

   1) 100 2) 150 3) 180 4) ∞

2. In how many ways can the letters of the word 'AEROPLANE' be arranged so that 'R' and 'O' are always together?

   1) $8! \times 2!$ 2) $8! \times 4$ 3) $8!$ 4) $\frac{8!}{2!}$

3. Sush went to a beach with her friends. Before arriving at the beach Sush entered into a quarrel with one of her friends. After the quarrel they decided that they would never stand or sit adjacent to each other. Keeping this in mind one of her friends realised that if they were to sit in a row at the beach they could still sit in 480 ways. How many friends had accompanied Sush to the beach?

   1) 5 2) 6 3) 4 4) 7

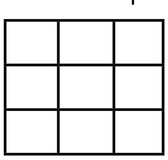
4. From 6 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done, if the committee is to include atleast one lady?

   1) 246 2) 252 3) 120 4) 492

5. How many numbers between 4000 and 9000 can be formed by using the digits 3, 4, 7 and 9 if each digit may be repeated?

   1) 64 2) 128 3) 24 4) 48

6. Six A's are to be placed in the 3×3 square grid as shown below such, that each row and each column has at least one square filled. In how many ways can this be done, if only one A can be placed in one square?

   1) 84 2) 82 3) 78 4) None of these

7. In how many ways can 3 identical Physics books and 3 identical Chemistry books be arranged in a row such that all Chemistry books are together?

   1) 144 2) 4 3) 24 4) 20

8. How many 5-digit numbers (greater than 50000) with distinct digits exist such that there is exactly one 6 and one 9 in the number and the 9 is to the left of 6?

   1) 2100 2) 1260 3) 2430 4) 3360

9. A committee of 8 persons has to be formed from 15 persons A, B, C, D .... O. Either all or none of A, E and I are to be selected. If E is selected. B, H and K cannot be selected. F and G are always to be selected. In how many ways can the committee be formed?

   1) 245 2) 35 3) 119 4) None of these

10. The coach of the Indian Cricket team has to select and fit in 11 players out of Rahul, Pathan, Sachin, Dhoni, Yuvraj, Anil, Gautam, Ajit, Jay, Sourav, Kaif, Amol, Wasim and Laxman in 11 batting slots (numbered 1, 2, ..., 11). If it is certain that Sachin will be picked and Sourav will not be picked, in how many ways can the coach set a batting line-up?

    1) $66 \times 11!$ 2) ${}^{12}P_{10}$ 3) ${}^{13}P_{11}$ 4) None of these

11. In a railway compartment there are 2 rows of seats facing each other with accommodation for 5 in each. 4 wish to sit facing forward and 3 facing towards the rear while 3 others are indifferent. In how many ways can the 10 passengers be seated?

    1) 120 2) 43200 3) 720 4) 14400

12. A necklace is to be made using six red, three blue and three green beads such that, no two red beads are adjacent to each other. All the beads of the same colour are identical. In how many ways can this necklace be formed?

    1) 2 2) 18 3) 3 4) 24

DIRECTIONS for questions 13 to 20: Solve as directed.

13. In how many ways can 7 chocolates of type A, 6 chocolates of type B and 5 chocolates of type C be distributed among 4 children such that each child receives at least one chocolate of each type?

14. How many numbers less than 1000 have all their digits even or zero and the sum of their digits as 10?

15. Invitation letters for a conference have to be sent to five participants. Each letter has a specific envelope in which it is supposed to be enclosed. In how many ways can only one letter be enclosed in the right envelope while the remaining four letters be enclosed in the wrong envelopes (i.e. the envelopes other than the one in which the letter is supposed to be enclosed)?

16. How many numbers can be formed using the digits 1, 2, 3, 4, 5, 6 and 7 that are greater than 10000 and less than 20000 such that no digit is repeated?

17. There are 6 bulbs B1 to B6 that need to be connected to one each of six switches S1 to S6. The first bulb cannot be connected to S1 or S2. The fourth bulb must be connected to either the S5 or S6. In how many ways can the six bulbs be connected to the six switches?

18. Between 95 and 750 (both excluded), how many natural numbers have at least one 5 as one of the digits?

19. WXYZ is a parallelogram. There are 5 distinct points (A, B, C, D, E) on WX, 4 distinct points (F, G, H, I) on XY, 5 distinct points (J, K, L, M, N) on YZ and 4 distinct points (O, P, Q, R) on ZW. How many triangles can be formed using all the above mentioned 22 points such that one of the vertices of the triangle is W?

20. How many numbers greater than 100 (with distinct digits) can be formed using the digits of the number 1269 such that they are divisible by 4?

QA-4.5 FACTORIALS, BINOMIAL THEOREM, REMAINDERS — THEORY

Factorials

The product of the first 'n' natural numbers is n! (Pronounced as n factorial).

2! = 2 × 1 = 2

3! = 3 × 2 × 1 = 6

The factorial of every number greater than 1 will have one or more prime numbers.

A factorial of a number (except of the number 1) is either a prime number or a combination of 1 or more prime numbers and sometimes powers of those prime numbers.

Highest power of a number in a factorial

Question types based on highest power of a number in factorial are as discussed below.

Case 1: To find the powers of a prime number 'p' contained in n!, where n is a natural number.

Highest power of prime number 'p' in n! = $[n/p] + [n/p^2] + [n/p^3] + [n/p^4] + \ldots$

Where the value of the term in the [ ] is the greatest integral value less than or equal to the terms n/p, n/p^2, n/p^3, n/p^4, ...

(i) Find the highest power of 2 in 20!

Highest power of 2 in 20!

$\Rightarrow [20/2] + [20/4] + [20/8] + [20/16]$

$\Rightarrow 10 + 5 + 2 + 1 = 18$

Case 2: To find the highest power of a composite number C in n! where n is a natural number.

Let C = $(x)^a (y)^b (z)^c$ ... where x, y, z, etc. are prime factors of C.

If the powers of x, y, z are the same then, we find the largest prime factor of C and then find its highest power contained in n!. The highest power of the largest prime factor of C in n! is the highest power of C in n!.

(i) Find the highest power of 30 in 75!.

$30 = 2^1 \times 3^1 \times 5^1$

Since 5 is the largest prime factor, highest power of 5 in 75!

$\Rightarrow \left[\frac{75}{5}\right]+\left[\frac{75}{5^2}\right]+\left[\frac{75}{5^3}\right]=15+3+0=18$

$\therefore$ The highest power of 30 in 75! is 18

Case 3: If the powers of x, y, z are not the same then, first find the individual highest power of x^a, y^b, z^c that exactly divide the factorial and then choose the smallest value amongst them. This is illustrated through an example given below.

(i) Find the highest power of 24 in 16!.

$24 = 2^3 \times 3^1$ $\therefore$ Prime factors of 24 are 2 and 3

Highest power of 2 in 16! $=\left[\frac{16}{2}\right]+\left[\frac{16}{2^2}\right]+\left[\frac{16}{2^3}\right]+\left[\frac{16}{2^4}\right]=8+4+2+1=15$

$\therefore$ Highest power of 2^3 in 16! $= \left[\frac{15}{3}\right] = 5$

Highest power of 3 in 16! $= \left[\frac{16}{3}\right] + \left[\frac{16}{3^2}\right] = 5 + 1 = 6$

Choosing the smallest value amongst 5 and 6 we get the highest power of 24 that divides 16! i.e., 5.

SOLVED EXAMPLES

Q : Find the highest power of 2 in 50!.

A : The highest power of 2 in 50! = [50/2] + [50/4] + [50/8] + [50/16] + [50/32]
= 25 + 12 + 6 + 3 + 1 = 47

Q : Find the highest power of 30 in 50!

A : 30 = 2 × 3 × 5
Now 5 is the largest prime factor of 30 so we find the highest power of 5 in 50!.
The highest power of 5 in 50! = [50/5] + [50/25] = 10 + 2 = 12
Hence the highest power of 30 in 50! = 12

Q : Find the number of zeroes in 50!

A : We get a zero at the end of a number when we multiply that number by 10. So, to calculate the number of zeroes at the end of 50!, we have to find the highest power of 10 present in the number. Since 10 = 2 × 5, we have to find the highest power of 5 in 50!
The highest power of 5 in 50! = [50/5] + [50/25] = 10 + 2 = 12
So, the number of zeroes at the end of 50! = 12

Q : What is the largest power of 3 contained in 100!?

A : The largest power of 3 contained in 100! is calculated as follows

$\frac{100}{3}$ = 33 multiples of 3 $\quad$ $\frac{100}{9}$ = 11 multiples of 9

$\frac{100}{27}$ = 3 multiples of 27 $\quad$ $\frac{100}{81}$ = 1 multiples of 81

Therefore total number of 3's = 33 + 11 + 3 + 1 = 48

Remainders

A natural number 'm' when divided by a natural number 'n' will yield the remainder that is a whole number and less than n i.e., 0, 1, 2, ... n – 1

Methods to find remainders

Remainder Theorem, Cyclicity of remainders and Congruence are some of the methods that can be applied to find remainders. To learn more on the same let us go to each of the methods one-by-one with examples.

1. **Cyclicity of Remainders**

Successive powers of a number when divided by the same number leave remainders that follow a cyclical pattern. (i.e., the repitition of the remainders occur in a particular pattern). Using this pattern, the remainder of the division of any power of that number (by the same number) can be found out.

Example

(i) Find the remainder when 2^{54} is divided by 7.
2^1, 2^2, 2^3, 2^4, 2^5, 2^6 and 2^7 when divided by 7 will leave remainders of 2, 4, 1, 2, 4, 1 and 2 respectively.
So the pattern repeats after 3 steps ($\because$ remainder by 2^1 and 2^4 is the same)
The remainder of 2^{54} when divided by 7 will be the same as the remainder of 2^3 when divided by 7. Hence the remainder of 2^{54} is 1.

(ii) Find the remainder when 5^{75} is divided by 15.
Following cyclicity method:
5^1, 5^2, 5^3 and 5^4 when divided by 15 will leave remainders of 5, 10, 5 and 10 respectively.
$\therefore$ The pattern repeats after 2 steps. Every alternate power of 5 will give the same remainder when divided by 15.
$\therefore$ The remainder of 5^{75} when divided by 15 will be the same as the remainder of 5^1 when divided by 15 i.e. 5.

2. **Congruence**

If two numbers a and b have the property that their difference a – b is integrally divisible by a number n (i.e., $\frac{a-b}{n}$ is an integer), then a and b are said to be "congruent modulon."
The number n is called the <u>modulus</u>, and the statement "a is congruent to b (modulo n)" is written mathematically as:
$a \equiv b \pmod{n}$ where $a - b$ is divisible by n. i.e., $a - b = nk$.

Example

(i) $24 \equiv 3 \pmod 7$ where $24 - 3 = 21$ is divisible by 7.
Similarly, $16 \equiv -2 \pmod 3$
If a – b is not integrally divisible by n, then we say "a is not congruent to b (modulo n)," which is written $a \not\equiv b \pmod m$

(ii) $25 \not\equiv 12 \pmod 7$ as $25 - 12 = 13$ is not divisible by 7.

Properties of Congruence

(i) $a \equiv a \pmod n$

For any integer a, $a - a = 0 = 0.n$ $\therefore a \equiv a \pmod n$.

(ii) If $a \equiv b \pmod n$, then $b \equiv a \pmod n$

Suppose $a \equiv b \pmod n$, then $a - b = qn$ for some integer q.

$b - a = -kn = (-k)n$. $\therefore b \equiv a \pmod n$.

(iii) If $a \equiv b \pmod n$ and $b \equiv c \pmod n$, then $a \equiv c \pmod n$

Suppose $a \equiv b \pmod n$ and $b \equiv c \pmod n$, then there are integers q_1 and q_2 such that $a - b = q_1n$ and $b - c = q_2n$.

It follows that $a - c = (a - b) + (b - c) = q_1n + q_2n = (q_1 + q_2)n$, as consequence $a \equiv c \pmod n$.

(iv) If $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $pa + qc \equiv pb + qd \pmod n$.

$a + c \equiv b + d \pmod n$ and $a - c \equiv b - d \pmod n$

Also $pa \equiv pb \pmod n$.

Since $a \equiv b \pmod n$ and $b \equiv c \pmod n$ there are integers q_1 and q_2 such that $a - b = q_1n$ and $b - c = q_2n$.

It now follows that $(pa + qc) - (pq + qp) = p(a - b) + q\ (c - d)$

$= pq_1n + qq_2n = (pq_1 + qq_2)n$.

As a result $(pa + qc) - (pb + qd)$ is divisible by hence $pa + qc \equiv pb + qd \pmod n$.

Taking $p = q = 1$, we get, $a + c \equiv b + d \pmod n$.

Next, choosing $p = 1$ and $q = 1$, we have $a - c \equiv b - d \pmod n$.

If we take $q = 0$, the statement reduces to $pa \equiv pb \pmod n$.

Example

If $a + 7 \equiv 2 \pmod 5$, then the remainder when a is divided by 5 is:

If $a \equiv b \pmod n$ then

$(a + c) \equiv (b + c) \pmod n$

$\therefore a + 7 \equiv 2 \pmod 5$

i.e., $a + 7 \equiv -5 + 7 \pmod 5$

$\therefore a \equiv -5 \pmod 5$

$\therefore a \equiv\ 0 \pmod 5$

(v) If $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $ac \equiv bd \pmod n$

If $a \equiv b \pmod n$, then, $ca \equiv cb \pmod n$.

Applying similar logic to $c \equiv d \pmod n$, we get $cb \equiv db \pmod n$.

They by (iii) above we conclude that $ca \equiv db \pmod n$, which is the same as $ac \equiv bd \pmod n$.

