Introduction 1

Introduction

Welcome to the fascinating world of algebra! Though it deals with study of "unknown", it should not remain unknown to you. Many students find it difficult to transit from number-based problems in arithmetic to alphabet based problems in algebra. However, do not panic; the concepts remain similar. It is only that the presentation differs. Make yourself comfortable in algebra by studying this chapter.

Algebra is usually found tough by the students because of the overwhelming variety of concepts and also because of the language of the variables. Surely, the following looks daunting simply because of the appearance:

$y = f(x_1, -f(x_2, -f(x_3, -f(x_4, \ldots, -f(x_{n-1}, x_n)))\ldots)$ where $f(a, b) = (a^2 + b^2)^{1/2}$. This chapter aims to remove this fear of algebra and elucidate even to the un-initiated that while Algebra seems difficult, there are just very few concepts, that too all interrelated, to master.

Once one looks beyond the appearance, even the above difficult expression boils down to just $y = (x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2)^{1/2}$.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

- Variables
- Expressions
- Operations on Algebraic Expressions
- Polynomials
- Functions
- Graphs

Variable

Introduction

Welcome to the fascinating world of algebra! Though it deals with study of "unknown", it should not remain unknown to you. Many students find it difficult to transit from number-based problems in arithmetic to alphabet based problems in algebra. However, do not panic; the concepts remain similar. It is only that the presentation differs. Make yourself comfortable in algebra by studying this chapter.

Algebra is usually found tough by the students because of the overwhelming variety of concepts and also because of the language of the variables. Surely, the following looks daunting simply because of the appearance:

$y = f(x_1, -f(x_2, -f(x_3, -f(x_4, \ldots, -f(x_{n-1}, x_n)))\ldots)$ where $f(a, b) = (a^2 + b^2)^{1/2}$. This chapter aims to remove this fear of algebra and elucidate even to the un-initiated that while Algebra seems difficult, there are just very few concepts, that too all interrelated, to master.

Once one looks beyond the appearance, even the above difficult expression boils down to just $y = (x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2)^{1/2}$.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

- Variables
- Expressions
- Operations on Algebraic Expressions
- Polynomials
- Functions
- Graphs

Variable

A **variable** is a symbol used to denote a quantity.

For example, in the expression $2x^2 + 3x - 1$, `x' is the variable.

Expression

An **algebraic expression** is a combination of numbers and one or more than one variable.

For example, $2x + 3$, $3y^2 - 6y + 3$, $3x - y + 5z$.

A **term** is a number or a product of number and one or more variables.

For example, the expression $3x^2 - 2x + 1$ has a total of three terms viz. $3x^2$, $-2x$ and 1.

The **coefficient** is a number that is multiplied with a variable or more than one variable.

For example, in the expression $3x^2 - 2x + 1$, the coefficient of x^2 and x is 3 and -2 respectively.

Operations on Algebraic Expressions

Arithmetic operations on Algebraic expressions:

Addition & Subtraction:

The addition or subtraction of algebraic expression follows the same rules as the arithmetic addition/subtraction. We can add the coefficients of the terms with the same degree and in the same variables only.

For example,

$(2x + 3y) + (5x + 4y) = 7x + 7y = 7(x + y)$

For example,

$(3x^2 + 4xy + 7z^2) - (5xy - 4z^2) = 3x^2 + 4xy + 7z^2 - 5xy + 4z^2 = 3x^2 - xy + 11z^2$.

Multiplication

The multiplication of any 2 algebraic expressions follows the distributive property of multiplication and the index rules.

For example,

$(2x + 3y)(5x + 4y) = (2x)(5x + 4y) + (3y)(5x + 4y) = 10x^2 + 8xy + 15xy + 12y^2 = 10x^2 + 23xy + 12y^2$

For example,

$(3x^2 + 4xy + 7z^2)(5xy - 4z^2) = (3x^2 + 4xy + 7z^2)(5xy) + (3x^2 + 4xy + 7z^2)(-4z^2)$

$= 15x^3y + 20x^2y^2 + 35xyz^2 - 12x^2z^2 - 16xyz^2 - 28z^4$

$= 15x^3y + 20x^2y^2 + 19xyz^2 - 12x^2z^2 - 28z^4$

Division

Just as in the normal arithmetic, division and multiplication of algebraic expressions follow similar rules. The standard method is called synthetic division.

For example, divide the expression

$f(x) = 3x^2 + 4x + 3$ by $(x - 2)$.

$$\begin{array}{r|l} & 3x+10 \\ \hline x-2 & 3x^2+4x+3 \\ & \underline{3x^2-6x} \\ & 10x+3 \\ & \underline{10x-20} \\ & 23 \end{array}$$

Just as in normal arithmetic division, the quotient of the division process is $3x + 10$ and the remainder is 23.

Polynomial

The word `poly' means **many** and the word `nomial' means **terms.** So polynomial is an algebraic expression, which consists of many terms involving powers of the variable.

For example, $5x - 5$, $x^2 + 5x + 6$, $x^3 + x^2 - 24$.

The general form of a polynomial is

$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1} + a_nx^n$

Where $a_0, a_1, a_2 \ldots a_n$ are real numbers and `n' is a non-negative integer.

Polynomial may be in more than one variable.

For example,

$x^2 + y^2 - 4, \quad 2x^2 + y^2 + z$.

Operations with Polynomials

Since a polynomial is an algebraic expression, therefore the operations on algebraic expressions hold true for polynomials also.

Types of Polynomials

Polynomials can be classified as follows:

1. By coefficients:

Here the basis of classification is the nature of the coefficients.

a. In $5x^2 + 3x + 6$, the coefficients are 5, 3 and 6 and all are integers. So, this is called polynomial over the set of integers.

b. In $\frac{2}{3}x^2 - \frac{4}{5}x + 2x - 3$, the coefficients are $\frac{2}{3}, -\frac{4}{5}, 2$ and -3 and all are rational numbers.

So, this is called polynomial over the rational numbers.

c. In $\sqrt{3}x^2 - 11x + \sqrt{3}$, the coefficients are real numbers.

So, this is called polynomial over the real numbers.

2. By the number of terms:

Here the basis of classification is the number of terms in the expression.

a. **Monomial:** Consisting of a single term.

For example, $\sqrt{5}x, -8y^2, 15m^2$.

b. **Binomials:** Consisting of two terms.

For example, $2x + 3y, x - 5, y^2 + 3y$.

c. **Trinomials:** Consisting of three terms.

For example,

$5x^3 - 3x^2 + 2, x + y - 2, x^5 + x^4 + x^2$

d. **Polynomials:** Consisting of more than three terms are generally termed as polynomials (even though Monomial, Bionomial and Trinomials are also Polynomials). For example,

$x + y + z + m, x^5 + x^4 + x^3 + x^2, a + 2b + 2c + 3d + 4e$.

3. By degree:

The highest exponent of any monomial of the given polynomial is called the degree of the polynomial.

For example, $x^6 - 3x^4 + 12$. The highest exponent of a term is 6.

So this polynomial is of degree 6.

$5y^3$ is a monomial of degree 3.

$4x^2 y^3 z^2$ is a monomial of degree $2 + 3 + 2 = 7$.

$\sqrt{5}\, x^3 y^3$ is a monomial of degree $3 + 3 = 6$.

5 is a monomial of degree 0 (because 5 can be written as $5x^0$).

$-\frac{4}{15}mn^3$ is a monomial of degree $3+1 = 4$.

$y^7 - 5x^2 + 3y^2 = 8$ is a **polynomial** of degree 7.

An equation in one variable and with degree `n' can have at maximum `n' real roots. The number of real roots can be less than `n' also because few of them may be Imaginary.

Linear polynomial:

A polynomial of degree one is called a linear polynomial.

For example, $5x + 3y$, $5x - 4$, $\frac{1}{3}x$, $x + \frac{5}{2}$ are linear polynomials.

Quadratic polynomial:

A polynomial of degree two is called a quadratic polynomial.

For example, $x^2 + 3x + 5$, $x^2 - 5x + 12$ are quadratic polynomials.

Example 1:

Is the equation $5x^2 - 3\sqrt{x} + 10$ a polynomial?

Solution:

No, because for a polynomial the condition is power of `x' or a variable must be a non-negative integer. Here, in $3\sqrt{x}$, `x' has a power $\frac{1}{2}$ which is not an integer.

Factorisation of Polynomial

The expression $x^2 - 5x + 6$ can be written as product of $(x - 2)$ and $(x - 3)$. This process of writing an expression as a product of different expressions is called as Factorisation. The expressions $(x - 2)$ and $(x - 3)$ are called factors of $x^2 - 5x + 6$. Just as in Arithmetic, in Algebra also a factor of an expression can completely divide the expression i.e. the remainder is zero. The process of factorization is useful to find the roots of any equation.

Thus $x^2 - 5x + 6 = 0 \Rightarrow (x - 2)(x - 3) = 0$. The product of two numbers can be 0 only if atleast one of them is zero. Thus, the LHS of the above equation will become 0 if $x = 2$ or 3 and the equation will be satisfied.

List of some common formulas	
Expanded form	Factored form
$x^2 + 2xy + y^2$	$(x + y)^2$
$x^2 - 2xy + y^2$	$(x - y)^2$
$x^2 - y^2$	$(x + y)(x - y)$
$x^3 + 3x^2y + 3xy^2 + y^3$	$(x + y)^3$
$x^3 - 3x^2y + 3xy^2 - y^3$	$(x - y)^3$
$x^3 + y^3$	$(x + y)(x^2 - xy + y^2)$
$x^3 - y^3$	$(x - y)(x^2 + xy + y^2)$

Functions

Functions are just a language of Algebra. We can refer to the expression $x^2 - 8x + 15$ as y i.e. $y = x^2 - 8x + 15$. Now what will be the numerical value of y? Obviously the value of y also keeps changing as the value of x changes i.e. the numerical value of y depends on the value assumed by x. If $x = -2$, $y = 35$, if $x = 0$, $y = 15$ and if $x = 2$, $y = 3$. This dependency is expressed as $y = f(x)$ and is read as y is a function of x. In $y = f(x) = x^2 - 8x + 15$, the right most side explicitly states how y is related to x.

Thus functions is nothing new and even the quadratic expression that we are so used to can be expressed in
the language of functions. In this example, if we need to find the value of y when $x = 5$, we simply mean to evaluate $f(5)$. To evaluate $f(5)$, we just substitute the value of x as 5 in the expression $x^2 - 8x + 15$ and find that $f(5) = 0$

$y = f(a, b) = a^2 + b^2$. This example introduces you to a function that is dependent on two variables viz. a and b. Thus the value of y depends on both the values of a and b and to evaluate y, we need both these values.

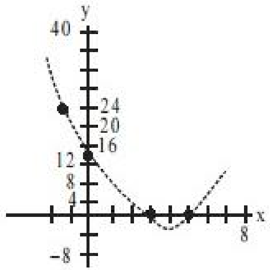
Graphs

Graphs are a very handy tool to understand algebraic expressions and concepts. So please get used to graphs and see how it simplifies a lot of concepts.

In our example of $y = f(x) = x^2 - 8x + 15$, we can plot a two-dimensional graph with the values of x on the X-axis and the corresponding value of y on the Y-axis. The following table gives the corresponding value of y for a few values of x:

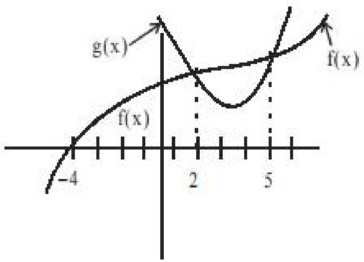
x =	y =
−2	35
−1	24
0	15
1	8
2	3
3	0
4	−1
5	0
6	3
7	8
8	15

If we join all (x, y) pairs, from the above data, with a smooth free hand sketch we get the graph of f(x).



In the above graph the value of y i.e. of f(x) is plotted on the vertical axis. We will be using graphs extensively to understand Algebraic expressions. Thus it would be worthwhile to understand graphs fully. In any graph of
f(x) v/s x, for a particular point, the vertical distance from the X axis will denote the value of f(x) and the horizontal distance from the Y axis will denote the corresponding `x' value.

If you have understood this thoroughly, the observations following the graph below should be self evident.



Observations:

1. For all values of $x < -4$, f(x) will be negative and for all values of $x > -4$, f(x) will be positive.

2. g(x) will always be positive

3. The value of f(2) = g(2) and so is f(5) = g(5).

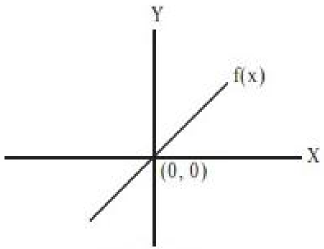
4. For all values of x such that $2 < x < 5$, $f(x) > g(x)$ and for all other values of x, $f(x) < g(x)$

Lets understand the curves of some of the most common algebraic functions. We will also learn how to draw free hand sketch for important algebraic expressions.

<u>Elementary Curves:</u>

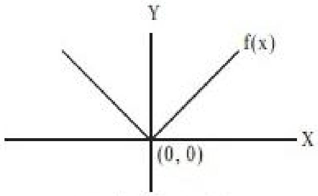
f (x) = x

The graph is a straight line passing through point (0,0) of the X-Y plane.

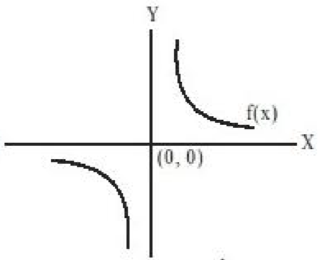


graph of $f(x) = x$

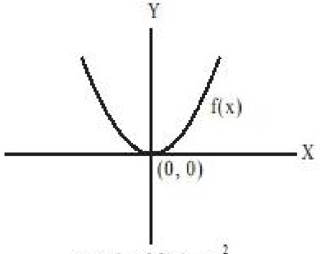
Some other curves are also drawn here:



graph of $f(x) = |x|$



graph of $f(x) = \frac{1}{x}$



graph of $f(x) = x^2$

Remainder Theorem:

To identify whether a given expression is a factor of another expression, we can take help of Remainder Theorem.

According to the remainder theorem, when any expression f(x) is divided by (x - a), the remainder is f(a). (a is any constant in this example).

Thus when the expression, $x^3 + x^2 + 4$ is divided by x + 1, the remainder is $(-1)^3 + (-1)^2 + 4$ i.e. 4.

Factor Theorem:

We can also use the remainder theorem or more specifically the Factor theorem to identify if an expression is a factor of another expression. As already seen, an expression is said to be a factor of another expression only when the remainder is 0 when the latter is divided by the former.

Thus, (x - a) is a factor of f(x) if and only if f(a) = 0.

Factor theorem also helps us in factorising higher degree equations. Consider the equation

$f(x) = x^3 + 6x^2 - 19x - 24 = 0.$

By hit and trial (basically substituting values of x as -2, -1, 1 or 2), we see that

$f(-1) = (-1)^3 + 6(-1)^2 - 19(-1) - 24 = -1 + 6 + 19 - 24 = 0.$

Thus, we can deduce that (x + 1) is a factor of f(x). i.e. $x^3 + 6x^2 - 19x - 24 = (x + 1) \times g(x)$, where g(x) is another algebraic expression in variable x.

Using common sense we can gather that g(x) is a quadratic expression. Why?

Only $(x + 1)(ax^2 + bx + c)$ will have a term in x^3, x^2, x and a constant.

Thus, $x^3 + 6x^2 - 19x - 24 = (x + 1)(ax^2 + bx + c)$.

By visual check we can ascertain that $a = 1$ and $c = -24$. How?

Equating coefficient of x^3 on RHS and LHS, we get $a = 1$ (on the RHS only $x \times ax^2$ will result in term in x^3). Equating the constant terms of RHS and LHS, we get $c = -24$ (only $1 \times c$ will result in a constant on RHS).

Thus, now we have $x^3 + 6x^2 - 19x - 24 = (x + 1)(x^2 + bx - 24)$.

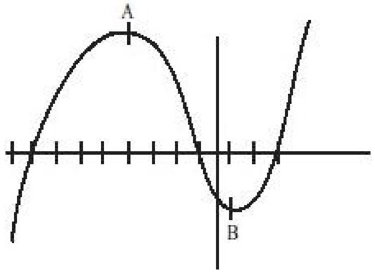
To find b, equate the coefficient of either x^2 or of x of the two sides of the equation. Equating the coefficient of x, we have $-19 = b - 24$ giving us $b = 5$ (when the LHS is expanded, the only terms in x are $bx - 24x$).

Thus finally, $x^3 + 6x^2 - 19x - 24 = (x + 1)(x^2 + 5x - 24)$.

The quadratic expression $x^2 + 5x - 24$ can further be factorised as $(x + 8)(x - 3)$ and thus the original equation becomes $(x + 1)(x + 8)(x - 3) = 0$ and the roots of this are -1, -8 or 3.

Once roots are known, one should also gather an idea of how will the graph of f(x) v/s x will look like. Obviously since the value of $f(x) = 0$ when $x = -1$ or -8 or 3, the graph will cut the X axis at these values of x.

Also simple visual check of f(x) tells us that as x assumes larger and larger positive values, f(x) will assume higher positive values as x^3 and $6x^2$ will increase in larger proportion than the decrease in $-19x$. Similarly as x assumes lower and lower values of x (basically tending towards $-\infty$), f(x) will also tend towards $-\infty$ as x^3 will be negative. Thus the graph of f(x) will look like



Note in the above figure, points A is called the local Maxima and point B is called the local Minima. The maximum and minimum possible value of f(x) will be $+\infty$ and $-\infty$ respectively. Maxima, Minima, Maximum and Minimum will be studied later.

Do you know about the Chinese Remainder Theorem?

Suz zi wrote the following problem in approximately late third century.

"We have a number of things, but we do not know exactly how many. If we count them by threes we have two left over. If we count them by fives, we have three left over. If we count them by sevens we have two left over.

How many things are there?

Example 2:

Is $(x + 5)$ a factor of $x^3 + 4x^2 - 5x + 3$?

Solution:

Applying the remainder theorem, let us find $f(-5)$ which is equal to $(-125) + (100) + (25) + 3 = 3$. Since, $f(-5)$ is not equal to 0, $(x + 5)$ is not a factor of the given expression.

Example 3:

If $(x + 7)$ and $(x - 4)$ are the only factors of $f(x)$, then find $f(x)$.

Solution:

Since, there are only 2 factors of $f(x)$, it must be of the form $C(x + 7)(x - 4)$. C is any constant.

So, the answer is $C(x^2 + 3x - 28)$.

Example 4:

Is $(a + b)$ a factor of $(a^3 + b^3)$?

Solution:

Applying the remainder theorem and defining the function in terms of the variable `a', we have $f(a) = a^3 + b^3$.

Since $f(-b) = 0$ in this expression, $a + b$ is a factor of $a^3 + b^3$.

Suppose there are two first degree expressions $(x - a)$ and $(x - b)$ which are both factors of another expression $g(x)$, then $g(x) = f(x)(x - a)(x - b)$.

The form and degree of $f(x)$ would depend on the degree of the original expression $g(x)$. For example, if $g(x)$ has a degree 4, then $f(x)$ would have a degree 2.

Test Your Understanding

Level - I

1. If $(x + 2)(x - a) = px^2 + qx + 8$, then what are the values of the constants a, p and q?

2. If $(x + 1)(2x - 2)(3x + 3) = ax^3 + bx^2 + cx + d$, then what are the values of a, b, c and d?

3. Factorise

a. $x^3 + 2x^2 - 11x - 12$

b. $x^3 - 2x^2 - 9x + 18$

c. $6x^3 - 5x^2 - 2x + 1$

4. What is the value of k for which the expression $x^3 + kx^2 + 3x + 4$ is divisible by $(x + 3)$?

5. What is the value of k and l if the expression $kx^3 + 4x^2 + lx + 5$ is divisible by $(x^2 - 1)$?

6. What is the common factor of the polynomials $3x^2 + 5x - 2$ and $3x^2 - 7x + 2$?

7. What is the common factor of the polynomials $2x^2 - x$ and $4x^2 + 8x + 3$?

8. `a' being a constant, would $(x + a)$ be a factor of $x^n + a^n$ when

a. `n' is even

b. `n' is odd

9. `a' being a constant, would $(x - a)$ be a factor of $x^n - a^n$ when

a. `n' is even

b. `n' is odd

Level - II

10. The graph of $x^3 - x^2 + 2x$ will cut the X axis at how many points? What will be the co-ordinates of the point/s at which the graph intersects the X axis?

Equations and Inequalities 2

Introduction

This chapter deals with concepts of Linear, Quadratic and Higher Degree Equations and Inequations and shows you how to solve them.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

- Linear, Quadratic and Higher Degree Equations.
- Common Roots of two equations.
- Graphs.
- Linear and Quadratic Inequalities.
- Concepts in Binoimial Theorem.

Equations

An **equation** sets two expressions (involving one or more than one variable) equal to each other.

For example, $3x + 4y = 3x^2 - 6$, $3x - 6 = 2x$, $4y^2 - 3y - 3 = 0$ etc.

What is an identity?

An identity is an equation that is true for all values of the variables.

For example,

$3x + 9 - 2(x + 2) = x + 5$

Remember,

While an identity is true for all values of x, an equation may be true for only some values of x, or for no values of x.

A **linear equation** is an equation of degree one.

For example, $3x = 2x - 6$, $4y - 3 = y + 6$, $3x - 3 = 4 + x$ are linear equations in **one** variable.

$3x + 4y = x + 32$, $5y - 7 = 4x + 3$ are linear equations in **two** variables because `x' and `y' are two variables involved in the equation.

Solutions of a Linear Equation in Two Variables

Solution of linear equation in two variables means a pair of values (one value for `x' and one value of `y') that satisfy the given linear equation.

For example, in the equation $3x + y = 13$, one solution is $x = 1$ and $y = 10$ because when we substitute $x = 1$ and $y = 10$ in the equation we find that LHS (Left Hand Side) = $3x + y = 13$ which is equal to RHS (Right Hand Side) of the equation.

Note: There will be more than one solution for the above equation ($x = 1$ and $y = 10$ is one solution.)

Example 1:

If $3x + 15 = 60$, find x.

Solution:

$3x + 15 = 60$.

So, $3x = 60 - 15 = 45$

$x = \frac{45}{3} = 15$

$x = 15$

Example 2:

Six years ago the age of Ram was thrice the age of Shyam and after six years Ram will become twice the age of Shyam. Find the present age of Shyam.

Solution:

Let the present age of Ram is x years, and the present age of Shyam is y years.

Then, $x - 6 = 3(y - 6)$

$\Rightarrow x - 6 = 3y - 18$

$\Rightarrow x - 3y = -12$

And $(x + 6) = 2(y + 6)$

$\Rightarrow x + 6 = 2y + 12$

$\Rightarrow x - 2y = 6$ ...(i)

$x - 3y = -12$ ...(ii)

Solving equation (i) and (ii)

$y = 18$ and $x = 42$

So Ram's present age is 42 years and Shyam's present age is 18 years.

Example 3:

The sum of a two-digit number and the two digit number obtained by interchanging the digits of this number is 33. Find the number(s)?

Solution:

Let the two digit number is xy; then the other number becomes yx.

So, their sum = xy + yx

$= 10x + y + 10y + x$

$= 11x + 11y$

$= 11(x + y) = 33$

$\Rightarrow x + y = 3$

The value of x is either 1 or 2. Similarly, the value of y is either 2 or 1.

So the possible numbers are 12 and 21.

Graph of a Linear Equation

Linear equation in one variable

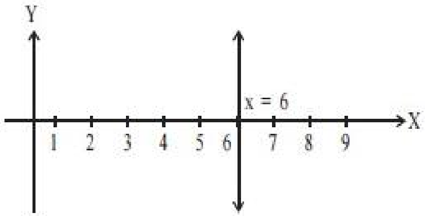
Consider the equation $2x - 5 = 7$

$2x - 5 = 7 \Rightarrow 2x = 7 + 5 \Rightarrow x = 6$

So, there is only one value i.e. $x = 6$ which satisfy the given equation. Hence, the graph of $x = 6$ will be as follows.

Here the graph would be nothing but the line *l*.

So, the solution of a linear equation when plotted in the X-Y plane will be represented by a line.

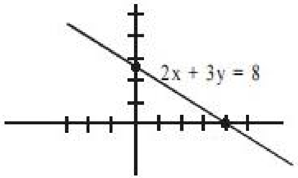


Linear equation in two variable

Consider the equation $2x + 3y = 8$. It can be re-written as $y = \frac{8 - 2x}{3}$. Thus, we can find the corresponding value of y for any value of x and the following table gives few values of (x, y) that satisfy this equation.

x	y
–3	4.66
–2	4
–1	3.33
0	2.66
1	2
2	1.33
3	0.66
4	0
5	–0.66

When the above points are plotted on a graph with values of x on the X axis and the value of y on the Y axis, we get a graph as follows:



Since, the graph is a straight line, the equation is called a Linear equation. Further any point on the line satisfies the given equation and hence the corresponding co-ordinates (x, y) is a solution to the equation. Thus, the equation $2x + 3y = 8$ has infinite solutions. Take any value of x and corresponding value of y can be found by substituting the value of x in the equation and finding the value of y.

Certain other observations worth noting are the integral solutions (x and y both being Integers) to the equation. From the table we see that the integral solutions are (-2, 4), (1, 2), (4, 0). We observe that the values of x increases in steps of 3 and the corresponding values of y decrease by 2. And this behaviour should be obvious by seeing the equation. $2x + 3y = 8$ is of the form A + B = constant and if A increases by k, B has to decrease by k. Thus in $2x + 3y = 8$, if x increases by 1, 2x will increase by 2 and correspondingly 3y will decrease by 2. If 3y decreases by 2, y would decrease by 2/3. If we want both x and y to remain Integers, the increase and corresponding decrease in either term has to be the LCM of 2 and 3 i.e. 6. If 2x increases by 6, 3y would decrease by 6 and this would mean an increase of 3 in x and a decrease of 2 in y.

Simultaneous Equation

Simultaneous Linear Equation in Two Variables:

It is commonly understood that if an equation has 2 variables, we need two equations to find the solution set. Well, this is not entirely correct. We would need two equations to identify a unique solution set that satisfies both the equation. And also there are conditions that the two equations should meet.

Consider the equation $6x - 4y = 8$. As already seen, such an equation has infinite solution (for any value of x, a value of y can be found that satisfies this equation). If we consider the two equations $6x - 4y = 8$ and $9x - 6y = 12$, would we now get a unique solution set for (x, y)? Not really, because though it seems that we have two equations, but in reality, the second equation is exactly same as first equation, as equation
2 divided by 1.5 gives us equation 1.

Thus, no new information is provided by the second equation. Infact both the equations are the same because multiplying both sides of an equation by a constant does not change the equation. Such equations are called Dependent Equations. Thus, any solution of equation 1 will also be a solution of equation 2 and thus we will again have infinite solutions satisfying both the equations simultaneously.

If we change the second equation such that now we have $6x - 4y = 8$ and $9x - 6y = 10$, the two equations are distinct and not the same i.e. the equations are Independent. Yet we would not be able to find a unique solution because when we try to eliminate one variable, both variables get eliminated and we will have

Eqn 1×3 will give $18x - 12y = 24$

Eqn 2×2 will give $18x - 12y = 20$

Subtracting we have $0 = 4$

Such equations are called Inconsistent equation.

Only when the equations are Independent and Consistent, will we get a unique solution to the set of equations and that too if there are as many equations as the number of variables.

The above discussion can be summarized for two simultaneous equations in two variables as given below. But rather than remembering the following table, just re-read through the above discussion and the following figure would become evident.

For the two simultaneous equations,

$$ax + by = c$$
$$px + qy = r$$

where a, b, c, p, q and r are constants

$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$	$\frac{a}{p} = \frac{b}{q} \neq \frac{c}{r}$	$\frac{a}{p} \neq \frac{b}{q}$
Dependent Equations The same equation Just one line Infinite Solutions	Inconsistent Equations Two parallel lines No Solutions	Two intersecting lines Unique Solutions

Example 4:

If $2x + y = 35$ and $3x + 4y = 65$, then find the value of x.

Solution:

$2x + y = 35$ ... (i)

$3x + 4y = 65$ ... (ii)

Multiplying the equation (i) by 4, we will get

$8x + 4y = 140$ ... (iii)

Subtracting equation (ii) from equation (iii),

$$\begin{array}{r} 8x + 4y = 140 \\ \underline{-3x \pm 4y = -65} \\ 5x = 75 \end{array}$$

So, $x = \frac{75}{5} = 15$

Example 5:

Five years ago, A was thrice as old as B and 10 years hence A shall be twice as old as B. Find the present age of A?

Solution:

According to the given condition,

$A - 5 = (B - 5)3$

$\Rightarrow A - 5 = 3B - 15$

$\Rightarrow A + 10 = 3B$ and $\frac{A+10}{3} = B$

And $A + 10 = (B + 10)2$

$\Rightarrow A + 10 = 2B + 20$

$\Rightarrow \frac{A-10}{2} = B$

So, $\frac{A-10}{2} = \frac{A+10}{3}$

$3A - 30 = 2A + 20$

$A = 30 + 20 = 50$

So the present age of A is 50 years.

Example 6:

A person has only 25-paisa and 50-paisa coins. In total he has 40 coins and total amount with the person is Rs. 12.50. Find the number of 50-paisa coins he has.

Solution:

Suppose that he has x number of 25-paisa coins and y number of 50 paisa coins.

Then $x + y = 40$ ...(i)

And $\frac{1}{4}x + \frac{1}{2}y = 12.50$

$\Rightarrow x + 2y = 50$ ...(ii)

Subtracting equation (i) from equation (ii),

$$\begin{array}{r} x + 2y = 50 \\ \underline{\overset{-}{x} \pm y = \overset{-}{40}} \\ y = 10 \end{array}$$

So he has 10 (50-paisa) coins.

Example 7:

For what value of k, the following system of equations does not have any solution?

$2x + 3y = 5, kx - 5y = 7$

Solution:

For no solution

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

$$\frac{2}{k} = \frac{3}{-5} \neq \frac{5}{7} \Rightarrow k = -\frac{10}{3}$$

Example 8:

How many integral solutions does the equation $2x + 3y = 8$ have such that $100 < x < 200$

Solution:

$x = 4 - \frac{3}{2}y$

x = 4, y = 0

x = 7, y = - 2

x = 10, y = -4

As given above, for integral solution, the value of x increases in steps of 3, with the first integral value above 100 being 103. Hence, there are 33 integral solutions for the given range of x.

Example 9:

How many integral solutions does the equation 7x - 4y = 1 have such that the product x × y < 0

Solution:

$y = \frac{7x - 1}{4}$. For x > 0, y > 0. Also, for x < 0, y < 0.

Hence, there are no integral solutions for the given condition.

Example 10:

Find for what value(s) of k would there be an unique solution for the given set of equations 2x - 3y = 1 and kx + 5y = 7.

Solution:

If two equations ax + by = M and cx + dy = N have a unique solution, then $\frac{a}{c} \neq \frac{b}{d}$.

So, in the given problem, $k \neq \frac{-10}{3}$.

Example 11:

Find the value(s) of `k' for which there is no solution for the given set of equations

2x - ky = -3 and 3x + 2y = 1.

Hint:

If two equations ax + by = M and cx + dy = N have no solution, then $\frac{a}{c} = \frac{b}{d} \neq \frac{M}{N}$

Solution:

3x + 2y = 1

If there are no sets of solutions, then

$\frac{2}{-k} = \frac{3}{2}$ or $k = -\frac{4}{3}$

For $k = -\frac{4}{3}$, the two lines in the coordinate planes are parallel to each other.

Example 12:

Find the value of `k' for which there are infinite solutions for the given set of equations

5x + 2y = k and 10x + 4y = 3.

Hint:

If two equations ax + by = M and cx + dy = N have infinite solutions, then $\frac{a}{c} = \frac{b}{d} = \frac{M}{N}$

Solution:

For the two sets of equation to have infinite solutions, we have $\frac{5}{10} = \frac{2}{4} = \frac{k}{3}$. Hence, $k = \frac{3}{2}$

Example 13:

What is the solution of the following system of simultaneous equations $x + y + z = 6$,

$x + 2y + 3z = 14$ and $x + 3y + z = 10$?

Solution:

$x + y + z = 6 \cdots (i)$

$x + 2y + 3z = 14 \cdots (ii)$

$x + 3y + z = 10 \cdots (iii)$

From (i), we get $z = 6 - x - y$

Substitute it in (ii) and (iii).

We have from (ii)

$x + 2y + 18 - 3x - 3y = 14$ or $2x + y = 4$

Similarly, from (iii)

$x + 3y + 6 - x - y = 10$ or $2y = 4$ or $y = 2$

On solving, we get $y = 2$, $x = 1$, $z = 3$.

Simultaneous Equations in Three or more Variables

The general method of solving `n' equations in `n' variables is the process of elimination. We eliminate a variable by taking two equations at a time and reduce the number of variables subsequently.

For example,

$x + y + 2z = 1$ ... (1)

$2x + y + z = -1$ ... (2)

$x - y + z = 0$ ... (3)

Can be solved for x, y and z by taking (1) and (2) together eliminating y and taking (2) and (3) together and eliminating y. Thus, Eliminating `y'

(1) - (2) $\Rightarrow -x + z = 2$ ... (4)

(2) + (3) $\Rightarrow 3x + 2z = -1$ ... (5)

Hence, the system has reduced into two-variables and two-equations and can be solved easily. However, we can still eliminate one of the two variables.

Eliminating `x'.

(5) + 3 × (4) $\Rightarrow 5z = 5$

$\Rightarrow z = 1$

Substituting $z = 1$ in (4), we get

$-x + 1 = 2 \Rightarrow x = -1$

Substituting $z = 1$ and $x = -1$ in (1) or (2) or (3),

$x + y + 2z = 1$

$\Rightarrow -1 + y + 2 = 1$

$\Rightarrow y = 0$

Hence, the solution is $x = -1$, $y = 0$, $z = 1$.

Example 14:

Solve the following system of equations for a, b, c and d

$a + b + c + d = 10$

$a - 2b + 3c - 4d = -10$

$2a + 3b + c - d = 7$

$-a + 2b + c + d = 10$

Solution:

Here,

$a + b + c + d = 10$ ... (1)

$a - 2b + 3c - 4d = -10$ ... (2)

$2a + 3b + c - d = 7$ ... (3)

$-a + 2b + c + d = 10$ ... (4)

Eliminating `a' we get the following

(1) - (2) $\Rightarrow 3b - 2c + 5d = 20$

(3) - 2 × (2) $\Rightarrow 7b - 5c + 7d = 27$

(3) + 2 × (4) $\Rightarrow 7b + 3c + d = 27$

Now,

$3b - 2c + 5d = 20$ ... (5)

$7b - 5c + 7d = 27$ ... (6)

$7b + 3c + d = 27$ ... (7)

Now, it is a system of three equations in three variables.

Eliminating `b' we get the following

7 × (5) - 3 × (6) $\Rightarrow c + 14d = 59$

(7) - (6) $\Rightarrow 8c - 6d = 0$

$c + 14d = 59$ ... (8)

$8c - 6d = 0$ ... (9)

Two-equation-two variable form

Eliminating `c' we get the following

8 × (8) - (9) $\Rightarrow 118d = 472$

$\Rightarrow d = 4$

Substituting $d = 4$ in (9)

$8c = 6d \Rightarrow c = 3$

Substituting $c = 3$ and $d = 4$ in (5)

$3b - 2c + 5d = 20$

$\Rightarrow 3b - 6 + 20 = 20$

$\Rightarrow b = 2$

Substituting $b = 2$, $c = 3$ and $d = 4$ in (1),

$a + b + c + d = 10$

$\Rightarrow a + 2 + 3 + 4 = 10 \Rightarrow a = 1$

Hence, the solution is a = 1, b = 2, c = 3, d = 4

Binomial Theorem

We know that

$$(x+1)^2 = x^2 + 2x + 1$$

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

$$\begin{aligned}(x+1)^4 &= (x+1)^2(x+1)^2 \\ &= (x^2+2x+1)(x^2+2x+1) \\ &= x^4+4x^3+6x^2+4x+1\end{aligned}$$

Binomial theorem gives the method of expanding the general expression given by $(x + y)^n$

$$(x+y)^n = K_0x^n + K_1x^{n-1}.y^1 + K_2x^{n-2}.y^2 + \ldots K_nx^0.y^n$$

where $K_0, K_1, K_2, \ldots K_n$ are constants (called coefficients of binomial expansion)

Note that:

(1) Sum of exponents of x and y in any term = n

(2) Any term is given by

$$T_{r+1} = K_rx^{n-r}.y^r = (r+1)^{th} \text{ Term}$$

(3) K_r = binomial coefficient of (r + 1)th term $= {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Example 15:

Find the term involving x^6 in the expansion of $(1 + x)^{12}$.

Solution:

$$T_{r+1} = {}^nC_rx^r \cdot 1^{(n-r)}$$

$$\Rightarrow T_{6+1} = {}^{12}C_6 \times x^6 = 924\,x^6$$

Example 16:

Find the number of terms in the expansion of $(1 + 2x + 3x^2)^6$

a. 18

b. 13

c. 7

d. None of these

Solution:

Recall that the sum of exponents of any term = 6

Here, all terms are positive. Hence, lowest power of x would be zero and highest power would be when;

$(x^2)^6 = x^{12}$.

$\therefore$ We will have terms involving integral exponents in x from 0 to 12.

$\Rightarrow$ 13 terms in all.

Example 17:

If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$ are in arithemetic progression, then find the value of `n'.

Solution:

Here $2 \times {}^nC_2 = {}^nC_1 + {}^nC_3$

$$\Rightarrow 2 \times \frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$$\Rightarrow (n-1) = 1 + \frac{n^2 - 3n + 2}{6}$$

$$\Rightarrow n^2 - 9n + 14 = 0$$

$$\Rightarrow n^2 - 7n - 2n + 14 = 0 \Rightarrow n = 7 \text{ or } 2$$

But n has to be greater than 2.

Hence n = 7.

Quadratic Equations

Equations of degree 2 are called quadratic equations.

The general format of the quadratic equation is $ax^2 + bx + c = 0$.

Where, a, b and c are real numbers and a is not equal to zero. a, b and c are known as constants or coefficients. In the given format, a is the coefficient of x^2 and b is the coefficient of x.

Since, quadratic equation is a polynomial of second degree in one variable, so it will have two solutions.

For example, $x^2 = 4$, so x will have two solutions, +2 or -2.

Solution of a quadratic equation is known as `roots'. So we can say that root is such a value which when placed in place of variable, the value of the equation becomes zero. Both the roots of a quadratic equation may be the same number and the roots may be complex or imaginary.

How to find the roots? There are two ways of finding out the roots of a quadratic equation.

1.Factorisation method: Suppose the equation is $x^2 - 5x + 6 = 0$

This equation can be further written like

$\Rightarrow x^2 - 3x - 2x - 6 = 0$

$\Rightarrow x(x - 3) - 2(x - 3) = 0$

$\Rightarrow (x - 2)(x - 3) = 0$

$\Rightarrow x - 2 = 0$ or $x - 3 = 0$

$\therefore x = 2$ or $x = 3$

Hence, roots of the equation are 2 and 3.

How to factorise? Consider the equation, $ax^2 + bx + c = 0$.

Split `b' in two parts so that their sum becomes equal to b and the product becomes equal to c.

For example, $x^2 - 5x + 6 = 0$

Here, a = 1, b = -5 and c = 6

$\Rightarrow x^2 - 3x - 2x + 6 = 0$

Thus, -5 has been broken into -3 and -2 so that their sum is -5 and the product is 6.

$\Rightarrow x(x - 3) - 2(x - 3) = 0$

$\Rightarrow x = 2, 3$

2.Formula: The roots of a quadratic equation, $ax^2 + bx + c = 0$ is generally denoted by a (alpha) and b (Beta).

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

In the equation $x^2 - 5x + 6 = 0$,

$$\alpha = \frac{-(-5)+\sqrt{(-5)^2-4\times1\times6}}{2\times1} = \frac{5+\sqrt{25-24}}{2} = \frac{5+1}{2} = 3$$

$$\beta = \frac{-(-5)-\sqrt{(-5)^2-4\times1\times6}}{2\times1} = \frac{5-\sqrt{25-24}}{2} = \frac{5-1}{2} = 2$$

α and β = 3 and 2.

General Form:

$ax^2 + bx + c = 0$, where $a \neq 0$

Such an equation has two roots, usually denoted by a and b.

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a};\ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Accordingly,

Sum of roots: $\alpha + \beta = -\frac{b}{a}$ and Product of roots: $\alpha \times \beta = \frac{c}{a}$.

Thus, if the quadratic equation has roots equal in magnitude and opposite in sign, then b = 0 (as sum of roots is 0) i.e. the equation would not have the term containing "x". Similarly if the roots are reciprocals of each other [i.e. $\alpha = \frac{1}{\beta}$] then c = a.

The general form of the equation can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Thus if the roots of a quadratic equation are α and β, the equation can be re-constructed as

x^2 - (sum of roots) $\times$ x + (product of roots) = 0

The equation $ax^2 + bx + c = 0$, $a \neq 0$ has *rational co-efficientsa, b, c*. The nature of roots of this equation can be predicted by the value of the expression $(b^2 - 4ac)$ which is called the Discriminant of the equation and is denoted by D or δ or Δ. We will discuss the various possible values of D and accordingly the nature of roots of the quadratic.

- $D < 0 \Rightarrow$ roots are imaginary and of the form p + iq and p - iq.

e.g., $x^2 + 4x + 6 = 0$. The D of this equation is

$D = 4^2 - 4 \times 1 \times 6 = 16 - 24 = -8$ i.e. $D < 0$.

So this equation must have imaginary roots.

As we can see:

$x^2 + 4x + 6 = 0$

or $(x + 2)^2 + 2 = 0$

or $(x + 2)^2 = -2$

For no real values of x, the square of (x + 2) will result in a negative number.

Hence, the roots of this equation cannot be real as has been deduced by the value of D also.

• $D = 0 \Rightarrow$ roots are equal to each other

e.g., $x^2 + 4x + 4 = 0$

The D of this equation is

$D = 4^2 - 4 \times 1 \times 4 = 0$

As $D = 0$, the roots of this equation must be equal.

As we can see $x^2 + 4x + 4 = 0$ or $(x + 2)^2 = 0$, which gives $x = -2$ as the root.

• $D > 0$ and D is a perfect square $\Rightarrow$ the roots are rational but unequal.

e.g., $x^2 - 5x + 6 = 0$

The D of this equation is

$D = (-5)^2 - 4 \times 1 \times 6 = 1$

The roots of this equation should be rational and unequal.

As we can see $x^2 - 5x + 6 = 0$ or

$(x - 2)(x - 3) = 0$ which gives $x = 2$ and $x = 3$ as the roots which are rational and unequal.

• $D > 0$ and D is not a perfect square $\Rightarrow$ the roots are irrational and of the form p + and p - $\sqrt{q}$

e.g., $x^2 - 2 = 0$

The D of this equation is

$D = (0)^2 - 4 \times 1 \times (-2) = 8$

$D > 0$ but not a perfect square

Hence, the roots of this equation should be irrational.

As we can see $x^2 - 2 = 0$

or, $\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = 0$

or, $x = \sqrt{2}$ and $x = -\sqrt{2}$ are the two roots which are irrational and of the form $p + \sqrt{q}$ and $p - \sqrt{q}$

(Here, $p = 0$ and $q = 2$)

Example 18:

If the sum of a number and its reciprocal adds to $\frac{5}{2}$, then what is the number?

Solution:

Let the number be x.

Then, $x + \left(\frac{1}{x}\right) = \frac{5}{2}$ or $2x^2 - 5x + 2 = 0$

or $2x^2 - 4x - x + 2 = 0 \Rightarrow (2x - 1)(x - 2) = 0$;

Hence, $x = \frac{1}{2}$ or 2.

Example 19:

Let a, b, c be three consecutive terms of a G.P. Can you comment on the nature of roots of the equation $ax^2 + bx + c = 0$.

Solution:

As a, b, c are in G.P.

Let $a = \frac{p}{r}, b = p$ and $c = pr$

Now, the D of the equation is

$D = p^2 - 4 \times \frac{p}{r} \times pr = p^2 - 4p^2 = -3p^2$

As p is a real number D, here is always negative.

⇒ this equation will always have imaginary roots. (Complex conjugate roots)

Example 20:

Find the roots of the quadratic equation $x^2 - 7x + 12 = 0$.

Solution:

Method 1:

The standard procedure would be to find a and b, where

$$\alpha = \frac{\left(7 + \sqrt{(7^2 - 4(1)(12))}\right)}{2(1)} = 4,\ \beta = \frac{\left(7 - \sqrt{(7^2 - 4(1)(12))}\right)}{2(1)} = 3$$

So, the two roots of the quadratic expression are 3 and 4.

Method 2:

An alternative method of solving this quadratic equation would be to split the coefficient of x in such a way that the product of these two numbers is equal to the product of the constant and the coefficient of x^2.

So $x^2 - 7x + 12$ can be written as

$x^2 - 3x - 4x + 12 = x(x - 3) - 4(x - 3) = (x - 3)(x - 4)$.

So the factorised form of $x^2 - 7x + 12 = 0$ is

$(x - 3)(x - 4) = 0$.

Hence, the value of x can be either 3 or 4.

Example 21:

If the sum of the squares of two consecutive integers is equal to 313, then what is the smaller number?

Solution:

Let the two numbers be x and x + 1, then

$(x)^2 + (x + 1)^2 = 313$

$2x^2 + 2x - 312 = 0$ or $x^2 + x - 156 = 0$

Hence, $(x + 13)(x - 12) = 0$

So, the value of x are $x = -13$ or $x = 12$. The consecutive integers are (-13, -12) or (12, 13).

So, the smaller number will be either -13 or 12.

Example 22:

Find the nature of the roots of the equation $6x^2 + 7x + 2 = 0$.

Solution:

In the equation, $6x^2 + 7x + 2 = 0$,

$a = 6, b = 7$ and $c = 2$

$D = b^2 - 4ac = 7^2 - 4 \times 6 \times 2 = 49 - 48 = 1$ since $D > 0$ and D is a perfect square.

So, its roots are real, unequal and rational.

Example 23:

Find the roots of the equation $6x^2 + 7x + 2 = 0$.

Solution:

$6x^2 + 7x + 2 = 0$

$\Rightarrow 6x^2 + 3x + 4x + 2 = 0$

$\Rightarrow 3x(2x + 1) + 2(2x + 1) = 0$

$\Rightarrow (3x + 2)(2x + 1) = 0$

So, $3x + 2 = 0$ or $2x + 1 = 0$.

$\therefore x = \frac{-2}{3}, \frac{-1}{2}$

Example 24:

If sum of the two roots of a quadratic equation is 5 and their product is 6, then form the quadratic equation.

Solution:

To form a quadratic equation we use the formula $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

where, α , β are the roots of the quadratic equation.

So, the equation is $x^2 - 5x + 6 = 0$

Example 25:

If one of the roots of a quadratic equation is +4 and the other root is -4, then form the quadratic equation.

Solution:

Here, $\alpha = +4$ and $\beta = -4$

So, $\alpha + \beta = 0$ and $\alpha \times \beta = -16$

So, the equation is

$x^2 - (\alpha + \beta)x + \alpha \times \beta = 0$

$\Rightarrow x^2 - 0 \times x + (-16) = 0$

$\therefore x^2 - 16 = 0$

Example 26:

If one root of a quadratic equation is 6 and the other root is 4, then form the quadratic equation.

Solution:

Here, $\alpha = 4$ and $\beta = 6$

So $\alpha + \beta = 4 + 6 = 10$ and

So, the equation is $\alpha \times \beta = 4 \times 6 = 24$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\therefore x^2 - 10x + 24 = 0$

Example 27:

Find the factors of the equation $12x^2 - 30x + 18 = 0$.

Solution:

$12x^2 - 30x + 18 = 0$ can be written as

$\Rightarrow 6(2x^2 - 5x + 3) = 0$

$\Rightarrow 2x^2 - 2x - 3x + 3 = 0$

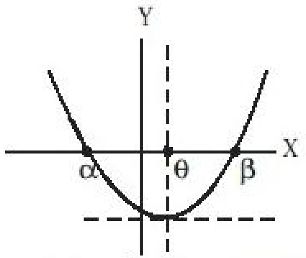
$\Rightarrow 2x(x - 1) - 3(x - 1) = 0$

$\therefore (2x - 3)(x - 1) = 0$

So, its factors are $(2x - 3)$ and $(x - 1)$.

Algebraic curve: $y = ax^2 + bx + c$

When plotted, the curve $y = ax^2 + bx + c$ has the shape of a parabola as shown below.



Curve: $ax^2 + bx + c$ $(a > 0)$

Observe the following properties of the above curve:

1. As $a > 0$ the parabola opens upwards.

2. As the parabola opens upwards, it has a definite **minima** only. As x increases in both positive and negative X-axis, the value of y approaches towards ∞.

This curve has **nomaxima**.

(Note: $\pm\infty$ is not a maxima/minima.)

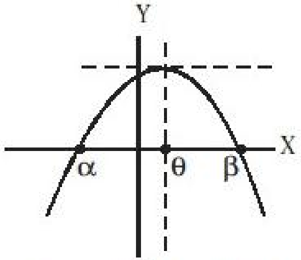
3. At $(\alpha, 0)$ and $(\beta, 0)$ the curve $ax^2 + bx + c = 0$ cuts the X-axis and accordingly $x = \alpha$, and $x = \beta$ are the roots of this equation.

4. At $x = \Theta$ the curve attains its minima.

5. The parabola is always symmetric about the line passing through (q, 0) and parallel to the Y-axis. a, b and q are related as below:

$$\theta = \frac{\alpha + \beta}{2}$$

If we had $a < 0$, the curve would have been inverted as shown below.



Curve: $ax^2 + bx + c$ $(a < 0)$

Observe the following properties of the above curve:

1. As $a < 0$ the parabola opens downwards.

2. As the parabola opens downwards, it has a definite **maxima** only. As x increases in both positive and negative X-axis, the value of y approaches towards $-\infty$.

This curve has **nominima**.

(Note: $\pm\infty$ is not a maxima/minima.)

3. At $(\alpha, 0)$ and $(\beta, 0)$ the curve

$ax^2 + bx + c = 0$ cuts the X-axis and accordingly $x = \alpha$, and $x = \beta$ are the roots of this equation.

4. At $x = \Theta$ the curve attains its maxima.

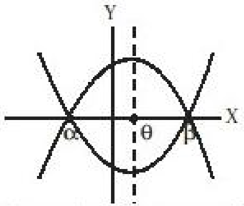
5. $\theta = \frac{\alpha+\beta}{2}$

With both the roots (α and β) common, how many different quadratic equations can be formed?

Solution: If α and β are roots, then two quadratic equation are possible which are

$y = (x-\alpha)(x-\beta)$

$y = -(x-\alpha)(x-\beta)$



Both equations have roots α, β

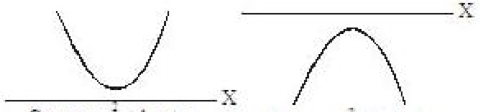
Graphical interpretations of D, the discriminant

$y = ax^2 + bx + c$ and $D = b^2 - 4ac$

Depending upon the value of D the position of the quadratic curve in the X-Y plane changes as we will see.

Case I: $D < 0$

If $D < 0$, there are no real roots of the equation, which means the curve **will never cut** the X-axis. Depending upon the sign of "a" we get either an upwards or an inverted parabola as shown below.

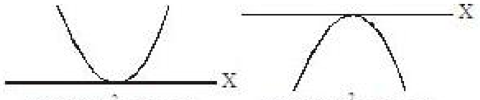


Curve: $ax^2 + bx + c$ $a > 0$ and $D < 0$

Curve: $ax^2 + bx + c$ $a < 0$ and $D < 0$

Case II: $D = 0$

If $D = 0$, the roots of the equation are equal, which means the curve will **touch** the x-axis. Again, depending upon the sign of "a" we get either an upwards or an inverted parabola as shown below.

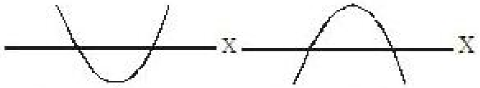


Curve: $ax^2 + bx + c$ $a > 0$ and $D = 0$

Curve: $ax^2 + bx + c$ $a < 0$ and $D = 0$

Case III: $D > 0$

If $D > 0$, the roots of the equation are real and distinct. Which means the curve will **cut** the X-axis at two distinct points. Again, depending upon the sign of "a" we get either an upwards or an inverted parabola as shown below.

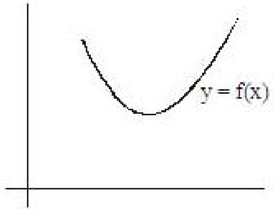


Curve: $ax^2 + bx + c$ $a > 0$ and $D > 0$ Curve: $ax^2 + bx + c$ $a < 0$ and $D > 0$

After this discussion we can observe that a quadratic curve meets the X-axis :

1. Never 2. Only once 3. Twice

More on quadratic polynomials



Here, f(x) is a quadratic expression in x. From the graph, we can see that f(x) does not cut X-axis, and hence, is always positive.

If $f(x) = x^2 + a x + b$ then, f(x) can be written as

$$f(x) = x^2 + 2 \times \frac{a}{2} \times x + \left(\frac{a}{2}\right)^2 + b - \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$$

Here, $\left(x + \frac{a}{2}\right)^2$ will always be non-negative. Thus, if we have $b > \frac{a^2}{4}$, the expression will always be positive for all real values of x. The theory would be clear by some examples.

Example 28:

The expression $(x^2 - 3x + 8)$ is

a. always positive b. always negative

c. always zero d. Either positive or zero or negative

Solution: (a)

Here $f(x) = x^2 - 3x + 8$

$$= \left[x^2 - 2 \times x \times \frac{3}{2} + \left(\frac{3}{2}\right)^2\right] + \left[8 - \left(\frac{3}{2}\right)^2\right] = \left(x - \frac{3}{2}\right)^2 + \frac{23}{4}$$

$\therefore$ Even if $x - \frac{3}{2} = 0$, we will always have f(x) as positive because $\frac{23}{4}$ is added to the square of $\left(x - \frac{3}{2}\right)$.

Example 29:

Find the maximum and the minimum possible value of the expression $(2x^2 - 12x + 24)$.

Solution:

$2x^2 - 12x + 24 = 2(x^2 6x + 12)$

$$= 2\left(x^2 - 2 \times 3x + 3^2 + 12 - 3^2\right) = 2(x - 3)^2 + 6$$

Hence, minimum possible value would occur when x = 3; maximum possible value would be infinitely high.

$\Rightarrow \min(2x^2 - 12x + 24) = 6$

Example 30:

For which of the following value of k does the following pair of equations yield a unique solution of x such that the solution is positive?

$x^2 - y^2 = 0$ and $(x-k)^2 + y^2 = 1$

a. 2 b. 0 c. $\sqrt{2}$ d. $-\sqrt{2}$ e. None of these

Solution: (c)

$y^2 = x^2$

$2x^2 - 2kx + k^2 - 1 = 0$

$D = 0$

$\Rightarrow 4k^2 = 8k^2 - 8 \Rightarrow 4k^2 = 8$

$k^2 = 2 \Rightarrow k = \pm\sqrt{2}$ with $k = +\sqrt{2}$ gives the equation $\Rightarrow 2x^2 - 2\sqrt{2}x + 1 = 0$; root is : $\frac{-b}{2a} = +\frac{1}{\sqrt{2}}$

but with $k = -\sqrt{2}$, the equation is $2x^2 + 2\sqrt{2}x + 1 = 0$ whose root is $-\frac{1}{\sqrt{2}}$ which is -ve.

So we will reject $k = -\sqrt{2}$.

Only answer is: $k = +\sqrt{2}$ only.

Alternative method:

Graph based.

$x^2 - y^2 = 0$ & $(x - k)^2 + y^2 = 1$ are plotted below:

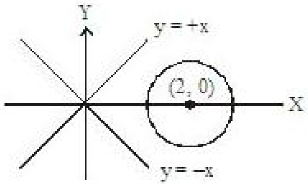
We are solving for a unique positive x.

$x^2 - y^2 = 0$ is a pair of straight lines

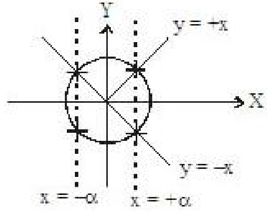
$y = x$ & $y = -x$

$(x - k)^2 + y^2 = 1$ is a circle with center (k, 0) & radius 1.

(a) k = 2; clearly, no solution



(b) k = 0



$x = \alpha, -\alpha$

two solutions.

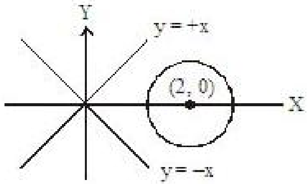
rejected.

(c) $k = +\sqrt{2}$

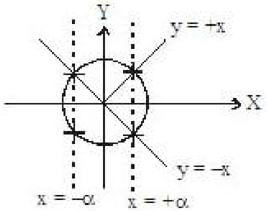
unique value of x & a positive one as shown.

$(x - k)^2 + y^2 = 1$ is a circle with center (k, o) & radius 1.

(a) k = 2; clearly, no solution



(b) k = o



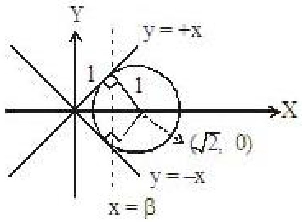
x = α, -α

two solutions.

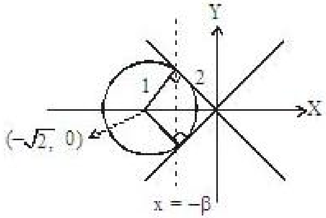
rejected.

(c) $k = +\sqrt{2}$

unique value of x & a positive one as shown.



(d) $k = -\sqrt{2}$, also gives the unique value of x but it is negative one.



In Europe, paper sizes are measured in A sizes, with Ao being the largest with an area of 1 m². The A sizes have a special relationship between them. If we take a sheet of A1 paper and then folding it in half (along its longest side) we get A2 paper. Folding it in half again gives A3, and so on....

However, the paper is designed so that proportions of length & breadth of each of the A sizes is the same -
that is, each piece of paper has the same shape.

This proportion is given by the quadratic equation $\left(\frac{x}{y}\right)^2 = 2$, *where x and y are the lengths of the two sides.*

Higher degree equations

While we will not deal with finding the solution to higher degree equation, yet simple observation can give us certain properties about the roots.

Consider the cubic equation

$ax^3 + bx^2 + cx + d = 0$.

The equation would have 3 roots (equal to the degree of the equation). Some of them can be imaginary. If the roots are denoted as α, β and γ, we have

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

The above can be extended for higher degree equations as well. For an `n' degree equation,

Sum of roots = $-\dfrac{\text{co-efficient of } x^{n-1}}{\text{co-efficient of } x^{n}}$

Sum of roots taken two at a time = $\dfrac{\text{co-efficient of } x^{n-2}}{\text{co-efficient of } x^{n}}$

Sum of roots taken three at a time = $-\dfrac{\text{co-efficient of } x^{n-3}}{\text{co-efficient of } x^{n}}$

And, sum of roots taken `r' at a time = $(-1)^r \dfrac{\text{coefficient of } x^{n-r}}{\text{coefficient of } x^{n}}$

Thus, product of roots = $(-1)^n \dfrac{\text{constant term}}{\text{co-efficient of } x^{n}}$.

Use positive sign for equations with even degree and minus sign for equations of odd degree.

Example 31:

If a, b, g are then the roots of $x^3 - 4x^2+x+6 = 0$, find the equation whose roots are

(i) $2\alpha, 2\beta, 2\gamma$

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

(iii) $\alpha^2, \beta^2, \gamma^2$

(iv) $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$

Solution:

α, β, γ are the roots of $x^3 - 4x^2 + x + 6 = 0$

$\Rightarrow (x-\alpha)(x-\beta)(x-\gamma) = x^3 - 4x^2 + x + 6$

Thus $\alpha+\beta+\gamma=4$

$\alpha\beta+\beta\gamma+\gamma\alpha=1$

$\alpha\beta\gamma=-6$

(i) Roots are $2\alpha, 2\beta, 2\gamma$

$\Rightarrow$ The equation would be $(x-2\alpha)(x-2\beta)(x-2\gamma)=0$

$$\Rightarrow \left(\frac{x}{2}-\alpha\right)\left(\frac{x}{2}-\beta\right)\left(\frac{x}{2}-\gamma\right)=0 \Rightarrow \left(\frac{x}{2}\right)^3-4\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)+6=0$$

$$\Rightarrow x^3-8x^2+4x+48=0$$

(ii) Roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma} \Rightarrow$ the equation would be $\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\left(x-\frac{1}{\gamma}\right)=0$

$$\Rightarrow (\alpha x-1)(\beta x-1)(\gamma x-1)=0$$

$$\Rightarrow \frac{1}{x^3}(\alpha x-1)(\beta x-1)(\gamma x-1)=0$$

$$\Rightarrow \left(\alpha-\frac{1}{x}\right)\left(\beta-\frac{1}{x}\right)\left(\gamma-\frac{1}{x}\right)=0$$

$$\Rightarrow \left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)=0$$

$$\Rightarrow \left(\frac{1}{x}\right)^3-4\left(\frac{1}{x}\right)^2+\left(\frac{1}{x}\right)+6=0$$

$$\Rightarrow 6x^3+x^2-4x+1=0$$

(iii) Roots are $\alpha^2, \beta^2, \gamma^2$

Product of roots = $\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=(-6)^2=36$

Sum of roots = $\alpha^2+\beta^2+\gamma^2$

$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=4^2-2(1)=14$$

Sum of product of two roots taken at a time $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

$$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)=1^2-2(-6)(4)=+49$$

Hence the equation is $x^3-14x^2+49x-36=0$

(iv) Roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\gamma\beta}$

Product of roots = $\frac{\alpha\beta\gamma}{\alpha^2\beta^2\gamma^2}=\frac{1}{\alpha\beta\gamma}=\frac{-1}{6}$

Sum of roots = $\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}=\frac{1}{\alpha\beta\gamma}(\alpha^2+\beta^2+\gamma^2)=\frac{1}{(-6)}\times 14=\frac{-7}{3}$

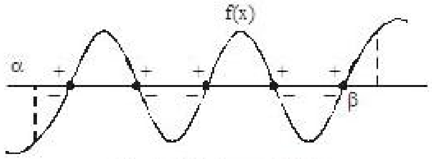
Sum of products of two roots taken at a time $=\frac{1}{\gamma^2}+\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$$=\frac{1}{\alpha^2\beta^2\gamma^2}\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)=\frac{1}{36}(49)=\frac{49}{36}$$

Hence, the equation is $x^3+\frac{7}{3}x^2+\frac{49}{36}x+\frac{1}{6}=0$

$$\Rightarrow 36x^3+84x^2+49x+6=0$$

Real roots of a polynomial equation, between x = a and x = b



Graph of polynomial f(x)

Let $x = \alpha$ and $x = \beta$ are two points on the curve, f(x). If we know the signs of f(α) and f(β), we can deduce

(i) whether f(x) has crossed X-axis between α and β or not. That means whether f(x) has any roots between α and β or not.

AND

(ii) the possible number of roots between α and β

Case I: If $f(\alpha) \times f(\beta) < 0$, then $f(x) = 0$ must have real root(s) between α and β. Infact in this case f(x) will always have an odd number of real roots between α and β like 1 roots, 3 roots, 5 root etc.

Case II: If $f(\alpha) \times f(\beta) > 0$, then $f(x) = 0$ has either

(i) no real roots between α and β

OR

(ii) An even number of real roots between α and β like 2 roots, 4 roots, 6 roots etc.

Example 32:

If $f(x) = x^3 - 4x + p$, and f(0) and f(1) are of opposite sings, then which of the following is necessarily true?

a. $-1 < p < 2$ b. $0 < p < 3$ c. $-2 < p < 1$ d. $-3 < p < 0$ e. $-5 < p < 4$

Solution: (b)

We have $f(0) = 0^3 - 4(0) + p = p$

and $f(1) = 1^3 - 4(1) + p = p - 3$

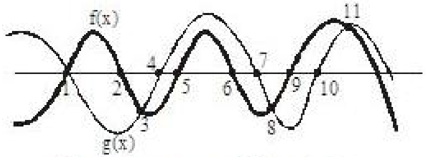
If `p' and `p - 3' are of opposite signs, then

$p(p - 3) < 0$.

Hence, $0 < p < 3$.

Common roots of two different polynomial equation:

Consider two equations $f(x) = 0$ and $g(x) = 0$ (their degrees may or may not be equal). Roots, common to both these equations are those values of x for which both f(x) and g(x) evaluate to zero.



Simultaneous curves of f(x) and g(x)

Observe the following points:

(i) Points 1, 4, 7 and 10 are roots of $g(x) = 0$

(ii) Points 1, 2, 5, 6 and 9 are roots of $f(x) = 0$

(iii) Points 1, 3, 8 and 11 are points of intersection of the two curves f(x) and g(x) or in other words these points are solutions of the equation $f(x) - g(x) = 0$.

(iv) Points 1 is a root which is common to both these equations.

In order to find the common roots of $f(x) = 0$ and $g(x) = 0$ we have to find the solution of the equation $f(x) - g(x) = 0$. The roots of this equation gives all the points of intersections of the two curves f(x) and g(x). These points of intersections, include the points of common roots also i.e. the points at which $f(x) = g(x) = 0$.

Example 33:

The number of roots common between the two equations $x^3 + 3x^2 + 4x + 5 = 0$ and $x^3 + 2x^2 + 7x + 3 = 0$ is

a. 0 b. 1 c. 2 d. 3 e. Cannot be determined

Solution: (a)

Here $f(x) = x^3 + 3x^2 + 4x + 5 = 0$ and $g(x) = x^3 + 2x^2 + 7x+3= 0$

To find the common roots we have to solve the equation $f(x) - g(x) = 0$

i.e. $(x^3 + 3x^2 + 4x + 5) - (x^3 + 2x^2 + 7x + 3) = 0$

$x^2 - 3x + 2 = 0$

$x = 2,\ x = 1$

These are the points of intersection of f(x) and g(x). Whether these points are also the common root will have to be checked by putting these values in $f(x) = 0$ and $g(x) = 0$.

For $x = 2$:

$f(2) = g(2) = 43 \neq 0$. Hence 2 is not a common root but only a point of intersection.

For $x = 1$:

$f(1) = g(1) = 13 \neq 0$. Again 1 is not a common root but only a point of intersection.

Hence, we find that the two equations do not have any common root between them.

Finding solutions of equations by intersection of Graphs:

The solution of an equation $f(x) - g(x) = 0$ can be found out by plotting f(x) and g(x) and finding the points of intersection of the two curves.

