CATapult Courseware

Module 2
Quantitative Ability

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QA-2.1 ADVANCED LINEAR AND QUADRATIC EQUATIONS

THEORY

This chapter is a continuation of QA 1.4. In this chapter, we will discuss the advanced concepts of Linear and Quadratic equations.

Linear equation in 2 variables

If a, b, c are real numbers where $a \neq 0$, $b \neq 0$; then $ax + by = c$ is a linear equation in two variables of degree 1. The values of 'x' and 'y', for which both sides of the equation assume the same value are called the solution of the equation.

Example

$2x + 4y = 24$.

The value $x = 6$ and $y = 3$ satisfies the above equation.

Therefore, the pair (6, 3) is a solution of the equation.

(4, 4) and (8, 2) are also solutions of the above equation

Therefore, for a linear equation with two variables (with no other conditions to be fulfilled), there exist an infinite number of solutions;

Note: In a solution set (x, y) the value of 'x' is written first, followed by the value of 'y'.

Quadratic Equations

Relation between roots and coefficients

1. In the equation, $ax^2 + bx + c = 0$, if we divide both sides of the equation by 'a', $(a \neq 0)$ then we get the equation as:

   $$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{--------} \quad \text{(i)}$$

   Now in the same equation, if it is known that the two roots of the equation $ax^2 + bx + c = 0$ are α and β, then,

   $$ax^2 + bx + c = (x - \alpha)\ (x - \beta) = 0$$

   $$= x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad \text{--------} \quad \text{(ii)}$$

   Comparing the co-efficients of the variables in (i) and (ii) we get:

   $$\frac{b}{a} = -(\alpha + \beta) \text{ and } \frac{c}{a} = (\alpha\beta)$$

   $$\text{sum of the roots} = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of x}}{\text{Coefficient of } x^2}$$

   $$\text{product of the roots} = \alpha.\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Example

$2x^2 - 5x + 6 = 0$

Sum of the roots = $\alpha + \beta = -\frac{b}{a} = \frac{5}{2}$; Product of the roots = $\alpha.\beta = \frac{6}{2} = 3$

Note: For polynomial equations of degree 'n' > 2 where

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots\ldots + a_n = 0$ and

$a_0, a_1, a_2 \ldots\ldots\ldots a_n \neq 0$

then sum of roots = $\frac{-a_1}{a_0}$

Product of roots = $(-1)^n \frac{a_n}{a_0}$

2. The signs of the roots of a quadratic equation depending upon the signs of a, b and c.

	Signs of roots	Signs of a, b and c	Example	Roots are
i	Both roots are negative	a, b, c all have the same sign	$x^2 + 5x + 6$	−3 and −2
ii	One root positive and other negative	a and c have different signs	$x^2 + 6x - 16$	−8 and 2
iii	Both roots are positive	a and c have the same sign and differ in sign from b	$x^2 - 7x + 12$	3 and 4

3. If c = a, the roots are reciprocals of each other.
4. If b = 0, the roots are equal in magnitude, but opposite in sign.
5. If one root of a quadratic equation with rational coefficients is irrational, the other root must be its irrational conjugate.

Example

If $\alpha = m + \sqrt{n}$, then $\beta = m - \sqrt{n}$

6. If one root of a quadratic equation with real coefficients is a complex number the other root must be its conjugate complex number.

Example

if $\alpha = m + in$, then, $\beta = m - in$

To form a quadratic equation with the given roots

If α, β are the two roots of a quadratic equation, then

$x - \alpha = 0$ and $x - \beta = 0$

$(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ is the required equation.

Hence, if the roots of a quadratic equation are given, the required equation is:

x^2 − (**Sum of roots**)x + (**Product of roots**) = 0

Nature of the roots of a quadratic equation

We know that the roots of the equation $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

The nature of the roots depends on the expression $b^2 - 4ac$.

Because, the value of $b^2 - 4ac$ enables us to determine the nature of the roots of the quadratic equation, it is called the discriminant of the quadratic. It is denoted by Δ.

Nature of roots	Condition
1. Complex conjugate numbers	$\Delta < 0$
2. Real numbers	$\Delta \geq 0$
3. Real and equal	$\Delta = 0$
4. Real and unequal	$\Delta > 0$
5. Rational and unequal	$\Delta > 0$ Δ is a perfect square
6. Irrational and unequal	$\Delta > 0$ Δ is not a perfect square

Symmetric expression of roots

If α and β are the roots of $ax^2 + bx + c = 0$, then an expression involving α and β is called symmetric if interchanging α and β does not change the expression.

i.e., $f(\alpha, \beta) = f(\beta, \alpha)$

The expression $f(\alpha, \beta) = \alpha^2 - \beta$ is not symmetric because $f(\beta, \alpha) = \beta^2 - \alpha \neq \alpha^2 - \beta = f(\alpha, \beta)$

Some of the symmetric functions of α and β are

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta)$$

Example

If the sum of the roots of a quadratic equation is 3 and the sum of the squares of the roots is 29, then find the equation.

Let α and β be the two roots.

$\alpha + \beta = 3$ and $\alpha^2 + \beta^2 = 29$

$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$3^2 = 29 + 2\alpha\beta$

$\therefore\ 2\alpha\beta = -20$

$\therefore\ \alpha\beta = -10$

The quadratic equation is $x^2 -$ (Sum of roots)x + Product of roots = 0

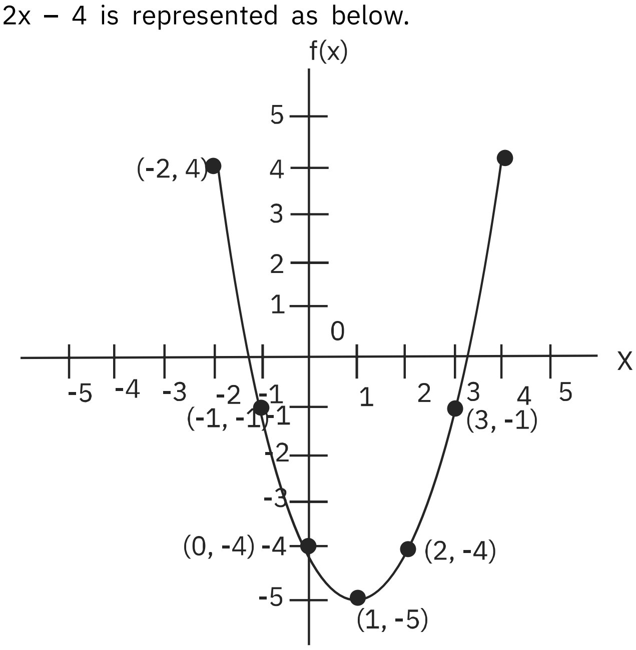
i.e., $x^2 - 3x - 10 = 0$

Graphical Representation of a Quadratic Equation

If we draw the graph of $y = ax^2 + bx + c$ in the xy plane, it is in the form of a parabola.
In the graph below, $y = x^2 - 2x - 4$
Given below are values of y for given values of x.

x	−2	−1	0	1	2	3	4
y	4	−1	−4	−5	−4	−1	4

The graph of $y = x^2 - 2x - 4$ is represented as below.



The shape of the parabola in a quadratic equation depends on the value of the numerical coefficients of the variables in the equation.

1. If $a > 0$, then the parabola opens upwards and if $a < 0$, then the parabola opens downwards
2. If $b^2 - 4ac < 0$, then the parabola does not intercept the x-axis
3. If $b^2 - 4ac > 0$, then the parabola intercepts the x-axis at exactly two different points
4. If $b^2 - 4ac = 0$, then the parabola intercepts the x-axis at exactly one point

High Power Equations reducible to Quadratic form

Some high power equations are not quadratic equations, but they can be reduced to a quadratic form.

Example

(i) Consider the equation $2y^4 - 5y^2 + 2 = 0$. The degree of the equation is 4 and the equation will have 4 roots. This equation can be reduced to a quadratic equation by substituting $y^2 = x$.

$\therefore\ 2x^2 - 5x + 2 = 0$

$\therefore\ 2x^2 - 4x - x + 2 = 0$ $\qquad$ $\therefore\ 2x(x - 2) - 1(x - 2) = 0$

$\therefore\ (2x - 1)(x - 2) = 0$ $\qquad$ $\therefore\ x = \frac{1}{2}$ or $x = 2$

If $x = \frac{1}{2}$, then $y^2 = \frac{1}{2}$ and $y = +\frac{1}{\sqrt{2}}$

If $x = 2$, then $y^2 = 2$ and $y = \pm\sqrt{2}$

$\therefore$ The roots of the equation are $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, -\sqrt{2} + \sqrt{2}$

(ii) $3\left(x^2 + \frac{1}{x^2}\right) + 4\left(x - \frac{1}{x}\right) - 6 - 0$

Let, $x - \frac{1}{x} = m$ $\qquad$ ... (i)

$m^2 = \left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$ $\qquad$ ... (squaring (i))

$\therefore\ m^2 + 2 = x^2 + \frac{1}{x^2}$ $\qquad$ ... (ii)

Substituting (i) and (ii) in the equation:

$3(m^2 + 2) + 4m - 6 = 0$

$\therefore\ 3m^2 + 4m = 0$

$\therefore\ m(3m + 4) = 0$

$\therefore\ m = 0$ or $m = -\frac{4}{3}$

$\therefore\ x - \frac{1}{x} = 0$ or $x - \frac{1}{x} = -\frac{4}{3}$

$\therefore\ x^2 - 1 = 0$ or $3x^2 + 4x - 3 = 0$

$\therefore\ (x - 1)(x + 1) = 0$ or $x = \dfrac{-4 \pm \sqrt{16 - 4 \times 3 \times (-3)}}{2 \times 3}$

$\therefore\ x = \pm 1$ or $x = \dfrac{-4 \pm \sqrt{52}}{6}$

$\therefore\ x = +1, -1, \dfrac{-2 + \sqrt{13}}{3}, \dfrac{-2 - \sqrt{13}}{3}$

Concept Builder 1

1. Minesh and Happy together had 90 apples. Both of them ate 10 apples each and the product of the number of apples they have now is 248. Formulate a quadratic equation to find the number of apples they start with.
2. Find the roots of: $x^4 - 25x^2 + 144 = 0$
3. The product of the roots of the equation $ax^2 - 4x + (4a + 1) = 0$ is 2. Find the value of a.
4. Formulate a quadratic equation whose roots are 3 and −5.
5. If Mukesh is 5 years younger to Vishal and product of their ages is 1400. Find the age of Mukesh.

Directions for questions 6 to 8: Find the nature of the roots of the following equations

6. $2x^2 - 3x + 9 = 0$
7. $x^2 + 9x - 14 = 0$
8. $x^2 - 18x + 81 = 0$.

Answer Key

1. $x^2 - 90x + 1048 = 0$
2. $x = \pm 4$ and ± 3
3. $a = \frac{-1}{2}$
4. $x^2 + 2x - 15 = 0$
5. 35 years
6. Complex conjugates
7. Real and unequal
8. Real and equal

SOLVED EXAMPLES

Q : Solve $\frac{1}{4}(2x - 1)^2 = 4$

A : $(2x - 1)^2 = 16$

$2x - 1 = \pm 4$

i.e., $2x - 1 = 4$ or $2x - 1 = -4$

i.e., $x = \frac{5}{2}$ or $x = -\frac{3}{2}$

Q : Solve $a(x^2 + 1) = x(a^2 + 1)$

A : $ax^2 + a = x(a^2 + 1)$

$ax^2 - (a^2 + 1)x + a = 0$

$\therefore\ x = \frac{(a^2+1) \pm \sqrt{(a^2+1)^2 - 4 \times a \times a}}{2a} = \frac{(a^2+1) \pm (a^2-1)}{2a}$

$\therefore\ x = \frac{a^2+1+a^2-1}{2a}$ or $x = \frac{a^2+1-a^2+1}{2a}$ $\therefore\ x = a$ or $x = \frac{1}{a}$

Q : If one root of the equation $4x^2 - 13x + k = 0$ is twelve times the other, find k.

A : $a = 4$, $b = -13$ and $c = k$

Let α, β be its roots.

$\therefore\ \beta = 12\alpha$

Now, $\alpha + \beta = -\frac{b}{a} = -\frac{-13}{4} = \frac{13}{4}$

$13\alpha = \frac{13}{4}$ $\therefore\ \alpha = \frac{1}{4}$

$\alpha.\beta = \frac{c}{a} = \frac{k}{4}$

$\therefore\ 12\alpha^2 = \frac{k}{4}$

$\therefore\ 12\left(\frac{1}{4}\right)^2 = \frac{k}{4}$

$\therefore\ k = 3$

Q : If α and β are the roots of the equation $5x^2 - 3x - 2 = 0$, find the value of $\alpha^3 + \beta^3$.

A : $5x^2 - 3x - 2 = 0$

$\alpha + \beta = \frac{3}{5}$, $\alpha\beta = -\frac{2}{5}$

Now, $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \left(\frac{3}{5}\right)^3 - 3\left(-\frac{2}{5}\right)\left(\frac{3}{5}\right) = \frac{27}{125} + \frac{18}{25} = \frac{117}{125}$

Q : Find 'a' so that the sum of the roots of equation $ax^2 + 4x + 6a = 0$ may be equal to their product.

A : $ax^2 + 4x + 6a = 0$

Sum of the roots = $-\frac{4}{a}$

Product of the roots = $\frac{6a}{a} = 6$

Sum of the roots = Product of the roots $-\frac{4}{a} = 6$

$\therefore\ a = -\frac{4}{6} = -\frac{2}{3}$

Q : If a = b = c, find the nature of the roots of : $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

A : $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

Substituting b = a and c = a

$x^2 - 2ax + a^2 + x^2 - 2ax + a^2 + x^2 - 2ax + a^2 = 0$

$3x^2 - 6ax + 3a^2 = 0$

$\Delta = (-6a)^2 - 4 \times 3 \times 3a^2 = 36a^2 - 36a^2 = 0$

$\therefore$ The roots are real and equal.

Q : Solve $\frac{x^2 - x + 1}{x^2 + x + 1} = \frac{a^2 - a + 1}{a^2 + a + 1}$

A : $(x^2 - x + 1)(a^2 + a + 1) = (x^2 + x + 1)(a^2 - a + 1)$

$\therefore\ x^2a^2 + ax^2 + x^2 - a^2x - ax - x + a^2 + a + 1$

$= x^2a^2 - ax^2 + x^2 + a^2x - ax + x + a^2 - a + 1$

$\therefore\ 2ax^2 - 2a^2x - 2x + 2a = 0$

$\therefore\ 2ax^2 - 2x(a^2 + 1) + 2a = 0$

$\therefore\ 2(x - a)(ax - 1) = 0 \quad \therefore\ x = a$ or $x = \frac{1}{a}$

Q : For which values of 'm' will the equation $x^2 - 2(5 + 2m)x + 3(7 + 10m) = 0$ have equal roots?

A : $\Delta = [-2(5 + 2m)]^2 - 4 \times 1 \times 3(7 + 10m)$

$= 4(25 + 20m + 4m^2) - 84 - 120m$

$= 100 + 80m + 16m^2 - 84 - 120m$

$= 16m^2 - 40m + 16$

$= 8(2m^2 - 5m + 2) = 8(2m - 1)(m - 2)$

The equation will have equal roots if $\Delta = 0$ i.e., $8(2m - 1)(m - 2) = 0$

$\therefore\ m = \frac{1}{2}$ or $m = 2$

Q : Find the roots of $(x - 1)(x - 3)(x - 4)(x - 6) + 8 = 0$

A : $(x - 1)(x - 3)(x - 4)(x - 6) + 8 = 0$

$\therefore (x - 1)(x - 6)(x - 3)(x - 4) + 8 = 0$

$\therefore (x^2 - 7x + 6)(x^2 - 7x + 12) + 8 = 0$

Substitute $x^2 - 7x = y$

$\therefore (y + 6)(y + 12) + 8 = 0$

$\therefore y^2 + 6y + 12y + 72 + 8 = 0$

$\therefore y^2 + 18y + 80 = 0$

$\therefore (y + 10)(y + 8) = 0$

$\therefore y = -10 \quad \text{or} \quad y = -8$

$\therefore x^2 - 7x = -10 \quad \text{or} \quad x^2 - 7x = -8$

$\therefore x^2 - 7x + 10 = 0$ or $x^2 - 7x + 8 = 0$

$\therefore (x - 2)(x - 5) = 0$ or $x = \dfrac{-(-7) \pm \sqrt{49 - 32}}{2}$

$\therefore x = 2$ or $x = 5$ or $x = \dfrac{7 + \sqrt{17}}{2}$ or $\dfrac{7 - \sqrt{17}}{2}$

Q : In a certain family, eleven times the number of children is greater by 12 than twice the square of the number of children. How many children are there?

A : Let x be the number of children.

$11x = 2x^2 + 12$

$2x^2 - 11x + 12 = 0$

$(x - 4)(2x - 3) = 0 \qquad \therefore x = 4$ or $x = \dfrac{3}{2}$

Since the number of children cannot be $\dfrac{3}{2}$, there are 4 children in the family.

Q : A two digit number is equal to three times the product of the digits, and the digit in the tens place is less by 2 than the digit in the units place. Find the number.

A : Let x be the digit in the tens place and x + 2 be the digit in the units place.

$\therefore 10x + (x + 2) = 3x(x + 2)$

$\therefore 10x + x + 2 = 3x^2 + 6x$

$\therefore 3x^2 - 5x - 2 = 0$

$\therefore (x - 2)(3x + 1) = 0$

$\therefore x = 2$ or $x = -\dfrac{1}{3}$

The digit of a number cannot be $-\dfrac{1}{3}$ $\therefore$ The digit in tens place is 2.

$\therefore$ The number is 24.

Q : The sum of a certain number and its positive square root is 90. What is the number?

A : Let x^2 be the number.

$\therefore\ x^2 + x = 90$

$\therefore\ x^2 + x - 90 = 0$

$\therefore\ (x + 10)(x - 9) = 0$

$\therefore\ x = -10 \quad \text{or} \quad x = 9$

As the square root is positive, the number cannot be −10.

$\therefore$ The number is 81.

CLASS EXERCISE

Teaser

A poor family is unable to afford proper lighting, and so they restrict themselves to using exactly one candle every night. After 5 candles are burnt, the leftovers and stubs from those 5 can be used to make one new candle. If the family has 50 candles on the morning of 1st January 2012, and do not purchase any more, on which night will they use up their last candle?

Linear Equations with Multiple Solutions

1. Find all non-negative integer solutions for (x, y) where:

   a) $x + y = 4$ * b) $2x + 3y = 18$

   c) $11x + 2y = 30$ * d) $4x + 6y = 36$

2. How many non-negative integer solutions for x and y exist if:

   a) $x + y = 100$ * b) $2x + 5y = 100$

   c) $4x - y = 100$ * d) $3x + 6y = 100$

3. Find solutions for the following (where x, y and z are all natural numbers)

   a) $x + 3y + 10z = 25$ b) $x + 7z - 20 = 10 - 2y + z$

4. Kamlesh goes to a shop with Rs 100 to buy ice-creams and milkshakes. An icecream costs Rs 12 while a milkshake costs Rs 9. If Kamlesh gets Rs 7 back as change, how many different combinations of ice-creams and milkshakes could he have bought? What is the minimum amount Kamlesh could get back as change?

Quadratic Equations

Relation between the coefficients and the sum and product of roots:

- A quadratic (in x) with roots α and β is of the form $k(x - \alpha)(x - \beta)$
- For the quadratic polynomial $ax^2 + bx + c$, Sum of roots = $\frac{-b}{a}$ and Product of roots = $\frac{c}{a}$
- A quadratic polynomial can have at the most 2 distinct roots.

Note: The coefficients of a quadratic polynomial/equation are assumed to be rational numbers unless mentioned otherwise.
The quadratic equation can be written as $x^2 - Sx + P = 0$, where S denotes the sum of the roots and P denotes product of the roots.
The condition for the quadratic equation whose roots are equal in magnitude but opposite in sign is Sum = 0 i.e b = 0.
The condition for the quadratic whose roots are reciprocal of each other is when Product = 1 i.e c = a.

5. α and β are the roots of the equation $x^2 - 5x + 8 = 0$. Find the equations whose roots are:

   a) $(\alpha+1)$ and $(\beta+1)$ b) α^2 and β^2 c) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ d) α^3 and β^3

6. For the given polynomials, fill the table below:

Polynomial	Discriminant	Roots	Nature of roots
	($\Delta = b^2 - 4ac$)		
$x^2 - 5x - 6$			
$6x^2 - x - 2$			
$x^2 - 3$			
$x^2 + 4x - 6$			
$x^2 + 4x + 4$			
$x^2 + 4$			

Formula for finding the roots of any quadratic polynomial:

The roots of the quadratic polynomial $ax^2 + bx + c$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Relation between the discriminant and nature of roots:

The discriminant (Δ) of the quadratic polynomial $ax^2 + bx + c$ is defined as $= b^2 - 4ac$.