Example

What is the value of x? [Given $6! \equiv x \pmod 7$]

$6! = 6 \times 5 \times 3 \times 4 \times 2 \times 1$

$\therefore 6 \equiv -1 \pmod 7$

$5 \times 4 \equiv -1 \pmod 7$

$3 \times 2 \equiv -1 \pmod 7$

$\therefore 6 \times 2 \times 5 \times 3 \times 4 \equiv -1 \times -1 \times -1 \pmod 7$

$6! \equiv -1 \pmod 7$

$\therefore x = -1$

(vi) If $a \equiv b \pmod n$, then $a^m \equiv b^m \pmod n$ for any positive interger m.
If $a \equiv b \pmod n$, i.e. $(a - b)$ is divisible by n.
For $a^m \equiv b^m \pmod n$. $a^m - b^m$ should be divisible by n. And $(a - b)$ is factor of $a^m - b^m$.
Then, $a^m \equiv b^m \pmod n$

Example
Find the value of x if $1331 \equiv x^3 \pmod 3$
$1331 \equiv x^3 \pmod 3$
$11^3 \equiv x^3 \pmod 3$
$\therefore\ 11 \equiv x \pmod 3$
$\therefore\ 11 - x$ is divisible by 3
$\therefore\ x = 2$

(vii) If a and b leave the same non-negative remainder upon division by n,
Then $a \equiv b \pmod n$.
$a = q_1 n + r$ and $b = q_2 n + r$, then $a - b = (q_1 - q_2)n$. i.e., $a \equiv b \pmod n$.
Example
$22 \equiv 1 \pmod 7$
$15 \equiv 1 \pmod 7$
$\therefore\ 22 \equiv 15 \pmod 7$ as $(22 - 15$ is divisible by 7).

(viii) If p is prime number the $a^p \equiv a \pmod p$

Example
$7^{23} \equiv 7 \pmod{23}$ as 23 is prime number.

Linear Congruence Theorem
If a and b are any integers and n is a positive integer, then the congruence
$ax \equiv b \pmod n$ has a solution x if and only if greatest common divisor (a, n) divides b.
For example, there is no integer x with
$4x \equiv 3 \pmod 6$ but there exists an integer x with
$4x \equiv 2 \pmod 6$ as GCD of 4 and 6 is divisible by 2.

3. Remainder Theorem

If a natural number 'n' is divided by a natural number 'm' and can be brought in the form:
$\frac{x.(a^p)^q}{a^p - 1}$, such that
i. $n = x.(a^p)^q$
ii. $m = a^p - 1$
iii. x, a, p & q are all natural numbers
iv. $x < m$
then the remainder of the division of 'n' by 'm' is 'x'

Example
$$\frac{2^{87}}{7} \Rightarrow \frac{1.(2^3)^{29}}{(2^3 - 1)} \Rightarrow \text{Remainder} = 1$$
$$\frac{2^5}{7} \Rightarrow \frac{2^2(2^3)}{(2^3 - 1)} \Rightarrow \text{Remainder} = 2^2 = 4$$

$\frac{40}{7} \Rightarrow \frac{5.(2^3)^1}{(2^3-1)} \Rightarrow$ Remainder = 5

Note: In the above expression if x ≥ m, then divide 'x' by 'm' such that the final remainder is less than m,

Example

$\frac{400}{15} \Rightarrow \frac{25(2^4)^1}{2^4-1} \Rightarrow$ Remainder = 25

$\frac{25}{15} \Rightarrow$ Remainder = 10

So, the remainder of 400 divided by 15 is 10.

Binomial Theorem

We know that $(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and so on.

Binomial Theorem tells us how to generalize the formula for $(a + b)^n$.

According to binomial theorem—

$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots\ldots.. + {}^nC_{n-2}\, a^2b^{n-2} + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$

Thus we can know the following from binomial theorem expansion—

1) Expansion of $(a + b)^n$ contains 'n + 1' terms.

2) Each term in the expansion of $(a + b)^n$ is of the form $K.a^x.b^y$. where K is a constant, both x and y are such that $0 \le x, y \le n$ and $x + y = n$.

3) P^{th} term in the expansion of $(a + b)^n$ is ${}^nC_{P-1}.a^{n-p+1}.b^{P-1}$

The principle of binomial theorem can be used to calculate last two digits of a power of a number as follows—

Calculate last two digits of 31^{37}.

Using binomial theorem, we can write—

$31^{37} = (30 + 1)^{37}$

$= 30^{37} + {}^{37}C_1 30^{36}.1^1 + {}^{37}C_2.30^{35}.1^2 + \ldots\ldots. + {}^{37}C_{35}.30^2.1^{35} + {}^{37}C_{36}.30^1.1^{36} + 1^{37}$

We can see that all the terms except last two terms contain at least two zeros. Therefore, sum of all those terms will end in two zeroes. Therefore, in order to calculate last two digits of the number, we need to consider only last two terms in the expansion: ${}^{37}C_{36}.30^1.1^{36} + 1^{37}$

Now, ${}^{37}C_{36}.30^1.1^{36} = \frac{37!}{36!\,1!}.30^1.1^{36} = 37 \times 30 = 1110$

$\therefore\ {}^{37}C_{36}.30^1.1^{36} + 1^{37} = 1110 + 1 = 1111$

$\therefore$ Last two digits of 31^{37} are 11.

Example

(i) Expand $(a + b)^8$.

Using binomial theorem,

$(a + b)^8 = {}^8C_0.a^8 + {}^8C_1a^7b^1 + {}^8C_2a^6b^2 + {}^8C_3a^5b^3 + {}^8C_4a^4b^4 + {}^8C_5a^3b^5 + {}^8C_6a^2b^6 + {}^8C_7a^1b^7 + {}^8C_8b^8$

$= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 5ba^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

(ii) Consider expansion of $(a + b)^{25}$. How many terms are there in the expansion? What is the 15^{th} term of the expansion?

Expansion of $(a + b)^{25}$ contains 26 terms.

15^{th} term in the expansion is ${}^{25}C_{14}.a^{11}b^{14}$

(iii) What are the last two digits of the number 77^{32}?

$77^{32} = (70 + 7)^{32}$

$70^{32} + {}^{32}C_1 70^{31}7^1 + {}^{32}C_2 70^{30}.7^2 + \ldots\ldots\ldots\ldots\ldots\ldots {}^{32}C_{30}.70^2.7^{30} + {}^{32}C_{31}.70.7^{31} + 7^{32}$

We need to consider only the last two terms in the expansion as all other terms have at least two zeroes.

${}^{32}C_{31}.70.7^{31} = 32 \times 70 \times 7^{31} = 2240.7^{31}$

This product will end in one zero. The digit in the ten's place of this product is the digit in the unit's place of 224×7^{31}. Using cyclicity, we know that 7^{31} ends in 3.

Therefore, 224×7^{31} ends in 4×3 or in 2.

$\therefore$ ${}^{32}C_{31}.70.7^{31}$ ends in 20.

We need to calculate last two of 7^{32}

7^1 ends in 07

7^2 ends in 49

7^3 ends in 43

7^4 ends in 01

7^5 ends in 07 and the cycle repeats.

$\therefore$ 7^{32} ends in 01.

$\therefore$ ${}^{32}C_{31}.70.7^{31} + 7^{32}$ ends in $20 + 01 = 21$

$\therefore$ Last two digits of 77^{32} are 21.

(iv) What are the last two digits of the number 24^{38}?

$24^{38} = (20 + 4)^{38}$

$= 20^{38} + {}^{38}C_1 20^{37}.4^1 + {}^{38}C_2.20^{36}.4^2 + {}^{38}C_3.20^{35}.4^3 + \ldots\ldots + {}^{38}C_{37}.20^1.4^{37} + 4^{38}$

We need to consider only the last two terms of the expansion.

${}^{38}C_{37}.20^1.4^{37} = 38 \times 20 \times 4^{37} = 760 \times 4^{37}$. Unit's place of $4^{37} = 4$.

$\therefore$ ${}^{38}C_{37}.20^1.4^{37}$ ends in 760×4 or in 40.

Now, $4^{38} = 2^{76}$. 2^{10} ends in 24 and $2^{20} = (2^{10})^2$ ends in 76.

$2^{30} = 2^{20} \times 2^{10}$, which ends in 76×24 or in 24. Similarly $2^{60} = 2^{30} \times 2^{30}$ ends in $24 \times 24 = 76$.

2^{70} ends in 76×24 or 24

Now, $2^{76} = 2^{70} \times 2^6.2^{70}$ ends in 24 and 2^6 ends in 64.

$\therefore\ 2^{76}$ ends in 24×64 or in 36.

$\therefore\ {}^{38}C_{37}.20^1.4^{37} + 4^{38}$ ends in $40 + 36 = 76$

$\therefore\ 24^{38}$ ends in 76.

Some interesting results

1. If we consider three consecutive positive integers, say $(n - 1)$, n and $(n + 1)$, then the product of these three numbers i.e. $n(n^2 - 1)$ is always divisible by 6.
2. If 'n' is an odd prime number greater than 3, then $(n^2 - 1)$ will always be divisible by 24.
   (example, for $n = 5$, $(n - 1) \times (n + 1) = 4 \times 6 = 24$)
3. For any 'n' consecutive positive integers, the product is divisible by n and n!.
4. For every positive integer n, $n^3 - n$ is divisible by 3, $n^5 - n$ is divisible by 5, $n^{11} - n$ is divisible by 11, $n^{13} - n$ is divisible by 13. In general, if 'p' is a prime number then for any natural number 'n', $(n^p - n)$ is divisible by 'p'.

Concept Builder

1. Find the remainder when 2^{622} is divided by 7.
2. Find the remainder when x is divided by 7 for
   a) $x = 8^{200}$ b) $x = 2^{300}$ c) $x = 2^{200}$
3. Find the remainder when x is divided by 9 for
   a) $x = 8^{841}$ b) $x = 2^{300}$ c) $x = 2^{400}$
4. Find the remainder when
   a) 19^{300} is divided by 17. b) 19^{345} is divided by 17.
   c) 13^{173} is divided by 84. d) 7^{177} is divided by 57.
5. Find the number of zeros in:
   a) 75! b) 200!
6. The highest power of 36 in 80!
7. Find the last digit of $2^{2!} \times 4^{4!} \times 6^{6!}$.
8. How many zeroes will be there at the end of the product of the first 100 natural numbers?

Answer key

1. 2
2. a) 1 b) 1 c) 4
3. a) 8 b) 1 c) 7
4. a) 16 b) 2 c) 13 d) 1
5. a) 18 b) 49
6. 18 7. 4 8. 24

CLASS EXERCISE

Teaser

A hundred people are standing in a square formation with ten rows and ten columns. The tallest person in each row is asked to raise his hand, and out of these ten people the shortest is found to be Tom. Next, the shortest person in each column is asked to raise his hand, and out of these ten people, the tallest is found to be Jerry. Who is taller, Tom or Jerry?

Factorial

1. Find the value of $\frac{100! \times 51!}{102! \times 50!}$.
2. Find the least value of n for which 400 divides n!
3. Find the highest power of 2 that divides 20!
4. * Find the highest power of 6 that divides 100!
5. Find the highest power of 100 that divides 100!
6. Find the number of zeroes at the end of:
   a) 20!
   b) 80!
   c) 1001!
7. * Can you find a value of 'n' such that:
   a) n! is divisible by 1 lakh but not by 10 lakh?

Highest power of a in n! is given by $\left[\frac{n}{a}\right]+\left[\frac{n}{a^2}\right]+\left[\frac{n}{a^3}\right]+\ldots\ldots.$ where a is a prime number
Where [x] is the greatest integer less than x.
Number of zeroes at the end of n! is equal to highest power of 5 in n! or
$\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\ldots\ldots.$

The Binomial Theorem

Observe:
$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$
$(a + b)^4 = (a + b)(a + b)(a + b)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b) = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
Do you see a pattern? How are the coefficients decided?

The Binomial Theorem *(or rather, just the binomial expansion)***:**
$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots \quad \ldots + {}^nC_{n-1}\, a^1b^{n-1} + {}^nC_n\, b^n$

Some facts worth observing
In the binomial expansion of $(a + b)^n$:
- There are in all 'n + 1' terms
- The sum of the powers of 'a' and 'b' in any term is 'n'.
  E.g., a^n, a^2b^{n-2}, $a^{n-3}b^3$, a^rb^{n-r}
- The coefficient of the a^rb^{n-r} is same as the coefficient of $a^{n-r}b^r$, which is ${}^nC_r = {}^nC_{n-r}$

8. In the expansion of $(a + b)^6$:
   a) How many terms are there in all?
   b) What is the middle term?
   c) What is the coefficient of a^2b^2?
9. Write the first and last 3 terms in the expansion of:
   a) $(m + 1)^{10}$
   b) $(a - 1)^{10}$
   c) * $(2x + 3y)^5$
10. In the expansion of $(x + 2)^{20}$:
   a) What is the coefficient of x^{18}?
   b) How many terms are not divisible by x^5?
   c) How many terms are not divisible by 4?
11. Calculate last two digits of:
   a) 51^8
   b) 49^{49}

Pascal's Triangle

$$
\begin{array}{c}
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

12. Calculate the following values using Pascal's triangle:
   a) ${}^7C_4$
   b) ${}^6C_2$
   c) ${}^8C_5$

Remainders

13. Fill in the blanks with the greatest possible natural number:
   a) The product of any three consecutive natural numbers is always divisible by ______.
   b) The product of any five consecutive natural numbers is always divisible by ______.
   c) If 'p' is a prime number greater than 3, then $p^2 - 1$ is definitely divisible by ______.
   d) If 'p' is a prime number greater than 7, then $(p + 3)(p - 7)$ is definitely divisible by ______.

Note:
If the number 'n' leaves remainder 'r' when divided by 'a' and 'm' leaves remainder 's' when divided by 'a' then:

- 'm + n' leaves the same remainder as 'r + s' when divided by a
- 'mn' leaves the same remainder as 'rs' when divided by a
- 'n^k' leaves the same remainder as 'r^k' when divided by a

14. Observe that 1234 leaves remainder 2 when divided by 7.
    Find what remainders the following numbers will leave when divided by 7:
    a) 1234 + 1235 + 1236
    b) 1234 × 1235
    c) 1234^2
    d) 1234^3
    e) 1234^{99}
    f) 1234^{100}
    g) 1234^{1234}

15. Find the remainder when 17^{17} is divided by:
    a) 8
    b) 9
    c) 34
    d) * 64
    e) * 14

Challengers

1. Find the number of zeroes at the end of:
   * 10! when written in base 3

2. Find the last two digits of:
   a) 79^{14}
   b) 61^{31}

3. Find the remainder when:
   48^{45} is divided by 35

4. Find the remainder when 123^{123} is divided by:
   a) 4
   b) 5
   c) 6
   d) 7
   e) 8
   f) 9
   g) 10

5. Find the remainder when divided by 25:
   a) 11^{100}
   b) 2^{222}

6. * What will be the remainder when 25^{75} is divided by 14?