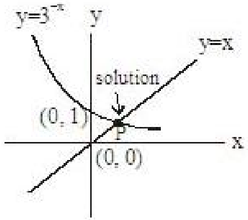
Example 34:

How many solutions exists for the equation $x.3^x = 1$.

Solution:

$x.3^x = 1$ can also be written as $x = 3^{-x}$ or $(x) - (3^{-x}) = 0$

As we know how to plot the curves $y = x$ and $y = 3^{-x}$ we can find the solution by graphical method as below:



As we can see, the two curves meet only once, at point P. Hence, the equations $x.3^x = 1$ has only one solution.

Miscellaneous Equations:

Example 35:

Find the roots of the equation $\dfrac{(7-x)\sqrt{(7-x)} + (x-5)\sqrt{(x-5)}}{\sqrt{(7-x)} + \sqrt{(x-5)}} = 2$

Solution:

Here, $\dfrac{(7-x)\sqrt{(7-x)} + (x-5)\sqrt{(x-5)}}{\sqrt{(7-x)} + \sqrt{(x-5)}} = 2$

$$\Rightarrow \frac{(7-x)^{\frac{3}{2}} + (x-5)^{\frac{3}{2}}}{(7-x)^{\frac{1}{2}} + (x-5)^{\frac{1}{2}}} = 2 \left[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)\right]$$

$$\therefore (7-x) - \sqrt{(7-x)} \times \sqrt{(x-5)} + (x-5) = 2 \Rightarrow \sqrt{(7-x)(x-5)} = 0 \Rightarrow x = 7 \text{ or } 5$$

Example 36:

Find the roots of the equation $\sqrt{16 + a - 5x} + \sqrt{4a + 4 - 5x} = 3\sqrt{4 + a - 2x}$

Solution:

Put, 4 - x = p and a - x = q

Then $\sqrt{4p+q}+\sqrt{4q+p}=3\sqrt{p+q}$

Squaring both sides, we get

$$(4p+q)+(4q+p)+2\sqrt{(4p+q)}\sqrt{(4q+p)}=9(p+q)$$

$$2(p+q)=\sqrt{(4p+q)(4q+p)}$$

Again squaring, we get

$$4p^2+4q^2+8pq=4p^2+4q^2+17pq$$

$$\Rightarrow 9pq=0 \Rightarrow p=0 \text{ or } q=0$$

$$\therefore x=4 \text{ or } a$$

Example 37:

Find the roots of the equation $\frac{x-5}{x+5}+\frac{x+2}{x-2}=2\frac{(x+4)}{(x-4)}$.

Solution:

The given equation can be written as

$$\left(\frac{x-5}{x+5}-1\right)+\left(\frac{x+2}{x-2}-1\right)=\frac{2(x+4)}{(x-4)}-2$$

$$\Rightarrow \frac{-10}{x+5}+\frac{4}{x-2}=\frac{16}{x-4} \Rightarrow \frac{-5x+10+2x+10}{(x+5)(x-2)}=\frac{8}{(x-4)} \Rightarrow \frac{-3x+20}{(x+5)(x-2)}=\frac{8}{(x-4)}$$

$$\Rightarrow (-3x+20)(x-4)=8(x+5)(x-2) \Rightarrow -3x^2+(12+20)x-80=8x^2+24x-80$$

$$\Rightarrow 11x^2-8x=0 \Rightarrow x(11x-8)=0 \Rightarrow x=0 \text{ or } \frac{8}{11}$$

Inequalities

When algebraic expressions are related using any of <, >, ≥, ≤ it results in an Inequality. Certain rules of inequality needed to solve even the most simple of problems are as follows:

When any number is added or subtracted from both sides of an inequality, the sign of inequality remains same

If 5x - 4 > 4x -1, we can surely say that 5x - 4x > -1 + 4 i.e. x > 3 because we are essentially adding 4 and subtracting 3x from both sides. Thus, one can transpose terms from one side to other side by changing their signs and the inequality sign will remain the same.

However, if both sides of an inequality are multiplied or divided by the same number, the inequality sign does not always remain the same. If the number multiplied or divided with is positive, then the sign remains the same but if the number is negative the inequality sign reverses.

Thus $\frac{x}{y}>\frac{3}{4}$ cannot be restated as 4x > 3y.

This is because here we are multiplying both sides by 4y and since we do not know the sign of 4y, we cannot be sure of the inequality remaining the same. For more proof, though $\frac{-2}{-1}>\frac{3}{4}$, yet -8 > -3 is wrong.

Thus, when we cross multiply, we have to be careful and should know the sign of the expression we are multiplying with. In the above example if we knew that x is negative, we can infer that y is negative (x/y is positive and if x is negative, y has to be negative) and then we can be sure that 4x < 3y. Note that the inequality sign has changed as we are multiplying with 4y which is negative.

Notations: For all real numbers x and y

1. $x < y$ means that x is less than y.

2. $x > y$ means that x is greater than y.

3. $x \geq y$ means that x is less than or equal to y.

4. $x \leq y$ means x is greater than or equal to y.

5. $x \ngtr y$ means that x is not greater than y.

6. $x \nless y$ means that x is not less than y.

7. $x >> y$ means that x is much greater than y.

8. $x << y$ means that x is much less than y.

Properties: For real numbers x, y, z and w.

1. If $x > y$, then $-x < -y$.

For example,

$6 > 2$ but $-6 < -2$

Proof: $x + (-x) = 0 = y + (-y)$

$\Rightarrow x + (-x) = y + (-y)$

Now, if x is greater then y, then (-x) should be less than (-y)

2. If $x < y$, then $-x > -y$

3. If $x > y$ and $y > z$, then $x > z$

4. If $x < y$ and $y < z$, then $x < z$

5. If $x > y$, then $x + z > y + z$ and $x - z > y - z$

6. If $x < y$, then $x - z < y - z$ and $x + z < y + z$

7. If $x > y$ and $z > w$, then $x + z > y + w$

but $x - z$ is not necessarily greater than $y - w$

For example,

$10 > 5$ and $-3 > -20$ but $10 - (-3) = 13$ and $5 - (-20) = 25$

$\therefore 13 < 25$

8. If x and y are both positive or both negative

i. If $x < y$ then $\frac{1}{x} > \frac{1}{y}$.

ii. If $x > y$ then $\frac{1}{x} < \frac{1}{y}$

9. If $\frac{x}{z} < \frac{y}{w}$, then $xw > yz$ provided

$z, w > 0$.

For example, $\frac{6}{5} > \frac{4}{7}$.

So, $6 \times 7 = 4 \times 5$

Similarly for the inequality $x^2 - 5x > 3x - 15$ if we go ahead and solve as follows:

$x(x - 5) > 3(x - 5)$

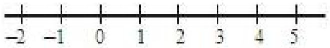
$\therefore x > 3$

we would be wrong. Don't believe it? Take $x = 4$ i.e. a value which as per our solution should satisfy the inequality. Yet when we substitute $x = 4$ in the inequality we find that $16 - 20 > 12 - 15$ i.e. $-4 > -3$ which we know is false.

The error is that when we cancel out $(x - 5)$ from both sides, essentially we are dividing both sides by $(x - 5)$ and since we do not know the sign of $(x - 5)$ we cannot be sure if the inequality remains the same or reverse. We will shortly see how to solve such an inequality.

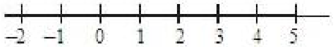
Certain other tools that are very handy and would be used very frequently are:

Consider $x > 3$. On a number line the set of all values of x that satisfy this inequality is represented by the bold line in the following figure:



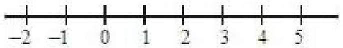
Similarly $x < 5$ is represented on the number line as follows:

Consider the two inequalities simultaneously $x > 3$ AND $x < 5$. This is basically the intersection of the above two line graphs and the solution set to these two inequalitites simultaneously is:



Similarly the solution for $x > 5$ AND $x < 3$ will be a null set i.e. no value of x will satisfy both the inequalities simultaneously.

However the solution for $x > 5$ OR $x < 3$ is all values of x represented by the bold line in the following line graph.



The difference is because of OR and not AND

Similarly the solution set of $x > 3$ OR $x > 5$ will be the union i.e. $x > 3$ but the solution set of
$x > 3$ AND $x > 5$ will be the intersection $x > 5$

Practice understanding of inequalities on number lines. This helps you in visualizing the number relationship and would eventually help you do it mentally.

Consider solving the following inequality $(x - 2)(x + 4)(x - 5) > 0$. This is of the type $a \times b \times c > 0$. This is possible if all three of a, b and c is positive OR if one of them is positive and other two are negative. Rather than trying all possibilities, a simpler way also exists. $(x - 2)$ will be positive if $x > 2$ and negative if $x < 2$. Similarly $(x + 4)$ will be positive if $x > -4$ and negative if $x < -4$. Also $(x - 5)$ will be positive if $x > 5$ and negative if $x < 5$.

Combining the above, we will have

If $x > 5$, all three $(x - 2)$, $(x + 4)$ and $(x - 5)$ will be positive and hence the product will be positive.

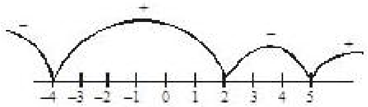
If $2 < x < 5$, $(x - 2)$ and $(x + 4)$ will remain positive but $(x - 5)$ will become negative and hence the product will be negative.

If $-4 < x < 2$, only $(x + 4)$ will be positive and other two $(x - 2)$ and $(x - 5)$ will be negative. Thus the product will be positive.

Finally if $x < -4$, all the terms will be negative and the product will also be negative.

Since we need the product to be positive, the solution will be $x > 5$ or $-4 < x < 2$

Rather than writing all the above, we can simply plot the points where the terms will change the signs, on a number line as follows :



The number line represents all the values of x from $-\infty$ to $+\infty$. However, it is now broken in various regions, four regions for this inequality. Now we have to identify those values of x that satisfy the given inequality, namely $(x - 2)(x + 4)(x - 5) > 0$.

For the right most region i.e. $x > 5$, all terms will be positive and hence the product will be positive. For the second region from the right, one term will turn negative and thus the product will be negative in this range of x. For the third region from right side, two terms will turn negative making the product positive for this range of x. And so on.

If we recollect we could not solve the inequality $x(x - 5) > 3(x - 5)$ by dividing both sides with
$(x - 5)$ (unless you took conditions of $x - 5$ being positive or negative). So how do we solve this inequality in a faster way? The inequality can be re-written as $x(x - 5) - 3(x - 5) > 0$ i.e. $(x - 5)(x - 3) > 0$.

And as just seen, the solution set to this inequality will be $x > 5$ or $x < 3$

It would be worthwhile to rewind to the start of the chapter where we had plotted the graph of y v/s x where $y = x^2 - 8x + 15$ i.e.

$y = (x - 5)(x - 3)$.

We see that y is + for all values of $x > 5$ or $x < 3$ and is negative for $3 < x < 5$. This is exactly what we fouad out from the above treatise.

The above method can be used to solve any quadratic or higher degree inequality. In fact the inequality $x^2 > 9$ is also a quadratic inequality and can be written as $x^2 - 9 > 0$, i.e.

$(x + 3)(x - 3) > 0$ and the solution set is $x < -3$ or $x > 3$. Using common sense also we can ascertain that this will be the solution as square of any number will be always positive.

Notice that the signs are same in alternate ranges. However, the sign will be the same in consecutive ranges, if the corresponding equation has any equal roots.

Solve this question from CAT 2001.

If $x > 5$ and $y < -1$, then which of the following statements is true?

1. $(x + 4y) > 1$ 2. $x > -4y$

3. $-4x < 5y$ 4. None of these

Example 38:

If $x^2 - x - 30 > 0$, then find the range of values of x.

Solution:

$(x + 5)(x - 6) > 0 \Rightarrow$ X does not lie between -5 and 6.

Try drawing the graph of the expression and interpreting the results.

Example 39:

Solve for the following conditions.

$x^2 - 4x + 3 \le 0$ and $x \ge 2$

Solution:

If $x^2 - 4x + 3 \le 0$, then $x^2 - 3x - x + 3 \le 0$

$\Rightarrow (x - 3)(x - 1) \le 0$ ...(i)

So from (i), we get $1 \le x \le 3$.

Combining with $x \ge 2$, we have $2 \le x \le 3$.

Example 40:

If $x^2 - 7x + 12 < 0$, then find the range of values of x.

Solution:

$x^2 - 7x + 12 = (x - 3)(x - 4) < 0 \Rightarrow 3 < x < 4$

Example 41:

Solve for the following conditions.

$x^2 + 8x - 33 \ge 0; x^2 \ge 36$

Solution:

(a) $x^2 + 11x - 3x - 33 \ge 0 \Rightarrow (x + 11)(x - 3) \ge 0$

Hence, $x \ge 3$ or $x \le -11$.

(b) $x^2 \ge 36 \Rightarrow x^2 - 36 \ge 0 \Rightarrow (x + 6)(x - 6) \ge 0$

Hence, $x \le -6$ (or) $x \ge 6$.

Combining the solutions from (a) and (b), we get $x \le -11$ (or) $x \ge 6$.

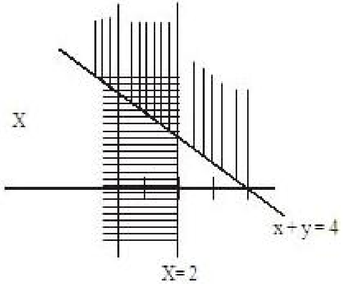
Thus, for positive a, the solution to $x^2 < a^2$ is $-a < x < a$ and the solution for $x^2 > a^2$ is $x < -a$ or $x > a$.

Solve for the conditions $x + y \ge 4$ and $x = 2$.

Now x can only take a value equal to 2. Substituting $x = 2$ in the inequality, we get $y \ge 2$. Thus the solution set for the two conditions is $x = 2$ and $y \ge 2$.

What would be the solution if the second equation was also an inequality i.e. the conditions were $x + y \ge 4$ and $x \le 2$? In this case, x can assume values 2 and less. If x takes a value of 2, y can take values 2 and higher values. As x decreases from 2, y would increase and thus y can assume values greater than 2 i.e. the range of values of y is given by $y \ge 2$. However we cannot express the solution to these conditions as $x \le 2$ and $y \ge 2$. Why? Because not all values of the pair (x, y) that satisfy the conditions $x \le 2$ and $y \ge 2$ will satisfy the given conditions $x + y$ ³ 4 and $x \le 2$. Consider $x = 1$ and $y = 2.5$. Though these values satisfy the so called solution, they do not satisfy $x + y \ge 4$.

Solution set to simultaneous inequations are more appropriately expressed as an area on a graph and only in very specific situations can be expressed in algebraic inequalities. The solution to the above conditions is given by the shaded region in the following graph. Thus though all solutions to the conditions satisfy
$x \le 2$ and $y \ge 2$, not all points satisfying $x \le 2$ and $y \ge 2$ are solutions to the conditions.



The region with horizontal bars is the solution set of the inequality $x \le 2$. This region is found by plotting the line $x = 2$ and then finding the appropriate region (to the left). The region with vertical bars is the region represented by $x + y$ ³ 4. This region is identified by plotting the line $x + y = 4$ and then again finding the appropriate region (above the line). The intersection of these two regions is the simultaneous solution to the two inequalities.

In order to check whether the inequality represents the area `above' or `below' the line, select any point, `above' the line and replace the value of x and y with it. If it satisfies the inequality, the area is `above' the line, otherwise, it is below the line.

Example 42:

Find the solution set for the following conditions. $3x + y \geq 8$; $y = 4$

Solution:

$3x + y \geq 8$. So, $3x + 4 \geq 8 \Rightarrow 3x \geq 4$

$\Rightarrow x \geq \frac{4}{3}$. So the solution set is represented by the conditions $x \geq \frac{4}{3}$ and $y = 4$.

Example 43:

Solve for x, $x + 2y > 3$ and $x = 3y$.

Solution:

Substituting for x in the inequality $5y > 3$, hence $y > \frac{3}{5}$. So $x > \frac{9}{5}$.

Example 44:

For what values of y would the following inequalities be consistent $x + y > 4$ and $x < 6$?

Solution:

The largest value of x has a limit of 6. If $x + y > 4$, the smallest possible value of y can be -2. Hence, $y > -2$.

Example 45:

Solve for $\frac{x}{a} < 4$; $\frac{a}{y} > 6$; $a^2 = 4$; $ab^2 = -8$ (x, y, a, b are all real)

Solution:

From $ab^2 = -8$, since $b^2 \geq 0$, it means $a < 0$ and since $a^2 = 4$, $a = -2$.

Applying this to $\frac{x}{a} < 4$, we get $x > -8$ [$\because$ $a = -2$ and hence negative, if $\frac{x}{a} < 4$ or $x > -4a$]

Also $\frac{a}{y} > 6$, since $a = -2$. For $\frac{a}{y}$ to be positive, y must be < 0.

So $a < 6y$ or $\frac{a}{6} < y \Rightarrow \frac{-2}{6} < y$ or $y > \frac{-1}{3}$. Hence, $\frac{-1}{3} < y < 0$

Solution set is $x > -8$; $\frac{-1}{3} < y < 0$.

Test Your Understanding

Level - I

1. For what values of k is the system of equations $kx + 5y = 3$ and $3x + 4y = 9$ has a unique solution?

2. Solve for x if $4\left(x + \frac{1}{x}\right) + 8\left(x - \frac{1}{x}\right) - 29 = 0$.

3. If $x + \frac{1}{(x+2)} = \frac{7}{24}$, then what is the value of x?

4. If $x + y = 9$, $xy^2 + yx^2 = 9$, then what is the value of xy?

5. If one of the roots of the quadratic equation $5x^2 - px + 15 = 0$ is 3, then what is the value of p?

6. If a, b are the roots of $x^2 - x - 3 = 0$, then form the equation whose roots are $(3a + 1)$ and $(3b + 1)$.

7. If the sum of the roots of a quadratic is 24 and their difference is 8, then what is the quadratic equation?

8. $4x^2 - 11x + 2k = 0$ and $x^2 - 3x - k = 0$ have a common root.

a. What is the value of k?

b. What is the common root?

9. If one of the roots of the cubic equation is $x^3 - x^2 - 4x + 4$ is -2, then find the other roots of the equation.

10. Let $f(x) = \frac{1 - tx + tx^2}{1 + tx + tx^2}$ where $t \in [0,1]$

Which of the following cannot be the value of f(x)?

a. -1 b. 1 c. 2 d. 5 e. none of these

11. If a, b and c are the roots of the equation $x^3 - 3x^2 + 2x + 1 = 0$, then the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is

a. 1 b. 2 c. 3 d. - 2 e. -1

12. In the equation $2^{x+5} = 2^{x+3} + 6$, the value of x is

a. 2 b. 4 c. -2 d. 3 e. - 4

13. How many values of `x' exist, if $\frac{(x-1)(x-3)}{(x-4)} = \frac{(x-2)(x-5)}{(x-7)}$?

14. Solve the system for x, y and z.

$2x + 5y = 29$

$3x + 2z = 6$

$5z - 2y = -10$

15. Find the constant term in the expansion of $\left(x^2 - \frac{1}{x}\right)^9$.

16. Find the solution set of x for the following inequalities, where `x' is a real number.

a. $2x - 5 < 3x$ and $4 - \frac{x}{3} < \frac{2x}{5} - 6$

b. $x + 5(4 - 2x) > 4 - 2x$ or $3 - \frac{x}{3} > 3$

17. Solve the following (independent of each other) for x.

a. $\frac{1}{x} > \frac{1}{5}$ b. $\frac{1}{x} < \frac{1}{5}$ c. $\frac{1}{x} > -\frac{1}{5}$ d. $3 - \frac{3}{x} > 0$

18. If $\frac{x}{y} = \frac{y}{z}$, then which of the following is true?

a. $xz > y^2$ b. $y^2 = xz$ c. $xy > y^2$ d. $xy < y^2$ e. $y^2 \geq xz$

19. If $x^2 - 3x - 10 \leq 0$ and x is an even positive integer, then find the value(s) of x.

20. $f(x) = (x^2 - 2x - 8)(x - 1)$. If $0 \leq x \leq \frac{3}{2}$, then find the range of values of x for which $f(x) \geq 0$.

21. If $f(x) = (x+1)(x)(x-2)(x-4) \geq 0$, then find the range of values of x.

22. Find the range of values of x that satisfies $x \times (x - 2) \times (2 - x) \times (1 - x)^2 < 0$

23. What is the minimum and maximum possible value of $\frac{x}{y}$ for each of the following conditions?

a. $3 \leq x\ 10$ and $5 \leq y\ 15$

b. $-10 \leq x \leq -3$ and $5 \leq y \leq 15$

c. $-10 \leq x \leq -3$ and $-15 \leq y \leq -5$

d. $-3 \leq x \leq 3$ and $-5 \leq y \leq 5$

e. $-3 \leq x \leq 3$ and $5 \leq y \leq 15$

Level - II

24. If the polynomial $ax^4 + bx^3 + cx^2 + dx + e$ has the property that the product of its roots is $\frac{1}{3}$ times of the sum taking two roots at a time, then find the relationship between e and c.

25. If $a + b + c = 3$, $a^2 + b^2 + c^2 = 6$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, where a, b, c are all non-zero, then \`abc' is

a. 3 b. $\frac{1}{3}$ c. $\frac{1}{2}$ d. $\frac{2}{3}$ e. $\frac{3}{2}$

26. If $|x^2 - x - 5| + (3x + 6) = 0$, then how many values of x exists?

27. Solve for \`x' if $8^{|x-2|} - 8^{(x-3)} = |8^{x-3} - 4| + 4$

28. If the coefficients of (2r + 1)th and (r - 3)th terms in the expansion of $(1 + x)^{23}$ are equal, then find the value of \`r'

29. If a and b are positive variables such that $ab < a + b$, then which of the following is a sufficient and necessary condition

a. $a < 1$ and $b < 1$ b. $a < b$ and $b > 1$ OR $b > a$ and $a > 1$

c. $\frac{1}{a}+\frac{1}{b} > 1$ d. $a < 0$ and $b < 0$ e. None of these

Coordinate Geometry and Graphs 3

Introduction

Co-ordinate Geometry is based on your basic knowledge of plane Geometry and that of graphing techniques you learnt in Algebra earlier.

It is nothing special except that you use algebraic equations in two variable to represent various geometric shapes.

The same mathematics theorems and corollaries that you proved in Geometry will be dealt with in co-ordinate geomatry.

However this is not an important topic from CAT perpective and have made a few appearences in the previous years' papers.

Learning Objectives

- Basics of co-ordinate geometry.
- Fundamental formulae
- Straight lines.

Coordinate Geometry and Graphs

Coordinate geometry is used to represent algebraic relations on graphs.

We shall be dealing with two-dimensional problems, where there are two variables to be handled.

The variables are normally denoted by the ordered pair (x, y).

The horizontal axis is the X -axis and the vertical axis is the Y-axis. If the coordinates of a point on the XY plane is (x, y), it implies that it is at a perpendicular distance of x from the Y-axis and at a perpendicular distance y from the X-axis. The point of intersection of the X and Y-axis is called the origin and the coordinates of this point is (o, o).

Some fundamental formulae:

1. Distance between the points (x_1, y_1) and (x_2, y_2) is $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) = $\frac{1}{2}$

$[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

3. The points that divide the line joining two given points (x_1, y_1) and (x_2, y_2) in the ratio m : n internally and externally are $\left(\frac{mx_2 \pm nx_1}{m \pm n}, \frac{my_2 \pm ny_1}{m \pm n}\right)$

Note: It would be '+' in the case of internal division and '-' in the case of external division.

4. The coordinate of the mid-point of the line joining the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

5. Centroid of a triangle. The centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

6. Slope of the line joining the points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$; $(x_1 \neq x_2)$. The slope is also indicated by m.

7. If the slopes of two lines be m_1 and m_2, then the lines will be

(i) parallel if $m_1 = m_2$

(ii) perpendicular if $m_1m_2 = -1$

Standard forms

1. All straight lines can be written as $y = mx + c$, where m is the slope of the straight line, c is the Y intercept or the Y coordinate of the point at which the straight line cuts the Y-axis.

2. The equation of a straight line passing through (x_1, y_1) and having a slope m is $y - y_1 = m(x - x_1)$.

3. The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

4. The point of intersection of any two lines of the form $y = ax + b$ and $y = cx + d$ is same as the solution arrived at when these two equations are solved.

5. The length of perpendicular from a given point (x_1, y_1) to a given line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}\right| = p$, where p is the length of perpendicular. In particular, the length of perpendicular from origin (0, 0) to the line $ax + by + c = 0$ is $\frac{c}{\sqrt{a^2 + b^2}}$

Example 1:

A line passes through the mid-point of the line joining the points (-3, -4) and (-5, 6) and has a slope of $\frac{3}{4}$. Find the equation.

Solution:

Slope $m = \frac{3}{4}$, mid-point of the line joining (-3, -4) and (-5, 6)

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3-5}{2}, \frac{-4+6}{2}\right) = (-4, 1)$

$\therefore$ The equation of the line is $y - y_1 = m(x - x_1)$, $y - 1 = (x + 4)$, i.e. $3x - 4y + 16 = 0$

Example 2:

Find the equation of the line through (2, -4) and parallel to the line joining the points (2, 3) and (-4, 5).

Solution:

Slope of the line joining (2, 3) and (-4, 5) is

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{-4-2} = \frac{2}{-6} = -\frac{1}{3}$

$\therefore$ Slope of any line parallel to it $= -\frac{1}{3}$, point $(x_1, y_1) = (2, -4)$ and $m = -\frac{1}{3}$

$\therefore$ The equation is $y - y_1 = m(x - x_1)$

$y + 4 = -\frac{1}{3}(x - 2)$, i.e. $3y + 12 = -x + 2$, $x + 3y + 10 = 0$

Example 3:

The vertices of a triangle are (1, 3), (-2, 4) and (3, -5). Find the equation of the altitude from (1, 3) to the opposite side.

Solution:

Let the vertices be A(1, 3), B (-2, 4) and C(3, -5), slope of BC $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-4}{3+2} = \frac{-9}{5}$

$\therefore$ Slope of altitude through A = $\frac{5}{9}$, since the altitude passes through A(1, 3).

Its equation is $y - y_1 = m(x - x_1)$, i.e. $y - 3 = \frac{5}{9}(x - 1)$, i.e. $9y - 27 = 5x - 5$,

i.e. $5x - 9y + 22 = 0$

Example 4:

Find the equation of the perpendicular bisector of the line joining (5, 6) and (2, -2).

Solution:

Let the point be A(5, 6) and B (2, -2), mid-point = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\therefore$ Mid-point of AB = $\left(\frac{5+2}{2}, \frac{6-2}{2}\right) = \left(\frac{7}{2}, \frac{4}{2}\right) = \left(\frac{7}{2}, 2\right)$

Slope of AB = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-6}{2-5} = \frac{-8}{-3} = \frac{8}{3}$

$\therefore$ Slope of perpendicular = $-\frac{3}{8}$

Now equation is $y - y_1 = m(x - x_1)$, i.e.

$$y - 2 = -\frac{3}{8}\left(x - \frac{7}{2}\right)$$

$$\Rightarrow 8y - 16 = -3x + \frac{21}{2} \Rightarrow 3x + 8y - 16 - \frac{21}{2} = 0$$

$6x + 16y - 32 - 21 = 0$,

$6x + 16y - 53 = 0$.

Example 5:

What is the equation of the line which joins the points A(-1, 3) and B(4, -2)?

Solution:

Equation is given by $\boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}$

i.e. $\frac{y-3}{-2-3} = \frac{x+1}{4+1} \Rightarrow \frac{y-3}{-5} = \frac{x+1}{5}$

$y - 3 = -(x + 1) \Rightarrow y - 3 = -x - 1$

$x + y - 2 = 0$.

Example 6:

Find the equation of the median from A in ΔABC with vertices A (5, 6), B (-11, 2) and C (6, -11).

Solution:

A is (5, 6), B is (-11, 2) and C is (6, -11), let D be the mid-point of BC, D = $\left(\frac{-11+6}{2}, \frac{2-11}{2}\right) = \left(\frac{-5}{2}, \frac{-9}{2}\right)$, median AD passes through A(5, 6)

$\therefore$ Equation is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$,

i.e. $\frac{y-6}{-\frac{9}{2}-6} = \frac{x-5}{-\frac{5}{2}-5} \Rightarrow \frac{y-6}{-\frac{21}{2}} = \frac{x-5}{-\frac{15}{2}}$

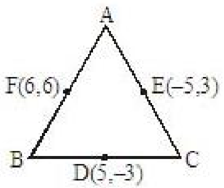
$-5(y - 6) = -7(x - 5)$

$7x - 5y - 5 = 0$

Example 7:

The mid-points of three sides of a triangle are (5, -3), (-5, 3) and (6, 6). Find the equation of the sides of the triangle.

Solution:



Let the triangle be ABC. D, E and F be the mid-points of BC, CA and AB respectively. D is (5, -3), E is (-5, 3) and F is (6, 6). The line joining the mid-points of two sides is parallel to the third side. So slope of BC = Slope of EF = $\frac{6-3}{6+5} = \frac{3}{11}$.

Also D (5, -3) is a point on BC.

$\therefore$ Equation of line BC is $y + 3 = \frac{3}{11}(x - 5)$, $11y + 33 = 3x - 15$, $3x - 11y - 48 = 0$

DF || AC, so slope of AC = Slope of DF = $\frac{6+3}{6-5} = \frac{9}{1} = 9$, E (-5, 3) is a point on AC

$\therefore$ Equation of AC is $y - 3 = 9(x + 5)$, i.e. $y - 3 = 9x + 45$,
$9x - y + 48 = 0$

DE || AB, so slope of AB = Slope of DE = $\frac{3+3}{-5-5} = \frac{6}{-10} = \frac{-6}{10}$,
F (6, 6) is a point on AB

$\therefore$ Equation of AB is $y - 6 = \frac{-6}{10}(x - 6)$

$10y - 60 = -6x + 36 \Rightarrow 6x + 10y - 96 = 0$

i.e. $3x + 5y - 48 = 0$

Example 8:

Find the equation of the line passing through (2, -1) and whose intercepts on the axes are equal in magnitude but opposite in sign.

Solution:

Let the intercepts on the axes be a and - a. The intercept form of the equation is

$\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a} + \frac{y}{-a} = 1$, i.e. $x - y = a$

The line passes through (2, -1)

$\therefore$ Substituting (2, -1) in the equation

$2 - (-1) = a$, $2 + 1 = a$, i.e. $a = 3$

The required equation is $x - y = 3$ or $x - y - 3 = 0$

Example 9:

Find the equation of the line passing through the point of intersection of $4x - y - 3 = 0$ and $x + y - 2 = 0$ and perpendicular to $2x - 5y + 3 = 0$

Solution:

The two lines are $4x - y - 3 = 0$ and $x + y - 2 = 0$. Solving the two equations, $5x = 5$, $x = 1$.

Put $x = 1$ in $4x - y - 3 = 0$,

i.e. $4(1) - y - 3 = 0$, $-y + 1 = 0$,

i.e. $y = 1$

$\therefore$ The point of intersection is (1, 1). The required line passes through (1, 1) and is perpendicular to $2x - 5y + 3 = 0$.

Slope of the line $2x - 5y + 3 = 0$ is $\frac{2}{5}$.

$\therefore$ Slope of perpendicular is $-\frac{5}{2}$. Equation of the required line is $y - y_1 = m(x - x_1)$,

i.e. $(y-1) = -\frac{5}{2}(x-1)$, $2y - 2 = -5x + 5$,

i.e. $2y + 5x = 7$

Test Your Understanding

Level - I

1. Find the point of intersection of the lines.

(i) $2x - 3y = 6$; $x + y = 3$

(ii) $3x + 5y = 6$; $5x - y = 10$

(iii) $2x + 3y = 8$; $2x - 3y = 4$

(iv) $4x - 5y = 26$; $3x + 7y = -2$

2. A line passes through (3, 4) and (8, 5). Find a point common to this line and $2x + y + 1 = 0$.

3. Find the coordinates of the circumcentre of the triangle whose vertices are

(i) A (3, 1), B (2, 2) and C (2, 0)

(ii) A (0, 0), B (-4, 0) and C (0, 4)

4. Draw the graphs of the following relations.

a. $f(x) = x^2$ b. $g(x) = 3x$ c. $h(x) = f(x) + g(x)$ [as in a, b]

Level - II

5. Find the point of intersection of the lines $2x + y - 1 = 0$ and $x - 3y + 3 = 0$ and also find the equation of the line parallel to X-axis which passes through the point of intersection of these lines.

6. Find the equation of the line parallel to Y-axis and passing through the point of intersection of the lines $x + y - 5 = 0$ and $2x + 3y = 13$.

7. The lines $x + y - 5 = 0$ and $3x - y + 1 = 0$ are the diameters of a circle. Find the radius of the circle if the point (0, 1) lies on the circle.

8. A straight line passes through the point of intersection of $2x + y = 3$ and $3x - y = 7$ and is parallel to
$2x - y + 5 = 0$. Find its equation.

9. Find the equation of the line through the point of intersection of the lines $x + y = 3$ and $2x + y = 5$ and bisecting the line segment joining the points (1, 5) and (-5, 1).

10. Obtain the equation of the line which is concurrent with the lines $x - y - 2 = 0$ and $3x + 4y + 15 = 0$ and is perpendicular to the line joining the points (2, 3) and(1, 1).

11. How do you represent the following functions on graphs?

(i) $f(x) = 2x + |x|$

(i) A (3, 1), B (2, 2) and C (2, 0)

(ii) A (0, 0), B (-4, 0) and C (0, 4)

4. Draw the graphs of the following relations.

a. $f(x) = x^2$ b. $g(x) = 3x$ c. $h(x) = f(x) + g(x)$ [as in a, b]

Level - II

5. Find the point of intersection of the lines $2x + y - 1 = 0$ and $x - 3y + 3 = 0$ and also find the equation of the line parallel to X-axis which passes through the point of intersection of these lines.

6. Find the equation of the line parallel to Y-axis and passing through the point of intersection of the lines $x + y - 5 = 0$ and $2x + 3y = 13$.

7. The lines $x + y - 5 = 0$ and $3x - y + 1 = 0$ are the diameters of a circle. Find the radius of the circle if the point (0, 1) lies on the circle.

8. A straight line passes through the point of intersection of $2x + y = 3$ and $3x - y = 7$ and is parallel to
$2x - y + 5 = 0$. Find its equation.

9. Find the equation of the line through the point of intersection of the lines $x + y = 3$ and $2x + y = 5$ and bisecting the line segment joining the points (1, 5) and (-5, 1).

10. Obtain the equation of the line which is concurrent with the lines $x - y - 2 = 0$ and $3x + 4y + 15 = 0$ and is perpendicular to the line joining the points (2, 3) and(1, 1).

11. How do you represent the following functions on graphs?

(i) $f(x) = 2x + |x|$

(ii) $f(x) = \begin{cases} x & \text{for } -3 \le x \le 3 \\ |x| & \text{for the remaining values of } x \end{cases}$

(iii) $f(x) = x^2 + 2x - 8$ for all $x \in R$

(iv) $f(x) = [x]$ where [ ] represents the greatest integer function for $x > 0$

12. Find the area of the figure bounded by the graphs $|x| = 2$; $|y| = 3$.

13. Find the general equation of the graphs in figure (a) and figure (b).

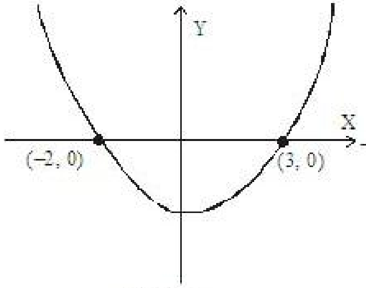


Figure a

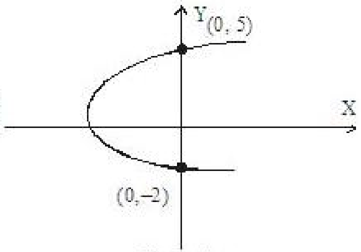


Figure b

14. Two ants start simultaneously from the points where the line $y = 3x + 4$ cuts the X and Y-axis. They move towards each other. If they meet when the y coordinate is 2, find the ratio of the speeds of the two ants.

15. A car travels along the line $y = 4x$, where x indicates the time in seconds and y the speed in metre per second (m/s). Find the distance covered by the car in 10 s.

16. Draw the graph of $|[x]|$ when | | indicates the modulus function and [ ] indicates the greatest integer value function.

Functions 4

Introduction

This chapter deals with the concepts related to Functions.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

- Basics of functions
- Modulus functions
- Greatest integer functions
- Logarithm and exponential functions

Functions

Functions are used to denote the relationship between variables.

Thus y = 2x + 3 can be expressed in language of functions as y = f(x) = 2x + 3. This is read as y is a function of x i.e. the value of y depends on the value of x and the relation is then described.

Any function f(x) should basically be considered as a black box where the input is x and the output is f(x). What happens in the black box is defined by the function, e.g. in the above example the black box computes 2 × input + 3. If one needs to compute f(2), one must consider the input x = 2. if one wants to compute f(-1.5), one must just substitute -1.5 in place of x.

Function can be visualized as an input - output machine as

$$\xrightarrow[x]{} \boxed{f} \longrightarrow f(x)$$

Every function has a domain and a co-domain. The function maps every element of domain to an element in co-domain. The elements of domain are called pre-image of the elements of co-domain and elements of co-domain are called image of the elements of domain.

Range - The set of elements in co-domain, which have a pre-image in domain, is called range.

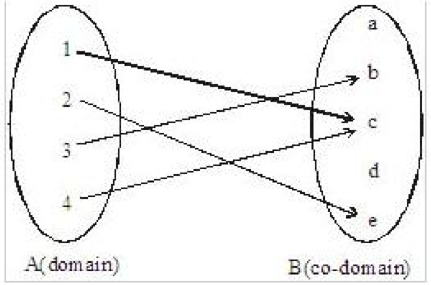
Conditions for a function to be defined:

(i) All the elements of domain must have an image in co-domain, though it is not necessary that every element of co-domain has a pre-image in domain.

(ii) Every element of domain can have only one image in co-domain i.e. an element in domain cannot have more than one image, though the elements in co-domain can have more than one pre-images in domain.

Example 1:

A = {1, 2, 3, 4} and B = {a, b, c, d, e}. Let function f : A → B (read as function f maps set A to set B) i.e. A is domain and B is co-domain.



In this example domain is A = {1, 2, 3, 4} and co-domain is B = {a, b, c, d, e}.

We can see that all elements in domain have an image in co-domain but not all the elements of co-domain have a pre image. Since only b, c and d have a pre-image, Range = {b, c, d}.

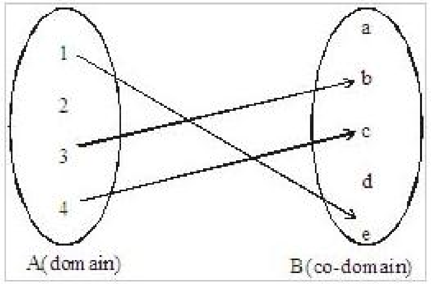
We can say that given function f is a valid function. Why?

(i) All the elements of domain have an image in co-domain.

(ii) Every element of domain has exactly one image in co-domain.

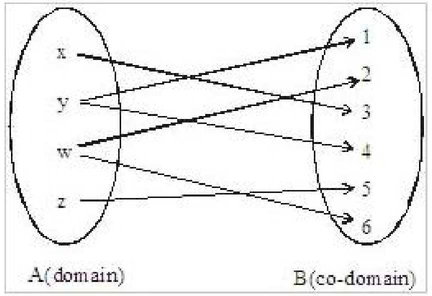
Now consider the following cases:

(i)



In this case function f : A → B is not defined as '2' does not have an image.

(ii)



Here again the function is not valid as y and w have two images.

Types of functions

One to One Function: A function is called one to one if each element of range has a unique pre-image in domain. One to one functions are also called injective functions.

Many to One Function: A function is called many one if at least one element of co-domain has more than one pre-images in domain.

Onto Function: A function is called onto if every element of co-domain has at least one pre-image in domain. Onto functions are also called surjective functions.

Into Function: A function is called into if there is at least one element in co-domain, which does not have a pre-image in domain. We can also say that if a function is not onto then it is into.

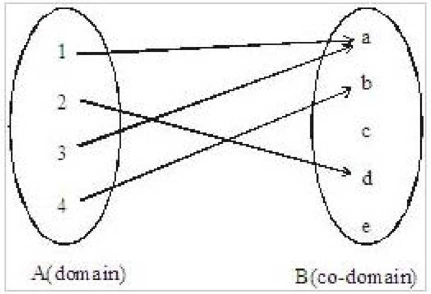
Invertible or Bijective Function: If a function is both one to one and onto, then it is called a bijective function or invertible function.

Example 2:

Which of the following functions $f : A \rightarrow B$ are valid?

In case of valid functions, classify them as one-one, many-one, onto, into or bijective function.

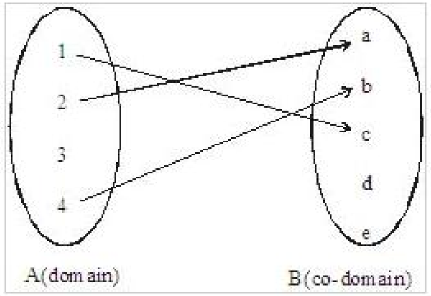
(i)



The above function is valid.

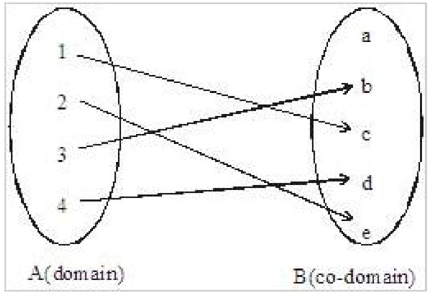
Given function is many-one as element 'a' in co-domain has two pre-images '1' and '3'.

(ii)



This is not a valid function as 3 does not have an image.

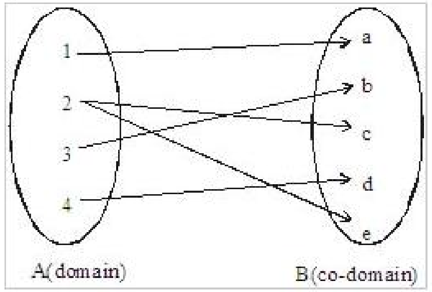
(iii)



The above function is valid.

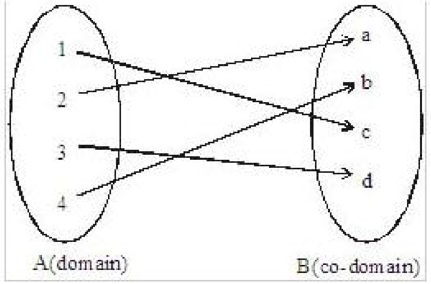
Since all elements of range have only one pre-image, the given function is one-one.

(iv)



Not a valid function as element (2) in domain has two images c & e in co-domain.

(v)



The above function is valid.

Given function is both one-one and onto and hence bijective function.

Identity function:

The function, $f(x) = x$, which assigns every element x to the same element is defined as identity function.

Domain of a function: Values of x for which we get real values of $f(x)$.

Range of a function:

Range of values of $f(x)$ for all values of x in the domain.

Example 3:

A function $f(x)$ is defined as $f(x) = x^2 + x$ for all x Î R. For what values of x is $f(x) = 0$?

Solution:

$f(x) = x^2 + x$

$f(x) = x(x + 1) = 0$ when $x = 0$ or $x = -1$

So the values of x are 0 or -1.

Example 4:

A function $f(x)$ is defined as $f(x)= x^2 + 2$, for all x Î Z and $0 \leq x \leq 5$.

(a) What is the domain of f ?

(b) What is the range of f ?

Solution:

The domain is defined as the set of values of x and the range is the set of values of $f(x)$ in the given problem

(a) Domain = $\{0, 1, 2, 3, 4, 5\}$

Since $0 \leq x \leq 5$ and it belongs to the set of integers.

(b) Range $\Rightarrow f(0)$ to $f(5)$ for all integer values of x.

We know that

$f(0) = 2; f(1) = 3; f(2) = 6;$

$f(3) = 11; f(4) = 18; f(5) = 27.$

So range = $\{2, 3, 6, 11, 18, 27\}$.

Example 5:

If $f(A, B) = A * B = A + B + A^2 B^3 - A^3 B^2$, find the value of $f(1, 2)$.

Solution:

By definition

$f(1, 2) = 1 * 2 = 1 + 2 + 1^2 2^3 - 1^3 2^2 = 11 - 4 = 7.$

Example 6:

$f(m, n) = m \# n = m + n + 2mn$. What is the value of $f(3, 4)$?

Solution:

$3 \# 4 = 3 + 4 + 2 \times 3 \times 4 = 7 + 24 = 31.$

Example 7:

It is defined that $m \# n = m + n + mn$. If for any m, there is a number q such that $m \# q = m$, then q =

Solution:

Since $m \# q = m$, we get

$m + q + qm = m$ or $q(1+ m) = 0$

$\Rightarrow m = -1$ or $q = 0$.

In previous examples, we have seen two functions both of which have only one definition over the entire domain i.e. the set of all possible values of x (the input of the function). Then there are functions, which do not have the same definition over the entire range of the independent variable, x.

Here is one example: $f(x) = \begin{cases} 2x+3, x < 0 \\ x+2, 0 \le x < 2 \\ 1-x, x \ge 2 \end{cases}$

And, the Modulus function, $g(x) = \begin{cases} x, \text{if } x \ge 0 \\ -x, \text{if } x < 0 \end{cases}$

As you can see, The definition of f(x) changes depending upon what the value of `x' is. Some other examples of representation of functions are:

i. $f(x) = \max(7x + 3, 0, 2 - 3x)$

ii. $f(x) = \min\{(g(x), h(x)\}$

Example 8:

Can you express f(x) in terms of x from the following:

$2f(x) - f\left(\frac{1}{x}\right) = x^2$

Solution:

$2f(x) - f\left(\frac{1}{x}\right) = x^2$ ... (1)

Replace "x" by $\left(\frac{1}{x}\right)$ in the above equation.

$2f\left(\frac{1}{x}\right) - f(x) = \left(\frac{1}{x^2}\right)$ ... (2)

Multiplying equation (1) by 2 and adding it to equation (2).

We get $4f(x) - 2f\left(\frac{1}{x}\right) + 2f\left(\frac{1}{x}\right) - f(x) = 2x^2 + \left(\frac{1}{x^2}\right)$

Or, $3f(x) = 2x^2 + \frac{1}{x^2}$

giving $f(x) = \frac{2}{3}x^2 + \frac{1}{3x^2}$

Example 9:

If $f(x) = \log\left\{\frac{(1+x)}{(1-x)}\right\}$, then f(x) + f(y) is

a. f(x + y) b. $f\left\{\frac{(x+y)}{(1+xy)}\right\}$ c. $(x+y)f\left\{\frac{1}{(1+xy)}\right\}$ d. $\frac{f(x)+f(y)}{(1+xy)}$ e. None of these

Solution: (b)

$$f(x) + f(y) = \log\left(\frac{1+x}{1-x}\right) + \log\left(\frac{1+y}{1-y}\right)$$

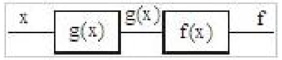
$$= \log\left(\frac{(1+x)\ (1+y)}{(1-x)(1-y)}\right) = \log\left(\frac{1+x+y+xy}{1+xy-x-y}\right)$$

$$= \log\left(\frac{1+xy+x+y}{1+xy-(x+y)}\right) = \log\left(\frac{1+\left(\frac{x+y}{1+xy}\right)}{1-\left(\frac{x+y}{1+xy}\right)}\right) = f\left(\frac{x+y}{1+xy}\right)$$

Composite Functions

Consider two functions $f(x) = x^2 - 1$ and $g(x) = 5 - 2x$.

The expression f(g(x)) can simply be considered as follows:



Thus one could calculate f(g(x)) as
$f(5 - 2x) = (5 - 2x)^2 - 1$ or as $f(g(x)) = [g(x)]^2 - 1 = (5 - 2x)^2 - 1$.

Please note that g(f(x)) will be different from f(g(x)) and would evaluate to $5 - 2(x^2 - 1)$.

f(g(x)) and g(f(x)) are called composite functions and are sometimes also represented as fog(x) and gof(x) respectively.

Example 10:

$f(x) = x^2$; $g(x) = 2x + 4$

Find (a) fof (x), (b) gof(x), (c) fog(x).

Solution:

(a) $fof(x) = f(f(x)) = f(x^2) = x^4$

(b) $gof(x) = g(f(x)) = g(x^2) = 2(x^2) + 4$

(c) $fog(x) = f(g(x)) = f(2x + 4) = (2x + 4)^2 = 4x^2 + 16x + 16$

Example 11:

f(x) = |x|

g(x, y) = x + y

h(x, y) = x - y

Find f(g(h(3, 4), 1)) - f(h (g(3, 4), -1))

Solution:

f(g(h(3, 4), 1)) = f(g (-1, 1)) = f (0) = 0.

f(h (g (3, 4), 1)) = + f(h (7 -1)) = f(8) = 8.

Hence, the answer is 0 - 8 = -8.

Example 12:

For all non-negative integers x and y, f(x, y) is defined as below.

$f(0, y) = y + 1$

$f(x + 1, 0) = f(x, 1)$

$f(x + 1, y + 1) = f(x, f(x + 1, y))$

Then what is the value of f(1, 2)?

a. 2 b. 4 c. 3 d. 1 e. Cannot be determined

Solution:

$f(1, 2) = f(0, f(1, 1));$

Now $f(1, 1) = f[0, f(1, 0)] = f[0, f(0, 1)] = f[0, 2] = 3$

Hence, $f(1, 2) = f(0, 3) = 4$

Iterative Functions

Just as we could have performed the function g on f(x), we could perform the function f on f(x). Thus $f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1$.

Extending this idea further, one could also have computed f(f(f(....f(x)))...) any number of times. This is called a nested function or an iterative function. Such an expression is usually denoted as follows:

If $f^1(x) = 2x - 1$ and $f^n(x) = f^1(f^{n-1}(x))$ for $n \geq 2$.

Please note the superscript is not the function raised to a power as in Indices. In this specific example it just serves to denote the number of time the function is performed iteratively, as explained below.

Suppose for the above function we have to calculate $f^4(2)$.

Since $f^n(x) = f^1(f^{n-1}(x))$, substituting n = 4, we have $f^4(x) = f^1(f^3(x))$.

But since we do not know $f^3(x)$, we have to again put n = 3 and evaluate $f^3(x) = f^1(f^2(x))$.

Thus we are reducing the superscript by 1 each time. Similarly $f^2(x) = f^1(f^1(x))$. Now since we know $f^1(x)$, we can calculate $f^2(x)$ and then substituting backwards evaluate $f^4(x)$.

By substituting backwards we will get

$f^4(x) = f^1(f^3(x)) = f^1[f^1(f^2(x))] = f^1\{f^1[f^1(f^1 (x))]\}$

Thus $f^4(x)$ is nothing but performing the function 4 times iteratively.

Thus, $f^1(2) = 2(2) - 1 = 3$.

$f^2(2) = f^1(f^1(2)) = f^1(3) = 2(3) - 1 = 5$

$f^3(2) = f^1(f^2(2)) = f^1(5) = 2(5) - 1 = 9$

$f^4(2) = f^1(f^3(2)) = f^1(9) = 2(9) - 1 = 17$

The way a person climbs the stairs is given by f(n) = n, where n is the nth step he takes and f(n) defines the number of steps he climbs in the nth step. If the person finishes climbing the staircase in his 4th step, find the total number of steps in the stairs.

Example 13:

$f^1(x) = 2x - 1$ and $f^n(x) = f^1(f^{n-1}(x))$ for $n \geq 2$. Find $f^{10}(2)$.

Solution:

$f^1(2) = 2 \times 2 - 1 = 3$

$f^2(2) = f^1(f^1(2)) = f^1(3) = 2 \times 3 - 1 = 5$

$f^3(2) = f^1(f^2(2)) = f^1(5) = 2 \times 5 - 1 = 9$

$f^4(2) = f^1(f^3(2)) = f^1(9) = 2 \times 9 - 1 = 17$

$f^5(2) = f^1(f^4(2)) = f^1(17) = 2 \times 17 - 1 = 33$

A simpler and a much less complicated way (atleast about the way it is written) would be to understand the function as twice the variable minus 1. Also for successive f^n, perform the function (i.e. twice the variable minus 1) on preceding term.

$f^1(2)\ 2 \times 2 - 1 = 3$

$f^2(2)\ 2 \times 3 - 1 = 5$

$f^3(2)\ 2 \times 5 - 1 = 9$

$f^4(2)\ 2 \times 9 - 1 = 17$

$f^5(2)\ 2 \times 17 - 1 = 33$

Hence, $f^k(2) = 2^k + 1$ and $f^{10}(2) = 2^{10} + 1 = 1025$

A very similar type of problem as above is when f_1, f_2, f_3, ...are used to denote terms of an series.

Say $f_1 = 0$, $f_2 = 1$ and in general $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Calculate f_7?

$f_{n+2} = f_{n+1} + f_n$ should be read as any particular term is equal to sum of the previous two terms. Why previous two terms? Because f_{n+1} and f_n are the previous two terms of f_{n+2} for any value of n.

Thus one can easily calculate the series forward (instead of reverse substitution) as

f_1	0
f_2	1
f_3	1
f_4	2
f_5	3
f_6	5
f_7	8

Example 14:

$f_0 = 1$

$f_n = 2\ f_{n-1}$ if n is odd.

$f_n = f_{n-1}$ if n is even.

What is the value of (i) $f_4 + f_5$ (ii) f_{20}?

Solution:

(i) $f_4 + f_5 = f_3 + 2f_4 = 2f_2 + 2f_3$

$= 2(f_2 + f_3) = 2(f_1 + 2f_2)$

$= 2(2f_0 + 2f_1)$

$= 2(2 + 2 \cdot 2f_0)$

$= 2(2 + 4) = 12$

(ii) $f_{20} = f_{19} = 2f_{18} = 2f_{17}$

$= 2^2(f_{16}) = \cdots$ so on.

The final value would be $2^{10} f_0 = 2^{10}$.

Example 15:

If $f(x) = -x$, then find $f^n(x)$.

Solution:

$f(x) = -x$

$f^2(x) = f\{f(x)\} = f\{-x\} = -(-x) = +x$

$f^3(x) = f\{f^2(x)\} = f(x) = -x$

$f^4(x) = f\{f^3(x)\} = f(-x) = +x$ ..and so on....

$$f^n(x) = \begin{cases} x, & n \text{ is even} \\ -x, & n \text{ is odd} \end{cases}$$

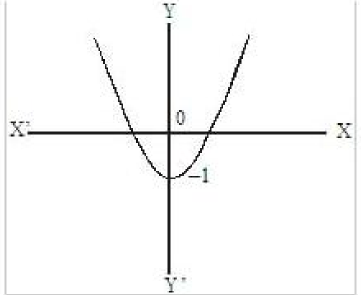
Even or Odd Functions

A function $f(x)$ is defined as an even function if $f(x) = f(-x)$. E.g. $f(x) = x^2 - 1$.

Since, $f(3) = 9 - 1 = 8$ and also $f(-3) = 9 - 1 = 8$, this is an even function.

The graph of an even function will be symmetric about the Y axis

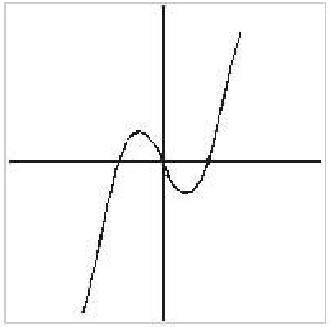
This is so because in the graph $f(x)$ v/s x, $f(x)$ is plotted as the vertical distance and since for an even function we will have $f(1) = f(-1)$ and also $f(-2) = f(2)$ and so on, every point has to be reflected across the Y axis.



A function $f(x)$ is defined as an odd function if $f(x) = -f(-x)$.

E.g. $f(x) = x^3 - x$. Since $f(3) = 27 - 3 = 24$ and also $f(-3) = -27 - (-3)1 = -24$, this is an odd function.

In the graph of an odd function, the first and third quadrants will be reflections of each other and so will the second and fourth quadrant.



Functions can be defined by the user in any which way. Lets take a look at an unusual function:

It is **NOT** necessary that functions always have to be one of the two types: EVEN or ODD. There are functions that are neither Even nor ODD for example:

$f(x) = 2x + 3$ is neither Even nor Odd. You may check it yourself. Put $x = 3$ and -3 and see, we get $f(3) = 9$ and $f(-3) = -3$. Clearly $f(x)$ does not satisfy either EVEN or ODD functions' definition.

Note: Every real function can be expressed as the sum of an EVEN and an ODD function.

Periodicity of the Functions: If a function $f(x)$, repeats its value after a definite increment (or decrement) in the value of x, then we say that the function $f(x)$ is a periodic function. In the language of algebra if $f(x)$ is periodic then we must have a real number P, such that:

$$\mathbf{f(x + P) - f(x) = 0}$$

For a function, which is, periodic, the minimum possible real number, say p, for which the above relation holds true, is called the period of that function.

Question: Can you relate P and p?

Answer: P will always be a multiple of p

Example 16:

Given that $f(x - a) + f(x) = 0$. Find that least possible number `P' such that: $f(x + P) = f(x)$

Solution:

$f(x - a) = -f(x)$ replacing "x" by "x - a"

$f(x - 2a) = -f(x - a) = f(x)$ replacing x by "x +2a"

$f(x) = f(x+2a)$

So the minimum possible value is 2a and it is the period of the function f(x).

Example 17:

Let g(x) be a function such that $g(x + 1) + g(x - 1) = g(x)$ for every real x. Then for what value of p is the relation $g(x + p) = g(x)$ necessarily true for every real x?

a. 5 b. 3 c. 2 d. 4 e. 6

Solution (e):

$g(x + 1) + g(x - 1) = g(x)$

$g(x + 2) + g(x) = g(x + 1)$

Adding these two equations we get

$g(x + 2) + g(x - 1) = 0$

$\Rightarrow g(x + 3) + g(x) = 0$

$\Rightarrow g(x + 4) + g(x + 1) = 0$

$\Rightarrow g(x + 5) + g(x + 2) = 0$

$\Rightarrow g(x + 6) + g(x + 3) = 0$

$\Rightarrow g(x + 6) - g(x) = 0$

$\Rightarrow p = 6$

Functions can be defined by the user in any which way.

Lets take a look at an unusual function.

Direction for examples 18 to 21: Answer the examples based on the given information.

The pages of a book are numbered 0, 1, 2, $\cdots$ up to M, $M > 0$. There are four categories of instructions that direct a person in positioning the book at a page. The instruction types and their meanings are as follows.

1. OPEN: Position the book at page number 1.

2. CLOSE: Position the book at page number 0.

3. FORWARD, n: From the current page, move forward by n pages; if, in this process, page number M is reached, stop at M.

4. BACKWARD, n: From the current page, move backward by n pages; if, in this process, page number 0 is reached, stop at page number 0.

In each of the following questions, you will find a sequence of instructions formed from the categories above. In each case, let n_1 be the page number before the instructions are executed, and n_2 be the page number at which the book is positioned after the instructions are executed. Treat each sequence of instructions independently of the others.

(Based on a problem which appeared in CAT 1990)

Example 18:

FORWARD, 30 ; BACKWARD, 12, which of the following statements is true?

(a) $n_1 = n_2$ if $M = 12$ and $n_1 = 0$ (b) $M = 25$ provided $n_1 > 0$

(c) $n_1 > 35$ provided $M = 1000$ (d) $n_1 = 40$ provided $M = 30$

Solution:

Let us consider option (a).

If $n_1 = 0$ and $M = 12$, the instructions given above will first position the book at the last page, i.e. 12 (according to instruction type 3), and then will position it at page 0 again (according to instruction type 4).

Therefore, $n_2 = 0$, i.e. $n_1 = n_2$. Therefore, (a) is correct.

Similarly, it can be worked out that none of the other choices are correct.

Example 19:

BACKWARD, 7; FORWARD, 7 which of the following statements is true?

(a) $n_1 = n_2$ provided $n_1 \geq 7$ (b) $n_1 = n_2$ provided $n_1 > 0$

(c) $n_2 = 7$ provided $M > 0$ (d) $n_1 > n_2$ provided $M > 0$

(e) None of these

Solution:

Consider option (a).

If $n_1 \geq 7$, the above instructions would take it 7 pages backward and then 7 pages forward, and because $n_1 \geq 7$, the instructions would bring the book back to the same page. Therefore, in effect, $n_1 = n_2$. None of the other options are true.

Example 20:

FORWARD, 15; FORWARD 15, which of the following statements is true?

(a) $n_2 - n_1 = 30$ only if $n_1 = 0$

(b) $n_2 - n_1 = 30$ if $M > 30$ and $n_1 = 1$

(c) $n_2 - n_1 = 15$ if $M = 31$ and $n_1 = 0$

(d) $n_2 > n_1$ if $M > 0$

(e) None of these

Solution:

Options (a), (c) and (d) can all be proved to be incorrect.

Considering option (b), if $n_1 = 1$, then the instructions given above will position the book at page $(1 + 15 + 15) = 31$, provided $M > 30$ which is given.

In this case, $n_2 - n_1 = 31 - 1 = 30$.

Therefore, (b) is true.

Example 21:

FORWARD, 5; BACKWARD, 4, which of the following statements is true?

(a) $n_2 = n_1 + 4$ provided $1 < n_1 < 7$

(b) $n_2 = n_1$ provided $M < 6$

(c) $n_2 = n_1 + 1$ provided $M - n_1 \geq 5$

(d) $n_2 - n_1 < 0$ provided $M > 0$

(e) None of these

Solution:

Options (a), (b) and (d) are all incorrect.

Consider option (c). If the difference between M and n_1 is at least 5, the instructions given above will take the book 5 pages forward and then 4 pages backward, i.e. 1 page forward in effect.

Therefore, n_2 will be one page ahead of n_1, i.e. $n_2 = n_1 + 1$. Thus, the correct answer is (c).

Modulus Function

The absolute value of any number is its magnitude irrespective of its sign. Absolute value of x is denoted as $|x|$. Thus $|2.5| = 2.5$ and $|-3.7| = 3.7$.

By definition $|\mathbf{x}| = \sqrt{x^2} = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$

Note that, in the definition, though it seems that absolute value can be negative, it is not as the negative is only of a negative number and would become positive.

Thus $|-3.7| = -(-3.7)$ as $-3.7 < 0$.

Consider $|x - 3|$. We cannot say that $|x - 3| = x - 3$, or for that matter $3 - x$, as we do not know the sign of
$(x - 3)$. The correct way to handle such an expression would be:

If $x - 3 \geq 0$ i.e. $x \geq 3$, then $|x - 3| = x - 3$ and if $x - 3 < 0$ i.e. $x < 3$, then $|x - 3| = 3 - x$.

Consider the equality $|x - 3| = 5$. If $x > 3$, then $x - 3 = 5$ i.e. $x = 8$. Since, 8 is > 3, this is an acceptable solution. Also if $x < 3$, $3 - x = 5$ i.e. $x = -2$. Since, $-2 < 3$, this is also acceptable and hence the solution to $|x - 3| = 5$ will be $x = -2$ or 8.

However if we had $|x - 3| = -7$, we would have if $x > 3$, $x - 3 = -7$ i.e. $x = -4$. Since, -4 is not greater than 3, this would have not resulted in any solution. Similarly if $x < 3$, $3 - x = -7$ i.e. $x = 10$. Since, 10 is not less than 3, this would also not give any solution. Thus, there would be no x for which $|x - 3| = -7$ and obviously so as irrespective of x, $|x - 3|$ has to be positive.

Some properties of modulus functions are as follows:

a. $|x| + |y|$ is greater than or equal to $|x + y|$ b. $|x| - |y|$ is less than or equal to $|x - y|$

c. $|x|\ |y| = |xy|$ d. $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$

e. $(|x| - |y|) \le |x \pm y| \le (|x| + |y|)$

Equations involving Modulus:

How many solutions would the equation

$x^2 + 5|x| + 6 = 0$ have? Is it 0, 2 or 4?

Lets find out.

If $x > 0$, $= |x|$ and the equation would be written as $x^2 + 5x + 6 = 0$. Solving the equation we would get
$x = -3$ or $x = -2$. Please note that these are solutions to the equation $x^2 + 5x + 6 = 0$

and not to $x^2 + 5|x| + 6 = 0$.

$x^2 + 5|x| + 6 = 0$ will be $x^2 + 5x + 6 = 0$ only if $x > 0$. But the solutions $x = -2$ or -3 are not positive. Thus this branch does not give us any solution.

Consider $x < 0$. In this case $|x| = -x$ and the given equation would transform to $x^2 - 5x + 6 = 0$. The solutions to this equation are $x = 3$ and 2. Since, these values do not satisfy the condition $x < 0$, even this does not give us any solution.

Thus, $x^2 + 5|x| + 6 = 0$ would have no solution. This is also very apparent since, irrespective of the value or sign of x, x^2 is positive and so is $5|x|$ and thus $x^2 + 5|x| + 6$ has to be greater than or equal to 6. Thus, for no value of x would $x^2 + 5|x| + 6$ be equal to 0.

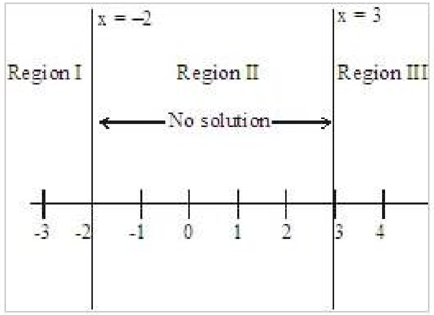
Most equations involving modulus can be solved in a similar manner by assuming the quantity within modulus sign as alternately positive and negative.

Example: Solve for x if $|x-3| + |x+2| = 6$

Since this equation involves two terms with modulus, we need to worry about the sign of

$(x - 3)$ and $(x + 2)$ simultaneously. The best way to do this is as learnt while solving quadratic inequalities.

Consider the ranges of x on the number line:



These particular values of x are plotted because as x crosses these values, $(x - 3)$ and $(x + 2)$ change their signs.

Region I	**Region II**	**Region III**
$-(x-3) - (x+2) = 6$	$-(x-3) + (x+2) = 6$	$(x-3) + (x+2) = 6$
$\Rightarrow -2x + 1 = 6$	$\Rightarrow 3 + 2 = 6$	$\Rightarrow 2x - 1 = 6$
$\Rightarrow x = -2.5$	$\Rightarrow 5 = 6$	$\Rightarrow x = 3.5$

These solutions have to be reconfirmed with the conditions on x as the given equation $|x-3| + |x+2| = 6$ is equal to the working equations only for those ranges of x.

