A Puzzle of Knights

For our first post, let's look at a time-honoured puzzle, of the kind often encountered in interviews:

Q. King Arthur's famous Round Table could hold 300 knights. One day a damsel in distress came to the court and asked for help. Each and every knight wanted to go and so King Arthur had to select one knight. He numbered all the knights in order from 1 to 300 (with 1 being himself) and made them count in order, every alternate knight having to leave the table; the last knight remaining would get to go on the quest. The knight with which number was eventually chosen?

1) 1 2) 49 3) 89 4) 99

We could obviously solve this by writing the 300 numbers in detail, and removing every alternate one, but that would simply take too much time.

Another approach would be to list out the people remaining after each 'pass' round the circle:
1st pass: 1, 3, 5, 7…..297, 299
2nd pass 1, 5, 9, 13…..293, 297
3rd pass 1, 9, 17, 25….289, 297
4th pass 9, 25….and so on.
However this would be pretty cumbersome and with a significant chance of going wrong, not to mention time-consuming! (Besides, what if there were 3000 people instead of 300?)

Let's try and use a pattern to solve this.

If there were only one person, obviously he would be the winner. If there were two, again #1 would get to go. For three, it would be #3, and for four again #1. Taking a few more numbers and checking we find a pattern:

Number of people	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Last man standing	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

If the number of people at the start is a power of 2, then the person left at the end will be #1. From then on till the next power of 2, each extra person increments the 'winning' number by 2.

So if there are 13 people, we would look at the power of 2 just before 13 (which is 8) and say: for 8 people, #1 would have won, for the remaining 5, add 5*2 = 10 and hence #11 will win.

Similarly for, say, 81 people, we would do 1 + (81 - 64)*2 = 35. Hence the 35th person would win.

And so, when there are 300 knights, all we need to do is 1 + (300-256) * 2 = 89. Hence the knight whose number is 89 will be selected for the mission. Hence (3)

(If there were 3000 knights, then the knight with the number 1 + (3000-2048) * 2 = 1905 would be chosen)

SEQUENCES – 1

Sequences

A sequence is, in essence, an ordered list of objects (usually numbers). From a function point of view, it may be considered as a function defined on a countably ordered set (in practical terms, this usually means the domain of definition is the set of Natural Numbers). Typically the terms of a sequence are written as T_1, T_2, T_3, T_4…. or as a_1, a_2, a_3, a_4… Let's start of by looking at the most common types of sequences, the Arithmetic Progression (AP) and the Geometric Progression (GP).

The AP or Arithmetic Progression

An Arithmetic Progression or AP is a sequence of terms in which the difference between any two consecutive terms is fixed. For example 17, 24, 31, 38, 45, 52…. is an AP. Traditionally an AP is described in terms of its first term "a" and the common difference "d" between two subsequent terms. Thus the 2nd term, T_2, will be a + d, the 3rd term, T_3, will be a + 2d, the 4th term, T_4, will be a + 3d, the 5th term, T_5, will be a + 4d….and extending this logic, the nth term can be written as $\mathbf{T_n = a + (n - 1)d}$

A little thought will enable us to recognise that an AP can be completely described by knowing either any two terms or by knowing one of the terms and the "common difference". For example, if an AP has third term 35 and common difference 6, then the 2nd term will be 29 and the 1st will be 23 and thus the AP will be 23, 29, 35, 41…. and so on. Similarly an AP with 5th term 21 and 8th term 27 will have a common difference 2 (as the difference from the 5th to the 8th term, which is nothing but 3d, is 6) and from that we can work back to find the first term as 13 and the AP as 13, 15, 17, 19, 21, 23…

A common requirement is to find the sum of the first n terms of an AP (denoted by S_n):

$$S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n = a + a+d + a+2d + \ldots + a+(n-1)d \quad \text{- (I)}$$

But we can also say

$$S_n = T_n + T_{n-1} + \ldots + T_2 + T_1 = a + (n-1)d + a+(n-2)d + a+(n-3)d + \ldots + a \quad \text{- (II)}$$

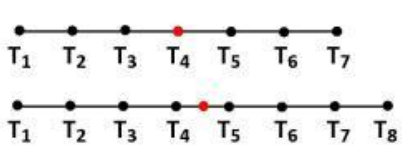
Adding (I) and (II) we get

$$2\,S_n = 2a + (n-1)d + 2a + (n-1)d + 2a + (n-1)d \ \ldots \text{(n times)} = n * [2a + (n-1)d]$$

Hence we have $S_n = \frac{n}{2} * [2a + (n-1)d]$

We could also have derived this in another way – the whole point of an AP is that the terms are equally spaced. This implies that the average terms can be visualised very easily as follows:
(a) if the AP has an odd number of terms, the average will be the middle term
(b) if the AP has an even number of terms, the average will be the average of the two middle terms

However there is another way of looking at it; in both the above cases the average of the AP will be the average of the 1st and last terms.



$$\Longrightarrow \text{Average} = T_4 = \frac{T_1+T_7}{2}$$

$$\Longrightarrow \text{Average} = \frac{T_4+T_5}{2} = \frac{T_1+T_8}{2}$$

Now since the first term is a and the last is a + (n – 1)d, the average will be $\frac{a + a + (n-1)d}{2}$; hence the sum of the terms will be n times this i.e. $n * \frac{2a + (n-1)d}{2} = \frac{n}{2} * [2a + (n-1)d]$

SEQUENCES – 2

Summations

Using the formula for an AP, we can derive a formula for the sum of the first n natural numbers. Note that the set of natural numbers too is an AP with a = 1 and d = 1. So the sum to n terms of this AP will be n * (average of 1st and last term) = n * $\frac{1+n}{2}$ or in a more standard form $\sum_1^n n = \frac{n*(n+1)}{2}$

In a similar manner we can find formulae for $\sum_1^n n^2$ and $\sum_1^n n^3$; these turn out to be

$$\sum_1^n n^2 = \frac{n*(n+1)*(2n+1)}{6}$$

$$\sum_1^n n^3 = \frac{n^2*(n+1)^2}{4} = (\sum_1^n n)^2$$

So for example,

The sum of the first 200 natural numbers will be $\frac{200*201}{2} = 20100$

The sum of the first 20 perfect squares will be $\frac{20*21*41}{6} = 2870$

The sum of the first 10 perfect cubes will be $\frac{10^2*11^2}{4} = 3025$

Note that the proofs of the above two formulae are somewhat complex and certainly not needed from an exam point of view. However, for demonstration purposes, let's prove the former of the above formulae using mathematical induction; the latter also can be proved similarly:

For n = 1: $\sum_1^1 1^2 = \frac{1*(1+1)*(2*1+1)}{6} = \frac{1*2*3}{6} = 1$. So the formula gives the right answer here.

Let us assume the formula to be true for some n i.e. $\sum_1^n n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n*(n+1)*(2n+1)}{6}$

So for n+1, $1^2 + 2^2 + \ldots + n^2 + (n+1)^2$

$$= \frac{n*(n+1)*(2n+1)}{6} + (n+1)^2$$

$$= \frac{n*(n+1)*(2n+1)+6*(n+1)^2}{6}$$

$$= \frac{(n+1)\,[n*(2n+1)+6*(n+1)]}{6}$$

$$= \frac{(n+1)\,[2n^2+7n+6]}{6}$$

$$= \frac{(n+1)\,(n+2)\,(2n+3)}{6}$$

$$= \frac{(n+1)\,[(n+1)+1]\,[2(n+1)+1]}{6}$$

Hence if it is true for any n, the formula also holds for n+1. Also it is true for n = 1 and hence by the principle of induction it must be true for all natural numbers n.

In the next post, we'll examine some simple examples of APs and Summations.

SEQUENCES – 3

Q. Find the sum of the series 8, 14, 20, 26….up to 20 terms

This is an AP with a = 8 and d = 6. The 20th term will be 19 steps from the 1st ; $T_{20} = 8 + 19 * 6 = 122$.
So the sum will be $20 * \frac{8+122}{2}$ **= 1300.**

Q. Find the sum of the first 15 numbers which leave a remainder 2 when divided by 10

We can figure out the first few numbers as 2, 12, 22, 32 and realise that this is again an AP with a = 2 and d = 10 so the 15th term will be 2 + 14 * 10 = 142. Hence the sum will be $15 * \frac{2+142}{2}$ **= 1080.**

Q. Find the sum of the natural numbers from 51 to 100

We can easily do this by finding the sum of the first 100 natural numbers and subtracting the sum of the first 50 natural numbers

$\sum_1^{100} n - \sum_1^{50} n = \frac{100 * 101}{2} - \frac{50 * 51}{2} = 5050 - 1275 =$ **3775**

Q. Find the sum of the first 100 even numbers

This is an AP of the form 2, 4, 6, 8….which has a = 2, d = 2 and the 100th term will therefore be 200.
So the sum will be $100 * \frac{2+200}{2}$ **= 10100.**

Alternatively, $2 + 4 + 6 + 8 + \ldots + 200 = 2 (1 + 2 + 3 + 4 + \ldots + 100) = 2 * \frac{100 * 101}{2} = 10100$

Q. Find the sum of the first 100 odd numbers

This is an AP of the form 1, 3, 5, 7….which has a = 1, d = 2 and the 100th term will therefore be 199.
So the sum will be $100 * \frac{1+199}{2}$ **= 10000.**

Alternatively, we can do this as the sum of the first 200 natural numbers minus the sum of the first 100 even numbers, which is $\frac{200 * 201}{2} - 2 * \frac{100 * 101}{2} = 20100 - 10100 =$ **10000**

Another way we could do this is to observe the pattern followed by adding odd numbers:
$1 = 1 = 1^2$ $\quad 1 + 3 = 4 = 2^2$ $\quad 1 + 3 + 5 = 9 = 3^2$ $\quad 1 + 3 + 5 + 7 = 16 = 4^2$
So the sum of the first n odd numbers turns out to be n^2 and so for n = 100, the sum is 100^2 **= 10000**

Q. Find the sum of the first 20 even perfect squares

The sum of the first 20 perfect squares = 4 + 16 + 36 + 64 + …. + 1600
$= 4 (1^2 + 2^2 + 3^2 + 4^2 + \ldots + 20^2)$
$= 4 (\frac{20 * 21 * 41}{6}) = 4 * 2870 =$ **11480**

SEQUENCES – 4

Some minor but useful things to note about APs and their behaviour:

- Generally most APs we encounter tend to have "a" as well as "d" positive. But both a and d could be negative, or zero (if d is zero the AP becomes the same number repeating!)
- Questions on APs can often be solved be visualisation of the AP as a set of equally spaced numbers.
- An AP is always a divergent series, i.e. the sum to infinite terms will be either ∞ or $-\infty$ depending on whether d is positive or negative.
- If we know the sum of n terms of an AP, we can find out only the middle term (for odd n) or the average of the middle two terms (for even n)
- One useful approach to dealing with AP-based questions (especially those which talk of individual terms such as T_7 or T_9 is to treat the terms in terms of a and d so replace T_7 by "a + 6d" and T_9 by "a + 8d"
- Another handy approach, when one talks of the sum or product of consecutive terms of an AP, is to start from the middle term. So for example if we want to add 5 terms in AP, rather than the standard convention of a, a + d, a + 2d, a + 3d, a + 4d we could take the middle term as, say, m and then name the terms as m – 2d, m – d, m, m + d, m + 2d. This would give us the sum as 5m instead of the more cumbersome 5a + 10d.

Let's look at some examples which test one's comfort level with the behaviour of APs:

Q. If in an Arithmetic Progression with general term a_n, it is known that $a_7 + a_9 = a_3 + a_4 + a_{10}$ then what is the first term of the AP?

At first glance one would be tempted to say that the question in unsolvable. However, let us try:
We are given that $a_7 + a_9 = a_3 + a_4 + a_{10}$
So we can say that (a + 6d) + (a + 8d) = (a + 2d) + (a + 3d) + (a + 9d)
So 2a + 14d = 3a + 14d
Which means that a = 0. But a is nothing but the first term of the AP and hence we can conclude that **the first term of the AP is 0**

Q. If the sum of the first 5 terms of an Arithmetic Progression is equal to the sum of the first 10 terms, then which term of the AP must necessarily be zero?

We can apply formulae here as follows
$\frac{5}{2} * [2a + 4d] = \frac{10}{2} * [2a + 9d]$ ➔ 5a + 10d = 10a + 45d) ➔ 5a + 35d = 0 ➔ a+ 7d = 0 ➔ $\mathbf{T_8 = 0}$

But we can also do this logically by visualising the AP. If the sum of the first 5 terms = sum of the first 10 terms, then $(T_1 + T_2 + \ldots + T_5) = (T_1 + T_2 + T_3 + \ldots + T_{10})$ which implies that the sum of the 6th to 10th terms $(T_6 + T_7 + T_8 + T_9 + T_{10})$ must be zero. But that means the average of those 5 terms also must be zero. And the average of 5 terms in AP is the middle i.e. 3rd term which means **T_8 must be 0**

SEQUENCES – 5

The GP or Geometric Progression

A Geometric Progression or GP is a sequence of terms in which the ratio between any two consecutive terms is fixed. For example 8, 24, 72, 216, 648, 1944…. is a GP. Traditionally a GP is described in terms of its first term "a" and the common ratio "r" between two subsequent terms. Thus the 2nd term, T_2, will be a*r, the 3rd term, T_3, will be $a*r^2$, the 4th term, T_4, will be $a*r^3$, the 5th term, T_5, will be $a*r^4$….and extending this logic, the nth term can be written as $\mathbf{T_n = a * r^{n-1}}$

Just as in the case of an AP, a GP too can be completely described by knowing either any two terms or by knowing one of the terms and the "common difference". For example, if a GP has fourth term 40 and common ratio 2, then the 3rd term will be 20, the 2nd term 10 and the 1st term 5 and thus the GP will be 5, 10, 20, 40, 80, 160…. and so on. Similarly a GP with 3th term 18 and 5th term 162 will have a common difference 3 (as the ratio of the 5th term to the 3rd, which is nothing but r^2, is 9) and from that we can work back to find the first term as 2 and the GP as 2, 6, 18, 54, 162…

A common requirement is to find the sum of the first n terms of a GP (denoted by S_n):

$$S_n = a + ar + ar^2 + ar^3 \dots ar^{n-2} + ar^{n-1} \quad \text{- (I)}$$

But we can also say

$$rS_n = ar + ar^2 + ar^3 \dots ar^{n-2} + ar^{n-1} + ar^n \quad \text{- (II)}$$

Subtracting (I) from (II) we get

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} \dots \cancel{ar^{n-2}} + \cancel{ar^{n-1}} + ar^n$$

$$-S_n = -a - \cancel{ar} - \cancel{ar^2} - \cancel{ar^3} \dots - \cancel{ar^{n-2}} - \cancel{ar^{n-1}}$$

$$(r-1)\,S_n = ar^n - a = a(r^n - 1)$$

Hence we have $S_n = \dfrac{a(r^n - 1)}{r-1}$

Another way of approaching this derivation could be to remember a standard result in algebra, which states that $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1})$.

From this we can show that $r^n - 1 = (r - 1)(r^{n-1} + r^{n-2} + r^{n-3} + \dots + r + 1)$

Thus we can write S_n as follows:

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}$$

$$= a\,(1 + r + r^2 + r^3 + \dots + r^{n-2} + r^{n-1})$$

$$= a\frac{(r^n - 1)}{r-1}$$

We will mainly concern ourselves with GPs in which a and r are positive (though it is technically possibly to define a GP with a or r negative as well, as we shall see). If r is greater than 1, then the GP will be increasing, while if r is less than 1 it will be decreasing. If r = 1 then, of course, the GP reduces to the same term repeated (equivalent to an AP with d = 0)

SEQUENCES – 6

Infinite Sums

In the previous posts, we have examined how to find the sum of an AP or a GP to a finite number of terms. What if we want to find the sum to infinity?

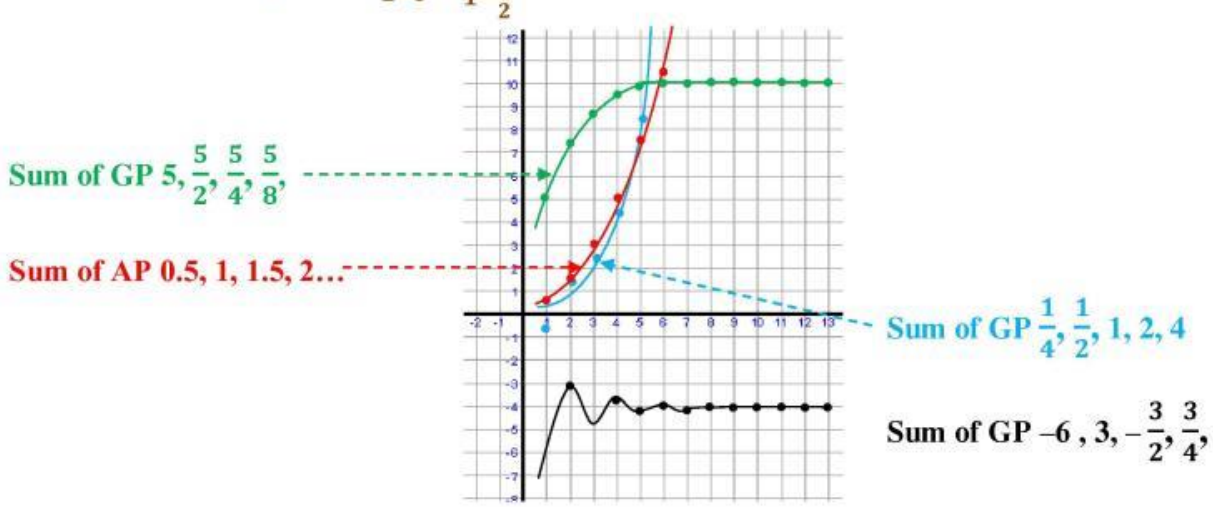
If we look at an AP, we will see that if d is positive, the terms will go on increasing and tend to ∞ as n approaches ∞. So the infinite sum will also tend to ∞. Similarly if d is negative, the terms will tend towards $-\infty$ and so the infinite sum will also tend to $-\infty$. Thus it makes no sense to talk of the sum to infinity of an Arithmetic Progression, as any AP will automatically be "divergent". (For example if we take an AP with a = 1 and r = -0.5, we get a sequence 1, 0.5, 0, -0.5, -1, -1.5, -2….and the sum after each term will be 1, 1.5, 1.5, 1, 0, -1.5, -3.5, -6 and so on.)

If we consider a GP with r > 1, then each term will have a greater magnitude than the previous one and so the infinite sum again will tend to ∞. So this also results in a divergent series. (For example if we take a GP with a = 4 and r = 1.5, we get a sequence 4, 6, 9, 13.5, 20.25… and the sums after each term will be 4, 10, 19, 32.5, 52.75 and so on.)

It turns out that if we take a GP with negative r but with a magnitude greater than 1 (i.e. $r < -1$), each term will have a magnitude greater than the previous one, and hence the sum will keep oscillating between positive and negative, but with increasing magnitude and so it too will be divergent. (For example if we take a GP with a = 4 and r = -3, we get a sequence 4, -12, 36, -108, 324, -972 and so on, which will give us sums after each term of 4, -8, 28, -80, 244, -728 and so on.)

However, if we take a GP with $0 < r < 1$, then each term turns out to be smaller than the one before it. It can be shown that in this case, the result is a convergent series, i.e. the sum to infinite terms nevertheless converges to a finite value (a similar result can be demonstrated for $-1 < r < 0$, the difference being that the sum in that case oscillates round the final value).

The sum of a GP to infinite terms can be obtained now from the formula derived in the previous post i.e. $S_n = \frac{a(r^n - 1)}{r-1}$. If we put $n = \infty$, we will get $S_\infty = \frac{a(r^\infty - 1)}{r-1}$. But since $r < 1$, r^∞ will tend to 0 and so the above will reduce to $\frac{a(0-1)}{r-1}$ which is better written as $\frac{a}{1-r}$. (For example if we take a GP with a = 3 and r = ½, we will get $S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6$)



SEQUENCES – 7

Let's look at some examples of problems involving GPs:

Q. Find the sum of the first 8 terms of the GP 7, 14, 28….

The sum of the GP to n terms is given by $S_n = \frac{a(r^n - 1)}{r-1} = \frac{7(2^8 - 1)}{2-1}$ = 7 * 255 = **1785**

Q. On the first day of its release, the movie Krusssh 3 makes 90 crores. On each subsequent day, it makes exactly a third as much as it did on the previous day. How much will the movie make by the end of its run?

Here the GP is reducing, and we need to find the infinite sum of 90 * $(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \ldots)$ which will be $\frac{90}{1-\frac{1}{3}}$ = **135 crores**

Q. A ball is dropped from a height of 80 feet. Every time it bounces to a height of $\frac{3}{4}$ th of what it had achieved on the previous bounce. How far will it have travelled before it comes to rest?

Here the situation is slightly different as during the first downward journey the ball covers a distance of 80 m but between subsequent bounces it moves up as well as down. In each bounce it will travel to a height of $3/4^{th}$ of that of the previous one, so between the first and the second bounce it will travel 60 + 60 = 120 m, between the second and third it will travel 45 + 45 = 90 m, and so on.

Thus the required distance will be 80 + 2 * 60 + 2 * 45 +… = 80 + 2 * 60 * $(1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots)$ = $80 + \frac{120}{1-\frac{3}{4}}$ = **560 feet**

Q. In an Arithmetic Progression, the 8th, 12th and 17th terms are in Geometric Progression. What is the ratio of the first and tenth terms?

This question appears confusing because it involves an AP, some of whose terms are in GP. First of all let us remember that the 8th, 12th and 17th terms of an AP may be written as a + 7d, a + 11d and a + 16d. Also let us note that when three terms are in Geometric Progression, the middle term will be the Geometric Mean of the other two (just as when 3 terms are in Arithmetic Progression, the middle term will be the Arithmetic mean of the other two)

So in this case we can say that the (8th term * 17th term) = $(12^{th}$ term$)^2$. This gives us
(a + 7d) (a + 16d) = (a + 11d)(a + 11d)
So $\cancel{a^2}$ + 23ad +$112d^2$ = $\cancel{a^2}$ + 22ad + $121d^2$
Which gives us a = 9d.
Hence the first term is 9d while the tenth term is a + 9d = 9d + 9d = 18d. So the ratio of the first and tenth terms is 9d : 18d = **1 : 2**

SEQUENCES – 8

The Arithmetico-Geometric Sequence

An Arithmetico-Geometric Sequence is a sequence in which each term is the product of the corresponding terms of an AP and a GP. (For example if we have an AP 2, 5, 8, 11, 14… and a GP 8, 4, 2, 1, ½… then the corresponding AGP would be 16, 20, 16, 11, 7… and if we had an AP 3, 4, 5, 6… and a GP 2, 4, 8, 16….then the corresponding AGP would be 6, 16, 40, 96….)

If the common ratio of the GP is greater than 1, then the AGP will be divergent and its sum will tend to infinity. If, however, the common ratio of the GP is less than 1, then eventually the sum of AGP will converge to a finite value. Let us see how to tackle such a problem:

Q. Find the sum to infinity of the sequence $\frac{2}{3}+\frac{7}{9}+\frac{12}{27}+\frac{17}{81}+\frac{22}{243}+\frac{27}{729}+\dots$

Let us write the required sum as S.

$$S=\frac{2}{3}+\frac{7}{9}+\frac{12}{27}+\frac{17}{81}+\frac{22}{243}+\frac{27}{729}+\dots$$

Now note that the ratio of the GP is $\frac{1}{3}$. Consequently, we find the value of 3S.

$$3S=2+\frac{7}{3}+\frac{12}{9}+\frac{17}{27}+\frac{22}{81}+\frac{27}{243}+\frac{32}{729}+\dots$$

Now subtract S from 3S (combining the terms which share a denominator):

$$3S-S=2+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+\frac{5}{81}+\frac{5}{243}+\frac{5}{729}+\dots$$

…but excepting the first term, the rest of this very neatly forms an infinite GP with $r=\frac{1}{3}$ and so we can find its sum as

$$2S=2+5*(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\dots)=2+5*(\frac{\frac{1}{3}}{1-\frac{1}{3}})=2+5\,(\frac{1}{2})=4.5$$

Hence **S = 2.25**

Let's see one more example, as this can be a tricky concept:

Q. Find the sum to infinity of the sequence $\frac{4}{7}+\frac{6}{49}+\frac{8}{343}+\frac{10}{2401}+\dots$

$$S=\frac{4}{7}+\frac{6}{49}+\frac{8}{343}+\frac{10}{2401}+\dots$$

$$7S=4+\frac{6}{7}+\frac{8}{49}+\frac{10}{343}+\frac{12}{2401}+\dots$$

$$7S-S=4+\frac{2}{7}+\frac{2}{49}+\frac{2}{343}+\frac{2}{2401}+\dots$$

$$6S=4+2*(\frac{1}{7}+\frac{1}{49}+\frac{1}{343}+\frac{1}{2401}+\dots)=4+2*(\frac{\frac{1}{7}}{1-\frac{1}{7}})=4+2\,(\frac{1}{6})=\frac{13}{3}$$

Hence $\mathbf{S=\frac{13}{18}}$

SEQUENCES – 9

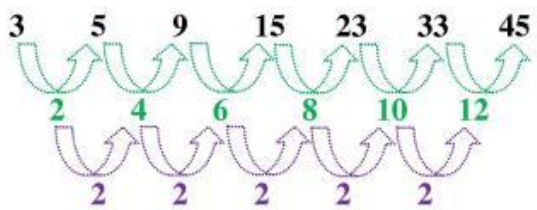
Quadratic Sequences:

Q. Find the 50th term of the sequence 3, 5, 9, 15, 23, 33, 45…

1) 2543 2) 2345 3) 2453 4) 2435 5) 2534

The first thing would be to identify the pattern of the sequence – patterns are useful in mathematics.

We can easily see that the differences between successive terms form an AP, and so the "second level differences" are constant (since they will just be the common difference of the AP)



It turns out that the nth term of such a sequence will be a quadratic expression of the form $\mathbf{an^2 + bn + c}$.

(Those who are comfortable with the concept of differentiation will not find this at all surprising – if we take a quadratic $ax^2 + bx + c$ and differentiate it once we will get a linear term, while differentiating it twice yields a constant; but differentiation is nothing but the act of finding the difference between successive terms when the distance between terms is negligible. The only difference here is that we are finding the difference for a discrete function rather the differentiation for a continuous function)

Thus the challenge of finding the nth term, T_n, reduces to that of finding the constants a, b and c in the above expression. To find 3 variables, we need 3 equations, which we can find as follows: in the given sequence, $T_1 = 3$, $T_2 = 5$ and $T_3 = 9$. But since we have noted that $T_n = an^2 + bn + c$, we can write:

$T_1 = a * 1^2 + b * 1 + c = a + b + c = 3$ …(I)

$T_2 = a * 2^2 + b * 2 + c = 4a + 2b + c = 5$ …(II)

$T_3 = a * 3^2 + b * 3 + c = 9a + 3b + c = 9$ …(III)

(II) - (I) gives $3a + b = 2$ …(IV)

(III) - (II) gives $5a + b = 4$ …(V)

(V) - (IV) gives $2a = 2$ i.e. $a = 1$ which when substituted back into the previous equations yields $b = -1$ and $c = 3$. Hence the nth term of the given sequence will be $\mathbf{T_n = n^2 - n + 3}$ (we can test this easily enough, for example putting $n = 6$ gives us $T_6 = 6^2 - 6 + 3 = 33$ which in fact is the 6th term of the given sequence).

Now the question requires us to find the 50th term – this can be found easily enough as $T_{50} = 50^2 - 50 + 3 = 2453$. **Hence (3)**

SEQUENCES – 10

Q. Find the sum of the first 20 terms of the sequence 4, 5, 7, 10, 14, 19…

As in the last post, we can see that the second level differences are the same and hence it is a quadratic sequence and so we can safely conclude that the nth term will be of the form $T_n = an^2 + bn + c$

$T_1 \quad = a * 1^2 + b * 1 + c \quad = a + b + c \quad = 4 \quad \ldots(I)$

$T_2 \quad = a * 2^2 + b * 2 + c \quad = 4a + 2b + c \quad = 5 \quad \ldots(II)$

$T_3 \quad = a * 3^2 + b * 3 + c \quad = 9a + 3b + c \quad = 7 \quad \ldots(III)$

(II) - (I) gives $\quad 3a + b = 1 \quad \ldots(IV)$

(III) - (II) gives $\quad 5a + b = 2 \quad \ldots(V)$

(V) - (IV) gives $\quad 2a = 1$ i.e. $a = \frac{1}{2}$ which when substituted back into the previous equations yields $b = -\frac{1}{2}$ and $c = 4$. Hence the nth term of the given sequence will be $\mathbf{T_n = \frac{1}{2}n^2 - \frac{1}{2}n + 4}$ (to check – putting n = say 5 gives $T_5 = \frac{1}{2}5^2 - \frac{1}{2}5 + 4 = 14$ which matches the given information)

But now we want the sum to 20 terms of the sequence; this means we want $\sum_1^{20}(\frac{1}{2}n^2 - \frac{1}{2}n + 4)$

$= \frac{1}{2}\sum_1^{20} n^2 - \frac{1}{2}\sum_1^{20} n + \sum_1^{20} 4$

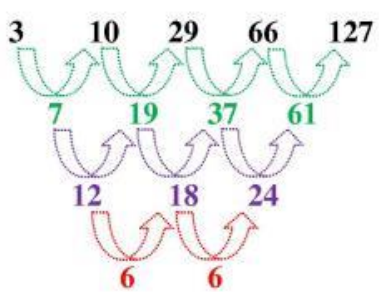
$= \frac{1}{2} * \frac{20*21*41}{6} - \frac{1}{2} * \frac{20*21}{2} + 20 * 4$

$= 1435 - 105 + 80$

$= 1410$

Q. Find the 20th term of the sequence 3, 10, 29, 66, 127…

If we examine the differences between successive terms, we will eventually find that the third level differences are constant in this sequence:



This implies a cubic sequence; so the nth term T_n can be expressed in the form $an^3 + bn^2 + cn + d$. The four unknowns a, b, c and d could be found by using 4 equations (generated using T_1, T_2, T_3 and T_4). But this is likely to be a long and cumbersome process. In such situations the alert test-taker will always look for an alternative approach – in this case a look at the terms of the sequence 3, 10, 29, 66, 127 will show them to be 1 + 2, 8 + 2, 27 + 2, 64 + 2, 125 + 2…. i.e. $1^3 + 2$, $2^3 + 2$, 3^3+2, 4^3+2, $5^3 + 2$ which means the nth term will be $n^3 + 2$ and the 20th term will be $20^3 + 2$ = **8002.**

SEQUENCES – 11

Powers of 2

A sequence of particular interest is the GP 1, 2, 4, 8, 16….i.e. the sequence of consecutive powers of 2.

The nth term in the series can easily be seen to be 2^{n-1}, while we can find the sum of the first n terms (using the standard formula for a GP) will be $\frac{1(2^n-1)}{2-1} = 2^n - 1$.

In other words, the sum of the first n terms is 1 unit less than 2^n; this turns out to be a useful property. For example $1 + 2 + 4 + 8 + 16 \ldots + 256 = 512 - 1 = 511$. Similarly if we want the sum of $2^5 + 2^6 + 2^7 + 2^8 + 2^9$ we can do it as $(2^{10} - 1) - (2^5 - 1) = 992$.

There is a wide range of problems which can be solved efficiently using powers of 2. For example:

Q. A man has 200 gold coins. He wishes to distribute them into bags such that he can pay any sum from 1 to 200 by handing over some bags. What is the minimum number of bags required?

There has to be a bag containing 1 coin (so that he can pay a sum of 1 coin).
Now to pay a sum of 2 coins, he can either have a bag with 2 coins or a second bag with 1 coin. But since we want to minimise the total number of bags, we will put 2 coins in the second bag. This incidentally enables us to pay a sum of 3 coins as well (2 + 1).
To pay a sum of 4 coins, we will place 4 coins in the third bag. This will enable us to pay sums of 5 (4 +1), 6 (4 +2) and 7 (4 + 2 + 1) coins as well.
The next bag should therefore contain 8 coins. We can proceed in this manner; by now it should be evident that the most efficient distribution we are able to make is to put successive powers of 2 into each bag. After filling 1, 2, 4, 8, 16, 32 and 64 coins in 7 bags, we have accounted for 127 coins; the remaining 73 coins can be put into an 8th bag.

Now if the bill is anything between 1 and 127 he can pay it using the first 7 bags, while if it exceeds 127, he can first hand over the 8th bag and then pay the balance amount using some combination of the first 7 (for example a bill of 150 could be split as 73 + 77 = 73 + (64 + 8 + 4 + 1) and a bill of 186 could be split as 73 + 113 = 73 + (64 + 32 + 16 + 1).

Q. A goldsmith has a pan balance and a set of 8 weights. He puts the weights in one pan, and the jewellery to be weighed in the other; and claims that he can weigh any positive integral weight up till 255 g. What must be the denominations of the weights?

As in the above case, we will find that the most efficient distribution of weights (to cover the largest possible range) occurs when the weights are powers of 2. Hence let us try weights of **1, 2, 4, 8, 16, 32, 64 and 128 g**. On checking we will find that these will enable us to weigh each and every number up till (256 – 1) i.e. 255 and hence these must be the required denominations.

SEQUENCES – 12

There's an interesting variant on yesterday's last question which goes as follows:

Q. A goldsmith has a pan balance and a set of 6 weights. Using these, he claims that he can weigh any positive integral weight up till 364 g. What must be the denomination of these weights?

In this case, clearly powers of 2 will not be sufficient – 6 weights with powers of 2 will enable us to weigh only up till 63 g. But if we look at the problem carefully, we realise that a condition has been removed. In the earlier question, the weights were to be placed in one pan only. In this case we can place weights in both the pans, thus allowing not only addition but also subtraction. Effectively, this adds an extra "degree of freedom" to the problem; and it rather elegantly turns out that instead of powers of 2, we can use powers of 3 here!

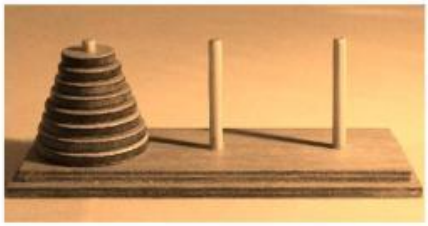
If we consider weights of 1, 3, 9, 27, 81 and 243g, we can make any required combination as follows

$1 = 1$	$2 = 3 - 1$	$3 = 3$	$4 = 3 + 1$	$5 = 9 - 3 - 1$
$6 = 9 - 3$	$7 = 9 - 3 + 1$	$8 = 9 - 1$	$9 = 9$	$10 = 9 + 1$
$11 = 9 + 3 - 1$	$12 = 9 + 3$	$13 = 9 + 3 + 1$	$14 = 27 - 9 - 3 - 1$ and so on….	

The Tower of Hanoi

Let's look at a popular test and interview problem, an ancient logical puzzle most popularly known as "Tower of Hanoi" (though another version of the problem statement refers to it as the Tower of Brahma!) The basic problem statement is very simple – Wikipedia states it thus:

Q. There are three rods, and a number of discs of different sizes which can slide onto any rod, (as shown in the adjacent figure, which has 8 discs). The puzzle starts with the discs in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.



The objective of the puzzle is to move the entire stack to another rod, obeying the following simple rules:

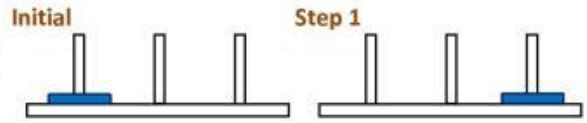
- **Only one disc may be moved at a time.**
- **Each move consists of taking the upper disc from one of the stacks and placing it on top of another stack.**
- **No disc may be placed on top of a smaller disc.**

From a mathematical perspective, what we need to ask ourselves is – **"what is the minimum number of moves required to move the discs from rod 1 to rod 3?"** As this is a rather tricky and interesting puzzle, I'll not reveal the solution instantly; rather I shall leave it for my readers to think about over the weekend (Hint: start with a smaller number of discs, say 2 or 3, observe the pattern, and then try to extend the same logic to a larger number such as 8).

SEQUENCES – 13

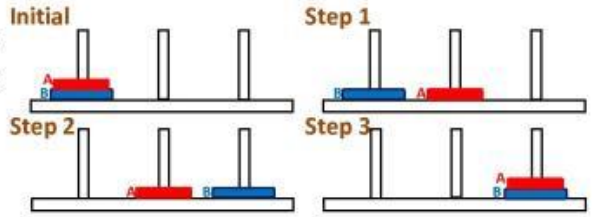
Tower of Hanoi – Solution

Let's look at a simple case with just 1 disc. Clearly, it will require just one move, from stack 1 to stack 3.



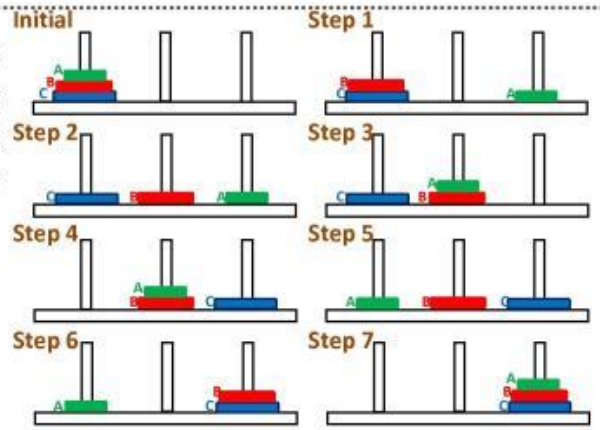
If we have two discs (smaller A and larger B), we can manage it in 3 moves, but the first move would be A to stack 2, then B to stack 3, and finally A to stack 3.

In total, therefore, 3 steps would be needed



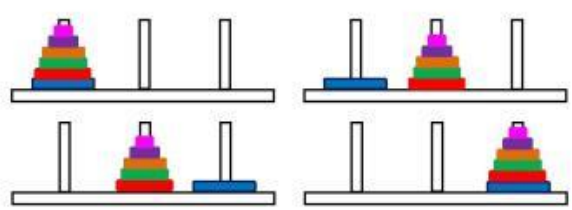
In the case of three discs (let us say A, B and C in increasing order of size), then we would start by moving A to stack 3, then B to stack 2, then A to stack 2, then C to stack 3, then A to stack 1, then B to stack 3 and finally A to stack 3.

In total, 7 steps would be required.



If we look at the series, we find that for the number of discs n = 1, 2, 3….the optimal number of moves m turns out to be 1, 3, 7… i.e. **when n increases by 1, m becomes 1 more than double its previous value**. So if n were to become 4, m would be 2 * 7 + 1 = 15 and for n = 5 m would be 2 * 15 + 1 = 31. Let us understand why this should be so:

Suppose that for n discs, the number of moves required is k. Now if we have n + 1 discs, let us first move the upper n discs to stack 2. This will require k steps. Now we move the largest ($n+1^{th}$) disc to stack 3, requiring only 1 move. And finally we move the other n discs to stack 3, using up another k moves. Thus the total moves required will be k + 1 + k = 2k + 1.



Now if we look carefully at the values of m we've obtained, we will find that 1, 3, 7, 15, 31 etc can be written as $2^1 - 1$, $2^2 - 1$, $2^3 - 1$, $2^4 - 1$, $2^5 - 1$ and so on. In other words if there are n discs, the number of moves required will be $2^n - 1$. (We can check this – if the number of discs becomes (n+1), then the number of moves should become $2 * (2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$. Thus we can formalise the solution obtained, using the method of Induction)

Therefore, in a Tower of Hanoi problem with n discs, the optimal solution will involve **$2^n - 1$ steps**. Also, if n is odd, the 1st move should be to stack 3 while if n is even, the 1st move should be to stack 2.

SEQUENCES – 14

Let us look at a few questions on sequences:

Q. What is the sum of the series $\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$ …. to 98 terms?

1) $\frac{1}{2}$ **2) $\frac{51}{100}$** **3) $\frac{49}{100}$** **4) None of the above**

The first step in attacking such questions is always to identify the pattern. Here the successive denominators are 6, 12, 20, 30, 42…looking at them closely we realise that they are 2 * 3, 3 * 4, 4 * 5, 5 * 6, 6 * 7 and so on.

Now if we look at, for example, $\frac{1}{30}$ i.e. $\frac{1}{5*6}$ we can write it as $\frac{6-5}{30}$, which is nothing but $\frac{1}{5}-\frac{1}{6}$.
Similarly $\frac{1}{42}$ i.e. $\frac{1}{6*7}$ is nothing but $\frac{1}{6}-\frac{1}{7}$ and in general $\frac{1}{n*(n+1)}$ is nothing but $\frac{1}{n}-\frac{1}{n+1}$

So then we can write $\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$ …. to 98 terms as

$$\frac{1}{2*3}+\frac{1}{3*4}+\frac{1}{4*5}+\frac{1}{5*6}+\frac{1}{6*7}\ \ldots.+\frac{1}{98*99}+\frac{1}{99*100}$$
$$=\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}\ \ldots.+\frac{1}{98}-\frac{1}{99}+\frac{1}{99}-\frac{1}{100}$$
$$=\frac{1}{2}-\frac{1}{100}=\frac{49}{100}.$$ **Hence 3.**

Let's look at another similar question:

Q. What is the sum of the series $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ …. to 10 terms?

1) $\frac{175}{132}$ **2) $\frac{36}{55}$** **3) $\frac{175}{264}$** **4) $\frac{209}{312}$**

Here the successive denominators are 3, 8, 15, 24, 35 which are 1 * 3, 2 * 4, 3 * 5, 4 * 6, 5 * 7...

Now if we look at, for example, $\frac{1}{24}$ i.e. $\frac{1}{4*6}$ we can write it as $\frac{1}{2}*\frac{6-4}{24}$, which is $\frac{1}{2}*(\frac{1}{5}-\frac{1}{6})$
Similarly $\frac{1}{35}$ i.e. $\frac{1}{5*7}$ turns out to be $\frac{1}{2}*(\frac{1}{5}-\frac{1}{7})$ and in general $\frac{1}{n*(n+2)}$ is nothing but $\frac{1}{2}*(\frac{1}{n}-\frac{1}{n+2})$

So then we can write $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ …. to 10 terms as

$$\frac{1}{1*3}+\frac{1}{2*4}+\frac{1}{3*5}+\frac{1}{4*6}+\frac{1}{5*7}\ \ldots.+\frac{1}{9*11}+\frac{1}{10*12}$$
$$=\frac{1}{2}*(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}\ \ldots.+\frac{1}{9}-\frac{1}{11}+\frac{1}{10}-\frac{1}{12})$$
$$=\frac{1}{2}*(\frac{1}{1}+\frac{1}{2}-\frac{1}{11}-\frac{1}{12})=\frac{1}{2}*\frac{175}{132}=\frac{175}{264}.$$ **Hence 3.**

SEQUENCES – 15

Let's look at a couple of slightly trickier questions in the next two posts.

Q. What can be said about A, where A is given by the product $\frac{1}{2} * \frac{3}{4} * \frac{5}{6} * \frac{7}{8} * \ldots * \frac{97}{98} * \frac{99}{100}$?

1) A = $\frac{1}{10}$ 2) A < $\frac{1}{10}$ 3) A > $\frac{1}{10}$ 4) None of the above

Consider A = $\frac{1}{2} * \frac{3}{4} * \frac{5}{6} * \frac{7}{8} * \ldots * \frac{97}{98} * \frac{99}{100}$

and B = $\frac{2}{3} * \frac{4}{5} * \frac{6}{7} * \frac{8}{9} * \ldots * \frac{96}{97} * \frac{98}{99} * 1$

Now comparing them term by term,

$\frac{2}{3} > \frac{1}{2}$

$\frac{4}{5} > \frac{3}{4}$

$\frac{6}{7} > \frac{5}{6}$

…

$\frac{98}{99} > \frac{97}{98}$

$1 > \frac{99}{100}$

So we may safely conclude that $\frac{2}{3} * \frac{4}{5} * \frac{6}{7} * \frac{8}{9} * \ldots * \frac{96}{97} * \frac{98}{99} * 1 > \frac{1}{2} * \frac{3}{4} * \frac{5}{6} * \frac{7}{8} * \ldots * \frac{97}{98} * \frac{99}{100}$

i.e. B > A.

But B * A = $\frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \frac{4}{5} * \ldots * \frac{98}{99} * \frac{99}{100} = \frac{1}{100}$

Since B > A this means B must be greater than $\sqrt{\frac{1}{100}}$ i.e. $\frac{1}{10}$ while A must be less than $\frac{1}{10}$

Hence (2)

Before we move to tomorrow's question, let me pose you a little conundrum. Consider the following infinite series **h = $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots$** to ∞ terms. Each individual term gets smaller and smaller, tending eventually to 0. My question is, **will the sum of this series be finite or infinite?**

Think about it, the answer can be a little counter-intuitive!

SEQUENCES - 16

Q. Let $k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots + \frac{1}{255} + \frac{1}{256}$. Then which of the following is the most appropriate statement?

1) $1 < k < 5$ 2) $5 < k < 9$ 3) $9 < k < 13$ 4) $13 < k < 17$

At first glance, this seems a pretty daunting question. There are, after all, 256 terms and each one is only slightly smaller than the previous one. However, there is an interesting way of finding approximate limits for this series, and it is hinted at by the fact that 256 is a power of 2.

Let us digress for a moment to yesterday's poser: $h = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots\infty$, often known as the Harmonic Series. One of the triumphs of 17th century mathematicians was the proof that this series diverges (i.e. its sum tends towards infinity as we add infinite terms) despite the fact that the individual terms tend towards 0. One such proof is as follows: consider a sequence s as shown

$$h = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ldots\infty$$

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \ldots \infty$$

Comparing term-by-term, h > s. However, s is nothing but $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots \infty$ which is ∞. Hence h too must be ∞.

We can use a modification of that proof to solve today's problem in an elegant manner:

Let us introduce a sequence p as follows

$$k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{254} + \frac{1}{255} + \frac{1}{256}$$

$$p = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{128} + \frac{1}{128} + \frac{1}{256}$$

Comparing term-by-term, k < p. But if we compute p by adding blocks of the same colour together we find that $p = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \frac{1}{256}$ which is roughly 8.004.

Now let us introduce another sequence q as follows:

$$k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{254} + \frac{1}{255} + \frac{1}{256}$$

$$q = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots + \frac{1}{256} + \frac{1}{256} + \frac{1}{256}$$

Comparing term-by-term, k > q. But if we compute q by adding blocks of the same colour together we find that $q = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ which is exactly 5.

Hence k must be more than 5 and less than 8.04 which means out of the answer choices, $5 < k < 9$ is the most appropriate. **Hence 2.**

SEQUENCES – 17

Sequences – CAT Scan

Now that XAT-14 is done, let's return to sequences for a bit and check out the kind of stuff CAT has asked in the past. CAT questions on sequences have tended to be not particularly difficult, but have focused on testing the basic understanding for the most part. These question were very popular in the early 2000s, the frequency had dropped in the last few years but in 2012 and 2013 again plenty of questions made an appearance, with Geometric Progressions being a particular favourite.

Q. There are 8436 steel balls, each with a radius of 1 cm, stacked in a pile, with 1 ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the 4th and so on. The number of horizontal layers in the pile is: **CAT 2003**

1) 34 **2) 38** **3) 36** **4) 32**

The sequence we want is 1 + 3 + 6 + 10 +…. The first question is to identify the series. If we look carefully we will see that

1 = 1

3 = 1 + 2

6 = 1 + 2 + 3

10 = 1 + 2 + 3 + 4

…and hence $T_n = 1 + 2 + 3 + 4 + \ldots + n$ which we know is $\sum_1^n n = \frac{n*(n+1)}{2}$

Hence the summation of this series will be $\sum_1^n(\sum_1^n n) = \sum_1^n \frac{n*(n+1)}{2} = \frac{1}{2}\sum_1^n n^2 + n$

$= \frac{1}{2}\left\{\frac{n*(n+1)*(2n+1)}{6} + \frac{n(n+1)}{2}\right\} = \frac{1}{2}\left\{\frac{n*(n+1)*(2n+4)}{6}\right\}$ but this is given to be 8436

Now we could always solve a cubic equation, or factorise 8436 *12 and equate to $n(n+1)(2n+4)$ but it would be simpler to plug in options and eliminate ➔ We see that 8436 * 12 ends in 2. If n = 34, then 34 * 35 * 72 ends in 0. Similarly 38 * 39 * 80 ends in 0, while 36 * 37 * 76 ends in 2 and 32 * 33 * 68 ends in 8. Hence only n = 36 satisfies. **Hence (3).**

Q. The sum of 3rd and 15th elements of an AP is equal to the sum of 6th, 11th and 13th elements. Then which term is necessarily equal to 0? **CAT 2003**

1) 1st **2) 9th** **3) 12th** **4) None of these**

We have seen a similar question earlier and so this should be easy now:

We are given that $a_3 + a_{15} = a_6 + a_{11} + a_{13}$

So we can say that (a + 2d) + (a + 14d) = (a + 5d) + (a + 10d) + (a + 12d)

So 2a + 16d = 3a + 27d

Which means that a + 11d = 0. But this is nothing but the twelfth term of the AP and hence we can conclude that **the 12th term of the AP is 0. Hence (3)**

SEQUENCES – 18

Q. In a certain examination paper, there are n questions. For j = 1, 2, 3…n, there are $2^{(n-j)}$ students who answered j or more questions wrongly. If the total number of wrong answers is 4095, then the value of n is **CAT 2003**

1) 12 **2) 11** **3) 10** **4) 9**

A high-risk approach would be to note the number 4095 as suspiciously close to a familiar number = 4096 i.e. 2^{12}. This is unlikely to be a coincidence and hence there is a good chance that the answer would be 12. Of course this is a risky trick to try and not advisable – but spotting it would give you a couple of clues as to how to start. Firstly, the pattern probably involves powers of 2; secondly the answer is likely to be 12. So let us try to check the pattern for a smaller number.

Suppose n = 5. Then we can generate the following table:

j	$2^{(5-j)}$ ➔ at least j wrong	Exactly j wrong	Total wrong answers
1	16	16 – 8 = 8	8 * 1 = **8**
2	8	8 – 4 = 4	4 * 2 = **8**
3	4	4 – 2 = 2	2 * 3 = **6**
4	2	2 – 1 = 1	1 * 4 = **4**
5	1	1	1 * 5 = **5**
			Total = 31

*If 16 got **at least** 1 wrong and 8 got **at least** 2 wrong, then 16 – 8 people must have got **exactly** 1 wrong*

So for n = 5 the answer turns out to be $2^5 - 1 = 31$. It is not too large a leap to conclude that the answer $4095 = 2^{12} - 1$ will occur for n = 12. (We could test it for another value, say n = 4 or n = 6, but the canny CAT aspirant will have learned to trust her instincts by the time of the exam. One does not have time to do a full mathematical thesis to prove the solution, after all – nor does one need to prove it!). **Hence (1)**

Q. The 288th term of the series a, b, b, c, c, c, d, d, d, d, e, e, e, e, e, f, f, f, f, f, f….is **CAT 2003**

1) u **2) v** **3) w** **4) x**

The series is pretty self-explanatory; the nth letter of the alphabet occurs n times. So by the time the writing of the nth letter has been completed, the total number of letters written will be $\sum_1^n n = \frac{n*(n+1)}{2}$. At this point trial and error would probably serve best. u, v, w and x are the 21st, 22nd, 23rd and 24th letters. So by u, $\sum_1^n 21 = \frac{21*22}{2}$ = 231 terms will be written. Writing v will add 22 more bringing us to 253. Writing w will add 23 more, bringing the total to 276, and the 288th letter will therefore be x. **Hence (4)**

SEQUENCES – 19

Q. Consider the sequence of numbers a_1, a_2, a_3, a_4, a_5 ... to infinity, where $a_1 = 81.33$, $a_2 = -19$ and $a_j = a_{j-1} - a_{j-2}$ for $j > 3$. What is the sum of the first 6002 terms of this sequence? **CAT 2004**

1) -100.33 2) -30.00 3) 62.33 4) 119.33

6002 numbers is a lot to add up. It is clear that there must be some pattern in the problem, or else it would be impossible to solve in the typical under-2-minute window available. Let us look at the sequence definition more closely. We are given that

$$a_j = a_{j-1} - a_{j-2}$$

i.e. each term is defined in terms of two previous terms. This means that if we have two terms (which we do – a_1 and a_2 are given!) then we can find all the further terms. But the terms given are cumbersome to deal with so for the moment let us simplify the problem by calling $a_1 = p$ and $a_2 = q$:

$$\begin{aligned}
a_1 &= p \\
a_2 &= q \\
a_3 = a_2 - a_1 &= q - p \\
a_4 = a_3 - a_2 &= -p \\
a_5 = a_4 - a_3 &= -q \\
a_6 = a_5 - a_4 &= p - q \\
a_7 = a_6 - a_5 &= p \\
a_8 = a_7 - a_6 &= q \\
a_9 = a_8 - a_7 &= q - p
\end{aligned}$$

Note that these terms are the negatives of a_1, a_2, and a_3

Note that these terms are the same as a_1, a_2, a_3…

It is evident that the sequence is repeating from the seventh term onward, and hence every set of 6 terms will be the same. But the sum of these six terms is 0 (since the second set of 3 terms are just the negatives of the first set). So when we take the first 6000 terms of the sequence, they can be grouped in 1000 such sets of 6 which will add up to 0. Also the 6001st and 6002nd term will be the same as the 1st and 2nd. Hence the sum of 6002 terms will just be the same as $a_1 + a_2$ which is **81.33 – 19 = 62.33. Hence (3).**

Q. Let g(x) be a function such that g(x+1) + g(x−1) = g(x) for every real x. Then for what value of p is the relation g(x+p) = g(x) necessarily true for every real x? **CAT 2005**

1) 5 2) 3 3) 2 4) 6

The expression g(x+1) + g(x−1) = g(x) can be rewritten as g(x+1) = g(x) – g(x−1) which actually boils down to saying that a given term is the difference of the previous two terms (which is exactly the same situation as in the previous problem, just expressed differently). Now we have already done the requisite hard work there, so we are immediately in a position to state that this sequence will repeat after every 6 steps or in other words g(x+6) will be the same as g(x) i.e. p = 6. **Hence (4)**

SEQUENCES – 20

Q. Consider a sequence where the nth term, $t_n = \frac{n}{n+2}$, n = 1, 2, 3 …. Then the value of $t_3 \times t_4 \times t_5 \times \ldots \times t_{53}$ equals: **CAT 2006**

1] $\frac{2}{495}$ 2] $\frac{2}{477}$ 3] $\frac{12}{55}$ 4] $\frac{1}{1485}$ 5] $\frac{1}{2970}$

This is a relatively simple question (though some people found it unpalatable because they could not find a "concept" or "formula" to relate it to!) – just substitute and you're good to go! Since we are given that $t_n = \frac{n}{n+2}$, we can easily write the given value as:

$t_3 \times t_4 \times t_5 \times \ldots \times t_{53}$

$= \frac{3}{\cancel{5}} \times \frac{4}{\cancel{6}} \times \frac{\cancel{5}}{\cancel{7}} \times \frac{\cancel{6}}{\cancel{8}} \times \ldots \times \frac{\cancel{51}}{\cancel{53}} \times \frac{\cancel{52}}{54} \times \frac{\cancel{53}}{55}$

$= \frac{3 \times 4}{54 \times 55} = \frac{2}{495}$. **Hence (1)**

Q. Consider the set S = {1, 2, 3, …, 1000}. How many arithmetic progressions can be formed from the elements of S that start with 1 and end with 1000 and have at least 3 elements? **CAT 2006**

1] 3 **2] 4** **3] 6** **4] 7** **5] 8**

Again, this was a problem which was not all that difficult, but was hard to classify into a standard format and hence led to much confusion. The key point is that we have here an AP with the first term $T_1 = 1$ and the last or n^{th} term $t_n = 1000$. If we recollect the standard form for the nth term of an AP i.e. $\mathbf{T_n = T_1 + (n-1)d}$ then we will see that here $\mathbf{1000 = 1 + (n-1)d}$ i.e. $\mathbf{(n-1)d = 999}$

This means that d has to be a factor of 999 which gives possible values of d as 1, 3, 9, 27, 37, 111, 333 and 999. However, if n = 999 then we will have the AP of 2 elements only (1, 1000) and hence we neglect this case, leaving us with 7 possible APs. **Hence (4)**

Q. If the sum of the first 11 terms of an Arithmetic Progression equals that of the first 19 terms, then what is the sum of the first 30 terms? **CAT 2004**

1) 0 **2) -1** **3) 1** **4) Not unique**

This, again, is similar to a problem we had encountered earlier (see the post Sequences – 4). Since the sum of the first 11 terms equals that of the first 19 terms, the sum from the 12th to the 19th terms must be 0. Thus the sum from the 11th to the 20th term, or from the 10th to the 21st….or from the 1st to the 30th must be 0. **Hence (1)**

SEQUENCES – 21

Let $a_1 = p$ and $b_1 = q$, where p and q are positive quantities. Define **CAT 2007**

$$a_n = pb_{n-1}, \quad b_n = qb_{n-1}, \quad \text{for even } n > 1$$

$$\text{and} \quad a_n = pa_{n-1}, \quad b_n = qa_{n-1}, \quad \text{for odd } n > 1$$

1. Which of the following best describes $a_n + b_n$ for even n?

(1) $q(pq)^{\frac{1}{2}n-1}(p+q)$ (2) $qp^{\frac{1}{2}n-1}(p+q)$ (3) $q^{\frac{1}{2}n}(p+q)$
(4) $q^{\frac{1}{2}n}(p+q)^{\frac{1}{2}n}$ (5) $q(pq)^{\frac{1}{2}n-1}(p+q)^{\frac{1}{2}n}$

2. If $p = \frac{1}{3}$ and $q = \frac{2}{3}$, then what is the smallest odd n such that $a_n + b_n < 0.01$?

(1) 7 (2) 13 (3) 11 (4) 9 (5) 15

1. Let us observe the pattern involved here:

$a_1 = p$ and $b_1 = q$

Applying the given rules we will find that

$a_2 = pq$ and $b_2 = q^2$

$a_3 = p^2q$ and $b_3 = pq^2$

$a_4 = p^2q^2$ and $b_4 = pq^3$

$a_5 = p^3q^2$ and $b_5 = p^2q^3$

$a_6 = p^3q^3$ and $b_6 = p^2q^4$

….and so on.

So for even n, we get

$a_n = p^{\frac{n}{2}}q^{\frac{n}{2}}$ and $b_6 = p^{\frac{n}{2}-1}q^{\frac{n}{2}+1}$

So $a_n + b_n = p^{\frac{n}{2}}q^{\frac{n}{2}} + p^{\frac{n}{2}-1}q^{\frac{n}{2}+1}$

which can be written as $p^{\frac{n}{2}}q^{\frac{n}{2}} + p^{\frac{n}{2}-1}q^{\frac{n}{2}+1}$ i.e.$q(pq)^{\frac{1}{2}n-1}(p+q)$. **Hence (1)**

2. Let us again look at the pattern observed above.

For odd n, we have

$a_n = p^{\frac{n+1}{2}}q^{\frac{n-1}{2}}$ and $b_6 = p^{\frac{n-1}{2}}q^{\frac{n+1}{2}}$

So $a_n + b_n = p^{\frac{n+1}{2}}q^{\frac{n-1}{2}} + p^{\frac{n-1}{2}}q^{\frac{n+1}{2}} = (p+q)(p^{\frac{n-1}{2}}q^{\frac{n-1}{2}})$

Now if $p = \frac{1}{3}$ and $q = \frac{2}{3}$, $p + q = 1$ and so

$a_n + b_n = (p^{\frac{n-1}{2}}q^{\frac{n-1}{2}})$

So $a_3 + b_3 = (pq) = \frac{2}{9} > 0.01$

$a_5 + b_5 = (p^2q^2) = \frac{4}{81} > 0.01$

$a_7 + b_7 = (p^3q^3) = \frac{8}{729} > 0.01$

$a_9 + b_9 = (p^2q^2) = \frac{16}{6561} < 0.01$. **Hence (4)**

SEQUENCES – 22

Q. If $a_1 = 1$ and $a_{n+1} - 3a_n + 2 = 4n$ for every positive integer n, then a_{100} equals **CAT 2005**

1) $3^{99} - 200$ 2) $3^{99} + 200$ 3) $3^{100} - 200$ 4) $3^{100} + 200$

We are told that $a_{n+1} - 3a_n + 2 = 4n$. This can be rearranged as $a_{n+1} = 3a_n + 4n - 2$. Looking at this we can see that a given term is defined in terms of exactly one previous term (a_{n+1} in terms of a_n) and hence knowing any one term we could generate all further terms.

However, it is difficult to solve this in a rigorous mathematical manner (at least, given the merciless time constraints in an exam like the CAT!). So a better approach would be to look for patterns. An important thing is to realise that the answer choices all follow a pattern. Let us try and convert those patterns in terms of n, given that the answer choices represent a_{100}.

Option 1 ➔ If $a_{100} = 3^{99} - 200$, then $a_n = 3^{n-1} - 2n$
Option 2 ➔ If $a_{100} = 3^{99} + 200$, then $a_n = 3^{n-1} + 2n$
Option 3 ➔ If $a_{100} = 3^{100} - 200$, then $a_n = 3^n - 2n$
Option 4 ➔ If $a_{100} = 3^{100} + 200$, then $a_n = 3^n + 2n$

Now given that $a_1 = 1$ and $a_{n+1} = 3a_n + 4n - 2$ we can find that
$a_2 = 3 * 1 + 4 * 1 - 2 = 5$
$a_3 = 3 * 5 + 4 * 2 - 2 = 21$ and so on.
Plugging in n = 3 into the four options, we find that
$3^{n-1} - 2n = 3$ $3^{n-1} + 2n = 15$ $a_n = 3^n - 2n = 21$ $a_n = 3^n + 2n = 33$
So only the third option matches the requirement. **Hence (3)**

Note: A really alert candidate might plug in n = 1 and see that the options give
$3^{n-1} - 2n = -1$ $3^{n-1} + 2n = 3$ $a_n = 3^n - 2n = 1$ $a_n = 3^n + 2n = 5$
and hence mark (3) right away! After all, if the statement is true for some general "n", it has to be true for each and every value of n…including n = 1.

SEQUENCES – 23

Q. A function f(x) satisfies f(1) = 3600, and f(1) + f(2) + ... + f(n) = n^2f(n), for all positive integers n > 1. What is the value of f(9) ? **CAT 2007**

(1) 80 **(2) 240** **(3) 200** **(4) 100** **(5) 120**

Let us look for the pattern:

f(1) is given to be 3600

Now f(1) + f(2) = 2^2 f(2)
∴ 3600 + f(2) = 4 f(2)
∴ f(2) = 3600/3 = 1200

f(1) + f(2) + f(3) = 3^2 f(3)
∴ 3600 + 1200 + f(3) = 9 f(3)
∴ f(3) = 4800/8 = 600

f(1) + f(2) + f(3) + f(4) = 4^2 f(4)
∴ 3600 + 1200 + 600 + f(4) = 16 f(4)
∴ f(4) = 5400/15 = 360

Let us observe the pattern so far. We see that f(1), f(2), f(3), f(4)…. are 3600, 1200, 600, 360…i.e. $\frac{3600}{1}, \frac{3600}{3}, \frac{3600}{6}, \frac{3600}{10} \ldots$

But 1, 3, 6, 10 is a standard series of summations. So the next term f(5) will be $\frac{3600}{15}$ = 240 (since $\sum 5$ = 15), and the required ninth term f(9) will be $\frac{3600}{45}$ = 80 (since $\sum 9$ = 45). **Hence (1)**

SEQUENCES - 24

Q. A group of 630 children is arranged in rows for a group photograph session. Each row contains three fewer children than the row in front of it. What number of rows is not possible?

CAT 2007

1] 3 2] 4 3] 5 4] 6 5] 7

Probably the simplest way to do this would be to simply check the options.

Suppose there are 3 rows, with the last row containing n people. The other rows would have n+3 and n+6 people and so the total number of people should be n + n + 3 + n + 6 = 3n + 9. Putting 3n + 9 = 630 gives us 3n = 621 which gives n = 207.

If there are 4 rows, the 4th will add n + 9 giving a total of 4n + 18 = 630 => 4n = 612 ⇒ n = 153

If there are 5 rows, the 5th will add n + 12 giving a total of 5n + 30 = 630 => 5n = 600 ⇒ n = 120

If there are 6 rows, the 6th will add n + 15 giving a total of 6n + 45 = 630 => 6n = 585 ⇒ n = 97.5

If there are 7 rows, the 7th will add n + 18 giving a total of 7n + 63 = 630 => 7n = 567 ⇒ n = 81

Since the number of people has to be an integer, 6 rows is not possible. **Hence (4)**

Q. On January 1, 2004 two new societies, S_1 and S_2, are formed, each with n members. On the first day of each subsequent month, S_1 adds b members while S_2 multiplies its current number of members by a constant factor r. Both the societies have the same number of members on July 2, 2004. If b =10.5n, what is the value of r?

CAT 2004

1] 2.0 2] 1.9 3]1.8 4]1.7

From Jan 1 to July 2, a total of 6 "time periods" have passed.

So S_1 will have n + 6b members while S_2 will have $n*r^6$ members.

Since b = 10.5n, n + 6b = n + 6*10.5 n = n + 63n = 64 n

Hence $n*r^6 = 64n \Rightarrow r^6 = 64$ i.e. r = 2.

Hence (1)

This brings us to the end of our series of posts on sequences. Now let us take a break from maths and look instead at logic for a while (but never fear, we'll return to maths eventually)

Since interview season is in the air, we thought we should look at some classic and typical puzzles which often turn up in interviews (whether MBA interviews or job interviews). The way we plan to go about it is, in every post there will be a puzzle, which you can think about and try to solve. The next post will contain the solution – as well as the next puzzle. So here goes the first in the series:

Q. There are 100 coins in a pile. Jack and Jill take turns to pick coins from the pile. In a turn, each person can pick a minimum of 1 and a maximum of 7 coins. The person who picks the last coin loses the game. Jack offers Jill the first move. Assuming that both of them play logically, how many coins should Jill pick in order to guarantee her win?

RECURSIVE EXPRESSIONS 1

Algebra: Recursive expressions

Q. Find the value of x if $x = \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}}$

1] $\frac{-3-\sqrt{13}}{2}$ **2]** $\frac{-3+\sqrt{13}}{2}$ **3]** $\frac{\sqrt{13}}{2}$ **4] Both (1) and (2)**

Now the standard way of solving such a problem is to make an algebraic equation out of it. For example, in this question, one could start by first noting that x should be positive and then writing:

$$x = \cfrac{1}{3+\left(\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}\right)}$$

But since the part in the bracket is the same as x, we could write

$x = \frac{1}{3+x}$ which is equivalent to

$x^2 + 3x - 1 = 0$ which when solved gives

$x = \frac{-3\pm\sqrt{13}}{2}$

But as we had earlier noted, x has to be positive and hence $x = \frac{-3+\sqrt{13}}{2}$

However, this method can be time-consuming and error-prone, especially for those who are not that fond of algebra. So let's look at an alternative:

Without even writing anything, we could roughly estimate the value of x as follows:

$x = \frac{1}{3+\text{(something)}}$ which means that x is less than $\frac{1}{3}$ i.e. 0.333…, but also

$x = \cfrac{1}{3+\cfrac{1}{3+(\text{something})}} = \cfrac{1}{3+(\text{less than } \frac{1}{3})} = \cfrac{1}{\text{less than } \frac{10}{3}}$ ➔ x is greater than $\frac{3}{10}$ i.e. 0.3

Now all we need to do is to check which answer choice lies between 0.3 and 0.333. :

Noting that $\sqrt{13}$ is approximately 3.6 (since $36^2 = 1296$) we get

1] $\frac{-3-\sqrt{13}}{2} \cong -3.3$ 2] $\frac{-3+\sqrt{13}}{2} \cong 0.3$

3] $\frac{\sqrt{13}}{2} \cong 1.8$ 4] Both (1) and (2) ≅ - 3.3 or 0.3

And hence the answer is (2).

While this takes much space to write out, it takes (with practice) less than a minute to apply and can be used to solve a variety of similar problems, as we shall see in the next post.

RECURSIVE EXPRESSIONS 2

In today's post we'll continue from where we left off last time:

First, let us consider a sum which would require more than a quadratic equation to solve:

Q. Find the value of 'x' if x = $\dfrac{1}{2+\dfrac{1}{4+\dfrac{1}{2+\dfrac{1}{4+\cdots}}}}$

1] $\sqrt{6}-2$ **2]** $2-\sqrt{2}$ **3]** $2+\sqrt{2}$ **4]** $\sqrt{6}+2$

We could start by writing x = $\dfrac{1}{2+\dfrac{1}{4+(\dfrac{1}{2+\dfrac{1}{4+\cdots}})}}$ $= \dfrac{1}{2+\dfrac{1}{4+x}}$ but then it starts getting complicated.

Instead, we could say x is less than $\frac{1}{2}$ = 0.5 and also x is $\dfrac{1}{\text{less than } 2\frac{1}{4}}$ i.e. greater than $\frac{4}{9}$ i.e. 0.444

Now looking at the answer choices: $\sqrt{6} \cong 2.45$ and $\sqrt{2} \cong 1.41$

1] $\sqrt{6}-2 \cong 0.45$ 2] $2-\sqrt{2} \cong 0.59$ 3] $2+\sqrt{2} \cong 3.41$ 4] $\sqrt{6}+2 \cong 4.45$

Hence (1) is clearly the answer.

Here's one more, from CAT 2005:

Q. Let x = $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-\cdots}}}}$ **to infinity, then x equals**

1] 3 **2]** $\dfrac{\sqrt{13}-1}{2}$ **3]** $\dfrac{\sqrt{13}+1}{2}$ **4]** $\sqrt{13}$

Here, we could start by saying that x has to be greater than $\sqrt{4}$ i.e. 2. Also x has to be less than $\sqrt{4+\sqrt{4}}$ i.e. $\sqrt{6}$. Hence it should lie between 2 and approximately 2.45. Checking the answer choices (noting that $\sqrt{13}$ is approximately 3.6):

1] 3 2] $\dfrac{\sqrt{13}-1}{2} \cong 1.3$ 3] $\dfrac{\sqrt{13}+1}{2} \cong 2.3$ 4] $\sqrt{13} \cong 3.6$

We can immediately see that only option (3) lies within the required range. Hence (3)

THE SQUARE ROOT FUNCTION

The Square Root Function

Q. Is there a difference between the statements (a) $x^2 = 16$ and (b) $x = \sqrt{16}$?

When we say $x^2 = 16$, there could be two values of x i.e. ±4. However, when we say $x = \sqrt{16}$, then the convention followed is that there is only one value of x i.e. +4. In other words, the square root function returns only a positive value. At first glance, this seems counter-intuitive, but there are several reasons why this convention is followed. Here are some ways of looking at it, which might help clear up the mystery:

Let me try to explain. There are several ways of looking at it:

a) Simple logic: If we see an equation $x^2 = 7$, then we say $x = \pm\sqrt{7}$, right? Why would we do so if $\sqrt{7}$ was already implying both the plus and minus values?

b) Function Approach: A function, to be properly defined, should only give one output for a given input. So to define the function "square root", we have to take only one value as output and that is the positive one by convention.

c) Algebra Logic: If we say $x^2 = 25$, it is a quadratic and so should have 2 roots. However, the equation $x = \sqrt{25}$ is a linear equation and hence should have only one root.

There is another related point which is quite relevant for us: in problems where we solve by squaring, very often we introduce an extra (and incorrect) root as a result, which we have to manually remove.

For example, we saw in the last post that when we try to solve the equation $x = \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}}$ by making it $x = x = \dfrac{1}{3+x}$ and creating a quadratic, we will get two values $x = \dfrac{-3\pm\sqrt{13}}{2}$ but quite clearly x is a positive quantity and hence we should only take $x = \dfrac{-3+\sqrt{13}}{2}$. The other is an extra, spurious root which was introduced when we squared the equation.

(If you are not convinced try the following chain of logic and see for yourself that it is flawed:
Let us solve the equation $x = 4$ by squaring:
$x = 4$
Hence $x^2 = 16$
Hence $x = \pm 4$
So $4 = -4$
....which of course is absurd!)

ALGEBRA – 1

Let's spend a while looking at some simple but interesting question-types in Algebra:

Q. If x + y = 100, what is the maximum value of xy? (where x and y are non-negative)

Note that if x + y = 100, the average of x and y is 50. So suppose x is (50 + k). Then y will be (50 – k). Therefore xy will be $(50 + k)(50 - k) = 2500 - k^2$.
Now when will this be maximised? Clearly when k is 0, since k^2 cannot go below 0.
But if k is 0, then both x and y will become 50. Hence we observe that the maximum occurs when the numbers are equal.

Q. If xy = 100, what is the minimum value of x + y? (where x and y are non-negative)

We can use the logic of the above question (in reverse) to realise that the minimum value of x + y should occur when x and y are equal i.e. both are 10. Thus x + y will be 20.

We could also have taken a different approach; we know that the Arithmetic Mean of a set of numbers is always greater than or equal to their Geometric Mean (becoming equal when the numbers are equal). So if we know that xy = k then we can say that $\frac{x+y}{2} \geq \sqrt{xy}$ or $x + y \geq 2\sqrt{xy}$, which tells us that the minimum value will occur when $x = y = \sqrt{k}$, and that this minimum value will be $2\sqrt{k}$.

Q. If x + y + z = 24, what is the maximum value of xyz? (where x, y and z are non-negative)

Logically, as above, we can conclude that the maximum value of xyz will occur when x = y = z = 8 and the value will be 8^3 i.e. 512

Q. If xyz = 20, what is the minimum value of x + y + z? (where x, y and z are non-negative)

Applying the logic above, we can easily see that the minimum value will occur when $x = y = z = \sqrt[3]{20}$ and hence $x + y + z = 3\sqrt[3]{20}$.

Thus we can draw the following conclusions:

For a group of positive numbers, if their sum is fixed, then their product will be maximum when all the numbers are equal.

If a + b + c…. (n numbers) = k then max (a * b * c…) = $\left(\frac{k}{n}\right)^n$

For a group of positive numbers, if their product is fixed, then their sum will be minimum when all the numbers are equal

If a * b * c…. (n numbers) = k then min (a + b + c + …) = $n\sqrt[n]{k}$

ALGEBRA – 2

Let us apply the logic derived last time to slightly more complicated problems:

Q. If x + y + z = 10, what is the maximum value of x^2y^2z? (where x, y and z are non-negative)

1] 100 **2] 625** **3] 512** **4] $\frac{100000}{243}$**

The thing to note here is that if we want to maximise a term like x^2y^2z we could equally well maximise $2*x^2y^2z$ or $5*x^2y^2z$ or $\frac{x^2y^2z}{6}$.

So since we want x twice, let us write x as $\frac{x}{2}+\frac{x}{2}$, and similarly treat y as $\frac{y}{2}+\frac{y}{2}$

Thus x + y + z = 10 becomes $\frac{x}{2}+\frac{x}{2}+\frac{y}{2}+\frac{y}{2}+z=10$

Then we can say that $\frac{x}{2}*\frac{x}{2}*\frac{y}{2}*\frac{y}{2}*z$ i.e $\frac{x^2y^2z}{16}$ will be maximised when $\frac{x}{2}=\frac{y}{2}=z$

Now since x + y + z = 10, we can say 2z + 2z + z = 10 or z = 2. Thus y = 4 and x = 4.

The maximum value of x^2y^2z will be 4^2*4^2*2 = **512. Hence (3)**

Q. If 5x + 3y + 4z = 60, what is the maximum value of xy^3z^2? (where x, y and z are non-negative)

1] 80000 **2] 50000** **3] 65536** **4] 32768**

Taking a similar approach to that of the above problem, we can say that if we want to maximise xy^3z^2 then we will have to write 5x + 3y + 4z = 60 as

$5x+\frac{3y}{3}+\frac{3y}{3}+\frac{3y}{3}+\frac{4z}{2}+\frac{4z}{2}=60$ and so xy^3z^2 will be maximised when

$5x*\frac{3y}{3}*\frac{3y}{3}*\frac{3y}{3}*\frac{4z}{2}*\frac{4z}{2}$ is maximised, which will happen when

$5x=\frac{3y}{3}=\frac{4z}{2}$.

Now a little jugglery will give us x = 2, y = 10 and z = 5 which means the maximum value of xy^3z^2 will be $2*10^3*5^2$ = **50000. Hence (2)**

Q. What is the minimum value of a + b + c, given that a^2bc^4 = 1024? (where a, b and c are non-negative)

1] 7 **2] 6** **3] 4** **4] 8**

Writing a + b + c as $\frac{a}{2}+\frac{a}{2}+b+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}$,

We can see that the minimum will occur when $\frac{a}{2}=b=\frac{c}{4}$.

Substituting these values we find that $a^2bc^4=(2b)^2b(4b)^4=1024$ which gives b = 1.

Hence a = 2 and c = 4.

Thus the minimum value of a + b + c will be 2 + 1 + 4 = **7. Hence (1)**

ALGEBRA – 3

Let us now try and approach the type of sum which involves a positive number and its reciprocal.

Q. What is the minimum value of $x + \frac{1}{x}$ (given that $x > 0$)

1] 1.75 **2] 2** **3] 3** **4] None of the above**

Note that x and $\frac{1}{x}$ are reciprocals of each other and hence their product will be 1.

Since the product is fixed, the sum will be minimum when $x = \frac{1}{x}$, which of course will happen at $x = 1$.

Thus the minimum value would be $1 + 1 =$ **2. Hence (2)**

(Again, alternatively, we could have approached this from an AM – GM approach; the AM of x and $\frac{1}{x}$ must be greater than or equal to their GM, which is 1. Hence the total of x and $\frac{1}{x}$ must be at least 2)

Q. If $x + \frac{1}{x} = 4$ then find the value of $x^2 + \frac{1}{x^2}$

1] 14 **2] 15** **3] 16** **4] 18**

Note that $(a + b)^2 = a^2 + 2ab + b^2$. So if a and b are reciprocals, ab will be 1. Let us use this fact.

$x + \frac{1}{x} = 4$. Squaring both sides, we get

$x^2 + 2 * \cancel{x} * \frac{1}{\cancel{x}} + \frac{1}{x^2} = 16$

i.e. $x^2 + \frac{1}{x^2} =$ **14. Hence (a)**

Q. If $x + \frac{1}{x} = 4$ then find the value of $x^3 + \frac{1}{x^3}$

1] 50 **2] 64** **3] 60** **4] 52**

$x + \frac{1}{x} = 4$. Cubing both sides, we get

Squaring both sides, we get

$x^3 + 3 * x + 3 * \frac{1}{x} + \frac{1}{x^3} = 64$. Re-arranging terms, we get

$x^3 + \frac{1}{x^3} + 3 * (x + \frac{1}{x}) = 64$ but we know that $x + \frac{1}{x} = 4$. Hence

$x^3 + \frac{1}{x^3} + 3 * 4 = 64$ which gives us

$x^3 + \frac{1}{x^3} =$ **52. Hence (4)**

ALGEBRA – 4

The logic we saw in the last post can be extended to a variety of other problems:

Q. If $x + \frac{1}{x} = 4$ then find the value of $x^4 + \frac{1}{x^4}$

1] 256 2] 196 3] 194 4] 252

$x + \frac{1}{x} = 4$. Squaring both sides, we get

$x^2 + 2 + \frac{1}{x^2} = 16$

i.e. $x^2 + \frac{1}{x^2} = 14$. Again squaring both sides, we get

$x^4 + 2 + \frac{1}{x^4} = 196$

i.e. $x^4 + \frac{1}{x^4}$ = **194. Hence (3)**

Q. If $x + \frac{1}{x} = 4$ then find the value of $x^5 + \frac{1}{x^5}$

1] 1024 2] 729 3] 724 4] 720

$x + \frac{1}{x} = 4$. Using binomial expansion and raising both sides to the fifth power, we get

$x^5 + \frac{1}{x^5} + 5 * (x^3 + \frac{1}{x^3}) + 10 * (x + \frac{1}{x}) = 1024$ but we know that $x + \frac{1}{x} = 4$ and $x^3 + \frac{1}{x^3} = 52$. Hence

$x^5 + \frac{1}{x^5} + 5 * 52 + 10 * 4 = 1024$

i.e. $x^5 + \frac{1}{x^5}$ = **724. Hence (3)**

Q. If $x + \frac{1}{x} = 4$ then find the value of $x^6 + \frac{1}{x^6}$

1] 2704 2] 2700 3] 2706 4] 2702

$x + \frac{1}{x} = 4$. Cubing both sides, we get

$x^3 + \frac{1}{x^3} + 3 * (x + \frac{1}{x}) = 64$ but we know that $x + \frac{1}{x} = 4$. Hence

i.e. $x^3 + \frac{1}{x^3} = 52$. Squaring both sides, we get

$x^6 + 2 + \frac{1}{x^6} = 2704$.

i.e. $x^6 + \frac{1}{x^6}$ = **2702. Hence (4)**

ALGEBRA – 5

Another interesting question type involving reciprocals, which can be neatly solved algebraically:

Q. Find the number of positive integer solutions for $\frac{20}{x} + \frac{30}{y} = 1$.

$\frac{20}{x} + \frac{30}{y} = 1.$

We can multiply this throughout by xy to get

$20y + 30x = xy$

$xy - 30x - 20y = 0$

Adding 20 x 30 = 600 to both sides, we get

$(x - 20)(y - 30) = 600$

Now it will suffice to just list out the factors of 600. We know that $600 = 2^3 * 3 * 5^2$ which means that it will have (3+1)(1+2)(2+1) = 4 * 2 * 3 = 24 factors. Each of these could be a value for (x - 20) and would yield a corresponding value for (y – 30) and so we should be able to find a total of **24 combinations of (x, y)**

For example if we take (x – 20)(y – 30) = 1 * 600 we will get a solution (21, 630). Similarly if we take (x – 20)(y – 30) = 2 * 300 we will get a solution (22, 330) and so on.

Note: If we just want integer solutions, we will get another 24 cases where (x – 20)(y – 30) = -1 * -600 or -2 * -300 and so on. So there should be another 24 solutions. However, there is a catch; one case would be (x – 20)(y – 30) = -20 * -30 which would give x = y = 0 which would not be a valid solution to the given equation. Hence there would be 24 + 24 – 1 = 47 integer solutions.

Q. Find the number of positive integer solutions for $\frac{5}{x} + \frac{8}{y} = \frac{1}{6}$.

$\frac{5}{x} + \frac{8}{y} = \frac{1}{6}$

Multiplying throughout by 6xy gives us

$30y + 48x = xy$

$xy - 48x - 30y = 0$

$(x - 30)(y - 48) = 30 * 48 = 1440$

Now $1440 = 2^5 * 3^2 * 5$ has 36 factors and hence there will be **36 positive integer solutions**

Again, if we are seeking integer (rather than positive integer) solutions, we will find that there are 71 such solutions.

ALGEBRA – 6

Q. Find the number of positive integer solutions for $\frac{12}{x} - \frac{8}{y} = \frac{1}{5}$.

Multiplying throughout by 5xy, we get

$60y - 40x = xy$

$xy + 40x - 60y = 0$

$(x - 60)(y + 40) = -2400$

$(60 - x)(y + 40) = 2400$

Now $2400 = 2^5 * 3 * 5^2$ has 36 factors and hence there will be 36 solutions.

However, this will include cases where $y + 40 > 0$ but $y \leq 0$ viz. $(y + 40) = 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 25, 30, 32$ and 40. Removing these 17 cases leaves us with **19 positive integer solutions.**

Note: In this case, there would be 36 more solutions with $(60 - x)$ and $(y + 40)$ being negative numbers so we would have 71 integer solutions in all.

Sometimes there might be additional conditions, in which case we might be able to find more efficient alternative approaches; consider, for example, this question from CAT 2007:

Q. How many pairs of positive integers satisfy $\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$ where n is an odd integer less than 60?

1) 6 **2) 4** **3)7** **4) 5** **5) 3**

We can use the method above, which would yield $(m - 12)(n - 4) = 48$, and then list out cases.

Alternatively, we could multiply the given equation by 12 and note that

$\frac{12}{m} + \frac{48}{n} = 1.$

Now if n is less than 48, m would be negative. So n has to be greater than 48. Also, we are told that n is odd and less than 60. So n can be 49, 51, 53, 55, 57 or 59.

$\frac{12}{m} + \frac{48}{49} = 1 \rightarrow \frac{12}{m} = \frac{1}{49} \rightarrow$ m is an integer

$\frac{12}{m} + \frac{48}{51} = 1 \rightarrow \frac{12}{m} = \frac{1}{17} \rightarrow$ m is an integer

$\frac{12}{m} + \frac{48}{53} = 1 \rightarrow \frac{12}{m} = \frac{5}{53} \rightarrow$ m is **not** an integer

$\frac{12}{m} + \frac{48}{55} = 1 \rightarrow \frac{12}{m} = \frac{7}{55} \rightarrow$ m is **not** an integer

$\frac{12}{m} + \frac{48}{57} = 1 \rightarrow \frac{12}{m} = \frac{3}{19} \rightarrow$ m is an integer

$\frac{12}{m} + \frac{48}{59} = 1 \rightarrow \frac{12}{m} = \frac{11}{59} \rightarrow$ m is **not** an integer

Hence there will be **3 solutions**.

ALGEBRA – 7

This week, let's look at linear equations where the variables outnumber the equations.

Consider the equation $x + y = 10$. Can we find solutions (x, y) to this? Yes, but listing them will be impossible; for every value of x, there will be a corresponding $y = 10 - x$ so that we will get literally infinite solutions. Even if we restrict x and y to be integers, there will be infinite possible solutions.

However, on imposing still more stringent restrictions, we can arrive at a question which will have a finite, limited solution set. For example, we could seek positive integer solutions (i.e. x and y are Natural Numbers), or as is more commonly required, non-negative integer solutions (i.e. x and y are Whole Numbers).

Q. Find the number of non-negative integer solutions to $x + y = 10$

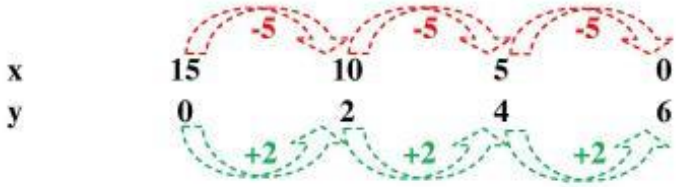
Clearly neither x nor y can exceed 10 (as then the other one would go negative), so we can start listing values of x as 0, 1, 2, till 10 and for each we will get a corresponding value of y 10, 9, 8,….0. Thus we will have 11 pairs viz (0, 10), (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1) and (10, 0).

It gets a bit trickier if x and y have coefficients; let's try and understand the pattern:

Q. Find the number of non-negative integer solutions to $2x + 5y = 30$

Looking at $2x + 5y = 30$, we can see that x cannot go beyond 15 and y cannot go beyond 6. But not all values can occur; some values of x between 0 and 15 will give non-integral values for y and vice versa.

Let's first try to do this by trial and error. Let's start with y as there are less values to check
If $y = 0$, $2x = 30 \Rightarrow x = 15$
If $y = 1$, $2x = 25 \Rightarrow$ x is non-integral
If $y = 2$, $2x = 20 \Rightarrow x = 10$
If $y = 3$, $2x = 15 \Rightarrow$ x is non-integral
If $y = 4$, $2x = 10 \Rightarrow x = 5$
If $y = 5$, $2x = 5 \Rightarrow$ x is non-integral
If $y = 6$, $2x = 0 \Rightarrow x = 0$
If $y = 7$, $2x = -5 \Rightarrow$ x is negative
So in this manner we can identify all possible solutions, but it is obviously cumbersome. Can we identify a pattern? Look at the solutions obtained



From one viable solution to the next, x changes by 5 while y changes by 2 in the opposite direction. But from where do we get these numbers, 2 and 5? They are the coefficients of y and x respectively, in the original equation! So x changes by the coefficient of y, while y changes by the coefficient of x!

ALGEBRA – 8

Last time we identified the pattern of the solutions to equations like $2x + 5y = 30$. Let us try and understand why this kind of pattern occurs.

Suppose there is an ordered pair of values for x and y, say 10 and 2, for which we have a solution to $2x + 5y = 30$.

Let the next solution for x occur at $(10 – k)$ and for y at $(2 + m)$. Hence $2(10 – k) + 5(2 + m) = 30$
Hence $30 – 2k + 5m = 30 \Rightarrow 5m = 2k$ = an integer value (since m and k are integers).

Now since 2 and 5 are co-prime, the smallest viable integer value is their LCM i.e. 10, and hence the smallest possible values for the "changes" k and m are 5 and 2 respectively.

Thus if we have an equation $ax + by = c$ where a, b, c are constants and a and b are co-prime, then viable solutions for x will change in steps of "b" and those for y will change in steps of "a". So for a question like

Q.Find the number of non-negative integer solutions to $3x + 10y = 200$

we could try to first find one solution (in this case, 200 is not divisible by 3 but is divisible by 10 so we can find a solution $y = 20$ and hence $x = 0$ just by observation) and work from there as follows:

x	**0**	**10**	**20**	**30**	**40**	**50**	**60**
y	**20**	**17**	**14**	**11**	**8**	**5**	**2**

Note that we can extend this logic to find all integers, not just non-negative cases; for example we could extend x into the negative with solutions like (-10, 23), (-20, 26), (-30, 29)…and similarly y could go negative giving us solutions like (70, -1), (80, -4), (90, -7)…

What if there is no obvious solution?

Q. Find the number of non-negative integer solutions to $11x + 7y = 100$

In this case we will have to do some trial and error to locate one solution. However, the question arises; should we do trial and error in x or in y? A little thought will show us that the variable with the larger coefficient makes most sense to start with; for example in the above equation we would rather start with x. The reason is simple – here x has a solutions every 7 steps and y every 11, so if we start our trial and error we know that within 7 steps we will find a solution while the same procedure for y could take up to 11. Applying this, and putting $x = 1, 2, 3…$ we find that at $x = 4$, we get an integer value for $y = 8$. Now reducing x by 7 or y by 11 yields a negative value and hence there is just one non-negative integer solution in this case.

Similarly if we try to solve, say, $13x + 3y = 200$ we will find that there are 5 non-negative integer solutions: (2, 58), (5, 45), (8, 32), (11, 19) and (14, 6).

ALGEBRA – 9

What if the constant in the equation is an extremely huge number, but we want to just find how many solutions exist (rather than actually listing them out)?

Q. Find the number of non-negative integer solutions to 5x + 12y = 10000

Obviously in this case we cannot list out all the individual values! Still we can figure out how many values will be there without too much trouble. One obvious solution is by putting y = 0 which gives x = 2000. Now y will have to increase, so x must decrease in steps of 12. So we just need to see how many times we can subtract 12 from 2000 before it goes negative…

Now $\frac{2000}{12}$ is 166.66…and so we will have 166 more solutions for a total of 167.

Q. Find the number of non-negative integer solutions to 3x + 7y = 2000

We can quickly find one solution as (662, 2). After that we check $\frac{662}{7}$ which implies 94 more solutions, for a total of 95.

Let us now look at some variants where we need to take care

Q. Find the number of non-negative integer solutions to 3x + 9y = 90

Clearly a solution for this would be (0, 10). Other solutions would exist at (9, 7), (18, 4) and (27, 1). Hence there should be 4 solutions. But look carefully – if we put y = 0, we get x = 30 and hence (30, 0) should also be a solution! Further examination will reveal still other solutions….

What are we missing here? The problem is that the coefficients, 3 and 9, are not co-prime. Which means their LCM is not 3 * 9 = 27, but instead is just 9. So if we need to change 3x and 9y by the LCM (as discussed in yesterday's post), we need to change x and y by 3 and 1 respectively.

To avoid confusion, perhaps the simplest thing to do in such a question would be to just divide the entire equation by the common factor (3 in this case). This would give us x + 3y = 30, which will yield, with minimal effort, 11 solutions (from (0, 10), (3, 9)….till (30, 0))

Q. Find the number of non-negative integer solutions to 5x – 2y = 40

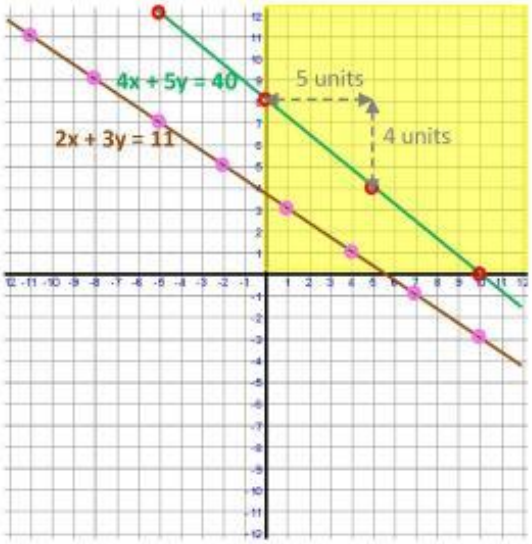
Here again we can immediately see a solution where x = 8 and y = 0. But if we try to find the next solution by increasing y by 5, what will be the effect on x? We will find that x will decrease by -2 (i.e. it will increase by 2!) to give 10, so the next solution will be (10, 5). The next one after that will be (12, 10) and so on. So then, when will we be able to stop? The answer, actually, is "never" – there will be **infinite** non-negative integer solutions to this question!

In the next post we shall try to understand why we get infinite solutions in the case of an ax – by = k form even though we get finite solution in the case of ax + by = k.

ALGEBRA – 10

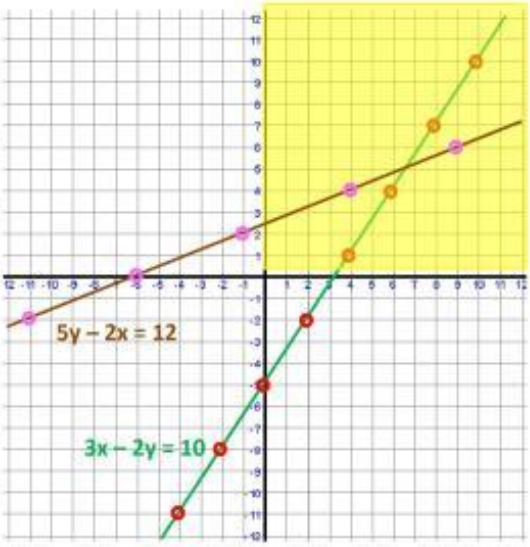
Let us examine graphically the solutions to equations of the form $ax \pm by = k$ to understand their behaviour under various circumstances:

Consider the case of **$4x + 5y = 40$**



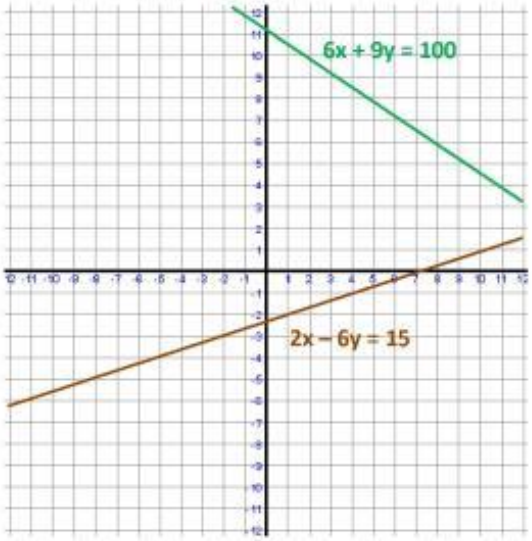
If we try to plot this line, we will find that it has a negative slope and will appear as shown by the green line alongside. The red dots indicate the points where both x and y are integers; in this case we can clearly see that since there is only a finite portion of the line lying within the first quadrant (the yellow region in the figure, where both x and y are positive), there will be a finite number of non-negative solutions to such an equation (3 in this case).

A similar situation is shown for **$2x + 3y = 11$** (the brown line with pink dots) which has 2 non-negative integer solutions.



However, if we consider a line like **$3x - 2y = 10$**, it turns out to have a positive slope. Thus it will appear as shown alongside (the green line with red dots). As can be clearly observed from the figure, once the line enters the first quadrant it will stay there forever and that is why we would get infinite solutions for such an equation.

A similar situation is shown for **$5y - 2x = 12$** (the brown line with pink dots) which also has infinite non-negative integer solutions.



However, there's another thing we should be cautious about. There need not always be solutions to such equations; all that we have discussed above is valid if solutions exist in the first place! To demonstrate, let us try to find the number of non-negative integer solutions for $6x + 9y = 100$ and $2x - 6y = 15$.

If we try to find a solution by trial and error, we would normally expect to find one for $6x + 9y = 100$ within 6 steps for y. But if we check y = 0, 1, 2, 3, 4 and 5 we will not find an integer value for x.

The issue is that the coefficients of x and y are not co-prime, and additionally the common factor is not present on the left-hand side. So $6x + 9y = 100$ when divided by 3 gives us $2x + 3y = 33.\overline{33}$, in which case x and y cannot both be integers! Graphically, it is a line which does not pass through any gridpoints, as shown.

ALGEBRA – 11

The logic we have seen in the last few posts can be extended to more than 2 variables as well; usually the best way to approach such a question is to freeze one variable, make cases and proceed.

Q. Find the number of non-negative integer solutions for $x + 3y + 8z = 20$

In such a question, it makes sense to start with the one having the highest coefficient (z in this case). The reason is simple; if we plan to list out cases it makes sense to take the option which gives us the minimum cases to list. In this case, none of x, 3y or 8z can exceed 20, since if that were to happen one or more of the other variables would go negative. So x could range from 0 to 20 (21 cases), y from 0 to 20/3 (7 cases) and z from 0 to 20/8 (3 cases). Let's list these 3 cases out independently:

If $z = 0$, $x + 3y = 20$ which gives us 7 cases (20, 0), (17, 1), (14, 2), (11, 3), (8, 4), (5, 5), (2, 6)

If $z = 1$, $x + 3y = 12$ which gives us 5 cases (12, 0), (9, 1), (6, 2), (3, 3), (0, 4)

If $z = 2$, $x + 3y = 4$ which gives us 2 cases (4, 0), (1, 1)

Hence in total we will have $7 + 5 + 2 =$ **14 solutions** of the form (20, 0, 0), (17, 1, 0)…(1, 1, 2).

(Note that we don't actually need to list out all the pairs for (x, y), we can use the approach mentioned in the "Algebra – 9" post to directly get the number or (x, y) pairs in each case.)