PRACTICE EXERCISE-1

DIRECTIONS for questions 1 and 2: Solve as directed.

1. In the expansion of $(a + 3)^{100}$:
   a) Find the coefficient of a^{99}.
   b) How many terms are divisible by 81?
   c) What is the middle term?

2. Find the remainder when the following numbers are divided by 11:
   a) 2^5 b) 2^{300}
   c) 2^{729} d) 3^5
   e) 3^{21} f) 9^{99}
   g) $10^{10^{10}}$ h) 20^{10}
   i) 21^2 j) 21^{21}
   k) 21^{1000} l) $21^{21^{21}}$
   m) $21^{22^{23}}$

DIRECTIONS for questions 3 to 11: Choose the correct alternative.

3. If n! is divisible by 66, find the least value n can have.
   1) 11 2) 66 3) 132 4) None of these

4. When 2^{144} is divided by 15, the remainder would be:
   1) 1 2) 13 3) 0 4) 5

5. Find the maximum power of 10 that will divide 125! completely.
   1) 25 2) 28 3) 30 4) 31

6. If (a, n)! is defined as the product of n consecutive numbers starting from a, where a and n are both natural numbers, and if H is the HCF of (a, n)! and n!, then what can be said about H?
   1) H = a! 2) H = n! 3) $H \geq n!$ 4) None of these

7. The product of the first 25 odd numbers can be given as:

   1) $\frac{49!}{2^{25}(24!)}$ 2) $\frac{50!}{2^{25}(25!)}$ 3) $\frac{50!}{2^{24}(24!)}$ 4) $\frac{49!}{2^{24}(25!)}$

8. What is the remainder when $(1!)^3 + (2!)^3 + (3!)^3 + (4!)^3 + (5!)^3 + \ldots + (n!)^3$ is divided by 5, where n is the largest 3-digit number?

   1) 1 2) 2 3) 3 4) 4

9. $n(n^2 - 1)$, where $n \in N$, n is odd and $n > 1$, is always divisible by:

   1) 32 2) 24 3) 36 4) 48

10. Which one of the following statements is true?

    1) $n(n^4 - 1)$ is divisible by 30 only if n is odd.
    2) $n(n^4 - 1)$ is divisible by 30 only if n is composite.
    3) $n(n^4 - 1)$ is divisible by 30 only if n is prime.
    4) $n(n^4 - 1)$ is divisible by 30 if n is a natural number other than 1.

11. Which of the following is the largest?

    1) 2500! × 2500! 2) 5000!
    3) 3600! × 1400! 4) 4000! × 1000!

DIRECTIONS for questions 12 to 17: Solve as directed.

12. If k is the constant term in the expansion of $\left(x + \frac{1}{2x}\right)^{10}$, what is the value of 40k?

13. What are the last three digits of 1011 × 2012 × 3013 × 4014 × 5015 × 6016?

14. If [x # y] denotes the remainder when x is divided by y, evaluate $[100^{25}$ # $101] - [100^{25}$ # $99]$.

15. What is the remainder when 6^{66} is divided by 1297?

16. What is the remainder when $(2016^{2017} + 2017^{2016})$ is divided by 7?

17. What is the remainder when $(1!)^4 + (2!)^4 + (3!)^4 + (4!)^4 + \ldots + (2592!)^4$ is divided by 2592?

DIRECTIONS for questions 18 to 22: Choose the correct alternative.

18. How many terms in the expansion of $\left(\sqrt[6]{x}+\sqrt[9]{y}\right)^{54}$ are free from radical $\left(\sqrt{\ }\right)$ signs ?

1) 3 2) 4 3) 6 4) 9

19. The remainder when 7^{84} is divided by 342 is:

1) 0 2) 1 3) 49 4) 341

20. The number of digits in $4^{4^2}.5^{5^2}$ (in base 10 – form) is:

1) 30 2) 29 3) 28 4) 27

21. The product 5 × 10 × 15 × 20 × 25 × 30 × 35 × 40 × 45 × 50 will end with:

1) 5 zeroes 2) 6 zeroes 3) 7 zeroes 4) 8 zeroes

22. Which of the following does not completely divide 200!?

1) 72^{41} 2) 72^{44} 3) 72^{47} 4) 72^{50}

PRACTICE EXERCISE-2

DIRECTIONS for questions 1 to 5: Solve as directed.

1. Find the last 2 digits of:

a) 99^{999}

b) 101^{202}

c) 43^{4343}

d) 75^{2020}

e) 37^{486}

2. What is the right most non zero digit in

a) 10! b) 16! c) 24!

3. Number of zeroes in 20! when written in

a) Base 6 b) Base 8 c) Base 9

4. How many zeros at the end of 20! + 21! + 22! + 23! + 24!

5. What is the remainder when divided by 32 of

a) 49^{25} b) 125^{38}

DIRECTIONS for questions 6 to 14: Choose the correct alternative.

6. What is the remainder when 3^{58} is divided by 6?

1) 0 2) 1 3) 2 4) 3

7. The factorial of any number can be decomposed into a product of prime numbers. Let the maximum power of 2 in the decomposition of the factorial of a number (n) be k. Find n, if k = 9.

1) 10 2) 12 3) 13 4) No such n exists

8. x = remainder when (1! + 2! + 3! + ... 100!) is divided by 15. Find x.

1) 4 2) 3 3) 6 4) 9

9. Find the remainder when $3^{41} + 7^{41}$ is divided by 13.

1) 7 2) 10 3) 11 4) 5

10. Let $n! = 1 \times 2 \times 3 \times \text{---} \times n$ for integer $n \geq 1$. If $p = 1! + (2 \times 2!) + (3 \times 3!) + \text{----} + (12 \times 12!)$ then, $p + 3$ when divided by $13!$ leaves a remainder of:

    1) 1   2) 2   3) 0   4) 11

11. What is the remainder when $(12570)^{20}$ is divided by $(243)^4$?

    1) 0   2) 3   3) 7   4) Cannot be determined

12. The tens' digit of $(45407)^{356}$ will be:

    1) 1   2) 0   3) 4   4) None of these

13. In the expansion of $(a + b + 2)^6$, what will be the coefficient of a^2b^2?

    1) 90   2) 180   3) 360   4) 720

14. The product of numbers from 110 to 126 (both inclusive) is perfectly divisible by:

    1) 19! but not 20!   2) 20! but not 21!
    3) 21! but not 22!   4) 22!

DIRECTIONS for questions 15 to 22: Solve as directed.

15. Let n be the smallest number such that n! is divisible by 784. What is the highest power of 12 that divides n!?

16. Find the remainder when $\left(\left((16)^{17}\right)^{18}\right)^{19}$ is divided by 7.

17. Find the number of zeroes at the end of 120!.

18. Find the remainder when $(7! \times 6! + 8! \times 7!)$ is divided by 9!.

19. The last two digits of $25^{63} \times 63^{25}$ are:

20. The third digit from the right of the number 101^{76} is:

21. Which least natural number when added to 7^{84} gives a resulting number that is divisible by 11?

22. a, b and c are the least possible consecutive natural numbers such that each one of them is divisible by some perfect square other than 1. What is the remainder when $\left[(a)^b\right]^c$ is divided by 5?

QA-4.6 | Probability

THEORY

Introduction

The word probability comes from the word probable, meaning 'likely but uncertain'. Thus the probability of getting a certain result or that of happening of a phenomenon refers to the likelihood of its occurrence.

When a coin is tossed, there are two possibilities – you may get a head or a tail. If the coin is unbiased, the likelihood of getting a head and that of getting a tail will be the same. Thus, the probability of getting head on tossing a coin is $\frac{1}{2}$.

Remember that from the above it does not strictly follow that if a coin is tossed two times, one toss will result in a head and another toss will result in a tail. The probability $\frac{1}{2}$ only indicates that if you go on tossing the coin for an infinite number of times under the same condition, the coin will turn up heads half of the times.

Definition of terms

Here are a few terms often used in the mathematical study of probability.

Random Experiment

An **experiment** or a **random experiment** is an action or happening in which all the possibilities are known but the exact result or outcome is unpredictable.

Here are some examples of random experiments:

Tossing a coin: there are just two possibilities: head or tail.
Rolling a six sided die: there are six possibilities: turning up of any of the six sides.
Picking up a ball from a collection of 4 red balls and 6 black balls: there are ten possibilities: picking up any one of the ten balls.

Trial and Outcome

Every time an experiment is conducted or a phenomenon takes place it is referred to as a **trial** of that experiment and each result is referred to as an **outcome** of that trial. In other words, an outcome is the result of a single trial.

If we consider an experiment of rolling a six-sided die, each throw or rolling of the die is a trial. If the faces of the die are numbered 1 through 6, getting any of the numbers 1 through 6 is an outcome. Thus this experiment has altogether six possible outcomes 1, 2, 3, 4, 5 and 6.

Sample Space

The set of all possible results or outcomes is called the **sample space** and is usually denoted by S.

Event

A predefined set of outcomes constitutes an **event**.

In the case of rolling a six-sided die as above, 'getting an even number' can be treated as an event that will comprise of the outcomes 2, 4 and 6. Similarly, 'getting 4' can also be treated as an event but it will consist of only one outcome, which is 4. If we consider 'getting 7' as an event the number of outcomes in that event will be 0 (zero), because there is no such possible outcome. Thus, the **sample space** constitutes the **universal set** in respect of the various event-sets.

Equally likely outcomes

When you are considering the rolling of a six-sided die, if the die is unbiased, the likelihood of each of the six possible outcomes is the same (and equal to 1/6). As such, the outcomes are **equally likely**. Similarly, in the case of tossing an unbiased coin, the outcomes head and tail are equally likely.

Probability of an event

In case of an experiment where all the possible outcomes are equally likely, the probability of an event E is ***P(E)*** = $\frac{\text{The number of outcomes comprising the event E}}{\text{The total number of all the possible outcomes}} = \frac{n(E)}{n(S)}$

Favorable outcomes

When we want to find the probability of an event E, the outcomes constituting the event E are referred to as the **favorable outcomes**.

Thus, ***Probability*** = $\frac{\text{The number of favourable outcomes}}{\text{The total number of all the possible outcomes}}$

Example

(i) If a coin is tossed once and we want to find the probability of getting a head then:
- (a) What is the experiment?
- (b) What is the total number of possible outcomes? That is, how many outcomes are there in the sample space?
- (c) What is the event?
- (d) What is the number of outcomes in the event? (or How many favorable outcomes are there?)
- (e) What is the probability of getting a head?

- (a) 'Tossing the coin once' is the experiment.
- (b) Just 2: head and tail.
- (c) 'getting head' is the event.
- (d) Just 1: 'getting head'
- (e) $\frac{\text{number of outcomes in the event (d)}}{\text{number of outcomes in the sample space (b)}} = \frac{1}{2}$

(ii) If a six-sided die is rolled once, what is the probability of getting
- (a) an even number?
- (b) 4?
- (c) 7?

- (a) $\frac{3}{6} = \frac{1}{2}$
- (b) $\frac{1}{6}$
- (c) $\frac{0}{6} = 0$

Using set notations the workings for the above results can be written as follows:

$P(\{2, 4, 6\}) = \frac{3}{6} = \frac{1}{2}$

$P(\{4\}) = \frac{1}{6}$

$P(\{7\}) = \frac{0}{6} = 0$

Certain Events and Impossible Events

Consider the following:

There are some red marbles in a bowl. If a marble is picked up at random, what is the likelihood that it is red? You know that the marble picked up has to be red. We say that the probability is 1 or 100%.

In the above case, what is the likelihood that the marble picked up is black? Again, you know that the marble cannot be black. We say that the probability is 0 or 0%.

However, if there were a few red marbles and a few black marbles in the bowl, the likelihood of a marble picked up at random being red (or black) would be uncertain.

The foregoing discussion leads us to the following concepts relating to probability.

The probability of an impossible event is 0 or 0% and the probability of a certain (that is sure to happen) event is 1 or 100%.

All other events E are possible but uncertain and have probabilities ranging between 0 and 1; i.e., $0 < P(E) < 1$.

Since probability is a fraction it can also be expressed as a percentage.

Probability only tells us about the possibility of occurrence of an event. It does not tell us anything with certainty about the happening or not happening of the event, except when the probability is 1 or 0.

Note that an impossible event is a null set.

Mutually Exclusive Events

If there are two events E and F and there is no outcome common to both, the two events are called **mutually exclusive events.** In other words, if the sets E and F are disjoint, the events E and F are mutually exclusive. Conversely, if two events E and F are mutually exclusive, the sets E and F are disjoint.

Example

Let us consider the experiment of 'rolling a six-sided die once'. Let event E be 'getting an odd number', event F be 'getting an even number' and event G be 'getting a prime number'. The two events E and F are mutually exclusive but events F and G are not mutually exclusive as the outcome '2' is an element common to both F and G.

Mutually exclusive and exhaustive events

If there are two or more mutually exclusive events E, F, G, H, ... such that outcomes of all such events together comprise the sample space, the events are called **mutually exclusive and exhaustive events.** The sum of the probabilities of all such mutually exclusive and exhaustive events equals 1.

For mutually exclusive and exhaustive events E, F, G, H, ..., we have $P(E) + P(F) + P(G) + P(H) + \ldots = 1$

Event 'Not E'

The event 'Not E' with relation to a sample space S is the set of all the outcomes that are not in the set of the outcomes constituting the event E. P(Not E), therefore, refers to the probability of the event E not happening or not taking place.

Probability of the event 'Not E' may be denoted as P(Not E) or P(E´).
Set E´is the complement of the set E with respect to the universal set S. Therefore:

Event E and Event 'Not E' are always mutually exclusive and exhaustive events.

n(E) + n(E') = n(S)
P(E) + P(Not E) = 1 or
P(E') = 1 – P(E)

Example

If two coins are tossed, what is the probability of getting at least one head?