Starting from the rightmost range, since $3.5 > 3$, $x = 3.5$ is a valid solution.

Since $5\ ^{1}\ 6$, no value of x in the range $-2 < x < 3$ satisfies the given equation

Since $-2.5 < -2$, hence $x = -2.5$ is also a valid solution.

Thus the solution to $|x-3|+|x+2|=6$ is $x = -2.5$ or 3.5

Check: For $x = -2.5$, $|x-3|+|x+2|=|-5.5|+|-0.5|=6$ and for $x = 3.5$,

$|x-3|+|x+2|=|0.5|+|5.5|=6$

While solving the equations involving modulus function, be careful about the critical points, that is, values of x at which the value of the expression within modulus symbol changes from positive to negative or vice versa.

Example 22:

Solve for $|2x + 3| + x = 10$. `x' is a real number.

Solution:

If $2x + 3 < 0$, then $|2x + 3| = -(2x + 3)$

So, $-2x - 3 + x = 10$

$\Rightarrow -x - 3 = 10$

$x = -13$

This satisfies $2x + 3 < 0$

If $2x + 3 > 0$, then $x > \frac{-3}{2}$.

So, $2x + 3 + x = 10$

Also, $3x = 7$ or $x = \frac{7}{3}$.

Since the conditions for x is consistent in either case, we have the two solutions for x as

$x = -13$ or $x = \frac{7}{3}$.

Example 23:

Solve for x, y if $|x + 3y| = 5$; $|y| = 4$. `x' and `y' are real numbers.

Solution:

$|y| = 4$

For, $y = 4$, we get $|x + 12| = 5$, therefore

$x + 12 = 5$ or $-x - 12 = 5$

Therefore, $x = -7$ or $x = -17$

Thus, we get $(-7, 4)$, $(-17, 4)$

For $y = -4$

$|x - 12| = 5$, therefore $x - 12 = 5$ or $-x + 12 = 5$

Therefore, $x = 17$ or $x = 7$

Thus, we get $(7, -4)$, $(17, -4)$

Inequalities involving Modulus

Consider the simple inequality $|x| < 3$. Logicwise, this means all those values of x which have a magnitude less than 3, irrespective of sign. This would simply be all values of x satisfying

$-3 < x < 3$. Lets see how we come to the same solution using maths. If $x \geq 0$, then the equation would mean $x < 3$. Thus the solution for $x \geq 0$ and $x < 3$ will be $0 \leq x < 3$. Also, for $x < 0$, we would have $-x < 3$, i.e. $x > -3$.

Thus, the solution would be $-3 < x < 0$. Since, x could be < 0 OR > 0, we would have the solution as $-3 < x < 0$ OR $0 \leq x < 3$ i.e. $-3 < x < 3$.

Thus for positive a, $|x| < a$

$\Rightarrow -a < x < a$ and $|x| > a$

$\Rightarrow x < -a$ or $x > a$

Solve for x if $|2x-4| < 6$

$|2x-4| < 6 \Rightarrow -6 < 2x - 4 < 6$

i.e. $-2 < 2x < 10$ (adding 4 to each)

i.e. $-1 < x < 5$

Example 24:

Solve for the following.

$|2x + 3| \leq 4$; $x + 2y = 4$; $y \geq 3$. `x' and `y' are real numbers.

Solution:

$|2x+3| \leq 4;\ x+2y=4;\ y \geq 3$

If $y \geq 3$, then $2y \geq 6$.

Hence, $x \leq -2$ ...(i)

If $|2x+3| \leq 4$, then $-4 \leq (2x+3) \leq 4$

or $\frac{-7}{2} \leq x \leq \frac{1}{2}$ ...(ii)

Combining (i) and (ii), we get $\frac{-7}{2} \leq x \leq -2$.

When x takes the maximum value, i.e. $x = -2$, y takes the minimum value, i.e. 3.

When x takes the minimum value, $x = \frac{-7}{2}$, y takes the maximum value, i.e. $2y = 4 + \frac{7}{2}$ or $y = \frac{15}{4}$

Hence, $3 \leq y \leq \frac{15}{4}$ and $\frac{-7}{2} \leq x \leq -2$

Solve this problem from CAT 2000.

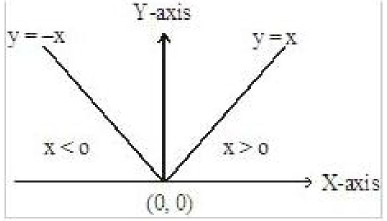
The area bounded by the three curves

$|x + y| = 1$, $|x| = 1$ and $|y| = 1$, is equal to

1. 4 2. 3 3. 2 4. 1

Graphs and Modulus:

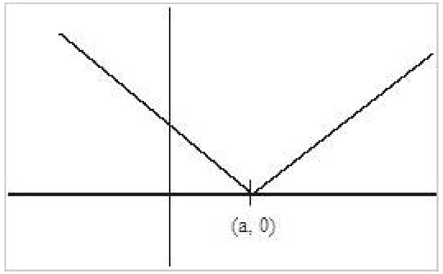
The graph of $y = |x|$ looks as follows:



The lines are at 45° and have a slope of 1.

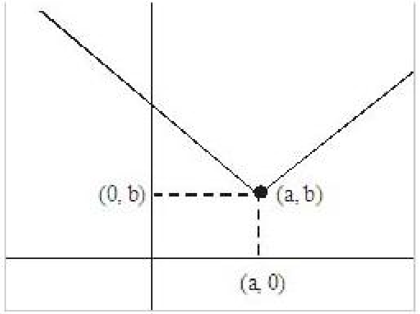
The graph of $y = |x - a|$ can easily be arrived by shifting the above graph horizontally to intersect X axis at (a, o).

For positive a, it would look as follows:



Similarly, $y = |x| + b$ can be arrived from graph of by just adding b to all points i.e. by moving the graph vertically by b units.

Combining the two movements, the graph of $y = |x - a| + b$ simply means considering the origin to be (a, b) and draw a graph similar to $|x|$.



Example:

How many integral solutions (x, y) exist satisfying the equation $|y| + |x| \le 4$?

$|y| + |x| = 4$ is basically four equations, viz.

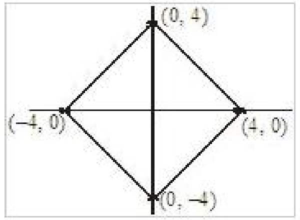
For x > o and y > o i.e. in first quadrant, x + y = 4

For x < o and y > o i.e. in second quadrant, -x + y = 4

For x < o and y < o i.e. in third quadrant, -x - y = 4

For x > o and y < o i.e. in fourth quadrant, x - y = 4

Plotting these lines in the respective quadrants, we can derive the graph of $|y| + |x| = 4$ as



The graph of $|y|+|x|\le 4$ will just be all the points within the lines. Now one can easily find the number of points with integral co-ordinates. The answer will be 9 + 2 (7 + 5 + 3 +1) = 41.

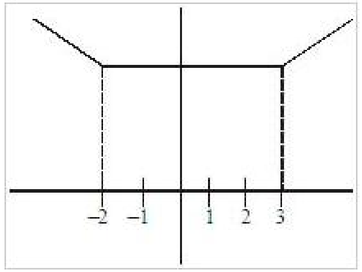
Example:

What will be the least value of $|x-3|+|x+2|$?

The following table gives the value of y for various values of x.

For x =	y =
–5	11
–4	9
–3	7
–2	5
–1	5
0	5
1	5
2	5
3	5
4	7
5	9
6	11

The graph of (y v/s) x can be plotted as follows:



Thus the minimum value that the expression can take is 5 and this is for all values of x such that

$-2 \le x \le 3$.

The above graph could also have been understood algebraically as follows:

For x > 3, the expression would be

$y = (x - 3) + (x + 2) = 2x - 1$. $y = 2x + 1$ is a upward sloping line as slope 2 is positive.

For $-2 \le x \le 3$, the expression would be

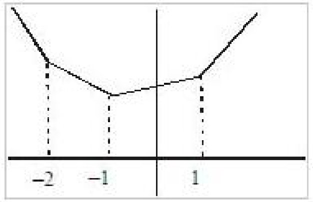
$y = -(x - 3) + (x + 2) = 5$

For x < -2, the expression would be

$y = -(x - 3) - (x + 2) = -2x +1$.

$y = -2x + 1$ is a downward sloping line as slope -2 is negative.

If the above is understood, one can quickly ascertain that the graph of the expression $|x+2|+|x-1|+|x+1|$ would look like:



Thus $|x+2|+|x-1|+|x+1|$ would take the least value when x = -1. By substituting this value of x, the least value of $|x+2|+|x-1|+|x+1|$ can be found to be $|1|+|-2|+|0| = 3$.

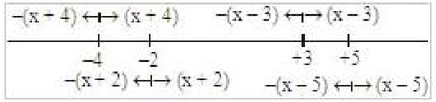
Example 25:

Solve for `x' if $|x+2|+|x-3|+|x+4|+|x-5| = 18$

Solution:

We know that $|x-a| = \begin{cases} x-a \text{ if } x \geq a \\ a-x \text{ if } x < a \end{cases}$

So the modulus functions $|x+2|$, $|x-3|$, $|x+4|$ and $|x-5|$ are defined as follows:



If $x < -4$, the equation is

$-(x + 2) - (x - 3) - (x + 4) - (x - 5) = 18$

$\Rightarrow -4x + 2 = 18 \Rightarrow -4x = 16 \Rightarrow x = -4$

But $x < -4$, so there is no solution

If $-4 \leq x < -2$, the equation is

$-(x + 2) - (x - 3) + (x + 4) - (x - 5) = 18$

$-2x + 10 = 18$ $-2x = +8 \Rightarrow x = -4$, which is admissible because it is falling in the interval $-4 \leq x < -2$.

If $-2 \leq x < 3$, the equation is $(x + 2) - (x - 3) + (x + 4) - (x - 5) = 18$

14 cannot be equal to 18. So there is no solution.

If $3 \leq x < 5$, the equation is $(x + 2) + (x - 3) + (x + 4) - (x - 5) = 18$

$\Rightarrow 2x + 8 = 18 \Rightarrow x = 5$.

But $3 \leq x < 5$, there is no solution.

If $x \geq 5$, the equation is $(x + 2) + (x - 3) + (x + 4) + (x - 5) = 18$

$4x = 20 \Rightarrow x = 5$.

Hence, the solution is $x = 5$ or -4

Can you draw the graph for the following equation?

$y = |x^2 - 5|$

Hint: Draw the parabola graph for the equation
($y = x^2 - 5$) first.

Greatest Integer Function

The **greatest integer function** is denoted by $y = f(x) = [x]$.

It denotes the greatest or largest integer less than or equal to x.

In general, if n is an integer and x is any real number between n and n + 1,

i.e. if $n \leq x < (n + 1)$, then $[x] = n$.

For example, $[3.4] = [3.9] = [3.9999] = 3$.

Similarly, $[-0.64] = [-1 + 0.36] = -1$ and so on.

There is one other way of defining the greatest integer function [x], which is very useful in some problems.

Any real number x consists of some integral part and some fractional part. For example, 2.5 consists of 2 as its integral part and 0.5 as its fractional part. All real numbers can be expressed in the same way:

$2.5 = 2 + 0.5$

$3.7 = 3 + 0.7$

$-2.5 = -3 + 0.5$

$4 = 4 + 0$

Generally any real number x can be represented as shown below:

$\mathbf{x = I + f}$, where I is the integral part of x and f is the fractional part and $0 \leq f < 1$.As it should be clear now, [x] always results in I. So that,

$[-0.75] = [-1 + 0.25] = -1$

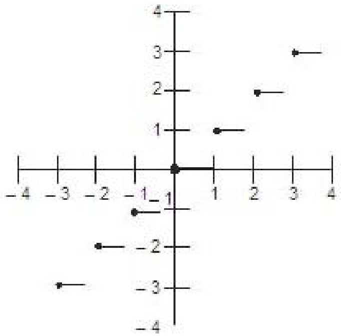
$[5.75] = [5 + 0.75] = 5$

$[2] = [2 + 0.0] = 2$

$[-5] = [-5 + 0.0] = -5$

Again for any real value of x, $f(x) = [x]$, represents an integer, hence as x takes up values from $-\infty$ to ∞, $f(x) = [x]$ will take all integral values. Hence the **range** of the function [x] is the set of all integers.

For the greatest integer function, the graph looks as follows.



The function above is also called a step function since the graph looks like steps.

Example 26:

If f(x) = [x] is the greatest integer function, find f(x) for

(i) x = -0.4, (ii) x = 0.9.

Solution:

(i) $x = -0.4 = -1 + 0.6$

Therefore, $[x] = [-0.4] = [-1 + 0.6] = -1$

(ii) $x = 0.9$. Therefore, $[x] = [0.9] = 0$

Example 27:

If $f(x) = 2[x]$, $g(x) = x^2$, find $f(g(x))$ for $x = 4.5$.

Solution:

$f(g(x)) = f(x^2) = f(4.5^2) = f(20.25) = 2\,[20.25] = 2(20) = 40$

Example 28:

Solve for $[x]^2 \le 9$ and $2x + 3 \le 5$.

Solution:

(1) $[x]^2 \le 9$ then $-3 \le [x] \le 3$

or $-3 \le x < 4$ ...(i)

(2) $2x+3 \le 5$ or $x \le 1$ ...(ii)

Hence, the value of x satisfying both is $-3 \le x \le 1$.

Example 29:

If $|x - 2| = x - 2$ and $[x] = 3$ and x is an integral multiple of 0.9, then find x.

Solution:

$[x] = 3$, hence $3 \le x < 4$ ... (i)

$|x - 2| = x - 2$. This means that

$x - 2 > 0$ or $x > 2$ ... (ii)

x is an integral multiple of 0.9.

Hence, the value of x must be 3.6 (i.e. 4×0.9).

Example 30:

Find the real values of x that satisfy the equation $5\{x\} = x + [x]$, where $\{x\}$ and $[x]$ denote the fractional part of x and greatest integer less than or equal to x respectively.

Solution:

Let $\{x\} = f$ and $[x] = I$

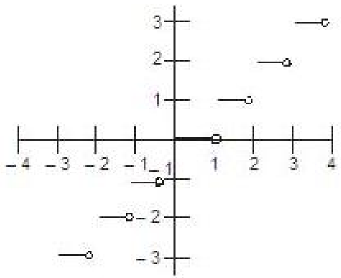
$\therefore x = \{x\} + [x] = f + I$

$5\{x\} = x + [x] \Rightarrow 5f = I + f + I = 2I + f \Rightarrow I = 2f$

Now, $0 \le f < 1$, therefore $I = 0$ and 1

$\therefore x = 0, 1.5$.

Step function: The function $f(x) = [x]$ has jump discontinuities at all integral points, and hence it is also called STEP function.



One may define a similar function called the **Least Integer Function**, which evaluates to the least integer greater than or equal to x and it is generally represented as **]x[.**

By definition:

]2.5[= 3

]-3.5[= -3

] -4[= - 4

Example 31:

Can you evaluate the expression:

[x] -] x [? Where the symbols [ ] and] [have the meaning as discussed above.

Solution:

0 if x is an integer and -1, otherwise.

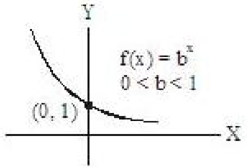
Logarithms and Exponentials

Exponential Functions:

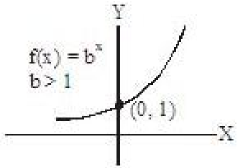
The exponential function is defined as $f(x) = b^X$

The quantity **b**, called the base, is a constant (independent of x). Also, b > 0 and b is not equal to 1. This means that b either lies strictly between 0 and 1 **OR** is greater than 1. x is a real number and f(x) always results in a positive value.

Case I: 0 < b < 1



Case II: b > 1



Example: Which of x and y is greater, if $a^X > a^Y$ **?**

Solution: Cannot be determined as we do not know, whether **`a'** lies between 0 and 1 or **`a'**

is greater than 1. If **`a'** lies between 0 and 1, x < y and if a >1, x > y.

Note: Please, remember the limitations on the values of base as given in the definition.

Logarithmic Functions:

The logarithmic function is the inverse of exponential function.

Definition: If any number N is expressed in the form a^x, (where a > 0 and a = 1), then the index X is called the logarithm of the number N to the base `a'.

Thus, if $N = a^X$, then $X = \log_a N$, where `a' is a positive number not equal to 1.

E.g. $1000 = 10^3$

$\Rightarrow \log_{10} 1000 = 3 \log_{10} 10 = 3$

E.g. $16 = 2^4 \log_2 16 = 4$

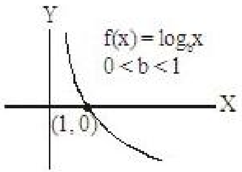
Logarithms of negative numbers are not defined.

The logarithmic function is defined as

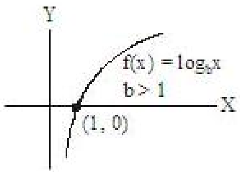
$f(x) = \log_b x$

The quantity **b,** called the base, is a constant (independent of x). Also b>0 and b is not equal to 1. This means that b either lies strictly between 0 and 1 **OR** is greater than 1. x is always positive and f(x) results in a real number(positive, zero or negative.)

Case I: $0 < b < 1$



Case II: $b > 1$



Example 32:

Which of x and y is greater if $\log_a x > \log_a y$?

Solution:

Cannot be determined as we do not know, whether `**a'** lies between 0 and 1 or `**a'** is greater than 1. If `**a'** lies between 0 and 1, x < y and if a > 1, x > y.

Note: Please, remember the limitations on the values of base b, and x as given in the definition.

Properties of Logarithms

1. $\log_a (XY) = \log_a X + \log_a Y$

2. $\log_a\left(\frac{X}{Y}\right) = \log_a X - \log_a Y$

3. $\log_a (X^k) = k \log_a X,$

4. $\log_{a^k} X = \frac{1}{k}\log_a X$

$\log_a \sqrt[k]{X} = \frac{1}{k} \log_a X$

$\log_{a^{1/k}} X = k \log_a X$

5. $\log_a 1 = 0$ [As $a^0 = 1$]

6. $\log_X X = 1$

7. $\log_a X = \frac{1}{\log_X a}$

8. $\log_a X = \frac{\log_b X}{\log_b a}$

9. $a^{(\log_a X)} = X$

10. When base is not mentioned, it will be taken as 10.

Two of the most common base used are 10 and e. e is another popularly used irrational number and is equal to 2.718.... Logs to base e are called Natural Logarithms.

Logarithms to base 10 can be used to find the number of digits in any number.

$log_{10} N = 23.46 \Rightarrow N = 10^{23.46}$

10^a will have (a + 1) digits and thus $10^{23.46}$ will have 24 digits.

Common mistake made by beginners is

$log(X + Y) = logX + logY$ or

$log(X + Y) = logX \times logY$

both of which are wrong.

Exponential Equations and their solutions:

If the equation is in the form $a^{f(x)}$ or $\{g(x)\}^{f(x)}$, then it is called exponential equation.

a. If the exponential equation can be put in the form $a^{f(x)} = a^{g(x)}$, $a \neq 1$, then f(x) = g(x) will give the solution.

Example 33:

Solve $\left(3\sqrt{5}\right)^{2x^3} = (2025)^{3x^2}$.

Solution:

Here, $(45)^{x^3} = (45)^{6x^2} \Rightarrow x^3 = 6x^2$

$\Rightarrow x^3 - 6x^2 = 0$

$\Rightarrow x^2 (x - 6) = 0$

$\Rightarrow x = 0$ or 6

b. If the exponential equation cannot be put in the form $a^{f(x)} = a^{g(x)}$, then select an exponential as `y'. The equation will then change into a polynomial equation in `y'.

c. In the exponential a^x, if a > 0, then no negative value of a^x is possible.

Example 34:

Solve for `x', if $5^x \times \left(\sqrt{32}\right)^{\frac{x}{x+3}} = 50$

Solution:

Here, $5^x \times \left(\sqrt{32}\right)^{\frac{x}{x+3}} = 50$

$\Rightarrow 5^x \times 2^{\frac{5x}{2x+6}} = 5^2 \times 2^1$

Equating the power of 2 or 5 from both sides, we get, x = 2 or,

$\frac{5x}{2x+6} = 1 \Rightarrow 5x = 2x + 6 \Rightarrow x = 2$

Example 35:

If $8 \times 5^{2x} + 9^{(x+0.5)} = 14 \times 15^x$, then how many values of x are possible?

Solution:

Here $8 \times 5^{2X} + 9^{(X+0.5)} = 14 \times 15^X$

$\Rightarrow 8 \times 5^{2X} + 3^{(2X+1)} = 14 \times 5^X \times 3^X$

$\Rightarrow 8 \times 5^{2X} - 14 \times 5^X \times 3^X + 3 \times 3^{2X} = 0$

$\Rightarrow 8\times\left(\frac{5}{3}\right)^{2x} - 12\times\left(\frac{5}{3}\right)^{x} - 2\times\left(\frac{5}{3}\right)^{x} + 3 = 0$

$$\Rightarrow 4\times\left(\frac{5}{3}\right)^x\left(2\left(\frac{5}{3}\right)^x-3\right)-1\left(2\times\left(\frac{5}{3}\right)^x-3\right)=0$$

$$\Rightarrow\left[2\left(\frac{5}{3}\right)^x-3\right]\left[4\times\left(\frac{5}{3}\right)^x-1\right]=0$$

$$\therefore 2\left(\frac{5}{3}\right)^x=3 \text{ Or } 4\left(\frac{5}{3}\right)^x=1$$

$$\therefore\left(\frac{5}{3}\right)^x=\frac{3}{2} \text{ or } \left(\frac{5}{3}\right)^x=\frac{1}{4}$$

$$\therefore x=\log_{(5/3)}\frac{3}{2} \text{ or } x=\log_{(5/3)}\frac{1}{4}.$$

Hence, two values are possible.

Logarithmic equations and their solutions

If the equation is of the form $y = g[\log_a f(x)]$, then it is called a logarithmic equation.

a. If the logarithmic equation can be put in the form $\log_a f(x) = \log_a h(x)$, then it will give the solutions for only those values of x which will give the solutions for $f(x) = h(x)$, and which also make both $f(x)$ and $h(x)$ positive.

b. If the logarithmic equation can be put in the form

$$(\log_a x)^n+(\log_a x)^{n-1}\ldots=k,$$

then by substituting the logarithmic expression as y, the equation will change into a polynomial equation in y.

c. If the bases of the logarithmic expressions are also the functions of x, the solutions must make the values of bases greater than o but not equal to1.

Example 36:

If $\log_{10}(5^x - 1) + (x - 1) = (1 + x)\log_{10}2$, then what is the value of x?

Solution:

Here

$$\Rightarrow \log_{10}\left(5^x-1\right)+\left(x-1\right)=\left(1+x\right)\log_{10}2$$

$$\Rightarrow \log_{10}\left(5^x-1\right)=\log_{10}10+\log_{10}2+x\left(\log_{10}2-1\right)$$

$$\Rightarrow \log_{10}\left(5^x-1\right)=\log_{10}20+x\left(\log_{10}2-\log_{10}10\right)$$

$$\Rightarrow \log_{10}\left(5^x-1\right)=\log_{10}20+x\log_{10}\frac{1}{5}$$

$$\Rightarrow \log_{10}\left(5^x-1\right)=\log_{10}20-\log_{10}5^x$$

$$\Rightarrow \log_{10}\left(5^x-1\right)+\log_{10}5^x=\log_{10}20$$

$$\Rightarrow 5^x\left(5^x-1\right)=20$$

$$\Rightarrow 5^{2x}-5^x-20=0$$

$$\Rightarrow 5^{2x}-5\times5^x+4\times5^x-20=0$$

$$\Rightarrow 5^x\times\left(5^x-5\right)+4\left(5^x-5\right)=0$$

$$\Rightarrow\left(5^x-5\right)\left(5^x+4\right)=0$$

$$\Rightarrow\left(5^x-5\right)=0 \text{ or } 5^x+4=0$$

As 5^x is always positive , then $5^x = -4$ is not possible.

$\therefore 5^x - 5 = 0 \Rightarrow 5^x = 5^1 \Rightarrow x = 1$

Example 37:

Solve for `x' if

$\left[\log_{|x-5|^2} 5\right] \times \log_5(x^2 - x - 6) = 1$

Solution:

Here $\left[\log_{|x+5|^2} 5\right] \times \log_5(x^2 - x - 6) = 1$

or $\log_{|x+5|^2}(x^2 - x - 6) = 1$ $[\because \log_a b \times \log_b p = \log_a p]$

$\Rightarrow (x^2 - x - 6) = |x+5|^2$ ... (i)

For logarithm functions to be defined,

$|x+5|^2 > 0;\ |x+5|^2 \neq 1;\ (x^2 - x - 6) > 0$

$\Rightarrow x \neq -5;\ |x+5| \neq 1;\ (x-3)(x+2) > 0$

$\Rightarrow x \neq -5;\ x \neq -4, -6;\ \ x < -2 \text{ or } x > 3$ ...(ii)

Now from (i),

$(x^2 - x - 6) = (x+5)^2$ $\left[\because |x|^2 = x^2\right]$

or $x^2 - x - 6 = x^2 + 10x + 25$

$\Rightarrow 11x = -31 \Rightarrow x = \frac{-31}{11}$

So, $x = \frac{-31}{11}$ satisfies all the conditions of (ii)

Hence the solution is $x = \frac{-31}{11}$.

Example 38:

If $\log_5 \log_3\left(\sqrt{x+3} + \sqrt{x}\right) = 0$, then what is the value of x?

Solution:

Here, $\log_5 \log_3\left(\sqrt{x+3} + \sqrt{x}\right) = 0$

$\Rightarrow \log_3\left(\sqrt{x+3} + \sqrt{x}\right) = 5^0 = 1$

$\Rightarrow \sqrt{x+3} + \sqrt{x} = 3^1 = 3$

$\Rightarrow \sqrt{(x+3)} = 3 - \sqrt{x}$

$\Rightarrow x + 3 = 9 + x - 6\sqrt{x}$

$\Rightarrow 6\sqrt{x} = 6$

$\Rightarrow \sqrt{x} = 1$

$\Rightarrow x = 1$

Example 39:

Solve for `x' , if $11^{\log_{10} x} = 242 - x^{\log_{10} 11}$

Solution:

Here $11^{\log_{10} x} = 242 - x^{\log_{10} 11}$

$\Rightarrow 11^{\log_{10} x} = 242 - 11^{\log_{10} x}$ [Remember: $a^{\log_{10} b} = b^{\log_{10} a}$]

$\Rightarrow 2 \times 11^{\log_{10} x} = 242 \Rightarrow 11^{\log_{10} x} = 11^2$

$\Rightarrow \log_{10} x = 2$

$\Rightarrow x = 10^2 = 100$

Exponential inequalities

1. If $x > 0$ and $a > b > 1$; then $a^x > b^x$

2. If $a > 1$ and $x > y > 0$; then $a^x > a^y$

3. If $0 < a < 1$ and $x > y > 0$, then $a^x < a^y$

Properties of Logarithmic inequalities:

1. If $a > 1$ and $0 < \alpha < \beta \Leftrightarrow \log_a \alpha < \log_a \beta$

2. If $0 < a < 1$, then $0 < \alpha < \beta \Leftrightarrow \log_a \alpha > \log_a \beta$.

3. If $a > 1$, $\alpha > 1$, then $\log_a \alpha > 0$.

4. If $0 < a < 1$, $0 < \alpha < 1$, then $\log_a \alpha > 0$.

5. If $0 < a < 1$, $\alpha > 1$, then $\log_a \alpha < 0$.

6. If $a > 1$, $0 < \alpha < 1$, then $\log_a \alpha < 0$.

7. If $a > 1$, $\alpha > 1$ and $\alpha > a$, then $\log_a \alpha > 1$.

8. If $a > 1$, $\alpha > 1$ and $\alpha < a$, then $0 < \log_a \alpha < 1$.

9. If $0 < a < 1$, $0 < \alpha < 1$ and $\alpha > a$, then $0 < \log_a \alpha < 1$.

10. If $0 < a < 1$, $0 < \alpha < 1$ and $\alpha < a$, then $\log_a \alpha > 1$.

Example 40:

If $\log_{0.5}(x - 5) < \log_{0.25}(x - 5)$, the interval in which x lies is

Solution:

Let $x - 5 > 0 \Rightarrow x > 5$

Now $\log_{0.5}(x - 5) < \log_{0.25}(x - 5)$

$\Rightarrow \log_{0.5}(x - 5) < \log_{(0.5)^2}(x - 5)$

$\Rightarrow \log_{0.5}(x - 5) < \frac{1}{2}\log_{0.5}(x - 5)$

$\Rightarrow 2\log_{0.5}(x - 5) < \log_{0.5}(x - 5)$

$\Rightarrow \log_{0.5}(x - 5)^2 < \log_{0.5}(x - 5)$

$\Rightarrow (x - 5)^2 > (x - 5)$ [The equality is reversed since base lies between 0 and 1]

$\Rightarrow (x - 5)^2 - (x - 5) > 0$

$\Rightarrow (x - 5)(x - 6) > 0$ ... (i)

Since $x > 5$, the inequality will hold if $x > 6$ i.e. if x lies in the interval $(6, \infty)$.

Example 41:

Find the solution set of the inequality $\log_{(x^2-2)}(2x - 6) < 1$

Solution:

Here, $\log_{(x^2-2)}(2x - 6) < 1$

Hence, $x^2 - 2 > 0$ and $2x - 6 > 0$

$\Rightarrow x^2 > 2$ are $x > 3$

Also since the value of $\log_{(x^2-2)}(2x-6)$ is less than 1

$x^2 - 2 > 2x - 6$

$x^2 - 2x + 4 > 0$

discriminant, $b^2 - 4ac = (-2)^2 - 4 \times 1 \times 4 = -12$

Which means roots are imaginary and therefore x lies in the interval $(-\infty, \infty)$

Combining all the three equations we get

$x > 3$

Therefore, the solution set is $(3, \infty)$

Example 42:

Solve for `x', if $\log_{0.3}\log_5\left(\frac{x^2-2}{x^2+2}\right) > 0$

Solution:

If $0 < a < 1$ and $\log_a x > 0$ then $0 < x < 1$

So, $0 < \log_5\left(\frac{x^2-2}{x^2+2}\right) < 1$

$\Rightarrow 1 < \frac{x^2-2}{x^2+2} < 5$

But $\frac{x^2-2}{x^2+2}$ is always lesser than 1.

Hence, no solution is possible.

Test Your Understanding

Level - I

1. If $f(x) = 4x^2 + 2$ and the domain of x is $1 \le x \le 4$, find the range of f(x).

2. If $f(x) = 3x^2 - 1$, $g(x) = x + 2$, find f(g(x)).

3. $f(a, b) = a^2 - b^2$, $g(a, b) = (a + b)$. Find the relationship when $f(a, b) \ge g(a, b)$.

4. Let set A = {red, blue} and set B = {triangle, square, circle, rhombus, rectangle}.

(a) How many one to one functions are there from A to B?

(b) How many one to one functions are there from B to A?

(c) How many onto functions are there from A to B?

(d) How many onto functions are there from B to A?

(e) How many into functions are there from B to A?

(f) How many bijective functions are there from A to B?

5. Find the domain and range of following functions.

(i) $f(x) = 2x + 3$ (ii) $f(x) = x^2 - 2$ (iii) $f(x) = \frac{1}{x}$ (iv) $f(x) = -\sqrt{x}$

(v) f(x) = log x (vi) f(x) = Sin x (vii) $f(x) = 2^x$

6. $f(x) = 3x - 4,\ 0 \le x \le 5$; $g(x) = x^2 + 1,\ -\infty \le x \le \infty$.

Find the domain of x for which $f(f(g(x)))$ is defined.

7. A function is defined as

Max(x, y) = The larger value of x and y.

Min(x, y, z) = The smaller value of $\frac{(x+y)}{2}$ and z.

If Min(x, y, z) = z, then which of the following is true?

a. x > z

b. y > z

c. Max(x, y) > z

d. Max(x, y) < z

e. z > (x + y)

8. Let A (x, y) = Maximum of x, y;

B(x, y) = Minimum of x, y ;

$C(x, y) = \frac{(x+y)}{2}$.

a. What is the value of B

(A (A(2, 3), C(4, 6)), C (B(1, 2), A (3, 2)))?

b. For what value of x will the value of the expression A(B(C(6, 8), A(5, x))), C(B(5, a), A(4, 6)) = 6?

c. Which of the following is the largest?

(i) A(C(a, b), B(a, b)) (ii) C(A(a, b), B(a, b)) (iii) B(C(a, b), A(a, b))

(iv) B(A(a,b), C(a.b)) (v) All are equal

d. If C(x, C(A(x, y), B(x, y))) = $\frac{7x}{4}$

What is the relationship between x and y?

9. If Sqr (A, B) = A^2 - B^2; Add (A , B) = A + B; @ (A, B) = $\frac{A+B}{2}$

Find $\frac{Sqr(Add(A, B), @(A, B))}{Add(A, B)}$.

10. If a and b are the two positive integers and r denotes the remainder obtained when a is divided by b, then f(a, b) is defined by f(a, b) = a, if b = 0 and f(a, b) = f(b, r), if a > b, then find the following.

(a) Value of f(27, 18)

(b) Value of f(6, 4)

(c) Value of f(15, 9)

Directions for questions 11 to 131: Answer the questions based on the following information.

The following functions have been defined for three numbers A, B and C.

@ (A, B) = Average of A and B.

*(A, B) = Product of A and B.

/ (A, B) = A divided by B.

11. If A = 2 and B = 4, the value of

@ (/ (* (A, B), B), A) would be

a. 2 b. 4 c. 6 d. 16 e. 8

12. The sum of A and B is given by

a. *(@ (A, B), 2) b. / (@ (A, B), 2) c. @ (* (A, B), 2) d. @ (/ (A, B), 2) e. *(/ (A, 3), 2)]

13. The sum of A, B and C is given by

a. *(@ (*(@ (B, A), 2), C), 3) b. * (@ (* (@ (B, A), 2), C), 2)

c. / (* (@ (* (B, A), 2), C), 3) d. (/ (@ (/ (@ (B, A), 2), C), 2)

e. None of these

Directions for questions 14 to 16: The questions are based on the following information.

md(x) = |x|

mn(x, y) = Minimum of x and y

Ma(a, b, c, ...) = Maximum of a, b, c, ...

(Based on a problem which appeared in CAT 1994)

14. The value of Ma[md(a), mn(md (b), a), mn(ab, md (ac))], where

a = -2, b = -3, c = 4 is

a. 2 b. 6 c. 8 d. -2 e. 4

15. Given that a > b, then the relation

Ma[md(a), mn(a, b)] = mn[a, md(Ma(a, b))] does not hold if

a. a < 0, b < 0 b. a > 0, b > 0

c. a > 0, b < 0, |a| < |b| d. a > 0, b < 0, |a| > |b|

e. a > 0, b > 0, |a|, > |b|

16. The value of mn[md(b), Ma(md(a), b), mn(ab, md (bc))], where a = 6, b = -9 and c = 10, is

a. 9 b. 6 c. - 54 d. - 90 e. None of these

17. A and B are two sets with n elements in each set. How many invertible functions are possible from A to B?

a. n^2 b. n c. n! d. 2n e. none of these

Directions for questions 18 and 19:

The questions are based on the data given below.

le(x, y) = Least of (x, y)

mo(x) = |x|

me(x, y) = Maximum of (x, y)

18. Find the value of me((a + mo(le(a, b))), mo(a + me(mo(a), mo(b)))), at a = -2 and b = -3.

a. 1 b. 0 c. 5 d. 3 e. 4

19. Which of the following must always be correct for a, b > 0?

a. mo(le(a, b))≥me(mo(a), mo(b)) b. mo(le(a, b)) > me(mo(a), mo(b))

c. mo(le(a, b)) < le(mo(a), mo(b)) d. mo(le(a, b)) = le(mo(a), mo(b))

e. mo(le(a, b)) ≤ le(mo(a), mo(b))

Directions for questions 20 and 21:

Solve the questions using the following data.

p and q are any numbers and x and y are any non-negative integers. Let certain operations be defined as given below.

$\frac{x}{y} = x$ divided by y.

lo(p) = The greatest integer less than or equal to p.

gr(p) = The smallest integer greater than or equal to p.

rem(x, y) = The remainder when x is divided by y.

20. gr(4.6) - lo(2.0) = ?

a. 4 b. 3 c. 2 d. 1 e. 0

21. rem(16, 7) - 2 = ?

a. 2 b. 1 c. 0 d. -1 e. -2

22. If $(28)^{8+3x} = \left(56\sqrt{7}\right)^{3x+3}$, then the value of x is

23. If $\left(2+\sqrt{3}\right)^{x^2-5}+\left(2-\sqrt{3}\right)^{x^2-5} = 4$, then the value of x is

24. $50^{\log_{50} 20}$ is equal to

a. 50 b. 20 c. 52 d. 25 e. None of these

25. $\log_{10}10 + \log_{100}100 + \log_{10}1000 + \log_{100}10000 + \log_{100}100000$ is equal to

a. 10 b. 9 c. 9.1 d. 9.5 e. 8.5

26. Find the values of the following.

a. $\log_2 4$ b. $\log_{25} 5$ c. $\log_8 16$ d. $\log_{36} \sqrt{216}$

27. Find the values of the following.

a. $\log_5 10 \times \log_{10} 15 \times \log_{15} 20 \times \log_{20} 25$

b. $\log_9 16 \times \log_{32} 27$

28. If $2 \log_x (x - 2) = \log_x 4$, then find the value of x.

29. If $\log_x 0.0016 = 4$, then x is equal to

a. 2 b. 0.2 c. 0.02 d. $\frac{1}{2}$ e. 200

30. Solve the following equations.

a. $\log\frac{5}{4} + \log 14 - \log\frac{7X}{3} = -1$ b. $\frac{1}{2}\log(2X+2) + \log\sqrt{(3X+4)} = 1 + \log 2$

31. Find the value of $\log_{49} 343$.

a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. 3 d. $\frac{3}{2}$ e. $\frac{1}{3}$

32. If log 2 = 0.301, the number of digits in 2^{64} is

a. 18 b. 19 c. 20 d. 21 e. 22

33. If $\log_x\left(\frac{27}{64}\right) = -\frac{1}{3}$, then find the value of x.

a. $\frac{4}{3}$ b. $\frac{3}{4}$ c. $\left(\frac{4}{3}\right)^9$ d. $\left(\frac{3}{4}\right)^9$ e. None of these

34. Find the value of $\log_{25} 27 \times \log_9 125$.

a. $\frac{9}{4}$ b. $\frac{4}{9}$ c. $\frac{3}{2}$ d. $\frac{2}{3}$ e. $\frac{1}{2}$

35. If $\log_{10} 3 = 0.4771$, then $\log_{10} 81$ is approximately equal to

a. 1.908 b. 1.648 c. 1.828 d. 1.988 e. None of these

36. If $\log_{10}(x-2)^2 = 4$, then find the value of x.

a. 100 b. 102 c. 0 d. 2 e. 101

37. If $\log_{10} X = 5$, then find the number of digits in number X.

a. 4 b. 5 c. 6 d. 7 e. None of these

38. Find the value of y in $\log_{10} y = 10^{\log_{100} 16}$.

Level - II

39. Suppose for any real number x, [x] denotes the greatest integer less than or equal to x. Let L(x, y) = [x] + [y] + [x + y] and R(x, y) = [2x] + [2y]. Then it is impossible to find any two positive real numbers x and y for which

a. L(x, y) = R(x, y) b. L(x, y) $\neq$ R(x, y)

c. L(x, y) < R(x, y) d. L(x, y) > R(x, y)

40. Find the range for values of y if x is real number where $y = \sqrt{x^2 - 6x + 13}$.

a. $y \geq 2$ b. $y \leq 2$ c. $0 \leq y \leq 2$ d. $y > 2$ e. None of these

41. Find the range for values of y if x is real number where $y = \sqrt{40 - x^2 + 6x}$.

a. $y \geq 7$ b. $-7 \leq y \leq 7$ c. $y \leq 7$ d. $0 \leq y \leq 7$ e. None of these

42. Solve for x, y if |x| + 3y = 7, 2x + |y - 10| = 3. (`x' and `y' are real numbers.)

43. Solve for x, if |x - 2| ≤ 2 and |x + 3| ≥ 4. `x' is a real number.

44. Solve for x, if $|x^2 + 3x| + x^2 - 2 \geq 0$. `x' is a real number.

45. If $x^2 + y^2 = 1$ and $|x + y| = \sqrt{2}$, then solve for x and y. (`x' and `y' are real numbers.)

46. What is the minimum possible value of

a. $|x-2|+|2x+1|$

b . $|x-2|+|2x+1|-|x-1|$

47. What is the area enclosed within the graph of

a. |x + y| = 3 and |x - y| = 3

b. $y = |x+3|-4$ and the x axis

c. $|2y| = x;\ x = 3$

48. Solve for `x' if

$\log_{(x-7)}\left(2x^2 - x - 91\right) = 5 - 2\log_{(2x-13)}(x^2 - 14x + 49)$

49. If $\log_7\left(2 \times 7^{x-3} - 1\right) + 6 = 2x$, then how many values of `x' are possible?

50. Solve for `x' if $\log_{(2x-1)}(3x^2 - 2x - 1) < 1$

51. Solve for `x' if $\log_{125}(x-2)^3 - \log_{0.2}(x-2) > 6$.

41. Find the range for values of y if x is real number where $y = \sqrt{40 - x^2 + 6x}$.

a. $y \ge 7$ b. $-7 \le y \le 7$ c. $y \le 7$ d. $0 \le y \le 7$ e. None of these

42. Solve for x, y if $|x| + 3y = 7$, $2x + |y - 10| = 3$. (`x' and `y' are real numbers.)

43. Solve for x, if $|x - 2| \le 2$ and $|x + 3| \ge 4$. `x' is a real number.

44. Solve for x, if $|x^2 + 3x| + x^2 - 2 \ge 0$. `x' is a real number.

45. If $x^2 + y^2 = 1$ and $|x + y| = \sqrt{2}$, then solve for x and y. (`x' and `y' are real numbers.)

46. What is the minimum possible value of

a. $|x-2|+|2x+1|$

b . $|x-2|+|2x+1|-|x-1|$

47. What is the area enclosed within the graph of

a. $|x + y| = 3$ and $|x - y| = 3$

b. $y = |x+3|-4$ and the x axis

c. $|2y| = x;\ x = 3$

48. Solve for `x' if

$\log_{(x-7)}(2x^2 - x - 91) = 5 - 2\log_{(2x+13)}(x^2 - 14x + 49)$

49. If $\log_7(2 \times 7^{x-3} - 1) + 6 = 2x$, then how many values of `x' are possible?

50. Solve for `x' if $\log_{(2x-1)}(3x^2 - 2x - 1) < 1$

51. Solve for `x' if $\log_{125}(x-2)^3 - \log_{0.2}(x-2) > 6$.

Progressions 5

Introduction

This chapter deals with concepts related to progressions and series.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

• Arithmetic, Geometric and Harmonic Progressions

Progression

Sequences:

A sequence is a function whose domain is N (the set of all the natural numbers) and range is a subset of R (the set of real numbers). Simply put, a sequence is a function f (n), where n is always a natural number.

Series:

If $a_1, a_2, a_3, a_4, \dots a_n, \dots$ is a sequence then

$a_1 + a_2 + a_3 + a_4 \dots + a_n + \dots$ is a series.

Progressions:

It is not necessary that the terms of a sequence always follow a certain pattern or be described by some explicit rule for the n^{th} term. ***A sequence in which all terms follow a pattern is called a progression.***

1. Arithmetic progression

Arithmetic progression is a sequence of numbers in which the difference between any two successive numbers is always constant.

For example 1, 6, 11, 16, ··· so on;

1, -1, -3, -5, ··· ;

In the examples above, the difference between any two successive numbers is equal to 5 and -2 respectively. This difference is called the common difference.

The general form of expressing this series is a, a + d, a + 2d, a + 3d, ··· so on.

The standard notations are as follows.

a = The first term,

d = Common difference,

T_n = The nth term

ℓ = The last term,

S_n = Sum of n terms,

1. $T_n = a + (n-1)d$

2. $S_n = \left(\frac{a+\ell}{2}\right) \times n$

In the formula above, you would also observe that $\left(\frac{a+\ell}{2}\right)$ would give you the average of the terms of the progression.

3. Deriving from the formula above, we also have

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

In an AP, S_n is given by solution of a quadratic equation in "n" as shown below:

$$\frac{d}{2}n^2 + (a - \frac{d}{2})n - S_n = 0$$

The first term "a" and the common difference "d" are always defined for any AP. When it comes to finding the total number of consecutive terms, which will sum up to S_n, you are supposed to solve a quadratic equation in "n".

Can the roots of the above equation be negative? If they can be, will they have any significance?

Lets solve this problem:

Example 1:

The first term of an AP is -3 and the common difference is +2. How many term of this series will give a sum of 12?

Solution:

Using the S_n formula,

$$12 = \frac{n}{2}\{2 \times (-3) + (n-1) \times (2)\} \text{ or } n^2 - 4n - 12 = 0$$

Solving we get, n = 6 and n = -2.

n = 6 tells us that starting from the first term, "-3", if we keep on adding till the 6^{th} term, we will get a total of 12. What does n = -2 tell us?. Let us first write down the first 6 terms of this series: -3, -1, 1, 3, 5, and 7

If we add up, the **LAST TWO TERMS** from the rear end of this series, we will again get the same sum i.e. 12. And hence, If we add up two terms from rear, we will get the same sum.

Arithmetic mean

If a and b are any two numbers, m is called the arithmetic mean of a and b and is given by $m = \frac{(a+b)}{2}$.

Arithmetic mean can also be found if there are more than two terms. For instance, the arithmetic mean of a, b, c and d is equal to $\frac{(a+b+c+d)}{4}$.

2. Geometric progression

If the ratio of any two successive terms in the same order is equal throughout a sequence, then the sequence is said to be in a geometric progression.

The general form of a GP with n terms is a, ar, ar^2, ..., ar^{n-1}. Thus, if

a = The first term,

r = The common ratio

T_n = The nth term and

S_n = The sum of n terms we have the following

1. $T_n = ar^{n-1}$

2. $S_n = a\frac{(1-r^n)}{(1-r)}$, where r < 1

3. $S_n = \frac{a(r^n - 1)}{(r-1)}$, where r > 1

The expression can be re-written as

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{a-(n^{th}\text{term} \times r)}{1-r}$$

The latter form might be quite useful in some problems. There is another way of expressing

$$S_n = \left(\frac{a}{r-1}\right)r^n + \left(-\frac{a}{1-r}\right)$$

$$S_n = Ar^n + B$$

In a GP, S_n is given by the solution of a polynomial of degree n.

Sum of infinite terms in a GP is a finite quantity.

Amazed ! But this is true when common difference is a pure fraction ($-1 < r < 1$). This is possible since the latter terms of the series become insignificant and the series is said to be converging to a finite value.

Geometric mean

If a and b are any two terms, G is called the geometric mean of a and b and is given by $G = \sqrt{ab}$.

GM is defined only for a set of positive numbers.

If a GP has infinite terms and $-1 < r < 1$, the sum to infinity (S_∞) is $S_\infty = \frac{a}{1-r}$.

To make problems easier, the following could be used as standard convention.

1. Three numbers in an AP should be taken as a - d, a, a + d.

2. Four numbers in an AP should be taken as a - 3d, a - d, a + d, a + 3d.

3. Three numbers in a GP should be taken as a/r, a, ar.

4. Four numbers in a GP should be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

3. Harmonic progression

A series of quantities is said to be in harmonic progression when their reciprocals are in an AP.

For example,

$$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$$

$$\frac{1}{a}, \frac{1}{(a+d)}, \frac{1}{(a+2d)}, \ldots$$

$\frac{1}{2}, \frac{-1}{3}, \frac{-1}{8}, \frac{-1}{13}, \ldots$ are in HP as their reciprocals are in AP.

To find the harmonic mean of any set of numbers, find the arithmetic mean of the reciprocals of all the numbers. The reciprocal of the arithmetic mean so found is the harmonic mean of the numbers. For any set of n positive numbers, the following relationship always holds true.

$$AM \geq GM \geq HM$$

The relationship would be equal when all the numbers are identical.

Harmonic progression derives its name from harmony in musical notes. As the great mathematician Leibnitz said "Music is the Mathematics of one who does not know that he is counting".

Let us now look at a few solved examples.

Example 2:

Find the sum of all even numbers from 10 to 300, excluding the multiples of 4.

Solution:

The sequences of all even numbers is 10, 12, 14, ...296, 298, 300. There are in all 146 such numbers. Sum of all such even numbers

$= \left(\frac{10+300}{2}\right) \times 146$ ... (i)

The sequence of all multiples of 4 from 10 to 300 is 12, 16, 20, ..., 292, 296, 300. There are in all 73 such numbers.

The sum of all multiples of 4 $= \left(\frac{300+12}{2}\right) \times 73$ ... (ii)

Hence, sum of all the required numbers = (i) - (ii) = 11,242

Example 3:

The sum of the first two terms of a GP is -3 and the sum of the first 4 terms is -15. Find the sequence.

Solution:

Let the terms be $a, ar, ar^2, ar^3, \ldots$ so on

$a + ar = -3 \Rightarrow a(1+r) = -3$ ...(i)

Similarly, $a(1+r+r^2+r^3) = -15$

$\Rightarrow a(1+r) + ar^2(1+r) = -15$

$a(1+r)(1+r^2) = -15$ ... (ii)

From (i), we get $1 + r^2 = 5$

Hence, $r = \pm 2$

So a = -1 when r = 2 or a = 3 when r = -2.

∴ The terms are -1, -2, -4, -8, 0, ..., so on

or 3, -6, 12, -24, ..., so on.

Example 4:

The 5th term of an AP is 27. The common difference is 4. Find the sum of the first 15 terms of this series.

Solution:

a + 4d = 27, d = 4

Hence, a = 11

The 15th term is 11 + 4 (14) = 67

Hence, sum to first 15 terms $= \left(\frac{67+11}{2}\right) \times 15 = 585$

Example 5:

If $a_1 = 1$ and $a_{n+1} = 2a_n + 5$, n = 1, 2 ..., then a_{100} is equal to

a. $(5 \times 2^{99} - 6)$

b. $(5 \times 2^{99} + 6)$

c. $(6 \times 2^{99} + 5)$

d. $(6 \times 2^{99} - 5)$

e. None of these

Solution:

$a_1 = 1, a_2 = 7, a_3 = 19, a_4 = 43.$

The difference between successive terms is in series 6, 12, 24, 48, ..., i.e. they are in GP. Hence,

$$a_{100} = a_1 + a\left(\frac{r^n - 1}{r - 1}\right) = 1 + 6\frac{(2^{99} - 1)}{(2-1)} = 6 \times 2^{99} - 5$$

Example 6:

What is the value of the following expression?

$$\left(\frac{1}{(2^2-1)}\right) + \left(\frac{1}{(4^2-1)}\right) + \left(\frac{1}{(6^2-1)}\right) + \cdots + \left(\frac{1}{(20^2-1)}\right)$$

a. $\frac{9}{19}$ b. $\frac{10}{19}$ c. $\frac{10}{21}$ d. $\frac{11}{21}$ e. $\frac{13}{21}$

Solution:

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{19.21} = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{19} - \frac{1}{21}\right)$$

$$= \frac{1}{2} - \frac{1}{42} = \frac{(21-1)}{42} = \frac{20}{42} = \frac{10}{21}$$

Example 7:

The nth element of a series is represented as $X_n = (-1)^n X_{n-1}$ If $X_0 = x$ and x > 0, then which of the following is always true?

a. X_n is positive if n is even

b. X_n is positive if n is odd

c. X_n is negative if n is even

d. Both (b) and (c)

e. None of these

Solution (e):

$X_0 = x, X_1 = -x, X_2 = -x, X_3 = x, X_4 = x,$

$X_5 = -x, X_6 = -x...$

⇒ Choices (a), (b), (c) are incorrect.

Example 8:

Let S denotes the infinite sum $2 + 5x + 9x^2 + 14x^3 + 20x^4 + \ldots,$

where |x| < 1 and the coefficient of x^{n-1} is $\frac{1}{2}n(n+3), (n = 1, 2, \ldots)$. Then S equals:

a. $\frac{2-x}{(1-x)^3}$ b. $\frac{2-x}{(1+x)^3}$ c. $\frac{2+x}{(1-x)^3}$ d. $\frac{2+x}{(1+x)^3}$ e. $\frac{2-x}{(x-1)^3}$

Solution (a):

Coefficient of $x^n = \frac{1}{2}(n+1)(n+4)$

$S = 2 + 5x + 9x^2 + 14x^3 + \ldots$

$xS = 2x + 5x^2 + \ldots$

$S(1-x) = 2 + 3x + 4x^2 + 5x^3 + \ldots$

Let $S_1 = S(1-x) \Rightarrow S_1 = 2 + 3x + 4x^2 + \ldots$

$xS_1 = 2x + 3x^2 + \ldots \quad S_1(1-x) = 2 + x + x^2 + \ldots$

$S_1(1-x) = 2 + \frac{x}{1-x}$

$S(1-x^2) = 2 + \frac{x}{1-x} \Rightarrow S = \frac{2-x}{(1-x)^3}$

Example 9:

Consider the sequence of numbers $a_1, a_2, a_3, \ldots$ to infinity where $a_1 = 81.33$ and $a_2 = -19$ and
$a_j = a_{j-1} - a_{j-2}$ for $j \geq 3$.

What is the sum of the first 6002 terms of this sequence?

a. -100.33 b. -30.00 c. 62.33 d. 119.33 e. 82.33

Solution (c):

Given $a_1 = 81.33$; $a_2 = -19$

Also: $a_j = a_{j-1} - a_{j-2}$, for $j \geq 3$

$\Rightarrow a_3 = a_2 - a_1 = -100.33$

$a_4 = a_3 - a_2 = -81.33$

$a_5 = a_4 - a_3 = 19$

$a_6 = a_5 - a_4 = +100.33$

$a_7 = a_6 - a_5 = +81.33$

$a_8 = a_7 - a_6 = -19$

Clearly onwards there is a cycle of 6 and the sum of terms in every such cycle = 0. Therefore, when we add $a_1, a_2, a_3\ldots$ upto a_{6002}, we will eventually be left with $a_1 + a_2$ only i.e. 81.33 - 19 = 62. 33.

Example 10:

If the sum of the first 11 terms of an arithmetic progression equals that of the first 19 terms, then what is the sum of the first 30 terms?

a. 0 b. -1 c. 1 d. 2 e. Not unique

Solution (a):

Given $t_1 + t_2 + \ldots + t_{11} = t_1 + t_2 + \ldots + t_{19}$ (for an A.P.)

$\Rightarrow \frac{11}{2}[2a + (11-1)d] = \frac{19}{2}[2a + (19-1)d]$

$22a + 110d = 28a + 342d$

$16a + 232d = 0$

$2a + 29d = 0$

$\Rightarrow \frac{30}{2}[2a + (30-1)d] = 0$

$\Rightarrow S_{30\ \text{terms}} = 0$

Example 11:

For a Fibonacci sequence, from the third term onwards, each term in the sequence is the sum of the previous two terms in that sequence. If the difference in squares of 7th and 6th terms of this sequence is 517, what is the 10th term of this sequence?

a. 147 b. 76 c. 123 d. 117 e. Cannot be determined

Solution:

Let the 6th and the 7th terms be x and y.

Then 8th term = x + y

Also $y^2 - x^2 = 517 \Rightarrow (y + x)(y - x) = 517$

$= 47 \times 11$

So $y + x = 47$

$y - x = 11$

Taking y = 29 and x = 18, we have 8th term = 47,

9th term = 47 + 29 = 76 and 10th term = 76 + 47 = 123.

Example 12:

The sum of 3^{rd} and 15^{th} elements of an arithmetic progression is equal to the sum of 6^{th}, 11^{th} and 13^{th} elements of the same progression. Then which element of the series should necessarily be equal to zero?

a. 1^{st} b. 9^{th} c. 12^{th} d. 10^{th} e. None of these

Solution (c):

Let the 1st term be `a' and common difference be `d' then we have 3^{rd} term = a + 2d

15^{th} term = a + 14d

6^{th} term = a + 5d

11^{th} term = a + 10d

13^{th} term = a + 12d

Since, sum of 3^{rd} and 15^{th} term = sum of 6^{th}, 11^{th} and 13^{th} term, therefore we have

$2a + 16d = 3a + 27d$

$= a + 11d = 0$

Which is the 12^{th} term.

Example 13:

The 288^{th} term of the series a, b, b, c, c ,c , d, d,d ,d ,e ,e , e, e, e, f, f, f, f, f, f... is

a. u b. v c. w d. x e. y

Solution (d):

The number of terms of the series forms the sum of first n natural numbers i.e. $\frac{n(n+1)}{2}$.

Thus the first 23 letters will account for the first $\frac{23 \times 24}{2}$ = 276 terms of the series.

The 288th term will be the 24th letter which is x.

Example 14:

If $\log_3 2$, $\log_3 (2^x - 5)$, $\log_3 (2^x - 7/2)$ are in arithmetic progression, then the value of x is equal to

a. 5 b. 4 c. 2 d. 3 e. 1

Solution (c):

Using log a - log b = $\log\left(\frac{a}{b}\right)$, $\frac{2}{y-5} = \frac{y-5}{y-3.5}$, where y = 2^x

Solving we get y = 4 or 8 i.e. x = 2 or 3. It cannot be 2 as log of negative number is not defined (see the second expression).

Example 15:

The infinite sum $1+\frac{4}{7}+\frac{9}{7^2}+\frac{16}{7^3}+\frac{25}{7^4}+\cdots$ equals

a. $\frac{27}{14}$ b. $\frac{21}{13}$ c. $\frac{63}{37}$ d. $\frac{256}{147}$ e. $\frac{49}{27}$

Solution (e):

Let $S = 1+\frac{4}{7}+\frac{9}{7^2}+\frac{16}{7^3}+\frac{25}{7^4}$ ... (i)

$\therefore \frac{1}{7}S = \frac{1}{7}+\frac{4}{7^2}+\frac{9}{7^3}+\frac{16}{7^4}$ ... (ii)

(i) - (ii) gives,

$S\left(1-\frac{1}{7}\right) = 1+\frac{3}{7}+\frac{5}{7^2}+\frac{7}{7^3}+\frac{9}{7^4}$ ... (iii)

$\frac{1}{7}\times S\left(1-\frac{1}{7}\right) = \frac{1}{7}+\frac{3}{7^2}+\frac{5}{7^3}+\frac{7}{7^4}$ ... (iv)

(iii) - (iv) gives,

$$S\left(1-\frac{1}{7}\right)-\frac{1}{7}S\left(1-\frac{1}{7}\right) = 1+\frac{2}{7}+\frac{2}{7^2}+\frac{2}{7^3}+\frac{2}{7^4}\cdots$$

$$\therefore S\left(1-\frac{1}{7}\right)\left(1-\frac{1}{7}\right) = 1+\frac{2}{7}\left[1+\frac{1}{7}+\frac{1}{7^2}+\cdots\infty\right]$$

$$\therefore S\left(1-\frac{1}{7}\right)^2 = 1+\frac{2}{7}\times\frac{1}{1-\frac{1}{7}}$$

$$\therefore S\left(\frac{6}{7}\right)^2 = 1+\frac{2}{7}\times\frac{7}{6}$$

$$\therefore S\times\frac{36}{49} = 1+\frac{1}{3}$$

$$\therefore S = \frac{49}{36}\times\frac{4}{3}$$

$$S = \frac{49}{27}$$

Example 16:

Find the sum to infinite terms of the series $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \infty$ terms.

Solution:

$a = 2,\ r = \frac{1}{2} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$

Example 17:

What is the sum of `n' terms in the series $\log m + \log\left(\frac{m^2}{n}\right) + \log\left(\frac{m^3}{n^2}\right) + \log\left(\frac{m^4}{n^3}\right) + \cdots ?$

a. $\log\left[\frac{n^{(n-1)}}{m^{(n-1)}}\right]^{\frac{n}{2}}$ b. $\log\left[\frac{m^m}{n^n}\right]^{\frac{n}{2}}$ c. $\log\left[\frac{m^{(1-n)}}{n^{(1-m)}}\right]^{\frac{n}{2}}$ d. $\log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{\frac{n}{2}}$ e. None of these

Solution (d):

Sum of $\log m + \log\left(\frac{m^2}{n}\right) + \log\left(\frac{m^3}{n^2}\right) + \cdots + n$

terms such problem must be solved by taking the value of number of terms. Let's say 2 and check the given option. If we look at the sum of 2 terms of the given series it comes out to be

$\log m + \log\frac{m^2}{n} \Rightarrow \log\frac{m \times m^2}{n} = \log\left(\frac{m^3}{n}\right)$

Now look at the option and put number of terms as 2, only option (d)validates the above mentioned answer.

As $\log\left[\frac{m^{(n+1)}}{n^{(n-1)}}\right]^{\frac{n}{2}} \Rightarrow \log\left[\frac{m^3}{n}\right]^1 \Rightarrow \log\left(\frac{m^3}{n}\right)$

Example 18:

Find the sum to n terms of the series

3 + 6 + 10 + 16 + ... so on.

Solution:

The sequence could also be written

$(2+4+6+8+10+\ldots)+(1+2+4+8+\ldots)$

The first sequence is an AP and the second sequence is a GP.

So the sum to n terms is $\frac{n}{2}[4+(n-1)2]+1\times\left(\frac{2^n-1}{2-1}\right) \Rightarrow n(n+1)+2^n-1$

Example 19:

Find the sum to n terms of the sequence

5 + 55 + 555 + 5555 + ... so on.

Solution:

$S_n = 5+55+555+5555+\ldots \text{ so on} = \frac{5}{9}[9+99+999+9999+\ldots \text{ so on}]$

$= \frac{5}{9}[(10-1)+(10^2-1)+(10^3-1)+(10^4-1)+ \quad \ldots \text{ so on}]$

$\Rightarrow \frac{5}{9}[10+10^2+10^3+10^4+\ldots \text{ n terms}] - \frac{5}{9}[1+1+1+\ldots \text{ n terms}]$

$\Rightarrow \frac{5}{9}\times 10\cdot\left(\frac{10^n-1}{10-1}\right) - \frac{5n}{9} \Rightarrow \frac{50(10^n-1)}{81} - \frac{5n}{9}$

Special series

There are some sequences that can be written in terms of the nth term. For instance, if the sequence is like $1^2, 2^2, 3^2, 4^2, 5^2, \ldots n^2$, so on. The nth term can be written n^2.

The summation of all the terms of such a sequence could be written $\sum n^2$.

Similarly, there could be the sequence like 1, 2, 3, 4, ..., n.

The sum to n terms of such a sequence is written $\sum n$.

If the sequence is $1^3, 2^3, 3^3, 4^3, \ldots n^3$, the sum to n terms of such a sequence is written $\sum n^3$.

The standard results are as follows.

(1) $\sum n = \frac{n(n+1)}{2}$

(2) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum n^3 = \frac{n^2(n+1)^2}{4}$

Example 20:

Find the sum of the sequence to 6 terms

$1^3, 2^3, 3^3, \ldots 6^3$.

Solution:

$$S_6 = \sum n^3 = \frac{n^2(n+1)^2}{4}$$

n = 6

$$S_6 = \frac{6^2(6+1)^2}{4} = 441$$

Example 21:

If the general term of a sequence is written $2n + n^2$, find the sum of the sequence to 5 terms.

Solution:

$T_n = 2n + n^2$

$S_n = \Sigma T_n = \Sigma 2n + \Sigma n^2$

n = 5

Hence, $S_5 = \frac{2(5)(6)}{2} + \frac{5(6)(11)}{6} \Rightarrow 30 + 55 = 85$

Test Your Understanding

Level - I

1. Find the sum to 20 terms of the series

$5 + 55 + 555 + \cdots$

2. The ratio between sum of n terms of two AP is (3n + 8) : (7n + 15). Find the ratio between their sums to the 12th terms.

3. In a certain colony of cancerous cells, each cell divides into two every hour. What will be the total number of cells produced in a span of 10 hours, if we start with a single cell?

Level - II

$n = 6$

$$S_6 = \frac{6^2(6+1)^2}{4} = 441$$

Example 21:

If the general term of a sequence is written $2n + n^2$, find the sum of the sequence to 5 terms.

Solution:

$T_n = 2n + n^2$

$$S_n = \Sigma T_n = \Sigma 2n + \Sigma n^2$$

$n = 5$

Hence, $S_5 = \frac{2(5)(6)}{2} + \frac{5(6)(11)}{6} \Rightarrow 30 + 55 = 85$

Test Your Understanding

Level - I

1. Find the sum to 20 terms of the series

$5 + 55 + 555 + \cdots$

2. The ratio between sum of n terms of two AP is $(3n + 8) : (7n + 15)$. Find the ratio between their sums to the 12th terms.

3. In a certain colony of cancerous cells, each cell divides into two every hour. What will be the total number of cells produced in a span of 10 hours, if we start with a single cell?

Level - II

4. A ball is dropped from a height of 81 m. Each time it strikes the ground it rebounds to two-thirds the distance through which it last fell.

a. Find the total distance that the ball travels when it reaches the top most position after the 4th bounce.

b. Find also the total distance that the ball travels before coming to rest.

5. A and B set out to meet each other from two places 165 km apart. A travels 15 km on the first day, 14 km on the second day, 13 km on the third day and so on. B travels 10 km on the first day, 12 km on the second day, 14 km on the third day and so on. When will they meet, if they started moving towards each other at the same time?

6. What is the sum to n terms of the series $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \cdots$?

Level - III

7. What is the sum to infinite terms of the series $4 + \frac{8}{3} + \frac{16}{9} + \ldots$?

8. Find the sum to n terms of the following.

(a) $1.2.4 + 2.3.7 + 3.4.10 + \cdots$ n terms

(b) $2 + 10 + 30 + 68 + \cdots$

9. Find the sum of the series $\frac{1}{8} + \frac{3}{8^2} + \frac{6}{8^3} + \frac{10}{8^4} + \ldots$

a. $\frac{8}{49}$ b. $\frac{512}{343}$ c. $\frac{16}{49}$ d. $\frac{64}{343}$ e. $\frac{24}{343}$

10. Find the sum of `n' terms of the series:

$2 \times 3 \times 4 + 4 \times 5 \times 6 + 6 \times 7 \times 8 + \ldots..$

Maxima and Minima 6

Introduction

This chapter deals with the concepts related to maxima and minima.

Learning Objectives

By the end of the chapter, you should be able to solve problems related to

- Finding the maximum or the minimum value of an expression, function, etc.

Maxima, Minima, Maximum & Minimum

Maxima and Minima of f(x):

Maxima and minima tends to be a topic that many students skip as they club it with differentiation/calculus. It need not be done, maxima and minima of all quadratic expressions can be found without use of calculus.