We observe that:

$\Delta > 0 \Rightarrow$ 2 distinct real roots (may be rational or irrational)

$\Delta = 0 \Rightarrow$ Single (repeated) real (rational) root

$\Delta < 0 \Rightarrow$ No real roots (roots are complex conjugates)

Moreover,

If Δ is a perfect square then the roots are Rational

If Δ not a perfect square then the roots are Irrational

7. Solve the following:

a) The sum of the roots of a quadratic equation is 3 while the sum of the squares of its roots is 7. Find the equation.

b) Find a quadratic polynomial with integral coefficients having both sum and product of roots = 4.5.

c) For what range of values of k will the polynomial $2x^2 + kx + 2$ have:

i) 2 real roots

ii) 1 (repeated) real root

iii) no real roots

d) How many quadratic polynomials with leading coefficient 1 have '1 + i' as one complex root?

1) 0 2) 1 3) 2 4) Infinitely many

e) If the roots of the quadratic equation $3x^2 + 5x + 3(4p - 1) = 0$ is reciprocal of each other. Find the value of p.

* f) Find the quadratic equation whose roots are 2 more than the roots of the equation

$x^2 - 6x + 4 = 0$.

* g) if α and β are the roots of the equation $2x^2 + 5x + 6 = 0$, then find the quadratic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.

h) A man sold a certain number of cows for Rs 12000. Had he sold five cows less for the same sum, he would have received Rs 80 more per cow. Find the number of cows he sold.

* i) If P and Q tried to solve the quadratic equation $x^2 + bx + c = 0$. P by mistake took the wrong value of b and found the roots to be 12, 2. Q did a similar mistake by taking the wrong value of c and found the roots to be 2, 8. Find the actual roots of the equation.

j) If m is real then the nature of roots of the quadratic equation $x^2 + \sqrt{6}(m-1)x - 3m$ is

If $\Delta > 0$ (equation has a real and non zero root)		
Sum of the roots (S)	Product of the roots (P)	Sign of the roots
+	+	+ , +
−	+	−, −
+	−	+, −(positive root is greater in magnitude)
−	−	−, + (negative root is greater in magnitude)

8. Ram and Rahim are solving a quadratic equation independently. Ram wrote the coefficient of x incorrect and gave the solution set as {3, −5} while Rahim wrote the constant term incorrect and gave the solution set as {−4, −6}. Find the actual quadratic equation.

1) $x^2 + 15x - 10 = 0$
2) $x^2 + 10x - 15 = 0$
3) $x^2 - 10x + 15 = 0$
4) $x^2 - 15x + 10 = 0$

9. Find the number of positive integer solutions for $\frac{4}{x} + \frac{10}{y} = 1$.

1) 4 2) 6 3) 8 4) 10

10. Find the value of x if $x = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}}}$

1) $\frac{-3 - \sqrt{13}}{2}$ 2) $\frac{-3 + \sqrt{13}}{2}$ 3) $\frac{\sqrt{13}}{2}$ 4) Both (1) and (2)

*11. Find the number of positive integer solutions for $\frac{2}{x} + \frac{3}{y} = \frac{1}{6}$.

1) 12 2) 16 3) 18 4) 9

*12. Let $4 = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$ to infinity, then x could equal

1) 4 2) 8 3) 12 4) 16

*13. Anaya purchased 19 pencils at Rs.3 each. He paid the shopkeeper using notes of Rs.10, Rs.5 and Rs.2 denominations. How many different ways can he make the payment if he uses at least one note of each denomination.

1) 17 2) 16 3) 15 4) 14

*14. For a quadratic equation $ax^2 + ax + 1 = 0$, $a \in I$; $0 < a < 6$, how many values can ‘a’ not take, if the roots are real and distinct?

1) 2 2) 3 3) 4 4) 5

Challengers

1. Aakash goes to the post office to send a parcel. He finds that he needs to put stamps worth Rs. 42 on the parcel. The post office has stamps in three denominations; Rs 2, Rs 7 and Rs 11. In how many different ways can Aakash put the stamps?

   1) 4 2) 9 3) 11 4) 16

2. If 3p – 8q = 20 and p and q are integers, which of the following must be false?

   1) if p is positive, q can be negative
   2) if p is negative, q cannot be positive
   3) there is a solution for q such that $197 \le q \le 203$
   4) there is a solution for p such that $197 \le p \le 203$

3. A stick is broken in two parts such that the ratio of the smaller part to longer part is same as the ratio of longer part to that of the entire stick. Find the ratio of length of smaller part to longer part.

   1) $\frac{\sqrt{5}+1}{2}$ 2) $\frac{\sqrt{5}-1}{2}$ 3) $\frac{-\sqrt{5}+1}{2}$ 4) none of these

4. Let $x = \sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-\ldots}}}}$ to infinity, then x equals

   1] 3 2] $\frac{\sqrt{13}-1}{2}$ 3] $\frac{\sqrt{13}+1}{2}$ 4] $\sqrt{13}$

5. How many pairs of positive integers satisfy $\frac{1}{m}+\frac{4}{n}=\frac{1}{12}$ where n is an odd integer less than 60?

   1] 6 2] 4 3] 3 4] 5

PRACTICE EXERCISE - 1

DIRECTIONS for question 1: Solve as directed.

1. Find all possible non-negative integer solutions for:
   A. $2x + 11y = 70$ B. $4x + 10y = 40$
   C. $6x + 9y = 100$ D. $x + 2y + 7z = 10$

DIRECTIONS for questions 2 to 11: Choose the correct alternative.

2. One root of a quadratic equation is $3 + \sqrt{2}$. The quadratic equation is
   1) $x^2 - 6x - 7 = 0$ 2) $x^2 + 6x - 7 = 0$
   3) $x^2 + 6x + 7 = 0$ 4) $x^2 - 6x + 7 = 0$

3. For what values of k will the equation: $x^2 - (3k - 1)x + 2k^2 + 2k = 11$ have equal roots.
   1) 9, −5 2) −9, 5 3) 9, 5 4) −9, −5

4. Two students, A and B, were asked to solve a quadratic equation. A made a mistake only in writing the correct coefficient of x. B made a mistake only in writing the constant term. A gave the solution set as {−3, −3} and B gave the solution set as {3, 7}. Find the actual quadratic equation if the coefficient of x^2 was 1.
   1) $x^2 + 10x + 9 = 0$ 2) $x^2 + 10x - 9 = 0$
   3) $x^2 - 10x + 9 = 0$ 4) $x^2 - 10x - 9 = 0$

5. Find the real solution set for x, if $x^2 + \frac{1}{9x^2} + x + \frac{1}{3x} + 1 = \frac{7}{3}$
   1) $-1 \pm \frac{\sqrt{6}}{3}$ 2) $3 \pm \sqrt{\frac{3}{6}}$ 3) $3 \pm \sqrt{\frac{3}{6}}i$ 4) None of these

6. If α and β are the roots of the equation $5x^2 + 17x + 6 = 0$, find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.
   1) $\frac{289}{25}$ 2) $\frac{25}{36}$ 3) $\frac{302}{36}$ 4) $\frac{229}{36}$

7. Let α and β be the roots of the quadratic equation $4x^2 + 16x + 15 = 0$. Find the value of $\alpha^3 + \beta^3$.
   1) 109 2) 19 3) −19 4) 38

8. One root of $x^2 + kx - 8 = 0$ is square of the other. Then the value of k is
   1) 2 2) 8 3) −8 4) −2

   (Past CAT question)

9. Given the quadratic equation $x^2 - (A - 3)x - (A - 2)$, for what value of A will the sum of the squares of the roots be zero?
   1) −2 2) 3 3) 6 4) None of these

   (Past CAT question)

10. If the roots x_1 and x_2 of the quadratic equation $x^2 - 2x + c = 0$ also satisfy the equation $7x_2 - 4x_1 = 47$, then which of the following is true?
    1) $c = -15$
    2) $x_1 = -5, x_2 = 3$
    3) $x_1 = 4.5, x_2 = -2.5$
    4) None of these

(Past CAT question)

11. Determine the number of solutions (x, y) of the system of equations
    $x^2 - xy + y^2 = 21$
    $x^2 + 2xy - 8y^2 = 0$ where x and y are real numbers.
    1) 2 2) 3 3) 4 4) Infinitely many

DIRECTIONS for questions 12 to 14: Solve as directed.

12. How many pairs of integers (x, y) are possible for $x^2 - y^2 = 287$?

13. If $(4x - 6)$ and $(5y - 2)$ are two consecutive numbers (in that order) that are prime, what is the sum of all the possible values of xy?

14. How many pairs of values of (x, y) satisfy the following two equations?
    $x + |y| = 6$ and $|x| + y = 4$?

DIRECTIONS for questions 15 to 19: Choose the correct alternative.

15. Iqbal dealt some cards to Mushtaq and himself from a full pack of playing cards and laid the rest aside. Iqbal then said to Mushtaq. "If you give me a certain number of your cards, I will have four times as many cards as you will have. If I give you the same number of cards, I will have thrice as many cards as you will have". Of the given choices, which could represent the number of cards with Iqbal?
    1) 9 2) 31 3) 12 4) 35

16. If x and y are integers, then the equation $5x + 17y = 87$ has
    1) No solution for $x < 350$ and $y < 0$
    2) No solution for $x > 250$ and $y > -80$
    3) A solution for $200 < x < 250$
    4) A solution for $-69 < y < -66$

17. The number of real roots of the equation $\frac{A^2}{x} + \frac{B^2}{(x-1)} = 1$ where A and B are real numbers not equal to zero simultaneously is
    1) None 2) 1 3) 2 4) 1 or 2

(Past CAT question)

18. Aziz bought apples and bananas such that the difference between the total sum he paid to buy apples and bananas is Rs. 40 (total price of apples being more than the total price of bananas). It is known that the price per kg of apple is Rs. 40 more than that of bananas (both being integer values) and the quantity of apples bought was 2 kg less than the quantity of bananas bought (both being integer values). Which of the following cannot be the total sum he paid to buy apples and bananas?

    1) Rs. 200 2) Rs. 440 3) Rs. 560 4) Rs. 760

19. If $x^2 + 5y^2 + z^2 = 2y(2x + z)$, then which of the following statements are necessarily true?

    A. $x = 2y$ B. $x = 2z$ C. $2x = z$

    1) Only A 2) Only B and C 3) Only A and B 4) None of these

(Past CAT question)

DIRECTIONS for question 20: Solve as directed.

20. Bholenath bought a certain number of mangoes to be sold in the market. On each day of his business, he would sell one-third of his available stock, give half of the remaining mangoes to his son and then take 10 mangoes to his own house. Whatever was left at the end of a day after these transactions would become the available stock for the next day. It was found that he could do business only for 3 days and there was no stock available for the fourth day. What is the number of mangoes that he had bought initially?

PRACTICE EXERCISE - 2

DIRECTIONS for questions 1 and 2: Solve as directed.

1. Find the number of possible solutions for $x + 2y = 40$ where
   A. x and y are positive integers
   B. x and y are non-negative integers
   C. x and y are even natural numbers
   D. x and y are natural numbers and $x > y$
   E. x and y are natural numbers and $x \leq 3y$

2. In how many ways can a person pay back a loan of Rs 150, using three types of notes: Rs 10, Rs 25 and Rs 50?

DIRECTIONS for questions 3 to 19: Choose the correct alternative.

3. The sum of all the roots of two equations is 10. If one equation is $x^2 - 7x = 6$ what could the other equation be?
   1) $x^2 - 3x - 100 = 0$ 2) $2x^2 + 55 = 6x$
   3) $3x^2 - 9x = 2\sqrt{2}$ 4) All of these

4. Given that $x^2 + \frac{5}{x^2} + x - \frac{\sqrt{5}}{x} = 2\sqrt{5}$, which of the following may be a value of x?
   1) $5^{\frac{1}{4}}$ 2) 5^4 3) $25^{\frac{1}{4}}$ 4) $5 \times 2^{\frac{1}{4}}$

5. Find the value of m for which $4x^2 + (3m - 3)x + (2m - 1) = 0$ has real and equal roots.
   1) $\frac{5}{9}$ 2) 5 3) 5, $\frac{5}{9}$ 4) None of these

6. If the roots of the quadratic equation $\ell x^2 + mx + n = 0$ are equal in magnitude and opposite in sign, then
   1) $\ell = 0$ 2) $m = 0$ 3) $n = 0$ 4) None of these

7. If the equation $x^2 - 2kx - 2x + k^2 = 0$ has equal roots, the value of k must be
   1) zero 2) either zero or $-\frac{1}{2}$
   3) $-\frac{1}{2}$ 4) either $\frac{1}{2}$ or $-\frac{1}{2}$

8. The number of triplets (a, b, c) where a, b, c are positive integers, which satisfy the simultaneous equations:
   (i) $ab + bc = 44$ (ii) $ac + bc = 23$ is/are
   1) 1 2) 2 3) 4 4) Indefinitely many

9. If $\left(m^2 + \frac{9}{m^2}\right) + \left(m + \frac{3}{m}\right) = 0$, find m.
   1) -3 2) 2 3) Both [1] & [2] 4) None of these

10. Given that $x^2 - 4(2m + 1)x + 3(m - 2) = 0$ has equal roots. What can be the values of m?
    1) $\frac{-13 \pm \sqrt{471}}{7}$ 2) 0 3) $\frac{-13 \pm \sqrt{471}\,i}{32}$ 4) $\frac{-13 \pm \sqrt{471}}{32}$

11. A real number and its square root add up to 132. Find the number.
1) 144 2) 121 3) Both (1) & (2) 4) 225

12. One of the two roots of a quadratic equation is complex. Which of the following is definitely true?
1) Product of the roots is always real 2) Sum of the roots is always real
3) Both (1) and (2) 4) Neither (1) nor (2)

13. If a quadratic equation has one of its root as 5 + 3i, what could be the equation?
1) $x^2 = 10x - 17$ 2) $x^2 = 10x - 34$ 3) $x^2 = 34$ 4) None of these

14. Mr. Iyer distributed Rs.24 amongst his grandchildren. Had there been 4 grandchildren less, each would have got a rupee more. How many grandchildren does Mr. Iyer have?
1) 6 2) 10 3) 12 4) 8

15. The sum of a father's age and his son's age is 100 years. Also, one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old is the son?
1) 60 years 2) 40 years 3) 30 years 4) 45 years

16. If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, having real but unequal roots, then which of the following should be known to find the value of $\left(\alpha^2 + \frac{1}{\alpha}\right)\left(\beta^2 - \frac{1}{\beta}\right)$, given that coefficients of the above quadratic equation are known?
1) Both α and β are different in polarity. 2) $\alpha > \beta$
3) $\alpha\beta$ is perfect square. 4) No additional information needed.

17. What is the value of $5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + ...}}}$?
1) $\frac{5+\sqrt{29}}{2}$ 2) $\frac{5 \pm \sqrt{29}}{2}$ 3) $\frac{5-\sqrt{29}}{2}$ 4) 7

18. Stephen was solving a couple of linear equations in 'x' and 'y' simultaneously. By mistake, he interchanged the coefficients of x and y in each equation and got the solution as x = 3 and y = −2. What is the actual solution to the original two equations?
1) x = −2 and y = 3 2) x = 3 and y = −2
3) Infinite solutions 4) Cannot be determined

19. Which of the following is/are true for a quadratic equation $ax^2 + bx + c = 0$, whose roots are α and β?
I. If $\alpha = m + \sqrt{n}$, then $\alpha + \beta = 2m$ II. If $\alpha = m + i\sqrt{n}$, then $\alpha - \beta = 2i\sqrt{n}$
III. If $\alpha + \beta = 0$, then $c = a\alpha^2$ IV. If $\alpha\beta = 1$, then $\alpha^2\beta + \beta^2\alpha = -\frac{b}{c}$
1) I and II 2) I, II and IV 3) I, II and III 4) All of these

DIRECTIONS for question 20: Solve as directed.

20. What is the sum of all the roots of the equation $x^2 - 2|x| - 8 = 0$?

QA-2.2 | POLYNOMIALS

THEORY

Arithmetic operations on Algebraic Expressions

1. Addition and Subtraction of Algebraic Expressions

To find the sum of a polynomial add the numerical coefficients of like terms and annex the common letter or letters

Example

$3xy + 6xy + 9xy = 18xy;$

$-5a^2b - 7a^2b - 8a^2b = -20a^2b$

$-13x + 7x - 8x + 5x = -21x + 12x = -9x$

If any number of terms are enclosed within brackets preceded by a '+' sign or a '–' sign, the brackets may be removed by multiplying the sign of each term inside the bracket with the respective sign before the bracket. If there are no similiar terms, the operation is algebraically complete

Example

$(2p - 3q + r) + (p + 2q - 3r) - (2p - 2q - 3r) = p + q + r$

2. Multiplication of Algebraic Expressions

To multiply one polynomial with another:

(i) The product of two factors with like signs is positive, and of those with unlike signs is negative.

$(+a) \times (+b) = +ab$ $\quad$ $(-a) \times (+b) = -ab$

$(+a) \times (-b) = -ab$ $\quad$ $(-a) \times (-b) = ab$

(ii) Multiply each term of one polynomial by each term of the other. Add the like terms thus obtained.

Example

(i) $(x + 5y)(x + 3y) = x(x + 3y) + 5y(x + 3y) = x^2 + 3xy + 5xy + 15y^2$
$= x^2 + 8xy + 15y^2$

(ii) $(a - 3b)(a + 2b) = a(a + 2b) - 3b(a + 2b) = a^2 + 2ab - 3ab - 6b^2$
$= a^2 - ab - 6b^2$

3. Division of Polynomials

(i) Division Method

Step 1: Arrange the terms of the dividend in the descending order of powers.

Step 2: Divide the first term of the dividend by the first term of the divisor and write this result as the first term of the quotient.

Step 3: Multiply the divisor by the first term of the quotient and then subtract from the dividend.

Step 4: Now divide the result obtained in step 3 by the first term of the divisor and write this result as the second term of the quotient.

Step 5: Repeat step 4 till a remainder whose degree is less than that of the divisor is obtained.

Example

$$
\begin{array}{r|l}
 & a^2 - 4a + 3 \\
\hline
a - 2 & a^3 - 6a^2 + 11a - 6 \\
 & a^3 - 2a^2 \\
\hline
 & \quad -4a^2 + 11a - 6 \\
 & \quad -4a^2 + 8a \\
\hline
 & \qquad\qquad 3a - 6 \\
 & \qquad\qquad 3a - 6 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

$\therefore\ a^3 - 6a^2 + 11a - 6 = (a - 2)\ (a^2 - 4a + 3)$

(ii) Synthetic division

Synthetic division can be used only if the divisor is in the form x + a.

Step 1: Write the dividend in the coefficient form (use zero for missing terms) in the first row. If the divisor is of the form x + a, put (–a) in the first row to the left of the first coefficient. Below the horizontal line in the third row, write the first coefficient.

Step 2: Multiply the first coefficient in the third row by the divisor (here –4) and put the product in the second row below the second coefficient of the first row and write their sum in the third row.

Step 3: Repeat above step for the next coefficient

Step 4: The last number in the third row is the remainder and the remaining numbers in the third row, in order, are the co-efficients of the quotient (left to right).

Example

Divide $6x^3 + 5x^2 + 5x + 12$ by $x + 4$

$$\begin{array}{r|rrrrl} -4 & 6 & 5 & 5 & 12 & \text{First row} \\ & & -24 & 76 & -324 & \text{Second row} \\ \hline & 6 & -19 & 81 & \boxed{-312} & \text{Third row} \end{array}$$

Here the quotient is $6x^2 - 19x + 81$ and the remainder is -312

Note: The polynomial $6x^3 + 5x + 2$ in the coefficient form is written as 6, 0, 5, 2

Concept Builder 1

1. Is $x^2 + \sqrt{x} + 3$ a polynomial?
2. Simplify:$(4p + 3q - 4r) - (8p - 5q) + (4p - 3q - 4r) + (-5q + 8r)$
3. Find the remainder when $a^3 - 9a^2 + 26a - 24$ is divided by $a - 4$
4. When $3x^2 - kx + 2$ is divided by $(x - 3)$ it leaves a remainder 5. Find the value of k
5. Find the degree of the polynomial $x^2 + 3x^2 + 7x^3 + 12$
6. Check whether $x + 2$ is factor of $x^2 - 3x - 10$
7. Find the quotient when $2x^5 - 3x^2 + 4x - 17$ is divided by $x - 3$.

Answer Key

1. NO
2. 0
3. 0
4. $k = 8$
5. 3
6. Yes
7. $2x^4 + 6x^3 + 18x^2 + 51x + 157$

Properties of Polynomials

1. Zeros of a Polynomial

Consider the polynomial $p(x) = 5x^3 + 3x^2 + x + 3$

If we replace x by 1 in p(x), we get

$p(1) = 5(1)^3 + 3(1)^2 + 1 + 3$

$= 5 + 3 + 1 + 3 = 12$

Similarly $p(0) = 5(0) = 5(0)^3 + 3(0)^2 + 0 + 3$

$= 0 + 0 + 0 + 3 = 3$

and $p(-1) = 5(-1)^3 + 3(-1)^2 + (-1) + 3$

$= -5 + 3 -1 + 3 = 0$

As $p(-1) = 0$, we say that -1 is a zero of the polynomial. It is also called as the root of the polynomial.

You can see that the zero (root) of the polynomial p(x) is obtained by equating it to 0. i.e. $p(x) = 0$ A polynomial can have more than one zero (root).