The first coin can show head or tail, the second coin can also show head or tail. Therefore, the sample space has 2 × 2 = 4 possible outcomes. The event 'at least one head' is nothing but all the outcomes excluding 'no head'.
Therefore, we can say that
P(at least one head) = 1 – P(no head)
There is only one way in which we can have no head, which is TT.
Therefore, the probability of 'no head' is $\frac{1}{4}$; or P(no head) = $\frac{1}{4}$.
Thus, P(at least one head) = 1 – P(no head) = $1 - \frac{1}{4} = \frac{3}{4}$.

Alternately, you may list out all the possible outcomes HH, HT, TH and TT and count the favorable outcomes and all the possible outcomes. Thus, n(E) = 3, n(S) = 4 and P(at least one head) = $\frac{3}{4}$.
However, if the number of tosses is very large, it may not be possible to adopt such a method.

Event 'E and F'

Event 'E and F' is the set of outcomes in both E and F. In other words, event 'E and F' is event '$E \cap F$,' and P(E and F) is $P(E \cap F)$.

If E and F are two mutually exclusive events there is no outcome common to both and the event 'E and F' is impossible. The sets E and F are disjoint and $n(E \cap F) = 0$.
Therefore:

For mutually exclusive events $P(E \cap F) = 0$

Example

In a class, 11 students study French and 12 students study German. 5 students study neither of the languages and there are altogether 26 students in the class. If a student is selected at random, what is the probability that the student studies both the languages?

If F is the set of outcomes 'the student studies French' and G is the set of outcomes 'the student studies German', the set $F \cap G$ is the set of outcomes 'the student studies French and German'. Then:

P(student studies both French and German) = $P(F \cap G) = \frac{n(F \cap G)}{n(S)}$

Now, $n(F \cap G)$ = n(F) + n(G) + Neither – Total = 11 + 12 + 5 – 26 = 2

Therefore, the probability that the randomly picked up student studies both the languages is $\frac{2}{26}$ $= \frac{1}{13}$.

Event 'E or F'

Event 'E or F' is the set of outcomes in E or F or both. In other words, event 'E or F' is event E $\cup$ F and P(E or F) is P(E $\cup$ F).

Since we know

$n(E \cup F) = n(E) + n(F) - n(E \cap F)$ by dividing throughout by n(S) it follows that

P(E or F) or $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Example

In the above problem, what is the probability that a randomly chosen student studies either French or German?

The probability of F or G is $P(F \cup G) = P(F) + P(G) - P(F \cap G)$

$$= \frac{11}{26} + \frac{12}{26} - \frac{2}{26} = \frac{21}{26}$$

This can also be arrived at from the relationships below:

$n(F \cup G)$ = Total − Neither = 26 − 5 = 21 and $P(F \cup G) = \frac{n(F \cap G)}{n(S)} = \frac{21}{26}$.

If events E and F are mutually exclusive events $P(E \cap F) = 0$.

Therefore:

For mutually exclusive events P(E or F) or $P(E \cup F) = P(E) + P(F)$

Example

In a pack of 52 cards there are 4 suits with 13 cards each. Each suit has one Ace, one King, one Queen, one Jack and 9 other cards numbered from 2 to 10. If a card is picked up at random from a well-shuffled pack of cards, what is the probability that it is a 2 or a Jack?

These two are mutually exclusive events as a card cannot be a 2 and a Jack at the same time. There are four 2's and 4 Jacks. Therefore,

$$P(2 \text{ or Jack}) = P(2) + P(\text{Jack}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

'Odds For' & 'Odds Against'

Sometimes, probability is also viewed in terms of '*Odds For*' and '*Odds Against*' an event.

Odds in favor of an event $= \frac{\text{Number of Favourable Cases}}{\text{Number of Unfavourable Cases}} = \frac{P(E)}{P(E')}$

Odds against an event $= \frac{\text{Number of Unfavourable Cases}}{\text{Number of Favourable Cases}} = \frac{P(E')}{P(E)}$

Conditional Probability

The **conditional probability** of an event B in relationship to an event A is the probability of occurring of the event B, given that event A has already occurred and is usually denoted by P(B|A).

$$P(B|A) = \frac{n(A \cap B)}{(A)} = \frac{P(A \cap B)}{P(A)} \text{ and}$$

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)}$$

It follows that:

P(A ∩ B) = P(A) P(B/A) = P(B)P(A|B)

Example

Let us consider the experiment of rolling a six-sided die (with sides marked 1 through 6) once. Let event E be 'getting a number less than 3' and let event F be 'getting an odd integer'.

Here, event E = {1, 2} and F = {1, 3, 5}

Let us find the probability of the event E

$$P(E) = \frac{n(E)}{6} = \frac{2}{6} = \frac{1}{3}$$

Now let us find the probability of occurring of the event E, given that event F has already occurred

$$P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{1}{3}$$

Thus, we find that P(E) = P(E|F)

Similarly,

$$P(F) = \frac{n(F)}{6} = \frac{3}{6} = \frac{1}{2}$$

$$P(F|E) = \frac{n(E \cap F)}{n(E)} = \frac{1}{2}.$$

Thus, P(F) = P(F|E)

We find that the probability of happening of either event is independent of happening of the other event. Therefore, these two events E and F (as defined here) are *independent events*.

Independent events

Two events are said to be independent if the probability of either of the events is not affected by the occurrence or non-occurrence of the other event.

This can be stated mathematically as follows:

If P(E) = P(E|F), the event E is independent of the event F.

In the case of the above two events viz., 'getting a number less than 3' and 'getting an odd integer', we find that these two events are independent events.

We also observe that

$$P(E \text{ and } F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{6}$$

$$P(E)P(F) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

Hence, P(E and F) = P(E)P(F)

This relationship is valid for any two independent events, and therefore:

For two independent events E and F, P(E and F) or P(E ∩ F) = P(E)P(F).

Consequently:

For two independent events P(E or F) or P(E ∪ F) = P(E) + P(F) – P(E)P(F).

Example

(i) A six-sided die with sides marked 1 through 6 is rolled once. Event A is that the number showing up is greater than 3. Event B is that the number is even. Verify whether the events are independent of one another.

P(A) = P(getting 'greater than 3') = $\frac{3}{6} = \frac{1}{2}$ and P(B) = P(even) = $\frac{3}{6} = \frac{1}{2}$

Now $P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{3}$

Also $P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{2}{3}$

Thus, you find that $P(A) \neq P(A|B)$ and

also $P(B) \neq P(B|A)$

Therefore, these two events are not independent events.

(ii) A six-sided die with sides marked 1 through 6 is rolled twice. What is the probability that the sum of the numbers facing up is 11?

You can get the sum of 11 only in two ways, 6 + 5 or 5 + 6.

The result of each throw is independent of the other and also (6 + 5) and (5 + 6) are mutually exclusive results.

Therefore, P(getting 11) = P(getting '6 and 5' or '5 and 6')

= P (getting '6 and 5') **+** P(getting '5 and 6')

= P(getting 6) × P(getting 5) + P(getting 5) × P(getting 6)

$= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$ or $\frac{1}{36} + \frac{1}{36}$, i.e., $\frac{1}{18}$

The two relationships that you will need most frequently in respect of probability problems are the following:

For two mutually exclusive events E and F, we have:
P(E or F) = P(E) + P(F)

and

For two independent events E and F, we have:
P(E and F) = P(E)P(F)

Concept Builder

1. How many elements are there in the sample space if a coin is tossed ‘n’ times?
2. In the experiment of throwing a die, event A is defined as ‘getting a prime number’ and another event B is defined as ‘getting a composite number’. (a) Are they Mutually Exclusive? (b) Are they Mutually Exclusive and Exhaustive?
3. If a card is drawn at random from a pack of 52 cards, then the number of favorable cases for that card being a spade is:
4. If P(E) = 0.25, find P(E') ?
5. The probability that a given problem will be solved by Ronak is $\frac{3}{5}$ and the same problem will be solved by Nikita is $\frac{4}{5}$. Find the probability of the problem being solved by both of them?
6. If P(E) = $\frac{3}{7}$, what are the Odds in Favor for the event (E)?
7. If P(E') = $\frac{5}{6}$, what are the Odds Against the event (E)?

Answer Key

1. 2^n
2. (a) Yes, (b) No
3. 13
4. 0.75
5. $\frac{12}{25}$
6. $\frac{3}{4}$
7. $\frac{5}{1}$

SOLVED EXAMPLES

Q : Find the probability that the birthday of a child is on a Saturday or Sunday.

A : S = {Sun, Mon, Tue, Wed, Thur, Fri, Sat} $\therefore$ n(S) = 7

Let A be the event that the child is born on a Sunday or a Saturday.

$\therefore$ n(A) = 2

$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$

Q : Find the probability of getting a multiple of 2 in the throw of a die.

A : S = {1, 2, 3, 4, 5, 6} $\therefore$ n(S) = 6

Let A be the event that the die shows a multiple of 2.

A = {2, 4, 6} $\therefore$ n(A) = 3

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

Q : Find the probability that the sum of the scores obtained when two fair dice are thrown, is odd.

A : S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)

(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)

(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)

(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)

(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)

(6, 1) (6, 2) (6, 3) (6, 4), (6, 5) (6, 6)}

$\therefore$ n(S) = 36

Let A be the event that the score on the two dice is odd.

$\therefore$ A = {(1, 2) (1, 4) (1, 6) (2, 1) (2, 3) (2, 5) (3, 2) (3, 4) (3, 6) (4, 1) (4, 3) (4, 5) (5, 2) (5, 4) (5, 6) (6, 1) (6, 3) (6, 5)}

$\therefore$ n(A) = 18

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$

Note:
When a fair die is thrown, n(S) = 6, when two fair dice are thrown, n(S) = 6^2 = 36, when three fair dice are thrown, n(S) = 6^3 = 216.

Q : A committee of 5 students is to be chosen from 6 boys and 4 girls. Find the probability that the committee consists of exactly 2 girls.

A : 5 students can be selected from 10 in ${}^{10}C_5$ ways.

$\therefore\ n(S) = {}^{10}C_5 = \dfrac{10!}{5!5!} = \dfrac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$

Let A be the event that the committee includes exactly 2 girls and 3 boys. The two girls can be selected in ${}^4C_2$ ways and the 3 boys can be selected in ${}^6C_3$ ways.

$\therefore\ n(A) = {}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

$\therefore\ P(A) = \dfrac{n(A)}{n(S)} = \dfrac{120}{252} = \dfrac{10}{21}$

Q : A bag contains 7 red, 5 blue, 4 white and 4 black balls. Find the probability that a ball drawn at random is red or white.

A : A ball can be selected from 20 in ${}^{20}C_1 = 20$ ways.

$\therefore\ n(S) = 20$

Let A be an event that the ball drawn is red or white.

$\therefore\ n(A) = {}^7C_1 + {}^4C_1 = 7 + 4 = 11$ $\qquad \therefore\ P(A) = \dfrac{11}{20}$

Q : Two cards are drawn at random from a pack of 52 cards. What is the probability that one of them is a Jack card and the other is an Ace card?

A : Two cards can be selected in ${}^{52}C_2$ ways.

$\therefore\ n(S) = {}^{52}C_2 = 26 \times 51$

Let A be the event that the two cards selected are a Jack and an Ace.

A Jack card can be selected in ${}^4C_1$ ways and an Ace card can also be selected in ${}^4C_1$ ways.

$\therefore\ n(A) = {}^4C_1 \times {}^4C_1 = 16$

$\therefore\ P(A) = \dfrac{n(A)}{n(S)} = \dfrac{16}{26 \times 51} = \dfrac{8}{663}$

Q : In a shooting competition, the probability that the target is hit by A is $\dfrac{2}{5}$, by B is $\dfrac{2}{3}$ and by C is $\dfrac{3}{5}$. If all of them fire independently at the same target, calculate the probability that only one of them will hit the target.

A : $P(A) = \dfrac{2}{3}$ $\qquad P(A') = \dfrac{3}{5}$

$P(B) = \dfrac{2}{3}$ $\qquad P(B') = \dfrac{1}{3}$

$P(C) = \frac{3}{5}$ $P(C') = \frac{2}{5}$

Probability that only one of them hits the target.

= Probability that A hits the target but not B and C.

+ probability that B hits the target but not A and C.

+ probability that C hits the target but not A and B.

$= P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$

$$= \left[\frac{2}{5} \times \frac{1}{3} \times \frac{2}{5}\right] + \left[\frac{2}{3} \times \frac{3}{5} \times \frac{2}{5}\right] + \left[\frac{3}{5} \times \frac{3}{5} \times \frac{1}{3}\right] = \frac{4}{75} + \frac{12}{75} + \frac{9}{75} = \frac{25}{75} = \frac{1}{3}$$

Q : A box contains 6 white balls and 3 black balls, and another box contains 4 white balls and 5 black balls. Find the probability that the ball randomly selected from one of the boxes, is a white ball?

A : Let A be the event that the first box is selected, B be the event that the second box is selected and C be the event that a white ball is selected.

$P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$

$P(C/A) = \frac{6}{9}$; $P(C/B) = \frac{4}{9}$

The probability of selecting the first box and getting a white ball

$= P(A \cap C) = P(A).\ P(C/A) = \left(\frac{1}{2}\right)\left(\frac{6}{9}\right) = \frac{6}{18}$

The probability of selecting the second box and getting a white ball

$= P(B \cap C) = P(B).\ P(B/C) = \left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{4}{18}$

$\because$ The event $(A \cap C)$ and $(B \cap C)$ are mutually exclusive events.

$\therefore P[(A \cap C) \cup (B \cap C)] = \frac{6}{18} + \frac{4}{18} = \frac{10}{18} = \frac{5}{9}$

Alternatively,

Probability of selecting any bag is $\frac{1}{2}$.

Probability of getting a white ball will be $= \frac{1}{2} \times \frac{{}^6C_1}{{}^9C_1} + \frac{1}{2} \times \frac{{}^4C_1}{{}^9C_1} = \frac{5}{9}$

Q : In a garden, 40% of the flowers are roses and the rest are carnations. If 25% of the roses and 10% of the carnations are red, find the probability that a red flower selected at random is a rose.