Let us look at a variety of problems that involve conventional as well as unconventional methods to solve the maxima and minima problems.

To emphasize, maxima-minima is not necessarily a topic from calculus, but, as you see it here, it is possible to solve such questions by algebraic methods.

Example 1:

What is the minimum possible value of the expression x^2+6x+4?

Solution:

The expression can be written

$x^2 + 6x + 9 - 9 + 4 = (x + 3)^2 - 5$.

The smallest value of such an expression would be (0) - 5 = -5.

Example 2:

What is the maximum possible value of the expression $6 - 4x - x^2$ and at what value of x is the value of the expression maximum?

Solution:

This can be written $10-(x+2)^2$.

The maximum value of this expression is 10.

The value of x = -2.

Example 3:

What is the maximum possible value of the function $f(x) = \frac{2x^2+3x+4}{x^2+x+3}$?

Solution:

Let $\frac{2x^2+3x+4}{x^2+x+3} = m$

Then $x^2(m - 2) + x(m - 3) + (3m - 4) = 0$.

Since x is real, we have $(m - 3)^2 - 4(3m - 4)(m - 2) \geq 0$ or $11m^2 - 34m + 23 \leq 0$.

$$\therefore \frac{34-\sqrt{34^2-44\times 23}}{22} \le m \le \frac{34+\sqrt{(34)^2-(44)(23)}}{22}$$

$$1 \le m \le \frac{23}{11}$$

We get the maximum and the minimum possible values of expression from the inequality above. Solving the problem with differentiation may not be a good method to do with.

Given sum, the maximum product OR given product, the minimum sum:

The following discussion is valid only for positive variables. Infact this property can be used to identify in which problems the following method should be used.

For positive variables, if the sum of the variables is a constant, the product of the variables will have the maximum value when all the variables are equal.

Thus if a + b + c = 15 and a, b, c are positive, then the product a × b × c will be maximum when
a = b = c. Thus each will be equal to 5 and the maximum product will be 125.

However if the maximum value of a × (b + 1) × (c + 2) is asked (given that a + b + c = 15 and a, b, c are positive), it would not be 5 × 6 × 7 = 210.

Here since the product of a, (b + 1), and (c + 2) is being asked, we need to consider the sum of a,
(b + 1) and (c + 2) and also a = (b + 1) = (c + 2).

Since a + b + c = 15, a + (b + 1) + (c + 2) = 18 and for maximum product of a × (b + 1) × (c + 2), a = (b + 1) =
(c + 2) = 6. Thus maximum product = 6^3 = 216.

The converse of the above is also true i.e.

For positive variables, if the product of the variables is a constant, the sum of the variables will have the minimum value when all the variables are equal.

If p, q, r are positive variables, what is the least value of $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}$?

Since $\frac{p}{q}\times\frac{q}{r}\times\frac{r}{p}=1$, $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}$ will take the least value when $\frac{p}{q}=\frac{q}{r}=\frac{r}{p}=1$ and the least value will be 3 i.e. $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}\ge 3$

Similarly for positive x, $x+\frac{1}{x}\ge 2$

If $ab^2c^3 = 27 \times 2^8$, find least value of a + b + c (given that a, b and c are positive).

To find least value of a + b + c, we need to know the product of a, b and c.

Substituting a = x, b = 2y and c = 3z, we have $x \times 4y^2 \times 27z^3 = 27 \times 2^8$ and we have to calculate the least value of x + 2y + 3z.

Thus $ab^2c^3 = 27 \times 2^8 \Rightarrow x \times y^2 \times z^3 = 2^6$.

Since we have to find least value of x + y + y + z + z + z and we know that

$x \times y \times y \times z \times z \times z = 2^6$, the least value will be when all are equal to 2. Thus least sum will be 12.

The above could also have been done with introducing new variables as:

$$a\times\frac{b}{2}\times\frac{b}{2}\times\frac{c}{3}\times\frac{c}{3}\times\frac{c}{3}=\frac{ab^2c^3}{4\times 27}=\frac{2^8\times 27}{4\times 27}=2^6$$

We have to find least value of a + b + c i.e. of $a+\frac{b}{2}+\frac{b}{2}+\frac{c}{3}+\frac{c}{3}+\frac{c}{3}$.

Hence all these are equal to 2 and least sum = 12.

Example 4:

If a + b +c + d = 1, then what is the maximum value of (1 + a) (1 + b) (1 + c) (1 + d)? Given a, b, c, d are all positive real numbers.

Solution:

We are given that: a + b +c + d = 1

Adding 4 on each side we get:

a + b +c + d + 4 = 1 + 4 = 5

Or

(a + 1) + (b + 1) + (c + 1) + (d + 1) = 5

We have four terms whose sum is constant; their product will be maximum if all of these four terms are equal:

i.e. if (a + 1) = (b + 1) = (c + 1) = (d + 1) = $\frac{5}{4}$ and the maximum value of the expression is $\left(\frac{5}{4}\right)^4$

Example 5:

If x + y + z = 6, then find the minimum value of the expression

$$\frac{1}{(x-1)(y-1)}+\frac{1}{(y-1)(z-1)}+\frac{1}{(z-1)(x-1)}.$$

(\`x', \`y' and \`z' are positive real numbers)

Solution:

$$\frac{1}{(x-1)(y-1)}+\frac{1}{(y-1)(z-1)}+\frac{1}{(z-1)(x-1)} = \frac{(z-1)+(x-1)+(y-1)}{(x-1)(y-1)(z-1)}$$

$$= \frac{(x+y+z)-3}{(x-1)(y-1)(z-1)} = \frac{3}{(x-1)(y-1)(z-1)} \text{ ... (i)}$$

The maximum value of $(x-1)(y-1)(z-1)$ can be found out:

$x+y+z=6$

$\Rightarrow (x-1)+(y-1)+(z-1) = 6 - 3 = 3$

$\Rightarrow$ The maximum value of $(x-1)(y-1)(z-1)$ will be = $\left(\frac{3}{3}\right)^3$ = 1

So that

$$\frac{1}{(x-1)(y-1)}+\frac{1}{(y-1)(z-1)}+\frac{1}{(z-1)(x-1)}$$

$= \frac{3}{(x-1)(y-1)(z-1)}$ has a minimum value of $\frac{3}{1}$ = 3 only.

NOTE: The Sum Product relations originate from the property of Arithmetic Mean (AM) and the Geometric Mean (GM) which is:

AM ≥ GM.

The above property holds for any number of positive quantities. When all the quantities involved are equal, we get **AM = GM.**

Please note that, **AM ≥ GM** is applicable for positive quantities only. You can not apply this property where negative quantities are involved e.g. consider the following case:

Lets take three numbers : -4, 8, -16

GM = $[(-4)(8)(-16)]^{1/3} = 8$

AM = $\frac{(-4+8-16)}{3} = -4$

Clearly, **AM < GM**

So, AM ≥ GM does not hold when any of the quantities is negative.

Example 6:

Find the minimum value of the expression:

$(x + 0)(x + 1)(x + 2)(x + 3)$

Solution:

Observe this carefully:

$(x + 0)(x + 3) = x^2 + 3x$ and $(x + 1)(x + 2) = x^2 + 3x + 2$

Put $x^2 + 3x = t$

So that the expression becomes: $t(t + 2) = t^2 + 2t = t^2 + 2t + 1 - 1 = (t + 1)^2 - 1$

The minimum value of this expression occurs when $t + 1 = 0$ i.e. when $t = -1$

And the minimum value is $(-1) \times (-1 + 2) = -1$.

(Note: We continued to work with variable "t" only as there was no need to assign -1 to $x^2 + 3x$ in order to find the minimum.)

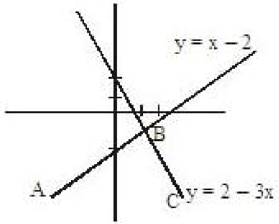
Min or Max of f(x), where f(x) = min or max of (g(x), h(x),...)

$y = \min(2 - 3x, x - 2)$. Find maximum value of y.

Now, $y = 2 - 3x$ or $y = x - 2$ depending on which of $(2 - 3x)$ or $(x - 2)$ is lesser for any particular value of x.

Graphical solution:

Plotting the graph of $y = 2 - 3x$ and $y = x - 2$ in the same Cartesian plane, we will have the following figure:



For $x = -1$, $2 - 3x = 5$ and $x - 2 = -3$.

Since $-3 < 5$, y will assume the value -3. On the graph, for $x = -1$, we have two points on the two lines. The vertical distance represents the value of $2 - 3x$ and $x - 2$ and y will assume the lower of them (the minimum). Thus for $x < 1$, y will assume values along line $x - 2$ i.e. the line segment AB.

For $x > 1$, $2 - 3x$ is lower than $x - 2$ and hence y will assume values along this line i.e. line segment BC. For $x = 1$, both $2 - 3x$ and $x - 2$ will have value equal to -1 and y will also be -1.

Thus graph of y will be the ABC. This graph has the maximum value = -1 i.e. the y coordinate of B. Coordinates of B can be easily obtained without drawing the graph by just equating $2 - 3x = x - 2$ i.e. $x = 1$. Substitute this value of x in either $2 - 3x$ or $x - 2$ (both are equal).

Tabular method:

The following table captures the value of y for different values of x.

x =	2 – 3x	x – 2	y
–2	8	–4	–4
–1	5	–3	–3
0	2	–2	–2
1	–1	–1	–1
2	–4	0	–4
3	–7	1	–7
4	–10	2	–10

Observations:

2 - 3x decreases as x increases. This is because co-efficient of x is negative.

x - 2 increases as x increases. This is because co-efficient of x is positive.

For lower values of x, 2 - 3x is less than x - 2. But as 2 - 3x decreases and x - 2 increases after a certain x (1 in this case) x - 2 becomes lesser than 2 - 3x.

Since y take the lower value of 2 - 3x and x - 2 for any particular x, y assumes values of 2 - 3x initially and is increasing.

But after x = 1 (where 2 - 3x = x - 2), y assumes values of x - 2 and starts decreasing.

Thus y assumes maximum value at that value of x when 2 - 3x = x - 2 i.e. at x = 1.

Substituting this value of x, we get maximum value of y as - 1.

Example:

$f(x) = \max(3x + 4, 4x + 5)$ for $-5 \le x \le 10$.

Find the largest and the smallest value of f(x).

This example is slightly different from the earlier one as in this example 3x + 4 and 4x + 5, both have the co-efficient of x as positive and hence both will continuously increase. Thus as x increases, irrespective of which expression (3x + 4 or 4x + 5) f(x) takes, one thing is certain that f(x) will also increase. Thus f(x) will have the least value at the least x and the maximum value at the highest value of x.

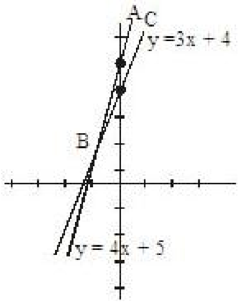
Minimum value of f(x) will be when x = -5.

$f(-5) = \max(-15 + 4, -20 + 5) = -11$

Maximum value of f(x) will be when

$x = 10$. $f(10) = \max(30 + 4, 40 + 5) = 45$.

The graph of the above function will look like:



The graph of f(x) will be the bold line in the above graph (Line segment ABC)

Solve this question from CAT `03:

Let $g(x) = \max(5 - x, x + 2)$

The smallest possible value of g(x) is

1. 4.0 2. 4.5

3. 1.5 4. None of these

Answer : Choice (4) 7 / 2

Test Your Understanding

Level - I

1. If a and b are both positive, then the minimum value of $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)$ is

a. 1 b. 2 c. 3 d. 4 e. 5

2. What is the minimum value of the expression $\frac{(p^2+p+1)(q^2+q+1)(r^2+r+1)}{pqr}$, where p, q and r are all positive?

3. If a, b, c and d are four positive numbers such that their sum is 4, then what is the maximum value of
$(1 + a)(1 + b)(1 + c)(1 + d)$?

4. What is the minimum value for a function $4x^2 - 6x + 1$?

5. What is the maximum value of $\left(\frac{1}{(x^2-6x+2)}\right)$?

6. What are the maximum and the minimum values of the expression $\frac{(x^2+14x+9)}{(x^2+2x+3)}$?

7. Given that an equilateral triangle, a square and a circle have the same perimeter, which figure would have the largest area?

8. $f(x) = \min(x^2 - 2, x - 2)$. What is the maximum value of f(x) in the interval [-1, 1]?

Level - II

9. A square piece of cardboard of side 20 units is taken and four equal small squares are removed from the corners. The sides are then turned up to make a cuboid of maximum volume. What would be the dimensions of such a cuboid?

10. $f(x) = \max(x + 1, 3, 4 - 2x)$. What is the minimum value of f(x)? In this problem if $f(x) = \max(x + 1, 1, 4 - 2x)$, what would the answer have been?

Practice Exercises

Introduction

There are 8 practice exercises out of which 4 are of level-1, 3 are of level 2 and 1 is of level 3 to strengthen your fundamentals. While solving the exercises make sure that each and every concept is understood properly.

Practice Exercise 1 - Level 1

1. Find the value of x in the equation $\frac{1}{x-1}+\frac{1}{x-2}=\frac{3}{x-3}$. [`x' is a real number]

a. $\pm\sqrt{2}$ b. $\frac{1}{2}$ c. $\pm\sqrt{3}$ d. $\frac{1}{3}$ e. $\frac{1}{4}$

2. Six years ago Mohan was 3 times as old as Shyam. At present Mohan is 1.5 times as old as Shyam. Find the present age of Mohan?

a. 12 years b. 11 years c. 8 years d. 48 years e. 60 years

3. Find the value of y which satisfies the pair of equations

$2x + 4y = 6$ and $3x + 15y = 25$

a. $\frac{4}{3}$ b. $\frac{4}{5}$ c. $\frac{5}{6}$ d. $\frac{6}{5}$ e. $\frac{16}{9}$

4. If $x^2+\frac{1}{x^2}=79$, then find the value of $x+\frac{1}{x}$. [`x' is a real number]

a. 7 b. 8 c. 11 d. ± 9 e. None of these

5. If $x^2+\frac{1}{x^2}=10$, then find the value of $x^4+\frac{1}{x^4}$. [`x' is a real number]

a. 100 b. 50 c. 49 d. 98 e. 102

6. If (x - 5) is a factor of $2x^2 + 2px - 2p = 0$, then find the value of p.

a. -4 b. $-\frac{25}{4}$ c. $\frac{25}{4}$ d. 4 e. 5

7. If $x^4 + 2x^3 - 3x^2 + x - 1$ is divided by x - 2, then find the remainder.

a. 12 b. 14 c. 16 d. 18 e. 21

8. Find the sum of the series S, where

S = 101 + 104 + 107 + ... + 161.

a. 2751 b. 2300 c. 2851 d. 2900 e. 2500

9. Find the value of 4 log 2 + log 6

a. log 60 b. log 90 c. log 84 d. log 96 e. log 48

10. Find the value of

$\log_{10}5 + \log_{10}4 + \log_{10}30 - \log_{10}6$.

a. 1 b. 3 c. 2 d. 2.5 e. None of these

11. If (x + p) is the HCF of $(x^2 + bx + a)$ and$(x^2 + cx + d)$, then find the value of p.

a. $\frac{d-a}{c-b}$ b. $\frac{b-c}{c-d}$ c. $\frac{b+c}{c+d}$ d. $\frac{d+a}{b+c}$ e. $\frac{ab}{cd}$

12. Find the LCM of $15x^2y^3 (x^2 - y^2)$ and $25x^4y (x - y)$.

a. $60x^4y^3 (x-y)$ b. $75 x^4y^3(x^2 - y^2)$ c. $240x^4y^4$

d. $375 x^2y^2 (x^2 - y^2)$ e. $150 x^4y^4 (x - y)$

13. Find the HCF of $2x^3 + x^2 - 3x$ and $x^3 - x$.

a. $x(x - 1)$ b. $x(x + 1)$ c. x^2 d. $x^2 + 1$ e. $x (x^2 - 1)$

14. If $ab + bc + ca = 40$ and $a + b + c = 10$, then find the value of $a^2 + b^2 + c^2$.

a. 10 b. 40 c. 30 d. 20 e. 50

15. In $2x + 3y = 8$ and $5x + Ky = 3$, find the value of K so that the given system of equations has infinite number of solutions.

a. $\frac{2}{15}$ or $\frac{8}{9}$ b. 15 or 9 c. 5 or 3 d. $\frac{15}{2}$ or $\frac{9}{8}$ e. Data Inconsistent

16. The sum as well as the product of the roots of a quadratic equation is 8, then the quadratic equation is

a. $x^2 + 4 = 0$ b. $x^2 - 4 = 0$ c. $x^2 + 4x + 4 = 0$ d. $x^2 - 8x + 8 = 0$ e. $x^2 - 4x - 4 = 0$

17. If $2^{x+5} = 2^{x+3} + 6$, then find the value of x

a. -2 b. -3 c. 1 d. 2 e. - 1

18. If a, b, c and d are in AP where common difference = 11 and a = 2, then find the arithmetic mean of a, b, c and d.

[Assume $d > c > b > a$]

a. 15.4 b. 16.2 c. 18.5 d. 13.1 e. None of these

19. For what real value of c will the equation $9x^2 - 48x + c = 0$ have equal roots?

a. 24 b. 44 c. 54 d. 64 e. 132

20. Find the sum of the roots of the equation $\sqrt{5}y^2 - 4\sqrt{5}y - 10 = 0$ [`y' is a real number].

a. $\frac{\sqrt{5}}{2}$ b. 4 c. $\frac{4}{5}$ d. $\frac{3}{10}$ e. $4\sqrt{5}$

21. Find the mean proportional of 4, 16 and 64.

a. 42 b. 24 c. 20 d. 16 e. 32

22. If there is a polynomial f(x) such that f(-3) = 0, then which of the following is a factor of
f (x)?

a. $x - 3$ b. $x \pm 3$ c. $x + 3$ d. $x \pm \sqrt{3}$ e. $x + \sqrt{3}$

23. Which of the following equation is equivalent to the quadratic equation $x^2 - 5x + 6 = 0$?

a. $(x + 3) (x - 2) = 0$ b. $(x - 3) (x - 2) = 0$ c. $(x + 2) (x - 3) = 0$

d. $(x + 2) (x + 3) = 0$ e. None of these

24. If the equation $2x^2 + 14x - 15 = 0$ is divided by $(x - 4)$, then find the remainder

a. 65 b. 0 c. 73 d. 45 e. 4

25. Find the sum of the series 1 + 4 + 9 + 16 + 25 + ...+ 100.

a. 295 b. 385 c. 425 d. 625 e. 525

26. If roots of the equation $x^2 - 5x + 1 = 0$ are a and b, then find the equation whose roots are $(3a + 1)$ and $(3b + 1)$.[x, a, b are real numbers].

a. $x^2 - 17x + 25 = 0$ b. $x^2 + 17x - 25 = 0$ c. $x^2 + 25x + 17 = 0$

d. $x^2 - 17x - 25 = 0$ e. None of these

27. The squares of two consecutive positive integers differ by 1987. What is the sum of these two integers?

a. 1986 b. 3974 c. 512 d. 2648 e. None of these

28. How many integral solutions exist for the equation 5x - y -120 = 0, such that the values that `x' assumes have opposite signs as compared to the corresponding values of y?

a. 22 b. 23 c. 24 d. 25 e. 21

29. If |x - 2| = 9 and |y + 9| = 8, then find the minimum possible value of xy.

[where, x and y are real numbers]

a. -11 b. -207 c. -187 d. -154 e. None of these

30. If $g(x) = \frac{x^2+1}{x}$ and $f(x) = x^{-1}$, then which of the following is true? [where `x' is a real number.]

a. $f(g(x)) = \frac{1}{g(f(x))}$ b. $\frac{f(g(x))}{g(f(x))} = 1$

c. $f(g(x)) - g(f(x)) = 1$ d. f(g(x) + g(f(x) = 1 e. None of these

31. If X * Y = X² - Y², X $ Y = (X - Y)² and X @ Y = X + Y, then find the value of $\left(m * \frac{1}{m}\right) @ \left(m \$ \frac{1}{m}\right)$.

a. $2(m^2 - 1)$ b. -4 c. 2 d. $\frac{2}{m^2}$ e. $2(1 - m^2)$

32. Given that

$f(x) = \frac{1}{x+4}$; $g(x) = \sin[\log(x-3)]$. Find the value of f [g(x)] at x = 6.

a. $\frac{1}{\sin(\log 3)+4}$ b. $\sin\left\{\log\left[\left(\frac{1}{4}\right)-3\right]\right\}$ c. sin(log3) + 4

d. $\sin\left\{\log\frac{1}{\sin(\log 3)}\right\}$ e. $\sin\left(\frac{1}{\log 3+4}\right)$

33. If $f(x, y, z) = \frac{x+y+z}{3}$, then

a. $f(x, y, z) \geq \frac{|x|+|y|+|z|}{3}$ b. $f(x, y, z) \geq \max(x, y, z)$

c. $|f(x, y, z)| \leq \max(x, y, z)$ d. $|f(x, y, z)| \leq \frac{|x|+|y|+|z|}{3}$ e. None of these

34. If x = 0.75, then what is the value of the expression $(1 + x + x^2) + \frac{x^3}{(1-x)}$?

a. 0.25 b. 4 c. 1.75 d. 1 e. 1.5

35. If $x = 3+\sqrt{3}$, then what is the value of $x^2 + \frac{9}{x^2}$?

a. $\left(15+\frac{3\sqrt{3}}{2}\right)$ b. $\left(18+\frac{3\sqrt{3}}{2}\right)$ c. $\left(27+\frac{3\sqrt{3}}{2}\right)$ d. $\left(18+\frac{9\sqrt{3}}{2}\right)$ e. $\left(15+\frac{9\sqrt{3}}{2}\right)$

36. Find the sum of all the possible values of K for which the equation $2(K - 1)x^2 + Kx + (K + 3) = 0$ has equal roots (K ≠ 1).

a. $\frac{-16}{7}$ b. -2 c. $\frac{-11}{5}$ d. $\frac{-24}{7}$ e. $\frac{-13}{7}$

$f(x) = \frac{1}{x+4}$; $g(x) = \sin[\log(x-3)]$. Find the value of f [g(x)] at x = 6.

a. $\frac{1}{\sin(\log 3)+4}$ b. $\sin\left\{\log\left[\left(\frac{1}{4}\right)-3\right]\right\}$ c. sin(log3) + 4

d. $\sin\left\{\log\frac{1}{\sin(\log 3)}\right\}$ e. $\sin\left(\frac{1}{\log 3+4}\right)$

33. If $f(x, y, z) = \frac{x+y+z}{3}$, then

a. $f(x, y, z) \geq \frac{|x|+|y|+|z|}{3}$ b. $f(x, y, z) \geq \max(x, y, z)$

c. $|f(x, y, z)| \leq \max(x, y, z)$ d. $|f(x, y, z)| \leq \frac{|x|+|y|+|z|}{3}$ e. None of these

34. If x = 0.75, then what is the value of the expression $(1 + x + x^2) + \frac{x^3}{(1-x)}$?

a. 0.25 b. 4 c. 1.75 d. 1 e. 1.5

35. If $x = 3+\sqrt{3}$, then what is the value of $x^2 + \frac{9}{x^2}$?

a. $\left(15+\frac{3\sqrt{3}}{2}\right)$ b. $\left(18+\frac{3\sqrt{3}}{2}\right)$ c. $\left(27+\frac{3\sqrt{3}}{2}\right)$ d. $\left(18+\frac{9\sqrt{3}}{2}\right)$ e. $\left(15+\frac{9\sqrt{3}}{2}\right)$

36. Find the sum of all the possible values of K for which the equation $2(K - 1)x^2 + Kx + (K + 3) = 0$ has equal roots ($K \neq 1$).

a. $\frac{-16}{7}$ b. -2 c. $\frac{-11}{5}$ d. $\frac{-24}{7}$ e. $\frac{-13}{7}$

37. It is given that $\frac{4p+9q}{p} = \frac{5q}{p-q}$ and p and q are both positive. Calculate the ratio of p to q.

a. 3 : 2 b. 9 : 4 c. 4 : 9 d. 2 : 3 e. 1 : 2

38. There are three statements. Find out which statement(s) is/are true.

Statement I : a = b if $a^x = b^x$

Statement II : x = y if $a^x = a^y$, where a, b, x and y are all real numbers.

Statement III : x = 2y if $a^x = a^{2y}$, where a, b, x any y are all real numbers.

a. I b. II c. II and III d. I and II

e. Neither of the given statements is true

39. The possible values of the coefficient a for which $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have at least one common root are

a. a = 1 and 2 b. a = 1 and -2 c. a = -1 and 2 d. a = -1 and -2 e. None of these

40. If $-7(3^{x+1})+5^{x+2} = -(3^{x+4})+5^{x+3}$, then find the value of x.

a. -1 b. 0 c. 3 d. -5 e. 2

Practice Exercise 2 - Level 1

1. If p + q = 1 and the ordered pair (p, q) satisfies 3x + 2y = 1, then it also satisfies

a. 3x + 3y = 3 b. 5x + 4y = 4 c. 5x + 5y = 4 d. Both (a) and (b) e. Both (b) and (c)

2. If yz : zx : xy = 1 : 2 : 3, then find the value of $\frac{x}{yz} : \frac{y}{xz}$.

a. 1 : 4 b. 3 : 1 c. $\sqrt{2}:1$ d. 1 : $\sqrt{2}$ e. 4 : 1

3. If the average of m numbers is a, and on adding x to the m numbers, the average of the (1 + m) numbers is b, then find the value of x.

a. m(b - a) + b b. m(b + a) + a c. m(a - b) + a d. m (a - b) + b e. m (a + b) + b

4. Given that a < b < c and a, b, and c are all positive, consider the following inequalities:

I. $\frac{ab}{c} > b + a$

II. a - b < b - c

III. $\frac{b}{a} > \frac{c}{b}$

Which of the following inequalities is always true?

a. I Only b. II Only c. I and III d. II and III e. None of these

5. The roots of the equation $x^2(2a^2 + 2b^2) + x(2a + 2b) + 1 = 0$ (a and b are real and distinct) are

a. real and unequal b. real and equal c. imaginary

d. real and irrational e. None of these

6. The roots of the equation $12x^2 + mx + 5 = 0$ will be in the ratio 3 : 2, if m equals

a. $\frac{1}{12}$ b. $\frac{5}{\sqrt{10}} \times \sqrt{12}$ c. $\frac{5}{12} \times \sqrt{10}$ d. $\pm 5\sqrt{10}$ e. $\frac{5}{12}$

7. If $x^3 - ax^2 + bx + 10$ is perfectly divisible by (x + 5) and $x^4 + x^3 + bx^2 - ax + 42$ is perfectly divisible by (x - 3), then find the value of (a - 3b).

a. $\frac{450}{7}$ b. 50 c. 100 d. 60 e. $\frac{75}{7}$

8. If x = 5 and y = z for x, y and z belong to the set of real numbers, then find the minimum possible value of $x^2 + y^2 + z^2 - (xy + yz + zx)$.

a. 0 b. 25 c. 50 d. - 5 e. - 10

9. If $y = 3^{x-1} + 3^{-x-1}$, where x is real, then the least value of y is

a. 2 b. $\frac{2}{3}$ c. 6 d. $\frac{1}{3}$ e. $\frac{2}{5}$

10. If $\frac{a}{b} < 1$ and $-1 < a < 0$, then which of the following is always true?

a. a < b < 0 b. b > a c. b < a for b < 0 d. a > b for all b e. None of these

11. If the value of $\log_{10} 2.5$ is n, then the value of $\log_{10} 2$ is

a. $\frac{4n}{5}$ b. $n - \frac{n}{5}$ c. $\frac{(1-n)}{2}$ d. $\frac{(3-n)}{2}$ e. $\frac{n-1}{2}$

12. A man born in 1900s realised that in 1980 his age was the square root of the year of his birth. When was he born?

a. 1929 b. 1949 c. 1936 d. 1946 e. 1940

13. If p + q = 2 and the ordered pair (p, q) satisfies 3p + 2q = 1, then the values of p and q are respectively

a. $\frac{1}{2}$ and $\frac{3}{2}$ b. 0 and $\frac{1}{4}$ c. -3 and 5 d. 5 and -3 e. $\frac{-3}{2}$ and $\frac{1}{4}$

14. If $a! = \frac{1}{2}b!$, then what is the value of a?

a. 2 b. 1 c. Zero

d. 3 e. Cannot be determined

15. If the equation 4x - 8 + a = bx - 1 has an integer solution (a, b) Î R, then a and b could take which of the following values?

a. a = 3, b = 1 b. a = 2, b = 4 c. a = 4, b = 2 d. a = 6, b = 2 e. a = 4, b = 1

16. |x| = y + 5 is equivalent to

a. x + y + 5 = 0 b. $\begin{matrix} x + y - 5 = 0; & \text{for } x \geq 0 \\ \text{or} & \\ x - y - 5 = 0; & \text{for } x < 0 \end{matrix}$

c. x - y - 5 = 0 and x > 0 d. $\begin{matrix} x - y - 5 = 0; & \text{for } x \geq 0 \\ \text{or} & \\ x + y + 5 = 0; & \text{for } x < 0 \end{matrix}$

e. None of these

17. If 3x + 4y : x + 2y = 9 : 4, then 3x + 5y : 3x - y is equal to

a. 7 : 2 b. 5 : 1 c. 6 : 1 d. 5 : 2 e. 7 : 1

18. If a, b and c are real numbers such that a + b + c = 0, then find the value of $a^3 + b^3 + c^3$

a. 1 b. $ab^2 + bc^2 + ca^2$ c. 3abc d. 0 e. None of these

19. What is the maximum possible value of $\frac{x}{y}$ for which $(x - 2)^2 = 9$ and $(y - 3)^2 = 25$?

a. $\frac{1}{2}$ b. $\frac{5}{8}$ c. $-\frac{1}{8}$ d. $-\frac{5}{2}$ e. $-\frac{5}{8}$

20. Under the usual two-dimensional coordinate system, the equation $|x| + |y| = 1$, where x and y are real numbers, describes

a. A rhombus which is not a square

b. A parallelogram which is not a rhombus

c. A square whose sides are not parallel to the coordinate axes

d. A square with sides parallel to the coordinate axes

e. None of these

21. Two clubs, Y and Z, have together x members. Y has y members, and Z has z members. If it is known that some people belong to both the clubs, which expression gives the number of people who belong to both the clubs?

a. x + y - z b. y + z - x c. 2(y + z) - x d. 2x - (y + z) e. y + z - 2x

22. If $rs - \frac{9}{rs} = 8$, the possible value of `r' can be

a. $-\frac{1}{s}$ b. $\frac{9}{s}$ c. $-\frac{6}{s}$ d. Both (a) and (c) e. Both (a) and (b)

23. If m and n are any positive integers then

a. $m^{100} + n^{100} > (m + n)^{100}$

b. $m^{100} + n^{100} \geq (m + n)^{100}$ and the equality can hold for some choices of m and n

c. $m^{100} + n^{100} < (m + n)^{100}$

d. $m^{100} + n^{100} \leq (m + n)^{100}$ and the equality can hold for some choices of m and n

e. None of these

24. If X + Y + Z = 25, (X, Y, Z > 0), then the maximum value of (X + 2)(Y+ 3)(Z) will be

a. 1500 b. 1000 c. $\frac{25^3}{27}$ d. 2000 e. 1200

25. For what value of A are the following equations consistent?

2x + 3y + 4 = 0, 3x + 4y + 6 = 0, 4x + 5y - A = 0

a. 8 b. -6 c. 7 d. -7 e. -8

26. Find the maximum and the minimum values of the function $\frac{X^2-X+1}{X^2+X+1}$ for real values of X.

a. 3 and -3 b. $\frac{1}{3}$ and -$\frac{1}{3}$ c. 3 and $\frac{1}{3}$ d. $\frac{1}{2}$ and -2 e. 2 and $\frac{-1}{2}$

27. Find the sum of the numbers between 100 and 200, the sum of whose digits is divisible by 9.

a. 1665 b. 1674 c. 1683 d. 1692 e. 1695

28. The set of values of m and n that does not satisfy < < $\frac{1}{8}$ is

a. (2, 17) b. (-2, -17) c. (3, 25) d. (-3, -26) e. (3, 28)

29. Let X < 0, 0 < Y < 1, Z > 1. Which of the following is true?

a. $(X^2 - Z^2)$ has to be positive b. $\frac{Z}{Y}$ can be less than 1 c. YZ > 1

d. $(Y^2 - Z^2) < 0$ e. None of these

30. What are the values of N such that $\frac{N}{(N+1)} > 1$?

a. N = 1 b. -1 < N < 0 c. N > - 1 d. N < -1 e. None of these

31. Let C be a positive integer such that C + 7 is divisible by 5. The smallest positive integer n (> 2)

such that $C + n^2$ is divisible by 5 is

a. 4 b. 5 c. 3 d. 6 e. `n' does not exist

32. If one root is thrice the other root of the quadratic equation $ax^2 + bx + c = 0$, then

a. $3b^2 = 16ac$ b. $3c^2 = 8bc$ c. $c^2 = 9b$ d. $4a^2 = 3\ bc$ e. $3b^2 = 10\ ac$

33. The range of k for which the roots are imaginary for the equation $x^2 + x(k + 1) + 8 = 0$ is

a. $k < 4\sqrt{2} + 1$ b. $-(4\sqrt{2} + 1) \leq k \leq 4\sqrt{2} - 1$

c. $k > 4\sqrt{2} - 1$ d. $k < 2\sqrt{2}+1$

e. $-\left(4\sqrt{2}+1\right) < k < 4\sqrt{2}-1$

34. In the equation $2X^2 + aX + b = 0$, the sum of the roots is 5 and the product is 6. Then `a' and `b' have the respective values

a. 1 and 6 b. 10 and 12 c. -10 and 12 d. -10 and -12 e. None of these

29. Let $X < 0, 0 < Y < 1, Z > 1$. Which of the following is true?

a. $(X^2 - Z^2)$ has to be positive b. $\frac{Z}{Y}$ can be less than 1 c. $YZ > 1$

d. $(Y^2 - Z^2) < 0$ e. None of these

30. What are the values of N such that $\frac{N}{(N+1)} > 1$?

a. $N = 1$ b. $-1 < N < 0$ c. $N > -1$ d. $N < -1$ e. None of these

31. Let C be a positive integer such that C + 7 is divisible by 5. The smallest positive integer n (> 2)

such that $C + n^2$ is divisible by 5 is

a. 4 b. 5 c. 3 d. 6 e. `n' does not exist

32. If one root is thrice the other root of the quadratic equation $ax^2 + bx + c = 0$, then

a. $3b^2 = 16ac$ b. $3c^2 = 8bc$ c. $c^2 = 9b$ d. $4a^2 = 3bc$ e. $3b^2 = 10ac$

33. The range of k for which the roots are imaginary for the equation $x^2 + x(k + 1) + 8 = 0$ is

a. $k < 4\sqrt{2} + 1$ b. $-(4\sqrt{2} + 1) \le k \le 4\sqrt{2} - 1$

c. $k > 4\sqrt{2} - 1$ d. $k < 2\sqrt{2}+1$

e. $-\left(4\sqrt{2}+1\right) < k < 4\sqrt{2}-1$

34. In the equation $2X^2 + aX + b = 0$, the sum of the roots is 5 and the product is 6. Then `a' and `b' have the respective values

a. 1 and 6 b. 10 and 12 c. -10 and 12 d. -10 and -12 e. None of these

35. If one of the roots of the cubic expression $x^3 - ax^2 + 11x - 6$ is 3, then what are the other roots?

a. 6 and 2 b. -1 and 2 c. 1 and 2 d. -2 and -1 e. -2 and 6

36. How many solutions does $x^2 + 5|x| + 6 = 0$ have?

a. 4 b. 3 c. 2 d. 1 e. 0

37. A horse can eat 3a + 2b bags of corn in a week. In how many weeks will $6a^2 + 13ab + 6b^2$ bags be consumed by him?

a. a + 3b b. 3b + a c. 2a + 3b d. 3(a + b) e. 3a + b

38. The solution to the equations $6x - 3y = 1$ and $6x^2 + 9xy - 6y^2 = 0$ is

a. $x = \frac{2}{15}, y = -\frac{1}{15}$ b. $x = -\frac{2}{15}, y = -\frac{1}{15}$ c. $x = \frac{2}{15}, y = \frac{1}{15}$

d. $x = -\frac{2}{15}, y = \frac{1}{15}$ e. None of these

Practice Exercise 3 - Level 1

1. When $x^2 + 4xy + 4y^2$ takes a minimum possible value, then

a. x = -2y b. x = 2y c. 2x = y d. -2x = y e. None of these

2. A man finds that it costs Rs. (300 + 4N) per day to make N articles. If the selling price of each article is Rs. 7, what is the minimum possible number of articles to be produced per day if he has to make a profit?

a. 300 b. 150 c. 100 d. 101 e. 201

3. A man is $8X^2$-year-old and his son is X^3-year-old. When the man turns $5X^3$, his son becomes X^4-year-old. How old is the man now, if he is less than 100-year-old?

a. 32 b. 40 c. 54 d. 46 e. 50

4. The solution set of the inequality $x^2 - x - 12 < 0$ and $|x + 2| < 3$ is

a. $-5 < x < 1$ b. $-3 \le x \le 3$ c. $-3 < x < 1$ d. $-4 < x < 2$ e. None of these

5. At how many distinct points in the X-Y plane do the two curves

$y = x^5 + 2x^4 + 8x^3 + 17x^2 + 10x - 36$ and $y = x^5 + 2x^4 + 5x^3 - 4x^2 - 20x - 36$, intersect each other?

a. 0 b. 1 c. 2 d. 3 e. 4

6. What is the sum of the 11 terms of an AP whose 3rd term is 10 and the 9th term is 20?

a. 135 b. 165 c. 330 d. 150 e. 210

7. Which of the following statements is correct about the root (s) of the equation $x^2 - |x - 1| + 1 = 0$?

a. One of the roots lies between -1 and 0 and two others lie between 0 and 2.

b. One of the roots lies between -2 and 0 and another one lies between 0 and 1.

c. Exactly one root lies between -2 and 1.

d. Exactly one root lies between 0 and 2.

e. Exactly two roots lie between -3 and 3.

8. A torch burns fuel at the rate of 60 L per hour. A workman lights a torch every 10th minute after the first is lit. What should be the minimum amount of fuel that the first torch must have in order that it is still burning when the last torch is lit at the end of the 2nd hour after it?

a. 200 L b. 210 L c. 120 L d. 130 L e. 160 L

9. ABC is a right angled triangle. The coordinates of A, B, C are (0, 0), (-10, 10) and (10,10) respectively. How many points on/inside the triangle have integer coordinates?

a. 121 b. 169 c. 256 d. 196 e. 145

Directions for questions 10 and 11: Answer the questions based on the following information.

The following are certain functions that have been pre-defined.

H(a, b, c) = Greatest common divisor of a, b, c

L(a, b, c) = Least common multiple of a, b, c

A(a, b, c) = Average of a, b, c

Min(a, b, c) = Smallest value among a, b, c

Max(a, b, c) = Largest among a, b, c

10. Which of the following is true? (a, b, c are distinctly different positive numbers.)

a. H(a, b, c) × L(a, b, c) = abc b. H(a, b, c) > A(a, b, c)

c. H(a, b, c) > Min(a, b, c) d. H(a, b, c) < A(a, b, c) < L(a, b, c) e. None of these

11. If Max(a, b, c) = Min(a, b, c), then

a. A(a, b, c) = H(a, b, c) b. A(a, b, c) = L(a, b, c)

c. A(a, b, c) = Min(a, b, c) d. Both (a) and (c) e. (a), (b) and (c)

12. If f(x) = 3[x], g(x) = x^2 + 3, then find the value of f(g(x)) for x = 4.5.

a. 60 b. 63 c. 57 d. 69 e. 61

13. If h * k denotes the HCF of h and k, and h D k denotes the LCM of h and k, where h and k are positive integers, what is the value of [231 D (12 * 42)] * 49?

a. 6 b. 1 c. 7 d. 8 e. 9

14. If f(x) = $2x^2 + 3x + 4$ and g(x) = $5 - x^2$, then min[f(x)] - max[g(x)] is

a. 0 b. 5 c. $\frac{17}{8}$ d. $-\frac{15}{8}$ e. $-\frac{17}{8}$

15. The function f(x) is defined as the min(2x - 3, 3 - 2x) for all the values of x. The maximum possible value of f(x) is at which value of x?

a. $x = \frac{3}{2}$ b. x = 0 c. x = 1 d. x = 2 e. $x = \frac{5}{2}$

16. In question 15, what is the value of f(x) when it takes a maximum value?

a. $\frac{3}{2}$ b. 3 c. 4 d. 0 e. 2

17. F(x) = $3x^2 - 2x + 4$; G(x) = x + 1. Find F(G(F(1))).

a. 69 b. 100 c. 96 d. 120 e. 80

18. If log (x + 4) = 2 log (x + 2), then x is equal to

a. 0 b. -3 c. 1 d. 2 e. -2

19. It is given that $4x^2 + 2xy + 9y^2 = 109$ and $x^2 - y^2 + 2xy = 7$. Find the value of $|3x + 2y| + 6$, if x and y are real numbers.

a. 20 b. 19 c. 22 d. 16 e. 18

20. When expanded, $(1 + 2x^2 + x^4)^{100}$ will have

a. 10,201 terms b. 12,101 terms c. 201 terms d. 101 terms e. 301 terms

21. Find the sum of 2 + 22 + 222 + ··· + up to n terms.

a. $\frac{2}{9}\left[\frac{10(10^n-1)-n}{9}\right]$ b. $\frac{2}{9}\left[\frac{10}{9}(10^n-1)-n\right]$

c. $\frac{2}{9}\left[10(10^n-1)-n^2\right]$ d. $\frac{2}{81}\left[10(10^n-1)-n\right]$ e. None of these

22. The sum of all odd numbers between 2 and 1000, which are divisible by 3, is

a. 83667 b. 83500 c. 83433 d. 83543 e. 83694

23. If $1 . 3 . 3^2 . 3^3 . 3^4 \cdots 3^x = 9^5$, then find the value of x.

a. 3 b. 4 c. 5 d. 6 e. 7

24. A soldier was awarded for each wound inflicted upon in a battle. For the first wound he was awarded Re 1, for the second Rs. 2, for the third Rs. 4 and for the fourth Rs. 8 and

so on. He received in total Rs. 65, 535. Find the number of wounds he received.

a. 14 b. 15 c. 16 d. 13 e. 12

25. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 10)$, then x lies in the interval

a. (2, 9) b. (1, 2) c. (-2, -1) d. (-2, 9) e. None of these

26. If $f(x) = (a - x^n)^{1/n}$, where a, x and n are all distinct real numbers, then f[f(x)] is

a. a - x b. a c. a - x^n d. x e. None of these

27. If $f(x) = x^2$ and $f[g(x)] = g[f(x)]$, then which of the following can never be equal to g(x)?

a. $\sqrt{x}$ b. x^2 c. x d. 2x e. None of these

28. If α and β are roots of $x^2 + 7x + 12 = 0$, then the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

a. $x^2 - 50x + 49 = 0$ b. $x^2 + 50x - 49 = 0$ c. $x^2 - 10x + 3 = 0$

d. $x^2 - 10x + 4 = 0$ e. $x^2 - 10x + 9 = 0$

29. If a, b and c are all real numbers, then the minimum possible value of ab + bc + ca is

a. 0 b. $-\frac{1}{2}$ c. $-\frac{3}{2}$ d. $\frac{1}{2}$ e. $-\infty$

30. For which of the following values of `k' does the given pair of equations yield a real solution?

$x^2 + y^2 = 3$.

$y = kx + 2$.

a. $\frac{-2}{5}$ b. $\frac{1}{2}$ c. $\frac{-3}{7}$ d. $\frac{3}{5}$ e. $\frac{-4}{9}$

31. If $(x^2 - y^2) = 16$ and $xy = -15$, then find the value of (x + y), if (x + y) is a positive number.

a. 3 b. 5 c. 2 d. 1 e. 4

32. If $f(x) = 2x^3 - x + 2k$, and f(1) and f(2) are of opposite signs, then which of the following is necessarily true?

a. $-7 < k < 1$ b. $-5 < k < \frac{1}{2}$ c. $-7 < k < -\frac{1}{2}$ d. $-5 < k < \frac{3}{2}$ e. $-6 < k < \frac{-3}{2}$

33. If $f(x) = x^3 - x^2 + x - f(x)$, where x is a whole number less than or equal to 15. For how many x's, f(x) will
not be a whole number?

a. 7 b. 8 c. 5 d. 6 e. 4

34. The equation whose roots are three times the roots of the equation $2x^3 - 10x^2 + 13x + 7 = 0$, is

a. $6x^3 - 30x^2 + 39x + 21 = 0$ b. $2x^3 - 30x^2 + 117x + 189 = 0$

c. $2x^3 - 30x^2 + 117x + 63 = 0$ d. $2x^3 + 30x^2 - 117x - 93 = 0$

e. $6x^3 + 30x^2 + 39x - 21 = 0$

35. For how many integer values of x does $2[x]^2 \le 32$ and $3x + 2 \ge -1$ hold true?

a. 8 b. 7 c. 6 d. 5 e. 9

36. Consider the numbers $f(n) = n^3 + 2n$, where n is a positive integer. How many of the following statements are then true?

Statement I: f(n) is divisible by 3 for all odd integers n.

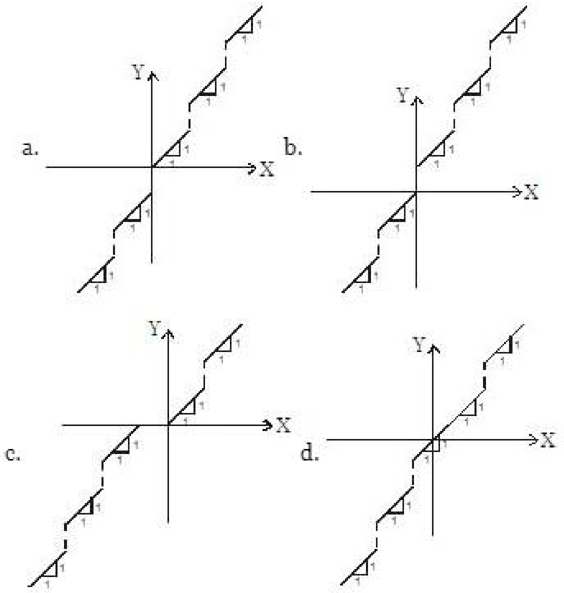
Statement II: f(n) is divisible by 3 for all n.

Statement III: f(n) is divisible by 6 for all even integers n.

a. 0 b. 1 c. 2

d. 3 e. Cannot be determined

37. Find the graph of $y = x + [x]$.



e. None of these

Direction for questions 38 to 40: In these questions the following functions are defined.

Min(a, b) = Least of a, b

Max(a, b) = Larger of a, b

G(a, b) = GCD of a, b

L(a, b) = LCM of a, b

P(a, b) = Product of a, b

38. The following expression has minimum value of expression L(1, P(3, Max(3, b)))

a. 9 when $b \leq 3$ b. $-\infty$ when $b > 3$ c. 0 when b = 0

d. 8 when $b \leq 3$ e. None of these

39. P(7, Max(3, Min(L(3, 5), P(17, 1))))

The value of the above expression is

a. 117 b. 105 c. 119 d. 109 e. 127

40. Max[Min{G(a, b), L(a, b)}, Min{L(a, b), P(a, b)}]

The above expression is equivalent to

a. G(a, b) b. L(a, b) c. P(a, b)

d. G (a, b) + L (a, b) e. Cannot be determined

Practice Exercise 4 - Level 1

1 If $y = f(x) = 3x + 9$, then $x = g(y)$, then, $g(y)$ is

a. $3y + 9$ b. $\frac{y-9}{3}$ c. $\frac{y}{3} - 9$ d. $3y - 9$ e. $3 - \frac{y}{9}$

2. If $x = \frac{5}{a+b}$ and $y = 5(a + b)$, and if $x, y > 0$ then $(x + y)$ is

a. Always less than 5 b. Always greater than or equal to 10

c. Always between 5 and 10 d. Always greater than or equal to 15

e. None of these

Direction for question 3: Answer the questions based on the following information.

$a \delta b = a^2 - b^2$

$a * b = a^2 + b^2$

$a \# b = |a + b|$

$a \Delta b = |a - b|$

3. The expression below reduces to

$\{(5 \ \delta \ 4) \ \Delta \ ((2 \ \Delta \ -1) \ * \ (2 \ \# \ -2))\}$

$\# \ \{(73 \ \Delta \ 37) \ \delta \ ((3 \ \# \ 3) \ * \ 0)\}$

a. 0 b. 72 c. 182 d. 36 e. 48

Direction for questions 4 and 5: Answer the questions based on the following information.

There are 4 positive integers p, q, r and s. All of them are two-digit numbers. Two of them are odd and two are even.

4. Which of the following is necessarily even?

a. $p + q + r - s$ b. pq c. $p + q - r$ d. $p + q + r$ e. None of these

5. Which of the following cannot be equal to zero?

a. $pq - 2s$ b. $p + q - 2r - 2s$ c. $p - q - r - s$ d. $pqr - pq$ e. $p - q + r - s$

6. If $f(x) = x^2$ and $g(x) = 2x$, where x is a whole number then, for how many values of x, $f(x) = g(x)$?

a. 1 b. 2 c. 3 d. 4 e. 5

7. Three times the first of three consecutive odd integers is 3 more than twice the third. What is the third integer?

a. 15 b. 9 c. 11 d. 5 e. 7

8. If $\dfrac{1}{x + \dfrac{1}{y + \dfrac{1}{z - \dfrac{1}{2}}}} = \dfrac{3}{4}$, then find $x + y + z$. [Given that $(x = y = z)$] and x is a real number.

a. 6 b. 4 c. 3 d. 1 e. 5

9. A function $f(x)$ is such that $f(x) + f(y) = f(xy)$. Which of the following could be $f(x)$?

a. a^x b. $\sqrt{x}$ c. x^2 d. $\log_a x$ e. x^3

10. Six numbers are in arithmetic progression such that their sum is 3. The first number is four times the third number. The fifth number is equal to

a. -15 b. -2 c. 9 d. -4 e. -7

11. The sum of two integers is 10 and the sum of their reciprocals is $\frac{5}{12}$. Then the larger of these integers is

a. 2 b. 4 c. 6 d. 8 e. 3

12. If $\log_{10} a + \log_{10} b = \log_{10}(a+b)$, then

a. a = b = 3 b. a = b = 1 c. $a = \frac{b^2}{1-b}$ d. $a = \frac{b}{b-1}$ e. a + b = 1

13. If $\log \frac{u+v}{2} = \frac{1}{2}$ (log u + log v), then

a. u = v b. u > v c. u < v d. $u^2 = v^2$ e. None of these

14. $f(x) = x^3 + kx^2 + hx + 6$ such that (x + 1) and (x - 2) are factors of f(x). Then (k, h) =

a. (3, -2) b. (-4, 1) c. (-4, -1) d. (4, -1) e. (-1, 4)

15. Which of the following could be the quadratic equation for which one root is $1\frac{1}{2}$ times the other root
and the difference between the roots is 1?

a. $x^2 + 3x + 3 = 0$ b. $x^2 + 4x + 3 = 0$ c. $x^2 - 5x + 6 = 0$ d. $x^2 + x - 6 = 0$ e. $x^2 - 3x - 3 = 0$

16. Two porters Vedalamani and Seetharamaiah were carrying suitcases. If one suitcase was transferred from Vedalamani to Seetharamaiah, Vedalamani would have half as many suitcases as Seetharamaiah; and if one suitcase was transferred from Seetharamaiah to Vedalamani, both would have an equal number of suitcases. The number of suitcases carried by Seetharamaiah was

a. 4 b. 5 c. 6 d. 7 e. 8

17. The number of solution(s) of the simultaneous equations given below

$V = 3\log_e K$ and $V = \log_e(3K)$ is / are

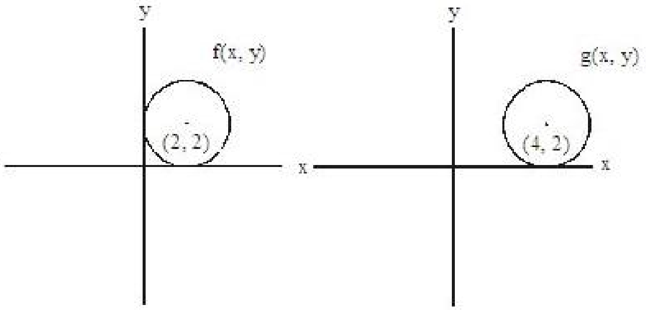
a. Three b. One c. Two d. Zero e. Four

18. Lata has same number of sisters as she has brothers, but her brother Shyam has twice as many sisters as he has brothers. How many children are there in the family?

a. 7 b. 6 c. 5 d. 4 e. 8

Direction for question 19: The graphs of two functions f(x, y) and g(x, y) are given. Choose the answers.

19.



Which of the following is true?

a. f(x, y) = g(x, y) b. f(x - 2, y) = g(x, y)

c. f(x, y + 2) = g(x + 2, y) d. f(x + 2, y) = g(x, y)

e. b or c

20. After plucking a certain number of fruits in a garden, a visitor has to walk back through 3 gates. At each gate he has to offer half the number of fruits in his possession to the guard and with a sense of gratitude the guard at each gate will return one fruit to him. When the visitor came out of the garden he possessed the same number of fruits as he had gathered from the tree. What was the number of fruits plucked by him?

a. 2 b. 18 c. 1000

d. 500 e. No unique number

21. Three consecutive positive even numbers are such that thrice the first number exceeds double the third by 2. Then, the third number is

a. 10 b. 14 c. 16 d. 12 e. 11

22. If a - b = 3 and $a^3 - b^3 = 117$, then a + b =

a. 5 b. 7 c. 9 d. 11 e. 8

23. How many integer solutions are there for the equation 3x + 6y = 7?

a. 2 b. 3 c. 1 d. 4 e. None of these

24. An aircraft flies at a speed of 300 km/hr when 100 passengers are sitting in it. For every extra passenger, the speed reduces by 1 km/hr. What is the number of passengers it should carry so that the product of speed and the extra passengers is maximum?

a. 150 b. 300 c. 250 d. 200 e. 240

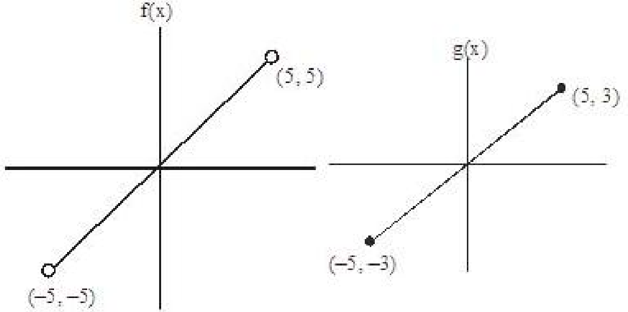
25. Out of two-thirds of the total number of basket ball matches, a team has won 17 matches and lost 3 of them. What is the maximum number of the remaining matches that the team can lose and still win more than three-fourths of the total number of matches, (given that no match can end in a tie)?

a. 4 b. 6 c. 5 d. 3 e. 7

Directions for questions 26 and 27: Answer the questions based on the information given below.

f(x) and g(x) are defined in the domain [-5, 5].

26.



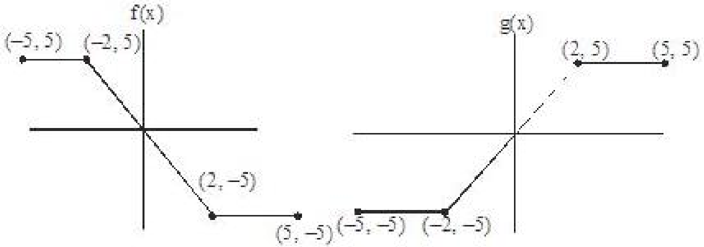
Which of the following is true?

a. f(x) > g(x) for $-5 \le x \le 5$ b. f(x) < g(x) for $-5 \le x \le 5$

c. f(x) > g(x) for $0 < x \le 5$ d. f(x) > g(x) for $-5 \le x \le 0$

e. f(x) < g(x) for $0 < x \le 5$

27.



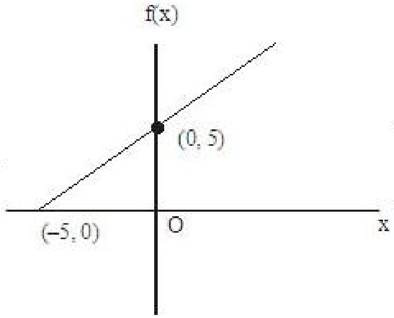
Which of the following is true for all values of x in the domain $-5 \le x \le 5$?

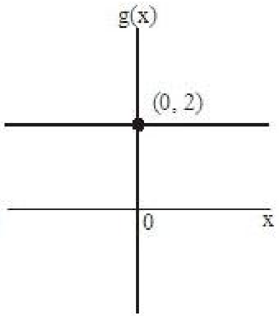
a. f(x) = -g(x) b. f(x) = g(-x) c. f(-x) = g(x) d. All of these e. None of these

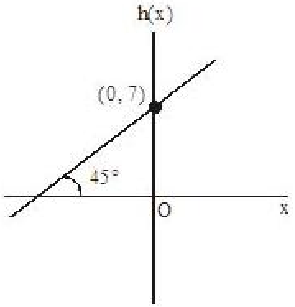
Directions for questions 28 and 29: Answer the questions based on the following information.

Find the relation among f(x), g(x) and h(x) in the graphs given below.

28.



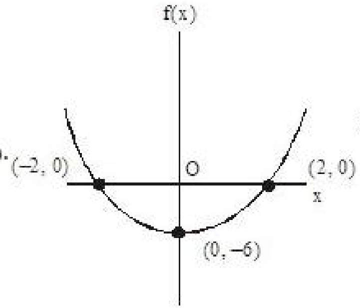


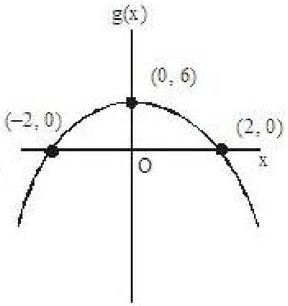


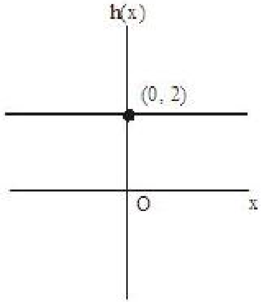
a. f(x) + g(x) = h(x) + 2 b. f(x) - g(x) = 2h(x) c. f(x) + g(x) = h(x)

d. f(x) = h(x) + g(x) e. f(x) + h(x) = g(x)

29.







a. 2f(x) + g(x) = h(x) b. f(x) + g(x) - h(x) = 2

c. f(x) + g(x) = h(x) - 2 d. 2f(x) - g(x) = h(x)

e. None of these

Practice Exercise 5 - Level 2

1. Find the sum of the series

$11 + 21 + 31 + 41 + \dots + 201$

a. 1950 b. 2020 c. 2420 d. 2300 e. 2120

2. Find the range of values of x which satisfy the inequality $2|x| + |3x| \geq 5$. [\`x' is a real number]

a. $x \geq 1$ b. $x \leq -1$ c. $-1 \leq x \leq 1$ d. x = 1 e. Both (a) and (b)

3. Find the range values of x which satisfy the inequality $x^2 - 5x - 14$ 0. [\`x' is a real number]

a. $x < 5$ b. $x \geq 5$ c. $-2 < x \leq -7$

d. $-2 \leq x \leq 7$ e. $x \leq -2$ and $x \geq 7$

4. The sum of the squares of three natural numbers which are in the ratio of 2 : 3 : 4 is 725. Find the largest number.

a. 24 b. 10 c. 15 d. 20 e. 25

5. A man is engaged in the condition that the day he works he will get Rs. 5 and the day he does not work he will have to pay a penalty of Rs. 7. After 20 days he got Rs. 52. Find the total number of days he worked?

a. 14 b. 16 c. 12 d. 13 e. 15

6. The sum of the squares of two numbers is 3341 and the difference of their squares is 891. Find the two numbers.

a. 46, 35 b. 45, 36 c. 11, 40 d. 36, 15 e. 25, 36

7. The area of a rectangle is 255 m^2. If its length is decreased by 1 m and its breadth is increased by 1 m, then it becomes a square. Find the perimeter of the square.

a. 45 m b. 50 m c. 55 m d. 64 m e. 32 m

8. A has twice as much money as B has. B has thrice as much money as C has. A, B and C all together have a sum of Rs. 100. How much money A has?

a. Rs.10 b. Rs. 20 c. Rs. 30 d. Rs. 60 e. Rs. 40

9. If $(x - 6)(4x + 3) = 0$ (\`x' is a real number), then find the value of 2x.

a. 12 or $-\frac{3}{2}$ b. 12 or 0 c. 6 or $-\frac{3}{4}$ d. 6 only e. $\frac{3}{2}$ only

10. The sum of the digits of a two-digit number is 7 and the digit at the ten's place is 25% less than the digit at the unit's place. Find the number.

a. 30 b. 45 c. 43 d. 34 e. 25

11. If one root of the equation $ax^2 + bx + c = 0$ is c, then find the other root.

a. $\frac{1}{b}$ b. $\frac{1}{c}$ c. $\frac{1}{a}$ d. $\frac{a}{c}$ e. 1

12. Find the equation whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

a. $x^2 + 4x + 2 = 0$ b. $x^2 - 6x + 3 = 0$ c. $x^2 + 6x - 3 = 0$ d. $x^2 - 4x + 2 = 0$ e. $x^2 - 4x - 2 = 0$

13. What must be subtracted from the numerator as well as denominator of the fraction $\frac{2}{x}$ such that it becomes equal to 40?

a. $\frac{16 - 3x}{5}$ b. $\frac{40x - 2}{39}$ c. $\frac{12x - 5}{3}$ d. $\frac{18 - 2x}{7}$ e. $\frac{20x - 1}{19}$

14. If 5 + 2 |x| = 12, find the value of x. [`x' is a real number.]

a. 3.5 b. -2.5 c. ±3 d. $\pm\frac{9}{2}$ e. $\pm\frac{7}{2}$

15. Find the sum of the series 4 + 8 + 16 + 32 + . . . till 10 terms.

a. $6\times 2^{10}-1$ b. $4\times 2^{10}+1$ c. 6×2^{10} d. $4(2^{10}-1)$ e. $2^{10}-1$

16. Find the HCF of x^2-6x+9 and x^2-5x+6.

a. (x - 3) b. (x - 2) c. $(x-3)^2$ d. (x - 3) (x - 2) e. 1

17. If one root of the equation $x^2 + kx - 8 = 0$ is square of the other root, then find the value of `k'. [where, `x' and `k' are real numbers.]

a. -4 b. 2 c. 4 d. 6 e. -2

18. If the equation $x^3-4x^2+bx+6=0$ has one of its roots as 1, then find the other roots.

[where, x and b are real numbers.]

a. $\frac{7\pm\sqrt{14}}{2}$ b. $\frac{3\pm\sqrt{15}}{2}$ c. $\frac{3\pm\sqrt{18}}{2}$ d. $\frac{3\pm\sqrt{33}}{2}$ e. None of these

19. How many pairs of natural numbers satisfy the condition that the sum of their reciprocals is $\frac{1}{12}$?

a. 16 b. 6 c. 8 d. 15 e. None of these

20. If the three real roots of equation

$$x^3+\left(\frac{1}{\sqrt{p}+\sqrt{q}}\right)x^2+\left(\frac{1}{\sqrt{p}-\sqrt{q}}\right)x-1=0$$

(where $p \neq q$)

are the second, fifth and the eighth terms of a geometric progression, then which of the following is true?

a. $q=0$ b. p + 2q = 0 c. q + 3p = 0 d. p + q = 0 e. None of these

21. Assume r, s and t be three distinct integers between 0 and 10. If $\frac{1}{r}+\frac{1}{s}-\frac{3}{t}=\frac{2}{5r}$, then find the number of distinct values of (r + s - t).

a. 1 b. 2 c. 3 d. 4 e. 5

22. What is the least integral value of `p' for which, the following inequality holds true for any real value of `x'?

$$x^2-(2p-2)x+\left[\left(\frac{3}{4}\right)p^2+\left(\frac{7}{3}\right)p-\frac{25}{2}\right]>0$$

a. 5 b. 6 c. 4 d. 3 e. 7

23. For how many integral values of `m' does the graph of the quadratic function $f(x) = x^2 + 3x + 2$ (where, x is a real number) and the line y = mx have no points of intersection?

a. 2 b. 3 c. 4 d. 5 e. None of these

24. If $\left(x^2-4\sqrt{3}x\right)^2=25\left(x-2\sqrt{3}\right)^2-36$, then how many real values of x satisfy the given equation?

a. 0 b. 1 c. 2

d. 4 e. Cannot be determined

25. For which of the following functions is $\frac{f(a)-f(b)}{a-b}$ constant for all the real numbers a and b, where $a \neq b$?

a. f(x) = 4x + 7 b. f(x) = x + x^2 c. f(x) = cosx d. $f(x) = \log_e x$ e. None of these

26. Given that $2a^2b + a(2b^2 + 1) + b = 135$ and $a + b = 15$. If a and b are the roots of a particular quadratic equation, then find this quadratic equation.

a. $a^2 - 15a + 15 = 0$ b. $a^2 + 15a + 4 = 0$ c. $a^2 - 9a + 15 = 0$

d. $a^2 - 15a + 4 = 0$ e. $a^2 + 15a - 15 = 0$

27. Find the sum of all the real values of `x' that satisfy the equation $243^{\log_{81} x} - 2x = 2^{\log_{16} x + 2} - 8$.

a. 16 b. 20 c. 12 d. 18 e. None of these.