Example

Find p(0), p(1), p(2) for $p(x) = 2 + x + 2x^2 - x^3$

$p(0) = 2 + 0 + 2(0)^2 - (0)^2 = 2$

$p(1) = 2 + 1 + 2(1)^2 - (1)^3 = 2 + 1 + 2 - 1 = 4$

$p(2) = 2 + 2 + 2(2)^2 - 2^3$

$= 2 + 2 + 8 - 8 = 4$

Example

Find the zeros of the following polynomials.

i) $p(x) = 3x - 2$ ii) $q(x) = x^2 - 7x + 12$

i) Let $p(x) = 0$

$\therefore\ 3x - 2 = 0$

$\therefore\ x = \frac{2}{3}$

$\frac{2}{3}$ is the zero (root) of p(x).

ii) Let $q(x) = 0$

$\therefore\ x^2 - 7x + 12 = 0$

$\therefore\ (x - 4)(x - 3) = 0$

$\therefore\ x = 3, 4$

3 and 4 are the zeros (roots) of q(x).

2. Remainder Theorem

Let p(x) be any polynomial of degree greater than or equal to one and let a be any real number. If p(x) is divided by the linear polynomial x − a, the remainder is p(a).

Example

Find the remainder when $x^4 - 3x^3 + 4x^2 - x + 7$ is divided by $x - 1$.

$p(x) = x^4 - 3x^3 + 4x^2 - x + 7$ and zero of $x - 1$ is 1.

$\therefore\ p(1) = (1)^4 - 3(1)^3 + 4(1)^2 - 1 + 7 = 1 - 3 + 4 - 1 + 7 = 8$

So, by Remainder Theorem, 8 is the remainder when $x^4 - 3x^3 + 4x^2 - x - 7$ is divided by $x - 1$.

Example

Check whether the polynomial $p(a) = 4a^3 + 4a^2 - a - 1$ is a multiple of $2a + 1$.

p(a) is a multiple of 2a + 1 if 2a + 1 divides p(a), leaving the remainder zero.

Now, $2a + 1 = 0 \quad \therefore\ a = -\frac{1}{2}$

Then, $p\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1$

$= -\frac{1}{2} + 1 + \frac{1}{2} - 1$

$= 0$

$\therefore$ The remainder is 0.

$\therefore$ 2a + 1 is a factor of the polynomial p(a) i.e. p(a) is a multiple of 2a + 1.

3. Factor Theorem

If p(x) is a polynomial of degree $n \geq 1$ and a is any real number, then

(i) x − a is a factor of p(x), if p(a) = 0

(ii) p(a) = 0 if (x − a) is a factor of p(x).

Example

Examine whether x − 2 is a factor of polynomial $p(x) = x^3 - 2x^2 + 7x - 14$

The zero of x − 2 is 2.

Then $p(2) = (2)^3 - 2(2)^2 + 7(2) - 14$

$= 8 - 8 + 14 - 14 = 0$

So, by Factor Theorem, x − 2 is a factor of polynomial $p(x) = x^3 - 2x^2 + 7x - 14$.

Example

Find k if x + 3 is a factor of $2x^3 - 5x^2 + 3x + k$

As x + 3 is a factor of $p(x) = 2x^3 - 5x^2 + 3x + k$

$p(-3) = 0$

Now $p(-3) = 2(-3)^3 - 5(-3)^2 + 3(-3) + k$

$= -54 - 45 - 9 + k$

$= -108 + k$

$\therefore\ -108 + k = 0$

$\therefore\ k = 108$

Example

Factorise $x^3 - 7x^2 + 14x - 8$

Let $p(x) = x^3 - 7x^2 + 14x - 8$

Factors of 8 are ± 1, ± 2, ± 4 and $+8$

$p(1) = (1)^3 - 7(1)^2 + 14(1) - 8 = 0$

So, $(x - 1)$ is a factor of $p(x)$

Again, $x^3 - 7x^2 + 14x - 8$

$= x^2(x - 1) - 6x(x - 1) + 8(x - 1)$

$= (x - 1)(x^2 - 6x + 8)$

$= (x - 1)(x - 2)(x - 4)$

Example

Factorise $x^3 + 6x^2 + 11x + 6$

Let $p(x) = x^3 + 6x^2 + 11x + 6$

Factors of 6 are ± 1, ± 2, ± 3, ± 6

All the terms are positive. $\therefore$ The value of x cannot be positive. Let us try $x = -1$

$p(-1) = (-1)^3 + 6 - 11(-1) + 6$

$= -1 + 6 - 11 + 6 = 0$

So, $(x + 1)$ is a factor of $p(x)$.

Again, $x^3 + 6x^2 + 11x + 6$

$= x^2(x + 1) + 5x(x + 1) + 6(x + 1)$

$= (x + 1)(x^2 + 5x + 6)$

$= (x + 1)(x + 2)(x + 3)$

Example

Factorise $x^3 + 15x^2 + 23x - 231$

Let $p(x) = x^3 + 15x^2 + 23x - 231$

Factors of 231 are ± 3, ± 7, ± 11

$p(3) = (3)^3 + 15(3)^2 + 23(3) - 231$

$= 27 + 135 + 69 - 231 = 0$

So, $(x - 3)$ is a factor of $p(x)$

Again, $x^3 + 15x^2 + 23x - 231$

$= x^2(x - 3) + 18x(x - 3) + 77(x - 3)$

THEORY
CLASS EXERCISE
PRACTICE EXERCISE

$= (x - 3)(x^2 + 18x + 77)$

$= (x - 3)(x + 7)(x + 11)$

4. Factorisation and HCF & LCM of polynomials

Factorisation of polynomials

An expression is said to be resolved into factors when those expressions of which it is the product are found.

Example

$x^2 - 5x + 6$ can be expressed as a product of $(x - 2)$ and $(x - 3)$

We can also say that '$x - 2$' and '$x - 3$' are factors of the term $x^2 - 5x + 6$.

HCF of polynomials

When two or more polynomials are factorised, the product of all the common factors is the HCF of the polynomials.

LCM of polynomials

When two or more polynomials are factorised, the product of all the factors with highest powers is the LCM of the polynomials.

Example

Find the HCF and LCM of $14xy^3$, $22x^2y$ and $26x^3y^4$.

$14xy^3 = 2 \times 7 \times x \times y^3$

$22x^2y = 2 \times 11 \times x^2 \times y$

$26x^3y^4 = 2 \times 13 \times x^3 \times y^4$

$\text{HCF} = 2 \times x \times y = 2xy$

$\text{LCM} = 2 \times 7 \times 11 \times 13 \times x^3 \times y^4 = 2002x^3y^4$

Important algebraic formulae

1. $(x + a)(x + b) = x^2 + (a + b)x + ab$
2. $(a + b)(a - b) = a^2 - b^2$
3. $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$
4. $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
5. $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
6. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
7. $(a + b)^2 - (a - b)^2 = 4ab$
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
9. $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd$
10. $(a + b)^3 = a^3 + 3ab(a + b) + b^3$
11. $(a - b)^3 = a^3 - 3ab(a - b) - b^3$
12. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 - ab + b^2)$
13. $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + ab + b^2)$
14. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$
    If $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

In general

15. $(x + y)^n = x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots\ldots\ldots\ldots + {}^nC_n y^n$
    $(x - y)^n = x^n - {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ (-1)^n\, {}^nC_n\, y^n$
    Note : Formulae (15) is also known as **binomial therom**
    where ${}^aC_b = \dfrac{a!}{b!\,(a-b)!}$, Also ${}^nC_n = \dfrac{n!}{n!\,(n-n!)} = 1$
16. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots. + b^{n-1})$ for all n.
17. $a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots. - b^{n-1})$ if n is even.
18. $(a^n - b^n)$ is divisible by $(a + b)$ when n is even and always divisible by $(a - b)$
19. $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \ldots. + b^{n-1})$ if n is odd.
    $(a^n + b^n)$ is divisible by $(a + b)$ when n is odd.

SOLVED EXAMPLES

Q : Subtract the sum of 11p − 8q + 3r and 12p − 9q + 4r from 15p − 2q + 12r

A : 15p − 2q + 12r − [(11p − 8q + 3r) + (12p − 9q + 4r)]
= 15p − 2q + 12r − [23p − 17q + 7r]
= 15p − 2q + 12r − 23p + 17q − 7r
= −8p + 15q + 5r

Q : When $x^3 + kx^2 + x + 1$ is divided by x − 1, the remainder is 6. Find k.

A : p(1) = 6
$\therefore\ 1^3 + k \times 1^2 + 1 + 1 = 6$
$\therefore\ 1 + k + 1 + 1 = 6$
$\therefore\ 3 + k = 6$
$\therefore\ k = 3$

Q : Multiply $\sqrt{2}\,x + \sqrt{3}\,y$ by $\sqrt{3}\,x + \sqrt{5}\,y$

A : $(\sqrt{2}\,x + \sqrt{3}\,y)(\sqrt{3}\,x + \sqrt{5}\,y) = \sqrt{2}\,x\ \sqrt{3}\,x + \sqrt{5}\,y + \sqrt{3}\,y\ \sqrt{3}\,x + \sqrt{5}\,y$

$= \sqrt{6}\,x^2 + \sqrt{10}\,xy + 3xy + \sqrt{15}\,y^2$

$= \sqrt{6}\,x^2 + (\sqrt{10} + 3)\,xy + \sqrt{15}\,y^2$

Q : Simplify $\frac{2c+3}{12} - \frac{7c+5}{3} + \frac{3c-9}{18}$

A : $\frac{2c+3}{12} - \frac{7c+5}{3} + \frac{3c-9}{18} = \frac{3(2c+3)}{3 \times 12} - \frac{12(7c+5)}{3 \times 12} + \frac{2(3c-9)}{2 \times 18}$

$= \frac{6c+9}{36} - \frac{84c+60}{36} + \frac{6c-18}{36} = \frac{6c+9-84c-60+6c-18}{36}$

$= \frac{-72c-69}{36} = \frac{3(-24c-23)}{36} = \frac{-24-23}{12}$

Q : Find the value of 511^2.

A : $(500 + 11)^2 = 250000 + 11000 + 121 = 261121$ [As $(a+b)^2 = a^2 + 2ab + b^2$]

Q : If $x + \frac{1}{x} = 15$, find $x^2 + \frac{1}{x^2}$.

A : $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2 = 225 - 2 = 223$.

Q : If $c = (a^2 + b^2 - 2ap)^{1/2}$, find c when a = 21, b = 13, p = 5.

A : $c = (441 + 169 - 210)^{1/2} = (400)^{1/2} = 20$.

Q : Find the square root of $4x^4 + 9y^4 + 16z^4 + 12x^2y^2 - 16x^2z^2 - 24y^2z^2$.

A : The given expression is $4x^4 + 4x^2(3y^2 - 4z^2) + 9y^4 - 24y^2z^2 + 16z^4$,

i.e. $(2x^2)^2 + 2(2x^2)(3y^2 - 4z^2) + (3y^2 - 4z^2)^2 = [2x^2 + (3y^2 - 4z^2)]^2$

Hence, the required square root is $2x^2 + 3y^2 - 4z^2$.

Q : Find the square root of $x^6 - 2x^5 + 3x^4 + 2x^3(y - 1) + x^2(1 - 2y) + 2xy + y^2$.

A : The expression only contains y^2 and y, we therefore arrange it according to powers of y, and have $y^2 + 2y(x^3 - x^2 + x) + (x^6 - 2x^5 + 3x^4 - 2x^3 + x^2)$.

Now, if the expression is a complete square, then the last term must be the square of half the coefficient of y; and it is easy to verify that $(x^3 - x^2 + x)^2 = x^6 - 2x^5 + 3x^4 - 2x^3 + x^2$.

So the given expression can be written as $y^2 + 2y(x^3 - x^2 + x) + (x^3 - x^2 + x)^2$

Hence, the required square root is $y + x^3 - x^2 + x$.

Q : Find the factors of 81p − 54.

A : $81p - 54 = 27 \times 3p - 27 \times 2 = 27(3p - 2)$. Hence, the factors of (81p − 54) are 27 and (3p − 2).

Q : Find the HCF and LCM of $25c^2 - 16d^2$ and $25c^2d - 20cd^2$.

A : $25c^2 - 16d^2 = (5c)^2 - (4d)^2 = (5c - 4d)(5c + 4d)$

$25c^2d - 20cd^2 = 5cd(5c - 4d)$

HCF = 5c − 4d

LCM = 5cd(5c − 4d)(5c + 4d)

Q : Find the remainder when $17^{24} - 13^{24}$ is divided by 8

A : $x = 17^{24} - 13^{24}$

$= (17 - 13)\ (17^{23} + 17^{22} \times 13 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 17 \times 13^{22} + 13^{23})$

$= 4\ (17^{23} + 17^{22} \times 13 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 17 \times 13^{22} + 13^{23})$

The expression in the bracket is divisible 2 (Since all the terms inside the bracket are odd and there are even number of terms)

$\therefore$ x is divisible by both 4 and 2

$\therefore$ x is divisible by $4 \times 2 = 8$

$\therefore$ Remainder = 0

Alternatively,

$\Rightarrow 17^{24} - 13^{24}$

$\Rightarrow (289)^{12} - (169)^{12}$

$\Rightarrow (289 - 169)[289^{11} + 289^{10} \times 169 + ... + 289 \times 169^{10} + 169^{11}]$

$\Rightarrow 120[289^{11} + 289^{10} \times 169 + ...]$

As 120 is divisible by 8, $\therefore$ x is divisible by 8

Hence, the remainder is zero.

Concept Builder 2

1. Simplify $(2a + 3)^2 - (3a - 4)^2$
2. Find the HCF & LCM of $12x^3y^2z$, $28x^2y^3z^2$
3. $(x - 3)(x + 2)(x + 6) = ?$
4. $(3x^4 + 6x^2 - 15) \div (x - 3) = A \times (x - 3) - (B)$
   Find the value of A and B
5. If $(x - 8)$ is one factor of $x^2 - 12x + p$. Find the other factor
6. Find the remainder when $(9^{16} - 5^{16})$ is divided by 14

Answer Key

1. $-5a^2 + 36a - 7$
2. HCF $= 4x^2y^2z$
   LCM $= 84x^3y^3z^2$
3. $x^3 + 5x^2 - 12x - 36$
4. $A = 3x^3 + 9x^2 + 33x + 99$
   $B = 282$
5. $x - 4$
6. 0

Roots of Equations of Higher Order

For determining the roots of equations of degree 3 or more, we can use the following method. Let α, β & γ be the roots of an equation. The equation can be written as:

$(x - \alpha)(x - \beta)(x - \gamma) = 0$

$[x^2 - (\alpha + \beta)x + \alpha\beta](x - \gamma) = 0$

$\therefore\ x^3 - (\alpha + \beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha + \beta)\gamma x - \alpha\beta\gamma = 0$

$\therefore\ x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$

We can observe that,

coefficient of $x^2 = -(\alpha + \beta + \gamma) = -$ sum of the roots

coefficient of $x = (\alpha\beta + \alpha\gamma + \beta\gamma)x =$ sum of the product of roots taken 2 at a time

constant term $= -\alpha\beta\gamma = -$ product of the roots

For example, consider the equation having roots as –1, 2 & –3.

$(x + 1)(x - 2)(x + 3) = 0$

$x^3 + 2x^2 - 5x - 6 = 0$

Sum of roots $= (-1 + 2 - 3) = -2$

Sum of product of roots taken 2 at a time

$= (-1)(2) + (-1)(-3) + (2)(-3)$

$= -2 + 3 - 6 = -5$

product of roots $= (-1)(2)(-3) = 6$

Nature of the roots of a polynomial equation

The number of roots of an equation is equal to or less than the degree of the equation. If the degree of the equation is ‘n’ then the number of roots $\le$ n.

The roots can be positive or negative or complex numbers.

- Number of complex roots will be an even number as for every complex root the conjugate of it will always be a root of the equation. Thus, the number of complex roots will be 2, 4, 6, 8, ...
- To find out the number of positive or negative roots of an equation, we should count the number of sign changes in the equation.
- Let the equation be $f(x) = 0$.

  Count the number of sign changes in this equation. If the sign changes ‘k’ times the number of positive roots will be k, k–2, k–4,....

For example consider the equation

$f(x) = x^3 - 2x^2 + 4x - 5 = 0$

	x^3	$-2x^2$	$+4x$	-5
sign ------------→	+	−	+	−
sign change -----→	1	2	3	

As there are 3 sign changes, the number of positive roots is 3 or 1.

– To find out the number of negative roots, count the number of sign changes of the equation $f(-x) = 0$

$$f(-x) = -x^3 - 2x^2 - 4x - 5 = 0$$

$$-x^3 - 2x^2 - 4x - 5$$

sign − − − −

sign changes (No sign change)

As there is no sign change, the number of negative roots is 0.

The following table summarizes the possible cases.

Nature of root & Total no. of roots	positive	negative	complex
3	3	0	0
3	1	0	2

SOLVED EXAMPLES

Q : Find the nature of the roots of the equation $x^5 - 4x^4 + 3x^3 + x^2 - 7x + 15 = 0$

A : Let $f(x) = x^5 - 4x^4 + 3x^3 + x^2 - 7x + 15$

There are 4 sign changes $\therefore$ No. of positive roots = 4, 2 or 0

$f(-x) = -x^5 - 4x^4 - 3x^3 + x^2 + 7x + 15$

There is 1 sign change $\therefore$ No. of negative roots = 1

The remaining roots will be the complex roots.

This can be summarized as follows:

Positive	Negative	Complex	Total
4	1	0	5
2	1	2	5
0	1	4	5

Q : α, β & γ are the roots of the equation $x^3 - 3x^2 - 13x + 15 = 0$. Find the following:

(i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta\gamma$ (iii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iv) $\alpha^2 + \beta^2 + \gamma^2$ (v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

A : (i) $\alpha + \beta + \gamma$ = sum of the roots

$= -$ coefficient of $x^2 = -(-3) = 3$

(ii) $\alpha\beta\gamma$ = product of roots = $-$ constant term = $-(15) = -15$

(iii) $\alpha\beta + \alpha\gamma + \beta\gamma$ = sum product = coefficient of x = -13

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (3)^2 - 2(-13)$

$= 9 + 26 = 35$

(v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \frac{-13}{-15} = \frac{13}{15}$

Q : Find the roots of the equation $x^3 - 4x - 11x + 30 = 0$ and find the equation whose roots are reciprocals of the roots of this equation.

A : $f(x) = x^3 - 4x^2 - 11x + 30 = 0$

There are 2 sign changes.

So, the number of positive roots = 2 or 0.

$f(-x) = -x^3 - 4x^2 + 11x + 30 = 0$

There is 1 sign change. So, the number of negative roots = 1

Let the three roots be α, β & γ.

$\alpha\beta\gamma = -$ product $= -30$

$\alpha + \beta + \gamma = -$ coefficient of $x^2 = -(-4) = 4$

$\alpha\beta + \alpha\gamma + \beta\gamma =$ coefficient of $x = -11$

By trial and error, we get $\alpha = 2$, $\beta = 5$, $\gamma = -3$

$\alpha + \beta + \gamma = 2 + 5 - 3 = 4$

$\alpha\beta\gamma = (2)(5)(-3) = -30$

$\alpha\beta + \alpha\gamma + \beta\gamma = (2)(5) + (2)(-3) + (5)(-3)$

$= 10 - 6 - 15 = -11$

Let required equation has roots

$\alpha' = \frac{1}{\alpha}$, $\beta' = \frac{1}{\beta}$, $\gamma' = \frac{1}{\gamma}$

Product $= \alpha'\beta'\gamma' = \frac{1}{\alpha\beta\gamma} = \frac{1}{-30} = -\frac{1}{30}$

Sum $= (\alpha' + \beta' + \gamma') = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{(-11)}{-30} = \frac{11}{30}$

Sum product $= \alpha'\beta' + \alpha'\gamma' + \beta'\gamma'$

$= \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}$

$= \frac{4}{-30} = -\frac{4}{30}$

$\therefore$ The required equation is

$x^3 - \left(\frac{11}{30}\right)x^2 + \left(-\frac{4}{30}\right)x - \left(-\frac{1}{30}\right) = 0$

$\therefore 30x^3 - 11x^2 - 4x + 1 = 0$

Q : Find the roots of the equation $x^4 - 7x^3 + 8x^2 + 28x - 48$.

A : $f(x) = x^4 - 7x^3 + 8x^2 + 28x - 48$

There are 3 sign changes. So, there are 3 or 1 positive roots.

$f(-x) = x^4 + 7x^3 + 8x^2 - 28x - 48$

There is 1 sign change. So, there is 1 negative root.

Let the 4 roots be a, b, c and d.

$\therefore$ abcd = constant term = -48

$a + b + c + d = -\text{coefficient of } x^3 = -(7) = 7.$

By trial and error we can get,

$a = 2, b = -2, c = 3, d = 4$

$= (2)(-2) + (-2)(3) + (3) + (3)(4) + (2)(4) + (2)(3) + (-2)(4)$

$= -4 - 6 + 12 + 8 + 6 - 8 = 8 = \text{coefficient of } x^3$

The roots of the equation are 2, -2, 3, 4.

Note: To find the roots of the equation by using this method, the coefficient of the term with the highest degree should be made equal to 1.

CLASS EXERCISE

Teaser

I have two dice. I wish to number their faces in such a way that the dice can be placed alongside to show any date of the year. Which numbers should I write on each die?

Note that both the dice must be used while displaying a date – e.g. If I write 0, 1, 2, 3, 4, 5 on one die and 6, 7, 8, 9, 1, 2 on the other, then I will not be able to display the dates 03, 04, etc.