Q. Find the number of positive integer solutions for $5x + 3y + 2z = 30$

Here we have to find positive integer solutions and not non-negative integer ones (this can also be asked occasionally, though it is rarer). We just need to eliminate the cases which give any of the variables equalling 0). Now x can take values 1, 2, 3, 4 or 5 (if x becomes 6, $y = z = 0$)

If $x = 1$, $3y + 2z = 25$ which gives us 4 cases for (y, z) ➔ (7, 2), (5, 5), (3, 8), (1, 11)

If $x = 2$, $3y + 2z = 20$ which gives us 3 cases for (y, z) ➔ (6, 1), (4, 4), (2, 7)

If $x = 3$, $3y + 2z = 15$ which gives us 2 cases for (y, z) ➔ (3, 3), (1, 6)

If $x = 4$, $3y + 2z = 10$ which gives us 1 case for (y, z) ➔ (2, 2)

If $x = 5$, $3y + 2z = 5$ which gives us 1 case for (y, z) ➔ (1, 1)

Overall this gives us $4 + 3 + 2 + 1 + 1 =$ **11 cases**

If we have 2 equations in 3 variables, we can approach them as follows:

Q. Find the number of non-negative integer solutions for $x + y + z = 20$ and $x + 3y + 8z = 70$

Subtracting $x + y + z = 20$ from $x + 3y + 8z = 70$ gives us $2y + 7z = 50$. This gives us solutions for (y, z) as (25, 0), (18, 2), (11, 4) and (4, 6). However, we need to also find the corresponding values for x; since $x + y + z = 20$, in the first of the above 4 cases we will get a negative value for x and hence we should eliminate that case. This leaves us with **3 solutions** for (x, y, z): (0, 18, 2), (5, 11, 4) and (10, 4, 6)

ALGEBRA – 12

All of this theory is fine, but why would we need to know the answer to such questions? And why do we mostly encounter questions asking for non-negative integer solutions (as opposed to positive integer solutions)? The answer lies in the fact that the above methods can be applied to solve problems based on certain "real-life" situations:

Q. A person has coins of 1 yen, 3 yen and 7 yen. In how many ways can he pay a bill of 40 yen?

1] 40 2] 44 3] 47 4] 49 5] 50

Let the number of 1 yen coins be x, of 3 yen coins be y and of 7 yen coins be z. Then we can say:

$x + 3y + 7z = 40$.

Also, we can say that x, y and z cannot be negative or fractional (though some of them could be 0!) and hence we are looking for non-negative integer solutions. Considering cases for z from 0 to 5:

If $z = 0$, $x + 3y = 40$ ➔ 14 solutions

If $z = 1$, $x + 3y = 33$ ➔ 12 solutions

If $z = 2$, $x + 3y = 26$ ➔ 9 solutions

If $z = 3$, $x + 3y = 19$ ➔ 7 solutions

If $z = 4$, $x + 3y = 12$ ➔ 5 solutions

If $z = 5$, $x + 3y = 5$ ➔ 2 solutions

Thus in all there will be $14 + 12 + 9 + 7 + 5 + 2 = 49$ solutions. **Hence (4)**

Q. Vartika writes an exam with 40 questions. A correct answer carries 4 marks, a wrong answer carries a penalty of 1 marks, while an unanswered question scores no marks. If Vartika scores 17 marks, what is the maximum number of questions she could have gotten right?

1] 9 2] 12 3] 11 4] 10 5] 14

We can create two equations here, by assuming variables c = number of correct answers w = number of wrong answers and u = number of unattempted questions (where all of c, w and u have to be non-negative integers):

$c + w + u = 40$

$4c - w = 17$

Solutions for the second equation can easily be found as (5, 3), (6, 7), (7, 11), (8, 15), (9, 19), (10, 23, (11, 27), (12, 31), (13, 35) and so on. As we know, there would be infinite solutions for this. However, there is another limitation as well; since $c + w + u = 40$, and u cannot go negative, $c + w$ cannot exceed 40. Hence we find that the above solutions after (11, 27) are not valid and hence there are only 7 ways in which the given score can be achieved. Out of these, the one with the highest number of correct answers is the last, which has 11 questions correct, 27 wrong, and the remaining 2 unattempted.

Of course, we could also have solved this by plugging in each of the answer choices and seen which are possible and which are not, starting from the largest. 14 correct would get 56 marks, which means we would need to get $56 - 17 = 39$ wrong (which is not possible given total questions are 40), 12 correct would require 31 wrong (infeasible), 11 correct would require 27 wrong (feasible). **Hence (3)**

ALGEBRA – 13

Linear Equations – CAT Scan 1

Q. The number of positive integer valued pairs (x, y), satisfying 4x - 17y = 1 and x < 1000 is:
CAT 1999

1] 59 2] 57 3] 55 4] 58

A quick glance at the answer choices should assure us that we do NOT want to list down all the possible cases. But with our knowledge from the last few posts, we can easily tackle this one:

4x - 17y = 1 ➔ x will have a solution at every 17 steps and y at every 4 steps. So let us do trial and error with y.
If y = 0, 4x = 1 ➔ x is non-integral
If y = 1, 4x = 18 ➔ x is non-integral
If y = 2, 4x = 35 ➔ x is non-integral
If y = 3, 4x = 52 ➔ x = 13

(**Note:** we could have shortened the above procedure still further by realising that since 4x = 1 + 17y, and since 4x is definitely even, y must be odd so that 1 + 17y would be even)

Thus we have solutions starting from (13, 3) and further solutions can be found by adding 17 to x and 4 to y as (30, 7), (47, 11)…or in general (13 + 17k, 3 + 4k). Now we can quickly check that the largest k for which 13 + 17k does not cross 1000 is 58 as 13 + 17 * 58 = 999. So we will have 58 more solutions after the 1st, making 59 in all. **Hence (1)**

Q. Suppose you have a currency, named Miso, in three denominations: 1 Miso, 10 Misos and 50 Misos. In how many ways can you pay a bill of 107 Misos ? **CAT 2007**

1] 17 2] 16 3] 18 4] 15 5] 19

Let the number of units of 1 Miso be x, of 10 Misos be y and of 50 Misos be z.

Then we can write the following equation:

x + 10y + 50z = 107

Now to solve this we can consider cases for z

If z = 0, x + 10 y = 107 ➔ (107, 0), (97, 1) … (7, 10) = 11 cases
If z = 1, x + 10 y = 57 ➔ (57, 0), (47, 1) … (7, 5) = 6 cases
If z = 2, x + 10 y = 7 ➔ (7, 0) = 1 case

Hence in total we have 11 + 6 + 1 = 18 ways to pay the bill. **Hence (3)**

ALGEBRA – 14

Q. A test has 50 questions. A student scores 1 mark for a correct answer, $-\frac{1}{3}$ for a wrong answer, and $-\frac{1}{6}$ for not attempting a question. If the net score of a student is 32, the number of questions answered wrongly by that student cannot be less than **CAT 2003**

1] 6 **2] 12** **3] 3** **4] 9** **5] 10**

One way of solving this would be to create the relevant equations:

Let the number of correct answers be c, the number of wrong answers be w and the number of unattempted questions be u. Note that c, w and u must all be non-negative integers!

Then $\mathbf{c + w + u = 50}$ … (1)

But we also know that $c - \frac{1}{3}w - \frac{1}{6}u = 32$
which can be written as $\mathbf{6c - 2w - u = 192}$ … (2)

Adding these two equations, we get

$7c - w = 242$

Looking for non-negative integer solutions to this, we will quickly be able to identify solutions such as (35, 3), (36, 10), (37, 17), (38, 24) and so on. Of course, we also need u to be non-negative; checking back with c + w + u = 50 shows us that only the first two cases above are valid. Hence the minimum number of wrongly answered question must be 3. **Hence (3)**

Of course, we could also have plugged in values and checked, but in this case, with the fractional penalties, that would be somewhat confusing, to say the least!

Q. The number of solutions of the equation 2x + y = 40 where both x and y are positive integers and x ≤ y is: **CAT 2006**

1] 7 **2] 13** **3] 14** **4] 18** **5] 20**

Again, this can be solved quite simply: we can instantly see a solution by putting x = 0 viz (0, 40)

After that further solutions would be arrived at by adding 1 to x and subtracting 2 from y as follows: (0, 40), (1, 38), (2, 36), (3, 34) and so on of the form (k, 40 - 2k). (We discount (0, 40) as we are looking for positive integers.

Now we can easily see that if we want k to be less than 40 – 2k, then k cannot exceed 13.

Thus we will get 13 solutions from (1, 38) to (13, 14). **Hence (2)**

ALGEBRA – 15

Q. Rs. 960 is formed with coins of denomination Rs. 5, Rs. 2 and Re. 1. If the number of coins of Rs 2 and Re 1 is interchanged then the amount reduces by Rs. 40. If the total no. of coins is 300, find the no. of coins of Rs. 5. **CAT 2001**

1)100 2)140 3) 60 4) 150

Let the number of coins of 5, 2 and 1 be x, y and z respectively. We can create the following equations:

$x + y + z = 300$

$5x + 2y + z = 960$

$5x + y + 2z = 920$

Solving the above, we would get x = 140, y = 100 and z = 60. **Hence (2)**

If 3 equations in 3 variables are not to your taste, however, you could solve this in another way:

We are told that if the number of coins of Rs 2 and Re 1 are interchanged the value drops by 40. Now replacing a Rs 2 coin with a Re 1 coin reduces the value by 1, so there must be 40 more coins of Rs 2 initially (and 40 more of Re 1 finally). So if we take the number of Re 1 coins as a, the number of Rs 2 coins will be a + 40 and hence the Rs 5 coins will be 300 – a – (a + 40) = 260 – 2a. Thus we can write a single equation:

$5 (260 – 2a) + 2 (a + 40) + a = 960$

which gives 1380 - 7a = 960 i.e. a = 60. Hence the number of Rs 5 coins must be 260 – 2a = 120.

Q. If x and y are integers then the equation 5x + 19y = 64 has **CAT 2003**

1] no solution for x < 300 and y < 0 **2] no solution for x > 250 and y > –100**
3] a solution for 250 < x < 300 **4] a solution for -59 < y < -56**

Let's look at the equation 5x + 19y = 64. There are no obvious solutions, so we need to do trial and error with y to find a solution.

If y = 0, 5x = 64 ➔ non-integral solution
If y = 1, 5x = 45 ➔ x = 9

Hence (9, 1) is a solution

If we seek further solutions by increasing x, we will find solutions like (28, -4), (47, -9) and so on. By increasing y on the other hand we will get solutions such as (-10, 6), (-29, 11) and so on.

(28, -4) invalidates the first option. If we go further we will find a solution like (256, - 64) which will invalidate option 2. And checking the pattern of negative y-values will give us -4, -9, -14, -19…-54, -59 so the 4th option is also false. Hence (3)

Alternatively, we know that x has to have a solution every 19 steps; so 3 must be true.

ALGEBRA – 16

Q. Answer questions 1 and 2 based on the given information: **CAT 2004**

In an examination there are 100 questions divided into three groups A, B and C such that each group contains at least one question. Each question in group A carries 1 mark, each question in group B contains 2 marks and each question in group C carries 3 marks. It is known that the questions in group A together carry a least 60% of the total marks.

Analysis: Denoting the number of questions of types A, B and C as a, b, and c respectively (which means the total marks are a + 2b + 3c), we can say that the conditions to be satisfied are:

$a + b + c = 100$ and

$a \geq \frac{3}{5}(a + 2b + 3c)$

Also, since there is at least 1 question of each type, a, b and c are all integers

1. If group B contains 23 questions, then how many questions are there in group C?

1] 1 **2] 2** **3] 3** **4] Cannot be Determined**

$a + 23 + c = 100$. So $a + c = 77$. Also total marks must be $a + 46 + 3c$.

We want to find possible values for c. So let us put $a = 77 - c$:

$77 - c \geq \frac{3}{5}(77 - c + 46 + 3c)$

$77 - c \geq \frac{3}{5}(123 + 2c)$

$385 - 5c \geq 369 + 6c$

$16 > 11c$

So c can only be 1. **Hence (1)**

Alternatively, we could just try putting values for c, out of the answer options:

If c = 1 => a = 76 => total score = 125. Now $\frac{3}{5}$ of 125 = 75 so this satisfies the required condition.

If c = 2 => a = 75 => total score = 127. Now $\frac{3}{5}$ of 127 > 75 so this does not satisfy.

2. If group C contains 8 questions and group B carries at least 20% of the total marks, which of the following best describes the number of questions in group B?

1] 11 or 12 **2] 12 or 13** **3] 13 or 14** **4] 14 or 15**

$a + b + 8 = 100$. So $a + b = 92$. Also the total marks = $a + 2b + 24$

Now $a \geq 60\%$ of total and $b \geq 20\%$ of the total. At this point it probably makes sense again to try out the options:

If b = 11 ➔ a = 81 ➔ total score = 127. 2b = 22 < 20% of 127. ➔ Unviable

If b = 12 ➔ a = 80 ➔ total score = 128. 2b = 24 < 20% of 128 ➔ Unviable

If b = 13 ➔ a = 79 ➔ total score = 129. 2b = 26 > 20% of 129. Also a = 79 > 60% of 129. ➔ Viable

If b = 14 ➔ a = 78 ➔ total score = 130. 2b = 28 > 20% of 130. Also a = 78 = 60% of 130. ➔ Viable

If b = 15 ➔ a = 77 ➔ total score = 131. 2b = 30 > 20% of 131 but a = 77 < 60% of 131. ➔ Unviable

Hence (3)

TIME & WORK - 1

Time and Work

In the next few posts, we'll take a look at one of the simpler concepts in Arithmetic, Time and Work. Conceptually such problems tend to be relatively straightforward; however, the fact that they tend to occur a lot in speed-based tests means that the emphasis needs to be not just on solving them, but on solving them fast. So I'll focus on a variety of approaches rather than a single solution method. I will usually also not spend much time on the standard (board exam) approaches!

Time and Work problems have several variants. There can be a situation where every person in the problem has equal efficiency. There can also be a situation wherein every individual involved works with a different efficiency. There could be a third, hybrid situation, wherein the workers involved fall into multiple categories (for example men, women and children), and all workers within a particular category are equally efficient but two workers from different categories have different efficiencies. We'll look at each of these situations, with some trickier examples as well, and also related concepts like the typical "Pipes and Cisterns" problems (which, while they look very different, share the same underlying logic)

Let's start with the simplest question type, wherein all the people in the problem are equally efficient.

Q. If 12 men can do a piece of work in 30 days, how long will 20 men take to do the same work?

Method 1: We can see that the more the men, the fewer will be the days required (in other words, it is a straightforward case of inverse proportion). In this case, the number of men has become 20 instead of 12 i.e. $\frac{5}{3}$ times. Thus the number of days should become $\frac{3}{5}$ times i.e. $\frac{3}{5} * 30 =$ **18 days.**

Method 2: We can introduce the idea of man-days (a man-day being the work done by a man in a day). Here 12 people working for 30 days means that the work is 12 * 30 = 360 man-days. If 20 men have to do the same work, it will therefore take $\frac{360}{20} =$ **18 days.**

Q. If 18 men can do a piece of work in 40 days, how long will 24 men take to do twice the same work?

Method 1: Again, it is a case of inverse proportion. The number of men has become 24 instead of 18 i.e. $\frac{4}{3}$ times. Thus the number of days to do the original work should become $\frac{3}{4}$ times i.e. $\frac{3}{4} * 40 = 30$ days. However the work has doubled, and so the days required should also double giving **60 days.**

Method 2: 18 people working for 40 days means that the original work is 18 * 40 = 720 man-days. So double the work would be equivalent to 1440 man-days and the time taken for 24 men to do this would be $\frac{1440}{24} =$ **60 days.**

In the next couple of posts we'll see some slightly more complicated problems:

TIME & WORK - 2

Q. A piece of work would take 30 men 25 days. The 30 men started doing the work, but after 15 days, some men had to leave. As a result, the work ended up being completed in a total of 30 days. How many men left after 15 days?

Approach 1:

30 men, 25 days $\Longrightarrow$ 750 man-days is the total required for the work.
For 15 days, the full 30 men have worked $\Longrightarrow$ 450 man-days were completed.
The balance of 300 man-days was completed in 30 – 15 = 15 days.
Hence the number of men must have been $\frac{300}{15}$ = 20 (out of the original 30)
So 10 men must have left.

Approach 2:

We may assume the 30 men did their work properly for the first 15 days. The work remaining, which should have taken 10 days more, instead took 15 days. But if the time for a given work became $\frac{3}{2}$, the number of people must have become $\frac{2}{3}$. So, the original 30 workers must have become $\frac{2}{3}$ * 30 = 20, which means that 10 workers must have left.

(Note: There could be other approaches too, such as percentages. I am just showing two approaches to give some idea of the diverse possibilities)

Q. A piece of work would take 25 men 50 days. The 25 men started doing the work, but after some days, 10 men had to leave. As a result, the work ended up being completed in a total of 60 days. After how many days did the 10 men leave?

Approach 1:

25 men, 50 days $\Longrightarrow$ 1250 man-days is the total required for the work.
Supposed these 25 men worked for n days, then they did 25 * n man-days of work.
After that, the remaining 15 men must have worked for (60 – n) days, doing 15 * (60 – n) man-days.
Since the work is then complete, we can say that 25 * n + 15 * (60 – n) = 1250
Solving we get n = 35 days

Approach 2:

Let us assume that the 10 men left k days from the end. At that point, the number of men changed from 25 to 15, i.e. it became $\frac{3}{5}$. So the number of days taken should have become $\frac{5}{3}$. In the process 10 extra days were taken. So $\frac{5}{3}$ k – k = 10 i.e. k = 15. Hence the men left 15 days from the end of the original deadline, i.e. after 35 days.

WEIGHTED AVERAGES - 1

Weighted Averages: Alternative Approaches

Weighted average problems can often involve an inordinate amount of calculation. Sometimes, though, it is possible to reduce the calculative grunt-work by understanding how weighted averages behave. Let's start with a simple problem and try to look at different methods of solving it:

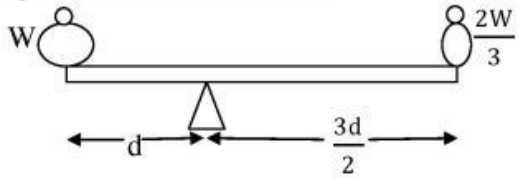
Q. The average score of the 27 boys in a class is 70 while the average score of the 18 girls is 80. Find the average score of the entire class?

Method 1: The basic logic of averages can be applied here, by finding (total score of the whole group) upon (total number of people), as follows: $\frac{27 \text{ x } 70+18 \text{ x } 80}{27+18} = \frac{1890+1440}{45} = \frac{3330}{45} = 74.$

Method 2: The calculation involved in the above can be significantly reduced by the technique of Weighted Average , as follows: $\frac{27 \text{ x } 70+18 \text{ x } 80}{27+18} = \frac{9\ (3 \text{ x } 70 + 2 \text{ x } 80)}{9\ (3+2)} = \frac{370}{5} = 74$

Notice that there are four numerical quantities in the problem, and the two we are averaging are '70' and '80'. These values have to remain untouched, as any change to these would change the final answer as well. However, the other two values, 27 and 18, are not as sacrosanct. They occur in the numerator as well as denominator and so we can cancel out common factors (in this case 9). We need not have written out the expression in so much detail – rather we could have said that the weights assigned to '70' and '80' are 3 and 2 respectively and directly computed $\frac{3 \text{ x } 70 + 2 \text{ x } 80}{3+2} = \frac{370}{5} = 74.$

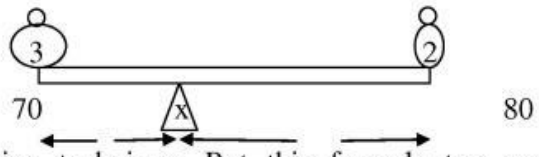
Method 3: The behaviour of weighted averages may be easily visualised as similar to that of a see-saw. If two children of significantly different weights sit on a seesaw, they can still balance – provided that the lighter one sits further from the pivot. If the lighter kid is 2/3 of the weight, he should sit at 3/2 of the distance as compared to the heavier one.



We can use the same kind of logic here, with the weights of the kids being replaced by the 'weights applied to the two quantities', and the distances from the pivot being replaced by the 'distances from the mean of the two quantities being averaged'. Let's see how to apply this in the above question:

We are averaging 70 and 80 to give some x.

So we could write $\frac{80-x}{x-70} = \frac{3}{2}$



This gives us the standard and well-known alligation technique. But this formula too can be cumbersome to apply and is best used only in fairly complex problems (say, those involving mixtures).

Instead, we could just divide the gap between 70 and 80 (i.e. 10 units) in the inverse ratio of the weights (i.e. 2:3) to get 4 and 6. The answer would directly be 70 + 4 (or 80 – 6) = 74.

Note that the answer would always be between the two quantities being averaged, and closer to the one with the higher weight.

Let's try and apply this in a few more examples:

Q. A man buys 21 kg of rice at Rs 29 per kg and 28 kg more at Rs 22 per kg and mixes it all together. What is the cost per kg of the resultant mixture?

We are mixing 29 and 22 in the ratio 21 : 28 i.e. 3 : 4. Let's write it like this
29 ------ x ------- 22
4k : 3k
Dividing |29-22| i.e. 7 in the ratio 4 : 3 we get 4 and 3 so the answer would be 29 – 4 (or 22 + 3) = 25.

Q. A group of 45 soldiers has an average height of 167 cm while another group of 75 soldiers has an average height of 191 cm. What is the average height of the entire troop?

Here we are averaging 167 and 191 in the ratio 45 : 75 i.e. 3 : 5
167 ------ x ------- 191
5k : 3k
Dividing |167 – 191| i.e. 24 in the ratio 5 : 3 we get 15 and 9, so the answer would be 167 + 15 = 182.

Q. Two groups of students took a test. The 34 students of the 1st group averaged 238 marks (out of 450). The 85 students of the 2nd group averaged 259 marks. What was the overall average?

Here we are averaging 238 and 259 in the ratio 34 : 85 i.e. 2 : 5
238 ------ x ------- 259
5k : 2k
Dividing |259 – 238| i.e. 21 in the ratio 5 : 2 we get 15 and 6, so the answer would be 238 + 15 = 253.

In the next post, we'll try and see how this method can be applied to a variety of problems.

WEIGHTED AVERAGES - 2

Weighted Averages: Variants

Q. Joy invests a total of Rs 65000, partly in mutual fund A and partly in mutual fund B. At the end of a year, A gives a return of 12% and B a return of 18.5%; consequently Joy gets a return of Rs 9750. How much did she invest in A?

We could solve this by creating an algebraic equation, and solving for x, as follows: let the amount in A be x. Hence the amount in B is (65000 – x). Hence 0.12 x + 0.185 (65000 – x) = 9750 which when solved would give x as 35000.

But we could also directly use weighted averages. Overall she is earning 9750 / 65000 = 15%

12 ------ 15 ------- 18.5

3 : 3.5

So Joy's investment in A and B must be in the ratio 3.5 : 3 = 7 : 6 and hence she must have invested 7/13 of 65000 = 35000 in A and the remaining 30000 in B.

Q. A man was assigned to find the average age of a class of 13 students. By mistake he included the 35-year old teacher as well and hence the average age went up by 2 years. Find the actual average age of the class.

We could solve this, again, by assuming the correct average to be x and creating an equation like 13 x + 35 = 14 (x + 2) which when solved would yield x = 7.

But we could also use the above approach. The 'ratio of differences from the mean' should be 1 : 13 (or 2 : 26) since the number of students is 13 times the number of teachers.

x ------ x + 2 ------- 35

2 : 26

Hence x + 2 = 35 – 26 = 9 and x = 7.

Q. Two jars contain 41% and 56% acid respectively. If 3.4 litres are drawn from the 1st jar, how many litres from the 2nd jar should be added so that the resultant concentration is 50%?

41 ------ 50 ------- 56

9 : 6

Hence the ratio if mixing should be 6 : 9 or 2 : 3 which means if we have 3.4 litres of the 1st, we need 3.4 x 3/2 = 5.1 litres of the 2nd.

Q. A jug contains 60 litres of 60% milk. How many litres of water should be added to make the concentration of milk 48%?

We may assume the water to be '0% milk' and then apply the usual logic:

60 ------ 48 ------- 0

12 : 48

Hence the ratio of mixing should be 4 : 1. Thus, to the original 60 litres we have to add 15 litres water.

WEIGHTED AVERAGES - 3

Weighted Averages: Variants

Q. A tank contains 700 gallons of 54% salt solution. On heating it, water starts evaporating. How much water should evaporate such that the remaining solution contains 63% salt?

Here the usual problem is reversed; the initial solution is the average one.

Remaining Initial Water Evaporated

63 ------ 54 ------- 0

9 : 54

Hence the ratio of evaporated water to the remaining solution must be 9 : 54 i.e. 1 : 6 i.e. 1/7 of the total (that is to say 100 gallons) must evaporate.

Q. In a certain pasture there are some cows, horses and sheep. On average, a cow grazes 42 sq ft of grass in a month, the corresponding number for horses and sheep being 54 and 25 respectively. The combined average for cows and horses is 47 sq ft a month and that for all the animals is 33 sq ft a month. What fraction of the total number of animals are sheep?

Cows Horses

42 ------ 47 ------- 54

5 : 7

Hence ratio of cows to horses is 7 : 5. Now let us repeat the process with cows and horses combined:

C + H Sheep

47 ------ 33 ------- 25

14 : 8

So the ratio of (cows + horses) to sheep is 8 : 14 or 4 : 7., So for every 7 cows there are 5 horses. And for these (5 +7) = 12 cows and horses there will be 21 sheep. Hence fraction of sheep is 21/(5 + 7 + 21) = 7/11.

Q. In what ratio should a solution of water and alcohol in the ratio 2: 3 be mixed with a solution of water and alcohol in the ratio 9: 1 to get a solution with a 11: 9 ratio of water and alcohol?

At first glance this appears to be a full-fledged alligation question. However, we can solve it quite easily by equalising the denominators of the fractions involved. Suppose we consider water – we are mixing $\frac{2}{5}$ with $\frac{9}{10}$ to get $\frac{11}{20}$. Equalising the denominators, we are mixing $\frac{8}{20}$ with $\frac{18}{20}$ to get $\frac{11}{20}$. Thus

8 ------ 11 ------- 18

3 : 7

Hence they should be mixed in the ratio 7 : 3.

Q. Given points A (13, –8) and B (1, 10), find a point that divides AB internally in the ratio 5 : 1.

At first glance there seems to be no correlation between weighted averages and coordinate geometry: But if we write it thus:

13 ------ x ------- 1 and –8 ------ y ------- 10

1 : 5 1 : 5

Solving we get x = 11 and y = –5 and the required point will be (11, –5)

WEIGHTED AVERAGES – CAT SCAN

Weighted Average – CAT Scan

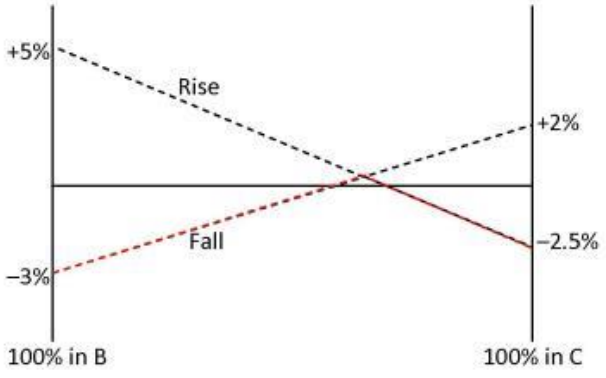
Let's check out a CAT questions from the past which could be solved using weighted averages:

Shabnam is considering three alternatives to invest her surplus cash for a week. She wishes to guarantee maximum returns on her investment. She has three options, each of which can be utilized fully or partially in conjunction with others.
Option A: Invest in a public sector bank. It promises a return of +0.10%.
Option B: Invest in mutual funds of ABC Ltd. A rise in the stock market will result in a return of +5%, while a fall will entail a return of -3%.
Option C: Invest in mutual funds of CBA Ltd. A rise in the stock market will result in a return of -2.5%, while a fall will entail a return of +2%.

1. The maximum guaranteed return to Shabnam is
(1) 0.25% **(2) 0.10%** **(3) 0.20%** **(4) 0.15%** **(5) 0.30%**

2. What strategy will maximize the guaranteed return to Shabnam?
(1)100 % in option A **(2) 36 % in option B and 64% in option C**
(3) 64 % in option B and 36% in option C **(4) 1/3 in each of the three options**
(5) 30 % in option A, 32% in option B and 38% in option C

Let us try and do this using a combination of concepts – graphing and weighted averages. Let us leave out A for the moment as that is a fixed return. Let's make a graph on which the horizontal axis represents the percentage of investment in B v/c C and the vertical represents the percentage profit earned.



Note that the minimum guaranteed return (marked by the red dotted line) is maximised at the intersection of the two lines. To find this point, we need simply note that the two triangles formed are similar, and that their bases are in the ratio (5 – (-3)) : (2 – (-2.5) i.e. 8 : 4.5 or 16 : 9. Hence the investment percentage also should be the inverse of 16 : 9 i.e 36% in option B and 64% in option C. This will give us a return of 0.2% whether the stock market rises or falls (for example for a fall the return will be 0.36 * (–3) + 0.64 * 2 = 0.2) and since this exceeds the 0.1% obtained by investing in A, this would be our optimal solution. Thus the answers to the questions would be 1 – (3) and 2 – (2)

FIXED & VARIABLE COST – 1

Fixed and Variable Costs - 1

Q. A telephone service provider charges a price of Rs X for a new connection, and allows 50 free calls within that. On every extra call beyond the 50th one, the charge is Rs 2. Manoj has 2 connections, on one of which he makes Y calls, and on the other, Z calls, (where both Y and Z are greater than or equal to 50) resulting in 2 bills of Rs 190 and Rs 230 respectively. Had he made all the calls on a single connection instead, his bill would have come to just Rs 370.

1) What can be said about the value of X?

1) $75 \le X < 110$ 2) $110 \le X < 140$ 3) $140 \le X < 175$ 4) $175 \le X < 250$

2) What can be said about the value of Y?

1) $15 \le Y < 45$ 2) $45 \le Y < 75$ 3) $75 \le Y < 105$ 4) $105 \le Y < 150$

3) What can be said about the value of Z?

1) $15 \le Z < 45$ 2) $45 \le Z < 75$ 3) $75 \le Z < 105$ 4) $105 \le Z < 150$

One way of solving such a question would be to create 3 equations (since there are 3 variables):

In the first situation, on the first connection, Manoj has to pay the initial charge (X) and an extra charge of 2 Rs per call for (Y – 50) calls, while on the second, he has to pay X and then Rs 2 per call for (Z – 50) calls. In the second situation, where he takes a single connection, he has to pay X and then the extra charge of Rs 2 per call for (Y + Z – 50) calls. Hence we can write:

$X + 2(Y - 50) = 190$ … (i)
$X + 2(Z - 50) = 230$ … (ii)
$X + 2(Y + Z - 50) = 370$ … (iii)

Solving these three we could find values for X, Y and Z.

However, we would ideally not want to get into a time-consuming exercise like solving 3 simultaneous equations – so let's try to develop another approach:

Let's look at the difference between the two situations carefully; in the second situation, he pays X less (as 1 connection instead of 2) and spends for 50 calls more (which works out to Rs 100) and yet overall spends (190 + 230) – 370 = 50 less. Hence X – 100 = 50 which means the connection cost X = 150.

So for the first connection he pays 190 -150 = 40 Rs extra for calls, meaning he makes 20 extra calls. The total calls, Y must therefore be 50 + 20 = 70. Similarly for the second connection he pays 230 -150 = 80 Rs extra for calls, meaning he makes 40 extra calls, which gives Z as 50 + 40 = 90.

In the next post, we will try to apply such logic to a couple of variants on this theme.

FIXED & VARIABLE COST – 2

Fixed and Variable Costs - 2

Q. Omkar was hired as a servant. His promised salary for the year was Rs 16000 and a set of new clothes. However, after 7 months he had to return to his native place and hence he was paid a pro-rated amount of Rs 8500 and the clothes. What was the value of the clothes?

If we wanted to solve this question using equations, we could do so by saying: let Omkar's monthly salary be x and the price of the clothes be y. Then
16000 + y = 12x …. (i) and also
8500 + y = 7x ……. (ii)
Solving, we can find x and y.

However, we could also solve this problem without needing to write anything, by looking at the difference between the two situations: He has worked for 5 months less in the second case, and his pay is correspondingly 16000 - 8500 = 7500 less, thus his monthly pay must be Rs 1500 and hence for a year he should be getting 1500 x 12 = Rs 18000. Since he gets 16000 in cash, the remaining 2000 must be the value of the suit.

Here's another problem, from CAT 1999:

Q. Total expenses of a boarding house are partly fixed and partly varying linearly with the number of boarders. The average expense per boarder is Rs. 700 when there are 25 boarders and Rs. 600 when there are 50 boarders. What is the average expense per boarder when there are 100 boarders?

1) 550 **2) 560** **3) 540** **4) 560**

We can assume the fixed expense to be f and the variable expense to be v, giving us two equations:
f + 25v = 700 x 25 = 17500 …. (i)
f + 50v = 600 x 50 = 30000 …. (ii)
Solving these equations gives us f = 5000 and v = 500, so that when there are 100 boarders, we should have a total cost of 5000 + 100 x 500 = 55000 and hence an average cost of 55000 / 100 = 550.

Another approach (slightly trickier to visualise) could be to recognise that when the number of boarders is doubled (from 25 to 50) the fixed cost per person should have halved (the per-boarder additional cost will remain the same of course). Since this caused a reduction of 100, it follows that the initial fixed cost per person must have been 200 out of 700 (which reduced to 100 out of 600) and so the variable per-boarder cost must have been 500. And now when the number of people is again doubled from 50 to 100, the fixed cost per person must again be halved, from 100 to 50, giving us a per-person average cost of 500 + 50 = 550.

FACTORS OF A NUMBER - 1

Number Theory: Number of Factors

In this post, we'll look at a concept in Number Theory: finding out the number of factors for a given number using the Fundamental Principle of Counting:

Q. How many factors does the number 84 have?

Let's start by trying to list out the factors manually: 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84. Since 84 is a small number, we can do this without too much difficulty. However, we run the risk of inadvertently missing out on a factor. One way of checking is to note that all factors occur in pairs, thus: 1 x 84, 2 x 42, 3 x 28, 4 x 21, 6 x 14, 7 x 12. So chances are we have covered them all in this enumeration (unless we have missed a pair!).

But what if it were a larger number? Say 1200 or 1728? Manual listing would be both time-consuming and error-prone, and hence we need a reliable and quick mathematical approach. The Fundamental Principle of Counting can be adapted to provide just such a method:

The number 84 can be written, in its prime factorised form, as $2^2 \times 3^1 \times 7^1$. Any factor of 84 will therefore contain a power of 2, a power of 3 and a power of 7 (for example: $12 = 2^2 \times 3^1 \times 7^0$, $21 = 2^0 \times 3^1 \times 7^1$, $1 = 2^0 \times 3^0 \times 7^0$, $42 = 2^1 \times 3^1 \times 7^1$ and so on). We note that the power of each of these primes can range from a minimum of 0 to a maximum of the highest power present in 84. Thus to make a factor of 84, we have to choose a power of 2 (which can be 0 or 1 or 2 i.e. 3 ways) AND a power of 3 (2 ways) AND a power of 7 (2 ways) and hence there will be 3 x 2 x 2 = 12 factors.

In a similar manner, 1200 (which can be expressed as $2^4 \times 3^1 \times 5^2$) will have 5 x 2 x 3 = 30 factors and 1728 ($2^6 \times 3^3$) will have 7 x 4 = 28 factors.

Q. How many odd factors does the number 84 have?

To get odd factors, we can only choose a power of 2 in one way (i.e. 2^0) and so for 84 ($2^2 \times 3^1 \times 7^1$) we will have 1 x 2 x 2 = 4 odd factors.

In similar fashion 1200 ($2^4 \times 3^1 \times 5^2$) will have 1 x 2 x 3 = 6 odd factors while 1728 ($2^6 \times 3^3$) will have 1 x 4 = 4 odd factors.

Q. How many even factors does the number 84 have?

By now it should be obvious that to calculate the number of even factors we can choose any power of 2 except 2^0. Hence, for 84 ($2^2 \times 3^1 \times 7^1$) we will have 2 x 2 x 2 = 8 even factors.

In similar fashion 1200 ($2^4 \times 3^1 \times 5^2$) will have 4 x 2 x 3 = 24 even factors while 1728 ($2^6 \times 3^3$) will have 6 x 4 = 24 even factors.

In the next post, we'll look at how to find the sum of factors of a number, using the same basic logic.

FACTORS OF A NUMBER – 2

Number Theory: Sum of Factors

Q. What is the sum of factors of the number 84?

We can, of course, list out all the factors and add them (in this case, the answer we get would be 224); but this will only work for a small number like 84.
A better method would be to extend the logic of finding the number of factors, as follows:
$84 = 2^2 \times 3^1 \times 7^1$. Consequently we observed that it must have 12 factors Let us list out these factors:

$1 = 2^0 \times 3^0 \times 7^0$
$2 = 2^1 \times 3^0 \times 7^0$
$3 = 2^0 \times 3^1 \times 7^0$
$4 = 2^2 \times 3^0 \times 7^0$
$6 = 2^1 \times 3^1 \times 7^0$
$7 = 2^0 \times 3^0 \times 7^1$
$12 = 2^2 \times 3^1 \times 7^0$
$14 = 2^1 \times 3^0 \times 7^1$
$21 = 2^0 \times 3^1 \times 7^1$
$28 = 2^2 \times 3^0 \times 7^1$
$42 = 2^1 \times 3^1 \times 7^1$
$84 = 2^2 \times 3^1 \times 7^1$

Now if we observe carefully, we can use algebraic expansion to show that the expression $(2^0 + 2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)$ will expand to give the same result as the sum of all the above terms. Hence the sum of all the factors of 84 could directly be written as $(1 + 2 + 4)(1 + 3)(1 + 7) = 224$.

In similar fashion 1200 ($2^4 \times 3^1 \times 5^2$) will have sum of factors $(1 + 2 + 4 + 8 + 16)(1 + 3)(1 + 5 + 25) = 31 \times 4 \times 31 = 3844$ while 1728 ($2^6 \times 3^3$) will have sum of factors $(1 + 2 + 4 + 8 + 16 + 32 + 64)(1 + 3 + 9 + 27) = 127 \times 40 = 5080$.

Q. What is the sum of even factors of the number 84?

$(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1) = 6 \times 4 \times 8 = 192$

Similarly for 1200 it would be $(2 + 4 + 8 + 16)(1 + 3)(1 + 5 + 25) = 30 \times 4 \times 31 = 3720$ while for 1728 it would be $(2 + 4 + 8 + 16 + 32 + 64)(1 + 3 + 9 + 27) = 126 \times 40 = 5040$.

Q. What is the sum of odd factors of the number 84?

$(2^0)(3^0 + 3^1)(7^0 + 7^1) = 1 \times 4 \times 8 = 32$

Similarly for 1200 it would be $(1)(1 + 3)(1 + 5 + 25) = 1 \times 4 \times 31 = 124$ while for 1728 it would be $(1)(1 + 3 + 9 + 27) = 1 \times 40 = 40$.

TIME, SPEED & DISTANCE – 1

Q. Two boats P and Q start simultaneously, at constant speeds p and q, from the opposite sides of a river A and B. When they meet for the first time, they are 120 m from A. When they reach the opposite bank, they immediately turn around and head back, and this time they meet at a point 90 m from B.

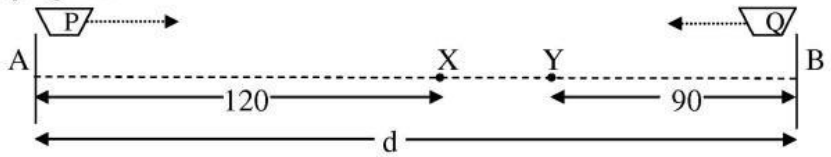
a) What is the width of the river?

1] 210 m **2] 250 m** **3] 270 m** **4] 300 m**

b) What is the ratio of the speeds p and q?

1] 1 : 1 **2] 2 : 3** **3] 3 : 4** **4] 4 : 5**

This problem could be solved by standard techniques of Time Speed and Distance. For example, let X be the first meeting point and Y be the second, and let the distance AB be d. We can draw the following figure:



Now by the first meeting, P has traversed a distance of 120 m, while Q will have covered (d-120) m. Similarly, by the time of their second meeting, P will have covered (d+90) m from the start of the trip, while Q will have covered (2d – 90) m. Since each boat is travelling at a fixed speed, the ratio of their speeds will be the same as the ratio of the distances covered in a fixed time. Hence if the speeds are p and q, then we can say that:

$$\frac{p}{q} = \frac{120}{d-120} = \frac{d+90}{2d-90}$$

Solving the equality, we can see that 120 (2d – 90) = (d + 90) (d – 120)

⇨ $240d - 10800 = d^2 - 30d - 10800$

⇨ $d^2 - 270d = 0$

⇨ d = 270 or 0

But clearly d = 0 is not the correct answer here, and hence d must be 270 and hence $\frac{p}{q} = \frac{120}{150} = \frac{4}{5}$

However, with a little care this problem can be solved orally. Not that by the first time the two boats meet, between them they will have covered a distance d. By the second time they meet, they will have covered a total distance of 3d (since they have to go apart to the starting points and then return). Hence, since they are travelling at fixed speeds, they will take 3 times as long, and hence each will separately cover 3 times the distance, for the second meeting.

Thus we can say that, since P travelled 120 m for the first meeting, he must have travelled 120 x 3 = 360 m for the second, and since he has returned 90 m to accomplish this, the distance d must be 360 – 90 = 270 m. And thus Q must have travelled 270 - 120 = 150 m for the first meeting which gives us the ratio of speeds as $\frac{p}{q} = \frac{120}{150} = \frac{4}{5}$..

TIME, SPEED & DISTANCE – 2

Let's see another question on Time, Speed and Distance:

Two towns A and B are 180 km apart. Car 1 and car 2 start travelling from A and B, respectively towards each other. Their speeds are in the ratio 1 : 2 and they start at 7:00 a.m. and 8:00 a.m., respectively. After meeting at C, they return to their starting positions and again start travelling towards each other. In order to meet car 1 for the second time at C, car 2 halts at C. For how much time does it halt?

1] 1 hour **2] 1.5 hours** **3] 2 hours** **4] 3 hours**

This problem could be solved by traditional and formulaic methods, i.e. by painstakingly writing out a set of equations and then solving them.

However, it could also be solved in a startlingly efficient manner, by just realising that whatever distances they had covered till their first meeting have to be covered twice more before the second meeting. So if Car 1, having started an hour earlier, takes an hour longer than Car 2 for the first meeting, it will take two hours longer than Car 2 to keep the second appointment and hence Car 2 will have to wait for 2 hours.

Let's look at a question from CAT 2004

Two boats, traveling at 5 and 10 kms per hour, head directly towards each other. They begin at a distance of 20 kms from each other. How far apart are they (in kms) one minute before they collide?

1] 1/12 **2]1/6** **3]1/4** **4]1/3**

This was a reasonably easy problem and most people who tried it managed to solve it. However, the most common approach was as follows:

The relative speed of the boats is 5 + 10 = 15 kmph and hence they would take 20/15 = 4/3 hours (or 80 minutes) to meet. Hence one minute before would be 79 minutes after they had started, in which time the first boat would have covered 5 x $\frac{79}{60}$ km while the second boat would have covered 10 x $\frac{79}{60}$ km, giving a total distance covered (by both boats towards each other) of 15 x $\frac{79}{60} = \frac{79}{4}$ km. Since the original distance is 20 km the new separation is 20 - $\frac{79}{4} = \frac{1}{4}$ km.

A much easier option would be to work **backward** from the meeting; to say that in every hour before they met, the boats would have been 10 + 5 i.e. 15 km further apart and hence in 1 min i.e. $\frac{1}{60}$ hours they would be 15 x $\frac{1}{60} = \frac{1}{4}$ km apart!

ESCALATORS - 1

Escalators: some basics

A topic where plenty of people (even those fond of arithmetic) get befuddled is escalators. Superficially, escalator problems are similar to problems on 'boats and streams', but they tend to be problematic because (a) there isn't a set of standard formulae (b) the wording tends to be confusing (usually deliberately) and (c) quite often, it is not stated explicitly in which direction the escalator is moving.

One thing which can often help is the realisation that the escalator's speed remains a constant throughout the problem. Let me explain with the help of a problem:

Q. Tom and Jerry are running up an ascending escalator. Jerry runs three times as fast as Tom. In all, Jerry climbs 45 steps in reaching the top, while Tom climbs only 24 steps (as he is slower, and is therefore carried by the escalator for a longer time). If the escalator is switched off, how many steps would be visible?

1) 69 2) 75 3) 80 4) 100

Let Tom's speed be T and Jerry's J, and let the number of steps in the stationary escalator be n. Even though the escalator's speed is not explicitly mentioned, let us assume it to be a constant E. This will give us a basis for comparison.

Now when Tom climbs 24 steps, the escalator climbs (n – 24) steps. So we can say $\frac{T}{E} = \frac{24}{n-24}$

Similarly, we can note that $\frac{J}{E} = \frac{45}{n-45}$

But we also know that J = 3T. Hence we can say that

$\frac{45}{n-45} = \frac{3 \times 24}{n-24}$ or $\frac{5}{n-45} = \frac{8}{n-24}$

Solving this, we get n = 80.

Another way (which works quite well as long as one is quick with basic arithmetic calculations) is to substitute the answer choices into the equation obtained i.e. $\frac{45}{n-45} = \frac{3 \times 24}{n-24}$.

Q. An escalator in a mall has 60 steps. Graeme, climbing along with it, takes 15 steps by the time he reaches the top. Jacques climbs at twice Graeme's speed. How many steps will he take to reach the top?

1) 18 2) 24 3) 30 4) 33

Note that Graeme climbs 15 steps while the escalator climbs 45. So the ratio of speeds $\frac{G}{E}$ is $\frac{1}{3}$.

Now since Jacques' speed is twice that of Graeme, we can see that $\frac{J}{E}$ must be $\frac{2}{3}$ and hence J will climb $\frac{2}{5}$ of the total steps i.e. 24 steps. Hence (2)

In the next post, we will try a couple of minor variants on escalator problems, just for practice.

ESCALATORS - 2

Escalators: variants

Let us try out another couple of problems involving escalators so that we can be comfortable if confronted with variants on the theme:

Q. While climbing an escalator, Vernon noticed that he took 30 steps, and that it took him 45 seconds to reach the top. The next time, he climbed the same escalator at double the speed and it took him only 27 seconds. How many steps are visible on the stationary escalator?

1) 45 2) 54 3) 60 4) 72

At Vernon's normal speed, it takes him 45 seconds for 30 steps. So in 27 seconds he would have gone $\frac{27}{45}$ x 30 = 18 steps. At double the speed, he must have taken 36 steps in those 27 seconds. Now the problem is the same as the one from the previous post, and we can say that

$\frac{36}{n-36} = \frac{2 \times 30}{n-30}$ or $\frac{3}{n-36} = \frac{5}{n-30}$

Solving this, we get n = 45. Hence (1)

Q. B moves by taking 3 steps forward and 1 step backward (each step in 1 sec). He walks up a stationary escalator and reaches the top of the escalator in 118 seconds. However on a moving escalator he takes 40 seconds to reach the top of the escalator. Find the speed of the escalator (in steps/second).

1] 2 steps/sec 2] 1 step/sec 3] 4 steps/sec 4] 3 steps/sec

Here we have to be a little trickier; we can see that B's motion in every completed block of 4 seconds is fixed - 3 Forward (F) and 1 Backward (B) step - and results in 2 F.

Now in the first case, we see that in 118 seconds he will have completed 29 blocks of 4 seconds plus an extra 2 seconds which means the distance covered will be 29 x 2F + 2F (since in the extra two seconds both steps will be F) = 60 F. Thus the escalator has 60 steps.

When the escalator is moving, B still moves for 40 seconds or 10 blocks of 4 seconds in which he will traverse a distance of 10 x 2F = 20F. But since he reaches the top (a total of 60F) the remaining 40F must be covered by the escalator in the same time of 40 seconds. Hence the speed of the escalator must be 1 step/ sec. Hence (2)

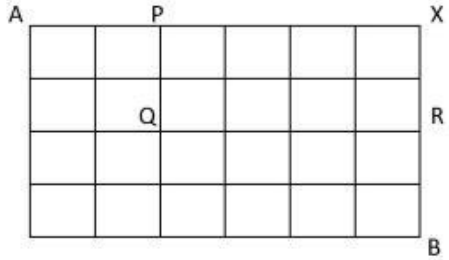
Another approach could be as follows: When the escalator is moving, the time taken is roughly 1/3rd of the original time. So the speed (of B and escalator together) must be three times the earlier speed and hence the escalator must be moving at twice the speed of B. Now B's speed is 2 steps in every 4 seconds or 0.5 step/sec, so the escalator speed (using answer choices) must be 1 step/sec.

SHORTEST PATH - 1

Number of Shortest Paths

A common problem one encounters is finding the number of 'shortest paths'. Today we'll look at the standard method of approaching the problem. A simple example to start with:

Q. In the figure below, what is the number of shortest paths along the grid from A to B?



Let's start by assuming each small line segment joining two adjacent paths to be of length 1 unit. So to get from A to B by the shortest path, we would have to traverse 10 units. One such route could be from A to X (6 units 'right') and then X to B (4 units 'down'). Another could be A to P to Q to R to B (2 units right, 2 down, 4 right and 2 down). Numerous such paths could be traced out, but the key thing to realise is that every such path will involve exactly 10 steps, and exactly 6 of these will be in the horizontal or 'right' direction while the remaining 4 will be in the vertical or 'down' direction. Hence the problem becomes equivalent to *"in how many* ways can we choose 6 out of 10 steps to be in the horizontal direction?*"* to which the answer is of course ${}^{10}C_6$ (or ${}^{10}C_4$) = 210 ways.

Another way of looking at it is to write the path representing a horizontal step by H and a vertical one by V so that the path AXB would be HHHHHHVVVV while the path APQRB could be written as HHVVHHHHVV. Every such ten-letter path would lead us from A to B covering the shortest distance i.e. 10 units, so this problem could also be treated as equivalent to *"how many words of 10 letters can be made using 6 'H's and 4 'V's?"* to which the answer would be $\frac{10!}{6!\,4!}$ which is nothing but ${}^{10}C_6 = 210$ ways. In general if there are m horizontal and n vertical steps, the number of shortest paths is ${}^{m+n}C_m$.

Let's try one more question:

Q: An ant is standing at the top right corner of a chessboard and wants to move to the bottom left corner by walking only along the edges of the squares. In how many ways can it do so, covering the minimum possible distance in the process?

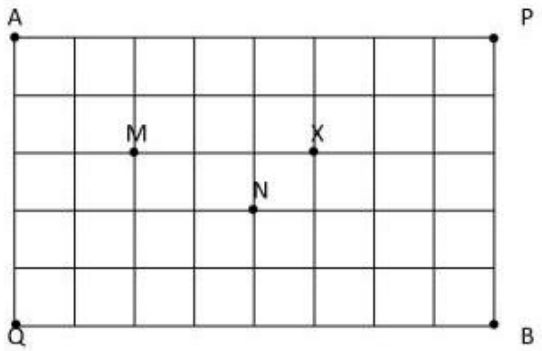
If the side of a chessboard square is x, then the ant will have to cover a total distance of 16x, out of which 8x will be 'horizontal' and 8x 'vertical'. Hence the number of shortest distance paths will be $\frac{16!}{8!\,8!}$ or ${}^{16}C_8$ (which is 12870, if one cares to calculate it; however the answer choices will probably be in ${}^{n}C_r$ format as the examiner intends to test conceptual understanding and not calculation!)

In the next post, we'll solve a more complicated problem along the same lines.

SHORTEST PATH - 2

Following up on our last post, here's a somewhat more complex question:

Q. The grid below is a network of roads in a town. Travel is permitted only along the gridlines. The distance between any two adjacent points on the grid is 100 feet.
(a) Find the number of shortest routes from A to B
(b) Find the number of shortest routes from P to Q via X
(c) Find the number of shortest paths from A to Q via N
(d) Find the number of shortest paths from A to B and back, going via X both ways
(e) Find the number of shortest routes from A to B which visit both X and N along the way



Part (a) is straightforward; we have to travel 13 units out of which 8 would be horizontal and 5 vertical. Using the formula derived in the last post the required answer is ${}^{13}C_8$.

Part (b) can be visualised as: first go from P to X and then go from X to Q. One can go from P to X in ${}^{5}C_3$ ways and from X to Q in ${}^{8}C_5$ ways. Since both these processes have to happen one after the other, i.e. P to X and X to Q, the total ways would be ${}^{5}C_3 \times {}^{8}C_5$.

For part (c), again we would apply the same logic, going from A to N and then from N to Q, giving us a total of ${}^{7}C_3 \times {}^{6}C_2$ ways.

For part (d), we could find the number of ways from A to B via X (${}^{7}C_5 \times {}^{6}C_3$) and square it, as there will be the same number of routes to return. Thus the required answer will be $(21 \times 20)^2 = 420^2$ =176400.

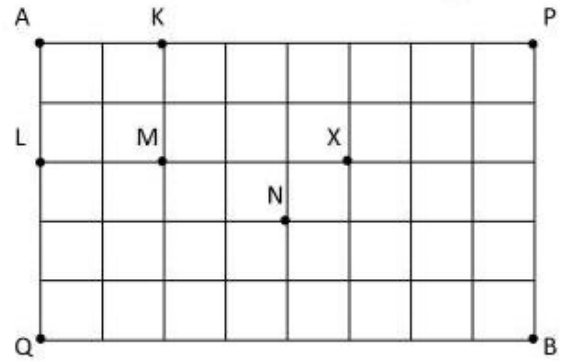
For part (e), the shortest route will not be of length 13 as there is no route of length 13 from A to B that contains both N and X. So one has to either traverse A – N – X – B or A – X – N – B, a distance of 15 units in either case. The number of ways to traverse the route A – N – X – B is (${}^{7}C_3 \times {}^{2}C_1 \times {}^{6}C_3$) while the number of ways along A – X – N – B is (${}^{7}C_2 \times {}^{2}C_1 \times {}^{6}C_4$) which means that the total number of ways is the sum of these two cases (35 x 2 x 20) + (21 x 2 x 15) which gives 2030.

In the next post, we'll look at a couple of slightly more complex questions based on the same grid.

SHORTEST PATH - 3

Continuing from the last post, let's look at a couple more possibilities based on the same grid:

Q. The grid below is a network of roads in a town. Travel is permitted only along the gridlines.



(f) If a straight road is built joining M and N, find the number of shortest routes from A to B.
(g) If a giant mall is constructed on the area AKML and all the roads in the interior of this area are destroyed to make room for it, then how many shortest routes will be there from A to B?

In part (f), a little thinking would tell us that the shortest route is now less than 13 units in length as the straight route MN is shorter than a route from M to N along the grid. Hence we could break the required route up into 3 parts – A to M, M to N and N to B. The first stretch, A to M, can happen in ${}^4C_2$ ways, M to N can be traversed in only 1 shortest way, while N to B can be covered in ${}^6C_2$ ways.
Thus the total number of ways is ${}^4C_2 \times 1 \times {}^6C_2$.

For part (g), we have to list possibilities. Note that when we are going from A to B, points like X and N are mutually exclusive, i.e. the same 'shortest path' cannot contain both. In such problems, the best way is to find a set of such mutually exclusive points which cover, between them, every possible path. In this case, every path must pass through K or through L, but no path passes through both. Number of paths through K would be ${}^{11}C_5$, while number of paths through L would be ${}^{11}C_3$. Hence since we go through K or L, we need to add to find the total possible paths as ${}^{11}C_5 + {}^{11}C_3$.

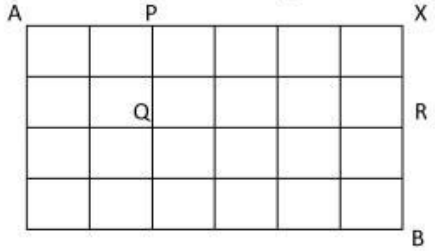
In the next post, we'll look at a slightly slower method, but one which can be used where this would fail (for example a grid with missing roads at multiple places)

SHORTEST PATH - 4

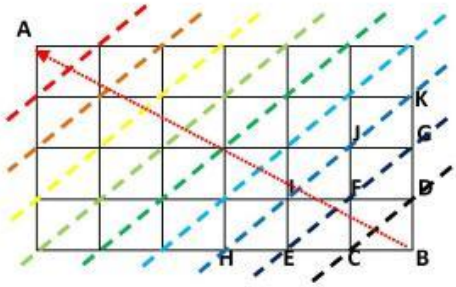
Number of Shortest Paths: Alternative Approach

In today's post, we'll look at another, albeit slower, way to solve the same problem we did 3 days ago:

Q. In the figure below, what is the number of shortest paths along the grid from A to B?



Let's try and develop an alternative method. In this method we'll try to reverse-engineer, i.e. to start from the end and go back to the starting point.



Start with the end point, B. Now move backwards one row at a time in the direction indicated by the red arrow (the order is colour-coded in rainbow colours, violet to red, for ease of visualisation) – so in the second round the points covered will be C and D, in the third round E, F and G, in the fourth H, I, J and K, and so on. In each step, assign a value to each point which will be the number of ways to get to B from that point. This is actually straightforward – since we can only move to the right and down, the value at any point (say J) will be the sum of the values to the right and below it (G and F), as long as there is a path connecting it to those points.

So here C and D will each have the value 1. E, F and G will be assigned values 1, 2, and 1. H, I, J and K will get values 1, 3, 3 and 1 and so on. The final table will end up looking something like this:

A

210	126	70	35	15	5	1
84	56	35	20	10	4	1
28	21	15	10	6	3	1
7	6	5	4	3	2	1
1	1	1	1	1	1	1

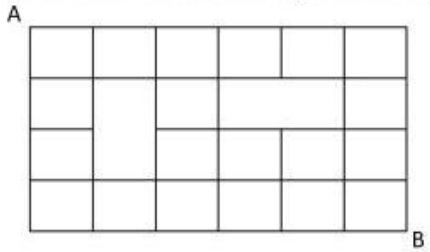
B

Obviously, for this particular problem, the earlier method is quicker. So why do we need another approach? The answer is very simple: the earlier approach is restricted and can be applied only for an unbroken grid. In the next post, we'll examine how to apply this alternative approach in a couple of such non-standard situations.

SHORTEST PATH - 5

Continuing from where we left off in the previous post, what happens if there are gaps in the grid?

Q. In the figure below, what is the number of shortest paths along the grid from A to B?



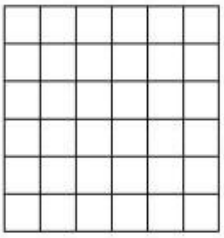
Immediately we can see that the earlier method, the direct formula based on ${}^{n}C_{r}$, will not work here. But the reversed method we introduced in the last post can still be applied even in a case like this where we have a broken grid:

135	87	52	23	9	5	1
48	35	29	14	4	4	1
13	6	15	10	6	3	1
7	6	5	4	3	2	1
1	1	1	1	1	1	1

So the number of shortest paths would be 135.

Let's look at another problem, superficially dissimilar but boiling down to the same kind of thing:

Q. There is a 6 x 6 chessboard and a king is placed on the bottom left corner square. In how many ways can the king reach the bottom right corner in exactly 5 moves? (In one move, a chess king can move 1 space horizontally, vertically, or diagonally from the square it currently occupies.)



The only difference here is that the king moves along the squares rather than the grid-intersections. Thus we could represent it as follows (note that the king can only move up to the third row).

		3	1		
	12	5	2	1	
21	9	4	2	1	1

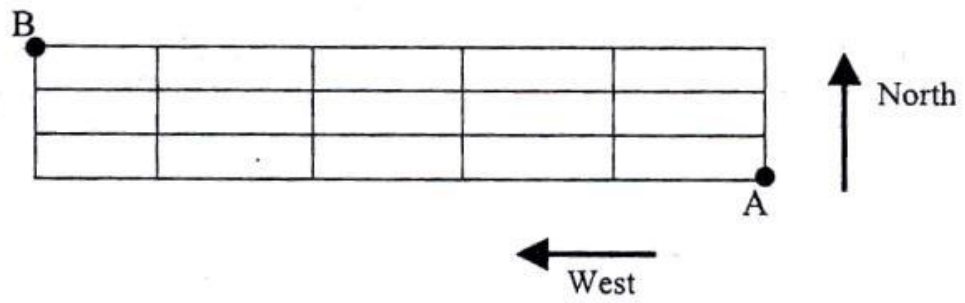
Considering all possible squares the king could visit, we can see that there are only 21 possible paths.

SHORTEST PATH – CAT SCAN

Number of Shortest Paths: CAT Scan

Let's check out some problems on this concept from past CATs; here's one from CAT 2004:

In the adjoining figure, the lines represent one-way roads allowing travel only northwards or only westwards. Along how many distinct routes can a car reach point B from point A?

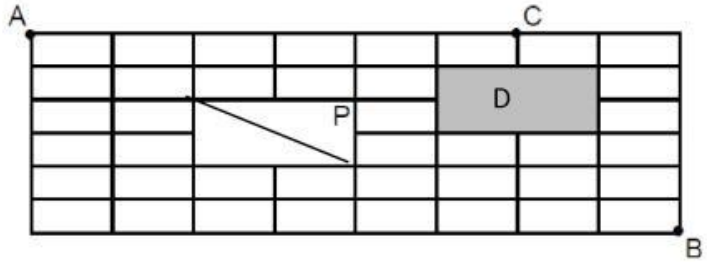


1) 15 2) 56 3) 120 4) 336

From our discussions this week we can see that the answer would be ${}^{8}C_{3} = 56$. Hence (2)

And a couple more from CAT 2008:

The figure below shows the plan of a town. The streets are at right angles to each other. A rectangular park (P) is situated inside the town with a diagonal road running through it. There is also a prohibited region (D) in the town.



Neelam rides her bicycle from her house at A to her office at B, taking the shortest path. Then the number of possible shortest paths that she can choose is
(1) 60 (2) 75 (3) 45 (4) 90 (5) 72

Neelam rides her bicycle from her house at A to her club at C, via B taking the shortest path. Then the number of possible shortest paths that she can choose is
(1) 1170 (2) 630 (3) 792 (4) 1200 (5) 936

In the first question the answer would be ${}^{4}C_{2} \times 1 \times {}^{6}C_{2} = 6 \times 15 = 90$. Hence (4)

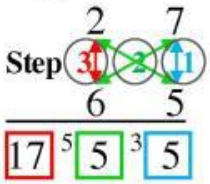
In the second question, the ways from B to C would be 1 (via the top right corner of the figure) + 2 x 6 (via the upper right corner of region D) = 13. Hence the total ways would be 90 x 13 = 1170. Hence (1) (Note that the answer has to be a multiple of 90, which eliminates the last 3 options anyway!)

MULTIPLICATION - 1

Multiplication - 1

We know the standard technique of multiplication which we learnt in school. But is it the quickest or the most efficient way? Not always. Let's look another approach – a Vedic Math style method:

Suppose we have to multiply 27 * 65. We can do it as follows



Hence 27 * 65 = 1755.

The steps involved are –
(1) 7*5 = 35, write 5 and carry 3
(2) 2*5 + 6*7 = 52 + the carried 3 = 55, write 5, carry 5
(3) 2*6 = 12 + the carried 5 = 17

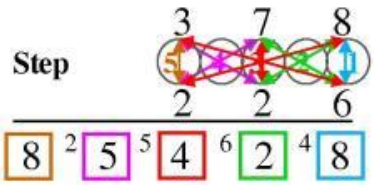
Why does this work? The method uses the properties of base systems. Remember that the digits of a number in base n, counting from the last, represent the number of "1"s, "n"s, "n^2"s, "n^3"s and so on. In our usual base 10, this means ones (or units), tens, hundreds, thousands and so on.

So for the first step above, we ask ourselves – for the units digit of the answer, what powers of 10 will multiply to give a 1? Only a 1 * 1. So the last digit is straightaway got by multiplying the last digits of the original number. The carry-over is the number of tens.

For the second step, we ask, what powers will give a ten? Either 1 * 10 or 10 * 1. So we multiply unit's place of one number with ten's place of the other and vice versa, and add them (remember, an OR becomes an addition according to the principle of counting). We also add the tens brought from the carry-over of the previous step. The carry-over from this step would be a number of hundreds.

Similarly in the third step, we would get hundred's place by 100 * 1, 10 * 10 or 1 * 100. In the above case, only 10 * 10 exists (as neither number has a hundreds place) and hence we would only use this, and add the carry-over, and since this is the last step, we would write it out in full.

Now this could easily be extended to 3 digit multiplication as well, for example to multiply 378 * 226:



Hence 378 * 226 = 85428.

(Note: The above **looks** complicated, but can be mastered within a dozen or so tries)

MULTIPLICATION - 1

Multiplication – 2

A couple of useful tricks which might help in multiplication:

1) To multiply (or divide) by 5: This is equivalent to multiplying (or dividing) by 10/2. For example:

218 * 5 = 109*10 = 1090 428/5 = 856/10 = 85.6

1387 * 5 = 693.5 * 10 = 6935 1257/5 = 2514/10 = 251.4

This can be extended to multiplying or dividing by 25 as well (for example 347*25 = 347/4*100 = 86.75*100 = 8675, while 724/25 = 724*4/100 = 2896/100 = 28.96)

2) To multiply numbers whose difference is even: These can often be done as (a+b)(a-b)

23 * 37 = (30+7)(30-7) = 900 – 49 = 851

26 * 18 = (22+4)(22-4) = 484 – 16 = 468

3) Approximation: We don't always need an exact or accurate answer. A reasonable approximation is good enough, in many DI-style questions. So how to approach this? Let's try some examples:

a) 7436 * 2507

Let's approximate this as 7436 * 2500. (Note that the error introduced here is 7/2500 ≈ 0.28%)

Now 25 = 100 /4 so let's do it as 7436 * 10000/4 = **18590000**

The exact answer is 18642052, so we have an error of just around 5/1800

b) 5872 * 2176

Neither number is very close to an "easy" round number. Let's analyse this then. First of all, if I want to find a * b and a is increased, what should happen to b to keep the value of a * b same? It should logically decrease. So here if we change, say, the second number from 2176 to 2000, we will need to increase the first number. But by how much?

One tempting thing would be to increase by the same value (i.e. 176) but a little thinking will show us that this doesn't make sense. Let me demonstrate using a smaller example: if we want to do 51 x 11, reducing 11 to 10 and increasing 51 to 52 will give us 520 – the correct answer is 561.

The thing to realise is that instead of changing by the same **quantity** in opposite directions, we should change by the same **percentage**. So since 51 is roughly 5 times 11, the change also should be 5 times. Thus if we were to change 11 (by subtracting 1) to 10, we must change 51 (by adding 5) to 56, and with this we would get 560, a pretty accurate answer.

Coming back to our original problem then, 5872 is very roughly 3 times 2176 and so if we change 2176 by -176 ≈ -180, we should change 5872 by ~+540 to 6412. Hence our approximation become 6412 * 2000 = **12824000**

The exact answer is 12777472, so we have an error of around 5/1200.

REMAINDERS -1

Over the next few posts, we'll discuss the theory of remainders (starting from the ground up)

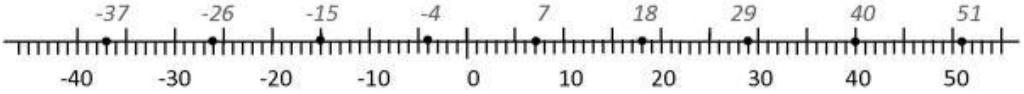
Remainders – Some basics

We know that when a number **(m, the dividend)** is divided by another smaller number **(n, the divisor)** it gives us a quotient **(say k)** and a remainder **(r)** which is less than n. If a number m is divided by a number n (where $n \leq m$) and it leaves a remainder r, then we say **m is of the form nk + r**.

For example, when 18 is divided by 11, it leaves a remainder 7; in other words 18 = 11 * 1 + 7. We can thus say that 18 is of the form 11k +7. Similarly 45 can be written as of the form 11 * 4 + 1 and thus is of the form 11k+1.

If we try to find other numbers of the form 11k + 7, we will find that they repeat after every 11 steps. For example by taking k = 1, 2, 3… we can generate a set of numbers 18, 29, 40, 51 and so on which are of the form 11k + 7. Note that, technically speaking, this can be extended on the lower side as well by taking k as zero or negative, thus numbers like 7, – 4 , – 15 etc can also be thought of as belonging to the 11k + 7 category.

On the number line, we can see this "equally-spaced" pattern:



Since there are 11 possible remainders (0 to 10) when dividing a number by 11, we can find 11 such categories (numbers of the form 11k, 11k+1, 11k+2 ….11k+9 or 11k+10) when dividing by 11.

Similarly, if we were to divide by 5, we would get 5 categories viz 5k, 5k+1, 5k+2, 5k+3 and 5k+4. Moreover, the numbers in each of these categories would be at intervals of 5 steps from each other on the number line.

In general, if we are dividing by n, we find that there are n possible categories with remainders from 0 to (n – 1) and the numbers in each of these categories will be at intervals of n steps.

Note: If we want to write a number in terms of two different divisors, it is a good idea to use different variables for the quotients. For example, if we want to write 18 in terms of 11, 13 and 5 we can write it as 11k + 7, 13m + 5 and 5p +3 (here we are using different symbols for the quotients, such as k, m, p, to avoid confusion as they may have different values in each case).

Thus if a number leaves a remainder 3 when divided by 5 and 4 when divided by 7, we can say that it is of the forms 5k+3 as well as 7m+4.

In the next post we shall see how to find such a number!

REMAINDERS -2

Combining Remainders and LCM

We observed last time that when we look for, say, a number that leaves a remainder 3 when divided by 7, we will find one such number at every 7 steps. Similarly if we look for a number which leaves 2 when divided by 11, we will find such a number once every 11 steps. Now what if we are asked to find a number that satisfies both the above criteria?

Q. Find a number that leaves a remainder 3 when divided by 7 and 2 when divided by 11.

We can quickly realise that if numbers satisfying the first criterion (7k + 3 type) occur every 7 steps and numbers satisfying the second one (11m + 2 type) occur every 11 steps, then numbers satisfying both criteria will occur once every LCM(7, 11) i.e. every 77 steps. This also means that the first such number will be between 0 and 76. Later numbers will be found by adding 77 to the first such number.

A simple way to go about finding such a number is to take all the numbers less than 77 satisfying one of the conditions and find which of them also meets the other requirement. But which condition should we start with - (7k + 3) or (11m + 2)? A sensible approach would be to start with the larger value (here 11) since that will give us less numbers to check (in 77 numbers, going every 7 steps would give us 11 candidates while going every 11 steps would give us 7 candidates!). So let us generate numbers of the form 11m + 2 (by just putting m = 0, 1, 2…) up till 77, and see which of them also happens to be of the form 7k + 3: 2, 13, 24, 35, 46, 57 and 68 out of which 24 = 7 * 3 + 1. Hence our answer would be 24.

Let's try some more:

Q. Find the least 3-digit number that leaves remainder 7 and 4 respectively when divided by 18 and 15.

The number must be of the form 18k + 7 as well as 15m + 4. LCM of 18 and 15 is 90. Candidates of the 18k+7 type below 90 are 7, 25, 43, 61, 79, of which 79 satisfies the given conditions. Further such numbers will be of the form 90n + 79 (i.e.79, 169, 259…) and the first 3-digit one is 169.

Q. Find a number that leaves remainder 2 and 3 respectively with 6 and 11, and is divisible by 5.

There are 3 conditions: 6k + 2, 11m + 3 and 5n. For the first two conditions, trying numbers up till 66 of the 11m + 3 form i.e. 3, 14, 25, 36, 47, 58, we find that the first number is 14. Further such numbers will be of the form 66p + 14 and checking such numbers (14, 80, 146…) for a number satisfying the third condition (i.e. divisible by 5) we get the required answer as 80.

In general, if a number when divided by A leaves x and when divided by B leaves y, then the first such number will lie within [0, L-1] where L is the LCM of A and B.

REMAINDERS -3

Variations on remainders

Q. A number when divided by 36 leaves a remainder 23. What remainder will it leave when divided by 9?

Let the number be 36k + 23. When this is divided by 9, the first part will leave no remainder (as 36k is perfectly divisible by 9); so we only need to look at the remainder of the second part i.e. 23 when divided by 9, which is 5.

Thus if we know the remainder of a number N when divided by A, we can find the remainder when N is divided by B, as long as B is a factor of A

Let's see a few more examples:

Q. A number when divided by 24 leaves a remainder 17. What remainder will it give when divided by: (a) 8 (b) 12 (c) 6 (d) 9 (e) 3 (f) 4

Let the number be 24k+ 17. Hence when divided by
(a) 8, it will leave 1
(b) 12, it will leave 5
(c) 6, it will leave 5
(d) 9, we will not be able to determine a unique answer as 24k will also leave a non-zero remainder
(e) 3, it will leave 2
(f) 4, it will leave 1

Q. A number leaves remainder 5 and 3 respectively when divided by 4 and 9. What will be the remainder when it is divided by 6?

The number is of the form 9k+5 and also 4m+3. Hence there will be a solution within LCM(9,4) i.e. 36 numbers. Checking numbers up till 36 of the form 9k+5 i.e. 5, 14, 23 and 32 we find that 23 also satisfies 4m+3 form. Hence the required number must be of the form 36n+23. When this is divided by 6, it will give a remainder of 5. **Hence (3)**

Q. When n is divided by d it leaves a remainder of 52. When 5n is divided by d, the resultant remainder is 4. How many values of d are possible?

We have n = kd +52 (Note: we may safely assume n and d as natural numbers in such problems)
So 5n = 5kd + 260 which when divided by d gives a reminder 4.
Since 5kd is clearly divisible by d, this implies that 260 leaves a remainder 4 with d, or d is a factor of 256. However, d also must be greater than 52 since the original remainder with d was 52! Hence the only possibilities are 64, 128 and 256, i.e. **3 values.**

What happens when B is a multiple (rather than a factor) of A? In this case we get multiple answers, as we shall see in the next post.

REMAINDERS -4

Variations on remainders

Q. A number is divisible by 7. What could be the remainder when it is divided by 28?

Let's just check what happens here. Consider multiples of 7 – 7, 14, 21, 28, 35, 42, 49, 56, 63, 70…
Now if we find the remainder of each of these when divided by 28, we get 7, 14, 21, 0, 7, 14, 21, 0, 7…
In other words, the remainder could only be 0, 7, 14 or 21 (in other words 7*0, 7*1, 7*2 and 7*3) and it goes in a cyclic pattern.

Similarly when a multiple of 13 is divided by 78, the remainders could be 0, 13, 26, 39, 52 or 65.

Note that if we wish to take say 7*15 and divide that by 28, we could do it by taking 7 common in the numerator and denominator (but not cancelling them – this is the place where people generally go wrong!), finding the remainder of 15/4, and multiplying that by 7 as follows:

$$\text{Rem}\left(\frac{7*15}{28}\right) \equiv \underbrace{\text{Rem}\left(\frac{7*15}{7*4}\right) \equiv 7*\text{ Rem }\left(\frac{15}{4}\right)}_{\text{(Note this step! 7 is not cancelled...)}} \equiv 7*3 = 21$$

In general, when a number divisible by k is divided by nk, the remainders would be 0, k, 2k…(n-1)k. This may seem like a very simple thing, but can come in startlingly useful in some types of remainder problems. For example, if we want to know what is the remainder when 7^{30} is divided by 21, we can take the 7 common and find the remainder when 7^{29} is divided by 3 (something we shall learn to do in a subsequent post), and multiply that by 7.

Q. What will be the remainder when 7*2013 is divided by 35?

This will be the same as 7*(remainder when 2013 is divided by 5) = 7*3=21.

Q. A number when divided by 7 leaves 3. What could be the remainder when it is divided by 28?

As above, let's check what could happen – taking numbers of the form 7k+3 i.e. 3, 10, 17, 24, 31, 38… we get remainders with 28 to be 3, 10, 17, 24, 3, 10, 17, 24, 3, 10…

So the required possible remainders will be all numbers of the form 7k+3, less than 28.

Q. A number when divided by 12 leaves a remainder 8. What could be the remainder when half the number is divided by 30?

If the original number is 12k + 8, half the number would be 6k + 4. When this is divided by 30 it would of course give all possible remainders of the form 6k+4 less than 30 i.e. 4, 10, 16, 22 and 28.

Now we are ready to try more advanced things with remainders, which we shall do next week:

REMAINDERS -5

Remainders of sums and products

Suppose we are discussing division by 13. Let us consider a few numbers:
18 (of the form 13k + 5) **30** (of the form 13p + 4) **55** (of the form 13q+ 3)

Now, what would happen to the remainder if we divide the sum of 18 and 30 by 13? The sum, 48, when divided by 13 gives us 9, which is the same as the sum of the individual remainders 4 and 5. We can try the same thing with the sum of 30 and 55 (85 divided by 13 gives 7, which is the same as 4 + 3)

What about the remainder of the products? 30 * 55 = 1650 which when divided by 13 gives a remainder 12 – the same as 4 * 3, the product of the individual remainders. Similarly 18 * 30 = 540, which when divided by 13 gives us 7. But if we multiply the individual remainders, 5 * 4 = 20. Since this is greater than 13, we divide it by 13 (remember, the remainder cannot be greater than the divisor!) which gives us the expected remainder of 7.

This seeming coincidence is easy to explain. If we add 18 and 30, we are adding numbers of the forms 13k + 5 and 13p + 4, which will yield 13k + 13p + 5 + 4. Since 13k + 13p is clearly divisible by 13, we only need the remainder when 5 + 4 is divided by 13.

Similarly when we multiply the two, basic algebraic expansion will give the product as (13k * 13p) + (13k * 4) + (13p * 5) + (5 * 4), and since all but the last term are clearly divisible by 13, we only need the remainder when (5 * 4) is divided by 13.

Thus we have two simple rules when dividing a sum or product of two numbers by a fixed number n

The remainder of the sum of two numbers a and b when divided by n is equivalent to the sum of their individual remainders when divided by n.

The remainder of the product of two numbers a and b when divided by n is equivalent to the product of their individual remainders when divided by n

Q. What will be the remainder when 18 + 30 + 55 is divided by 13?

We can just add up all the remainders as above giving 5 + 4 + 3 = **12**

Q. What about the remainder when 18 * 30 * 55 is divided by 13?

We can just multiply all the remainders as above giving 5 * 4 * 3 = 60 which in turn, divided by 13 gives **8**

Q. And when 18 * 30 * 55 * (18 +30) * (30 + 55) is divided by 13?

Proceeding as above, we can say 5 * 4 * 3 * (5 + 4) * (4 + 3). Rather than solve the whole thing, we can simplify in steps: this is equivalent to 60 * 62 ≡ 8 * 11 ≡ **10**

REMAINDERS -6

Remainders of powers

Q. What will be the remainder when 18^{99} is divided by 13?

Remember that the remainder of product is the product of remainders. When divided by 13

- $18 \equiv 5$
- 18^2 i.e. 18 * 18 gives 5*5 ≡ 25 ≡ 12
- 18^3 i.e. 18^2 * 18 gives 12*5 ≡ 60 ≡ 8
- 18^4 i.e. 18^3 *18 gives 8*5 ≡ 40 ≡ 1
- 18^5 i.e. 18^4 *18 gives 1*5 ≡ 5

After this, the pattern will repeat, the remainders of subsequent powers of 18 will be successively 54, 12, 8, 1, 5, 12, 8, 1…thus we can say that when divided by 13 any number of the form 18^{4k+1} will give a remainder 5, any number of the form 18^{4k+2} will give a remainder 12, any number of the form 18^{4k+3} will give a remainder 8, and any number of the form 18^{4k} will give a remainder 1. Thus 18^{99} which is of the form 18^{4k+3} will give a remainder 8.

Note that we could also have started by saying that $18^{99} \equiv 5^{99}$ i.e. they are equivalent, when divided by 13. Then we could have proceeded with powers of 5.

Once we know the pattern 5, 12, 8, 1 ➔ 18^4 gives 1, we could also use the laws of indices to say $18^{99} = (18^4)^{24}*18^3$and since remainder of product = product of remainders, this is equivalent to $(1)^{24}*(8) \equiv 8$.

Some more examples:

Q. What is the remainder when 30^{3030} is divided by 13?

When divided by 13, $30^{3030} \equiv 4^{3030}$. Now looking at powers of 4, the remainders with 13 are **4**, (4*4) ≡ **3**, (3*4) ≡ **12**, (12 *4) ≡ **9**, (9*4) ≡ **10**, (10*4) ≡ **1**. Hence 4^6 gives 1. So we can write $4^{3030} \equiv (4^6)^{505} \equiv 1^{505} \equiv 1$.

Q. What is the remainder when 2013^{2014} is divided by 7?

When divided by 7, 2013 ≡ 4. Hence $2013^{2014} \equiv 4^{2014}$. The pattern for 4 with 7 is **4**, 4*4 ≡ **2**, 2*4 = **1**. So $4^{2014} \equiv (4^3)^{671}*4^1 \equiv 1*4 \equiv 4$.

Q. What is the remainder when 3^{45678} is divided by 16?

The pattern for 3 with 16 is **3**, 3*3 ≡ **9**, 9*3 = **11**, 11*3 = 1. So $3^{45678} \equiv (3^{4k+2}) \equiv 1*3^2 \equiv 9$.

Finding a repeating step can be a long procedure; can we reduce it a little bit? Let us examine this in the next post.

REMAINDERS -7

Using Negative Remainders

Let's examine the remainder of a power more closely: will we always get a repeating pattern? The short answer is "yes".

Suppose we are dividing by 13. The possible remainders are 0, 1, 2, 3….12 i.e. 13 values. Either the 1st remainder will be 0 (in which case it repeats immediately!) or it will be non-zero (in which case it will repeat in a maximum cycle of 12 steps (as there are only 12 different non-zero remainders possible). Similarly when dividing by 17, we could have a maximum cycle of 16 steps. Still, 16 steps is a lot of calculation. Is it possible to reduce this further? Yes, if we relax the definition of a remainder a trifle.

When we divide 29 by 8, we say that the remainder is 5. What this means is that 29 can be written as
29 = 8 * 3 **+ 5**. But if we think about this, 29 could also be written as
29 = 8 * 2 **+ 13** or
29 = 8 * 1 **+ 21** or
29 = 8 * 0 **+ 29** or even
29 = 8 * 4 – **3** or
29 = 8 * 5 – **11** and so on
So we could say (non-rigorously) that the remainder of 29, when divided by 8, could be equivalent to 29 or 21 or 13 or 5 or -3 or -11….and here we observe a pattern; the difference between two consecutive remainders in this series is the divisor 8. And the **formal Remainder** would be the one between 0 and 7. A more general way to put it would be to say that when dividing by 8, the remainder $29 \equiv 21 \equiv 13 \equiv 5 \equiv -3 \equiv -11$.

Similarly when dividing by say 11, remainder $40 \equiv 29 \equiv 18 \equiv 7 \equiv -4 \equiv -15$ and so on.

So if by any method we could get a remainder as -3 when dividing by 8, we just need to add 8 to get 5.

So how do we use this? One possible way is in sums like the one below:

Q. What is the remainder when 2011 * 2012 * 2013 * 2014 is divided by 13?

We could solve this by noting that 2002 is divisible by 13 (it is 13 *154) and hence the remainders of the given elements are 9 * 10 * 11 * 12. However, multiplying these and diving by 13 is a cumbersome process; we could also just note that 2015 is also divisible by 13 and hence (–4)* (–3)* (–2)* (–1)/13 = 24/13 = **11**.

Another way would be in solving "remainder of power" sums more quickly:

Q. What is the remainder when 8^{789} is divided by 9?

We can note that when 8 is divided by 9 it gives $8 \equiv -1$ as the remainder. So $8^{789} \equiv (-1)^{789} \equiv -1 \equiv 8$.

We can extend this to considerably more complex sums as well, as we shall see next time.

REMAINDERS -8

Remember that in "remainder of power" sums, we look for the pattern of powers until we get a 1. The reason is simple; 1 is the only natural number for which we can instantly find any power. Is there any other such number if we relax the criterion of "natural number"? Yes there is; powers of (-1) also can be found at a glance. Hence we can put the idea of a negative remainder to good use, as follows:

Q: What will be the remainder when 11^{2013} is divided by 17?

Now we can see the cycle of 11 with 17 as:

$11^1/17 \equiv$ **11** $\quad 11^2/17 \equiv 11*11/17 \equiv$ **2** $\quad 11^3/17 \equiv 2*11/17 \equiv$ **5**

$11^4/17 \equiv 5*11/17 \equiv 4 \quad 11^5/17 \equiv 4*11/17 \equiv$ **10** $\quad 11^6/17 \equiv 10*11/17 \equiv$ **8**

$11^7/17 \equiv 8*11/17 \equiv$ **3** $\quad 11^8/17 \equiv 3*11/17 \equiv$ **16**

….but 16 ≡ -1 when dividing by 17. So we can take an 8-step-repeating logic and solve as follows:

$\frac{11^{2013}}{17} \equiv \frac{(11^8)^{251} * 11^5}{17} \equiv \frac{(-1)^{251} * 11^5}{17} \equiv \frac{-1*10}{17} \equiv \frac{-10}{17} \equiv \frac{7}{17}$. Hence the required remainder would be **7**.

Let's see a few other examples.

Q: What will be the remainder when 7^{567} is divided by 11

The cycle of 7 with 11 will be 7, 5, 2, 3, 10 (≡ -1) So we can solve this as:

$\frac{7^{567}}{11} \equiv \frac{(7^5)^{113} * 7^2}{11} \equiv \frac{(-1)^{113} * 7^2}{11} \equiv \frac{-1*5}{11} \equiv \frac{-5}{11} \equiv \frac{6}{11}$. Hence the remainder is 6.

Q: What will be the remainder when 23456^{34567} is divided by 13?

23456 when divided by 13 gives 4. The cycle of 4 with 13 is 4, 3, 12 (≡ -1). So we can do this as

$\frac{23456^{34567}}{13} \equiv \frac{4^{34567}}{13} \equiv \frac{(4^3)^{11522} * 4^1}{13} \equiv \frac{(-1)^{11522} * 4^1}{13} \equiv \frac{1*4}{13} \equiv \frac{4}{13}$. Hence the remainder is **4**.

Q: What will be the remainder when 19^{2013} is divided by 17?

19 when divided by 17 gives 2. The cycle of 2 with 17 is 2, 4, 8, 16 (≡ -1). So we can do this as

$\frac{19^{2013}}{17} \equiv \frac{2^{2013}}{17} \equiv \equiv \frac{(2^4)^{503} * 2^1}{17} \equiv \frac{(-1)^{503} * 2^1}{17} \equiv \frac{-1*2}{17} \equiv \frac{-2}{17} \equiv \frac{15}{17}$. Hence the remainder is **15**.

Q: What will be the remainder when $5^{555} + 3^{333}$ is divided by 26?

We can find the separate remainders as above and add them. 5^2 gives -1 while 3^3 gives 1 so

$\frac{5^{555} + 3^{333}}{26} \equiv \frac{(5^2)^{277} * 5^1 + (3^3)^{111}}{26} \equiv \frac{(-1)^{277} * 5^1 + 1^{111}}{26} \equiv \frac{-1*5+1}{26} \equiv \frac{-4}{26} \equiv \frac{22}{26}$. Hence **22**.

Next week, let's take a break from Remainders and look at a couple of other useful concepts.

REMAINDERS -9

The Binomial Theorem

We know that

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and with a little trouble we can work out that

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Let us try to generalise this:

Suppose we need to find $(a + b)^7$. This can be written, in expansion, as

$$(a + b)^7 = (a + b)\,(a + b)\,(a + b)\,(a + b)\,(a + b)\,(a + b)\,(a + b)$$

Note that every term in the expansion will be of the 7th order, as we will be picking either an "a" or a "b" from each bracket. Hence the resultant would be something like

$$m_1 * a^7 + m_2 * a^6b^1 + m_3 * a^5b^2 + m_4 * a^4b^3 + m_5 * a^3b^4 + m_6 * a^2b^5 + m_7 * a^1b^6 + m_8 * b^7$$

where m_1, m_2 etc are some coefficients.

So how do we find these coefficients?

To get an a^7, we need to select an "a" from all 7 brackets. This can happen in only 1 way. So $m_1 = 1$
To get a^6b^1, we need to select b from 1 of the 7 brackets, which can happen in 7 ways. So $m_2 = 7$
To get a^5b^2, we need to select b from 2 of the 7 brackets, which can happen in ${}^7C_2$ ways. So $m_3 = {}^7C_2$
Proceeding in this manner, we can update the above expression as

$$(a + b)^7 = {}^7C_0\, a^7 + {}^7C_1\, a^6b^1 + {}^7C_2\, a^5b^2 + {}^7C_3\, a^4b^3 + {}^7C_4\, a^3b^4 + {}^7C_5\, a^2b^5 + {}^7C_6\, a^1b^6 + {}^7C_7\, b^7$$

This logic, of course, can be extended to any power of the binomial expression (a + b).

We can also note that $(a + b)^n$ will consist of exactly n + 1 terms; all except the last are divisible by "a", and all except the first are divisible by "b". This also enables us to answer certain questions like

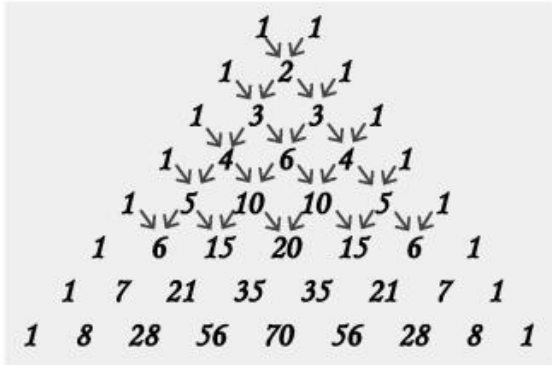
Q. How many terms in the expansion of $(p+q)^{15}$ are divisible by p^6?

Notice that $(p+q)^{15}$ will have 16 terms out of which p^0 to p^5 i.e. 6 terms will not be divisible by p^6.

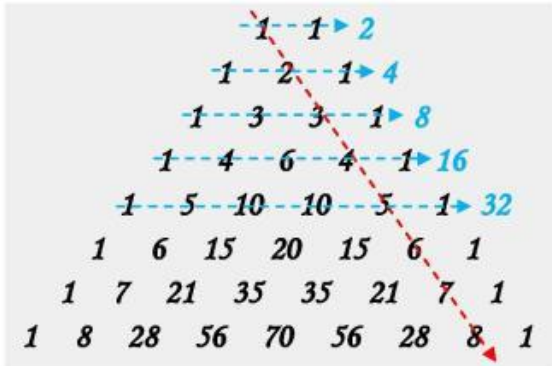
REMAINDERS -10

Pascal's Triangle: Introduction

Pascal's Triangle is a versatile tool which can be of help in a variety of problems, from Compound Interest to Combinatorics. It is an infinite triangular array of numbers, starting with two '1's in the first row, and with each subsequent row containing one number more than the previous one. Each row starts and ends with a 1, and each number in between is computed by adding the two numbers diagonally above in the previous row as shown in the adjacent figure.

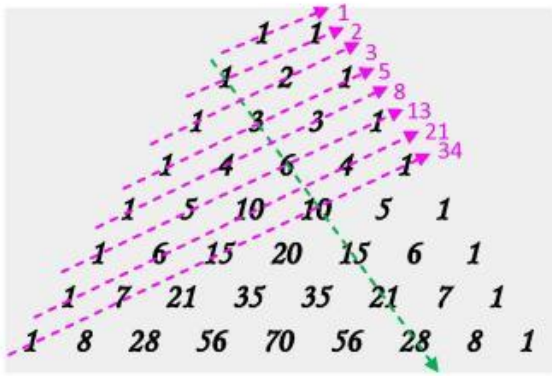


Many useful series can be generated using this triangle, for example:



1) N (Natural numbers: 1, 2, 3, 4, 5…)
These can directly be read along the red arrow.

2) 2^n (Powers of 2: 2, 4, 8, 16, 32…)
These are the row sums, along the blue arrows. Thus the sum of the numbers in the nth row is 2^n.



3) Σn (Summation of Natural Numbers: 1, 3, 6, 10, 15…)
These can be read along the green arrow.
For example, $\Sigma 5 = 15$ is the 5th number along the arrow.

4) F_n (Fibonacci sequence: 1, 2, 3, 5, 8, 13)
These numbers can be found as the sums along the 'shallow diagonals', the pink arrows in the figure.

5) 11^n (Powers of 11: 11, 121, 1331, 14641…)
These numbers can be obtained directly by reading off each row. If a number is greater than 10, however, it needs to be carried over.
For example, 11^6 is computed by taking the 6th row i.e. 1 6 15 20 15 6 1 and adding up as shown alongside; thus $11^6 = 1771561$.

```
1 – – – – – –
  6 – – – – –
  1 5 – – – –
    2 0 – – –
      1 5 – –
          6 –
            1
-----------
1 7 7 1 5 6 1
```

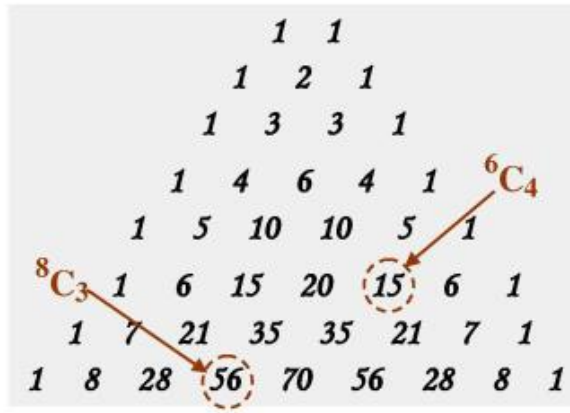
This is particularly useful when calculating 10% compound interest (a fairly common question) as we need to calculate powers of $(1 + \frac{10}{100})^n = 1.1^n$
So in 6 years at 10%, the C.I. earned is 77.1561%

REMAINDERS -11

Pascal's Triangle, ^nC_r values and Binomial Coefficients

One of the most useful properties of Pascal's triangle is the fact that the elements can directly give us ^nC_r values. Specifically, the elements of the nth row, read in order, will give us $^nC_0$, $^nC_1$, $^nC_2$ … ^nC_n.

For example if we need $^6C_4$, we can go to the 6th row and take the 5th value (remember, we start from $^6C_0$!). Thus $^6C_4 = 15$.



In a similar manner we can find, say, $^8C_3 = 56$

From the triangle, we can see several properties of ^nC_r:

1) $^nC_0 = {}^nC_n = 1$

2) $^nC_1 = {}^nC_{n-1} = n$

3) $^nC_r = {}^nC_{n-r}$

4) $^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

5) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n$

Since Binomial expansion can be written using ^nC_r values, Pascal's triangle can also help to quickly work out basic binomial expansions as follows:

In general, we know that

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$

But since $^nC_0$, $^nC_1$, $^nC_2$ … ^nC_n are directly given by the nth row of the triangle, we can quickly write the expansion using this table. For example for $(a + b)^5$ we could look at the 5th row of the triangle and write:

$$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

and similarly for $(a+b)^7$ we would get

$$(a + b)^7 = a^7 + 7\,a^6b^1 + 21\,a^5b^2 + 35\,a^4b^3 + 35\,a^3b^4 + 21\,a^2b^5 + 7\,a^1b^6 + b^7$$

REMAINDERS -12

Pascal's Triangle: Examples

Q. A certain sum of money is invested for 7 years at 10% compound interest. If it yields an interest of Rs 18974, find the original sum.

1) Rs 200000 2) Rs 25000 3) Rs 20000 4) Rs 24000

Note that in 7 years, the original sum P will become P * $(1.1)^7$ = 1.9487171 P. But this is the total amount. Hence the compound interest accrued (Rs 18974) will be equal to 0.9487171P.
This means that P is $\frac{18970}{0.9487171}$ which equals Rs 20000. Hence (3)

Q. In how many years will a sum of Rs 128,000, invested at 10% per annum compound interest, earn an interest at least equal to the principal?

1) 6 years 2) 7 years 3) 8 years 4) 9 years

For CI to be equal to P, the amount has to be 2P. In other words P * $(1.1)^n$ must be $\geq$ 2P or $(1.1)^n \geq 2$.
(Note that this is irrespective of the actual value of P – the number 128,000 is thrown in to confuse!)
As seen above, $(1.1)^7 \cong 1.9$ which is just less than 2. Also $(1.1)^8$ will comfortably exceed 2. Hence the sum will double sometime during the 8th year. Hence (3).

Q. A team of 6 people has to be chosen from 7 men and 8 women, such that there are at least 2 women in the team. In how many ways can the selection be done?

1) 4830 2) 7200 3) 5005 4) 5630

Note that the selection has to be done out of 8 (for women) and 7 (for men). Let us therefore note the 7th and 8th lines of Pascal's Triangle, viz:

	1	*7*	*21*	*35*	*35*	*21*	*7*	*1*
1	*8*	*28*	*56*	*70*	*56*	*28*	*8*	*1*

We can consider the following cases:
2W 4M can be selected in ${}^8C_2$ x ${}^7C_4$ = 28 x 35 = 980 ways
3W 3M can be selected in ${}^8C_3$ x ${}^7C_3$ = 56 x 35 = 1960 ways
4W 2M can be selected in ${}^8C_4$ x ${}^7C_2$ = 70 x 21 = 1470 ways
5W 1M can be selected in ${}^8C_5$ x ${}^7C_1$ = 56 x 7 = 392 ways
6W 0M can be selected in ${}^8C_6$ x ${}^7C_0$ = 28 x 1 = 28 ways
Hence the total number of ways would be 980 +1960 +1470 + 392 +28 = 4830. Hence (a).

Alternative method: Note that if we had not used Pascal's triangle, we would have had to calculate 10 ${}^nC_r$ values in addition to the above calculation, which would easily have added another minute or two to the solution time. However, an alert person could also observe that the number of ways to select 6 people from 15 is ${}^{15}C_6$ which with a little calculation yields 5005. Hence the required answer must be less than this, and hence, out of the given options, 4830 must be correct.

REMAINDERS -13

Let us now return to remainders for a bit:

Remainders from the perspective of the Binomial Theorem:

Q. What is the remainder when 17^{1717} is divided by 8?

We can write 17 as 16 + 1 i.e. 8k+1 and hence 17^{1717} is equal to $(8k+1)^{1717}$ which when expanded will have all but the last term divisible by 8. Hence the overall number will be of the form 8m+1 and the remainder will be **1**.

Q. What is the remainder when 17^{1717} is divided by 9?

We can write 17 as 18 – 1 i.e. 9k - 1 and hence 17^{1717} is equal to $(9k - 1)^{1717}$ which when expanded will have all but the last term divisible by 9. Hence the overall number will be of the form 9m – 1 which when divided by 9 will give a remainder **8** (Note that this way of looking at it gives a certain legitimacy, as it were, to the concept of negative remainders!)

Q. What is the remainder when 23^{111} is divided by 121?

We can write 23 as 22 + 1 i.e. 11k + 1 and hence 23^{111} is equal to $(22 + 1)^{111}$ which when expanded will have all but the **last two** terms divisible by 121 (which is 11^2). The resultant will be of the form $121m + 111 * 22 * 1^{110} + 1^{111}$ which is 121m + 2443 which when divided by 121 gives remainder **23**.