28. If $a^x = b^y = ab$, then find the value of y.

a. $\frac{x-1}{x}$ b. $\frac{x}{x+1}$ c. $\frac{x}{x-1}$ d. $\frac{x}{1-x}$ e. $\frac{x+1}{x}$

29. Suppose, a function f is defined over the set of natural numbers as follows:

f(1) = 1, f(2) = 1, f(3) = -1, and f(n) = f(n - 1) f(n - 3) whenever n > 3. Find the value of f(694) + f(695).

a. -2 b. -1 c. 1 d. 2 e. 3

30. The number of integers n for which the value of $\log_{10}\left[\frac{n(10-n)}{16}\right]$ becomes negative is

a. 2 b. 4 c. 1 d. 3 e. 5

31. Simplify:

$$\frac{x^3 - abc + (ab - bc - ca)x + (a + b - c)x^2}{(a-c)x + x^2 - ac}$$

a. (x - b) b. (x + b) c. $\frac{(x+a)(x-b)}{(x-c)}$ d. $\frac{(x+a)}{(x-a)}(x-b)$ e. $\frac{x+b}{x-b}$

32. v is inversely proportional to the square of m, and m is inversely proportional to the square of t. What is the relationship between v and t?

a. $v \propto t^2$ b. $v \propto t^4$ c. $v \propto \frac{1}{t}$ d. $v \propto \sqrt{t}$ e. $v \propto t^3$

33. Given that a and b are real numbers and a = b + 1, then find the largest value of x that satisfies the inequality given below.

$$\left(\sqrt{a}-\sqrt{b}\right)^{1-\frac{1}{\sqrt{x}}} \geq \left(\sqrt{a}+\sqrt{b}\right)^{\sqrt{x}+1} \quad (x ¹ 0)$$

a. $3+2\sqrt{2}$ b. $3\sqrt{2}-1$ c. $3-2\sqrt{2}$ d. $3\sqrt{2}+1$ e. None of these

34. If $16 \leq n \leq 36$ and $A = \left(\frac{2n^3 + 2n^2 + \sqrt{n}(n+3) - 3}{4n\sqrt{n}(n\sqrt{n}+3) + 9}\right)$, then A necessarily satisfies which of the following?

a. $0.301 \leq A \leq 0.318$ b. $0.412 \leq A \leq 0.424$ c. $0.692 \leq A \leq 0.614$

d. $0.508 \leq A \leq 0.511$ e. None of these

35. The time period of a pendulum varies directly with the square root of the length of the pendulum and inversely as the acceleration due to gravity. If the length of the pendulum is made four times the original and the acceleration due to gravity is halved, then what is the ratio of the time periods of the pendulum in these two cases?

31. Simplify:

$$\frac{x^3 - abc + (ab - bc - ca)x + (a + b - c)x^2}{(a - c)x + x^2 - ac}$$

a. (x - b) b. (x + b) c. $\frac{(x+a)(x-b)}{(x-c)}$ d. $\frac{(x+a)}{(x-a)}(x-b)$ e. $\frac{x+b}{x-b}$

32. v is inversely proportional to the square of m, and m is inversely proportional to the square of t. What is the relationship between v and t?

a. $v \propto t^2$ b. $v \propto t^4$ c. $v \propto \frac{1}{t}$ d. $v \propto \sqrt{t}$ e. $v \propto t^3$

33. Given that a and b are real numbers and a = b + 1, then find the largest value of x that satisfies the inequality given below.

$$(\sqrt{a} - \sqrt{b})^{1 - \frac{1}{\sqrt{x}}} \geq (\sqrt{a} + \sqrt{b})^{\sqrt{x}+1} \quad (x \, 1 \, 0)$$

a. $3 + 2\sqrt{2}$ b. $3\sqrt{2} - 1$ c. $3 - 2\sqrt{2}$ d. $3\sqrt{2} + 1$ e. None of these

34. If $16 \leq n \leq 36$ and $A = \left(\frac{2n^3 + 2n^2 + \sqrt{n}(n+3) - 3}{4n\sqrt{n}(n\sqrt{n} + 3) + 9}\right)$, then A necessarily satisfies which of the following?

a. $0.301 \leq A \leq 0.318$ b. $0.412 \leq A \leq 0.424$ c. $0.692 \leq A \leq 0.614$

d. $0.508 \leq A \leq 0.511$ e. None of these

35. The time period of a pendulum varies directly with the square root of the length of the pendulum and inversely as the acceleration due to gravity. If the length of the pendulum is made four times the original and the acceleration due to gravity is halved, then what is the ratio of the time periods of the pendulum in these two cases?

a. 1 b. $\frac{1}{4}$ c. $\frac{1}{16}$ d. $\frac{1}{8}$ e. $\frac{1}{3}$

36. If $4^{a-5+b} = 2^{a+b\times} \; 2^{b\times} \; 2^{a-4} - 63$, then find the sum of a and b.

a. 2 b. 3 c. 4 d. 5 e. 6

37. How many real solutions (x, y) (i.e., both x and y are real numbers) does the following set of equations have?

$x^2 + 4 = y^3$ and $x + y = 16$

a. 3 b. 2 c. 1 d. 0 e. 4

38. Find the value of following expression $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)...\left(1 - \frac{1}{50^2}\right)$.

a. 0.51 b. 0.55 c. 0.6 d. 0.61 e. 0.57

39. If x, y, z are any real numbers, then find the minimum possible value of $x^2 + 2y^2 + z^2 + 2yz$ subject to $x + 2y + z = -6$

a. - 6 b. 6 c. - 12 d. 12 e. 8

40. Consider the number p = 0.787878 .... Then which of the following is true?

a. p is an irrational number b. $p^2 > 0.64$ c. $p^3 < \frac{27}{64}$

d. $p^4 < 0.4$ e. None of these

Practice Exercise 6 - Level 2

1. $2x > 2^{x-1} + 2^{x-2} + \cdots$ till 2^0. If x is an integer, then x can take the values

a. 1 b. 2 c. 3 d. 4 e. Both (a) and (b)

2. Let p be any root, real or complex, of the equation $x^n + x^{n-1} + x^{n-2} + \ldots + 1 = 0$. Find the value of $(p^{2n+2} + 3)(p^{3n+3} - 4)$.

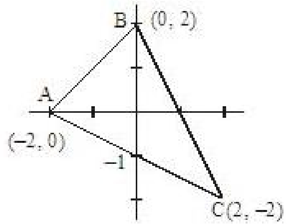
a. 12 b. - 12 c. 24 d. - 24 e. - 20

3. Find the sum of the series

$$\frac{1}{(3\times7)}+\frac{1}{(7\times11)}+\frac{1}{(11\times15)}+\ldots.$$

a. $\frac{1}{3}$ b. $\frac{1}{6}$ c. $\frac{1}{12}$ d. $\frac{1}{24}$ e. $\frac{1}{4}$

4. The enclosed region is denoted by which of the following inequalities?



a. $y < 2 + x, y < 2 - 2x, 2y + x > -2$ b. $x < 2 + y, 2x < 2 - x, 2y > x + 2$

c. $x + y < 2, 2x + y < 2, 2y + x < 2$ d. $y > 2 + x, y < 2x - 2, 2y + x > -2$

e. None of these

5. Given that x is a number greater than one, an increasing sequence would be:

a. $\log_{10}x, \log_3x, \log_ex, \log_2x$ b. $\log_3x, \log_ex, \log_{10}x, \log_2x$

c. $\log_2x, \log_ex, \log_3x, \log_{10}x$ d. $\log_{10}x, \log_2x, \log_ex, \log_3x$

e. $\log_{10}x, \log_ex, \log_2x, \log_3x$

6. Find the value of

$$S = 2^{\frac{1}{4}}\times4^{\frac{1}{8}}\times8^{\frac{1}{16}}\times16^{\frac{1}{32}}\times\ldots\infty$$

a. 1 b. 2 c. $\frac{1}{2}$ d. 4 e. $\frac{1}{4}$

7. If the roots of $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$ are equal in magnitude and opposite in sign, then the product of the roots is

a. 1 b. $\frac{(p+q)}{2}$ c. $(p^2 + q^2)$ d. $-\frac{1}{2}\left(p^2+q^2\right)$ e. $-(p^2 + q^2)$

8. Let $f(x) = \frac{4^x}{4^x+2}$, then $f\left(\frac{1}{1997}\right)+f\left(\frac{2}{1997}\right)+\ldots+f\left(\frac{1996}{1997}\right)$ is equal to:

a. 995 b. 997 c. 998 d. 996 e. 999

9. Sum of `n' successive positive integers x, (x + 1), ..., {x + (n - 1)} is 1000. Then, which of the following is not true about x? (n > 1)

a. x can be 55. b. x can be 40. c. x can be 198. d. x can be 28. e. None of these

10. Consider a function `f' on natural numbers such that f(1) + f(3) + f(5) + ... infinite terms
= b and f(n) = a {f(n + 2) + f (n + 4) + ...infinite terms}. Find the value of f(3).

a. $\frac{a}{1+a}$ b. $\frac{ab}{1+a}$ c. $\frac{ab}{(1+a)^2}$ d. $\frac{a}{(1+a)^2}$ e. None of these

11. Ramu Khakha, a farmer, has 9 cows whose ages are 1, 2, 3 ... 9 years respectively. Each cow gives milk daily, the quantity of which is proportional to the cow's age. In how many ways can Ramu Khakha form three groups of the cows, such that the cows in each group gives the same quantity of milk each day?

a. 6 b. 7 c. 8 d. 9 e. 10

12. Honda, a car manufacturing company, manufactures `p' Honda CRVs and `q' Honda Accords in a particular day, where 8q + 16p = pq + 96 and $0 \le p \le 6$. The profit on each Honda CRV is Rs. 4 lakhs and that on each Accord is Rs. 2 lakhs. Find the maximum profit that can be made by Honda in a day.

a. Rs. 24 lakhs b. Rs. 36 lakhs c. Rs. 32 lakhs d. Rs. 40 lakhs e. Rs. 28 lakhs

13. $\frac{p}{a}+\frac{q}{b}+\frac{r}{c} = 1$ and $\frac{a}{p}+\frac{b}{q}+\frac{c}{r} = 0$, where a, b, c, p, q and r are non-zero real numbers.

Then $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2} =$

a. 0 b. 6 c. 3 d. 9 e. 1

14. In solving a problem on quadratic equation, one student makes a mistake in the constant term of the equation and gets -3 and -2 for the roots. Another student makes a mistake in the coefficient of first degree term and finds -1 and -2 for the roots. The correct equation was

a. $X^2 + 5X + 2 = 0$ b. $X^2 - 5X + 2 = 0$ c. $X^2 + 5X - 2 = 0$

d. $X^2 - 5X - 2 = 0$ e. None of these

15. A book has 98 sheets each having a page on either side, where pages are numbered from 1 to 196. A student tore 35 successive sheets from the book. The sum of the page numbers on the remaining sheets was found and noted down on a piece of paper. Which one of the following could be the number which was noted down? (represent the digits that got rubbed off the paper accidentally)

a. 5 ** 2 b. 6 2 * 6 c. 1 1 1 * 7 d. 1 1 3 * 1 e. None of these

16. At $x = \frac{7}{2}$, f(x) = ax² + bx + c attains a maximum value. If the product of the roots of the equation f(x) = 0 is 10, then find the value of (a × b × c).

a. 70 b. - 70 c. $\frac{49}{4}$ d. $-\frac{49}{4}$ e. Data Insufficient

17. In the X-Y plane, points A, B, C, D and E have co-ordinates (2, 0), (0, -2), (-2, 0) $\left(\frac{-3}{2}, \frac{\sqrt{7}}{2}\right)$ and (2, -4) respectively. How many distinct circle(s) can pass through at least 3 of these points?

a. 5 b. 7 c. 6 d. 10 e. None of these

18. Find the total distance covered by an elastic ball before it comes to rest, if it is dropped from a height of 20 m, and after each fall, it rebounds to 50% of the height from which it falls.

a. 120 m b. 100 m c. 60 m d. 80 m e. 90 m

19. Given, $-3.5 \le a \le 7.5$ and $-7.5 \le b \le 5.5$.
If c = -a × |b|, then find the least value of a × c.

a. - 196.875 b. - 309.375 c. - 421.875 d. - 504.225 e. -358.50

Directions for questions 20 and 21: Answer the questions based on the following information.

A series of positive integers has the following terms.

First term: 1; second term: (2 + 3); third term: (4 + 5 + 6), ... so on.

20. The last number in the nth term of the series is

a. $\frac{(n^2-n+2)}{2}$ b. $\frac{(n^2+n)}{2}$ c. $\frac{(2n^2+3)}{4n}$ d. $\frac{n^2+2n}{2}$ e. $\frac{2n^2-n+2}{2}$

21. The sum of the numbers of the nth term is

21. The sum of the numbers of the nth term is

a. $n^2 + n$ b. $\frac{n(n^2+1)}{2}$ c. $\frac{(n+n^2)}{3n}$ d. $\frac{n^2-n}{2}$ e. None of these

22. The polynomial equation $x^3 - 3x^2 + Ax + B = 0$ has 3 real and positive roots. Find the minimum possible value of B.

a. 1 b. 3 c. -1 d. 2 e. -2

23. The cost C of a registered parcel is calculated thus: registration charges = Rs. 10, the charge for the first 500 g = Rs. 3. The charge for every subsequent 100 g is Re 1. If the weight of a consignment, which can be split into as many parts, is 2.1 kg, find the cost incurred by an intelligent person in registering the same.

a. Rs. 29 b. Rs. 37 c. Rs. 53 d. Rs. 43 e. Rs. 41

24. In the previous question, if the charge for the first 1,000 g is Rs. 5 and the charges for every 50 g thereafter is Re 1, what is the minimum cost in sending the package?

a. Rs. 37 b. Rs. 32 c. Rs. 43 d. Rs. 27 e. Rs. 36

25. If $f(x) = \log \frac{1+x}{1-x}$ and $g(x) = \frac{3x+x^3}{1+3x^2}$, then $f[g(x)] =$

a. $[f(x)]^3$ b. $-f(x)$ c. $[f(x)]^3$ d. $3[f(x)]$ e. $2\,[f(x)]$

26. X, Y, Z and W are all integers. Now if XYZW = -1, then (x - y)(x - z)(x - W) is

a. 0 or 1 b. 8 or -8 c. 0, 8, or -8 d. 0 e. None of these

27. If $x = 1 + a + a^2 + \cdots + \infty$ and $y = 1 + b + b^2 + \cdots + \infty$, where a and b are proper fractions, then find the sum of the series $1 + ab + a^2b^2 + \cdots + \infty$.

a. $\frac{x^2+y^2}{x-y}$ b. $\frac{xy}{x+y+1}$ c. $\frac{xy}{x+y-1}$ d. $\frac{xy}{x^2-y^2}$ e. $\frac{xy}{x-y+1}$

28. If pth term of an HP is qr and qth term is pr, then the rth term of the HP is

a. pqr b. $\frac{1}{pq}$ c. pq d. $\frac{1}{pqr}$ e. $\frac{2}{pqr}$

29. How many real solutions exist for the equation $\frac{x^2-5}{5-6|x-3|} = 1$?

$\left(x \neq \frac{13}{6}, \frac{23}{6}\right)$

a. Four b. Three c. Two d. One e. Zero

30. Find the GM of $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^{n-1}}$.

a. $\left[\frac{1}{3}\right]^{\frac{n-2}{2}}$ b. $\left[\frac{1}{3}\right]^{\frac{n-1}{2}}$ c. $\left[\frac{1}{3}\right]^{\frac{(n-1)(n-1)}{2}}$ d. $\left[\frac{1}{3}\right]^{\frac{(2-n)(n-1)}{2}}$ e. None of these

31. If α and β are the roots of the equation $x^2 - 9x + 5 = 0$, then find the equation whose roots are $\left(\frac{\alpha-\beta}{2\beta}\right)$ and $\left(\frac{\beta-\alpha}{2\alpha}\right)$.

a. $10x^2 - 122x - 61 = 0$ b. $x^2 - 122x - 61 = 0$

c. $20x^2 - 122x - 61 = 0$ d. $10x^2 + 61x - 61 = 0$

e. None of these

32. S_1, S_2 and S_3 are the respective sums of n, 2n and 3n terms of the same arithmetic progression whose first term is \`a' and the common difference is \`d'. If $R = S_3 - S_2 - S_1$, then R is dependent on

a. \`a' and \`d' b. \`n' and \`d' c. \`a' and \`n' d. \`a' e. \`d'

33. The nth term of a series is given as $T_n = (3n - x)$. The sum of the terms between the 10th term and the 20th term is

a. 405 - 9x b. 630 - 10x c. 450 - x

d. 605 - 9x e. Cannot be determined

34. It is given that $x^2 + y^2 = z + 1$, $y^2 + z^2 = x + 1$, $z^2 + x^2 = y + 1$. Find xyz.

a. 1 b. $-\frac{1}{8}$ c. 1 or $\frac{1}{8}$ d. 1 or $-\frac{1}{8}$ e. None of these

35. Two real functions are defined below:

$f(x + 1) = f(x) + 1$

and $g(x - 1) = g(x) - 1$

If f(0) = n and g(0) = m then find the value of $\frac{f(f(f(f(\ldots f(n)))))_{m \text{ times}}}{g(g(g(g(\ldots g(m)))))_{n \text{ times}}}$.

a. $\frac{1+\frac{1}{m}}{1+\frac{1}{n}}$ b. $\frac{m+1}{n+1}$ c. $\frac{1+\frac{1}{n}}{1+\frac{1}{m}}$ d. $\frac{n+1}{m+1}$ e. $\frac{1+\frac{n}{m}}{1+\frac{m}{n}}$

36. Any real number x, can be expressed as x = [x] + {x}, where [x] is the greatest integer less than or equal to x and {x} is the fractional part of x. How many real numbers exist that satisfy the following equations?

$[x]^2 - 12[x] + 20 < 0$ ...(i)

and

$\{x\}^2 - 0.8\{x\} + 0.15 = 0$ ...(ii)

a. 7 b. 28 c. 21 d. 17 e. 14

37. If xy = r, $xz = r^2$ and $yz = r^3$, also x + y + z = 13 and $x^2 + y^2 + z^2 = 91$, then find the value of $\frac{z}{y}$.

a. 3 b. $\frac{7}{3}$ c. 4 d. $\frac{13}{3}$ e. $\frac{5}{3}$

38. Sum of first n terms of an AP is n(n - 1). Then sum of the squares of these terms is

a. $n^2(n - 1)^2$ b. $\frac{n}{6}(n-1)(2n-1)$

c. $\frac{2}{3}n(n-1)(2n-1)$ d. $\frac{2}{3}n(n+1)(2n+1)$ e. $\frac{n}{6}(n+1)(2n+1)$

39. Given that $g(h(x)) = 2x^2 + 3x$ and $h(g(x)) = x^2 + 4x - 4$ for all real values x. Which of the following could be the value of g(-4)?

a. 1 b. - 1 c. 2 d. - 2 e. - 3

a. $\frac{1+\frac{1}{m}}{1+\frac{1}{n}}$ b. $\frac{m+1}{n+1}$ c. $\frac{1+\frac{1}{n}}{1+\frac{1}{m}}$ d. $\frac{n+1}{m+1}$ e. $\frac{1+\frac{n}{m}}{1+\frac{m}{n}}$

36. Any real number x, can be expressed as x = [x] + {x}, where [x] is the greatest integer less than or equal to x and {x} is the fractional part of x. How many real numbers exist that satisfy the following equations?

$[x]^2 - 12[x] + 20 < 0$ ...(i)

and

$\{x\}^2 - 0.8\{x\} + 0.15 = 0$ ...(ii)

a. 7 b. 28 c. 21 d. 17 e. 14

37. If $xy = r$, $xz = r^2$ and $yz = r^3$, also $x + y + z = 13$ and $x^2 + y^2 + z^2 = 91$, then find the value of $\frac{z}{y}$.

a. 3 b. $\frac{7}{3}$ c. 4 d. $\frac{13}{3}$ e. $\frac{5}{3}$

38. Sum of first n terms of an AP is n(n - 1). Then sum of the squares of these terms is

a. $n^2(n-1)^2$ b. $\frac{n}{6}(n-1)(2n-1)$

c. $\frac{2}{3}n(n-1)(2n-1)$ d. $\frac{2}{3}n(n+1)(2n+1)$ e. $\frac{n}{6}(n+1)(2n+1)$

39. Given that $g(h(x)) = 2x^2 + 3x$ and $h(g(x)) = x^2 + 4x - 4$ for all real values x. Which of the following could be the value of g(-4)?

a. 1 b. - 1 c. 2 d. - 2 e. - 3

40. What is the sum of the series?

$3 + 8 + 15 + 24 + 35 + \cdots + 9999$

a. 338350 b. 338530 c. 338150 d. 338305 e. 338250

41. The maximum possible sum of the series

$\left(20+\frac{1}{4}\right)+\left(18+\frac{2}{4}\right)+\left(16+\frac{3}{4}\right)+\left(14+\frac{4}{4}\right)+\left(12+\frac{5}{4}\right)+ \ldots$ is

a. $\frac{255}{2}$ b. $\frac{495}{4}$ c. $\frac{253}{2}$ d. $\frac{495}{2}$ e. $\frac{257}{2}$

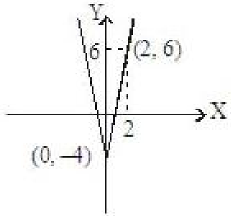
42. Let {x} and [x] denote the fractional and integral parts of a real possible number x respectively. Solve for x, if 6{x} = x + 2[x].

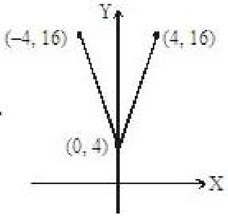
a. 2.6 b. 1.5 c. 2.5

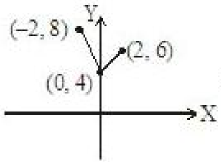
d. 1.6 e. 1.9

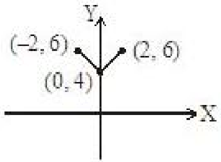
Practice Exercise 7 - Level 2

1. Find the graph of y = |x| + 4.

a.


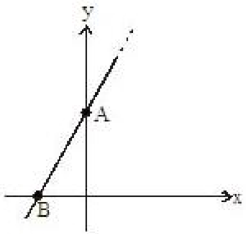
b.


c.


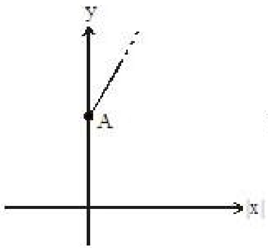
d.


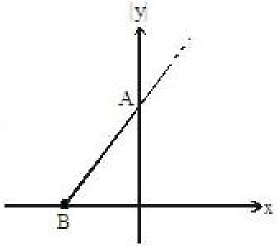
e. None of these

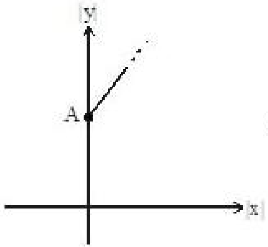
2. For different values of real numbers x and y, a graph of y against x is plotted, as shown in the diagram below:



Which of the following is an incorrect graph?

a.


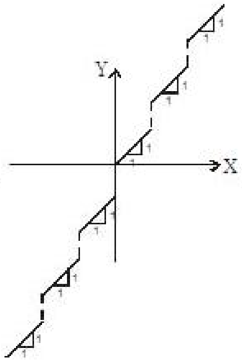
b.


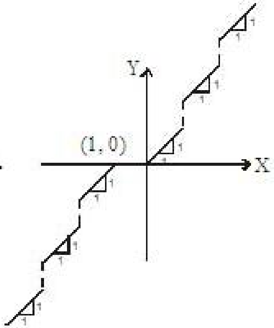
c.


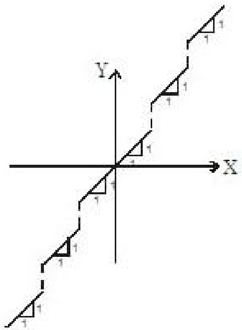
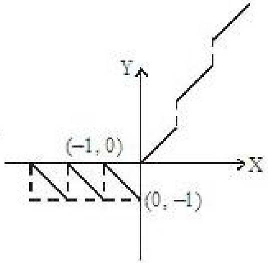
d. Both (b) and (c)

e. None of these

3. Find the graph of y = |x| + [x].

a.


b.


c.  d. 

e. None of these

4. A, B, C and D are four different numbers selected from 1, 2, 3, 4, 5, 6, 7, 8 and 9 so as to minimize $\frac{A}{B}+\frac{C}{D}$. The minimum possible value of $\frac{A}{B}+\frac{C}{D}$ is

a. $\frac{3}{17}$ b. $\frac{2}{17}$ c. $\frac{17}{72}$ d. $\frac{25}{72}$ e. $\frac{1}{3}$

5. If $f(x) = x - 3$, $g(x) = x^2$ and $h(x)=\frac{1}{x}$, then for $-1 < x < 1$,

a. $f(g(h(x))) = h(f(g(x)))$ b. $f(g(h(x))) > h(f(g(x)))$

c. $f(g(h(x))) > f(h(g(x)))$ d. $f(x) = g(x)$

e. None of these

6. Let $f(x) = |x|$, $g(x) = [x]$ (greatest integer function), then

a. $f(g(x)) > g(f(x))$ b. $f(g(x)) = g(f(x))$ c. $f(g(x)) = -g(f(x))$

d. Either (a) or (b) e. Either (b) or (c)

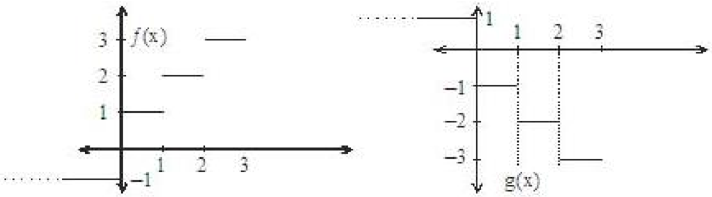
7. If $f(x) = |x|$, $h(x) = x - 3.5$, $p(x) = x^2$, $q(x)=\sqrt{x}$

$g(x) = [x]$, then

a. $p(1 - h(3)) > q(f(-9) - 1)$ b. $h(q(16)) = h(g(p(2)))$ c. $h(g(p(2))) = 1$

d. Both (a) and (b) e. Both (b) and (c)

8. In the given graphs, f(x) can be described as



a. $f(x)=\begin{cases}-1, & x<0\\ [x+1], & x>0\end{cases}$ b. $f(x)=\begin{cases}-x, & x<0\\ [x], & x>0\end{cases}$

c. $f(x)=\begin{cases}-x, & x<0\\ [x+1], & x>0\end{cases}$ d. $f(x)=\begin{cases}1, & x<0\\ [x+1], & x>0\end{cases}$

e. None of these

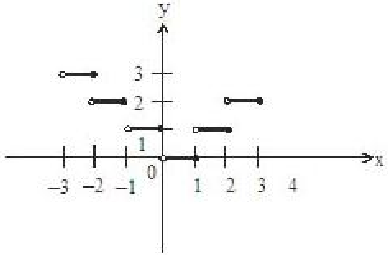
9. Referring to question 8, which equation defines the relationship between f(x) and g(x)?

a. $g(-x) = f(x)$ b. $g(x) = f(-x)$ c. $g(x) = -f(x)$

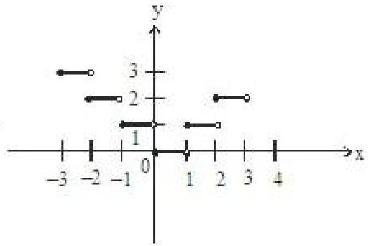
d. $g(x)=\frac{f(x)}{2}$ e. None of these

10. Which of the following graphs represents $y = |[x - 1]|$

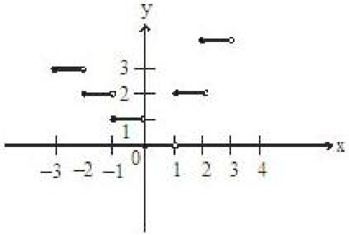
a.



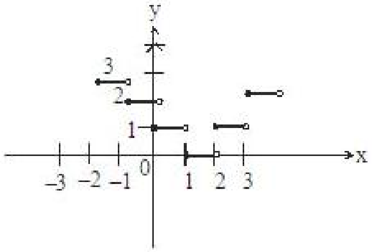
b.



c.

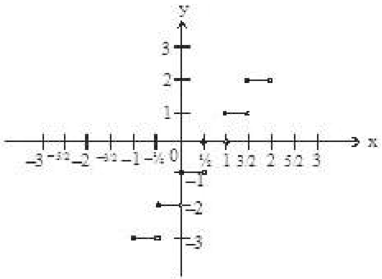


d.



e. None of these

11. Which of the following equation represents the graph given below?



a. $y=[2x+1]$ b. $y=[2x-1]$ c. $y=[2x+1]+1$ d. $y=[2x+1]+\frac{1}{2}$ e. $y=[2x-1]+\frac{1}{2}$

12. What is the maximum possible value of $(3x + 2y)$ given that $0 \le x \le 4$, $0 < y < 3$, $2x + 3y \le 10$?

a. 12.5 b. 8.5 c. $\frac{40}{3}$ d. $\frac{25}{3}$ e. $\frac{50}{3}$

13. The system of equations $(3 - a)x + 9y = 0$, $x + (3 - a)y = 0$, has a solution for x, y, with at least one of x and y nonzero, if and only if

a. $a = 0$ b. $a = 0$ or $a = 6$ c. $a = 0$ or $a = - 6$

d. $a = 0$ or $a = 6$ or $a = - 6$ e. None of these

Direction for questions 14 and 15: Answer the questions based on the following information.

There are 12 rooms in a circular building numbered 1 to 12. Ashok is in room number 1. The rooms are numbered in clockwise fashion. The following steps are performed.

$$S_n = \begin{cases} \text{Move n steps clockwise, if n is odd} \\ \text{Move n steps anticlockwise, if n is even.} \end{cases}$$

14. In which room is Ashok if the value of n is increased to 8 from 0 in steps of 1?

a. Room 4 b. Room 8 c. Room 5 d. Room 7 e. Room 9

15. If n = 8 and is reached in steps of 2, what is the position of Ashok?

a. Room 9 b. Room 8 c. Room 5 d. Room 6 e. Room 7

16. If $f(x) = x^2 - \frac{1}{x^2}$, then

a. $f(x) = -f(x)$ b. $f(x) = f\left(\frac{1}{x}\right)$ c. $f(x) = -f\left(\frac{1}{x}\right)$ d. $f(x) = f(x^2)$ e. None of these

17. If $f(x) = \frac{a^x + a^{-x}}{2}$, then

a. f(x + y) + f(x - y) = f(x) + f(y) b. f(x + y) + f(x - y) = 2f(x) f(y)

c. f(x + y) + f(x - y) = f(x) - f(y) d. f(x + y) + f(x - y) = f(x + y) f(x - y)

e. None of these

18. If $f(x) = (x - 2)^3$, where x is a natural number less than 10 and $x = \frac{1}{2}f(x)$, then what is the value of x?

a. 3 b. 6 c. 4 d. 2 e. 5

19. $y = f(x)$, $f(x) = g^2(x)$ and $g^{-1}(x) = 2x + 3$. What is the value of x, if y = 5? [$g^n(x) = g\ (g^{n-1}(x))$, where n is a natural number greater than 1.]

a. 26 b. 27 c. 28 d. 30 e. 29

20. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$ then $a^a b^b c^c$ is

a. 0 b. 1 c. 0.5 d. 2 e. 0.75

Directions for questions 21 and 22: Answer the questions based on the following information.

Hakim Chacha owned plenty of land, which easily encompassed an area 1000 × 200 sq. ft. He wanted to give some part of it to his servant Gopal. But he did not gift it directly. He supplied the material that could form a fence of length 100 ft only. Then he allowed Gopal to take any part with four sides that could be enclosed with the help of the given fencing material.

21. What is the maximum possible land Gopal can take away from Hakim Chacha? (All the four sides are fenced.)

a. 100 sq. ft b. 2,500 sq. ft c. 750 sq. ft d. 625 sq. ft e. 600 sq.ft

22. Gopal had intimate knowledge of the land. Hence, he selected the site having a natural fencing of rocks on one side, because he could utilize the given material only on three sides of the plot. What is the maximum possible land that Gopal can claim now?

a. 1,250 sq. ft b. 2,500 sq. ft c. 625 sq. ft d. 1,000 sq. ft e. 800 sq.ft

23. A fruit packer packs x oranges per box into y boxes. If he packs 20 more oranges into each box, he would require 3 boxes less and if he packs 5 less in each box, he would require 1 box more. x is equal to

a. 80 b. 100 c. 90 d. 40 e. 70

24. Bhola, sick of his boring life, decides to explore the Sahara desert. With how many assistants can he make a 6-day crossing on foot of the Sahara if he and the available assistants can each carry only enough food and water to last 4 days ? [No man should die.]

a. 2 b. 3 c. 4 d. 5 e. Not possible

25. A cylinder of the maximum possible size is made out of a solid wooden cube. How much material is lost in this process in percentage, approximately?

a. 20% b. 22% c. 24% d. 23%

e. Cannot be determined

26. If $2^{2x+5} = 3 \times 2^{x+2} - 1$, what is(are) the possible value(s) of x?

a. 2 or 3 b. -2 or 3 c. -3 or -2 d. -3 or 2 e. None of these

27. In solving an equation of the form ax - b = 0 (a and b have 1 as the only common factor). A made a mistake of copying `b' and got as the root whereas B made a mistake of copying `a' and got as the root. The correct root is

a. $\frac{7}{5}$ b. $\frac{3}{8}$ c. $\frac{8}{3}$ d. $\frac{3}{5}$

e. Cannot be determined

28. $f(x) = \frac{1}{x}$; $f^n(x) = f(f^{n-1}(x))$. If $x \neq 0$ and $f^{100}(x) = 40$, then what is the value of $f^1(x)$?

a. 40 b. $\frac{1}{40}$ c. $\frac{-1}{40}$ d. Data insufficient e. None of these

29. Every month, Pappu's mother gives him a fixed sum of money for his monthly expenses. He gets "a" number of 5 Rs. coins and "b" number of 2 Rs. coins, every month. Last month, instead of asking his mother, he asked his father for the pocket money. If his father gave him "b" number of 5Rs. coins and "a" number of 2 Rs. coins then which of the following can never be true?

a. Last month, Pappu got 60 Rs lesser for his monthly expenses.

b. Last month, Pappu got 50 Rs more for his monthly expenses.

c. Last month, Pappu got 90 Rs lesser for his monthly expenses.

d. Last month, Pappu got 30 Rs more for his monthly expenses.

e. None of these

30. A group of bees, equal in number to the square root of half of the whole swarm, is alighted on a jasmine bush. One little bee circles about a lotus as it is attracted by the buzzing of a sister-bee that is so careless as to fall into the trap of the lotus flower. The remaining bees, being four less than $\frac{1}{9}^{th}$ of the total number in the swarm, flew away. How many bees were there in the swarm?

a. 98 b. 72 c. 128 d. 50 e. 70

31. A farmer divides his herd of n cows among his four sons so that one son gets one-half the herd, the second one-fourth, the third one-fifth and the fourth gets 7 cows. Find the number of cows.

a. 180 b. 140 c. 240 d. 100 e. 160

32. If a, b and c are three arbitrary positive real numbers satisfying the equation $3ab + 4bc + 5ca = 180$, then the maximum possible value of abc is

a. 3600 b. 120 c. 90 d. 60 e. 2400

33. I bought 14 fruits, some of them are apples and the rest are bananas. If I had doubled the number of apples and increased the number of bananas by 18, then the number of bananas would be greater. If I had doubled the number of bananas without changing the number of apples, the number of apples would be greater. How many apples did I buy?

a. 8 b. 9 c. 10 d. 11 e. 12

34. An egg seller sells x dozen eggs. If he had sold 100 dozen eggs more at 20 paise a dozen less or if he had sold 120 dozen eggs less at 30 paise a dozen more, he would have received the same amount for his eggs. How many dozen eggs does he sell?

a. 1,600 b. 700 c. 2,100 d. 1,000 e. 1500

35. $(5^2 + 1)(5^4 + 1)(5^8 + 1)(5^{16} + 1)$ ... till nth term, on simplification is (where n is a natural number).

a. $5^{2^{n+1}}$ b. $5^{2^{n+1}} - 1$ c. $\frac{5^{2^{n+1}} - 1}{12}$ d. $\frac{5^{2^{n+1}} - 1}{24}$ e. $5^{2^{n-1}}$

36. If $a + b + c = 0$, where $a \neq b \neq c$, then $\frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ac} + \frac{c^2}{2c^2 + ab} =$

a. 0 b. 1 c. -1 d. abc e. a + b + c

37. How many distinct integral values of `a' satisfy the equation $2^{2a} - 3(2^{a+2}) + 2^5 = 0$?

a. 0 b. 1 c. 2 d. Data Insufficient e. None of these

38. If $\frac{1}{x}+\frac{1}{y}=a$ and $xy=\frac{1}{b}$, find $\frac{1}{x^2}+\frac{1}{y^2}$.

a. $\frac{1}{a^2}-\frac{2}{b}$ b. $a^2-\frac{2}{b}$ c. $a^2 - 2b$ d. $\frac{1}{a^2}-2b$ e. $\frac{1}{a^2}+2b$

39. Find the value of k in the equation $x^3 - 6x^2 + kx + 64 = 0$ if it is known that the roots of the equation are in geometric progression.

a. 24 b. 16 c. -16 d. -24 e. -18

Directions for questions 40 and 41: Answer the questions based on the following information.

f(x) = [x] indicates the largest integer less than or equal to x

g(x) = {x} indicates the smallest integer greater than or equal to x

h(x) = |x| indicates the absolute value of x

40. If f(x) - g(x) = 0 and h(x) - x = 6, find x.

a. 3 b. -3 c. -2 d. None of these e. Data insufficient

41. $[g(x)]^2 - [h(x)]^2 > 0$, then x - h(x) is

a. > 0 b. < 0 c. = 0 d. ≥ 0

e. Cannot be determined

42. If a, b and c are three positive integers in geometric progression, then the roots of the equation $ax^2 + 4bx + 2c = 0$ are

a. Imaginary b. equal c. rational d. real e. Irrational

43. The min value of $f(x)=\frac{x^3+x+2}{x}$, where x is a positive real number is.

a. $\frac{1}{4}$ b. 1 c. 2 d. 4 e. $\frac{1}{3}$

44. For x > 0 and y > 0, let

$$f(x, y) = \begin{cases} y^{-1}, & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

For y > 0, define

g(y) = f(2, y) f (4, y) f(6, y) f(8, y). Then

a. g(y) attains its maximum at y = 5 b. g(y) attains its maximum at y = 8

c. g(y) attains its minimum at y = 10 d. Both (b) and (c)

e. Both (a) and (c)

45. Find the co-ordinates of points of intersection of the curves $3x^3 - 5x^2 + 10x - 1 = 0$ and $x^3 + 2x^2 + 3x + 1 = 0$.

a. $\left(\frac{1}{2}, 3\right)$, (1, 7) and (2, 23) b. $\left(\frac{1}{2}, \frac{25}{8}\right)$, (1, 7) and (2, 24)

c. $\left(\frac{1}{2}, 3\right)$, (1, 7) and (2, 24) d. $\left(\frac{1}{2}, \frac{25}{8}\right)$, (1, 7) and (2, 23)

e. Cannot be determined

46. If |x| - |y| = 13, then which of the following cannot be the value of x - y?

a. Imaginary b. equal c. rational d. real e. Irrational

43. The min value of $f(x) = \frac{x^3 + x + 2}{x}$, where x is a positive real number is.

a. $\frac{1}{4}$ b. 1 c. 2 d. 4 e. $\frac{1}{3}$

44. For $x > 0$ and $y > 0$, let

$$f(x, y) = \begin{cases} y^{-1}, & \text{if } x \le y \\ 0 & \text{otherwise} \end{cases}$$

For $y > 0$, define

$g(y) = f(2, y)\, f(4, y)\, f(6, y)\, f(8, y)$. Then

a. g(y) attains its maximum at y = 5 b. g(y) attains its maximum at y = 8

c. g(y) attains its minimum at y = 10 d. Both (b) and (c)

e. Both (a) and (c)

45. Find the co-ordinates of points of intersection of the curves $3x^3 - 5x^2 + 10x - 1 = 0$ and $x^3 + 2x^2 + 3x + 1 = 0$.

a. $\left(\frac{1}{2}, 3\right)$, $(1, 7)$ and $(2, 23)$ b. $\left(\frac{1}{2}, \frac{25}{8}\right)$, $(1, 7)$ and $(2, 24)$

c. $\left(\frac{1}{2}, 3\right)$, $(1, 7)$ and $(2, 24)$ d. $\left(\frac{1}{2}, \frac{25}{8}\right)$, $(1, 7)$ and $(2, 23)$

e. Cannot be determined

46. If $|x| - |y| = 13$, then which of the following cannot be the value of $x - y$?

a. -18 b. -9 c. -17 d. -13 e. -29

47. If $x^2 - 20x + 91$ 0, then which of the following can never be the value of $x^3 - 3x^2 + 3x - 1$?

a. 125 b. 729 c. 512 d. 216 e. None of these

48. Which of the following is correct about the following equation?

$(x - 1)2^x = 1$

a. The equation has exactly two real roots at $x \pm 1$.

b. The equation has exactly one real root at $x = 1$.

c. The equation has exactly one real root between $x = -1$ and 1.

d. The equation has no real root.

e. None of these

Practice Exercise 8 - Level 3

1. Find the range of values of `x' that satisfy the inequality $x^2 - 4x + 3 \leq 0$ and $x \geq 2$. [`x' is a real number]

a. $x \leq 0$ b. $1 \leq x \leq 3$ c. $x \geq 3$ d. $x \geq 0$ e. $2 \leq x \leq 3$

2. The sum of three fractions is $2\frac{11}{24}$. The ratio of the largest fraction to the smallest fraction is $\frac{7}{6}$, which is $\frac{1}{3}$ more than the middle fraction. Find the fractions.

a. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ b. $\frac{4}{3}, \frac{8}{7}, \frac{5}{6}$ c. 2, 4, 5 d. $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}$ e. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

3. If (x - 2) and (x + 1) are the factors of the expression $(2x^4 - ax^3 + x + b)$, then find the value of b

a. $\frac{-14}{3}$ b. -11 c. $\frac{14}{3}$ d. +11 e. 22

4. How many positive integral values of x and y satisfy the equation 2x + 3y = 10?

a. 1 b. 5 c. 3 d. 2 e. 4

5. If $|2x| + 5|x| \leq 30$, then find the range of values of x. [`x' is a real number.]

a. $-5 < x < 5$ b. $x < \frac{30}{7}$ c. $-\frac{30}{7} \leq x \leq \frac{30}{7}$ d. $x > \frac{-30}{7}$ e. $x \leq -5$

6. Find the number of real values of x that satisfy the equation $\log_x (2x^2 - 3x - 5) = 2$.

a. 0 b. 1 c. 2 d. 3 e. Infinitely many

7. If $\log_x y + \log_y x = 3$, then find the value of the expression

$$\log_{x^2 y}\left[\frac{x}{y^3}\right] + \log_{y^2 x}\left[\frac{y}{x^3}\right].$$

a. $\frac{-17}{11}$ b. $\frac{17}{11}$ c. $\frac{14}{11}$ d. $\frac{-14}{11}$ e. $\frac{15}{11}$

8. Find the number of real values of x satisfying the equation

$$|3-x|^{\log_7 x^2 - 7\log_x 49} = (3-x)^3.$$

a. 0 b. 1 c. 2 d. 3 e. More than 3

9. If $\frac{x^3 - 2x^2 - x + 2}{2x + 3} < 0$, then which of the following values of `x' does not satisfy the given inequality?

a. $-\sqrt{2}$ b. $\sqrt{2}$ c. $\frac{\sqrt{3}}{2}$ d. $\sqrt{3}$ e. 1.9

10. Let P denote the expression $\frac{3 \times 5 \times 7 \times \ldots \times 99}{2 \times 4 \times 6 \times \ldots 100}$, , then

a. $\frac{1}{3} < P < \frac{1}{2}$ b. $\frac{1}{5} < P < \frac{1}{3}$ c. $\frac{1}{10} < P < \frac{1}{5}$ d. $\frac{1}{15} < P < \frac{1}{10}$ e. None of these

11. Let a set be

$$S_{2006} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{2006}\right\}$$

Two numbers (a, b) are chosen from the set and replaced by a single number (a + b + a × b) and this operation is done successively 2005 times in this set. What is the only number left in the set S_{2006} finally?

a. 2006 b. 2005 c. 2005 × 2007! d. 2005 × 2006! e. 2006 × 2007!

12. The product of n positive real numbers $a_1, a_2, a_3 \ldots a_n$ is K. Find the minimum possible value of

$(a_1 + 2a_2 + 3a_3 + 4a_4 + \ldots\ldots + na_n)$

a. $[(n+1)!\times K]^{1/n}$ b. $n(n!\times K)^{1/n}$

c. $(n+1)\times((n+1)!\times K)^{1/n}$ d. $(n+1)\times(n!\times K)^{1/n}$

e. None of these

13. Which of the following value(s) of x, satisfy the following inequality? (given, $|x| < 1$)

$$\left\{\log_{(1+x+x^2+\ldots\infty)}(x)\right\} > \left\{\log_{\left(2-\frac{x}{2}+\frac{x^2}{8}-\ldots\ldots\infty\right)}(x)\right\}$$

a. $\frac{1}{9}$ b. $\frac{2}{9}$ c. $\frac{4}{9}$ d. $\frac{1}{3}$ e. $\frac{5}{9}$

14. Find the greatest integer less than or equal to $(\sqrt{3}+1)^8$.

a. 4521 b. 4125 c. 4251 d. 4215 e. 4512

15. Given, $f(x) = \frac{x+1}{x-1}$ for all real values of If the functions f(x), f(f(x), f(f(f(x))), f(f(f(f(x)))) etc. are expressed as $f^1(x)$, $f^2(x)$, $f^3(x)$, $f^4(x)$ and so on, then which of the following is not a factor of the integer "P" where $P = f^1(2) \times f^2(3) \times f^3(4) \ldots\ldots\times f^{10}(11)$.

a. 693 b. 847 c. 363 d. 945 e. 545

16. How many integral pairs (x, y) satisfy the following set of equations and inequalities?

$$x - 2y = 0$$
$$x + 2y \le 6$$
$$y - 3x \le 3$$
$$x - 3y \le 6$$

a. 6 b. 3 c. 5 d. 4 e. 2

17. An arithmetic series consists of 2n terms, and the first term equals the value of the common difference. If a new series is formed taking the 1st, 3rd, 5th, ..., (2n - 1)th terms of the old series, find the ratio of the sum of new series to that of the sum of the terms of the old series.

a. $\frac{n+1}{2(2n+1)}$ b. $\frac{n}{2n+1}$ c. $\frac{1}{2}$ d. $\frac{n}{n+1}$ e. $\frac{2n}{2n+1}$

18. Find the sum of the series:

$1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 8 \times 9 \times 10$

a. 1980 b. 1880 c. 1989 d. 1889 e. 1990

DIRECTIONS for Questions 19 and 20: Answer the questions on the basis of the information given below.

A company `XYZ' engages three small companies P,Q and R to manufacture 100 cricket bats per day. Every
person in companies P,Q and R can manufacture 2, 3 and 8 cricket bats per day respectively. Companies P, Q and R charge Rs. 20, Rs. 25 and Rs. 15 respectively for every bat manufactured by them. The number of persons involved in manufacturing bats for company `XYZ' from company P is not greater than the total number of persons involved in manufacturing bats from companies Q and R. At least one person from each of the companies P, Q and R is involved in making bats.

19. Find the minimum possible expenditure incurred by the company `XYZ' to manufacture 100 bats.

a. Rs. 2460 b. Rs. 2380 c. Rs. 2310 d. Rs. 1590 e. None of these

20. Find the difference between the number of persons involved from company Q and R in manufacturing bats for company `XYZ' such that the expenditure incurred by the company XYZ is minimum in manufacturing 100 bats.

a. 11 b. 10 c. 9 d. 8 e. 7

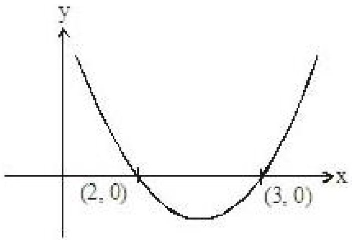
21. A piece of land with very good irrigation facilities yields three crops a year: A, B and C. If in an year, A's yield was 1/5 of its usual yield, B's was 1/2 of its usual yield, and C's was 2/5 of its usual yield, the total production of that year was 800 tonnes. Had it been $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$ of the usual yield respectively, the total yield would have been 1,200 tonnes. What is the normal yield of all the three put together?

a. 4,800 tonnes b. 2,800 tonnes c. 14,750 tonnes d. 22,400 tonnes e. 1,800 tonnes

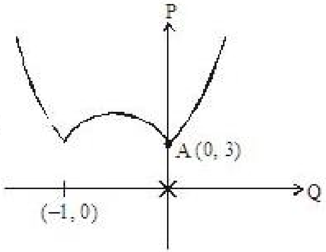
22. If Log 2, Log $(2^x - 1)$ & Log $(2^x + 3)$ are in A.P. then x equals.

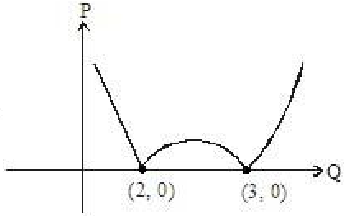
a. 2 b. 3 c. $Log_2 5$ d. $Log_5 2$ e. 5

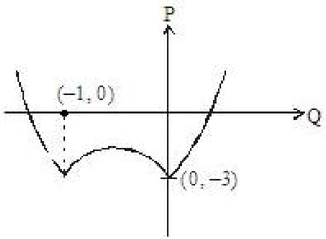
23. Real numbers x and y are related as y = (x - 2) (x - 3). When y is plotted against x; we get the following graph :-

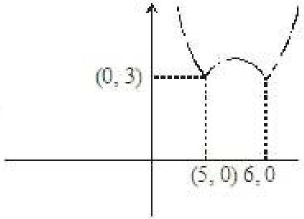


If another two real numbers, P and Q follow the relation P - 3 = |(Q - 5)(Q - 6)| then which of the following is the correct graph of P against Q ?

a.


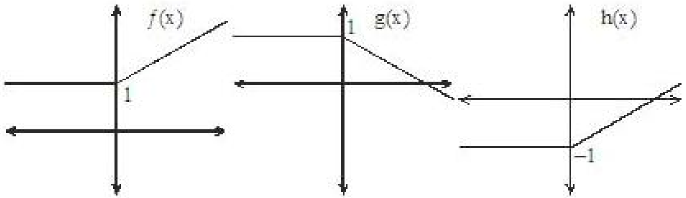
b.


c.


d.


e. None of these

24. In the following graphs all inclined lines make an angle 45° with positive x-axis.



What best describes their relationship?

a. f(x) = 1 - g(x) b. 1 - g(x) = h(x) c. f(x) = h(x) + 2

d. f(x) = h(x) + g(x) e. None of these

25. Referring to question 24, which of the following is true?

a. f(-x) = g(x) b. f(x) = -g(x) c. f(x) = -g(-x)

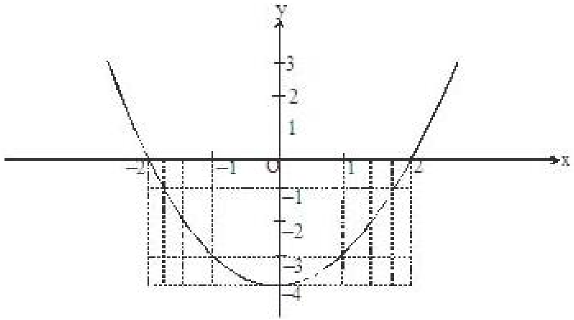
d. f(x) + g(x) = 1 e. None of these

26. A graph is a set of points connected by edges. Every edge connects a pair of points. Thus a triangle is a graph with three edges and three points. The degree of a point is the number of edges directly connected to it. Then the number of points of odd degree will be

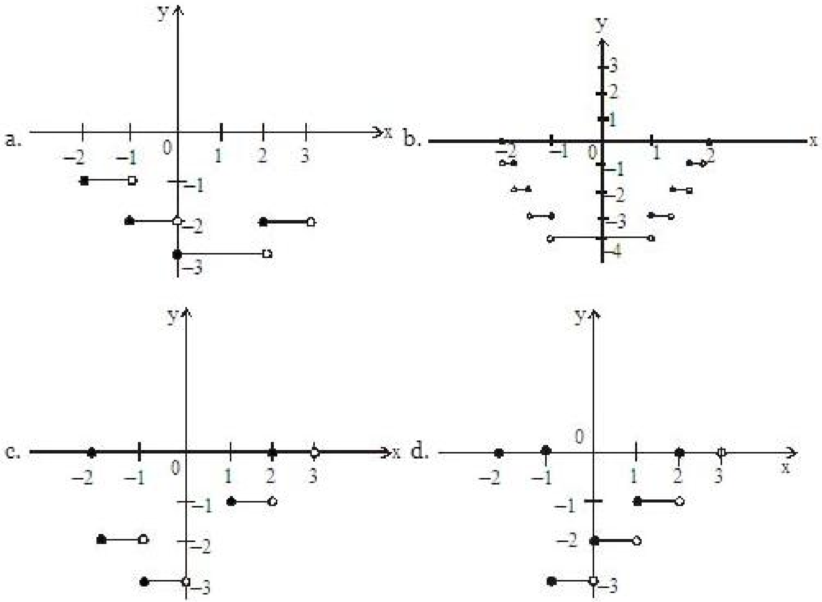
a. Odd only b. even only c. any integer

d. always zero e. Cannot be determined

27. The following graph shows the curve y against x.

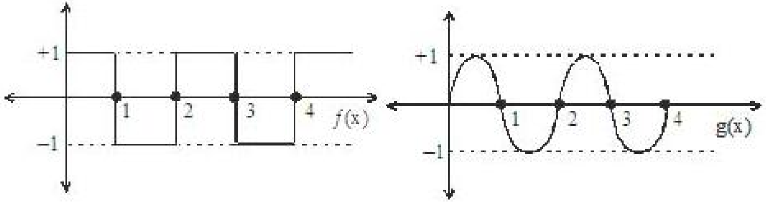


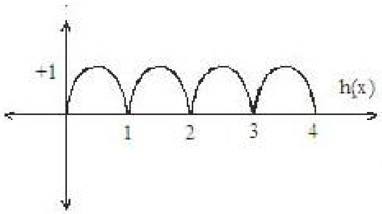
If y = f(x) then, which of the following graphs, shows the curve y = [f(x)] for $-2 \le x \le 2$?

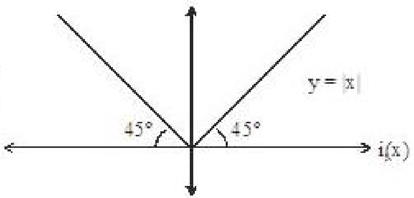


e. None of these

Direction for questions 28 and 29: Answer the questions based on the following figures.



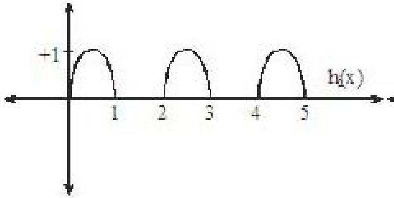


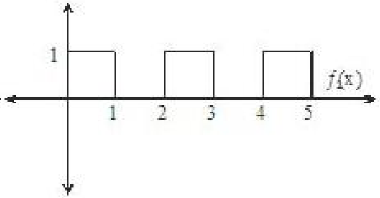


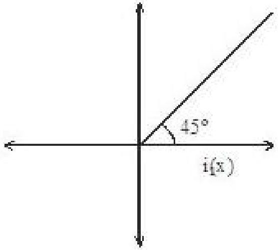
28. In the above graphs, $h_1(x)$ is given by

a. $h_1(x) = g(x) \times f(x)$ b. $h_1(x) = i_1(g(x))$ c. $h_1(x) = g(i_1(x))$

d. Both (a) and (b) e. Both (b) and (c)

29. 





In the above figures, g(x) is same as previous questions. Now $h_2(x)$ is given by

a. $h_2(x) = 1 - i_2(g(x))$ y b. $h_2(x) = i_2(g(x))$

c. $h_2(x) = f_2(x) \times g(x)$ d. Both (b) and (c)

e. Both (a) and (c)

30. A foreign currency dealer charges a commission of Rs 5 for exchanging up to US \$ 10, and an extra Rs 3 for every additional US \$ 20 or part thereof. If $\langle k \rangle$ denotes the smallest integer greater than or equal to k and [k] is the largest integer not exceeding k, then the amount Y of commission in Rupees, as a function of the amount X (>0) of US Dollars exchanged, is

a. $5+3\left\langle\frac{X-10}{20}\right\rangle$ b. $5+3\left\langle\frac{x+10}{20}\right\rangle$

c. $5+3\left[\frac{(X-10)}{20}\right]$ d. $5+3\left[\frac{(X+10)}{20}\right]$ e. None of these

31. If α and β are roots of the equation $x^2+5x-5=0$, then $\left(\frac{1}{\alpha+1}\right)^3+\left(\frac{1}{\beta+1}\right)^3$ equals

a. $\frac{-322}{27}$ b. $\frac{4}{27}$ c. $\frac{-4}{27}$ d. $\frac{322}{27}$ e. None of these

Answers Key

Chapter 1

1. $a = -4, p = 1, q = 6$

2. $a = 6, b = 6, c = -6, d = -6$

3. *(a)* $(x + 1)(x + 4)(x - 3)$ *(b)* $(x - 2)(x + 3)(x - 3)$

(c) $(x - 1)(3x - 1)(2x + 1)$

4. $k = \frac{32}{9}$

5. The values of k and *l* are inconsistent.

6. $(3x - 1)$

7. No common factor exists

8. *(a)* No *(b)* Yes

9. *(a)* Yes *(b)* Yes

10. One point viz $(0, 0)$

Chapter 2

1. $k \neq \frac{15}{4}$

2. $x = \frac{29+\sqrt{1033}}{24}, \frac{29-\sqrt{1033}}{24}$

3. $x = \frac{-41+\sqrt{721}}{48}, \frac{-41-\sqrt{721}}{48}$

4. $xy = 1$

5. $p = 20$

6. $x^2 - 5x - 23 = 0$

7. $x^2 - 24x + 128 = 0$

8. *(a)* $k = 0, \frac{-17}{36}$ *(b)* $0, \frac{17}{6}$

9. $x = 1$ and $x = 2$

10. a

11. d

12. c

13. Only one solution.

14. $x = 2, y = 5, z = 0$

15. 84

16. *(a)* $x > 13\frac{7}{11}$ *(b)* $x < \frac{16}{7}$

17. *(a)* $0 < x < 5$ *(b)* $x < 0$ or $x > 5$ *(c)* $x > 0$ OR $x < -5$ *(d)* $x < 0$ or $x > 1$

18. b

19. $x = 2$ or 4

20. $f(x) \geq 0$ for $0 \leq x \leq 1$

21. $x \geq 4, \; 0 \leq x \leq 2 \; \& \; x \leq -1$

22. $x > 0$ but x 1, 2

23. *(a)* $\frac{1}{5}, 2$ *(b)* $-2, \frac{-1}{5}$ *(c)* $\frac{1}{5}, 2$ *(d)* $-\infty, \infty$ *(e)* $\frac{-3}{5}, \frac{3}{5}$

24. $e = \frac{c}{3}$

25. e

26. No Real solution for x.

27. $x = \{-1$ and $-3\}$

28. 9th term

29. e

Chapter 3

1. *(i)* (3, 0); *(ii)* (2, 0); *(iii)* $\left(3, \frac{2}{3}\right)$; (iv) (4, -2)

2. (-2, 3)

3. *(i)* (2, 1) *(ii)* (-2, 2)

5. (0, 1); $y = 1$

6. $x = 2$

7. $\sqrt{10}$

8. $2x - y = 5$

9. $x + 2y = 4$

10. $x + 2y + 7 = 0$

12. 24 sq. units.

13. Figure (a) represents $f(x) = K(x + 2)(x - 3)$, where K is any positive number.

Figure (b) represents $f(y) = I(y + 2)(y - 5)$, where I is any positive number.

14. 1: 1

15. 200 m.

Chapter 4

1. $6 \le f(x) \le 66$

2. $3x^2 + 12x + 11$

3. $(a+b)(a-(b+1)) \ge 0$

4. (a) 20 (b) 0 (c) 0 (d) 30 (e) 2 (f) 0

5. (i) domain = R, range = R (ii) domain = R, range = [-2, $+\infty$ [

(iii) domain = all real numbers except 0. range = all real numbers except 0.

(iv) domain = $R^+ \cup 0$, range = $R^- \cup 0$

(v) domain = R^+, range = R

(vi) domain = R, range = [-1, 1]

(vii) domain = R, range = R^+

6. $-\sqrt{2} \le x \le \frac{-1}{\sqrt{3}}$ or $\frac{1}{\sqrt{3}} \le x \le \sqrt{2}$

7. c

8. *(a)* 2 *(b)*6 *(c)* (v)*(d)* $4x = y$

9. $\frac{3}{4}(a+b)$

10. *(a)* 9 *(b)* 2 *(c)*3

11. a **12.** a **13.** a

14. b **15.** a **16.** c

17. c **18.** a **19.** d

20. b **21.** c **22.** $\frac{7}{3}$

23. $x = \pm 2, \ \pm\sqrt{6}$

24. b **25.** d

26. *(a)* 2 *(b)* $\frac{1}{2}$ *(c)* $\frac{4}{3}$ *(d)* $\frac{3}{4}$

27. (a) 2 (b) 1.2

28. 4 **29.** b

30. (a) 150 (b) 7

31. $\frac{3}{2}$ **32.** c **33.** $\left(\frac{4}{3}\right)^9$

34. $\frac{9}{4}$ **35.** 1.908

36. 102 **37.** 6

38. 10000 **39.** d

40. a **41.** d

42. x = -2.8 and y = 1.4

43. $1 \le x \le 4$

14. b **15.** a **16.** c

17. c **18.** a **19.** d

20. b **21.** c **22.** $\frac{7}{3}$

23. $x = \pm 2, \ \pm\sqrt{6}$

24. b **25.** d

26. *(a)* 2 *(b)* $\frac{1}{2}$ *(c)* $\frac{4}{3}$ *(d)* $\frac{3}{4}$

27. (a) 2 (b) 1.2

28. 4 **29.** b

30. (a) 150 (b) 7

31. $\frac{3}{2}$ **32.** c **33.** $\left(\frac{4}{3}\right)^9$

34. $\frac{9}{4}$ **35.** 1.908

36. 102 **37.** 6

38. 10000 **39.** d

40. a **41.** d

42. x = -2.8 and y = 1.4

43. $1 \le x \le 4$

44. $x \le \frac{-2}{3}$; $x \ge \frac{1}{2}$

45. The points of (x, y) are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

46. *(a)* 2.5 *(b)* 1

47. *(a)* 18 *(b)* 16 *(c)* $\frac{9}{2}$

48. One

49. One

50. $1 < x < \frac{4}{3}$

51. $x > 127$

Chapter 5

1. $S = \frac{5}{9}\left[\frac{10(10^{20}-1)}{9} - 20\right]$

2. $\frac{4}{9}$

3. 2045

4. *(a)* 325 m *(b)* 405 m

5. 6 days

6. $\frac{n(n+1)(n+2)}{6}$

7. 12

8. *(a)* $\frac{n(n+1)}{12}[(9n+7)(n+2)]$

(b) $\frac{n(n+1)(n^2+n+2)}{4}$

9. $\frac{64}{343}$

10. $2(n^4 + 4n^3 + 5n^2 + 2n)$

Chapter 6

1. d

2. 27

3. 16

4. $-\frac{5}{4}$

5. No Maxima

6. 4, - 5

7. Circle

8. -1

9. $\frac{10}{3}, \frac{40}{3}, \frac{40}{3}$

10. 3, 2

Chapter 7

Practice Exercise 1 - Level 1

1	c	**2**	a	**3**	e	**4**	d	**5**	d	**6**	b	**7**	e	**8**	a	**9**	d	**10**	c
11	a	**12**	b	**13**	a	**14**	d	**15**	e	**16**	d	**17**	a	**18**	c	**19**	d	**20**	b
21	d	**22**	c	**23**	b	**24**	c	**25**	b	**26**	a	**27**	e	**28**	b	**29**	c	**30**	a
31	a	**32**	a	**33**	d	**34**	b	**35**	e	**36**	a	**37**	a	**38**	e	**39**	b	**40**	a

Practice Exercise 2 - Level 1

1	a	**2**	e	**3**	a	**4**	e	**5**	c	**6**	d	**7**	b	**8**	a	**9**	b	**10**	c
11	c	**12**	c	**13**	c	**14**	e	**15**	e	**16**	d	**17**	e	**18**	c	**19**	b	**20**	c
21	b	**22**	e	**23**	c	**24**	b	**25**	e	**26**	c	**27**	c	**28**	e	**29**	d	**30**	d
31	e	**32**	a	**33**	e	**34**	c	**35**	c	**36**	e	**37**	c	**38**	a				

Practice Exercise 3 - Level 1

1	a	**2**	d	**3**	a	**4**	c	**5**	d	**6**	b	**7**	e	**8**	c	**9**	a	**10**	d
11	e	**12**	d	**13**	c	**14**	e	**15**	a	**16**	d	**17**	b	**18**	a	**19**	e	**20**	c
21	b	**22**	a	**23**	b	**24**	c	**25**	e	**26**	d	**27**	d	**28**	a	**29**	e	**30**	d
31	c	**32**	c	**33**	b	**34**	b	**35**	c	**36**	d	**37**	a	**38**	a	**39**	b	**40**	b

Practice Exercise 4 - Level 1

1	b	**2**	b	**3**	a	**4**	a	**5**	d	**6**	b	**7**	a	**8**	c	**9**	d	**10**	d
11	c	**12**	d	**13**	a	**14**	b	**15**	c	**16**	d	**17**	b	**18**	a	**19**	b	**20**	a
21	b	**22**	b	**23**	e	**24**	c	**25**	a	**26**	c	**27**	d	**28**	c	**29**	c		

Practice Exercise 5 - Level 2

1	e	**2**	e	**3**	d	**4**	d	**5**	b	**6**	a	**7**	d	**8**	d	**9**	a	**10**	d
11	c	**12**	d	**13**	b	**14**	e	**15**	d	**16**	a	**17**	e	**18**	d	**19**	c	**20**	e
21	b	**22**	a	**23**	d	**24**	d	**25**	a	**26**	d	**27**	b	**28**	c	**29**	d	**30**	a
31	b	**32**	b	**33**	a	**34**	d	**35**	b	**36**	d	**37**	c	**38**	a	**39**	d	**40**	d

Practice Exercise 6 - Level 2

1	e	**2**	b	**3**	c	**4**	a	**5**	a	**6**	b	**7**	d	**8**	c	**9**	b	**10**	c
11	c	**12**	c	**13**	e	**14**	a	**15**	d	**16**	a	**17**	c	**18**	c	**19**	c	**20**	b
21	b	**22**	c	**23**	a	**24**	b	**25**	d	**26**	c	**27**	c	**28**	c	**29**	c	**30**	b
31	c	**32**	b	**33**	a	**34**	d	**35**	a	**36**	e	**37**	a	**38**	c	**39**	b	**40**	e
41	a	**42**	d																

Practice Exercise 7 - Level 2

1	d	**2**	e	**3**	d	**4**	d	**5**	e	**6**	d	**7**	b	**8**	a	**9**	c	**10**	d
11	b	**12**	c	**13**	b	**14**	e	**15**	c	**16**	c	**17**	b	**18**	c	**19**	e	**20**	b
21	d	**22**	a	**23**	a	**24**	a	**25**	b	**26**	c	**27**	e	**28**	b	**29**	b	**30**	b
31	b	**32**	d	**33**	c	**34**	d	**35**	d	**36**	b	**37**	c	**38**	c	**39**	d	**40**	b
41	c	**42**	d	**43**	d	**44**	b	**45**	d	**46**	b	**47**	a	**48**	e				

Practice Exercise 8 - Level 3

1	e	**2**	d	**3**	a	**4**	a	**5**	c	**6**	b	**7**	d	**8**	d	**9**	c	**10**	d
11	a	**12**	b	**13**	e	**14**	c	**15**	e	**16**	e	**17**	b	**18**	a	**19**	d	**20**	c
21	b	**22**	c	**23**	d	**24**	c	**25**	e	**26**	b	**27**	b	**28**	d	**29**	c	**30**	a
31	b																		

Explanations: Fundamentals of Algebra

Chapter 1

Level - I

1. $(x + 2)(x - a) = px^2 + qx + 8$

$= x^2 + 2x - ax - 2a = px^2 + qx + 8$

$= x^2 + (2 - a)x - 2a = px^2 + qx + 8$

Equating the coefficients of x^2, x and constant terms on both the sides, $p = 1$; $q = 2 - a$ $-2a = 8$

Solving, we get $a = -4$, $p = 1$, $q = 6$

2. $(x + 1)(2x - 2)(3x + 3) = ax^3 + bx^2 + cx + d$

$= 6(x + 1)(x - 1)(x + 1) = ax^3 + bx^2 + cx + d$

$= 6(x^2 - 1))(x + 1) = ax^3 + bx^2 + cx + d$

$= 6(x^3 + x^2 - x - 1) = ax^3 + bx^2 + cx + d$

$= 6x^3 + 6x^2 - 6x - 6 = ax^3 + bx^2 + cx + d$

Equating the coefficients of like terms on both the sides, $a = 6$, $b = 6$, $c = -6$ and $d = -6$.