A : Suppose there are 100 flowers.

Number of roses = 40 and number of carnations = 60

25% of 40 = 10 roses are red and 10% of 60 = 6 carnations are red.

Let A be the event that the flower is red and B be the event that the flower is a rose.

$\therefore$ $A \cap B$ is the event that the flower is a red rose.

$n(A) = 16;\quad \therefore P(A) = \frac{16}{100}$

$n(A \cap B) = 10;\quad \therefore P(A \cap B) = \frac{10}{100}$

P(B/A) = probability that the selected flower is a rose given that the flower is red in colour.

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{10/100}{16/100} = \frac{10}{16} = \frac{5}{8}$

Q : Two sisters A and B appeared for an audition. The probability of selection of A is $\frac{1}{5}$ and that of B is $\frac{2}{7}$. Find the probability that both of them are selected.

A : Let A be the event that A is selected and B be the event that B is selected.

$\therefore P(A) = \frac{1}{5}$ and $P(B) = \frac{2}{7}$

Let C be the event that both are selected.

$\therefore C = A \cap B;\qquad \therefore P(C) = P(A \cap B)$

$\therefore$ P(C) = P(A). P(B) as A and B are independent events = $\frac{1}{5} \times \frac{2}{7} = \frac{2}{35}$

Q : Nikita draws a ball randomly from a bag containing 4 white and 5 black balls. What are the Odds against it being a black ball?

A : Total favourable outcomes (i.e. getting a black ball) = 5

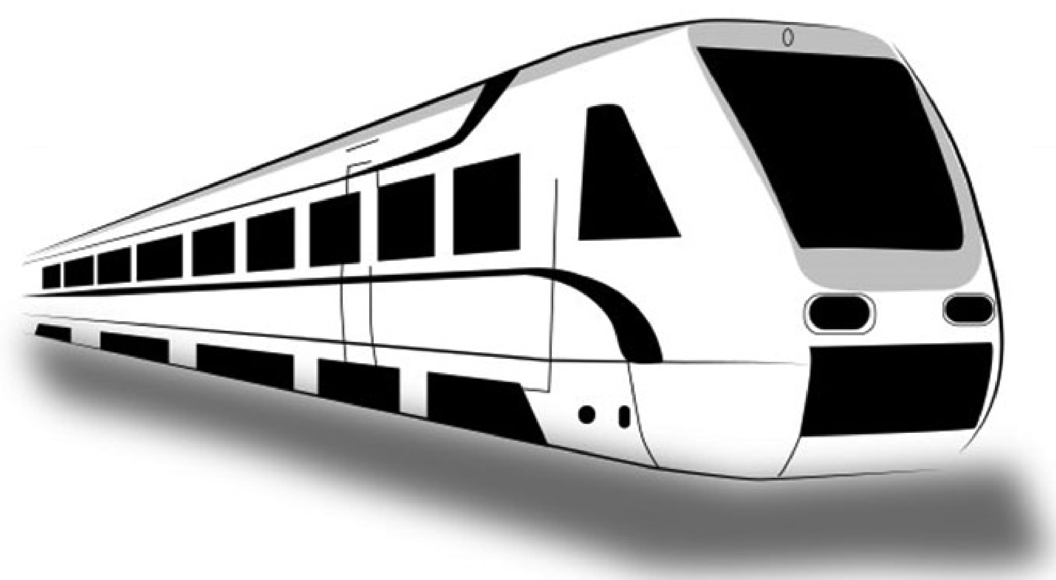
Total Unfavourable outcomes (i.e. getting a ball other than black color) = 4

So, Odds against the ball drawn being black = $\frac{4}{5}$

CLASS EXERCISE

Teaser

At platform 9¾, a train arrives punctually every 20 minutes travelling towards Hogwarts. Also, a similar train arrives every 20 minutes travelling away from Hogwarts. Luna arrives at the platform every day at some time between 9 and 10 am and catches the first train that turns up. She expects to reach Hogwarts only on half the trips. But surprisingly, 9 times out of ten she finds herself reaching Hogwarts. How can this be possible?



(Hint: No magic whatsoever is involved)

Probability: Basics

1. If a fair coin is tossed, what is the probability that it shows a tail?
2. What is the probability that my friend Saurabh was born on a Sunday?
3. Jenny, Johnny and Ginny celebrate their birthdays in a few days. What is the probability that none of them will celebrate their birthday on a weekend?
4. If a fair die with 6 faces number 1 to 6 is rolled, what is the probability that
   a) it shows a 4?
   b) It shows an odd number?
   c) It shows an even number?
   d) It shows a prime number?
   e) It shows a composite number?
   f) It shows either an odd or an even number?
   g) It shows either a prime or an even number?
   h) It shows either a prime or a composite number?

Sample Space: The set of all possible outcomes of an experiment

Note that the elements in a sample space must be
(a) **Mutually exclusive:** there should be no overlap
(b) **Exhaustive:** they should cover all the possibilities
(c) **Equiprobable:** they should be equally likely to occur

Probability of an event A = P(A) = $\frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{n(A)}{n(S)}$

Two events A and B are said to be **Mutually Exclusive** if no outcome present in A is also present in B. In this case, $A \cap B = \phi$ and so $n(A \cap B) = 0$; hence **$P(A \cap B) = 0$**

Two events A and B are said to be **Complementary Events** ($A = B^c$) if between them they cover all possible outcomes (i.e. they are exhaustive). In this case $n(A \cup B) = n(S)$ and hence **$P(A \cup B) = 1$**

5. *In a class there are 8 boys and 5 girls. What is the probability that a randomly chosen student is a boy?
6. *If a number between 1 and 100 (both included) is randomly picked, what is the probability that it is:
   a) a prime
   b) a perfect square
   c) a prime or a perfect square
   d) an even perfect square

Conditional Probability, Dependent and Independent Events

7. A letter is chosen from the English Alphabet. What is the probability that:
   a) it is a vowel?
   b) it is one of the first ten alphabets (A to J)?
   c) it is a vowel falling within the first ten alphabets?
   d) it is a vowel, given that it is one of the first ten alphabets?
   e) it is one of the first ten alphabets, given that it is a vowel?

Answer questions 8 – 10 based on the following information: A bag contains 2 white, 3 blue, 4 red, 5 green and 6 yellow marbles. A person picks marbles from it at random without looking inside.

8. If one marble is picked from the bag, what is the probability that
   a) it is white, given that it is not red?
   b) it is not blue, given that it is not green?

9. If two marbles are picked with replacement from the bag, what is the probability that
   a) the first is blue and the second is red?
   b) one of them is blue and one is red?
   c) both of them are red?

10. If two marbles are picked without replacement from the bag, what is the probability that
    a) the first is blue and the second is red?
    b) one of them is blue and one is red?
    c) both of them are red?

The probability that an event B occurs, given that an event A has occurred, is called the **Conditional Probability** of B given A and is calculated as $\mathbf{P(B/A) = \frac{P(A \cap B)}{P(A)}}$

Thus the probability of A and B both occurring is $\mathbf{P(A \cap B) = P(A) \times P(B/A)}$

If $\mathbf{P(B/A)}$ is equal to $\mathbf{P(B/A^C)}$ – i.e., the probability of B is unaltered whether A happens or not - then A and B are **Independent Events**. In this case we can say $\mathbf{P(A \cap B) = P(A) \times P(B)}$

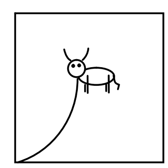
11. A, B, C and D are among the participants in a chess tournament, which ends with only one winner. Their respective probabilities of winning are $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{8}$. What is the probability that:
    a) At least one of A and B wins
    b) Neither A nor B wins
    c) A loses
    d) A wins, given that B does not win
    e) Both A and B win

12. A, B, C and D write the TAM (Test for Admission to Management). Their respective probabilities of passing are $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{8}$.
    a) At least one of A and B passes
    b) Neither A nor B passes
    c) A fails
    d) A passes, given that B does not pass
    e) Both A and B pass

13. *The Wisden Cricket Team of the year contains 3 Indians, 3 South Africans, 2 Sri Lankans, 2 Australians and 1 Englishman. What is the probability that
    a) the captain and the vice-captain are both Indian?
    b) the captain and the wicket-keeper are both Indian?
    c) neither the captain nor the vice-captain is an Indian?

14. *Three people P, Q and R fire one shot each at a target. Their probabilities of hitting the target are 0.5, 0.2 and 0.1 respectively. What is the probability that:
    a) Only P and Q will hit the target?
    b) All three will hit the target?
    c) The target will be hit?
    d) P and R will hit the target?
    e) At least 2 of them will hit the target?

15. A cow is tethered by a rope of length 10 m to a post at the corner of a square field with a side of 10 m. A daisy blooms in the field. Given that the cow likes daisies, what is the probability that the daisy will be eaten?



16. A point (p, q) is taken within the region defined by $0 \le x \le 2$, $0 \le y \le 2$. What is the probability that
    a) $p + q \ge 2$
    b) $p \ge q$
    c) $pq \le 4$
    d) $p \ge q$ given that $q \le 1$

17. The odds on Brazil in the world cup are 1 : 6 in favour and on Germany in the European championship are 1 : 8 in favour. What is the probability that they will both win their respective tournaments?

18. *The police come to know that a notorious thief was at Ramgarh village at 9 a.m. They manage to cordon off a circular area of radius 5 km round the village by 10 am. If it is known that the thief can travel not more than 15 km in this time, then what is the probability that he will be caught in the cordon?

19. *What is the probability that two kings placed on a regular 8 x 8 chessboard, will be attacking each other? (A king can attack any square adjacent to its position vertically, horizontally or diagonally)

20. *The odds against Riya getting into Hardvar Institute of Management are 6 : 1 and the odds against Raima getting into Hardvar Institute of Management are 7 : 2. What are the odds in favour of them both getting in?

21. *Priyanka applied to 3 colleges in the US for a scholarship. Her chances of getting a scholarship in MIT are $\frac{1}{8}$, in CalTech are $\frac{1}{6}$ and in Michigan are $\frac{1}{5}$. What is the probability that

    a) she will get a scholarship
    b) she will get a scholarship from MIT and CalTech but not Michigan?
    c) she will get a scholarship from exactly 2 colleges?
    d) she will get a scholarship from at most 2 colleges?

Challengers

1. A die with 6 faces marked 1 to 6 is biased in such a way that the probability of any face appearing is proportional to the number on that face. If the die is rolled twice, find the probability that it will show two sixes?

   1) $\frac{1}{30}$ 2) $\frac{2}{7}$ 3) $\frac{1}{36}$ 4) $\frac{4}{49}$

2. A real number R between – 3 and 3 is chosen at random. What is the probability that $R < R^2$?

   1) 1 2) $\frac{5}{6}$ 3) $\frac{1}{6}$ 4) $\frac{1}{2}$

3. During a war, a plane is sent to drop bombs on a bridge. The pilot knows that a direct hit is required to destroy the bridge. He also knows that each of his bombs has a probability of only 20% of scoring a direct hit. What is the minimum number of bombs he should drop so that his chances of successfully completing his mission are greater than 60%?

   1) 3 2) 4 3) 5 4) 6

4. Find the probability that the product of the digits of a 3-digit number is odd:

   1) $\frac{5}{36}$ 2) $\frac{1}{8}$ 3) $\frac{1}{2}$ 4) $\frac{27}{1000}$

5. A committee of 6 people has to be selected from a group of 5 men and 5 women. What is the probability that:

   a) the committee will contain at least 3 women?

   1) $\frac{31}{42}$ 2) $\frac{1}{2}$ 3) $\frac{10}{21}$ 4) $\frac{6!}{5!5!}$

   b) the committee will contain at least 1 man?

   1) $\frac{20}{21}$ 2) $\frac{41}{42}$ 3) $\frac{5}{6}$ 4) None of these

PRACTICE EXERCISE-1

DIRECTIONS for questions 1 to 6: Refer to the data below and answer the questions that follow:

In a class, there are 30 boys and 20 girls. 20 of the boys and 5 of the girls drink coffee. Two students X and Y are picked at random.

1. What is the probability that X is a girl who does not drink coffee?

2. What is the probability that X drinks coffee, given that Y drinks coffee?

3. What is the probability that X drinks coffee, given that X is a girl?

4. What is the probability that X is a girl, given that X drinks coffee?

5. What is the probability that X and Y both drink coffee?

6. What is the probability that X and Y are a girl and a boy who both drink coffee?

DIRECTIONS for questions 7 to 21: Choose the correct alternative.

7. Find the chance of throwing more than 15 in one throw with 3 dice.

   1) $\frac{1}{54}$ 2) $\frac{17}{216}$ 3) $\frac{5}{108}$ 4) Cannot be determined

8. A wardrobe consists of 4 shirts, 5 T-shirts and 6 trousers. Of these items of clothing, 3 are selected. What is the probability that one from each is selected?

   1) $\frac{76}{91}$ 2) $\frac{3}{455}$ 3) $\frac{24}{91}$ 4) None of these

9. Jitu invites Sudhir for a party. Sudhir knows that on the way to Jitu's house he'll come across four lanes. There are five houses at the end of each lane. One of these five houses is Jitu's. Find the probability that the first house that Sudhir checks is not Jitu's.

   1) $\frac{1}{20}$ 2) $\frac{19}{20}$ 3) $\frac{4}{5}$ 4) $\frac{1}{5}$

10. In a group consisting of 12 boys, there are 6 boys two each of whom share the names Suresh, Mahesh and Jayesh. The rest of the six boys have six different names, other than the names mentioned. Four boys from the group are selected for a party such that the boys with the same names are always together (i.e., either all of them go or none of them goes). Find the probability that the boys named Suresh go for the party.

    1) $\frac{3}{41}$ 2) $\frac{10}{41}$ 3) $\frac{17}{63}$ 4) $\frac{15}{63}$

11. In a certain survey it was found that 20% of the women and 40% of the men smoke. Women constitute 45% of the population surveyed. If a person is selected at random and it was found that he or she smokes, determine the probability that the person is a male.