Polynomials

1. Expand / Simplify:

a) $(x - 2)(x - 3) - (x - 1)(x + 1)$

b) $(x + 2)(x - 3) + (3 - x)(2 + x)$

*c) $(x + 1)(x + 2)(x + 3)$

2. Find quotient and remainder when:

a) $x^3 + x^2 + 1$ is divided by $x + 1$

b) $x^4 - x^3 + x^2 - 1$ is divided by x^2

c) $x^4 - x^3 + x^2 - 1$ is divided by $x - 1$

d) $x^4 - x^3 + x^2 - 1$ is divided by $x^5 - 1$

e) x^3 is divided by $x + 1$

f) $14x^5 - 2x^3 + 27$ is divided by $x^3 - 1$

g) $x^8 + x^7 + 1$ is divided by $x^2 + x + 1$

Remainder theorem/ Factor theorem

$(x - a)$ is a factor of the polynomial $p(x)$ if and only if 'a' is a root of $p(x)$, i.e. $p(a) = 0$.

3. Answer the following:

a) For $p(x) = x^3 + x - 2$, find $p(0)$, $p(3)$, $p(-1)$ and $p(1)$. Which of {0, 3, −1, 1} are roots of $p(x)$?

b) Is $p(x) = 2x^2 - 3x + 1$ divisible by the following?
i) $x + 1$ ii) $x - 1$

c) Find the remainder when $p(x) = x^4 - x^3 + x^2 - x + 1$ is divided by:
i) $x - 1$ ii) $x + 2$

d) Is $2x - 1$ a factor of $2x^2 + 13x - 7$?

4. Factorize:

a) $x^3 + x^2 - 5x + 3$

b) $x^3 - 19x - 30$

c) $2x^2 - 3$

d) $3x^3 + 5x^2 + 8x + 6$

e) $x^3 - 15x^2 + 71x - 105$

5. a) Find LCM of $12y^3 - 27y$ and $12y^2 - 6y - 18$.

b) Find the HCF of $y^3 - 4y$ and $4y(y^3 + 8)$

Applications of Algebraic formulae

Some useful formulae

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $(a + b)^3 = a^3 + 3ab(a + b) + b^3$
- $(a - b)^3 = a^3 - 3ab(a - b) - b^3$
- $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + ab + b^2)$

In general, for any natural number n,

- $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-1})$ only for odd n
  i.e. (a + b) is a factor of $a^n + b^n$ only if n is odd
- On the other hand, $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$ for all n
  i.e. (a − b) is always a factor of $a^n - b^n$

6. Which of the following are factors of $a^6 - b^6$?

i) $(a - b)$
ii) $(a + b)$
iii) $(a^2 - b^2)$
iv) $(a^2 + b^2)$
v) $(a^3 - b^3)$
vi) $(a^3 + b^3)$
vii) $(a^4 - b^4)$
viii) $(a^4 + a^2b^2 + b^4)$

7. State true or false:

i) $x + 1$ is a factor of $x^7 + 1$
ii) $x - 1$ is a factor of $x^8 - 1$
iii) $x + 1$ is a factor of $x^7 - 1$
iv) $x + 1$ is a factor of $x^8 - 1$
v) $x^3 - 2$ is not a factor of $x^6 - 4$
vi) $x^2 + y^2$ is a factor of $x^{10} + y^{10}$
vii) $x^2 + 1$ is a factor of $x^3 + 1$
viii) $x^6 + y^6$ cannot be factorized

8. Which of these is certainly a factor of $a^{24} + 1$?

i) $a^{12} + 1$
ii) $a^8 + 1$
iii) $a^6 + 1$
iv) $a^4 + 1$
v) $a^3 + 1$
vi) $a^2 + 1$

Some more useful formulae

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

9. State true or false:

i) $11^5 + 12^5 + 13^5$ is divisible by 24
ii) $1000^3 + 1001^3 + \ldots + 1019^3 + 1020^3$ is divisible by 101

10. Simplify: $(x^{7/4} - x^{-5/4})(x^{7/4} + x^{-5/4})(x^{7/2} + x^{-5/2})(x^7 + x^{-5})$

11. If a, b, c are such that $\frac{a+b}{c} = -1$ and abc = $\frac{1}{12}$ then find the value of $a^3 + b^3 + c^3$.

12. What is the remainder when $16^3 + 17^3 + 18^3 + 19^3$ is divided by 70?

13. What is the minimum value of $x + \frac{1}{x}$ (given that x > 0)?
 1) 0 2) 1 3) 2 4) None of the above

14. If $x + \frac{1}{x} = 5$ then find the value of $x^2 + \frac{1}{x^2}$
 1) 20 2) 23 3) 25 4) 27

15. If $x + \frac{1}{x} = 5$ then find the value of $x^3 + \frac{1}{x^3}$
 1) 100 2) 115 3) 125 4) 110

16. *If $x + \frac{1}{x} = 5$ then find the value of $x^4 + \frac{1}{x^4}$
 1) 527 2) 525 3) 529 4] 523

Roots of Higher Degree Polynomials

A polynomial with leading coefficient 1 and exactly 2 roots a and b can be written as
$x^2 - (a + b)x + (ab)$

A polynomial with leading coefficient 1 and exactly 3 roots a, b and c, it can be written as
$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - (abc)$

A polynomial with leading coefficient 1 and exactly 4 roots a, b, c and d, it can be written as
$x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - (abc + abd + acd + bcd)x + (abcd)$

In general, a polynomial with leading coefficient 1 and exactly n roots $a_1, a_2, a_3 \ldots a_n$ can be written as
$x^n -$ (sum of roots 1 at a time) x^{n-1} + (sum of roots taken 2 at a time) $x^{n-2} \ldots + (-1)^n$ (product of all n roots)

17. α, β and γ are the roots of the equation $x^3 - 7x^2 + 6x - 13 = 0$. Find the following:
 a) $\alpha + \beta + \gamma$ b) $\alpha\beta\gamma$ c) $\alpha\beta + \alpha\gamma + \beta\gamma$
 d) $\alpha^2 + \beta^2 + \gamma^2$ e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

18. Find the roots of $x^4 + 6x^3 + 11x^2 + 6x$

19. Find the roots of the equation $x^3 + 5x^2 + 2x - 8 = 0$ and find the equation whose roots are reciprocals of the roots of this equation.

Challengers

1. What is the remainder when $x^{2016} + x - 1$ is divided by $x - 1$?

2. Find the roots of:

   a) $x^4 - 2x^2 + 1$ b) $4x - 4\sqrt{x} + 1$ c) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 3$

3. If $p(x) = ax^2 + bx + c$ is a quadratic polynomial such that $p(1) = p(2)$ and $p(4) = 0$ then can you find the value of:

   a) $m \neq 4$ such that $p(m) = 0$

   b) $p(5)$

   c) $\frac{p(0)}{p(3)}$

4. Which of the following range best describes the value of $M = \frac{99^{99} - 98^{99}}{99^{98} + 98^{98}}$?

   1) $M \leq \frac{1}{2}$ 2) $\frac{1}{5} < M \leq \frac{1}{2}$ 3) $\frac{1}{2} < M \leq 1$ 4) $M > 1$

5. Which of the following is definitely a factor of $17^{2016} + 12^{2016} - 6^{2016} - 1^{2016}$?

   1) 198 2) 253 3) 121 4) 143

6. If $x^3 + \frac{1}{x^3} = 198$ then find the value of $x + \frac{1}{x}$.

   1) 5.5 2) 6 3) 6.25 4) 7

PRACTICE EXERCISE - 1

DIRECTIONS for questions 1 to 4: Choose the correct alternative.

1. If $x - y = 1$, evaluate $x^3 - y^3 - 3xy$.
   1) 0 2) 1 3) 2 4) $x^2 - y^2$

2. If $x = b + c$, $y = c - a$, $z = a - b$, find $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.
   1) $a + b + c$ 2) $4b^2$ 3) abc 4) $a^2 + b^2$

3. Find $x^3 + y^3 + z^3 - 3xyz$, when $x + y + z = 9$ and $xy + yz + zx = 11$.
   1) 384 2) 192 3) 432 4) 48

4. Find the remainder when $x^3 - 7x + 8$ is divided by $x - 3$.
   1) −8 2) 8 3) −2 4) 14

DIRECTIONS for questions 5 to 8: Solve as directed.

5. If $x + \frac{1}{x} = 3$, calculate the value of $x^4 + \frac{1}{x^4}$

6. If $3x + 4y = 40$ and $xy = 5$, find the value of $27x^3 + 64y^3$.

7. Match each expression to its factor:

i) $p^{12} + q^{12}$	a) $p^9 - q^9$
ii) $p^{35} + q^{35}$	b) $p^4 + q^4$
iii) $p^{10} - q^{10}$	c) $p^6 + q^6$
iv) $p^{36} - q^{36}$	d) $p^4 - q^4$
v) $p^{27} - q^{27}$	e) $p^7 + q^7$
vi) $p^{30} + q^{30}$	f) $p^5 + q^5$

8. p, q and r are the roots of the equation. $2x^3 - 7x^2 + 5x + 14 = 0$, then find the following:
   a) pqr b) $p + q + r$ c) $pq + pr + qr$
   d) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ e) $p^2 + q^2 + r^2$

DIRECTIONS for questions 9 to 17: Choose the correct alternative.

9. Find the $\left(a+\frac{1}{a}\right)^2 \times \left(a+\frac{1}{a}-3\right)^2 \times \left(a+\frac{1}{a}-1\right)^2$ given that $a^6 + \frac{1}{a^6} = -2$. $(\sqrt{a} > 0)$

   1) 2 2) $6(12-4\sqrt{3})$ 3) $36(7-4\sqrt{3})$ 4) None of these

10. What is the product of $(1 - x + x^2)(1 - x^2 + x^4)(1 + x + x^2)$?

    1) $1 + x^2 + x^4$ 2) $1 + x^3 + x^6$ 3) $1 + x^4 + x^8$ 4) $1 + x^5 + x^{10}$

11. Let a polynomial $P(k) = a_0k^4 + a_1k^3 + a_2k^2 + a_3k^2 + a_3k + a_4$ satisfy $P(0) = P(1) = P(2) = P(-1) = 0$ and $P(-2) = 12$. Then $P(3)$ equals:

    1) $\frac{1}{3}$ 2) $-\frac{1}{2}$ 3) 12 4) 2

12. y_1, y_2, y_3 ... are all distinct integers and represent the entire set of integers. P is a non negative number such that:

    $P = (x - y_1)(x - y_2)(x - y_3)$.... upto ∞, where x is an integer. Which of the following value can x take so that P has the least value?

    I. 0 II. 1 III. −1

    1) I only 2) II only 3) I and II only 4) any of I, II or III

13. $(19^{43} + 17^{43})$ is definitely divisible by which of the following:

    1) 48 2) 36 3) 72 4) 24

14. $54^3 - 33^3 - 21^3$ is atleast divisible by which of the following:

    1) 33 and 21 2) 54 and 33 3) 54 and 21 4) 21, 33 and 54

15. If $x - \sqrt{2}$ is a factor of $x^4 - x^2 - 2$, find the other factor.

    1) $x^3 + \sqrt{2}x^2 + x + \sqrt{2}$ 2) $x^3 + x + \sqrt{2}$

    3) $-\sqrt{2}x^2 - x - 2$ 4) $x^3 + x^2 - \sqrt{2}$

16. If $a^2 + b^2 - ab = bc + ca - c^2$, where a, b and c are non-zero real numbers, then find the value of $\frac{a^4+b^4+c^4}{a^3b+b^3c+c^3a}$.

    1) 0 2) 1 3) abc 4) $\frac{1}{abc}$

17. If the roots of the equation $x^3 + ax^2 - bx + c = 0$ are α, β and γ, while the roots of the equation $x^3 - dx^2 + ex - 72 = 0$ are $\alpha + 3$, $\beta + 3$ and $\gamma + 3$, what is the value of $9a + 3b + c$?

1) 45 2) 99 3) −45 4) −99

DIRECTIONS for questions 18 to 20: Solve as directed.

18. What is the remainder when $x + 2x^2 + 3x^3 + 4x^4 + ... + 100x^{100}$ is divided by $x + 1$?

19. How many distinct real values can y take in the equation $y(y + 3)(3y + 5)(3y + 14) - 54 = 0$?

20. Find the remainder when $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2 - 8^2 + 9^2 - 10^2 + 11^2$ is divided by 22

PRACTICE EXERCISE - 2

DIRECTIONS for question 1: Solve as directed.

1. Which of the following are roots of: $x^6 - x^5 - 15x^4 + 3x^3 + 38x^2 - 2x - 24$?

a) 1 b) −1
c) 2 d) −2
e) 3 f) −3
g) 4 h) −4
i) $\sqrt{2}$ j) $\sqrt{3}$

DIRECTIONS for questions 2 to 15: Choose the correct alternative.

2. Which of the following values of (x, y) satisfies $(20+x)^{\frac{1}{3}} + (20-y)^{\frac{1}{2}} = x + y - 4$.

1) (4, 7) 2) (44, 11) 3) (7, 4) 4) (–19, 19)

3. If $xy = p$, $xz = p^2$ and $yz = p^3$, also $x + y + z = 13$ and $x^2 + y^2 + z^2 = 91$, then $\frac{z}{y}$ is equal to:

1) 3 2) $\frac{7}{3}$ 3) 13 4) $\frac{13}{3}$

4. Solve for ‘a’ if ‘a’ is a positive rational number: $\left(a^4 - \frac{5^2}{12^2}\right)\left(a^4 - \frac{1^2}{12^2}\right) = -\frac{1}{162}$

1) $\frac{1}{2}$ 2) $\frac{1}{3}$ 3) $\frac{1}{4}$ 4) $\frac{1}{5}$

5. The difference between the squares of two positive numbers is 48 and the difference between the reciprocals of their squares is $\frac{3}{800}$. Find the product of the two numbers.

1. 160 2. $40\sqrt{3}$ 3. 80 4. $80\sqrt{2}$

6. What is the product of $x + 1 + \frac{1}{x}$, $x + \frac{1}{x} - 1$, $x^2 - 1 + \frac{1}{x^2}$ and $x^4 - 1 + \frac{1}{x^4}$?

1) $x^8 - \frac{1}{x^8} + 1$ 2) $\frac{x^8 + x^4}{x^8}$ 3) $x^8 - \frac{1}{x^8} - 1$ 4) $x^8 + \frac{1}{x^8} + 1$

7. $P = a^4 - ab$ and $Q = b^2 - a^4$
a and b are natural numbers and a is a factor of b ($a > 1$, $b > 1$). Find HCF of P and Q.

1) a^3 2) b^2 3) ab 4) a^2

8. If $a + \sqrt{b} = p + \sqrt{q}$, what is the relation between the expressions I and II, where a, b, p and q are integers such that none of them is a proper square of any integer?

I. $a(bq - p^2) + p(a^2 - bq)$ II. $b(ap - q^2) + q(b^2 - ap)$

1) I > II 2) I < II 3) I = II 4) Cannot be determined

9. If $R = \frac{40^{55} - 38^{55}}{40^{54} + 38^{54}}$, then

1) $0 < R \leq 1$ 2) $R > 2$ 3) $1 < R < 2$ 4) $R < 0$

10. One of the factors of $x^4 + 3x^3 - 2x^2 - 3x + 1 = 0$ is $\left(x - \frac{a + \sqrt{13}}{2}\right)$. Given that (x + 1) and (x − 1) are two of the factors, find a.

1) −1 2) 1 3) 0 4) 3

11. Let $p(n) = (3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n$. Then, find the ratio $\frac{[p(n)]}{p(n)}$ (for all n), where [x] greatest integer less than or equal to x.

1) 1
2) less than 1
3) greater than 1
4) Cannot be determined

12. The equations $x^3 - 5x^2 - 2x + 24 = 0$ and $x^3 + ax^2 - 7x - 6 = 0$ share two common roots. Find the value of a.

1) 0 2) 1 3) −2 4) 3

13. How many negative roots exist for the equation $(x^2 + 4)^2 + 20x^2 = 9x(x^2 + 4)$?

1) 0 2) 1 3) 2 4) 3

14. The GCD and LCM of three quadratic polynomials are (x + 3) and ($x^4 - 12x^3 + 29x^2 + 102x - 360$) respectively. Which of the following isn't one of the polynomials?

1) $x^2 - x - 12$ 2) $x^2 - 2x - 15$ 3) $x^2 - 3x - 18$ 4) $x^2 + x - 12$

15. If α, β and γ are the roots of the equation $x^3 + 6x^2 - 51x - 210 = 0$, find the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

1) $210x^3 + 6x^2 - 51x - 1 = 0$
2) $210x^3 + 51x^2 - 6x - 1 = 0$
3) $210x^3 + x^2 - 6x - 51 = 0$
4) $210x^3 + 6x^2 - x - 51 = 0$

DIRECTIONS for questions 16 to 20: Solve as directed.

16. If a and b are the remainders when $2x^4 + 9x^3 - 15x + 5 = 0$ is divided by (x − 1) and (x − 3) respectively; what would be the least number to be added to/subtracted from (a + b) so that the resulting number is a multiple of 63?

17. If α, β and γ are the roots of the equation $x^3 + 8x^2 - 15 = 0$, find the value of

$$\frac{1}{\alpha^2 - \beta\gamma} + \frac{1}{\beta^2 - \alpha\gamma} + \frac{1}{\gamma^2 - \alpha\beta}$$

18. What is the remainder when $1 + 10^2 + 10^4 + 10^6 + 10^8 + \ldots + 10^{100}$ is divided by 99?

19. Find the product of all the possible roots of the equation $4x^2 + 7x - 7x^{-1} + 4x^{-2} = 19$.

20. If 'a' and 'b' are the roots of the equation $x^2 - 5x + 3$ and 'c' and 'd' are the roots of the equation $x^2 - 3x + 1$, find the value of $a^4 - c^4 + b^4 - d^4$.

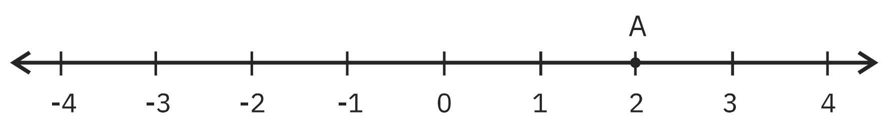
QA-2.3 CARTESIAN COORDINATE SYSTEM, EQUATION OF A LINE

THEORY

Cartesian Co-ordinate System

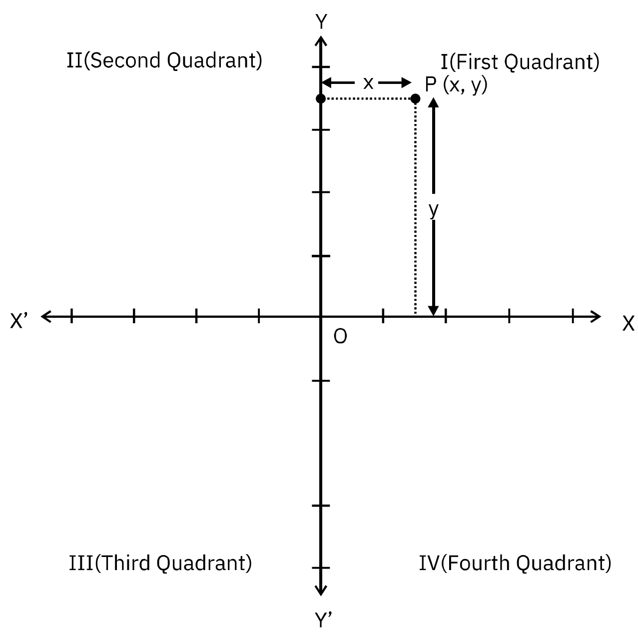
All real numbers can be represented on a line called number line. The number zero corresponds to the origin. The positive real numbers +x correspond to the points on the number line whose distance from the origin is x units towards the right of the origin. The negative real numbers –x correspond to the points on the number line whose distance from the origin is x units towards the left of the origin.

Co-ordinate Geometry or Cartesian Geometry is that branch of geometry in which two numbers i.e. the co-ordinates are used to indicate the position of a point in a plane. Co-ordinate geometry was introduced in 1637 by René Descartes.



The number associated with a point on the number line is termed as a co-ordinate of that point. Co-ordinate of A is 2.

Let X'OX and Y'OY be two coplanar number lines intersecting at the origin at right angles.



The horizontal line **X'OX is called the x-axis** and the vertical line **YOY'** is called the **y-axis**. 'O' the point of intersection is called the origin. On the x-axis, the positive numbers are to the right and negative numbers are to the left of the origin. On the y-axis, the positive numbers are above and negative numbers are below the origin.

The x and y axes divide the plane into four quadrants XOY, YOX', X'OY' and Y'OX referred to as the **first, second, third and fourth quadrant**, respectively. Any point in the plane is either in one of these quadrants or on the x or the y-axis.

Let P be a point in the first quadrant and PM be the perpendicular on the x-axis and PN be the perpendicular on the y-axis. The lengths PN and PM are called the rectangular cartesian co-ordinates or simply the co-ordinates of P.

The length PN is **called the abscissa or x co-ordinate** and is denoted by the letter x. The length **PM is called the ordinate** or y co-ordinate and is denoted by the letter y. The co-ordinates of a point are written in round brackets, the first number is the x co-ordinate and the second number is the y co-ordinate.