3. (a) $x^3 + 2x^2 - 11x - 12$

Putting $x = -1$, $(-1)^3 + 2(-1)^2 - 11(-1) - 12 = 0$

$\therefore$ By Remainder Theorem, $(x + 1)$ is a factor of given polynomial

–1	1	2	–11	–12
		–1	–1	12
	1	1	–12	0

$\therefore$ Other factor of given polynomial is $x^2 + x - 12$

$\therefore$ Given polynomial $= (x + 1)(x^2 + x - 12) = (x + 1)(x^2 + 4x - 3x - 12) = (x + 1)(x + 4)(x - 3)$

(b) $x^3 - 2x^2 - 9x + 18$

Putting $x = 2$, $(2)^3 - 2(2)^2 - 9(2) + 18 = 0$

$\therefore$ $(x - 2)$ is a factor of given polynomial.

2	1	–2	–9	18
		2	0	–18
	1	0	–9	0

$\therefore$ Other factor of given polynomial is $x^2 - 9$.

$\therefore$ Given polynomial $= (x - 2)(x^2 - 9)$

$= (x - 2)(x + 3)(x - 3)$

(c) $6x^3 - 5x^2 - 2x + 1$

Putting $x = +1$

$6(1)^3 - 5(1)^2 - 2(1) + 1 = 0$

$\therefore$ $(x - 1)$ is a factor

$$\begin{array}{c|cccc} 1 & 6 & -5 & -2 & 1 \\ & & 6 & 1 & -1 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

$\therefore$ $(6x^2 + x - 1)$ is the other factor which is equal to

$(6x^2 + 3x - 2x - 1) = 3x(2x + 1) - 1(2x + 1) = (2x + 1)(3x - 1)$

$\therefore$ Given polynomial $= (x - 1)(2x + 1)(3x - 1)$.

4. Applying the remainder theorem,

If $f(x) = (x^3 + kx^2 + 3x + 4)$ is divisible by $(x + 3)$, then $f(-3) = 0$

i.e. $(-3)^3 + k(-3)^2 + 3(-3) + 4 = 0$

or $k = \frac{32}{9}$.

5. If $f(x) = kx^3 + 4x^2 + lx + 5$ is divisible by $(x^2 - 1)$, it must be divisible by both $(x - 1)$ and $(x + 1)$.

So applying remainder theorem, we have

$f(1) = 0$ and $f(-1) = 0$

$k + 4 + l + 5 = 0$

or $k + l = -9$ ... (i)

$-k + 4 - l + 5 = 0$

or $k + l = 9$ ... (ii)

Since (i) and (ii) are inconsistent, $f(x)$ cannot be divisible by $(x^2 - 1)$, for real values of x, K and L.

6. $3x^2 + 5x - 2 = (x + 2)(3x - 1)$

$3x^2 - 7x + 2 = (x - 2)(3x - 1)$

Common factor $= (3x - 1)$

7. $2x^2 - x = x(2x - 1)$

$4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Hence, no common factors exists.

8. $x^n + a^n$ is divisible by $x + a$ only when n is odd.

9. $x^n - a^n$ is always divisible by $(x - a)$

Level - II

10. The graph $y = x^3 - x^2 + 2x$ will cut x-axis when $y = 0$

$\therefore x^3 - x^2 + 2x = 0 \Rightarrow x(x^2 - x + 2) = 0$

$\Rightarrow x = 0$ [$\because x^2 - x + 2$ has imaginary roots]

$\therefore$ There is only one real root of the given cubic equation

$\therefore$ The given graph will intersect x-axis at only one point viz. (0, 0)

Chapter 2

Level - I

1. For the equations to be independent,

condition is $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ $\therefore \frac{k}{3}=\frac{5}{4}$ or $k=\frac{15}{4}$

2. On solving, we get $12x^2 - 29x - 4 = 0$

$\therefore x = \frac{-(-29)\pm\sqrt{(-29)^2-4(12)(-4)}}{2\times 12}$

$\therefore x = \frac{29\pm\sqrt{1033}}{24}$

$\therefore x = \frac{29+\sqrt{1033}}{24}, \frac{29-\sqrt{1033}}{24}$

3. $x+\frac{1}{x+2}=\frac{7}{24}$

On solving, we get $24x^2 + 41x + 10 = 0$

$\therefore x = \frac{-41\pm\sqrt{(41)^2-4(24)(10)}}{2\times 24}$

$x = \frac{-41\pm\sqrt{1681-960}}{48}$

$x = \frac{-41+\sqrt{721}}{48}, \frac{-41-\sqrt{721}}{48}$.

4. $x + y = 9$; $xy^2 + yx^2 = 9$

Therefore, $xy(x + y) = 9$ or $xy(9) = 9$, i.e. $xy = 1$

5. Roots must satisfy the equation.

$\therefore 5(3)^2 - p(3) + 15 = 0 \therefore p = 20$

6. Given $a + b = \frac{-(-1)}{1}=1$ and $ab = \frac{-3}{1}=-3$

Equation whose roots are $(3a + 1)$ and $(3b + 1)$ is

x^2 - x(Sum of roots) + (Product of roots) = 0

i.e. $x^2 - x(3a + 3b + 2) + (9ab + 3a + 3b + 1) = 0$

$\Rightarrow x^2 - 5x - 23 = 0$

7. Let the roots be α, β.

$\alpha + \beta = 24$, $\alpha - \beta = 8$. Then $\alpha = 16$ and $\beta = 8$

$\therefore$ Equation is $x^2-(\alpha+\beta)x+\alpha\beta=0$

i.e. $x^2-24x+128=0$.

8. $4x^2 - 11x + 2k = 0$, $x^2 - 3x - k = 0$

Let the common root be α. Now putting the value of x as α in both the equations, we get

$4\alpha^2-11\alpha+2k=0$ and $\alpha^2-3\alpha-k=0$

On solving, we get

$\alpha=0, \frac{17}{6}$ and $k=0, \frac{-17}{36}$.

9. If one of the roots is -2, then

$(x + 2)$ must be a factor of $f(x) = x^3 - x^2 - 4x + 4$. Hence, $(x + 2) . g(x) = f(x)$

$$\therefore g(x) = \frac{f(x)}{(x+2)} = (x^2 - 3x + 2)$$

Also g(x) = (x - 1)(x - 2)

Hence, f(x) is also divisible by (x - 1) and (x - 2), and the other roots are x = 1 and x = 2.

10. a $f(x) = \frac{1 - tx + tx^2}{1 + tx + tx^2} = \frac{1 - \frac{t}{4} + t\left(x - \frac{1}{2}\right)^2}{1 - \frac{t}{4} + t\left(x + \frac{1}{2}\right)^2}$

Since $t \in [0, 1], \left(1 - \frac{t}{4}\right)$ is always positive.

So numerator and denominator of above expression is also positive.

Hence f(x) cannot be negative.

11. d $(x - a)(x - b)(x - c) = x^3 - 3x^2 + 2x + 1$

Equating the coefficients on LHS and RHS,

-(a + b + c) = -3; (ab + bc + ac) = 2 and -abc = 1

Now $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ac}{abc} = -2$

12. c $2^{x+5} = 2^{x+3} + 6$

$\Rightarrow 2^5 2^x = 2^x \times 2^3 + 6 \Rightarrow 2^x(24) = 6$

$\Rightarrow 2^x = \frac{6}{24} \Rightarrow 2^x = 2^{-2}$ and x = -2

13. The equation can be rewritten as:

$$\frac{(x-1)(x-3)}{x-4} - x = \frac{(x-2)(x-5)}{(x-7)} - x$$

$$\Rightarrow \frac{x^2 - 4x + 3 - x^2 + 4x}{x-4} = \frac{x^2 - 7x + 10 - x^2 + 7x}{(x-7)} \Rightarrow \frac{3}{(x-4)} = \frac{10}{(x-7)} \Rightarrow 3x - 21 = 10x - 40$$

$$\Rightarrow 7x = 19 \Rightarrow x = \frac{19}{7}$$

Hence, only one value of x exists.

14. 2x + 5y = 29 ... (i)

3x + 2z = 6 ... (ii)

5z - 2y = -10 ... (iii)

Eliminating `x', 3 × (i) - 2 × (ii)

15y - 4z = 75

So, we have

15y - 4z = 75 ... (iv)

5z - 2y = -10

Eliminating `y', 2 × (iv) + 15 × (iii)

67z = 0 ⇒ z = 0

Substituting z = 0 in (ii) and (iii), we get

x = 2 and y = 5

Hence, the solution is x = 2, y = 5 and z = 0

15. $T_{r+1} = {}^9C_r \left(x^2\right)^{9-r} \cdot \left(\frac{-1}{x}\right)^r$

Constant term must have exponent of x as zero.

$\Rightarrow 2(9-r)-r=0$

$\Rightarrow 18-3r=0 \Rightarrow r=6$

$\Rightarrow$ Constant term $= T_{6+1} = {}^9C_6\left(x^2\right)^{9-6}.\left(\frac{-1}{x}\right)^6 = {}^9C_6 = 84$

16. (a)
$$\begin{array}{c|c} 2x-5<3x & 4-\frac{x}{3}<\frac{2x}{5}-6 \\ \Rightarrow x>-5 & \Rightarrow 4+6<\frac{2x}{5}+\frac{x}{3} \\ & \Rightarrow \frac{11x}{15}>10 \\ & \Rightarrow x>\frac{150}{11} \\ & \Rightarrow x>13\frac{7}{11} \end{array}$$

Taking common values, the solution set is $x>13\frac{7}{11}$.

(b)
$$\begin{array}{l|l} x+5(4-2x)>4-2x & 3-\frac{x}{3}>3 \\ \Rightarrow x+20-10x>4-2x & \Rightarrow 0>\frac{x}{3} \\ \Rightarrow 16>7x & \Rightarrow x<0 \\ \Rightarrow x<\frac{16}{7} & \end{array}$$

Combining both solutions set is $x<\frac{16}{7}$.

17. (a) $\frac{1}{x}>\frac{1}{5}$

We cannot cross multiply since we don't know whether x is positive or negative

$$\begin{array}{r|l} \text{If } x>0 & \text{If } x<0 \\ 5>x & \Rightarrow 5<x \\ \Rightarrow x<5 & \Rightarrow x>5 \\ \therefore 0<x<5 & \text{No solution.} \end{array}$$

$\therefore$ Solution set for given inequation is $0 < x < 5$.

(b) $\frac{1}{x}<\frac{1}{5}$

$$\begin{array}{l|l} \text{If } x>0 & \text{If } x<0 \\ 5<x & 5>x \\ \Rightarrow x>5 & \Rightarrow x<5 \\ \therefore x>5 & \therefore x<0 \end{array}$$

$\therefore$ Solution set for given inequation is $x < 0$ or $x > 5$.

(c) $\frac{1}{x}>\frac{-1}{5}$

$$\begin{array}{r|r} \text{If } x>0 & \text{If } x<0 \\ x>-5 & x<-5 \\ \text{Combining we get} & \text{Combining we get} \\ x>0 & x<-5 \end{array}$$

$\therefore$ Solution set for given inequation is $x > 0$

OR $x < -5$.

(d) $3-\frac{3}{x}>0$

$1-\frac{1}{x}>0$

$\frac{-1}{x}>-1$

$\frac{1}{x}<1$

$$\begin{array}{r|r} \text{If } x>0 & \text{If } x<0 \\ x>1 & x<1 \\ \text{Combining we get} & \text{Combining we get} \\ x>1 & x<0 \end{array}$$

∴ Solution set for given inequation is $x < 0$ or $x > 1$.

18. Correct answer is (b).

19. $x^2 - 3x - 10 \le 0 \Rightarrow (x-5)(x+2) \le 0$

Hence, $-2 \le x \le 5$

Since $x \in I^+$ and $x \in$ even numbers, the possible values of $x = 2$ and 4.

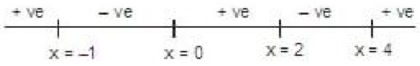
20. $f(x) = (x^2 - 2x - 8)(x - 1) \Rightarrow (x-4)(x+2)(x-1)$

$(x-4)(x+2)(x-1)^2$ 0 Þ x ³ 4 or -2 £ x £ 1

But since 0 £ x £ $\frac{3}{2}$, the required range of values of x is 0 £ x £ 1.

21. Let us solve f(x) using the number-line.

The roots of the expression are $x = -1$, $x = 0$; $x = 2$ and $x = 4$.



As in the figure, if

(a) $x \ge 4,\ f(x) \ge 0$

(b) $2 \le x \le 4,\ f(x) \le 0$

(c) $0 \le x \le 2,\ f(x) \ge 0$

(d) $-1 \le x \le 0,\ f(x) \le 0$

(e) $x \le -1,\ f(x) \ge 0$

As $f(x) \ge 0$ is given possible values of x are $x \ge 4$, $0 \le x \le 2$ and $x \le -1$.

22. $-x(x - 2)(x - 2)(x - 1)^2 < 0$

$\Rightarrow x(x - 2)^2 (x - 1)^2 > 0$

$(x - 2)^2 (x - 1)^2$ is always > 0

So sign of LHS depends on the value of x only.

$\therefore x(x - 2)^2 (x - 1)^2 > 0$ when $x > 0$ but $x = 1, 2$

23. (a) $3 \le x \le 10$ and $5 \le y \le 15$

Minimum value of $\frac{x}{y}$ will be

when x is minimum and

when y is maximum

$$\min\left(\frac{x}{y}\right) = \frac{3}{15} = \frac{1}{5}$$

max value of will be, when x is max and y is min.

$$\max\left(\frac{x}{y}\right) = \frac{10}{5} = 2$$

(b) $-10 \le x\ -3$ and $5 \le y \le 15$

By putting values intelligently we can find out

$$\min\left(\frac{x}{y}\right) = \frac{-10}{5} = -2\ ;\ \max\left(\frac{x}{y}\right) = \frac{-3}{15} = \frac{-1}{5}$$

(c) $-10 \le x\ -3$ and $-15 \le y \le -5$

$\min\left(\frac{x}{y}\right)=\left(\frac{-3}{-15}\right)=\frac{3}{15}=\frac{1}{5}$; $\max\left(\frac{x}{y}\right)=\left(\frac{-10}{-5}\right)=2$

(d) $-3 \le x \le 3$ and $-5 \le y \le 5$

$\min\left(\frac{x}{y}\right)=\left(\frac{-3}{0}\right)=-\infty$; $\max\left(\frac{x}{y}\right)=\left(\frac{3}{0}\right)=\infty$

(e) $-3 \le x \le 3$ and $5 \le y \le 15$

$\min\left(\frac{x}{y}\right)=\frac{-3}{5}$; $\max\left(\frac{x}{y}\right)=\frac{3}{5}$

Level - II

24. Product of all the roots = $\frac{e}{a}$

Sum of the roots taking 2 at a time = $\frac{c}{a}$

If $\frac{e}{a}=\frac{1}{3}\left(\frac{c}{a}\right)$, we get $e=\frac{c}{3}$

25. e $a + b + c = 3$, $a^2 + b^2 + c^2 = 6$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$

$(a + b + c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$\Rightarrow (a+b+c)^2 = a^2+b^2+c^2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\times abc$

$3^2 = 6 + 2(1) \times abc$

$\Rightarrow abc = \frac{3}{2}$

26. Here $|x^2-x-5| = -(3x+6)$

as $|y| \ge 0$, so the above equation can have solution if $(3x + 6) \le 0$

Also $|y| = a \Rightarrow y = \pm a$ where $a \ge 0$

Thus $(x^2-x-5) = \pm(3x+6)$

Now $(x^2-x-5) = -(3x+6)$

$\Rightarrow x^2 + 2x + 1 = 0 \Rightarrow (x + 1)^2 = 0 \Rightarrow x = -1$

Again $(x^2 - x - 5) = (3x + 6)$

$\Rightarrow x^2 - 4x - 11 = 0$

$\Rightarrow x=\frac{4\pm\sqrt{16+44}}{2}=\frac{4\pm2\sqrt{15}}{2}=2\pm\sqrt{15}$

When $x = -1$, $3x + 6 = 3 \times (-1) + 6 = 3 > 0$

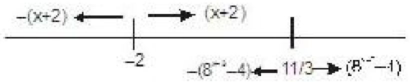
$x = 2 + \sqrt{15}$, $3x + 6 = (6+3\sqrt{15}+6) > 0$

and $x = 2 - \sqrt{15}$, $3x + 6 = 6 + 6 - 2\sqrt{15} > 0$

Hence , no value of x exists.

27. Here the moduli are $|x+2|$ and $|8^{x-3}-4|$.

Their definition are as follows:



If $x < -2$, the equation is

$8^{-(x+2)} - 8^{(x-3)} = -(8^{x-3}-4)+4$

$\Rightarrow 8^{-(x+2)} = 8 \Rightarrow -(x+2) = 1 \Rightarrow x = -3$

$\therefore$ x = -3, which satisfies x < -2.

If $-2 \le x < \frac{11}{3}$, the equation is $8^{(x+2)} - 8^{(x-3)} = -(8^{x-3} - 4) + 4$

$\Rightarrow 8^{(x+2)} = 8 \Rightarrow x + 2 = 1 \Rightarrow x = -1$

$\therefore$ x = -1, which satisfies $-2 \le x < \frac{11}{3}$

If $x \ge \frac{11}{3}$, the equation is $8^{(x+2)} - 8^{(x-3)} = (8^{x-3} - 4) + 4$

$\Rightarrow 8^{(x+2)} = 2 \times 8^{x-3} \Rightarrow 2^{3x+6} = 2 \times 2^{3x-9}$

$\Rightarrow 2^{3x+6} = 2^{3x-8} \Rightarrow 3x + 6 = 3x - 8$

$\Rightarrow 6 = -8$ which is never possible.

Hence the solution set is {3, -1}.

28. Here $T_{r-3} = {}^nC_{r-4} \times x^{r-4}$

$\therefore$ coefficient is ${}^nC_{r-4}$

$T_{2r+1} = {}^nC_{2r} \times x^{2r}$

$\therefore$ coefficient is ${}^nC_{2r}$

Now ${}^nC_{r-4} = {}^nC_{2r} \Rightarrow r - 4 + 2r = n$ or r - 4 = 2r

$\Rightarrow 3r - 4 = 23$ or r = -4 (not possible)

$\Rightarrow 3r = 27 \Rightarrow r = 9$

29. e Consider the value of a and b satisfying the given condition a + b > ab

Suppose a = 1.5 and b = 1

Now option (a) doesn't satisfy

Option (b) doesn't satisfy

Option c: $\frac{1}{a} + \frac{1}{b} > 1$

itself is other form of a + b > ab

Hence answer is none of these.

Chapter 3

Level - I

1. (i) Solving the two equations simultaneously, we get $x = 3$ and $y = 0$

Hence, point of intersection = $(3, 0)$

(ii) Solving the two equations simultaneously, we get $x = 2$ and $y = 0$

Hence, point of intersection = $(2, 0)$

(iii) Solving the two equations simultaneously, we get the point of intersection to be $\left(3, \frac{2}{3}\right)$.

(iv) Solving the two equations simultaneously, we get the point of intersection to be $(4, -2)$.

2. Equation of the line passing through $(3, 4)$ and $(8, 5)$

$$y - 4 = \frac{5-4}{8-3}(x-3) \Rightarrow 5y - x = 17$$

Common point between the two given lines can be obtained by solving the equations of the two lines simultaneously.

Hence, the common point = $(-2, 3)$

3. (i) Let $O(x, y)$ be the circumcentre of ΔABC.

Then $AO = BO = CO$ = Radii of the circumcircle

$AO^2 = (x - 3)^2 + (y - 1)^2$

$BO^2 = (x - 2)^2 + (y - 2)^2$

$CO^2 = (x - 2)^2 + (y - 0)^2$

Then $(x - 3)^2 + (y - 1)^2 = (x - 2)^2 + (y - 2)^2 \Rightarrow x - y = 1$

Also $(x - 2)^2 + (y - 2)^2 = (x - 2)^2 + (y - 0)^2 \Rightarrow y = 1$

Hence, $x = 2$

Therefore, circumcentre is $(2, 1)$.

(ii) As we worked in the previous question,

$AO^2 = BO^2 = CO^2$

Therefore, $(x - 0)^2 + (y - 0)^2 = (x + 4)^2 + (y - 0)^2$

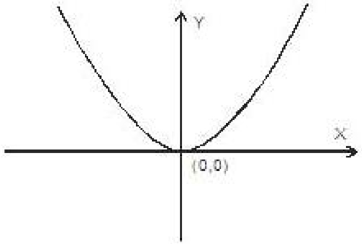
$\Rightarrow x = -2$

Also, $(x + 4)^2 + (y - 0)^2 = (x - 0)^2 + (y - 4)^2$

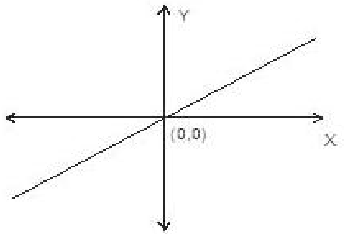
$\Rightarrow y = -x = 2$

Therefore, circumcentre = $(x, y) = (-2, 2)$

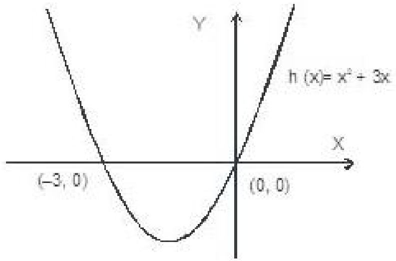
4. (a) $f(x) = x^2$



(b) $g(x) = 3x$



(c) $f(x) + g(x) = h(x) = x^2 + 3x = x(x + 3)$



The resulting graph is also a parabola.

Level - II

5. The point of intersection of the two lines can be obtained by solving the two equations simultaneously.

Point of intersection = (0, 1)

Since the line passing through (0, 1) is parallel to the x-axis, it is of the form y = a

Hence, equation of the line is y = 1

6. $x + y - 5 = 0$

$2x + 3y = 13$

Solving the equations simultaneously, we get

$x = 2$ and $y = 3$

Hence, the equation of the line parallel to Y-axis and passing through (2, 3) is x = 2

7. Since the lines are the diameters of a circle, they will intersect at the centre.

Solving $x + y - 5 = 0$ and $3x - y + 1 = 0$, we get

$x = 1$ and $y = 4$

So (1, 4) are the coordinates of the centre.

Radius = The distance between (1, 4) and (0, 1) $= \sqrt{(0-1)^2 + (1-4)^2} = \sqrt{10}$ units

8. Point of intersection of the lines $2x + y = 3$ and $3x - y = 7$ is (2, -1).

Hence, (2, -1) is a point on the line whose slope = Slope of the line $2x - y + 5 = 0$, i.e. 2.

∴ Equation of the line passing through (2, -1), and parallel to $2x - y + 5 = 0$ is

$(y + 1) = 2(x - 2) \Rightarrow 2x - y = 5$

9. Point of intersection of $x + y = 3$ and $2x + y = 5$ is (2, 1).

Mid-point of line joining (1, 5) and (-5, 1) is $\left(\frac{1-5}{2}, \frac{5+1}{2}\right)$, i.e. (-2, 3)

Equation of the line passing through (2, 1) and (-2, 3) is

$y - 3 = \frac{1-3}{2-(-2)}(x+2)$

$y - 3 = -\frac{1}{2}(x+2)$

$\Rightarrow 2y - 6 = -x - 2 \Rightarrow x + 2y = 4$

10. Point of intersection of the two lines = (-1, -3)

Hence, (-1, -3) is a point on the required line.

Slope of the line joining (2, 3) and (1, 1) = $m_1 = \frac{2}{1} = 2$

If the slope of the required line = m_2

Then $m_1 m_2 = -1$

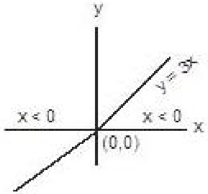
$$\Rightarrow m_2 = -\frac{1}{2}$$

Therefore, equation of the line is

$$y+3 = -\frac{(x+1)}{2} \Rightarrow 2y+6 = -x-1$$

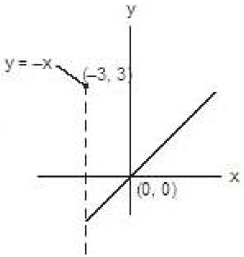
$$\Rightarrow x+2y+7=0$$

11. (i)

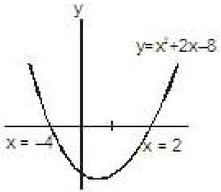


At x = 0, y = 0

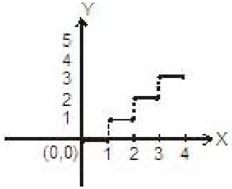
(ii)



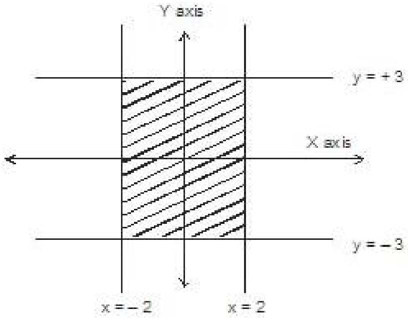
(iii) $f(x) = x^2 + 2x - 8 = (x + 4)(x - 2)$



(iv)



12.

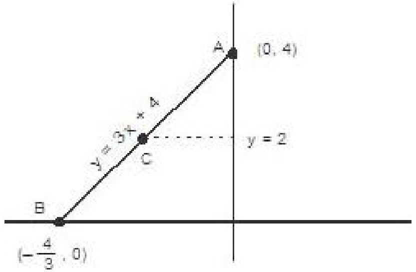


The area bound by the graphs is the shaded region. The area = 6 × 4 = 24 sq. units.

13. Figure (a) represents f(x) = K(x + 2)(x - 3), where K is any positive number.

Figure (b) represents f(y) = l(y + 2)(y - 5), where l is any positive number.

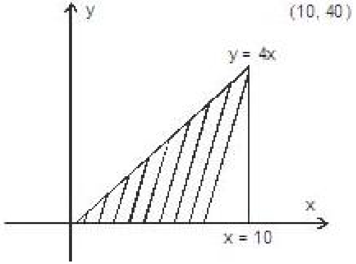
14. If the two ants meet when the Y coordinate is 2. Then



they must be meeting mid-way between A and B.

Hence, ratio of speeds = 1 : 1

15.

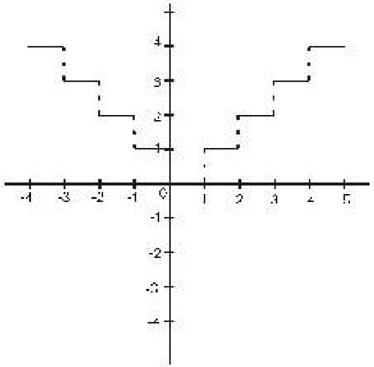


The total distance covered is equal to the area of the shaded region.

Area of th e shaded region = $\frac{1}{2} \times 10 \times 40 = 200$

Hence, distance covered = 200 m.

16.



Chapter 4

Level - I

1. f(x) is a strictly increasing function.

f(x) at x = 1 ⇒ 6

f(x) at x = 4 ⇒ 66

Hence, the range of f(x) is $6 \le f(x) \le 66$.

2. $f(g(x)) = f(x+2) = 3(x+2)^2 - 1$

$\Rightarrow 3(x^2+4+4x) - 1 \Rightarrow 3x^2 + 12x + 11$

3. If $f(a,\ b) \ge g(a,\ b)$, then

$(a^2 - b^2) \ge (a+b) \Rightarrow (a+b)(a-b-1) \ge 0$

i.e. $(a+b)(a-(b+1)) \ge 0$

For the relationship to be true, $(a+b)(a-(b+1)) \ge 0$

4. (a) Here A is domain and B is co-domain. In a one to one function every element of Range has exactly one distinct image in domain.

Number of choices to select image for red = 5

Number of choices to select image for blue = 4

Number of one to one functions from A to B = 4 × 5 = 20

(b) Here domain is B and co-domain is A. Since there are 5 elements in domain and only 2 elements in co-domain, we cannot have distinct image for all elements of domain and hence there is no one to one function from B to A.

(c) For onto functions, number of elements in co-domain should be less than or equal to number of elements in domain. In this case, domain A has 2 elements and co-domain B has 5 elements which does not satisfy the above criteria.

Hence there will be no onto function from A to B.

(d) Here domain B has 5 elements and co-domain A has 2 elements. For onto functions, every element of domain should be mapped to one of the two elements of co-domain such that there is no element in co-domain, which is not mapped to at least one element of domain.

The situation is quite similar to where we have to distribute 5 distinct objects in 2 groups such that none of the group is empty.

Since each of the five objects has 2 choices,

Number of ways to distribute 5 objects in 2 groups = 25 = 32

Number of cases where one of the two groups is empty = 2

Required number of ways = 32 - 2 = 30

Hence there will be 30 onto functions from B to A.

(e) We know that a function, which is not onto, is into.

Total number of functions from B to A = 2^5 = 32 (because every element of domain has 2 choices for mapping).

Number of onto functions = 30 (from last question)

Number of into functions = 32 - 30 = 2

(f) For bijective functions, both domain and co-domain should have equal number of elements, which is not the case here.

Hence no bijective function is there from A to B.

5. (i) Domain is R as f(x) will be real for all real numbers x.

Range is also R as f(x) can have all real values.

(ii) Domain is R as f(x) will be real for all real numbers x.

Range is $[-2, +\infty)$ as minimum value of f(x) is -2 and there is no maximum value

(iii) Domain is all real numbers except 0 as f(x) will be real for all values of x except 0.

Range is also all real numbers except 0 as f(x) can have all real values except 0.

(iv) Domain is $R^{-} \cup 0$ as f(x) will be real only when x is positive or 0.

Range is $R^{-} \cup 0$ as f(x) will either be 0 or it can be any negative real number.

(v) Domain is R^{+} as f(x) will be real only when x is positive.

Range is R as f(x) can have all real values.

(vi) Domain is R as f(x) will be real for all real values of x.

Range is $[-1, 1]$ as f(x) can vary from -1 to +1.

(vii) Domain is R as f(x) will be real for all real values of x.

Range is R+ as f(x) can have all positive real values.

6. $f(g(x)) = 3(x^2 + 1) - 4 = 3x^2 - 1$

$0 \le g(x) \le 5$ or $0 \le x^2 + 1 \le 5$ $-2 \le x \le 2$

Now $f(3x^2 - 1)$ is to be defined.

$0 \le 3x^2 - 1 \le 5$ or $\frac{1}{3} \le x^2 \le 2$

On solving, we get

$\left[-\sqrt{2} \le x \le \frac{-1}{\sqrt{3}}\right]$ or $\left[\frac{1}{\sqrt{3}} \le x \le \sqrt{2}\right]$

Solution set $\left[-\sqrt{2} \le x \le \frac{-1}{\sqrt{3}}\right]$ or $\left[\frac{1}{\sqrt{3}} \le x \le \sqrt{2}\right]$

7. (c) z is smaller than $\frac{(x+y)}{2}$, which is less than the larger of the values x, y.

8. (a) B(A(A(2, 3), C(4, 6)), C(B(1, 2), A(3, 2)))

= B(A(3, 5), C(1, 3)) = B(5, 2) = 2

(b) For what value of x is

A(B(C(6, 8), A(5, x)), C(B(5, a), A(4, 6))) = 6

Consider C(B(5, a), A(4, 6)).

This can take a maximum value of 5.5 (which happens when a < 5).

Consider B(C(6, 8), A(5, x)).

This has to take a value of 6.

Hence, A(5, x) has to be 6 since c(6, 8) = 7.

Therefore, the value of x = 6.

(c) All are equal.

(d) C(x, C(A (x, y), B(x, y))) = $\frac{7x}{4}$

$C(x, C(x, y)) = \frac{7x}{4}$ if $x > y$

or $C(x, C(y, x)) = \frac{7x}{4}$ if $x < y$

In either case we arrive at $C(x, \frac{x+y}{2}) = \frac{7x}{4}$

i.e. $\frac{3x+y}{4} = \frac{7x}{4} \therefore 4x = y$

9. Sqr (Add (A, B), @ (A, B))

$$\Rightarrow \frac{(A+B)^2 - \left(\frac{A+B}{2}\right)^2}{\text{Add}(A, B)} \Rightarrow \frac{(A+B)^2\left(1-\frac{1}{4}\right)}{\text{Add}(A, B)} \Rightarrow \frac{\frac{3}{4}(A+B)^2}{\text{Add}(A, B)}$$

Hence, the required fraction = $\frac{3}{4}(A + B)$

10. The function is

f(a, b) = a if b = 0

f(a, b) = f(b, r) if a > b and r = rem (a divided by b)

(a) f(27,18) = f(18, 9) = f(9, 0) = 9

(b) f(6, 4) = f(4, 2) = f(2, 0) = 2

(c) f(15, 9) = f(9, 6) = f(6, 3) = f(3, 0) = 3.

11. a According to the given rules,

@ (/ (* (A, B), B), A) = @ (/ (* (2, 4), 4), 2)

= @ (/ (8, 4), 2)

= @ (2, 2) = 2

Thus, the correct answer is (a) 2.

12. a According to the given rules, only option (a) is equal to the sum of A and B.

Let us see this: A + B = 2 + 4 =6.

* (@ (A, B), 2) = * (@ (2, 4), 2) = * (3, 2) = 6

Also * (@ (A, B), 2) = * ((A + B) / 2, 2)

= [(A + B) / 2] × 2 = A + B

Thus, the correct answer is (a).

13. a According to the given rules, only option (a) is equal to the sum of A, B and C, and is the correct answer.

14. b Here, according to the given values, the equation is:

Ma [md (-2), mn (md (-3), -2), mn (6, md (-8))] = Ma [2, mn (3, -2), mn (6, 8)] = Ma [2, -2, 6] = 6

Thus, the correct answer is (b).

15. a Given $\text{Ma}[|a|, b] = \text{mn}[a, |a|]$

For a > 0, LHS = RHS = a

For a < 0, LHS = |a|

RHS = a

Hence, the answer is (a).

16. c According to the given values, the equation

mn [md (b), Ma (md (a), b), mn (ab, md (bc))]

= mn [md (-9), Ma (md (6), -9), mn (-54, md(-90))]

= Mn [9, Ma (6, -9), mn (-54, 90)] = Mn [9, 6, -54] = -54

Thus, the correct answer is (c), -54.

17. c In case of invertible functions every element of domain should be mapped to exactly one element of co-domain where both domain and co-domain has equal number of elements.

This case is quite similar to arrange n distinct objects in n distinct positions.

Hence number of invertible functions from A to B = n!.

18. a If a = -2 and b = -3, then the expression is:

me((-2 + mo(le(-2, -3)), mo(-2 + me(mo(-2), mo(-3))))

= me((-2 + mo(-3)), mo(-2 + me(2, 3)))

= me((-2 + 3), mo(-2 + 3)) = me(1, mo(1)) = me(1, 1) = 1

Thus, the correct answer is (a).

19. d This problem can be solved by simulation. Assume the values for a and b in the given range.

Let a = 2 and b = 3.

You will find that option (d) holds good.

Let us substitute these values in option (d) to get

mo(le(2, 3)) = le(mo(2), mo(3))

= mo(2) = le(2, 3) = 2 = 2

Thus, the correct answer is (d).

20. b gr (4 . 6) = 5, lo (2 . 0) = 2. So 5 - 2 = 3

Thus, the correct answer is (b).

21. c rem (16, 7) = 2. So 2 - 2 =0

Thus, the correct answer is (c).

22. Here $(28)^{8+3x} = \left(56\sqrt{7}\right)^{3x+3}$

$\Rightarrow \left[\left(2\sqrt{7}\right)^2\right]^{8+3x} = \left[\left(2\sqrt{7}\right)^3\right]^{3x+3}$

$\Rightarrow 16 + 6x = 9x + 9 \Rightarrow 3x = 7 \Rightarrow x = \frac{7}{3}$

23. Here $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 4-3 = 1$

$\therefore \left(2-\sqrt{3}\right) = \frac{1}{\left(2+\sqrt{3}\right)}$

$\therefore$ The equation is $\left(2+\sqrt{3}\right)^{x^2-5} + \left(\frac{1}{2+\sqrt{3}}\right)^{x^2-5} = 4$

Put $\left(2+\sqrt{3}\right) = a$

So, $a^{x^2-5} + \left(\frac{1}{a}\right)^{x^2-5} = 4$

or, $\left(a^{x^2-5}\right)^2 - 4 \times a^{x^2-5} + 1 = 0$

$\therefore a^{x^2-5} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} = (2+\sqrt{3});(2+\sqrt{3})^{-1}$

$\Rightarrow x^2 - 5 = 1, -1$

or $x^2 = 6, 4$

or $x = \pm 2 , \pm\sqrt{6}$

24. b Let $50^{\log_{50} 20} = x$

$\Rightarrow \log_{50} x = \log_{50} 20$

$\Rightarrow x = 20$ (removing logarithm from both sides)

25. d $\log_{10} 10 + \log_{100} 100 + \log_{10} 10^3 + \log_{100} (100)^2 + \log_{10^2} 10^5$ = 1 +1+3+2+2.5=9.5

26. (a) 2 (b) $\frac{1}{2}$ (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

27. (a) 2 (b) 1.2

28. 4

29. b $\log_x 0.0016 = 4 \Rightarrow (0.2)^4 = x^4 \therefore x = 0.2$

30. (a) 75 (b) 7

31. $\frac{3}{2}$

32. c $2^{64} = x \therefore$ logx = 64 log2 = 64 × 0.301 = 19.264 Characteristic = 19

$\therefore$ Number of digits in 2^{64} = 19 +1 = 20

33. $\left(\frac{4}{3}\right)^9$ 34. $\frac{9}{4}$

35. 1.908 36. 102

37. 6

38. $10^{\log_{100}16}$ can be written as $10^{\log_{10}4}$ and $10^{\log_{10}4} = 4$

$\therefore \log_{10} y = 4$

$\Rightarrow y = 10^4 = 10000$

Level - II

39. d $x - 1 \le [x] \le x$

$2x + 2y - 3 \le L(x,y) \le 2x + 2y \Rightarrow a - 3 \le L \le a$

$2x + 2y - 2 \le R(x,y) \le 2x + 2y \Rightarrow a - 2 \le R \le a$

Therefore, $L \le R$

Note: Choice (b) is wrong, otherwise choice (a) and choice (c) are also not correct. Choose the numbers to check.

40. a $y = \sqrt{x^2 - 6x + 13} \Rightarrow y = \sqrt{(x-3)^2 + 4}$

Since $(x - 3)^2 \ge 0$, we can say that minimum value of f(x) is 2 and there is no maximum value. Hence $y \ge 2$.

41. d $y = \sqrt{40 - x^2 + 6x} \Rightarrow y = \sqrt{49 - (x-3)^2}$

Since $(x-3)^2 \ge 0$, the maximum value of f(x) will be 7 and minimum value is 0 as square root of a number cannot be negative. Hence $0 \le y \le 7$.

42. |x| + 3y = 7

2x + |y - 10| = 3

(a) When $x \geq 0$ and $y \geq 10$,

x + 3y = 7 and 2x + y = 13

$\Rightarrow y = \frac{1}{5}$; which is less than 10 and hence not acceptable.

Hence, no solution is possible.

(b) When $x \geq 0$ and y < 10,

x + 3y = 7 and 2x + 10 - y = 3

$\Rightarrow$ y = 3 and x = -2 (not acceptable again)

(c) When x < 0 and $y \geq 10$,

-2x + 6y = 14 and 2x + y = 13

$\Rightarrow$ y = $3\frac{6}{7}$ (not acceptable)

No solution is possible.

(d) When x < 0 and y <10,

-2x + 6y = 14 and 2x - y = -7

$\Rightarrow$ y = 1.4 and x = -2.8 which is acceptable.

Therefore, the unique values of x and y are -2.8 and 1.4.

43. $|x-2| \leq 2$ and $|x+3| \geq 4$ $-2 \leq (x-2) \leq 2$

Hence, $0 \leq x \leq 4$ ... (i)

Also $(x+3) \geq 4$ or $x+3 \leq -4$

or, $x \geq 1$ or $x \leq -7$ ... (ii)

Combining (i) and (ii), we have the solution $1 \leq x \leq 4$.

44. $|x^2+3x| + x^2 - 2 \geq 0$

(a) $|x^2+3x|$ = x^2 + 3x, when $x^2+3x \geq 0$, i.e.

$x(x+3) \geq 0$ which means either

$x \geq 0$ or $x \leq -3$ ... (i)

Under these circumstances,

$x^2+3x+x^2-2 \geq 0$

i.e. $2x^2+3x-2 \geq 0$

$x \leq -2$; $x \geq \frac{1}{2}$ ... (ii)

Combining (i) and (ii), the solution is $x \leq -3$ or $x \geq \frac{1}{2}$

(b) $|x^2+3x| = -(x^2+3x)$ when $x^2+3x < 0$

i.e. x(x + 3) < 0, it means -3 < x < 0 ... (iii)

Under these circumstances,

$-x^2-3x+x^2-2 \geq 0$ i.e. $3x+2 \leq 0$

or $x \leq -\frac{2}{3}$ ... (iv)

Combining (iii) and (iv), solution is $-3 < x \leq -\frac{2}{3}$.

Solution set combining a, b is $x \le \frac{-2}{3}; x \ge \frac{1}{2}$

45. **Method 1:**

$x^2 + y^2 = 1$

$|x + y|^2 = (x + y)^2 \Rightarrow x^2 + y^2 + 2xy = 2$

Hence, $2xy = 1$ or $xy = \frac{1}{2} \Rightarrow y = \frac{1}{2x}$

Substituting in $|x + y| = \sqrt{2}$, we have $\left|x + \frac{1}{2x}\right| = \sqrt{2}$

or $x + \frac{1}{2x} = \sqrt{2}$ or $x + \frac{1}{2x} = -\sqrt{2}$

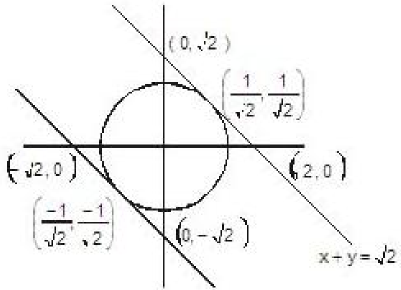
On solving, we get

$(2x^2 - 2\sqrt{2}x + 1) = 0$ or $2x^2 + 2\sqrt{2}x + 1 = 0$

On solving, we get $x = \frac{1}{\sqrt{2}}$ or $\frac{-1}{\sqrt{2}}$

Hence, $y = \frac{1}{\sqrt{2}}$ or $\frac{-1}{\sqrt{2}}$

Hence, the points of (x, y) are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$



Method 2:

$x^2 + y^2 = 1$ is a circle with the origin as the centre and the radius is 1.

$|x + y| = \sqrt{2}$ is a set of two straight lines

$x + y = \sqrt{2}$ and $x + y = -\sqrt{2}$.

Each of these lines is at a perpendicular distance of 1 from the origin. Hence, they must be tangent to the circle. The points of tangency are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

46. (a) $y = |x - 2| - |2x + 1|$

$y = |x - 2| - 2\left|x - \left(-\frac{1}{2}\right)\right|$

For $x < -\frac{1}{2}$

$y = (2 - x) - (2x + 1) = 2 - x - 2x - 1$

$y = 1 - 3x$

For $-\frac{1}{2} \le x < 2$

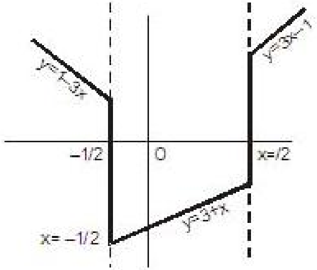
$y=(2-x)+2x+1=3+x$

For 2 ≤ x

$y=x-2+2x+1=3x-1$

When we draw the graph of

$y=|x-2|+|2x+1|$, we get the following curve.



Clearly, the minimum value is at $x=-\frac{1}{2}$ and it is $y=\left(3-\frac{1}{2}\right)=\frac{5}{2}$

(b) |x - 2| + |2x + 1| - |x - 1| = f(x)

The above graph can be understood algebraically as follows:

For $x\geq 2$, the expression would be

y = x - 2 + 2x + 1 - x + 1

y = 2x

For $1\leq x<2$, the expression would be

y = -x + 2 + 2x + 1 - x + 1

y = 4

For $\frac{-1}{2}\leq x\leq 1$, the expression would be

y = -x + 2 + 2x + 1 + x - 1

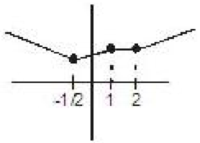
y = 2x + 2

For $x<\frac{-1}{2}$, the expression would be

y = -x + 2 - 2x - 1 + x - 1

y = -2x

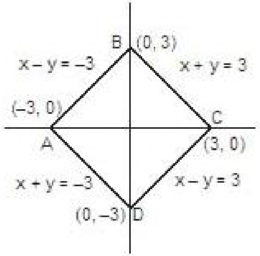
The graph for the following function is as shown below:



By visualizing graph, we get the inference that minimum value is at $x=-\frac{1}{2}$ and it is 2(x + 1) = $2\left(-\frac{1}{2}+1\right)=1$

47. (a) x + y = 3 $\begin{cases} \because |a| = a & a>0 \\ \quad\;\; = -a & a<0 \end{cases}$

x + y = -3; x - y = 3; x - y = -3



The figure formed is a square ABCD as product of slopes of consecutive sides is -1 and

side of square = AB

AB = $\sqrt{(0-(-3))^2+(3-0)^2}$

AB $= \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

Area of square = AB² = $(3\sqrt{2})^2$ = 18 units

(b) y = |x + 3| - 4 and x axis

|x + 3| = y + 4

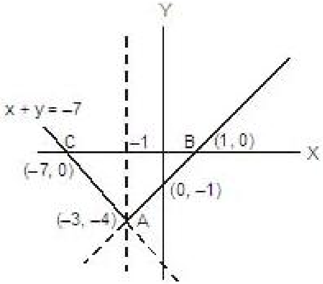
Now, when x + 3 > 0 ⇒ x + 3 = y + 4

⇒ x - y = 1 ...(i)

When x + 3 < 0 ⇒ -(x + 3) = y + 4

⇒ -x - 3 = y + 4 ⇒ y + x = -7 ...(ii)

So area is to be found of x - y = 1, x + y = -7 and x - axis.



Intersection point of x - y = 1 and x + y = -7

x - y = 1

x + y = -7

2x = -6

x = -3

y = -4

So it is a triangle with vertices at A, B and C.

Its area is given by

$= \frac{1}{2}(BC)\times(AB) = \frac{1}{2}(7+1)\times(4)$ = 16 sq. units

(c) |2y| = x; x = 3

Now if

y > 0

2y = x

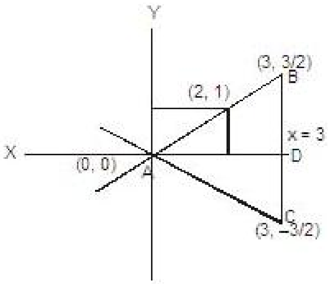
$y = \frac{x}{2}$ ...(i)

If y < o

-2y = x

$y = \frac{-x}{2}$ ...(ii)

x = 3 ...(iii)



Intersection of x = 3 & $y = \frac{x}{2}$ gives $y = \frac{3}{2}$

Intersection of x = 3 & $y = \frac{-x}{2}$ gives $y = \frac{-3}{2}$

Vertices of triangle formed

(o, o), $\left(3, \frac{3}{2}\right)$, $\left(3, \frac{-3}{2}\right)$

Area = $\frac{1}{2}(BC)\times(AD) = \frac{1}{2}\left(\frac{3}{2}+\frac{3}{2}\right)\times(3) = \frac{1}{2}(3)\times(3) = \frac{9}{2}$ square units

48. Here

$\log_{(x-7)}\left(2x^2 - x - 91\right) = 5 - 2\log_{(2x+13)}(x^2 - 14x + 49)$

$\Rightarrow \log_{(x-7)}(x-7)(2x+13) = 5 - 2\log_{(2x+13)}(x-7)^2$

$\Rightarrow \log_{(x-7)}(x-7) + \log_{(x-7)}(2x+13) = 5 - 4\log_{(2x+13)}(x-7)$

$\Rightarrow 1 + \log_{(x-7)}(2x+13) = 5 - 4\log_{(2x+13)}(x-7)$

Assume

$\log_{(x-7)}(2x+13) = y$, so $\log_{(2x+13)}(x-7) = \frac{1}{y}$

Now the equation is y = 4 - $\frac{4}{y}$

⇒ $y^2 - 4y + 4 = 0$

⇒ $(y-2)^2 = 0 \Rightarrow y = 2$

Now, $\log_{(x-7)}(2x+13) = 2$

= 2x + 13 = $(x - 7)^2$

= 2x + 13 = $x^2 - 14x + 49$

= $x^2 - 16x + 36 = 0$

⇒ $x = 8 \pm 2\sqrt{7}$

But at $x = 8 - 2\sqrt{7}$, Base x - 7 is negative.

So only one value of x exists.

49. Here, $\log_7\left(2\times7^{x-3} - 1\right) + 6 = 2x$

$\Rightarrow \log_7\left(2\times7^{x-3} - 1\right) = 2x - 6$

$\Rightarrow \log_7\left(2\times 7^{x-3}-1\right)=\log_7 7^{2(x-3)}$

Take $7^{x-3} = a$

$\therefore a^2 - 2a + 1 = 0$

$\Rightarrow (a - 1)^2 = 0$

$\Rightarrow \left(7^{x-3}-1\right)^2=0 \Rightarrow 7^{x-3}=1$

$\Rightarrow x - 3 = 0$

$\Rightarrow x = 3$

Hence only one value of `x' exists.

50. Here, $\log_{(2x-1)}(3x^2-2x-1)<\log_{(2x-1)}(2x-1)$

Case I:

If $2x - 1 > 1$ and $3x^2 - 2x - 1 > 0$

$\Rightarrow x>1$ and $(3x + 1)(x - 1) > 0$

$\Rightarrow x>1$ and $x<\frac{-1}{3}$ or $x > 1$

$\therefore x > 1$

$\therefore (3x^2 - 2x - 1) < 2x - 1$

$3x^2 - 4x < 0$

$\Rightarrow x(3x - 4) < 0$

$\Rightarrow 0 < x < \frac{4}{3}$

So, the solution is $1<x<\frac{4}{3}$.

Case II:

If $0 < 2x - 1 < 1$ and $3x^2 - 2x - 1 > 0$

$\Rightarrow \frac{1}{2}<x<1$ and $x=\frac{-1}{3}$ or $x > 1$

So no value of x exists.

Hence the solution set is $1<x<\frac{4}{3}$

51. Here $\log_{(5)^3}(x-2)^3-\log_{1/5}(x-2)>6$

$\Rightarrow \frac{1}{3}\log_5(x-2)^3-\log_{5^{-1}}(x-2)>6$

$\Rightarrow \log_5(x-2)+\log_5(x-2)>6$

$\Rightarrow 2\log_5(x-2)>6 \Rightarrow \log_5(x-2)>3$

$\therefore x-2>0 \Rightarrow x>2$ and $(x-2)>5^3 \Rightarrow x>127$

Chapter 5

Level - I

1. S = 5 + 55 + 555 + ... 20 terms

S = 5(1 + 11 + 111 + ... 20 terms)

$S = \frac{5}{9}\times(9+99+999+\ldots\ 20\ \text{terms})$

$S = \frac{5}{9}[(10-1)+(100-1)+(1000-1)+\ \ldots\ 20\ \text{terms}]$

$S = \frac{5}{9}\left[(10+10^2+10^3+\ \ldots\ 20\ \text{terms}-20\right]$

$S = \frac{5}{9}\left[\frac{10(10^{20}-1)}{(10-1)}-20\right]$

$S = \frac{5}{9}\left[\frac{10(10^{20}-1)}{9}-20\right]$

2. $\frac{(S_1)n}{(S_2)n} = \frac{3n+8}{7n+15}$

$\therefore \frac{(S_1)12}{(S_2)12} = \frac{3\times12+8}{7\times12+15} = \frac{44}{99} = \frac{4}{9}$

3. 1 cell divides into 2 cells or 2^1 in 1st hour.

Now these 2 cells divide into 4 cells or 2^2 in 2nd hour.

Now these 4 cells divide into 8 cells or 2^3 in 3rd hour and so on. In 10 hr they will become 2^{10}.

The series becomes: 2, 2^2 . 2^3, 2^4, ..., 2^{10}

$\text{Sum} = a\left(\frac{r^n-1}{r-1}\right)$

Sum = $2\times\frac{(2^{10}-1)}{2-1}$ = 2046

Hence, the number of cells produced = 2045

Level - II

4. Ball drops 81 m, then rebounds $81\times\frac{2}{3}\text{m}$, then drops $81\times\frac{2}{3}\text{m}$, then rebounds $81\times\left(\frac{2}{3}\right)^2$, then drops $81\times\left(\frac{2}{3}\right)^2$, then rebounds and so on.

(a) Distance travelled

$= 81+2\times81\times\frac{2}{3}+2\times81\times\left(\frac{2}{3}\right)^2+2\times81\times\left(\frac{2}{3}\right)^3+81\times\left(\frac{2}{3}\right)^4$

= 81 + 108 + 72 + 48 + 16 = 325 m

(b) Ball comes to rest after infinite bounces

$81+2\times81\times\frac{2}{3}+2\times81\times\left(\frac{2}{3}\right)^2+\ \ldots\ \alpha\ \text{terms}$

$81+2\times81\times\frac{2}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 81+2\times81\times\frac{2}{3}\times3$ = 405 m

5. A travels 15 km, 14 km, 13 km ... on respective days and B travels 10 km, 12 km, 14 km, etc., on respective days.

Total distance travelled by A = 15 + 14 + 13 + ... n terms = $\frac{n}{2}[31-n]$

Total distance travelled by B = 10 + 12 + 14 + ... n terms = $\frac{n}{2}[18+2n]$

Therefore, total distance = $\frac{n}{2}[31-n]$ + $\frac{n}{2}[18+2n]$ = 165

Solving, n = 6 days

6. 1 + (1 + 2) + (1 + 2 + 3) + ...

If sum of n terms is, say, S_n and nth term = T_n,

1st term = 1; 2nd term = 3; 3rd term = 6

nth term = $\frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n)$

Sum $S_n = \frac{1}{2}\left[1^2+2^2+3^2+\dots n^2\right]+\frac{1}{2}[1+2+3+\dots n]$

$$= \frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4} = \frac{n(n+1)}{4}\left[\frac{2n+1}{3}+1\right] = \frac{2n(n+1)(n+2)}{12} = \frac{n(n+1)(n+2)}{6}$$

Level - III

7. $4+\frac{8}{3}+\frac{16}{9}+$ ... to infinite terms

$4+4\times\frac{2}{3}+4\times\left(\frac{2}{3}\right)^2+\dots$ r = $\frac{2}{3}$ < 1

$\therefore$ Sum of infinite series = $\frac{4}{1-\frac{2}{3}}=12$

8. (a) 1 . 2 . 4 + 2 . 3 . 7 + 3 . 4 . 10 + ... n terms

$T_n = n(n+1)(3n+1)$

$T_n = 3n^3+4n^2+n$

$\therefore S_n = 3\Sigma n^3+4\Sigma n^2+\Sigma n$

$$= 3\left(\frac{n(n+1)}{2}\right)^2+\frac{4}{6}n(n+1)(2n+1)+\frac{n(n+1)}{2} = \frac{n(n+1)}{12}[(9n+7)(n+2)]$$

(b) 2 + 10 + 30 + 68 + ...

S = $1+1^3+2+2^3+3+3^3+4+4^3+\dots$

S = $1+2+3+\dots+1^3+2^3+3^3+4^3+\dots$

S = $\frac{n(n+1)}{2}+\left(\frac{n(n+1)}{2}\right)^2$

On solving, we get $\frac{n(n+1)(n^2+n+2)}{4}$

9. d Let $S=\frac{1}{8}+\frac{3}{8^2}+\frac{6}{8^3}+\frac{10}{8^4}+\frac{15}{8^5}+\dots$

$\Rightarrow 8S=1+\frac{3}{8}+\frac{6}{8^2}+\frac{10}{8^3}+\frac{15}{8^4}+\dots$

$\Rightarrow 8S-S=1+\frac{3-1}{8}+\frac{6-3}{8^2}+\frac{10-6}{8^3}+\frac{15-10}{8^4}+\dots$

$\Rightarrow 7S=1+\frac{2}{8}+\frac{3}{8^2}+\frac{4}{8^3}+\frac{5}{8^4}+\dots$

$\Rightarrow 8\times7S=8+2+\frac{3}{8}+\frac{4}{8^2}+\frac{5}{8^3}+\dots$

$\Rightarrow (8-1)\times7S=8+(2-1)+\frac{3-2}{8}+\frac{4-3}{8^2}+\frac{5-4}{8^3}+\dots$

$=8+\left(1+\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\dots\right)$

$\therefore S_n = 3\Sigma n^3 + 4\Sigma n^2 + \Sigma n$

$= 3\left(\frac{n(n+1)}{2}\right)^2 + \frac{4}{6}n(n+1)(2n+1) + \frac{n(n+1)}{2} = \frac{n(n+1)}{12}[(9n+7)(n+2)]$

(b) 2 + 10 + 30 + 68 + ...

$S = 1 + 1^3 + 2 + 2^3 + 3 + 3^3 + 4 + 4^3 + \ldots$

$S = 1 + 2 + 3 + \ldots + 1^3 + 2^3 + 3^3 + 4^3 + \ldots$

$S = \frac{n(n+1)}{2} + \left(\frac{n(n+1)}{2}\right)^2$

On solving, we get $\frac{n(n+1)(n^2+n+2)}{4}$

9. d Let $S = \frac{1}{8} + \frac{3}{8^2} + \frac{6}{8^3} + \frac{10}{8^4} + \frac{15}{8^5} + \ldots$

$\Rightarrow 8S = 1 + \frac{3}{8} + \frac{6}{8^2} + \frac{10}{8^3} + \frac{15}{8^4} + \ldots$

$\Rightarrow 8S - S = 1 + \frac{3-1}{8} + \frac{6-3}{8^2} + \frac{10-6}{8^3} + \frac{15-10}{8^4} + \ldots$

$\Rightarrow 7S = 1 + \frac{2}{8} + \frac{3}{8^2} + \frac{4}{8^3} + \frac{5}{8^4} + \ldots$

$\Rightarrow 8 \times 7S = 8 + 2 + \frac{3}{8} + \frac{4}{8^2} + \frac{5}{8^3} + \ldots$

$\Rightarrow (8-1) \times 7S = 8 + (2-1) + \frac{3-2}{8} + \frac{4-3}{8^2} + \frac{5-4}{8^3} + \ldots$

$= 8 + \left(1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \ldots\right)$

$\Rightarrow 49S = 8 + \frac{1}{1-\frac{1}{8}} = \frac{64}{7} \Rightarrow S = \frac{64}{343}$

10. $S = 2\times3\times4 + 4\times5\times6 + 6\times7\times8 + \ldots$

$T_n = 2n(2n+1)(2n+2)$

$S = \Sigma T_n = \Sigma[2n(2n+1)(2n+2)]$

$= \Sigma\left[8n^3 + 12n^2 + 4n\right] = 8\Sigma n^3 + 12\Sigma n^2 + 4\Sigma n$

$= 8\left[\frac{n(n+1)}{2}\right]^2 + 12\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\frac{n(n+1)}{2} = 2(n^2+n)^2 + 2\left[2n^3 + 3n^2 + n\right] + 2n^2 + 2n$

$= 2\left[n^4 + 2n^3 + n^2 + 2n^3 + 3n^2 + n + n^2 + n\right] = 2\left[n^4 + 4n^3 + 5n^2 + 2n\right]$

Chapter 6

Level - I

1.d $a > 0, b > 0$

Therefore, $(a+b)\left[\frac{1}{a}+\frac{1}{b}\right]=2+\frac{a}{b}+\frac{b}{a}=2+\left(x+\frac{1}{x}\right)$

Where $\frac{a}{b}=x$

Since the minimum value of $\left(x+\frac{1}{x}\right)$ is 2, minimum value of the given expression is 4.

2. The expression is $\left(1+p+\frac{1}{p}\right)\left(1+q+\frac{1}{q}\right)\left(1+r+\frac{1}{r}\right)$

This expression is minimum if $p = q = r = 1$

The minimum value of the expression $= 3 \times 3 \times 3$

$= 27$

Alternative Method

The expression will have minimum value when $p = q = r = 1$.

Therefore, minimum value $= \frac{(1+1+1)}{1} \times \frac{(1+1+1)}{1} \times \frac{(1+1+1)}{1} = 27$

3. **Method 1:**

$a + b + c + d = 4$

Then $(a + 1) + (b + 1) + (c + 1) + (d + 1) = 8$

Now since AM $\geq$ GM,

$\frac{(a+1)+(b+1)+(c+1)+(d+1)}{4} \geq \sqrt[4]{(a+1)(b+1)(c+1)(d+1)}$ or $\frac{8}{4} \geq \sqrt[4]{(a+1)(b+1)(c+1)(d+1)}$

$\therefore (a + 1)(b + 1)(c + 1)(d + 1) \leq 16.$

Method 2:

If $a + b + c \ldots$ is constant, then the product $abcd$ is maximum when $a = b = c = \ldots$

$\therefore (a + 1) = (b + 1) = (c + 1) = (d + 1)$

Given $(a + 1) + (b + 1) + (c + 1) + (d + 1) = 8$

$4(a + 1) = 8$

$a + 1 = 2$

Therefore, maximum value $= 2 \times 2 \times 2 \times 2 = 16$

4. $4x^2 - 6x + 1 = \left[2x-\frac{3}{2}\right]^2 - \frac{5}{4}$

This expression will obtain minimum value when $\left[2x-\frac{3}{2}\right]^2$ is equal to zero.

Therefore, minimum value $= \frac{-5}{4}$

5. The value of this expression tends to infinity. Hence the expression has no maxima.

6. $\frac{x^2+14x+9}{x^2+2x+3} = m$ (Say)

Solving, $x^2(1-m)+2x(7-m)+(9-3m)=0$

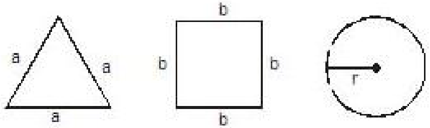
For real values of x, $b^2 - 4ac \geq 0$

$\therefore [2(7-m)]^2 - 4(1-m)(9-3m) \geq 0$

On solving, we get $(m + 5)(m - 4) \leq 0$

i.e. m lies between -5 and 4.

$\therefore$ The maximum value of the expression = 4 and the minimum = -5

7. 

Suppose all of these figures have same perimeter say "x"

1. For equilateral triangle

$3a = x, \; a = \frac{x}{3}$

2. For square

$4b = x$

$b = \frac{x}{4}$

3. For circle

$2\pi r = x, \; r = \frac{x}{2\pi}$

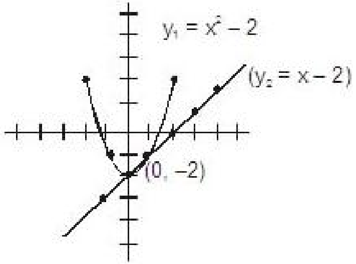
Area of Equilateral Triangle	Area of Square	Area of Circle
$\frac{\sqrt{3}}{4}\left(\frac{x^2}{9}\right)$ units	$\frac{x^2}{16}$ units	$\frac{\pi \times x^2}{4\pi^2}$ units

Comparing areas of all three, it signifies that circle has the largest area.

8. $f(x) = \min(x^2 - 2, x - 2)$

$y_1 = x^2 - 2$

$y_2 = x - 2$



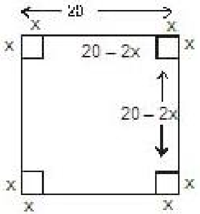
The graph f(x) is shown in bold.

In the interval [-1, 1] f(x) is maximum at x = 1 and

its value is -1.

Level - II

9. **Method 1:**



We get a cuboid of dimension (20 - 2x), (20 - 2x) and x units volume V = (20 - 2x)(20 - 2x)(x)

The dimensions are (20 - 2x), (20 - 2x) and x.

The product of these three terms has to be maximum.

If this were true, then the product of the terms (20 - 2x), (20 - 2x) and 4x would also be maximum.

Using the property:

If A + B + C = K = Constant, then ABC is maximum when A = B = C = $\frac{k}{3}$

We find that (20 - 2x) = 4x = $\frac{40}{3}$

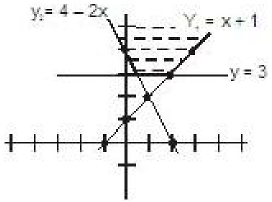
Therefore, the dimensions are $\frac{10}{3}, \frac{40}{3}, \frac{40}{3}$ for the volume of the cuboid to be maximum.