Q. What is the remainder when $6^{100} + 8^{100}$ is divided by 49?

We can start by noticing that 6 = 7 - 1 and 8 = 7 + 1, while $49 = 7^2$.

So $6^{100} + 8^{100}$ is nothing but $(7 - 1)^{100} + (7 + 1)^{100}$ which is 49 m – 100 * 7 + 1 + 49 p + 100 * 7 +1 which when divided by 49 leaves **2**.

Q. What is the remainder when $6^{99} + 8^{99}$ is divided by 49?

Again, we can use the fact that 6 = 7 - 1 and 8 = 7 + 1, while $49 = 7^2$.

So $6^{99} + 8^{99}$ is nothing but $(7 - 1)^{99} + (7 + 1)^{99}$ which is 49 m + 99 * 7 -1 + 49 p + 99 * 7 +1 which when divided by 49 is equivalent to 2 * 99 * 7 ≡ 2 * 1 * 7 ≡ **14**.

In this manner, the basic logic of binomial expansion can be applied to a wide variety of "remainders" problems; practice would help us to identify in which type of problems it would be the optimal solution methodology!

In the next post, we'll look at remainders with some specific numbers.

REMAINDERS -14

Remainders with specific numbers:

While the techniques already discussed are useful in a wide variety of situations, there are some divisors where we can get an answer startlingly fast, by bypassing the standard techniques entirely.

Remainders with 10 and its powers

Note that the remainder when divided by 10 is nothing but the last digit of the number. Similarly the remainder with 100 is just the last 2 digits and so on. [Note that the same logic can be extended to remainders when dividing by 5 as well (since 10 is divisible by 5)] For example:

Q. What is the remainder when $72^{73^{74}}$ is divided by 10?

Note that the last digit of powers of 2 has a 4-step cycle (2, 4, 8, 6) and hence we start by checking the remainder when 73^{74} is divided by 4. Using the binomial theorem, 73^{74} is equal to $(72+1)^{74}$ and hence is of the form 4k+1 hence $72^{73^{74}}$ is of the form 72^{4k+1} which will end in 2. Hence the remainder when divided by 10 is 2.

Q. What is the remainder when $17^{19^{21}}$ is divided by 5?

When we calculate, we will find that $17^{19^{21}}$ is of the form 17^{4k+3} and will give a last digit of 3. So it is of the form 10k+3 and when divided by 5 it will leave **3**.

Remainder with 9, 99, 999…

Q. Find the remainder when 14285 is divided by 9?

14285 can be written as 1 * 10000 + 4 * 1000 + 2 * 100 + 8 * 10 + 5 which is the same as
(1 * 9999 + 1) + (4 * 999 + 4) + (2 * 99 + 2) + (8 * 9 + 8) + 5 or
(1 * 9999 + 4 * 999 + 2 * 99 + 8 * 9) + (1 + 4 + 2 + 8 + 5).
Now the first bracket is divisible by 9 and so we only need to check the remainder for the second one which is the remainder of 20 with 9 which is **2**. (Note that this is the basis of the divisibility test for 9!)

Q. Find the remainder when 17281331 is divided by 99?

If we want to check the remainder with 99, we can similarly consider 2 digits at a time as follows:
17281331 ≡ 17 * (999999+1) + 28 * (9999+1) + 13 * (99+1) + 31
≡ (17 * 999999 + 28 * 9999 + 13 * 99) + 17 + 28 + 13 + 31 ≡ 0 + 89 = 89.

Q. Find the remainder when 12345123451234512345…200 digits is divided by 99?

We can break this up, 2 digits at a time, as (12 + 34 + 51 + 23 + 45) + (12 + 34 + 51 + 23 + 45) + (same sequence repeated 20 times) which means the remainder when divided by 99 will be the same as that of 12 + 34 + 51 + 23 + 45) * 20 ≡ 165 * 20 ≡ 33.

REMAINDERS -15

Remainders: Some more complex approaches

Statutory Warning: When studying for exams like the CAT, often one comes across theorems and methods with esoteric (and often scary) titles such as "Wilson's Theorem". Remember that the CAT rarely tests very complicated concepts; rather it prefers to examine how clearly the candidate has understood the basics, and so it will not usually require you to know these theorems. However, knowing some of these techniques could help to get a clearer understanding of how the basics behave and so we will occasionally take a brief look at some of them. There are many such concepts in Remainders – we'll examine two of them; Fermat's Little Theorem and the Chinese Remainder Theorem. If you find these interesting, you could explore further on your own – for example the Euler's Totient Function is a popular and handy way to solve Remainder problems.

Fermat's Little Theorem

Fermat's Little Theorem states that:

If p is a prime number, then for any integer a, the number $a^p - a$ is an integer multiple of p.

So for example, $3^{17} - 3$ will be divisible by 17.

Logically this means that the remainder when a^p is divided by p (where p is a prime) will be the same as that of a when divided by p.

Q. What is the remainder when 21^{47} is divided by 47?

Since 47 is prime, the remainder will be **21.**

Q. What is the remainder when 1001^{19} is divided by 19?

Since 19 is prime, it will be equivalent to the remainder when 1001 is divided by 19 which is **13**.

A useful corollary of this is that

If a is not divisible by p, then $a^{p-1} - 1$ is an integer multiple of p which may be restated as

If a is not divisible by p, a^{p-1} leaves a remainder of 1 when divided by p

For example 3^{16} when divided by 17 will give a remainder 1.

Q. What is the remainder when 1001^{56} is divided by 29?

Since 29 is prime, and co-prime to 1001 = 7*11*13, the remainder of 1001^{28} with 29 will be 1. So the remainder of 1001^{56} which is $(1001^{28})^2$ with 29 will be 1^2 i.e. **1.**

REMAINDERS -16

The Chinese Remainder Theorem approach

How would we approach a problem like the following? :

Q. What is the remainder when 9^{2013} is divided by 77?

Obviously trying to find the full pattern with 77 is not a good idea (even with negative remainders, we might need 38 steps to get an answer – and then dividing by 77 is not exactly a cakewalk anyway!). But we can observe that 77 = 11 * 7.

Let's start by separately finding the remainders of 9^{2013} with 7 and with 11 (using the earlier methods):

$\frac{9^{2013}}{11} \equiv \frac{(9^5)^{402} * 9^3}{11} \equiv \frac{(1)^{402} * 9^3}{11} \equiv \frac{1*3}{11} \equiv \frac{3}{11}$ i.e. 9^{2013} is of the form 11k+3 and also

$\frac{9^{2013}}{7} \equiv \frac{(9^3)^{671}}{7} \equiv \frac{1^{671}}{7} \equiv \frac{1}{7}$ i.e. 9^{2013} is of the form 7m +1

Now if we combine these two results, as seen in an earlier post, 9^{2013} is of both forms 11k+3 and 7m +1; so taking numbers of the form 11k+3 i.e. 3, 14, 25, 36, 47, 58, 69 we find that 36 satisfies both. Hence the required remainder is **36.**

This is a surprisingly easy method to apply after a little practice; let's do a few more examples:

Q. What is the remainder of 7^{6666} when divided by 65?

65 = 13 * 5.

$\frac{7^{6666}}{13} \equiv \frac{(7^6)^{1111}}{13} \equiv \frac{(-1)^{1111}}{13} \equiv \frac{-1}{13} \equiv \frac{12}{13}$ i.e. 7^{6666} is of the form 13k +12

Also 7^{6666} ends in 9 and hence will be of the form 5m+4.

Combining the two, investigate 12, 25, 38, 51, 64 and we find that **64** satisfies both conditions.

Q. What is the remainder when $17^{21^{225}}$ is divided by 117?

117 = 13 * 9

$\frac{17^{21^{25}}}{13} \equiv \frac{(17)^{6k+3}}{13} \equiv \frac{(17^6)^k * 17^3}{13} \equiv \frac{1*12}{13} \equiv \frac{12}{13}$ i.e. $17^{21^{225}}$ = 13k + 12

$\frac{17^{21^{25}}}{9} \equiv \frac{(-1)^{21^{25}}}{9} \equiv \frac{-1}{9} \equiv \frac{8}{9}$ i.e. $17^{21^{225}}$ = 9m + 8

The first value satisfying both is **116**.

Q. What is the remainder when 21^{1001} is divided by 91?

91 = 13 * 7

$\frac{21^{1001}}{13} \equiv \frac{(21^4)^{250} * 21^1}{13} \equiv \frac{(1)^{250} * 8^1}{13} \equiv \frac{1*8}{13} \equiv \frac{8}{13}$ i.e. 21^{1001} is of the form 13k + 8

and also 21^{1001} is divisible by 7 and so is of the form 7m.

So examining 13k+8 numbers i.e. 8, 21, 34, 47, 60, 73, 86 we find that **21** should be our answer.

LAST DIGIT - 1

This week, let's look at how to find the last digit of numbers.

Let's start with a simple question – what will decide the last digit of any multiplication?

Q. What is the last digit of 2043563 * 746856347

It can be found simply by taking the last digits of the individual numbers being multiplied. For example the last digit of **2043563 * 746856347** will have the same last digit as **3 * 7** i.e. **1**. Why does this happen? We could write **2043563 * 746856347** as **(2043560 + 3) * (746856340 + 7)**. Now when expanded algebraically, 3 out of the 4 terms here will end in a 0 and hence only the 4th i.e. **3 * 7** will give us a non-zero last digit.

What if we desire the last 2 digits? We can do this also in a similar manner, using the last 2 digits of each number: **2043563 * 746856347** as **(2043500 + 63) * (746856300 + 47)** will end in the same two digits as 63 * 47 = 2961. So the last 2 digits will be **61**.

Note that the last digit can also be looked at as the remainder when that number is divided by 10. Similarly the last 2 digits can be considered as equivalent to the remainder when the number is divided by 100. So we can occasionally use the techniques of remainders (described in an earlier series of blog-posts!) to answer such questions. We can also modify the idea of a negative remainder. For example:

Q. What are the last two digits of 7365593 * 453870397 * 45006703?

We can treat 93 as -07 (wrt 100!) and 97 as -03 and solve this as (-07) * (-03) * (03) giving **63**.

Q. What are the last two digits of 234120312 * 2233557799 * 843562788?

This will be the same as the last two digits of (12) * (-1) * (-12) = 144 => the last 2 digits are **44**.

Q. What are the last two digits of 62378596 * 235681989 * 43578236597?

This will be the same as (-4) * (-11)* (-3) = -132. Hence we should get -32. But since we do not want a negative answer, we treat this as -32 + 100 (as is the standard practice for negative remainders!) and get the answer as **68**.

Q. Find the last non-zero digit of 18!

We can solve this manually by just multiplying the last digit of each successive number – if at any point it should become 0, then take the second last digit – as follows:
1 * 2 = **2** * 3 = **6** * 4 = **4** * 5 = **2** * 6 = **2** * 7 = **4** * 8 = **2** * 9 = **8** * 10 = **8** * 11 = **8** * 12 = **6** * 13 = **8** * 14 = **2** * 15 = **3** * 16 = **8** * 17 = **6** * 18 = **8.** Hence the last non-zero digit is **8.**

(Note that there is a fancy formula which is used to find the last non-zero digit of factorials of large numbers. I am not going to bother with that here – frankly, it is a question unlikely to appear in an exam like CAT. And I am more interested in examining the logic behind an approach, not in formulae.)

LAST DIGIT – 2

Last digit of Powers – the Rule of Cyclicity

Q. Find the last digit of $(3432875082097)^{2015}$

Now that we know that it is only the last digits of the individual numbers being multiplied which decides the last digit of the product. Hence here we only need concern ourselves with the 7 at the end of the base.

If the number ends in 7, then its square will end in 7 * 7 = 9 while its cube will end in 7^2 * 7 i.e 9 * 7 = 3 and so on. But surely we can't do this for 2000+ steps? Fortunately, we don't need to – if at any point the last digit repeats, then after that the entire sequence will repeat (since we are just doing the same operation of multiplying by 7 in each and every step!). Proceeding with our calculation above, the 4th power of the number will end in 7^3 * 7 i.e 3 * 7 = 1, the 5th power will end in 7^4 * 7 i.e 1 * 7 = 7 and voila – we have a repeated number!

So we find the pattern of last digits for powers of (xxx7) as 7, 9, 3, 1, 7, 9, 3, 1, 7, 9, 3, 1…. and so after every 4 steps we get 1. We need the 2015th number in this sequence; since 2012 is divisible by 4, the 2012th number will be 1 and 3 steps further will be 7, 9, 3. Hence the required last digit is **3.**

The question might now arise – what if we have to go a very long way before this kind of repetition happens? As it turns out, we need not worry about this – there are only 10 different last digits possible (0 – 9) and so it will repeat within a maximum of 10 steps for sure. And if we actually check every possible last digit, every single one turns out to repeat in a maximum of 4 steps i.e. the last digit of x^5 will always be the same as that of x. The following table will demonstrate this fact:

If x ends in	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
then x^2 ends in	0	1	4	9	6	5	6	9	4	1
then x^3 ends in	0	1	8	7	4	5	6	3	2	9
then x^4 ends in	0	1	6	1	6	5	6	1	6	1
then x^5 ends in	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
then x^6 ends in	0	1	4	9	6	5	6	9	4	1

For a number ending in 0, 1, 5 or 6: the last digit of any power will remain **0, 1, 5, 6** respectively

For a number ending in 4 or 9: The last digit of any power will only depend on whether the power is odd or even – **$(xxx4)^{odd}$ ends in 4, $(xxx4)^{even}$ ends in 6, $(xxx9)^{odd}$ ends in 9 and $(xxx9)^{even}$ ends in 1**.

For a number ending in 2, 3, 7, 8: We can find the remainder of the power when divided by 4 (knowing the divisibility test for 4, this can be found using just the last 2 digits of the power!). If the remainders are 1, 2, 3 or 0, we raise the last digit to 1, 2, 3 or 4 and note its last digit.

Examples:

$(26343)^{198}$ ➔ 3^2 i.e. it will end in 9

$(18656\mathbf{8})^{144}$ ➔ 8^4 i.e. it will end in 6

$(11844)^{2233}$ ➔ 4^{odd} i.e. it will end in 4

$(175392)^{212356\mathbf{7}}$ ➔ 2^3 i.e. it will end in 8

$(10340849)^{783548}$ ➔ 9^{even} i.e. it will end in 1

$(14285\mathbf{7})^{31415926}$ ➔ 7^2 i.e. it will end in 9

LAST DIGIT – 3

Occasionally, the examiner could ask the last 2 digits of a power. Let's see different approaches to attack this kind of problem:

One way is to use Binomial Expansion. This approach works particularly well for numbers ending in one or 9 (i.e. close to a multiple of 10)

Q. Find the last two digits of $(861)^{447}$

Let us write $(861)^{447}$ as $(860 + 1)^{447}$. On binomial expansion, we will get 448 terms, but all of them except the last two will be xyz * 10^2 which will give the last two digits as 00. Also the penultimate term will be ${}^{447}C_1 * 860 * 1^{446}$ which will end in 20. And the last term will be 1^{447} which is 1.

Thus $(861)^{447}$ = ...00 + 447 * 860 * 1^{446} + 1^{447} = …00 + …20 + 1 = …21 ➔ last 2 digits are **21**

Q. Find the last two digits of $(1729)^{1111}$

Let us write $(1729)^{1111}$ as $(1730 – 1)^{1111}$. Again, we need only look at the last 2 terms. The penultimate term will be ${}^{1111}C_1 * 1730 * (-1)^{1110}$ which will end in 30. The last term will be $(-1)^{1111}$ which is -1.

$(1729)^{1111}$ = ...00 + 1111 * 1730 * $(-1)^{1110}$ + $(-1)^{1111}$ = …00 + …30 - 1 = …29 ➔ last 2 digits are **29**

Q. Find the last two digits of $(7689)^{7688}$

Let us write $(7689)^{7688}$ as $(7690 – 1)^{7688}$. The penultimate term will be ${}^{7688}C_1 * 7690 * (-1)^{7687}$ which will end in -20 i.e. + 80 (remember the negative remainder logic in Monday's post!). The last term will be $(-1)^{7688}$ which is 1.

$(7689)^{7688}$ = ...00 + 7688 * 7690 * $(-1)^{7687}$ + $(-1)^{7688}$ = …00 + …80 + 1 = …81 ➔ last 2 digits are **81**

Once we are comfortable with this, we can look for ways to make it even smoother. One neat trick is possible when the base ends in 1. Consider a number $(xxxa1)^{yyyb}$ i.e. a number whose last two digits are a and 1 raised to another number whose last digit is b. If we do the binomial expansion we will get the penultimate term of $(xxxa0 + 1)^{yyyb}$ as (yyyb) * (xxxa0) * 1. Now if we think about this, we will find the last digit to be 0 and the second last digit to be given by the last digit of a * b. Also the last term of the expansion will be 1. So adding these, we can now proceed to answer such a question almost immediately – the last digit will be 1 and the second last digit will be governed by a * b. Let's see some examples

Last two digits of $(12431)^{31567}$ ➔ 7*3 gives 1 ➔ **11**
Last two digits of $(65561)^{32768}$ ➔ 8*6 gives 8 ➔ **81**
Last two digits of $(428571)^{2718}$ ➔ 8*7 gives 6 ➔ **61**
Last two digits of $(591)^{91298357}$ ➔ 7*9 gives 3 ➔ **31**

LAST DIGIT - 4

The method used in the previous post focussed on a base which ends in 1 or 9. But what if there is some other number? The last two digits of a power could also be attacked from the cyclicity perspective, with a little extra effort.

Remember that when we take powers of any number, the last digit cycles every 4 steps (at most). Similarly, since there are only 100 possible last 2 digits (00 – 99), we can rest assured that the last two digits also would show a cyclical repetition and when we test this practically, it turns out that every possible combination of last 2 digits will give us a repetition within 20 steps. However, 20 steps is a lot, and so we should look for ways to make life simpler – in most cases between 4 to 8 calculations will suffice (though even that will take a good 2-3 minutes!) Let's see some examples:

Q. Find the last two digits of $(2007)^{2007}$

Let us observe the pattern for the last 2 digits of powers of 07: (Note: This was a CAT 2008 question)

07 * 07 = **49** * 07 = **43** * 07 = **01** * 07 = **07** and hence it is repeating after 4 steps 07, 49, 43, 01. Hence we can write $\mathbf{(2007)^{2007}}$ as $\mathbf{(07)^{4k+3}}$ and this will have the same last 2 digits as $\mathbf{07^3}$ i.e. **43.**

Q. Find the last two digits of $(5668)^{7263}$

68 * 18 = **24** * 18 = **32** * 18 = **76** * 18 = **68** which is a repeat. So the pattern is 68, 24, 32, 76, 68, 24, 32, 76 and so on. So $(5668)^{7263} \equiv (68)^{7263} \equiv (68)^{4k+3} \equiv (68)^{3} \equiv$ **32.**

So some numbers repeat fairly quickly – another such is 43 which follows the pattern 43, 49, 07, 01… However, there are others which take longer, for example:

Q. Find the last two digits of $(121212)^{2353}$

12 * 12 = **44** * 12 = **28** * 12 = **36** * 12 = **32** * 12 = **84** * 12 = **08** * 12 = **96** * 12 = **52** * 12 = **24**
This doesn't seem to be repeating (it turns out to repeat only after 20 steps!) So what should we do? We could look for a pattern (remember, the last digit already has a pattern) as follows:

12	**44**	**28**	**36**
+20	+40	-20	+60
32	**84**	**08**	**96**
+20	+40	-20	+60
52	**24**	**88**	**56**
+20	+40	-20	+60
72	**64**	**68**	**16**
+20	+40	-20	+60
92	**04**	**48**	**76**
+20	+40	-20	+60
12	**44**	**28**	**36**

The pattern repeats in 20 steps. So we can say that $(121212)^{2353} \equiv (12)^{2353} \equiv (12)^{20k+13} \equiv (12)^{13} \equiv$ **72**

(Note that the CAT and other exams are exceedingly unlikely to ask a question like this. It is just being covered here for demonstration purposes, to show what can be done. The techniques we have discussed in the last two posts could be extended, with suitable modifications, even to "last 3 digits")

P n C - 1

For the next few posts, we'll concentrate on Combinatorics: starting from the Fundamental Principle of Counting and going on to the basic formulae and applications of Permutations and Combinations.

When we wish to count the number of ways in which an event (or a series or group of events) may occur, we can do it manually if the numbers involved are small. However, as the numbers involved get larger, it becomes impossible to manually enumerate the possibilities. The Fundamental Principle of Counting gives us a couple of very basic but elegant ways to approach enumeration problems.

The Fundamental Principle of Counting

If an action A can be performed in "m" ways, and for each of those "m" cases, another action B can be performed in "n" ways, then the number of ways of performing both A and B is m x n.

If an action A can be performed in "m" ways, and for each of those "m" cases, another action B can be performed in "n" ways, the number of ways of performing either A or B (but not both!) is m + n.

Let's look at it through an example:

Q: In order to go from A to C, one has to pass through B. There are 6 ways to go from A to B (car, bus, bullock-cart, train, ship and plane) and 4 ways from B to C (car, bus, train and plane). In how many ways can one go

(a) from A to C?
(b) From A to C and back?
(c) From B to either A or C?
(d) **From B to either A or C and back?**

(a) There are 6 ways (car, bus, bullock-cart, train, ship and plane) to traverse the first leg (from A to B). For each of those ways, there are 4 ways to traverse the second leg (from B to C); for example for the first leg by car, we have (car, car) (car, bus) (car, train) and (car, plane) as our options for the trip. This gives us, in total 6 * 4 = 24 ways to go from A to C (A to B **AND** B to C ➔ we **MULTIPLY**)

(b) There are (as seen above) 24 ways to go from A to C and for each of those there are 24 ways of returning. So overall we will have 24*24 = 576 ways of travelling from A to C and back.

(c) There are 6 ways to go from B to A and there are 4 ways to go from B to C. Hence there are 6 + 4 = 10 ways to go from B to A or C. (B to A OR B to C ➔ we ADD)

(d) There are 10 ways to go from B to either A or C. But we cannot start with this.The FPC clearly states that "for each of those 'm' cases….B can be performed in 'n' ways". However, in the cases where the person has gone to A, there are 6 ways to return and when he has gone to C there are 4 ways. So we treat it as 2 separate cases – either he goes to A AND returns OR he goes to C AND returns which gives us 6 * 6 + 4 * 4 = 36 + 16 = 52 ways in total.

P n C - 2

Some examples using the Fundamental Principle of Counting:

Q. Aditi wants to dress up for a party. She has 8 saris, 6 skirts, 7 tops, and 10 pairs of shoes. If she wants to wear either a sari or a skirt and top, along with a pair of shoes, in how many ways can she dress for the party?

She wears [(skirt and top) or sari] and shoes hence [(6 * 7) + 8] * 10 = **500 ways**

Q. In the above question, in how many ways can Aditi dress if she may choose to go barefoot?

In this case, for the last choice (of shoes), Aditi has not 10 but 11 options (any of the ten pairs, or none of them) and so overall number of ways will be [(6 * 7) + 8] * 11 = **550 ways**

Q. There are ten people in a class. In how many ways can we award

(a) the first and second rank in the class

(b) the first rank in History and the first rank in Geography?

The difference between the two sub-questions is that in the first case, repetition is not allowed (as one person cannot get both first and second rank) while in the second case, repetition is allowed. Hence:
(a) There are 10 ways in which the first rank can be awarded, and for each of those there are 9 ways to award the second rank. Hence 10 * 9 = **90 ways**
(b) There are 10 ways to award the first rank and for each of those there are again 10 ways to award the second rank. Overall, therefore, there are 10 * 10 = **100 ways**

Q. In a hotel, there are 6 Starters, 4 Soups, 8 Indian Main Courses, 12 Chinese Main Courses, and 5 Desserts. In how many ways can a diner choose:

(a) one item of each available category?

(b) a three course meal with a starter or a soup, one main course and one dessert?

(a) One item of each category can be chosen as Starter and Soup and Indian and Chinese and Dessert i.e. 6 * 4 * 8 * 12 * 5 = **11520 ways**
(b) The three course meal can be selected as (Starter or Soup) and (Indian or Chinese) and Dessert i.e. (6 + 4) * (8 + 12) * 5 = 10 * 20 * 5 = **1000 ways**

Q. Ameena attempts 4 multiple choice questions, each having 5 options. In how many ways can she do this if:

(a) she attempts all the questions

(b) she may or may not attempt all the questions

(a) The first question can be attempted in 5 ways. For each of those ways the second can be attempted in 5 ways meaning 25 ways of attempting the first two questions. For each of those the third has 5 ways to attempt and so also with the fourth. So we have 5 * 5 * 5 * 5 = $\mathbf{5^4}$ **= 625 ways**
(b) The first question can be attempted in 6 ways here (any of the five options, or left unattempted) so as above there would be 6 * 6 * 6 * 6 = $\mathbf{6^4}$ **= 1296 ways**

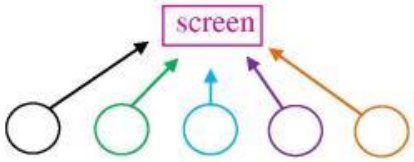
P n C – 3

Arranging 'n' distinct objects in 'n' distinct positions in a straight line

Suppose we have to arrange 3 distinct objects (say A, B and C) in 3 places. We can easily see, by listing, that this can be accomplished in 6 ways – ABC, ACB, BAC, BCA, CAB and CBA.

However, if instead we had to arrange 5 people (say A, B, C, D, E) in 5 places, listing would be an onerous task. In this case we could instead say that A has 5 ways to seat himself, for each of these, B can choose from the remaining 4 seats, C then has 3 ways, D has 2 and E has 1. So overall, we have 5 * 4 * 3 * 2 * 1 = 5! = 120 ways of arranging the 5 people in the 5 positions.

A point to note – when we have to arrange 5 people in 5 seats in a straight line, it is obvious that the **people are all distinct**. But a point which sometimes gets overlooked is that **the seats are also distinct**. The characteristic of a seat is the view it commands, and 5 seats in a line will each have a slightly different view (or a slightly different position relative to the others). For example in a theatre, the seats in the middle rows generally afford a better view of the screen than the seats at the side.

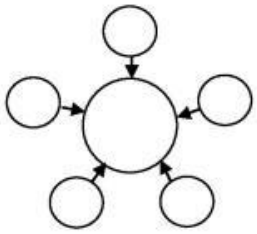


Thus we could equally well have solved this question by considering the point of view of the seats; the first seat could be occupied by any of A, B, C, D or E while the subsequent seats would have 4, 3, 2 and 1 possible occupants, giving 5 * 4 * 3 * 2 * 1 = 5! = 120 ways of arranging the 5 people.

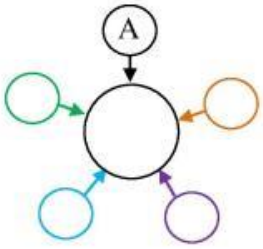
Similarly, if we had had 12 people to arrange in a line, we would have got 12! ways of doing so.

In general, the number of ways of arranging 'n' distinct objects in a straight line is n!

Arranging n distinct objects in n distinct positions in a circle



Suppose we have 5 people to be arranged around a circular table. Note that, unlike in the earlier case of a straight line, the view from **each seat here is identical** – a circular table and 4 other seats placed symmetrically. So the first person arriving at the table will really have not 5 options, but rather only 1 option.



However, once the first person to arrive (say A) has seated himself, **the remaining 4 seats become distinct** relative to him and hence the next arrival (say B) has 4 options, and the subsequent arrivals have 3, 2 and 1 option respectively. So overall, we have 1 * 4 * 3 * 2 * 1 = 4! or 24 ways.

Similarly, if we had had 12 people to arrange in a circle, the number of ways would similarly be 11!

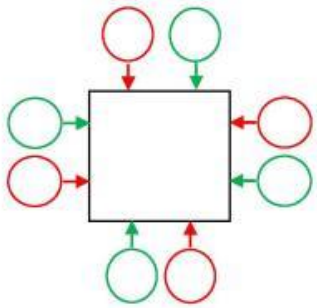
In general, the number of ways of arranging 'n' distinct objects in a circular manner is (n-1)!

P n C - 4

Let's look at some variants on the two cases we considered yesterday, in which we have some arrangement other than the linear or circular one. In each case, notice that we only need to check the first person's options separately – the rest of the people will be arranged in (n-1)! ways. For example:

What happens if we want to arrange 8 people round a square table with 2 seats on each side?

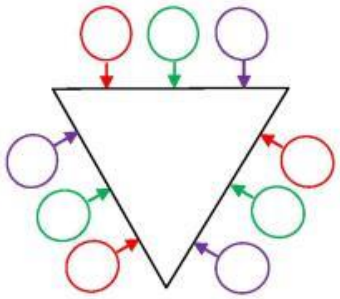
Here the first person can have 2 different positions, either to the left or to the right of a side of the table (see the adjacent figure- a person sitting in any of the green seats would have the same view; similarly a person sitting in any of the red seats would have the same view). Again, after the first person sits, the remaining 7 seats would be distinct and hence the remaining people could be arranged in 7! ways.



So overall there are 2 * 7! arrangements.

What if we want to arrange 9 people on an equilateral triangular table with 3 seats on each side?

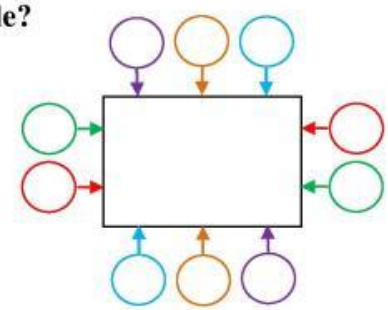
Here the first person can have 3 different positions, either to the left, to the right or in the middle of a side of the table (see the adjacent figure). After this, the remaining 8 seats would be distinct and hence the remaining people could be arranged in 8! ways.



So overall there are 3 * 8! arrangements.

What if we want to arrange 10 people on a 2 x 3 rectangular table?

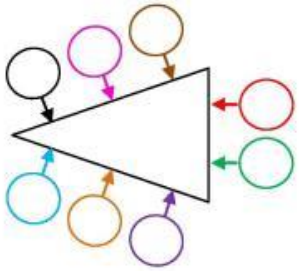
Here the first person can have 5 different positions (2 on the shorter side and 3 on the longer side).



So overall there are 5 * 9! arrangements.

What about arranging 8 people on an isosceles triangular table with sides having 3, 3 and 2 seats?

If we observe carefully, we find that here **all 8 seats are distinct** in the view they provide! Hence the first person can have 8 different positions.



So overall there are 8 * 7! i.e. 8! arrangements, the same as there would have been in the straight line case!

So far we have considered the case where all the objects being arranged are distinct. What happens when some of them are identical? Let us examine this situation in the next week's posts.

P n C – 5

Arranging 'n' objects, out of which 'k' are identical, in 'n' positions in a straight line

Suppose we have to arrange 6 people in a line for a photo. But 2 of them happen to be identical twins. How many different arrangements would be there?

We need to find 'x', 'x' being the number of possible arrangements of 6 people (A, A, B, C, D and E)

Let us consider one of these x arrangements, say **B A D C A E**

In this case, the two **A**s occupy the 2nd and 5th positions. Now suppose the photographer decides to make it easier to distinguish the twins, and gives them each a different coloured jacket to wear, then there could be two ways to get the above arrangement:

Hence for every one of the x arrangements, we would get 2 cases and hence we would now have 2x ways of arranging the people.

However, now all the 6 people are distinct and hence we would get 6! ways. So 2x = 6! and so $\mathbf{x} = \frac{6!}{2}$

What if there had been 3 identical triplets? In how many ways could the arrangement happen?

Let there be 'x' ways in total; consider one case, say **A B C A D A**:

If we make the three **A**s distinct, we get 3! = 6 different ways of arranging the above group:

For each of the x cases we have 3! ways to arrange them if the 3 are distinct. So x*3! = 6! and $\mathbf{x} = \frac{6!}{3!}$

What if there are ten people including one set of twins and one set of triplets?

In this case we would say that x * 2! * 3! = 10! and hence **x would be** $\frac{10!}{2!\,3!}$

Let us see how this concept can be applied to a variety of situations:

- The number of ways to arrange 8 marbles, 3 of which are blue and the rest green: $\frac{8!}{3!\,5!}$
- The number of ways in which we can arrange the letters of the name DHANASHREE: $\frac{10!}{2!\,2!\,2!}$
- The number of ways to arrange 2 blue, 3 green, 4 red and 5 yellow notebooks: $\frac{14!}{2!\,3!\,4!\,5!}$

P n C – 6

Permutations

Arranging r out of n distinct objects in r distinct positions ($n \geq r$) [with repetition not allowed]

Let us consider 2 questions.

a) In how many ways can we arrange 5 people in 8 seats in a straight line?
b) In how many ways can we arrange 5 out of 8 people in 5 seats in a straight line?

Are these two cases different? As it turns out, they are not (from a combinatorics point of view)!

Suppose we have 5 people, but 8 seats available. Examining this problem from the point of view of the people gives us 8 * 7 * 6 * 5 * 4 but examining it from the point of view of the seats instead gives us 5 * 4 * 3 * 2 * 1. Which is correct? A little thought will show us that the second one is wrong – it implicitly assumes that the 6th, 7th and 8th seats are empty (which need not be true!).

Similarly if we have 5 seats and 8 people, and we need to seat 5 of the people, we will need to look at it from the point of view of the seats, and will get 8 * 7 * 6 * 5 * 4 ways.

In either case, we can observe that we look at it from the point of view of that type of object which is fewer in number – the Economic principle of "scarcity implies demand/value" can be applied here.

Now writing 8 * 7 * 6 * 5 * 4 can get very cumbersome. Instead, we could note that this is just $\frac{8*7*6*5*4*3*2*1}{3*2*1}$ or $\frac{8!}{3!}$. This is usually called ${}^{8}P_3$ – the P stands for **Permutation** or arrangement.

In general, the number of ways of arranging 'r' out of 'n' distinct objects is ${}^{n}P_r = \frac{n!}{(n-r)!}$

Note that Permutation can be looked at as a special case of the formula in the previous post; if we want to arrange 5 people out of 8, we could say that 5 people have distinct positions and the remaining 3 have identical positions and hence it is equivalent to "arrange 8 objects, 3 of which are identical" $\Rightarrow \frac{8!}{3!}$

Q. There are ten movies showing at the Bottels Film Festival. Prizes are to be awarded to the top three movies. In how many ways can the jury award these prizes?

The first prize can be awarded in 10 ways, the second in 9 and the third in 8. So overall we have 10 * 9 * 8 = **720 ways** (or ${}^{10}P_3 = \frac{10!}{7!} = 720$)

Q. Angrywords Bookstore has a shelf where they display 4 bestsellers in a row every day. If they have 6 books to choose from, can they display a different arrangement for every day of the year?

The total possible arrangements of 4 books out of 6 will be ${}^{6}P_4 = \frac{6!}{2!}$ i.e. 360. Since there are 365 days in a year, the bookstore will have to repeat an arrangement at some point. Hence the answer will be **No**.

P n C – 7

Permutations: Variations

What happens if we wish to arrange r people in n places in a circle? Or in some other arrangement?

First of all, we should recall that once one person sits, the remaining places all become distinct. So again, the options available to the first person become crucial.

In how many ways can 10 people be seated

(a) in 12 seats in a straight line?

(b) around a circular table having 12 seats?

(c) at a square table with 3 seats on each side?

(d) at an equilateral triangular table with 4 seats on each side?

(e) around a rectangular table with 4 and 2 seats on the longer and shorter sides?

(a) Here we can see that all the seats are distinct and so the number of arrangements will be $^{12}\mathbf{P}_{10}$.

(b) Here the first person has only 1 option as all the seats are effectively identical to begin with. After that we have 11 seats and 9 people so we can arrange them in $^{11}\mathbf{P}_{9}$ ways

(c) Here the first person has 3 choices which would provide him a different view (the left, middle or right positions on any side) after which the remaining 9 people have to be arranged in 11 distinct seats and hence the total number of ways would be $\mathbf{3 * {}^{11}P_9}$.

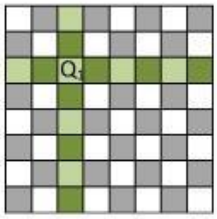
(d) Here the first person has 4 options and hence we would have $\mathbf{4 * {}^{11}P_9}$ ways.

(e) Here the first person has 6 options and hence we would have $\mathbf{6 * {}^{11}P_9}$ ways.

In how many ways can 3 chess queens of different colours be placed on a chessboard?

The first thing to note here is – don't waste your time wondering how there can be chess queens of 3 different colours. Understand the **intent** of the question-setter, which is to indicate that the 3 queens are distinct. Since there are 64 squares on a standard chessboard we will have **64 * 63 * 62** or $^{64}\mathbf{P}_{3}$ **ways**.

In how many ways can 3 chess queens of different colours be placed on a chessboard such that no two are in the same row or column??



The first queen can be placed in 64 ways. Now that row and column is eliminated, leaving 7 rows and columns i.e. 49 places for the second queen. (We could also have done this in a more cumbersome way, by removing 8 + 8 – 1 i.e. 15 squares from 64). Similarly the third queen could be placed in 36 ways (6 rows and columns). Overall, therefore, the number of ways will be **64 * 49 * 36**.

P n C – 8

Allotting n distinct objects to r distinct groups [with repetition allowed]

Let us consider a very similar looking question to the one we started with last Wednesday:

In how many ways can 8 people be sent to 5 classrooms?

At first glance, one might think this should be ${}^{8}P_{5}$. But there is one basic difference here – the situation (unlike with people and seats) is not a one-to-one case – more than one person could go to a single classroom. Let us apply the Fundamental Principle of Counting: the 1st person has 5 options, for each of those, the 2nd person also has 5 options, and so on, giving us, in total, 5 * 5 *….(8 times) or $\mathbf{5^8}$ **ways**.

A confusion that often arises here is: **should it be 5^8 or 8^5?** For example if we have 5 different chocolates (say A, B, C, D and E) to be distributed among 8 children, then should we start from the point of view of chocolates or children? Let us examine this in some detail:

If we look at one chocolate, it can go to only one kid, right? But if we take one kid he can get many chocolates. So that is the first reason to go for chocolates in this question. The first chocolate has 8 choices, but the first kid has not 5 options but many more – A, B, C, D, E, AB, AC, AD, ABCDE, none and all such possible combinations (as we shall see in a later post, he in fact has 2^5 i.e. 32 choices!)

But there is a second, even more important reason. If the first chocolate goes to some kid, the next chocolate still has 8 choices, independent of the first. But if we look from the kids' point of view, the second kid's number of choices will depend on what the first kid got (for example if the first kid got all the chocolates the second guy has 0 choices!)

Now the Fundamental Principle of Counting states that if A can be done in 'm' ways and **for each of those ways** B can be done in 'n' ways, then total ways of doing A and B are m x n. The phrase in bold is one we sometimes tend to ignore as a mere technicality, but it is most crucial to the application of the FPC. So in the case where we are looking from the kids' point of view, we cannot even apply the FPC, because different cases have different number of ways ahead. However we can do it for the chocolates as the number of ways for each chocolate is independent of what happened to the earlier ones.

So when we have to send r distinct things to n distinct places with no limits, we have n^r ways of doing it (basically, check which can have only one destination associated with it and call those r, and which can accept more than one and call those n) (note: the objects are not being arranged within the group!)

In general, the number of ways of sending 'r' distinct objects to 'n' places (with repetition) is n^r

For example:

- Put x children in y buses: $\mathbf{y^x}$
- Send m children to n classrooms: $\mathbf{n^m}$
- Distribute p (distinct) apples to q children: $\mathbf{q^p}$
- Give k different novels to l children: $\mathbf{l^k}$
- Let g children have h distinct desserts (without splitting any dessert): $\mathbf{g^h}$

(So it is not that people will always be 'r', they could equally well be 'n' depending on the context.)

P n C – 9

Let's look at some examples of how we can apply the FPC and the ${}^{n}P_{r}$ and n^{r} formulae:

Forming Words

The English alphabet contains 26 alphabets (21 consonants and 5 vowels).

(a) How many distinct 4-letter words can be formed?

The first letter can be chosen in 26 ways. For each of these, the second letter can be chosen in 26 ways, the third also in 26 ways and so on. Hence 26 * 26 * 26 * 26 = $\mathbf{26^4}$

(b) How many distinct 4-letter words can be formed having all letters distinct?

The first letter can be chosen in 26 ways. For each of these, the second letter can be chosen in 25 ways, the third in 24 ways and so on. Hence 26 * 25 * 24 * 23 = $\mathbf{{}^{26}P_4}$

(c) How many distinct 4-letter words can be formed with all letters vowels?

The first letter can be chosen in 5 ways. For each of these, the second letter can be chosen in 5 ways, the third also in 5 ways and so on. Hence 5 * 5 * 5 * 5 = $\mathbf{5^4}$

(d) How many distinct 4-letter words can be formed with all letters distinct consonants?

The first letter can be chosen in 21 ways. For each of these, the second letter can be chosen in 20 ways, the third in 19 ways and so on. Hence 21 * 20 * 19 * 18 = $\mathbf{{}^{21}P_4}$

(e) How many distinct 4-letter words can be formed with the first letter as a vowel?

The first letter can be chosen in 5 ways. For each of these, the second letter can be chosen in 26 ways, the third also in 26 ways and so on. Hence 5 * 26 * 26 * 26 = $\mathbf{5 * 26^3}$

(f) How many distinct 4-letter words can be formed with only the first letter as a vowel?

The first letter can be chosen in 5 ways. For each of these, the second letter can be chosen in 21 ways, the third also in 21 ways and so on. Hence 5 * 21 * 21 * 21 = $\mathbf{5 * 21^3}$

(g) How many distinct words of up to 4 letters can be formed?

We can form words of length 1, 2, 3 or 4. Using the logic of question (a) we get $\mathbf{26^4 + 26^3 + 26^2 + 26^1}$

(h) How many distinct words of up to 4 distinct letters can be formed?

Using the logic of question (b) we get $\mathbf{{}^{26}P_4 + {}^{26}P_3 + {}^{26}P_2 + {}^{26}P_1}$

P n C – 10

Forming Numbers

We can apply the same logic to numbers as well, with an additional condition that the first digit ≠ 0.

(a) How many 4-digit numbers exist?

The first digit can be chosen in 9 ways and each of the rest in 10 ways ⇒ 9 * 10 * 10 * 10 = **9000**
(We could also have done this by 9999 (numbers of up to 4 digits) – 999 (upto 3 digits) = 9000)

(b) How many 4-digit numbers exist with all digits odd? And with all digits even?

Odd digits are 5 in number (1, 3, 5, 7, 9) so all digits can be odd in 5 * 5 * 5 * 5 = **625 numbers**
Even digits are also 5 but include a 0 so all digits can be even in 4 * 5 * 5 * 5 = **500 numbers**

(c) How many 4-digit numbers exist with all digits prime and distinct?

Prime digits are 2, 3, 5, 7 and distinct ⇒ no repetition, so we have 4 * 3 * 2 * 1 = **24** such numbers.

(d) How many 4-digit odd numbers exist? And how many 4-digit even numbers?

Here, only the last digit must be odd. Also the first digit has a condition (≠ 0). (In general, when there are conditions, we should take care of those first). Hence 9 * 10 * 10 * 5 = **4500** such numbers exist. For even numbers too the same holds true.

(e) How many 4-digit numbers exist with all digits distinct?

The first digit can be filled in 9 ways (except 0). The second digit also can be filled in 9 ways (as 0 is acceptable, but not the digit used in the first position. The other two places can be filled in 8 and 7 ways respectively, giving 9 * 9 * 8 * 7 = **4536 numbers.**

(f) How many 4-digit odd numbers exist with all digits distinct? And how many even numbers?

In the first case, odd numbers, the last digit can be filled in 5 ways (1, 3, 5, 7, 9). After that the first digit can be filled in 8 ways as it cannot be 0 or the digit already used in last position. The remaining two digits can be filled in 8 and 7 ways (they can have 0, but not the digits already used) so overall we will have 8 * 8 * 7 * 5 = **2240** 4-digit odd numbers with all digits distinct.

For even numbers it is a bit trickier. Again the last digit could have 5 possibilities (0, 2, 4, 6, 8). But if the last digit is 2, 4, 6 or 8, the first digit will have 8 possibilities as above. But if the last digit is 0, the first digit will have 9 possibilities. Hence we must take these as two separate cases –
Either the last digit is 0, in which case we will have 9 * 8 * 7 * 1 = **504** numbers
or the last digit is 2, 4, 6 or 8, in which case we will have 8 * 8 * 7 * 4 = **1792** numbers
Thus overall we will have 504 + 1792 = **2296** 4-digit even numbers with all digits distinct.

(Note: we could also have done this as All – Odd = 4536 – 2240 = 2296)

P n C – 11

Combinations

Selecting r out of n distinct objects ($n \geq r$)

Let us consider the following question:

Suppose we have 3 openings for the post of software engineer. Ten people (A, B, C, D, E, F, G, H, I, J) apply for the posts. In how many ways can we select 3 applicants out of the ten?

Note that here the order of selection is not important. A selection of B, E and G is the same as a selection of G, E and B.

Let the number of ways to do the required selection be x. Now supposing after selection, the three chosen people were assigned to **different** roles. In how many ways could they be assigned? 3 * 2 * 1 = 3! ways, of course. So for each of the x selections, we would have 3! ways of arranging the selected people i.e. x * 3! ways in total. However, if we were to directly allot the people to their roles at the time of selection itself then this could have been done in 10 * 9 * 8 or ${}^{10}P_3$ ways. So we can say that x * 3! = ${}^{10}P_3$ or $x = \frac{{}^{10}P_3}{3!}$. This is usually called ${}^{10}\mathbf{C_3}$ – the C stands for **Combination** or selection.

In general, the number of ways of selecting 'r' out of 'n' distinct objects is ${}^{n}C_r = \frac{n!}{r!\,(n-r)!}$

Note that Combination can be looked at as a special case of the formula for arranging objects where some are similar; if we want to select 4 people out of 11, for example, it is equivalent to dividing those 11 people into two groups of size 4 and 7, where neither group is arranged internally. So effectively we can say that for the purposes of the problem, the 4 selected people are identical and the 7 rejected people are also identical. Hence it is equivalent to "arrange 11 objects, 4 of which are identical, and the remaining 7 of which are also identical" or "how many words from AAAAAAABBBB" $\Rightarrow \frac{11!}{7!\,4!}$

Q. 20 people participate in the office lottery for 3 identical prizes. In how many ways can the winners be chosen?

The order of the 3 winners does not matter here; hence the answer would be ${}^{20}\mathbf{C_3}$ **= 1140 ways**

Q. A contestant in the Mastermind quiz attempts 10 rapid fire questions and gets 6 right. In how many ways can this happen?

We need to choose 6 questions to be right (and 4 to be wrong) and so we will have ${}^{10}\mathbf{C_6}$ **= 210 ways**

Q. In how many ways can 4 elephants be seated on 9 large seats in a straight line?

Note that elephants are identical (to the majority of people, anyway – which makes it a reasonable assumption from a CAT point of view). So this question is asking how many ways to choose 4 seats which will need to be replaced later, after an elephant sits on them! This can be done in ${}^{9}\mathbf{C_4}$ **= 126 ways**

P n C – 12

When and how to use the ^nP_r and ^nC_r formulae

It is tempting to assume that we can just use ^nP_r whenever the word arrange is used and ^nC_r whenever the word select is used; but it is not always as straightforward as that. We need to observe the situation and apply common sense - if the objects being selected are being arranged among themselves or are being assigned to distinguishable positions, then we use ^nP_r; however, if the objects are themselves indistinguishable or the positions being selected for are identical, then ^nC_r is our logical choice. For example look at the following two questions:

Q. In how many ways can we arrange 7 red marbles and 3 yellow marbles in a row?

We may assume that all marbles of a given colour are effectively identical; hence the question is asking us to choose 7 places out of 10 in which to put red marbles. Hence **$^{10}C_7$ = 120 ways**

Q. In how many ways can we select a president and treasurer from among 15 members of a club?

Despite the word "select", the two posts are different and hence we need to use **$^{15}P_2$ = 210 ways**

Let's look at a few basic examples:

Q. A rowing team consists of 8 rowers seated in a boat.

a) In how many ways can the 8 rowers be selected from 12 available candidates?
b) In how many ways can 8 rowers out of 12 available candidates be arranged in a boat?
c) In how many ways can 8 rowers be selected out of 12 available candidates if A (the captain) and B (the vice-captain) are always selected?
d) In how many ways can 8 rowers out of 12 available candidates be arranged in a boat, if A (the captain) and B (the vice-captain) are always selected?

(a) 8 rowers can be selected in **$^{12}C_8$ ways**
(b) 8 rowers out of 12 can be arranged in **$^{12}P_8$ ways**
(c) We need to select 6 out of the remaining 10 candidates, which can be done in **$^{10}C_6$ ways**
(d) The tempting answer would be $^{10}P_6$ ways, but a little thought will show us that this is wrong as it arranges only the 6 additional rowers. In fact, we need to **select 6** more rowers to join A and B, but we need to **arrange all 8** within the boat. Hence we do the two steps separately, thus: **$^{10}C_6$ * 8! ways**

Q. There are 8 people in a room. Every person shakes hands with everybody else. How many handshakes are there?

The order of a handshake is unimportant. A shaking hands with B is the same as B shaking hands with A. Hence we just need to see in how many ways we can select 2 different people from 8 i.e. **$^8C_2$ or 28.**

Q. There are 8 people in a room. Every person slaps everybody else. How many slaps are there?

The order of a slap is important. A slapping B is not the same as B slapping A. Hence we just need to see in how many ways we can select and arrange 2 people from 8 i.e. **$^8P_2$ or 56.**

P n C – 13

Arranging people with additional conditions

What do we do if two people insist they want to be together? Or two people insist they want to be apart? A good rule of thumb is, if two or more people want to be together, grant their wish right at the start and then treat them as a single entity. If two or more people want to be apart, though, arrange everybody else first and put these fussy people in the gaps.

Q. In how many ways can we arrange 6 men and 4 women such that all the women are together?

Let us treat the women as a single entity. They can be arranged among themselves in 4! ways. Now these women, and the 6 men, form 7 objects to be arranged in a straight line, which can be done in 7! ways. Hence overall **7! * 4! ways**

Groups Women

Q. In how many ways can we arrange 6 men and 4 women such that all the men are together?

In this case the men can be arranged among themselves in 6! ways, and that group and 4 women can be arranged in 5! ways and hence overall it will be **5! * 6! ways**

Groups Men

Q. In how many ways can we arrange 6 men and 4 women such that all the women are together and all the men are together?

Here we will get two groups so **2! * 4! * 6! ways**

Groups Women Men

Q. In how many ways can we arrange 6 men and 4 women such that no two women are together?

First let us arrange 6 men (say ABCDEF), which can be done is 6! ways. Now this creates 7 spaces, in each of which we can put one (but not more!) of the women as follows: X A X B X C X D X E X F X
Since 4 women can be put in 7 possible seats in $^{7}P_4$ ways, overall we will get **6! * $^{7}P_4$ ways**

Men Women

Let us practice this concept with another example:

There is a bookshelf containing 8 books of the Adventure genre, 7 of the Crime genre and 5 of the Thriller genre (all distinct). In how many ways can we arrange the books such that:

(a) no two Crime books are together? **13! * $^{14}P_7$**

(b) no two Adventure books are together? **12! * $^{13}P_8$**

(c) all the Crime books are together? **7! *14!**

(d) all the books of each particular genre are together? **3! * 8! * 7! * 5!**

Crime C+A Thriller

(e) all the Crime books are together but no two Thriller books are together? **7! * 9! * $^{10}P_5$**

P n C – 14

PnC: Applications

In the last several posts we have covered many concepts in Permutations and Combinations. There's still plenty to go, but first let's do a few examples to revise and reinforce the concepts covered so far and their usage.

Q: In how many ways can we rearrange the letters of the name GABBAR:

As seen earlier, the word GABBAR can be arranged in $\frac{6!}{2!\,2!} = 180$ ways. But the question is asking for **rearrangements** and hence we would have to do $\frac{6!}{2!\,2!} - 1 =$ **179 ways.**

This is a common trap in competitive exams as it gives them a chance to test your command of English as well as Maths. Another similar question would be:

Q: A combination lock has 4 digits and each digit can take values from 1 to 5. How many trials would be needed to determine the correct combination?

The total number of possible combinations is 5 * 5 * 5 * 5 = 625. However, while it would take 625 trials to actually open the lock, it would take only 624 trials to find out the correct combination; since if after 24 trials the lock has not opened the 625th one must be correct! So the answer would be **624**.

The number of ways to rearrange objects is one less than the number of ways to arrange them

Q: In how many ways can we divide 23 people into 3 distinct groups of 10, 8 and 5?

We could do this by first selecting 10 from the 23 available people, then 8 from the remaining 13, and finally 5 from the remaining 5. This would give us ${}^{23}C_{10} * {}^{13}C_8 * {}^{5}C_5$ or $\frac{23!}{13!\,10!} * \frac{13!}{8!\,5!} * \frac{5!}{5!\,0!} = \frac{\mathbf{23!}}{\mathbf{10!\,8!\,5!}}$

We can notice, though, that the numbers in the denominator are exactly 10, 8 and 5 i.e. the group sizes – this is no coincidence, as we could see by visualising the problem in another way:

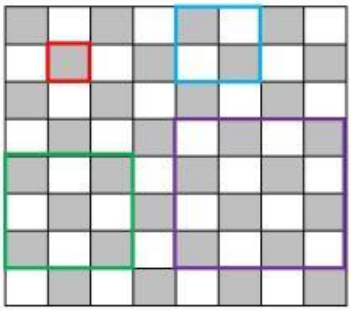
Suppose we want to place 10 people in group A, 8 in group B and 5 in group C, we first make all 23 people stand in a line. Then we write out 23 post-it notes, 10 bearing the letter A, 8 B and 5 C. Then we walk down the line and randomly stick a post-it note on each person's forehead to denote the group to which that person is allotted. Now the numbers of ways of allotting the 23 people will be the same as the possible arrangements of the post-it note – which will be the same as the number of ways of arranging a word formed from 10 As, 8 Bs and 5 Cs i.e $\frac{\mathbf{23!}}{\mathbf{10!\,8!\,5!}}$.

In general, the number of ways of dividing a+b+c objects into groups of a, b and c is = $\frac{(a+b+c)!}{a!\,b!\,c!}$

P n C – 15

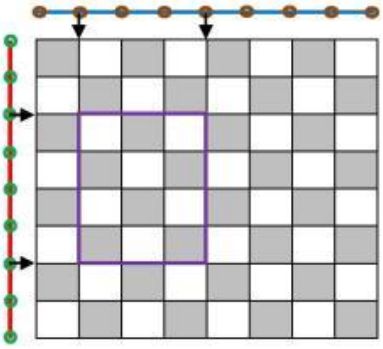
PnC: Applications

Q. How many squares are there on a regular chessboard?



Let us check squares of different sizes in the adjacent figure.
There will be 8 * 8 = 64 squares of size 1 x 1 (such as the red square)
There will be 7 * 7 = 49 squares of size 2 x 2 (such as the blue square)
There will be 6 * 6 = 36 squares of size 3 x 3 (such as the green square)
There will be 5 * 5 = 25 squares of size 4 x 4 (such as the purple square)
…and so on until there will be just the one square of size 8 x 8.
Overall, therefore, we get 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = **204** squares on a chessboard.

Q. How many rectangles are there on a regular chessboard?



Enumerating, as in the above case, would take too long as there are too many sizes of rectangle available. Instead, let us see in how many ways we could choose a horizontal side and in how many ways we could choose a vertical side. To get a vertical side lying along the line marked in red, for example, we could pick any two of the vertices (the points marked in green). This could be done in ${}^{9}C_2$ = 36 ways Similarly one could choose a horizontal length by selecting any two of the brown points on the horizontal blue line. For example if we select the 3rd and 7th vertical point and the 2nd and 5th horizontal one, the rectangle formed would be the one shown. Any such selection of a horizontal and a vertical line would give us a unique rectangle. So overall we would have ${}^{9}C_2 * {}^{9}C_2$ = 36 * 36 = **1296** rectangles.

Q. How many squares are there on a 6 x 9 rectangular grid?

We could proceed in a manner similar to the above:
There will be 6 * 9 = 54 squares of size 1 x 1, 5 * 8 = 40 of size 2 x 2, 4 * 7 = 28 of size 3 x 3, 3 * 6 = 18 of size 4 x 4, 2 * 5 = 10 of size 5 x 5 and 1 * 4 = 4 of size 6 x 6. So overall 54 + 40 + 28 + 18 + 10 + 4 = **154** squares

Q. How many rectangles are there on a 6 x 9 rectangular grid?

We can choose a horizontal side in ${}^{10}C_2$ = 45 ways and a vertical one in ${}^{7}C_2$ = 21 ways, so overall we will have 45 x 21 = **945** rectangles.

In general, if we have an m x n grid of tiles ($m \geq n$) then:

The number of squares present will be m*n + (m-1)(n-1) + (m-2)*(n-2) + … + (m-n+1)*(1)

The number of rectangles present will be ${}^{m+1}C_2 * {}^{n+1}C_2$

P n C – 16 b

PnC: Applications

Q. In how many ways can we arrange 6 people A, B, C, D, E and F in a straight line such that B always comes in front of E?

A slow way of doing this is to count cases. If B is in 1st position, E has 5 options, B in 2nd position means E has 4 options, and so on giving us 5 + 4 + 3 + 2 + 1 = 15 ways or arranging B and E. Now for each of these cases, the remaining 4 people can be arranged in 4! ways as there are conditions imposed on them. So overall we would have 15 * 4! = 360 ways.

There is, however, a far simpler way to solve this question; you just need to look at it in the right way. Overall there are 6! i.e. 720 ways of arranging the 6 people. Now in these 720 ways, who among B and E is more likely to be in front more often? We can quickly realise that there is no earthly reason why one of them should get a preference and hence B will be ahead of E in half the cases (i.e. 360) and E ahead of B in the remaining half. So straightaway we could say $\frac{6!}{2}$ **= 360 ways.** This "symmetry" based approach is amazingly useful on occasion.

Q. In how many ways can we arrange 6 people A, B, C, D, E and F in a straight line such that B always comes in front of E, who comes in front of C?

In any arrangement, B, C and E can be arranged among themselves in 3! = 6 equally likely ways out of which only 1 (B – E – C) is acceptable. So we can say that 1/6th of all cases satisfy the condition. Thus the required number of ways is $\frac{6!}{6}$ **= 120 ways.**

Q. In how many ways can we arrange 6 people A, B, C, D, E and F in a straight line such that B always comes between C and E?

In any arrangement, B, C and E can be arranged among themselves in 3! = 6 equally likely ways out of which only 2 (E – B – C and C – B – E) are acceptable. Hence 1/3rd of possible cases = $\frac{6!}{3}$ **= 240 ways.**

Q. Three friends P, Q, and R go to an amusement park. Each of the three is accompanied by his father and his paternal grandfather. In how many ways can these nine people stand in the queue at the entry gate if no father wants to stand ahead of his son in the queue?

Consider P and his family. We can select any 3 out of the 9 positions for them in 9C3 ways. There will be only 1 way to arrange them. Similar for Q and his family, we can select 3 out of the remaining 6 places and again there would be just 1 way to arrange them. The remaining 3 places would go to R and his family. Thus we would get ${}^{9}C_3 * {}^{6}C_3 * {}^{3}C_3$ or $\frac{9!}{6!\,3!} * \frac{6!}{3!\,3!} * \frac{3!}{3!\,0!}$ **= 1680 ways**

Again, if we are alert, we can realise that we're just dividing 9 distinct positions into 3 groups P, Q, R with no arrangement within a group – so we could directly have said $\frac{9!}{3!\,3!\,3!}$ **= 1680 ways**

P n C – 17

PnC: Applications

A common question is to find the position of a word among all possible alphabetical arrangements of that word. For example:

Q. If all possible arrangements of the letters of the word TRIANGLE are written out in alphabetical order, in which position would the original word appear?

1) 39999 **2) 39981** **3) 39980** **4) 39982**

First of all let us note the total number of possible words. Since all 8 letters are distinct, there will be 8! = 40320 possibilities. Out of these, the words starting with any letter (say A) will be 7! as we can arrange the remaining 7 distinct letters after that fixed letter. Similarly words starting with, say, AE, will be 6!, with AEG will be 5! and so on.

Now when we arrange them in alphabetical order, first we will get words starting with A, then E, G, I, L, N, R and T in that order. So before T, we will have 7! x 7 words.

Within the words starting with T, we will have 6! words each starting with A, E, G, I, L and N. giving us 6 x 6! words before TR.

Within words starting with TR, we will have 5! each for A, E and G giving 3 x 5!

Proceeding in this manner, the words before TRIANGLE would be (7 x 7! + 6 x 6! + 3 x 5! + 0 x 4! + 3 x 3! + 1 x 2! + 1 x 1!) = 39981. And hence TRIANGLE would be in position 39982. Hence 4.

How about if the word has repeated letters?

Q. If all possible arrangements of the letters of the word GEOMETRY are written out in alphabetical order, in which position would the original word appear?

1) 5307 **2) 5308** **3) 5305** **4) 5306**

The order of letters is EEGMORTY, and the total number of words will be $\frac{8!}{2!} = 20160$

In the first step here, words starting with each of G, M, O, R, T and Y will be $\frac{7!}{2!} = 2520$ as E would be repeated. However, words starting with E will be 7! = 5040 as all the other 7 letters would then be distinct. This logic can be carried to every step.

Thus before G we get 7!, before GE we add 0 (as E is the first letter), and then we proceed normally (since the repeated letter is gone) giving us (1 x 7! + 0 x 6! + 2 x 5! + 1 x 4! + 0 x 3! + 1 x 2! + 0 x 1!) = 5306 words before GEOMETRY. Thus the position of GEOMETRY will be 5307. Hence (1)

Try it for yourself with a tougher one like NONAGON (it will be in the 266th place)

PnC: Applications

A variant on similar lines, but using numbers instead of letters, will be finding the sum of all possible numbers with a given set of digits.

Q. What will be the sum of all the possible five-digit numbers that can be formed with the digits 1, 2, 3, 4 and 5 used once each?

1) 39999990 2) 3999990 3) 3999960 4) 39999960

First of all, let us note that there will be 5! i.e. 120 such numbers. Now consider the units digit. Each of 1, 2, 3, 4 and 5 will occur there in exactly $1/5^{th}$ of the total cases (by symmetry, since no digit is more likely than any other) i.e. 24 times.* Hence the sum of all the "units place" will be (1+2+3+4+5) x 24. Similarly for the tens place, hundreds place and so on. In other words, the overall total will be (1+2+3+4+5) x (24) x (1+10+100+1000+10000) = 15 x 24 x 11111** = 3999960. Hence (3)

* Note that we could also have computed this by fixing each digit and arranging the other digits in 4! ways. However, the symmetry approach simplifies the modified cases such as when there is a 0.

** This has become a standard formula (n-1)! * 1111...*n times* * *sum* of digits. But it will need to be modified if the question changes (as in the examples below) so do understand where it comes from!

Q. What will be the sum of all the possible five-digit numbers that can be formed with the digits 1, 2, 3, 4 and 5 used any number of times each?

1) 104165625 2) 104625625 3) 3999960 4) 101465625

Here there could be 5^5 = 3125 possible numbers. Each digit will occur in each position 625 times. So the total would be (625) x (1+2+3+4+5) x (11111) which comes to 104165625. Hence (1)

Q. What will be the sum of all the possible five-digit numbers that can be formed with the digits 0, 1, 2, 3 and 4 used once each?

1) 2599990 2) 2599980 3) 2599900 4) 3999960

Here the 0 cannot come as the first digit. So the total possible numbers are 4 x 4! = 96. Now in the 1st position (i.e. the ten thousands place) each of 1, 2, 3 and 4 will occur 24 times. However in the other places they will occur only $3/4^{th}$ of 24 i.e. 18 times (since in $1/4^{th}$ of the cases, they will be in the first position themselves!***). Hence the total will be (240000+18000 +1800+180+18) (1+2+3+4) = 2599980. Hence (2).

***Note that 0, not being in the first place, will occur 24 times in each of the other positions. If we want to check this in detail, we can note that 1, for example, will come in the units place 6 times each in numbers starting in 2, 3 or 4. But 0 will occur 6 times each in numbers starting with 1, 2, 3 or 4.

P n C – 19

PnC and Base systems: Applications

Here's a problem which we shall first solve by the standard method, using the Fundamental Principle of Counting. However, we shall also look at a couple of alternative elegant approaches.

Q. How many numbers, not greater than 200000, can be made using the digits 0, 1, 2, 3 and 4?
1) 10000 2) 6250 3) 100000 4) 3125

The possible numbers could have from 1 to 6 digits. Also, the first digit of any number cannot be 0.

- 1 digit: 4 numbers (1, 2, 3 and 4).
- 2 digit: 4 x 5 = 20 numbers (4 ways to select the first digit, 5 for the second one)
- 3 digit: 4 x 5^2 = 100 numbers (4 ways to select the first digit, 5 for each subsequent one)
- 4 digit: 4 x 5^3 = 500 numbers (4 ways to select the first digit, 5 for each subsequent one)
- 5 digit: 4 x 5^4 = 2500 numbers (4 ways to select the first digit, 5 for each subsequent one)
- 6 digit: 1 x 5^5 + 1 = 3126 (if the first digit is 1, there are 5 ways to select each subsequent digit, while if the first digit is 2, there will be just 1 number i.e. 200000)

Hence there will be a total of 4 + 20 + 100 + 500 + 2500 + 3126 = 6250 numbers satisfying the required criteria. Hence (2)

However, this is a long and cumbersome method. We could save ourselves a good bit of trouble by recognising that we are looking for all number using 5 digits (i.e. 0, 1, 2, 3 and 4) and there is one other place in maths where we have this situation – numbers written in base 5.

Hence effectively we are writing all possible numbers in base 5 up till $(200000)_5$. But $(200000)_5$, when converted into base 10, is 2 x 5^5 = 6250. Hence there must be **6250** numbers.

We could also have done this in yet another way, by treating all the numbers as 6 digit numbers but allowing the first digit to become 0. In this case, we could choose the first digit in 2 ways (0 or 1) and all the subsequent ones in 5 ways each, giving us 2 x 5^5 = 6250 numbers. From this tally, we must remove one number (i.e. 000000) but we would also add back one number (200000) so the resultant answer would still be **6250**.

Q. How many numbers between 0 and one million can be formed using 0, 7 and 8? (Cat 02)
1) 486 2) 1086 3) 728 4) None of these

As seen above we could have solved it by saying that it has to be maximum 6 digits, first non-zero, so 2 + 2 * 3 + 2 * 3 * 3 + 2 * 3 * 3 * 3 + 2 * 3 * 3 * 3 * 3 + 2 * 3 * 3 * 3 * 3 * 3 = **728**

However we could also have done it by saying that we have 6 digits _ _ _ _ _ _ each of which could be 0, 7, or 8, which could be done in $\underline{3} * \underline{3} * \underline{3} * \underline{3} * \underline{3} * \underline{3} = 3^6 = 729$ ways out of which all except 1 (000000) are valid solutions. Hence 729 – 1 = **728**

Or we could have said that we need all numbers of up to 6 digits in base 3, which is $(222222)_3$ = **728**

P n C – 20

PnC and Set Theory: Applications

Here's a question involving Set theory and Function terminology as well as PnC.

Q. The total number of relations that can be defined from a set A to set B is 4096. If m and n are the number of elements in the sets A and B respectively, then find the number of values the ordered pair (m, n) can take.

1) 3 **2) 4** **3) 6** **4) 8**

Let's start analysing this by looking at a simpler case.

If we define a function from a set A with 3 elements (a, b, c) and a set B with 4 elements (p, q, r, s), then how can we find the number of relations? Remember that for a function to be defined from a set A to set B, every element of set A must be involved and must connect to exactly one element of set B. However, an element of set B may be connected to multiple elements from A, or even to no element.

We can look at this problem, then, as a P&C question, where each element of A has to be associated with exactly one element of B (but an element of B could be connected with any number of elements of A, even none). This is equivalent to a question like "In how many ways can n students be sent to r classrooms" which would yield the answer r^n. So in the above case each of the elements of A could have 4 options from B ($a \Rightarrow p$, $a \Rightarrow q$, $a \Rightarrow r$ or $a \Rightarrow s$) and hence for the 3 elements of A, the total ways would be $4^3 = 64$

Now we can conclude that in general, if A has 'm' elements and B has 'n' elements, then the total number of possible relationships will be m^n.

Now the above question reduces to asking, 'how many values of (m, n) are there such that m^n is 4096?' and given that m and n have to be natural numbers, the cases are:
(m, n)
(1, 4096)
(2, 64)
(3,16)
(4, 8)
(6, 4)
(12, 2)

Hence there will be 6 cases.

Next week we will take a break from PnC and look at some other topics – a few one-off questions on Geometry would make a good change. When we return to PnC, we shall cover the Theory of Partitioning (which tells us how to distribute identical objects into distinct groups), the theory of Derangement (arranging objects so that each and every one is in the wrong place) and other concepts.

P n C – 21

Theory of Partitioning – 1 (Distributing identical objects into distinct groups)

Most of the basic formulae in permutation and combination deal with the distribution of **distinct** objects. However, there are situations where we wish to count the number of ways of distributing **identical** objects (typically into distinct groups). The Theory of Partitioning deals with such cases. Let's begin with a simple example:

Q. Divide 4 identical balls into 3 boxes A, B and C.

To divide the given 4 balls into 3 groups, it suffices to place them in a straight line and insert two walls or 'partitions'. We can observe that each of the possible distributions may be treated as a distinct arrangement of the 4 balls (identical) and 2 partitions (identical); for example -

4 0 0	is equivalent to	**O O O O I I** (where O is a ball, I a partition)
3 1 0	is equivalent to	**O O O I O I**
3 0 1	is equivalent to	**O O O I I O**
2 2 0	is equivalent to	**O O I O O I**
1 1 2	is equivalent to	**O I O I O O**
1 0 3	is equivalent to	**O I I O O O**
0 3 1	is equivalent to	**I O O O I O**

…and so on.

Thus the original problem is equivalent to arranging 4 identical balls and 2 identical partitions, which can be done in $\frac{6!}{4!2!} = {}^{6}C_2$ i.e. **15 ways.**

In general, to divide n identical objects into r distinct groups, we would have to make (r-1) partitions and hence the number of ways would be $\frac{(n+r-1)!}{n!\,(r-1)!} = {}^{n+r-1}C_{r-1}$. Let's see some examples:

Q. In how many ways can 30 packets of milk be divided among 10 children?

10 children implies 9 partitions. Hence the number of ways should be $\frac{(30+9)!}{30!\,9!}$ or $\mathbf{{}^{39}C_9}$ **ways.**

Q. In how many ways can 200 newspapers be distributed among 7 delivery boys?

Here there will be 6 partitions and the answer will be $\mathbf{{}^{206}C_6}$ **ways.**

Q. 5 men went fishing and returned with a total of 12 fish. In how many ways could they have caught these fish, given that each man may catch any number of fish?

Here there will be 4 partitions and the answer will be $\mathbf{{}^{16}C_4}$ **ways.**

P n C – 22

Theory of Partitioning – 2 (Identical objects into distinct groups, with no empty groups)

When distributing identical objects into distinct groups, what if there is an additional condition that **no group may be empty**? Consider the example we solved in the previous post, where we had to distribute 4 balls into 3 boxes. If we impose the new condition, we find just 3 cases which satisfy it:

2 1 1	is equivalent to	O O I O I O
1 2 1	is equivalent to	O I O O I O
1 1 2	is equivalent to	O I O I O O

Note that two partitions cannot be together. Also no partition can be at the beginning or the end. Hence we can place the partitions only in the intermediate spaces between the balls, and that too only one to a space. So for 4 balls, there will be 3 intermediate spaces O _ O _ O _ O in which we have to put 2 (identical) partitions which can be done in $^{3}\mathbf{C_2 = 3}$ ways.

In general, to divide n identical objects into r distinct groups such that no group is empty, we can place (r-1) partitions in (n-1) spaces; hence the number of ways would be $^{n-1}C_{r-1}$. Let's see some examples:

Q. In how many ways can 30 packets of milk be divided among 10 children, such that no child leaves empty-handed?

By the above logic, the number of ways should be $^{29}C_9$.

Q. In how many ways can 200 newspapers be distributed among 7 delivery boys such that each gets at least one paper?

Here the answer will be $^{199}C_6$ ways.

Q. 5 men went fishing and returned with a total of 12 fish. In how many ways could they have caught these fish, given that each man caught at least one fish?

In this case the answer will be $^{11}C_4$ ways.

An **alternate way** of looking at this concept exists: We could first meet the condition (by placing 1 object in each of the r groups) and then distribute the remaining (n-r) objects into r groups by the earlier formula – this will give us $^{(n-r)+r-1}C_{r-1}$ i.e. $^{n-1}C_{r-1}$ directly.

The advantage of this method is that we can extend it to variants where 'each group must get at least k items' and so on.

P n C – 23

Theory of Partitioning – 3 (More examples and simple variants)

Q. Manoj went to the market to buy fruits. It is known that he bought 20 fruits in all, comprising apples, bananas and chickoos. In how many ways could he have bought the fruits?

Here we need to distribute 20 fruits into 3 groups (apples, bananas or chickoos) and hence the answer would be ${}^{22}C_2$.

Q. How many non-negative integer solutions exist for x + y + z = 20?

This is just the same problem as above; we need to distribute 20 '1's into 3 groups and hence the answer would be ${}^{22}C_2$.

Q. How many positive integer solutions exist for x + y + z = 20?

At first glance this looks similar to the previous problem. However the words positive and non-negative should be a red flag to any experienced test-taker; non-negative adds the possibility of 0 as an answer. Here, groups cannot be empty and so the answer would be ${}^{19}C_2$.

Q. In how many ways can 40 Kitkats be given to 8 children such that each gets at least 3 Kitkats?

First give 3 Kitkats to each of the 8 children. Thus 24 Kitkats are accounted for and the remaining 16 can be then distributed in ${}^{16+8-1}C_{8-1} = {}^{23}C_7$ ways.

Q. 5 men went fishing and returned with a total of 12 fish. In how many ways could they have caught these fish, given that each man caught at least 2 fish?

We could start by first giving 2 fish to each man (thus 10 out of the 12 fish are distributed). The remaining 2 fish can then be distributed to the 5 men in ${}^{2+5-1}C_{5-1} = {}^{6}C_4$ ways.

Q. 5 men went fishing and returned with a total of 12 fish. In how many ways could they have caught these fish, given that Jacob caught at least 3 fish and all the others caught at least one fish each?

We can first give Jacob 3 fish and the remaining 4 men 1 fish each. That takes care of 7 fish. The remaining 5 can be distributed in ${}^{5+5-1}C_{5-1} = {}^{9}C_4$ ways.

Q. Manoj went to the market to buy fruits. It is known that he bought 20 fruits in all, comprising apples, bananas and chickoos. In how many ways could he have bought the fruits, given that he bought at least 3 of each type??

Initially let him buy 3x3 = 9 fruits, satisfying the condition. Then the remaining 11 fruits need to be distributed into 3 groups which can be done in ${}^{13}C_2$ ways.

Theory of Partitioning – 4 (Distribution of some or all of n identical items into distinct groups)

Let's look at another variant of the distribution of identical objects: What if we have to distribute 'up to n identical items' instead of 'exactly n items'?

Q. 20 students appear for an exam in 6 subjects. No two people score the same marks in any subject. In each subject, the top scorer is awarded a gold coin, subject to the condition that his or her score is at least 60% (if no one has scored 60%, no coin is awarded for that subject). A person may win more than one coin. In how many ways can the coins be awarded?

One way of approaching this kind of problem would be to consider that the number of coins awarded could be any number from 0 to 6. Thus we have the following cases:

6 coins awarded among 20 people: ${}^{6+20-1}C_{20-1} = {}^{25}C_{19}$ ways

5 coins awarded among 20 people: ${}^{5+20-1}C_{20-1} = {}^{24}C_{19}$ ways

4 coins awarded among 20 people: ${}^{4+20-1}C_{20-1} = {}^{23}C_{19}$ ways

3 coins awarded among 20 people: ${}^{3+20-1}C_{20-1} = {}^{22}C_{19}$ ways

2 coins awarded among 20 people: ${}^{2+20-1}C_{20-1} = {}^{21}C_{19}$ ways

1 coin awarded among 20 people: ${}^{1+20-1}C_{20-1} = {}^{20}C_{19}$ ways

0 coins awarded among 20 people: ${}^{0+20-1}C_{20-1} = {}^{19}C_{19}$ ways

However, this is long and cumbersome. There is a much more elegant (and, more importantly from a test-taking perspective, much quicker!) approach - introduce a dummy person, like so:

Let's assume that in addition to the 20 people there is 1 dummy with exactly 60% in every subject. So if none of the 20 get a coin, then it goes to the dummy. Thus the problem reduces to the much simpler one of distributing 6 coins among 21 people which can be done in ${}^{6+21-1}C_{21-1} = \mathbf{{}^{26}C_{20}}$ **ways.**

(To convince yourself, you can compute the two results – both yield the numerical answer of 230230)

Q. The King of Siam has several white elephants. On his birthday, he declares that he plans to distribute up to 20 of his elephants to his 9 favourite ministers as a reward for their loyal service. In how many ways could he do this?

The king could give anything from 0 to 20 elephants among the 9 ministers. Effectively, we have to distribute 20 elephants among 10 people (the tenth or dummy minister being the king himself in this case, as he would retain the remaining elephants!) and so it could be done in ${}^{20+10-1}C_{10-1} = \mathbf{{}^{29}C_{9}}$ **ways.**

Q. 5 men went fishing and returned with a total of not more than 12 fish. In how many ways could they have caught these fish, given that each man may catch any number of fish?

Now this is easy: 12 fish can be distributed among 6 people in ${}^{12+6-1}C_{6-1} = \mathbf{{}^{17}C_{5}}$ **ways.**

P n C – 25

Theory of Partitioning – 5 (Distribution with upper limits on group size)

Q. How many two digit numbers exist the sum of whose digits is 13?

By enumeration, we could easily find that there are only half a dozen numbers satisfying this, viz. 94, 85, 76, 67, 58 and 49. However, for 3- or 4-digit numbers enumeration would become infeasible and hence we need to seek an alternative.

Let the 2 digit number be ab. So a + b = 13 with the condition that neither a nor b can exceed 9.
First let us calculate the cases without any condition; a + b = 13 has ${}^{13+2-1}C_{2-1} = {}^{14}C_1 = 14$ solutions. Now let us introduce a' such that a' = a – 10. So a + b = 13 implies a'+ b = 3. These cases which give a solution for a' $\geq$ 0 need to be eliminated (as a cannot be 10 or more). Similarly taking b = b' + 10 means we have to eliminate the solutions of a + b' = 3. Each of these yields 4 cases (and both a and b cannot simultaneously exceed 9) and hence we would have 14 – 4 – 4 = **6 solutions.**

Q. 5 men went fishing and returned with a total of 12 fish. In how many ways could they have caught these fish, given that no man caught more than 7 fish?

As above, ${}^{16}C_4 - 5 \times ({}^{8}C_4)$ = **1470 ways**

Q. How many non-negative solutions exist for x + y + z = 12 such that none of x, y, z exceeds 5?

With no conditions, x + y + z = 12 yields ${}^{14}C_2$ solutions. Now if x exceeds 5, we introduce x' = x – 6 we get x' + y + z = 6 and remove those cases - similarly for y and z - for a total of 3 x ${}^{8}C_2$ solutions. However it is also possible for two variables, say x and y, to simultaneously cross 5; here we put x' + y' + z = 0 - similarly for x and z and for y and z for a total of 3 x ${}^{2}C_2$ solutions - these cases have been subtracted twice in the earlier step and hence must be added back once (the logic mirrors that of a Venn diagram). Hence the total number of solutions would be ${}^{14}C_2 - 3 \times {}^{8}C_2 + 3 \times {}^{2}C_2$ = **10.**

Q. In how many ways can a person score a total of 130 marks in an exam consisting of 3 sections of 50 marks each?

Let the sectional marks be a, b and c. With no conditions, a + b + c = 130 $\Rightarrow$ ${}^{132}C_2$ ways. Now if any one of a, b or c reaches 51, we get a' + b + c = 79 i.e. ${}^{81}C_2$ cases for each of the three variables. When two out of a, b, c cross 50 together, we have 3 cases like a' + b' + c = 28 which yield ${}^{30}C_2$ ways each.

Hence the final answer would be ${}^{132}C_2 - 3 \times ({}^{81}C_2) + 3 \times ({}^{30}C_2)$ which when computed gives **231 ways.**

Alternative solution: Since he loses a total of only 20 marks, an easier way would be to distribute those 20 marks among the 3 sections which can be done, without need of limits, in ${}^{22}C_2$ = **231 ways.**

Distributing Identical Objects into Identical Groups

In the Theory of Partitioning, we concerned ourselves with distributing identical objects into distinct groups. What if the groups are identical? This becomes a more complex problem, and becomes an order of magnitude more complicated when the objects being distributed are distinct….

Q. In how many ways can 7 identical balls be divided among 3 identical boxes?

The simplest way to do this is to list out the possibilities:

(7, 0, 0) (6, 1, 0) (5, 2, 0) (5, 1, 1) (4, 3, 0) (4, 2, 1) (3, 3, 1) (3, 2, 2)

So it can be done in 8 ways.

Q. In how many ways can the above distribution be carried out, such that no box is empty?

Again, the best way is listing out; it turns out that there are four possibilities, as follows:

(5, 1, 1) (4, 2, 1) (3, 3, 1) (3, 2, 2)

A variant problem would be to find the number of ways to distribute distinct people into identical groups of fixed size.

Q. In how many ways can 30 rugby players form two 15-member teams to play each other?

At first glance, this seems to be obvious – there are ${}^{30}C_{15}$ ways of selecting one team and then the other should be automatically selected, right? But wait a moment…. try this question first: **In how many ways can 4 tennis players (A, B, C and D) be divided into two doubles teams to play each other?**

By the same logic we applied above, the answer should be ${}^{4}C_{2} = \frac{4!}{2!\,2!} = 6$ ways. But since it is a small group of people, let us manually list down the possible match-ups:

AB v/s CD AC v/s BD AD v/s BC

….and try as we might, we cannot find another distinct match-up. Hence there are only 3 cases and not 6. Where did we go wrong?

The thing is that, when we do ${}^{4}C_{2}$, i.e. selecting 2 out of 4, we are dividing the 4 into groups of 2 and 2, but those two groups are distinct – one is being selected and the other rejected. Here, the two groups are identical, and hence the answer should further be divided by 2! to account for this: $\mathbf{{}^{4}C_{2} / 2! = 3}$

Similarly in the original question, when we need to divide 30 players into two groups of 15 to play each other, there will be $\mathbf{{}^{30}C_{15} / 2! = \frac{30!}{15!\,15!\,2!}}$ **ways**

(Note: if the groups were of size 16 and 14, they would be distinct and hence ${}^{30}C_{16}$ would work there!)

P n C – 27

Distinct Objects into Identical Groups

With the insight gained in the previous post, we can now attack one of the trickiest questions:

Q. In how many ways can 7 people be divided into 3 identical groups?

People are distinct, and hence we will have to treat this differently. First we will have to list out the distributions for 7 identical objects, and then for each distribution, we will have to figure out the number of ways for distinct people to be distributed.

(7, 0, 0) - 1 way

(6, 1, 0) - $\frac{7!}{6!}$ **= 7 ways**

(5, 2, 0) - $\frac{7!}{5!\,2!}$ **= 21 ways**

(5, 1, 1) - $\frac{7!}{5!\,2!}$ **= 21 ways**

To take care of the two identical groups containing 1 and 1

(4, 3, 0) - $\frac{7!}{4!\,3!}$ **= 35 ways**

(4, 2, 1) - $\frac{7!}{4!\,2!}$ **= 105 ways**

(3, 3, 1) - $\frac{7!}{3!\,3!\,2!}$ **= 70 ways**

To take care of the two identical groups containing 3 and 3

(3, 2, 2) - $\frac{7!}{3!\,2!\,2!\,2!}$ **= 105 ways**

To take care of the two identical groups containing 2 and 2

Thus the total number of ways is 1 + 7 + 21 + 21 + 35 + 105 + 70 + 105 = **365 ways**

If we have the additional condition that **no group should be empty**, then we only take the 4th, 6th, 7th and 8th cases above giving us 21 + 105 + 70 + 105 = 301 cases. The solutions to this situation "dividing distinct objects into identical groups such that no group is empty" (or, as Wikipedia prefers to define it, "the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets") are called Stirling Numbers.

For homework, try to do the simpler case of dividing 5 people into 3 identical groups. If you have understood and applied the procedure correctly, you should get 41 cases there.

P n C – 28

Let us examine another question which at first glance looks very similar to a concept covered earlier:

Q. In how many ways can 7 rings of different colours be worn on the 5 fingers of one hand?

1) 7^5 2) 5^7 3) ${}^{11}P_7$ 4) ${}^{7}C_5$

Superficially, this problem appears similar to that of distributing distinct objects into distinct groups (such as sending 7 children to 5 classrooms). Thus the tempting answer would be, each ring has 5 choices and so the 7 rings would have 5 * 5 * 5 * 5 * 5 * 5 * 5= 5^7 ways.

However, the problem is not quite the same. In the above problem, an implicit assumption is that the order of the objects within the group is not important. However, in the rings-and-fingers question, **the arrangement of rings within a finger is also important**. So this is more like "in how many ways can we arrange 7 children in 5 classrooms?" than "in how many ways can we send 7 children to 5 classrooms"

Let us look at 2 ways of solving such problems:

Approach 1: First imagine that the rings are all identical. In this case the problem reduces to distributing 7 identical objects into 5 distinct groups, which can be solved (using the Theory of Partitioning) by the direct formula ${}^{7+5-1}C_{5-1}$ i.e. ${}^{11}C_4$. Now once the rings have been allocated their positions respective to the fingers, since they are actually distinct, they can be arranged among themselves in 7! ways. Thus the total number of ways will be ${}^{11}C_4 * 7! = \frac{11!}{7!\,4!} * 7! = \frac{11!}{4!} = {}^{11}P_7$ **ways.**

In fact we could also have looked at it (remembering the underlying logic of the Theory of Partitioning) as the number of ways of arranging 7 distinct rings and 4 identical partitions, or in other words arranging 11 objects out of which 4 are identical, which directly gives us $\frac{11!}{4!} = {}^{11}P_7$ **ways**

Approach 2: The 1st ring has 5 options (any of the 5 fingers). Now the second ring has 6 options as it can go above or below the first ring on the same finger, as well as to any of the other 4 fingers. Similarly the third ring has 7 options, the fourth 8 and so on until the 7th ring which has 11 options. Hence overall the 7 rings have $5 * 6 * 7 * 8 * 9 * 10 * 11 = \frac{11!}{4!} = {}^{11}P_7$ **ways**

In general, the number of ways of distributing n objects to r groups such that the order within the groups is also important is $r * (r + 1) * (r + 2) *\ldots* (r + n - 1) = \frac{(n+r-1)!}{(r-1)!} = {}^{n+r-1}P_n$

Q. There are 5 queues for tickets at the local multiplex. 12 men walk in to buy tickets. In how many ways can they stand in the queues?

From the above logic, the number of ways will directly be $5 * 6 * 7 * 8 *\ldots* 16 = {}^{16}P_{12}$ **ways.**

P n C – 29 (A)

Derangements - Introduction

Let's start with the formal definition: a **derangement** is a permutation of the elements of a set such that none of the elements appear in their original position.

Suppose a professor administers a test to four students (A, B, C and D). He grades the sheets (a, b, c and d) and hands them back. In how many ways can he give each student his sheet? Only one way, of course, as shown in the table alongside.

	Possible distributions – all right
A	a
B	b
C	c
D	d

Now suppose the professor administers another test to these students, and this time he decides to let the students grade each other's tests. Obviously, none of the students can grade his/her own paper. Then in how many ways can he distribute the sheets, one to a student, such that none receive their own sheet?

In this case there will be more than one possible solution – in fact if we try to list out the possibilities, it turns out there will be 9 different ways in which we can "derange" the sheets, as listed in table alongside.

	Possible distributions – all wrong								
A	b	b	b	c	c	c	d	d	d
B	a	c	d	d	d	a	a	c	c
C	d	d	a	a	b	d	b	a	b
D	c	a	c	b	a	b	c	b	a

For a 4-membered set, with a total of 4! or 24 possible arrangements, it is already quite a task to list out and enumerate the possibilities. For a 5-membered set, with 120 possible arrangements, it will be even more so, and hence we need to figure out a way to find the number of derangements for larger numbers, without actually listing them out. But first, let us look at a typical problem of the kind often encountered in interviews:

Q. A manager has to send 10 letters to 10 clients. She writes the 10 letters and addresses the 10 envelopes, and then calls in her secretary and asks her to put the letters in the envelopes and mail them. The secretary, being in a hurry to leave for the day, doesn't realise that the letters are all different, and haphazardly stuffs a letter into each envelope without checking if it is correct, and mails the lot. What is the probability that she managed to get exactly 9 of the letters in the correct envelope?

This is a trick question, of course. The snag being that it is impossible to get exactly 9 letters out of 10 correct, because then the 10th one also would be correct. So the required probability is 0. However, there is a relevant learning from this – the fact that if n objects are being allocated to n positions, it is impossible to get exactly (n – 1) correct. This is one of the key pieces of information we need to be able to understand how to proceed with derangements.

Q. In the above question, in how many ways can the secretary get exactly 6 letters correct?

The 6 correct letters can be chosen from the ten in ${}^{10}C_6 = 210$ ways. Also, for each such selection, the remaining 4 letters must all go to the wrong envelopes – which means a derangement of these 4 items. But we have seen above that 4 items can be deranged in 9 ways and hence she can get exactly 6 letters right in 210 * 9 = 1890 ways.

P n C – 29 (B)

Now we are ready to figure out how to approach derangements. Let us take a recursive approach, using the value obtained for a smaller number in the derivation for a larger one.

Let's start with the simplest case, **1 object in 1 position**. Clearly this cannot go wrong. So the number of derangements is 0.

D(1) = 0

Similarly for **2 objects in 2 positions**, either both are right or both are wrong. So there is 1 derangement possible.

D(2) = 1

3 objects in 3 positions: let us look at the ways in which we could get 3, 2 or 1 correct.

Total arrangements	= 3!	= 6
3 correct	only one way	= 1
2 correct	not possible	= 0
1 correct	${}^{3}C_{1}$ (ways of choosing the 1 correct) * 1 (derangements of other 2)	= 3
Hence 0 correct	**total arrangements – 1, 2 and 3 correct**	**= 2**

D(3) = 2

4 objects in 4 positions: let us look at the ways in which we could get 4, 3, 2 or 1 correct.

Total arrangements	= 4!	= 24
4 correct	only one way	= 1
3 correct	not possible	= 0
2 correct	${}^{4}C_{2}$ (ways of choosing the 2 correct) * 1 (derangements of other 2)	= 6
1 correct	${}^{4}C_{1}$ (ways of choosing the 1 correct) * 2 (derangements of other 3)	= 8
Hence 0 correct	**total arrangements – 1, 2, 3 and 4 correct**	**= 9**

D(4) = 9

5 objects in 5 positions: let us look at the ways in which we could get 5, 4, 3, 2 or 1 correct.

Total arrangements	= 5!	= 120
5 correct	only one way	= 1
4 correct	not possible	= 0
3 correct	${}^{5}C_{3}$ (ways of choosing the 3 correct) * 1 (derangements of other 2)	= 10
2 correct	${}^{5}C_{2}$ (ways of choosing the 2 correct) * 2 (derangements of other 2)	= 20
1 correct	${}^{5}C_{1}$ (ways of choosing the 1 correct) * 9 (derangements of other 2)	= 45
Hence 0 correct	**total arrangements – 1, 2 and 3 correct**	**= 44**

D(5) = 44

We can continue in this manner to find D(6), D(7) and so on. Clearly, this is a cumbersome approach, and hence tomorrow we'll take a look at how to solve this problem more efficiently.

Derangements - 2

Now we know how to find the number of derangements. However, this is a time-consuming and cumbersome approach, ill-suited for exam purposes. Can we shorten it? Let us see…

Option 1) A recursive approach to finding Derangements:

Assume that at a dance, there are n men numbered 1, 2 ..., n and n women also numbered 1, 2, ..., n. We have to find the number of ways in which no man gets a partner having the same number. Let us assume that the first man dances with woman i (where $i \neq 1$, obviously). There are $n - 1$ ways for the first man to make such a choice.

There are now two possibilities, depending on whether or not man i dances with woman 1 in return:

Case 1: Man i does not dance with woman 1. This case is equivalent to solving the problem with $n - 1$ men and $n - 1$ women: each of the remaining $n - 1$ men has precisely 1 forbidden choice from among the remaining $n - 1$ women (i's forbidden choice is woman 1).

Case 2: Man i dances with woman 1. Now the problem reduces to $n - 2$ men and $n - 2$ women.

From this, we can logically derive the following relation:

Number of derangements of n objects = $\mathbf{D(n) = (n - 1)[D(n - 1) + D(n - 2)]}$

Option 2) Direct formula:

Using the above logic, we can derive a convoluted direct formula for the derangement of n objects:

Number of derangements of n objects = $\mathbf{D(n) = n!} \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \ldots + \frac{(-1)^n}{n!} \right\}$

Option 3) Another recursive approach to finding derangements:

If we observe the derangements of 1, 2, 3…we observe a pattern which can be used to find the derangements of further numbers. Let's start with D(1) which is 0:

$$\underset{D(1)}{\mathbf{0}} \times 2 + 1 = \underset{D(2)}{\mathbf{1}} \times 3 - 1 = \underset{D(3)}{\mathbf{2}} \times 4 + 1 = \underset{D(4)}{\mathbf{9}} \times 5 - 1 = \underset{D(5)}{\mathbf{44}} \times 6 + 1 = \underset{D(6)}{\mathbf{265}} \times 7 - 1 = \underset{D(7)}{\mathbf{1854}}$$ and so on

Option 4) Memorise!

An exam like the CAT is unlikely to ask you the number of derangements of any number beyond 6 or 7, so it might be simplest to just memorise the first few derangement values: **0, 1, 2, 9, 44, 265, 1854**…

The Necklace Problem

One interesting case we have not considered so far is the arrangement of beads in a necklace.

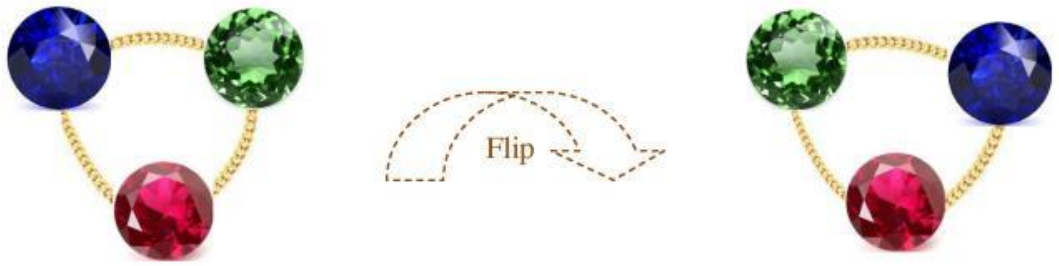
Q. There are 8 beads of different colours and 8 identical lengths of chain with which to connect them. How many distinct necklaces can be formed?.

At first glance, this seems identical to the problem of arranging 8 people in a circle – the answer should be 7! or 5040 ways. But is this really the case? Let's examine a simpler case – where there are just three precious stones (let's say a ruby (red), an emerald (green) and a sapphire(blue)) to be arranged.

The possible arrangements, according to the above logic, should be (3-1)! = 2! or 2 necklaces. Let us try and figure out these two:

If we consider positions from the top left as 1, 2, 3 in the clockwise direction, then 1 possible arrangement will be Sapphire, Emerald, Ruby (**SER**) while the other would be Sapphire, Ruby, Emerald (**SRE**). Note that the other arrangements are equivalent to these – **ERS** and **RSE** can be obtained by rotating **SER**, while **RES** and **ESR** can be obtained by rotating **SRE**.

But…let us think about it; are these two arrangements really different? If we take one of these necklaces and flip it over (upside-down) it will just give us the other necklace! So the same necklace, i.e. the same order of the constituent elements, has been counted twice, once in the clockwise direction and once in the counter-clockwise direction. Hence the actual number of necklaces would be only half of the expected (i.e. only 1 in this case).



In the original question, similarly, every one of the 7! or 5040 cases has been counted twice and hence the actual number of distinct necklaces possible is only half of 5040 = 2520 necklaces.

What is the difference between this case and the case of 8 people arranged around a table? The people cannot be turned upside down, or rather they **should** not be (i.e. they have an inherent sense of "up" and "down", but the beads or jewels can be turned upside down.

Q. A sheet of paper of size 6 cm x 1 cm is divided into 6 squares of size 1 cm x 1 cm. This sheet is to be painted with 6 given colours so that each square is a different colour. In how many ways can this be done?

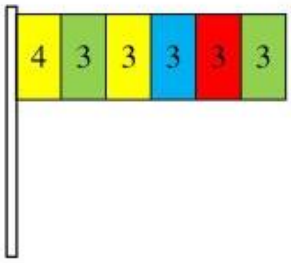
Realise that here, too, the sheet can be turned upside down, so the number of ways will be $\frac{6!}{2}$ = 360. (This is a problem many would get wrong even if they know the necklace logic, as they often remember that as a "formula" applicable only in a circular situation! The logic is what matters…)

FPC and PnC: CAT Scan

Let's look at a few actual questions from past CATs on the concepts we've discussed:

Q. A new flag is to be designed with six vertical stripes, using some or all of the colours yellow, green, blue and red. Then, the number of ways this can be done such that no two adjacent stripes have the same colour is (CAT 04)

1) 12 x 81 2) 16 x 192 3) 20 x 125 4) 24 x 216



The first stripe can be coloured in 4 ways. The next strip can be coloured in 3 ways (any colour except the one used in the first strip). The third also can be in 3 ways (any colour except that of the second strip) and so on. So overall we have 4 * 3 * 3 * 3 * 3 * 3 = 12 x 81 ways. **Hence (1)**

Q. There are 6 tasks and 6 persons. Task 1 cannot be assigned either to person 1 or to person 2; task 2 must be assigned to either person 3 or person 4. Every person is to be assigned one task. In how many ways can the assignment be done? (CAT 06)

1) 144 2) 180 3) 192 4) 360 5) 716

The most stringent condition is on task 2, which can be assigned to 3 or 4. The next is on task 1, which can be assigned to 3, 4, 5 or 6. The other tasks have no restrictions except "one task per person". So we can assign task 2 in 2 ways. For each of these we can assign task 1 in 3 ways (since 1 person out of 3 and 4 is already allotted to task 2). And for each of these, the remaining 4 tasks can be assigned to the remaining 4 people in 4 * 3 * 2 * 1 ways. So 2 * 3 * 4! = 144 ways in all. **Hence (1)**

Q. In a tournament, there are n teams T1, T2, ... Tn, with n > 5. Each team consists of k players, k > 3. The following pairs of teams have one player in common: T_1 and T_2, T_2 and T_3, … T_{n-1} and T_n, and T_n and T_1. No other pair of teams has any player in common. How many players are participating in the tournament, considering all the n teams together? (CAT 07)

1) n(k-1) 2) k(n-1) 3) n(k-2) 4) k(n-2) 5) (n-1)(k-1)

The total number of players across all the teams (counting repetitions) is n*k. But out of these, the number of players being counted twice is equal to the number of teams i.e. n. Hence since they have been counted twice, they should be subtracted once, and hence the required number of players will be n*k – n = n(k – 1). Hence (1)

(Note that this could also have been done by taking some actual numerical values for n and k. For example if we assume 8 teams with 5 members each, we would get 40 players out of whom 8 players represent exactly 2 teams and hence we would have 40 – 8 = 32 distinct players.)