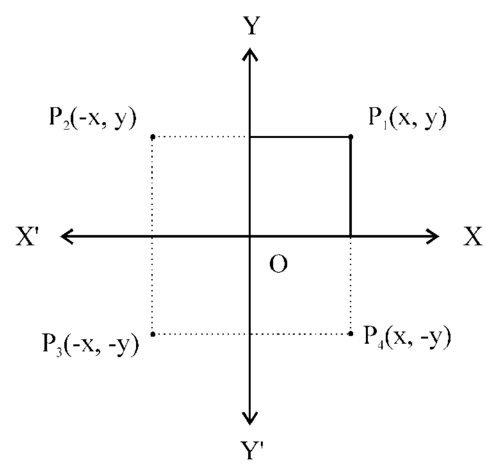
The x and y co-ordinates of a point in the quadrants are:

Quadrant	x-coordinate	y-coordinate	Coordinates
First Quadrant	+	+	(+, +)
Second Quadrant	–	+	(–, +)
Third Quadrant	–	–	(–, –)
Fourth Quadrant	+	–	(+, –)

The co-ordinates of the origin are (0, 0).
The x co-ordinate of every point on the y-axis is zero.
The y co-ordinate of every point on the x-axis is zero.

Reflection of a point

The reflection of a point $P_1(x, y)$ in the first quadrant in the y-axis is $P_2(–x, y)$ and in the x-axis is $P_4(x, –y)$. $P_3(–x, –y)$ is the double reflection of $P_1(x, y)$, first in the y-axis and then in the x-axis. $P_3(–x, –y)$ is also the reflection of $P_1(x, y)$ across the origin.



Distance formula

The distance between two points (x_1, y_1) and (x_2, y_2) = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Example

Find the value of 'a' if the distance between A(a, 2) and B(5, a) is $\sqrt{5}$ units.

$\ell(AB) = \sqrt{(5-a)^2 + (a-2)^2}$

$\sqrt{5} = \sqrt{25 - 10a + a^2 + a^2 - 4a + 4}$

$5 = 29 - 14a + 2a^2$

$2a^2 - 14a + 24 = 0$

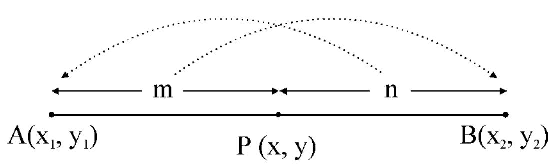
$a^2 - 7a + 12 = 0$

$(a - 3)(a - 4) = 0 \quad \therefore\ a = 3$ or $a = 4$

> **Note:** The distance of the point (x_1, y_1) from the origin (0, 0) = $\sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{x_1^2 + y_1^2}$

Section formula

Internal Division



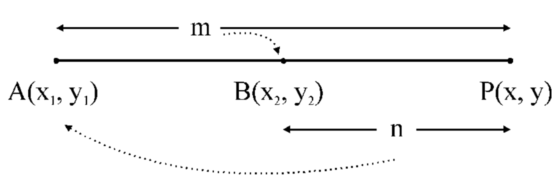
If point P divides the segment AB internally in the ratio m : n (i.e. PA : PB = m : n), where A ≡ (x_1, y_1) and B ≡ (x_2, y_2) then the co-ordinates (x, y) of P are given by:

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

External Division

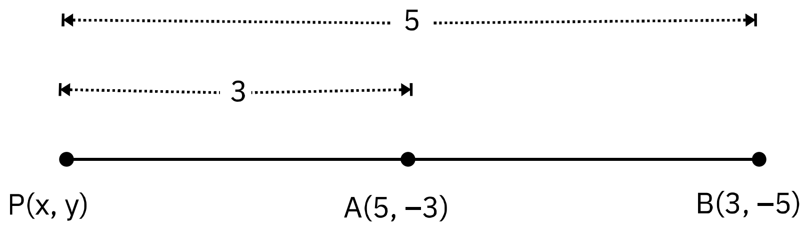
If P(x, y) divides the segment AB externally in the ratio m : n (i.e. PA : PB = m : n), where A ≡ (x_1, y_1) and B ≡ (x_2, y_2) and m > n then the co-ordinates (x, y) of P are given by

$$P(x, y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$



Example

Find the co-ordinates of the point which divides the segment AB where A ≡ (5, –3) and B ≡ (3, –5) externally in the ratio 3 : 5.



$$\frac{PA}{PB} = \frac{3}{5}$$

$$P(x,\ y) = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right) = \left(\frac{3 \times 3 - 5 \times (5)}{3 - 5}, \frac{3 \times (-5) - 5\,(-3)}{3 - 5}\right)$$

$$= \left(\frac{9 - 25}{-2}, \frac{-15 + 15}{-2}\right) = (8,\ 0)$$

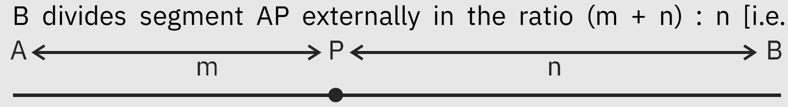
Note: External division with a positive ratio can be treated as an internal division with negative ratio.

Midpoint of a segment

If P is the midpoint of the segment joining the points A(x_1, y_1) and B(x_2, y_2), then the co-ordinates (x, y) of P are given by: $P(x,\ y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

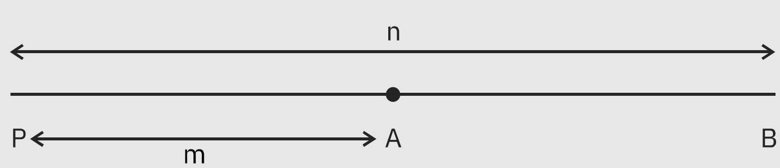
Note: In the section formula for internal division, if m = n, then the point is the midpoint of the segment.

1. If point P divides segment AB internally in the ratio of m : n (i.e. PA : PB = m : n), then B divides segment AP externally in the ratio (m + n) : n [i.e. BA : BP = (m + n) : n]



2. If point P divides segment AB externally in the ratio m : n (where m > n and PA : PB = m : n), then B divides segment PA internally in the ratio (m – n) : n [i.e. BA : BP = (m – n) : n]

3. If point P divides segment AB externally in the ratio m : n (where m < n) i.e. PA : PB = m : n, then A divides segment PB internally in the ratio m : (n – m) [i.e. AP : AB = m : (n – m)]



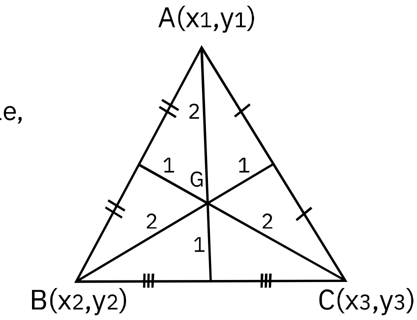
Application of co-ordinate geometry

Centroid of a Triangle

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle, then

The co-ordinates of the **centroid G(x, y)** of the ΔABC are:

$$G(x,y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$



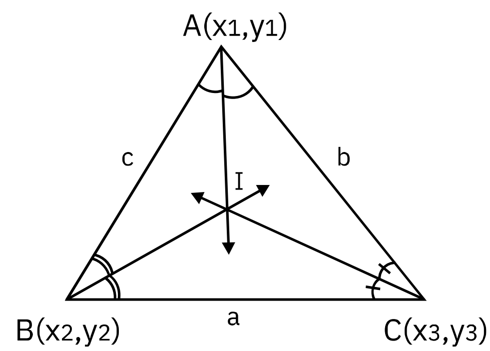
Note: The centroid is the point of intersection of the three medians of the triangle. The centroid divides each median in the ratio 2:1.

Incentre of a Triangle

The co-ordinates of the **incentre I(x, y)** of the ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$I(x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + by_3}{a + b + c}\right)$$

where, a, b and c are the lengths of three sides opposite to $\angle BAC$, $\angle ABC$ and $\angle ACB$.



Note: Incentre is the point of intersection of the three angle bisectors of the triangle.

Area of a triangle given co-ordinates of the three vertices

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the three vertices of a triangle then,

$$\text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\,|[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]|$$

Example

Find the area of the triangle whose vertices are (–1, 3), (2, 4) and (3,–2).

Here, $(x_1, y_1) = (-1, 3)$, $(x_2, y_2) = (2, 4)$, $(x_3, y_3) = (3, -2)$

$$\text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\,|[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]|$$

$$= \frac{1}{2}\,|[-1(4 - (-2)) - 2(3 - (-2)) + 3(3 - 4)]| = \frac{1}{2}\,|[-6 - 10 - 3]| = \frac{19}{2} = 9.5 \text{ sq.units.}$$

Some interesting results based on the above formulae

1. If the area = 0, then the 3 points are collinear.
2. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order 3.

   An array of 9 numbers is arranged in three rows and three columns.

   The value of the determinant is given by

   $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$
3. If one of the vertices of the triangle is at the origin and other two vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, then Area = $\left|\frac{x_1y_2 - x_2y_1}{2}\right|$

Conditions for a parallelogram

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ are four non-collinear points such that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$, then ☐ABCD is a parallelogram.

Conversely, if $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ form a parallelogram, then $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

Example

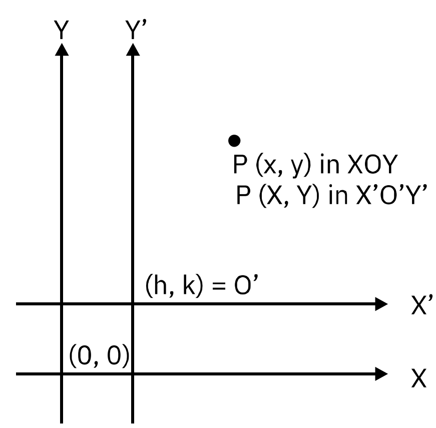
In a parallelogram ABCD, where A(1, 0), B(2, 5), C(1, 9), find the co-ordinates of D.
Let the co-ordinates of D be (x_4, y_4).
Since ☐ABCD is a parallelogram, $x_1 + x_3 = x_2 + x_4$
$\therefore\ 1 + 1 = 2 + x_4 \Rightarrow x_4 = 0$
Also, $y_1 + y_3 = y_2 + y_4$
$\therefore\ 0 + 9 = 5 + y_4 \Rightarrow y_4 = 4$
The co-ordinates of D are (0, 4).

Shift of Origin

Sometimes the origin (0, 0) is shifted to a point (h, k) to make the equation shorter. Then the old co-ordinates (x, y) of a point P will change to (X, Y), where X = x – h and Y = y – k.

XOY is the old frame of reference and X′O′Y′ is the new frame of reference.

The axes O′X′ and O′Y′ in the new frame of reference will be parallel to OX and OY in the old frame of reference.



Example

If the origin is shifted to (1, 2), the axes remaining parallel to the old axes, find the new co-ordinates of the point B(–3, –7).

Here, (h, k) = (1, 2)

$\therefore$ X = x − h = –3 − 1 = –4

Y = y − k = –7 − 2 = –9

$\therefore$ The new co-ordinates are (–4, –9)

Locus

A locus is a set of all points that satisfies a given geometrical condition. If a point moves according to some fixed rule, its co-ordinates will satisfy some corresponding algebraic relation and the path of the moving point is the locus of the point. If a point lies on a locus, its co-ordinates satisfy the equation.

Example

If the point (5, b) lies on the locus $x^3 + 3y^3 = 25x$, find b.

(5, b) lies on the locus and therefore should satisfy the equation.

$x^3 + 3y^3 = 25x$.

$(5)^3 + 3(b)^3 = 25 \times 5$

$3b^3 = 0 \quad \therefore\ b = 0$

Concept Builder 1

1. The point A(2, –3) is in the II^{nd} quadrant (T/F)
2. Find the distance between the points A(18, 3) and B (6, 8). Also find distance of point B(6, 8) from the origin.
3. Find the point P which divides the segment AB where A ≡ (2, 5) and B ≡ (4, 2) in the ratio 2 : 1 internally.
4. If point P divides segment QR internally in the ratio 3 : 2, then what is the ratio in which R divides segment QP externally?
5. Find the centroid of a triangle whose vertices are (6, 4), (3, 5) and (2, 8)
6. Find the area of the triangle whose vertices are (1, 3), (–7, 6) and (5, –1)
7. What will be the reflection of the point (3, 5) in the

   a) second quadrant b) third quadrant
8. If the origin gets shifted to (3, 2), then what will be the new coordinates of the point (6, –3)
9. The points A (4, 5), B(1, 1) and C(7, 9) are collinear (T/F).

Answer Key

1.	False	2.	13 units, 10 units	3.	$\left(\frac{10}{3}, 3\right)$ 4. 5 : 2
5.	$\left(\frac{11}{3}, \frac{17}{3}\right)$	6.	10 sq.units	7.	a) (–3, 5) b) (–3, –5)
8.	(3, –5)	9.	True		

Equation of a line

Any straight line on the co-ordinate plane can be described by the equation:

$y = mx + c$

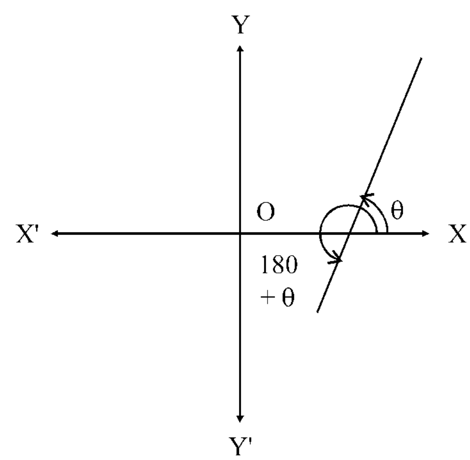
where: x, y are co-ordinates of a point on the line
m is the slope of the line
c is the y-intercept (i.e., where the line crosses the y-axis).

Slope (Gradient) of a line

The slope of a line is the ratio of rate of change of y to rate of change of x. It is denoted by 'm'

Slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Inclination of a line



If a straight line intersects the x-axis, the inclination of the line is defined as the measure of the angle, which the x-axis makes with the line in the anticlockwise direction.

If a straight line is parallel to the x-axis, its inclination is defined to be zero.

The slope of a line having inclination θ and not perpendicular to the x-axis is defined to be tan θ i.e. **$m = \tan\theta$.**

The slope of a line perpendicular to x-axis is not defined, as tan 90° is not defined. The slope of a line parallel to x-axis is zero as tan 0 = 0. Hence, for a line parallel to x-axis, m = 0 and for a line parallel to y-axis, m does not exist or vice versa.

If θ is acute, the slope is positive.

If θ is obtuse, the slope is negative.

Parallel and Perpendicular Lines

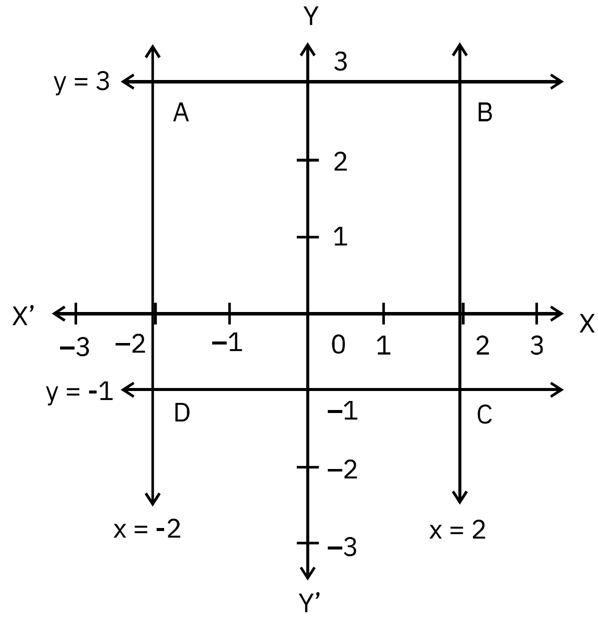
1. Two lines whose slopes are m_1 and m_2 are parallel to each other, if and only if $m_1 = m_2$, or both m_1 and m_2 do not exist.
2. Two lines whose slopes are m_1 and m_2 are perpendicular to each other if and only if either $m_1 \times m_2 = -1$ or if $m_1 = 0$ and m_2 does not exist.
   Thus, if the slope of a line is m, the slope of a line perpendicular to it is $-\frac{1}{m}$.

Equation of lines parallel to the axes

1. The equation of a line parallel to the x-axis and at a distance of k units above it is $y = k$.
   $y = k$ is also called the y intercept.

   Example

   $y = 3$ is a line passing through (0, 3) and parallel to the x-axis.



Note: Equation of x-axis is $y = 0$

2. The equation of a line parallel to the x-axis and at a distance of k units below it is $y = -k$.
   $y = -k$ is also called the y intercept.

   Example

   $y = -1$ is a line passing through (0, −1) and parallel to the x-axis.

3. The equation of a line parallel to the y-axis and at a distance of k units to the right of it is $x = k$. $x = k$ is also called the x intercept.

   Example

   $x = 2$ is a line passing through (2, 0) and parallel to the y-axis.

Note: Equation of y-axis is $x = 0$

4. The equation of a line parallel to the y-axis and at a distance of k units to the left of it is $x = -k$. $x = -k$ is also called the x intercept.

   Example

   $x = -2$ is a line passing through (−2, 0) and parallel to the y-axis.

Standard forms for the equation of a line

1. **The Point Slope form**: The equation of a straight line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.
2. **The Two Point form**: The equation of a straight line passing through the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$ is $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

   Example

   The point (a, –4) lies on the line joining the points (6, 2) and (12, 5). Find a.

   Equation of the line is $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

   $\frac{y-5}{x-12} = \frac{5-2}{12-6}$; $x - 2y = 2$

   (a, –4) satisfies this equation $\therefore a - 2 \times (-4) = 2$

   $a = 2 - 8 = -6$.
3. **Slope Intercept form**: The equation of a line having slope m and making an intercept c on y-axis is $y = mx + c$.

   Example

   If the equation of a line is $3x + 4y = 12$, then find the y-intercept and slope of the line.

   Converting equation to Slope-intercept form $y = mx + c$, $y = \frac{-3}{4}x + 3$

   Hence, $m = \frac{-3}{4}$, anc $c = 3$
4. **Double Intercept form**: The equation of a line making intercepts a and b, when $a \neq 0$ and $b \neq 0$ on the x and y axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.

   Example

   If the equation of a line is $7x + 3y = 21$, then find the x-intercept and the y-intercept of the line.

   Converting $7x + 3y = 21$ to the double intercept form i.e., $\frac{x}{a} + \frac{y}{b} = 1$

   $\Rightarrow \frac{x}{3} + \frac{y}{7} = 1$

   Here, $a = 3$ $b = 7 \Rightarrow$ x-intercept = 3, y-intercept = 7

Note: The x intercept of a line can be obtained by substituting $y = 0$ in the equation of the line. Similarly, the y intercept can be obtained by substituting $x = 0$ in the equation of the line.

5. **The Normal form**: If the perpendicular drawn from the origin to a line has inclination α and length p, then the equation of the line is $x \cos \alpha + y \sin \alpha = p$.

6. **The Symmetric form**: The equation of a line passing through a point (x_1, y_1) and making an angle of θ with x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$

7. **The General form**: All the different forms of equations of a straight line are nothing but linear equations of the first degree in x and y. Hence, the most general form of an equation is of the form $Ax + By + C = 0$, where A, B and C are real numbers.
Conversely, $Ax + By + C = 0$, always represents a straight line.
$Ax + By + K = 0$ represents a line parallel to $Ax + By + C = 0$ and $Bx - Ay + K = 0$ represents a line perpendicular to $Ax + By + C = 0$.
Slope $m = -\frac{A}{B} = \frac{-\text{Coefficient of x}}{\text{Coefficient of y}}$ x intercept $= -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of x}}$
y intercept $= -\frac{C}{B} = \frac{-\text{Constant term}}{\text{Coefficient of y}}$

Point of intersection of two lines

If $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are two straight intersecting lines, then co-ordinates of the point of intersection can be obtained by solving the equations simultaneously for x and y.

Angle between two lines

If m_1 and m_2 are the slopes of two lines such that $m_1 \times m_2 \neq 1$, then $\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ where θ is the acute angle between the two lines.
Also, if the equation of the two lines are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ then,

$\tan \theta = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}$

Example

Find the equation of the straight lines passing through the point (2, 3) and inclined at an angle of 45° to the line $3x + y - 5 = 0$.

Slope (m_2) of the given line is $-\frac{3}{1}$

If m_1 is the slope of the required line, $\tan 45° = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

i.e., $1 = \left|\frac{m_1 + 3}{1 - 3m_1}\right|$ $\therefore \left|\frac{m_1 + 3}{1 - 3m_1}\right| = 1$ or $\left|\frac{m_1 + 3}{1 - 3m_1}\right| = -1$; $\therefore m_1 = -\frac{1}{2}$ or 2.

(2, 3) is a point on the line.

So, the equation of the line is $y - 3 = -\frac{1}{2}(x - 2)$ or $y - 3 = 2(x - 2)$
i.e. $x + 2y - 8 = 0$ or $2x - y - 1 = 0$

Perpendicular distance of origin from a line

The perpendicular distance of origin from a line Ax + By + C = 0, is given by $\left|\frac{C}{\sqrt{A^2+B^2}}\right|$

Example

Find the shortest distance of origin from the line 3x + 4y + 10 = 0.