1) $\frac{22}{31}$ 2) $\frac{2}{5}$ 3) $\frac{9}{22}$ 4) $\frac{9}{31}$

12. In a three section test, the probability that Mr. X clears first section, second section and third section is $\frac{1}{5}$, $\frac{2}{3}$ and $\frac{3}{4}$ respectively. Find the probability that Mr. X clears exactly one section in the test.

1) $\frac{7}{20}$ 2) $\frac{1}{4}$ 3) $\frac{7}{36}$ 4) $\frac{13}{95}$

13. There are two bags, one red and the other black. The red bag contains 9 balls, of which 4 are yellow. The black bag contains 17 balls, of which 9 are yellow. The probability of selecting a red bag is $\frac{4}{7}$ while the probability of selecting a black bag is $\frac{3}{7}$. Find the chance that when a ball is selected at random, it is not yellow.

1) $\frac{152}{357}$ 2) $\frac{498}{1071}$ 3) $\frac{560}{1071}$ 4) $\frac{556}{1071}$

14. In a chess tournament, three matches are played between A and B. The probability that A wins a match is $\frac{1}{2}$ and the probability that A draws a match is $\frac{1}{3}$. Find the probability that A and B win alternately in the tournament such that there is no draw.

1) $\frac{1}{18}$ 2) $\frac{1}{24}$ 3) $\frac{1}{36}$ 4) $\frac{1}{48}$

15. What is the probability of choosing a two-digit prime number such that the sum of its digits is odd?

1) $\frac{3}{7}$ 2) $\frac{8}{21}$ 3) $\frac{10}{21}$ 4) $\frac{4}{7}$

16. The odds in favour of my clearing the 3 stages of admissions to MBA viz., CAT, GD and PI, in that order are 3 to 2, 2 to 3 and 4 to 1 respectively. What is the probability that I clear at least 2 of the 3 stages? (Given that selection to the next stage depends on clearing the previous one)

1) $\frac{82}{125}$ 2) $\frac{27}{50}$ 3) $\frac{6}{25}$ 4) $\frac{36}{75}$

17. A and B are among the participants of a dart shooting competition. The probability that A wins is $\frac{2}{9}$ and the probability that B does not hit the target in the 'bulls eye' is $\frac{1}{3}$. Assume that a 'win' is hitting the 'bulls eye'. The probability of dead heat (i.e. all of them win) is $\frac{1}{27}$. If there are 8 contestants in the competition and probability of all others (except A and B) winning is equal; then what is the probability that no one wins?

1) $\frac{7}{108}\left(\sqrt[6]{4}-1\right)$ 2) $\frac{7}{108}\left(\sqrt[3]{2}-1\right)^6$ 3) $\frac{7}{27}\left(\sqrt[6]{4}-1\right)$ 4) $\frac{7}{27}\left(\sqrt[3]{2}-1\right)^6$

18. In a company, the total number of employees is 100. A “Grade A” employee can avail of any combination of a house loan, personal Loan and an automobile loan. A “Grade B” Employee can avail of any combination of a personal loan and an automobile loan. A “Grade C” employee can only avail of an automobile loan. There are no loan privileges for “Grade D” employees. Also, the probability of obtaining at least two loans by an employee selected at random is $\frac{1}{2}$ and the probability of obtaining at least one loan is $\frac{7}{10}$. What is the probability that an employee selected at random can obtain exactly one loan?

   1) 0.1 2) 0.2 3) 0.3 4) 0.4

19. Suppose all the teams batting first in one-day international cricket score between 200 and 399 runs (both included) and each score between 200 and 399 is equally likely, what is the probability that Team A scored the same number of runs in two matches when batting first?

   1) 1%
   2) 0.5%
   3) 2%
   4) More information is needed to answer this question

20. Two squares are chosen at random on a 8 × 8 chessboard. What is the probability that they are adjacent and touching each other diagonally?

   1) $\frac{43}{504}$ 2) $\frac{7}{144}$ 3) $\frac{7}{72}$ 4) $\frac{61}{504}$

21. An unbiased die is thrown thrice. What is the probability that the sum of the numbers appearing on it is a perfect square?

   1) $\frac{17}{108}$ 2) $\frac{4}{27}$ 3) $\frac{5}{36}$ 4) $\frac{11}{72}$

PRACTICE EXERCISE-2

DIRECTIONS for question 1: Solve as directed.

1. Given below are two different cases. In each case, it is known that Ricky's chances of winning are $\frac{2}{5}$ while Shane's chances of winning are $\frac{2}{7}$. Find the probability of the given events in each case.

Case I:
Ricky and Shane each participate in a separate episode of "Who wants to be a Millionaire".

Case II:
Ricky and Shane are among the candidates for Captaincy. Only one person can be elected Captain.

Event	Probability in Case I	Probability in Case II
a. Ricky loses		
b. Both Ricky & Shane win		
c. Exactly one of them wins		
d. At least one of them wins		
e. Ricky wins given that Shane loses		
f. At least one of them loses		

DIRECTIONS for questions 2 to 7: Refer to the data below and answer the questions that follow:

If two fair dice are rolled at random, what is the probability that:

2. The sum of the numbers they show will be a perfect square

3. The product of the numbers they show will be a perfect square

4. The sum of the numbers they show will be a prime

5. The product of the numbers they show will be a prime

6. The sum of the numbers they show will be greater than 10

7. The product of the numbers they show will be less than 21

DIRECTIONS for questions 8 to 19: Choose the correct alternative.

8. In a race where 12 horses are running, the chance that horse A will win is $\frac{1}{6}$, that B will win is $\frac{1}{10}$ and that C will win is $\frac{1}{8}$. Assuming that a dead heat is impossible, find the chance that one amongst three of them will win.

1) $\frac{47}{120}$ 2) $\frac{1}{480}$ 3) $\frac{1}{160}$ 4) $\frac{1}{240}$

9. There are 2 boxes containing numbers varying from 1 to 30 each. Now, Rajeev and Kapil draw 1 number each from different boxes. Find the probability that the number drawn by both is the same in the first draw.

   1) 1 2) $\frac{1}{6}$ 3) $\frac{1}{15}$ 4) $\frac{1}{30}$

10. In a football match between team A and team B, the probability that Sachin from team A scores the goal is 3 times to the probability of Saurav from team B scoring the goal, which is equal to P. $\left(P \le \frac{1}{3}\right)$. At the end of the match, for team B to win the game, Saurav has to score a goal and Sachin has to miss it. What is the probability that team B wins the match?

    1) $P(1 - 3P)$ 2) $3P^2$ 3) P^3 4) $P^2(1 - 3P)^2$

11. There are 16 bags with two pockets each. Eight bags having 2 gold and 3 silver coins in one pocket while 3 gold and 2 silver in the other. The other eight bags have 4 gold and 6 silver coins in 1 pocket and 6 gold and 4 silver coins in the other. 1 pair of each different type of bags (i.e., bags carrying different number of coins) is placed in 8 rooms. If one person happens to enter one of rooms and he picks one of the bags and picks a coin from one of the pocket, what is the probability that it is a gold coin?

    1) $\frac{1}{64}$ 2) $\frac{1}{32}$ 3) $\frac{1}{2}$ 4) $\frac{25}{32}$

12. A and B are two events such that P(A) = 0.2, P(B) = 0.5 and P(B/A) = 0.5. Find the relation between $P(A \cap B)$, $P(A \cup B)$, P(A/B)

    1) $P(A \cap B) > P(A \cup B) > P(A/B)$ 2) $P(A \cap B) > P(A \cap B) > P(A/B)$
    3) $P(A \cup B) > P(A/B) > P(A \cap B)$ 4) None of these

13. A team of 5 members is to be chosen from 4 Englishmen and 6 Indians. Find the probability that the team contains exactly 2 Indians.

    1) $\frac{10}{21}$ 2) $\frac{5}{42}$ 3) $\frac{5}{7}$ 4) $\frac{5}{21}$

14. A, B and C are playing a game of darts. The probability that A will hit bullseye is $\frac{1}{3}$. The probability that B will hit bullseye is $\frac{1}{4}$. The probability that C will hit bullseye is $\frac{1}{2}$. What is the probability that exactly two of them will hit bullseye?

    1) $\frac{1}{4}$ 2) $\frac{1}{3}$ 3) $\frac{1}{2}$ 4) $\frac{2}{3}$

15. The probability that a man will get a plumbing contract is $\frac{2}{3}$. The probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting at least one contract is $\frac{4}{5}$, what is the probability that he will get both the contracts?

    1) $\frac{4}{9}$ 2) $\frac{14}{45}$ 3) $\frac{7}{5}$ 4) $\frac{7}{9}$

16. There are two boxes. The first box contains 8 cubes and 6 spheres and the second box contains 6 cubes and 12 spheres. One object is transferred at random from the first box to the second. Find the probability that an object selected from the second box is a cube.

1) $\frac{7}{19}$ 2) $\frac{56}{133}$ 3) $\frac{67}{133}$ 4) $\frac{46}{133}$

17. In a casino, 3 games are played.

I. Betting on tossing of 2 unbiased coins.
II. Betting on drawing a card from a pack of 52 cards.
III. Betting on the sum of the scores of the top face of 2 dice.

Each of the above events occur at random.

A 'mechanical robot', who has 3 arms conducts the 3 events simultaneously. What is the probability of getting at least 1 head, a picture card and a sum below 9 (in the game of dice)?

1) $\frac{5}{52}$ 2) $\frac{1}{4}$ 3) $\frac{1}{36}$ 4) $\frac{1}{8}$

18. Two bags, A and B, consist of black and white balls only. The probability that a ball taken out from bag A is black and the probability that a ball taken out from bag B is white are both equal to 0.25. The balls of both the bags are emptied into bag C (which was initially empty). The probability that a ball taken out from bag C is black is $\frac{7}{15}$. Which of the following cannot be the total number of balls in bag C?

1) 120 2) 240 3) 300 4) 360

19. A die is rolled thrice. What is the probability that the number appearing in the second turn is the difference of the numbers appearing in the first and third turns?

1) $\frac{5}{36}$ 2) $\frac{4}{27}$ 3) $\frac{7}{54}$ 4) $\frac{1}{36}$

DIRECTIONS for questions 20 and 21: Refer to the data given below and answer the questions that follow.

Both Rajesh and Ramesh have one 5 paise, one 10 paise and one 20 paise coin each. They play a game in which each selects a coin randomly from their existing coins without the knowledge of other. If the sum of the coins is an odd amount, then Rajesh wins Ramesh's coin and if it is an even amount then Ramesh wins Rajesh'scoin.

20. Find the probability that Rajesh wins one odd and one even coin in the first two games.

1) $\frac{7}{36}$ 2) $\frac{2}{9}$ 3) $\frac{1}{4}$ 4) $\frac{11}{36}$

21. Find the probability that Ramesh wins all three coins belonging to Rajesh in three consecutive games.

1) 0.2 2) 0.25 3) 0.5 4) 0.1

QA-4.7 | APPLICATIONS OF P&C, PROBABILITY THEORY

In this module so far, we have studied Permutations and Combinations as well as Probability. In this class, we will study the applications of P & C and Probability.

Different case of Partitioning

1. **Dividing 'm + n' objects into 2 groups of 'm' and 'n' objects respectively**

   Consider a set of 3 boys A, B and C who are to be split up into two groups. In how many ways can this be achieved?

   The obvious choices are

   (a) A alone and B, C in the other group.
   (b) B alone and A, C in the other group.
   (c) C alone and B, A in the other group.

   i.e., 3 ways; but 3 = $\frac{3!}{2!\,1!}$.

   Thus, in general, the number of ways in which (m + n) things can be divided into two groups containing m and n things respectively = ${}^{m+n}C_n$ = $\mathbf{\frac{(m+n)!}{m! \times n!}}$

Note: This is true only if the groups are not numbered.

2. **Dividing '2m' objects into equal groups of 'm' objects**

   Now suppose we have to divide 4 boys A, B, C, D into two groups containing an equal number of boys; this can be done as follows:

   (a) A, B in a group and C, D in another.
   (b) A, C in a group and B, D in another.
   (c) A, D in a group and B, C in another.

   These are the only possible options; thus in this instance, we have 3 ways in all.

   But 3 = $\frac{4!}{2(2!)^2}$.

   Thus, in general, **if n things are divided into two groups each containing m things then the number of ways to divide the n things are $\mathbf{\frac{(2m)!}{2(m!)^2}}$**.

Note: As a general case we have, the number of ways to divide n things into different groups, one containing p things, another q things and so on is $\frac{(p+q+r+\ldots)!}{p! \times q! \times r! \times \ldots}$
where n = p + q + r .....

3. The number of selections of ‘k’ consecutive things out of ‘n’ things in a row = n − k + 1

10 One-Rupee coins are placed in a row. How many different ways can you select 5 consecutive coins?
The total number of ways = 10 − 5 + 1 = 6 ways.

4. Out of ‘n’ things, if ‘p’ things are alike and of one kind, ‘q’ are alike and of another kind, ‘r’ are alike and of a third kind and the rest t things are different, the total number of selections that can be made
= $(p + 1)(q + 1)(r + 1)2^t - 1$, where $n = p + q + r + t$

Consider a club has 3 Indians, 2 Britishers, 1 American and 1 African. At least one of these persons is to be selected for a tour. How many different selections are possible?
We note that
(a) None or one or two or all of the Indians may be selected ≡ 4 choices.
(b) None or one or both the Britishers may be selected ≡ 3 choices.
(c) Either the American or the African or both or none may be selected ≡ 4 choices.
Thus, by the Fundamental Principle of counting the number of choices
= $4 \times 3 \times 4 = (3 + 1)(2 + 1)2^2$.
However, note that this number includes the possibility wherein none of the persons is selected.
Hence, the number of different selections = $(3 + 1)(2 + 1)2^2 - 1$.

SOLVED EXAMPLES

Q : In how many ways can three balls be selected out of 4 identical green balls and 4 identical red balls?

A : Here, the total number of ways will not be $^8C_3$, as all the objects are not distinct.

Following combinations of three balls are possible:

1] All balls red

2] All balls green

3] 1 red and 2 green

4] 2 red and 1 green

Total number of ways = 4.

Q : There are 3 copies each of 4 different books. In how many ways can they be arranged on a shelf?