10. f(x) = max(x + 1, 3, 4 - 2x)

$y_1 = x + 1$

$y_2 = 3$

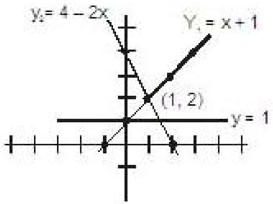
$y_3 = 4 - 2x$



In the area representing the f(x) = max (x + 1, 3, 4 - 2x)

Minimum value of f(x) will be 3.

Now if f(x) = max (x + 1, 1, 4 - 2x)



min of max(x + 1, 1, 4 - 2x) will be 2, which is clearly shown in graph.

In this case, the answer would have been 2.

Chapter 7 - Practice Exercises

Exercise 1 - Level I

1. c $\frac{1}{x-1}+\frac{1}{x-2}=\frac{3}{x-3} \Rightarrow \frac{(x-2)+(x-1)}{(x-1)(x-2)}=\frac{3}{x-3}$

$\Rightarrow \frac{x-2+x-1}{x^2-2x-x+2}=\frac{3}{x-3}$

$\Rightarrow 3x^2 - 9x + 6 = 2x^2 - 6x - 3x + 9$

$\Rightarrow x^2 + 6 - 9 = 0$

$\Rightarrow x^2 - 3 = 0$

$\therefore x^2 = 3$ and $x = \pm\sqrt{3}$

2. a Let the present age of Mohan be `x' years and the present age of Shyam be `y' years

So (x - 6) = (y - 6) 3 and x - 6 = 3y - 18

x = 3y - 12 ...(i)

Also, x = y × 1.5, $\Rightarrow$ y = $\frac{x\times 2}{3}$

Substituting the value of $y=\frac{2x}{3}$ in (Eq. 1) we get

$\Rightarrow x = 3\times\frac{2x}{3}-12$ and 3x = 6x - 36

x = 12 years and y = 8 years

3. e 2x + 4y = 6 ...(i)

3x + 15y = 25 ...(ii)

Multiplying the equation (i) by 1.5, we get

3x + 6y = 9 ...(iii)

Subtracting (iii) from (ii)

$$\begin{array}{r} 3x + 15y = 25 \\ -3x \pm 6y = -9 \\ \hline 9y = 16 \\ y = \frac{16}{9} \end{array}$$

4. d $\left(x+\frac{1}{x}\right)^2 = x^2+2x\left(\frac{1}{x}\right)+\frac{1}{x^2}$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 79+2 = 81$

$\therefore x+\frac{1}{x} = \sqrt{81} = \pm 9$

5. d $\left(x^2+\frac{1}{x^2}\right)^2 = x^4+\frac{1}{x^4}+2$

$\Rightarrow (10)^2 = x^4+\frac{1}{x^4}+2$

$\Rightarrow 100-2 = x^4+\frac{1}{x^4}$

$\therefore x^4+\frac{1}{x^4} = 98$

6. b x - 5 is a factor of the given expression then 5 is the root of the expression (Remainder theorem).

So $2x^2 + 2px - 2p = 0$

$2.(5)^2 + 2\ p.5 - 2p = 0$

$50 + 10\ p - 2p = 0$

$8\ p = -50$ and $p = \frac{-50}{8} = \frac{-25}{4}$

7. e Putting the value of x = 2 in the expression

$x^4 + 2x^3 - 3x^2 + x - 1$

$(2)^4 + 2.2^3 - 3.2^2 + 2 - 1 = 16 + 16 - 12 + 2 - 1$

$= 34 - 13 = 21$

Had x - 2, a factor (means 2 as a root) then the remainder would have become zero.

8. a In the series 101 + 104 + ... + 161

a = 101 and d = 3

nth term = a + (n - 1) d

$\Rightarrow 161 = 101 + (n - 1)3$

$\Rightarrow 161 - 98 = 3n$

$\Rightarrow 3n = 63$

$\Rightarrow n = 21$

So the sum is = $\frac{n}{2} = \{2a + (n-1)\ d\}$

$= \frac{21}{2}\ \{2\times101 + (21-1)\,3\}$

$= \frac{21}{2}\ (262) = 21\times131 = 2751$

Shortcut: $\frac{a+l}{2}\times x = \frac{101+161}{2}\times21$

$= 131 \times 21 = 2751$

9. d $4 \log 2 + \log 6 = \log 2^4 + \log 6$

$(\log n^m = m \log n)$

$= \log (16 \times 6) = \log 96$

$(\log m + \log n = \log m \times n)$

10. c $\log 5 + \log 4 + \log 30 - \log 6$

$\log 5 \times 4 \times 30 - \log 6 = \log 600 - \log 6$

$\log \frac{600}{6} = \log 100 = 2$

11. a x + p is the HCF of both expression, means x + p is the factor and -p is root. So substituting -p in place of x.

$(-p)^2 + b(-p) + (a) = 0$ and $(-p)^2 + c(-p) + (d) = 0$

or $p^2 - bp + a = p^2 - cp + d = 0$

So $cp - bp = d - a$ and $p\ (c - b) = d - a$ and $p = \frac{d-a}{c-b}$

12. b LCM of 15 and 25 is 75

LCM of $x^2\ y^3$ and x^4y is x^4y^3

LCM of $x^2 - y^2$ and $x - y$ is $x^2 - y^2$

So the required LCM is $75\ (x^4y^3)\ (x^2 - y^2)$

13. a $2x^3 + x^2 - 3x$ and $x^3 - x$

$2x^3 + x^2 - 3x = 2x^3 + 3x^2 - 2x^2 - 3x$

$= x^2 (2x + 3) - x (2x + 3)$

$= (x^2 - x) (2x + 3)$

$= x (x - 1) (2x + 3)$

$x^3 - x = x (x^2 - 1) = x (x + 1) (x - 1)$

So the highest common factor is $x (x - 1)$

14. d $(a + b + c)^2 = a^2 + b^2 + c^2 + 2 (ab + bc + ca)$

$(10)^2 = a^2 + b^2 + c^2 + 2 (40)$

$100 - 80 = a^2 + b^2 + c^2$

So $a^2 + b^2 + c^2 = 20$

15. e For infinite solutions, $ax + by = m$ and $cx + dy = n$

$\frac{a}{c} = \frac{b}{d} = \frac{m}{n}$

So in the given equation,

$\frac{2}{5} = \frac{3}{K} = \frac{8}{3}$, which is not possible.

Hence, option (e).

16. d Quadratic equation $= x^2 - (\alpha + \beta) x + \alpha \times \beta = x^2 - 8x + 8 = 0$.

17. a $2^{x+5} = 2^{x+3} + 6$

$\Rightarrow 2^x \times 2^5 = 2^x \times 2^3 + 6 \Rightarrow 2^x \times 2^5 - 2^x \times 2^3 = 6$

$\Rightarrow 2^x \times 2^3 (2^2 - 1) = 6 \Rightarrow 2^x \times 8 \times 3 = 6$

$\Rightarrow 2^x = \frac{6}{24} = \frac{1}{4}$ and

$2^x = \frac{1}{4} = 4^{-1} = (2^2)^{-1} = 2^{-2}$

$\Rightarrow 2^x = 2^{-2}$

$\therefore x = -2$

18. c $a = 2$, $b = (a + d = 2 + 11)$ 13, $c = a + 2d = 2 + 22 = 24$ and $d = a + 3d = 2 + 33 = 35$

$\frac{a + d}{2} = \frac{b + c}{2} = 18.5$

19. d For equal roots $D = 0$

So $D = b^2 - 4ac = (-48)^2 - 4 \times 9 \times c = 0$

$\Rightarrow 2304 - 36c = 0$

$\Rightarrow 36c = 2304 \therefore c = \frac{2304}{36} = 64$

20. b Sum of the roots of a quadratic equation $= -\frac{b}{a} = \frac{-(-4\sqrt{5})}{\sqrt{5}} = \frac{4\sqrt{5}}{\sqrt{5}} = 4$

21. d Mean proportion of a, b, c is $\sqrt[3]{abc}$

So, mean proportion of 4, 16 and 64 is $\sqrt[3]{4 \times 16 \times 64} = \sqrt[3]{4 \times 4 \times 4 \times 4 \times 4 \times 4} = 4 \times 4 = 16$

22. c $x + 3$ for a polynomial $f(x)$

If $(x + 3)$ is a factor, then -3 is its root and $f(-3) = 0$

If $(x - 3)$ is a factor, then 3 is its root and $f(3) = 0$

(Since, if $(x - a)$ is a factor of a polynomial then `a' is its root).

23. b $x^2 - 5x + 6 = 0$

So $x^2 - 2x - 3x + 6 = 0$

So $x(x - 2) - 3(x - 2) = 0$

So $(x - 3)(x - 2) = 0$

24. c In $2x^2 + 14x - 15$ take the value of $x = 4$

$2(4)^2 + 14 \times 4 - 15$

$32 + 56 - 15 = 73,$

Note: Here, you can say that $2x^2 + 14x - 15 - 73$ is exactly divisible by $x - 4$ or 4 is the root of this equation.

25. b The series is the sum of the squares of first 10 numbers = $\frac{n(n+1)(2n+1)}{6} = \frac{10 \times 11 \times 21}{6} = 385$

26. a a and b are roots of $x^2 - 5x + 1 = 0$

$\therefore ab = \frac{1}{1} = 1$ and $a + b = \frac{-(-5)}{1} = 5$

Now, equation with roots $(3a + 1)$ and $(3b + 1)$ is

$x^2 - [(3a + 1) + (3b + 1)]x + (3a + 1)(3b + 1) = 0$

$\therefore x^2 - [3(a + b) + 2]x + [(9ab + 3(a + b) + 1] = 0$

$\therefore x^2 - [3(5) + 2]x + [9(1) + 3(5) + 1] = 0$

$\therefore x^2 - 17x + 25 = 0$

27. e $a^2 - b^2 = 1987$

$\Rightarrow (a - b) \times (a + b) = 1987$

But, $(a - b) = 1$

$\Rightarrow (a + b) = 1987$

28. b $5x = 120 + y$

Or, $x = 24 + \frac{y}{5}$

For `x' to be an integer `y' must be a multiple of 5.

Since the values that `x' assumes have opposite signs as compared to the corresponding values of `y', `y' will have to be negative.

So all multiples of 5 from - 5 to -115 (both inclusive) will give positive integral values of `x'

Therefore number of integral solutions = 23.

29. c $|x - 2| = 9$

$\therefore x - 2 = \pm 9$

$\therefore x = 11$ or $x = -7$

$|y + 9| = 8$

$\therefore y + 9 = \pm 8$

$\therefore y = -1$ or $y = -17$

$\therefore$ Minimum possible value of $xy = 11 \times (-17) = -187$

30. a $f(g(x)) = \dfrac{1}{\dfrac{x^2+1}{x}} = \dfrac{x}{x^2+1}$

$\therefore\ g(f(x)) = \dfrac{\left(\dfrac{1}{x}\right)^2 + 1}{\dfrac{1}{x}} = \dfrac{x^2+1}{x} = \dfrac{1}{f(g(x))}$

31. a $\left(m * \dfrac{1}{m}\right) @ \left(m \$ \dfrac{1}{m}\right) = \left(m^2 - \dfrac{1}{m^2}\right) @ \left(m - \dfrac{1}{m}\right)^2$

$= m^2 - \dfrac{1}{m^2} + \left(m - \dfrac{1}{m}\right)^2 = m^2 - \dfrac{1}{m^2} + m^2 + \dfrac{1}{m^2} - 2 = 2(m^2 - 1)$

32. a $f[g(x)] = f\{\sin[\log(x-3)]\} = \dfrac{1}{\sin[\log(x-3)] + 4}$

So at x = 6. It is $\dfrac{1}{\sin(\log 3) + 4}$

33. d $|x|$, $|y|$, $|z|$ are always positive,

whereas $f(x, y, z) = \dfrac{x+y+z}{3}$ can be positive or

negative depending upon the values of x, y, z.

$\dfrac{|x|+|y|+|z|}{3}$ can be either equal to or more than $|f(x, y, z)|$.

34. b $(1+x+x^2) + \dfrac{x^2}{(1-x)} = \dfrac{(1-x)(1+x+x^2)+x^2}{(1-x)} = \dfrac{(1^2-x^2)+x^2}{(1-x)} = \left[\dfrac{1}{1-x}\right]$

For x = 0.75, required value $= \dfrac{1}{1-0.75} = \dfrac{1}{0.25} = 4$

Alternative method:

We see that for $x = 0.75 = \dfrac{3}{4}$

$1 + x + x^2 = 1 + \dfrac{3}{4} + \dfrac{9}{16} > 2$

$\therefore$ Only one option is higher than 2.

35. e $x = 3 + \sqrt{3}$

$\dfrac{1}{x} = \dfrac{1}{3+\sqrt{3}} = \dfrac{1}{3+\sqrt{3}} \times \dfrac{3-\sqrt{3}}{3-\sqrt{3}}$ or $\dfrac{1}{x} = \dfrac{3-\sqrt{3}}{9-3} = \left(\dfrac{3-\sqrt{3}}{6}\right)$

$x^2 + \dfrac{9}{x^2} = (3+\sqrt{3})^2 + \dfrac{(3-\sqrt{3})^2}{4} = (9+3+6\sqrt{3}) + \left(\dfrac{9+3-6\sqrt{3}}{4}\right) = 15 + \dfrac{9\sqrt{3}}{2}$

36. a For unique solution $\Rightarrow K^2 = 4 \times 2(K-1)(K+3)$

$\Rightarrow K^2 = 8(K^2 + 2K - 3) \Rightarrow 7K^2 + 16K - 24 = 0$

Two values of K are nothing but the roots of this equation.

Required Sum $= \dfrac{-16}{7}$

37. a $\dfrac{4p+9q}{p} = \dfrac{5q}{(p-q)}$

$4p^2 - 4pq + 9pq - 9q^2 = 5pq$

$4p^2 = 9q^2$

or $\dfrac{p^2}{q^2} = \dfrac{9}{4}$ or $\dfrac{p}{q} = \dfrac{3}{2}$

38. e $a^x = b^x$ might not mean a = b, if x = 0

$a^x = a^y$ might not mean x = y, if a = 1

We see that for $x = 0.75 = \frac{3}{4}$

$1 + x + x^2 = 1 + \frac{3}{4} + \frac{9}{16} > 2$

$\therefore$ Only one option is higher than 2.

35. e $x = 3 + \sqrt{3}$

$\frac{1}{x} = \frac{1}{3+\sqrt{3}} = \frac{1}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$ or $\frac{1}{x} = \frac{3-\sqrt{3}}{9-3} = \left(\frac{3-\sqrt{3}}{6}\right)$

$x^2 + \frac{9}{x^2} = (3+\sqrt{3})^2 + \frac{(3-\sqrt{3})^2}{4} = (9+3+6\sqrt{3}) + \left(\frac{9+3-6\sqrt{3}}{4}\right) = 15 + \frac{9\sqrt{3}}{2}$

36. a For unique solution $\Rightarrow K^2 = 4 \times 2(K-1)(K+3)$

$\Rightarrow K^2 = 8(K^2 + 2K - 3) \Rightarrow 7K^2 + 16K - 24 = 0$

Two values of K are nothing but the roots of this equation.

Required Sum $= \frac{-16}{7}$

37. a $\frac{4p+9q}{p} = \frac{5q}{(p-q)}$

$4p^2 - 4pq + 9pq - 9q^2 = 5pq$

$4p^2 = 9q^2$

or $\frac{p^2}{q^2} = \frac{9}{4}$ or $\frac{p}{q} = \frac{3}{2}$

38. e $a^x = b^x$ might not mean $a = b$, if $x = 0$

$a^x = a^y$ might not mean $x = y$, if $a = 1$

Similarly the statement III is not true

Hence, neither of the statements is true.

39. b $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$

$\frac{x^2}{a^2-1} = \frac{x}{1-a} = \frac{1}{1-a}$

$\Rightarrow x = 1$ or $x = \frac{a^2-1}{1-a} = -(a+1)$

$\Rightarrow -(a+1) = 1 \Rightarrow a = -2$

If $a = 1$, then the two equations become identical

$\therefore$ Possible values of co-efficient a are 1 and -2

40. a $-7 \times 3^{x+1} + 3^{(x+4)} = 5^{x+3} - 5^{x+2}$ or $3^{x+1}\left[-7+3^3\right] = 5^{x+2}[5-1]$

or $3^{x+1}\left[-7+3^3\right] = 5^{x+2}[4]$ or $20 \times 3^{x+1} = 4 \times 5^{x+2}$

or $20 \times 3^{x+1} = 20 \times 5^{x+1}$ or $\left(\frac{3}{5}\right)^{x+1} = 1$ or $x + 1 = 0$

So $x = -1$

Alternative method:

It is best to work with options here. We see that $x = -1$ satisfies the above given expression. Therefore, the answer is choice (a).

Exercise 2 - Level I

1. a $p + q = 1 \Rightarrow q = 1 - p$

Since the ordered pair (p, q) satisfies $3x + 2y = 1$, $3p + 2q = 1$ or $3p + 2(1 - p) = 1$

$\therefore p = -1$ and $q = 2$

Therefore, $3x + 3y = 3$ is satisfied.

2. e $\frac{x}{yz} : \frac{y}{xz} = \frac{x}{yz} \times \frac{xz}{y} = \frac{x^2}{y^2}$

Now $\frac{xz}{yz} = \frac{2}{1}$ or $\frac{x}{y} = \frac{2}{1}$

So $\frac{x^2}{y^2} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$

3. a Let's assume the sum of m numbers = A

So $\frac{A}{m} = a$ or $A = am$ ... (i)

According to the second condition,

$\frac{A + x}{(m + 1)} = b$ or $A + x = b(m + 1)$ ... (ii)

Here putting the value of x from the previous equation,

$am + x = b(m + 1)$

or $x = b(m + 1) - am$

or $x = m(b - a) + b$

4. e I. $ab > cb + ca$. Not true

II. $2b > a + c$. Not necessarily true

Take $a = 9$, $b = 3$ and $c = 1$

III. Similarly $b^2 > ac$ is not necessarily true.

5. c $x^2(2a^2 + 2b^2) + x(2a + 2b) + 1$

Discriminant $= [2(a + b)]^2 - 4[2(a^2 + b^2)]$

$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$

$= -4a^2 - 4b^2 + 8ab$

$= -4(a^2 + b^2 - 2ab)$

$= -4(a - b)^2$

Since discriminant < 0 (because $(a - b)^2$ can never be negative).

Therefore, the roots are imaginary.

6. d Let a, b be the roots of given equation.

So here $\alpha + \beta \text{(sum of the roots)} = \frac{-m}{12}$

$\alpha\beta = \frac{5}{12}$

Since, $\frac{\alpha}{\beta} = \frac{3}{2} \therefore 3\beta = 2\alpha$ or $\alpha = \frac{3}{2}\beta$

Now $\frac{3}{2}\beta^2 = \frac{5}{12}$

$\beta^2 = \frac{5}{12} \times \frac{2}{3} = \frac{5}{18}$ ... (i)

$\therefore \frac{3}{2}\beta+\beta=\frac{-m}{12} \Rightarrow \frac{5}{2}\beta=\frac{-m}{12} \Rightarrow \beta=\frac{-m}{30}$ ... (ii)

So $\left(\frac{-m}{30}\right)^2=\frac{5}{18} \Rightarrow m^2=250$

Hence, $m=\pm 5\sqrt{10}$

7. b $x^4+x^3+bx^2-ax+42$ is divisible by (x - 3).

Putting x = 3, according to remainder theorem,

$(3)^4+(3)^3+b(3)^2-a(3)+42=0$

Therefore, 81 + 27 + 9b - 3a + 42 = 0

a - 3b = 50

We need not use first information.

8. a Putting x = 5, y = z in $x^2 + y^2 + z^2 - (xy + yz + zx)$,

we get $(5 - y)^2$. The minimum value of which is definitely zero.

9. b We know that AM ≥ GM, i.e.

$\frac{3^x+3^{-x}}{2} \geq \sqrt{3^x \times 3^{-x}}$ ie $\frac{3^x+3^{-x}}{2} \geq 1$

$\frac{1}{3}\left[3^x+3^{-x}\right] \geq 2\times\frac{1}{3}$ or $\left[3^{x-1}+3^{-x-1}\right] \geq \frac{2}{3}$

So the least value is $\frac{2}{3}$

10. c Work with options.

(a) If we put b = -2, $\frac{a}{b}$ < 1 but b < a

(b) If we put a = -0.9, b = -2, $\frac{a}{b}$ < 1 but b < a

(c) If b > 0, $\frac{a}{b}$ will be a negative number and will be always less than 1

If b < a, then $\frac{a}{b}$ will be a positive number but $\frac{a}{b}$ will be less than 1.

11. c $\log_{10} 2.5 = n$

$\Rightarrow \log_{10} 10 - (\log_{10} 5 - \log_{10} 2) = 1 - n$

$\Rightarrow \log_{10} 4 = 1 - n \Rightarrow \log_{10} 2 = \frac{1-n}{2}$

12. c Age = 44, 44^2 = 1936

The only square between 1900 and 2000 is 1936.

13. c p + q = 2; 3p + 2q = 1

Then p = -3 and q = 5

14. e $a! = \frac{1}{2}b!$,

b! = 2a!

This is possible only when

b = 2 and a = 1

or

b = 2 and a = 0

∴ b is always equal to 2.

Hence value of a cannot be determined.

15. e $4x - 8 + a = bx - 1$

$\therefore x(4 - b) = 7 - a$

$x = \frac{7-a}{4-b}$, if x is an integer, then a = 4, b = 1 is a possible option.

Do this question using the options.

16. d $|x| = y + 5$

When $x < 0$, $-x = y + 5$ or $x + y + 5 = 0$

When x ³ 0, $x = y + 5$ or $x - y - 5 = 0$

17. e $\frac{3x+4y}{x+2y} = \frac{9}{4}$

$\therefore 12x + 16y = 9x + 18y$ or $3x = 2y$

$\therefore \frac{3x+5y}{3x-y} = \frac{2y+5y}{2y-y} = \frac{7}{1}$

18. c $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\therefore$ If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

19. b $(x - 2)^2 = 9 \therefore x = +5$ or -1

$(y - 3)^2 = 25 \therefore y = -2$ or 8

$\therefore$ Maximum value of $\frac{x}{y} = \frac{5}{8}$

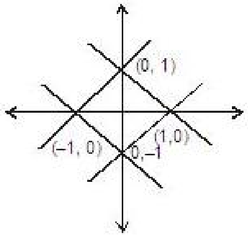
20. c $|x| + |y| = 1$ represent four lines which are as follows.

$y = -x = 1$ ...(i)

$y = x + 1$ ...(ii)

$y = -x - 1$ ...(iii)

$y = x - 1$ ...(iv)

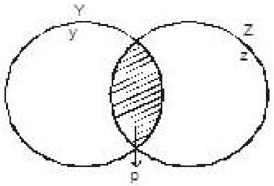


Hence a square whose sides are not parallel to the co-ordinates axes. Option (c) is the correct choice.

21. b The number common to both the clubs = $(y + z) - x$

Alternative method:

This can also be done by using Venn diagrams.



We know that $y + z = x + p$

$\{n(A) + n(B) = n(A \cap B) + n(A \cup B)\}$

$\therefore p = (y + z) - x$

22. e If $rs - \frac{9}{rs} = 8$, then rs = -1 or rs = 9

23. c $(m + n)^{100} = m^{100} + n^{100}$ + sum of same possible integer which is obviously greater then $m^{100} + n^{100}$.

Hence option (c) is the correct choice.

24. b (X + 2) + (Y + 3) + Z = 25 + 5 = 30

∴ The product is maximum,

if $(X + 2) = (Y + 3) = Z = \frac{30}{3} = 10$

∴ max[(x + 2) (y + 3) (z)] = 1000

25. e 2x + 3y + 4 = 0 and 3x + 4y + 6 = 0

Solving, we get x = -2 and y = 0

∴ 4x + 5y - A = 0 must be satisfied for the equations to be consistent.

Hence, A = -8

26. c Let $\frac{X^2 - X + 1}{X^2 + X + 1} = m$

Then $X^2(m - 1) + X(m + 1) + (m - 1) = 0$

If X Î R, then $(m + 1)^2 - 4(m - 1)^2 \geq 0$

$[(m + 1) + 2(m - 1)][(m + 1) - 2(m - 1)] \geq 0$

or $(3m - 1)(m - 3) \leq 0$

Hence, $\frac{1}{3} \leq m \leq 3$

27. c All the numbers, which are divisible by 9, have their sum of digits divisible by 9. Hence, the sum of all numbers from 100 to 200 having this property is

$(108 + 117 + 126 + ... + 198) = \frac{n}{2}$[First term + Last term] $= \left[\frac{108 + 198}{2}\right] \times 11 = 1683$

28. e Use the choices. From option (e),

$\frac{1}{9} < \frac{3}{28} < \frac{1}{8} \Rightarrow \frac{3}{27} < \frac{3}{28} < \frac{3}{24}$

29. d Use the choices.

30. d $\frac{N}{N+1} > 1$ or $\frac{N}{N+1} - 1 > 0$

or $\frac{N - N - 1}{N + 1} > 0$ or $\frac{-1}{N+1} > 0$

Hence, $\frac{1}{N+1} < 0$ or N + 1 < 0 or N < -1

31. e C + 7 is divisible by 5. Hence, C ends in either 3 or 8. For $C + n^2$ to be divisible by 5, n^2 has to end in either 2 or 7. Square of any number does not end in 2 or 7. Therefore, n cannot be found.

32. a $a + 3a = 4a = -\frac{b}{a}, 3a^2 = \frac{c}{a}$

$\therefore \left(\frac{-b}{4a}\right)^2 = \frac{c}{3a}$

Hence, $3b^2 = 16ac$

33. e $x^2 + x(k + 1) + 8 = 0$

If the roots are imaginary, then

$(k + 1)^2 - 4 \times 8 < 0$ or $(k + 1)^2 - (4\sqrt{2})^2 < 0$

$\therefore$ The value of k lies between $-(4\sqrt{2} + 1)$ and $(4\sqrt{2} - 1)$ excluding the extremes.

34. c Sum of the roots $= -\frac{a}{2} = 5 \therefore a = -10$

Product of the roots $= \frac{b}{2} = 6$ or $b = 12$

35. c $(x - 3)(x^2 + Cx + 2) = 0 = x^3 - ax^2 + 11x - 6$

$\therefore x^3 + Cx^2 + 2x - 3x^2 - 3Cx - 6 = x^3 - ax^2 + 11x - 6$

Equating the coefficients of like-powers, we get

$2 - 3C = 11$ or $C = -3$

$\therefore$ The other roots are roots of $x^2 - 3x + 2$,

i.e. $x = 1, x = 2$

36. e If we take $x \geq 0$, we get $x^2 + 5x + 6 = 0$

or $x = -2, -3$.

If we take $x < 0$, we get $x^2 - 5x + 6 = 0$ or

$x = 2, 3$

Thus, in none of the cases, we get any solution.

37. c $6a^2 + 13ab + 6b^2 = (3a + 2b)(2a + 3b)$

38. a $6x^2 + 9xy - 6y^2 = 0$

$\Rightarrow (6x - 3y)(x + 2y) = 0$

$\therefore x + 2y = 0$

$\therefore x = -2y$

If $6x - 3y = 1$, then $6(-2y) - 3y = 1$

or $-15y = 1, y = -\frac{1}{15}$

$\therefore x = \frac{2}{15}$

Alternative method:

Out of the four choices, only (a) satisfies the given equation $6x - 3y = 1$

Hence, (a) is the answer.

Exercise 3 - Level I

1. a $x^2 + 4xy + 4y^2 = (x + 2y)^2$

This takes a minimum value when $x + 2y = 0$

or $x = -2y$

2. d Manufacturing cost of N articles = 300 + 4N,

SP of N articles = 7N, profit = 3N - 300.

To break even, the number of articles to be produced is 100.

Hence, a minimum of 101 articles are to be produced per day.

3. a $5x^3 - 8x^2 = x^4 - x^3$

$\Rightarrow 5x - 8 = x^2 - x$ $(\because x \neq 0)$

$\therefore x = 4$ or 2

Man's age = $8 \times (2)^2 = 32$ years.

(Since, his age is less than 100 years)

4. c $x^2 - x - 12 < 0$

$\Rightarrow (x - 4)(x + 3) < 0$

or $-3 < x < 4$ ...(i) Also we have $|x + 2| < 3$

$\Rightarrow -3 < (x + 2) < 3$ or $-5 < x < 1$ ...(ii)

Combining (i) and (ii), we get $-3 < x < 1$.

5. d The two curves intersect each other at all those points which satisfy:

$(x^5 + 2x^4 + 8x^3 + 17x^2 + 10x - 36) - (x^5 + 2x^4 + 5x^3 - 4x^2 - 20x - 36) = 0$

Or, $3x(x^2 + 7x + 10) = 0$

Or at $x = 0, -2$ and -5. Hence, there are three points of intersection. So, option (d) is correct.

6. b 3rd term = a + 2d, 9th term = a + 8d

a + 2d = 10, a + 8d = 20

$\Rightarrow 2a + 10d = 30$

Sum of 11 terms = $\frac{11}{2}(2a + 10d) = \frac{11}{2}(30) = 165$

Alternative method:

Clearly, the sum has to be divisible by 11 {number of terms = 11} that eliminates (a).

Also the average of 11 terms is $\frac{10+20}{2} = 15$

{Average of all terms = Average of first and last

= Average of second and second last}

$\therefore$ Sum of 11 terms = $11 \times 15 = 165$

7. e **Case I:** For $x < 1$, the equation is:

$x^2 + x = 0$. The roots are $x = 0$ & $x = -1$.

Case II: $x \geq 1$, the equation is:

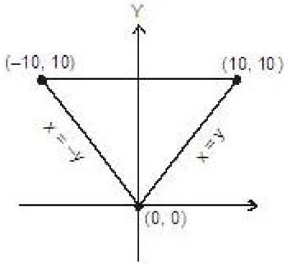
$x^2 - x + 2 = 0$. There are no real roots.

So the equation has exactly two roots: $x = 0$ & $x = -1$.

Hence, only option (e) is correct.

8. c $60 \times 2 = 120$ L

9. a The number of integral coordinates are
$(21 + 19 + 17 + \ldots + 1) = 121$.



10. d (a) is only true for 2 numbers,

i.e. $GCD(a, b) \times LCM(a, b) = a \times b$

but $GCD(a, b, c) \times LCM(a, b, c) \neq a \times b \times c$

(c) and (b) are false as the average of 3 numbers can be more than any one of them, i.e. average (abc) > a is possible but GCD(a, b, c) has to be $\leq$ a. {Assuming a < b and a < c}

Which leaves us with only choice (d) which is true.

11. e Evidently, if max(a, b, c) = min(a, b, c),

then a = b = c and all 3 are true.

12. d $f(g(x)) = 3[g(x)] = 3[x^2 + 3]$

$f(g(4.5)) = 3[(4.5)^2 + 3] = 3[23.25] = 69$

13. c 12 * 42 = HCF (12, 42) = 6

(231 D 6) $\Rightarrow 231 \times 2 = 462$

462 * 49 $\Rightarrow 7$

14. e $\min(2x^2 + 3x + 4)$

$$\Rightarrow \min\left(\left(\sqrt{2}x + \frac{3}{2\sqrt{2}}\right)^2 + \frac{23}{8}\right) = \frac{23}{8}$$

$\max(5 - x^2) = 5$

$$\therefore \frac{23}{8} - 5 = \frac{-17}{8}$$

15. a **Note:**

The function takes a maximum value when $2x - 3 = 3 - 2x$

or $4x = 6$, $x = \frac{3}{2}$, $f(x) = 0$

16. d $f(x) = 2 \times \frac{3}{2} - 3 = 0$

17. b $F[G(F(1)] = F[G(3 - 2 + 4)] = F[G(5)] = F(6) = 100$

18. a $(x + 2)^2 = x + 4$

$\Rightarrow x^2 + 4 + 4x = x + 4$, $x^2 + 3x = 0$

$\Rightarrow x(x + 3) = 0$, $\therefore x = -3$ or 0

But since $x + 2 > 0$. Therefore, $x = 0$

Alternative method:

Work with options, log of negative number is not defined. So x = -3 is not a solution.

19. e $4x^2 + 2xy + 9y^2 + 5(x^2 - y^2 + 2xy) = 9x^2 + 12xy + 4y^2 = (3x + 2y)^2 = 109 + 5 \times 7 = 144$

$\therefore (3x+2y)^2 = 144 \Rightarrow |3x + 2y| = 12$

$\therefore |3x + 2y| + 6 = 12 + 6 = 18.$

20. c $(1 + 2x^2 + x^4)^{100} = [(1 + x^2)^2]^{100} = (1 + x^2)^{200}$

When, expanded will have (200 + 1) terms i.e. 201 terms.

Hence, (c) is correct.

21. b $\therefore$ Series is 2 + 22 + 222 + ... to n times 2.

$2[1 + 11 + 111 + \ldots + n] = \frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

Alternative method:

By putting n = 0 in options, you would get 0. By putting n = 1 in options, you would get 2, etc.

22. a So according to the question, the series will be

$S = 3 + 9 + 15 + 21 + \ldots + 999$

Suppose 999 is nth term.

Now $999 = 3 + 6(n - 1)$ or $n = 167$

$S = \frac{167}{2}(999 + 3) = 83667$

23. b $\therefore 3^1 . 3^2 . 3^3 . 3^4 \ldots 3^x = 9^5$

So $3^{(1+2+3+4+\ldots x)} = (3^2)^5 = 3^{10}$

$\therefore 1 + 2 + 3 + 4 + \ldots x = 10$

$\Rightarrow \frac{x(x+1)}{2} = 10$

$\Rightarrow x^2 + x = 20 \Rightarrow x^2 + x - 20 = 0$

So x = 4 or x = -5

Hence, the answer is (b).

24. c According to the question, the series is

$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 65535$

$\frac{1(2^n - 1)}{2 - 1} = 65535 \left[\text{GP sum} = \frac{a(r^n - 1)}{r - 1}\right]$

So $(2^n - 1) = 65535$ or $2^n = 65536$ or $n = 16$

The answer is (c).

25. e $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-10)$

$\Rightarrow \log_{0.3}(x-1) < \log_{0.3}\sqrt{x-10}$

$\Rightarrow x - 1 < \sqrt{x-10}$

Put the values to check the answer, for all values given $\sqrt{x-10}$ is imaginary.

Hence, the answer is (e).

26. d $f[f(x)] = f[(a - x^n)^{1/n}] = [a - \{(a - x^n)^{1/n}\}^n]^{1/n}$

$= [a - a + x^n]^{1/n}$

$= x$

27. d $f(g(x)) = g(f(x))$ or $(g(x))^2 = g(x^2)$

Now try the options.

28. a $\alpha + \beta = -7$ and $\alpha\beta = 12$

$(\alpha + \beta)^2 + (\alpha - \beta)^2 = (-7)^2 - 4\alpha b + (\alpha + \beta)^2 = 50$

$(\alpha + \beta)^2 (\alpha - \beta)^2 = 49 \times 1 = 49$

$\Rightarrow$ Equation $x^2 - 50x + 49 = 0$

29. e Take $a = 1$, $b = 2$, $c = -10^{1000}$. We can take the values of a, b, c so that $ab + bc + ca$ keep decreasing. Hence, the minimum value will be $-\infty$.

30. d $x^2 + y^2 = 3$ ...(i)

$y = kx + 2$ ...(ii)

Putting, $y = kx + 2$ from (ii) in (i) we get:

$x^2 + (kx + 2)^2 = 3$

$\Rightarrow (k^2 + 1)x^2 + 4 - 3 + 4kx = 0$

$\Rightarrow (k^2 + 1)x^2 + 4kx + 1 = 0$

For real x : $(4k)^2 \geq 4(k^2 + 1).1$

$\Rightarrow 12k^2 \geq 4 \Rightarrow 3k^2 - 1 \geq 0$

$\Rightarrow \left(k - \frac{1}{\sqrt{3}}\right)\left(k + \frac{1}{\sqrt{3}}\right) \geq 0$; $k \leq \frac{-1}{\sqrt{3}}$ and $k \geq \frac{1}{\sqrt{3}}$

Only $k = \frac{3}{5}$ lies within the permissible limits.

31. c $xy = -15$ or $y = -\frac{15}{x}$

$x^2 - y^2 = 16$ or $x^2 - \left(-\frac{15}{x}\right)^2 = 16$ ... (i)

Solving (i), we get $x = 5$.

Therefore, $y = -3$ and $x + y = 2$

32. c We have, $f(x) = 2x^3 - x + 2k$

$\Rightarrow f(1) = 2(1)^3 - 1 + 2k = 2k + 1$

$\Rightarrow f(2) = 2(2)^3 - 2 + 2k = 2k + 14$

Since f(1) and f(2) are of opposite signs therefore

$(2k + 1)(2k + 14) < 0$

$\Rightarrow (2k + 1)(k + 7) < 0$

$\therefore -7 < k < -\frac{1}{2}$

33. b For any odd value of x, f(x) will not be a whole number, and there are 8 odd numbers in first 16 whole numbers.

34. b If we put $y = 3x$, then whatever values x takes, the values of y will be three times of that. $y = 3x$ or $x = \frac{y}{3}$

Now putting the value of x in the given equation, we get $2\left(\frac{y}{3}\right)^3 - 10\left(\frac{y}{3}\right)^2 + 13.\frac{y}{3} + 7 = 0$

Now replacing y with x, we get

$2x^3 - 30x^2 + 117x + 189 = 0$

35. c $2[x]^2 \le 32 \Rightarrow [x]^2 \le 16$

or, $-4 \le x < 5$

$3x + 2 \ge -1 \Rightarrow 3x \ge -3 \Rightarrow x \ge -1$

$\therefore -1 \le x < 5$

Number of integer values of x is 6.

36. d Expression $f(n) = -n^3 + 2n$ can be re-written as

$f(n) = n(n^2 + 2)$

$\Rightarrow n(n^2 - 1 + 3)$

$\Rightarrow (n-1)(n)(n+1) + 3n$

First term is the product these consecutive numbers which means it is divisible by 2, 3 as well as 6. Second term is obviously divisible by 3 for all positive integers. Therefore whole expressing is divisible by all values of 3. Also for n equal to an even number second expression becomes divisible by 6.

Therefore all the there statements are true.

Option (d) is the correct choice.

37. a For $0 \le x < 1$, $y = x$

$1 \le x < 2$, $y = x + 1$

$2 \le x < 3$, $y = x + 2$

For $-1 \le x < 0$, $y = x - 1$

$-2 \le x < 1$, $y = x - 2$

$-3 \le x < 2$, $y = x - 3$

This is satisfied by only choice (a).

So the answer is (a).

38. a When $b \le 3$, Max(b, 3) = 3

$\therefore$ Expression = 9

When $b > 3$, max(b, 3) = b

$\therefore$ Expression = $3b > 9$

{As $b > 3$}

39. b P(17, 1) = 17

L(3, 5) = 15

Min(15, 17) = 15

Max(3, 15) = 15

P(7, 15) = 105

40. b For all a, b

$G(a, b) < L(a, b) \le P(a, b)$

[G(a, b) = L(a, b)

$2 \le x < 3$, $y = x + 2$

For $-1 \le x < 0$, $y = x - 1$

$-2 \le x < 1$, $y = x - 2$

$-3 \le x < 2$, $y = x - 3$

This is satisfied by only choice (a).

So the answer is (a).

38. a When $b \le 3$, Max(b, 3) = 3

$\therefore$ Expression = 9

When $b > 3$, max(b, 3) = b

$\therefore$ Expression = $3b > 9$

{As $b > 3$}

39. b P(17, 1) = 17

L(3, 5) = 15

Min(15, 17) = 15

Max(3, 15) = 15

P(7, 15) = 105

40. b For all a, b

$G(a, b) < L(a, b) \le P(a, b)$

[G(a, b) = L(a, b)

only when a = b]

Exercise 4 - Level I

1. b $y = 3x + 9$

$\Rightarrow y - 9 = 3x$

$\Rightarrow x = \left(\frac{y-9}{3}\right)$

2. b $x = \frac{5}{a+b}$ and $y = 5(a + b)$

$\Rightarrow xy = 25$

$\Rightarrow (x+y) \geq 10$ {as AM $\geq$ GM}

$\Rightarrow (x + y)_{min} = 10$

$\Rightarrow$ Option (b) is correct.

3. a Given expression $= \{9 \Delta (3 * 0)\} \# \{36 \delta (6 * 0)\}$

$= \{9 \Delta 9\} \# \{36 \delta 36\} = 0 \# 0 = 0$

4. a Here we have to check all of the options given.

Check (b): It is not necessary because p and q both can be odd. Similarly, with (d) and (c), two of them, i.e. (p + q) can be odd or even. Hence, we are left with option (a).

5. d Since it is given that all of them are two-digit numbers, it is quite apparent that pqr - pq =
pq(r - 1) cannot be equal to zero. Hence, the answer is (d).

6. b We have to check that for what all x, $x^2 = 2x$

or $x^2 - 2x = 0$ or $x(x - 2) = 0$ or $x = 2$ or 0.

Hence, there are two such values.

7. a Let the 3 odd numbers be (x - 2), x and (x + 2).

It is given that $3(x - 2) = 3 + 2(x + 2)$

Hence, x = 13. So the third integer is (x + 2) = 15.

8. c $\frac{3}{4} = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x - \cfrac{1}{2}}}}$ ...[since x = y = z]

$= \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{2x-1}}} = \cfrac{1}{x + \cfrac{2x-1}{2x^2-x+2}}$

$\frac{3}{4} = \frac{2x^2 - x + 2}{2x^3 - x^2 + 4x - 1}$

$6x^3 - 11x^2 + 16x - 11 = 0$

For x = 1, 6 - 11 + 16 - 11 = 0

So x = 1 has to be a root of above cubic equation.

Hence, x + y + z = 3x = 3

9. d Use the choices.

(a) $a^x + a^y = a^{xy}$

(b) $\sqrt{x} + \sqrt{y} = \sqrt{xy}$

(c) $x^2 + y^2 = x^2y^2$

(d) $\log_3 x + \log_3 y = \log_3(xy)$

(e) $x^3 + y^3 = x^3y^3$

10. d Let the numbers of AP be

a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d,

where terms have their usual meaning.

Now 6a = 3

$\Rightarrow a = \frac{1}{2}$

Also a - 5d = 4(a - d)

$\Rightarrow d = -\frac{3}{2}$

$\therefore$ Fifth term, i.e. a + 3d = $\frac{1}{2} - \frac{9}{2} = -\frac{8}{2} = -4$

11. c The best to solve this question is the method of reverse substitution. Hence, the answer is (c), since $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

Observation: Since the sum of the two numbers is 10 and you are required to find the larger of them, your answer has to be more than 5.

So only verify for answers (c) and (d).

12. d ab = a + b. Therefore, $a = \frac{b}{b-1}$

13. a $\log \frac{u+v}{2} = \frac{1}{2}(\log u + \log v) = \frac{1}{2}\log(uv) = \log(uv)^{\frac{1}{2}}$

$\Rightarrow \frac{u+v}{2} = \sqrt{uv}$

$(u + v)^2 = 4uv$

$\Rightarrow u^2 + v^2 - 2uv = 0$

$\Rightarrow (u - v)^2 = 0$

$\Rightarrow u = v$

14. b Use remainder theorem and solve the simultaneous equations or put x = -1 and try choices combinations.

15. c Let p and q be the roots of the quadratic equation. From what is given, p = 1.5 q and p - q = 1 or

q - p = 1. From these equations, we get p = 3 and

q = 2 (or) p = -3 and q = -2.

Since the quadratic equation with roots p and q is given by (x - p) (x - q) it follows that $x^2 - 5x + 6$

$= 0$ (or) $x^2 + 5x + 6 = 0$.

16. d 2(V - 1) = S + 1 and V + 1 = S - 1

Solving both equations, we get S = 7

17. b $V = 3\log_e K = \log_e 3K$

or, $\log_e K^3 = \log_e 3K$

or, $K^3 = 3K$ or, $K^3 - 3K = 0$ or, $K(K^2 - 3) = 0$

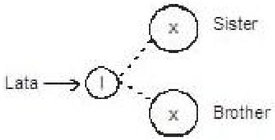
or, $K = 0, \sqrt{3}$ or $-\sqrt{3}$

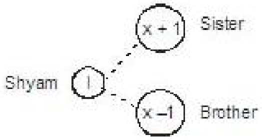
but K > 0 as log is defined for only positive numbers.

$\therefore K = \sqrt{3}$ is the only possible solution of the system of the given equations. Hence only one solution exists.

18. a Since Lata is a female and has same number of brothers and sisters, it is obvious that total number of sisters in the family is 1 more than brothers. So total number of children has to be odd. So (b) and (d) cannot be the answer. Also Shyam is a male and has twice the number of sisters as he has brothers. So total number of sisters in the family = 2 (Number of brothers - 1). The only answer that suits this is (a), 7 children (4 sisters and 3 brothers).

Alternative method:





$(x + 1) = 2(x - 1)$

$x = 3$

Total children = $2x + 1 = 7$

19. b $f(x, y) = (x - 2)^2 + (y - 2)^2 - 4$

and $g(x, y) = (x - 4)^2 + (y - 2)^2 - 4$

Or $g(x, y) = (x - 2 - 2)^2 + (y - 2)^2 - 4$

Or $g(x, y) = f(x - 2, y)$

20. a Let x is the total number of fruits , then

$$\frac{\frac{\frac{x}{2}+1}{2}+1}{2}+1=x$$

On solving, we get $\frac{x+14}{8}=x$

$\Rightarrow x = 2$

Alternative method:

Use choice. It is obvious that the number of fruits must be equal to 2(1) = 2.

21. b a, a + 2, a + 4

$3a = 2(a + 4) + 2$

$\Rightarrow a = 10$ and $a + 4 = 14$.

Alternative method:

Check through the choices.

22. b If a - b = 3, then

$(a - b)^3 = a^3 - b^3 - 3ab\,(a - b) = 117 - 3ab(3) = 27$

Therefore, $ab = 10 \Rightarrow a = 5 , b = 2$

Thus, $a + b = 7$

23. e $3x + 6y = 7$

$x = \frac{7-6y}{3}$

Try with some values of y and see that for any integer value of y there exists no integer value of x.

Also we can see that $3x + 6y$ is a multiple of 3 but not of 7. Thus, no integer solution is possible.

24. c Let x be the extra number of passengers.

Required product of speed and extra passengers

$= x(300 - x) = 300x - x^2$

$= 22500 - 22500 + 300x - x^2 = 22500 - (x - 150)^2$

If product of speed and extra passengers is maximum,

when $x - 150 = 0$

$\Rightarrow x = 150$

Number of passengers is 250.

25. a The team has played a total of $(17 + 3) = 20$ matches. This constitutes $\frac{2}{3}$ of the matches.

Hence, total number of matches played = 30.

To win $\frac{3}{4}$ of them, a team has to win 22.5, i.e. at least win 23 of them. In other words, the team has to win a minimum of 6 matches (since it has already won 17) out of remaining 10. So it can lose a maximum of 4 of them.

26. c It can be easily verified from the figure that

$f(x) > g(x)$ for $x \in (0, 5]$

27. d Put any value of x in f(x) and g(x) and check. Alternately, g(x) is derived by multiplying f(x) with -1. Hence, all the three choices are true.

28. c Take any value of x, and put into the functions. Then check the choices.

29. c Take any value of x and put into the functions. Then check the choices.

Exercise 5 - Level 2

1. e In the given series a = 11, l = 201 and d = 10 number of terms is not known, so the nth term

$a_n = a + (n - 1)d$

$201 = 11 + (x - 1)\,10 = 11 + 10x - 10 = 10x + 1$

$200 = 10\,x$

$x = 20$

So sum = $\left(\frac{a+l}{2}\right)\times n = \frac{(11+201)}{2}\times 20 = 212\times 10 = 2120$ $= 212 \times 10 = 2120$.

2. e Since x is in modulus so either x is positive or negative, if x ³ 0

$2x + 3x \geq 5$

$5x \geq 5$

and $x \geq 1$

If $x < 0$ then

$2\,(-x) + 3\,(-x) \geq 5$

$-2x - 3x \geq 5$

$-5x \geq 5$

$5x \leq -5$

$x \leq -1$

$\Rightarrow x \geq 1$ or $x \leq -1$

3. d $x^2 - 5x - 14 \leq 0$

$x^2 - 7x + 2x - 14 \leq 0$

$x\,(x - 7) + 2\,(x - 7) \leq 0$

$(x + 2)\,(x - 7) \leq 0$

So $-2 \leq x \leq 7$

[Because if $(x - a)\,(x - b) \leq 0$ then the value of x lies from a to b]

4. d Let the common factor of the number is x, so

$(2x)^2 + (3x)^2 + (4x)^2 = 725$

$29\,x^2 = 725$ So, $x^2 = 25$ and $x = \pm 5$

So the highest number is $4x = 4 \times 5 = 20$

5. b Suppose that he worked for x days then

$5x - (20 - x)7 = 52$

$5x - 140 + 7x = 52$

So $12x = 140 + 52$

$x = \frac{192}{12} = 16$

So he worked for 16 days.

6. a Let the number be x and y

$x^2 + y^2 = 3341$ and $x^2 - y^2 = 891$

So $2x^2 = 4232$

$x^2 = 2116$

$x = 46$

$2116 - 891 = y^2$

So $y^2 = 1225$ and $y = 35$

7. d Let length = x metres and breadth = y metres

So $xy = 255$ and $x - 1 = y + 1$

So $(y + 2)y = 255 = y^2 + 2y - 255 = 0$

$y^2 + 17y - 15y - 255 = 0$

$y(y + 17) - 15 (y + 17) = 0$

$(y - 15) (y + 17) = 0$

$y - 15 = 0$

$y = 15$

and $x = \frac{255}{y} = \frac{255}{15} = 17$

So side of the square = 15 + 1 = 16 (or 17 - 1 = 16)

So its perimeter = $16 \times 4 = 64$ metres

Alternative method:

$xy = 15 \times 17$. If length is decreased by 1 m and breadth is increased by 1 m, then it becomes an area equal to $(16)^2 = 256$.

Hence the perimeter is 64.

8. d A = 2B, B = 3C

$\frac{A}{B} = \frac{2}{1}\ \frac{B}{C} = \frac{3}{1}$

A : B : C = 6 : 3 : 1

Share of A = $\frac{6}{10} \times 100 = 60$

9. a $(x - 6) (4x + 3) = 0$, then out of two terms at least one term must be zero, so

$x - 6 = 0$ then $x = 6$ and $2x = 12$

and $4x + 3 = 0$

So $x = \frac{-3}{4}$ and $2x = \frac{-3}{4} \times 2 = \frac{-3}{2}$

10. d Let the two digit number be xy

So $x + y = 7$ and $x = \frac{3}{4}y$

So $\frac{3}{4}y + y = 7$

$7y = 28$

So $y = 4$ and $x = 3$ and the number is 34.

11. c $ax^2 + bx + c = 0$

Product of the roots = $\frac{c}{a}$

$\Rightarrow \alpha \times \beta = \frac{c}{a} \Rightarrow c \times \beta = \frac{c}{a}$

$\therefore \beta = \frac{1}{a}$

12. d Sum of the roots:

$\alpha + \beta = (2+\sqrt{2}) + (2-\sqrt{2}) = 4$

Product of roots $\alpha \times \beta = (2+\sqrt{2})\ (2-\sqrt{2}) = (2)^2 - (\sqrt{2})^2$ = 4 - 2 = 2

So the equation is $x^2 - (\alpha+\beta)x + \alpha \times \beta\ =\ x^2 - 4x + 2 = 0$

13. b Let `y' should be subtracted from the numerators as well as the denominators , then $\frac{2-y}{x-y} = \frac{40}{1}$

So 2 - y = 40x - 40y

So 39y = 40x - 2 and y = $\frac{40x-2}{39}$

14. e 5 + 2 |x| = 12

First case: When x > 0, then $|x| = x$

5 + 2x = 12

So, 2x = 12 - 5 = 7 and x = $\frac{7}{2}$

Second case: When x > 0, then $|x| = -x$

5 + 2 - (x) = 12

So 5 - 2x = 12 and x = $\frac{-7}{2}$

So $x = \pm\frac{7}{2}$

15. d The given series is a geometric progression.

Here, a = 4 and common ratio r = 2 and n = 10

So, sum of the series = $\frac{a(r^n-1)}{r-1} = \frac{4(2^{10}-1)}{1}$

16. a x^2 - 6x + 9

$\Rightarrow$ (x - 3)(x - 3)

and x^2 - 5x + 6

$\Rightarrow$ (x - 3) (x - 2)

So their HCF is x - 3

17. e x^2 + kx - 8 = 0

One root is square of the other.

$\therefore \alpha + \alpha^2 = -k$

$\alpha . \alpha^2 = -8$ or $\alpha = -2$

$\therefore$ -2 + 4 = -k

$\therefore$ k = -2

18. d Since 1 is one of the roots of the equation $x^3 - 4x^2 + bx + 6 = 0$, we have

$1^3 - 4(1)^2 + b.1 + 6 = 0 \Rightarrow b = -3$

So the equation is $x^3-4x^2-3x+6=0$

or $(x-1)(x^2-3x-6)=0$

∴ Other roots of this equation are roots of the quadratic equation $x^2-3x-6=0$. Roots of the equation $x^2-3x-6=0$ are $\frac{3\pm\sqrt{33}}{2}$.

19. c Let the 2 numbers be a, b.

$\Rightarrow \frac{1}{a}+\frac{1}{b}=\frac{1}{12}$

= 12a + 12b = ab

So, ab - 12a - 12b + 144 = 144

= a(b - 12) - 12 (b - 12) = 144

= (a - 12) (b - 12) = 144

144 has 15 factors. Hence either (a - 12) or (b - 12) has 15 possible values. Since we are interested in pairs and NOT ordered pairs, there are 8 possible pairs.

20. e Let the three roots be α, β and γ. The second, fifth and eight terms of any geometric progression will always be the three consecutive terms of some other geometric progression as well. Hence, we can write:

$\alpha=\frac{a}{r}$; $\beta=a$ and $\gamma=ar$.

also, $\alpha+\beta+\gamma=\frac{-\frac{1}{\sqrt{p}+\sqrt{q}}}{1}=-\frac{1}{\sqrt{p}+\sqrt{q}}$

or $\frac{a}{r}+a+ar=\frac{-1}{\sqrt{p}+\sqrt{q}}$

or $a\left(\frac{1}{r}+1+r\right)=-\frac{1}{\sqrt{p}+\sqrt{q}}$ ...(i)

and $\alpha.\beta+\beta\gamma+\gamma.\alpha=\frac{1}{\sqrt{p}-\sqrt{q}}$

$a^2\left(\frac{1}{r}+1+r\right)=\frac{1}{\sqrt{p}-\sqrt{q}}$ ...(ii)

and $\alpha.\beta\gamma=1$

or $a^3=1$ or a = 1 ...(iii) $[\because a \text{ is real}]$

From equations (i) and (ii)

$a=\frac{\sqrt{p}+\sqrt{q}}{\sqrt{q}-\sqrt{p}} \Rightarrow \sqrt{p}+\sqrt{q}=\sqrt{q}-\sqrt{p}$ or $p=0$

Hence (e) is the correct option.

21. b $\frac{1}{r}+\frac{1}{s}-\frac{3}{t}=\frac{2}{5r}$

$\Rightarrow\left(\frac{3s-t}{st}\right)r=\frac{3}{5}$

Now, st must be among 5, 10, 15, 20, 25, 30, 35, 40, 45.

Only, s = 2, t = 5, r = 6

and s = 5, t = 6, r = 2 are found to be valid.

Therefore, (r + s - t) = 3 or 1.

Therefore there are two possible values of (r + s - t)

$x^2 - 2(p-1)x + \left[\frac{3}{4}p^2 + \frac{7}{3}p - \frac{25}{2}\right] > 0$

22. a $\Rightarrow x^2 - 2(p-1)x + (p-1)^2 + \left[\frac{3}{4}p^2 + \frac{7}{3}p - \frac{25}{2}\right] - (p-1)^2 > 0$

$\Rightarrow [x-(p-1)]^2 + \left[-\frac{1}{4}p^2 + \frac{13}{3}p - \frac{27}{2}\right] > 0$

The first term

$[x-(p-1)]^2$ is always positive for any real values of x and p.

Now $\left[-\frac{1}{4}p^2 + \frac{13}{3}p - \frac{27}{2}\right] = \frac{-3p^2 + 52p - 162}{12}$

The least integral value of p for which $\frac{-3p^2 + 52p - 162}{12}$ is greater than zero is 5.

23. d Here, $f(x) = x^2 + 3x + 2$ and $y = mx$,

$\Rightarrow x^2 + 3x + 2 = mx$

$\Rightarrow x^2 + (3-m)x + 2 = 0$ ...(i)

For the graph of f(x) and the line y = mx to have no points of intersection, discriminant (D) of equation (i) must be negative.

$D = (3 - m)^2 - 4 \times 2 < 0$

$\Rightarrow (3 - m)^2 < 8$

$\Rightarrow -2\sqrt{2} < 3 - m < 2\sqrt{2}$

$\therefore 3 - 2\sqrt{2} < m < 3 + 2\sqrt{2}$

$\therefore$ Five integral values of m are possible i.e. 1, 2, 3, 4 and 5.

24. d Here $x^2(x-4\sqrt{3})^2 = 25(x-2\sqrt{3})^2 - 36$

Put $x - 2\sqrt{3} = y$ then $x = y + 2\sqrt{3}$

Therefore $(y+2\sqrt{3})^2(y-2\sqrt{3})^2 = 25y^2 - 36$

$\Rightarrow (y^2 - 12)^2 = 25y^2 - 36$

$\Rightarrow y^4 - 24y^2 + 144 = 25y^2 - 36$

$\Rightarrow y^4 - 49y^2 + 180 = 0$

$\Rightarrow (y^2 - 4)(y^2 - 45) = 0$

$\Rightarrow y = \pm 2$ or $\pm 3\sqrt{5}$

So, $x = 2 + 2\sqrt{3}, -2 + 2\sqrt{3}, 3\sqrt{5} + 2\sqrt{3}$

or $-3\sqrt{5} + 2\sqrt{3}$.

25. a If $f(x) = 4x + 7$, then $\frac{f(a) - f(b)}{a - b} = \frac{4a - 4b}{a - b} = 4$

26. d $2a^2b + a(2b^2+1) + b = 135$

$\Rightarrow 2a^2b + a + 2ab^2 + b = 135$

$\Rightarrow (a + b)(2ab + 1) = 135$

$\Rightarrow 15(2ab + 1) = 135$

$\Rightarrow ab = 4$ and $a + b = 15$

If a and b are the roots of the quadratic equation, then equation would be a^2 - (Sum of the roots) a + (Product of the roots) = $a^2 - 15a + 4 = 0$

27. b Let $243^{\log_{81} x} = y$

$\Rightarrow \log_{81} x \log 243 = \log y$

$\Rightarrow \log_{81} x = \log_{243} y \Rightarrow y = x^{\frac{5}{4}}$

$243^{\log_{81} x} - 2x = 2^{\log_{16} x + 2} - 8$

$\Rightarrow x^{\frac{5}{4}} - 2x = 4x^{\frac{1}{4}} - 8$

$\Rightarrow x(x^{\frac{1}{4}} - 2) = 4(x^{\frac{1}{4}} - 2)$

So, either x = 4 or $x^{\frac{1}{4}} - 2 = 0$

$\therefore x = 4 \text{ or } x = 16$

Sum of all the real values of `x' that satisfy the given equation is 20.

28. c $a^x = b^y = ab$

$\Rightarrow a^x = ab \Rightarrow a^{x-1} = b$

$\Rightarrow b^y = ab \Rightarrow a = b^{y-1} \Rightarrow b = a^{\frac{1}{y-1}}$

$\Rightarrow a^{x-1} = a^{\frac{1}{y-1}} \Rightarrow x - 1 = \frac{1}{y-1} \Rightarrow y = 1 + \frac{1}{x-1} = \frac{x}{x-1}$

29. d f (1) = f(8) = f(15) = ... = 1

f(2) = f(9) = f(16) = ... = 1

f(3) = f(10) = f(17) = ... = -1

f(4) = f(11) = f(18) = ... = -1

f(5) = f(12) = f(19) = ... = -1

f(6) = f(13) = f(20) = ... = 1

f(7) = f(14) = f(21) = ... = -1

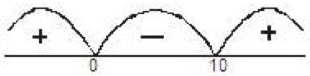
∴ f (694) + (695) = 1 + 1 = 2.

30. a $\log_{10}\left[\frac{n(10-n)}{16}\right] < 0$

$\Rightarrow 0 < \frac{n(10-n)}{16} < 1 \Rightarrow 0 < n(10-n) < 16$

$\Rightarrow n(10-n) > 0 \text{ and } n(10-n) < 16$

n (n - 10) < 0



⇒ 0 < n < 10

n (10 - n) < 16

⇒ (n - 2) (n - 8) > 0

⇒ n < 2 and n > 8

∴ 0 < n < 2 and 8 < n < 10

Therefore, there are 2 integer values i.e. n = 1 and

n = 9 for which $\log 10\left(\frac{n(10-n)}{16}\right) < 0$

31. b First of all arrange the given equation

$$\frac{x^3 + (a+b-c)x^2 + (ab-bc-ca)x - abc}{x^2 + (a-c)x - ac} = \frac{(x+a)(x+b)(x-c)}{(x+a)(x-c)} = (x+b)$$

Alternative method:

Working with choices, we can observe that the numerator will have 3 terms having a, b and c in them, and the denominator will have 2 terms having a and c in them.

Choice (d) can be eliminated as it has x + a in it.

Choice (c) can be eliminated as $x \pm c$ seems to have been cancelled by $x \pm a$ term.

Choice (b) for (x + b) to be the right choice, the numerator should reduce to zero for x + b = 0

$\Rightarrow$ x = (- b)

$N_r = -b^3 - abc + (ab - bc - ca)(-b) + (a + b - c)b^2 = 0$

Hence, the choice (b) is correct.

32. b $v \propto \frac{1}{m^2} \Rightarrow v = \frac{k}{m^2}$

$m \propto \frac{1}{t^2} \Rightarrow m = \frac{k_1}{t^2}$

$\therefore m^2 = \frac{k_1^2}{t^4}$

Now put this value into v = $\frac{k}{m^2}$

V = $\frac{k}{\frac{k_1^2}{t^4}} \Rightarrow \frac{k}{k_1^2} \times t^4$

Since $\left(\frac{k}{k_1^2}\right)$ is a constant, $v \propto t^4$.

33. a As a - b = 1

$\left(\sqrt{a} + \sqrt{b}\right) = \left(\sqrt{a} - \sqrt{b}\right)^{-1}$

So, we can write

$\left(\sqrt{a} + \sqrt{b}\right)^{-\left(1 - \frac{1}{\sqrt{x}}\right)} \geq \left(\sqrt{a} + \sqrt{b}\right)^{\sqrt{x}+1}$

As a and b are real & a = b + 1; $\sqrt{a} + \sqrt{b} > 1$

We must have

$-\left(1 - \frac{1}{\sqrt{x}}\right) \geq \left(\sqrt{x} + 1\right)$

Or $\sqrt{x} - \frac{1}{\sqrt{x}} + 2 \leq 0$

Or $x + 2\sqrt{x} - 1 \leq 0$

Or $t^2 + 2t - 1 \leq 0 \ldots. \left(\text{put } t = \sqrt{x}\right)$

Or $[t - (-1 - \sqrt{2})][t + (-1 + \sqrt{2})] \leq 0$

$\Rightarrow -1 - \sqrt{2} \leq t \leq -1 + \sqrt{2}$

$\Rightarrow 0 \leq t^2 \leq (-1 - \sqrt{2})^2$

Or $0 \leq x \leq 3 + 2\sqrt{2}$

The largest value of x is $3 + 2\sqrt{2}$.

34. d $A = \frac{2n^3 + 2n^2 + \sqrt{n}(n+3) - 3}{4n\sqrt{n}(n\sqrt{n} + 3) + 9}$

$= \frac{(2n\sqrt{n} + 3)(\sqrt{n}(n+1) - 1)}{(2n\sqrt{n} + 3)^2} = \frac{(n+1)\sqrt{n} - 1}{2n\sqrt{n} + 3}$

For n = 16, A = $\frac{67}{131}$

For n = 36, A = $\frac{221}{435}$

$0.508 \le A \le 0.511$

It is also observed that the value of A is monotonously increasing in the given range.

35. b Since time period $\propto \frac{\sqrt{\text{Length}}}{\text{Acceleration due to gravity}}$,

$T \propto \frac{\sqrt{l}}{g}$, $T_1 \propto \frac{\sqrt{l_1}}{g_1}$ and $T_2 \propto \frac{\sqrt{l_2}}{g_2}$

Where l = Length and

g = Acceleration due to gravity.

$$\therefore \frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}} \times \frac{g_2}{g_1} = \sqrt{\frac{1}{4}} \times \frac{\frac{1}{2}}{1} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

36. d $4^{a-5+b} = 2^{a+b} \times 2^{b} \times 2^{a-4} - 63$

$$4^{(a+b)-5} = 2^{2(a+b)-4} - 63$$

$$4^{(a+b)-5} - 4^{(a+b)-2} = -63$$

$$4^{(a+b)-2}\left[4^{-3} - 1\right] = -63$$

$$4^{(a+b)-2}\left[\frac{1}{64} - 1\right] = -63$$

$$4^{(a+b)-2}\left[-\frac{63}{64}\right] = -63 \Rightarrow 4^{(a+b)-2} = 64 = (4)^3$$

or (a + b) - 2 = 3

Therefore, (a + b) = 5

37. c Given equations are

$x^2 + 4 = y^3$ ...(i)

x + y = 16 ...(ii)

Solving (i) and (ii), we get

$y^3 - y^2 + 32y - 260$ ...(iii)

In equations (iii) 5 is the root of the equation as it satisfies it. Dividing $y^3 - y^2 + 32y - 260$ by (y - 5) gives a quadratic equation having imaginary roots.

Therefore given set of equations have one real solution i.e. (11, 5)

38. a $S = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\ldots\left(1 - \frac{1}{n^2}\right)$, Where n = 50

$$S = \left(\frac{1}{2} \times \frac{3}{2}\right)\left(\frac{2}{3} \times \frac{4}{3}\right)\left(\frac{3}{4} \times \frac{5}{4}\right)\ldots\left(\frac{n-1}{n} \times \frac{n+1}{n}\right)$$

$$\Rightarrow S = \frac{1}{2} \times \frac{51}{50} = \frac{51}{100} = 0.51$$

39. d $(x^2 + 2y^2 + z^2 + 2yz) = x^2 + (y + z)^2 + y^2$

This is minimum when x = y + z = y.