Perpendicular distance = $\left|\frac{C}{\sqrt{A^2+B^2}}\right| = \left|\frac{10}{\sqrt{3^2+4^2}}\right| = \frac{10}{5} = 2$

Perpendicular distance of a point from a line

If $P(x_1, y_1)$ is any point and Ax + By + C = 0 is a line, then the perpendicular distance of P from the line is given by $\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$.

If $Ax_1 + By_1 + C$ is positive, the point $P(x_1, y_1)$ lies on the origin side of the line and if $Ax_1 + By_1 + C$ is negative, then $P(x_1, y_1)$ lies on the non-origin side of the line.

Perpendicular distance between two parallel lines

$Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ are two parallel lines, and the perpendicular distance between them is given by $\left|\frac{C_1-C_2}{\sqrt{A^2+B^2}}\right|$.

Example

Find the distance between the parallel lines 3x + 4y − 10 = 0 and 6x + 8y + 15 = 0.

3x + 4y − 10 = 0 ...(i)

6x + 8y + 15 = 0; i.e., 3x + 4y + $\frac{15}{2}$ = 0 ...(ii)

...(making coefficients of x and y equal)

$\therefore$ Distance between the parallel lines = $\left|\frac{C_1-C_2}{\sqrt{A^2+B^2}}\right|$

$$= \left|\frac{\frac{15}{2}-(-10)}{\sqrt{3^2+4^2}}\right| = \left|\frac{\frac{15}{2}+10}{5}\right| = \frac{35}{10} = 3.5 \text{ units}$$

Condition for concurrency of three straight lines

If $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ and $A_3x + B_3y + C_3 = 0$ are three straight lines, then the three lines will be concurrent (i.e., intersect at a point) if $A_3(B_1C_2 - B_2C_1) + B_3(C_1A_2 - C_2A_1) + C_3(A_1B_2 - A_2B_1) = 0$.

THEORY

PRACTICE EXERCISE CLASS EXERCISE

Family of lines

If $u \equiv A_1x + B_1y + C = 0$ and $v \equiv A_2x + B_2y + C = 0$ are two intersecting lines, then $u + kv = 0$, where k is any constant, represents a straight line through the point of intersection of $u = 0$ and $v = 0$.

Equation of a circle

Let P(x, y) be any point on the circle whose centre is C(h, k) and radius is r. The equation of the circle is given by $\sqrt{(x-h)^2 + (y-k)^2}$ **= r**. This form is called the centre radius form.

Equation of a circle with centre as origin is $\sqrt{x^2 + y^2}$ **= r**

The general form of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$.

Note: The ratio of the distance of a point from a fixed point and a fixed straight line is called the eccentricity and denoted by 'e'.

Concept Builder 2

1. Find the equation of the line parallel to x-axis and 2 units below it
2. Find the equation of the line passing through (4, 5) and having slope = 3
3. Find the following for the equation $6x - 5y = 30$:
   a) x-intercept
   b) y-intercept
   c) slope
4. Find the angle between line $2x + 3y + 6 = 0$ and $6x + 9y + 7 = 0$
5. Find the shortest distance of origin from the line $12x + 9y + 5 = 0$
6. Find the distance between $3x + 4y + 7 = 0$ and $6x + 8y + 10 = 0$
7. Find the perpendicular distance of A(2, 2) from the line $15x + 8x + 5 = 0$.

Answer Key

1. $y = -2$ 2. $3x - y - 7 = 0$ 3. a) +5 b) −6 c) $\frac{6}{5}$

4. Parallel Lines (0°)   5. $\frac{1}{3}$ units   6. $\frac{2}{5}$ units   7. 3 units

Solved Examples

Q : Find the ratio in which x-axis divides the segment joining A(–3, –2) and B(4, –4). Also find the point of division. Is the division internal or external?

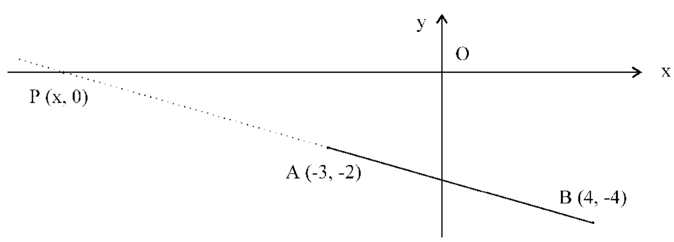
A : If P(x, 0) is the point of division and if the ratio is k : 1, then,

$$x = \frac{k \times 4 + 1(-3)}{k+1}; \qquad 0 = \frac{k(-4) + 1(-2)}{k+1}$$

$$x = \frac{4k-3}{k+1};\ 0 = \frac{-4k-2}{k+1}$$

$\therefore\ -4k - 2 = 0 \quad \therefore\ k = -\frac{1}{2}$

So the ratio is $k : 1 = -\frac{1}{2} : 1$ i.e., –1 : 2 i.e. 1 : –2



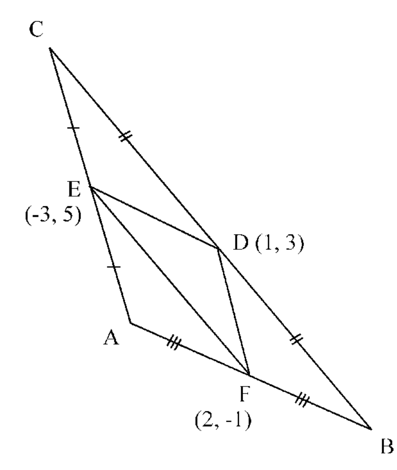
The negative sign implies that the division is external. Hence, the ratio is 1 : 2.

$$x = \frac{4\left(-\frac{1}{2}\right) - 3}{-\frac{1}{2} + 1} = \frac{-5}{\frac{1}{2}} = -10$$

$\therefore$ The point of division is (–10, 0).

Q : The midpoints of the sides of ΔABC are shown in the figure. Find the co-ordinates of B and A.

A : □BDEF is a parallelogram ...(midpoint theorem)



If the co-ordinates of B are (a, b)

$a - 3 = 1 + 2 \quad \therefore\ a = 6$

$b + 5 = 3 - 1 \quad \therefore\ b = -3$

$\therefore\ B \equiv (6, -3)$

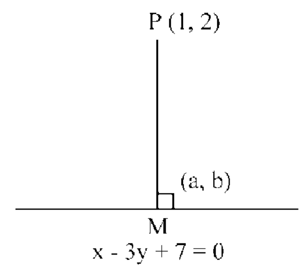
Similarly, AFDE is a parallelogram.

If the co-ordinates of A are (x, y)

$x + 1 = 2 - 3 \quad \therefore\ x = -2$

$y + 3 = -1 + 5 \quad \therefore\ y = 1 \qquad \therefore\ A \equiv (-2, 1)$

Q : Find the co-ordinates of the foot of the perpendicular from the point (1, 2) on the line x − 3y + 7 = 0.



A : If the foot of the perpendicular is (a, b) it satisfies the equation.

$\therefore$ a − 3b + 7 = 0 ...(i)

Slope of the given line is $-\left(-\frac{1}{3}\right) = \frac{1}{3}$

Slope of PM = $-\frac{3}{1}$; Also, slope of PM = $\frac{2-b}{1-a}$

$\therefore \frac{2-b}{1-a} = -\frac{3}{1}$; $\therefore$ 2 − b = −3 + 3a; $\therefore$ 3a + b = 5 ...(ii)

Solving (i) and (ii), a = $\frac{4}{5}$ and b = $\frac{13}{5}$ $\therefore$ Foot of the perpendicular is $\left(\frac{4}{5}, \frac{13}{5}\right)$.

Q : Find k if the points A(−1, 4); B(2, 5); C(3, k) are collinear.

A : If the points are collinear, Slope of AB = Slope of BC ...(they are one and the same line)

$\therefore \frac{4-5}{-1-2} = \frac{5-k}{2-3}$; $\therefore \frac{1}{3} = \frac{5-k}{-1}$; $\therefore$ k = $\frac{16}{3}$

Alternatively,

The area of the triangle formed by the points A, B and C should be zero, i.e.,

$$\frac{1}{2}\begin{vmatrix} -1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & k & 1 \end{vmatrix} = 0 \Rightarrow k = -\frac{16}{3}$$

Q : Find the equation of the line parallel to the line passing through (5, 7) and (2, 3) and having x intercept −4.

A : Slope of the given line = $\frac{7-3}{5-2} = \frac{4}{3}$

$\therefore$ The slope of the required line is also = $\frac{4}{3}$

One point on this line is (–4, 0).

$\therefore$ Equation of the line is y − 0 = $\frac{4}{3}$(x + 4); $\therefore$ 3y = 4x + 16

Q : Find the equation of the locus of a point P equidistant from the points A(–3, 2) and B(5,4).

A : Let the co-ordinates of P be (x, y). PA = PB

$\therefore \sqrt{(x+3)^2 + (y-2)^2} = \sqrt{(x-5)^2 + (y-4)^2}$

Squaring and simplifying, 16x + 4y = 28, i.e., 4x + y = 7

This is the equation of the locus of the point P.

Q : Find the acute angle between the lines $x\sqrt{3} + y = 7$ and $x - y\sqrt{3} + 5 = 0$.

A : Slope of the first line, $m_1 = -\sqrt{3}$

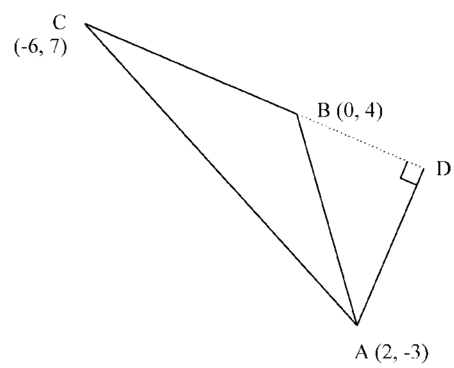
Slope of the second line, $m_2 = \frac{1}{\sqrt{3}}$

$\therefore\ m_1m_2 = (-\sqrt{3})\left(\frac{1}{\sqrt{3}}\right) = -1$

$\therefore$ The lines are perpendicular

$\therefore$ Measure of the angle between them = 90°.

Q : In ΔABC, A(2, –3), B(0, 4), C(–6, 7). Find the equation of altitude AD.



A : Slope of BC = $\frac{7-4}{-6-0} = -\frac{1}{2}$

$\therefore$ Slope of AD = $+\frac{2}{1}$ ...(as $m_1m_2 = -1$)

One point on AD is (2, –3)

$\therefore$ Equation of AD is $y + 3 = 2(x - 2)$

$\therefore\ 2x - y = 7$.

Q : What is the x intercept made by the line whose gradient is $\frac{2}{3}$ and y intercept is –6?

A : Here, $m = \frac{2}{3}$ and C = –6.

Hence the equation of the line is $y = \frac{2}{3}x - 6$

To find the x intercept, put y = 0

$\therefore\ 0 = \frac{2}{3}x - 6\ \therefore\ x = 9$

Q : One end of a diameter of a circle with centre C(–2, 0) is A(4, 4). Find the co-ordinates of the other end of the diameter.

A : Let B(x, y) be the other end of the diameter.

C is the midpoint of AB.

$-2 = \frac{x+4}{2}$ and $0 = \frac{y+4}{2}$

$-4 = x + 4$ and $0 = y + 4$

$x = -8$ and $y = -4$

$\therefore$ B(–8, –4) is the other end of the diameter.

CLASS EXERCISE

Teaser

Jayant is visiting Pune city and wants to visit several old acquaintances.

- Starting from his hotel, he drives ten km north to meet Jagdish.
- He then goes 4 km west to Jigar's house.
- After that he moves 3 km north to Jayashree's place, and then 6 km east to Jalan's home.
- He finally drives 13 km south to visit Jenny.

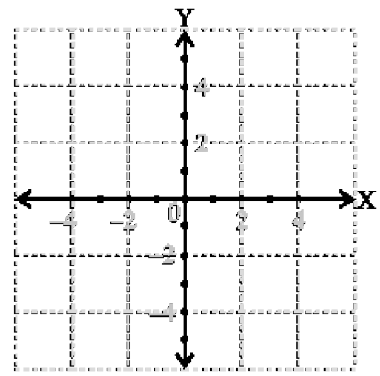
How far must he go to get back to his hotel, and in which direction?

The Cartesian Coordinate System

Consider the following points:

A (4, 3) B (–4, –3) C (–2, –6) D (2, –3) O (0, 0)

1) Plot the above points on the given coordinate plane:



2) What is the distance of
   (a) point A from point O?
   (b) point B from point D?
   (c) point C from point D?
   (d) point A from point D?

3) Find the coordinates of the mid-point of segment BD.

4) Find the coordinates of the mid-point of segment CD.

5) Find the coordinates of the point that divides segment AB internally in the ratio 3:1

6) Find the coordinates of the point that divides segment AB externally in the ratio 3:2

Distance Formula: Distance between two points (x_1, y_1) and (x_2, y_2) is given by: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Midpoint Formula: The midpoint (x, y) of the line segment joining two points (x_1, y_1) and (x_2, y_2) is given by: $x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}$

Section Formula (Internal): The point (x, y) dividing the line segment joining two points (x_1, y_1) and (x_2, y_2) internally in the ratio m:n is given by: $x = \frac{mx_2 + nx_1}{m + n}, y = \frac{my_2 + ny_1}{m + n}$

Section Formula (External): The point (x, y) dividing the line segment joining two points (x_1, y_1) and (x_2, y_2) externally in the ratio m:n (where $m \neq n$) is given by: $x = \frac{mx_2 - nx_1}{m - n}, y = \frac{my_2 - ny_1}{m - n}$

7) * Find the distance between the midpoints of the segments AB and CD

8) * Find the distance between the midpoints of the segments BD and AC

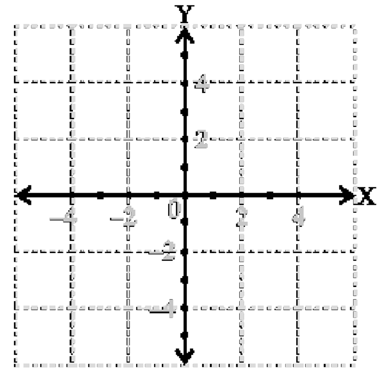
9) * Find the coordinates of the point which divides segment AO externally in the ratio 5:4

10) * Given P (21, 13) and Q (9, 19), find the point which divides PQ internally in the ratio 1:5

11) * Given R (7, –6) and S (–3, 8) find the point which divides segment RS externally in the ratio 3:2

Plotting Linear Equations

12) Consider the equation $y = -2x + 1$



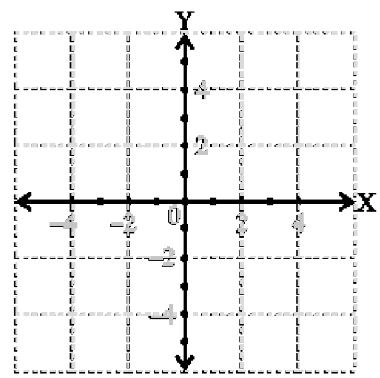
a) Given x, find y:

x	−3	−2	−1	0	1	2	3
y							

b) Plot the points from the table above on the given coordinate axes

c) Find the change in y for a unit change in x

13) Consider the equation $2y = x - 3$



a) Given x, find y:

x	−3	−2	−1	0	1	2	3
y							

b) Plot the points from the table above on the given coordinate axes

c) Find the change in y for a unit change in x

14) Find the solution for the above two equations $y = -2x + 1$ and $2y = x - 3$.

15) Find the slope of a line segment joining:

a) (3, 4) and (7, 6)

b) (−5, 3) and (−2, −6)

c) (4, 3) and (4, −2)

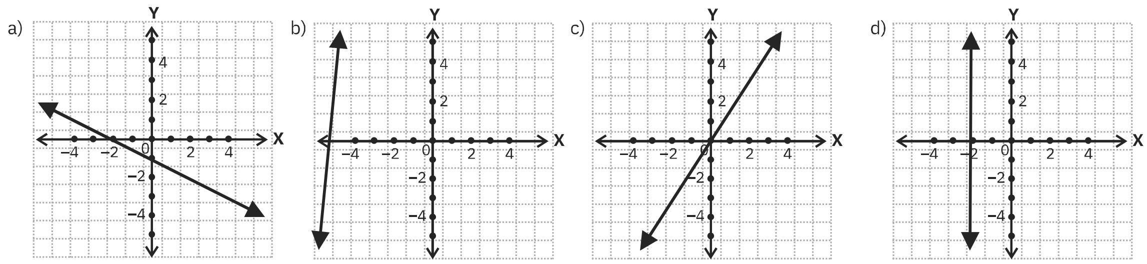
Slope of a line (m) is defined as the change in y for a unit change in x

Inclination of a line (θ) is defined as the angle between the line and the positive direction of the X-axis

Slope of a line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $m = \frac{y_2 - y_1}{x_2 - x_2}$

Slope in terms of inclination is given by $m = \tan\theta$

16) Identify which of the following lines have (i) a positive slope and (ii) a positive y-intercept



17) Find a unique line, if it exists, having the given properties:

a) Passing through the point (7, −2)

b) Passing through (2, 3) and (3, 1)

c) Passing through (5, 7) with slope 2

d) With slope 3 and y-intercept −2

e) Parallel to the X-axis and two units from the origin

f) With x intercept −4 and slope −1

g) Cutting the X and Y axes at 3 and 2 respectively

h) Through (–2, 3) and parallel to the Y-axis

i) Through (2, 5), (8, 3) and (–7, 8)

Point-Slope Form: A line with slope m through P(x_1, y_1)	:	$y - y_1 = m(x - x_1)$
Two Point Form: A line through P(x_1, y_1) and Q(x_2, y_2)	:	$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$
Slope-Intercept Form: A line with slope m and y-intercept c	:	$y = mx + c$
Two Intercept Form: A line with X-intercept ‘a’ and Y-intercept ‘b’	:	$\frac{x}{a} + \frac{y}{b} = 1$ (where a, b $\neq$ 0)
Standard form: The equation of a line can be written in the form	:	$ax + by + c = 0$
A line parallel to X – axis will always be of the form	:	y = constant
A line parallel to Y – axis will always be of the form	:	x = constant
A line passing through the origin will always be of the form	:	$ax + by = 0$

18) * Find the equation of a line:

a) Passing through (7, −4) and the origin

b) Passing through (3, 4) and parallel to the y-axis

c) Through the origin with a slope of 0

d) Having X-intercept = Y-intercept = 3

e) With slope −2 and lying entirely in the 2nd and 4th quadrants

19) * Find whether the following sets of points are collinear:

a) A (11, 2) B (14, 7) C (–1, –18)

b) P (5, 3) Q (0, 0) R (–7.5, –5)

c) W (4,5) X (13, 14) Y (–2, –3) Z (–8, –9)

d) L (5, 3) M (0, 1) N (–4, 3)

20) * For each of the following lines, identify the slope, y-intercept and x-intercept:

	Equation	Slope	Y intercept	X intercept
1	$3x + 7y - 42 = 0$			
2	$4x - 2.5y = 10$			
3	$y = 2.25x - 9$			
4	$x = 1.6y + 4$			
5	$3y = 21 - 3.5x$			

21) Match the following:

1	A line parallel to $3x + 4y = 7$	a	$x = y - 1$
2	A line through the origin and perpendicular to $2x - 3y + 4 = 0$	b	$y = 1.5x - 17$
3	A line though (1,1) parallel to the X-axis	c	$x + 3 = 0$
4	A line through (2,3) parallel to $x = y$	d	$3x = 0$
5	A line perpendicular to the x-axis through the origin	e	$y = -0.75x - 2$
6	A line perpendicular to $3x - 7y + 9 = 0$	f	$31 - 7x - 3y = 0$
7	A vertical line through (–3, 1)	g	$y - 1 = 0$
8	A line parallel to $3x = 2y$	h	$3x + 2y = 0$

If two lines with slopes m_1 and m_2 are parallel then	$m_1 = m_2$
If two lines with slopes m_1 and m_2 are perpendicular, then	$m_1m_2 = -1$
If two lines $ax + by + c = 0$ and $dx + ey + f = 0$ are parallel then	$\frac{a}{d} = \frac{b}{e}$
If two lines $ax + by + c = 0$ and $dx + ey + f = 0$ are perpendicular then	$ad = -be$ or $ad + be = 0$

22) * Find the distance of the line $4x - 3y + 10 = 0$ from the origin

23) * Find the distance of the point (5, 3) from the line $12x + 5y + 3 = 0$

24) * Find the distance between the two lines $x + 2y - 3 = 0$ and $x + 2y + 5 = 0$

The perpendicular distance of a point (x_1, y_1) from a line $ax + by + c = 0$ is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Corollary: The perpendicular distance of the origin from a line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$

The perpendicular distance between two parallel lines $ax + by + c_1$ and $ax + by + c_2$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

25) * Consider the triangle ABC formed by the points A (5, 3), B (8, 4) and C (5,11).

a) Find the centroid of ΔABC

b) Find the area of ΔABC

Given three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$:

The centroid of the triangle ABC is given by $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

The area of the triangle ABC formed by the given points is $\left|\frac{x_1(y_2 - y_3) - x_2(y_3 - y_1) + x_3(y_1 - y_2)}{2}\right|$

If the lengths of the sides opposite to A, B and C are a, b and c respectively, then the incentre of the triangle ABC is given by: $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$

26) Consider the circle given by $(x - 3)^2 + (y + 4)^2 = 25$

a) What is the centre of the circle?

b) What is the radius of the circle?