A : Since there are 3 things alike in each set of 4,

Number of arrangements = $\frac{12!}{3!\,3!\,3!\,3!} = \frac{12!}{(3!)^4}$

Q : In how many ways can 6 similar rings be worn in the 4 fingers of one hand?

A : n non-distinct things (rings) can be partitioned into r distinct groups (fingers), where some groups may be empty, in $^{n+r-1}C_{r-1}$ ways.

$\therefore$ 6 rings can be worn on 4 fingers in = $^{6+4-1}C_{4-1}$ = $^9C_3$ = 84 ways.

Binomial Probability

Suppose an unbiased die is rolled 100 times. On each throw, probability of getting 6 is same and that is $\frac{1}{6}$ while probability of getting 6 on any other throw is also equal to $\frac{1}{6}$ and is independent of probability of getting 6 on other throws. Such probability distributions are called binomial probability distribution. They have the following characteristics—

1) Experiment consists of 'n' number of trials (100 in this example)
2) Each trial can result in either success or failure (success is getting 6 in this example and failure is not getting 6)
3) Probability of success is same across all the trials.
4) Probability of success is in each trial is independent of probability of success in other trials.

Suppose there are 'n' trials, each having probability of success of 'p'. Probability of getting success on 'r' trials out of 'n' is given by—

$^nC_r\ p^r\ (1 - p)^{n-r}$

SOLVED EXAMPLES

Q : Each of the six friends have probability of getting into a reputed B-school of 0.2. What is the probability that two of the six friends will get into the school?

A : Each of the friends has same probability of getting into the B-School. The probability that one friend will get admission is independent of probability that other friends will get admission. Therefore, this is a question of binomial probability.

Required probability : $^6C_2(0.2)^2\ (1 - 0.2)^{6-2}$

$= {}^6C_2(0.2)^2\ (0.8)^4$

Q : The probability that it would rain on any of the first 10 days of July is same and that is equal to 0.8. What is the probability that it would rain on only one day out of first 10 days of July?

A : Required probability : $^{10}C_1\ (0.8)^1\ (0.2)^9$

Bayes' Theorem

Suppose P(A) and P(A′) are the probabilities that event A will happen and event A will not happen respectively. For both A and A′, event B may or may not happen. Then we can deonte the following conditional probabilities —

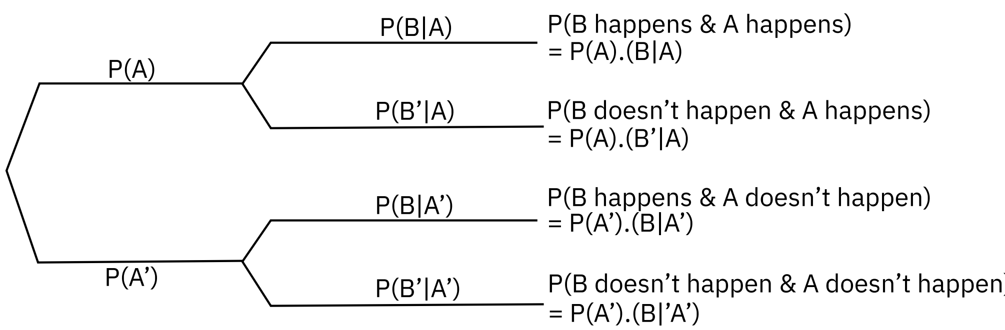
P(B|A) — Probability that B happens given that A has happened.

P(B′|A) — Probability that B doesn't happen given that A has happened.

P(B|A′) — Probability that B happens given that A has not happened.

P(B′|A′) — Probability that B does not happen given that A has not happened.

We can construct the tree diagram as follows—



$\therefore$ P(B happens) = P(B happens & A happens) + P(B happens & A doesn't happen)

= P(A).P(B|A) + P(A′).P(B|A′)

Similarly, P(A happens given B has happened) = P(A/B) = $\dfrac{P(A)\,P(B \mid A)}{P(A).P(B \mid A) + P(A')\,P(B \mid A')}$

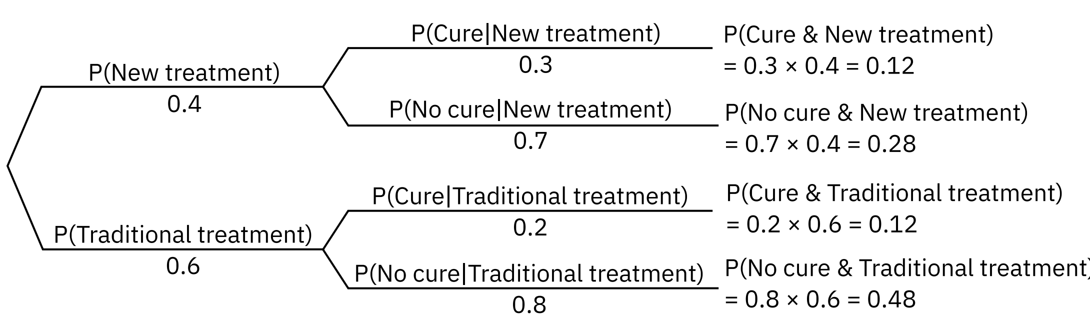
SOLVED EXAMPLES

Q : A new treatment for cancer can cure disease in 30% of patients while tranditional treatment can cure disease in 20% of patients. 40% patients are given new treatment while remaining 60% patients are given traditional treatment. Calculate—

(i) Probability that a randomly selected patient would have been cured.

(ii) Probability that a randomly selected patient has been cured using new treatment.

A : We can construct the tree diagram as follows — Note that patient will be cured only after only after giving treatment.



∴ Probability that a randomly selected patient would have been cured
= P(Cure & New treatment) + P(Cure & Traditional treatment)
= 0.12 + 0.12
= 0.24

Probability that a randomly selected patient would have been cured using new treatment

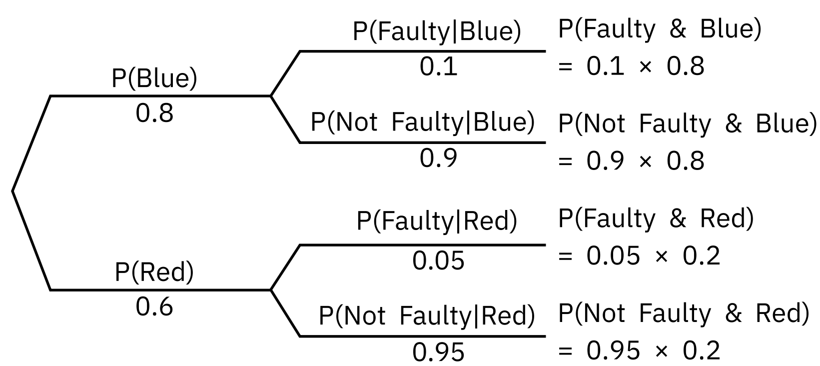
$$= \frac{0.3 \times 0.4}{0.3 \times 0.4 + 0.2 \times 0.6} = \frac{0.12}{0.24} = 0.5$$

Q : In a workshop, 80% screws manufactured are blue screws and 20% are red screws. Out of blue screws manufactured, 10% screws are faulty while out of red screws manufactured 5% screws are faulty. Calculate—

(i) The probability that a randomly selected screw is faulty

(ii) The probability that randomly selected faulty screw is blue.

A : We can construct the tree diagram as follows—



P(Faulty) = P(Faulty & Blue) + P(Faulty & Red)
= 0.1 × 0.8 + 0.05 × 0.2
= 0.08 + 0.01
= 0.09

$$\text{P(Blue/Faulty)} = \frac{0.1 \times 0.8}{0.1 \times 0.8 + 0.05 \times 0.2} = \frac{0.08}{0.09} = \frac{8}{9}$$

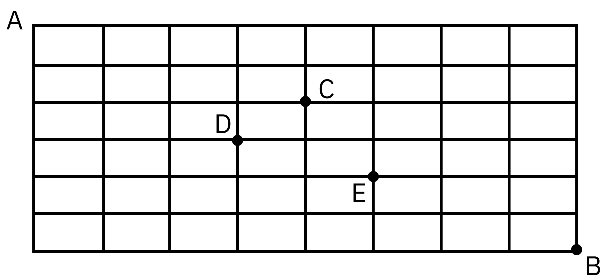
CLASS EXERCISE

Teaser

Mercedes and Edmond are playing a game. They toss 25 coins and count the number of heads and tails. If the number of heads is more, Mercedes wins, while if the number of tails is more, Edmond wins.

What is the probability that Mercedes wins, given that all the coins are unbiased?

Pathways and Routes



In a town in Canada, all roads are either in North-South direction or in East-West direction, forming a grid, as shown in the figure. Two adjacent roads in both North-South and East-West directions are equidistant such that each cell in the grid in a square as shown. Length of side of each square is 500 m.

1. What is the minimum distance covered to travel from A to B?

2. How many ways are there to get from A to B by covering shortest possible distance?

3. If a delivery van has to start from A, deliver a parcel to C and reach B, covering the minimum possible distance along the way in how many ways can this be managed?

4. If there is an underground subway joining D and E, and a man walks from A to B via road, takes the subway to E and then walks to B, how many minimum length routes can he take?

5. * How many shortest path's routes can a person take if she desires to go from A to B visiting both C and D along the way?

6. How many shortest paths routes are possible between A and B if point C is to be avoided?

Binomial Probability

7. If 4 non identical fair coins are tossed, what is the probability that there will be exactly 3 heads?

8. If 12 non identical fair dice are rolled, what is the probability that exactly 4 of them will show a perfect square?

9. If a person randomly attempts 50 questions in a multiple choice paper with each question having 5 options (only one of which is correct), what is the probability that she will get exactly 20 right?

If n trials are conducted, each of which has a probability of success p and a probability of failure q (where q = 1 – p) then the probability that exactly r out of n trials are successful is $\mathbf{{}^nC_r\, p^r\, q^{(n-r)}}$.

10. * If each of the 20 children at a party is equally likely to drink an orange drink, a lemon drink or a cola, what is the probability that exactly 3 of them have a cola?

11. * If the Hard Rock Café is frequented by twice as many men as ladies, what is the probability that exactly 3 out of the next 10 people to enter will be ladies?

Assorted Questions on Permutations, Combinations and Probability

12. How many distinct lines can be drawn through:
    a) 8 points, no three of which are collinear?
    b) 8 points, exactly 4 of which are collinear?

13. How many distinct triangles can be drawn with their vertices selected from:
    a) 8 points, no three of which are collinear?
    b) 8 points, exactly 4 of which are collinear?

14. Out of all the laptops produced by company X, 50% are made with monitors from company P (out of which 10% stop working in a year), 30% are made with monitors from company M (out of which 20% stop working in a year) and the remaining are made with monitors from company D (out of which 25% stop working in a year). If I buy a laptop from X and the monitor stops working in a year, what is the probability that that monitor came from company M?
    1) 37.5% 2) 30% 3) 33.33% 4) 40%

15. In a garden, 30% flowers are roses while remaining are carnations. 40% of roses and 60% of carnations are red in colour while remaining flowers are yellow in colour. What is the probability that a red flower picked at random will be a rose?

16. If three cards are drawn from a pack of cards, what is the probability that all are kings if
    a) the cards are drawn with replacement
    b) the cards are drawn without replacement

17. If three cards are drawn from a pack of cards, what is the probability that the first is a king, the second a queen and the third an ace if
    a) the cards are drawn with replacement
    b) the cards are drawn without replacement

18. If three cards are drawn from a pack of cards, what is the probability that one is a king, one a queen and one an ace if
    a) the cards are drawn with replacement
    b) the cards are drawn without replacement

19. In how many ways can 5 dimes and 6 nickels be put into 3 pockets?
    1) ${}^{11}C_3$ 2) ${}^{13}C_2$ 3) ${}^{7}C_2 \times {}^{8}C_2$ 4) ${}^{10}C_2$

Challengers

1. In how many ways can 22 people be divided into two cricket teams to play each other?
   1) $^{22}C_2$ 2) $^{22}C_{11}$ 3) $(^{22}C_{11})^2$ 4) $0.5 \times {}^{22}C_{11}$

2. If two points F and G are selected at random on the circumference of a circle with radius 1, what can be said about x, where x is the probability that the linear distance between F and G is less than 1?
   1) $0.22 \le x < 0.33$ 2) $0.33 \le x < 0.44$
   3) $0.44 \le x < 0.55$ 4) $0.55 \le x < 0.66$

3. In how many ways can we can we put 6 distinct rings in 10 fingers?
   1) $^{15}C_6$ 2) $^{10}P_6$ 3) 10^6 4) $^{15}P_6$

4. In the expansion of $(p + q + r)^{10}$, find the coefficient of $p^2q^5r^3$
   1) 2520 2) 1260 3) $^{10}C_5$ 4) None of these

5. How many numbers of at least 2 distinct digits exist such that the digits are strictly decreasing?
   1) 1011 2) 1012 3) 1013 4) 1014

6. There are 10 people in a room. Some or all of them shake hands. What is the probability that there will be 2 people in the room who have shaken hands with the same number of people?
   1) $\frac{729}{1024}$ 2) $\frac{^{9}C_2}{^{10}C_2}$ 3) $\frac{9}{10}$ 4) None of these

PRACTICE EXERCISE-1

DIRECTIONS for questions 1 to 17: Choose the correct alternative.

1. A father wishes to distribute Rs.1400 among his three sons such that the eldest one gets the maximum amount, the middle one gets more money than the youngest one. In how many ways can the father distribute the money if each of his sons gets an amount in a multiple of 100?

   1) 10 2) 26 3) ${}^{14}C_2$ 4) 28

2. In how many ways can seven faces of a pentagonal prism be painted with 7 different colours?

   1) 720 2) 5040 3) 144 4) 504

3. In an office cubicle there are six desks, each provided with a telephone having a specific extension number. A wireman forgets to make a note of the connections of the six telephones according to the extension numbers while changing the old wires with new ones. What is the probability that the wireman does not connect any wire to its respective telephone matching the extension number?