FPC and PnC: CAT Scan

Let S be the set of all pairs (i,j) where $1 \le i < j \le n$ and $n \ge 4$. Any two distinct members of S are called "friends" if they have one constituent of the pairs in common and "enemies" otherwise. For example if n = 4, then S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}. Here, (1, 2) and (1, 3) are friends, (1, 2) and (2, 3) are also friends, but (1, 4) and (2, 3) are enemies. **(CAT 2007)**

1. For general n, how many enemies will each member of S have?

1) $n - 3$ 2) $\frac{1}{2}(n^2 - 3n - 2)$ 3) $\frac{1}{2}(n^2 - 5n + 6)$ 4) $\frac{1}{2}(n^2 - 7n + 14)$ 5) $2n - 7$

2. For general n, consider any two members of S that are friends. How many other members of S will be common friends of both these members?

1) $2n - 6$ 2) $\frac{1}{2}(n^2 - 5n + 8)$ 3) $(n - 3)$ 4) $\frac{1}{2}(n^2 - 7n + 16)$ 5) $n - 2$

Solution – initial thoughts: For any value of n, the definition given i.e. "all pairs (i,j) where $1 \le i < j \le n$" means that each pair is a selection (without arrangement, be it noted!) out of the first n integers. So in other words there should be ${}^nC_2$ member of S. (Note that the $n \ge 4$ is not important, probably it is just given to discourage one from taking a nice small value and doing trial and error!)

So if we want to find the number of friends of any given member, we could consider a specific member - say (1,2). Now any other member containing 1 would be a friend – (1, 3), (1, 4), (1, 5)….(1, n) which gives us (n – 2) friends via 1. Similarly there would be another (n – 2) friends containing 2. Overall, therefore (1, 2) would have 2n – 4 friends.

1. We could find the number of strangers using this – since any other member must be either a friend or a stranger, the number of strangers would be ${}^nC_2 - (2n - 4) - 1$. But there is an easier way to approach it. Consider (1, 2). Any stranger would have to contain neither 1 nor 2 – which means it would need to be a selection of 2 numbers from 3 to n, which is nothing but ${}^{n-2}C_2$ or $\frac{1}{2}(n - 2)(n - 3)$ or $\frac{1}{2}(n^2 - 5n + 6)$.
Hence (3)

2. Again, let us start by considering two members that are friends. For simplicity, let us take (1, 2) and (1, 3). Now any member containing 1 will be a common friend of both these. So (1, 4), (1, 5) (1, 6) …(1, n) will all be common friends. This will give us n – 3 common friends. (The answer choice being present, a lot of people stopped here and marked it and didn't look more closely...but in CAT one has to be suspicious!)

However, there would be one more member who is a common friend of both (1, 2) and (1, 3), and that is (2, 3). So we would have (n – 3) + 1 = (n - 2) common friends. Hence (5).

P n C – 34

Theory of Partitioning – CAT Scan

Let's look at a couple of CAT questions which could be solved using the Theory of Partitioning:

Q. Suppose n is an integer such that the sum of the digits of n is 2, and $10^{10} < n < 10^{11}$. The number of different values for n is (CAT 04)

1) 11 2) 10 3) 9 4) 8

One way to solve it would be to actually list the numbers. Note that the numbers must be 11 digits long and the first digit is non zero. So the first such number would be 10000000001. After that we could keep moving the latter 1 to the left to get 10000000010, 10000000100, 10000001000….11000000000. Since there are ten digits after the first, and any one of them can be made 1, this gives us 10 numbers.

However (and this is where most people got it wrong) there is also one more number – 20000000000. So there would be 11 numbers meeting the required criteria. **Hence (1)**

Another way to approach it would be to use the Theory of Partitioning – we have to divide 2 identicals "units" into 11 groups such that the first group cannot be empty. We can satisfy the condition by first allotting one unit to the first group, and then we can distribute 1 unit to 11 groups with no conditions which can be done in ${}^{1+11-1}C_{11-1} = {}^{11}C_{10}$ = **11 ways.**

Q. What is the number of distinct terms in the expansion of $(a + b + c)^{20}$? (CAT 08)

1) 231 2) 253 3) 242 4) 210 5) 228

At first glance this looks like a question on Algebra, or perhaps Number Theory. However, it can be neatly solved using the logic of partitioning.

Let's first examine a simpler case: consider the expansion of $(a + b + c)^4$. When expanded, this will become $a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2b^1c^1 + 6a^2c^2 + \ldots + c^4$. We'll look into how to compute the coefficients for these terms in another post, for today let it suffice to note that each term in the above expansion is of the 4th order, i.e. it consists of 4 variables ('a's, 'b's or 'c's) being multiplied. To find out the number of distinct terms possible, then, all we need to do is find out the number of different 4th order terms which can be created using a, b, and c – in other words, we need to find out the number of ways in which 4 powers can be distributed into three groups ('a's, 'b's and 'c's). Now the theory of partitioning tells us that this can be done in ${}^{4+2}C_2$ or 15 ways and hence there would be 15 terms in the given expansion.

Taking a more complicated case – let us find the number of terms in $(w + x + y + z)^{12}$. Here it would be equivalent to finding the number of ways in which one could distribute 12 powers into 4 groups, which would be ${}^{12+3}C_3$ or ${}^{15}C_3$ i.e. 455 ways.

The original question is now a straightforward one - $(a + b + c)^{20}$, when expanded would have ${}^{20+2}C_2$ or ${}^{22}C_2$ i.e. 231 terms. **Hence (1)**

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Theory of Partitioning Postscript: The Multinomial Theorem:

We had looked earlier at the Binomial Theorem which enabled us to expand any power of the binomial expression (a+b) as follows:

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$

Now, we can extend that further to any power of a multinomial expression. For example, if we want to expand $(a + b + c + d)^{30}$ we would note the following:

- Every term in the expansion would have a power 30 (in other words the sum of the powers of the variables a, b, c and d would be 30)

- The total number of terms could be obtained by the logic used in the last post: we have to divide 30 (identical) powers into 4 (distinct) groups a, b, c and d so we can do this in ${}^{30+4-1}C_{4-1}$ ways which means that there would be ${}^{33}C_3$ or **5456** terms

- For each term we could find the coefficient by using the logic of arranging items, some of which are identical. For example if we had to find the coefficient of $a^{11}b^3c^6d^{10}$, we could say that we need to choose 11 brackets out of 30 to give an 'a', 3 out of the remaining 19 to give a 'b', 6 out of the remaining 16 to give a 'c' and the remaining 10 brackets to give a 'd'. This could be managed in ${}^{30}C_{11} * {}^{19}C_3 * {}^{16}C_6 * {}^{10}C_{10}$ or $\frac{30!}{11!\,3!\,6!\,10!}$ ways.

  Alternatively, we could look at it as, we have to divide 30 brackets among 4 groups a, b, c and d of size 11, 3, 6 and 10 respectively, which can be done in $\frac{\mathbf{30!}}{\mathbf{11!\,3!\,6!\,10!}}$ ways.

So we could expand $(a + b + c + d)^{30}$ as follows:

$$\frac{30!}{30!}a^{30} + \frac{30!}{29!\,1!}a^{29}b^1 + \frac{30!}{29!\,1!}a^{29}c^1 + \frac{30!}{29!\,1!}a^{29}d^1 + \frac{30!}{28!\,2!}a^{28}b^2 + \frac{30!}{28!\,1!\,1!}a^{28}b^1c^1 + \frac{30!}{28!\,2!}a^{28}c^2 \ldots + \frac{30!}{30!}d^{30}$$

$$(a + b + c + d + \ldots)^n = \sum \frac{n!}{w!\,x!\,y!\,z!}\, a^w\, b^x\, c^y\, d^z \ldots \text{ (where } w + x + y + z + \ldots \text{ adds up to n)}$$

We have done a lot of different stuff in PnC; which could get pretty confusing. In the next post, therefore, before winding up our series on this topic, we'll try to summarise many of the basic situations we have discussed by consolidating them into a single table.

P n C – 36

To sum up, here's a brief ready-reckoner of a whole bunch of cases in PnC:

Objects	Groups	Condition	Instructions	Formula
Distinct	Distinct	1 to 1 linear	Arrange **n** out of **n** in a line	$n!$
Distinct	Distinct	1 to 1 circular	Arrange **n** out of **n** in a circle	$(n-1)!$
Distinct	Distinct	1 to 1 other	Arrange **n** out of **n** in other arrangements	$(n-1)! * k$ (k = number of ways for 1st person)
Distinct	Distinct	1 to 1 necklace	Arrange **n** out of **n** in a circle (where up-down orientation is absent)	$\frac{(n-1)!}{2}$
Distinct	Distinct	1 to 1 linear	Arrange **r** out of **n** in a line	${}^{n}P_{r} = \frac{n!}{(n-r)!}$
Distinct	Distinct	1 to 1 circular	Arrange **r** out of **n** in a circle	${}^{n-1}P_{r-1}$
Distinct	Distinct	1 to 1 other	Arrange **r** out of **n** in other arrangements	${}^{n-1}P_{r-1} * k$ (k = number of ways for 1st person)
Some Distinct	Distinct	1 to 1	Arrange **p+q+r…** objects comprising **p** identical, **q** identical, **r** identical…	$\frac{(p+q+r\ldots)!}{p!\ q!\ r!\ldots}$
Some Distinct	Distinct	1 to 1	Divide **p+q+r…** objects into groups of size **p**, **q**, **r**…	$\frac{(p+q+r\ldots)!}{p!\ q!\ r!\ldots}$
Distinct	Distinct	1 to 1	Select **r** out of **n** objects	${}^{n}C_{r} = \frac{n!}{(n-r)!\ r!}$
Distinct	Distinct	Some together	Arrange **n** objects in a line such that **r** of them are together	$(n-r+1)!\ r!$
Distinct	Distinct	Some apart	Arrange **n** objects in a line such that no 2 out of a particular **r** are together	$(n-r)! * {}^{n-r+1}P_{r}$
Distinct	Distinct	Many to 1	Send **r** distinct objects to **n** distinct groups (No arrangement in groups)	n^{r}
Distinct	Distinct	Many to 1	Send **r** distinct objects to **n** distinct groups (Arrangement within groups)	${}^{n+r-1}P_{n}$
Identical	Distinct (no condition)	Many to 1	Distribute **n** identical objects into **r** distinct groups, groups may be empty	${}^{n+r-1}C_{r-1}$
Identical	Distinct (non-empty)	Many to 1	Distribute **n** identical objects into **r** distinct groups, no group empty	${}^{n-1}C_{r-1}$
Identical	Distinct (upper limit)	Many to 1	Distribute up to n identical objects into r groups, groups may be empty	${}^{n+r}C_{r}$
Distinct	Distinct	1 to 1 (all wrong)	Send **n** objects to **n** places so that all are wrongly placed (Derangement)	$n!(1-\frac{1}{1!}+\frac{1}{2!}\ldots\frac{(-1)^{n}}{n!})$
Identical	Identical	Many to 1	Send **n** identical objects to **r** identical groups	Manual enumeration
Distinct	Identical	Many to 1	Send **n** identical objects to **r** distinct groups	Manual enumeration

Note that this by no means covers all cases; just some of the most popular.

The Pigeonhole Principle

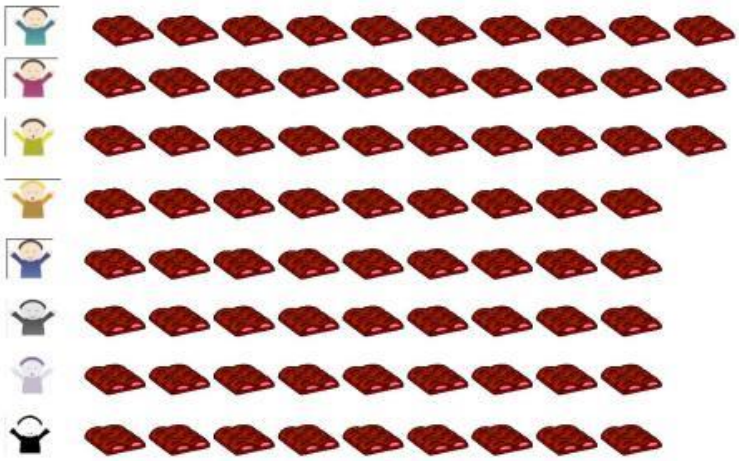
The Pigeonhole Principle states that if m pigeons are put into n pigeonholes, where m > n, then at least one pigeonhole must contain more than one pigeon.

At first glance this seems like an overly simple and obvious statement but it has profound implications (and varied applications) in the world of maths. It is a form of counting principle, and employs many of the same thought processes required in Permutations and Combinations and other applications of the Fundamental Principle of Counting. In slightly more complicated cases, it provides a kind of minimax criterion, the maximum guaranteeable number in a distribution or the minimum number of trials to guarantee a certain result.

This week, let's try and see some examples of how the pigeonhole principle can be used:

Q. 75 identical chocolates are distributed among 8 children. Find the maximum value of n for which the statement "there is at least one child who gets at least n chocolates" is sure to be true?

Let us try to distribute 75 chocolates as evenly as possible among 8 children



The most even distribution we can achieve involves giving 9 chocolates to each of the 8 (using up 72 chocolates) and then giving the remaining 3 chocolates, 1 each, to 3 of the kids. So whatever we do, there will be a kid somewhere who has at least 10 chocolates. Hence the maximum **guaranteeable** value of n is 10.

Note that this is not the maximum **possible** value of n – it could be that one kid gets all 75 chocolates!

Of course, once we understand this, we no longer have to actually do the distribution manually. In the example above, we could just have done $\left\lceil \frac{75}{8} \right\rceil = \lceil 9.375 \rceil = 10$ (where $\lceil x \rceil$ represents the least integer greater than or equal to x).

Effectively, what we are doing is looking at the worst-case scenario which does not satisfy the required conditions (in this case, it is "everybody getting 9"). This "worst-case analysis" is a common feature of Pigeonhole Principle Problems.

Q. There is a drawer in a dark room which contains a large numbers of socks, all of which are either blue or black (and otherwise identical). What is the minimum number of socks which should be picked to guarantee a pair of the same colour?

Let us start by seeing the worst-case scenario. How many socks can be picked so that there is at maximum 1 of each colour? Only 2, since there are only 2 colours available. Hence if we pick 3 socks, we are guaranteed that there will be at least one pair.

Q. There is a drawer which contains 12 pairs of blue gloves, 12 pairs of grey gloves and 12 pairs of black gloves. What is the minimum number of gloves that should be picked to ensure:

a) a pair of gloves of the same colour

b) at least one glove of each colour

c) at least one left-hand and at least one right-hand glove

d) a matching pair of gloves (of the same colour but for different hands)

e) at least 6 gloves of the same colour

a) The worst-case scenario is that we pick only one glove of each colour. Since there are 3 colours, the maximum we can pick without satisfying the conditions is 3. Hence we should pick **4 gloves**.

b) The worst-case scenario is that we pick only all gloves of two colours without picking the third. The maximum we can pick without satisfying the conditions is 48 (12 pairs each of 2 colours). Hence we should pick **49 gloves**.

c) The worst-case scenario is that we pick gloves of only one hand. There are 36 gloves for each hand; which is the maximum we can pick without satisfying the conditions. Hence we should pick **37 gloves**.

d) Since there are 36 pairs, we could end up picking 36 gloves (say, all the left-handed ones) without getting a pair. However, picking 37 gloves will ensure there is a pair. Hence we should pick **37 gloves**.

e) The worst-case scenario is that we pick not more than 5 gloves of each colour. Thus we can pick 5 gloves each of 3 colours i.e. not more than 15 gloves without satisfying the conditions. Hence we should pick **16 gloves**.

Exercise: There are 52 playing cards in a standard pack, consisting of 4 suits having 13 cards each. What is the minimum number of cards to be picked to ensure:

a) at least 1 face card (King, Queen or Jack)

b) at least 7 cards of the same suit

c) at least 10 cards of the same colour

d) at least 1 card from each suit

e) at least 4 non-face cards

You should get: (a) 41 (b) 25 (c) 19 (d) 40 (e) 16

P n C – 39

The pigeonhole principle leads us to some interesting situations and conclusions; for example the following questions which have been asked in interviews and tests off and on:

Q: Is it possible that there will be two people in Mumbai who have exactly the same number of hairs on their head? (Given that the average human being has 200000 hairs)

If the average human being has 200000 hairs, it is a reasonably safe assumption that no one will have more than 1000000 (i.e. 1 million) hairs on their head. Now the population of Mumbai is in the region of 20 million people, so if we try to allot each person a different number of hairs it is clearly impossible (as there are less available numbers than there are people!) Hence **it is not only possible, but certain** that there will be two (and in fact twenty) people in Mumbai with the same number of hairs on their head.

Q: What is the minimum number of people that should be present in a crowd to ensure that at least 6 people share the same birthday?

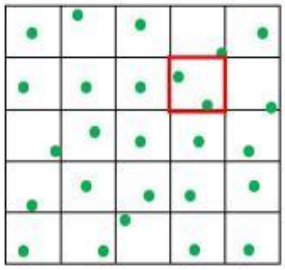
There are a total of 366 possible birthdays (including Feb 29th!) and so even if we have 366 * 5 = 1830 people, it could be possible that there are not more than 5 people born on any given day of the year. However, if we have 1831 people, then there would definitely be at least one day which is the birthday of at least 6 people. hence the minimum necessary number of people would be **1831**.

Q: There are 100 people in a room. Some or all of them shake hands. What is the probability that there will be two people in the room who have shaken hands with the same number of people?

Each person could shake hands with any number of people from 0 to 99. This gives us 100 possible values for the number of handshakes, and so at first glance it seems entirely possible that 100 people could each have a different number of handshakes. But a little careful thought will show us that "0 handshakes" and "99 handshakes" cannot occur simultaneously – if a person shakes hands with all 99 other people then there cannot be a person who has shaken hands with nobody! Hence there are only 99 possible values for the number of handshakes, and 100 people - which guarantees that there will be at least 2 people who have shaken hands with the same number of people. So the required probability is **1**.

Q. There is a square of dimensions 10 x 10. 26 points are randomly selected on it. Is it possible that no two of these points can be enclosed by a single 2 x 2 square?

At first glance this might seem to have nothing to with the pigeonhole principle. But let us observe carefully. The 10 x 10 square can be divided into smaller squares of the required size – 2 x 2 – and there will be exactly 25 such squares formed (as shown in the figure alongside). So if we try and place 26 points in these 25 squares, it is a foregone conclusion that there will be some square (such as the red one in the figure) which will contain two of them. Hence it is **not possible.**



In tomorrow's post we will see some more typical problems from a test perspective.

P n C – 40

Q. There are 126 boxes containing apples. The number of apples in each box lies between 120 and 150 (inclusive). Let F(n) indicate the number of boxes containing n apples and m indicate the maximum value of F(n) for any n. What could be the minimum possible value of m?

1) 4 **2) 6** **3) 5** **4) 7**

An integral part of solving a pigeonhole principle based problem is identifying the pigeons and pigeonholes. Here the pigeonholes are the values 120 to 150 to which each box can be allotted while the pigeons are the boxes themselves.

The number of "pigeonholes" in the problem is 31, while the number of "pigeons" is 126, hence the required answer is $\left\lceil\frac{126}{31}\right\rceil = \lceil 4.\text{xx}\rceil = 5$. **Hence 3**

Q. According to a survey, 78% of people like apples, 75% like bananas and 82% like cherries. What is the minimum percentage of people who like all three?

1) 32 **2) 35** **3) 39** **4) 78**

This can be neatly solved by a variant on the pigeonhole principle. Here the "pigeonholes" are 100% of the people, while the "pigeons" are the "likes" given by them, which are 78 + 75 + 82 = 235%. Now the question can be reworded as "given 235 pigeons and 100 pigeonholes, with not more than 3 pigeons per pigeonhole, what is the minimum number of pigeonholes guaranteed to have 3 pigeons?"

Now an even distribution will give us 200% likes allocated (2 per person) among 100% people. This still leaves 35% likes, which means that there will be a minimum of 35% people who will have 3 likes i.e. who will like all 3 fruits. **Hence (2)**

Another way to do this problem would be to identify the maximum number who people who do not like some fruit or other. 22% do not like apples, 25% do not like bananas and 18% do not like cherries. Let us assume there is no overlap in the above categories (in order to minimise those who like all three) and hence we have a maximum 22 + 25 + 18 = 65% who don't like some fruit or other. Hence the remaining 100 – 65 = 35%, at least, must like all three!

Q. A set S consists of 143 natural numbers, each of which is a perfect cube. The maximum number of elements of S that one can always find such that each of them leaves the same remainder when divided by 13 is

1) 11 **2) 31** **3) 33** **4) 29**

When divided by 13, all numbers could be divided into the categories 13k, 13k+1, 13k+2…..13k+12. When cubed, there turn out to be only 5 possible remainders with 13 (i.e. 0, 1, 5, 8 and 12). Thus we have to place 143 pigeons (numbers) into 5 pigeonholes (possible remainders) guaranteeing at least $\left\lceil\frac{143}{5}\right\rceil = \lceil 28.\text{xx}\rceil = 29$ numbers in at least one remainder. **Hence (4)**

Probability – 1

The **Probability** of an event means the likelihood that that specific event will take place. In English, we use the words probable and improbable fairly loosely - saying something is probable just means that it is more likely to occur than not, while saying that it is improbable means that is more likely to not happen than to happen.

However, in maths, we prefer to be more precise in our definition and hence we need to establish a sound mathematical foundation for the concept of Probability. Let us try to establish such a foundation:

Suppose a normal coin is tossed, and while it is in the air, you call out "Heads". What are the chances that you will win the toss?

Ignoring unrealistic results like the coin landing on its side, there are only two things that can happen – it can either show Heads or Tails. And in one case (Heads) you win, while in the other (Tails) you lose. We can hence say that, in 1 out of 2 cases, you will win. The probability of winning is $\frac{1}{2}$ (or 50%).

(**Note:** the above is true only in the case of a "fair" or "unbiased" coin. If you were to use the coin of Harvey Dent in the Dark Knight, or that of Jai in Sholay, the answer would be very different!)

Let's consider another example: suppose an unbiased cubic die with 6 faces (numbered 1, 2, 3, 4, 5, 6) is rolled and you bet on the number 4. The roll of a die has 6 possible outcomes, and in only 1 of them will you win the bet. Hence we can say that the Probability of winning is $\frac{1}{6}$ (or 16.66%).

In both the above cases, to find the probability, we took the total number of things that could happen, and we saw what fraction (or percentage) of them satisfied the required conditions. Now let us try to couch this in more formal terminology, through the following definitions:

Sample Space (S): Whenever any action is taken which has multiple possible outcomes, the set of all possible outcomes is described as the Sample Space. In the above example of a coin toss, the sample space could be defined as S = {H, T}. In the case of the die being rolled, we could define a sample space of S = {1, 2, 3, 4, 5, 6} or a sample space of S ={even, odd}.

Event: This is the set of "desired results" of the action. For example if a die is rolled and you bet on "odd numbers", then the desired Event or set of favourable outcomes is E = {1, 3, 5}.

(**Note:** a favourable event just means "the event whose probability is sought" - it might be quite unfavourable for the protagonist of the question! For example if we want to know "what is the probability that someone will die in the next episode of 'Game of Thrones'?" the favourable outcome is "someone dying" even though that is very evidently not favourable for the individual in question!)

Given the above two definitions, we can now proceed to define the probability of an event as follows:

The **Probability of an event E** can be computed as $\frac{\text{Number of Favourable Outcomes}}{\text{Total number of Outcomes}}$ i.e. $P(E) = \frac{n(E)}{n(S)}$

[where n(A) is the number of elements in set A)].

Probability – 2

Now that we have a basic definition in place, we need to understand how to apply it. In particular, it is very essential to define the Sample Space properly. Let's understand this through an example:

If a fair die is tossed, what is the probability that it will show
(a) a prime number?
(b) a composite number?

Let's try and list out some possible sample spaces and figure out whether they would be valid or not:

➢ If we take the standard sample space **{1, 2, 3, 4, 5, 6}** we will get the primes 2, 3 and 5 and the composites 4 and 6. So the probability of getting a prime is $\frac{3}{6}$ or $\frac{1}{2}$ while that for a composite is $\frac{2}{6}$ or $\frac{1}{3}$.

➢ What if we take the sample space as **{Prime, Composite}?** This would give us P(Prime) $= \frac{1}{2}$ but P(Composite) $= \frac{1}{2}$ as well. This doesn't match our earlier answer; something must be wrong!

One problem we can immediately recognise is that this sample space does not cover everything! The number 1 is not included in it. So it is not a valid sample space. Let's correct it:

➢ What if we take the sample space as **{Prime, Composite, Odd}?** This covers everything, and gives us P(Composite) $= \frac{1}{3}$; however, it gives us P(Prime) $= \frac{1}{3}$ as well. This also seems problematic.

Looking at this sample space, we see that some numbers occur more than once. For example 3 and 5 occur both in Prime and in Odd. Again, this is not a valid Sample Space.

➢ What if we take the sample space as **{Prime, Composite, 1}?** This covers everything, and that too exactly once each. But again it gives us P(Composite) $= \frac{1}{3}$ and P(Prime) $= \frac{1}{3}$ as well. (And it gives us P(1) $= \frac{1}{3}$ instead of the $\frac{1}{6}$ we ought to expect. So something is still not matching.

Here, the problem is that there is only one way to get a 1, two ways to get a composite, and three ways to get a prime. A prime is thrice as likely to occur as a 1. So a third requirement is that the elements of a Sample Space should also be equally likely to happen.

Thus, the elements of a valid sample space must satisfy the following three criteria:

(a) Exhaustive: Every possible outcome has to be included

(b) Mutually Exclusive: No outcome should occur more than once (i.e. there should be no overlap)

(c) Equiprobable: All the elements should be equally likely to occur

For example, in the case of the die above, the first possibility we considered **{1, 2, 3, 4, 5, 6}** is a valid Sample Space, satisfying all the three conditions. **{Even, Odd}** is another such valid Sample Space.

Probability – 3

The third condition we established for a sample space (equiprobability of the elements) can sometimes lead to slightly counter-intuitive results. We need to understand the underlying logic carefully. So let's spend a bit of time examining it more closely.

For example, an unwary person, when faced with a question like "what is the probability that Virat will score a century in the next match?" will say "Well, either he will score a century, or he won't, and therefore the probability should be $\frac{1}{2}$, right?"

However, one could refute that in several ways. For example:

"What would be the probability that Virat will score 500 in his next match?" OR

"But you could also say that he could score 0, or 1, or 2, or….or 100. So that gives us 101 cases. So should it be $\frac{1}{101}$…? And in fact, he could go on to score more than 100 as well…" OR

"So then, what is the probability of Ishant scoring a century in his next match (not in bowling, mind!)"

The problem being of course, that scoring a century does not have the same likelihood as not doing so. In fact, even the great Don Bradman scored a century every 3 innings or so and so in any given innings his probability of scoring a century would have been a bit above $\frac{1}{3}$.

In fact, in real life conditions, this is how probability is usually estimated - using historical data. And hence, while some questions will directly tell you the probability, some will be presented in the form of data. So if a question has a sentence like "2 out of every 5 bulbs made by company X are defective" or "Robin Hood hits the target three times out of every four arrows he shoots" we should understand that this does not mean that in the **next** 5 bulbs you pick, 2 will be faulty or that in the **next** four arrows shot by Robin, three will hit the target. Rather, we should understand that for **each** bulb, the probability that it is faulty will be $\frac{2}{5}$ and similarly for **each** arrow shot, the probability that it hits is $\frac{3}{4}$.

Consider two coins being tossed. What is the probability of getting two heads?

There could be two ways of generating a Sample Space:

a) {Two Heads}, {One Head and One Tail}, {Two Tails}
b) {H, H}, {H, T}, {T, H}, {T, T}

Now the first would give us a probability of $\frac{1}{3}$ while the second would give us a probability of $\frac{1}{4}$. Both cannot simultaneously be correct! But on closer examination we find that {Two Heads} can occur in only one way while {One Head and One Tail} can occur in two ways and hence they are not equiprobable - this invalidates the first sample space. The correct answer will be $\frac{1}{4}$. And this will be true whether the coins are distinct or identical!

Probability – 4

Let us reinforce the concept from the last post with a few examples:

If three identical coins are tossed, what is the probability of getting 2 Heads?

(Note that an erroneous sample space here would be {0 Heads}, {1 Head}, {2 Heads} {3 Heads})

The valid Sample Space would consist of the following 8 possibilities:
{H, H, H}, **{H, H, T}, {H, T, H}, {T, H, H}**, {H, T, T}, {T, H, T}, {T, T, H}, {T, T, T},

The three cases marked in red would be favourable. Hence the required probability will be $\frac{3}{8}$.

If two dice are rolled, what is the probability that the sum on their faces will be 8?

(Note that an erroneous sample space here would be {Sum = 2}, { Sum = 3} … {Sum = 12})

The valid sample space will consist of 36 cases (which I will list here for demonstration purposes - in the exam, as long as you recognise that there will be 6 ways for each die and hence a total of 6 x 6 = 36 cases (applying the Fundamental Principle of Counting) you should not need actual listing!):

{1, 1},	{1, 2},	{1, 3},	{1, 4},	{1, 5},	{1, 6},
{2, 1},	{2, 2},	{2, 3},	{2, 4},	{2, 5},	**{2, 6},**
{3, 1},	{3, 2},	{3, 3},	{3, 4},	**{3, 5},**	{3, 6},
{4, 1},	{4, 2},	{4, 3},	**{4, 4},**	{4, 5},	{4, 6},
{5, 1},	{5, 2},	**{5, 3},**	{5, 4},	{5, 5},	{5, 6},
{6, 1},	**{6, 2},**	{6, 3},	{6, 4},	{6, 5},	{6, 6},

Again, the favourable cases are marked in red - we can observe that the required probability will be $\frac{5}{36}$.

If three dice are rolled, what is the probability that the sum on their faces will be 16?

We can easily work out that the sample space will consist of 6 x 6 x 6 = 216 cases.
We can also carefully list out the favourable cases (which will add up to 16) as follows:
{6, 6, 4}, {6, 5, 5}, {6, 4, 6}, {5, 6, 5}, {5, 5, 6}, {4, 6, 6},
Hence the required probability will be $\frac{6}{216} = \frac{1}{36}$.

If two dice are rolled, what is the probability that the sum on their faces will be 10 or more?

As above the sample space would consist of 36 cases.
Now the sum on the dice could be 10 **{6, 4}, {5, 5}, {4, 6},** or 11 **{5, 6}, {6, 5},** or 12 **{6, 6}.**
So the required probability would be $\frac{6}{36} = \frac{1}{6}$.

Probability – 5

Complementary Events

Having defined an event E, we can also define a **Complementary Event** - denoted by **E'** or $\mathbf{E^c}$ or $\bar{E}$ - the set of "not desired results" or "unfavourable outcomes". In the case of rolling a die, for example, if the event E is defined as "getting a composite number", then E = {4, 6}, while E' = {1, 2, 3, 5}.

Note that all outcomes are either favourable or unfavourable, and hence we can say

n(E) + n (E') = n(S)

If we divide this throughout by n(S), we can say

$$\frac{n(E)}{n(S)} + \frac{n(E')}{n(S)} = \frac{n(S)}{n(S)}$$

which gives us

P(E) + P(E') = 1 or **P(E') = 1 - P(E)**

This means that if we have, let's say, 6 possible Mutually Exclusive and Exhaustive outcomes, and we want to find the probability that any one of 5 of these outcomes takes place, we could equally well solve it by finding the probability of the remaining outcome, and subtracting that from 1.

If two dice are rolled, what is the probability that the sum on their faces will be 4 or more?

As we saw last time, the sample space would consist of 36 cases.
Now the sum on the dice could be 4, 5, 6, 7, 8, 9, 10, 11 or 12. This would mean computing 9 cases and adding them, not a very comforting proposition.
However, we can see that the only possibilities which do not satisfy the given requirement are a sum of 2 or 3. This gives us only three cases **{1, 1}, {1, 2}, {2, 1}**
So the probability of the sum being 2 or 3 (i.e. of the required conditions **not** being met) is $\frac{3}{36} = \frac{1}{12}$.
The required probability would therefore be 1 - $\frac{1}{12} = \mathbf{\frac{11}{12}}$.

Three friends are planning their birthday parties. What is the probability that at least one of them will be on a weekend?

Here, we could have exactly one birthday on a weekend (3 ways, as any one of the three could be the one) or exactly two on weekends (again 3 ways) or all three on weekends (1 way) leading to a total of 7 cases to evaluate. But a simpler exercise would be to find the complementary probability - that none of them will have their birthday on a weekend.

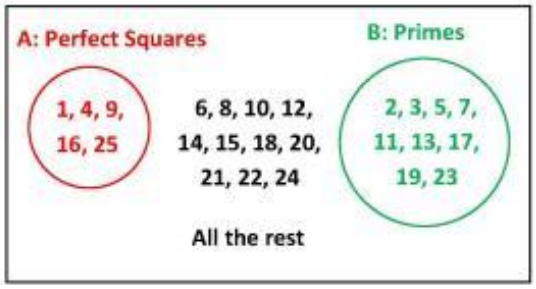
Now each of them has 7 possible days to celebrate their birthday and so the total number of ways for all three will be 7 * 7 * 7 = 343 ways. Each of them has 5 ways to celebrate their birthday on a non-weekend, so our numerator will be 5 * 5 * 5 = 125. Hence the probability that none of them will have a birthday on the weekend is $\frac{125}{343}$ and the requirede answer will therefore be 1 - $\frac{125}{343} = \mathbf{\frac{218}{343}}$

Probability – 6

Mutually Exclusive Events

In this post, let's look at how to calculate the probability for two Mutually Exclusive events resulting from a single trial. Let us also see how to approach it when the events are not Mutually Exclusive.

Q. A man picks a random number between 1 and 25 (inclusive). What is the probability that he picks a perfect square or a prime?



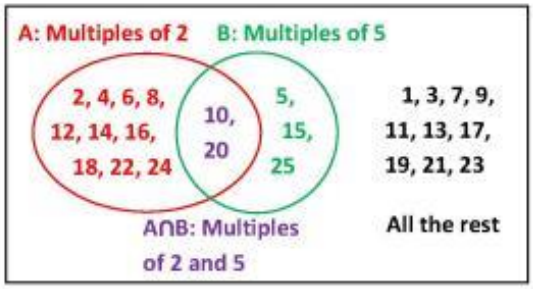
Let us define two events A and B as follows:
A : the event that the number is a perfect square
B : the event that the number is a prime

Note that the events A and B are mutually exclusive; a perfect square can never be a prime! Also note that in Set Theory parlance, we are attempting to find the probability of (A U B). For mutually exclusive events:

$n(A \cup B) = n(A) + n(B)$
dividing throughout by n(S), we get
$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$
$\therefore P(A \cup B) = P(A) + P(B)$

Thus if A and B are two mutually exclusive events, then **P(A OR B occurs) = P(A) + P(B)**

Q. A man picks a random number between 1 and 25 (inclusive). What is the probability that the number he picks is a multiple of 2 or of 5?



Let us define two events A and B as follows:
A : the event that the number is a multiple of 2
B : the event that the number is a multiple of 5

Note that the events A and B are not mutually exclusive; there are two numbers (10 and 20) which are multiples of 2 and of 5! Again, in Set Theory parlance, we are attempting to find the probability of (A U B).

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
dividing throughout by n(S), we get
$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$
$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Thus for any two events A and B then **P(A OR B occurs) = P(A) + P(B) - P(A ∩ B)**

Probability – 7

Conditional Probability

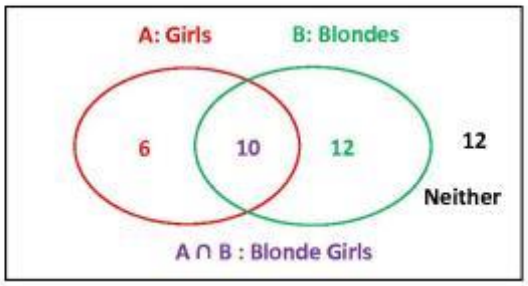
Now, let us examine what happens when we try to find the probability that an event A occurs, given that another event B has already taken place (often denoted by P(A/B)). Note that the wording of such a problem often gives the impression that A must occur after B, but that is not necessary.

Let us take the example of a class containing 40 students, 16 girls (10 blonde and 6 brunette) and 24 boys (12 blond and 12 brunette):

Q. What is the probability that a student chosen at random will be blonde, given that it is a girl?

Remember that the sample space consists of "all possible outcomes". The moment you see a "given that" or "if" in the question, it should be a red flag and you should ask yourself "is this restricting the possible outcomes?" Here, the words "given that it is a girl" means that our sample space is reduced to the 16 girls, and the favourable outcomes will then be the 10 blonde girls; the required probability will therefore be $\frac{10}{16}$ or 0.625.

We can represent the possibilities, as earlier, with a Venn diagram:



Let A be the set of Girls and B be the set of Blondes

Then "blonde, given girl" means we want P(B/A)

"Given Girl", means the sample space is reduced only to the set A (16 elements). Also the Event is the 10 elements in A ∩ B. Hence the required probability is

$$P(B/A) = \frac{n(A \cap B)}{n(A)}$$

Q. What is the probability that a student chosen at random will be a boy, given that it is blonde?

In this case, the words "given that it is a blonde" means that our sample space is reduced to the 22 blondes, and the favourable outcomes will then be the 12 blonde boys; the required probability will therefore be $\frac{12}{22}$ or 0.545.

Probability – 8

Let us try to extend the logic of the previous post. We had found that $P(B/A) = \frac{n(A \cap B)}{n(A)}$

If we divide both the numerator and the denominator by n(S), the entire sample space (40 in this case), it will not change the validity of the equation. But dividing $n(A \cap B)$ by n(S) gives $P(A \cap B)$, and similarly dividing n(A) by n(S) gives P(A). Hence we can say that

$$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{P(A \cap B)}{P(A)}$$

We can rearrange this result to say that $P(A \cap B) = P(A) * P(B/A)$ (or, for that matter, $P(B) * P(A/B)$. In the above example, this would be P(blonde girl) = P(girl) * P(blonde, given girl) $= \frac{16}{40} * \frac{10}{16} = \frac{1}{4}$. (which is the actual probability, as we could check from visual observation!)

In a similar manner we could also show that $P(A \cap B) = P(B) * P(A/B)$. In the above example, this would be P(blonde girl) = P(blonde) * P(girl, given blonde) $= \frac{22}{40} * \frac{10}{22} = \frac{1}{4}$.

In a case like this, the numbers involved are small enough that we could find the answer by observation. Where this approach comes in really handy is for cases involving bigger numbers:

Q. A person writes numbers from 1 to 100 on chits of paper and puts them in a bag. Another person draws two chits of paper one after another without replacement. What is the probability that both numbers are perfect squares?

The probability that the first number is a perfect square will be $\frac{10}{100}$ (since there are 10 perfect squares among the 100 numbers). Given that the first number picked was a perfect square, there will be only 9 perfect squares left in the remaining 99, and hence the probability that the second number will also be a perfect square will be $\frac{9}{99}$. Using the logic we derived above, P(1st and 2nd both perfect squares) = P(1st perfect square) * P(2nd perfect square, given 1st perfect square) $= \frac{10}{100} * \frac{9}{99} = \mathbf{\frac{1}{110}}$.

Q. A person writes numbers from 1 to 100 on chits of paper and puts them in a bag. Another person draws two chits of paper one after another with replacement. What is the probability that both numbers are perfect squares?

The probability that the first number is a perfect square will be $\frac{10}{100}$. Since the chit is being replaced in the bag, the probability that the second number will also be a perfect square will again be $\frac{10}{100}$. Hence P(1st and 2nd both perfect squares) = P(1st perfect square) * P(2nd perfect square, given 1st perfect square) $= \frac{10}{100} * \frac{10}{100} = \mathbf{\frac{1}{100}}$.

In the next post, we shall use the above example to understand the terms Dependent and Independent.

Probability – 9

Dependent and Independent Events

What is the essential difference in the two questions we solved at the end of the last post? In one case, the two chits were drawn without replacement, in the other they were drawn with replacement. This small change in wording leads to a significant change in the behaviour of the system, though.

Let the event of the 1st chit being a perfect square be A and that of the 2nd being a perfect square be B.

When we drew the two chits without replacement, the probability of B depended on whether A happened or not. When A happened (the first number picked was a perfect square) we found that P(B/A) was $\frac{9}{99}$. But if A^c had happened instead (the first number was **not** a perfect square), then there would have been 10 squares left in the 99 numbers and hence $P(B/A^c)$ would have been $\frac{10}{99}$. Since the probability of B occurring depends on whether A occurred or not, we can say that B is dependent on A.

However, when we drew the two chits with replacement, the sample space for the second pick would be the same irrespective of whether the first number was a perfect square or not – 100 chits, 10 bearing a perfect square. And hence, whether A occurred or not, the probability of B's occurrence would have been the same. In other words, $P(B/A) = P(B/A^c) = \frac{10}{100} = \frac{1}{10}$. In this case, we can say that B is independent of A. We can define $P(B/A) = P(B/A^c)$ as P(B), the probability that B will occur.

Now we can see that in the case of independent events, we can re-write the formula we created earlier $P(A \cap B) = P(A) * P(B/A)$ by calling P(B/A) as P(B). Thus we can say that for independent events, $P(A \cap B) = P(A) * P(B)$

Q. Frodo and Sam decide to appear for a test. Frodo's chances of clearing the test are $\frac{1}{4}$ while Sam's are $\frac{2}{7}$. What is the probability that both will pass the test?

Assuming the two of them do not copy from each other, the events of each of them passing are independent of each other as Frodo's passing or failing will have no bearing on Sam's. So we can just multiply the probabilities of their passing individually to get the probability that both pass as $\frac{1}{4} \times \frac{2}{7} = \mathbf{\frac{1}{14}}$

We can also now re-visit a question we had solved earlier by first principles:

Q. Three friends are planning their birthday parties. What is the probability that at least one of them will be on a weekend?

The date of birth of one friend will not have a significant correlation with the date of any other and hence all three birthdates would be independent. Now the probability that each of them is NOT on a weekend is $\frac{5}{7}$ and hence for all three to be not on weekends it is $\frac{5}{7} \times \frac{5}{7} \times \frac{5}{7} = \frac{125}{343}$ and so the probability that at least one will be on a weekend (the complementary case) it will be $1 - \frac{125}{343} = \mathbf{\frac{218}{343}}$.

Probability – 10

Let us now try and solve a few questions on the above concepts:

Q. Three archers are shooting at a difficult target. Green Arrow has a probability of G = $\frac{2}{5}$ of hitting it, Katniss Everdeen has a probability of K = $\frac{1}{2}$, and Hawkeye has a probability of H = $\frac{2}{3}$.

(a) What is the probability that all three hit the target?

(b) What is the probability that the target is hit?

(c) What is the probability that only Hawkeye hits the target?

(d) What is the probability that exactly one person hits the target?

(e) What is the probability that exactly two people hit the target?

We should note that the probability of any of the three hitting the target is unaffected by the performance of the others; i.e. the three events are independent. Also if G is $\frac{2}{5}$, G^c (the probability that Green Arrow misses) will be $\frac{3}{5}$. Similarly if K = $\frac{1}{2}$, K^c will be $\frac{1}{2}$ and if H = $\frac{2}{3}$, H^c will be $\frac{1}{3}$.

(a) The probability of all three hitting the target will be $G * K * H = \frac{2}{5} * \frac{1}{2} * \frac{2}{3} = \frac{2}{15}$

(b) The target could be hit by any one, or by any two, or by all three. We would have to consider numerous cases (7 in all) and add them up. Alternatively, we could just find the probability of the complementary event i.e. that all three miss and subtract it from 1. Hence the probability that the target is hit = $1 - G^c * K^c * H^c = 1 - \frac{3}{5} * \frac{1}{2} * \frac{1}{3} = 1 - \frac{1}{10} = \frac{9}{10}$

(c) The probability that only Hawkeye hits the target is $G^c * K^c * H = \frac{3}{5} * \frac{1}{2} * \frac{2}{3} = \frac{1}{5}$

(d) The probability that exactly one person hits the target is $G * K^c * H^c + G^c * K * H^c + G^c * K^c * H$
$= \frac{2}{5} * \frac{1}{2} * \frac{1}{3} + \frac{3}{5} * \frac{1}{2} * \frac{1}{3} + \frac{3}{5} * \frac{1}{2} * \frac{2}{3} = \frac{11}{30}$

(e) The probability that exactly two people hit the target is $G * K * H^c + G * K^c * H + G^c * K * H$
$= \frac{2}{5} * \frac{1}{2} * \frac{1}{3} + \frac{2}{5} * \frac{1}{2} * \frac{2}{3} + \frac{3}{5} * \frac{1}{2} * \frac{2}{3} = \frac{12}{30} = \frac{2}{5}$

Note that there are 4 broad cases here (mutually exclusive and exhaustive) – all three hit, two hit, one hits and none hit; their probabilities turned out to be $\frac{2}{15}$, $\frac{11}{30}$, $\frac{12}{30}$ and $\frac{1}{10}$ respectively. If we add up these probabilities, they come to 1.

Probability – 11

Q. During a war, the High Command decides that a bridge belonging to the enemy needs to be destroyed. They send a squad of bomber planes to drop bombs on the bridge. One direct hit from a bomber is enough to destroy the bridge. Each bomber has a 20% chance of scoring a direct hit. What is the minimum number of bombers that should be sent so that the probability of the bridge being destroyed is at least 60%?

A common mistake in such a problem is to say "for one bomb the chance is 20%, so for two bombs it will be 40% and for 3 it will be 60%…" The thing is, the events of the individual bombers scoring a hit are not mutually exclusive events and cannot be added up (if they could, then with 6 bombs we would get a probability of 120%, which is clearly absurd!). Instead, the events are independent; and hence we can solve this by using the logic of complementary probability – since one or more bombs will destroy the bridge, we can find the probability that all the bombs will miss and figure out when this will drop below 40%.

For 1 bomb, the probability of it missing will be 0.8
For 2 bombs, the probability of both missing will be $(0.8)^2 = 0.64$
For 3 bombs, the probability of all three missing will be $(0.8)^3 = 0.512$
For 4 bombs, the probability of all four missing will be $(0.8)^4 = 0.4096$
For 5 bombs, the probability of all five missing will be $(0.8)^5 = 0.32768$ (we need not calculate this last one accurately as it will clearly be less than 40%)
Hence if 5 bombers are sent, there will be a greater than 60% chance that the bridge will be destroyed.

Q. Pakistan and Bangladesh are planning a series of ODI matches in the year 2020. It is known that on average, Pakistan wins one out of every six matches against Bangladesh. What is the minimum number of games that should be scheduled such that Pakistan has a fair chance of winning at least one match?

Let us first notice a couple of words liable to cause confusion – "a fair chance". What they indicate is "a probability of 50% or more". We can use the same approach as last time – finding out for how many games the probability of losing every single game will drop below 50%. Now the probability of Pakistan winning a game is $\frac{1}{6}$ (and hence that of losing a game is $\frac{5}{6}$). We could write this as 83.33%, but then the calculation would become more difficult; this is one of the cases where we would probably prefer to go with the fractional form

If there is only 1 game, the probability of losing it would be $\frac{5}{6}$
If there were 2 games, the probability of losing both would be $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$ (which is >50%)
If there were 3 games, the probability of losing all three would be $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$ (which is >50%)
If there were 4 games, the probability of losing all four would be $\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$ (which is <50%)
Hence a minimum of 4 games would need to be played.

Probability – 12

Let's look at a couple of similar-looking questions which can illustrate a subtle distinction:

Q. There are 10 bulbs in a box. It is known that 4 of the bulbs are faulty. A room has 3 sockets for bulbs. If Watson takes 3 bulbs from the box and fits them into the sockets, then what is the probability that

a) all three bulbs will be lit?
b) the room will be lit?

a) The probability that the first bulb will be lit will be $\frac{6}{10}$. Given that it is lit, the probability that the second will also be lit is $\frac{5}{9}$, and then for the third one it will be $\frac{4}{8}$. Hence the required probability will be $\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \mathbf{\frac{1}{6}}$.

b) The simplest way to do this would be to calculate the complementary probability (i.e. that the room is not lit, that all the bulbs are faulty). Hence the required probability will be $1 - \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \mathbf{\frac{29}{30}}$.

Q. It is known that 40% of Bulbul brand bulbs are faulty. A room has three sockets for bulbs. If Sherlock buys 3 Bulbul bulbs and fits them into the sockets, then what is the probability that

a) all three bulbs will be lit?
b) the room will be lit?

How is this problem different from the previous one? In the previous case, we had a limited number of bulbs (10) from which we were choosing. So each bulb removed changed the probabilities for the next bulb. However, in this case, the number of bulbs is not limited in this manner so the events are effectively independent; the probability that a bulb is faulty (or not) will have nothing to do with whether the previous one was faulty (or not).

a) The probability that the first bulb will be lit will be $\frac{6}{10}$. Given that it is lit, the probability that the second will also be lit is again $\frac{6}{10}$, and then for the third one too it will be $\frac{6}{10}$. Hence the required probability will be $\frac{6}{10} \times \frac{6}{10} \times \frac{6}{10} = \mathbf{\frac{27}{125}}$.

b) Here the required probability will be $1 - \frac{4}{10} \times \frac{4}{10} \times \frac{4}{10} = \mathbf{\frac{117}{125}}$.

[Note that these two questions are, in a way, analogous to the Permutation cases of without repetition ($n \times (n-1) \times (n-2) \ldots = {}^{n}P_{r}$) and with repetition ($n \times n \times n \ldots = n^{r}$)]

In the next post we'll talk about probabilities in infinite sample spaces (aka geometric probability)

Probability – 13

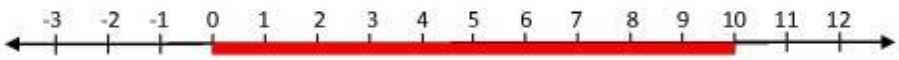
Probability in Infinite Sample Spaces

Q. A real number k is chosen between 0 and 10. What is the probability that $k^2 > 36$?

A common mistake here would be to assume the sample space to be {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the favourable outcomes to be {7, 8, 9, 10} giving a probability of $\frac{4}{11}$. That would have been valid, had the question asked for integer values of k, but in fact we are required to choose a real k.

If we try to create a sample space here, we will find that there are infinite possible outcomes as there are infinite real numbers between 0 and 10. So we will need to use a somewhat different approach here:

Let us consider a number line: the region marked in red covers the entire set of numbers we can select.

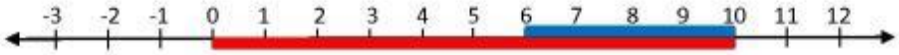


Now let us ponder a question: if I divide the region from 0 to 10 into two parts - "0 to 5" and "5 to 10", which part will contain more numbers? Technically, this seems like a silly question, since both contain infinite numbers, but we can intuitively realise that the number of numbers in both sets is the same (after all, if we choose a number x in the range "0 to 5", there will be a corresponding value x+5 in the range "5 to 10"!). Note that the two ranges are equal in length, when drawn on the number line! So the number of elements in "0 to 5" is half the total number in "0 to 10"

What if we were to choose the range "0 to 2"? We can visualise that this will contain 1/5th of the numbers in the "0 to 10" range. Not surprisingly, the range "0 to 2", when drawn on the number line, is exactly one-fifth of the length of "0 to 10".

Thus we can conclude that we can use the relative length of the line segment on the number line as a surrogate measure for the number of numbers included in the range.

Now returning to the original question, in the numbers between 0 and 10, any value of k above 6 will satisfy the condition $k^2 > 36$ and hence the favourable outcomes can be represented as shown in blue:



The "favourable event" is a line of length 4 (while the "sample space" is a line of length 10).
The required probability will therefore be $\mathbf{\frac{4}{10}}$.

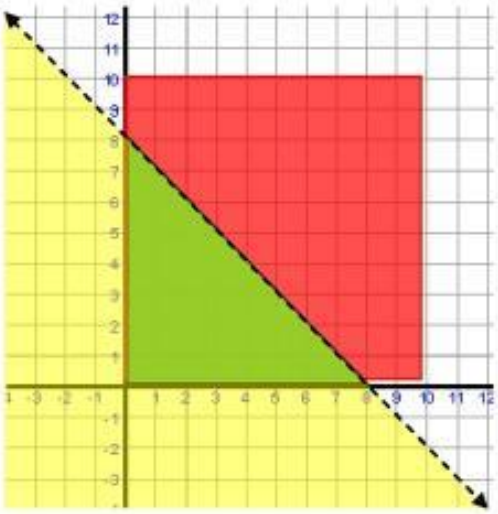
Q. A real number k is chosen between 0 and 10. What is the probability that $k^2 \geq 36$?

Will our answer change in this case? We have an additional solution of k = 6, after all! However, we are effectively adding one outcome in the numerator, and there are infinite outcomes in the denominator. The probability will therefore increase by $\frac{1}{\infty} \approx 0$ i.e. it will effectively stay the same!

Probability – 14

Q A point (x, y) such that $0 \le x \le 10$, $0 \le y \le 10$ is plotted on the x-y plane. What is the probability that $x + y < 8$?

Taking a cue from last time's question, here we can use area, rather than length, as the surrogate. Let us roughly plot the given information:

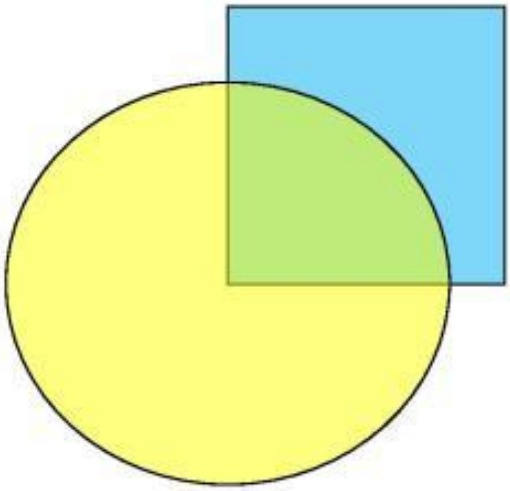


As can be seen in the adjoining figure, the first condition i.e. $0 \le x \le 10$, $0 \le y \le 10$ is represented by a square of side 10 units (shown in red). This is our "sample space".

The second condition, $x + y < 8$ can be represented by the yellow region shown, However, our favourable outcomes have to be a subset of the sample space and hence we only consider the region shown in green as our "event".

Now the area of the square is $10 * 10 = 100$ while that of the triangle is $\frac{1}{2} * 8 * 8 = 32$ and hence the required probability is $\frac{32}{100}$ or 0.32.

Q: A cow is tied to a post at one corner of a square field of side 12 m, using a rope of length 10 m. One day, a daisy grows in the field. Find the probability that the cow will be able to eat the daisy.



How does one find the sample space here? It's actually quite simple - the daisy grows somewhere in the field, so the area of the field (shown alongside in blue) covers all possible places where the daisy can grow. This is our sample space.

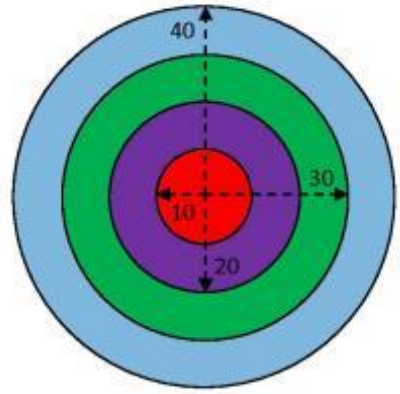
What about the favourable outcomes? (Favourable for the cow, and for the question, but not for the daisy, of course!). Should it be all the regions the cow can reach? Not quite - part of the area within a radius of 10 m of the cow (shown in yellow) is not within the field (which is our sample space!) and hence we have to neglect this area and consider only the area in green, i.e. "the area within the field that the cow can reach". Effectively, it is a case of conditional probability.

The field will be a square of side 12, while the area within it that the cow can reach will be a quarter of a circle of radius 10. The required probability will therefore be $\frac{\frac{1}{4}\pi 10^2}{12^2} = \frac{25\,\pi}{144} \approx 54.5\%$

Probability – 15

Q. A dartboard is formed of 4 concentric rings with radii 10, 20, 30 and 40 cm respectively. Given that a dart strikes the dartboard, what is the probability that it strikes somewhere on the outermost ring?

Here, again, we can use the area logic:



We are told that the dart has struck the dartboard. Hence our set of all possible outcomes, our sample space, is represented by the total area of the dartboard, which will be $\pi * 40^2$

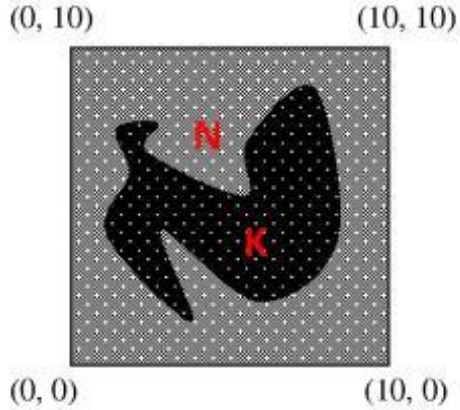
Also, the favourable outcome is that it strikes somewhere on the outermost ring, which as can be seen from the figure is the blue annular region, whose area will be $\pi * (40^2 - 30^2)$

Hence the required probability will be

$$\frac{\pi * (40^2 - 30^2)}{\pi * (40^2)}$$

which works out to $\frac{7}{16}$ or 0.4375.

Note: This approach can be inverted to solve certain problems in Geometry and related areas. For example, if one wants to find the area of a randomly shaped closed figure, one could enclose it in a square, and generate random points within the square. The fraction of points lying within the figure, when multiplied by the area of the square, will give the area of the figure.



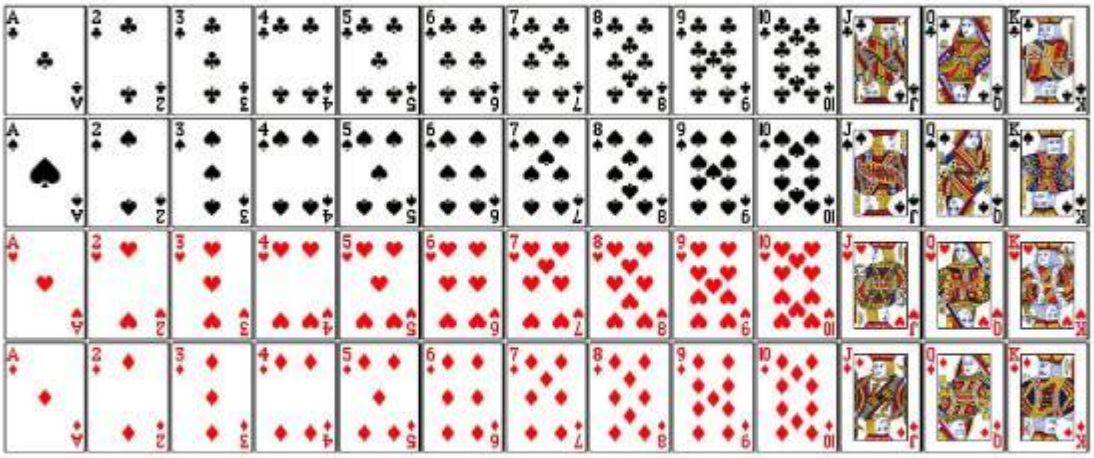
To demonstrate: suppose we need to find the area of the dark shape alongside. Let us say we can enclose it in a square of side 10 units, i.e. of total area 100 square units. Now let us generate N points within the square (for each one, generate two random numbers between 0 and 10 and assign them as the x- and y-coordinates respectively). Now count how many of these points lie within the boundaries of the figure whose area we are attempting to determine. Suppose there are K such points. Then the area of the figure can be roughly approximated by $\frac{K}{N}$ x 100. The greater the value of N, the more accurate our answer will turn out to be.

Nowadays, such methods are typically run using computer simulations and the number of points taken runs into the millions. This kind of technique of "repeated random sampling" is most useful in cases like the above one when it would be difficult or impossible to use other more formal mathematical methods. This is a variant of a class of numeric computation methods called Monte Carlo simulations.

In the next post(s) we'll discuss one of the most popular question types: picking cards from a pack.

Probability – 16

The "Standard Pack" of cards consists of 52 cards (as well as two Jokers, which are generally neglected from a perspective of Maths questions). There are 4 "suits" (Hearts, Clubs/Flowers, Spades and Diamonds) each containing 13 cards - an Ace or 1, nine cards numbered from 2 – 10, and 3 Face Cards (Jack, Queen and King).



Q. Two cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that both are kings?

The probability that the first card is a king will be $\frac{4}{52}$ (since there are 4 kings among the 52 cards). Given that the first card picked was a king, there will be only 3 kings left in the remaining 51, and hence the probability that the second card will also be a king will be $\frac{3}{51}$. Using the logic we derived above, P(1st and 2nd both kings) = P(1st king) * P(2nd king, given 1st king) = $\frac{4}{52} * \frac{3}{51} = \mathbf{\frac{1}{221}}$.

We could also have used combinations to solve such a problem in which we are picking two (or more) cards without replacement (which is an equivalent situation to "picking up two cards simultaneously) when the order is not specified. In this question, the denominator (sample space) would be the number of ways of choosing 2 cards out of 52 i.e. ${}^{52}C_2$. The numerator (event) would be the number of ways of picking 2 kings out of 4, i.e. ${}^{4}C_2$. Hence the required probability would be $\frac{{}^{4}C_2}{{}^{52}C_2} = \mathbf{\frac{1}{221}}$.

Q. Two cards are drawn (one after the other, with replacement) from a standard pack of 52 cards. What is the probability that both are kings?

The probability that the first card is a king will be $\frac{4}{52}$ (since there are 4 kings among the 52 cards). With replacement, the events become independent and the probability that the second card will also be a king will again be $\frac{4}{52}$. Hence, P(1st and 2nd both kings) = $\frac{4}{52} * \frac{4}{52} = \mathbf{\frac{1}{169}}$.

In this "with replacement" case, we cannot use combinations as repetition is allowed. Here the logic will be similar to n^r (as we had observed in the post "Probability – 12"). The "without replacement" situation is the more common one, and hence we shall spend a little more time on it over the next few posts, examining several minor variants:

Probability – 17

Q. Two cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that the first is a king and the second a queen?

The probability that the first card is a king will be $\frac{4}{52}$ (since there are 4 kings among the 52 cards). Given that the first card picked was a king, there will be 4 queens in the remaining 51, and hence the probability that the second card will be a queen will be $\frac{4}{51}$. So, P(1st king, 2nd queen) = $\frac{4}{52} * \frac{4}{51} = \mathbf{\frac{4}{663}}$.

Note: If the above problem had been posed as a "with replacement" question, then we would have solved it as $\frac{4}{52} * \frac{4}{52} = \mathbf{\frac{1}{169}}$.

Q. Two cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that one is a king and the other a queen?

At first glance this looks similar to the previous question. However, the key difference is that in the previous question the order (king, queen) was fixed while here the order is flexible. So we could either have a king followed by a queen (with a probability of $\frac{4}{663}$ as seen above), or a queen followed by a king (which also turns out to have a probability of $\frac{4}{663}$) resulting in an overall probability of $\mathbf{\frac{8}{663}}$.

Another way of looking at it would be: the first card could be either a king or a queen, giving a probability of $\frac{8}{52}$. If the first is a king, then the second needs to be a queen, and vice versa; either way the probability is $\frac{4}{51}$. The required probability will therefore be $\frac{8}{52} * \frac{4}{51} = \mathbf{\frac{8}{663}}$.

Note: If the above problem had been posed as a "with replacement" question, then we would have solved it as $\frac{4}{52} * \frac{4}{52} * 2 = \mathbf{\frac{2}{169}}$. Alternatively we could have said $\frac{8}{52} * \frac{4}{52} = \mathbf{\frac{2}{169}}$.

Here, too, we could have used combinations; the denominator (sample space) would be the number of ways of choosing 2 cards out of 52 i.e. ${}^{52}C_2$, while the numerator (event) would be the number of ways of picking a king AND a queen = ${}^{4}C_1 * {}^{4}C_1$. Hence the required probability would be $\frac{{}^{4}C_1 * {}^{4}C_1}{{}^{52}C_2} = \mathbf{\frac{8}{663}}$.

But in the case of an ordered question (such as the previous "1st king 2nd queen" question) we cannot apply combinations as the situation is that of a permutation. We could technically do it as $\frac{{}^{4}P_1 * {}^{4}P_1}{{}^{52}P_2} = \mathbf{\frac{4}{663}}$. But this would be confusing and prone to error. Apply it only if you are confident you know what you're doing!

What if we need to pick 3 cards? Let's look at a couple of such examples next time:

Probability - 18

Q. Three cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that the first is an ace, the second a king and the third a queen?

The probability for the first card to be an ace would be $\frac{4}{52}$, that for the second to then be a king would be $\frac{4}{51}$ and that for the third to subsequently be a queen would be $\frac{4}{50}$. So the required probability would be $\frac{4}{52} * \frac{4}{51} * \frac{4}{50}$.

Note: If the above problem had been posed as a "with replacement" question, then we would have solved it as $\frac{4}{52} * \frac{4}{52} * \frac{4}{52}$

Q. Three cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that one is an ace, one a king and one a queen?

The probability for 1st ace, 2nd king and 3rd queen would be $\frac{4}{52} * \frac{4}{51} * \frac{4}{50}$ as seen in the previous question. Since (ace, king, queen) can be in any order, this gives us 3! cases, each with the above probability, and hence the total probability would be $\frac{4}{52} * \frac{4}{51} * \frac{4}{50} * 3!$ (note that for more than 2 cards you won't have to find the actual number)

Alternatively we could choose the first card as any of (ace, king queen) in 12 ways giving us $\frac{12}{52}$; the second card can be from any of the remaining two types and so on which would yield us $\frac{12}{52} * \frac{8}{51} * \frac{4}{50}$

A third option would be to say that the required probability would be $\frac{{}^4C_1 * {}^4C_1 * {}^4C_1}{{}^{52}C_3}$ which if we examine it carefully will yield the same result as above.

Note: If the above problem had been posed as a "with replacement" question, then we would have solved it as $\frac{4}{52} * \frac{4}{52} * \frac{4}{52} * 3!$ or as $\frac{12}{52} * \frac{8}{52} * \frac{4}{52}$

Q. Three cards are drawn (one after the other, without replacement) from a standard pack of 52 cards. What is the probability that two are aces and one is a king?

The probability for 1st ace, 2nd ace and 3rd king would be $\frac{4}{52} * \frac{3}{51} * \frac{4}{50}$. Since (ace, ace, king) can be in any order, this gives us $\frac{3!}{2!}$ cases, each with the above probability, and hence the total probability would be $\frac{4}{52} * \frac{3}{51} * \frac{4}{50} * \frac{3!}{2!}$

Alternatively we could say that the required probability would be $\frac{{}^4C_2 * {}^4C_1}{{}^{52}C_3}$.

Note: If the above problem had been posed as a "with replacement" question, then we would have solved it as $\frac{4}{52} * \frac{4}{52} * \frac{4}{52} * \frac{3!}{2!}$

Probability - 19

The Tree Method - 1

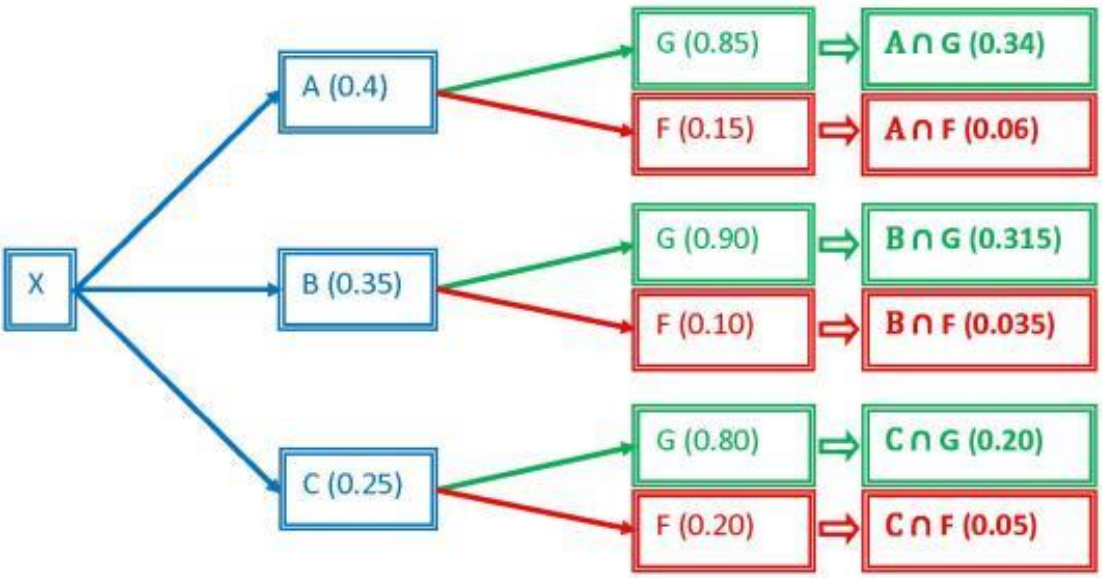
The standard formula we derived in an earlier post for dealing with Conditional Probability viz:

$P(A \cap B) = P(A) * P(B/A) = P(B) * P(A/B)$

is the basic form of what is known as Bayes' Theorem. This is a pretty efficient result. Probability experts routinely use it (and its corollaries) to achieve complex and useful results. However, it can be a little painful to apply for one who is not thoroughly clear with the basics. An alternative, more informal way of representing the same concepts would be to create a probability tree. Let me demonstrate:

Q: Three machines A, B & C make all the widgets in factory X. A makes 40% of the production, B makes 35% of the production and C makes the rest. 15% of A's production is faulty, 10% of B's production is faulty, and 20% of C's producton is faulty. At the end of a day's production run, a widget is randomly selected. What is the probability that it will be faulty?

Let us create a tree of the possible outcomes. Either A, B or C makes the widget; also in each case the widget is either good (G) or faulty (F). For ease of understanding, I will represent the "Good" branches in green and the "Faulty" ones in red.



Now the red branches represent all the cases of faulty widgets and hence the total probability of a faulty item will be $P(A \cap F) + P(B \cap F) + P(C \cap F) = 0.06 + 0.035 + 0.05 =$ **0.145** or 14.5%

Note: in the exam you definitely should NOT be writing all the details such as $A \cap G$ and $A \cap F$; you should only write the minimum necessary to solve the question, i.e. the relevant numbers only. For this specific problem, you could get away by only constructing the red (Failure) branches of the tree. One minor advantage of constructing the entire tree is that you get a check of your calculations – if you've done them correctly, the sum of the six numbers on the right will be 1.

Probability – 20

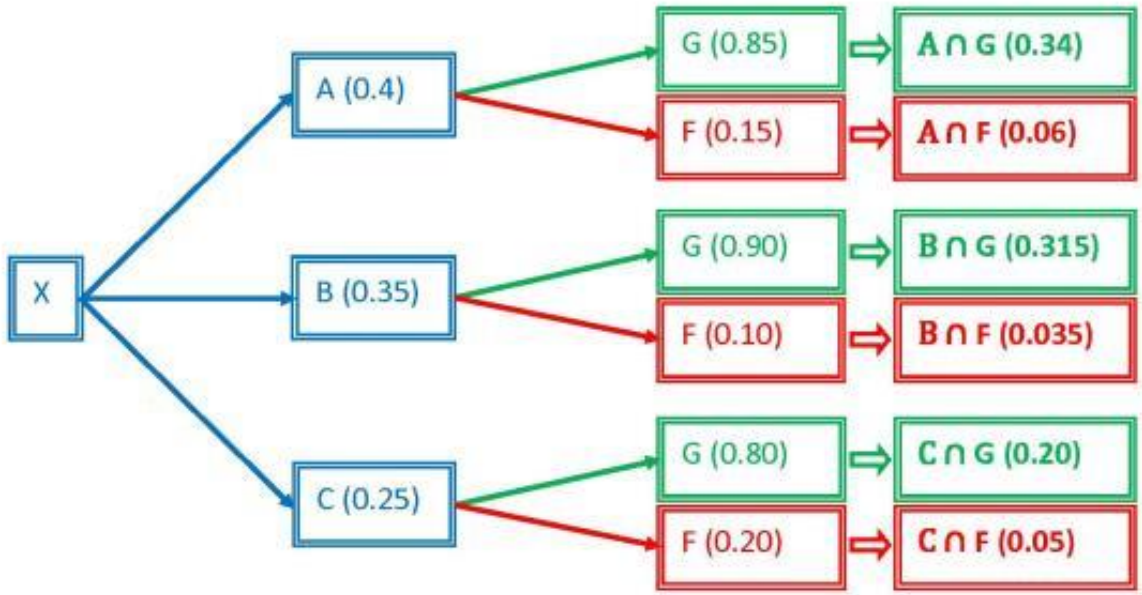
The Tree Method - 2

The Tree Method of the previous post is even more useful when dealing with a question of "Reverse Conditional Probability" which would otherwise require us to apply the Bayes theorem in reverse; a non-trivial exercise for most people. The typical statement of such a question usually tells us that a final result was achieved, and then asks the probability that a particular initial case actually happened. Again, let's look at an example, a continuation of yesterday's question:

Q: Three machines A, B & C make all the widgets in factory X. A makes 40% of the production, B makes 35% of the production and C makes the rest. 15% of A's production is faulty, 10% of B's production is faulty, and 20% of C's producton is faulty. At the end of a day's production run, a widget is randomly selected. What is the probability that it was made by A, given that it is faulty?

When faced with such a question, a lot of people make a very simple and tempting mistake – they think "Faulty or Good comes later; how could that affect the probability of which machine it was made on?" and confidently mark 0.4 (which of course will be one of the answers offered by any examiner worth his salt!). However, the words "given that it is faulty" should alert us to the fact that there is a conditon here, and that the sample space may therefore change.

Again let us create the tree:



Now "given that it is faulty" means that the "Good" cases are no longer part of our possible outcomes, our Sample Space. So our new Sample Space consists of just the "Faulty" cases (which, as we saw last time, add up to 0.06 + 0.35 + 0.05 = 0.145). Also the "favourable" outcomes are "Faulty from A" which is 0.06. Hence the required probability is $\frac{0.06}{0.145} = \frac{12}{29} = 0.413$

Probability – 21

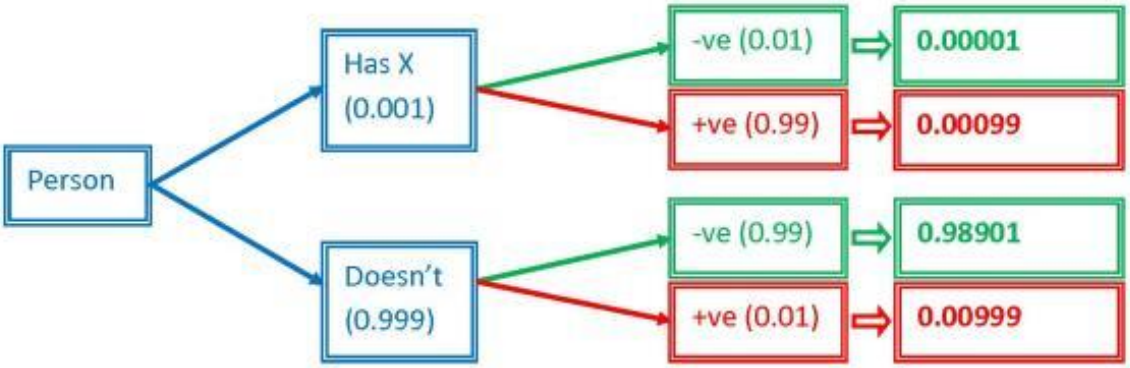
The Tree Method - 3

Let's look at a version of a famous "tough" problem:

Q. A certain disease X is known to affect one in a thousand people. But it is quite difficult to detect reliably. A new detection test Y has recently been devised; the makers claim that it is 99% accurate (i.e. if a person is tested, it will give a correct diagnosis in 99% of cases). A person Z is tested using test Y and it indicates that he is suffering from disease X. What is the probability that Z actually suffers from X?

The first gut reaction of someone encountering this problem for the first time (even many doctors, apparently!) is likely to be "0.99, of course! The test is 99% reliable after all…". However, it isn't quite as cut-and-dried as all that.

As before, let us make a tree to simplify things a bit. Here I will represent the cases where the test indicates a +ve with red:



In fact we find that the probability that Z actually suffers from X is $\frac{0.00099}{0.00099+0.00999} = \frac{99}{1098} = 0.0902$.

Or in other words, there is less than a 10% chance! How did this happen?

Even though the test is 99% accurate, it is 99% accurate for those not suffering from the disease as well; i.e. it indicates that 1% of them are actually suffereing from it. And since the number of people not suffering from the disease is very high (compared to those suffering), even 1% of that will be significant; the number of such "false positives" will outweigh the "true positives" in this case by a factor of 10.

Try the same exercise for a really rare disease where one in a lakh people suffer from it. You'll find that the probability that a person designated as positive is actually positive will be approximately 0.000989, i.e. just under 1 in 1000!

Sometimes, probability can be extremely counter-intuitive. It is situations like this which give probability its ogreish reputation and makes many people afraid to try even the simple problems!

Probability – 22

Multiple Identical Trials - Binomial Probability

Sometimes we have a series of trials (say, rolling a die multiple times) and the desired outcome in each trial is the same; i.e. the probability of success (and consequently of failure) in each trial remains constant. In such a situation, we could encounter questions like:

Q. Ratish is playing a game of chance at a fair. He rolls five fair dice and he wins if exactly three of them show a 6. What is his probability of winning the game?

Since the dice are fair, the probability of success (getting a 6) on any one die is $\frac{1}{6}$. Consequently the probability of failure (getting a non-6) is $\frac{5}{6}$. (Remember that to maintain the equiprobability condition of the sample space, we must treat all the dice as distinct). Now there are many possible sequence of results on the 5 dice which would win Ratish the game; let us consider one such sequence where he gets a 6 on the first three dice and a non-six (N) on the others. The five results are independent, so

$$P(6\ 6\ 6\ N\ N) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2$$

Now how many such sequences will be possible? Any sequence which has 6 thrice and N twice will work; we know from PnC that 666NN can be arranged in $\frac{5!}{3!\,2!} = {}^5C_3$ ways (alternatively, we could say that there are ${}^5C_3$ ways of picking which three out of the five trials will be successes). Each of these will have the same probability and hence the total probability of success will be $\mathbf{{}^5C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2}$

In general, if 'n' identical trials are conducted, each of which has a probability of success 'p' and a probability of failure 'q' (where q = 1 – p), then the probability that exactly 'r' out of n trials are successful is $\mathbf{{}^nC_r\, p^r\, q^{(n-r)}}$

Q. Deepika is writing a test which has 100 multiple-choice questions, each with 5 options. If she marks all the 100 questions randomly, what is the probability that she will get exactly 20 correct?

We can choose which 20 she gets correct in ${}^{100}C_{20}$ sequences and for each of those sequences the probability will be $\left(\frac{1}{5}\right)^{20} \times \left(\frac{4}{5}\right)^{80}$. Hence the required probability will be $\mathbf{{}^{100}C_{20} \times \left(\frac{1}{5}\right)^{20} \times \left(\frac{4}{5}\right)^{80}}$

Q. In Cafe Leopold, on average a third of the clientele are women. What is the probability that 4 out of the next 10 people to enter the room will be women?

The probability of a particular person being a woman is $\frac{1}{3}$ and hence of a man is $\frac{2}{3}$. Hence the required probability will be $\mathbf{{}^{10}C_4 \times \left(\frac{1}{3}\right)^4 \times \left(\frac{2}{3}\right)^6}$

Note that in the case of a fair coin being tossed, the probability of success equals that of failure. So for n coin tosses the denominator will always be 2^n.

Q. If 50 coins are tossed, what will be the probability of getting exactly 18 heads?

We can directly write this as ${}^{50}C_{18} \times \frac{1}{2^{50}}$

Q. If 8 dice are rolled, what is the probability of getting at least six "1"s?

There can be either six or seven or eight "1"s. We can calculate these three cases and add them to get:

$${}^{8}C_6 \times \left(\frac{1}{6}\right)^6 \times \left(\frac{5}{6}\right)^2 + {}^{8}C_7 \times \left(\frac{1}{6}\right)^7 \times \left(\frac{5}{6}\right)^1 + {}^{8}C_8 \times \left(\frac{1}{6}\right)^8 \times \left(\frac{5}{6}\right)^0$$

Q. The probability that Hawkeye hits a target is 2/5. If he shoots 5 arrows at the target, what is the probability that exactly 2 of them will hit the target?

The required probability will be ${}^{5}C_2 \times \left(\frac{2}{5}\right)^2 \times \left(\frac{3}{5}\right)^3$

Q. Tony and Bruce play a game in which they toss 15 fair coins. Tony wins if there are more heads and Bruce wins if there are more tails. What is the probability that Tony wins?

Tony will win if there are 8 or more heads. So we can consider cases of 8 heads, 9 heads, 10 heads etc till 15 heads which gives us a probability of

$$\frac{15C8 + 15C9 + 15C10 + 15C11 + 15C12 + 15C13 + 15C14 + 15C15}{2^{15}}$$

However, there is an easier way to look at it. The probability of 15 heads is the same as that of 15 tails (i.e. 0 heads), that of 14 heads is the same as that of 14 tails (i.e. 1 head) and so on. So the probability of Tony winning, P (15 or 14 or 13 or … or 9 or 8 heads) will be exactly the same as that of Bruce winning, P(0 or 1 or 2 or … or 6 or 7 heads). Since the two are mutually exclusive and exhaustive events, their probabilities must add up to 1 and so the required probability of Tony winning will be $\frac{1}{2}$

Q. Tony and Bruce play a game in which they toss 16 fair coins. Tony wins if there are more heads and Bruce wins if there are more tails. What is the probability that Tony wins?

This looks very similar to the previous problem. However there is one crucial difference – the number of coins is even and hence there is a possibility of an equal number of heads and tails i.e. a tie. The probability of Tony winning is still equal to that of Bruce winning, but they are no longer exhaustive events. The required probability would be slightly less than half, and could be expressed as:

$$\frac{1}{2} \times \left\{1 - \frac{16C8}{2^{16}}\right\}$$

Probability – 24

Using Symmetry in Probability

Sometimes, as in PnC, identifying the symmetry of a problem gives us a very efficient solution. We saw one example of this in the previous post, (the 15 coins question), let's look at some more:

Q. 10 friends all score different marks in an exam. What is the probability that Xavier scores more than Yatish?

Given that we know nothing about Xavier and Yatish, who is likely to score more? Both are equally likely. Also, all scores are different, so (X scores more than Y) and (Y scores more than X) and mutually exclusive as well as exhaustive. So the probability of either case will be $\frac{1}{2}$

Q. 10 friends all score different marks in an exam. What is the probability that Xavier scores more than Yatish, who scores more than Zeba?

Examining this on lines similar to the previous one, the three friends Xavier, Yatish and Zeba can be arranged in 3! = 6 ways. Only one of these 6 satisfies the given condition and hence the required probability will be $\frac{1}{6}$

Q. 10 friends all score different marks in an exam. What is the probability that Xavier scores more than both Yatish and Zeba?

As before, the three friends Xavier, Yatish and Zeba can be arranged in 3! = 6 ways. Two out of these 6 (X > Y > Z and X > Z > Y) satisfy the given condition and hence the required probability will be $\frac{2}{6} = \frac{1}{3}$

Q. 10 friends sit round a circular table. What is the probability that Vinay is sitting next to Waheeda?

Let's seat Vinay anywhere. Now there are 9 other seats, 2 of which are next to him. Hence $\frac{2}{9}$

Q. In a building with 8 floors (excluding the ground floor), Mr. 'X' along with 4 other people enters the lift on the ground floor. Given that each person gets off the lift on a different floor, what is the probability that Mr. X gets off the lift before exactly 3 people?

1] $\frac{1}{5}$ 2] $\frac{2}{45}$ 3] $\frac{32}{35}$ 4] $\frac{3}{7}$

Basically the question is asking, "what is the probability that X is the second person to get off?" Now, who is most likely to be the second person to get off? All are equally likely! Hence the required probability will be $\frac{1}{5}$

Probability – 25

Let's look at some more examples (often tricky) based on Probability:

Q. Two friends, Han and Indy, take turns rolling a fair die. Han rolls first. The first person to get a 6 wins the game. What is the probability that Indy will win the game?

The game, in theory, could go on till infinity; but if we examine the first few turns we will find that each probability reduces to an infinite Geometric Progression which will converge to a finite value.

For H to win on turn 1, the probability will be $\frac{1}{6}$

For I to win on turn 1 (H has to lose first!) the Probability will be $\frac{5}{6} \text{ x } \frac{1}{6} = \frac{5}{36}$

For H to win on turn 2, the probability will be $\frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{1}{6} \text{ x} = \frac{25}{216}$

For I to win on turn 2, the probability will be $\frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{1}{6} = \frac{125}{1296}$

For H to win on turn 3, the probability will be $\frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{5}{6} \text{ x } \frac{1}{6} = \frac{625}{7776}$ and so on…

So for H, the numbers are $\frac{1}{6} + \frac{1}{6} \text{ x } \frac{25}{36} + \frac{1}{6} \text{ x } \left(\frac{25}{36}\right)^2 + \ldots$till ∞, which will become $\frac{\frac{1}{6}}{1-\frac{25}{36}} = \mathbf{\frac{6}{11}}$.

Similarly for I, the numbers are $\frac{5}{36} + \frac{5}{36} \text{ x } \frac{25}{36} + \frac{5}{36} \text{ x } \left(\frac{25}{36}\right)^2 + \ldots$till ∞, which will become $\frac{\frac{5}{36}}{1-\frac{25}{36}} = \mathbf{\frac{5}{11}}$.

While perfectly logical, this approach is a little time-consuming. Can we find a better way? Let's see…

In every turn, B has the same probability of winning as A does, if he gets to play. But he has only a $\frac{5}{6}$ chance of getting to play in that turn. If A has a probability of winning of 'p', in a particular turn, then B has a probability, in that turn, of $\frac{5p}{6}$.

In other words, in every step the ratio of the respective probabilities of A and B will be 6 : 5. Can we then say that the total will also be in the ratio 6 : 5? Certainly we can! But eventually, if we play long enough, either A or B has to win. So the total probability of A winning and B winning will add up to 1. From here, it is but a tiny leap of logic to realise that **the probabilities for A and B will be $\frac{6}{11}$ and $\frac{5}{11}$**.

Q. There are 10 marbles in a bag, 9 blue and one red. Martha and Rose take turns picking one marble (with replacement) until one of them picks the red marble. Martha picks first. What is the probability that Rose will get the red marble?

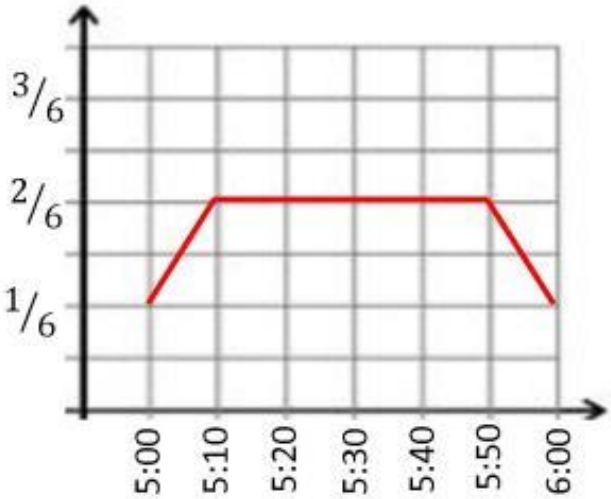
In each step, the probability that M does not pick the red (and hence R gets a chance to play) is $\frac{9}{10}$. Hence the ratio of probabilities for M and R will be 10 : 9 and so R's chances will be $\frac{9}{10+9} = \mathbf{\frac{9}{19}}$

Some problems lend themselves to elegant graphical solutions; let's examine a couple of those in the upcoming posts.

Probability – 26

Q. One day, Archie and Jughead each visit Pop Tate's. Each arrives at a random time between 5 and 6 pm. Each spends exactly ten minutes at Pop Tate's. What is the chance the two will meet?

Let's examine the problem from the point of view of one person, say Archie. He can arrive at any time between 5 and 6. Suppose he arrives at 5. Then he will meet Jughead only if Jughead arrives before 5:10 (i.e. a 10 minute range out of 60). If he arrives at 5:05, though, the range will be 5 to 5:15 i.e. 15 minutes. The range goes on increasing till 5:10 which gives a range of 20 minutes. After that, the range will remain at 20 minutes (ten before till ten after) till 5:50; if Archie arrives after 5:50 the range will decrease again. We can represent this as follows, given that a range of 10 minutes corresponds to a probability of $\frac{10}{60}$ i.e. $\frac{1}{6}$:



Now there are 6 equal slots of ten minutes each. The average probability during the first ten min (and during the last ten as well) is the average of $\frac{1}{6}$ and $\frac{2}{6}$ i.e. $\frac{3}{12}$ or $\frac{1}{4}$, while for the remaining four slots the average is just $\frac{2}{6}$ or $\frac{1}{3}$. So overall the probability is $(\frac{1}{4}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}) / 6 = \frac{11}{36}$

Q. One day, Betty and Veronica each visit Pop Tate's. Each arrives at a random time between 12 noon and 2 pm. Each spends exactly 15 minutes at Pop Tate's. What is the chance the two will meet?

At 12, the range is 15 minutes (and the probability $\frac{1}{8}$). At 12:15 the range is 30 minutes (and the probability $\frac{1}{4}$). So let us divide the problem into 8 equal intervals of 15 minutes. The average probability for the first slot will be the average of $\frac{1}{8}$ and $\frac{1}{4}$ i.e. $\frac{3}{16}$, and the same for the last slot. So the required probability will therefore be $(\frac{3}{16}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{3}{16}) / 8 = \frac{15}{64}$

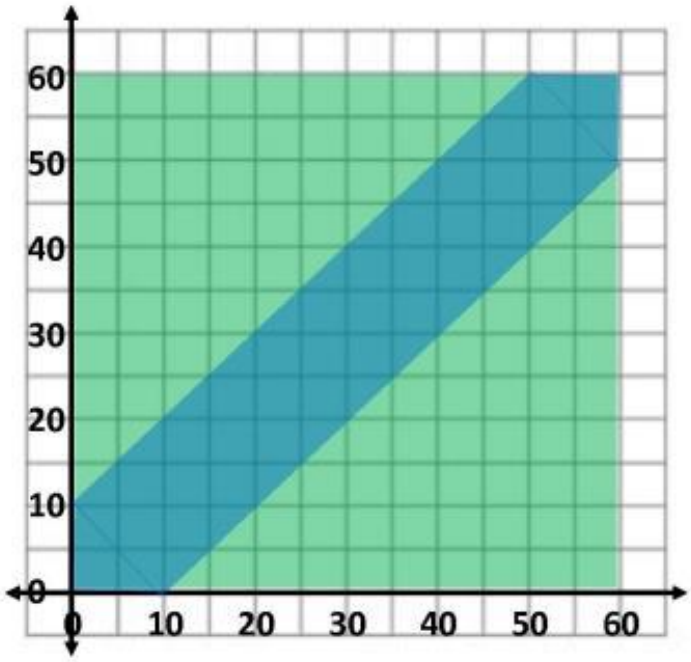
Probability – 27

There could be another way of solving last time's problem geometrically.

Q. One day, Archie and Jughead each visit Pop Tate's. Each arrives at a random time between 5 and 6 pm. Each spends exactly ten minutes at Pop Tate's. What is the chance the two will meet?

Let us designate Archie's arrival time by 'x' and Jughead's arrival time by 'y'. Let's assume time = 0 at 5 o'clock for them both. Now we can plot their arrival time on an X-Y coordinate plane.

Also, the cases where they meet will be the region between the lines y = x + 10 and y = x – 10.



Hence the required probability will be the area of the region marked in blue (favourable outcomes) upon that of the green square (all possible outcomes).

Now the area of the square is 60 x 60 = 3600. The area of the blue region can be calculated by subtracting the two isosceles right triangles from the sides. Each of them has area ½ x 50 x 50 = 1250 and hence the area of the blue region will be 3600 – 2 x 1250 = 1100. So the reuired probability must be $\frac{1100}{3600} = \frac{11}{36}$

This is a more elegant approach, but almost certainly more time-consuming. (In other words, use at your own risk!)

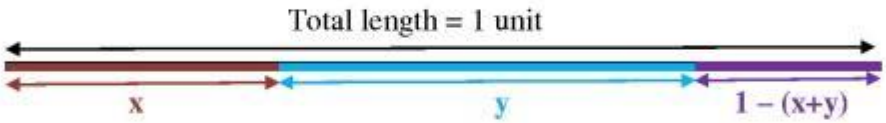
Note: In the next post, I will post another famous Probability puzzle that could be solved by using a graphical approach. I had already posted it as a stand-alone probability question two years ago as part of the XAT scan series (so some of my readers might have encountered it already), but it fits better here with the rest of probability and so I am re-posting it here for the sake of completeness.

Probability - 28

Q. If a stick is broken into 3 parts, what is the probability that the parts will form a triangle?

Let us assume the stick to be of length 1 (whatever the actual length is, we could multiply every variable in the problem by that length leaving the solution unchanged)

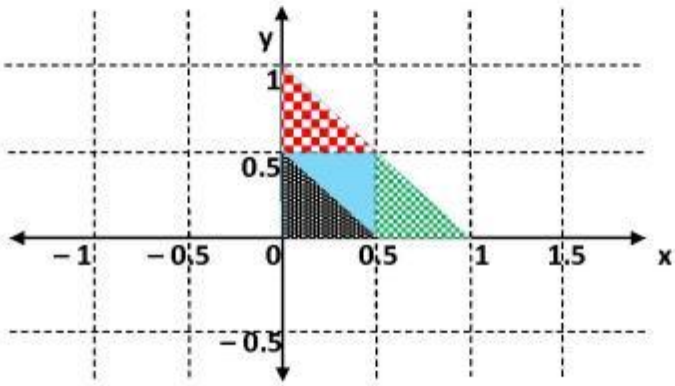
Also, let us assume the first two parts to be of length x and y. The third part would then be simply 1 – (x+y). Note that x and y could take any values as long as x + y is less than or equal to 1.



Now we want the three parts to form a triangle. What does this mean? Recall that one of the most basic properties of a triangle is that the sum of any two sides must be greater than the third. Now in the given situation, we have the sum of all three sides to be 1, so none of the three sides can exceed 0.5 (else it would be greater than the sum of the other two). This gives us the following additional conditions:

(i) $x < 0.5$
(ii) $y < 0.5$
(iii) $x + y > 0.5$ (else the third side would exceed 0.5!)

Solving these equations would be fairly cumbersome; so let us take a different approach and try to find the solution set graphically. In the figure below, the large triangle represents all the possible outcomes ($0 \le x \le 1$, $0 \le y \le 1$, $x + y \le 1$.



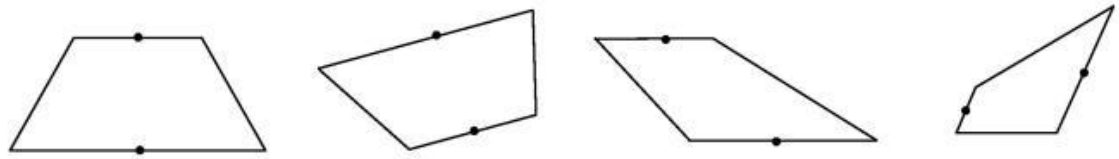
The small triangles show where the additional three conditions are violated – the red triangle shows the region disqualified when $y > 0.5$, the green one shows the region where $x > 0.5$, while the black one indicates the region where $x + y < 0.5$. The part which satisfies all the conditions, then, is the small blue triangle in the above figure, which is just $\frac{1}{4}$ of the area of the large triangle which forms our sample space. **Hence the required probability is $\frac{1}{4}$.**

Note: This problem could also be stated as **"if 3 points are selected on the circumference of a circle, what is the probability that they lie on the same semi-circle?"** or **"if a string is cut at two random points, what is the probability that no part so formed is longer than the sum of the other two?"**

GEOMETRY - 1

This week let us take a break from PnC and do some Geometry. Specifically, we'll focus on the **Trapezium** and its properties.

A trapezium is a quadrilateral where exactly one pair of sides is parallel (note that if the second pair were to become parallel too, it would be a parallelogram). The most common way of depicting a trapezium is as in figure (a) below, but note that the other figures are also trapezia (the parallel sides being marked with a •), and we should not get locked into visualising a trapezium in only that one way.



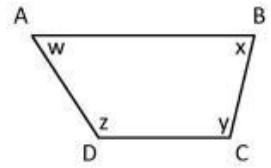
(a) (b) (c) (d)

We can construct a line joining the midpoints of the two non-parallel sides of a trapezium. For convenience, we shall refer to this as the median of the trapezium. The height of a trapezium is usually defined as the perpendicular distance between the two parallel sides.

A special case of a trapezium is the isosceles trapezium, wherein the two non-parallel sides are congruent (Fig (a) above is an example of an isosceles trapezium). In this case, the base angles (i.e. the two angles bordering any one of the parallel sides) are equal.

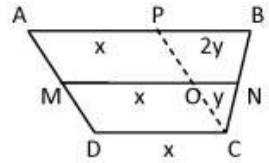
Properties of a trapezium:

In any trapezium, two pairs of adjacent angles are supplementary



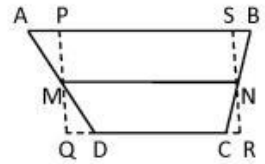
This is a straightforward result of the fact that two sides are parallel. In the adjacent figure AB ∥ CD and hence $\angle w + \angle z = \angle x + \angle y = 180°$ (since each is a pair of interior angles)

The length of the median of a trapezium is the mean of the lengths of the parallel sides.