27) Consider the circle $x^2 + y^2 + 4x - 6y - 36 = 0$

a) What is the centre of the circle?

b) What is the radius of the circle?

A circle can be written in the form $(x - a)^2 + (y - b)^2 = r^2$ where the centre is (a, b) and the radius is r

The standard form of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Here, centre = $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c}$

Challengers

1) Which of the following attributes will guarantee a unique line?

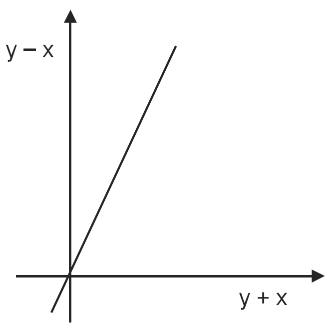
   a) Passing through (2, 2) and at a minimum distance of 1 unit from the origin?

   b) Through a given x-intercept and a given y-intercept?

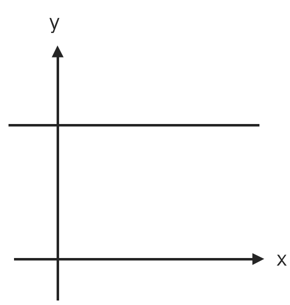
   c) Intersecting the X-axis at a given point and not touching the Y-axis?

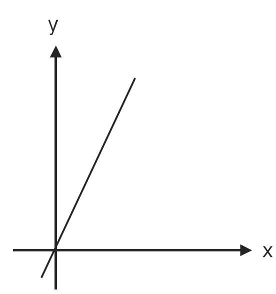
   d) With a slope of 1 and at a perpendicular distance of 1 unit from the point (4, 4)

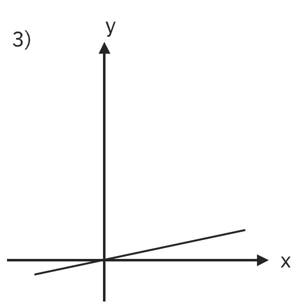
2) The graph of $y - x$ against $y + x$ is as shown below. (All graphs in this question are drawn to scale and the same scale has been used on each axis.)

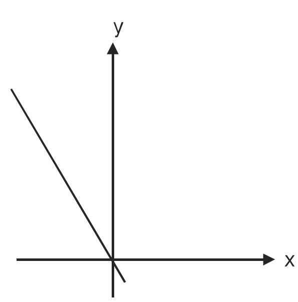


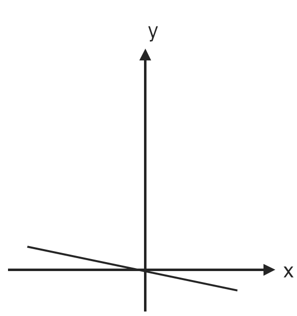
Which of the following shows the graph of y against x?

1)


2)


3)


4)


5)


3. Two ants Amar and Akbar are standing at the co-ordinates (3,7) and (9,1) respectively of a coordinate plane. A third ant Anthony is standing at a point (x, 0) on the X-axis. Both Amar and Akbar start crawling in a straight line towards Anthony at the same constant speed and reach him simultaneously. What is the value of x?

   1) 2 2) −2 3) 6 4) 3.25

4. Two ants Amar and Akbar are standing at the co-ordinates (3,7) and (9,1) respectively of a coordinate plane. Amar wants to visit the X-axis and then meet Akbar. What is the minimum distance he has to crawl to do this?

   1) $7 + \sqrt{37}$ 2) $1 + \sqrt{85}$ 3) 10 4) $2\sqrt{6}$

PRACTICE EXERCISE

DIRECTIONS for questions 1 and 2: Solve as directed:

1. a) What would be the equation of a line parallel to $4x + 7y = 8$ and passing through the origin?

   b) What would be the equation of a line perpendicular to $4x + 7y = 8$ and passing through the origin?

   c) What would be the equation of a line parallel to $4x + 7y = 8$ and passing though (2, 1)?

   d) What would be the equation of a line perpendicular to $4x + 7y = 8$ and passing through (2, 1)?

2. Match the following (More than one answer may be correct)

1	A line through (1,2) and (4,3)	a	$x - 3y - 17 = 0$
2	A line with slope −2, through the origin	b	$x - 1 = 0$
3	A line through (3, 5) with equal y- and x-intercepts	c	$15 - y = 2x$
4	A line perpendicular to $x - y = 4$	d	$5x - 8y = 20$
5	A line perpendicular to $y + 3x = 7$	e	$x + 17 = -y$
6	A line through (7, 5) with y intercept − 2	f	$2x + y = 0$
7	A line cutting the X-axis at 4 and the Y-axis at −2.5	g	$1 - y = 0$
8	A line passing through (1,1) and at a distance of 1 unit from the origin	h	$y = \frac{x+5}{3}$
9	A line perpendicular to $2x - 4y - 9 = 0$	i	$y = x - 2$
10	A line with its y-intercept twice its x-intercept	j	$x = 8 - y$

DIRECTIONS for questions 3 to 11: Choose the correct alternative.

3. In ΔABC, the co-ordinates of A, B and C are (4, 3); (6, −2); (k, −3) respectively and the triangle is right angled at A. Find k.

   1) 11 2) −11 3) $\frac{8}{5}$ 4) $-\frac{5}{8}$

4. Find the area of the triangle ABC whose vertices are A(4, 8); B(6, 0) and C(−2, 6).

   1) 13 sq. units 2) 39 sq. units 3) 26 sq. units 4) 20 sq. units

5. If P divides segment AB internally in the ratio 2 : 3 and if the co-ordinates of A and B are (–7, –2) and (5, 3) respectively. Find the co-ordinates of P.

1) $\left(-\frac{11}{5}, 0\right)$ 2) $\left(0, -\frac{11}{5}\right)$ 3) $\left(\frac{5}{11}, 0\right)$ 4) $\left(0, -\frac{5}{11}\right)$

6. What will be the equation of line cutting lines 2x + y = 4 and x + 2y = 4 at same point and parallel to line x + y = 0?

1) x + 3y = 8
2) 3x + y = 8
3) 3x + 3y + 8 = 0
4) 3x + 3y = 8

7. Find the coordinates of the centroid of a triangle formed by joining the mid-points of the sides of a triangle whose vertices are at (4, –3), (5, 0) and (–12, 6).

1) (–1, –1) 2) (1, 1) 3) (–1, 1) 4) (1, –1)

8. Find the value of k such that the line (4x + 6y + 7) + K(3x – 2y + 17) = 0 is parallel to y axis.

1) k = 3 2) k = 5 3) $k = \frac{-4}{3}$ 4) k = 0

9. At what point on line x – y = 3 does a perpendicular drawn from the line x – y = 1 at point (3, 2) intersect?

1) (4, 1) 2) (1, 4) 3) (1, 3) 4) (3, 1)

10. Find the equation of the straight line passing through the point (3, 4) and inclined at an angle of 45^o to the line 4x + y – 6 = 0.

1) 5y + 3x = 29
2) 3y – 5x + 3 = 0
3) Either (1) or (2)
4) Neither (1) nor (2)

11. In parallelogram ABCD, the co-ordinates of A, B and C are (1, 0); (4, 5); (1, 2) respectively. Find the co-ordinates of D.

1) (4, 3) 2) (4, 7) 3) (–2, –3) 4) (–2, 3)

DIRECTIONS for question 12: Solve as directed:

12. Find the centre and radius of the circles below:

a) $(x + 2)^2 + (y + 3)^2 = 9$

b) $x^2 + y^2 + 8x - 2y - 64 = 0$

DIRECTIONS for questions 13 to 16: Choose the correct alternative.

13. Which of the following points will not form a triangle with the points (1, 1) and (10, 16)?
 1) (2, 4) 2) (3, 5) 3) (4, 6) 4) (5, 7)

14. How far from the origin is the point of intersection of the diagonals of the rectangle formed by the lines $x + 2y = 5$, $x + 2y = -5$, $2x - y = -5$ and $2x - y = 10$?
 1) Between 1 and 1.1
 2) Between 1.1 and 1.2
 3) Between 1.2 and 1.3
 4) Between 1.3 and 1.4

15. Find the coordinates of the point on the line $7x + 8y = 77$ that is equidistant from the points (4, 2) and (8, 6).
 1) (11, 0) 2) (–1, 10.5) 3) (7, 3.5) 4) (3, 7)

16. If the points (2, 1), (x, 10) and (10, y) are collinear and xy = 104, which of the following points cannot be collinear with these three points?
 1) (4, 4) 2) (6, 7) 3) (18, 7) 4) (14, 7)

DIRECTIONS for question 17: Solve as directed:

17. If the points (a – 3, 2), (4, a), (a + 1, 8) and (10, b) are collinear, find b.

DIRECTIONS for questions 18 to 20: Choose the correct alternative.

18. $\triangle ABC$ is right-angled at B. AB lies along the line $2x + 3y - 7 = 0$. BC extended meets the y-axis at (0, 3). Find the coordinates of point B.
 1) $\left(-\frac{2}{13}, \frac{33}{13}\right)$ 2) $\left(-\frac{4}{13}, \frac{23}{13}\right)$ 3) $\left(-\frac{33}{13}, \frac{4}{13}\right)$ 4) $\left(-\frac{4}{13}, \frac{33}{13}\right)$

19. Amy the ant starts walking from the point (–3, 1). She reaches point A and then takes a right turn to reach the point (3, 3) such that the distances travelled before and after taking the turn are the same. Find the coordinates of point A. (Amy always walks along a straight line and on the x-y plane.)

1) (–1, 5) 2) (1, –1) 3) (–2, 3) 4) (2, –3)

20. Which of the following is the locus of a point which is always at a distance $\sqrt{5}$ units from the line $2y - x + 2 = 0$?

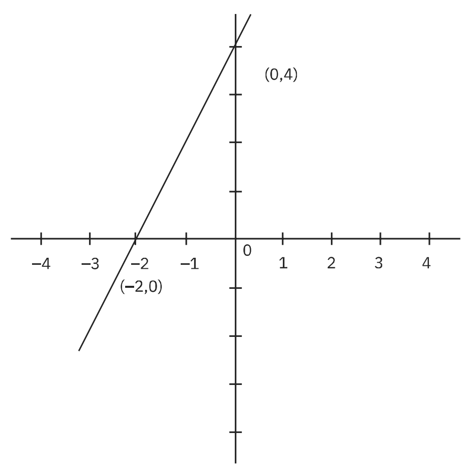
1) $2y - x + 7 = 0$
2) $2y - x - 3 = 0$
3) Both (1) and (2)
4) None of these

QA-2.4 Plotting of Algebraic Functions & Maxima and Minima

THEORY

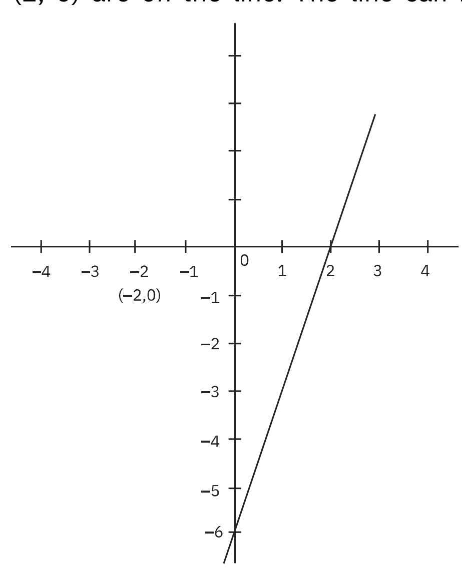
(A) Plotting of Linear Functions

Out of all algebraic functions, linear functions are simplest to plot. Consider a linear function such as y = 2x + 4. You need two points to plot a unique line. If you put x = 0, you can see that y = 4. Similarly, if you put y = 0, x = –2. Therefore, points (0, 4) and (–2, 0) definitely lie on the line. The line can be plotted as shown bellow—

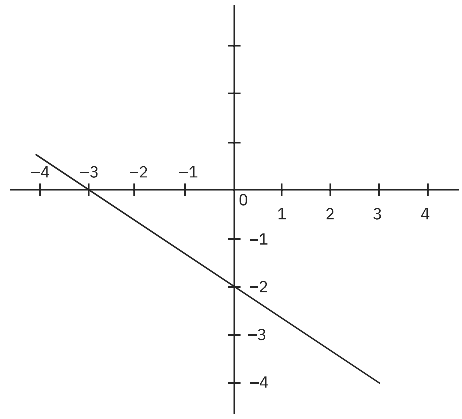


SOLVED EXAMPLES

1) Plot y = 3x – y – 6 = 0
   Put x = 0, y = –6, put y = 0, x = 2
   Therefore points (0, –6) and (2, 0) are on the line. The line can be plotted as shown below—



2) Plot $2x + 3y + 6 = 0$
Put $x = 0$, $y = -2$. Put $y = 0$, $x = -3$
Therefore points $(0, -2)$ and $(-3, 0)$ are on the line. The line can be plotted as shown below—

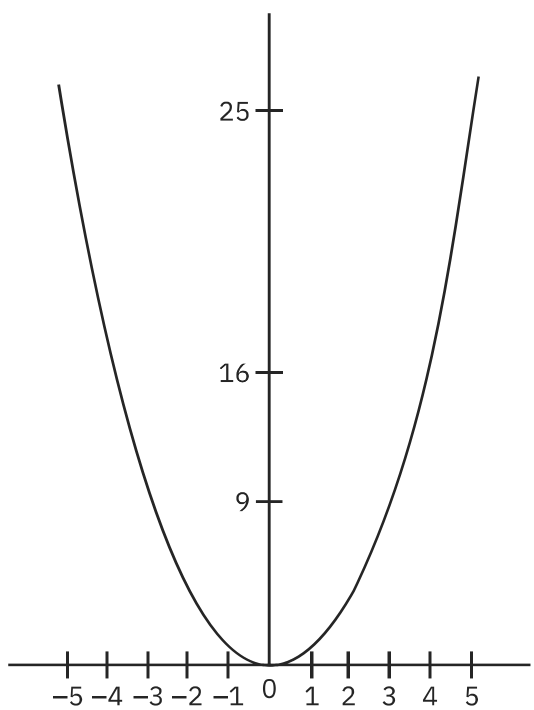


(B) Plotting of Square Functions

Consider function $y = x^2$. Since y is a square of a real number, it is positive. For all values of x. Different values of x and y can be tabulated as below—

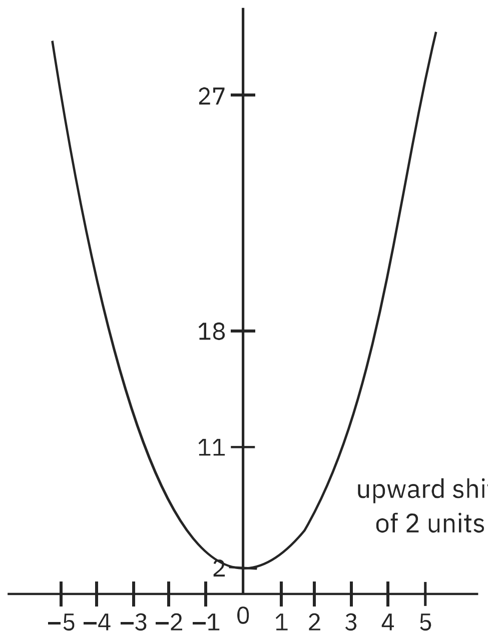
x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

The nature of the curve is as below—

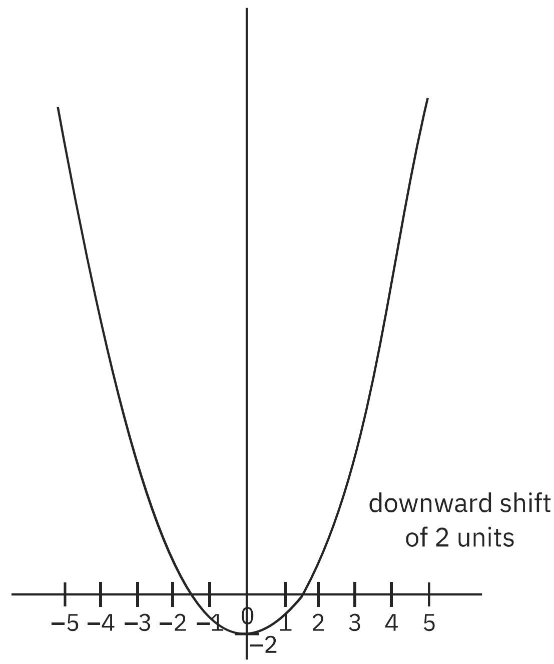


The curve $y = x^2 + k$ (where k is a positive constant) is the curve $y = x^2$ shifted upwards by k units. Similarly the curve $y = x^2 - k$ (where k is a positive constant) is the curve $y = x^2$ shifted downwards by k units.

The curves $y = x^2 + 2$ and $y = x^2 - 2$ are as shown below—

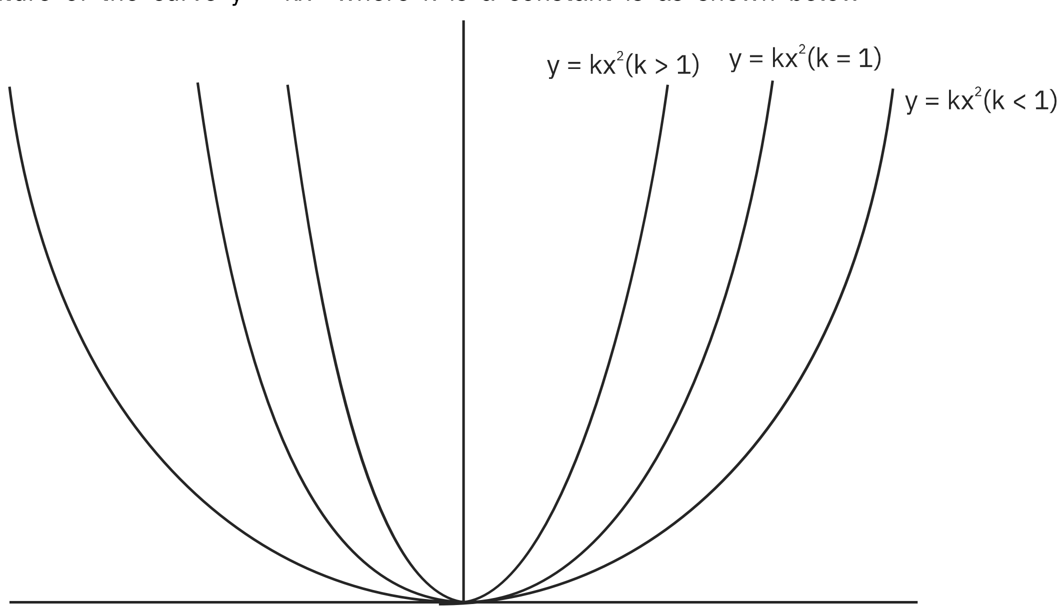


$y = x^2 + 2$



$y = x^2 - 2$

The nature of the curve $y = kx^2$ where k is a constant is as shown below—



In general if the value of $k > 1$, the curve becomes narrow and if $k < 1$, the curve becomes flat.

(C) Plotting of Quadratic Functions

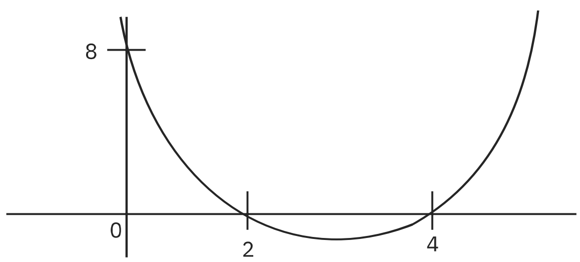
At times plotting the nature of the curve helps us solve the problems related to quadratics (especially inequalities) fast. In this section, we will study how to draw rough sketches of quadratic curves. Consider a quadratic polynomial, which can be factorised easily, such as $x^2 - 6x + 8$. As we can see, $x^2 - 6x + 8 = (x - 4)$ and $(x - 2)$.

The equation $x^2 - 6x + 8 = 0$ therefore, has two roots $x = 4$ and $x = 2$.

Therefore, the curve intersects x axis at $x = 4$ and $x = 2$.

If you put $x = 0$ in the expression, $x^2 - 6x + 8 = 8 > 0$. Therefore, the expression is positive at $x = 0$

Therefore nature of the curve is as follows—



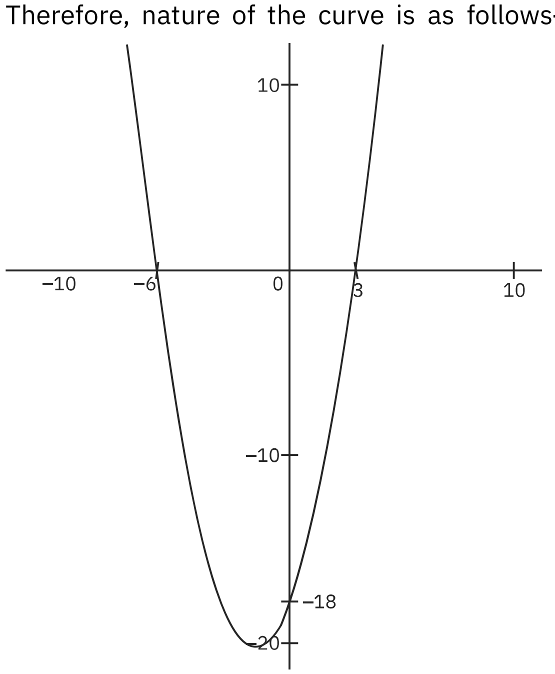
SOLVED EXAMPLES

Q : Plot $x^2 + 3x - 18$

A : $x^2 + 3x - 18 = (x + 6)(x - 3)$. Therefore the curve interescts x-axis at $x = -6$ and $x = 3$.