As x + 2y + z = -6,

x = -2

y = -2

& z = 0

$\Rightarrow$ The minimum value is $(-2)^2 + (-2)^2 + (-2)^2 = 12$.

37. c Given equations are

$x^2 + 4 = y^3$ ...(i)

$x + y = 16$ ...(ii)

Solving (i) and (ii), we get

$y^3 - y^2 + 32y - 260$ ...(iii)

In equations (iii) 5 is the root of the equation as it satisfies it. Dividing $y^3 - y^2 + 32y - 260$ by $(y - 5)$ gives a quadratic equation having imaginary roots.

Therefore given set of equations have one real solution i.e. (11, 5)

38. a $S = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\left(1-\frac{1}{n^2}\right)$, Where $n = 50$

$S = \left(\frac{1}{2}\times\frac{3}{2}\right)\left(\frac{2}{3}\times\frac{4}{3}\right)\left(\frac{3}{4}\times\frac{5}{4}\right)\ldots\left(\frac{n-1}{n}\times\frac{n+1}{n}\right)$

$\Rightarrow S = \frac{1}{2}\times\frac{51}{50} = \frac{51}{100} = 0.51$

39. d $(x^2 + 2y^2 + z^2 + 2yz) = x^2 + (y + z)^2 + y^2$

This is minimum when $x = y + z = y$.

As $x + 2y + z = -6$,

$x = -2$

$y = -2$

& $z = 0$

$\Rightarrow$ The minimum value is $(-2)^2 + (-2)^2 + (-2)^2 = 12$.

Hence, (d) is correct.

40. d $p = 0.787878$ ---

$\Rightarrow 100\, p = 78.7878$ ---

$\Rightarrow 99\, p = 78$

or $p = \frac{78}{99} = \frac{26}{33}$

$p^2 = 0.620$

And $p^4 = 0.38 < 0.40$

$\Rightarrow$ Option (d) is correct.

Exercise 6 - Level 2

1. e RHS of given inequality is in GP sum of which is $2^x - 1$, so we have $2x > 2^x - 1$, which is true for only x = 1 and 2.

2. b for n = 1, the equation becomes $x + 1 = 0 \Rightarrow x = -1$

$\Rightarrow p = (-1)$, so; $[p^{2(n+1)} + 3] \times [p^{3(n+1)} - 4]$

$= 4 \times (-3) = -12$.

$\Rightarrow$ Option (b) is correct.

3. c $S = \frac{1}{3\times7} + \frac{1}{7\times11} + \frac{1}{11\times15} + \ldots\ldots$

$4S = \frac{4}{3\times7} + \frac{4}{7\times11} + \frac{4}{11\times15} + \ldots\ldots = \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{11}\right) + \left(\frac{1}{11} - \frac{1}{15}\right) + \ldots\ldots$

or $4S = \frac{1}{3}$

$\Rightarrow S = \frac{1}{12}$

$\Rightarrow$ Option (c) is correct.

4. a For such type of problems, get the equations of the three lines.

Equation of AB is $-x + y = 2$

Similarly, equation of BC is $y = (2 - 2x)$

Equation of CA is $2y + x = -2$

$\therefore$ The equation of the enclosed area is

$-x + y < 2$

or $y < 2 + x$

$y < (2 - 2x)$

And $2y + x > -2$

Alternative method:

We see that origin (0, 0) lies in the enclosed area. This means x = 0 and y = 0 should satisfy all the
3 inequalities of a particular choice.

Choice (a): All three inequalities are satisfied.

Choice (b): $2y > x + 2 \Rightarrow 0 > 2$

Hence, the third inequality is not satisfied.

Choice (c): Although it satisfies all three equations but all the slopes are identical.

Hence, choice (a) is the appropriate choice.

5. a Since, `x' is a number greater than 1, it means that log x > 0.

Also, $10 > 3 > e > 2 \Rightarrow \log 10 > \log 3 > \log > \log 2$

$\therefore \log_2 x > \log x > \log_3 x > \log_{10} x$

6. b $S = 2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times ----$

$\Rightarrow S = 2^{\left[\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + ---\infty\right]}$

$\Rightarrow S = 2^p$, where

$p = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + ---\infty$

$\Rightarrow \frac{p}{2} = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{4}{32} + --- \infty$

$\left(p - \frac{p}{2}\right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + --- \infty$

$\Rightarrow \frac{p}{2} = \frac{\frac{1}{4}}{\frac{1}{2}} \Rightarrow p = 1 \Rightarrow S = 2^1 = 2.$

7. d $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r} \Rightarrow \frac{2x+p+q}{(x+p)(x+q)} = \frac{1}{r}$

$\Rightarrow 2rx + (p+q)r = x^2 + x(p+q) + pq$

$\Rightarrow x^2 + x(p+q-2r) + pq - (p+q)r = 0$

Since the roots of the equation are equal in magnitude and opposite in sign, therefore sum of the roots of the equation is zero.

$\Rightarrow p+q = 2r \Rightarrow r = \frac{p+q}{2}$

Product of roots $= pq - (p+q)\frac{(p+q)}{2}$

Hence (d) is the correct choice.

8. c Given that

$f(x) = \frac{4^x}{4^x + 2}$ ...(i)

Also $f(1-x) = \frac{4^1 - x}{4^{1-x} + 2}$ ... (ii)

Adding (i) and (ii) we get

$f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{4^{1-x}}{4^{1-} + 2} = 1$

$\therefore f\left(\frac{1}{1997}\right) + f\left(\frac{1996}{1997}\right) = 1$

$f\left(\frac{2}{1997}\right) + f\left(\frac{1995}{1997}\right) = 1$ and to on

Hence

$f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + ---- + f\left(\frac{1996}{1997}\right) = \frac{1996}{2} = 998$

Option (c) is the correct choice.

9. b Sum of n successive integers = (n × middle term)

The number of terms can be

(i) even

(ii) odd

(i) If the number of terms `n' is even, middle term must be interpreted as the arithmetic mean of the $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$th term.

⇒ Middle term will have 0.5 as the decimal.

Here, (n × middle term) = 1000

⇒ n = 16 and hence,

16 × 62.5 = 1000

⇒ The integers must be (55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70)

(ii) If the number of terms `n' is odd,

(n × middle term) = 1000

n is odd ⇒ middle term must be even

(n × middle term) = $2^3 \times 5^3$

⇒ n = 5 or 25 or 125

But for n = 125, not all terms would be positive integers.

⇒ The integers must be (198, 199, 200, 201, 202) OR

(28, 29, ..., 40, ..., 51, 52)

10. c f(1) = a[f(3) + f(5)+......] = a [b - f(1)].

So f(1) = $\frac{ab}{1+a}$ ...(i)

f(3) = a[b - f(3) - f(1)].

Substituting for f(1) from result (i), we get f(3) = $\frac{ab}{(1+a)^2}$

11. c Since the cows give milk proportional to their age let them give x, 2x , 3x ....9x liters per day. So we will have to split them into 3 groups such that each group has 15x.

This involves counting. The starting point is 9x and 8x have to be in different groups...

Hence the number of ways are :

(9,6) (8,7) (rest); (9,6) (8,5,2) (rest); (9,6) (8,4,3) (rest); (9,6) (8,4,2,1) (rest); (9,5,1) (8,7) (rest); (9,5,1) (8,4,3) (rest); (9,4,2) (8,7) (rest); (9,4,2) (8,6,1) (rest)...In all there are 8 ways.

12. c 8q + 16p = pq + 96 ⇒ $q = \frac{16(6-p)}{(8-p)}$

Let S be the profit made by Honda in a day

$S = 4p + 2q = 4p + \left[\frac{2\times16(6-p)}{(8-p)}\right] = \frac{4\{p(8-p)+8(6-p)\}}{(8-p)}$

$\frac{4(48-p^2)}{(8-p)} = 4x \text{ where } x = \frac{48-p^2}{8-p}$

⇒ $p^2 - px + 8x - 48 = 0 \Rightarrow x^2 - 32x + (4 \times 48) \geq 0$

⇒ $(x - 8)(x - 24) \geq 0 \Rightarrow x \leq 8$ or $x \geq 24$

Now $0 \leq p \leq 6$ and $p^2 - px + 8x - 48 = 0$

Sum of roots = x whose maximum value will be

p + p = 2p = 12

∴ Maximum value of x = 8 in the range

Maximum profit = 32 lakhs.

Alternative method:

$q = 16\frac{6-p}{8-p}$

Since p is some integer from 0 to 6, only 3 values of (p, q) hold

(i) p = 0, q = 12 ⇒ profit = Rs. 24 lakhs

(ii) p = 4, q = 8 ⇒ profit = Rs. 32 lakhs

(iii) p = 6, q = 0 ⇒ profit = Rs. 24 lakhs

$\therefore$ Maximum profit = Rs. 32 lakhs.

13. e $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1$

So $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\left(\frac{pq}{ab}+\frac{pr}{ac}+\frac{qr}{bc}\right)=1$ ... (I)

Also $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=0$ (or) aqr + prb + pqc = 0 ... (II)

(I) Can be written as

$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\left[\frac{pqc+prb+qra}{abc}\right]=1$

or $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1$. [From II]

14. a Sum of the roots = -5

Product of the roots = 2

Hence, the equations is $x^2 + 5x + 2 = 0$

15. d Sum of all page numbers = $\left(\frac{1+196}{2}\right)\times 196 = 19306$.

When you are tearing (35 sheets) 70 pages, the smallest page number would be odd and the largest would be even or vice versa.

Hence, $\frac{70}{2}$ (F + L) will end in 5.

So the last digit of the sum of the remaining page numbers must end in 1. So options (a), (b) and (c) are not possible.

Out of the options, only 11361 is possible.

16. a If $f(x) = ax^2 + bx + c$ attains a maxima, shape of the graph must be a parabola, opening downwards, as shown below:

Now, $f(x) = ax^2 + bx + c$

$$=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\left(\frac{b}{2a}\right)^2-\frac{c}{a}\right)\right]=a\left[\left(x+\frac{b}{2a}\right)^2-k\right]\left\{\text{Putting}\left(\left(\frac{b}{2a}\right)^2-\frac{c}{a}\right)=k,\text{ say}\right\}$$

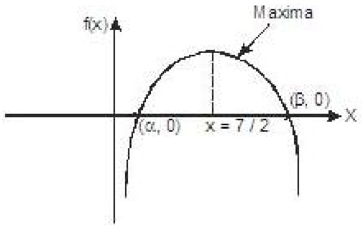
f(x) will be either maximum or minimum at $x=\frac{-b}{2a}$ depending upon whether a is negative or positive.

As f(x) attains maxima at $x=\frac{7}{2}$, 'a' must be negative and $\frac{-b}{2a}=\frac{7}{2}\Rightarrow\frac{-b}{a}=7$ ...(i)

Let the roots of f(x) be α & β. So that

$\alpha\times\beta=\frac{c}{a}=10$ (given)

& $\alpha+\beta=\frac{-b}{a}=7$, from (i)



Solving for α & β we get the roots as

$\alpha=2$ & $\beta=5$

$\Rightarrow$ the equation is $\pm k(x-2)(x-5)=0$

where k is constant. Since the value of k is not known, $a \times b \times c$ cannot be determined from the given information.

17. c Consider the quadrilateral ABCD. It can be seen that $\angle B = \angle D = 90°$.

Hence, it must be a cyclic quadrilateral.

So A, B, C and D are concyclic and there is exactly one circle passing through 4 of them.

Similarly, consider the points C, B and E. All these points C, B and E are collinear. In other words, there is no circle passing through C and B and E. For every triplet of points involving E, there is a circle. Hence, 5 circles in this case.

Total number of circles = 1 + 5 = 6.

18. c Total distance travelled

$$= 20 + 2\left[20\left(\frac{1}{2}\right) + 20 \times \left(\frac{1}{2}\right)^2 + \cdots \infty\right] = 20 + 2 \times \frac{1}{2}(20)\left(\frac{1}{1-1/2}\right) = 60 \text{ m}$$

19. c $-3.5 \le a \le 7.5$ and $-7.5 \le b \le 5.5$

$a \times c$

$= a \times (-a \times |b|)$

$= -a^2 |b|$

This expression is never positive.

Least value of $-a^2|b|$ occurs when a^2 and $|b|$ are maximum possible (both positive)

$\therefore$ Maximum value of $a \times c = -(7.5)^2 \times |-7.5|$

$= -(7.5)^3 = -421.875$

20. b The last term of the nth term = $1 + 2 + 3 + \cdots + n$

$$= \left(\frac{1+n}{2}\right)n = \frac{n^2+n}{2}$$

Short cut: Go by option in this type of question.

21. b The first term = $\left[\frac{(n-1)^2+(n-1)}{2}+1\right] = \frac{n^2+1-2n+n-1+2}{2} = \frac{n^2-n+2}{2}$

The last term = $\left(\frac{n^2+n}{2}\right)$

$\therefore$ Sum of the terms = $\frac{n}{2}\left[\frac{n^2-n+2}{2}+\frac{n^2+n}{2}\right] = \frac{n(n^2+1)}{2}$.

Alternative method:

Work with choices: only (b) satisfies for

n = 1, 2, 3, ...

22. c Let the roots be p, q, r.

Then $p + q + r = 3$; $pqr = -B$; $pq + qr + pr = A$.

Since the roots are all real and positive B must be negative and A must be positive.

Also, the maximum value of -B (which is minimum value of B) is when all the roots are equal to 1.
So $pqr = 1$. So $(pq)(qr)(pr) = 1$.

Hence minimum value of $pq + qr + pr$ (= A) is when each of pq, qr and pr is 1.

Minimum value of A = 3 and B = -1.

23. a The money to be spent for first 500 g of a parcel = Rs. 10 + Rs. 3 = Rs. 13 ... (i)

The money spent for subsequent 500 g blocks = Re 1 × $\frac{500}{100}$ = Rs. 5 ... (ii)

Now since (ii) < (i), we send it as a single package.

∴ Cost=Rs.10+Rs. 3 + Rs. $\left(\frac{2100-500}{100}\right)$ × Re 1 = Rs. 29.

24. b Money spent for first 1,000 g

= Rs. 10 + Rs. 5 = Rs. 15 ...(i)

Money spent for further 1,000 g

$= \frac{1000}{50} \times \text{Re } 1 =$ Rs. 20 ...(ii)

Now, since (ii) > (i), we divide the package into two of weights 1,000 g and 1,100 g.

Hence, money spent = Rs. 15 + Rs. 15 + Rs. 2

= Rs. 32

25. d f[g(x)] = $f\left[\frac{3x+x^3}{1+3x^2}\right] = \log\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}$

$= \log\left[\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x3}\right] = \log\frac{(1+x)^3}{(1-x)^3} = \log\left(\frac{1+x}{1-x}\right)^3 = 3\log\left(\frac{1+x}{1-x}\right) = 3[f(x)]$

26. c If XYZW = -1, then there are 8 sets of integral values that are possible.

(1, 1, 1, -1), (1, 1, -1, 1), (1, -1, 1, 1), (-1, 1, 1, 1), (-1, -1, -1, 1), (-1, -1, 1, -1), (-1, 1, -1, -1), (1, -1, -1, -1)

So (X - Y)(X - Z)(X - W) has to be 0, 8, or -8.

27. c $x = \frac{1}{1-a}$ and $y = \frac{1}{1-b}$.

So x - xa = 1 and y - yb = 1

or xa = x - 1 and y - 1 = yb

or $a = \frac{x-1}{x}$ and $b = \frac{y-1}{y}$

Now $ab = \frac{(x-1)(y-1)}{xy}$

$s = 1 + ab + a^2b^2 + \cdots + \infty$, $s = \frac{1}{1-ab} = \frac{1}{1-\frac{(x-1)(y-1)}{xy}} = \frac{xy}{x+y-1}$

28. c pth term of an AP = $\frac{1}{qr}$ = A + d(p - 1) and

qth term of an AP = $\frac{1}{pr}$ = A + d(q - 1)

rth term of an AP = ? = A + d(r - 1)

$\frac{1}{qr} - \frac{1}{pr}$ = d[(p - 1) - (q - 1)]

∴ $d = \frac{1}{pqr}$ since pth term = $\frac{1}{qr} = A + \frac{(p-1)}{pqr}$

⇒ $A = \frac{1}{pqr}$, rth term = A + d(r - 1) = $\frac{1}{pqr} + \frac{(r-1)}{pqr} = \frac{r}{pqr} = \frac{1}{pq}$

So in HP, rth term will be pq.

29. c $x^2 - 5 = 5 - 6\,|\,x - 3\,|$

$x^2 + 6\,|\,x - 3\,| - 10 = 0$

case I : $x \geq 3$

$x^2 + 6(x - 3) - 10 = 0$

$x^2 + 6x - 28 = 0$

which gives $x = \frac{-6 \pm \sqrt{148}}{2}$

$x = \frac{-6 + \sqrt{148}}{2}$ is an acceptable solution.

case II: $x < 3$

$x^2 - 6x + 8 = 0$

gives $x = 2$, $x = 4$

$x = 2$ is an acceptable solution.

So, a total of 2 real solutions exist for the given equation.

Hence (c)

30\. b $GM = \sqrt[n]{1 \times \frac{1}{3} \times \frac{1}{9} \times \ldots \frac{1}{3^{n-1}}} = \sqrt[n]{\frac{1}{3^{1+2+\ldots+n-1}}} = \sqrt[n]{\frac{1}{3^{\frac{(n-1)n}{2}}}} = \left(\frac{1}{3}\right)^{\frac{n-1}{2}}$

31\. c Here $x^2 - 9x + 5 = 0$

$\Rightarrow \alpha + \beta = 9$ and $\alpha\beta = 5$

Now $\frac{\alpha - \beta}{2\beta} + \frac{\beta - \alpha}{2\alpha} = \frac{2\alpha^2 + 2\beta^2 - 4\alpha\beta}{4\alpha\beta} = \frac{(\alpha - \beta)^2}{2\alpha\beta} = \frac{\left[(\alpha + \beta)^2 - 4\alpha\beta\right]}{2\alpha\beta} = \frac{81 - 20}{10} = \frac{61}{10}$

and $\left(\frac{\alpha - \beta}{2\beta}\right)\left(\frac{\beta - \alpha}{2\alpha}\right) = \frac{-(\alpha - \beta)^2}{4\alpha\beta} = \frac{-\left[(\alpha + \beta)^2 - 4\alpha\beta\right]}{4\alpha\beta} = \frac{-[81 - 20]}{20} = -\frac{61}{20}$

For the equation is

$x^2 - \frac{61}{10}x - \frac{61}{20} = 0$

$\Rightarrow 20x^2 - 122x - 61 = 0$

32\. b According to the given conditions,

$S_1 = \frac{n}{2}\left[2a + (n-1)d\right]$; $S_2 = \frac{2n}{2}\left[2a + (2n-1)d\right]$,

$S_3 = \frac{3n}{2}\left[2a + (3n-1)d\right]$

$\therefore R = S_3 - S_2 - S_1$

$= \frac{n}{2}\left[6a + 9nd - 3d - 4a - 4nd + 2d - 2a - nd + d\right] = 2n^2d$

Hence, R is dependent on only n and d.

33\. a Sum of the terms from the 11th term to the 19th term

$= \frac{n}{2}\left[11\text{th term} + 19\text{th term}\right]$

$= \frac{9}{2}\left[T_{11} + T_{19}\right] = \frac{9}{2}\left[3(11) - x + 3(19) - x\right]$

$= \frac{9}{2}\left[90 - 2x\right] = 405 - 9x$

Hence, the answer is (a).

34\. d $x^2 + y^2 = z + 1$ and $y^2 + z^2 = x + 1$

Subtracting, we get $x^2 - z^2 = z - x$

or $(x + z)(x - z) + (x - z) = 0$

or $(x - z)(x + z + 1) = 0$

Now either $x = z$ or $x + z + 1 = 0$

Similarly, we can prove that either $y = z$

or $y + z + 1 = 0$

Either way, we get $x = y = z$

Now we have $x^2 + y^2 = z + 1$

or $x^2 + x^2 = x + 1$

or $x = 1$ or $-\frac{1}{2}$ $\therefore$ $xyz = 1^3$ or $\left(-\frac{1}{2}\right)^3$

35. a $f(x + 1) = f(x) + 1$

$\Rightarrow f(1) = f(0 + 1) = f(0) + 1$

$f(2) = f(1 + 1) = f(1) + 1 = 1 + (1 + f(0))$

or $f(2) = 2 + f(0)$

$f(3) = 3 + f(0)$

... ...

$f(n) = n + f(0)$ ...(i)

Now, $g(x - 1) = g(x) - 1$

$\Rightarrow$ $g(0) = g(1) -1$ or $g(1) = 1 + g(0)$

$g(1) = g(2) -1$ or $g(2) = 1 + g(1)$

or $g(2) = 2 + g(0)$

$g(2) = g(3) -3$ or $g(3) = 3 + g(0)$

... ...

$g(m) = m + g(0)$ ...(ii)

Now, $f(f(n)) = (n + f(0)) + f(0) = n + 2f(0)$

$f(f(f(n))) = n + 3f(0)$

... ...

$f(f(f(f......f(n))))_{m\ time} = n + mf(0)$ ...(iii)

$= n + mn$

And $g(g(m)) = g(m + g(0)) = m + 2g(0)$

$g(g(g(m))) = m +3g(0)$

$g(g(g(......g(m))))_{n\ times} = m +ng(0)$

$= m + nm$ ...(iv)

The required ratio is $= \frac{n+mn}{m+nm} = \frac{1+\frac{1}{m}}{1+\frac{1}{n}}$

Hence (a) is the correct option.

36. e $[x]^2 - 12[x] + 20 < 0$

or, $([x] - 10)([x] - 2) < 0$

$\Rightarrow$ $2 < [x] < 10$

$\Rightarrow [x] = 3, 4, 5, 6, 7, 8, 9$ ...(i)

Also; $\{x\}^2 - 0.8\{x\} + 0.15 = 0$

or, $(\{x\} - 0.5)(\{x\} - 0.3) = 0$

$\Rightarrow \{x\} = 0.5$ or 0.3 ...(ii)

combining the two equations;

$x = [x] + \{x\}$

$= 3.5, 3.3, 4.5, 4.3, 5.5, 5.3, 6.5, 6.3, 7.5, 7.3, 8.5, 8.3, 9.5, 9.3.$

$\Rightarrow$ there are 14 real numbers that satisfy the given system. Hence option (e) is correct.

37. a $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

or $91 + 2(r + r^2 + r^3) = 169$

$\therefore r + r^2 + r^3 = 39$

Thus, $r = 3 \therefore \frac{z}{y} = \frac{xz}{xy} = r = 3$

38. c Here

$S_n = n(n - 1), S_1 = T_1 = 0$ ... (i)

$S_2 = T_1 + T_2 = 2$ ... (ii)

$S_3 = T_1 + T_2 + T_3 = 6$ ... (iii)

From these equations, we get

$T_1 = 0, T_2 = 2, T_3 = 4$, etc.

Therefore, the series is 0, 2, 4, 6, 8, ...

Sum of the squares of the first n terms will be

$0^2 + 2^2 + 4^2 + 6^2 + \ldots$

$= 4(0^2 + 1^2 + 2^2 + 3^2 + \ldots$ up to n terms

$= 4.\frac{n(n-1)(2n-1)}{6} = \frac{2}{3}n(n-1)(2n-1)$

Short cut: By substituting values of n, we can get the answer.

39. b $g(h(x)) = 2x^2 + 3x \Rightarrow g(h(g(x))) = 2(g(x))^2 + 3g(x)$.

Now $g(h(g(x)) = g(x^2 + 4x - 4)$

$\therefore g(x^2 + 4x - 4) = 2(g(x))^2 + 3g(x)$

If we put, $x^2 + 4x - 4 = x \Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow x^2 + 4x - x - 4 = 0$

$\Rightarrow x(x + 4) - 1(x + 4) = 0 \Rightarrow x = -4$ and $x = 1$

$\therefore 2(g(-4))^2 + 3g(-4) = g(-4)$

$\Rightarrow 2(g(-4))^2 = -2g(-4) \Rightarrow g(-4) = 0, -1$

$\therefore$ -1 could be a value of g(-4).

40. e Here the series is

$(1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + \cdots + (100^2 - 1)$

$= 1^2 + 2^2 + 3^2 + \cdots + 100^2 - 100 = \frac{100 \times 101 \times 201}{6} - 100 = 338250$

Therefore, the series is 0, 2, 4, 6, 8, ...

Sum of the squares of the first n terms will be

$0^2 + 2^2 + 4^2 + 6^2 + \ldots$

$= 4(0^2 + 1^2 + 2^2 + 3^2 + \ldots$ up to n terms

$= 4.\frac{n(n-1)(2n-1)}{6} = \frac{2}{3}n(n-1)(2n-1)$

Short cut: By substituting values of n, we can get the answer.

39. b $g(h(x)) = 2x^2 + 3x \Rightarrow g(h(g(x))) = 2(g(x))^2 + 3g(x)$.

Now $g(h(g(x)) = g(x^2 + 4x - 4)$

$\therefore g(x^2 + 4x - 4) = 2(g(x))^2 + 3g(x)$

If we put, $x^2 + 4x - 4 = x \Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow x^2 + 4x - x - 4 = 0$

$\Rightarrow x(x + 4) -1(x + 4) = 0 \Rightarrow x = -4$ and $x = 1$

$\therefore 2(g(-4))^2 + 3g(-4) = g(-4)$

$\Rightarrow 2(g(-4))^2 = -2g(-4) \Rightarrow g(-4) = 0, -1$

$\therefore -1$ could be a value of $g(-4)$.

40. e Here the series is

$(1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + \cdots + (100^2 - 1)$

$= 1^2 + 2^2 + 3^2 + \cdots + 100^2 - 100 = \frac{100\times101\times201}{6} - 100 = 338250$

41. a The series consists of two series $(20 + 18 + 16 + \ldots) + \frac{1}{4}(1+2+3+\ldots)$

Maximum sum [for maximum sum we will go up to 12 terms after that sum will get reduced.]

$= 2(10 + 9 + 8 + \ldots + 1 + 0 - 1) + \frac{1}{4}\times\frac{12\times13}{2} = 108 + \frac{39}{2} = \frac{255}{2}$

Here we do not have to consider negative terms, otherwise we will not get the maximum value.

42. d $\Rightarrow 6\{x\} = x + 2[x]$

$\because x = \{x\} + [x]$

$\Rightarrow 6\{x\} = [x] + \{x\} + 2[x]$

$\Rightarrow 5\{x\} = 3[x]$

$\Rightarrow \{x\} = \frac{3[x]}{5}$

Now, $0 < \{x\} < 1$

$\Rightarrow 0 < \frac{3[x]}{5} < 1$

$\Rightarrow 0 < 3[x] < 5$

$\therefore [x] = 1$ and $x = [x] + \{x\} = 1.6$

Exercise 7 - Level 2

1. d While looking at the graphs only, graph of (d) satisfies the conditions for both positive as well as negative values of x.

2. e When x is replaced by IxI, as in option (a); the horizontal axis takes all the real numbers greater than or equal to zero. Correspondingly, the portion lying in the - ve x - axis (in the given graph) is deleted to get the graph of y against IxI. Similarly, in option (b); by eliminating the portion of the given graph that lies in the negative y - axis; we get the graph of IyI against x. In option (c); both the - ve x axis and the -ve y axis are deleted, to get the IyI against IxI, graph. All the three graph are drawn correctly. Hence, (e).

3. d For $x \geq 0$, $f(x) = x + [x]$

For this, $0 \leq x < 1$, $f(x) = x$

$1 \leq x < 2$, $f(x) = x + 1$

$2 \leq x < 3$, $f(x) = x + 2$

For $x < 0$, $f(x) = -x + [x]$

$-1 \leq x < 0$, $f(x) = -x - 1$

$-2 \leq x < -1$, $f(x) -x - 2$

$-3 \leq x < -2$, $f(x) = -x - 3$

So only choice (d) satisfies all the conditions.

So answer is (d).

4. d The minimum value will be obtained when (B, D) are (8, 9) & (A, C) are (1, 2) The minimum value is

$$\left(\frac{A}{B}+\frac{C}{D}\right)_{min} = \left(\frac{1}{8}+\frac{2}{9}\right) = \frac{25}{72}$$

$\Rightarrow$ Option (d) is correct.

5. e $f(g(h(x))) = \frac{1}{x^2} - 3$ ... (i)

$f(h(g(x))) = \frac{1}{x^2} - 3$ ... (ii)

$\therefore$ Clearly, (c) is not true.

Now $h(f(g(x))) = \frac{1}{x^2 - 3}$ ... (iii)

$\therefore$ Clearly, (a) is not true and also in equations (i) and (iii), we see that (a) is true only at

$x^2 = \frac{3 \pm \sqrt{5}}{2} = 0.3$ or 3.6 (approximately)

$\therefore$ The graphs of (i) and (iii) intersect at $x = 0.5$ and $x = 1.9$. Hence, the answer is (e).

6. d For $x = 3.2$

$f(g(x)) = |[3.2]| = 3$

$g(f(x)) = [|3.2|] = 3$

For $x = -3.5$

$f(g(x)) = |[-3.5]| = 4$

$g(f(x)) = [|-3.5|] = 3$

Thus, either (a) or (b).

7. b $p(1 - h(3)) = 2.25$

$q(f(-9)-1)=\sqrt{8}=2\sqrt{2}=2.82$

Clearly, (a) is false.

h(q(16)) = 0.5

h(g(p(2))) = 0.5

Hence, (b) is the answer.

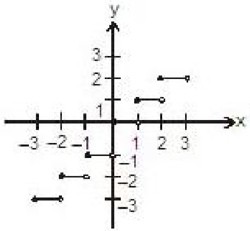
8. a For x < 0, the value of f(x) is constant at -1.

9. c The two graphs are mirror image of each other about the X-axis. Hence, g(x) = -f(x)

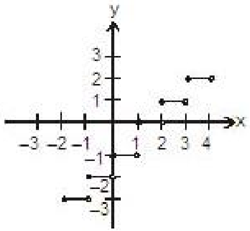
10. d If we know the graph of y = [x]; We can very easily, draw the graph of y = [x - 1] which is obtained by shifting the y = [x] graph by 1 unit in the forward direction. Once, y = [x - 1] graph is drawn; we can take the reflection of the negative y-axis portion of the graph in the positive y - axis portion of the along the x -axis. This gives us the graph of y = $|[x-1]|$.

The graph is drawn as shown below :-

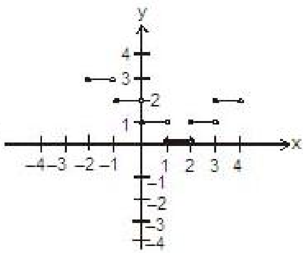
Step - I : y = [x]



Step - II : y=[x-1]



Step - III : y=|[x-1]|



11. b The given graph is derived by the graph of y against some linear expression in x. In other words; by modifying the graph of a straight line; we will get the graph. Note that the graph is made up of a number of steps. Each step has a hole at its end. In the graph if we join the solid points of each step, 1 then we will get the linear expression, as discussed. The solid points are $(0,-1);\left(\frac{1}{2},0\right),\left(\frac{-1}{2},-2\right)$. All these points lie on the straight line y = 2 x - 1.

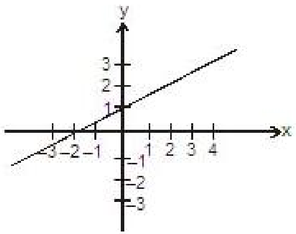
= The graph is representing the curve y = [2 x - 1]. Hence (b) is the correct option.

Conversely in order to get y = [a x + b] graph from y = a x + b, proceed as in this example.

Example:

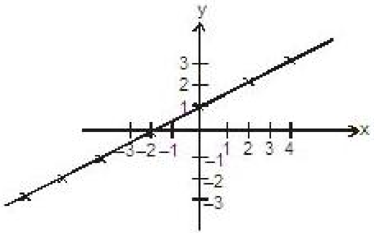
$y=\left[\frac{x}{2}+1\right]$ graph is drawn in the following steps.

Step I : Draw the straight line $y=\frac{x}{2}+1$

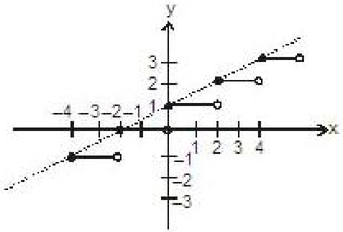


Step II : Draw straight lines,

$y = 0, y = 1, y = 2$, ---- & $y = -1, y = -2, y = -3$ etc. Mark the points on the curve $\left(y = \frac{x}{2} + 1\right)$ where these lines $(y = 0, y = 1, y = -1 \text{ etc.})$ cut the curves.



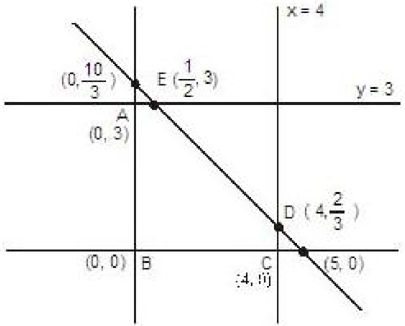
Step III : Now the graph of $y = \left[\frac{x}{2} + 1\right]$ is obtained as below :



12. c

x	0	5	$\frac{1}{2}$	4
y	$\frac{10}{3}$	0	3	$\frac{2}{3}$

Draw the graph of all five lines, i.e. $x = 4, x = 0, y = 0, y = 3$ and $2x + 3y = 10$ and find the desired region.



Here ABCDE is the desired region, to find the maximum value of $3x + 2y$ in this region (consider only points A, B, C, D, E).

$\therefore$ At point D, $3x + 2y = \frac{40}{3}$.

Hence, (c) is the answer.

13. b For $\frac{3-a}{1} = \frac{9}{3-a}$ to be equal a has to be either 0 or 6.

Hence (b) is the correct choice.

14. e We can consider clockwise movements as positive and anticlockwise movements as negative.

So $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4$

So Ashok moved by 4 steps in anticlockwise direction. So he will move to 9th room.

15. c Similar to question number 14.

$-2 - 4 - 6 - 8 = -20$

After -12 he will come back to his initial position.

So he will finally move by 8 steps in anticlockwise direction.

16. c $f(x) = x^2 - \frac{1}{x^2}$

$f(x) = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

$f\left(\frac{1}{x}\right) = \left(\frac{1}{x} + x\right)\left(\frac{1}{x} - x\right) = -\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

$f\left(\frac{1}{x}\right) = -f(x)$

So the answer is (c).

17. b $f(x) = \frac{a^x + a^{-x}}{2}$

If I substitute x = 0, f(0) = 1

Now in choices if I substitute x = y = 0, (c) and (d) choices get eliminated. And when we look choice (a) $f(x+y) = \frac{a^{x+y} + a^{-(x+y)}}{2}$

So power of a, i.e. x + y which cannot come by simply adding f(x) + f(y) it will come by multiplication of f(x) and f(y). So only choice (b) remains.

Alternative method:

$$f(x+y) + f(x-y) = \frac{a^{x+y} + a^{-(x+y)}}{2} + \frac{a^{x-y} + a^{-(x-y)}}{2}$$

$$= \frac{a^{x+y} + a^{-(x+y)} + a^{x-y} + a^{y-x}}{2} = \frac{a^y(a^x + a^{-x}) + a^{-y}(a^x + a^{-x})}{2} = 2\frac{(a^x + a^{-x})}{2} \cdot \left(\frac{a^y + a^{-y}}{2}\right) = 2\, f(x) \, . \, f(y)$$

So (b) is the choice.

18. c $x = \frac{1}{2} f(x)$ or $x = \frac{1}{2}(x-2)^2$ ... (i)

Now put the choices in (i) to get the answer.

19. e $g^{-1}(x) = 2x + 3$ or $g(2x + 3) = x$

or $g(x) = \frac{x-3}{2}$ or $g^2(x) = \frac{\frac{x-3}{2} - 3}{2} = \frac{x-9}{4}$

Now we have $y = \frac{x-9}{4}$ or x = 4y + 9

Therefore, x = 29 when y = 5

20. b $\log a^a b^b c^c = a \log a + b \log b + c \log c$ ... (i)

Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$ ... (ii)

Putting the values of log a, log b and log c from (ii)

in (i),

$\log\left(a^a b^b c^c\right) = a(b-c)k + b(c-a)k + c(a-b)k = 0$

or $a^a b^b c^c = 1$

21. d Maximum area is possible only when all the sides of the rectangle are equal (i.e. it is a square). Then since perimeter is 100, side must be 25.

Hence, area must be 625.

22. a Maximum area is possible when the ratio of the length to the width is 2 : 1. Then, since the sum of three sides is 100, length must be 50 and the width must be 25. Hence, area must be 1250.

23. a Total number of oranges = xy

Given (x + 20)(y - 3) = xy ... (i)

and (x - 5)(y + 1) = xy ... (ii)

Therefore, x = 80 and y = 15

Thus, (a) is the correct choice.

24. a Bhola could take 2 assistants. At the end of day one, one assistant returns distributing one day's ration to each of Bhola and his second assistant. At the end of day two the second assistant returns after giving away one day's ration to Bhola. Bhola can complete the journey all by himself now.

25. b Let the side of the cube be of `a' units.

Therefore, `a' will be the diameter of the cylinder.

Volume of cube is a^3 and the volume of cylinder will be $\pi \times \frac{a^2}{4} \times a$

Thus, lost volume = $\frac{a^3 - \pi \times \frac{a^3}{4}}{a^3} \times 100$

Therefore, loss percentage = 22% approximately.

26. c $2^{2x+5} - 3 \cdot 2^{x+2} + 1 = 0$

Therefore, $32 \cdot 2^{2x} - 12 \cdot 2^x + 1 = 0$

Substituting, $2^x = y$, we have $32y^2 - 12y + 1 = 0$

We have $(8y - 1)(4y - 1) = 0$, i.e. $y = \frac{1}{8}$ or $\frac{1}{4}$

So $2^x = \frac{1}{4}$ or $2^x = \frac{1}{8}$

So x = -2 or x = -3

Note: Students please note that the best possible method to solve this sum is the method of reverse substitution. i.e. using options.

27. e ax = b

When b is copied wrongly, let is copied as b'

So $ax' = b' \Rightarrow x' = \frac{b'}{a}$ or $\frac{7}{3} = \frac{b'}{a}$ ...(i)

and,

When a is copied wrongly;

$x'' = \frac{b}{a'}$ or $\frac{8}{5} = \frac{b}{a'}$ ...(ii)

The correct root cannot be determined from the two equations Hence (d).

28. b $f(x) = \frac{1}{x,}$

$f^2(x) = \frac{1}{\frac{1}{x}} = x$

$f^3(x) = f\left(f^2(x)\right) = \frac{1}{x}$

$f^4(x) = x$ ... ... and so on

which means:

$$f'(x)=\begin{cases}\frac{1}{x}; & n \text{ is odd}\\ x; & n \text{ is even}\end{cases}$$

it is given that $f^{100}(x) = 40$. As 100 is even; $x = 40,$

$\Rightarrow f'(40)=\frac{1}{40}.$

Hence (b) is correct.

29. b Let S = pocket money that he gets from his mother and S' = pocket money that he got from his father
So that we have S = 5a + 2b and S' = 2a + 5b

and also

S - S' = 3(a - b).

This means, (S- S'), whether it is negative or positive, must always be a multiple of 3. Hence, except for the option (b), all other are possible depending upon the exact values of the constants "a" and "b".

30. b Let X be the total number of bees in the swarm then $\sqrt{\frac{X}{2}}=\frac{X}{9}-2$ ...(i)

Squaring both sides $\frac{X}{2}=\left(\frac{X}{9}-2\right)^2$

$\frac{X}{2}=\frac{X^2}{81}+4-\frac{4X}{9} \Rightarrow X=72$

Note: One can simply put the options in equation (i) and check to get the answer.

31. b $\frac{n}{2}+\frac{n}{4}+\frac{n}{5}+7$ = n or n = 140

32. d abc is maximum, then $a^2b^2c^2$ is also maximum and 60 $a^2b^2c^2$ is also maximum.

⇒ (3ab)(4bc)(5ca) is also maximum.

And this will happen when

$3ab = 4bc = 5ca = \frac{180}{3}=60$

$\therefore 60a^2b^2c^2 = 60^3 \Rightarrow a^2b^2c^2 = 60^2 \Rightarrow abc = 60$

33. c 2a < b + 18 and 2b < a

2a < 14 - a + 18

$\Rightarrow a < 10\frac{2}{3}$

$2(14-a) < a$

$\Rightarrow a > 9\frac{1}{3}$

34. d If the price per dozen eggs is "p" paisa, then

x × p = Amount of x eggs

Also given that (x + 100)(p - 20) = xp ... (i)

and (x - 120)(p + 30) = xp ... (ii)

Solving (i) and (ii), we get x = 1000

35. d Take different values of n and check options.

36. b Assume some values of a, b and c such that

a + b + c = 0 and find the value of the expression that is given. So let a = 1, b = -1 and c = 0. We find that

$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab} = \frac{1}{2}+\frac{1}{2}+0=1$

37. c $2^{2a} - 3(2^{a+2}) + 2^5 = 0$

$\Rightarrow 2^{2a} - 12 \cdot 2^a + 32 = 0 \Rightarrow (2^a - 8)(2^a - 4) = 0 \Rightarrow a = 3$ or $a = 2$

38. c $\frac{1}{x^2}+\frac{1}{y^2}=\left(\frac{1}{x}+\frac{1}{y}\right)^2-2\times\frac{1}{xy} = a^2 - 2b$

39. d Let the roots be $\frac{a}{r}$, a and ar

$\therefore \frac{a}{r} + a + ar = 6$

$\frac{a^2}{r}+a^2+a^2r=k$

$a^3 = -64 \Rightarrow a = -4$

$\therefore -\frac{4}{r}+(-4)-4r=6$

Solve this to get $r = -2$ or $-\frac{1}{2}$

$\therefore k = \frac{16}{-2}+16-32=-24$

40. b If f(x) = g(x), then x is an integer. Only one integer-3 is satisfying the condition h(x) - x = 6

41. c If $[g(x)]^2 - [h(x)]^2 > 0$, then x is positive.

Hence, x - h(x) = 0

42. d Discriminant i.e. $b^2 - 4ac$

$= (4b)^2 - 4ac = 16b^2 - 8b^2 = 8b^2$ $(\because b^2 = ac)$

Also, $8b^2 > 0$.

$\therefore$ Roots are real.

43. d $\frac{x^3+x+2}{x}=x^2+1+\frac{1}{x}+\frac{1}{x}$

$\frac{x^2+1+\frac{1}{x}+\frac{1}{x}}{4}\geq\sqrt[4]{x^2\times1\times\frac{1}{x}\times\frac{1}{x}}$

$x^2+1+\frac{1}{x}+\frac{1}{x}\geq4$. Minimum value = 4

Hence (d) Ans.

Alternative method:

$f=\frac{x^3+x+2}{x}=x^2+1+\frac{2}{x}$

$f'=2x-\frac{2}{x^2}$

$f'=0\Rightarrow 2x=\frac{2}{x^2}\Rightarrow x^3=1$

as x is positive real number x = 1

$f''=2+\frac{4}{x^3}$

$f''>0$ for $x=1$

$\therefore$ f will be min at x = 1

$\therefore f_{min}=\frac{1^3+1+2}{1}=4$

44. b $g(y) = f(2,y) \times f(4,y) \times f(6,y) \times f(8,y)$

$\Rightarrow$ maximum value of g(y) will be obtained at y = 8, & it will be $\frac{1}{8^4}$.

Hence, (b) is correct.

45. d Here, $f(x) \equiv 3x^3 - 5x^2 + 10x - 1 = 0$ and

$g(x)\ x^3 + 2x^2 + 3x + 1 = 0$

To find the common roots we need to find the roots of the equation $f(x) - g(x) = 0$

$\Rightarrow (3x^3 - 5x^2 + 10x - 1) - (x^3 + 2x^2 + 3x + 1) = 0$

$\Rightarrow 2x^3 - 7x^2 + 7x - 2 = 0 \Rightarrow (2x - 1)(x - 1)(x - 2) = 0$

$\therefore x = \frac{1}{2}$, 1 or 2

$f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right) - 1 = \frac{25}{8}$ and $g\left(\frac{1}{2}\right) = \frac{25}{8}$

$f(1) = 3 - 5 + 10 - 1 = 7$ and

$g(1) = 1 + 2 + 3 + 1 = 7$

$f(2) = 23$ and $g(2) = 23$

Required co-ordinates are:

$\left(\frac{1}{2}, \frac{25}{8}\right)$, (1, 7) and (2, 23)

46. b $|x - y| \geq |x| - |y|$ or $|x - y|\ 13$

$\Rightarrow$ Either $x - y \geq 13$ or $x - y \leq -13$

i.e. x - y cannot be between 13 and -13.

47. a $x^2 - 20x + 91 \leq 0$

or $(x - 7)(x - 13) \leq 0$

$\therefore 7 \leq x \leq 13$

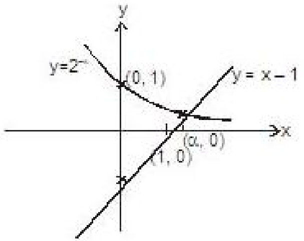
or $x^3 - 3x^2 + 3x - 1 = (x - 1)^3$

Minimum value of $(x - 1)^3$ will be $(7 - 1)^3 = 216$

48. e Equation can also be written as:-

$(x - 1) = \left(\frac{1}{2}\right)^x$

By plotting y = (x - 1) and $y = \left(\frac{1}{2}\right)^x$ on the same x-y plane, we can comment on the location and the number of the real roots of the equation.



As it is clear from the graph, the equation has exactly one real root and the root lies beyond

x = 1.

i.e $\alpha = 1$

i.e. x - y cannot be between 13 and -13.

47. a $x^2 - 20x + 91 \le 0$

or $(x - 7)(x - 13) \le 0$

$\therefore 7 \le x \le 13$

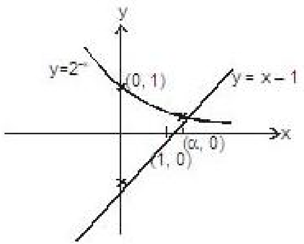
or $x^3 - 3x^2 + 3x - 1 = (x - 1)^3$

Minimum value of $(x - 1)^3$ will be $(7 - 1)^3 = 216$

48. e Equation can also be written as:-

$$(x - 1) = \left(\frac{1}{2}\right)^x$$

By plotting $y = (x - 1)$ and $y = \left(\frac{1}{2}\right)^x$ on the same x-y plane, we can comment on the location and the number of the real roots of the equation.



As it is clear from the graph, the equation has exactly one real root and the root lies beyond

x = 1.

i.e $\alpha = 1$

Hence, option (e) is the correct option.

Exercise 8 - Level 3

1. e $x^2 - 4x + 3 \le 0$

$x^2 - 3x - x + 3 \le 0$

$(x - 1)(x - 3) \le 0$

So $1 \le x \le 3$ but it is given that $x \ge 2$

So, $2 \le x \le 3$

2. d Let the numbers in the ascending order are x, y and z

So $x + y + z = \frac{59}{24}$

and $\frac{z}{x} = \frac{7}{6}$

and $y = \frac{7}{6} - \frac{1}{3} = \frac{5}{6}$

$\frac{z}{x} = \frac{7}{6}$

So, $x = \frac{6z}{7}$

$x + y + z = \frac{59}{24}$

$\frac{6z}{7} + \frac{5}{6} + z = \frac{59}{24}$ and $\frac{13z}{7} = \frac{59}{24} - \frac{5}{6} = \frac{39}{24} = \frac{13}{8}$

and $\frac{13z}{7} = \frac{13}{8}$ So $z = \frac{7}{8}$

and $\frac{z}{x} = \frac{7}{6}$ So, $\frac{\frac{7}{8}}{x} = \frac{7}{6}$ So $\frac{7}{8} \times \frac{1}{x} = \frac{7}{6}$

So $x = \frac{6}{8} = \frac{3}{4}$

$x = \frac{3}{4}, y = \frac{5}{6}$ and $z = \frac{7}{8}$

3. a If x - 2 and x + 1 are factors of the given expression means 2 and -1 are the roots.

$2(2)^4 - a(2)^3 + 2 + b = 0$

$\Rightarrow 34 + b = 8a$ ...(i)

and $2 + a - 1 + b = 0$

$\Rightarrow a = -(b + 1)$ ...(ii)

So $34 + b = -8b - 8$

$9b = -42$

$b = \frac{-42}{9} = \frac{-14}{3}$

4. a $2x + 3y = 10$

The only value of (x, y) that satisfy the equation is (2, 2).

Hence, option (a) is the correct choice.

5. c If x is positive, then

$2x + 5x \le 30$

$7x \le 30$

and $x \le \frac{30}{7}$

If x is negative then

$-2x + 5 - (x) \le 30$

$-7x \le 30$

$x \ge \frac{-30}{7}$

So the value of x

$\frac{-30}{7} \le x \le \frac{30}{7}$

6. b For the equation $\log_x(2x^2 - 3x - 5) = 2$ to hold valid $x > 0$ and $(2x^2 - 3x - 5) > 0$

$\Rightarrow 2x^2 + 2x - 5x - 5 > 0$

$\Rightarrow 2x(x+1) - 5(x+1) > 0$ or $(2x-5)(x+1) > 0$

Hence, $x > \frac{5}{2}$ or $x < -1$

But $x > 0$, $\therefore x > \frac{5}{2}$

Now, $2x^2 - 3x - 5 = x^2 \Rightarrow x^2 - 3x - 5 = 0$

Solving for x, we get $x = \frac{3 \pm \sqrt{9+20}}{2}$

$\therefore x = \frac{3+\sqrt{29}}{2}$

Therefore there is only one real value of x that satisfies the given equation.

7. d $\log_{x^2 y}\left[\frac{x}{y^3}\right] + \log_{y^2 x}\left[\frac{y}{x^3}\right]$

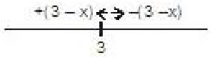
$\Rightarrow \left(\frac{\log x}{\log x^2 y} - \frac{\log y^3}{\log x^2 y}\right) + \left(\frac{\log y}{\log y^2 x} - \frac{\log x^3}{\log y^2 x}\right) \Rightarrow \left(\frac{1}{2+\log_x y} + \frac{1}{2+\log_y x}\right) - \left(\frac{3}{2\log_y x+1} + \frac{3}{2\log_x y+1}\right)$

$\Rightarrow \left(\frac{4+\log_x y+\log_y x}{5+2(\log_x y+\log_y x)}\right) - 3\left(\frac{2(\log_x y+\log_y x)+1}{5+2(\log_x y+\log_y x)}\right) \Rightarrow \frac{1-5(\log_x y+\log_y x)}{5+2(\log_x y+\log_y x)} = \frac{-14}{11}$

8. d Here

$|3-x|^{(\log_7 x^2 - 7\log_x 49)} = (3-x)^3$

By the definition of modulus, we get



Case I:

If $1 < x < 3$, the equation is

$(3-x)^{(\log_7 x^2 - 7\log_x 49)} = (3-x)^3$

$\Rightarrow (\log_7 x^2 - 7\log_x 49) = 3$

$\Rightarrow 2\log_7 x - \frac{14}{\log_7 x} = 3$

$\Rightarrow 2(\log_7 x)^2 - 3\log_7 x - 14 = 0$

$\Rightarrow \log_7 x\,(2\log_7 x - 7) + 2\,(2\log_7 x - 7) = 0$

$\Rightarrow (2\log_7 x - 7)(\log_7 x + 2) = 0$

$\Rightarrow 2\log_7 x - 7 = 0$ or $\log_7 x + 2 = 0$

$\Rightarrow x = 7^{\frac{7}{2}}$ or $x = \frac{1}{49}$

But 1 < x < 3, there is no solution in this range.

Case II:

If 0 < x < 1, then the equation is

$(3-x)^{(\log_7 x^2 - 7\log_x 49)} = (3-x)^3$

Similar to case I, we get $x = 7^{\frac{7}{2}}$ or $x = \frac{1}{49}$

So only $x = \frac{1}{49}$ satisfies the interval 0 < x < 1.

Case III:

If x > 3, then the equation is

$[-(3-x)]^{\left(\log_7 x^2 - 7\log_x 49\right)} = (3-x)^3$

$\Rightarrow (x-3)^{\left(\log_7 x^2 - 7\log_x 49\right)} = -(x-3)^3$

Here $LHS = RHS$ because L.H.S is positive and R.H.S. is negative.

So, no solution exists

If 3 - x = 0, then LHS = RHS, therefore x = 3 is also a valid solution.

If 3 - x = 1, then also LHS = RHS, therefore x = 2 is also a valid solution.

Therefore there are 3 real values of x that satisfy the given equation.

9. c $\frac{x^3-2x^2-x+2}{2x+3} < 0 \Rightarrow \frac{(x-1)(x+1)(x-2)}{2x+3} < 0$

$\Rightarrow \frac{(x-1)(x+1)(x-2)(2x+3)}{(2x+3)^2} < 0 \Rightarrow (x-1)(x+1)(x-2)(2x+3) < 0$

$\therefore -\frac{3}{2} < x < -1 \text{ or } 1 < x < 2$

Only option (c), i.e. $x = \frac{\sqrt{3}}{2}$ does not lie within the permissible values of x and hence is the correct choice.

10. d Given that

$P = \frac{3 \times 5 \times 7 \times --- \times 99}{2 \times 4 \times 6 \times --- \times 100}$

or

$P = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)---\left(1+\frac{1}{98}\right)\times\frac{1}{100}$

$= \frac{(1+k)}{100}$ Where k is the summation of some fractions obviously P is of the from 0.01 k

Only option (d) inches `P' in its range.

Hence (d) is the correct choice.

11. a Notice that a + b+ ab = (a + 1) (b + 1) - 1. If a, b are first two number chosen, then it will get replaced by

(a + b + ab) = (a + 1) (b + 1) - 1

If c is chosen along with it,

c + [(a + 1) (b + 1) - 1] + c [(a + 1) (b + 1) -1]

= (a + 1) (b + 1) (c + 1) -1

$\therefore S_{2006} = \left(\frac{1}{2}+1\right)\left(\frac{1}{3}+1\right)....\left(\frac{1}{2006}+1\right)-1 = \left[2\times\frac{3}{2}\times\frac{4}{3}\times.....\frac{2007}{2006}-1\right] = 2006$

12. b $(a_1) \times (2a_2) \times (3a_3) \times ... \times (na_n)$

$= (1 \times 2 \times 3 \times 4 \dots n) \times (a_1 \times a_2 \times a_3 \times \dots \times a_n) = n! \times K$

Their sum would be minimum when

$a_1 = 2a_2 = 3a_3 = 4a_4 = \dots = (n! \times k)^{1/n}$

$\Rightarrow (\text{Sum})_{min} = n \times (n! \times k)^{1/n}$

13. e As $|x| < 1$

$$1 + x + x^2 + \dots\dots\infty = \frac{1}{1-x}$$

and,

$$2 - \frac{x}{2} + \frac{x^2}{8} - \dots\dots\infty = \frac{2}{1+\frac{x}{4}} = \frac{1}{\frac{1}{2}+\frac{x}{8}}$$

The inequality can be written as:-

$$\log_{\frac{1}{1-x}}(x) > \log_{\frac{1}{\frac{1}{2}+\frac{x}{8}}}(x)$$

$$\text{Or } \frac{1}{\log_x\left(\frac{1}{1-x}\right)} > \frac{1}{\log_x\left(\frac{1}{\frac{1}{2}+\frac{x}{8}}\right)} \Rightarrow \log_x\left(\frac{1}{1-x}\right) < \log_x\left(\frac{1}{\frac{1}{2}+\frac{x}{8}}\right)$$

As $|x| < 1$, we must have:

$$\frac{1}{1-x} > \frac{1}{\frac{1}{2}+\frac{x}{8}} \Rightarrow 1 - x < \frac{1}{2} + \frac{x}{8}$$

Or $x > \frac{4}{9}$

$\Rightarrow$ x cannot be negative; the permissible range of values of x are:-

$$\frac{4}{9} < x < 1$$

$\Rightarrow$ Option (e) is correct.

14. c Let $(\sqrt{3}+1)^8 = I + F$

{Here I = integral part and $0 \le f < 1$ is fractional part

Let $G = (\sqrt{3}-1)^8$

$$I + F + G = (\sqrt{3}+1)^8 + (\sqrt{3}-1)^8 = 2\{{}^8C_0(\sqrt{3})^8 + {}^8C_2(\sqrt{3})^6 + {}^8C_4(\sqrt{3})^4 + 28(3)^1 + 8 \times (3)^0\}$$

$= 2\{8 \times (3)^4 + 28 \times (3)^3 + 70(3)^2 + 28(3)^1 + 8 \times (3)^0\}$

I + F + G = 4252, an integer ...(i)

$\Rightarrow$ F + G also, must be an integer, as both of then, individually are less than 1;

we must have F + G = 1 ...(ii)

From (i) and (ii)

we get I = 4251, which is the integral part or the greatest integer less than $(\sqrt{3}+1)^8$.

Hence, option, (c) is correct.

15. e $f^1(x) = \frac{x+1}{x-1}$ ..... $(x \neq 1)$

$$f^2(x) = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = x$$

$$f^3(x) = \frac{x+1}{x-1}$$

$f^4(x) = x$

… … … …

so for any given n,

$$f^n(x)=\begin{cases}\dfrac{x+1}{x-1} & \text{, when n is odd}\\ x & \text{, when n is even}\end{cases}$$

$$\Rightarrow P = f^1(2)\times f^2(3)\times f^3(4)..........\times f^{10}(11)$$

$$=(3)\times(3)\times\left(\frac{5}{3}\right)\times(5)\times\left(\frac{7}{5}\right)\times(7)\times\left(\frac{9}{7}\right)\times(9)\times\left(\frac{11}{9}\right)\times 11$$

$P = 3 \times 5 \times 7 \times 9 \times 11^2$

Now, $693 = 7 \times 9 \times 11$

$363 = 3 \times 11 \times 11$

$945 = 3 \times 5 \times 7 \times 9$

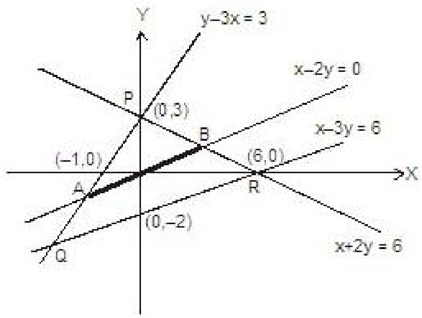
$847 = 7 \times 11 \times 11$

All are factor of P.

But 545 is not a factor of P.

Hence, (e) be is the correct option.

16. e The region of the X-Y plane that satisfies the given three inequalities bounds a closed region PQR as shown below:



The integral solution pairs that satisfy the inequalities (2), (3), (4) and also the equation (1) are all the integral points that lie on the segment AB of the line $x - 2y = 0$.

Solving $y - 3x = 3$ and $x - 2y = 0$ simultaneously, we get point A as (-1.2, -0.6). Similarly solving $x - 2y = 0$ and $x + 2y = 6$ point B is (3.0, 1.5). In all there are two integral pairs (0,0) and (2,1) on the line segment AB that satisfy the equation under the given constraint. Hence (e).

17. b First series = a, 2a, 3a, ..., so on.

Last term = a + a(2n - 1) = 2an

Second series = a, 3a, 5a, ..., a(2n - 1)

Sum of the old series = $\frac{2n}{2}[a+2an]=na[1+2n]$

$= an + 2an^2$

Sum of new series = $\frac{n}{2}[a+(2n-1)a]=an^2$

$\therefore$ Ratio = $\dfrac{an^2}{an+2an^2} = \dfrac{n}{1+2n}$

18. a Here $T_n = n(n + 1)(n + 2) = n^3 + 3n^2 + 2n$

$S_n = \sum (n^3 + 3n^2 + 2n) = \sum n^3 + 3\sum n^2 + 2\sum n$

$= \frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$

Putting n = 8, we get the sum to be 1980.

Note: Students can remember this formula for the given series as it might be useful.

For questions 19 and 20:

Let the number of persons involved in making bats for company XYZ from companies P, Q and R be `p', `q' and `r' respectively.

Therefore, 2p + 3q + 8r = 100

Also, $p \le q + r$

Let the total expenditure incurred by the company XYZ be E,

$\therefore$ 20 × 2p + 25 × 3q + 15 × 8r

= 5[8p + 15q + 24r]

= 5[4(2p + 3q + 8r) + 3q - 8r] = 5[400 + 3q - 8r]

$\Rightarrow$ E = 2000 + 5(3q - 8r)

For, 3q - 8r to be minimum, r should be maximum and p and q should be minimum.

Possible values of p, q and r are 3, 2 and 11 respectively

Minimum value of 3q - 8r = -82.

19. d E_{min} = 2000 + 5(-82) = 1590

20. c The difference between number of persons from companies Q and R is 9.

21. b $\frac{A}{5} + \frac{B}{2} + \frac{2C}{5} = 800$ ...(i)

$\frac{A}{2} + \frac{B}{4} + \frac{C}{3} = 1200$ ...(ii)

(i)×10 + (ii) 12

8(A + B + C) = 22400

A + B + C = 2800

22. c If Log a, Log b, Log c are in A.P. then a, b, c are in G.P. thus $b^2 = ac$

So, $(2^x - 1)^2 = 2(2^x + 3)$

$2^{2x} - 4 \times 2^x - 5 = 0 \Rightarrow (2^x - 5)(2^x + 1) = 0$

$2^x = 5$ or $2^x = -1$ (Not possible)

thus $x = \text{Log}_2 5$ Hence Ans. (c)

23. d The variable P, Q can be replaced by y & x :

$y - 3 = |(x-5)(x-6)|$

Or, $y - 3 = |(x-3-2)(x-3-3)|$

we observe that the y against x graph of the above function can be obtained, simply by simultaneously translating the graph of $y = |(x-2)(x-3)|$ by 3 units in the positive x - direction and 3 units in the +ve y direction.

Option (a) shows the graph of $y - 3 = |x(x+1)|$

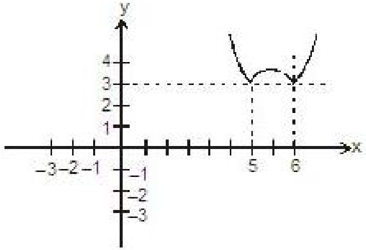
Option (b) shows the graph of $y = |(x-2)(x-3)|$

Option (c) shows the graph of $y + 3 = |x(x+1)|$

Hence. option (d) is correct.

The correct graph of $P-3=|(Q-5)(Q-6)|$ is

as shown below:-



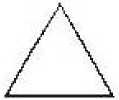
24. c $f(x)=\begin{cases} g(x) \\ h(x)+2 \end{cases}$ for $x<0$

$\left.\begin{array}{l} f(x)=x+1 \\ g(x)=-x+1 \\ h(x)=x-1 \end{array}\right\}$ for $x>0$

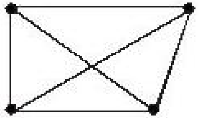
Hence, the answer is (c).

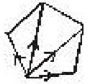
25. e None of these

26. b With 3 points the degree is 2.



The degree of 4 points is 3.



The degree with 5 points is. 4. 

The degree with 6 points is 5.

$\Rightarrow$ Degree will be odd, when the corrected points are even in number (e.g. 4, 6, 8 etc)

27. b For any function f(x), the graph of [f(x)] will always result in "steps" of integers.

In this problem to obtain the required graph, draw lines parallel to x - axis, from y = 0, y = 1, y = 2,
y = - 1, y = - 2 etc. Mark the points at which these lines cut the curve of y against f(x). Erase the curve y = f (x) to obtain "steps" of integers to get the graph of option (b), which is the correct option.

28. d g(x) varies from 0 to 1 to 0, at $x\in[0,1]$ and from 0 to -1 to 0, at $x\in[1,2]$. $f(x)$ is a constant function of values 1 and -1. Hence, $h_1(x)=g(x)\times f(x)$

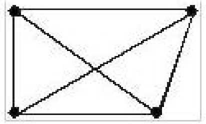
Also $i_1(x)$ is always positive for all x.

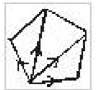
Hence, $h_1(x)=i_1(g(x))$ and nature of curve will be same as g(x).

29. c Similar to the earlier question except that $f_2(x)$ is a constant function of values 1 and 0.

Also for positive values of g(x), $i_2(g(x))$ will take the shape of g(x) curve and for negative values it will be zero. Hence, (c) is the right choice.

30. a Since <k> denotes the smallest integer grater than or equal to k and the foreign currency dealer charges an extra Rs. 3 for every additional US$20 or a put thereof



The degree with 5 points is. 4. 

The degree with 6 points is 5.

$\Rightarrow$ Degree will be odd, when the corrected points are even in number (e.g. 4, 6, 8 etc)

27. b For any function f(x), the graph of [f(x)] will always result in "steps" of integers.

In this problem to obtain the required graph, draw lines parallel to x - axis, from y = 0, y = 1, y = 2,
y = - 1, y = - 2 etc. Mark the points at which these lines cut the curve of y against f(x). Erase the curve y = f (x) to obtain "steps" of integers to get the graph of option (b), which is the correct option.

28. d g(x) varies from 0 to 1 to 0, at $x \in [0, 1]$ and from 0 to -1 to 0, at $x \in [1, 2]$. $f(x)$ is a constant function of values 1 and -1. Hence, $h_1(x) = g(x) \times f(x)$

Also $i_1(x)$ is always positive for all x.

Hence, $h_1(x) = i_1(g(x))$ and nature of curve will be same as g(x).

29. c Similar to the earlier question except that $f_2(x)$ is a constant function of values 1 and 0.

Also for positive values of g(x), $i_2(g(x))$ will take the shape of g(x) curve and for negative values it will be zero. Hence, (c) is the right choice.

30. a Since <k> denotes the smallest integer grater than or equal to k and the foreign currency dealer charges an extra Rs. 3 for every additional US$20 or a put thereof therefore the amount Y as a function of amount X > 0 is given by $Y = 5 + 3\left\langle \frac{X-10}{20} \right\rangle$

Checking by options, take X = $17

$$\Rightarrow Y = 5 + 3\left\langle \frac{7}{20} \right\rangle = \text{Rs. } 8$$

Only option (a) satisfies as for just $10 he will charge Rs. 5 and for the remaining $7, he will charge Rs. 3, making a total of Rs. 8.

31. b Here, α + β = -5 and αβ = -5

So,

$$\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} = \frac{\beta^3 + 1 + 3\beta(1+\beta) + \alpha^3 + 1 + 3\alpha(1+\alpha)}{(\alpha\beta + \alpha + \beta + 1)^3}$$