Consider the adjacent figure, where MN is the median of ABCD. Let's draw PC ∥ AD. Thus APCD will be a parallelogram and hence l(AP) = l(MO) =l(DC) = x (say). Also in ΔPBC, by midpoint theorem if l(ON) = y then l(PB) = 2y. Then l(MN) = x + y while l(AB) + l(DC) = 2x + 2y, i.e. l(MN) = ½ [l(AB) + l(DC)]

The area of a trapezium is given by (median) x (height)

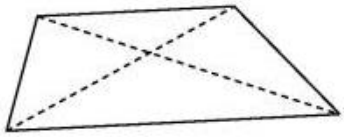


Consider the adjacent figure. Drop perpendiculars PQ and SR through M and N. It is easy to show that ΔAPM ≅ ΔDQM and ΔBSN ≅ ΔCRN. This in turn implies that the area of trapezium ABCD = the area of rectangle PQRS which is nothing but l(MN) x l(PQ) or median x height.

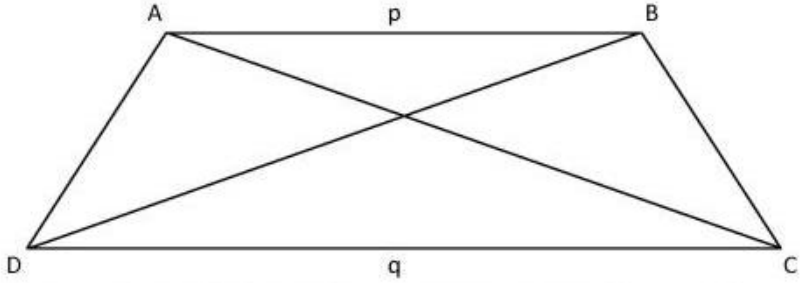
GEOMETRY - 2

Dividing a Trapezium into triangles

An interesting (and sometimes surprisingly useful) property of a trapezium is obtained by joining the diagonals. 4 triangles are formed in the figure and their areas turn out to be in a very specific ratio.

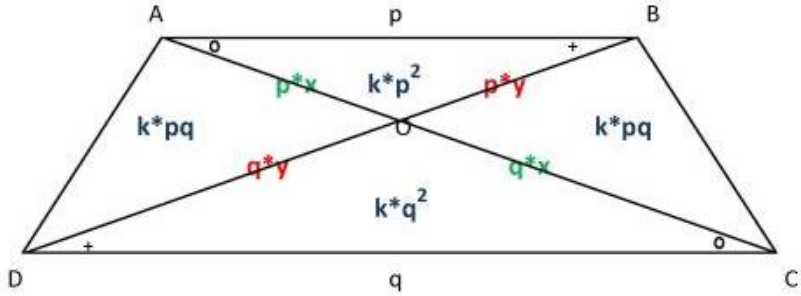


Consider a trapezium ABCD as shown in the figure below, with AB ∥ DC. Let the lengths of AB and CD be p and q respectively. If we draw the diagonals, can we find a relation between the areas of the four triangles so formed?



Using the properties of parallel lines and transversals, we can show that ∠OAB = ∠OCD and ∠OBA = ∠ODC and hence that ΔAOB ~ ΔCOD. Hence the ratio of the areas of the two similar triangles will be the square of the ratio of the sides i.e. $p^2 : q^2$. Let us take these as $k * p^2$ and $k * q^2$.

Also, the ratio of length of OA to that of OC will be p : q, as will the ratio of length of OB to that of OD. Now consider triangles AOB and AOD. If we consider the base line for both to be BD, then they have the same height and hence their areas will be in proportion to the bases i.e. p: q. Thus the area of ΔAOD will be $\frac{q}{p}$ * (area of ΔAOB) = $\frac{q}{p} * (k*p^2) = k*pq$. Similarly the area of ΔBOC will also be k*pq. The figure will appear as below.



In general, if we draw the diagonals of a trapezium whose parallel sides are in the ratio p : q (top : bottom) the ratio of areas of the 4 triangles formed (in order from top) will be $\mathbf{p^2 : pq : q^2 : pq}$

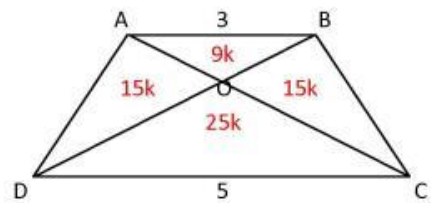
Over the next two days, we'll look at a few examples showcasing the trapezium and how to deal with it effectively.

GEOMETRY - 3

Q. A trapezium ABCD has AB ∥ DC. Also l(AB) = 3 and l(DC) = 5. The two diagonals intersect at a point O. If the area of ΔAOB is 18 square units, find the area of ABCD.

The standard approach to solve this problem would be by using the triangle AOB. Knowing that the side AB is 3 and the area is 18, we could find the height h; since $\frac{1}{2} * 3 * h = 18$, we find that h = 12. Using the similarity of triangles AOB and COD we could then show that the height of COD from base CD will be $\frac{5}{3} * 12 = 20$ and hence the total height of the trapezium is 12 + 20= 32 units. This would enable us to find the total area (using the formula for the area of a trapezium) as $\frac{1}{2} * (3+5) * 32 =$ **128**

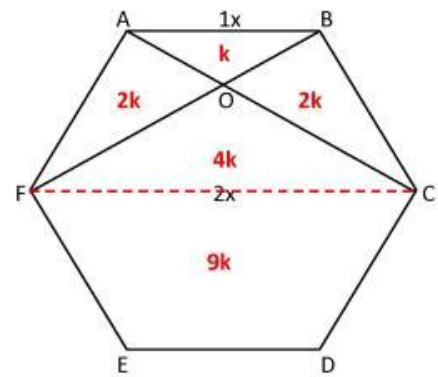
However, using the logic derived yesterday, this question becomes extremely easy. The ratios of the areas of the 4 triangles will be 9 : 15 : 25 : 15 (as shown in the figure) and hence if the smallest triangle has area 18, the others will be 30, 50 and 30 adding up to a total of **128 square units**.



Q. In a regular hexagon ABCDEF with side $6\sqrt{3}$, the diagonals AC and FB intersect at a point O. Find the ratio of areas of ΔABO and ⬠OCDEF .

This seems to be a more daunting problem than the previous one. Of course, given the length of the side we could find the lengths AO and BO, as well as OF and OC, and proceed from there.

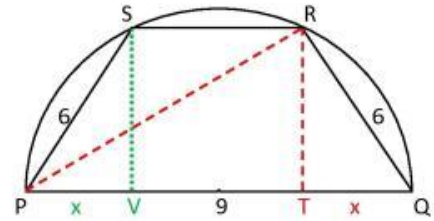
However, a simpler approach would be to join CF (as shown in the figure), dividing the hexagon into two equal halves. Now if we remember that the longest diagonal of a regular hexagon is twice the side then we can say that if l(AB) = x then l(FC) = 2x and then apply the logic discussed yesterday.



If we assume the area of ΔAOB to be k, the other 3 triangles become 2k, 2k and 4k as shown, and so the half-hexagon ABCF has an area k + 2k + 2k + 4k = 9k. Thus FCED will also have area 9k, which means that the area of ⬠OCDEF will be 4k + 9k = 13k. This gives us the required ratio as **13 : 1.**

Q. On a semicircle with diameter PQ of length 9 cm, a chord SR is parallel to the diameter. The length of QR is 6 cm. Find the length of SR?

We can note that the trapezium formed will be isosceles, which means that l(PS) will also be 6 cm. Let us join PR and also drop a perpendicular from R to PQ. Note that ∠PRQ will be 90° (angle inscribed in a semicircle). So we can show that ΔPRQ ~ ΔRTQ.



Hence $\frac{x}{6} = \frac{6}{9}$ which gives us x = 4. Now if we drop another perpendicular SV to PQ, then we can see that l(PV) will also be x i.e. 4 cm and so l(VT) will be 9 – 4 – 4 = 1 cm. But SRTV is now a rectangle and so l(SR) will also be **1 cm.**

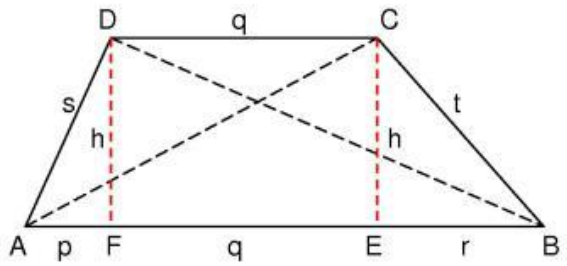
GEOMETRY - 4

Q. Consider a trapezium ABCD with AB ∥ CD and l(AB) > l(CD). Let J be the sum of the squares of the four sides and K be the sum of squares of the two diagonals. If l(AB) = 12 and J – K is 49, then find l(CD).

1) 6 2) 5 3) 7 4) Cannot be determined

How to start solving something like this? Well, the question itself gives some hints as to the lines on which to think – for one we have a trapezium (which means a readymade pair of parallel lines). Also, the question several times involves the "square of a side" which should immediately hint at the possibility of applying Pythagoras' theorem. So let's try and construct right angles.

Consider the trapezium ABCD as shown below. Let's drop perpendiculars from C and D to AB.



Now we know that J is the sum of squares of the sides. Hence we can write:

$J = s^2 + q^2 + t^2 + (p+q+r)^2 = h^2 + p^2 + q^2 + h^2 + r^2 + p^2 + q^2 + r^2 + 2pq + 2pr + 2qr$ … (i)

Similarly K is the sum of squares of the diagonals which means that

$K = l(AC)^2 + l(DB)^2 = (p+q)^2 + h^2 + (q+r)^2 + h^2 = p^2 + q^2 + 2pq + h^2 + q^2 + r^2 + 2qr + h^2$ … (ii)

Hence J – K will be (i) – (ii)
$= (\cancel{h^2} + \cancel{p^2} + \cancel{q^2} + \cancel{h^2} + \cancel{r^2} + p^2 + \cancel{q^2} + r^2 + 2pq + 2pr + \cancel{2qr})$
$- (\cancel{p^2} + \cancel{q^2} + 2pq + \cancel{h^2} + \cancel{q^2} + \cancel{r^2} + \cancel{2qr} + \cancel{h^2})$
$= p^2 + r^2 + 2pr$ which equals $(p+r)^2$ which means that if J – K = 49 then we must have p + r = 7.

Now l(AB) = p + q + r = 12 ⇒ q = 5. But this is nothing but the required value i.e. l(CD). So l(CD) = 5. **Hence (2)**

Trapezium: CAT Scan

Try this question, from CAT 2004, for practice:

Q. On a semicircle with diameter AD, chord BC is parallel to the diameter. Further, each of the chords AB and CD has length 2 cm, while AD has length 8 cm. What is the length of BC?

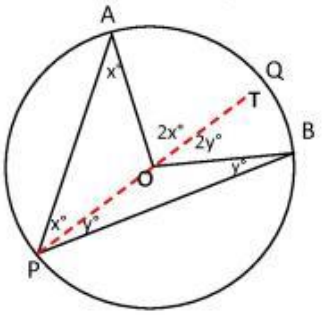
1) 7.5 cm 2) 7 cm 3) 7.75 cm d) None of the above

GEOMETRY – 5

Today we'll look at a useful theorem dealing with circles

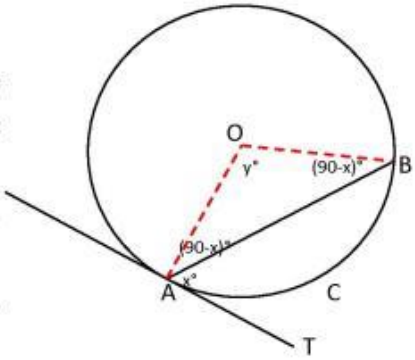
The Inscribed Angle Theorem

The Inscribed Angle Theorem states that the measure of an angle inscribed in a circle is half the measure of the intercepted arc (i.e. the angle subtended by that arc at the centre of the circle)



Proof:
Consider the figure shown alongside. An angle APB is inscribed in the circle with centre O, intercepting the arc AQB. The measure of arc AQB (i.e. the angle subtended at the centre) is ∠AOB. We are trying to show that ∠APB = ½ ∠AOB.
Let us start by joining PO and extending it a little further to point T as shown. Note that AO and OP are radii and hence ΔAOP is isosceles. So the base angles are equal (say x° each, as shown). Similarly ΔBOP is also isosceles and its base angles are also equal (say y° each)
Now using external angle theorem, ∠AOT = 2x and ∠BOT = 2y. So ∠AOB = 2x + 2y. But this is just double of ∠APB which is x + y.

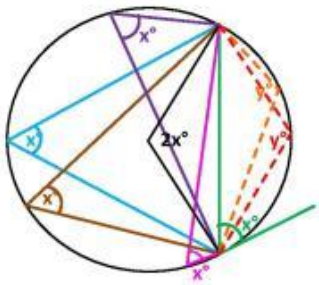
Corollary: The angle between a chord and a tangent at one end-point of the chord is equal to half the measure of the intercepted arc.



Proof:
Consider the figure shown alongside. In the circle with centre O, the chord AB and the tangent AT form an angle x°. They intercept an arc ACB which has a central angle of y°. We wish to show that y = 2x.
Let us draw OA and OB. Now OA is a radius and is hence perpendicular to the tangent at A. So ∠OAT = 90°.
Since ∠BAT = x°, we get ∠BAO = (90 – x)°. Also since OA and OB are equal (radii), ΔAOB is isosceles and so ∠ABO = (90 – x)°.
Thus ∠ OAB = y = 180 – (90 – x)° – (90 – x)° = 2x.

Visualisation

Once we are comfortable with this theorem, we should be able to grasp that any of the angles marked with solid lines in the figure alongside, which subtend an arc 2x°, will have the same value x°.



Also, any of the angles marked with a broken line will be equal, and we can show that y = (180 – x)° as they will be half of the reflex angle of 2x.

GEOMETRY - 6

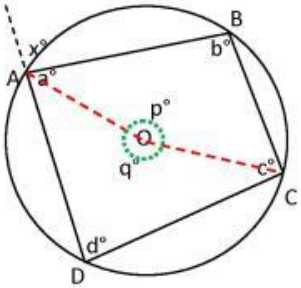
Circles and Quadrilaterals

Let us look at 2 interesting situations involving a circle and a quadrilateral – one wherein a quadrilateral is inscribed in a circle (i.e. all its vertices of the quadrilateral are on the circumference of the circle) and another wherein the quadrilateral circumscribes the circle (i.e. all the sides of the quadrilateral are tangents to the circle.

A quadrilateral inscribed in a circle is called a **cyclic quadrilateral**. Some useful properties:

- **Opposite angles of a cyclic quadrilateral are supplementary**
- **Each exterior angle of a cyclic quadrilateral is equal to the most remote interior angle**

Proof:



Consider the adjacent figure depicting a cyclic quadrilateral ABCD.

If we join A and C to the centre O of the circumcircle, then two angles p and q are formed at the centre. We can apply the inscribed angle theorem to show that b = ½ * p and that d = ½ * q. Thus (b+d) = ½ * (p+q). But since (p+q) = 360°, (b+d) must be 180°. Similarly we can show that (a+c) will also be 180°.

Now consider the exterior angle at A (marked as x in the figure). From the above proof, a+c = 180°. But a and x form a linear pair, and so a+x = 180°. Thus x must be equal to c.

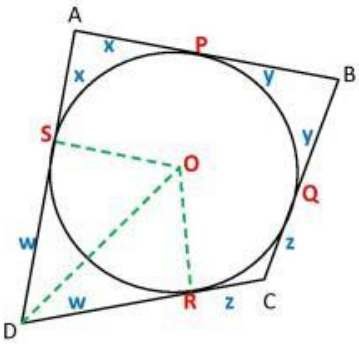
A couple of other useful results for which we shall not go into the proof:

- **Brahmagupta's Formula: Area of a cyclic quadrilateral of sides a, b, c, d = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semi-perimeter = ½ (a+b+c+d)**
- **Ptolemy's Theorem: The product of the lengths of the two diagonals of any cyclic quadrilateral is equal to the sum of the products of pairs of opposite sides**

A quadrilateral circumscribing a circle is called a **tangential quadrilateral**. Some useful properties:

- **The two pairs of opposite sides in a tangential quadrilateral add up to the same total length**
- **The four angle bisectors are concurrent, meeting at the centre of the incircle**

Proof:



Consider the adjacent figure depicting a tangential quadrilateral ABCD. Let the points of tangency of the 4 sides AB, BC, CD and DA to the circle be P, Q, R and S respectively.

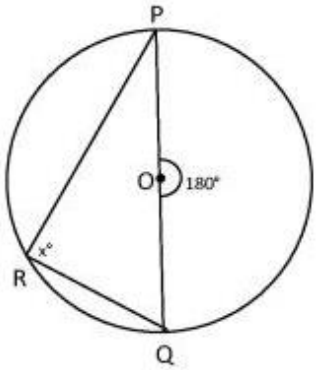
Now AP and AS are tangents from the same external point and hence will have the same length (say 'x'). Similarly we can consider the other lengths as shown and then it is a simple matter to show that AB + CD = AD + BC = x + y + z + w.

Also, joining the centre O of the circle to R, S and D, we can easily show that ΔDOS ≅ ΔDOR and hence that O lies on the angle bisector of ∠ADC.

GEOMETRY – 7

Inscribed Angle Theorem - Corollary

Corollary: Any angle inscribed in a semicircle is a right angle.

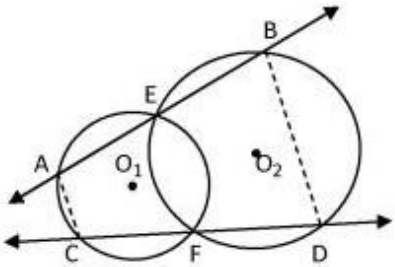


Proof:
Consider the figure. If ∠PRQ is x°, where PQ is a diameter and R is any point on the circumference of the circle, then by the inscribed angle theorem, we can say that x = ½ * 180 = 90°.

Some Examples:

Let's look at a few questions where we could apply concepts of circles and quadrilaterals, or the inscribed angle theorem.

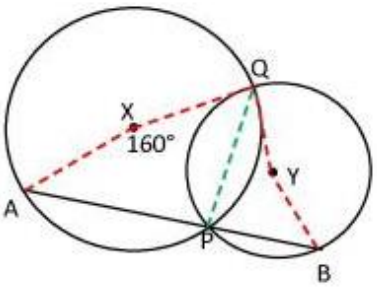
Q. In the figure, lines A-E-B and C-F-D pass through the points of intersection E and F of two circles with centres O_1 and O_2. Show that line AC is parallel to line BD.



We could solve this in a quick, elegant manner by one construction – join EF. This gives us two cyclic quadrilaterals ACFE and BDFE.
Now using the properties of cyclic quadrilaterals, we have:
∠ACF + ∠AEF = 180° and also ∠BDF + ∠BEF = 180°.
But ∠AEF + ∠BEF = 180° (linear pair)
So we can see that ∠ACF + ∠BDF = 180° as well. But since these are interior angles, we can say that AC ∥ BD.

Q. Two circles with centres X and Y intersect at P and Q. A and B are points on the circles with centres X and Y respectively such that A, P and B are collinear. If ∠AXQ = 160°, find the value of ∠QYB?

(1) 160° (2) 170° (3) 180° (4) 190° (5) None of these



In the figure, let us join PQ. From the inscribed angle theorem,
∠APQ = (½ the reflex angle of ∠AXQ) = ½ (360° – 160°) = 100°
So ∠QPB = 180 – 100 = 80°. Again using inscribed angle theorem,
∠QYB = 2(∠QPB) = 2 * 80 = 160°. Hence (1)

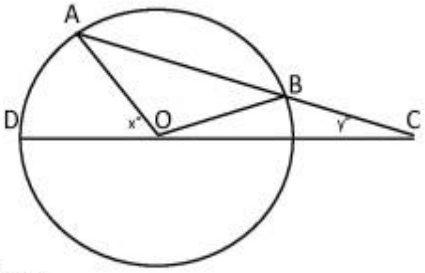
(Note that technically there are two angles QYB – one being 160° and the other being its reflex angle i.e. 200°. Either of them could have been in the answer choices.)

GEOMETRY - 8

Inscribed Angle Theorem: CAT Scan

Here are a couple of questions (from CAT '03 and CAT '04 respectively) to give a flavour of the kind of questions the CAT has posed on this concept in the past:

Q. In the figure given alongside, AB is a chord of a circle with centre O. AB is extended to C such that BC = OB. The straight line CO is produced to meet the circle at D. If ∠ACD = y° and ∠AOD = x° such that x = ky, then the value of k is ____.

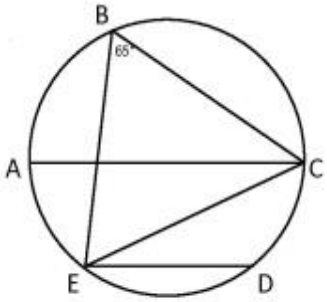


(1) 3 **(2) 2** **(3) 1** **(4) None of these**

Since BC = OB, ΔOBC is isosceles
So ∠BOC = ∠BCO = y°
But ∠ABO is an exterior angle to ΔOBC
So ∠ABO = y + y = 2y°
Now AO = OB (radii) and so ΔAOB is also isosceles
Thus ∠BAO = ∠ABO = 2y°
But ∠AOD is an exterior angle for ΔAOC
So ∠AOD = y + 2y = 3y°
Which means x = 3y or k = 3. **Hence (1)**

Note – ∠AOD is not an exterior angle for Δ AOB! This was a mistake committed in haste by a lot of test-takers that year…

Q. In the adjoining figure, chord ED is parallel to the diameter AC of the circle. If ∠CBE = 65°, then what is the value of ∠DEC?



(1) 35° (2) 55° (3) 45° (4) 25°

Let us construct a line segment AE.
Now if ∠EBC = 65° then ∠EAC = 65° (angles inscribed in the same arc)
Note that AC is a diameter and hence angle AEC = 90°
Hence ∠ ECA = 180 – 90 – 65 = 25°
But AC and ED are parallel lines
Hence ∠ACE = ∠DEC = 25°
Hence (4)

Next week, we'll return to Permutations and Combinations and learn how to deal with the distribution of identical objects.

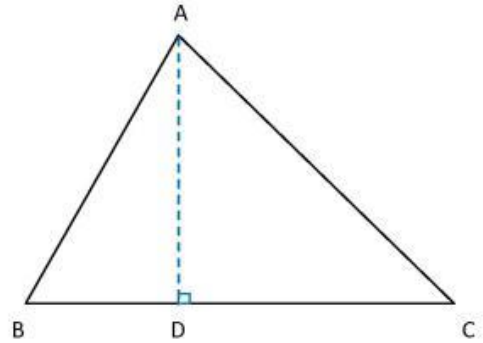
GEOMETRY – 9

Over the next few weeks, I'll try to look at a few concepts and some interesting questions in Geometry, since we have been neglecting it somewhat. Geometry is one of the more consistent elements in CAT over the years, typically accounting for around a quarter of the Quantitative Ability questions.

This week, I'll look at certain points and lines associated with a triangle: altitudes (and orthocentre), medians (and centroid), perpendicular bisectors (and circumcentre) and angle bisectors (and incenter). In the subsequent weeks we'll look at some theorems involving these, and at how to apply them to problems.

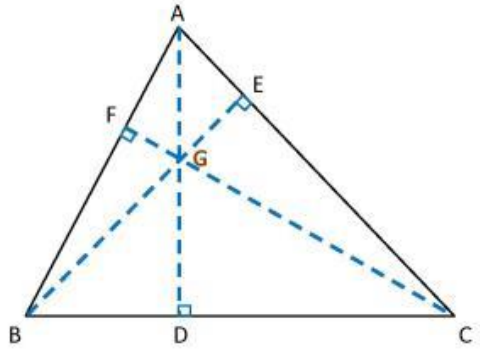
The Altitude

One of the useful types of lines to know in any triangle is the Altitude (or as we more commonly know it, the Height). An altitude may be defined as a perpendicular drawn from any one vertex to the opposite side.



The most useful property, from our point of view, is the fact that an altitude is a necessary part of the simple time-honoured formula for the area of a triangle: $\frac{1}{2}$ * base * height; where the base is any side of the triangle and the height is the altitude perpendicular to that side. For example the area of the above triangle can be expressed as $\frac{1}{2}$ * l(BC) * l(AD)

Since there are 3 vertices, there will be 3 altitudes. It turns out that all these three altitudes are concurrent i.e. they will meet in a single point; this point is called the **Orthocentre** of the triangle. of the points we are going to look at, this one is probably the least significant (from a CAT point of view, at any rate), but it is still a good idea to be comfortable with altitudes and their properties.

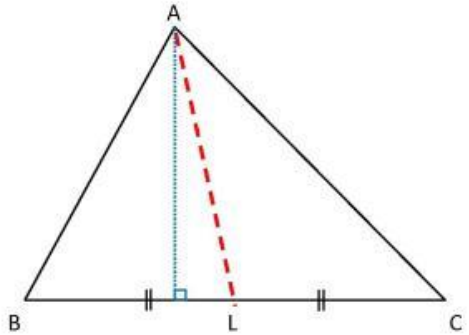


In the above figure, G is the orthocentre and $\frac{1}{2}$ * l(BC) * l(AD) = $\frac{1}{2}$ * l(AC) * l(BE) = $\frac{1}{2}$ * l(AB) * l(CF)

GEOMETRY – 10

The Median

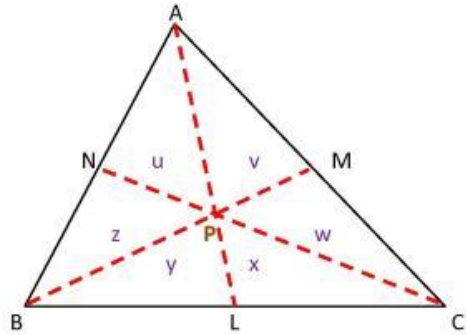
An extremely handy line to know is the median. A median may be defined as a line joining one vertex of a triangle to the midpoint of the opposite side (the red line in the figure below).



Perhaps the most useful property involving medians is the fact that a median divides the area of a triangle into two equal parts. So in the above figure, A(ΔABL) = A(ΔACL). This can easily be proved as follows: Since AL is a median, l(BL) = l(LC). So both triangles share a common base. Also, the height for both of them from that base is the same (the dotted blue line in the figure). Hence the area, being $\frac{1}{2}$ * base * height, will also be equal for both triangles.

Note: the two triangles are not congruent or similar, just equal in area.

Since there are three vertices, there will be three medians. Again, they turn out to be concurrent and the point in which all three intersect is called the **Centroid**.



In the above figure, P is the centroid. It can be shown that the 6 triangles formed have the same area. This can be proved as follows:

Let the areas of the 6 triangles be u, v, w, x, y and z as shown.

Now since AL and BM are medians, A(Δ ALC) = $\frac{1}{2}$ the area of the triangle = A(Δ BMC)

So v + w + x = y + w + x which implies **v = y**. Similarly we can show that **u = x** and **w = z**.

We can also observe that in Δ BPC, L is the midpoint of BC and hence PL is a median; this implies that A(Δ PLB) = A(Δ PLC), i.e. **x = y**. Similarly we can show that **v = w** and **u = z**.

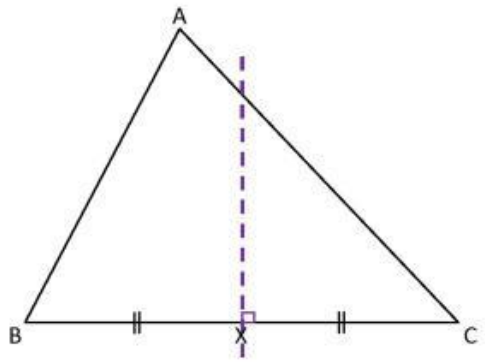
Putting the above two sets of results together, it is clear that all 6 triangles are equal in area.

Note: Once again, the 6 triangles are not congruent or similar, just equal in area.

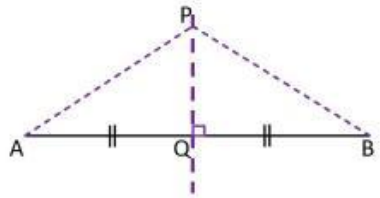
GEOMETRY – 11

The Perpendicular Bisector

Another useful line to consider in a triangle is the perpendicular bisector of a side.

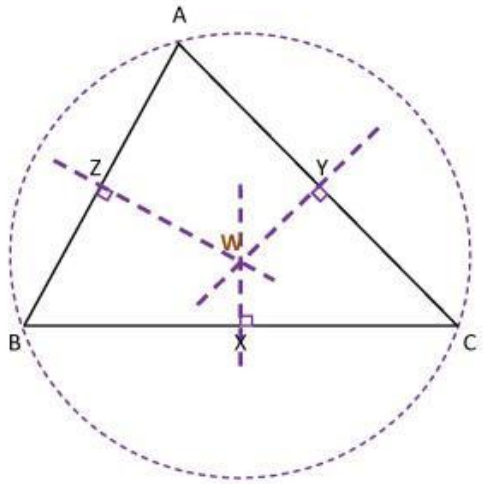


An interesting property of the perpendicular bisector of a line segment is that every point on the perpendicular bisector is equidistant from the endpoints of the line segment. Consider the figure below:



Consider the two right triangles Δ PQA and Δ PQB. Since PQ is the perpendicular bisector of AB, l(QA) = l(QB). Also PQ is a common side. So by hypotenuse-side test, the two triangles are congruent. And hence l(PA) = l(PB).

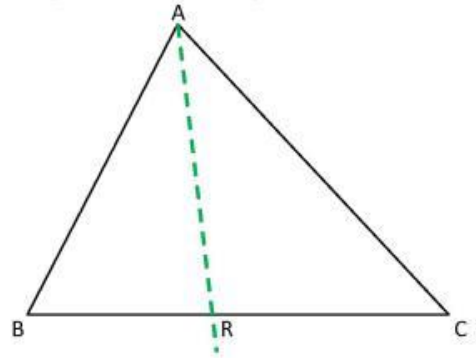
Now if we draw all the three perpendicular bisectors (see the figure below), any point on the perpendicular bisector WZ will be equidistant from A and B and any point on WY will be equidistant from A and C. So point W, which lies on both, must be equidistant from A, B and C. Using this point as a centre, therefore, we can draw a circle which passes through all three vertices (A, B and C) and hence may be said to circumscribe the triangle. This is called the **Circumcircle** of the triangle. The point W, then is called the **Circumcentre** and the distance AW (or BW or CW) is called the **Circumradius** (usually denoted by R)



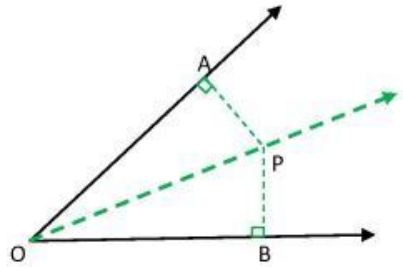
GEOMETRY – 12

The Angle Bisector

The fourth (and final) line we shall look at will be the angle bisector. As the name suggests, this is a line which passes through a vertex, bisecting the internal angle there.

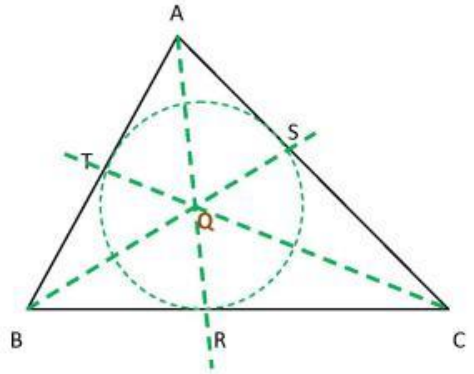


The noteworthy property of an angle bisector is that every point on it is equidistant from the two rays making up the angle. Consider the figure below:



In ∠ AOB, OP is the angle bisector. Let us draw perpendiculars from P to A and B. Consider the two right-angled triangles APO and BPO. We can see that ∠ POA = ∠ POB (since OP is an angle bisector), ∠ PAO = ∠ PBO and PO is a common side, and hence the two triangles are congruent. Thus we can say that l(PA) = l(PB) i.e. P is equidistant from both OA and OB.

If we draw the angle bisectors of all the three angles (see figure below), then we see that any point on AR will be equidistant from AB and AC, while any point on BS will be equidistant from BA and BA. Soa point lying on both AR and BS (i.e. their point of intersection, Q) will be equidistant from all three sides of the triangle and hence using this point as a centre, we can draw a circle which just touches all the three sides internally, i.e. which inscribes the triangle. This circle is called the **incircle** and the point Q is called the **incentre** and the perpendicular distance of Q from the sides is called the **inradius**.



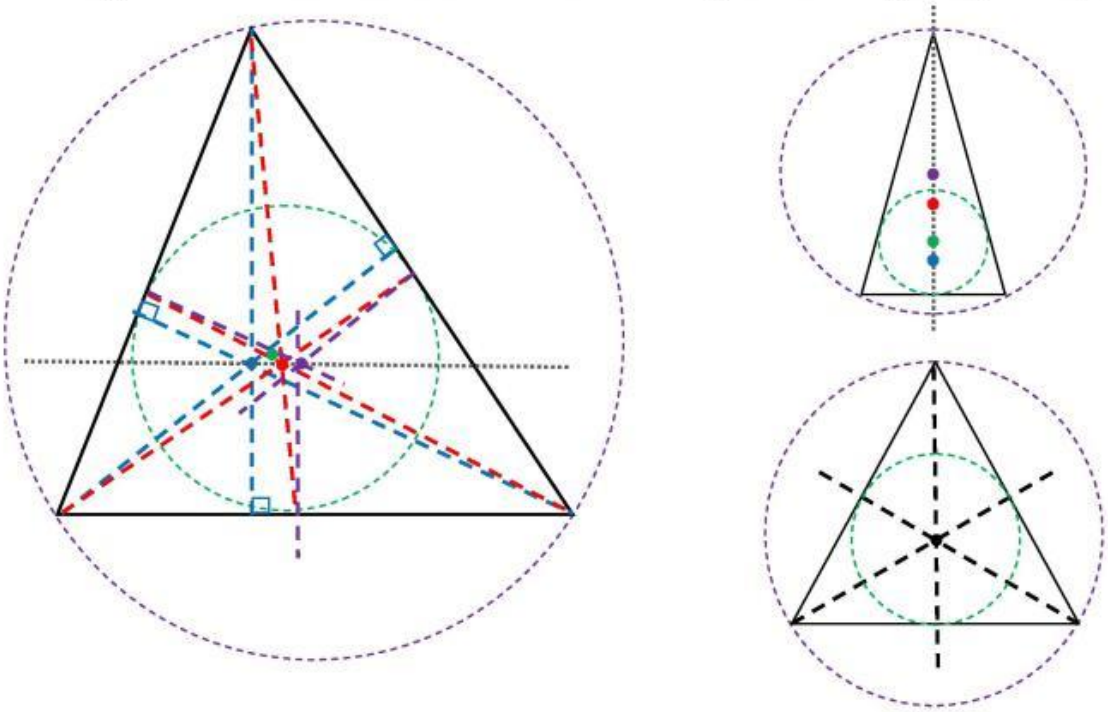
Note: In the above figure, QT, QS and QR are **not** inradii.

GEOMETRY – 13

Let us observe the behaviour of the 4 types of lines (and the 4 corresponding points) we saw last week.

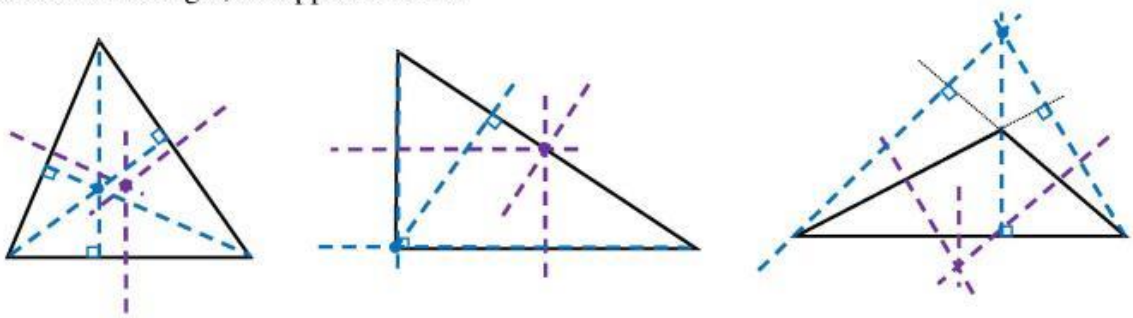
1) The Orthocentre, Circumcentre and Centroid of any triangle always lie on the same straight line, called the Euler Line of the triangle. (In case of an isosceles triangle, the Incentre too lies on it). In the special case of an Equilateral triangle, all these 4 points coincide and so there is no Euler line.

In the first figure below, the grey dotted line indicates the Euler line and the Circumcentre (purple), Orthocentre (blue) and Centroid (red) all lie on it. The Incentre (green), however, lies a little off it. The figures to the right show the situation for an isosceles and an equilateral triangle respectively.



2) The centroid always lies inside the circle, as does the incentre (obviously!). However, the other two points need not lie inside the circle.

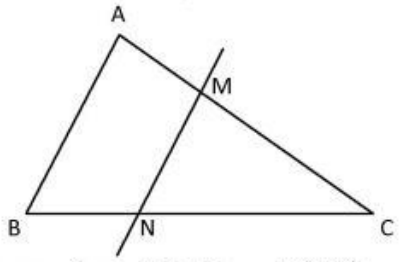
- In the case of an acute-angled triangle, the Circumcentre and Orthocentre also lie inside.
- In the case of a right-angled triangle, the Circumcentre and Orthocentre lie on the boundary (specifically, the Circumcentre is the midpoint of the hypotenuse while the Orthocentre is the vertex opposite the hypotenuse).
- In the case of an obtuse-angled triangle, the Circumcentre (purple) and Orthocentre (blue) lie outside the triangle, on opposite sides.



GEOMETRY – 14

Let us look at a couple of Triangle Theorems:

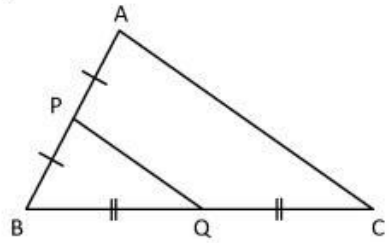
Firstly, let's examine what happens when we take a line parallel to the base of a triangle and intersecting the other two sides. Consider the triangle ABC below, with MN drawn parallel to AB:



Since MN is parallel to AB, we can see that ∠ CAB = ∠ CMN and that ∠ CBA = ∠ CNM. Thus the two triangles CAB and CMN are similar. From this we can see that $\frac{AC}{MC} = \frac{BC}{NC}$ and by applying the Dividendo rule to this we can show that $\frac{AM}{MC} = \frac{BN}{NC}$. Thus we can see that sides AC and BC are divided in the same ratio. This gives us the Basic Proportionality Theorem:

Basic Proportionality Theorem: A line parallel to one of the sides of a triangle divides the other two sides in the same ratio.

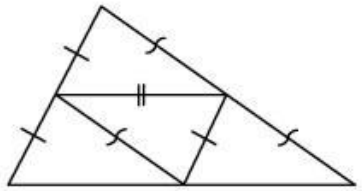
Now let us look at a similar theorem, but in the reverse direction; what happens when we join the midpoints of any two sides of a triangle? Consider the triangle ABC below, with P and Q being the midpoints of AB and BC respectively:



The two triangles ABC and PBM can easily be shown to be similar (since l(PB) is half of l(AB), l(QB) is half of l(CB) and ∠ ABC is a common angle). Thus l(PQ) also has to be half of l(AC). Also ∠ BPQ must be equal to ∠ BAC and so line PQ must be parallel to AC. Thus we have the Midpoint Theorem:

Mid-Point Theorem: A line joining the mid-points of any two sides of a triangle is parallel to the third side and half its length.

An interesting corollary of the Midpoint Theorem is the fact that if we join the midpoints of all three sides in any triangle, the four smaller triangles so formed will be congruent (and hence equal in area, each having a quarter of the area of the original triangle!)

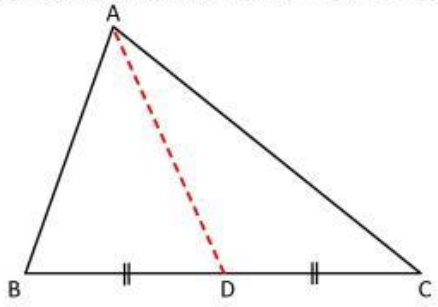


GEOMETRY – 15

The Theorem of Apollonius

One theorem which should spring to mind immediately on seeing a median is the Theorem of Apollonius.

Apollonius' Theorem: The sum of the squares of any two sides of any triangle equals twice the square on half the third side, together with twice the square on the median bisecting the third side.

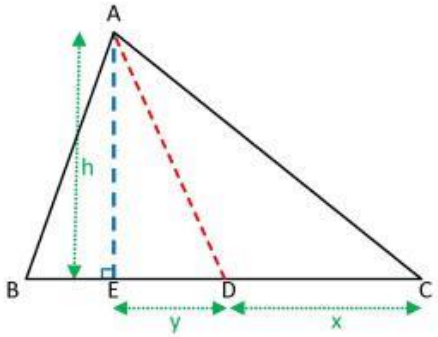


Specifically, in any triangle ABC, if AD is a median, then:

$$\mathbf{AB^2 + AC^2 = 2(AD^2 + BD^2)}$$

Note that since D is the midpoint of BC, we have BD = DC and hence the theorem can also be written as:

$$\mathbf{AB^2 + AC^2 = BD^2 + AD^2 + AD^2 + CD^2}$$

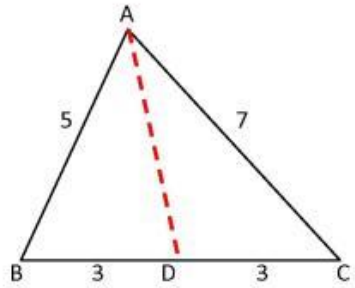


To prove it, let's start by dropping a perpendicular from A to BC. Let the height of this perpendicular be h. Also let l(BD) - and hence also l(DC) - be equal to x, and the distance from D to E (the foot of the perpendicular) be y.
It is clear that l(BE) will be (x – y) while l(CE) will be (x + y)

Now let us apply the Pythagoras Theorem twice; to AB and AC

$AB^2 + AC^2$
$= AE^2 + BE^2 + AE^2 + CE^2$
$= h^2 + (x - y)^2 + h^2 + (x + y)^2$
$= h^2 + x^2 \cancel{- 2xy} + y^2 + h^2 + x^2 \cancel{+ 2xy} + y^2$
$= 2\,(h^2 + x^2 + y^2)$
But note that $AD^2 = h^2 + y^2$ and $BD^2 = x^2$
So we can say that $AB^2 + AC^2 = 2\,(AD^2 + BD^2)$

Q. In Δ ABC, l(AB) = 5, l(BC) = 6 and l(AC) = 7. Find the length of the median from A to BC.



As above, let the median be AD.
We can directly apply Apollonius' Theorem to say that
$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$
so $25 + 49 = 2(AD^2 + 9)$
which gives us $AD^2 = 28$ or $AD = 2\sqrt{7}$

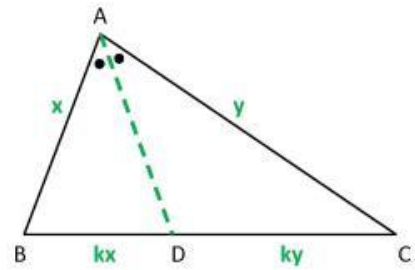
GEOMETRY – 16

The Angle Bisector Theorem

This is probably the most crucial theorem that should strike one on seeing an internal angle bisector.

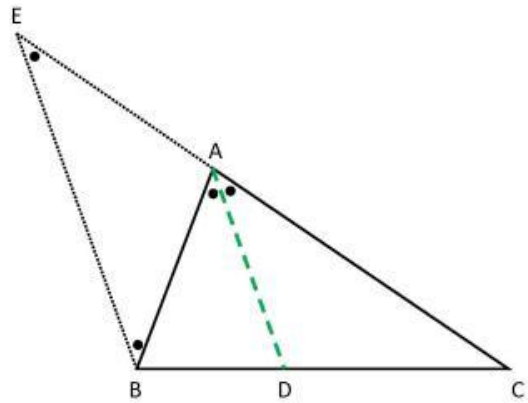
Internal Angle Bisector Theorem: An internal angle bisector in any triangle divides the opposite side in the ratio of the other two sides.

Specifically, if in a triangle ABC, AD is an angle bisector, then:



$$\frac{AB}{AC} = \frac{DB}{DC}$$

In order to prove this theorem, we will need to make a few constructions:



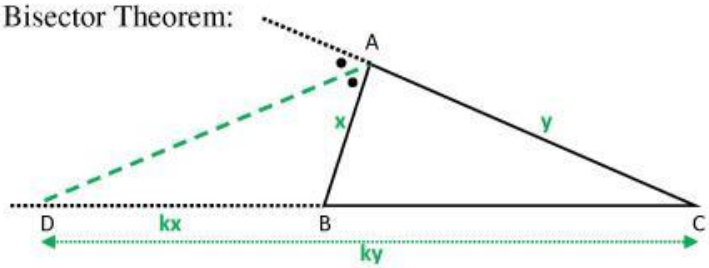
Firstly draw a line through B parallel to DA.
Extend CA to meet this line in a point E.
Note that AD ∥ BE.
Considering transversal AB, ∠ BAD = ∠ ABE
Considering transversal AE, ∠ BEA = ∠ DAC
But since AD is an angle bisector, ∠ BAD = ∠ DAC
Thus all the four angles marked in the figure are congruent.

From this we can observe two things:
Firstly, Δ ABE is an isosceles triangle and hence AB = AE
Secondly, Δ ADC ~ Δ EBC and hence $\frac{AE}{AC} = \frac{DB}{DC}$
Putting AE = AB from above, we get our required result $\frac{AB}{AC} = \frac{DB}{DC}$

There is also a corresponding External Angle Bisector Theorem:

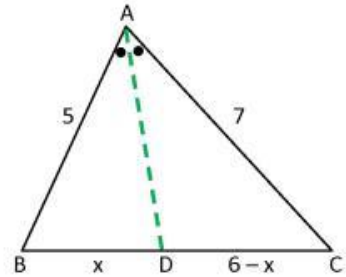
In the adjoining figure,

$$\frac{AB}{AC} = \frac{DB}{DC}$$



Q. In ΔABC, l(AB) = 5, l(BC) = 6 and l(AC) = 7. Angle bisector AD intersects BC at D. Find l(BD).

We know that $\frac{AB}{AC} = \frac{DB}{DC}$.
Thus $\frac{5}{7} = \frac{x}{6-x}$ which gives x = 2.5.



GEOMETRY – 17

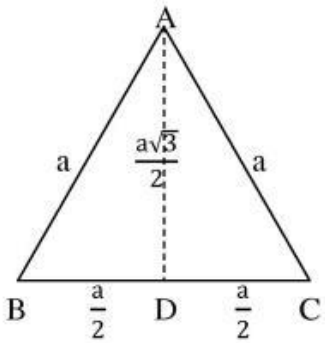
Some useful triangles:

The equilateral triangle

An equilateral triangle, as the name suggests, is a triangle in which all the three sides are equal. It turns out that all the three angles are also equal*, and since their sum has to be 180°, each angle of an equilateral triangle measures 60°.

* Note that this is not true for other polygons – for example a rhombus, a quadrilateral with all four sides equal, need not have all angles equal.

Consider an equilateral triangle ABC of side 'a' as shown:
Let's drop a perpendicular from A meeting BC at point D.
We can show that triangles ABD and ACD are congruent.



So $\angle ADB = \angle ADC = 90°$ and $BD = DC = \frac{a}{2}$

So $AD = \sqrt{a^2 - (\frac{a}{2})^2} = \frac{a\sqrt{3}}{2}$

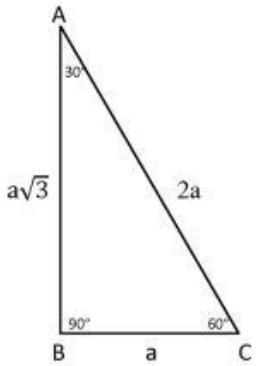
Now AD is the height of the triangle ABC

Hence the area of the triangle is $\frac{1}{2} * a * \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}\,a^2}{4}$

Note that in an equilateral triangle, the median, altitude, angle bisector and perpendicular bisector are all the same line.

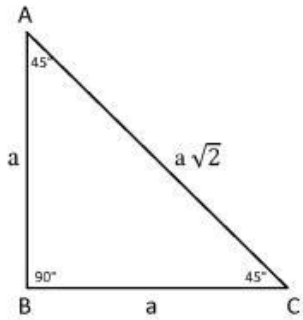
The 30° - 60° - 90° triangle

Another useful triangle is a right-angled triangle in which one of the acute angles is double the other. The measures of the three angles will be 30°, 60° and 90°. Note that such a triangle can be visualised as exactly half of an equilateral triangle (for example ABD in the figure above) and hence we can show that the ratio of sides, from smallest to largest, will be $1 : \sqrt{3} : 2$ (opposite to 30°, 60° and 90° respectively). The area of a 30-60-90 triangle with smallest side a can be shown to be $\frac{\sqrt{3}\,a^2}{2}$



The 45° - 45° - 90° triangle

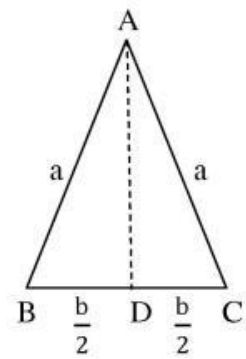
An isosceles right angled triangle will have its two base angles as 45° each. Using Pythagoras' theorem, the sides can easily be shown to be in the ratio $1 : 1 : \sqrt{2}$. If one of the shorter sides of a 45-45-90 triangle is a, then its area will be $\frac{a^2}{2}$.



Note that a 45-45-90 triangle can also be visualised as half of a square.

GEOMETRY – 18

The general isosceles triangle



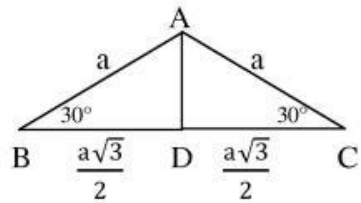
An isosceles triangle is one in which two of the sides are equal in length. Supposes AB = AC = a in the figure. BC = b can be called the base
If we draw a median AD, the two triangles formed will be congruent.
Thus the two base angles (opposite to the equal sides) will also be equal.
Also, ∠ADB and ∠ADC will be right angles and so AD will be an altitude.

Thus l(AD) will be $\sqrt{a^2 - \left(\frac{b}{2}\right)^2} = \sqrt{\frac{4a^2-b^2}{4}}$

The area of the triangle can then be computed as $\frac{1}{2} * b * \sqrt{\frac{4a^2-b^2}{4}} = \frac{b}{4}\sqrt{4a^2 - b^2}$

Note that in an isosceles triangle ABC with base BC, the median, altitude and angle bisector at vertex A and the perpendicular bisector of BC are all the same line.

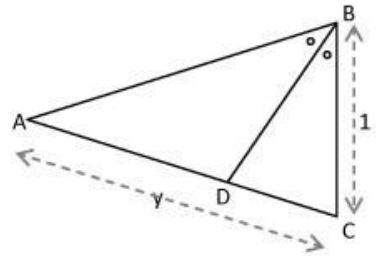
The 30-30-120 triangle



This triangle is worth noting as it crops up in a variety of figures and questions. It can be visualised as two 30-60-90 triangles with their shortest sides coinciding, and thus we can show that the sides will be in the ratio 1 : 1 : $\sqrt{3}$.

The area of a 30-30-120 triangle with smallest side a can be shown to be $\frac{\sqrt{3}\,a^2}{4}$ as it is technically equivalent to the corresponding equilateral triangle of side a.

The 36° - 72° - 72° Triangle (a.k.a. the Golden Triangle):



Consider an isosceles triangle ABC with the property that the angle bisector of a base angle (say B) produces a new triangle CDB which is similar to the original.

Suppose ∠ ABC = ∠ ACB = 2x. Then ∠ ABD and ∠ DBC will be x. But since, by definition, Δ ABC ~ Δ BCD, ∠ BAC must be x as well. Hence we can say x + 2x + 2x = 180 or x = 36 i.e. the Δ ABC is a 36°–72°–72° triangle.

Also since ∠ BAD = x = ∠ ABD, l(AD) is equal to l(BD) which in turn is equal to l(BC). So if we take l(AC) as y and l(BC) as 1 we will find that l(AD) will also be 1 and hence l(CD) will be (y - 1). But applying similarity we know that $\frac{AC}{BC} = \frac{BC}{CD}$ which means that $\frac{y}{1} = \frac{1}{y-1}$ which gives us $y = \frac{1+\sqrt{5}}{2}$. This ratio, which works out to approximately 1.618, is called the Golden Ratio. This triangle is hence called the Golden Triangle. (We shall revisit this triangle later, when we discuss the Fibonacci sequence)

GEOMETRY – 19

The Area of a Triangle

There are many formulae to find the area of a triangle. Let us note some of these. Some of them may be slightly obscure, but any and all of them could come in handy to solve a problem quickly (for example, CAT 2008 had a problem which was extremely easy if one remembered the formula for area in terms of circumradius – however most people I know left the problem because they didn't remember that formula at the crucial moment). I shall not go into details of the proofs, but just provide hints; my aim is simply to mention them and keep them top-of-mind. Those who are interested can find excellent proofs at proofwiki.org or similar sites.

1) Area in terms of Base and Height = $\frac{1}{2}$ * b * h

The area of a triangle in terms of one side taken as a base (b) and the altitude (h) perpendicular to it is given by $\frac{1}{2}$ * b * h. (This can be proved by using the formula for the area of a parallelogram.)

2) Area in terms of the Sides = $\sqrt{s(s-a)(s-b)(s-c)}$

The area of a triangle with sides a, b and c can be given by $\sqrt{s(s-a)(s-b)(s-c)}$ where s, the Semi-perimeter of the triangle, is given by s = $\frac{a+b+c}{2}$. This is known as **Heron's Formula.** (This can be proved by considering it as a special case of Brahmagupta's Formula for Cyclic Quadrilaterals, which itself is a pretty complicated proof!)

3) Area in terms of two sides and the angle between them = $\frac{1}{2}$ ab(sin C)

If two sides of a triangle are "a" and "b" and the angle between them is "C" then the area can be given as $\frac{1}{2}$ * a * b * sin C. (This can be obtained by realising that if we take b as the base, then the corresponding altitude is nothing but a * sin C)

4) Area in terms of Inradius and Sides = rs

The area of a triangle can be expressed in terms of its inradius (r) and its semi-perimeter (s) as r * s where s = $\frac{a+b+c}{2}$. (This can be proved by joining the incentre to each of the vertices and realising that the inradius will be an altitude for each of the three triangles formed)

5) Area in terms of Circumradius and Sides = $\frac{abc}{4R}$

The area of a triangle with sides a, b, c and circumradius R is given by $\frac{abc}{4R}$. (The proof for this is long and convoluted, involving the circumcircle, the inscribed angle theorem, and similar triangles. Definitely not for the faint-hearted!)

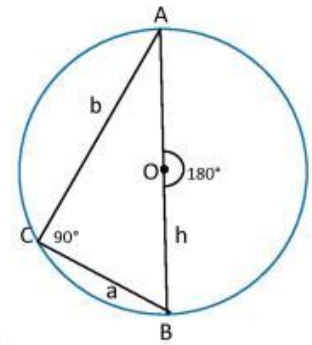
GEOMETRY - 20

Inradius and Circumradius: Right-angled Triangle

In a right-angled triangle, if we know the lengths of the sides (let us say a, b and h, with h as the hypotenuse), it turns out that we can find out both the circumradius and the inradius orally as follows:

- $\mathbf{R = \frac{h}{2}}$
- $\mathbf{r = \frac{a+b-h}{2}}$

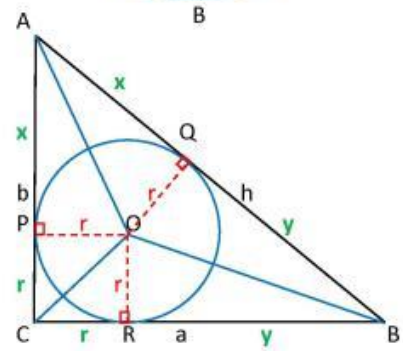
We saw earlier that an angle inscribed in a semi-circle is a right angle.



From the figure, it becomes clear that the converse is also true, and that if we draw the circumcircle for a right-angled triangle, the hypotenuse of the triangle will be a diameter of the circle.

This immediately gives us the result that for a right-angled triangle, the circumradius is half the hypotenuse i.e. $R = \frac{h}{2}$

Now let's turn to the incircle:

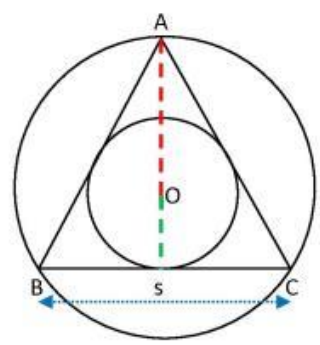


In the adjoining figure, let O be the incentre and OP, OQ and OR be the inradii. Note that since ∠OPC, ∠PCR and ∠CRO are all right angles, the figure OPCR must be a square. Hence l(PC) = l(CR) = r.

Also if l(AP) = x then l(AQ) must also be x (tangents from an external point) and similarly if l(BR) = y then l(BQ) will also be y.

We can see from the figure that a = r + y, b = r + x and h = x + y. Which means that if we compute a + b – h it works out to (r + y) + (r + x) – (x + y) = 2r. Hence we get our result i.e. $\mathbf{r = \frac{a+b-h}{2}}$

Inradius and Circumradius: Equilateral Triangle



In the case of an equilateral triangle, we saw earlier that all of the various points (incentre, circumcentre, orthocentre and centroid) are the same. We also know that a centroid divides its corresponding median in the ratio 2 : 1.

But from the figure it is clear that the median is the same as the height (which is $\frac{\sqrt{3}}{2}$ times the side) and that therefore the inradius and circumradius must be respectively $\frac{2}{3}$rd and $\frac{1}{3}$ rd of this. Thus gives the following relationships for an equilateral triangle of side s:

- $\mathbf{R = \frac{s}{\sqrt{3}}}$
- $\mathbf{r = \frac{s}{2\sqrt{3}}}$

GEOMETRY – 21

For a change from theory, let's solve some Geometry questions this week. These are interesting questions from varied sources and of varying levels of difficulty which I have encountered over the years, the idea is to just look at some of the ways in which the theory of Geometry can be applied efficiently…

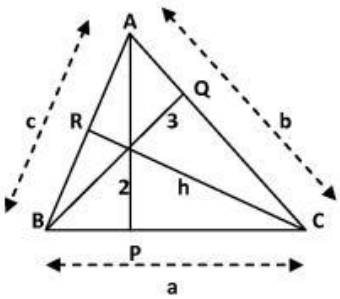
Q. Two altitudes of a triangle ABC are of length 2 cm and 3 cm. Which of the following cannot be the length of the third altitude?

1) 1.1 cm **2) 2.2 cm** **3) 3.3 cm** **4) 4.4 cm** **5) 5.5 cm**

On looking at this question, we should notice certain things:

Firstly, the question is asking *"which of the following cannot be the answer"* which straightaway tells us that there will not be a unique solution.

Secondly, the question involves altitudes. What property of altitudes can be of relevance to us at a basic level? The most logical conclusion we can jump to is – altitudes can be used to calculate the area of the triangle (and not much else). So let's try to see if that would help:



The area of a triangle is half the length any side multiplied by the length of the altitude from the opposite vertex. Hence in the figure above, we can say **A = 2 * a = 3 * b = h * c** (where A is half the area of the triangle)

Now, from the basic properties of a triangle, c has to lie between (a + b) and (a – b) (since the sum of any two sides has to be greater than the third, while the difference of any two sides has to be less than the third).

Hence (A/2 – A/3) < c < (A/2 + A/3)

i.e. A/6 < c < 5A/6.

Now h = A /c

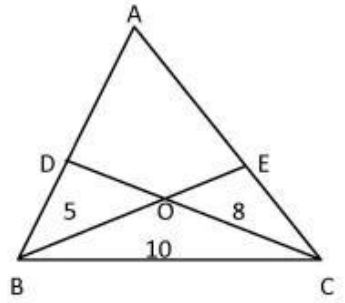
Hence we can say that $\frac{A}{A/6} > h > \frac{A}{5A/6}$

i.e. 6 > h > 1.2

Hence out of the given answer choices, h cannot be 1.1.

GEOMETRY – 22

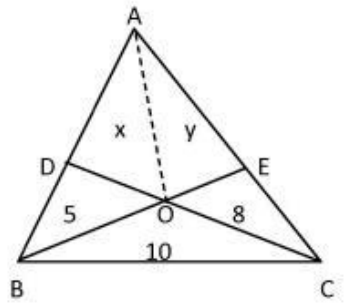
Q. Find the area of the triangle in the given figure (the numbers indicate areas).



1) 32 **2) 40** **3) 45** **4) 50** **5) 52**

There is a fairly direct formulaic way to solve such problems, using a variant of something called the Crossed Ladder Theorem. However, since I am not really an advocate of memorising abstruse formulae, I would use a different approach to solve this:

Let us join OA and take AOD as x and AOE as y (as shown in the figure below).



Note that triangles AOB and AOE stand upon the same baseline BE and have the same heights and hence their areas will be in the ratio of their bases i.e. A(ΔAOB) : A(ΔAOE) = BO : OE. Similarly we may note that A(ΔCOB) : A(ΔCOE) = BO : OE as well. From this we can write the following equation:

$$\frac{x+5}{y} = \frac{10}{8} \Rightarrow 4x + 20 = 5y$$

Similarly by considering the triangles on the baseline CD, we can generate another equation:

$$\frac{y+8}{x} = \frac{10}{5} \Rightarrow 2x = y + 8$$

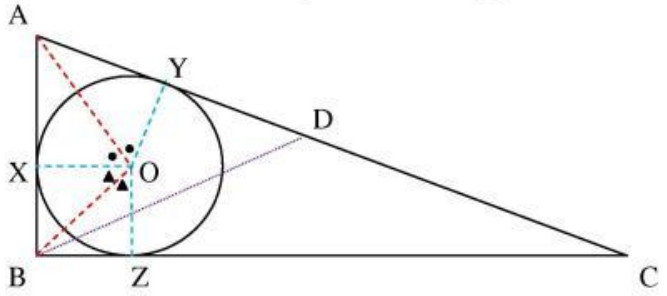
Solving, x = 10 and y = 12 and so the missing area is x + y = 22.

Thus the area of the triangle is 22 + 5 + 8 + 10 = 45 units. **Hence (3)**

GEOMETRY – 23

Two questions today

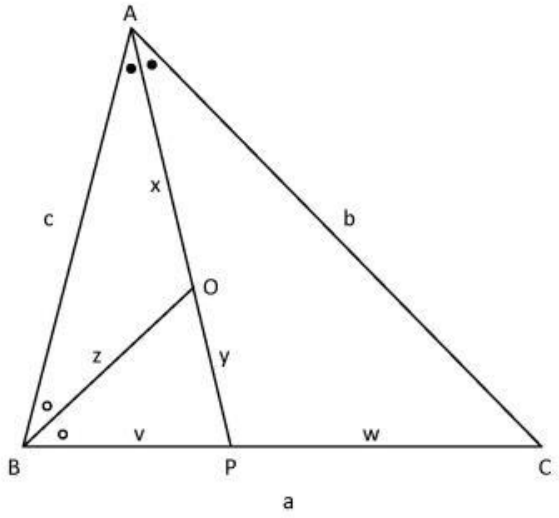
Q. In a right-angled triangle ABC, right angled at B, the angle subtended by AB at the in-centre is 110°. If D is the midpoint of the hypotenuse, find the measure of angle BDC.



Let O be the incentre. Note that AOX and AOY are congruent, as are BOX and BOZ. Also, all of them are right-angled triangles and ∠XOZ is 90°. So ∠BOX is 45° (out of the 110° of AOB) and hence ∠AOX must be 65°. Since ∠AXO = 90°, this makes OAX 25° and hence OAY is also 25°.

Now D, the midpoint of the hypotenuse, is also the circumcentre. So by inscribed angle theorem, ∠BDC = 2∠BAC = 2 x 50 = 100°.

Q. The lengths of the three sides of a triangle ABC are l(BC) = a = 4700 cm, l(CA) = b = 4935 cm and l(AB) = c = 6815 cm. The internal bisector of angle A meets BC at P, and this bisector passes through incentre O. What is the ratio of PO : AO?



In the figure:

Applying Angle Bisector Theorem in triangle ABC, $\frac{v}{w} = \frac{c}{b}$. Hence we can say that $v = a * \frac{c}{c+b}$

Applying Angle Bisector Theorem in triangle BAP, $\frac{y}{x} = \frac{v}{c} = \frac{a * \frac{c}{c+b}}{c} = \frac{a}{c+b}$

Hence OP / OA = a / (c+b) = 4700 / (4935 + 6815) = 2/5

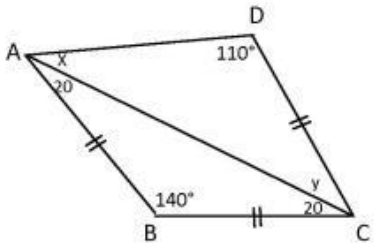
GEOMETRY – 24

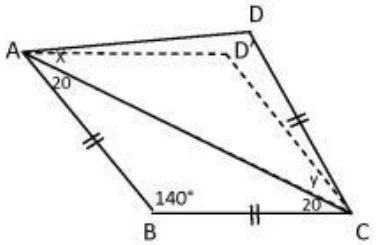
Visualisation is a very important tool in geometry (this cannot be emphasised enough!).

Q. In a quadrilateral ABCD, l(AB) = l(BC) = l(CD). Also, the measure of ∠ ABC is 140° while that of ∠ CDA is 110°. Find the measure of ∠ ACD?

1) 20° **2) 30°** **3) 40°** **4) 35°**

Consider the figure on the left below. Since ABC is isosceles, ∠BAC = ∠BCA = 20°. Also ∠CAD + ∠ACD = 70 i.e. x + y = 70.

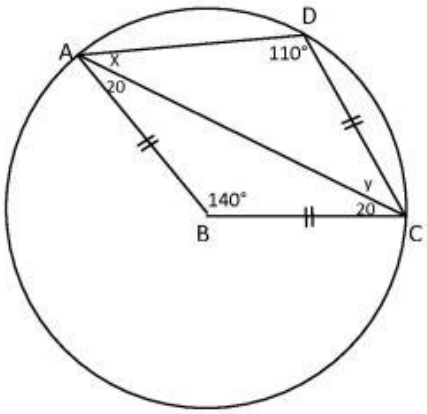




Now if we were to solve it by a formal mathematical approach (as we shall do below), this is a fairly tough problem. However, in the exam, with very little time available, how does one approach such a question? One way would be to reflect ABC about AB to get AD'C (as shown on the right), and realising that to keep CD the same length as CD' while reducing the angle from 140° to 110°, AD has to be longer than AD'. This immediately tells us that ∠DCA has to be greater than ∠DAC, i.e. y > x, but since y + x is 70, y must be greater than 35°. A cursory glance at the options can lead us to mark 40° (i.e. option (3)), safe in the knowledge than none of the others CAN be correct! Hence (3)

However, while this gives us the correct option, it does not prove that the required value is actually 40° (which would be an inconvenience, were there to be a "none of these" in the options!). Let's try to prove it formally, by another tricky 'observation'.

Note that l(BA) = l(BC) and ∠ABC = 140. Also, ∠ADC = 110° which, if we observe carefully, is half of the reflex angle of 140°. Hence, applying the inscribed angle theorem in reverse, we can say that D must lie on a circle centred at B with BA and BC as radii – in other words, BD is also a radius!



Now it is a simple matter to note that BDC is an equilateral triangle – which means that ∠BCD must be 60° and so ∠ACD must measure 60 – 20 = 40°. **Hence (3)**

Next week, we'll take a quick look at the **Fibonacci Sequence** and try and see how it is relevant from the perspective of Geometry.

GEOMETRY – 25

The Fibonacci sequence

In today's post, we'll encounter the Fibonacci sequence, a fairly famous number sequence that appears now and then in various problems. It is so called because Leonardo Fibonacci of Pisa introduced it to the western mathematical world (though it was known to Indian mathematicians for over two millennia!). Fibonacci numbers are very common in nature (papers have been written about their occurrence in flower petals, rabbit breeding, spiral snail shells and diverse other biological phenomena) and can also be used to generate mathematical patterns.

The standard version of the sequence begins with the numbers $F_0 = 0$ and $F_1 = 1$ (called the seed values of the series); each subsequent number is computed by adding the previous two numbers. Hence the third term will be 0 + 1 = 1, the fourth will be 1 + 1 = 2, the fifth 1 + 2 = 3, the sixth 2 + 3 = 5, and so on. The sequence will therefore appear thus:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610….

A generalised Fibonacci sequence can also be created starting with any two initial seed values (except 0, 0 of course!); for example if we start with the seed values 6 and 17, we get a sequence like:

6, 17, 23, 40, 63, 103, 166, 269, 435, 704…

In mathematical terms, the sequence F_n of Fibonacci numbers is defined (for all whole numbers) by the recurrence relation $\mathbf{F_n = F_{n-1} + F_{n-2}}$ where $F_0 = 0$ and $F_1 = 1$.

The Fibonacci sequence has some interesting properties. For example, every third number is even. In fact, in general, every k^{th} number of the sequence is a multiple of F_k – for example $F_4 = 3$, so $F_8 = 21$, $F_{12} = 144$ etc are all multiples of 3. Similarly $F_5 = 5$, so $F_{10} = 55$, $F_{15} = 610$ etc are all multiples of 5.

Another useful property of the sequence is that any two consecutive Fibonacci numbers are co-prime (i.e. GCD $(F_k, F_{k+1}) = 1$). In fact, if we consider a generalised Fibonacci sequence, then the GCD of two consecutive terms will be the GCD of the initial seed numbers F_0 and F_1. So in a Fibonacci sequence starting with 6 and 9 (i.e. 6, 9, 15, 24, 39, 63, 102…) the GCD of any two consecutive terms will be 3.

A lovely (and useful) property involves the ratio of successive terms: by calculation we can observe that the ratio between successive Fibonacci numbers converges to a constant value:

$\frac{1}{1} = 1$, $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} = 1.666$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.6153$

As we calculate further the end result converges to approximately 1.618. Even more remarkable, this holds true for any generalised Fibonacci sequence. For example, consider the sequence starting with 6 and 17 we had generated above:

$\frac{17}{6} = 2.833$, $\frac{23}{17} = 1.352$, $\frac{40}{23} = 1.739$, $\frac{63}{40} = 1.575$, $\frac{103}{63} = 1.634$, $\frac{166}{103} = 1.611$, $\frac{269}{166} = 1.620$

This number is called the Golden Ratio. In the next post we'll look at the mathematical derivation of this ratio.

GEOMETRY – 26

The Golden Ratio

In the last post we saw that the ratio of consecutive Fibonacci numbers tends to approximately 1.618. We can derive this value (non-rigorously) in the following manner, assuming we accept the fact that the ratio will eventually become constant:

Suppose that two consecutive terms of the sequence are 'a' and 'b'. The next term will then be 'a+b' and if we accept that the ratio between successive terms is fixed, we can say that $\frac{b}{a} = \frac{b+a}{b} = 1 + \frac{a}{b}$.
Putting $\frac{b}{a} = x$ in the above equation, we will get $x = 1 + \frac{1}{x}$ or $x^2 - x - 1 = 0$ which when solved gives $x = \frac{1+\sqrt{5}}{2}$ as the positive root. Now since $\sqrt{5} \cong 2.236$, this value comes to approximately 1.618.

This number, the Golden Ratio, was considered extremely aesthetically pleasing and hence has been popular in art and architecture since the times of the ancient Greeks; many works of art have been deliberately proportioned using this ratio – for example the Parthenon, a famous temple in Athens built almost 2500 years ago, has many dimensions which seem to follow this ratio. The Golden Ratio and the Fibonacci Sequence also occur in various mathematical figures.

Figure 1 below shows a tiling created using squares with sides as consecutive Fibonacci numbers. At every stage in the creation of such a figure, we will have a rectangle whose sides are two consecutive Fibonacci numbers (in this case the final one shown is a 34 x 21 rectangle)

Figure 2 shows why this happens: consider a Golden Rectangle with longer side x and shorter side y where $x : y = \frac{1+\sqrt{5}}{2}$. When such a golden rectangle is placed adjacent to a square with sides of length x, we get a similar golden rectangle with longer side x + y and shorter side x.

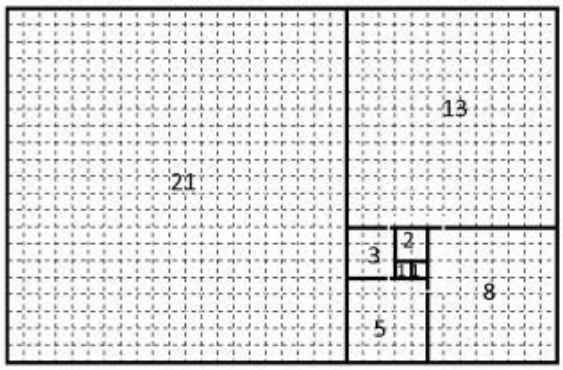


Figure 1

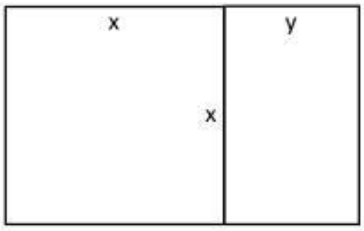


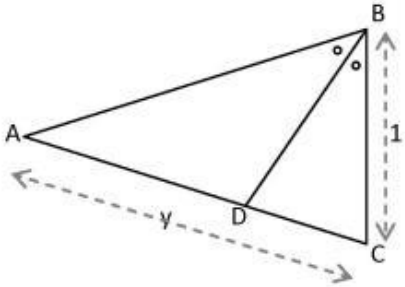
Figure 2

In the next post we'll look at a couple of related geometrical figures which could quite possibly come up in exams like the CAT or XAT: the so-called Golden Triangle and the Regular Pentagon.

GEOMETRY – 27

The Golden Triangle:

Consider an isosceles triangle ABC with the property that the angle bisector of a base angle (say B) produces a new triangle CDB which is similar to the original.



Suppose ∠ ABC = ∠ ACB = 2x. Then ∠ ABD and ∠ DBC will be x. But since, by definition, Δ ABC ~ Δ BCD, ∠ BAC must be x as well. Hence we can say x + 2x + 2x = 180 or x = 36 i.e. the Δ ABC is a 36°–72°–72° triangle.

Also since ∠ BAD = x = ∠ ABD, l(AD) is equal to l(BD) which in turn is equal to l(BC). So if we take l(AC) as y and l(BC) as 1 we will find that l(AD) will also be 1 and hence l(CD) will be (y - 1). But applying similarity we know that $\frac{AC}{BC} = \frac{BC}{CD}$ which means that $\frac{y}{1} = \frac{1}{y-1}$ which gives us $y = \frac{1+\sqrt{5}}{2}$ – i.e. the sides of the triangle are in the Golden Ratio. This triangle is hence called the Golden Triangle.

The Regular Pentagon and the Golden Ratio:

A regular pentagon is a 5-sided figure with all sides congruent and all angles equal. Each of the interior angles turns out to be 108°. It turns out that there is a significant relationship between this figure and the Golden Ratio we discussed in the previous post: in a regular pentagon, the ratio between a side and a diagonal is the golden ratio, while intersecting diagonals also section each other in the golden ratio.

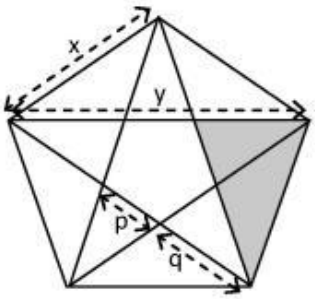


Figure 3

$$\frac{y}{x} = \frac{x}{q} = \frac{q}{p} = \frac{1+\sqrt{5}}{2}$$

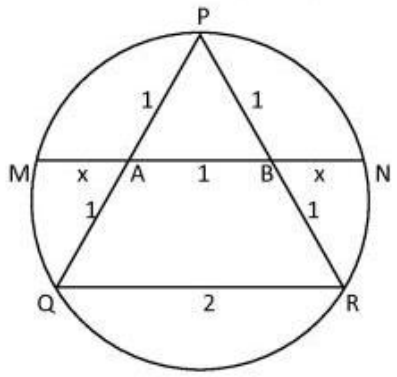
The above result can be easily proved by recognising that several golden triangles (i.e. having angles 36° – 72° – 72°) are formed within the figure (one of which has been highlighted).

GEOMETRY – 28

Q. An equilateral triangle PQR of side 2 units is inscribed in a circle. A chord MN of the circle is drawn such that it passes through the midpoints A and B of segments PQ and PR respectively. Find l(AM)?

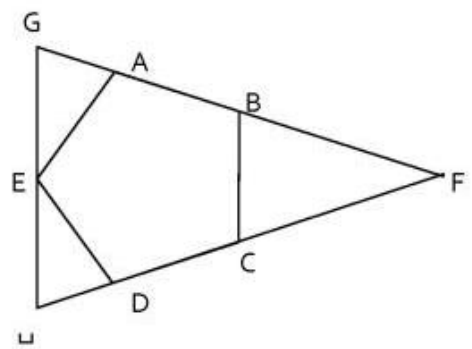
1) 1 **2) $\frac{1+\sqrt{5}}{2}$** **3) $\frac{\sqrt{3}}{2}$** **4) $\frac{\sqrt{5}-1}{2}$**

Consider the following figure (note that PAB will also be an equilateral triangle). Let MA be x. Hence BN, by symmetry, will also be x and AN will be (1 + x).



Now PQ and MN are two chords which intersect in the interior of the circle. Consequently, by the theorem of intersecting chords, l(AP) * l(AQ) = l(AM) * l(AN). Thus 1 x 1 = x (1+x) or $x^2 + x - 1 = 0$. Solving this gives us $x = \frac{\sqrt{5}-1}{2}$. Hence (4)

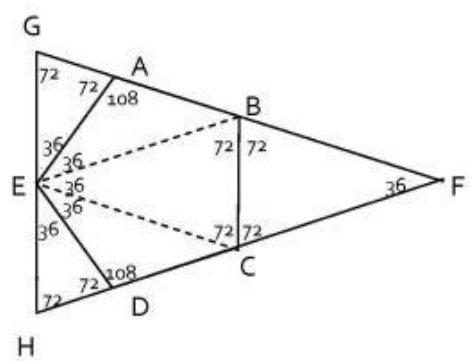
Q. A regular pentagon ABCDE has sides of length 1 cm. Sides AB and DC are extended to meet in a point F as shown in the figure. HG is drawn parallel to BC such that it meets the sides BA and CD (extended) in G and H respectively.



1) What is the ratio of l(GA) : l(AB)?

2) What is the ratio of l(AB) : l(BF)?

Let l(AG) be x.
If we draw BE and CE, all the angles can be easily worked out. Observe that ΔBCF and ΔBCE are congruent by the ASA test. Consequently, l(BE) = l(BF). But from this we can then prove that ΔBEG and ΔCEH are also congruent to ΔBCF.
∴ l(GB) = l(BF) = 1+x



Using similarity of ΔAEG and ΔGBE $\frac{l(AG)}{l(AE)} = \frac{l(AG)}{l(AE)}$

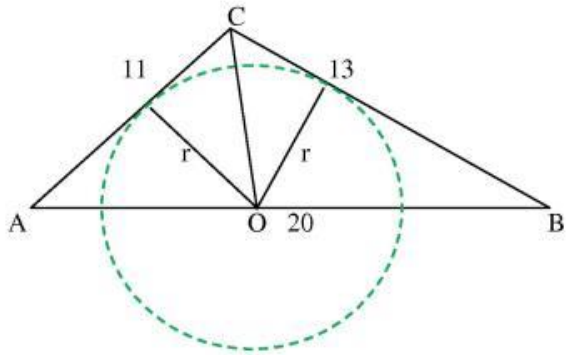
Hence $\frac{x}{1} = \frac{1}{1+x}$ or $x^2 + x - 1 = 0$. Thus $x = \frac{\sqrt{5}-1}{2}$ which is the answer to both the questions above!

GEOMETRY – 29

Some more examples on triangles this week:

Q) In a triangle ABC side AB = 20, AC = 11 and BC = 13. Find the diameter of the semicircle inscribed in ΔABC whose diameter lies on AB and that is tangent to AC and BC.

1) 9 **2) 10** **3) 11** **4) 13**



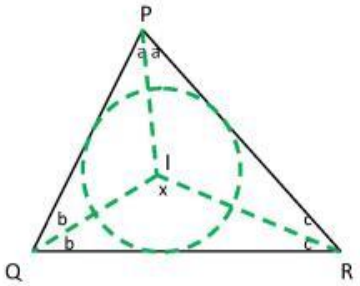
If we join C to O (the centre of the circle) we get two triangles COA and AOB with areas $\frac{1}{2}$ x 11 x r and $\frac{1}{2}$ x 13 x r. The total of these should equal total area of ΔABC which is $\sqrt{22 * 2 * 11 * 13}$ = 66.

Hence r = $\frac{66}{12}$ = 5.5 and so the required diameter = 11. **Hence (3)**

Q. In a triangle PQR with incentre I, if ∠P = 50°, find angle QIR.

1) 100° **2) 105°** **3) 110°** **4) 115°**

Suppose the three angles are P = 2a, Q = 2b and R = 2c (for convenience). So 2a + 2b +2c = 180 and a + b + c = 90. Also let the angle we want to find i.e. ∠QIR be x.



Now remember that the incentre is formed by the angle bisectors. Hence since I is the incentre, angle IQR = b and angle IRQ = c. Hence angle QIR = 180 - (b+c) = 180 - (90 - a) = 90 + a.

We can remember this as a general result: Angle formed at Incenter = 90 + $\frac{1}{2}$ (Angle at opposite vertex)

Hence in this particular case, the angle formed will be 90 + $\frac{1}{2}$ * 50 = 115°. **Hence (4)**

Q. The base of an isosceles triangle is root 2. The medians to the legs intersect each other at right angles. Then the area of the triangle is:

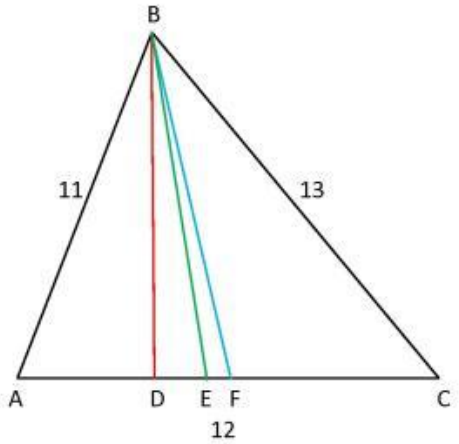
1) 75 **2) 1.5** **3) 2.5** **4) 3**

The area of the triangle formed by the base and two medians is $\frac{1}{2}$ * 1 * 1 = $\frac{1}{2}$.

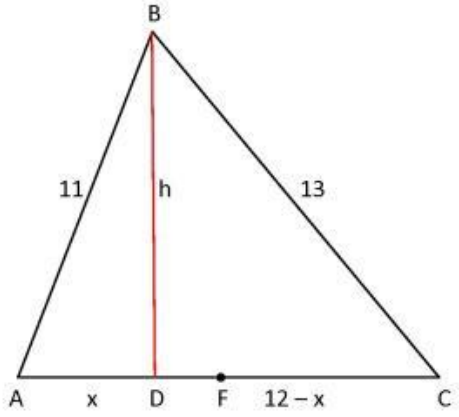
But this is $\frac{1}{3}$ of the total triangle (since the triangle is divided into 6 equal area parts by the medians) hence the area of the triangle will be 3 * $\frac{1}{2}$ = $\frac{3}{2}$ = 1.5. **Hence (2)**

GEOMETRY – 30

Q. Consider a triangle ABC such that AB = 11, AC = 12 and BC = 13. D, E and F are points on AC such that BD is an altitude, BE an angle bisector and BF a median. Find lengths DF and EF.

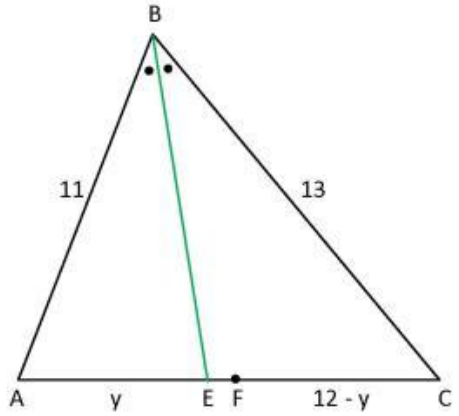


In the above figure, BD is an altitude, BE an angle bisector and BF a median. By definition, AF = FC = 6 (as it is a midpoint!)



Let distance AD be x and height BD be h. CD will be 12 – x.
Using Pythagoras theorem, $h^2 = 11^2 - x^2 = 13^2 - (12 - x)^2$
So $121 - \cancel{x^2} = 169 - 144 + 24x - \cancel{x^2}$
which gives x = 4.

Hence DF = 2 units.



Let AE be y and hence EC be 12 - y

Applying the angle bisector theorem,
$$\frac{y}{12 - y} = \frac{11}{13}$$
Which when solved gives y = 5.5
Hence EF = 0.5

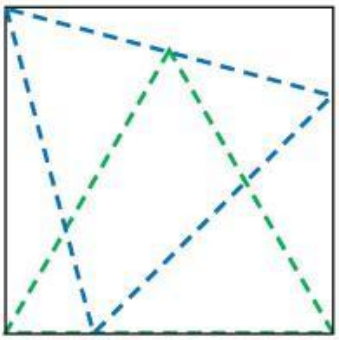
Interestingly, it turns out that this will prove true in general; if we take a triangle of sides (k – 1), k and (k + 1) and draw the above three lines to the side of length k, then the foot of the altitude will be 2 units from the foot of the median, while the foot of the angle bisector will be 0.5 units from it! (Try checking it for a triangle of sides, say, 5, 6 and 7)

GEOMETRY – 31

Q. The largest possible equilateral triangle is drawn inside a square of side 1. Find the side of the equilateral triangle.

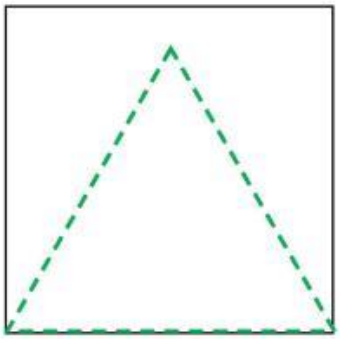
1) 1 **2)** $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **3)** $\frac{\sqrt{3}}{\sqrt{2}}$ **4)** $\frac{2\sqrt{2}}{\sqrt{3}+1}$

We can see from symmetry that there will be two candidates for the largest equilateral triangle – one which shares a side with the square and one which shares a vertex with the square as shown:



Let us examine each of these individually.

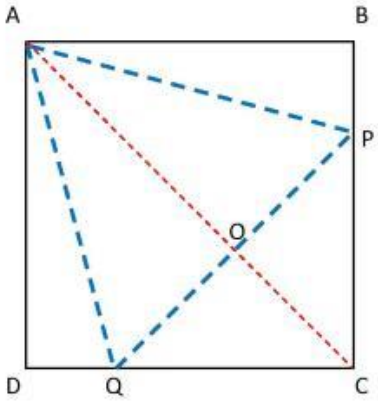
In the first case we can clearly see that the side of the equilateral triangle is just the side of the square i.e. 1. So we need to check the other case to see if the side obtained there is greater than 1.



(Incidentally, at this point we can eliminate option 2 as the value is less than 1).

Coming to the second case:

We can note that there is a 45° – 45° – 90° triangle (ΔPQC) and also an equilateral triangle (ΔAPQ) in the figure. Let us draw AC. By symmetry, we can easily show that AC is perpendicular to PQ.



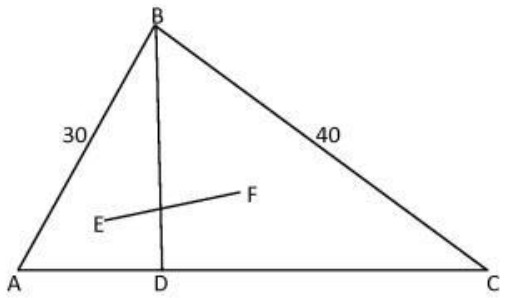
Let the side of the equilateral triangle be x. Then in ΔPQC, we can show that OC is an altitude to PQ and that l(OC) = l(OP) = l(OQ) = $\frac{x}{2}$.

Also in ΔAPQ, AO is an altitude and hence l(AO) will be $\frac{x\sqrt{3}}{2}$.

But we can also see that AC being the diagonal of a square of side 1, its length will be $\sqrt{2}$. So we can say that $\frac{x}{2}+\frac{x\sqrt{3}}{2}=\sqrt{2}$ which when solved gives us $\mathbf{x = \frac{2\sqrt{2}}{\sqrt{3}+1}}$ **. Hence (4).**

GEOMETRY – 32

Q. In the figure below, triangle ABC is right angled at B. BD is the altitude to AC. E and F are the incentres of the circles inscribed in triangles ABD and BCD respectively. Find l(EF).

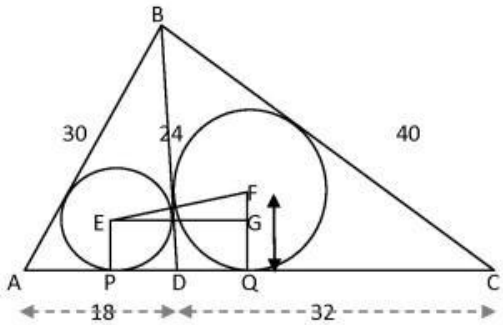


1) 14 **2) $2\sqrt{10}$** **3) $10\sqrt{2}$** **4) $4\sqrt{5}$**

First of all note that since l(AB) = 30 and l(BC) = 40, l(AC) must be 50.

Also note that if in a triangle ABC right angled at B, an altitude BD is drawn perpendicular to the hypotenuse AC, the triangles formed will be similar to each other (and to the original triangle). Using this we can say that $\frac{AD}{AB} = \frac{BD}{BC} = \frac{AB}{AC} = \frac{30}{50}$ and $\frac{BD}{AB} = \frac{CD}{BC} = \frac{BC}{AC} = \frac{40}{50}$ from which we can show that l(AD) = 18, l(BD) = 24 and l(DC) = 32.

Let us draw the incircles in the figure:



Now we can see that the inradius of the triangle with sides 18, 24, and 30 will be $\frac{18+24-30}{2} = 6$ while that of the triangle with sides 24, 32 and 40 will be $\frac{24+32-40}{2} = 8$.

Note that E and F are not at same level! So the circles will not touch each other and hence the distance between them will not just be 6 + 8 i.e. 14. However, we can construct a right angled triangle as shown in the figure. Now l(EP) and l(PD) must be 6, while l(FQ) and l(DQ) must be 8. So l(EG) will be equal to l(PQ) which is 6 + 8 i.e. 14 while l(FG) must similarly be 8 – 6 = 2.

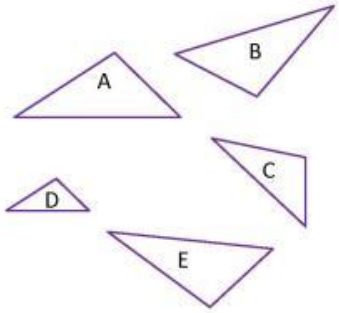
Hence l(EF) = $\sqrt{14^2 + 2^2} = \sqrt{200} = \mathbf{10\sqrt{2}}$**. Hence (3)**

(This is based on an actual question from CAT 2002)

GEOMETRY – 33

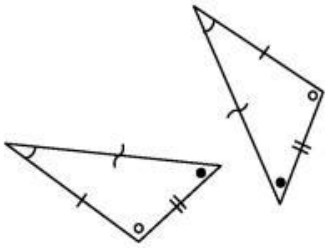
Similarity and Congruence

In Geometry, two figures are said to be similar if their shape is the same. Note that this shape could be rotated or even flipped and it would still be fine. They would be said to be congruent if the shape as well as the size is the same. So congruence is a more stringent condition – any pair of congruent figures is similar by definition.

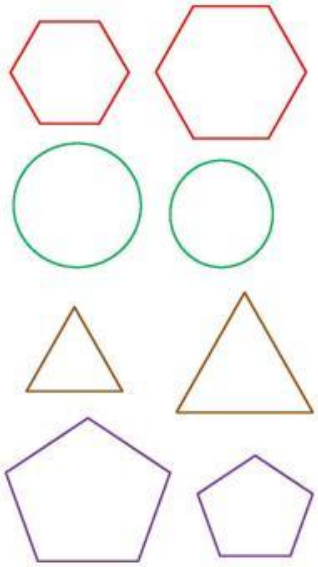


For example the 5 triangles shown alongside are all similar. A, B and E are congruent (but E is flipped), while C and D are of different sizes

A more mathematical way of looking at similarity and congruence: Two figures can be said to be Congruent if all their corresponding sides and angles (in order) are the same
Two figures can be said to be similar if all their angles (in order) are the same and all their corresponding sides are in the same ratio.
(The figure may not always **look** similar – one should test to make sure)

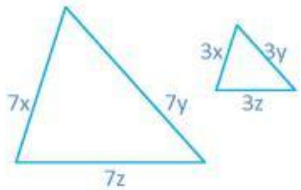


Note that, while we normally study similarity and congruence for triangles, any two figures could be tested to check for similarity and congruence. In the case of regular figures, this is easiest – any two regular figures with the same number of sides will be similar to each other. For example if we take two regular hexagons, or two circles, or two equilateral triangles, or two squares, or two regular pentagons, each pair of figures will be similar without any further checking required.

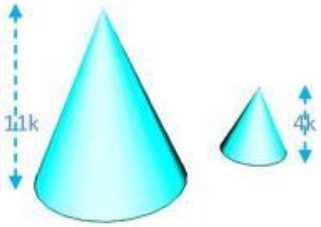


Once we identify two figures as similar with sides in the ratio (say) x : y, then straightaway all their linear dimensions will be in this same ratio x : y. Also, their area dimensions will be in the ratio $x^2 : y^2$ (remember that any area dimension is the product of two length dimensions – length * breadth or base * height or radius * radius…)

For example if two triangles are similar with sides in the ratio 3 : 7 then their perimeters, circumradii, inradii, medians or altitudes from corresponding vertices etc will all be in the ratio 3 : 7, while their areas will be in the ratio 9 : 49.



Similarly if two cones are similar with heights in the ratio 4 : 11 then their base radii, slant heights etc will also be in the ratio 4 : 11. Their surface areas, base surface areas, curved surface areas etc will be in the ratio 16 : 121. And their volumes will be in the ratio 64 : 1331.

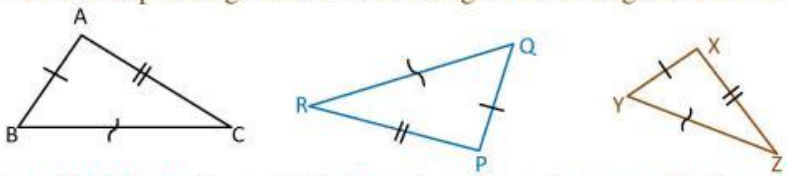


In the next post we'll look at tests for similarity/congruence in triangles.

GEOMETRY – 34

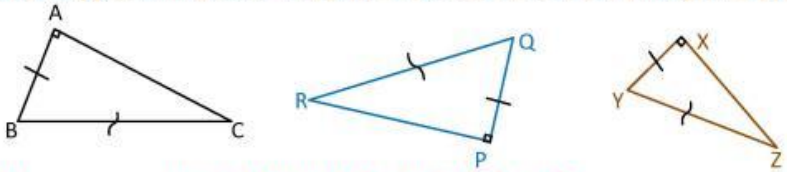
Triangles: Tests for Congruence and Similarity

SSS Test: If we check the three sides of two triangles, then the triangles are
Congruent if three pairs of sides of the two triangles are equal in length.
Similar if the corresponding sides of two triangles have lengths in the same ratio



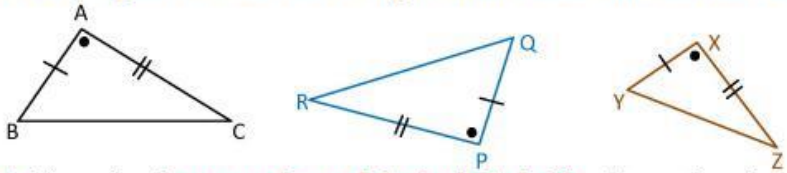
ABC ≅ PQR ~ XYZ

Hypotenuse-Side Test: If we check the sides of two right-angled triangles, then the triangles are
Congruent if the hypotenuses and one pair of shorter sides are equal in length
Similar if the hypotenuses and one pair of shorter sides have lengths in the same ratio.



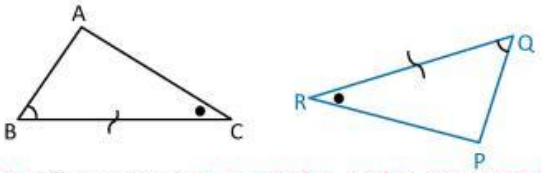
ABC ≅ PQR ~ XYZ

SAS Test: If we check two sides and the included angle of two triangles, then the triangles are
Congruent if the two pairs of sides are equal in length and the included angle is equal.
Similar if the two pairs of sides have lengths in the same ratio and the included angle is equal



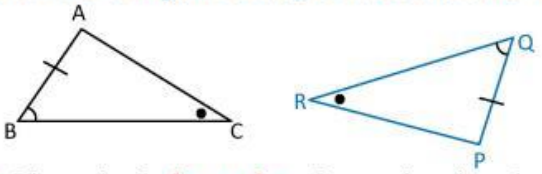
ABC ≅ PQR ~ XYZ

ASA Test: If we check two angles and the included side of two triangles, then the triangles are
Congruent if the two pairs of angles have the same measure and the sides are equal in length



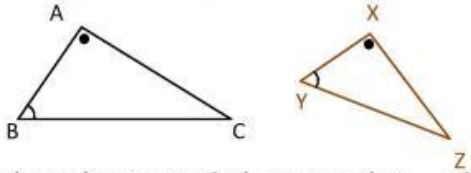
ABC ≅ PQR

AAS Test: If we check two angles and a corresponding non-included side of two triangles, then the triangles are
Congruent if the two pairs of angles have the same measure and the sides are equal in length



ABC ≅ PQR

AA Test: If we check the angles of two triangles, then the triangles are
Similar if two pairs of angles are the same.



ABC ~ XYZ

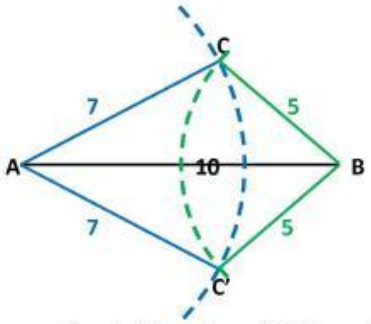
This list gives rise to an obvious question – why is there no ASS test? In the next post, we shall try to understand this.

GEOMETRY – 35

Why is there no ASS test?