   1) $\frac{1}{2}$ 2) $\frac{59}{6!}$ 3) $\frac{11}{120}$ 4) $\frac{53}{144}$

4. Probability of giving right answer by a student is $\frac{1}{3}$. If eight questions are asked to a student one after the other, what is the probability that a student gives at least two right answers?

   1) $1 - 4\left(\frac{2}{3}\right)^5$ 2) $1 - \frac{2}{9}$ 3) $\left(\frac{2}{3}\right)^7$ 4) $1 - 5\left(\frac{2}{3}\right)^8$

5. Vishwanathan Anand is playing a 9 match rapid exhibition against Magnus Carlssen. On average, Anand's probability of winning a game against Carlssen is 0.75. If Anand is 3 – 2 down after 5 games, and there are no ties, what is the probability that Anand will win the championship?

   1) $\frac{81}{256}$ 2) $\frac{3}{4}$ 3) $\frac{189}{256}$ 4) None of these

6. From a well shuffled pack of 52 cards, one card is drawn at random.

   P(A) = Probability that the card is red and P(B) = Probability that the card is a diamond or a spade and P(C) = Probability that the card is a king or a card of hearts. Then which of the following holds?

   1) $P(A) \geq P(B) > P(C)$ 2) $P(A) > P(C) > P(B)$
   3) $P(A) = P(B) > P(C)$ 4) $P(C) \geq P(A) = P(B)$

7. The ace, 2, 3, 4, 5, 6, 7 and 8 of spades are placed, face up, in a row on a table. Then a pack of eight cards containing the ace, 2, 3, 4, 5, 6, 7 and 8 of diamonds is shuffled and placed, face down, in front of the player. As each successive diamond is turned over, the corresponding spade is removed from the row. What is the probability that all the spades can be removed without a break (hole) occurring in the row of spades?

   1) $\frac{1}{315}$ 2) $\frac{1}{245}$ 3) $\frac{1}{225}$ 4) $\frac{1}{175}$

8. There are 20 people sitting around a rectangular dining table, 10 on one side facing the other 10 on the opposite side, in such a way that each person knows the person sitting exactly opposite to him and the two persons sitting beside him. Four persons are chosen at a random such that there are at least 3 pairs of people knowing each other. Find the probability that there are four pairs of people knowing each other.

   1) $\frac{3}{16}$ 2) $\frac{3}{19}$ 3) $\frac{9}{11}$ 4) $\frac{9}{87}$

9. In a multiple choice exam with 10 questions, each questions has 4 option. A student has not studied for the exam and decides to mark the answers randomly. What is the probability that she will get exactly 2 questions right?

   1) ${}^{10}C_2\ (0.25)^2(0.75)^8$ 2) ${}^{10}C_1\ (0.25)^2(0.75)^8$
   3) ${}^{10}P_2\ (0.25)^2(0.75)^8$ 4) ${}^{10}P_1\ (0.25)^2(0.75)^8$

10. In a University in the US, 60% of students are girls while remaining 40% are boys. Out of boys, 30% study Economics while out of girls, 40% study Economics. What is the probability that a randomly selected economics student is a girl?

   1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) $\frac{7}{16}$ 4) $\frac{9}{16}$

11. Kamlesh and Vimlesh are the two candidates who have reached the final round of interview for a job. The probability that Kamlesh will be selected is 0.4 and the probability that both Kamlesh and Vimlesh will be selected is at most 0.2. If the probability that neither of them will be selected is 0.1, which of the following can be the probability that Vimlesh will be selected?

   1) 0.45 2) 0.65 3) 0.75 4) 0.85

12. If a number is chosen from 1 to 15000, what is the probability that it has exactly 3 factors?

   1) $\frac{7}{3000}$ 2) $\frac{1}{500}$ 3) $\frac{1369}{15000}$ 4) None of these

13. A, B, C, D, E, F, G and H together take part in a race against each other. All the eight persons are equally capable of winning the race. What is the probability that A is ahead of C who in turn is ahead of E who in turn is ahead of G?

   1) $\frac{1}{40320}$ 2) $\frac{1}{96}$ 3) $\frac{1}{48}$ 4) $\frac{1}{24}$

14. There are 20 equidistant light poles around a circular garden. From the 20 light poles, if only a set of 3 light poles which form either an acute angled or obtuse angled triangle is lit together, then how many such sets of 3 light poles can be formed?

   1) 1140 2) 960 3) 980 4) 1130

15. Five wooden blocks of different sizes are placed next to each other along a row. What is the probability that all of them are placed neither in increasing nor in decreasing order of their sizes?

   1) $\frac{119}{120}$ 2) $\frac{59}{60}$ 3) $\frac{1}{120}$ 4) $\frac{1}{60}$

16. Three distinct numbers are simultaneously selected from the set {1, 2, 3, 4, 8, 9, 16, 27} at random. The numbers so selected are arranged in the ascending order. What is the probability that the three numbers are not in GP?

1) $\frac{3}{28}$ 2) $\frac{13}{28}$ 3) $\frac{25}{28}$ 4) $\frac{5}{56}$

17. Total 10 persons named P_1, P_2....P_{10} are seated around a rectangular table such that three persons are seated along each of the longer sides of the table and two persons are seated along each of the shorter sides of the table. What is the probability that two persons P_3 and P_9 are seated next to each other along the longer side of the table?

1) $\frac{1}{9}$ 2) $\frac{2}{9}$ 3) $\frac{8}{45}$ 4) $\frac{4}{45}$

DIRECTIONS for questions 18 to 20: Solve as directed.

18. 12 students (A, B, C, ... K, L) are sitting in a row in that order. In how many ways can you distribute a red rose each to 3 consecutive students, a pink rose each to 3 consecutive students and a white rose each to 3 consecutive students such that no student gets more than one rose?

19. A certain number of individuals who were invited to a party wanted to give a rose to each other. While they were doing so, the power went off. Since it took a lot of time for the power to resume, 6 people left the party without receiving any rose. It is known that each of these 6 people gave a rose to 6 other people. After the power resumed, the remaining individuals gave a rose to each other such that no person gave a rose to another person more than once during the party. If the total number of roses that was exchanged in the party was 192, how many people were invited to the party?

20. Gameathon is a game in which three players participate in a match. In a Gameathon tournament, only players from Russia and America participated. Each player played with every possible pair exactly once. The number of matches played in which all the three players were Russian was 84 while the number of matches played in which all the three players were Americans was 165. What was the total number of matches involving players both from Russia and America?

PRACTICE EXERCISE-2

DIRECTIONS for questions 1 to 15: Choose the correct alternative.

1. Ten sticks of length 10 cm and four sticks of length 6 cm are to be placed one beside the other so that no two shorter sticks are adjacent to each other. In how many ways can this be done so that the length of the entire arrangement is 108 cm? [Note that all the sticks need not be used.]

   1) 165 2) 120 3) 2^5 4) 330

2. Jenny cuts ten equilateral triangles from a paper such that every triangle is congruent to exactly one other triangle. She keeps these triangles in a box. In how many different ways can Jenny select four triangles from the box such that no two of them can be used in making a rhombus?

   1) ${}^{10}C_4$ 2) $2^4 \times {}^{10}C_4$ 3) ${}^{5}C_4$ 4) $2^4 \times {}^{5}C_4$

3. Three distinct unbiased dice are tossed simultaneously. What is the probability that the sum of the numbers appearing on the dice is divisible by 5?

   1] $\frac{43}{216}$ 2] $\frac{53}{216}$ 3] $\frac{35}{216}$ 4] $\frac{49}{216}$

4. Rajan has two boxes of chocolates. Each box contains 25 chocolates. He tells his five cousins to take the chocolates from these two boxes such that no one should take more than five chocolates from each box and each one should take 7 chocolates. The four cousins take the chocolates such that equal number of chocolates are left in both the boxes. If from each box the four cousins take different number of chocolates, then in how many ways can the last cousin choose his chocolates?

   1) $2({}^{25}C_2 \times {}^{25}C_5 + {}^{25}C_4 \times {}^{25}C_3)$ 2) $2({}^{11}C_2 \times {}^{11}C_5 + {}^{11}C_4 \times {}^{11}C_3)$
   3) $2({}^{14}C_2 \times {}^{14}C_5 + {}^{14}C_3 \times {}^{14}C_4)$ 4) ${}^{25}C_7$

5. Mohit and his friends are playing the mathematical scrabble. Mohit has six tiles of which three tiles bear number 2, two tiles bear number 3 and remaining one bears number 4. Mohit has to start the game by putting a three-digit number on the board using the tiles he has. In how many ways can Mohit do this?

   1) 18 2) ${}^{6}P_3$ 3) 19 4) 3^3

6. In a Claypin Lottery game, one has to purchase tickets with each ticket costing Rs.5. There is a single number out of 1 to 17 on each ticket. In all there are 4 different series of tickets. 4 tickets (1 from each series) are declared as winning tickets. Given below are some of the rules and regulations for winning prize money at the lottery game.

   I. For a participant, if exactly 4 of his tickets match the 4 winning tickets he will receive Rs.2 lakhs.
   II. If exactly 3 of his tickets match 3 of the 4 winning tickets he will receive Rs.50000.
   III. If exactly 2 of his tickets match 2 of the 4 winning tickets he will receive Rs.5000.
   IV. Nothing will be given for matching only one number. If one wants to make a profit, what is the minimum number of tickets he should buy? (Assume that any number of tickets can be sold/purchased for the game from any series. A participant can purchase a ticket of any number from any of the 4 series)

   1) ${}^{17}C_2$ 2) ${}^{17}C_3$ 3) 4913 4) None of these

7. Eight friends and their heights are as follows: A – 57 in., B – 54 in., C – 66 in., D – 63 in., E – 55 in., F – 59 in., G – 52 in., and H – 64 in. Five friends out of these eight are selected. If E is definitely selected, then what is the probability that the number of people taller than E is more than the number of people shorter than E?

   1) $\frac{25}{56}$ 2) $\frac{3}{7}$ 3) $\frac{15}{56}$ 4) $\frac{5}{7}$

8. A square paper sheet is folded 4 times to get the maximum number of triangles when the paper is unfolded. The triangles of the paper are formed by joining the lines which are formed due to the foldings of the paper. Find the number of ways of putting 19 identical balls in those triangles which do not contain any other triangle, in such a way that each triangle should contain at least one ball but no triangle contains exactly 3 balls.

   1) 16! 2) 576 3) 1632 4) 720

9. In a multiple choice exam with 10 questions, each question has 4 options. A student needs to get at least 3 questions correct in order to pass the exam. Tina has not studied for the exam and decides to mark the options randomly. What is the probability that she passes the exam?

   1) $\frac{219}{16}\left(\frac{3}{4}\right)^8$ 2) $1-\frac{219}{16}\left(\frac{3}{4}\right)^8$ 3) $\frac{21}{4}\left(\frac{3}{4}\right)^8$ 4) $1-\frac{21}{4}\left(\frac{3}{4}\right)^8$

10. In a city, 60% cars are blue and 40% cars are red. A criminal escaped in a car and only one person saw him driving away. However, the witness is colour blind and can recognize the colour of the car with only 70% accuracy. What is the probability that the car was actually blue if the witness said that it was blue?

   1) $\frac{2}{9}$ 2) $\frac{7}{9}$ 3) $\frac{9}{23}$ 4) $\frac{14}{23}$

11. Tushar’s son Tinku removed the first seven letters of the alphabet from his alphabet box and took five out of them and arranged them in a sequence. What is the probability that he arranged the letters to form the word FACED?

   1) $\frac{1}{120}$ 2) $\frac{1}{2520}$ 3) $\frac{7}{40}$ 4) $\frac{5}{120}$

12. Two numbers ‘x’ and ‘y’ (x > y) are randomly chosen from the set {1, 2, 3, 4 ... 9, 10}. What is the probability that $x^2 - y^2$ is divisible by 3?

   1) $\frac{6}{25}$ 2) $\frac{32}{45}$ 3) $\frac{5}{9}$ 4) $\frac{8}{15}$

13. Ten persons want to cross a river by a boat which can carry a maximum of six persons at a time. If the boat is to be used only for two round trips then in how many ways can these persons cross the river?

    1) 420 2) 504 3) 462 4) 924

14. 5 cards are randomly drawn out of a pack of 52 cards. What is the probability that exactly 4 out of these 5 cards belong to the same suit?

    1) $\frac{143}{1666}$ 2) $\frac{143}{833}$ 3) $\frac{143}{3332}$ 4) $\frac{143}{6664}$

15. In a hall, there are 13 lamps among which 2 each are of red, green, yellow and blue colours and other 5 lamps of different colours. Each one of these except the two red lamps has separate control switch. However, both the red lamps have one switch in common. Find the number of ways in which the switches can be put on to illuminate the hall with 4 lamps.

    1) 167 2) 139 3) 188 4) 385

DIRECTIONS for questions 16 to 18: Solve as directed.

16. How many whole number solutions are possible for the equation $x + y + z = 30$ such that $x < y < z$?

17. In how many ways can 3 distinct numbers be selected among 10, 11, 12, ... 18, 19 such that their product is divisible by 20?

18. In how many ways can the letters of the word KARAKORAM be rearranged so that the vowels are always together?

DIRECTIONS for questions 19 and 20: Refer to the data below and answer the questions that follow.

In a class of 5 girls and 6 boys, one monitor, 2 prefects and 3 coordinators are to be appointed. First the 6 coordinators are selected, three among these also have additonal responsibility of being a monitor and two prefects.

19. Out of two prefects, if one has to be a boy and other, a girl, In how many ways can the selection be done?

    1) ${}^{15}C_6 - ({}^{5}C_6 + {}^{16}C_6)$ 2) $[{}^{11}C_6 - ({}^{5}C_6 + {}^{6}C_6)] \times {}^{6}C_2$
    3) 15120 4) Data insufficient

20. Which of the following questions could not be solved?

    1) If 2 boys and 4 girls are selected in the group of six coordinators, in how many ways can a girl be selected as a monitor?
    2) What is the chance that the monitor is a boy if equal number of boys and girls get selected in the group of 6 coordinates?
    3) Is it true that there is a greater chance of a boy getting selected as a monitor?
    4) In how many ways can one select a monitor if equal number of boys and girls are left unselected as any of coordinators or prefects or a monitor?