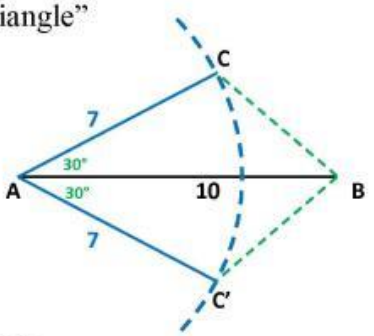
The underlying logic beneath the tests for Congruence (and also for Similarity) is that they are based on the **minimum information needed to draw a unique triangle**.

For example let us consider the SSS test. If we are informed that a triangle ABC has l(AB) = 10, l(AC) = 7 and l(BC) = 5, we could start by drawing AB. We could then mark an arc of radius 7 centered at A and another of radius 5 centered at B; the intersection point of these two arcs will give us point C (as shown in the figure)



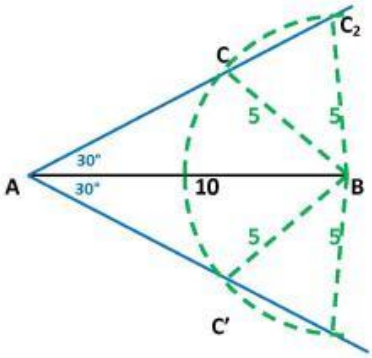
Note that there are two intersection points C and C'; however, the two resultant triangles ABC and BAC' are congruent and hence the information still results in a "unique triangle"

Similarly if we look at the SAS test. If we are informed that a triangle ABC has l(AB) = 10, l(AC) = 7 and ∠A = 30°, we could start by drawing AB. We could then mark an arc of radius 7 centered at A and draw a ray from A at an angle of 30° to AB; the intersection point of the arc and the ray will give us point C (as shown in the figure)



Here, too, there are two intersection points C and C'; but both are congruent.

But now let us look at the ASS test. If we are informed that a triangle ABC has l(AB) = 10, l(BC) = 5 and ∠A = 30°, we could start by drawing AB. We could then mark an arc of radius 5 centered at B and draw a ray from A at an angle of 30° to AB; the intersection point of the arc and the ray should give us the point C.



But as we can see from the figure that there are two triangles of distinct size and shape being formed in the figure – both ΔABC and ΔABC_2 satisfy the required conditions.

Thus we cannot always have a unique triangle defined by two sides and a non-included angle, and hence we do not have such a test. The easy way to remember this is that if you use an ASS test, an ASS is precisely what you will make of yourself!

Note that this problem does not arise with the SAA test, wherein we take two angles and a non-included side. This is because if two pairs of angles of angles are the same, the third pair automatically must also be equal since the sum of all three angles in a triangle is 180°. Technically, therefore, an SAA test reduces to an ASA test and indeed some textbooks in Geometry consider them as identical

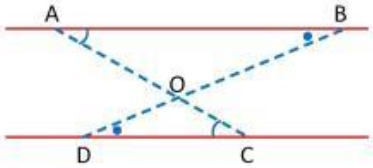
Note also that in some cases an ASS could result in a unique situation. The difference is that in the other tests, **in all cases** it results in a unique triangle while in an ASS it is only **in some cases** - and there is usually no way of knowing, a priori, whether the test will be valid or not. There is one clear exception - technically, the Hypotenuse-Side test is a special, valid case of an ASS situation!

GEOMETRY – 36

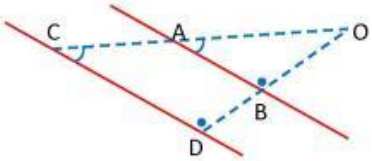
Spotting Similarity and Congruence:

Being able to spot Similarity (and Congruence) is of paramount importance to the visualisation of problems in Geometry; in my experience Similarity stands next only to Right-Angled triangles in its usefulness as a concept. And a key to recognising similarity is spotting equal angles. Let me demonstrate through some typical cases:

Parallel lines: The moment one sees two parallel lines, one should immediately look out for the possibility of similar triangles. This is because parallel lines and transversals create equal angles galore.

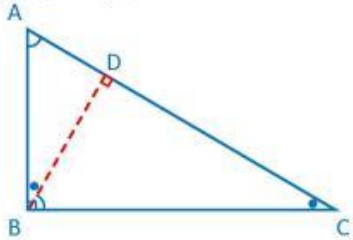


In both the situations alongside, AB and CD are parallel lines. Immediately we should recognise that ΔAOB and ΔCOD are similar as two pairs of angles are the same in each case.



This is an idea underlying a lot of proofs – for example we used this in proving the Basic Proportionality and Mid-point Theorems, as well as in showing that when the diagonals of a trapezium are drawn, the two triangles formed having their bases as the parallel sides are similar.

Altitude to a hypotenuse: Another very useful similarity-based result is that when we drop a perpendicular to the hypotenuse of a right triangle from the opposite vertex, the two triangles formed are similar to each other and to the original triangle.

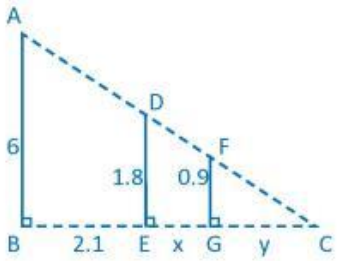


In the adjoining figure, for example, ΔABC is right angled at B. BD is dropped perpendicular to AC. Then we can see that if we compare ΔABC and ΔADB, they both have a common angle (A) and a right angle and thus are similar. Also if we compare ΔABC with ΔBDC, they both have a common angle (C) and a right angle and thus are similar. So all three triangles (ΔABC, ΔADB and ΔBDC) are similar.

Let's look at a simple CAT question based on similarity:

Q. A father and his son are waiting at a bus stop in the evening. There is a lamp post behind them. The lamp post, the father and his son stand on the same straight line. The father observes that the shadows of his head and his son's head are incident at the same point on the ground. If the heights of the lamp post, the father and his son are 6 metres, 1.8 metres and 0.9 metres respectively, and the father is standing 2.1 metres away from the post, then how far (in metres) is the son standing from his father? (CAT 04)

1] 0.9 2] 0.75 3] 0.6 4] 0.45



Since all the three (lamp-post, father and son) may be assumed to be vertical, they would be parallel to each other. Clearly, then, all the triangles formed (ΔABC, ΔDEC and ΔFGC) are similar to each other.

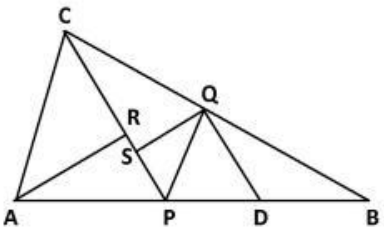
Thus $\frac{y}{0.9} = \frac{y+x}{1.8} = \frac{y+x+2.1}{6}$ which when solved gives us y = x = 0.45.

Hence (4)

GEOMETRY – 37

Let us look at a problem from CAT 2003 with an interesting twist:

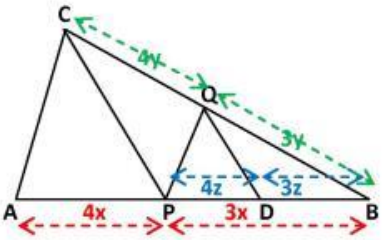
Q. In the figure (not drawn to scale), P is a point on AB, such that AP : PB = 4 : 3. PQ is parallel to AC and QD is parallel to CP. In ΔARC, ∠ARC = 90° and in ΔPSQ, ∠PSQ = 90°. The length of QS is 6 cm. What is the ratio of AP : PD? **(CAT 03)**



1) 10:3 **2) 2:1** **3) 7:3** **4) 8:3**

The figure is a pretty complicated one. There's also plenty of information, which makes it tempting to start plugging in values and slaving away at finding the lengths of each and every side. However, in such cases it is always a good idea to take a step back and examine the problem from a distance…

CP and QD are a pair of parallel lines, while AC and PQ are another. As we observed last week, parallel lines should throw us a big hint that there is the possibility of similar triangles. Let's search out some pairs of similar triangles:



ΔACB ~ ΔPQB

ΔBCP ~ ΔBQD

ΔACP ~ ΔPQD

ΔACR ~ ΔQPS

…and so on.

But for now, let us focus on the first two of these pairs.

In ΔACB and ΔPQB, $\frac{AP}{PB} = \frac{CQ}{QB} = \frac{4}{3}$

In ΔBCP and ΔBQD, $\frac{CQ}{QB} = \frac{PD}{DB} = \frac{4}{3}$

So PD = $\frac{4}{7}$ of PB which in turn is $\frac{3}{7}$ of AB i.e. PD = $\frac{4}{7} * \frac{3}{7} = \frac{12}{49}$ of AB

Also AP is $\frac{4}{7}$ of AB.

So the ratio of AP to PD must be $\frac{4}{7} : \frac{12}{49}$ = 7 : 3. **Hence (3)**

The interesting point to note is that **the entire latter half of the information** "In ΔARC, ∠ARC = 90° and in ΔPSQ, ∠PSQ = 90°. The length of QS is 6 cm." **proved irrelevant and inconsequential to solving the problem**. This led to two interesting types of errors:

a) People who tried to use all the information and quickly gave up due to information overload

b) People who solved the question without using all of that information, **and then didn't mark the answer,** reasoning thus: since the information has been provided, it must be necessary; so if I got an answer without using it, I must be wrong…

Only a few people actually had the courage of their convictions to mark the answer with confidence!

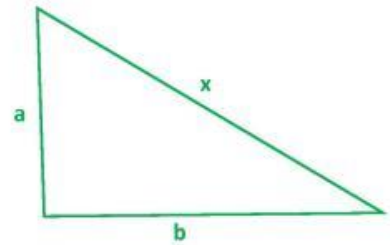
GEOMETRY – 38

Q. Consider obtuse-angled triangles with sides 8 cm, 15 cm and x cm. If x is an integer, then how many such triangles exist? **(CAT 08)**

1] 5 **2) 21** **3] 10** **4] 15** **5] 14**

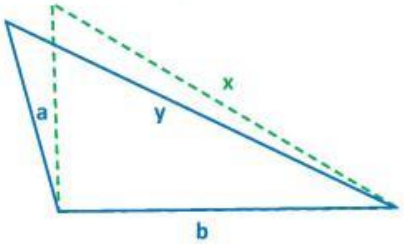
Identifying whether a triangle is right-angled is easy. How does one figure out if it is acute-angled or obtuse-angled? It turns out that the most overworked concept in Geometry, the theorem of Pythagoras, can be modified to give us the answer:

Let's look at a right angled triangle with sides a, b and x (x being the hypotenuse), as shown in the figure. Note that the angle between a and b is 90°

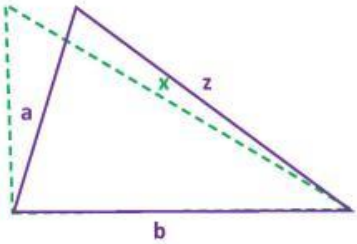


Quite clearly, here, we have $\mathbf{x = \sqrt{a^2 + b^2}}$

Now consider another triangle with two sides as a and b and the angle between them greater than 90°. If the third side is of length y, comparing it with the original triangle, we can clearly see that y > x. So we can say that $\mathbf{y > \sqrt{a^2 + b^2}}$



Consider a third triangle, this time with two sides as a and b and the angle between them less than 90°. If the third side is of length z, comparing it with the original triangle, we can clearly see that z < x. So we can say that $\mathbf{z < \sqrt{a^2 + b^2}}$



Thus if we have a triangle with three sides p, q, r, we start by identifying the longest side. Suppose this is p, then all we need to do is check p^2 v/s $q^2 + r^2$.

In the above problem, two sides are 15 and 8, so first of all the third side x should be an integer such that (15 - 8) < x < (15 + 8) i.e. 7 < x < 23.

Now there are two possible cases: either 15 will be the longest side of x will be the longest side.

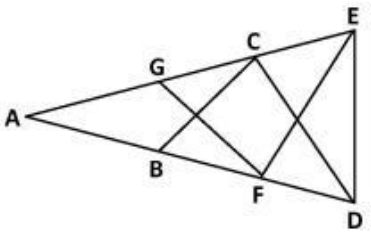
Case 1: If $x \leq 15$, then for the triangle to be obtuse, $15^2 > x^2 + 8^2$. This gives us $x^2 < 161$ and hence x must take values less than 13. But we also have the condition that x exceeds 7 so this gives us 5 values for x (8, 9, 10, 11 and 12)

Case 2: If x > 15, then for the triangle to be obtuse, $x^2 > 15^2 + 8^2$. This gives us $x^2 > 289$ and so x must be more than 17. However we also know that x cannot be 23 or more, and so we get 5 more values for x (18, 19, 20, 21 and 22).

Combining the above two cases, we see that there are 10 such triangles. **Hence (3)**

GEOMETRY – 39

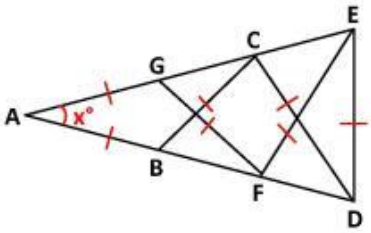
Q. In the figure l(AB) = l(BC) = l(CD) = l(DE) = l(EF) = l(FG) = l(GA). What is the approximate measure of ∠DAE? **(CAT 2000)**



1) 15° **2) 20°** **3) 25°** **4) 30°**

The figure is quite intimidating, no doubt. But on looking closely we can observe that since there are several line segments which are equal in length, we will correspondingly have many isosceles triangles. And the obvious property of an isosceles triangle is that the base angles are equal. Let us try to use this.

Let's designate the required angle ABC as x.



Consider ΔAGF. Since l(AG) = l(GF), ∠GFA = ∠GAF = x°. Now ∠FGC is an exterior angle to this triangle and hence must be equal to x + x = 2x°.

But since ΔGFE is also an isosceles triangle, ∠GEF = ∠GFE = 2x°. Now, ∠EFD is an exterior angle to ΔAFE and hence will be equal to x + 2x = 3x°.

Now ΔFED is also an isosceles triangle, so ∠EDF = ∠DFE = 3x°.

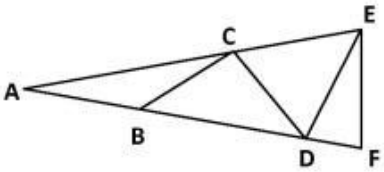
In a similar manner we can proceed starting from ΔABC and show that ∠DEC = 3x° (or we could say so directly, from symmetry).

Now looking at the large triangle AED, the sum of its angles must be 180°.

So x + 3x + 3x = 180 which gives us 7x = 180 or $x = \frac{180}{7} = 25.71°$.

Hence (3)

Try a simpler version of this question, for practice:



Q. In the figure l(AB) = l(BC) = l(CD) = l(DE) = l(EF). Also, l(AE) = l(AF). What is the measure of ∠FAE?

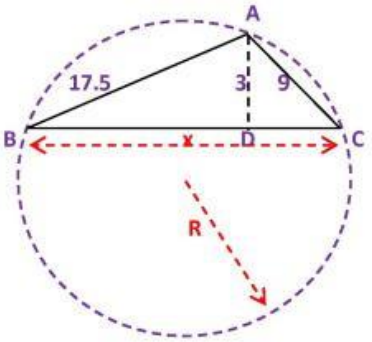
1) 15° **2) 20°** **3) 25°** **4) 30°**

Proceeding in a manner similar to that above, you should be able to get the answer as 20°.

GEOMETRY – 40

Q. In a triangle ABC, the lengths of the sides AB and AC equal 17.5 cm and 9 cm respectively. Let D be a point on the line segment BC such that AD is perpendicular to BC. If AD = 3 cm, then what is the radius (in cm) of the circle circumscribing the triangle ABC? **(CAT 08)**

1] 17.05 2] 27.85 3] 22.45 4] 32.25 5] 26.25



Consider the adjacent figure.

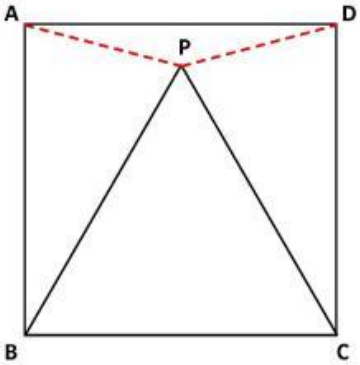
At first glance there seems to be no reasonable way to find the circumradius without significant effort. However a little thought will remind us that there are actually very few theorems or formulae involving the circumradius – and one of them is the formula for the area of the triangle.

So let us start by assuming the circumradius to be R. Let the length of BC be x as shown. Now we can say that the area of triangle ABC should be $\frac{1}{2}$ * l(BC) * l(AD). But also the area of the triangle must be $\frac{l(\mathrm{AB}) * l(\mathrm{BC}) * l(\mathrm{AC})}{4R}$. Equating the two, we get:

should be $\frac{1}{2} * \cancel{x} * 3 = \frac{17.5 * \cancel{x} * 9}{4R}$ => R = 26.25. **Hence (5)**

Q. An equilateral triangle BPC is drawn inside a square ABCD. What is the value of the angle APD in degrees? **(CAT 06)**

1] 75 2] 90 3] 120 4] 135 5] 150



This is a fairly simple question if one manages to visualise the figure properly – in addition to the square and equilateral triangle that are already mentioned in the question, we should realise that there are also 3 other triangles. ΔAPD is very easy to classify – it is an isosceles triangle. But what about ΔAPB and ΔDPC?

Since ABCD is a square, l(AB) = l(BC). Also since BPC is an equilateral triangle, l(BP) = l(BC). Putting these together, we find that l(AB) = l(BP), which means that ABP is an isosceles triangle.

Also, we know that since ABCD is a square, ∠ABC = 90°. And since BPC is an equilateral triangle, ∠PBC = 60°. This gives us ∠ABP as 30°. So ∠PAB + ∠APB = 150°. But since ABP is isosceles each of these angles must be 75°.

In a similar manner we can show that ∠DPC = 75°. And ∠ BPC = 60°.

Thus ∠ APD must be 360 – 75 – 75 – 60 = 150°. **Hence (5)**

GEOMETRY – 41

Let's spend a while looking at some 3-Dimensional Geometry for a change; beginning with a question type which was exceedingly popular in CAT in the 90s:

Cubes and Cubelets

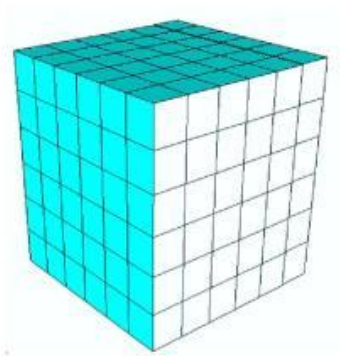
Q. A cube of side 6 cm is painted on all its faces, and then cut into identical smaller cubes of side 1 cm each. How many of the new cubes will have

(a) exactly three faces painted?

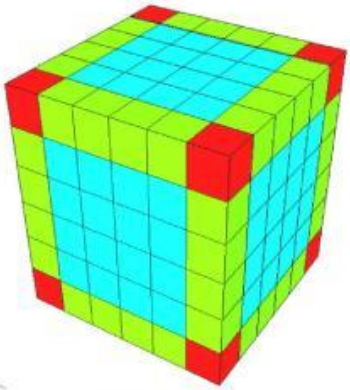
(b) exactly two faces painted?

(c) exactly one face painted?

(d) no faces painted?

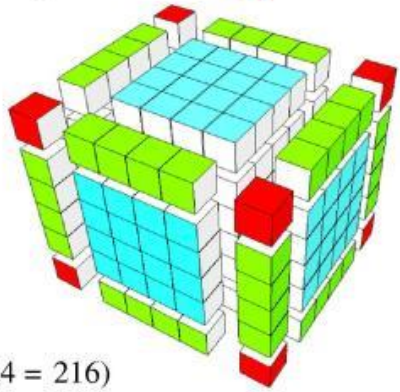


Let us look at the figure alongside. First of all we can note that the original cube will be cut into 6^3 i.e. 216 smaller cubes as shown. Now, in any of the smaller cubes, we can quickly realise that not more than 3 of the faces will be painted. Let's try and identify which of the cubes will have 3, 2 1 or 0 faces painted,



(a) The only cubes with 3 faces painted will be **at the vertices.** There will be **8** such cubes (marked with red in the figure)

(b) The cubes with exactly 2 faces painted will be those **on the edges but not on the vertices** (marked in green in the figure). On each edge, there will be (6 – 2) = 4 such and hence, since a cube has 12 edges, there will be in all 12 * 4 = **48** cubes.



(c) The cubes with exactly 1 face painted will be the ones **on the faces but not on the edges** (marked with blue in the figure). On each face, there will be $(6 - 2)^2$ = 16 such, and hence, since a cube has 6 faces, there will be in all 6 * 16 = **96** cubes

(d) Cubes with no faces painted will be **in the interior of the figure**. If we remove the outermost layer of small cubes (see the "exploded" view alongside), the remaining i.e. interior part will be a cube of side 6 - 2 i.e 4 and hence there will be 4^3 i.e **64** cubes with no sides painted.

(Note that as expected, the sum of all these results = 8 + 48 + 96 + 64 = 216)

In general, if an n x n x n cube is painted and then cut into n^3 smaller cubelets, then:

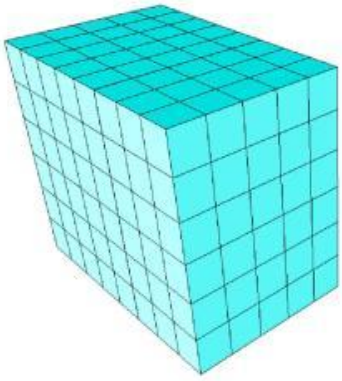
- The number of cubelets with **3 faces** painted will be **8**
- The number of cubelets with **2 faces** painted will be **12 * (n – 2)**
- The number of cubelets with **1 faces** painted will be **6 * $(n - 2)^2$**
- The number of cubelets with **0 faces** painted will be **$(n - 2)^3$**

GEOMETRY – 42

Cubes and Cubelets: Variants

Since the basic cube-cutting problem can be solved very simply by reducing to a standard formula, direct questions of this form have become less popular now that they used to be. However, the problem could easily be altered in various ways to make it slightly trickier. For example, a cuboid could be given instead of a cube:

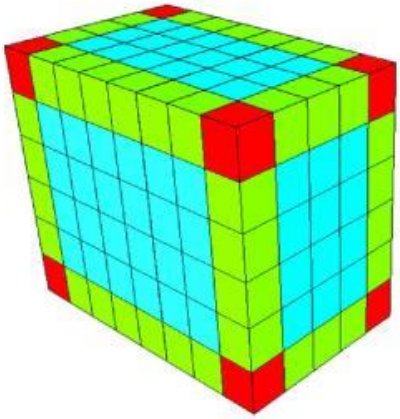
Q. A cuboid of 8 cm x 6 cm x 5 cm is painted on all its faces, and then cut into identical smaller cubes of side 1 cm each. How many of the new cubes will have
(a) exactly three faces painted?
(b) exactly two faces painted?
(c) exactly one face painted?
(d) no faces painted?



The original cuboid will be cut into 8 * 6 * 5 = 240 smaller cubes.

(a) There will again be **8** cubes with 3 faces painted

(b) For cubes with 2 faces painted – there will be 4 edges containing 6 such cubes, 4 containing 4 cubes, and 4 containing 3 cubes so in all 4 (6 + 4 + 3) = 52 cubes have 2 faces painted.



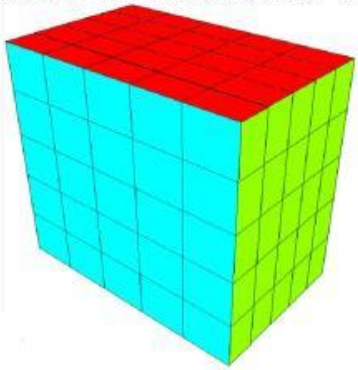
(c) For cubes with exactly 1 face painted - there will be 2 faces containing 6 * 4 such cubes, 2 containing 6 * 3 cubes, and 2 containing 4 * 3 cubes; so in all 2 * (24 + 18 + 12) = 108 cubes have one face painted.

(d) For cubes with no faces painted – the interior of the figure will be a smaller cuboid with dimension 6 * 4 * 3 and hence there will be 72 such cubes with no faces painted.

(Check: 8 + 52 + 108 + 72 = 240, as expected)

Another type of variant could be colouring the different faces of the cube in different colours. Try this:

Q. A cube of side 5 is coloured with three colours (red, blue and green) such that every pair of opposite faces is the same colour and then cut into smaller cubes of side 1. Find the number of cubes having:
(a) One face blue
(b) One face red and at least one other face painted
(c) One face green and exactly one other face painted
(d) One face blue and one face red
(e) Not more than one face painted
(f) No face red
(A figure has been provided to make your visualisation easier)



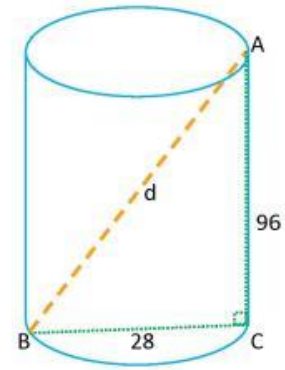
You should get these answers: (a) 50 (b) 32 (c) 24 (d) 20 (e) 81 (f) 75

GEOMETRY – 43

Another interesting type of problem is to find the shortest path from one point to another in a 3-dimensional scenario. For example, consider the following problem:

Q. A fly is sitting on the upper rim of a cylindrical jam-jar of radius 14 mm and height 96 mm. It sees a speck of jam at the opposite lower corner of the jar. What is the minimum distance it will have to fly to get the jam?

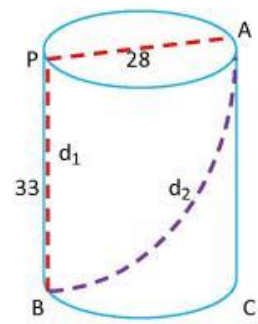
This is a relatively simple question; the shortest distance would be the straight line distance. Suppose the ant is at a point A and the jam at a point B (as shown in the adjacent figure).



In this case, as can be seen from the figure, the shortest distance will be $d = \sqrt{96^2 + 28^2} = \sqrt{10000} = 100$ mm

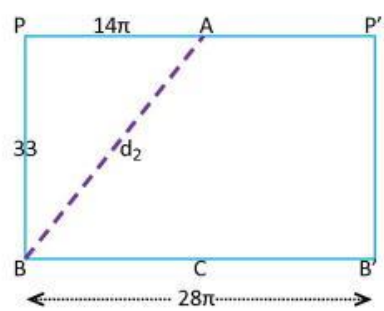
Q. An ant is sitting on the upper rim of a cylindrical jam-jar of radius 14 mm and height 33 mm. It sees a speck of jam at the opposite lower corner of the jar. What is the minimum distance it will have to crawl to get the jam?

In this case, the question is slightly trickier, as the ant has to crawl along the surface (a variant and equivalent case would be if the cylinder happened to be solid and one had to get from a point on its surface to an opposite point).



It turns out that there are two viable routes we need to consider in this case: one wherein the ant crosses the circular face of the jar diametrically and then moves vertically down the "wall" of the jar, and another where the ant moves in a curved path along the surface of the jar.

The first path, d_1, is easy to calculate – it will just be 28 + 33 = **61 mm.**



For the second path, d_2, we can use a simple trick. Let us remove the top and bottom of the cylinder and cut it open along PB. Now the resultant will be a rectangle, where B is one of the bottom corners while A is the midpoint of the top side as shown. We can find the length d_2 easily enough now, using the theorem of Pythagoras:

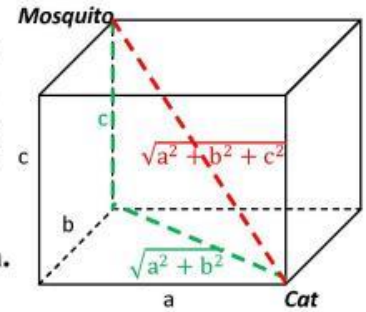
$d_2 = \sqrt{(14\pi)^2 + (33)^2} \approx \sqrt{(44)^2 + (33)^2} \approx$ **55 mm.**

So d_2 is less than d_1 and will be the shortest path route. However, this will not always be the case – in the case of a cylinder with a radius much large than its height, the path d_1 would prove shorter. In general, it is a good idea to check both cases!

GEOMETRY – 44

A related shortest-path question could be – what happens if instead of a cylinder we have a cuboid?

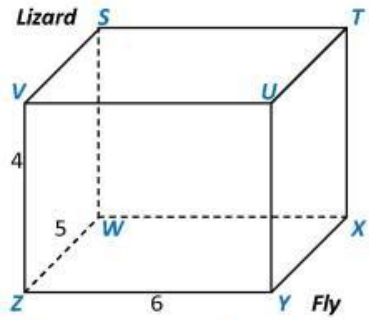
Q. A mosquito sitting at the top corner of a cuboidal room of dimensions 6 x 3 x 2 m sees a cat sleeping at the farthest bottom corner of the room. How much distance will the mosquito have to fly to bite the cat?



Again, the shortest path in a straight line is easy - the distance to be travelled will be just the body diagonal of the cuboid. In general if the dimensions of the cuboid are a, b and c, then we can show (using the Theorem of Pythagoras twice) that this distance will be $\sqrt{a^2+b^2+c^2}$

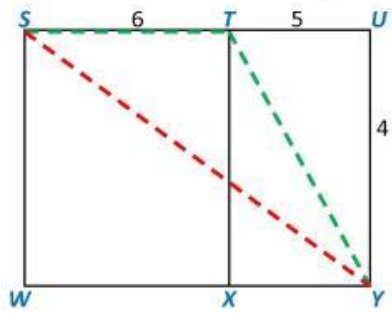
Thus in the above question, the distance will be $\sqrt{6^2+3^2+2^2}$ i.e. **7 m.**

Q. A lizard sitting at the top corner of a cuboidal room of dimensions 6 x 5 x 4 m sees a fly sleeping at the farthest bottom corner of the room. How much distance will the lizard have to crawl to catch the fly?



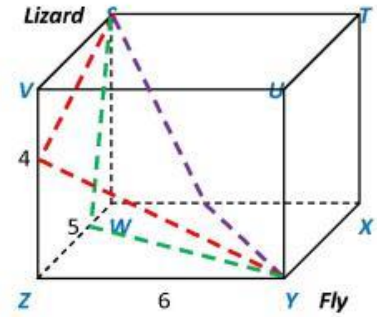
This problem is not as straightforward as it first appears.

Consider the vertices of the cuboid to be S, T, U, V, W, X, Y and Z as shown in the adjacent figure (the lizard being at S and the fly at Y)



The tempting option is for the lizard to travel along any of the routes S-W-Y or S-V-Y or S-T-Y. However, we can show that none of these will be the shortest route (for example if we consider the route S-T-Y, we can "unfold" the two adjacent sides SWXT and TXYU of the cuboid to form a rectangle SWYU – now the straight-line path SY of length $\sqrt{(6+5)^2+4^2} = \sqrt{137} \approx 11.7$ will clearly be shorter than the path STY = $6 + \sqrt{5^2+4^2} = 6 + \sqrt{41} \approx 12.4$)

However, this gives us a hint as to how we can approach the problem; there could be three such paths (created by "straightening out" any two adjacent faces) and if we check these three paths, the shortest one among them will be the shortest path of all.



Consider the adjacent figure:

The length of the red path (cutting VZ) = $\sqrt{(5+6)^2+4^2} = \sqrt{137}$

The length of the green path (cutting WZ) = $\sqrt{(4+6)^2+5^2} = \sqrt{125}$

The length of the purple path (cutting WX) = $\sqrt{(4+5)^2+6^2} = \sqrt{117}$

Clearly the last of these is the shortest, and hence the minimum distance will be $\sqrt{117} \approx 10.8$ m (the body diagonal in this case would be around 8.77 m).

In general if the dimensions of the cuboid are a, b, c where $a \geq b \geq c$, then the shortest path along the surface will be given by $\sqrt{(b+c)^2+a^2}$ (this can be proved algebraically, if we so desire)

GEOMETRY – 45

Similarity in 3 dimensions

We had observed earlier that if we can show 2 figures to be similar (i.e. their corresponding sides are all in a fixed ratio, while their corresponding angles are congruent) then **all** their linear dimensions are also in that same ratio. Furthermore, all their area dimensions are in the square of that ratio while all the volume dimensions (for 3-dimensional figures) are in the cube of that ratio.

Also, just as in the case of 2-dimensional figures, here too symmetrical, "regular" figures will automatically be similar. Any two spheres will be similar to each other, as will any two cubes or any two regular tetrahedrons. So for example, if we take two cubes of sides 4 and 20, the sides are in the ratio 1 : 5 and hence the areas will be in the ratio 1 : 25 and the volumes will be in the ratio 1 : 125. Similarly if we take two spheres with radii in the ratio 5 : 7 then their surface areas will be in the ratio 25 : 49 while their volumes will be in the ratio 125 : 343. This property can help us to solve certain questions quickly:

Q. A metal sphere of radius 5 is melted and the resultant metal is shaped into smaller spheres of diameter 2. How many such smaller spheres can be made?

1) 15 **2) 120** **3) 125** **4) 25** **5) 625**

We could say that the volume of a sphere is $\frac{4}{3}\pi r^3$ and hence the total volume of metal obtained from a sphere of radius 5 will be $\frac{4}{3}\pi (5)^3$ which is $\frac{500}{3}\pi$. Now the new smaller spheres are of diameter 2 i.e. radius 1 and hence if there are n such spheres, their total volume will be $n * \frac{4}{3}\pi (1)^3 = \frac{4n}{3}\pi$. Equating these two volumes will give us n as 125. **Hence (3)**

However we could have just used the idea of similarity to say that since the radii are in the ratio 5 : 1 the volumes will be in the ratio $5^3 : 1^3$ i.e. 125 : 1. So the larger sphere is 125 times the volume of the smaller ones and hence there will be 125 smaller spheres.

(Note that in all such questions of melting and reforming the **volumes will remain unchanged**. However, the **surface area might change drastically** so we cannot make any assumptions about it. In this case for example, the new surface area will be 5 times the original one.)

Q. A hemisphere of radius 4 is melted down and made into 108 smaller spheres of a fixed size. What will be the radius of each of the smaller spheres?

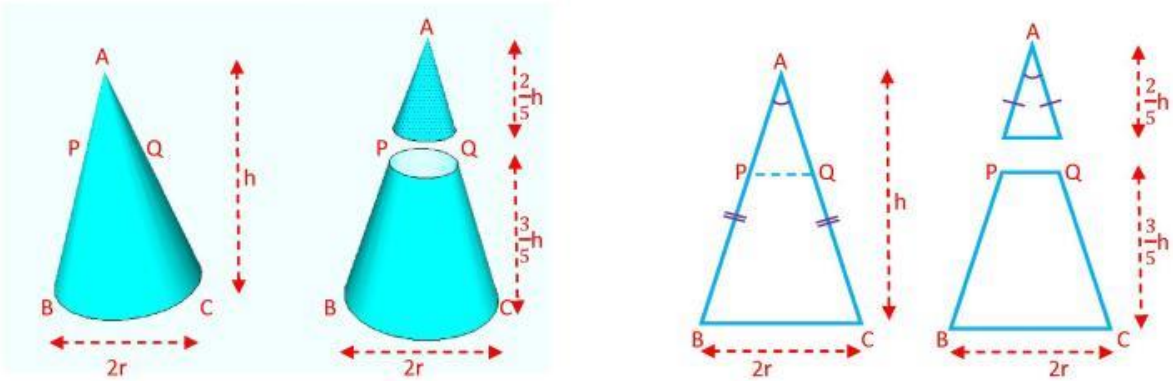
1) 1 **2) 0.33** **3) 0.5** **4) 0.66** **5) 0.16**

If the hemisphere is 108 times the volume of each small sphere, then twice the hemisphere would give us a sphere of 216 times the volume of each smaller sphere. So the radius of the larger sphere (and its hemisphere) would be $\sqrt[3]{216}$ i.e. 6 times the radius of the smaller sphere. Since the hemisphere has radius 4, the radius of each sphere must be 4/6 i.e 0.66. **Hence (4)**

GEOMETRY – 46

Similarity in the context of Cones

If we cut a cone (or indeed any pyramidal figure) into two parts by a plane parallel to its base, then it turns out that one of the parts (the one containing the vertex) will be similar to the original cone. For example, consider the following figure wherein a cone has been cut by a plane through P and Q, parallel to its base and at a height 3/5th of the original height of the cone. The figures on the right show the cross-sections of the same.



Looking at the triangles APQ and ABC, they are both isosceles. So two pairs of sides are automatically in the same ratio → $\frac{AP}{AB} = \frac{AQ}{A}$. Also the angle between them is the same, which makes the triangles similar. The heights being in the ratio $\frac{2}{5}$, the sides also will be in the same ratio. Thus PQ also will be $\frac{2}{5}$ of BC and hence the radius of the smaller cone will also be $\frac{2}{5}$ of that of the large cone.

This straightaway tells us that the two cones are similar. Also, their surface areas will be in the ratio $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$ while their volumes will be in the ratio $\left(\frac{2}{5}\right)^3 = \frac{8}{125}$.

Q. A conical vessel of height 420 cm is initially placed with its vertex downward
(a) The vessel is filled to a height of 280 cm with water. What fraction of the volume is filled?
(b) The vessel is filled to a height of 210 cm with water. What percentage of the volume is filled?
(c) The vessel is initially filled to a height of 66 cm using x gallons of water. How many gallons must be added so that that height of water in the vessel becomes 99 cm?
(d) The vessel is filled to half its capacity. What will be the height of water in it?

(a) The water filled in the vessel, in the given orientation, forms a cone similar to the original cone. So if the height of this cone is 280 (i.e. $\frac{2}{3}$ of the original) then the volume will be $\left(\frac{2}{3}\right)^3 = \mathbf{\frac{8}{27}}$ **of the original**

(b) This time the height is $\frac{1}{2}$ of the original, and so the volume will be $\left(\frac{1}{8}\right)^3 = \frac{1}{8} =$ **12.5% of the original**

(c) We need not use the original cone; since the two cones formed in the problem are similar to each other with heights in the ratio $\frac{3}{2}$, their volumes must be in the ratio $\frac{27}{8}$ = 3.375 so the total volume must be 3.375x gallons which means we need to add **2.375 gallons**

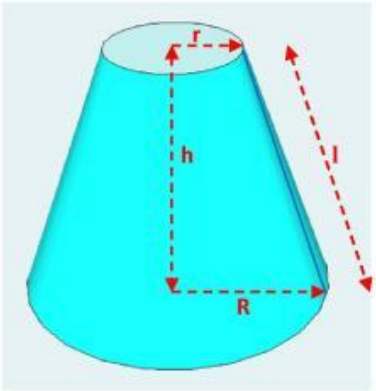
(d) If the volumes are in the ratio $\frac{1}{2}$, the heights must be in the ratio $\sqrt[3]{\frac{1}{2}}$ and so the height should be $\mathbf{\frac{420}{\sqrt[3]{2}}}$

GEOMETRY – 47

The Truncated Cone: The Frustum

As we saw in the previous post, when we cut a cone parallel to its base, one of the parts formed is similar to the original in shape. The other part, called the frustum of a cone, is what we shall concentrate on today.

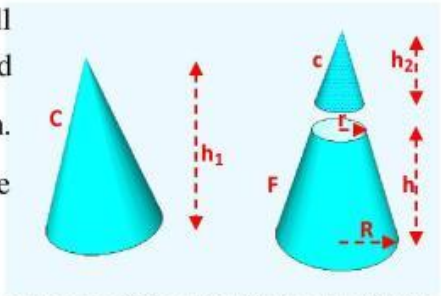
The frustum of a cone can be uniquely described using three values; the radii '**R**' and '**r**' of the circles at the base and the top, and the height '**h**', the perpendicular distance between the two parallel faces.



In addition, we can also define the "slant height", denoted by '**l**' in the figure, which can easily be calculated by $\sqrt{(R-r)^2+h^2}$.

The formula for the volume and surface area for a frustum can be calculated by looking at a frustum as the difference of two similar cones. It turns out that the formula we arrive at, while fairly complex, can be easily remembered by relating them to the corresponding formulae for the volume and surface area of a cone:

Notice that the radii, heights and slant heights of the two cones are all in proportion. Thus if we define the height of the large cone as h_1 and that of the smaller as h_2, we can say that $\frac{h_1}{h_2}=\frac{R}{r}$. Also $h_1 - h_2 = h$. Using this we can find the heights of the two cones, and since we already have their radii, we can find their volumes and surface areas.



Now, we can find out the volume of the frustum by subtracting the volume of the smaller cone from that of the larger one. for the surface area, we need a little more care – we have to add the bases surface areas of both cones, but for the curved surface areas we need to add that of the larger cone but subtract that of the smaller cone.

Eventually we can arrive at the following formulae for the volume and surface area of a frustum:

Volume of a Frustum $= \frac{1}{3}\pi h (R^2 + Rr + r^2)$	(Compare with $\frac{1}{3}\pi h r^2$ for a cone)
Base Surface Area of a Frustum $= \pi (R^2 + r^2)$	(Compare with πr^2 for a cone)
Curved Surface Area of a Frustum $= \pi l (R + r)$	(Compare with $\pi r l$ for a cone)

However, this turns out to be a pretty cumbersome set of formulae; even if memorised, they take a fair bit of time to apply. An alternative approach could be to directly treat the frustum as the difference of two cones; in the next post we shall see how to do this.

GEOMETRY – 48

Q. A conical vessel of height 420 cm is initially placed with its vertex upward
(a) The vessel is filled to a height of 280 cm with water. What fraction of the volume is filled?
(b) The vessel is filled to a height of 210 cm with water. What percentage of the volume is filled?
(c) The vessel is filled to half its capacity. What will be the height of water in it?

In this kind of situation, if we need to find the volume of the frustum, we can start by finding out the volume of the similar cone and subtracting it from the original cone:

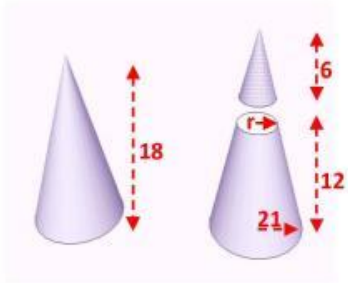
(a) The smaller cone has height 140 i.e. $\frac{1}{3}$ of the original. So its volume will be $\frac{1}{27}$ of the original which means the frustum will have a volume $\mathbf{\frac{26}{27}}$ **of that of the original**.

(b) The smaller cone has height 210 i.e. $\frac{1}{2}$ of the original. So its volume will be $\frac{1}{8}$ of the original which means the frustum will have a volume $\frac{7}{8}$ i.e. **87.5% of that of the original**.

(c) The smaller cone has to have a volume of $\frac{1}{2}$ of the original. So its height will be $\frac{1}{\sqrt[3]{2}}$ of the original which means the frustum will have a height of $\mathbf{(1 - \frac{1}{\sqrt[3]{2}}) * 420}$ i.e. approximately **86.6 cm**.

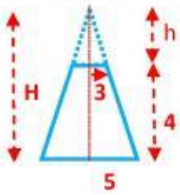
Q. A right circular cone of base radius 21 and height 72 is cut parallel to its base at a distance of 24 cm from its vertex. Find the volume and the surface area of the frustum so formed.

A little thought will show us that $\frac{r}{21}$ should be equal to $\frac{24}{72}$ which gives us r = 7. The slant height of the larger cone will be $\sqrt{24^2 + 7^2} = 25$ while that of the larger will be $\sqrt{72^2 + 21^2} = 75$ Now the volume of the smaller cone = $\frac{\pi}{3}$ (7* 7*24) = 1232. But since the height of the larger cone is thrice that of the smaller one, its volume will be 27 times as large and that of the frustum will be 26 times as large = 26 * 1232 = **32032**



The base surface area of the frustum will just be the sum of the bases of the two cones = $\pi * (R^2 + r^2)$= $\pi * (21^2 + 7^2)$ = **1540** while the curved surface area will be the difference of the curved surface areas of the two cones = π (RL – rl) = π (21 * 75 – 7 * 25) = **4400**

Which brings us to the question: what if the dimensions of the cones are not given, and we are provided only with the dimensions of the frustum? In that case too, we can use similarity to figure out the dimensions of the cones and proceed as in the above example; let us see a simple example of this. Suppose we have a frustum with base radii as 3 and 5 and height 4 and we need to find its volume:



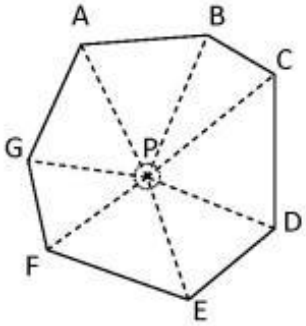
Let us extend the frustum as shown by adding a small cone. Let the height of the overall figure be H and that of the small added cone be h. Thus H = h + 4. Using similar triangles, we can see that $\frac{h}{h+4} = \frac{3}{5}$ which give us h = 6 and H = 10. After this the problem can easily be solved!

GEOMETRY – 49

Polygons

A polygon is a closed figure formed by straight lines. For example, a triangle is a polygon formed by three straight lines, while a quadrilateral is a polygon formed by four straight lines. Five lines would form a pentagon, six a hexagon, seven a heptagon, eight an octagon, nine a nonagon and ten a decagon. (There are names for figures with more sides as well – for example, a dodecagon has 12 sides, while an icosagon has 20 – but from an exam point of view, we should know the terminology up to ten sides)

Let us find the sum of all the interior angles of a polygon with 'n' sides.



Consider the seven-sided polygon (or heptagon) in the figure. Let us take a point P somewhere in the interior of the heptagon and join it to all the vertices of the heptagon. Seven triangles will be formed, which have one common vertex P. The sum of the angles of these triangles will be 7 x 180 = 1260. This includes 7 angles at point P (which add up to 360°) as well as 14 other angles (which add up to the required 7 angles of the septagon). So the sum of the angles of the septagon will be 1260 – 360 = 900°.

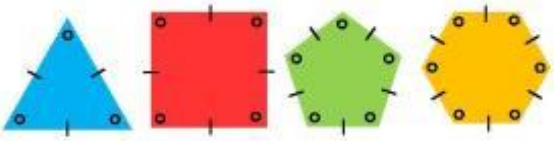
In the general case, if we had an n-sided polygon, we could conclude that there would be n triangles formed, and the sum of the angles of these n triangles would be 180n°. Again, the central angle (which would need to be subtracted) would be 360°. Consequently, the sum of all the angles of the polygon would be 180n – 360 = (n – 2) * 180°. Thus, for example, the sum of all the interior angles of a pentagon would be 3 * 180 = 540° while that for a decagon would be 8 * 180 = 1440°.

Note that for every interior angle, there would be a corresponding exterior angle, and that the sum of an interior-exterior pair would be 180°. So the sum of all the interior and exterior angles would be 180n°. As the interior angles sum up to 180 (n – 2)°, the sum of the exterior angles will be 180 n - 180 (n – 2) = 360°. Remarkably, this value is independent of n; the sum of the exterior angles for any polygon, irrespective of the number of sides, is always 360°.

Regular Polygons

A regular polygon is a special case of a polygon in which

I. all the sides are equal **and**
II. all the angles are equal.



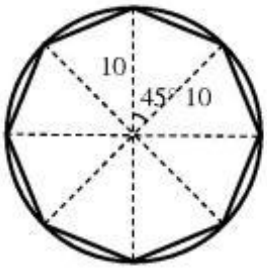
For example a regular triangle would be equilateral while a regular quadrilateral would be a square. (Note that both conditions are necessary! A quadrilateral with only all sides equal (a rhombus) or only all angles equal (a rectangle) will not be regular.)

We already know that the sum of all the interior angles of an n-sided polygon is (n – 2)180° while the sum of the exterior angles is 360°. So in a regular polygon with n sides, each interior angle will be $\frac{(n-2)}{n} * 180$ while each exterior angle will be $\frac{360}{n}$.

GEOMETRY – 50

An interesting fact is that any regular polygon can be inscribed in a circle.

Q. Find the area of a regular octagon inscribed in a circle of radius 10



Consider a regular octagon inscribed in a circle, as shown in the adjacent figure. It can be divided into 8 isosceles triangles as shown, in each of which the two congruent sides have length equal to the radius i.e. 10 units, while the angle between them is $\frac{360}{8} = 45°$.

We can remember that the area of a triangle with 2 sides a and b and included angle c will be given by $\frac{1}{2} * a * b * \sin c$. The area of each such isosceles triangle will therefore be $\frac{1}{2} * 10^2 * \sin 45 = \frac{1}{2\sqrt{2}} * 100$ and the area for the octagon will therefore be $8 * \frac{1}{2\sqrt{2}} * 100$ which is $2\sqrt{2} * 100 \approx 282.8$.

The above logic can be extended to any regular polygon of n sides inscribed in a circle of radius r. We will get n isosceles triangles, each with two adjacent sides of length r and the angle between them $\frac{360}{n}$. Thus the area of each such triangle will be $\frac{1}{2} * r^2 * \sin \frac{360}{n}$.

Thus the area of the polygon can be expressed as $\boxed{\frac{nr^2}{2} \sin \frac{360}{n}}$

Let's solve another question based on this logic:

Q. Ten circles are drawn, with radii 1, 2 … 10. Inside each circle a regular dodecagon (12-sided polygon) is inscribed. Find the sum of the areas of the dodecagons.

1) 385 **2) 385 π** **3) 1155** **4) 1155 π**

If the circles have radii 1, 2, 3, …, 10, then the corresponding regular dodecagons will have sides in the ratio 1 : 2 : 3 : … : 10 as well. Being similar figures, their ares will be in the ratio 1 : 4 : 9 : … : 100. Let us first calculate the area for the smallest dodecagon (which is inscribed in the circle of radius 1). The area will be $\frac{nr^2}{2} \sin \frac{360}{n} = \frac{12*1^2}{2} \sin \frac{360}{12} = 6 \sin 30 = 3$. Hence the total area of all the dodecagons will be $3 * (1 + 4 + 9 + \ldots + 100) = 3 * \frac{(10)(11)(21)}{6} = 1155$. Hence 3.

Alternatively, of course, we could have visualised the situation – each dodecagon will be very slightly smaller than the corresponding circle. The sum of the areas of the ten circles of radius 1, 2,…10 will be $\pi (1^2 + 2^2 + \ldots + 10^2) = 385\pi$ (which is approximately 1200). Hence our answer has to be slightly less than this, and only 1155 fits the bill.

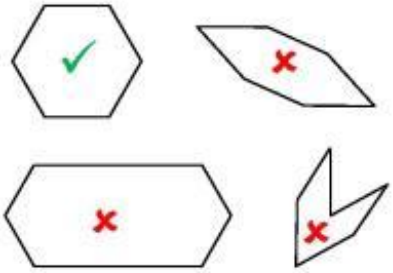
In the next few posts, we'll look at the regular hexagon and the regular octagon in more detail.

GEOMETRY – 51

The Regular Hexagon

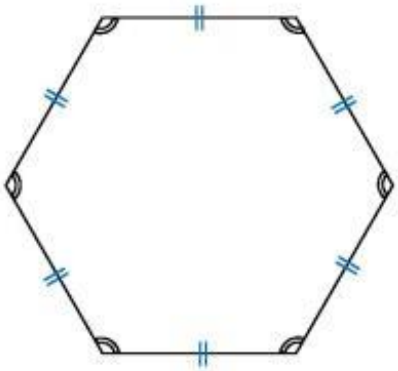
The regular hexagon is a very popular figure among question-setters, which is hardly surprising, as it is one of the most flexible figures and lends itself to a wide range of problems. It is also a figure which can often be tackled by skilful visualisation. So let us spend a little while with it:

A regular hexagon, as the name suggests, is a 6-sided figure in which all the sides are equal, and all the angles are equal.



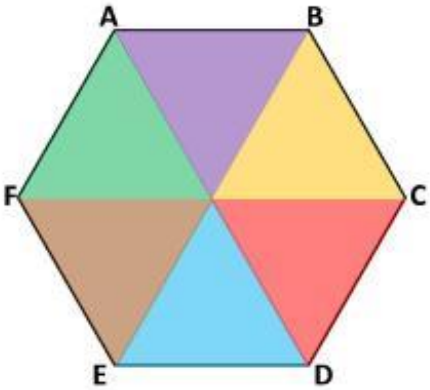
Note that either condition alone is not sufficient, we could have cases with all sides equal but not all angles, as well as cases with all angles equal but not all sides; neither of these would be a regular hexagon.

We know that the sum of all the angles of a polygon of n sides can be found as $180(n - 2)°$. In the case of a hexagon, $n = 6$ and so the sum of all the angles will be $180 * 4 = 720$.



If the hexagon is regular, then all the angles must be equal, and so each angle of a regular hexagon will measure 120°. (Correspondingly, each exterior angle will measure 60°).

An excellent starting point for visualising a regular hexagon is to see it as 6 congruent equilateral triangles (having the same side length as the hexagon) formed by drawing the 3 'long diagonals'.



With this perspective, it becomes fairly obvious that the area of the hexagon will be 6 times the area of the equilateral triangle. We have already seen that the area of an equilateral triangle of side 'x' is given by $\frac{\sqrt{3}x^2}{4}$, and so we can conclude that the area of a regular hexagon of side 'x' will be $6 * \frac{\sqrt{3}x^2}{4}$ i.e. $\frac{3\sqrt{3}x^2}{2}$.

It becomes immediately obvious that the length of each long diagonal (AD, BE and CF) is twice the side of an equilateral triangle, i.e. 2x.

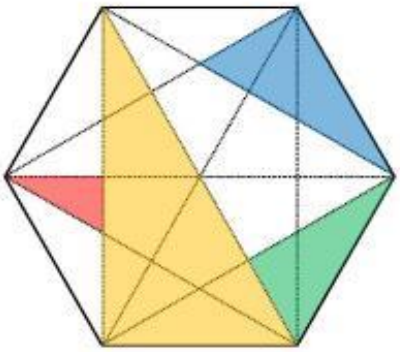
The length of the shorter diagonal (for example AE) can be seen as twice the height of the equilateral triangle, i.e. $2 * \frac{\sqrt{3}x}{2} = \sqrt{3}x$. Alternatively, we could realise that AFE is a 30-30-120 isosceles triangle, which means that the AE will be $\sqrt{3}$ times $AF = \sqrt{3}x$.

In the next post, we'll try some visualisation of sub-figures within a regular hexagon:

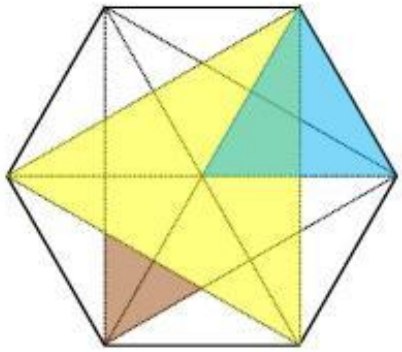
GEOMETRY – 52

If we draw all the diagonals of a regular hexagon, we can observe a number of other standard figures lurking within. For example we can see

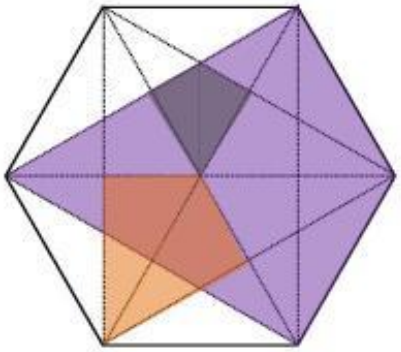
Right angled triangles…



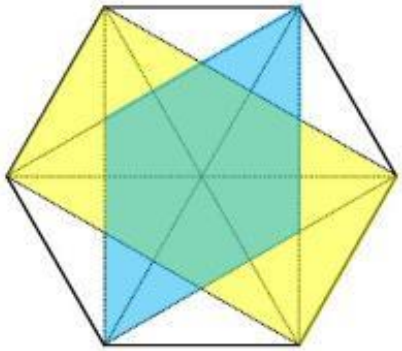
Equilateral triangles



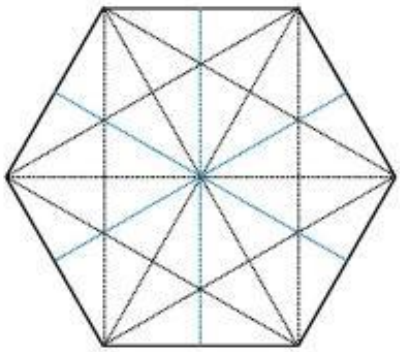
Kites…



Rectangles, Rhombi and a Hexagon

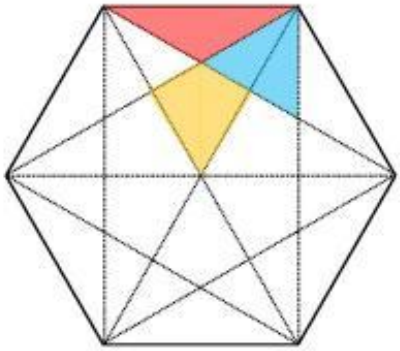


This plethora of figures makes a regular hexagon one of the best figures to practice visualisation. And it doesn't end at this; for example if we draw three more lines, joining the midpoints of opposite sides, we end up with a huge number of tiny triangles (36 to be precise) all of which are congruent!



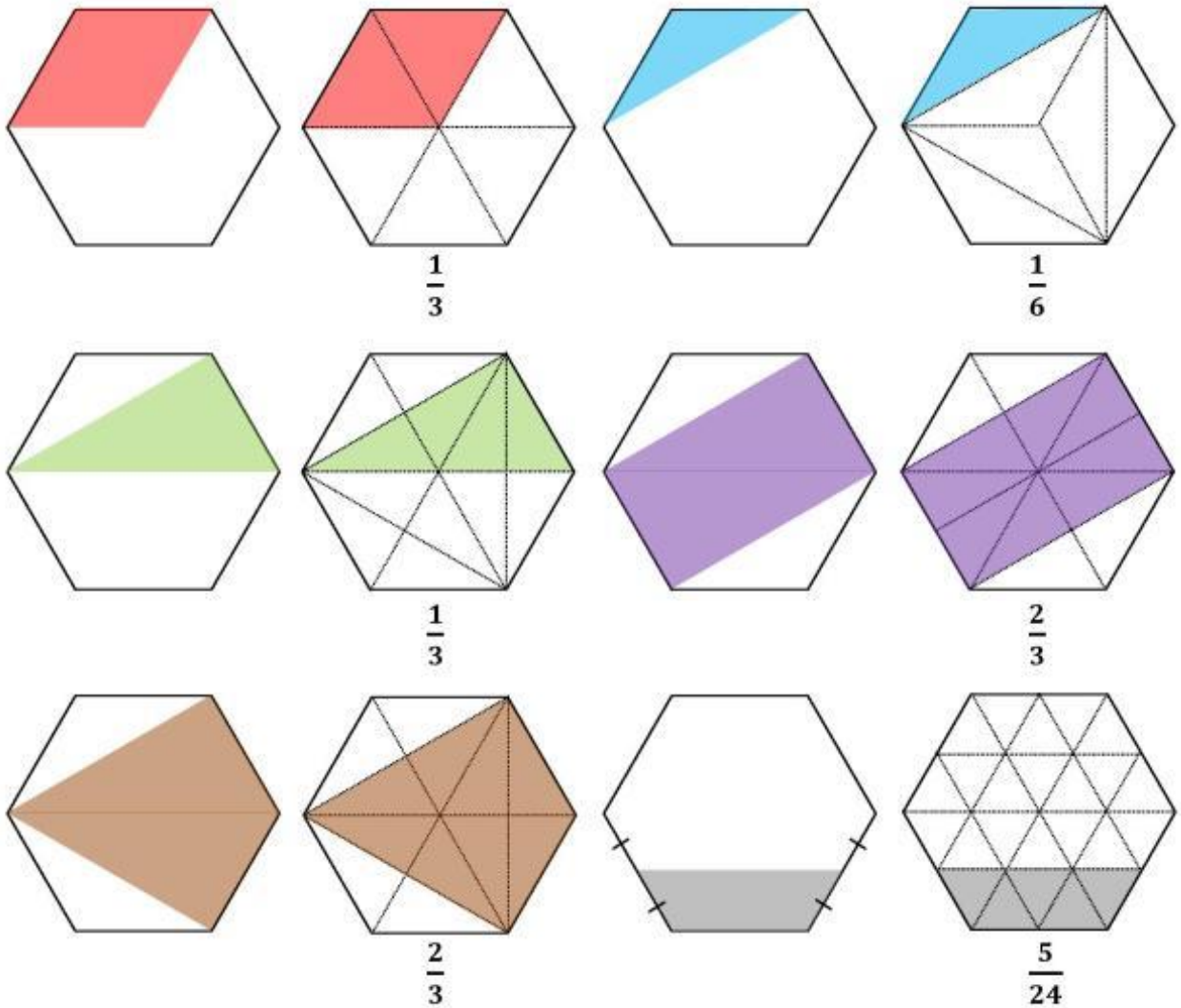
An excellent way to improve your practical Geometry skills is simply to try and find the relation between various sub-figures in a regular hexagon.

For example, try to prove (both through formulae and through visualisation) that each of the three figures alongside is equal in area.

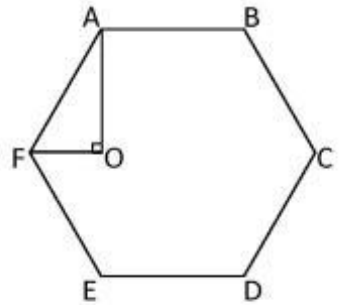


GEOMETRY – 53

With a little practice, given a smaller sub-figure within a regular hexagon, we can identify what fraction of the hexagon's area it will occupy, just by visualisation. For example:

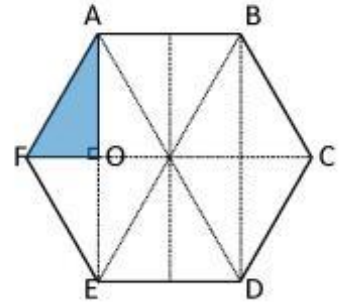


Let's look at a question from CAT:



Q. In the figure, ABCDEF is a regular hexagon and ∠AOF = 90°. FO is parallel to ED. What is the ratio of the area of the triangle AOF to that of the hexagon ABCDEF? **CAT 2003**

1] $\frac{1}{12}$ **2] $\frac{1}{6}$** **3] $\frac{1}{24}$** **4] $\frac{1}{18}$**

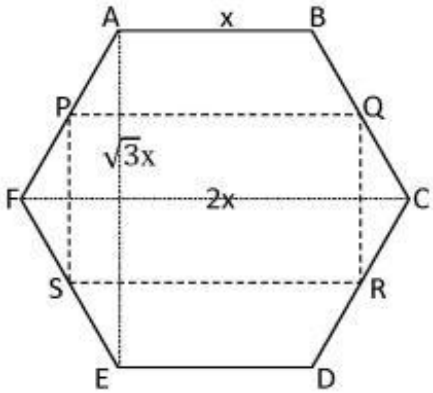


We can clearly see from the adjoining figure that the area of the given triangle is $\frac{1}{12}$ of that of the hexagon. Hence (1).

GEOMETRY – 54

Q. In a regular hexagon, a rectangle is drawn by joining the midpoints of four of the sides. What is the ratio of the area of the rectangle to that of the hexagon?

1] 1 : $\sqrt{3}$ **2] 1 : 2** **3] $\sqrt{3}$: 2** **4] 2 : 3**



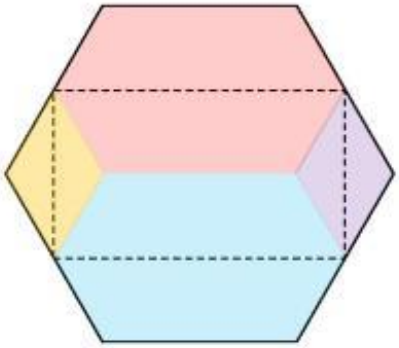
If we draw a rectangle as instructed, the resultant figure will look as shown alongside.

Suppose the side of the hexagon, l(AB) is "x". Then l(FC) the long diagonal, will be 2x, while l(AE), the shorter diagonal, will be $\sqrt{3}x$.

Now in trapezium ABCF, PQ is a median and hence l(PQ) will be $\frac{x+2x}{2} = \frac{3x}{2}$. Also in triangle AFE, by mid-point theorem, l(PS) will be half of l(AE) = $\frac{\sqrt{3}x}{2}$.

So the area of rectangle PQRS will be $\frac{3x}{2} * \frac{\sqrt{3}x}{2} = \frac{3\sqrt{3}x^2}{4}$. We know that the area of the hexagon is $\frac{3\sqrt{3}x^2}{2}$. So the area of the rectangle is half that of the hexagon and the required ratio is 1 : 2. Hence (2).

However, visualisation could give us an alternative approach:

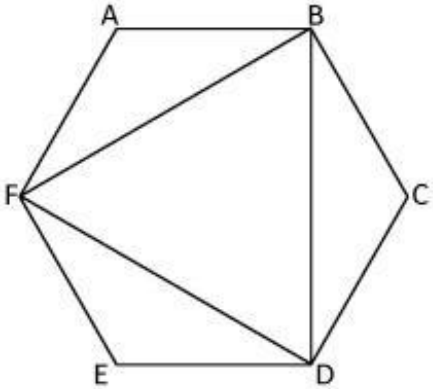


Let's imagine the sides of the rectangles to be creases and just fold the figure along those creases.

It is clear that the area within the rectangle is equal to the area outside the rectangle but within the hexagon. In other words, the area of the rectangle is half that of the hexagon. Hence (2)

Here's a CAT question for you to try; solve it using both approaches (formula and visualisation)

Q. Let ABCDEF be a regular hexagon. What is the ratio of the area of the triangle ACE to that of the hexagon ABCDEF? **CAT 2003**



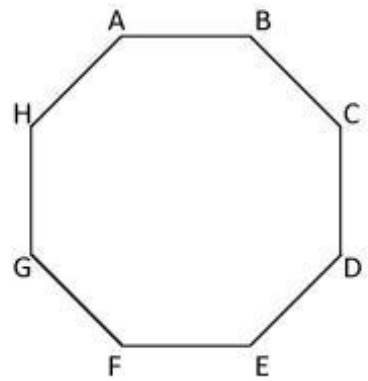
1] $\frac{1}{3}$ **2] $\frac{1}{2}$** **3] $\frac{2}{3}$** **4] $\frac{5}{6}$**

GEOMETRY – 55

The Regular Octagon

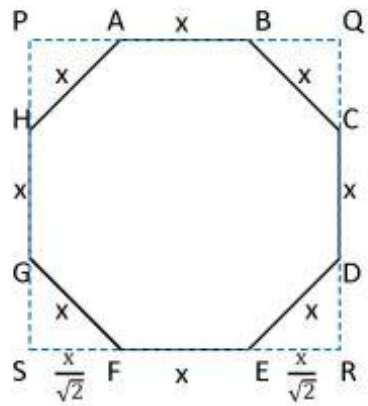
While not as popular among exam-setters as the regular hexagon, the regular octagon is also a useful figure to be comfortable with.

A regular octagon, as the name suggests, is an eight-sided figure with all sides and all angles equal (as shown alongside).



We know that the sum of all the angles of a polygon of n sides can be found as $180(n - 2)°$. In the case of an octagon, $n = 8$ and so the sum of all the angles will be $180 * 6 = 1080$. If the octagon is regular, then all the angles must be equal, and so each angle of a regular octagon will measure 135°. (Correspondingly, each exterior angle will measure 45°).

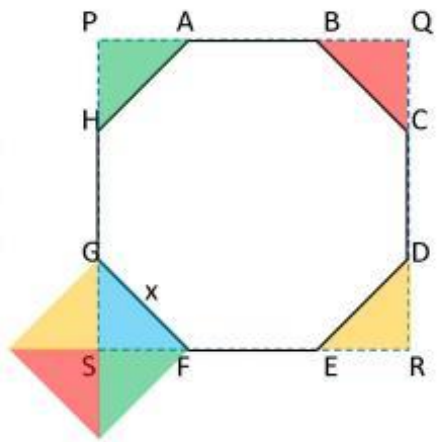
We could find the area of a regular octagon of side 'x' by dividing the octagon into smaller sub-figures. But an easier way to do it would be to enclose it in a square, as shown alongside.



Note that each of the four small triangles at the corners will be a 45° - 45° - 90° triangle. Since each of them has a hypotenuse x, the side will be $\frac{x}{\sqrt{2}}$. So the side of the square will be $\frac{x}{\sqrt{2}} + x + \frac{x}{\sqrt{2}} = x(1 + \sqrt{2})$

The area of the square will therefore be $\left[x(1 + \sqrt{2})\right]^2$
$= x^2(3 + 2\sqrt{2})$.

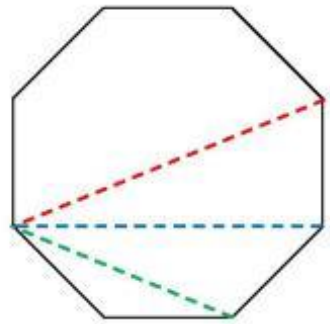
To find the area of the octagon, all we need to do is subtract the area of the four small triangles. Again, visualisation can help here as the four triangles can be combined to form a square of side 'x'. So their total area would be x^2.



Thus the area of the regular octagon would be
$x^2(3 + 2\sqrt{2}) - x^2 = x^2(2 + 2\sqrt{2})$.

Exercise for the reader:

A regular octagon has diagonals of three different sizes. See if you can figure out an approach to calculate the length of each of these, for a regular octagon of side 'x'.



(Hint: start by finding the middle one, blue in the adjacent figure)

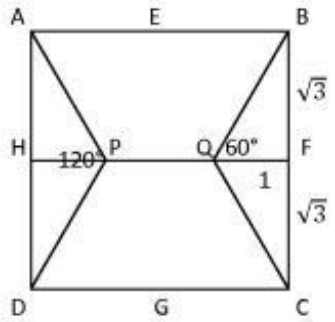
GEOMETRY – 56

In the next few posts I will cover a few random geometry questions (mostly based on questions which have appeared in CAT and other exams over the years), starting with this CAT 2008 question:

Q. Consider a square ABCD with midpoints E, F, G, H of AB, BC, CD and DA respectively. Let L denote the line passing through F and H. Consider points P and Q, on L and inside ABCD, such that the angles APD and BQC both equal 120°. What is the ratio of the area of ABQCDP to the remaining area inside ABCD?

1] $\frac{4\sqrt{2}}{3}$ **2]** $2+\sqrt{3}$ **3]** $\frac{10-3\sqrt{3}}{9}$ **4]** $1+\frac{1}{\sqrt{3}}$ **5]** $2\sqrt{3}-1$

First let us look at a conventional approach:



The figure will appear as shown alongside.

Note that BQF will be a 30° - 60° - 90° triangle

If we assume the length QF to be 1, then BF will be $\sqrt{3}$

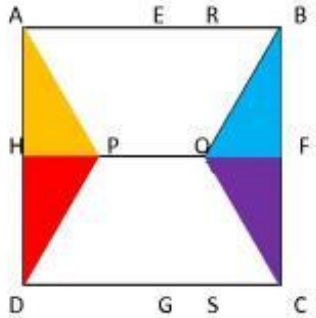
Thus BC, the side of the square, will be $2\sqrt{3}$

The area of the square will therefore be $(2\sqrt{3})^2 = 12$

The area of each triangle will be $\frac{1}{2} * 2\sqrt{3} * 1 = \sqrt{3}$. So the area of two triangles will be $2\sqrt{3}$

The required ratio will therefore be $12 - 2\sqrt{3} : 2\sqrt{3}$ which is nothing but $2\sqrt{3} - 1 : 1$. Hence (5)

Alternatively:

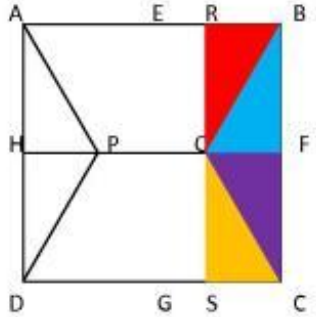


Note that BQF will be a 30° - 60° - 90° triangle

If we assume the length QF to be 1, then BF will be $\sqrt{3}$

Thus BC, the side of the square, will be $2\sqrt{3}$

So HF will also be $2\sqrt{3}$, which means HQ will be $2\sqrt{3} - 1$



Now, we can cut out triangles AHP and DHP and paste them as triangles QSC and QRB as shown.

Then the required area will be the ratio of the area of ARSD to that of RBCS, which will be nothing but the ratio of HQ to QF, i.e. $2\sqrt{3} - 1 : 1$

Hence (5)

GEOMETRY – 57

Q. A closed rectangular cuboid has a surface area of 399 cm^2 and the sum of the lengths of all the edges is 128 cm. This cuboid is inscribed in a sphere. Find the diameter of the sphere.

Let the sides of the cuboid be l, b and h. The diameter of the sphere which just encloses the cuboid would be the body diagonal i.e. $\sqrt{l^2 + b^2 + h^2}$. Thus we need to find $l^2 + b^2 + h^2$.

The surface area of the cuboid would be 2lb + 2lh + 2bh. We are told that this is 399. Also, the sum of the lengths of all the sides would be 4(l + b + h). Since this is given to be 128, (l + b + h) will be 32.

Now it is just a matter of finding a relation between the information known and the required answer; a little knowledge of algebra will give us the necessary hint. We know that

$(l + b + h)^2 = l^2 + b^2 + h^2 + 2lb + 2lh + 2bh$

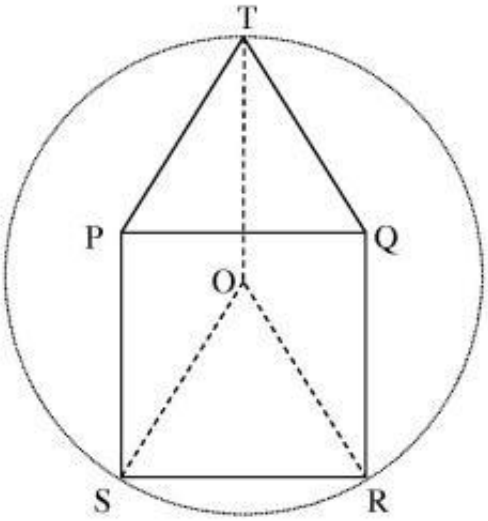
Thus $32^2 = l^2 + b^2 + h^2 + 399$

$\therefore\ l^2 + b^2 + h^2 = 1024 - 399 = 625$

Hence the required diameter = $\sqrt{l^2 + b^2 + h^2} = \sqrt{625} = 25$.

Q. A pentagon is formed by drawing an equilateral triangle PQT on the side PQ of a square PQRS, such that T lies outside the square. A circle is now drawn such that the vertices R, S and T of the pentagon lie on the circumference of the circle. If the side of the square is 5 cm, what is the radius of the circle drawn?

Let us try to use visualisation here.



Draw another equilateral triangle SOR below PTQ as shown. It will obviously be of the same size as triangle PTQ

Also SR ∥ PQ, so OT will be equal to PS and QR i.e. 5.

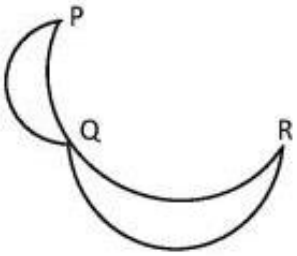
Also OS and OR will be equal to 5 as well.

So O is equidistant from all the three points R, S and T and hence is the circumcentre of the circle passing through these three points.

Thus the circumradius is 5.

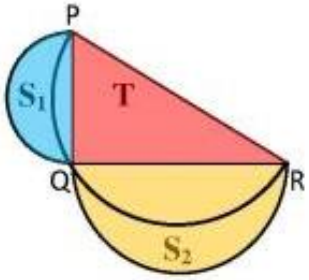
GEOMETRY – 58

Q. In the adjacent figure, three points P, Q and R are taken such that the lengths PQ, QR and PR are 5, 12 and 13 cm respectively. Using the segments PQ, QR and PR as diameters, three semi-circular arcs are constructed as shown. Find the area enclosed by the 3 arcs.

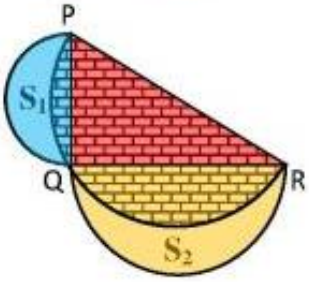


1) 30π cm^2 2) 36 cm^2 3) 30 cm^2 4) $12\pi - 6\sqrt{3}$ cm^2

The number 5, 12 and 13 should ring a bell – they form a Pythagorean Triplet. So a logical first step would be to construct the right-angled triangle.



Now, the complete figure obtained is effectively two semicircles S_1 and S_2 with diameters PQ and QR (shown in Blue and Orange alongside) and a right triangle T (shown in pink).



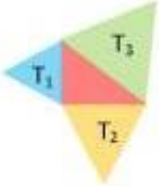
Also, the required area enclosed by the arcs can be obtained by taking this complete figure and subtracting the large semicircle S_3 with diameter PR (shaded in the figure).

Now we could note that the area of a semi-circle with diameter d is $\frac{\pi d^2}{8}$.

So the required area will be $S_1 + S_2 + T - S_3 = \frac{\pi\, 5^2}{8} + \frac{\pi\, 12^2}{8} + \frac{1}{2} * 5 * 12 - \frac{\pi\, 13^2}{8} = 30.$

Alternatively, we could extrapolate from the theorem of Pythagoras as follows: we know that two similar figures will have areas proportional to the squares of their corresponding sides. While the theorem of Pythagoras tell us that the sum of the areas of the squares constructed on the perpendicular sides of a right triangles equals the area of the square on the hypotenuse, the same could be true of any set of similar figures so drawn, be it a square, an equilateral triangle, or a semi-circle…

$Sq_1 + Sq_2 = Sq_3$



$T_1 + T_2 = T_3$

$S_1 + S_2 = S_3$

Thus the above expression $S_1 + S_2 + T - S_3$ reduces to T and hence the area enclosed by the arcs will just be equal to the area of the triangle i.e. 30.

BASE SYSTEMS - 1

Base Systems: Introduction to Base Conversion

Let's start with the basic (pun intended!) question: what is a base system?

Human beings have ten fingers. It is not surprising, then, that several civilisations independently developed number systems based on the number ten. So as we start counting from 1, 2, 3, 4…when we cross 9 we count the new number as 1 ten, and restart the count again as follows 10, 11, 12, 13. Similarly when we cross 99 (i.e. the count of tens has also crossed 9) we count it as 1 "ten-squared" (i.e. one hundred) and so on. Thus our system of counting is based on the number of ones, tens, hundreds, thousands and so on. Hence a number like 786 really implies $7 * 10^2 + 8 * 10^1 + 6 * 10^0$ and a number like 1729 would be $1 * 10^3 + 7 * 10^2 + 2 * 10^1 + 9 * 10^0$. Note that each of these terms contain the form a^m i.e. $\text{base}^{\text{index}}$ and since the base is always a power of 10, we call it a "base ten" or decimal system.

A base 9 number, then, would involve powers of 9 instead – for example 786 in base 9 (usually written as $(786)_9$ would work out to $7 * 9^2 + 8 * 9^1 + 6 * 9^0 = 567 + 72 + 6$ which is 645 in base 10. Similarly if we were to convert 143 in base 11 that is to say $(143)_{11}$ to base 10, we would get $1 * 11^2 + 4 * 11^1 + 3*11^0 = 121 + 44 + 3 = 168$ in base 10.

Thus to convert from any base to base 10, we use the powers of that base. If we write a 5-digit number $(abcde)_n$, then it will be equivalent to $an^4 + bn^3 + cn^2 + dn^1 + e$ in base 10.

To convert in the reverse direction, i.e. from base 10 to some other base n, we can successively divide the original (base 10) number by n and note down the remainders in the REVERSE order. So if we wish to convert 120 in base 10 to base 7, we could do

$120 \div 7 = 17$	remainder 1
$17 \div 7 = 2$	remainder 3
$2 \div 7 = 0$	remainder 2

$\Rightarrow (120)_{10} = (231)_7$

The logic is straightforward; when we divide 120 by 7, the remainder obtained is the number of units, while the quotient obtained is the number of complete 7s. Now when we divide it again by 7 (effectively dividing the original number by 49), the remainder will be the number of 7s left after dividing by 49 and so on. Thus we are generating, in reverse order, the units (7^0s), sevens (7^1s), forty-nines (7^2s) and so on.

Similarly if we wish to convert 87 in base 10 into base 4, we could do

$85 \div 4 = 21$	remainder 3
$21 \div 4 = 5$	remainder 1
$5 \div 4 = 1$	remainder 1
$1 \div 4 = 0$	remainder 1

$\Rightarrow (87)_{10} = (1113)_4$

BASE SYSTEMS – 2

Some useful observations about Base Systems

- **The apparent size of a number is inversely related to the size of the base in which it is written.**

  Observe that when we convert from a smaller base to a larger one, the number appears to reduce in size, while when we convert from a larger base to a smaller one, the number appears to increase. The smallest possible base is 2 (binary), but there is technically no upper limit.

- **Any number allowable in base n must contain only digits from 0 to (n-1) i.e. n possible digits.**

  When we convert to a base n, every step gives us a digit as a remainder when divided by n. Hence the maximum value of a digit can be n – 1 (the minimum value, naturally, can be 0). Thus a binary (base 2) number could contain only 0 or 1, an octal (base 8) number could have any of the digits 0, 1, 2, 3, 4, 5, 6 and 7, and of course, our standard decimal numbers consist of the ten digits 0 – 9.

  A consequence of this is that if we want to deal with a base larger than 10, we need more than the standard ten digits. We solve this difficulty by adding extra digits denoted by alphabets – for example in base 12 we would add 2 digits (a = 10 and b = 11), while in base 16 we would add 6 extra digits (from a = 10 to f = 15).

- **If a number converted to base n ends with 0, then the number will prove divisible by n.**

  Further, if a number in base n ends with 00, then it will be divisible by n^2 and so on. We can understand this from two different perspectives:

  One way to look at it is: when we convert a number from base 10 to base n we divide the number by n and write the remainder. So if the remainder is 0, then the original number must have been divisible by n. When we divide it again by n, we are effectively dividing the original number by n^2, so if we still get another 0, then the original number must be divisible by n^2.

  A second way to see could be: consider, say a 5-digit number in base n which ends with 00. This number $(abc00)_n$ could be written as $an^4 + bn^3 + cn^2 + 0n^1 + 0 = an^4 + bn^3 + cn^2$ which is clearly divisible by n^2.

  An alert observer might have realised that this is true in base 10 as well – a number ending in k zeroes will be divisible by 10^k.

- **If a number written in base n has k digits, then it lies between n^{k-1} and $n^k - 1$**

  Further, if the first digit is a, then it lies between $a * n^{k-1}$ and $(a+1) * n^{k-1} - 1$. For example, a 4 digit number in base 5 lies between 5^3 and $5^4 - 1$ i.e. between 125 and 624, while a 5 digit number in base 3 lies between 3^4 and $3^5 - 1$ i.e. between 81 and 242. Also a number like $(2abc)_6$ will lie between $2*6^3$ and $3*6^3 - 1$ i.e. between 432 and 647.

BASE SYSTEMS – 3

Converting from base 10 to some other base

The standard method i.e. division is quite time-consuming. Let's try to see if we can shorten it. Suppose we want to convert, let's say, 100 from base 10 to base 6. First of all, let us note that 100 lies between 36 i.e. 6^2 and 216 i.e. 6^3 it will be a 3 digit number in base 6. So we need to find (abc) such that a is the maximum possible number of occurrences of '36', b is the maximum possible number of occurrences of '6' in the remaining part, and c is the number of 'units' left.

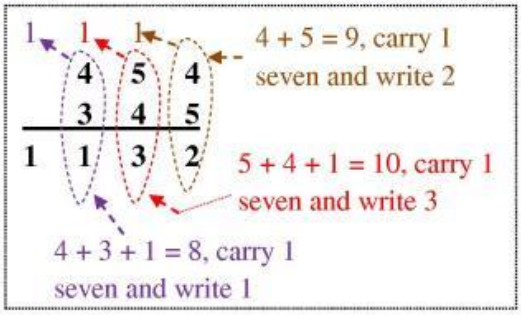
Now, '36' will occur twice, so a will be 2. The balance is 28 (from 100 – 36*2) in which '6' occurs 4 times, so b will be 4 and the balance 4 will be the number of units i.e. c. Hence $\mathbf{(100)_{10} = (244)_6}$

Similarly if we attempt to convert the number 142 from base 10 into base 5 it will be 125 (1 time) leaving 17 ⇒ 25 (0 times) leaving 17 ⇒ 5 (3 times) leaving 2 ⇒ 1 (2 times) and so $\mathbf{(142)_{10} = (1032)_5}$ and if we try to convert 128 into base 3 it will be 81 (1 time) leaving 47 ⇒ 27 (1 times) leaving 20 ⇒ 9 (2 times) leaving 2 ⇒ 3 (0 times) leaving 2 ⇒ 1 (2 times) and hence $\mathbf{(128)_{10} = (11202)_3}$

Addition of two numbers in the same base

Suppose we have to add two numbers in a certain base. The lazy way out would be to convert both to base 10, add them, and convert back, but this is definitely too time-consuming – let's see an alternative:

Suppose we have to add $(454)_7$ and $(345)_7$. Note that when adding two numbers in base 10, we "carry over" anything extra whenever the sum crosses 10. Why is this? Because in base 10, each successive digit position carries 10 times the value of the one to its right. So in base 7, we should carry over anything extra when the sum crosses 7. The box on the right demonstrates this technique.



Direct conversion between two bases, one of which is a power of the other

Consider a number $(abcdefgh)_2$ which has to be converted to base 8. The number can be expanded as $a*2^7 + b*2^6 + c*2^5 + d*2^4 + e*2^3 + f*2^2 + g*2^1 + h*2^0$. We can break this up, combining three terms at a time (from the right), as follows: $\mathbf{(a*2^7 + b*2^6) + (c*2^5 + d*2^4 + e*2^3) + (f*2^2 + g*2^1 + h*2^0)}$ which can be rewritten as $\mathbf{8^2 * (a*2^1 + b*2^0) + 8^1 * (c*2^2 + d*2^1 + e*2^0) + 8^0 * (f*2^2 + g*2^1 + h*2^0)}$.

In other words, we can combine 3 digits at a time from the number in base 2 to get a single base 8 digit.

So $\mathbf{(11010101011)_2}$ would be $\mathbf{(3253)_8}$ while $\mathbf{(1727)_8}$ would be $\mathbf{(1111010111)_2}$

Similarly we could convert from base 2 to base 16, 4 digits at a time: $\mathbf{(11010101011)_2}$ would be $\mathbf{(6AB)_{16}}$ while $\mathbf{(15D9)_{16}}$ would be $\mathbf{(1010111011001)_2}$

(We could also convert from base 3 to base 9 and vice-versa in the same manner, try it for yourself!)

BASE SYSTEMS – 4

Base Systems: Examples

Q. How many two digit numbers in base 9 get reversed when written in base 7?

1) 0 **2) 1** **3) 2** **4) 3**

Let the number in base 9 be ab. Now we are told that $(ab)_9 = (ba)_7$, i.e. $9a + b = 7b + a$ i.e. $8a = 6b$. Thus we get $\frac{a}{b} = \frac{3}{4}$. So we need two digits which are in the ratio 3 : 4 and hence $(ab)_9$ could be 34 or 68. However $(ba)_7$ also should be a valid number and hence the digit 8 cannot be there. So there is just 1 such number, i.e 34. **Hence (2)**

Q. How many numbers have exactly 3 digits when expressed in either base 5 or base 11?

1) 4 **2) 5** **3) 6** **4) more than 6**

Numbers of 3 digits in base 5 are those between 5^2 and $5^3 - 1$ i.e. between 25 and 124. Numbers of 3 digits in base 11 similarly lie between 121 and 1330. There are only 4 numbers (121-124) which satisfy both conditions. **Hence (1)**

Q. Given below is the sequence of all the natural numbers which don't contain the digit '0'. 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13… What is the 200th term of this sequence?

1) 238 **2) 240** **3) 236** **4)242**

Here, we are basically allowed to use only 9 digits, which makes it equivalent to counting in base 9. The 200th number in base 9 can be found by converting 200 from base 10 to base 9. $(200)_{10} = (242)_9$ and so the 200th term in the sequence is 242. **Hence (4)**

Q. The number $(2324X1013)_7$ is converted to base 10 and then divided by 6. It leaves no remainder. What could be the value of X?

1) 0 **2) 1** **3) 2** **4) 3**

Let us start by looking at a simpler example. Suppose we take the number $(234)_7$. What would be the remainder in base 6? We can note that $(234)_7 = 2 * 7^2 + 3 * 7^1 + 4$. But 7^2 and 7^1 both leave a remainder 1 with 6, and hence this remainder is just equivalent to 2 + 3 + 4 in base 6, i.e. 3. We can show, in general, that if we have a number $(abcd)_{n+1}$, the remainder in base n will be the same as that of a + b + c + d.

The above question, then, can be solved by noting that if $(2324X1013)_7$ leaves no remainder with 6, then 2 + 3 + 2 + 4 + X + 1 + 0 + 1 + 3 should be divisible by 6. Also $X \leq 6$. So X = 2. **Hence (3)**

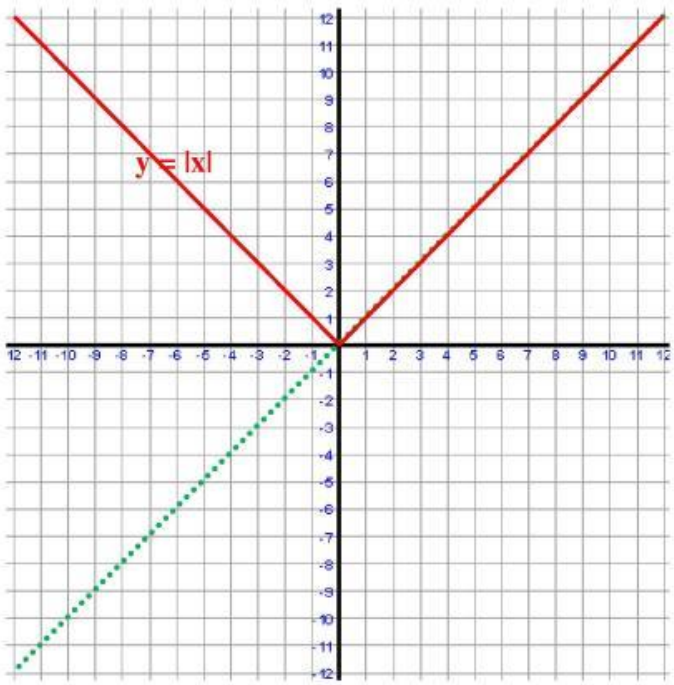
MODULUS – 1

Over the next few posts, we'll attempt to master the plotting of modulus functions.

(**Pre-requisites:** I am going to assume a working knowledge of how to plot a linear equation of the form y = mx + c. At some future point I might cover the basics of that in a series of posts, but for now I will take it as a given that the reader is aware of that concept)

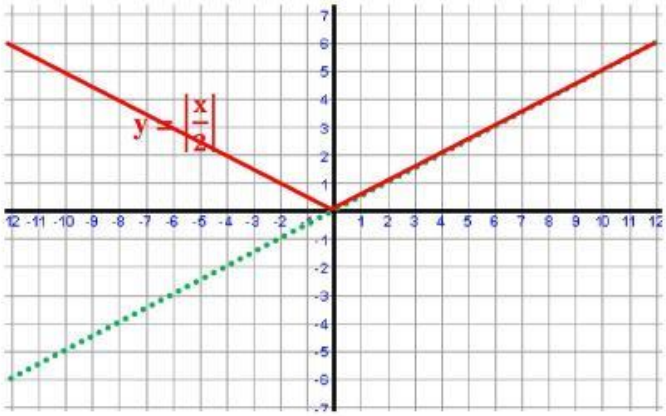
Plotting a modulus function

➢ Consider the function **y = |x|**. If we try to plot this on a co-ordinate plane, how will it look? Let us remember that a modulus function basically changes a negative value to a positive one. So we could start by plotting y = x (the dotted green line below), and wherever it drops below the x axis (i.e. wherever the function becomes negative) we reflect it above to get the modulus function (the red line):



A basic modulus function will always have the distinctive, characteristic V shape seen above. However, it could be modified by changing the various parameters of the function.

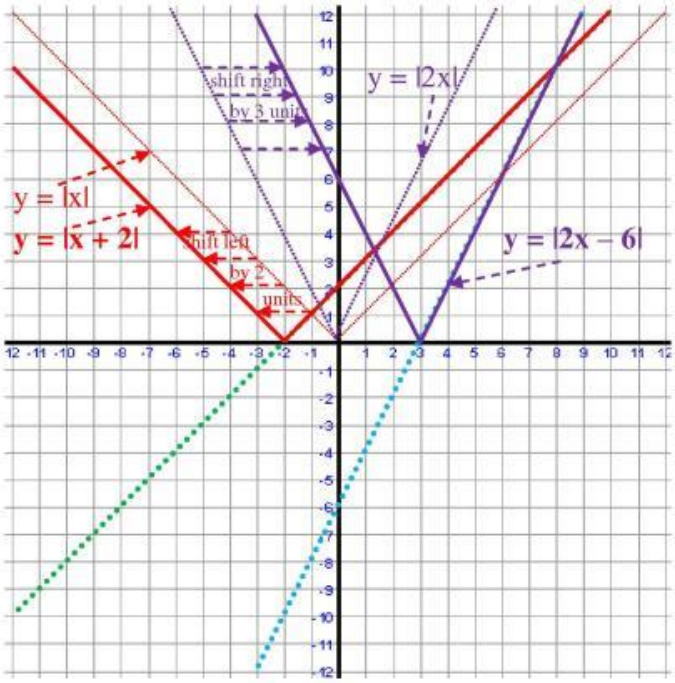
➢ For example if we changed the slope of the line to $\frac{1}{2}$ by making it $\mathbf{y = \left|\frac{x}{2}\right|}$ then it would look like this:



In the next few posts, we'll look at the effect of some other changes in the function.

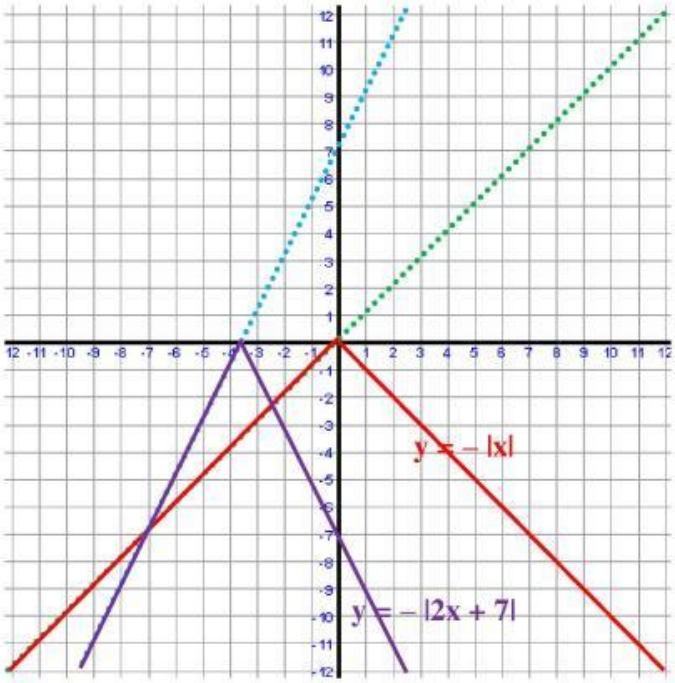
MODULUS – 2

➢ Let's try and plot **y = |x + 2|** and **y = |2x – 6|**



Note that the coefficient of x just gives the slope of the two lines comprising the modulus. Also note that the inflection point is just the point where y = 0. So for |2x – 6| it will change direction when 2x – 6 = 0 i.e. at x = 3. Using this, we can draw the modulus without having to draw the original line. Alternatively, we can just draw |x| or |2x| and shift it left or right by the appropriate amount as shown!

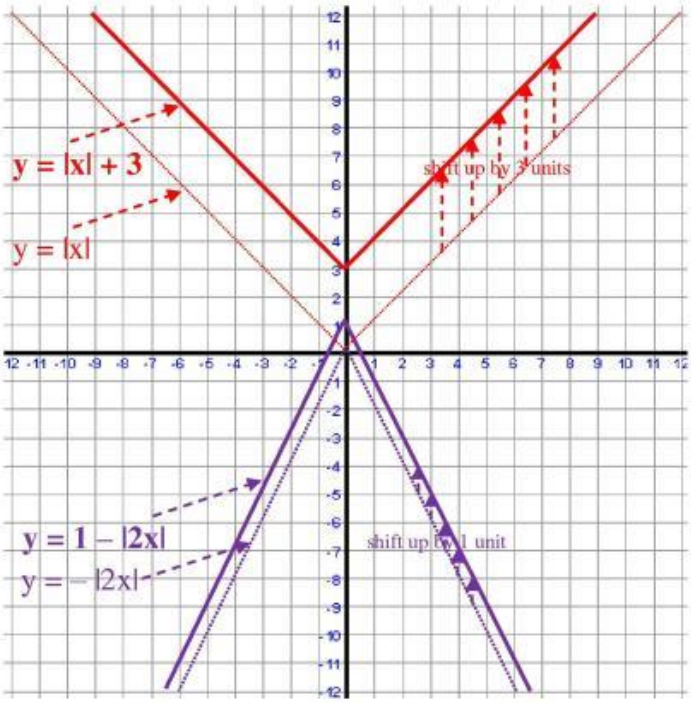
➢ How will **y = – |x|** and **y = – |2x +7|** look? Let's try to plot them:



We can observe that a negative modulus lies entirely below the x-axis (which makes sense – since |x| is always ≥ 0, - |x| will always be ≤ 0).

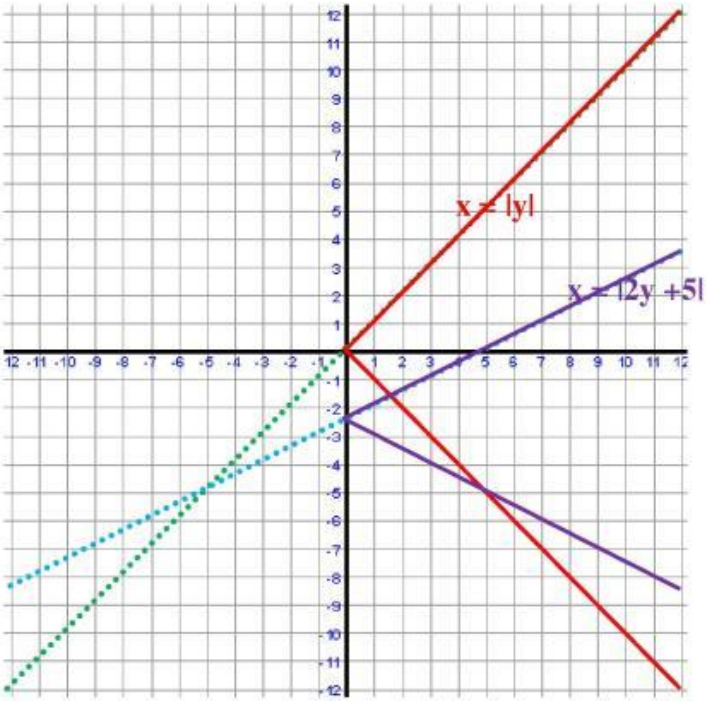
MODULUS – 3

➢ Let's try and plot **y = |x| + 3** and **y = 1 - |2x|**



Yesterday we saw that a constant **within** the modulus sign causes a **horizontal** shift in the graph. Today we can note that when there is a constant **outside** the modulus sign, it causes a **vertical** shift.

➢ Next, let's try and plot **x = |y|** and **x = |2y + 5|**



If we plot a function of the form **x = |y|**, then obviously x, being the modulus function, can never be negative. So we have the graph entirely on the right side of the Y-axis (similarly if we were to plot **x = – |y|**, the graph would lie entirely to the left of the Y-axis).

MODULUS – 4

➢ Find the maximum or minimum value, as applicable, of $\mathbf{y = |2x - 4| - 1}$ and $\mathbf{y = 1 - |\frac{x}{2} + 3|}$

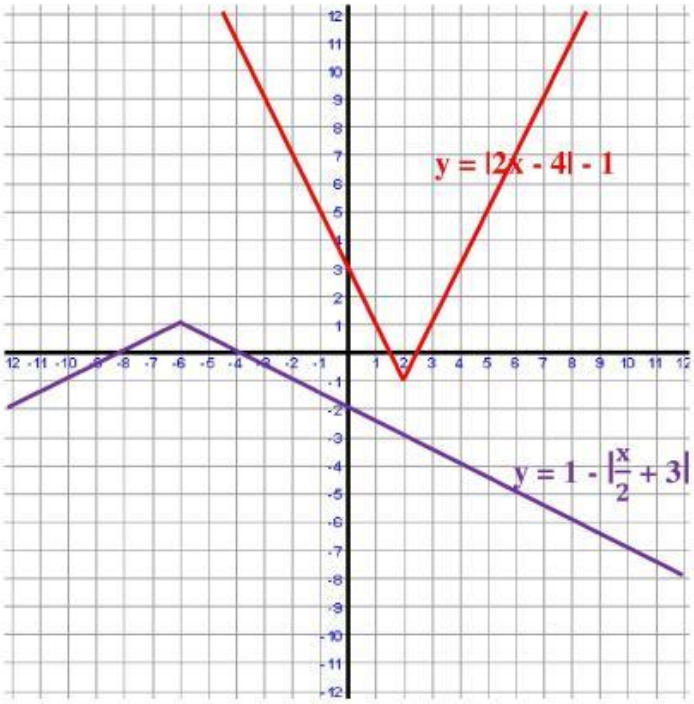
Let us now use the knowledge gained over the past few trials to analyse and plot these two curves directly – as we shall see, this will eventually enable us to find the maxima/minima without plotting the curve at all.

$\mathbf{y = |2x - 4| - 1}$

This curve is going to be upward facing (since there is a positive modulus) with a slope of 2. It will be shifted right by 2 and downwards by 1

$\mathbf{y = 1 - |\frac{x}{2} + 3|}$

This curve will be downward facing with a slope of $\frac{1}{2}$. It will be shifted left by 6 and upwards by 1.



But we could also observe that since each curve can be considered as being of the form **(modulus) + (constant)** and so will take its extremum value when the modulus is at an extremum. But a modulus always has an extremum value 0 (minimum for a positive modulus and maximum for a negative modulus), so the extremum will just be the constant part.

Hence the first curve, $\mathbf{y = |2x - 4| - 1}$ will have a **minimum value of –1 at x = 2** while the second curve, $\mathbf{y = 1 - |\frac{x}{2} + 3|}$ will have a **maximum value of 1 at x = –6**

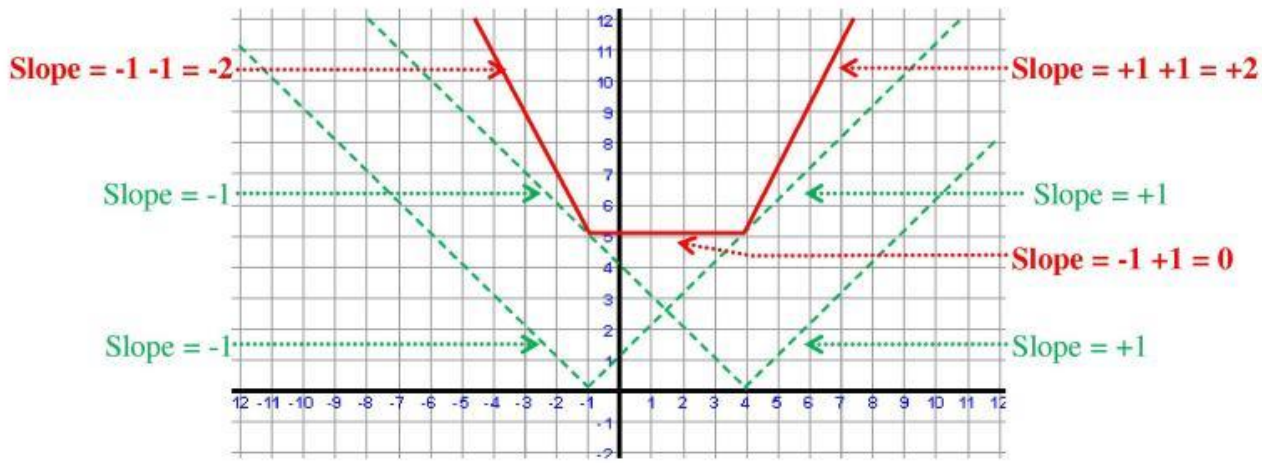
In general, $\mathbf{y = |ax + b| + c}$ will have a **minimum of c at** $\mathbf{x = -\frac{b}{a}}$ while $\mathbf{y = c - |ax + b|}$ will have a **maximum of c at** $\mathbf{x = -\frac{b}{a}}$

MODULUS – 5

Let's examine more complicated modulus functions this week - what happens when we add two or more modulus functions, what happens when we add and subtract modulus functions and other similar conundrums.

Find the minimum value of |x + 1| + |x – 4|

If we plot the two individual modulus functions (the green dotted lines in the figure) and the sum function (the solid red line) we can notice certain interesting patterns of behaviour:



The first thing to observe is that the red line changes direction twice. The values of x at which it changes direction are precisely the values at which the individual modulus functions become zero (i.e. the roots) – in this case, -1 and 4. As we shall see, this is not a coincidence….

Each modulus is made up of two parts, the part before the root having slope -1 and the part after the root having slope +1. Let us consider three ranges from x - before both roots, between the roots, and after both roots (in other words, from ∞ to -1, from -1 to 4 and from 4 to ∞).

In the region from ∞ to -1, the red line is the sum of two lines with a slope of -1. Hence it will be a line with slope -2. In between the roots i.e. between -1 and 4, one of the lines being added has a slope of -1 while the other has a slope of +1. So when added, it results in a slope of 0. And finally, in the region from 4 to ∞, both the lines being added have a slope of +1 and hence the resultant slope is +2

Also, at x = -1 (i.e. where the first modulus becomes 0), the value of the function equals that of the second modulus and at x = 4 (where the second one becomes zero), it takes the value of the first one. This is a very useful fact, as we shall see later.

Hence the resultant graph has a minimum value anywhere between the roots, and this value can be found by finding the value of the composite function at either of the roots as follows:

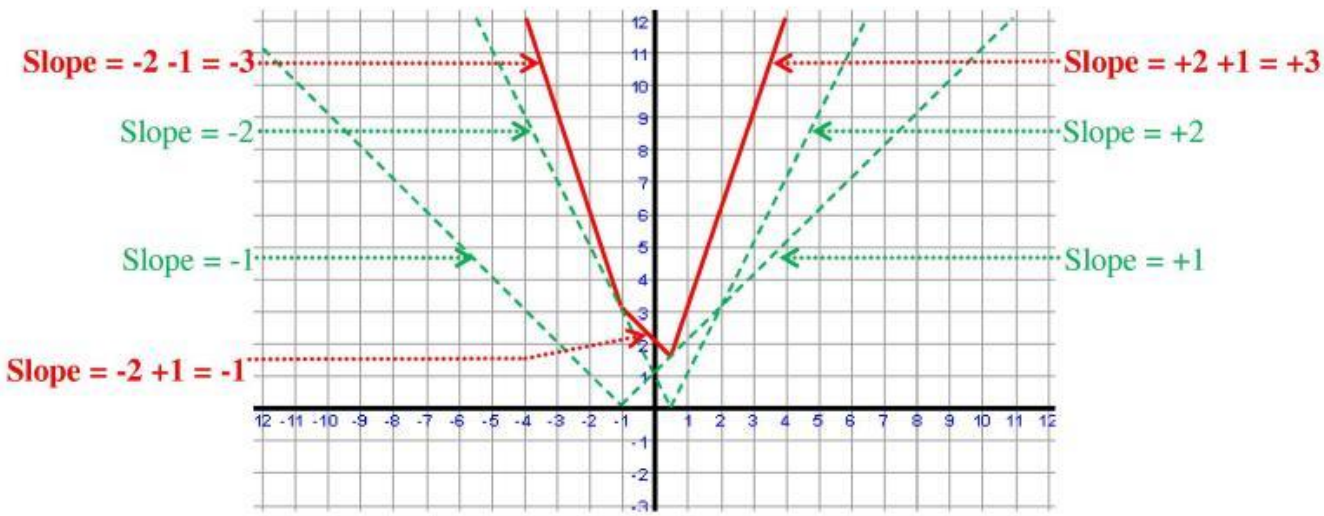
If we put x = -1 (or x = 4) then |x + 1| + |x – 4| = 5. Hence **the minimum value of |x + 1| + |x – 4| is 5 for any value of x between -1 and 4.**

What if the two modulus functions have different slopes? We shall examine that case tomorrow, and also discover that it is not necessary to even draw the graph to find the minimum value!

MODULUS – 6

Q. Find the minimum value of |x + 1| + |2x – 1|

As before, let us plot the individual modulus functions as well as their sum:



Since the two modulus functions have different slopes, the central line segment will not be a horizontal line in this case, but will have a definite slope (equal to the difference in slopes of the two original functions). However, we can see that, as in yesterday's case, the function increases towards infinity on either side of the roots and hence the minimum value will lie on that central line segment, that too at one end of it.

With some thought we could even work out at which end it should lie, but there is no need to do that; it would be quicker to just find the two endpoints and see which has the lower value. And this is easy to do – as we observed yesterday, the two endpoints of that line segment will lie neatly at those values of x at which the two original modulus functions become 0.

So in this case all we need to do is evaluate the function |x + 1| + |2x – 1| at x = -1 (which gives us 3) and at $x = \frac{1}{2}$ (which gives us a lower value, $\frac{3}{2}$). Hence **|x + 1| + |2x – 1| has a minimum of $\frac{3}{2}$ at $x = \frac{1}{2}$.**

Now let us see how to apply this quickly and efficiently:

Q. Find the minimum value of |3x + 5| + |5x + 3|

Actually plotting this would be quite a headache. But now that we know how such a function behaves, we can proceed as follows:

|3x + 5| = 0 at $x = -\frac{5}{3}$. At this value of x, |5x + 3| becomes $\frac{16}{3}$

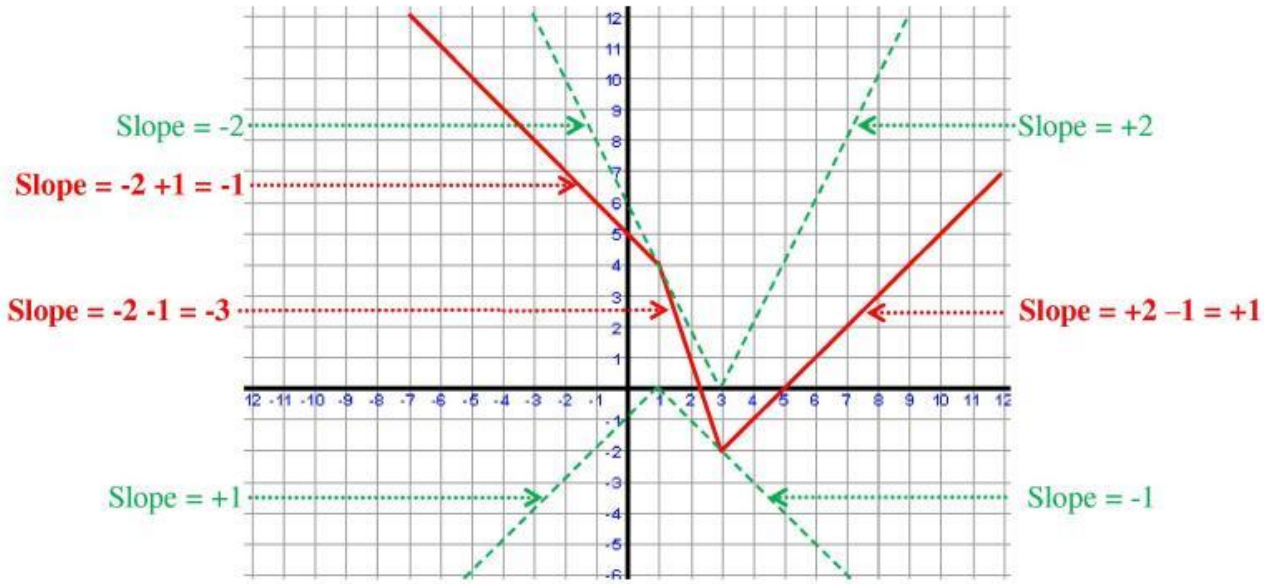
|5x + 3| = 0 at $x = -\frac{3}{5}$. At this value of x, |3x + 5| becomes $\frac{16}{5}$

Since $\frac{16}{5} < \frac{16}{3}$, we can say that **|3x + 5| + |5x + 3| has a minimum value of $\frac{16}{5}$ at $x = -\frac{3}{5}$**

MODULUS – 7

Q. Find the minimum value of |2x – 6| – |x – 1|

As before, let us plot the individual modulus functions as well as their sum:

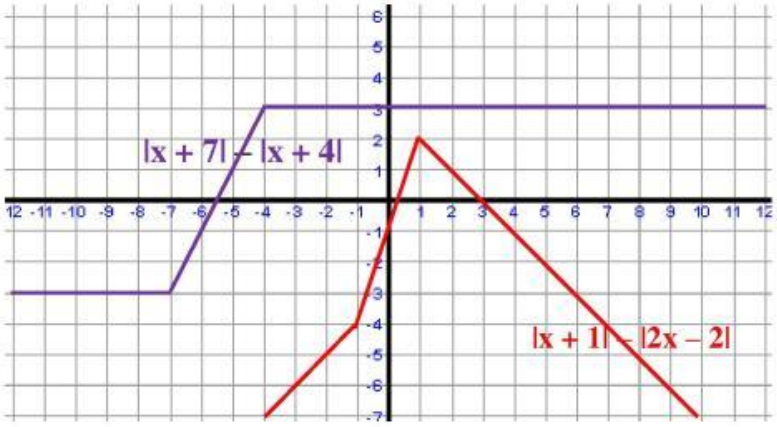


Since one of the modulus functions is positive and the other is negative, the figure shows a change in shape. Since the positive modulus has a larger slope, though, it will "overcome" the other one and for very large values of x (either positive or negative) it will tend towards +∞. Hence the overall function will have a minimum, and again, it will be at one of the ends of the central line segment, i.e. at one of the roots. |2x – 6 | = 0 gives us = 3 and |x – 1| = 0 gives x = 1.

Evaluating |2x – 6| – |x – 1| at x = 3 gives us –2, while evaluating |2x – 6| – |x – 1| at x = 1 gives us 4. **Hence the minimum value of |2x – 6| – |x – 1| is –2 at x = 3**

Note that if the slope of the negative modulus had been larger, then the function would have had a maximum rather than a minimum. Also if the slopes of both the positive and the negative moduli were equal, then the function would be horizontal except between the roots.

For example, try plotting |x + 1| - |2x – 2| and |x + 7| - |x + 4|. You should get something like this:



MODULUS – 8

Now we are ready to tackle a few slightly more intimidating looking problems:

Q. The function f(x) = |x – 1| + |2.8 – x| + |x – 3| (where x is a real number), can attain a minimum value of:

1) 2 **2) 3** **3) 2.8** **4) 1**

We notice that this is composed of 3 moduli of slope 1, and two of them are positive. So as x tends towards ± ∞, the value of this function will also tend towards ∞. So it will have a minimum, and this minimum must come at one of the inflection points or roots i.e. at x = 1 or 2.8 or 3. Putting x = 1 we get a value of 3.8, putting x = 2.8 we get a value of 2 and putting x = 3 we get a value of 2.2. **Hence the minimum value of |x – 1| + |2.8 – x| + |x – 3| is 2, at x = 2.8. Hence (1)**

Q. Find the minimum value of |x – 3| + |x – 2| + |x – 1| + |x| + |x + 1| + |x + 2| + |x + 3|

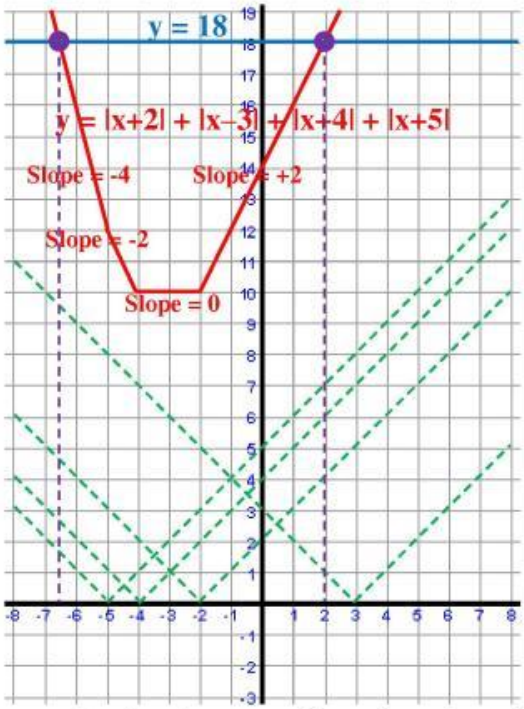
1) 10 **2) 12** **3) 13** **4) 11**

We can see that we are adding 7 different modulus functions, all with the same positive slope, so the minimum value should logically lie at the root of the middle one i.e. at x = 0. Putting x = 0 into the above expression results in a value of 12. **Hence (2)**

Q. Find the sum of the values of x such that |x+2| + |x–3| + |x+4| + |x+5| = 18

1) 2.5 **2) -3.5** **3) -4.5** **4) -5**

Rather than trying to do it in detail, we can just draw a rough figure and mark the slopes at various points, by just computing the values at each inflection point – i.e. -2, 3, -4 and -5. If we have observed the pattern in the previous few posts we would have realised that for an even number of moduli of slope 1, the resultant curve will be flat between the middle two roots. And that is exactly what we observe:



At x = -5, the value will be 12
At x = -4, the value will be 10
At x = -2, the value will again be 10
At x = 3, the value will be 20

Note that we have drawn an exact figure alongside, so we could just read the required x-values off the graph. However in the exam one would not have time to draw such an accurate graph. Still, it would be easy to find out as follows:

The first value is less than that at x = -5. The slope in this segment is -4 and so to go from the value of 12 (at x = -5) to 18, we will have to move 1.5 units left, i.e. to x = -6.5. The second value is between -2 and 3. This line segment has slope +2 and so to go from the value of 10 (at x = -2) to 18 we will need to move 4 units right i.e. to x = +2. **Hence the required values of x are -6.5 and +2. Hence (3)**

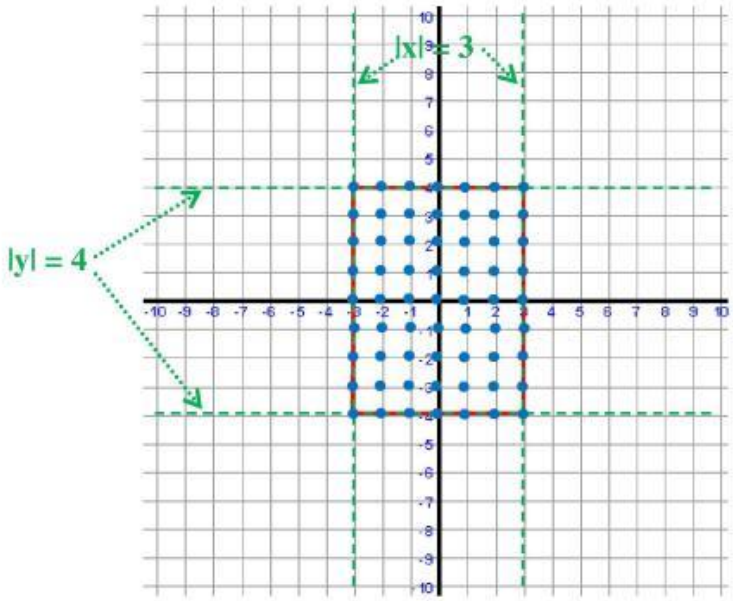
MODULUS – 9

What happens if we have modulus functions in x as well as in y simultaneously? Let's see what the graphs will look like, and the typical question-types associated with such graphs:

Let's start with an easy one:

Q. Consider |x| = 3, |y| = 4
(a) Find the area of the region enclosed
(b) Find the number of integer points within or on the boundary of the region so demarcated

Since the equation in |x| is entirely separate from the equation in |y| we can treat them separately. |x| = 3 gives us either x = 3 or x = -3. Similarly |y| = 4 gives us either y = 4 or y = -4. Plotting these 4 lines (in green below), we find that the area enclosed by them is a rectangle (in red below).



(a) The sides of the rectangle so obtained are 6 and 8 and hence the area will be **48.**

(b) We can plot the points with integer coordinates as shown in the figure (the blue dots) – we can see that there are **63** of them.

Alternatively, since we would rather not sit and plot them all, we could look for a pattern. In this case the pattern is easy, there are 7 values of x (from -3 to +3) and for each of these, there are 9 values of y (from -4 to +4) and hence by the fundamental principle of counting we can straightaway say that there will be 7 * 9 = **63** points.

Note that if the question had asked for the number of points with integer coordinates within the boundary (but not on it!) then we could have got only the interior points which would have been 5 * 7 = **35** in number.

Exercise: Find how many integer points will lie within the area bounded by |x| = 7, |y| = 9.
(You should get an answer of **221 points**)

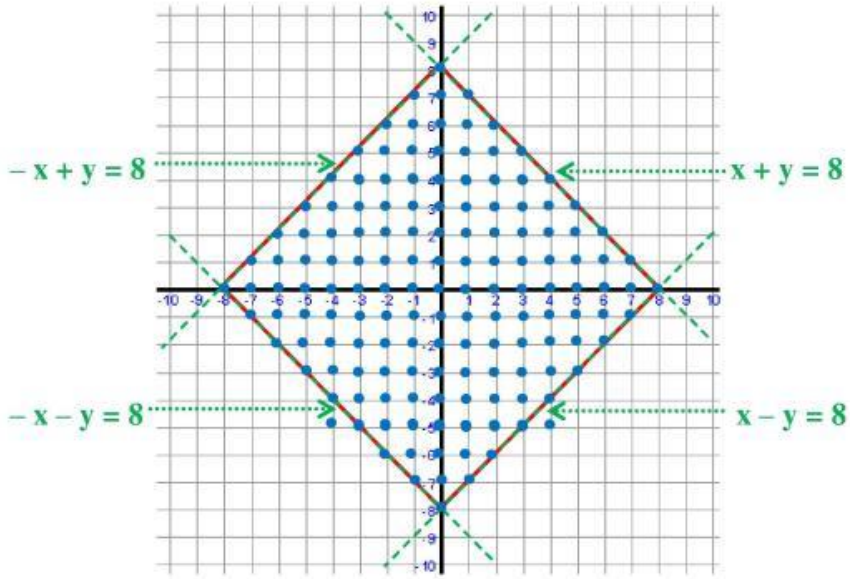
MODULUS – 10

Q. Consider |x| + |y| = 8
(a) Find the area of the region enclosed
(b) Find the number of integer points within or on the boundary of the region so demarcated

In this case we can approach it in two ways. One way is trial and error (and often proves fairly effective in an exam situation as it reduces the chances of confusion) - just take a few values of x and find the corresponding values for y. Since |x| + |y| = 8, and a modulus function can never take a negative value, we can see that neither |x| nor |y| can exceed 8. So let us plug in value of x from 8 downwards:

If we take x as 8, y becomes 0
If we take x as 7, y becomes 1 or -1
If we take x as 6, y becomes 2 or -2
…..
If we take x as 1, y becomes 7 or -7
If we take x as 0, y becomes 8 or -8
If we take x as -1, y becomes 7 or -7
…
If we take x as -8, y becomes 0

Alternatively, we can realise that if we have |x| = 8, it can be treated as x = ± 8, but also as ± x = 8
So |x| + |y| = 8 can be rewritten as ± x ± y = 8 which gives us 4 possible lines which we can proceed to plot (viz. x + y = 8, x – y = 8, – x + y = 8 and – x – y = 8). We can then proceed to plot it as follows:



(a) Very quickly we can see that the figure formed will be a square (in general it turns out to be a rhombus, as we shall see tomorrow) and the area can be found by ½ * product of diagonals. The diagonal lengths are determined by the limits of x and y and so both will be 16 here. So the required area is ½ * 16 * 16 = **128**.

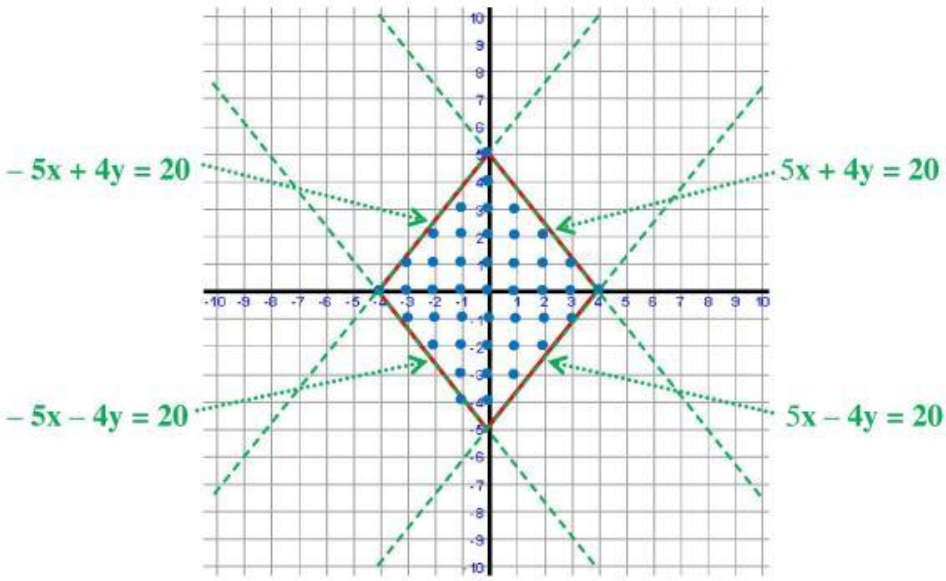
(b) Note that this is equivalent to asking the integer solutions for |x| + |y| ≤ 8
By putting x = -8, -7, -6…and so on we find that the number of possible values for y are 1, 3, 5…and so on. So the total number of points is 1 + 3 + 5 + 7 + … + 15 + 17 + 15 + … + 3 + 1 = **145 points**

MODULUS – 11

Q. Consider 5|x| + 4|y| = 20
(a) Find the area of the region enclosed
(b) Find the number of integer points within or on the boundary of the region so demarcated

Since 5|x| + 4|y| = 20, and a modulus function can never take a negative value, we can see that |x| cannot exceed 4 and |y| cannot exceed 5. Now that we know that it will be defined by 4 straight lines, we can quickly figure that at x = 4 or –4, y becomes 0 and at y = 5 or –5, x becomes 0. These 4 points will give us the limits of the rhombus and we can quickly plot it.

Alternatively, 5|x| + 4|y| = 20 can be rewritten as ± 5x ± 4y = 20 which gives us 4 lines (5x + 4y = 20, 5x – 4y = 20, – 5x + 4y = 20 and – 5x – 4y = 20). Plotting these we can get the required figure.



(a) Here we have a rhombus with diagonals 8 and 10 and so the area can be found by ½ * product of diagonals = ½ * 8 * 10 = **40**. Note that we don't even need to draw the figure once we have understood this logic – since the limits of x are 4 and -4 the diagonal along the X-axis will be of length 8 and similarly since the limits of y are 5 and -5 the diagonal along the Y-axis will be of length 10.

(b) We can solve 5|x| + 4|y| ≤ 20 by putting x = -4, -3, -2…and finding the number of values for y:

If we take x as 4 or -4, y becomes 0	➔	1 * 2 = 2 points
If we take x as 3 or -3, y ranges from 1 to -1	➔	3 * 2 = 6 points
If we take x as 2 or -2, y ranges from 2 to -2	➔	5 * 2 = 10 points
If we take x as 1 or -1, y ranges from 3 to -3	➔	7 * 2 = 14 points
If we take x as 0, y ranges from 5 to -5	➔	11 * 1 = 11 points

This gives us a total of 2 + 6 + 10 + 14 + 11 = **43 points** with integer coordinates.

Exercise: Find how many integer pairs (x, y) will satisfy **7|x| + 3|y| ≤ 28**
(You should get **79 pairs**)

Exercise: Find how many integer pairs (x, y) will satisfy **7|x| + 3|y| < 28**
(You should get **73 pairs**)

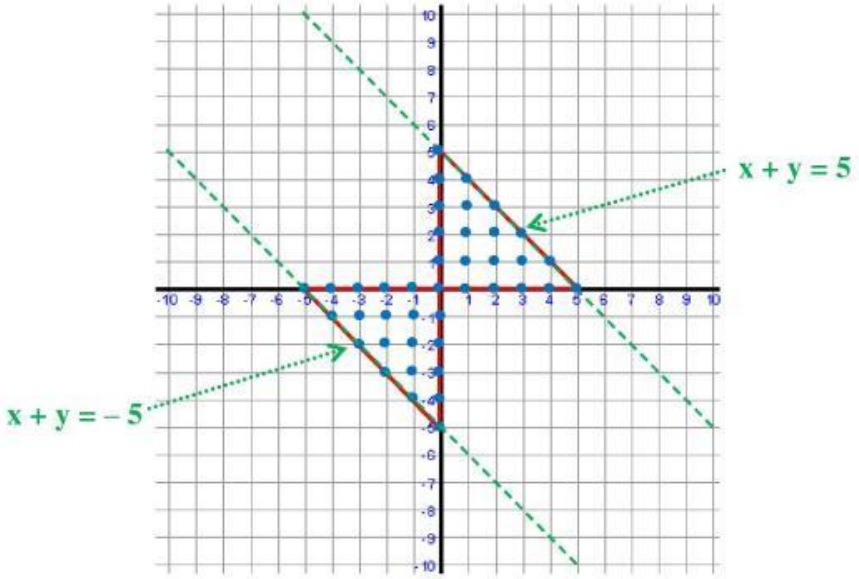
MODULUS – 12

Q. Consider $|x + y| = 5$
(a) Find the area of the region enclosed between this and the coordinate axes
(b) Find the number of integer points within or on the boundary of the region in (a) above

If we do trial and error in this case we will notice something interesting – neither x nor y is limited!

For $x = 0$, $y = 5$ or -5
For $x = 1$, $y = 4$ or -6
For $x = 2$, $y = 3$ or -7
For $x = 3$, $y = 2$ or -8
For $x = 4$, $y = 1$ or -9 and so on.

This can easily be explained algebraically when we realise that the equation $|x + y| = 5$ can be rewritten as $\pm (x + y) = 5$ which gives us 2 lines ($x + y = 5$ and $x + y = -5$). Plotting them gives us the following:



(a) We can see that the two lines do not enclose a finite area – which is why the question here has been altered to ask the area enclosed by the two lines and the co-ordinate axes. As we can see, the area formed consists of two triangles, each of area ½ * 5 * 5 and hence the total area will be **25**.

(b) Putting x = 5, 4, 3…–4, –5 we find there are 1 + 2 + 3 + 4 + 5 + 11 + 5 + 4 + 3 + 2 + 1 = **41 points**.

Exercise: How many integer values of a satisfy $|a - 2| + |a + 3| + |5 - a| + |a - 8| \leq 16$?
You should get **6 values** - 1, 2, 3, 4, 5, 6

Exercise: How many integer solutions exist for the region enclosed by $|5x + 3y| < 15$, $x > 0$ and $y > 0$
You should get **4 values** - (1,1), (1, 2), (1,3), (2,1)

Exercise: Find the number of non-negative integer solutions for $|5x| + |7y| \leq 35$
You should get **25 values**

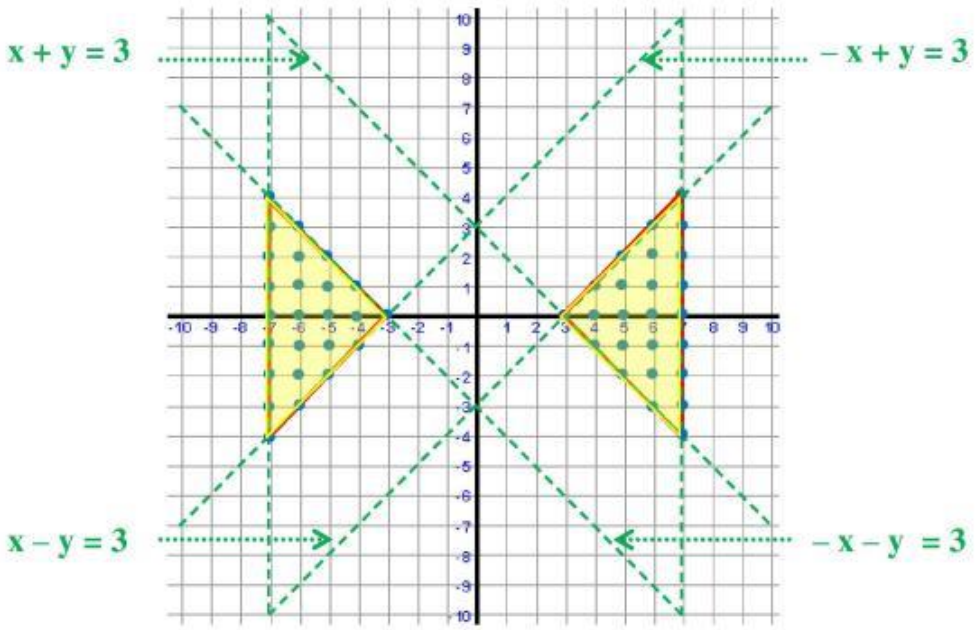
Exercise: Find the area enclosed by $9|x| + |11y| = 99$
You should get **198**

MODULUS – 13

Q. Find (a) the area and (b) the number of points with integer coordinates within or on the boundary of the region enclosed by |x| – |y| = 3 and |x| = 7

If we try to plot |x| - |y| = 3, we can see immediately that |x| = 3 + |y| and so must be greater than 3. So x cannot lie between 3 and -3. Trying out 3, 4, 5 and so on we can generate the required graph as shown below. This is an infinite graph, hence the need for a second limitation which is imposed by |x| = 7. The region enclosed by both is marked in pale yellow.

Alternatively, we could plot the lines x + y = 3, x – y = 3, – x + y = 3 and – x – y = 3, but here we have to be careful to identify the appropriate region – the butterfly-shaped region marked in pale yellow and demarcated by red lines in the figure:



(a) The required area will consist of two triangles of base 8 and height 4 and hence we can compute it as 2 * ½ * 8 * 4 = 32 square units.

(b) The number of points can be physically counted from the figure as being 50 in number. Alternatively, since we are unlikely to draw such a precise figure in the test, we could consider x = ±3, ±4, ±5, ±6 and ±7 and find the corresponding values for y:

For x = ±3, y = 0 => 1*2 = 2 values
For x = ±4, y lies between 1 and – 1 => 3*2 = 6 values
For x = ±5, y lies between 2 and – 2 => 5*2 = 10 values
For x = ±6, y lies between 3 and – 3 => 7*2 = 14 values
For x = ±7, y lies between 4 and – 4 => 9*2 = 18 values

which gives a total of **50 points**.

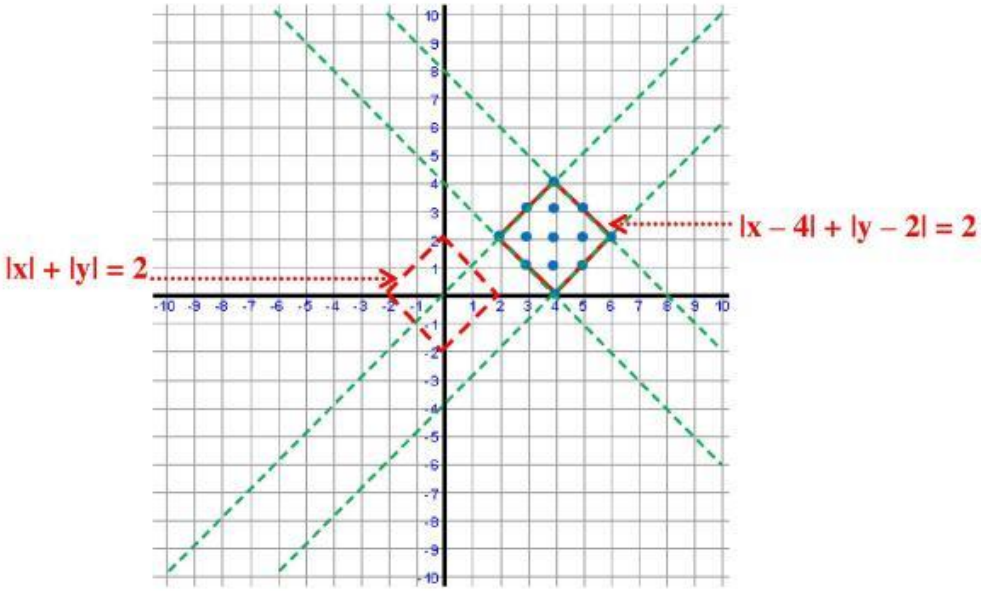
MODULUS – 14

Q. Find how many ordered pairs of values (x, y) satisfy |x – 4| + |y – 2| <= 2 where x and y are positive integers)

At first glance this looks like a pretty intimidating problem. But if we think carefully, we don't actually need to plot it point by point or to find the component lines forming it – we just need to find the number of points with integer coordinates.

Now if we observe |x – 4| + |y – 2| = 2, we can see that the shape will be the same as |x| + |y| = 2. The only difference is that |x| + |y| = 2 will be centred at (0, 0) while |x – 4| + |y – 2| = 2 will be centred at (4,2).

This is nothing but the concept of shifting the origin, if you think about it. So we will get the same rhombus shape which we could have got for |x| + |y| = 2:



So we can count as usual and find 1 + 3 + 5 + 3 + 1 = 13 points. However, since we are looking for positive integer solutions, one point (which has y = 0) will not be counted and hence our answer will have to be **12**.

Note again that we did not need to draw the figure; we could have started by looking at |x – 4| + |y – 2| = 2 and saying that since both are modulus functions, neither |x – 4| nor |y – 2| can exceed 2 and so x ranges between 2 and 6 while y ranges between 0 and 4. Plugging in values of x as 2, 3, 4, 5, 6 we will find 1, 3, 4, 3, and 1 values of y > 0 satisfying the given conditions. **Hence 12.**

Exercise:
Find the number of points with integer coordinates satisfying |x + 5| + |y – 3| ≤ 5
Find the area enclosed by |x + 5| + |y – 3| = 5

You should get the answers as **61 points** and **50 square units**

MODULUS – 15

Let's try a rather tough question today. Do try to solve it on your own before you see the answer!

Q. Find (a) the area and (b) the number of points with integer coordinates within or on the boundary of the region enclosed by $|x| + |y| + |x + y| = 4$

Trying to solve this algebraically is going to be a painful process, not to mention confusing; there is a pretty high chance of going wrong. Perhaps the best approach would be to do it slowly and steadily, by the trial-and-error method.

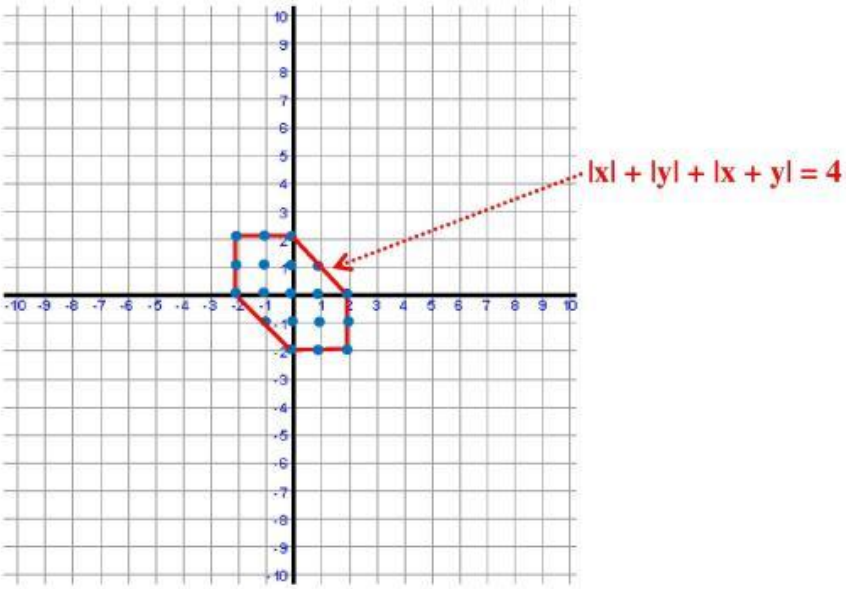
Looking at $|x| + |y| + |x + y| = 4$ a little experimentation will show us that neither x nor y can go above 2 or below – 2 (plug in x = 3, for example, and check for yourself!)

So let us put x = 2, 1, 0, – 1 and – 2 and see what happens to y.

For $x = 2$, $|2| + |y| + |y + 2| = 4$ => y ranges from – 2 to 0 ➔ 3 values
For $x = 1$, $|1| + |y| + |y + 1| = 4$ => y ranges from – 2 to 1 ➔ 4 values
For $x = 0$, $|0| + |y| + |y + 0| = 4$ => y ranges from – 2 to 2 ➔ 5 values
For $x = -1$, $|-1| + |y| + |y - 1| = 4$ => y ranges from – 1 to 2 ➔ 4 values
For $x = -2$, $|-2| + |y| + |y - 2| = 4$ => y ranges from 0 to 2 ➔ 3 values

This gives us 19 values.

Plotting the above results, we get an irregular hexagonal shape as shown below:



The area enclosed comprises two squares and two half-squares of side 2 and hence the required area will be 4 + 4 + 2 + 2 = **12 square units.**

Exercise: Find the area of the region formed by $|y - x| + |y| + |x| <= 6$.

You should get the answer **27**

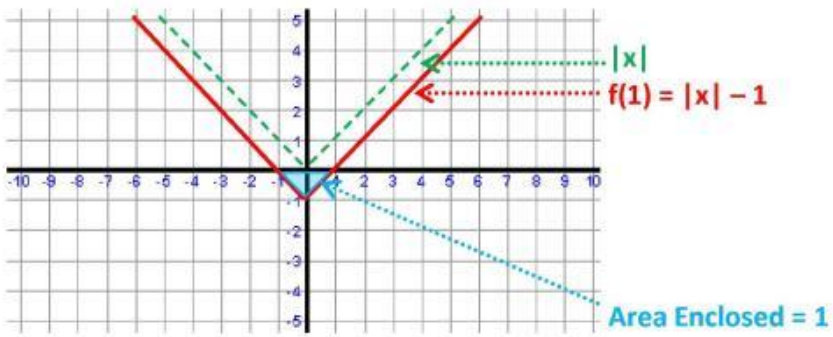
MODULUS – 16

Let's look at another intimidating looking question today. Again, try it for yourself first…

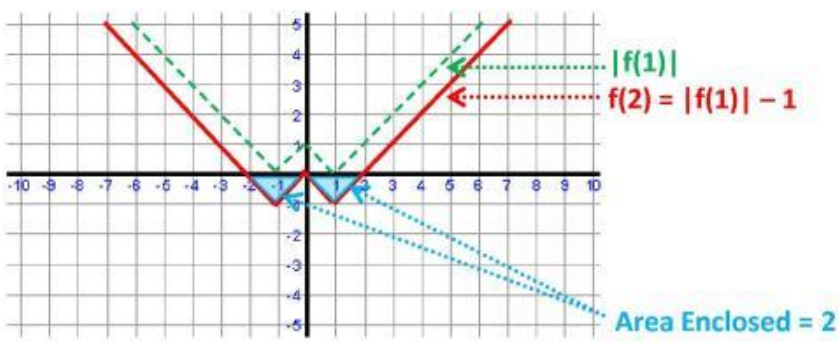
Q. If $f_1 = |x| - 1$ and $f_n = |f_{n-1}| - 1$ for all $n \in N$ then find the area bounded by f_{1947} and the coordinate axes

Let's try and get a feel of how the function works for small values.

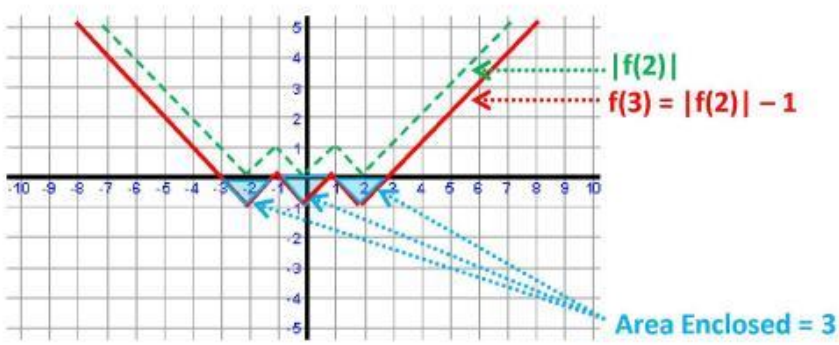
$f(1) = |x| - 1$



$f(2) = |f(1)| - 1$



$f(3) = |f(2)| - 1$



By now the pattern is pretty obvious and we can confidently conclude that the area bounded by f_{1947} must be **1947 square units**.

MODULUS – 17

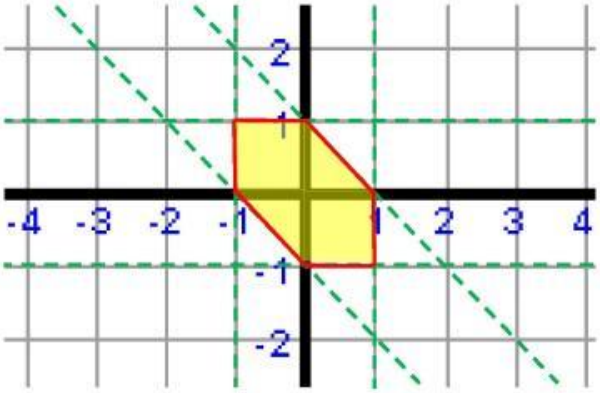
CAT Scan: Modulus Plotting

Let's look at a few questions the CAT has asked in the past on modulus plotting:

Q. The area bounded by the three curves $|x + y| = 1$, $|x| = 1$ and $|y| = 1$ is equal to

1) 4 **2) 3** **3) 2** **4) 1** **CAT 2000**

A quick plotting of the mentioned lines will give us the enclosed region:



As can be seen, the area enclosed by all 3 pairs of lines is 3 square units. **Hence (2)**

Q. The function $f(x) = |x – 2| + |2.5 – x| + |3.6 – x|$, where x is a real number, attains a maximum at _____?

1) x = 2.3 **2) x = 2.5** **3) x = 2.7** **4) None of these** **CAT 2003**

We can remember that when three modulus functions are added, the minimum value will occur at the middle root of the three (in this case, the roots in question are 2, 2.5 and 3.6). This will directly give us the answer as x = 2.5. **Hence (2)**

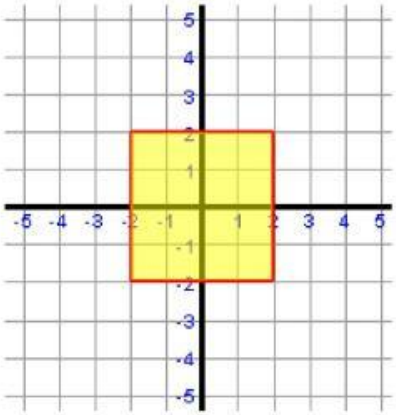
Q. In the X - Y plane, the area of the region bounded by the graph of $|x + y| + |x – y| = 4$ is

1) 8 **2)12** **3)16** **4) 20** **CAT 2005**

We can quickly see that neither x nor y can go beyond ±2

Putting in x = -2, -1, 0, 1 and 2, we find that in every case, y ranges from 2 to -2. Plotting gives us the figure alongside:

The area of the enclosed region is therefore 16. **Hence (3)**



In the next few posts, we will look at the plotting of circles, another useful skill.

MODULUS – 18

Co-ordinate Geometry: Plotting Circles

Like Modulus functions, circles also make the occasional appearance in Co-ordinate Geometry. Let's spend a little while getting familiar with them.

Let us remember that a circle is a set of points in a plane that are at a fixed distance (the radius) from a fixed point (the centre).

Also let us remember the distance formula in Co-ordinate Geometry, which tells us that the distance of a point (x, y) from another point (a, b) is given by $\sqrt{(x-a)^2 + (y-b)^2}$

So if we have a circle centred at a point - let's say (5, 6) and having radius 4, then any point (x, y) lying on that circle will be at a distance 4 from (5, 6) or in other words $(x-5)^2 + (y-6)^2 = 16$. Expanding and re-arranging the terms gives us $x^2 + y^2 - 10x - 12y + 45 = 0$

This is the general form of the equation of a circle: a quadratic in both x and y in which there is no term in xy and in which the coefficients of x^2 and y^2 are the same.

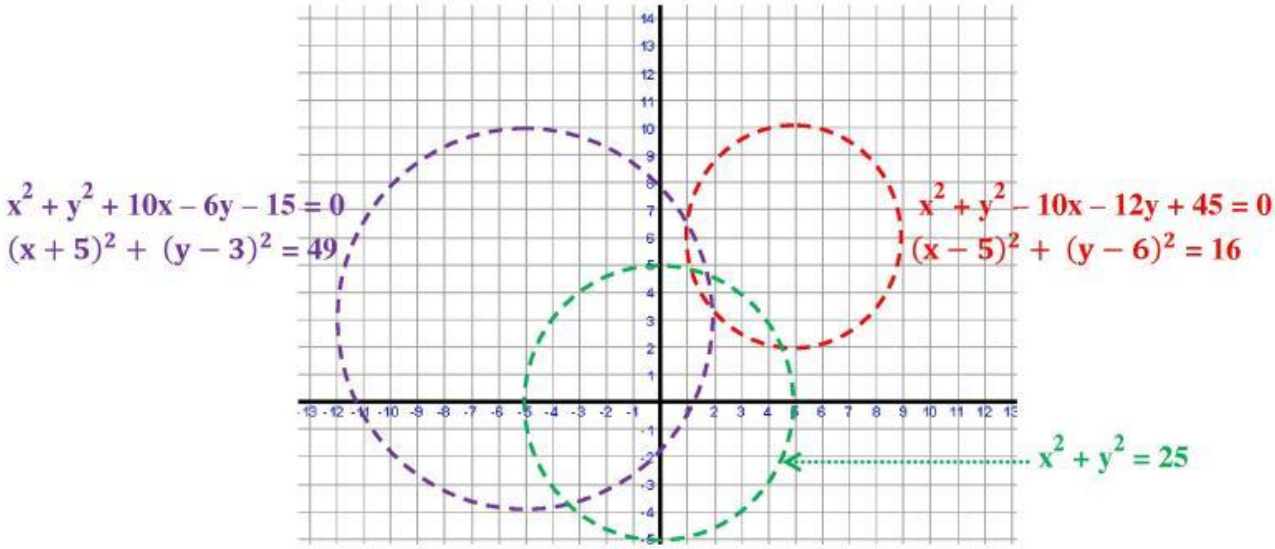
In general, the equation of a circle centred at a point (a, b) and with radius r can be written as

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \text{or}$$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

We should note that we can reverse-engineer the centre and radius from the equation of a circle by "completing the square". For example if we have an equation like $x^2 + y^2 + 10x - 6y - 15 = 0$ then we can re-write it as $(x^2 + 10x + 25) + (y^2 - 6y + 9) - 25 - 9 - 15 = 0$ which is $(x+5)^2 + (y-3)^2 = 49$ i.e. a circle of radius 7 centred at (–5, 3).

This also implies that if we have an equation of the form $x^2 + y^2 = r^2$, it will be a circle of radius r centred at the origin. So $x^2 + y^2 = 25$ will be a circle of radius 5 centred at the origin.



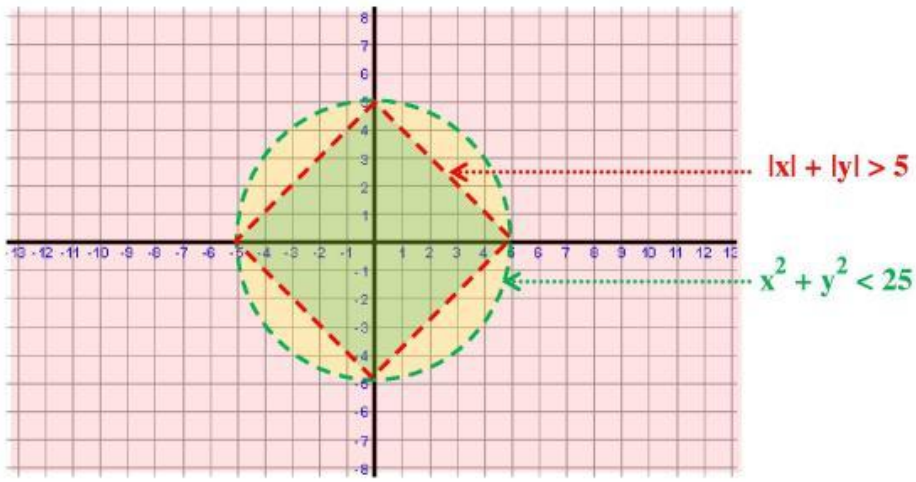
MODULUS – 19

A common question format which combines both the circle and the modulus could be something like:

Q. Find the area satisfying $|x| + |y| > 5$ and $x^2 + y^2 < 25$

Since we already know what both these functions will look like, let's jump directly to plotting them:

The region satisfying $|x| + |y| = 5$ is shown everything outside the rhombus/square (shown in red) and that satisfying $x^2 + y^2 < 25$ is within the circle (bounded in green). The required area is the intersection, the region shaded yellow below, which can be computed as the circle minus the square.

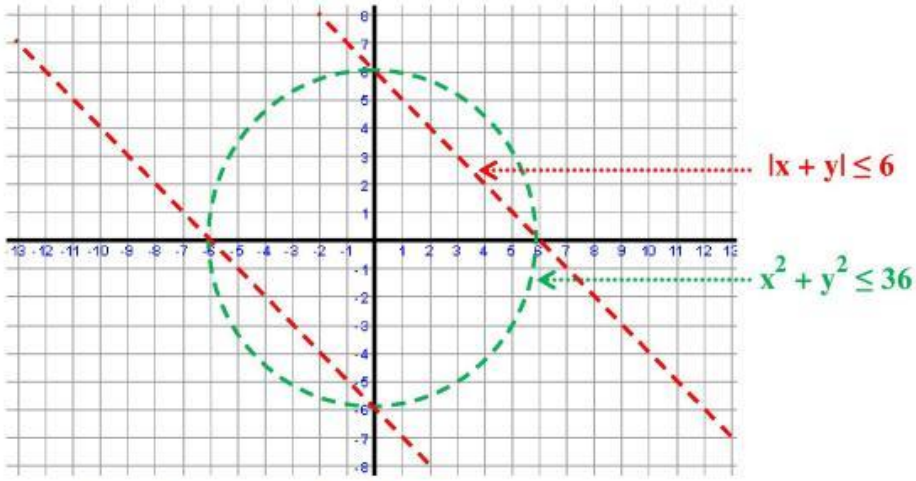


The required area is therefore (Area of circle of radius 5) – (Area of square of diagonal 10) = **$25\pi - 50$.**

Exercise

Q. Find the area satisfying $|x + y| \leq 6$ and $x^2 + y^2 \leq 36$

The figure for this would be something like the one below. I leave it to the reader to figure out which is the region satisfying the given conditions; you should get an answer **$18\pi + 36$**



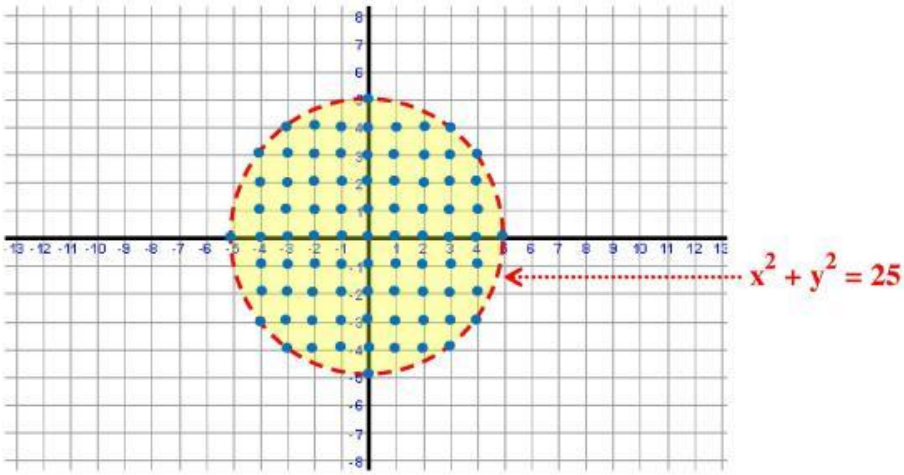
Q. Find the area satisfying $x^2 + y^2 \geq 16$ and $x^2 + y^2 \leq 36$

You should get an answer **20π**

MODULUS – 20

Q. How many points with integer coordinates lie within or on the boundary of $x^2 + y^2 = 25$?

If we draw the figure we will see that it looks something like this:



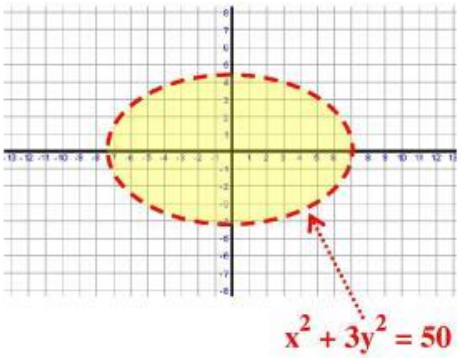
Of course, we can solve it even without drawing the figure. Just like in the modulus situation, we could observe that since x^2 and y^2 both cannot be negative, neither of them can exceed 25 and hence both x and y will be constrained to lie between -5 and 5. We can then plug in these values in $x^2 + y^2 \leq 25$:

When $x = \pm 5$, $y^2 \leq 0$ so y can only be 0 (2 * 1 = 2 points)
When $x = \pm 4$, $y^2 \leq 9$ so y can lie between – 3 and 3 (2 * 7 = 14 points)
When $x = \pm 3$, $y^2 \leq 16$ so y can lie between – 4 and 4 (2 * 9 = 18 points)
When $x = \pm 2$, $y^2 \leq 21$ so y can lie between – 4 and 4 (2 * 9 = 18 points)
When $x = \pm 1$, $y^2 \leq 24$ so y can lie between – 4 and 4 (2 * 9 = 18 points)
When $x = 0$, $y^2 \leq 25$ so y can lie between – 5 and 5 (1 * 11 = 11 points)
Thus there are a total of **81 points** satisfying the given criteria.

Q. How many points with integer coordinates lie within $x^2 + 3y^2 = 50$?

This is a rather more complicated figure (it will turn out to be an ellipse) but we need not plot it. We can see that x cannot go beyond ±7 and y cannot go beyond ±4. We could use either x or y as our starting point to list cases, but since y has less values, it would probably be the better option:

When $y = \pm 4$, $x^2 \leq 2$ ➔ x between – 1 and 1 (2 * 3 = 6 points)
When $y = \pm 3$, $x^2 \leq 23$ ➔ x between – 4 and 4 (2 * 9 = 18 points)
When $y = \pm 2$, $x^2 \leq 38$ ➔ x between – 6 and 6 (2 * 13 = 26 points)
When $y = \pm 1$, $x^2 \leq 47$ ➔ x between – 6 and 6 (2 * 13 = 26 points)
When $y = \pm 0$, $x^2 \leq 50$ ➔ x between – 7 and 7 (1 * 15 = 15 points)
This gives us a total of **91 solutions.**



(A rough figure is shown to give some idea of how it looks)

LOGARITHMS – 1

Logarithms - Basics

The logarithmic function may be thought of as the inverse of the exponential function (a^x).

A logarithm is usually defined to a particular base. We will remember that in the jargon of indices, the base in the number that is raised to a power (for example in a^x, a is called the base). Also in base systems, the base is the number whose power was raised to 0, 1, 2 etc to get the weights for the respective digits. In logarithms, too, the base can be thought of as "the number which is raised to something".

A logarithm may be defined as follows: The logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. In other words, if $m^b = a$ then we can say that $\log_m a = b$

Let us look at some examples to clarify this:

$\log_5 625 = 4$ ($\because 5^4 = 625$) $\log_2 512 = 9$ ($\because 2^9 = 512$) $\log_{10} 1000 = 3$ ($\because 10^3 = 1000$)

$\log_3 243 = 5$ ($\because 3^5 = 243$) $\log_4 1 = 0$ ($\because 4^0 = 1$) $\log_{99} 9801 = 2$ ($\because 99^2 = 9801$)

$\log_{\frac{1}{2}}\left(\frac{1}{32}\right) = 5$ ($\because \left(\frac{1}{2}\right)^5 = \frac{1}{32}$) $\log_{\frac{1}{13}}\left(\frac{1}{169}\right) = 2$ ($\because \left(\frac{1}{13}\right)^2 = \frac{1}{169}$) $\log_{\frac{1}{7}}\left(\frac{1}{2401}\right) = 4$ ($\because \left(\frac{1}{7}\right)^4 = \frac{1}{2401}$)

Logarithms can also be fractional….

$\log_{64} 4 = \frac{1}{3}$ ($\because 64^{1/3} = 4$) $\log_{256} 2 = \frac{1}{8}$ ($\because 256^{1/8} = 2$) $\log_{343} 7 = \frac{1}{3}$ ($\because 343^{1/3} = 7$)

…or even negative

$\log_3\left(\frac{1}{9}\right) = -2$ ($\because 3^{-2} = \frac{1}{9}$) $\log_{\frac{1}{2}} 8 = -3$ ($\because \left(\frac{1}{2}\right)^{-3} = 8$) $\log_{\frac{1}{27}} 3 = -\frac{1}{3}$ ($\because \left(\frac{1}{27}\right)^{-1/3} = 3$)

Note, however, that a logarithm is not defined for certain cases:

A logarithm for a negative number or for 0 is not defined

If we try to define something like $\log_2(-3)$, we will realise that it makes no sense – any positive based, raised to any power, will still remain positive. Also if we try to something like $\log_2 0$ we will find that the value will be undefined.

A logarithm is not defined to a base which is negative or 0

If we try to define something like $\log_{(-2)} 8$ we will find that there is no power to which we can raise -2 which will give 8. Also raising a negative number to a fractional value will be problematic.

A logarithm is not defined to base 1

If we try to define something like $\log_1 5$ we will have to solve an equation $1^x = 5$ which is patently absurd! Thus a log to base 1 is not defined.

Thus, in brief, $\log_m a$ is defined only if $a > 0$ and if $m > 0$ and $m \neq 1$

LOGARITHMS – 2

Some useful results involving logarithms

- **$\log_n n = 1$ and $\log_n 1 = 0$ for any n** - - - - - (1)

This is self-evident and can be proved using the basic definition of a logarithm

- **$\log_n xy = \log_n x + \log_n y$** - - - - - (2)
- **$\log_n\left(\frac{x}{y}\right) = \log_n x - \log_n y$** - - - - - (3)

If $\log_n x = a$ and $\log_n y = b$, then $x = n^a$ and $y = n^b$.
Now applying the laws of indices $xy = n^{a+b}$ which means that $\log_n xy = a + b = \log_n x + \log_n y$.
Similarly $\frac{x}{y} = n^{a-b}$ which means that $\log_n\left(\frac{x}{y}\right) = a - b = \log_n x - \log_n y$

(The above two identities are perhaps the most significant reason why logs came to prominence and remained popular – by using them it was possible to simplify multiplication and division. Recollect that in the 17th century, when John Napier introduced logs, there were no computers or calculators or even slide rules; all calculations had to be accomplished by hand. Thanks to logs, a multiplication/division could be accomplished by looking up two logs in a table, adding/subtracting them, and looking up the antilog. Since addition and subtraction are much easier than multiplication and division, logs became hugely popular among the mathematicians of the day.)

- **$\log_n x^a = a \log_n x$** - - - - - (4)

We can write $\log_n x^a = \log_n(x * x * x * \ldots \text{ a times})$ which, on applying result (2) above, becomes
$\log_n x + \log_n x + \log_n x + \ldots \text{a times} = a \log_n x$

- **$\log_n \sqrt[a]{x} = \frac{\log_n x}{a}$** - - - - - (5)

Let $\sqrt[a]{x} = y$. Then $x = y^a$. Thus by the result (4) above we can say $\log_n x = a \log_n y$
Thus $\log_n y = \frac{\log_n x}{a}$ but since $y = \sqrt[a]{x}$, this becomes the required $\log_n \sqrt[a]{x} = \frac{\log_n x}{a}$

- **$\log_b a = \frac{\log_n a}{\log_n b}$** - - - - - (6)

Suppose $\log_b a = c$.
Then $a = b^c$
Taking log to base n on both sides
$\log_n a = \log_n b^c$
Applying result (4) above
$\log_n a = c * \log_n b$
Hence $c = \frac{\log_n a}{\log_n b}$

LOGARITHMS – 3

Some more useful results and facts:

- $\mathbf{n^{\log_n x} = x}$ - - - - - (7)

Let $n^{\log_n x} = y$. Taking log to base n on both sides $\log_n(n^{\log_n x}) = \log_n y$.
Using result 4, $\log_n(n^{\log_n x}) = \log_n x * \log_n n$ which, since $\log_n n = 1$, gives us just $\log_n x$.
So since $\log_n x = \log_n y$ we find that y must be equal to x itself.

- $\mathbf{\log_n n^x = x}$ - - - - - (8)

Applying result (4) gives us that $\log_n n^x = x \log_n n$. But since $\log_n n = 1$, this will just be x.

The above two results (7) and (8) can also be derived logically; since logarithms and exponents are inverse functions, they will cancel each other out when applied with the same base in both cases.

- $\mathbf{\log_b a * \log_a b = 1 \text{ i.e } \log_b a = \frac{1}{\log_a b}}$ - - - - - (9)

This can easily be proved from result 6

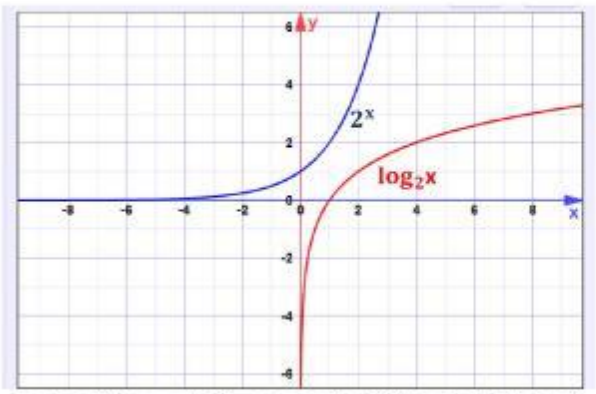
Typical Bases

Quite often, in an exam like the CAT, the base will not be specified. The question will just state something like "log 12". In this case, it is usually understood that the base is to be taken as 10

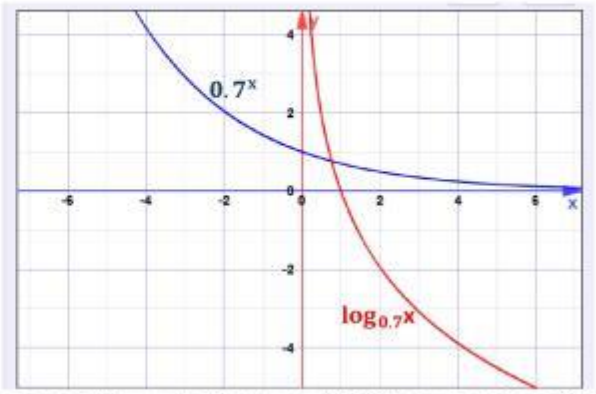
Note that in pure mathematics (including calculus) if the base is not specified, it is assumed to be not 10, but e (where e ≈ 2.718 is an irrational number known as Euler's number or Napier's constant and used as the base of "natural logarithms"). A logarithm to the base e is also often indicated by ln (standing for logarithm natural). Base 2 is also used (binary logarithms) mostly in computer science.

The Graph of a Logarithmic Function

Remember that a logarithmic function is not defined for 0 or for negative numbers. So it lies entirely to the right of the y axis. Compare this with its inverse, the exponential function, which never goes below the x-axis. The two graphs mirror each other across the line x = y.



Logarithm v/s Exponential Graphs: Base 2



Logarithm v/s Exponential Graphs: Base 0.7

LOGARITHMS – 4

Logarithms: Examples

Q. If $\log_q p + \log_q p^3 + \log_q p^5 + \log_q p^7 + \log_q p^9 = 100$ then what is the relation between p and q?

$\log_q p + \log_q p^3 + \log_q p^5 + \log_q p^7 + \log_q p^9 = 100$
so we can write
$\log_q p + 3 * \log_q p + 5 * \log_q p + 7 * \log_q p + 9 * \log_q p = 100$
i.e. $25 * \log_q p = 100$
so $\log_q p = 4$
Hence $p = q^4$

Q. If $\log_q p + \log_{\sqrt[3]{q}} p + \log_{\sqrt[5]{q}} p + \log_{\sqrt[7]{q}} p = 96$ then what is the relation between p and q?

This looks a little more intimidating, so let us take one term and examine it more closely:

$$\log_{\sqrt[3]{q}} p = \frac{\log p}{\log \sqrt[3]{q}} = \frac{\log p}{\log q^{\frac{1}{3}}} = \frac{\log p}{\frac{1}{3}\log q} = \frac{3 \log p}{\log q} = 3 * \log_q p$$

Similarly $\log_{\sqrt[5]{q}} p = 5 * \log_q p$ and so on Hence the given equation becomes

$\log_q p + \log_{\sqrt[3]{q}} p + \log_{\sqrt[5]{q}} p + \log_{\sqrt[7]{q}} p = 96$

$\log_q p + 3 * \log_q p + 5 * \log_q p + 7 * \log_q p = 96$
i.e. $16 * \log_q p = 96$
so $\log_q p = 6$
Hence $p = q^6$

Q. Find the value of $\log_y x^2 * \log_z y^2 * \log_x z^2$.

$\log_y x^2 * \log_z y^2 * \log_x z^2$

$$= \frac{\log x^2}{\log y} * \frac{\log y^2}{\log y} * \frac{\log z^2}{\log x}$$

$$= \frac{2 \cancel{\log x}}{\cancel{\log y}} * \frac{2 \cancel{\log y}}{\cancel{\log z}} * \frac{2 \cancel{\log z}}{\cancel{\log x}}$$

$= 8$

Q. Find the solutions for $x^2 - y^{\log_y 5x} + 6 = 0$

Remembering that $n^{\log_n x} = x$, we can say that $y^{\log_y 5x} = 5x$
So the given equation can then be simplified as follows:
$x^2 - y^{\log_y 5x} + 6 = 0$
➔ $x^2 - 5x + 6 = 0$
➔ $x = 2$ or $x = 3$

The above examples should have amply demonstrated that it is easy to set a difficult looking sum in logarithms, without it actually being a difficult one!

LOGARITHMS – 5

Q. Given that log 2 = 0.3, find the value of log 25

Since the base has not been mentioned, we may assume that it is in base 10.
Now the tools available to us are $\log_{10}2$ (given as 0.3) and $\log_{10}10$ (which of course is 1).
We can use these to compute log 5 as follows:
$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - 0.3 = 0.7$
Hence $\log 25 = \log 5^2 = 2 * \log 5 = 2 * 0.7 = 1.4$

Q. Given that log 3 = 0.477 and log 2 = 0.3, find the value of log 48

$\log 48 = \log (2^4 * 3) = 4 \log 2 + \log 3 = 4 * 0.3 + 0.477 = 1.677$

Q. What is the domain of definition of $f(x) = \log (9x - x^2 - 14)$

The question is essentially asking for what values the given function is defined. Now we know that a log is defined only for positive numbers so we need to identify the range of values for which
$9x - x^2 - 14 > 0$ i.e
$-(x - 2)(x - 7) > 0$ i.e
$(x - 2)(x - 7) > 0$.
Hence $\mathbf{2 < x < 7}$ would be the required range.

Q. For what real values is $\frac{1}{\log(x^2 - 6x + 9)}$ not defined?

The logarithmic function will not be defined where $x^2 - 6x + 9 \leq 0$ i.e. at $x = 3$
However the reciprocal function is not defined when the denominator becomes 0
➔ $\log (x^2 - 6x + 9) = 0$
➔ $x^2 - 6x + 9 = 1$
➔ $x^2 - 6x + 8 = 0$
➔ $x = 2$ or 4
Hence the given expression is not defined at x = 2, 3 and 4

Q. Find the value of x if log (x), log (x+8) and log (x+20) are in A.P.

An interesting property which will help here is to note if the logs of three numbers are in G.P., then their logs will be in A.P.

For example, let us consider 3 numbers x, y, z which are in GP. Suppose $x = a$, $y = a * r$ and $z = a * r^2$.
Now $\log x = \log a$, $\log y = \log a + \log r$ and $\log z = \log a + 2 \log r$ ➔ $\log x + \log z = 2 * \log y$ which means that log x, log y, log z are in A. P

So in the given question, for their logs to be in A.P., (x), (x+8) and (x+20) must be in G.P.
Thus $(x)(x+20) = (x+8)^2$ or $x^2 + 20x = x^2 + 16x + 64$ ➔ $4x = 64$ ➔ $x = 16$

LOGARITHMS – 6

Let's look at a log-based question type which has been pretty popular in the CAT in recent years:

Q. Given that log 2 = 0.301, find the number of digits in 2^{345}

1) 345 **2) 103** **3)104** **4) 102**

At first glance, it would seem that there is no correlation between the information given and the answer sought. But if we think about it, the required value, number of digits, is related to the base (i.e. decimal) in which we are operating. And since the base of the log is also not specified, we may assume that it is also given in base 10. This gives us a starting point from which we can approach the problem:

Since $\log_{10}2 = 0.301$, we can say that $\log_{10}2^{345} = 345 * \log_{10}2 = 345 * 0.301 = 103.845$.

Now let's make a few basic observations about numbers in base 10 and their log values.

$\log_{10}1 = 0$
$\log_{10}10 = 1$
$\log_{10}100 = 2$
$\log_{10}1000 = 3$
$\log_{10}10000 = 4$
…and so on.

This means that
for any number between 1 and 10 (i.e. any 1-digit number) the log will be 0.xx
for any number between 10 and 100 (i.e. any 2-digit number) the log will be 1.xx
for any number between 100 and 1000 (i.e. any 3-digit number) the log will be 2.xx
for any number between 1000 and 10000 (i.e. any 4-digit number) the log will be 3.xx
and so on.

So if the log of 2^{345} is 103.xx, it must be between 10^{103} and 10^{104} and must have **104 digits**. **Hence (3)**

Let's try a couple more of these to reinforce the concept:

Q. Given that log 5 = 0.699, find the number of digits in 5^{500}

Now we know that $\log 5^{500} = 500 \log 5 = 500 * 0.699 = 349.5$
Hence 5^{500} must lie between 10^{349} and 10^{350} and hence must have 350 digits.

Q. Given that log 2 = 0.301 and log 3 = 0.477, find the number of digits in 12^{20}

If log 2 = 0.301 and log 3 = 0.477, then $\log 12 = \log (2^2*3) = 2 \log 2 + \log 3 = 1.079$
So $\log 12^{20} = 20 \log 12 = 20 * 1.079 = 21.58$.
Hence 12^{20} must lie between 10^{21} and 10^{22} and hence must have 22 digits.

LOGARITHMS – 7

Let's look at a variant of yesterday's question, which has also made an appearance in the CAT recently

Q. Given that log 2 = 0.301, find the number of leading zeroes in 2^{-345}

1) 115 **2) 103** **3)104** **4) 102**

As in the previous question type, let us start by calculating the log of 2^{-345}

Since $\log_{10}2 = 0.301$, we can say that $\log_{10}2^{-345} = -345 * \log_{10}2 = -345 * 0.301 = -103.845$.

(I should probably mention here that the formal notation of logs would require the part after the decimal point (the **mantissa**) to be recorded as positive and hence technically this should be written as $-104 + 0.155$ or in log notation $\overline{-104}.155$. However, this only confuses the issue further and hence we shall leave the formal notation to the textbooks and proceed using –103.845)

With negative log values, the pattern to be observed will be slightly different

$\log_{10}1 = 0$
$\log_{10}0.1 = -1$
$\log_{10}0.01 = -2$
$\log_{10}0.001 = -3$
$\log_{10}0.0001 = -4$
…and so on.

This means that
for any number between 0.1 and 1 (i.e. any number with no leading zero) the log will be –0.xx
for any number between 0.01 and 0.1 (i.e. any number with 1 leading zero) the log will be –1.xx
for any number between 0.001 and 0.01 (i.e. any number with 2 leading zeroes) the log will be –2.xx
for any number between 0.0001 and 0.001 (i.e. any number with 3 leading zeroes) the log will be –3.xx
and so on.

Thus if $\log_{10}2^{-345} = -103.845$ then we can say that 2^{-345} must have **103 leading zeroes. Hence (2)**

Again, let's look at a couple of examples

Q. Given that log 3 = 0.477, find the number of leading zeroes in 3^{-33}

Since $\log_{10}3 = 0.477$, we can say that $\log_{10}3^{-33} = -33 * \log_{10}3 = -33 * 0.477 = -15.741$.
Hence 3^{-33} must lie between 10^{-15} and 10^{-16} and hence must have 15 leading zeroes

Q. Given that log 3 = 0.477 and log 5 = 0.699, find the number of leading zeroes in 15^{-100}

Since $\log_{10}3 = 0.477$ and $\log_{10}5 = 0.699$, $\log_{10}15$ must be $0.477 + 0.699 = 1.176$
So $\log_{10}15^{-100} = -100 * \log_{10}15 = -100 * 1.176 = -117.6$
Hence 15^{-100} must lie between 10^{-117} and 10^{-118} and hence must have 117 leading zeroes.

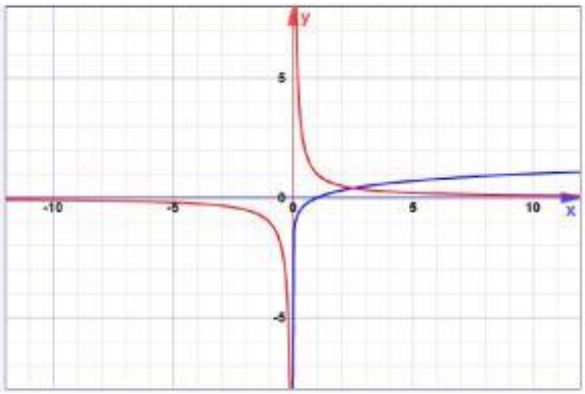
LOGARITHMS – 8

Logarithms: CAT Scan

Q. When the curves $y = \log_{10}x$ and $y = x^{-1}$ are drawn in the X-Y plane, how many times do they intersect for values of $x \geq 1$? **CAT 2003**

(1) Never **(2) Once** **(3) Twice** **(4) More than twice**

We could solve this by drawing rough graphs (an immensely useful skill!), which would immediately reassure us that the two functions would meet exactly once.



Another approach would be to recognise that $\log_{10}x$ is an increasing function, while x^{-1} is a decreasing function. Now at $x = 1$, $\log_{10}x$ is 0 and will increase continuously from there on, while x^{-1} is 1 and will decrease continuously from there on. So the two functions will meet exactly once. **Hence (2)**

Q. If $\log_3 2$, $\log_3 (2^x - 5)$ and $\log_3 (2^x - \frac{7}{2})$ are in arithmetic progression, then the value of x must be equal to **CAT 2003**

(1) 5 **(2) 4** **(3) 2** **(4) 3**

As we observed earlier, if the logs of three numbers are in AP the numbers must be in GP. Hence we can say that 2, $(2^x - 5)$ and $(2x - \frac{7}{2})$ must be in GP.

So $2 * (2^x - \frac{7}{2}) = (2^x - 5)^2$

➔ $2 * 2^x - 7 = 2^{2x} - 10 * 2^x + 25$

Note that 2^{2x} is the square of 2^x. So let us put $2^x = y$.

➔ $2y - 7 = y^2 - 10y + 25$

➔ $y^2 - 12y + 32 = 0$

i.e. $2^x = y = 4$ or 8.

But if $2^x = 4$, then $\log_3(2^x - 5) = \log_3(-1)$ is not defined and so we eliminate that option leaving only $2^x = 8$ or $x = 3$. **Hence (4)**

LOGARITHMS – 9

Q. If $\log_{10}x - \log_{10}\sqrt{x} = 2\log_x 10$, then a possible value of x is given by **CAT 2003**

(1) 10 **(2) $\frac{1}{100}$** **(3) $\frac{1}{1000}$** **(4) None of these**

$\log_{10}x - \log_{10}\sqrt{x} = 2\log_x 10$

➔ $\log_{10}x - \frac{1}{2}\log_{10}x = 2\log_x 10$

➔ $\frac{1}{2}\log_{10}x = 2\log_x 10$

➔ $\log_{10}x = 4\log_x 10$

Now let us remember that $\log_b a = \frac{1}{\log_a b}$

Suppose we represent $\log_{10}x$ as y then we can rewrite the equation as

➔ $y = \frac{4}{y}$

This of course gives us $y^2 = 4$ or $y = \pm 2$

So if $\log_{10}x = \pm 2$ then x must be either 100 or $\frac{1}{100}$. **Hence (2)**

Note that this question could also have been solved quite efficiently by just plugging in the answer choices. So even if one were unable to remember the mathematical logic one could still solve it.

Q. What is the sum of 'n' terms in the following series: **CAT 2003**

$$\log m + \log\frac{m^2}{n} + \log\frac{m^3}{n^2} + \log\frac{m^4}{n^3} + \ldots + \log\frac{m^n}{n^{n-1}}$$

(1) $\log\left[\frac{n^{(n-1)}}{m^{(n+1)}}\right]^{\frac{n}{2}}$ **(2)** $\log\left[\frac{m^m}{n^n}\right]^{\frac{n}{2}}$ **(3)** $\log\left[\frac{m^{(1-n)}}{n^{(1-m)}}\right]^{\frac{n}{2}}$ **(4)** $\log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{\frac{n}{2}}$

This is a typical CAT question which looks extremely intimidating. However, on applying the laws of logs logically, it yields its secrets:

$\log m + \log\frac{m^2}{n} + \log\frac{m^3}{n^2} + \log\frac{m^4}{n^3} + \ldots + \log\frac{m^n}{n^{n-1}}$ can be rewritten as:

$= \log m + 2\log m - \log n + 3\log m - 2\log n \ldots + n\log m - (n-1)\log n$ which on rearranging is

$= \log m\,(1 + 2 + 3 + \ldots + n) - \log n\,(1 + 2 + 3 + \ldots + (n-1))$

$= \log m\,(\frac{n(n+1)}{2}) - \log n\,(\frac{n(n-1)}{2})$

$= \log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{\frac{n}{2}}$. **Hence (4)**

LOGARITHMS – 10

Q. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$, then **CAT 2003**

(1) $M^9 = \frac{9}{N}$ **2) $N^9 = \frac{9}{M}$** **3) $M^3 = \frac{3}{N}$** **4) $N^9 = \frac{3}{M}$**

$\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$

Now $\frac{1}{3}\log_3 M + 3\log_3 N = \log_3(\sqrt[3]{M} * N^3)$

Also $0.008 = \frac{8}{1000} = \frac{1}{125} = 5^{-3}$, and so $\log_{0.008} 5 = -\frac{1}{3}$. So $1 + \log_{0.008} 5 = 1 - \frac{1}{3} = \frac{2}{3}$

So $\log_3(\sqrt[3]{M} * N^3) = \frac{2}{3}$

➔ $\sqrt[3]{M} * N^3 = (3)^{2/3}$

Cubing both sides

$M * N^9 = 9$

Which gives us $N^9 = \frac{9}{M}$. **Hence 2.**

Q. If $x \geq y$ and $y > 1$, then the value of the expression $\log_x\left(\frac{x}{y}\right) + \log_y\left(\frac{y}{x}\right)$ can never be **CAT 2005**

(1) −1 **(2) −0.5** **(3) 0** **(4) 1**

$\log_x\left(\frac{x}{y}\right) + \log_y\left(\frac{y}{x}\right)$

$= \log_x x - \log_x y + \log_y y - \log_y x$

$= 1 - \log_x y + 1 - \log_y x$

Now x and y are both greater than 1 and so $\log_y x$ and $\log_x y$ are both positive. They are also reciprocals of each other. But we know that for any positive value k the value of $(k + \frac{1}{k}) \geq 2$.

Thus $(\log_y x + \frac{1}{\log_y x}) \geq 2$

i.e. $(\log_y x + \log_x y) \geq 2$

So our original expression i.e.

$1 - \log_x y + 1 - \log_y x = 2 - (\log_y x + \log_x y) = 2 - (\text{a number} \geq 2) \leq 0$

So out of the options all are possible except 1. **Hence (4)**

WRITERS BLOCK – 1

Over the next few posts, I'd like to look at some very simple problems. The kind that everyone is capable of doing, really. But what I will try to emphasise is the approach. When confronted with a problem, most people start by writing stuff down. They wring an equation or three from the data, and then sit down, satisfied, to solve them, happy as a dog with a bone to chew. The order of precedence is always "first write and then think"

My contention is that this is a counter-productive strategy. **One should think first and then write**; indeed, very often once you start thinking you find that there is no need to write. Let's start with a few simple examples and then move on to slightly more complex ones. To explain it here in writing, I will have to take a variable or two, but try to visualise it without writing down anything.

Q. Mr A is thrice as old as Mr B and the difference between their ages is 24. Find Mr A's age?

Without writing, let us just imagine the ages as x and 3x. Now the difference will be 2x which is 24, thus x is 12 and so 3x, i.e. A's age, must be 36.

Q. X has 5/3 times as much money as Y, who has $40 less than X, How much does each have?

Let the amounts be 5x and 3x so the difference (2x) is $40 ⇒ x = $20 and 5x and 3x are **$100 and $40**

Q. Toby is multiplying two numbers. By mistake he reads one as 46 instead of 64 and hence the answer changes by 1836. Find the other number.

Let the number be n, so 64n – 46n = 1836 ⇒ 18n = 1836 so n must be **102**

Q) In a ballot process, 900 votes more were polled in favour of a resolution than were polled against it. It was also noticed that the number of votes polled against the resolution was 30 percent of the total number of votes polled. If all the votes polled were valid and each vote was either in favour of the resolution or against it, find the total number of votes polled.

If the number of votes **against** is 30%, the votes **for** must be 70% and the difference must be 40% of the total votes (since all votes are valid). So 40% = 900 which means 100% will be **2250.**

Q. Each of the four parties – A, B, C and D – contested in all the seats in an election. Each seat was won by exactly one of the four parties. Party A lost in 19 seats, Party B lost in 12 seats, Party C lost in 10 seats and Party D lost in 16 seats. What was the total number of seats being contested in the election?

We could make equations here using 4 variables a, b, c and d, or if options are provided, do a trial and error solution. But a simpler way of looking at it would be: every seat contested means exactly one win, but also exactly three losses (one for each of the other parties!). So since the total number of losses is 19 + 12 + 10 + 16 = 57, the number of seats must straightaway be 57 / 3 = **19**.

Some slightly tougher ones tomorrow…

WRITERS BLOCK – 2

Try each of these without writing **before** looking at the solutions:

Q. A says to B "I am four times as old as you were when I was as old as you are now". B replies "10 years back I was 9 years younger to you". How old is B?

1) 12 **2) 15** **3) 16** **4) 18**

Note that B's statement means that at any point in time, even today, B must be 9 years younger than A. Let the second part of A's statement "when I was as old as you are now" refer to a specific time t. Let B's age then be x, then A's age today is 4x and the age of B today (and of A at time t) must be halfway in between (think about it!) and is hence 2.5x. Thus the difference in their ages is 4x – 2.5x = 1.5x which is 9 years. Hence x is 6 and 4x is 24. And B's current age is 2.5x = **15 years.**

Q. The ratio of the monthly income of A and B is 3:4. The ratio of the monthly expenditure of A and B is 4:5. Which of the following represents a possible value of the ratio of their savings?

1) 9 : 10 **2) 3 : 4** **3) 13 : 20** **4) 4 : 5**

(Note the words **possible value** in the question, which implies that there are many possible values and we have to choose the one present in the answer choices. Straightaway this indicates that there is no point looking for a specific, unique answer – rather, we'll look for a logic or pattern.)

The incomes are in the ratio $\frac{3}{4}$ i.e. numerator is 75% of denominator. Now if we were to reduce the numerator and denominator both in the ratio $\frac{3}{4}$, then there would be no change in the ratio. However the reductions are in the ratio $\frac{4}{5}$ i.e. 80% which means the numerator is dropping, relatively, more than the denominator. Thus the new value should be less than $\frac{3}{4}$, and on checking the answer options, the only one such is **13 : 20.**

Q. The average age of Ram and Rita at the time of their marriage was 25 years. Two years after their marriage, a child was born to them. The present average age of all the three of them is 24 years. How many years is it since they got married?

1) 5 **2) 7** **3) 8** **4) 10**

We can do this very simply without equations, as follows:

Initially the average age of two people is 25. After two years it will be 27. but now there is a third person added (a child of age 0) so the total doesn't change! Thus the average must be 2/3rd of earlier i.e. 18 on the day the child is born.

Now at the end of the problem, this has gone up to 24, which means the age of each has increased by 6 ⇒ 6 years have passed. hence the total number of years since their marriage will be 2 + 6 = **8 years.**

Tomorrow we'll see some more examples of how to look at Arithmetic problems orally.

WRITERS BLOCK – 3

Some examples from Arithmetic (again, try to do them orally before seeing the answers!)

1. A can do a job in 20 days, while B can do same in 25 days. They started working together, and a few days later C also joined them and thus all of them completed work in a total of 10 days. If the total remuneration was Rs 1400, what is the share of C?

a) 128.57 **b) 140** **c) 350** **d) 142.85**

2. A contractor undertakes to dig a canal 10Km long in 180 days and employs 40 men. After 60 days he finds that only 2.5 km of the canal has been completed. To complete the work on the scheduled time how many more men should he hire?

a) 20 **b) 40** **c) 32** **d) 16**

3. A car and a bus are travelling along a straight road. The car overtakes the bus at point A at 9 am, reaches a point B at 1 pm and immediately turns back. It again meets the bus at point C at 2 pm. At what time will the bus reach B?

a) 6 pm **b) 3:15 pm** **c) 3:40 pm** **d) 4 pm**

4) Ram and Shyam are two friends whose houses are connected by a straight road. They walk with constant speeds (which are not necessarily the same). The speed of Shyam is 3 km/hr. One day, they start walking towards each other's houses at the same time. They meet for the first time 9 km from Shyam's house. Ram reaches Shyam's house first, and on doing so he immediately turns back and starts walking back to his own house. Some time later (but before Ram reaches his own house), Shyam reaches Ram's house, and he rests there for an hour before he starts walking back to his own house. They meet for the second time 30/7 km from Ram's house. What is the distance (in km) between the two houses?

a) 3/7 **b) 14** **c) 123/7** **d) 21**

1. A worked for 10 days, so he did 10/20 = 1/2 of the work. B also worked for 10 days, so he did 10/25 = 2/5 of the work So between them they have done 1/2 +2/5 = 9/10 (or 50% + 40% = 90%) of the work. Thus C has done a tenth of the work and will get a tenth of the money i.e. **140.**

2. The work done is 2.5, so that remaining is 7.5 i.e. thrice as much. The time gone is 60, so the time left is 120, i.e. twice as much. So to do 3 times the work in twice the time, he needs 3/2 times the men ⇒ 40 * 3/2 = 60 men ⇒ **20 more men**.

3. The car takes 4 hours for the journey from A to B. Let us call this distance 4x. So in 1 hour, to reach point C, it must have returned a distance x. So the bus has travelled 3x (A to C) in the same time that the car took for 5x (A to B to C) and hence the ratio of their speeds is 3 : 5. Once we have this, it is a simple matter to realise that their times for distance AB should be in the ratio 3 : 5 and hence the bus will take (4 hours) * 5/3 = 6 hours 40 min and will hence reach B at **3:40 pm.**

4. This is clearly a long and painful problem and cannot be solved without equations, right? Wrong.
Ram reaches the end first so he is walking faster than Shyam. Now when they first meet, Shyam walked 9 km, so Ram must have walked more than 9, so the total distance must be more than 18 km. Now look at the answer choices….and without any further ado, mark **21**.

WRITERS BLOCK – 4

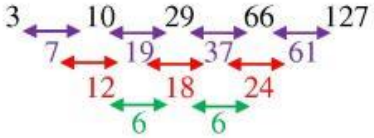
Observing patterns

Surprisingly often, we can solve a problem by taking a simple case or two and observing the pattern followed. Never forget that maths is about patterns, about logic, and hence a keen eye for those can save much effort on occasion.

1. What is the 12th term of the series 3, 10, 29, 66, 127….

a) 1568 **b) 1730** **c) 1632** **d) 1854**

If we look at this series we will find that the 3rd order differences come to be the same.



This means that it will be a cubic expression and we could solve it by assuming an nth term T_n of the form $an^3 + bn^2 + cn + d$, creating corresponding equations, and finding the coefficients. However, this would clearly be time-consuming. Instead, we could just observe closely and realise that the terms are: 3 = 1+2, 10 = 8+2, 29 = 27+2, 66 = 64+2 and so on ⇒ $T_n = n^3+2$ ⇒ $T_{12} = 1728 + 2 =$ **1730. Hence (b)**

2. What is the sum of the first 200 odd numbers?

a) 30000 **b) 40401** **c) 40000** **d) 39486**

We can use the formula for the sum of an Arithmetic Progression, of course. But a little observation will show us that 1 = **1**, 1 + 3 = **4**, 1 + 3 + 5 = **9**, 1 + 3 + 5 + 7 = **16** and all these totals are consecutive squares. So the sum of the first 5 odd numbers is 25 and that of the first 200 will be **40000. Hence (c)**

3. If P = (1111..... 2n times) - (88888..... n times) where n > 11, find the sum of the digits of P?

a) $n^2/3$ **b) n + 2** **c) 3n** **d) 9**

Let us see what happens when n = 1 ⇒ 11 – 8 = 3 ⇒ P = 3. Options (a) and (d) are eliminated.
Next try n = 2 ⇒ 1111 – 88 = 1023 ⇒ P = 6. Hence option (b) is also out. **Hence (c)**

4. The odd numbers are written in rows one after another in such a manner that the first row contains one number and each subsequent row contains one more number than the previous one:

1
3 5
7 9 11
13 15 17 19

…and so on. What will be the last number in the 100th row?

a) 10001 **b) 9099** **c) 10099** **d) 11099**

We can observe that the last number in the 1st row is 1, the 2nd is 5, the 3rd is 11, the 4th is 19 and the 5th is 29. Looking at this set of numbers, we can see that 1, 5, 11, 19, 29… can be rewritten in the form 1+0, 4+1, 9+2, 16+3, 25+4…and hence the nth row will have a last term $n^2 + (n - 1)$ ⇒ the last term of the 100th row would be $100^2 + (100 - 1) = 10099$.

WRITERS BLOCK – 5

What's the difference?

Very often, an elegant (and oral) solution can be found by comparing the two given solutions and just looking at the **difference** between them. Let's look at an example:

Q. By selling 200 oranges at 32 for a rupee, a man lost 40 %. How many must he sell for a rupee so as to gain 20%?

a) 16 **b) 50** **c) 12** **d) 20**

A typical solution to this would involve equations. For example, we could do something like this:

If he sells at 32 for a rupee, he will get 200/32 for 200 oranges. So his Loss % = 40 = 100(CP - 200/32)/CP ⇒ CP = 1000/96. Now, if Profit = 20%, then (SP - 1000/96)/(1000/96) = 1/5 ⇒ SP = 1200/96 = 200/16. Consequently, he will sell **16 for a rupee**.

However, we could do this even faster if we examine the difference between the two situations:

In the first case he is losing 40% ⇒ he is selling at **60% of CP**, in the second case gaining 20% ⇒ he is selling at **120% of the CP**. Thus the selling price has doubled, so the number of oranges per rupee must be halved ⇒ he must sell 16 **for a rupee.**

Q. Amit goes to the market to buy two dozen eggs. He finds that he has 14 rupees less so he buys only 20 eggs and returns with 2 rupees change. What is the price of an egg?

a) 6 **b) 7.5** **c) 8** **d) CBD**

The traditional approach to this would start with "let the price of an egg be x and the money with him be y" or some such ponderous statement. However, let's look at the difference between the two cases:

He bought 4 eggs less in the 2nd case, and the amount was (14+2) = 16 Rs less. So an egg costs **Rs 4.**

Q: A food processing company received a fresh consignment of dates from containing '100' kg of dates. After analysis these dates were found to contain 98% water. After 15 days, a sample of dates from the same consignment was found to contain 94% water. What must be the weight of the consignment of dates at that time?

a) 96.96 **b) 50** **c) 37.5** **d) 33.33**

Here, there is a slight twist. Let's ask ourselves: **what remains the same across the two situations?** The part which is not water (let's call it the pulp). So in the first case this constitutes 2% i.e. 2/100 of the total and in the second case the same pulp becomes 6% i.e. 6/100 of the new total. So if the fraction has tripled, with the numerator remaining the same, the denominator (as in the total quantity of dates) must have become 1/3rd. Hence the new quantity is 100 / 3 ≈ **33.33 kg.**

Let's try a few more of these tomorrow.

WRITERS BLOCK – 6

Here are a few more questions which could be solved by observing the difference in the two situations – as before, try to do them on your own, with little or no writing, before seeing the answers!

1. Three runners, A, B and C run a race with A finishing 20 m ahead of B and 34 m ahead of C, while B finishes 21 m ahead of C (Each runner running throughout at a constant speed). What was the length of the race?

a) 50 m **b) 60 m** **c) 72 m** **d) None of these**

2. A book was sold for a certain sum and there was a loss of 20%. Had it been sold for Rs. 12 more, there would have been a gain of 30%. What would be the profit if the book were sold for Rs. 4.8 more than what it was actually sold for?

a) 10% **b) 7.5%** **c) 3.125%** **d) None of these**

3. Two students appeared at an examination. One of them secured 9 marks more than the other and his score was 56% of the sum of their marks. The marks obtained by them are:

a) 39 & 30 **b) 22 & 28** **c) 42 & 33** **d) 56 & 47**

4. Samantha covers 700 km in 9 hours out of which she covers 300 km by bus and remaining by car. She takes 20 minutes more if she covers 500 km by bus and the remaining by car. Find the ratio of the speeds of bus and car?

a) 45 : 53 **b) 51 : 58** **c) 26 : 29** **d) 27 : 35**

1. Note that when A finished the race, B was 14 m ahead of C. But when B finished the race he was 21 m ahead of C. Now 14 is 2/3 of 21, so when A finished the race, B must have been at 2/3rd of the distance. But he is 20 m from the end at that point (since A beat him by 20 m); thus 1/3rd of the race must be 20, and the entire race must be **60 m** in length. **Hence (b)**

2. The difference of 50% = Rs 12 so the cost of the book must be Rs 24. So currently it is being sold at 20% less i.e. 4.8 less, so if the SP increases by 4.8 it is back to the CP and there is **0 profit. Hence (d)**

3. The two percentages are 56 and 44 and are separated by 12% of total which is 9 marks, so 4%= 3 marks (and 100% = 75 marks) which gives 44% = 11* 3 = **33** and 56% = 14*3 = **42** marks. **Hence (c)**

4. Let's see the difference between the cases; if she covers 200 kms **extra** by bus instead of by car, she takes 20 minutes more. So every 100 km by bus takes 10 minutes more as compared to 100 km by car. Now initially she covered 300 by bus and 400 by car in 9 hours or 540 minutes. So if the entire 700 km was by car instead, she would save 30 minutes ⇒ 510 minutes while if she covered the whole distance by bus it would take 40 min more ⇒ 580 minutes. But the distance being the same, speed is inversely proportional to time and so the ratio of speed of bus to car must be 51 : 58.

WRITERS BLOCK – 7

Sleight of Mind

Over the next couple of posts, let's look at some diverse examples where, just by being alert, one might be able to solve the problem startlingly fast. (Note that the basics – formulae, theorems, etc – still need to be known – there is no avoiding that!). In each case, I will also indicate (or hint at) the "standard" approach to solve them:

1. What will be the 20th digit from the left in the number $(60003)^4$?

1) 4 **2) 1** **3) 3** **4) 7**

2. A group of students is to be divided into some rows such that all the rows have an equal number of students. When 7 students are put in each row, 3 students are left; when 8 students are put in each row, 5 students are left; and when 9 students are put in each row, 7 students are left. If 12 students are put in each row, then how many students will be left?

1) 4 **2) 1** **3) 3** **4) 7**

3. From a point inside an equilateral triangle of side 3cm, 3 perpendiculars are drawn to meet the three sides. What is the sum of the length of these 3 perpendiculars?

1) 3 **2) $\frac{3\sqrt{3}}{2}$** **3) $2\sqrt{3}$** **4) $\frac{4\sqrt{3}}{3}$**

1. We know how to find the last digit, and often can manage the second-last as well. But the 20th? Immediately we need to recognise that there must be a trick. Also, it is the **20th from the left**, i.e. from the start. So let's look more closely at the number - $(60003)^4$ can be written as $(60000 + 3)^4$ which on binomial expansion would give $(60000)^4$ + (much smaller terms). Now $(60003)^4$ is going to be 6^4 followed by 16 '0's, i.e. 1296 followed by 16 zeroes ⇒ it will be a 20-digit number. And hence the question is really asking, what will be the last digit of $(60003)^4$? And that is 1. **Hence (2)**

2. The traditional way of doing this would be to say that the number must be of the form 7k+3 and 8m + 5 and 9n + 7. After doing the relevant calculations we will find that the number must be of the form 504p + 493, and when such a number is divided by 12 it will give 1. This would take a good 3 minutes.

However, if alert one might note that 12 = 4 * 3. Now if the original number is of the form 8m + 5, when divided by 4 it should leave 1. Also, if it is of the form 9n + 7, when divided by 3 it will again leave 1. So since the remainder with both 4 and 3 is 1, the remainder with 12 will also be 1. **Hence (2)**

3. We could solve it in detail, for example by using the fact that 3 triangles are formed whose areas add up to give the original area. But since there is no NOTA (none of the above) or CBD (cannot be determined) in the answer choices, we can say that the answer would be the same whatever point we chose! So let us just take the point as one of the vertices – two of the values would be 0, and the third would be the height of an equilateral triangle of side 3, which is $\frac{3\sqrt{3}}{2}$. Hence the total would also be $\frac{3\sqrt{3}}{2}$. **Hence (2)**

WRITERS BLOCK - 8

Here are a few more problems, along the same lines:

1. Find the remainder when $12^{13^{14}}$ is divided by 145.

1) 1 **2) 144** **3) 12** **4) 72**

2. 4 dice are thrown. In how many ways can a sum of 20 be obtained?

1) 1728 **2) 84** **3) 21** **4) 35**

3. There are six bottles of wine whose concentrations are 20%, 30%, 40%, 50%, 60% and 70%. Six 100 ml glasses are filled, each by taking 20 ml each from exactly five of the above bottles. In how many of these solutions, is the concentration of wine more than 45%?

1) 5 **2) 4** **3) 3** **4) 2**

4. A cylindrical hole of height 3cm is drilled through a sphere along its diameter. Find the volume of remaining part of the sphere.

1) 3π **2) 4.5π** **3) 7.5π** **4) 6π**

1. The traditional approach to this problem would be the Chinese Remainder Theorem (covered briefly in a previous post). In this we would express 145 as 29 * 5 and find the remainders independently, and eventually (after much effort) finding that $12^{13^{14}}$ leaves a remainder 12 when divided by 145.

However, an alert observer might notice that $12^2 = 144$ i.e. 145 – 1, and that $12^{13^{14}} = 12^{4k+1} = (12^2)^{2k} * 12^1$ i.e. the remainder with 145 will be 1 * 12 = 12. **Hence (3)**

2. The standard approach would be to solve the equation a + b + c + d = 20 given that none of a, b, c, d cross 6. This can be done as discussed in an earlier post on Theory of Partitioning (Distribution with upper limits on group size) by ${}^{19}C_3 - 4 * {}^{13}C_3 + 6 * {}^{7}C_3$ = **35 ways.**

However, it could be done much more easily by realising that the maximum possible total for 4 dice is 24 – so a total of 20 implies a shortfall of 4. Now we just have to distribute this shortfall among 4 dice with no limits of any kind, which can be done in ${}^{4+3}C_3 = {}^{7}C_3 = 35$ ways. **Hence (4)**

3. The average of all the 6 values is 45%. So each time we are leaving out one – if we leave out one of those which is less than 45%, the resultant average will be more than 45%, while if we leave out one of the others, it will be less than 45%. So 3 out of 6 will have concentration greater than 45%. **Hence (3)**

4. As in the triangle question from the previous post, we could find a formal geometrical solution. But again, with no CBD or NOTA options, we should realise that the value must be the same whatever sphere we choose. So let us take the simplest case of a hole of width 0 and height 3, which means the sphere itself must be of height (and diameter) 3. So its volume = $\frac{4\pi(3/2)^3}{3}$ = 4.5π. **Hence (2)**

(This is a very tricky problem to visualise, as it turns out the volume is independent of the radius! As far as I know, this question was originally popularised by one of the greatest creators and compilers of Mathematical puzzles ever – Martin Gardner. It is also called the "Napkin Ring Problem")

WRITERS BLOCK – 9

This week, let's try and solve a number of questions with minimal writing:

1) Consider a function f(x) such that f(x) = $[1+x][1+x^2][1+x^4][1+x^8]$……to ∞. The what can be said about m where m = $f\left(-\frac{1}{5}\right)$

1) m = $\frac{4}{5}$ 2) m = $\frac{5}{4}$ 3) m = $\frac{6}{5}$ 4) m = $\frac{5}{6}$

2) The sum of the first 'n' terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+$ ……. is equal to:

1] $2^n - n + 1$ 2] $n - 2^{-n} + 1$ 3] $n + 2^{-n} - 1$ 4] $2^n - n - 1$

3) Let n = 999 ... 99 be an integer consisting of a string of 2009 nines and no other digits Find the sum of digits of n^2.

1] 18072 2] 18081 3] 18080 4] 18073

4) Find the value of 'x' if x = $\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cdots}}}}$

1] $\sqrt{6}-2$ 2] $2-\sqrt{2}$ 3] $2+\sqrt{2}$ 4] $\sqrt{6}+2$

1) If we multiply f(x) by [1 – x] we get $[1-x][1+x][1+x^2][1+x^4][1+x^8]$……to ∞ which is nothing but $1 - x^{\infty}$. Now here |x| is less than 1, so x^{∞} will become 0 and f(x) * [1 – x] = 1.
So we can just write m $\left[1-\left(-\frac{1}{5}\right)\right]$ = 1 or m $\left(\frac{6}{5}\right)$ = 1 or **m = $\frac{5}{6}$. Hence (4)**

2) Put n = 1. The sum of the first 1 term should be $\frac{1}{2}$. Putting n = 1 in each of the answer choices we get (1) 2 (2) $\frac{3}{2}$ (3) $\frac{1}{2}$ and (4) 0, only one of which satisfies the required condition. **Hence (3)**

3) Let us consider a couple of simpler cases. Consider that 9^2 = 81 with sum of digits 9. Also, 99^2 = 9801 with sum of digits 18. And further, 999^2 = 998001 which has sum of digits 27. It seems only logical to conclude that 999…999^2 (2009 times) will have a sum of digits 2009 *9 = 18081. **Hence (2)**

4) This is a problem we had already discussed in the earliest days of this blog – rather than creating a 4th order equation, we can just observe two things: x has to be less than $\frac{1}{2}$ = 0.5 and also x must be $\frac{1}{(\text{less than } 2\frac{1}{4})}$ i.e. greater than $\frac{4}{9}$ i.e. 0.444

Now $\sqrt{6} \cong 2.45$ and $\sqrt{2} \cong 1.41$ and using these we can compute the answer choices approximately:

1] $\sqrt{6}-2 \cong 0.45$ 2] $2-\sqrt{2} \cong 0.59$ 3] $2+\sqrt{2} \cong 3.41$ 4] $\sqrt{6}+2 \cong 4.45$. **Hence (1)**

WRITERS BLOCK – 10

1. There are 6 students in a class and all scored different ranks in the exam. What is the probability that Harry scored higher than Mary?

1) $\frac{15}{36}$ **2)** $\frac{1}{2}$ **3)** $\frac{1}{3}$ **4)** $\frac{1}{6}$

2. 3 dice are thrown together. Exactly 1 die shows a 6. Find the probability that it is the 2nd dice.

1) $\frac{2}{7}$ **2)** $\frac{11}{36}$ **3)** $\frac{1}{3}$ **4)** $\frac{13}{36}$

3. 15 people are sitting around a circular table. What is the probability that A and B sit together?

1) $\frac{1}{7}$ **2)** $\frac{15C2}{15!}$ **3)** $\frac{13!\,2!}{15!}$ **4)** $\frac{2}{15}$

4. A box contains five red and five blue balls. A ball is picked from the box and is replaced by a ball of the other colour (for instance, if a blue ball is picked then it is replaced by a red ball and vice-versa). The process is repeated ten times and then a ball is picked from the box. What can be said about p, where p is the probability that this ball is red?

1) $0 < p \leq 0.2$ **2)** $0.2 < p \leq 0.4$ **3)** $0.4 < p \leq 0.8$ **4)** $p > 0.8$

5. Mickey and Minnie toss 5 fair coins. Mickey wins if he gets 3 or more heads. What is the probability that he wins the game?

1) 0.5 **2) 0.3333** **3) 0.75** **4) CBD**

1. We could use full-scale PNC to solve this, of course. But it might be simpler to observe the symmetry in the problem - there is no reason that either of Harry or Mary should get a preference, each of them is equally likely to score more than the other. So the probability should be $\frac{1}{2}$. **Hence (2)**

2. Again, symmetry says that there is no reason for any of the dice to be more likely to roll a six than the others. So, given that exactly one die shows a 6, each has a 1/3rd chance of being the one. **Hence (3)**

3. There are a total of 14! ways to seat 15 people in a circle. Out of these, A and B are together in 13!*2! ways. So the required probability would be $\frac{13! * 2!}{14!} = \frac{1}{7}$ ways. But there is a simpler way:

Let A be sitting in any of the seats. For each position he can occupy, there are 14 others seats B could take, 2 of which are next to A. Hence the probability is $\frac{2}{14}$ or $\frac{1}{7}$. **Hence (1)**

4. If we replaced "red" by "blue and "blue" by "red" throughout the problem, it would not change. So the problem is symmetric in terms of blue/red. Hence there will be equal probability that the final ball drawn will be blue or red i.e. 0.5. **Hence (3)**

5. Heads and tails are symmetrical; Mickey wins if he gets 0, 1 or 2 heads and loses if he gets 0, 1 or 2 tails – the two situations are equi-probable and hence again the probability will be 0.5. **Hence (1)**

WRITERS BLOCK – 11

1) Two towns A and B are 180 kms apart. Car 1 and car 2 start travelling from A and B, respectively towards each other. Their speeds are in the ratio 1 : 2 and they start at 7:00 a.m. and 8:00 a.m., respectively. After meeting at C, they return to their starting positions and again start travelling towards each other. In order to meet car 1 for the second time at C, car 2 halts at C. For how much time does it halt?

1] 1 hour **2] 1.5 hours** **3] 2 hours** **4] 3 hours**

2) B moves by taking 3 steps forward and 1 step backward (each step in 1 sec). He walks up a stationary escalator and reaches the top of the escalator in 118 seconds. However on a moving escalator he takes 40 secs to reach the top of the escalator. Find the speed of the escalator (in steps/second.)

1] 1 step/sec **2] 2 steps/sec** **3] 4 steps/sec** **4] 3 steps/sec**

3) A, B and C play a card game once every day for thirty days. The sole loser of every game gives the other two one rupee each. They keep a record of their wins and losses and at the end of thirty days, A and C together owe B Rs. 6. In how many games was B not the loser?

1] 18 **2] 20** **3] 22** **4] 24**

4) X can do a piece of work in 20 days working 7 hours a day. The work is started by X and on the second day, one man whose capacity to do the work is twice that of X, joined in. On the third day, another man whose capacity is thrice that of X, joined in and this process continued till the work was completed. In how many days will the work be completed, if everyone works for four hours a day?

1] 5 **2] 6** **3] 7** **4] 8**

1) Note that on the first trip to point C, car 1 took 1 hour more. So to go back to the start and return (i.e. to retrace the journey twice) car 1 would take 2 hours more. in other words, car 2 would have to have 2 hours. **Hence (3)**

2) B's speed is effectively 2 steps in 4 seconds (i.e. 0.5 step per second). Now when the escalator also moves, the time becomes 40 from 118, i.e. around $1/3^{rd}$. Thus the speed of (B + escalator) must be 3 times B's speed or the escalator must be 2 times B's speed i.e. 1 step per second. **Hence (1)**

3) If B had won all the games, he would have earned Rs 30. Every game he loses costs him Rs 3 (1 which he would have won and 2 which he has to give to the others). His actual winnings are Rs 6 i.e 24 less than they should have been which means, at a reduction of Rs 3 per loss, he must have lost 24/3 or 8 games. So he was not the loser in 22 games. **Hence (3)**

4) Let's find out everything in terms of X, treating an X-day as 4 hours worth of work by X. If X takes 20 days at 7 hours per day, it translates to 140 X-hours of work. At 4 hours a day, this works out to 35 X-days. Now on the 1st day, 1 X-day of work is being done, on the 2nd day, (1 + 2) = 3, on the 3rd day, (1 + 2 + 3) = 6, on the 4th day 10 and on the 5th, 15. And 1 + 3 + 6 + 10 + 15 = 35. So it will take 5 days in all. **Hence (1)**

WRITERS BLOCK – 12

1) From a 4-digit number, Raju subtracted 10 times the sum of its digits. From this result, he erased one of the digits. The remaining three digits were 0, 2 & 4. Find the digit erased.

1) 6 **2) 5** **3) 9** **4) 3**

2) A shopkeeper gives 20 successive discounts of 5%, 10%, 15% and so on. Approximately how much is the overall discount given?

1) Less than 95% **2) 95 – 98%** **3) 98 – 99.999%** **4) Over 99.999%**

3) In a building with 8 floors (excluding the ground floor), Mr. 'X' along with 4 other people enters the lift on the ground floor. Given that each person gets off the lift on a different floor, what is the probability that Mr. X gets off the lift before exactly 3 people?

1] $\frac{1}{5}$ 2] $\frac{2}{45}$ 3] $\frac{32}{35}$ 4] $\frac{3}{7}$

4) 20 children are standing in a line outside a ticket window. 10 of them have a one rupee coin each while the other 10 have a two rupee coin each. The entry ticket is priced at Rs 1. If all arrangements of the 20 children are equally likely, find the probability that 10th child will be the 1st to wait for change.

1) 210 / 10! **2) 1 / 20C10** **3) (20!–10!*10!)/20!** **4) None of these**

5) If the three altitudes drawn in a triangle ABC are 8cm, 15cm and 17cm long, which of the following is the inradius of that triangle?

1) 9 **2) 16** **3) 4** **4) None of these**

1) If the original number is **abcd** then it is equivalent to 1000a +100b +10c +d. When we subtract 10(a+b+c+d) from this, a little thought will show us that the resultant must be divisible by 9. So the sum of digits of the resultant must be divisible by 9 ⇒ the missing digit must be 3. **Hence (4)**

2) The 20th discount will be of 100%, which will bring the price to 0. **Hence (4)**

3) At first glance this is a pretty nasty question. But if we look at it closely, the number of floors etc is not really important – 5 people are getting off in order and the question is actually asking "what is the probability that Mr X is the 2nd person to get off?" Any of the 5 people in the lift should have an equal probability of being the second person to get off. So the required probability should be $\frac{1}{5}$. **Hence (1)**

4) This problem is definitely going to require a lot of calculation. So let's try and see if we can understand how the system behaves. The 1st person might have to wait if he has Rs 2. The 2nd person can wait only if the 1st person has also had to wait (since if the 1st one gave Rs 1, he would get change) so he cannot be the first person to wait. Similarly the third could be the first person to wait, but not the 4th. Thus any even numbered person can never be the first to wait. So the probability is 0. **Hence (4)**

5) The diameter of the incircle must be less than any of the altitudes (think about it!) and hence the inradius must be less than 4 (actually calculating it would be quite a chore!). **Hence (4)**

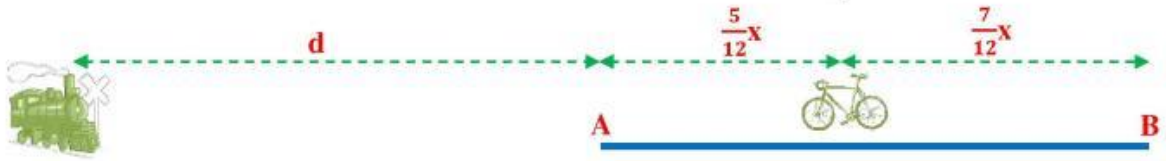
WRITERS BLOCK – 13

In each of the next few posts, we shall look at a single construct i.e. a fairly specific problem based on a certain logic. Most such problems are very hard to solve (even with writing!) the first time one encounters them. However, for each of these problems, if you have your basics strongly in place and if you are able to visualise the problem and internalise the approach, you can even solve it orally. In each case I will also try to briefly explain (or hint at) the standard method of solving it. (Many of these problems have become fairly famous over the years and so many of my readers might have seen many of them before, I ask your indulgence if I am telling you something you already know!)

Q. A man lives beside the railway track, between stations A and B, such that the ratio of his distances from A and from B is 5 : 7. He hears the whistle of a train in the distance and rushes on his cycle to catch it. If he cycles to station A, it turns out he reaches just in time to catch the train there, and if he cycles instead to B, then also he reaches exactly in time to catch the train there. Assuming the both the train and the cyclist move at their respective constant speeds, and the train stops for a negligible amount of time, find the ratio of speeds of the train and the cyclist.

1) 7 : 5 **2) 6 : 1** **3) 35 : 12** **4) 12 : 1**

The standard approach to this could be: the man is closer to A than to B so he would reach A first. So the train must also be closer to A than to B as it also reaches along with the man.



Let the distance between A and B be 'x' and the distance of train from A be 'd'.

Now since both reach A together, the man covering $\frac{5x}{12}$ and the train covering d, the ratio of their speeds must be $\frac{5x}{12d}$ …. (i)

Also in the second case, when both reach B together, the man covering $\frac{7x}{12}$ and the train covering d + x, the ratio of their speeds must be $\frac{7x}{12(d+x)}$ ….(ii)

Equating the two we get $\frac{5}{d} = \frac{7}{d+x}$ and so $d = \frac{5x}{2}$. Substituting this in (i), the ratio of speeds turns out to be **6 : 1. Hence (2)**

This relatively cumbersome method could be circumvented, though by a little careful observation. Let us look at the **difference** in the two situations (a technique we discussed last fortnight for some other problems)

Let's assume the distance between A and B to be 12 units. In the first case the man cycled 5 units and in the second case, he cycled 7, thus the **extra** distance cycled is 2 units. **In the same time,** the train ran 12 extra units (as it covered the extra distance from A to B in that time). So the ratio of speeds must be 12 : 2 i.e. **6 : 1. Hence (2)**

WRITERS BLOCK – 14

Q. There is a pool PQ of length l metres. A starts swimming from P and B simultaneously starts from Q, and they meet for the first time 13 m from P. They complete one lap and turn back and then meet for a second time 9 m from Q. If they both swim at constant speeds, what can be said about l?

1) 18 < l < 23 **2) 23 < l < 28** **3) 28 < l < 33** **4) 33 < l < 38**

The standard approach to this problem would go something like:

At the first meeting between the two, A will have travelled **13** m and B will have swum **(l – 13)** m. So the ratio of their speeds will be $\frac{13}{l-13}$

At the second meeting, A will have travelled **(l + 9)** m and B will have swum **(2l – 9)** m. So the ratio of their speeds will be $\frac{l+9}{2l-9}$

But since they are both travelling at constant speeds, we can say that these two ratios must be the same. Hence $\frac{13}{l-13} = \frac{l+9}{2l-9}$ or $\mathbf{26l - 117 = l^2 - 4l - 117}$ or $\mathbf{30l = l^2}$. But **l** cannot be 0, so **l = 30. Hence (3)**

However, another approach could be used to solve the problem without resorting to any equations.

For this, we need to first understand the mechanics of the situation. The two have started together from opposite ends, so when they meet, **together** they will have completed exactly the length of the pool **l**. By the time they move to the opposite ends, turn round and meet again, they will have covered an additional 2 lengths of the pool i.e. a total of 3**l** between them.

If this is clear, then we can proceed as follows:

By the time of their first meeting, when they have covered a total of **l**, A has covered 13 m. By their second meeting, therefore, when they have covered 3 times the distance, A should have covered 13 * 3 = 39 m. But we know that he has completed one full lap and returned 9 m at this point – which means that **l** has to be 39 – 9 **= 30 m. Hence (3)**

WRITERS BLOCK – 15

Q. A car travelling from A to B develops engine trouble at a point P. As a result, it travels thereafter at 5/6th of its usual speed and reaches B 37 minutes after the scheduled arrival. Had the trouble developed at point Q, 60 km further on from P, it would have reached 27 minutes after the scheduled arrival. Find the normal speed of the car.

1) 54 kmph **2) 60 kmph** **3) 72 kmph** **4) 75 kmph**

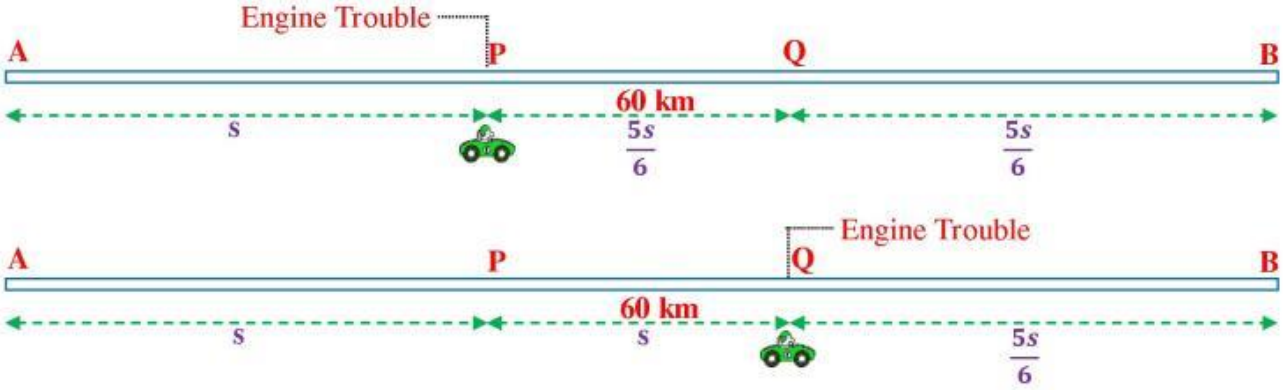
The standard approach to this, of course, would require numerous equations. So let us try to see how this kind of question could be approached orally. Let's start with a simpler case:

Suppose a train met with a accident and travels at 4/7th of its regular speed thereafter and hence reached its destination 36 minutes late. How much time would the train have taken to reach its destination from the site of the accident had it travelled at its regular speed?

If the train is travelling at 4/7th of its regular speed, it must take 7/4 of its regular time i.e. if it took time 4x earlier it should take 7x now. But the extra time is 36 minutes i.e 3x = 36 and hence 4x and 7x are 48 and 84 minutes. Hence the answer would be 48 minutes here.

Now let us try and see if we can extend this logic to the given situation:

Let us divide the route into three stretches: from A to P, from P to Q and from Q to B. Suppose the normal speed is "s". Let's see the two cases, paying particular attention to the difference between them:



Note that in both cases, the car is running at the same speed **s** on stretch PA and hence the time from P to A will be the same in both cases. Also on stretch QB it is running at $\frac{5s}{6}$ in both cases and so the time for this stretch QB too will be the same. The only place where a difference could creep in is stretch PQ.

Now in the first case, the car moves at 5/6th of its usual speed on that stretch and hence should take 6/5th of its usual time. But since we know that it takes 37 – 27 i.e. 10 minutes more in the first case, we can use the logic above to figure out that the times involved must be 50 and 60 minutes. So normally the car travels 60 km in 50 min (which is 72 kmph) and after the engine trouble, it travels 60 km in 60 min (which is 60 kmph). **Hence (3)**

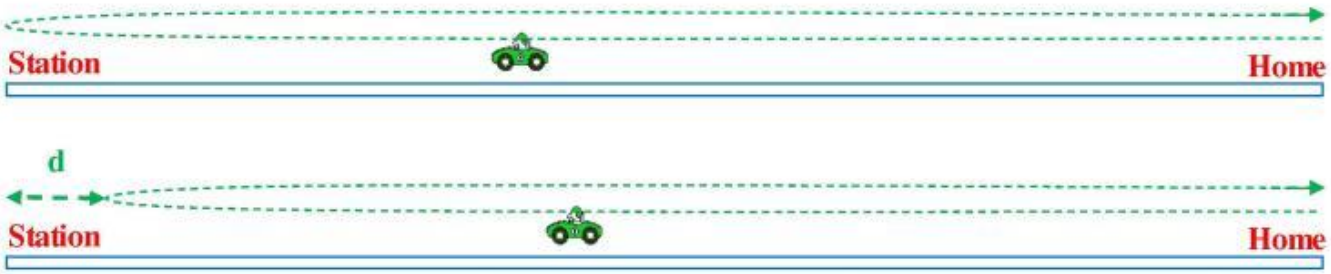
WRITERS BLOCK – 16

Q. A man normally catches the Intercity express every Friday from Mumbai to Pune, which reaches at 11 am. His father picks him up at station and they drive home. One Friday, he travels by the Indrayani express instead, arriving at Pune at 9:30 am. Rather than wait until 11, he starts walking towards home, and meets his father along the way. As a result, they reach home precisely 12 minutes earlier than usual. What is the ratio of speeds of the man and his father?

1) 1 : 14 **2) 1 : 15** **3) 1 : 12** **4) 2 : 25**

At first glance this seems like a somewhat dicey problem. The information provided is surely too little to solve it! But let us think about it a little…

Let us assume the man walked a distance d before he was picked up. Now this means that the father drove a distance of d less on his journey to the station, and also a distance d less on his journey back.



But since he saved 12 minutes in this exercise, he must be saving 6 minutes each way, and hence must be picking up his son 6 minutes earlier than usual (i.e. at 10:54 am). So the son walked from 9:30 to 10:54 i.e. for 84 minutes.

Now the father would have covered a distance d in 6 minutes while his son traverses the same distance in 84 minutes, and hence the ratio of their speeds of the man and his father must be **1 : 14. Hence (1)**

Once this logic is internalised, we will realise that we don't even need to know the specific timing at which he arrives. For example if faced with a problem like this:

Q. Julie's mother comes to pick her up from school. She starts from home in her car at such a time that she will reach the school when it gets over, exactly in time to pick up Julie. One day, Julie's school gets done 2 hours early and, rather than wait for 2 hours for her mother, Julie starts walking home. Her mom picks her up along the way and they reach home 30 minutes earlier than usual. Find the ratio of their speeds.

Now we can just say – if they reach home 30 minutes earlier, her mother must have saved 15 minutes each way, i.e. she must have turned back 15 minutes earlier. But that means Julie was walking for 120 – 15 = 105 minutes to cover the distance which her mother would have taken 15 minutes to cover. Consequently, the ratio of their speeds must be 105 : 15 i.e **7 : 1.**

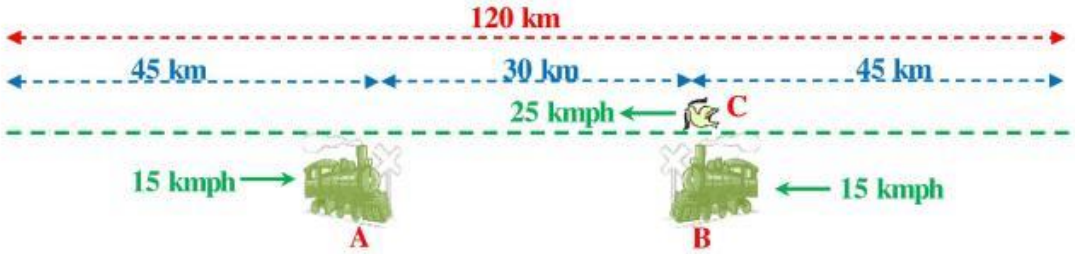
WRITERS BLOCK – 17

Q. Two trains A and B, travelling at speeds of 15 kmph are approaching each other on the same railway track. A bird C is sitting on the engine of train A. When the trains are 120 km apart, the bird starts flying at 25 kmph towards B. The moment it meets B, it turns back and flies towards A. It keeps this pattern up until the two trains collide. What is the total distance the bird will cover in the process?

1) 83.33 km **2) 120 km** **3) 75 km** **4) 100 km**

The standard way to solve the problem would be:

Let us look at the bird C and train B. The relative speed of the two is 25 + 15 = 40 kmph and so they will cover the distance of 120 km in 3 hours. In that time, A will have covered 3 * 15 = 45 km, B too will have covered 45 km and C will have covered 75 km.



Now C turns round, and the distance between C and A is 120 – 45 – 45 = 30 km while the relative speed is again 40kmph and so the time taken will be 30/40 = $\frac{3}{4}$ hour, in which time each of A and B will travel 11.25 km and C will travel 18.75 km. The distance remaining is 30 – 2 * 11.25 = 7.5 km.

Continuing in this fashion we find that each time, the time taken will become $\frac{1}{4}$ of that in the previous step, and so will the distance covered. Effectively this results in a infinite geometric series and so we can find the distance travelled by the bird as $75 + \frac{75}{4} + \frac{75}{16} + \frac{75}{64} + \ldots = 75 * (1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots)$

$= 75\left(\frac{1}{1-\frac{1}{4}}\right)$ **= 100 km.**

However, we could also have solved this problem in a much more straightforward manner by noting that the bird keeps flying till the trains collide. Now the trains would have a relative speed of 15 + 15 = 30 kmph which means they will cover the distance of 120 km separating them in $\frac{120}{30}$ i.e. 4 hours. This implies that the bird must be flying for 4 hours, at its constant speed of 25kmph, and must therefore have covered 25 * 4 **= 100 km. Hence (4)**

WRITERS BLOCK – 18

Let us look at a couple of other questions, based on the question we solved yesterday:

Q. Alexander and Porus are marching towards each other. Their armies move at speeds of 11 kmph and 9 kmph respectively. When they are 100 km apart, Alexander sends a threatening message to Porus by pigeon. Porus immediately send a defiant reply and the two keep exchanging messages with no delays till the two armies clash. If all the pigeons travel at a uniform speed of 13 kmph, how much distance will have been covered by the pigeons in total?

Note that in this case the individual speeds are fairly awkward numbers and thus if we were to try and derive the geometric series (the first approach we tried in the last problem) it would be a very troublesome affair. At the very first step the time taken would be 4.54 hours (not to mention the fact that the GP would start repeating after two steps as the two armies have different speeds!)

However, applying the secondary logic, we find this to be a problem of a few seconds only – the armies have a relative speed of 11 + 9 = 20 kmph and hence will take 5 hours to meet, in which time the pigeons will have covered a total of 13 * 5 = 65 km.

Q. A thief runs away from the local prison at 4 pm, running at 8 kmph. At 5 pm, a policeman starts chasing him at 10 kmph. The policeman is accompanied by a dog, which runs up to the thief at 15 km/h, then returns to the policeman, again runs up to the thief and returns to the policeman, and so on, till the policeman catches the thief. Calculate the total distance travelled by the dog in the forward direction.

1) 40 km **2) 50 km** **3) 55 km** **4) 60 km**

Using the logic we arrived at in yesterday's question, we can see that the dog will keep running until the policeman catches the thief. Now the thief has an initial lead of 8 km (in the 1 hour from 4 to 5 pm) and the policemen is catching up to him at a speed of 10 – 8 i.e. 2 kmph. So we can see that he will catch the thief in 4 hours. In this time, the dog will have run 4 * 15 = 60 km.

However, the question is asking how much distance the dog travels **in the forward direction**. To calculate this, we could (as in the previous problem) create a geometric series and consider only the alternate (i.e. forward) terms, adding them to infinity. This would be a truly cumbersome and time-consuming approach. Instead, we could consider a simpler way: in 4 hours, the dog has run a total of 60 km. In the same time, however, the policeman has covered 4 * 10 = 40 km and at the end of the story the dog and the policeman (and for that matter, the thief) are all at the same place. So for the dog to have covered 60 km and still end up only 40 km away from its starting point, it must have travelled half the extra distance forward, and the remaining half in the reverse direction (i.e of the extra 20 km that the dog has run as compared to the policeman, it must have run 10 forward and 10 back) and so the total distance in the forward direction must be 40 + 10 = **50 km. Hence (2)**

If this last step is tricky to follow, we can think of it using variables – let the distance in the forward direction be f and in the reverse direction be r. Then we have f + r = 60 and f – r = 40 which gives f = 50 and r = 10. So the required distance is **50 km**.

WRITERS BLOCK – 19

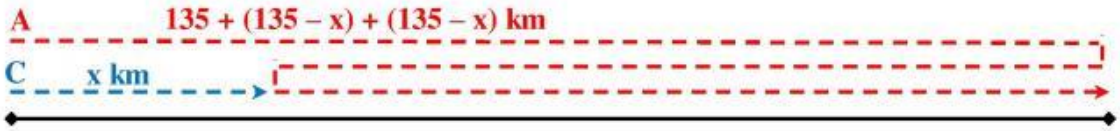
Q. There are three friends, A, B and C. A has a motorcycle (which can carry only two people at a time) and B and C can either ride with him or walk. A's motorcycle travels at a speed of 45 kmph while B and C walk at a steady 5 kmph. One day, the three friends decide to go to Kashipur, 135 km away, for a picnic. What is the earliest time they can all reach their destination, if they set out at 9 a.m.?

1) 3: 48 p.m. **2) 4 p.m.** **3) 4.48 p.m.** **4) 5.08 p.m.**

The first gut feeling for most people is that A should take B with him, ride to the end and then come back and pick up C who has meanwhile walked for a while. On reflection, however, we can see that there is a better option; A should drop B a little before the end, turn back and pick up C, timing the whole procedure in such a way that they reach the end at the same time as the walking B.

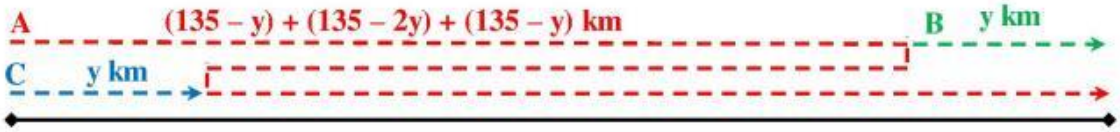
Let us examine these two cases. Note that the ratio of speed of A and B (or C) is 45 : 5 i.e. 9 : 1

In the first case, where A drops B to the end, and comes back to pick up C, let C have walked a distance x before he gets picked up, as shown in the figure below.



So in the time C has taken to travel x, A has travelled a distance of 135 + (135 - x) = 270 – x km. So we can say that 270 – x = 9x (since A travels 9 times as fast) and so 270 = 10x or x = 27. So in all, A must have travelled 135 + (135 – x) + (135 – x) = 405 – 2x km = 405 – 54 = 351 km. This will take a time of 351 / 45 = 7.8 hours or 7 hours 48 minutes.

In the second case, where A drops B some distance from the end and goes back for C, let us assume C walks a distance y. If we think about it, since B and C are travelling at the same walking speed, B also must walk the same distance y after A drops him to ensure that he reaches at the same time as A and C (the symmetry of the problem w.r.t. B and C also ensures this).



Here, in the time C takes to travel y, A will have travelled (135 – y) + (135 – 2y) = 270 – 3y. Hence, taking the ratio of speed into consideration, we can say 270 – 3y = 9y i.e. 270 = 12y or y = 22.5. Now we will see that for the entire trip, A has to cover (135 – y) + (135 – 2y) + (135 – y) = 405 – 4y = 315 km and hence will take 315 / 45 = 7 hours, which is less than in the other case.

Hence they will reach, at the earliest, 7 hours after 9 am i.e. **at 4 pm. Hence (2)**

WRITERS BLOCK – 20

Q. There are two vessels containing different qualities of milk - A and B having different prices per litre, their volumes being 40 litres and 60 litres respectively. An equal amount of milk Q was removed from both, and the milk removed from A was added to B and vice versa. As a result the two vessels now contain milk having the same price per litre. Find Q.

1) 20 **2) 24** **3) 25** **4) None of these**

The typical approach to this problem would go something like:
Initially, let the milk in A be of x% concentration and the milk in B of y%.
After the mixing, let both be of z% concentration.
Now when Q litres are taken out of A and replaced by an equal quantity from B, we can apply alligation to this mixture (i.e. Q litres from B and 40 – Q litres from A), giving us: $\frac{x-z}{z-y} = \frac{Q}{40-Q}$

Similarly when we apply alligation to vessel B, it tells us that $\frac{x-z}{z-y} = \frac{60-Q}{Q}$

Combining the two, we can say that $\frac{Q}{40-Q} = \frac{60-Q}{Q}$
which when solved gives us **Q = 24**

However, we could also solve the problem directly, in a single step, thus: $\mathbf{Q = \frac{60*40}{100}}$ **= 24 litres**

The logic is as follows:

Initially we have 40 litres of A and 60 litres of B so the total available (100 litres) should have these two in a ratio 40: 60. Hence at the end each of the two containers should also hold 40 : 60 ratio of the two types, i.e $\frac{40}{40+60}$ of each will be of the first type and $\frac{60}{40+60}$ of the second type.
So the first container contains $\frac{60}{40+60}$ * 40 of the second type which is nothing but Q, as Q litres of the second type have been brought to the first container!
Alternatively, using the second container, we would observe it to contain $\frac{40}{40+60}$ * 60 of the first type.
Either way, Q is directly $\frac{60*40}{60+40}$ or $\frac{60*40}{100}$

Let's look at another similar problem:

Q. There are two flasks M and N containing 130 ml and 180 ml of two different concentrations of acid. Now 'x' ml is withdrawn from each flask, and then the amount withdrawn from M is added to N and vice versa. Finally the concentrations in both flasks are found to be equal. Find 'x'.

1) 56.66 ml **2) 75.5 ml** **3) 77.5 ml** **4) 83.33ml**

Here we can immediately solve it by saying $\mathbf{x = \frac{130*180}{130+180}}$ **≈ 75.5 litres** (try solving it by alligation to get an idea of the time it would take!)

WRITERS BLOCK – 21

CAT Scan: A few questions from CAT and other tests which could be solved without writing

1. Find the sum $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots\ldots\ldots\ldots+\sqrt{1+\frac{1}{2007^2}+\frac{1}{2008^2}}$

1) $2008-\frac{1}{2008}$ **2)** $2007-\frac{1}{2007}$ **3)** $2007-\frac{1}{2008}$ **4)** $2008-\frac{1}{2007}$ **5)** $2008-\frac{1}{2009}$

2. If the sum of the first 11 terms of an arithmetic progression with first term a equals that of the first 19 terms, then what is the sum of the first 30 terms?

1) 11a **2)** 15a **3)** 11a + 209 **4)** $\frac{1}{2}(11a+15a^2)$ **5) None of these**

3. Consider the sets = (2, 3, 4,...., 2n+1}, where n is a positive integer larger than 2007. Define X as the average of the odd integers in S and Y as the average of the even integers in S. What is the value of X-Y?

1) 0 **2)** 1 **3)** $\frac{n+1}{2n}$ **4)** $\frac{1}{2}n$ **5)** 2008

4. p,q,r are positive numbers such that $p \neq q \neq r$ **and p+q+r = 1.Which of the following options is true?**

1) $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 0$ **2)** $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 9$ **3)** $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=9$ **4)** $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\geq 9$

1. Notice that the last term goes till 2008 (not surprisingly, as this was a question in CAT 2008!) and all the options are also in terms of values close to 2008. Let us look at a simpler case – the first term, which ends in a 2 rather than a 2008. $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}=\sqrt{\frac{9}{4}}=\frac{3}{2}$. Now look at the answer choices, replacing a 2008 by a 2 in each case (so 2007 becomes 1 and 2009 becomes 3):

1) $2-\frac{1}{2}=\frac{3}{2}$ 2) $1-\frac{1}{1}=0$ 3) $1-\frac{1}{2}=\frac{1}{2}$ 4) $2-\frac{1}{1}=1$ 5) $2-\frac{1}{3}=\frac{5}{3}$. Only (1) matches. **Hence (1)**

2. If the sum of the first 11 terms equals that of the first 19, then the sum of terms 12 to 19 must be 0 and so their average also must be 0. So the average (and sum) of terms 11 to 20 must also be 0, as must 10 to 21, 9 to 22…1 to 30. So the sum of the first 30 terms would be 0 i.e. none of these. **Hence (5)**

3. Note that there are an even number (2n) of consecutive integers – half of them (n) will be odd and half even. Also, each pair of numbers (2, 3), (4,5)… will have one odd and one even, with O = E + 1. So if the sum of all the n even numbers is S, the sum of all the odd ones will be S + n and hence the averages will be S/n and (S+n)/n i.e S/n + 1 i.e. the difference in averages will always be 1. **Hence (2)** (We could also have plugged in some small values and checked, the "positive number greater than 2007" condition is just there to intimidate you and prevent you from trying tricks like this!)

4. Again, we could plug in a few values which sum up to 1 and check – if we try p, q and r as $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{3}$, we find the value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ to be 9. Trying $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, we find the value to be 10. The only possible answer, then, is ≥ 9. **Hence (4)**

WRITERS BLOCK – 22

1. A group of 30 sets A_1, A_2, A_3, till A_{30} has 5 elements each. Another group of n sets B_1, B_2, till B_n has 3 elements each. Every element is a member of exactly 10 A-type sets and exactly 9 B-type sets. What is the value of n?

1) 27 **2) 30** **3) 45** **4) 90**

2. The distance between A and B is 100 km. A and B simultaneously start cycling towards each other at 14 km/hr and 11 km/hr. A housefly starts from A at 20 km/hr, meets B, returns, meets A till both of them meet. What is the distance covered by the fly?

1) 100 km **2) 59.3 km** **3) 75 km** **4) 80 km**

3. Three people stood for the club election. Every member of the club voted for exactly 1 person. 200 did not vote for A, 150 did not vote for B and 250 did not vote for C. How many members does the club have?

1) 250 **2) 300** **3) 450** **4) 600**

4. What can be said about the value of $3^{\sqrt{2}}$?

1) $0 < 3^{\sqrt{2}} < 1$ **2) $1 < 3^{\sqrt{2}} < 3$** **3) $3 < 3^{\sqrt{2}} < 6$** **4) $6 < 3^{\sqrt{2}} < 9$**

1. This question is tricky in wording but actually not nearly as complicated as one would expect. Let's look at A-type sets. 30 sets with 5 elements each implies 150 elements (if all of them were distinct). However, each of the elements present is repeated 10 times, and so to get 150 elements we need only 15 **distinct** elements. Since each of these elements occurs 9 times in the B-type sets, the B-type sets in total must contain 15 * 9 = 135 elements. But since each B-type set contains only 3 elements, there must be 45 such sets. So n = 45. **Hence (3)**

2. This is similar to the problem we looked at in Singles – 5, and so can be solved in a few seconds – the cyclists have a relative speed of 25 kmph, and hence will take 4 hours to cover 100 km and collide; during this time the bird will travel 20 * 4 = 80 km. **Hence (4)**

3. Each member of the club votes for 1 person. Hence he/she "does not vote" for 2 people. The total number of "did not vote"s is 200 + 150 + 250 = 600. So there must be 600/2 = 300 voters. **Hence (2)**

4. $3^{\sqrt{2}}$ is roughly equal to $3^{1.41}$. Now $3^{1.41}$ must be between 3^1 and 3^2 i.e. between 3 and 9. This eliminates the first two options. On thinking a bit, we would realise that $3^{1.5}$ would be nothing but $3\sqrt{3}$ which is 3 * 1.73 ≈ 5.19. So now we can say that $3^{1.41}$ must be less than $3^{1.5}$ and hence must be between 3 and 6. **Hence (3)**

WRITERS BLOCK – 23

1. If a man cycles at 10 km/hr, then he arrives at a certain place at 1 p.m. If he cycles at 15 km/hr, he will arrive at the same place at 11 a.m. At what speed must he cycle to get there at noon?

1) 11 km/hr **2) 12 km/hr** **3) 13 Km/hr** **4) 14 Km/hr**

2. Mayank, Mirza, Little and Jagbir bought a motorbike for $60. Mayank contributed half of the total amount contributed by others, Mirza contributed one-third of total amount contributed by others, and Little contributed one-fourth of the total amount contributed by others. What was the money paid by Jagbir?

1) $12 **2) $13** **3) $18** **4) $20**

3. A rich merchant had collected many gold coins. He did not want anybody to know about them. One day, his wife asked. "How many gold coins do we have?" After pausing a moment, he replied, "Well, if I divide the coins into two unequal numbers, then 48 times the difference of the numbers is equal to the difference of their squares". The wife looked puzzled. Can you help the merchant's wife by finding out how many gold coins the merchant has?

1] 48 **2] 96** **3] 32** **4] 36**

4. It takes 6 technicians a total of 10 hours to build a new server from Direct Computer, with each working at the same rate. If six technicians start to build the server at 11 am and one technician per hour is added beginning at 5 pm, at what time will the server be complete?

1] 6:40 pm **2] 7:00 pm** **3] 7:20 pm** **4] 8:00 pm**

1. If the speed has gone from 10 to 15 i.e 3/2 times, the time must be 2/3. Since this is saving 2 hours (reaching at 11 instead of 1), we can say that the times involved must be 4 hours and 6 hours and hence the starting time must be 7 a.m. and the distance must be 10 * 6 or 15* 4 i.e. 60. Now to reach by 12, this distance must be covered in 5 hours, and the speed should be 60/5 i.e. 12 km/hr. **Hence (2)**

2. If Mayank contributed half the amount of the others, he must have contributed $1/3^{rd}$ of the total. (If *you can't visualize this, think of it this way: M + 2M = T hence M = T/3)*. So he paid $20. Similarly Mirza paid $1/4^{th}$ of the total i.e. $15 and Little paid $1/5^{th}$ of the total i.e. $12. Together, they have paid $47 and hence Jagbir must have paid the balance i.e. $13. **Hence (2)**

3. This super-long question turned out to be super-easy for those who read it till the end and thought for half a second – if the numbers are a and b then $48(a - b) = (a - b)^2 = (a - b)(a + b) \Rightarrow a + b = 48$. **Hence (1)**

4. The total man-hours required are 6*10 = 60. From 11 a.m. till 5 p.m. 6*6 = 36 man-hours are done, leaving 24. Now in the subsequent hours, 7, 8 and so on man-hours are completed; we see that in 3 hours 7 + 8 + 9 = 24 man-hours will be accomplished and the work will be done. So the completion time is 8 p.m. **Hence (4)**

WRITERS BLOCK - 24

1) If the product of n positive real numbers is unity, then their sum is necessarily

1) a multiple of n **2) equal to n + 1/n** **3) never less than n** **4) a positive integer**

2) If $a_1 = 1$ and $a_{n+1} - 3a_n + 2 = 4n$ for every positive integer n, then a_{100} equals

1) $3^{99} - 200$ **2) $3^{99} + 200$** **3) $3^{100} - 200$** **4) $3^{100} + 200$**

3) The integers 1, 2, …, 40 are written on a blackboard. The following operation is then repeated 39 times: In each repetition, any two numbers, say a and b, currently on the blackboard are erased and a new number a + b – 1 is written. What will be the number left on the board at the end?

1) 820 **2) 819** **3) 781** **4) 780**

4) There is a tunnel connecting A and B. A cat is standing at 3/8th of the length of the tunnel. It hears the whistle of an approaching train and starts running towards the entrance of the tunnel, reaching just in time to avoid being crushed. If instead it had run towards the exit, then too it would have just reached in time to avoid being crushed. What is the ratio of speeds of the train and the cat?

1) 4 : 1 **2) 1 :2** **3) 8 : 1** **4) None of these**

1. Let's try it with n = 2. Take two numbers whose product is 1 (say ½ and 2). Their sum is 2½ which eliminates the 1st and 4th options. Now let's try n = 3. Consider three numbers whose product is 1 (for example ½, 1 and 2). The sum is 3½, which eliminates option 2 as well. **Hence (3)**

2. All the options for a_{100} are in terms of 100 or close numbers. So let us replace 100 by 2 throughout. Now $a_2 - 3a_1 + 2 = 4$ implies that a_2 is equal to 5. Now the corresponding options are:

1] $3^1 - 4 = \mathbf{-1}$ 2] $3^1 + 4 = \mathbf{7}$ 3] $3^2 - 4 = \mathbf{5}$ 4] $3^2 + 4 = \mathbf{13}$

And hence the answer should be the 3rd option. **Hence (3)**

Note: A really alert test-taker might go a step further and avoid even the minor calculation of a_2 by evaluating the options for n = 1:

1] $3^0 - 2 = \mathbf{-1}$ 2] $3^0 + 2 = \mathbf{3}$ 3] $3^1 - 2 = \mathbf{1}$ 4] $3^1 + 2 = \mathbf{5}$

and directly marking (3), as $a_1 = 1$ is given!

3. One way of doing this would be to write the numbers, say, 1 to 5 and see what happens.

Another way would be to realise that what we are effectively doing is adding together all the numbers over the course of several operations, and also subtracting 1 at every step. Since there are 39 steps, 39 must be subtracted. Hence we can say that the final total will be (1 + 2 + 3 + 4 +…+ 40) – 39 = 781. **Hence (3)**

4. This CAT 02 question is the same kind as discussed in Singles – 1 – now we can just say that if we assume the tunnel to be of length 8, the cat covers 2 units in the same time in which the train covers 8 and hence the ratio of speeds of train and cat should just be 4 : 1. **Hence (1)**

PUZZLES - 1

The best way to solve such a problem is to work backwards.

First, though, let us understand that a strategy in such a game would mean determining the appropriate response to each move by the opponent. So rather than looking at a move in isolation, let us look at a pair of moves together – Jack's move followed by Jill's response.

Jack can pick any number from 1 to 7 and in response Jill, too can pick any number from 1 to 7. The total can therefore range from 2 to 14. The question is – is there any total which Jill can guarantee whatever Jack plays? The answer is yes – a total of 8 can always be guaranteed across 2 consecutive turns. If Jack picks 1, Jill can pick 7 in response, if he picks 2, she can pick 6 and so on. We can easily check that no other total can be guaranteed.

Now in order to win, on her last move, Jill has to leave exactly 1 coin, thus forcing Jack to pick the last coin. But we know that Jill can guarantee 8 coins being removed in a pair of moves.

Hence, on her penultimate move, Jill should ensure that there are 1 + 8 = 9 coins left (so that after removing 8 coins, 1 will remain) This in turn means that on the move before, she needs to leave 9 + 8 = 17 coins and so on.

Working backwards, we see that at any stage she should leave a number of coins of the form 1 + 8k and the largest such number below 100 is 97. Thus she should pick 3 coins on her first turn, and subsequently, whatever number "x" Jack picks, Jill should respond by picking "8 – x" coins.

Let's reinforce this idea with another similar question:

Q. There are 100 coins in a pile. Humpty and Dumpty take turns to pick coins from the pile. In a turn, each person can pick a minimum of 1 and a maximum of 10 coins. The person who picks the last coin loses the game. Humpty offers Dumpty the first move. Assuming that both of them play logically, how many coins should Dumpty pick in order to guarantee his win?

In this case since they are picking from 1 to 10 coins, the total number guaranteeable across two turns is 11. So Dumpty should leave 11k+1 coins. Unfortunately, 100 is already of the form 11k + 1 and the next number of this form, 89, cannot be achieved. Whatever number (say "n") Dumpty picks, Humpty only needs to respond by picking "11-x" coins and he will eventually win.

Hence Dumpty cannot win the game!

Let's look at another problem now:

Q. Basanti has 25 coins. She plays a game with her friend in which they toss the coins in the air and count the number of heads. If there are more heads than tails, Basanti will win, but if there are more tails she will lose. What is the probability that Basanti wins the game? (Assume that all the coins are unbiased).

PUZZLES – 2

We can solve this mathematically by using the logic of binomial probability. Since the coins are unbiased, for each coin the probability of getting a head is $\frac{1}{2}$, and so is that of getting a tail. So the probability of getting r heads when 25 coins are tossed (and hence (25 – r) tails) is given by ${}^{25}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{(25-r)}$ which works out to ${}^{25}C_r\left(\frac{1}{2}\right)^{25}$ or $\frac{{}^{25}C_r}{2^{25}}$

Now for Basanti to win she has to get 13 or more coins showing heads; hence the probability would be $\frac{1}{2^{25}}\left({}^{25}C_{13} + {}^{25}C_{14} + {}^{25}C_{15} + \cdots + {}^{25}C_{23} + {}^{25}C_{24} + {}^{25}C_{25}\right)$

But knowing the properties of Combinations, we can show that $\left({}^{25}C_{13} + {}^{25}C_{14} + {}^{25}C_{15} + \cdots + {}^{25}C_{23} + {}^{25}C_{24} + {}^{25}C_{25}\right) = 2^{25-1} = 2^{24}$.

Hence the answer will be $\frac{2^{24}}{2^{25}} = \frac{1}{2}$

However, it is unlikely that you would be expected to do this in an interview – and if you tried, chances are that in the pressure of the situation, you would get it wrong! So let's look at it from a much simpler point of view, that of symmetry:

Heads and tails are symmetrical in this probably (as they are equiprobable on a given toss). So getting 25 heads should be equivalent (in probability terms) to getting 25 tails … which is the same as getting 0 heads. Similarly, 24 heads will be just as probable as 1 head and so on. So the probability of (13 or 14 or 15….or 25 heads) is the same as the probability of getting (12 or 11 or 10…or 0 heads). But since these two cases between them cover all possible outcomes, their total probability must be 1 and hence each of these two events must have probability $\frac{1}{2}$. (Note that this will work only if the total number of coins is odd – if even there will also be the case where there are an equal number of heads and tails!)

If we have understood this, then we can try and attack another similar question:

Q. Jai has exactly 100 coins while Veeru has 99. Both players throw all of their coins in the air simultaneously and observe the number that come up heads. The one who gets more heads will win. Assuming all the coins are fair, which of the two is more likely to win?

We can say that either Jai will get more heads or he will get more tails than Veeru. Since heads and tails are symmetric in this problem, again the required probability will be $\frac{1}{2}$. (Think about it – it is somewhat tricky to visualise!)

Here's the next puzzle:

Q. A gang of 9 pirates find a chest with 999 coins. They divide it as follows. First the eldest proposes a division. If at least 50% support him, then it is accepted. If not, he is killed, and the next oldest pirate proposes a division. This goes on until an acceptable division is proposed.

What is the maximum that the first pirate can get for himself, assuming all the pirates to be logical?

PUZZLES – 3

At first glance, it seems that the eldest pirate should give 1/9th of the coins to each of the others. That only seems fair, right? However, a quick check will show that this cannot be the right approach – if it was, then the other pirates should just kill him and then have 1/8th each (and so on). So one would expect the answer to be less than 111. However, it turns out to be more than 111, far more!

The thing we have to keep in mind in such problems is that the protagonists are "Perfectly Logical Beings" - which means that they are smart, logical, can do mental calculations astonishingly fast….and know that the others also share these qualities and can do the same calculations.

Let's consider the pirates, in decreasing order of age, as A, B, C, D, E, F, G, H and I. Since 9 pirates are not that easy to deal with, let's start with a simpler case.

Suppose there are just 2 pirates left, H and I. Now H gets to propose a division and he will just say 999 – 0 i.e. keep all 999 coins for himself. Since he needs "at least 50%", his vote alone is enough.

Now let us go a step back and assume 3 pirates left, G, H and I. G has to propose a division, and he knows that he needs one more pirate to support him. He also knows that if he dies, I will get 0 coins and that **I is also aware of this fact**! So all he needs to do is say 998 – 0 – 1 i.e. offer I a single coin and I will support him (something is better than nothing!)

When we have 4 pirates F, G, H and I, the division is in the hands of F and he needs one supporter. He knows that if he dies, H will get nothing. So he can say 998 – 0 – 1 – 0 i.e. offer 1 coin to H.

In the case of 5 pirates, E needs 2 more supporters and will have to say 997 – 0 – 1 – 0 – 1 offering a coin apiece to G and I.

For 6 pirates, D needs 2 supporters and will say 997 – 0 – 1 – 0 – 1 – 0 giving a coin each to F and H

For 7 pirates, C needs 3 more supporters and hence will say 996 – 0 – 1 – 0 – 1 – 0 – 1 giving a coin each to E, G and I

For 8 pirates, B needs 3 more supporters and hence will say 996 – 0 – 1 – 0 – 1 – 0 – 1 - 0 giving a coin each to D, F and H

For 9 pirates, A needs 4 more supporters and hence will say 995 – 0 – 1 – 0 – 1 – 0 – 1 – 0 – 1 giving a coin each to C, E, G and I

Thus A can keep 995 coins for himself! *(Note: if you should ever become a pirate, gentle reader, do not try this. In real life, most pirates tend to be depressingly illogical…)*

Q. The pirate ship "Black Diamond" sinks, but Captain Cook, Captain Kidd and Captain Flint manage to escape on a motorised lifeboat. They are a month from the nearest land and have enough food for only 20 days. So they decide that one of them must die. They put 9 identical coins (8 dated 1799 and 1 dated 1798) in a bag and take turns to pick. The one picking the 1798 coin must die. Kidd, being the youngest, is given the choice of picking 1st, 2nd or 3rd.

How can he maximise his chances of survival?

PUZZLES – 4

For convenience, let the 3 pirates be A, B and C (in order of picking).

The first reaction of many people to this question is to do a hasty (and superficial) probability calculation, saying that the chance for A to pick the coin on his first try is $\frac{1}{9}$, the chance for B is $\frac{1}{8}$ and that for C is $\frac{1}{7}$. Similarly on further rounds A will always have the smaller number etc etc … and hence it makes sense to pick first. However, this can easily be disproved as we see that if we add up $\frac{1}{9}+\frac{1}{8}+\frac{1}{7}+\frac{1}{6}$ … to 9 terms, the total comfortably crosses 1 (and probability can never be greater than 1!) So then, what are we doing wrong here?

The problem with this approach is – we are forgetting one essential thing; for B to pick the coin in his first turn, a necessary precondition is that A should not have already picked the coin in his first turn!

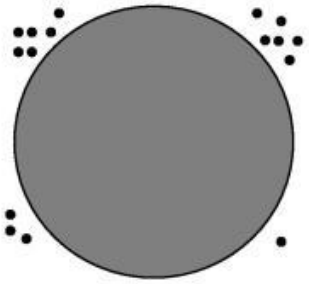
So while the probability for A is indeed $\frac{1}{9}$, the probability for B is P(A does not pick the coin **and** B picks it) $=\frac{\cancel{8}}{9} * \frac{1}{\cancel{8}} = \frac{1}{9}$. And the probability for C to get the coin on his first try will be P(A and B both don't pick the coin and C does) $= \frac{\cancel{8}}{9} * \frac{\cancel{7}}{\cancel{8}} * \frac{1}{\cancel{7}} = \frac{1}{9}$. In fact o each turn, the chance of the one whose turn it is to pick the coin will be the same i.e. $\frac{1}{9}$ and so, since each of them gets 3 chances, each of them will have a probability of dying of $3 * \frac{1}{9} = \frac{1}{3}$. So it doesn't matter which position he chooses.

We could have done this more logically, without so much calculation – when the three are picking in order, the 1st, 4th and 7th coins will go to A, the 2nd, 5th and 8th to B and the 3rd, 6th and 9th to C. So let us just throw the coins against a wall so that they come to rest in a straight line. Now pick the 1st, 4th and 7th and give them to A and so on. Now since the coins have been thrown randomly, the 1798 coin is equally likely to be in any of the 9 positions and hence the probability for each is $\frac{1}{9}$. This leads us to the same answer as above, that the order of picking doesn't matter.

Q. What if there are 20 coins and the same rules hold?

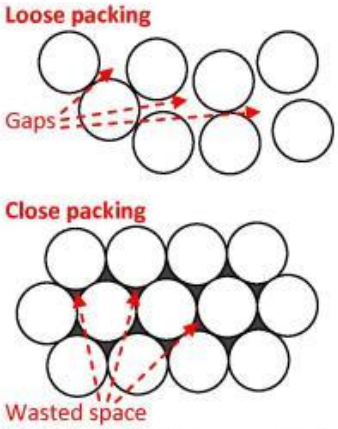
In this case, each coin has a probability $\frac{1}{20}$. But A will pick 7 of the coins, as will B, while C will pick only 6, and hence one should choose to be C (i.e. to pick last) to maximise his chances of survival.

Q. The figure shows a round table of diameter 2 m. Peter Pan and Wendy have an infinite supply of coins of diameter 1 cm. They each take turns to place a coin flat on the table such that (a) no coin overlaps any other coin and (b) no coin overhangs the table's edge. A coin once placed cannot be moved. The person who cannot place a coin loses. Wendy has to choose whether to play 1st or 2nd. What should she do to win?



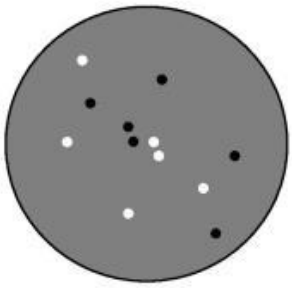
PUZZLES – 5

The first reaction of many people faced with this puzzle is to divide the total area of the table by the area of a coin. Since this value is even, they conclude that an even number of coins can be placed and hence one should choose to play second.



However this doesn't take into account two things – firstly, that the coins need not be touching each other; the opponent could always place a coin a couple of mm away from an existing one resulting in a loose packing (as shown above to the right). Secondly, even if the coins are as closely packed as possible, there would still be some "wasted space" left in between (as shown below to the right). So we cannot just divide the total area by the area of a coin and figure out how many coins might fit.

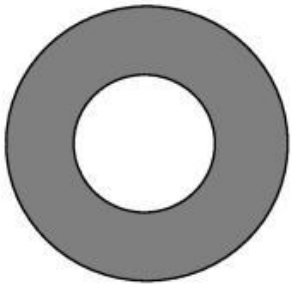
Having realised this, most people veer towards the thought process that there would be no "winning strategy" as such, one should wait till the table is almost full and then start strategising. While this could be attempted, it begs the question of when should we start strategising? When the table is 75% full? 90% full? 95% full?



It turns out, though, that there is a strategy that will work. Let the black symbols indicate the moves of the first player and the white indicate the moves of the second player. If we notice carefully the second player is exactly mirroring the moves of the first player and hence as long as the first player has a place to play, the second will also have a place opposite to him (the mirror image) on which he can play. On this logic, it would appear that one should choose to play second.

However, there is one point which does not have a mirror image – the centre of the table. So the ideal strategy turns out to be to play first, occupy the centre on your first move, and thereafter mirror the moves of your opponent. (This will work in any symmetrical table having a centre.)

Q. What is the table is a circle of radius 2 m with a circular hole of radius 1 m in the centre (as shown in the figure)?



In this case, there is no centre and hence every point on the table would have a perfect counterpoint or mirror image; thus here the ideal strategy would be to play second and just mirror your opponent's moves.

Q. The Joker has robbed Fort Knox and has stolen 81 specially manufactured notes of $1 million each, serially numbered from 1 to 81. He places them around himself in a circular manner in order and starts setting fire to every alternate note (#2, #4….) When the Batman arrives to stop him, there is only 1 note left. What is the number on the note?

PUZZLES – 6

This problem is actually related to our very first maths post (A Puzzle of Knights). The underlying logic is the same; so I shall just reiterate it here:

We need to look for the pattern in the problem:

If there were only one note, obviously it would be left.

If there were two, again #1 would remain. For three, it would be #3, and for four again #1. At this point no clear pattern is visible.

Taking a few more numbers and checking we find a pattern emerging:

Number of notes	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Last note remaining	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

If the number of notes at the start is a power of 2, then the note left at the end will be #1. From then on till the next power of 2, each extra note increments the 'winning' number by 2.

So if there are 13 notes, we would look at the power of 2 just before 13 (which is 8) and say: for 8 notes, #1 would remain, for the remaining 5, add 5*2 = 10 and hence #11 will remain.

Similarly for, say, 50 notes, we would do 1 + (50 - 32)*2 = 37. Hence the 37th note would remain.

And so, in this case where there are 81 notes, all we need to do is 1 + (81 - 64) * 2 = 35. Hence the 35th note will be the last one left.

Let's try another similar question:

Q. 500 children are playing a game. They stand in a large circle and start counting 1, 2, 1, 2, 1… Every child who says 2 leaves the circle and they keep counting until only one child is left. At what position was that child standing?

This is a similar situation to the earlier one and hence the remaining child could be found by **1 + (500 – 256) * 2 = 489** i.e the 489th child (as measured from the first to start counting) will be the one left.

Let us now look at some very standard problems, involving weights

Q. You have a standard weighing balance, but no weights. You are given 81 coins and told that 80 of them have the same weight, but one is a fake and thus is very slightly lighter. You have to identify the faulty one by using the balance as few times as possible. How many weighings will be required?

PUZZLES – 7

The question we need to need to ask ourselves here is: in a single weighing, how many coins can I discriminate between?

Obviously if I have two coins A and B, I will be able to solve this in a single weighing; place A on one side and B on the other, whichever goes down is heavier. But can we do better than this? (Note: the total number of coins, 81, should give us a hint. It is a familiar number, a power of 3…). On consideration, it is also possible to identify the faulty one among three coins (say A, B, C). Weigh A against B; we have two possibilities:

Case 1: Either A or B goes down ➔ it is the heavy coin

Case 2: A and B balance ➔ C is the heavy coin

So now our procedure becomes straightforward. Divide the 81 coins into 3 groups X, Y, Z of 27 each. Weigh X v/s Y. By this we will figure out which of X, Y and Z has the faulty coin. Now take this group and repeat the process, dividing into 3 groups P, Q, R of 9 coins each and weighing P v/s Q. In the third step we form 3 groups L, M, N or 3 coins each, and finally in a 4th step we divide the resultant group into 3 single cons A, B, C. Thus we can identify the faulty coin in **4 weighings.**

Let us examine another variant of the same question, wherein it is not known whether the faulty coin is heavier or lighter:

Q. You have a standard weighing balance, but no weights. You are given 81 coins and told that 80 of them have the same weight, but one is a fake and thus is the wrong weight. However you do not know whether it is lighter or heavier. You have to identify the faulty one by using the balance as few times as possible. How many weighings will be required?

In this case we start (as in the previous case) by dividing into 3 groups X, Y, Z of 27 coins each. We weigh X v/s Y. Now again we have two cases:

Case 1: One of them, say X, goes down. This means that either X contains a heavy coin or Y contains a light one. So we weigh X v/s Z. Now if X again goes down, then X is heavy, while if they balance then Y must be light.

Case 2: X and Y balance. That means Z has the faulty coin. Again, we now weigh X v/s Z. Z will go down if it is heavy and up if light.

Thus in 2 weighings we identify which group has the faulty coin as well as whether the faulty coin is lighter or heavier. After this, we proceed as in the previous example and hence take 3 more weighings to get our answer. Thus we need a total of one extra i.e. **5 weighings** in this case.

Q. You have an electronic weighing machine with a single pan, which can measure accurately any weight from 1 gm to 1000 gm (in steps of 1 gm). You have ten sacks of coins in front of you, each containing between 100 and 200 coins (there needn't be the same number in every bag!). You are informed that the coins in nine of the bags are identical, weighing 10 gm each, but the coins in the tenth one are all forgeries, weighing 9 gm each. How many weighings will be required to identify the sack containing the forgeries?

The first thing one might think of would be to weigh the sacks. However, there are two problems with this – firstly, the sacks may not contain the same number of coins each and secondly, each sack weighs more than the weight which can be measured by the machine. So we need to weigh individual coins from the sacks.

The second thought might be to try what we did in the previous problem, weighing and comparing. However, this problem differs from the previous one, in that we have a single-pan weighing machine (something like a spring balance) rather than a two-pan one. Of course, we could weigh one coin from each sack, and in ten weighings that would give us the answer. But can we do it in less?

One common error here is to assume that only 1 coin needs to be taken from each sack. Or indeed that the same number need to be taken from each sack. The problem can be rather elegantly solved by taking a different number of coins from each sack – 1 from the first, 2 from the second, 3 from the third and so on until, from the tenth sack, we take ten coins. Thus in total we have 55 coins. Were they are to be genuine, the weight would be 550 gm; but for each forgery present the weight would drop by 1 gram, Thus if the total weight obtained is, say 543 gm, then there are 7 forgeries and hence the seventh sack is the one containing the forgeries.

Let's see another variant on the same problem:

Q. You have an electronic weighing machine with a single pan, which can measure accurately any weight from 1 gm to 1000 gm (in steps of 1 gm). You have ten sacks of coins in front of you, each containing between 100 and 200 coins (there needn't be the same number in every sack!). You are informed that the coins in nine of the sacks are identical, weighing 1 gm each, but the coins in the tenth one are all forgeries, weighing 0.9 gm each. How many weighings will be required to identify the sack containing the forgeries?

We can try to apply the same logic as in the previous problem, but the snag here is that the machine has a "least count" of 1 gm, while the difference in weight of a genuine coin and a faulty one is only 0.1 gm, which is too small to measure accurately with the given balance.

This problem, however, can be elegantly circumvented by multiplying the previous solution by ten – take 10 coins from the first sack, 20 from the second, and so on, taking 100 from the last sack. Thus again the total should have been 550 gm if all the coins were genuine, and so if the machine reads, say, 546 gm, then the fourth sack will have the faulty coins.

Let's try yet another minor variant:

Q. You have an electronic weighing machine with a single pan, which can measure accurately any weight from 1 gm to 500 gm (in steps of 1 gm). You have ten sacks of coins in front of you, each containing between 100 and 200 coins (there needn't be the same number in every sack!). You are informed that the coins in nine of the sacks are identical, weighing 10 gm each, but the coins in the tenth one are all forgeries, weighing 9 gm each. How many weighings will be required to identify the sack containing the forgeries?

PUZZLES – 9

This question looks identical to the first problem of yesterday. But there is one small change – the weighing machine has an upper limit of 500 gm, and the solution we used for that problem requires us to be able to measure accurately up till 550 gm.

One possible way to handle it, of course, is to make two groups of 5 sacks each, and in each group, take 1 + 2 + 3 + 4 + 5 = 15 coins which will lie comfortably within the required limits. However, is it possible to manage a solution using just one weighing, as in the previous case?

It turns out that it is. We can take ten distinct numbers from 0 to 9 instead of from 1 to 10; i.e. take 1 coin from the first sack, 2 from the second….9 from the ninth and **none** from the tenth. Thus the total number of coins will be 45 and the expected weight will be 450; if the actual weight turns out to be, say, 442 then the 8th sack contains the fakes. Of course, in this case if the weight is the correct 450, then the tenth sack (from which we took no coins!) will be the faulty one.

Let's try one more example:

Q. You have an electronic weighing machine with a single pan, which can measure accurately any weight from 1 gm to 1000 gm (in steps of 1 gm). You have 20 identical sacks, each containing 4 coins. All coins are identical in appearance. However, 19 of the sacks contain 10 gm coins while the remaining sack contains 9 gm coins. What is the minimum number of weighings required to identify the sack containing 9 gm coins?

In this case, we have a limited number of coins in each sack. So we can distinguish or discriminate between 5 sacks at a time. Using this, we can proceed as follows:

Divide the 20 sacks into 5 groups of four each. From each sack in the first group, take 1 coin, from each in the second group take 2, from each in the third group take 3, from each in the fourth group, take 4 and from each in the fifth group take none. Weighing these 40 coins, we will identify which group contains the faulty sack. Again repeating the process for those 4 sacks, we will be able to find out which sack contains the faulty coins.

Let us now look again at some more general puzzles of various types:

There are 4 toggle switches, all currently at "off". One of them lights a bulb in the next room, the rest do nothing. From the room with the switches in it, you can't see whether the light in the next room is turned on or off. You may flip the switches as often and as many times as you like, but once you open the door of the next room to check on the light, you cannot flip any further switches.

How can you determine which switch controls the light?

PUZZLES – 10

Let's start by looking at the simplest version of the problem. If there were only two switches, say A and B, there would be no problem. One could switch one of them on (say A), walk into the room and check the status – if the bulb is on, A is the right switch, if not, B is the one.

What if there are 3 switches - A, B and C? (This, by the way, is the most common avatar of the given problem). Things get a little trickier here. But since you are given as much time as you like, a solution presents itself. It is based on the fact that light bulbs are not perfectly efficient convertors of energy; along with the light, some heat is also produced.

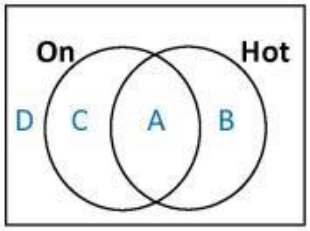
So a possible solution is: switch on A. Wait for ten minutes, then switch off A, switch on B, and walk in. Now if the light is on, the correct switch is B. If it is off, touch the bulb – if it is hot then A is the switch, while if it is cold, C must be the correct one.

Finally, let us arrive at our current conundrum – what if there are 4 switches A, B, C and D? We can simply extend the logic of the previous problem a little (assuming that a bulb will take at least a little while to heat up, which is a reasonable assumption).

Our solution then could be something like: Switch on A and B. Wait 10 minutes. Switch off B, switch on C, run into the room and touch the bulb. Now if the bulb is on and hot, it must be A. If it is off but hot, B must be the answer. Switch C would result in the bulb being on but still cold, while abulb which is off and cold would indicate switch D as our answer.

Note that with two criteria (hot/cold and on/off) we can distinguish a maximum of 4 different cases. This can also be represented tabularly (or as a Venn diagram, if that should be your preference) in a manner similar to that of a truth table.

	On	**Off**
Hot	A	B
Cold	C	D



Let's try another poser:

Q. You have a 10-volume encyclopaedia (with each volume having 1000 pages) which you are extremely fond of. One day you get a call from the IIMs and hence have to leave home for two years. Since the encyclopaedia is too bulky to carry with you, you reluctantly leave it at home, having carefully arranged it in your bookshelf. However, when you come back after two years, you discover, to your horror, that a bookworm has attacked your beloved encyclopaedia; eating its way in a straight line from the first page of the first book to the last page of the last book. How many papers has the bookworm chewed through in its odyssey?

PUZZLES – 11

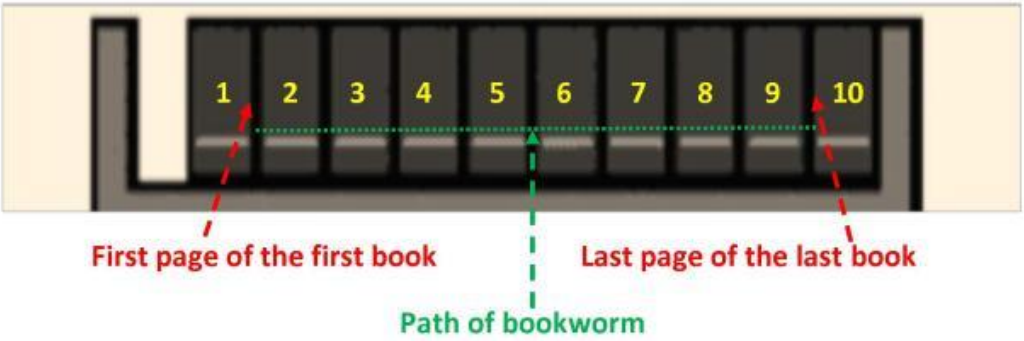
This has been an extremely popular interview question over the years, and for excellent reason – most people manage to get it wrong. The first gut reaction of many interviewees is to say "10000 pages, of course!" But let us examine that 'obvious' answer a little more closely:

The first thing one should consider is that when we talk of the number of pages in a book, every physical page i.e. every paper actually has two page numbers. So a 1000-page book would actually have 500 physical pages. So first, one needs to check this with the interviewer – does 100 pages mean 1000 physical pages or 1000 page numbers? The default assumption would have to be 1000 page numbers and hence 500 actual pages. So perhaps we should have not 10000, but 500 * 10 i.e. 5000 pages as our answer.

But what about the covers? Are they included in this tally or not? This is another thing which should be clarified – if they are included, well and good (but in real life, they usually are not, which adds another two pages per volume!). So our answer morphs again, becoming 502 * 10 = 5020 pages.

However, there is a yet more insidious trap here, which relies upon the "standard" arrangement of books in a bookshelf. Consider what happens when you are putting 10 books, **in order,** into a bookshelf:

We always place the books in order from left to right. Also, we always place them the right way up, and with the spine facing outward. Thus the first page of the first book is actually at the end of the book, right next to the last page of the second book, resulting in an arrangement something like the one shown below:

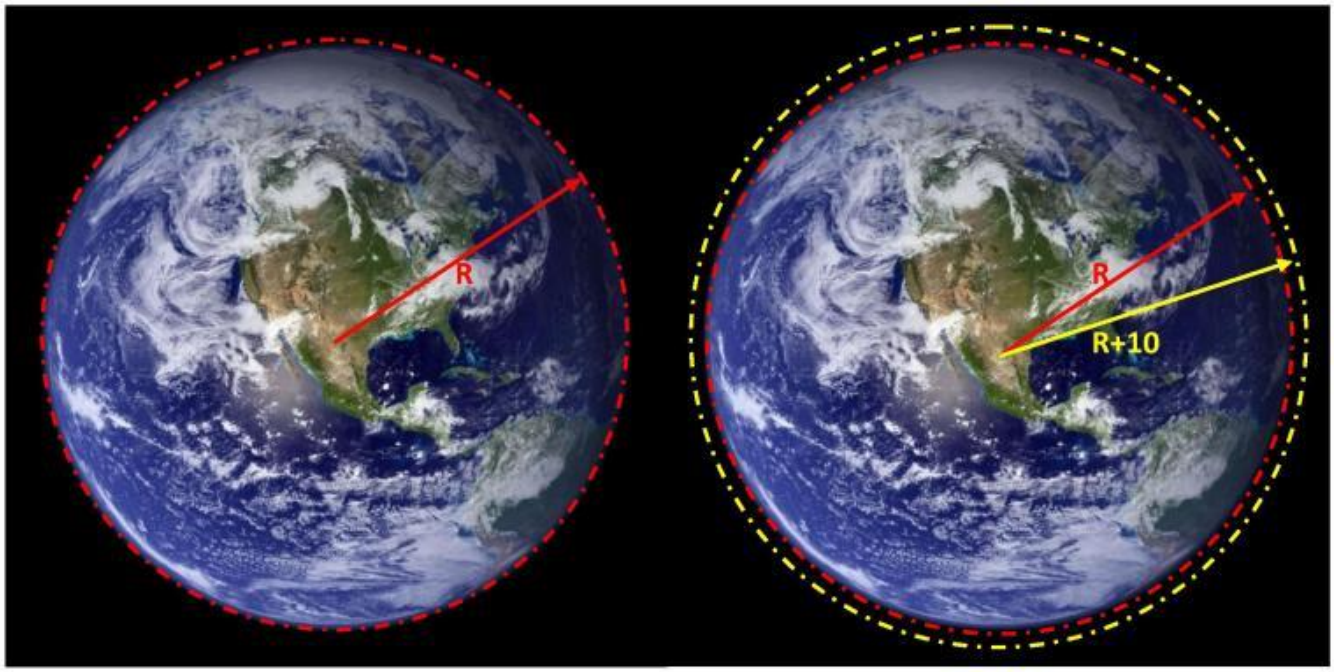


So in the first and last books, only 1 + 1 = 2 pages are eaten, while in the remaining the whole 502 * 8 = 4016 pages are eaten, resulting in a total of 4018 pages. (This answer would change if the examiner replies, for example, that the covers should be included in the page count)

Let's do a short puzzle this time round:

An eccentric individual makes it his life's work to tie a rope around the earth's equator. He buys a lot of rope and makes the attempt. A rival of his, not to be outdone, decides he wants to tie a rope around the earth's equator that is elevated from the ground by ten metres at all points along the rope. How much more rope does he need? (Assume the earth to be perfectly spherical with a radius of 6400 km).

We can, of course, find the answer by calculating how much rope is required by each of them separately. However, it would be simpler to just examine the difference between the two cases:



Suppose the radius of the earth is R metres. The first man will require a rope of length $2\pi R$ m to encircle the Earth. Now the second man is creating a circle of radius (R + 10) and hence the length of rope required will be 2π (R + 10).

The extra rope required will therefore be $\mathbf{2\pi * 10 \approx 62.8}$ **m**.

Let's look at another popular question next. There are many different versions of this puzzle extant; I have used the one from the site folj.com (which has a collection of excellent logical and lateral puzzles of varying difficulty) as it is fairly precisely worded.

You are the ruler of a medieval empire and you are about to have a celebration tomorrow. The celebration is the most important party you have ever hosted. You've got 1000 bottles of wine you were planning to open for the celebration, but you find out that one of them is poisoned. The poison exhibits no symptoms until death. Death occurs within ten to twenty hours after consuming even the minutest amount of poison.

You have over a thousand slaves at your disposal and just under 24 hours to determine which single bottle is poisoned. You have a handful of prisoners about to be executed, and it would mar your celebration to have anyone else killed.

What is the smallest number of prisoners you must have to drink from the bottles to be absolutely sure to find the poisoned bottle within 24 hours?

PUZZLES – 13

Let us start by noting that even a very minute quantity of the poisoned wine will kill – but will take the same 10-20 hours as a large amount would. So for example, we could feed a thousand prisoners one drop from a bottle and see which one died. This would be exceedingly wasteful though; since we have been given only a handful of prisoners we will not be able to implement it.

One thing we can note is that each prisoner can either drink from a particular bottle, or not drink. So we can consider employing a binary kind of logic.

Let us take a simpler case to understand this, suppose we have just 16 bottles instead of a thousand. Let's assign each bottle a number from 0 to 15, and write this number in binary. We will need 4 digits to do this, so let us take 4 prisoners and assign them to the thousand's place, hundred's place, ten's place and unit's place.

	Prisoner 1	Prisoner 2	Prisoner 3	Prisoner 4
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

Now after 20 hours we just need to check who lives and who dies. If for example the 1st, 2nd and 4th prisoner die then the bottle $(1101)_2$ i.e. 13 is poisoned, while if the 2nd and 3rd prisoner die then the bottle $(0110)_2$ i.e. 6 is poisoned.

So in the given case we need to take at least 10 prisoners (since 1000 is between $2^9 = 512$ and $2^{10} = 1024$, we will need 10 digits), assign each to a position and number the bottles from 0 to 999. Then proceeding as above, we will be able to identify the faulty bottle within the time limit!

There are 4 people who need to cross a bridge in the dark. They have only one torch, and have to use it while crossing as it is dangerous. The bridge can hold only 2 people at a time. The people (let's say A, B, C and D) take 1, 2, 5 and 10 minutes respectively to cross the bridge. What is the minimum time in which they could all cross the bridge?

The points to note are that only two people at a time can cross the bridge. Also, the torch needs to be brought back, so after each forward trip someone has to return as well.

One logical approach could be that the quickest person (A) should escort each of the others across the bridge and then come back to pick up the next person. This would give us:

A and B cross ⟹	2 min
A returns ⟸	1 min
A and C cross ⟹	5 min
A returns ⟸	1 min
A and D cross ⟹	10 min

Total Time	**19 min**

However, can we guarantee that this is the optimum time? We should consider another possibility, using the concept of bottlenecks. (A bottleneck, at the most basic level, is that element of a process which slows things down).

In this question, the bottlenecks are C and D, who take 5 minutes and 10 minutes respectively. If we send them on two separate trips, then that itself take up 15 minutes. So why not send them together?

A natural objection to this line of thought would be that someone has to bring the torch back! However, we can solve this; let A and B cross first, then one of them (say A) brings the torch back and C and D cross, then B brings the torch back and takes A along with him. Let's see what the resultant time is:

A and B cross ⟹	2 min
A returns ⟸	1 min
C and D cross ⟹	10 min
B returns ⟸	2 min
A and B cross ⟹	2 min

Total Time	**17 min**

So this solution saves us 2 minutes.

Let's try it again with different numbers:

There are 4 people who need to cross a bridge in the dark. They have only one torch, and have to use it while crossing as it is dangerous. The bridge can hold only 2 people at a time. The people (let's say A, B, C and D) take 3, 8, 11 and 17 minutes respectively to cross the bridge. What is the minimum time in which they could all cross the bridge?

Applying the bottleneck logic of the previous question gives us:

A and B cross ⟹	8 min
A returns ⟸	3 min
C and D cross ⟹	17 min
B returns ⟸	8 min
A and B cross ⟹	8 min

Total Time	**44 min**

Out of curiosity, let us also see what happens if we allow A to escort each of the others across:

A and B cross ⟹	8 min
A returns ⟸	3 min
A and C cross ⟹	11 min
A returns ⟸	3 min
A and D cross ⟹	17 min

Total Time	**42 min**

In this case the situation is reversed! Why does this happen? The thing is that when we go from the "A escorts everyone" solution to the "bottlenecks cross together" solution, we get return one trip of A alone replaced by one of B, and one forward trip (A + B) where the maximum time is that of B replaced by one (A + C) whose maximum time is that of C. So effectively 2B is replaced by A + C. So if the difference between B and C is more than that between A and B, the bottleneck solution will be optimal, while if the difference between A and B is more, the other solution will prove better!

Moral of the story: Take care when crossing rickety bridges (also, don't look down!)

97 baseball teams participate in a state tournament. The champion is chosen for this tournament by the usual elimination scheme. That is, the 97 teams are divided into pairs, and the two teams of each pair play against each other. The loser of each pair is eliminated, and the remaining teams are paired up again, etc. How many games must be played to determine a champion?

PUZZLES – 16

One way to do this question is to list out what happens in every round.

In round 1, 96 teams will play 48 matches (while the 97th will have a bye). This will result in a total of 49 teams going through to round 2.

In round 2, 48 teams will play 24 matches (while the 49th will have a bye). This will result in a total of 25 teams going through to round 3.

In round 3, 24 teams will play 12 matches (while the 25th will have a bye). This will result in a total of 13 teams going through to round 4.

In round 4, 12 teams will play 6 matches (while the 13th will have a bye). This will result in a total of 7 teams going through to round 5.

In round 5, 6 teams will play 3 matches (while the 7th will have a bye). This will result in a total of 4 teams going through to round 6.

In round 6, 4 teams will play 2 matches (no byes needed). 2 teams will be left to contest round 7

In round 7, the remaining two teams will play 1 match (the final)

In total, therefore, the number of matches played would be 48 + 24 + 12 + 6 + 3 + 2 + 1 = **96 matches**.

Obviously this is a long and cumbersome way to solve the problem – so is there a better way?

We can observe that every match played will result in one winner. But (and this is often ignored) it will also result in exactly one loser i.e. one team getting knocked out. Now since we have 97 teams, to determine a winner we need 96 of them to get knocked out, and hence we need **96 matches**.

Our next puzzle is rigidly mathematical:

You are given two identical eggs, and access to a 100-storey building. The aim is to find out the highest floor from which an egg will not break when dropped out of a window from that floor. If an egg is dropped and does not break, it is undamaged and can be dropped again. However, once an egg is broken, that's it for that egg.

If an egg breaks when dropped from floor n, then it would also have broken from any floor above that. If an egg survives a fall, then it will survive any fall shorter than that.

What is the minimum number of egg drops required to find the solution?.

PUZZLES – 17

Let's look at a simpler version of the problem first. Suppose we have a building of n floors, and only 1 egg. With only 1 egg, we would have very little choice – we would drop it from the 1st floor; if it survived, we would drop it from the 2nd floor; if it still made it through intact we would drop it from the 3rd floor, and so on. So with n floors, in a worst-case scenario we would require n trials.

But in the given problem, we have two eggs. This complicates matters, as there are several tempting ways to go from here, for example:

The Binary Sort Algorithm: Computer Engineers would naturally gravitate to this solution; indeed, if we had a dozen or so eggs, this would work better – drop the 1st egg at floor 50 – if it breaks, drop the 2nd at floor 25, else drop it at floor 75, and so on. At each step, we are halving the number of possibilities left. However, with only 2 eggs this will be inefficient – after the 1st drop, there is only 1 egg left and so it might take up to 49 more tests to find out the exact floor at which an egg will break.

The Square Root Approach: If we take the square root of 100, which is 10, we get another possibility: drop the first egg from the 10th, 20th, 30 etc floor till it breaks. Once it does (suppose it breaks on floor 60), get a more precise solution by dropping the second one from floor 51, 52, 53…59. This give us a worst case of 19 drop required (assuming the egg breaks on the 99th floor). That's a good deal more efficient than the previous solution.

However, it can be further improved. We can note that whether the first egg breaks on the 10th floor or the 90th, it will need another 10 trials to test. But for the 90th floor, it would require more trials with the first egg and hence more drops in total. So an improvement might be possible if we ensure that the gaps between drops of the first egg reduce as we go higher (so that if we need to drop the first egg more times, the second egg will have to be dropped fewer times)

Suppose we first drop the first egg from floor n, and it breaks. Then the second egg will require at maximum (n – 1) drops. So the total drops required would be 1 + (n - 1) = n. Now if, instead, the first egg survived the first drop, the second drop should be at a further (n-1) floors up, so that the second egg will need not more than (n - 2) drops to get a solution and the total drops will be 2 + (n - 2). Proceeding in this manner, the third drop for the first egg (assuming it survives the first two!) would be a further (n - 2) steps upward, and so on. The total number of floors we can check in this manner would be n + (n - 1) + (n - 2) + (n – 3) … + 3 + 2 + 1 and if this has to be at least 100, then $\frac{n(n+1)}{2} \geq 100$ and so the minimum possible n turns out to be 14. With this we can test up to 105 floors as follows, using not more than **14 total egg-drops**:

1st egg drop from floor #	14	27	39	50	60	69	77	84	90	95	99	102	104	105
Total drops of 1st egg	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2nd egg number of trials	13	12	11	10	9	8	7	6	5	4	3	2	1	0

The next puzzle is a lot less mathematical:

An explorer was trekking through a remote jungle when he was captured by logic-loving cannibals. He was brought before the chief and told, "You may now speak your last words. If your statement is true, then we will burn you at the stake. If your statement is false, we will boil you in oil." What should the explorer tell them in order to save his life?

PUZZLES – 18

In such a puzzle, we must assume that that the cannibals have a way to check the truth/falsehood of any statement (maybe access to Wikipedia) and therefore one cannot get away by just making a difficult – to-check statement such as "It is impossible to derive the analytic and algebraic topology of the locally Euclidean parameterization of an infinitely differentiable Riemannian manifold"*

The solution, therefore, has to be something which literally is neither true nor false, which is inherently self-contradictory. This kind of statement is what we call a paradox.

Technically, the guy could get away by making any paradoxical statement, for example

"This statement is a lie"

or

"The second half of this sentence will be a lie, but the first half of this sentence was the truth"

But then there is a chance that they might still boil him on a technicality. To be absolutely safe, in this specific case, the explorer should probably tell the cannibals something like:

"You are not going to burn me at the stake; instead, you are going to boil me in oil".

Clearly, if they boil him in oil, the statement would become true, so they cannot boil him in oil. However, if they burn him at the stake, the statement would become a lie, so they cannot burn him at the stake. Thus the cannibals would have no choice but to let him go (hopefully in one piece).

* In case you're wondering, that's paraphrased from Tom Lehrer's delightful song "Lobachevsky"

For next time, try another extremely logical puzzle involving truth and lies which has been popular in interviews:

You are walking along a road trying to get to a city A. You come to a fork in the road, and you know that one road leads to A and the other to B, but you have no idea which is which. There are two huge soldiers standing guard at the fork. One of them always tells the truth and the other always lies (again, you don't know which is which). You are allowed to ask one of the guards one question, and with that you have to determine which road goes to A. What should you ask?

And here's another variant of the same:

You are walking along a road trying to get to a city A. You come to a fork in the road, and you know that one road leads to A and the other to B, but you have no idea which is which. There is a huge soldier standing guard at the fork. He alternately speaks the truth and tells a lie (and you don't know whether his last statement was true or false). You are allowed to ask the guard one question, and with that you have to determine which road goes to A. What should you ask?

PUZZLES – 19

The problem in this puzzle is that we have two unknown things – the first being which roads leads to A and the second being which of the soldiers is the liar. It would seem that one question would be required just to identify who is the liar and who the speaker of truth! So a second question would then be needed to find the road to city A…but only one question is allowed.

However, the question does not require us to find out which of the soldiers is telling the truth; all we need to find out is how to get to A. So we need to ask ourselves – can we find this out **without** knowing which of the two soldiers is lying?

The key piece of logic is a simplified version of the truth tables used in formal mathematical logic. Note that a truth is equivalent to a positive while a lie is equivalent to a negative. Now positive * positive results in positive and similarly negative * negative is also positive. But positive * negative and negative * positive both yield the result negative. In other words, a combination of a truth and a lie in either order will always yield a lie.

So all we need do is to ensure that the question somehow manages to involve the answers of both people, even though it is asked to only one of them. A possible solution would be to ask any one soldier **"If I were to ask your companion which road goes to A, which road would he point to?"** Now if the soldier I address is the truthful one, he will point to where his companion would have pointed – but his companion, being a liar, would have pointed to B. On the other hand, if the soldier I addressed happened to be the liar, then he would not point out the route indicated by his companion (who, being truthful, would have pointed to A!) and hence will point to B. Either way, I will know that the guy is pointing towards the wrong road.

As for the alternate version of the problem, we would need to find an answer which would incorporate two consecutive questions; one possible way to do this would be to ask the soldier something like:
"If my next question to you were to be 'Which road leads to A' then what would you reply?"

You can check for yourself that in either case the soldier will indicate the road towards B!

To wind up our series on puzzles for this season, let's try a fairly tough one, based on probability:

A mall is having a promotional event. Every customer who comes in is asked to stand in a line. Now the mall manager walks down the line and asks each person to whisper a natural number from 1 to 100. The first person to say a number which someone else has already said earlier wins the bumper prize. At which position in the queue should you stand to maximise your chances of winning the prize?

PUZZLES – 20

First let us understand the pattern of probabilities.

The 1st person can never win. Hence P_1 = probability of 1st person winning = 0.

The 2nd person has only 1 chance in 100 of winning. Hence P_2 = probability of 2nd person winning = 1/100.

For the 3rd person to win, the 2nd person has to lose, and then the 3rd person has 2 chances in 100 of winning. Hence P_3 = probability of 3rd person winning = 99/100 x 2/100.

Note that $P_2 > P_1$, and $P_3 > P_2$ i.e. $P_2 - P_1 > 0$ and $P_3 - P_2 > 0$ and so on. We want the first n such that $P_{n+1} - P_n < 0$. We could keep multiplying, which would result in a discrete probability distribution, and then check for which value this is maximum. But since it is for a set of discrete values, it could be quite a long and tedious process, with lots of calculation (and no option of using short-cuts such as differentiation). An alternate way would be as follows:

In general for the nth person to win, the first (n – 1) have to lose, and then he has (n – 1) chances in 100 of winning. For convenience, let us suppose the probability for the first (n – 1) people to lose is k.

Hence P_n = probability of nth person winning = $k \times \frac{(n-1)}{100}$

Now probability that first n people will lose is $k\,[1 - \frac{(n-1)}{100}]$

Hence P_{n+1} = probability of $(n+1)^{th}$ person winning = $k\,[1 - \frac{(n-1)}{100}] \times \frac{n}{100}$

Hence $P_{n+1} - P_n$ turns out to be $k\,[1 - \frac{(n-1)}{100}] \times \frac{n}{100} - k\,[\frac{(n-1)}{100}]$

which equals $\frac{k}{100} \times [1 - \frac{n(n-1)}{100}]$

The earliest 'n' at which this will be < 0 is n = 11.Hence the customer should stand in the 11th position.

Here's another similar question for you to try:

There is a big line of people waiting outside a theatre for buying tickets. The theatre owner comes out and announces that the first person to have the same birthday as someone standing before him in the line gets a free ticket. Where will you stand to maximize your chance of winning?

On solving, you will find the optimum position to be the 20th in this case.

In general, it turns out that the optimum position in these conditions (i.e. we have k total options available and you have match one of the people before you will generally be just more than the square root of the total number of cases i.e. around $\sqrt{k} + 1$, since the factor of $[1 - \frac{n(n-1)}{k}]$ will become negative then)

From next time, we'll get back to maths for a while.