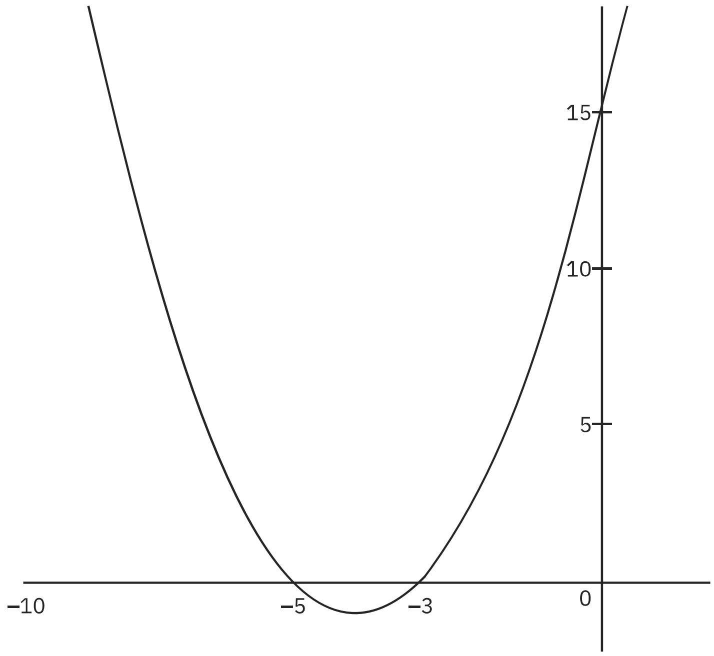
Putting $x = 0$, $x^2 + 3x - 18 = -18 < 0$

Therefore, nature of the curve is as follows—



Q : Plot $x^2 + 8x + 15$

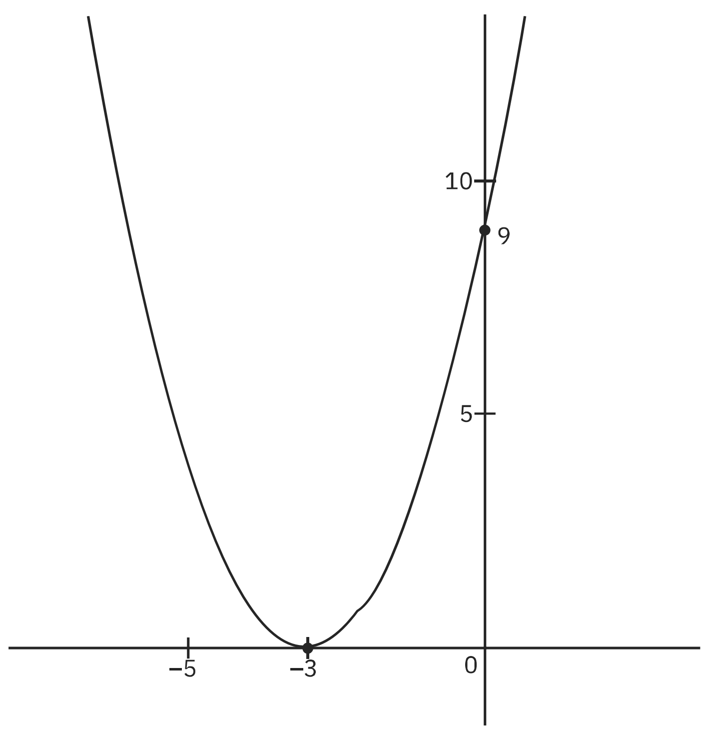
A : $x^2 + 8x + 15 = (x + 5)(x + 3)$. Therefore, the curve intersects x-axis at $x = -3$ and $x = -5$. Putting $x = 0$, $x^2 + 8x + 15 = 15 > 0$. Therefore, the nature of the curve is as follows—



Q : Plot $x^2 + 6x + 9$

A : $x^2 + 6x + 9 = (x + 3)^2$. Therefore, the curve touches the x-axis at $x = -3$

Therefore, the nature of the curve is as follows—



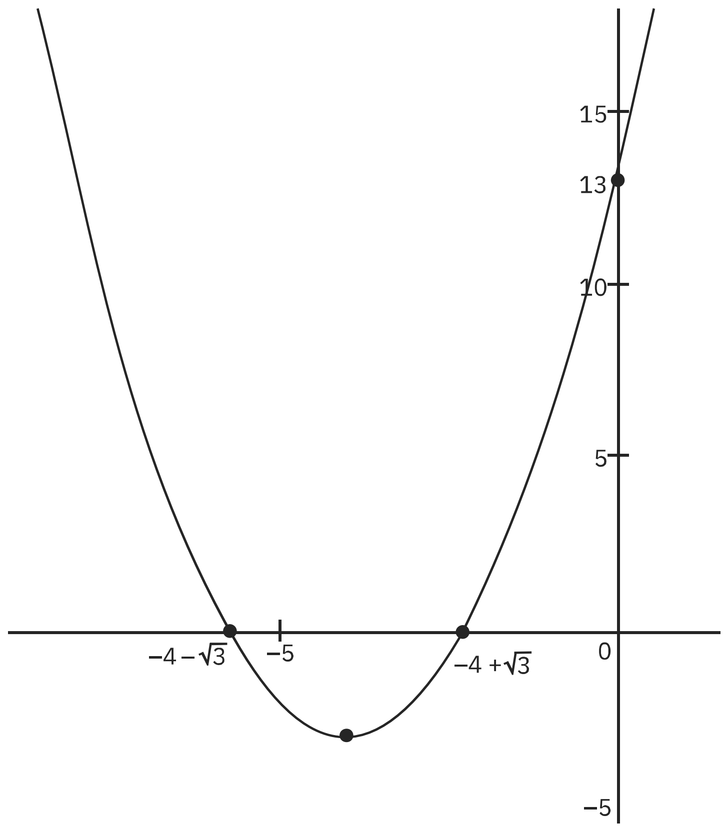
Now consider an expression such as $x^2 + 8x + 13$, which cannot be factorised easily. solving quadratic equation $x^2 + 8x + 13 = 0$, we get

$$x = \frac{-8 \pm \sqrt{64 - 52}}{2} = \frac{-8 \pm \sqrt{12}}{2} = -4 \pm \sqrt{3}$$

Therefore, the curve intersects x-axis at $x = -4 - \sqrt{3}$ and $-4 + \sqrt{3}$.

Putting $x = 0$ in $x^2 + 8x + 13$, the expression $= 13 > 0$ at $x = 0$.

Therefore nature of the curve is as follows—



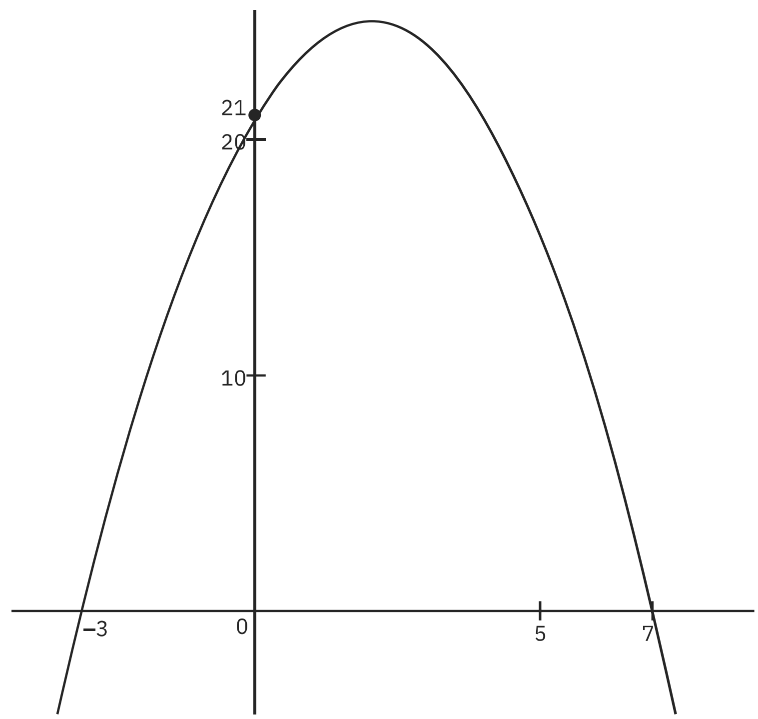
Now consider function $-x^2 + 4x + 21$

$-x^2 + 4x + 21 = -(x^2 - 4x - 21) = -(x - 7)(x + 3)$.

Therefore, the curve interescts x-axis at $x = 7$ and $x = -3$.

Putting $x = 0$, $-(x - 7)(x + 3) = 21 > 0$

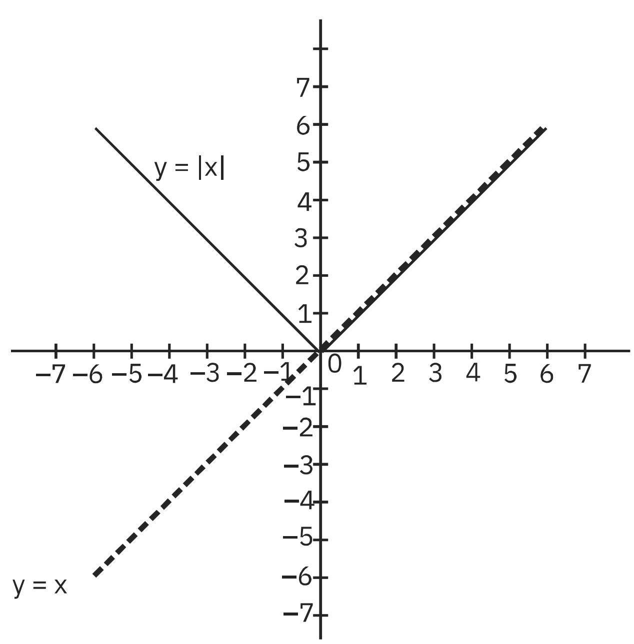
Therefore, nature of the curve is as follows—



In a function $ax^2 + bx + c$ (a, b, c are real numbers), if $a > 0$, the curve opens upwards while if $a < 0$, it opens downwards.

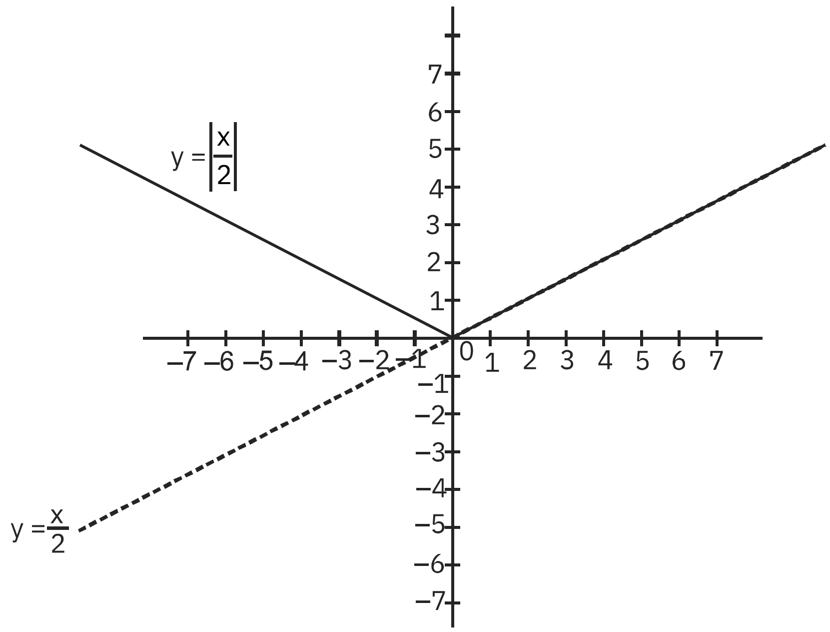
(D) Plotting of Modulus Functions

Consider the function $y = |x|$. A modulus function changes a negative value to a positive one. If we plot a line $y = x$ and wherever it drops below the x axis (i.e. wherever the function becomes negative), the mirror image of part appears above x-axis.

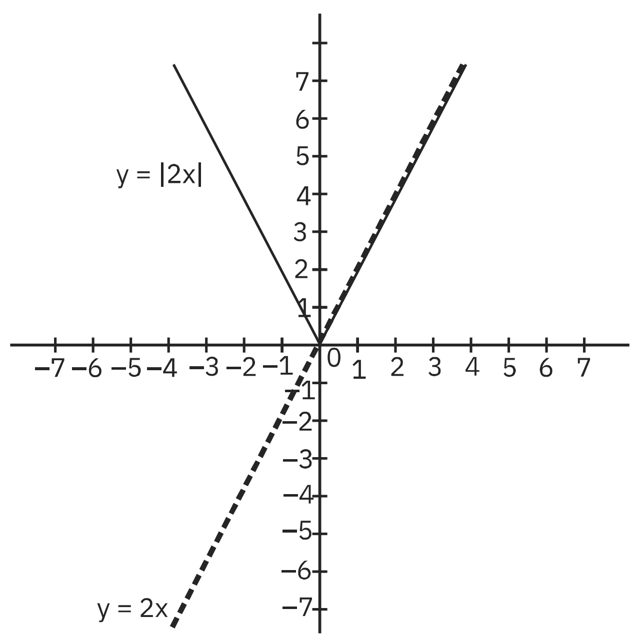


A basic modulus function will always have the distinctive, characteristic V shape seen above. However, it could be modified by changing the various parameters of the function.

- For example if we changed the slope of the line to $\frac{1}{2}$ by making it $y = \left|\frac{x}{2}\right|$ then it would be flatter than y = |x| as shown below:

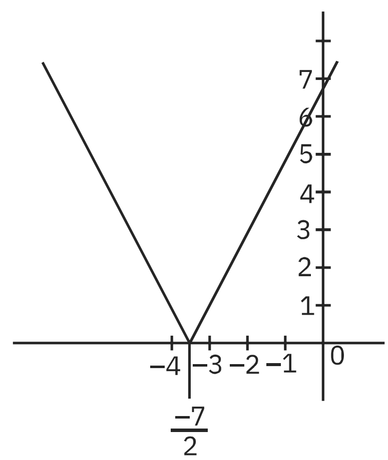


If we can change the slope of the line to 2 by making it y = |2x|, then it would be steeper than y = |x| as shown below—



In general, the graph of a line with higher slope is steeper and the graph of a line with lower slope is flatter.

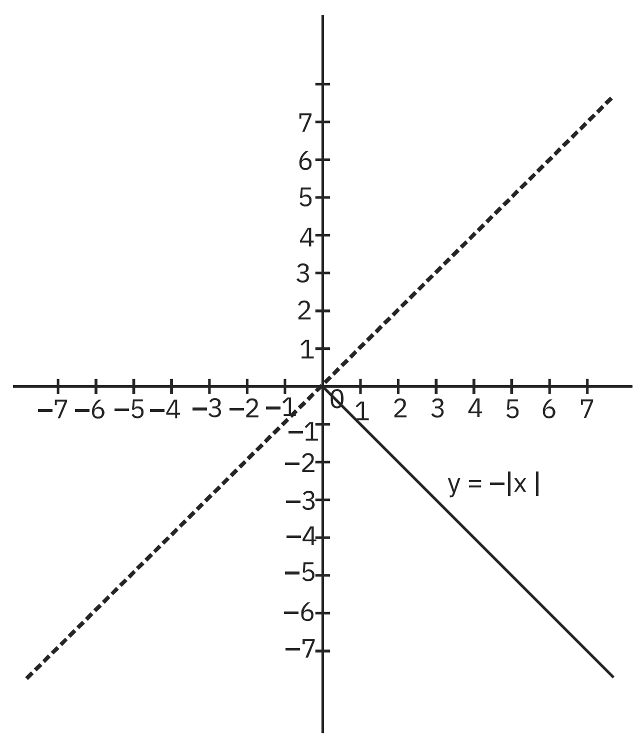
Now consider y = |2x + 7| touches x axis at $x = -\frac{7}{2}$ and has slope equal 2(steeper line). The nature of the graph is as shown below—



Now

Consider y = − |x|

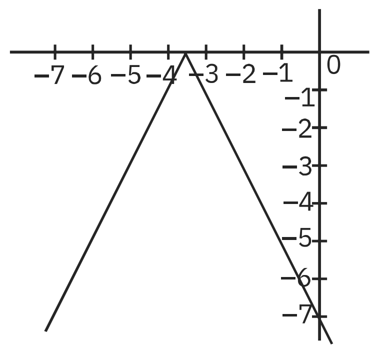
Just as graph of y = |x| is positive for all values of x, the graph of y = −|x| is negative for all values of x and is of the stape of 'inverted V' as shown below—



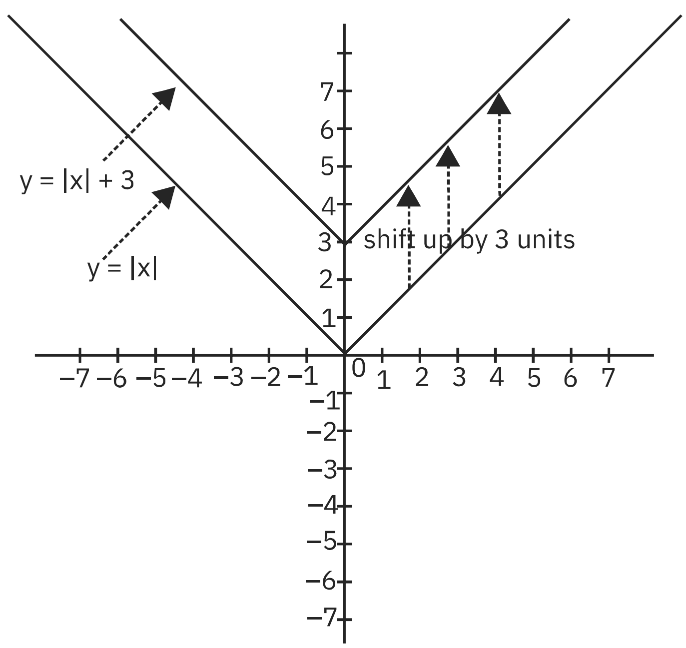
We can observe that a negative modulus lies entirely below the x-axis (since |x| is always > 0, −|x| will always be < 0).

Now consider y = −|2x + 7|. If x = $-\frac{7}{2}$, 2x = −7 and 2x + 7 = 0

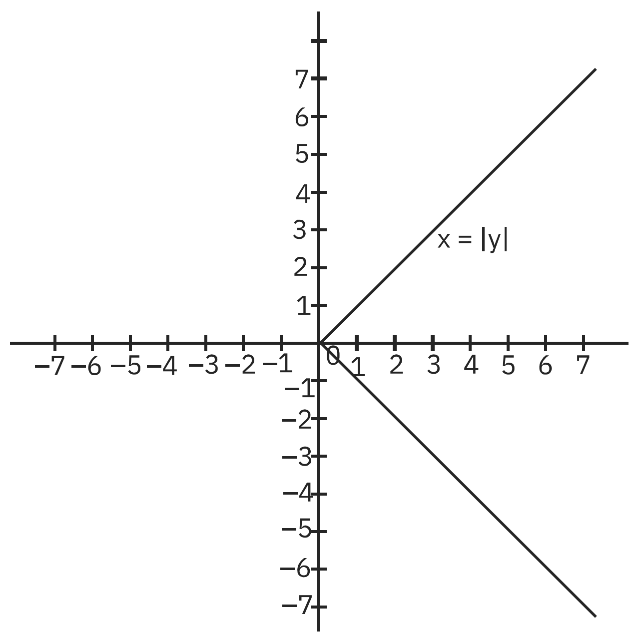
Therefore graph of y = −|2x + 7| touches x axis at x = $-\frac{7}{2}$ and has slope equal to 2 (steeper line). However because of negative sign, it is of the shape of an inverted V as shown below—



Consider y = |x| + 3



Consider x = |y|. You can see that we interchange x and y in the equation y = |x|. Therefore the nature of the graph is similar to that of y = |x| with x and y axes interchanged.



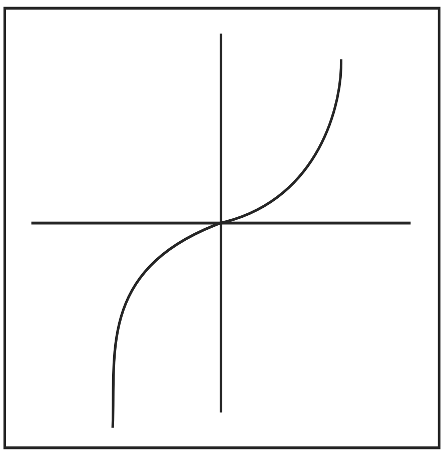
(E) Plotting of Cubic Function

Consider the function $y = x^3$.

Since y is a cube of a real number, it is positive when x is positive and negative when x is negative. Different values of x and y can be tabulated as below:

x	–3	–2	–1	0	1	2	3
y	–27	–8	–1	0	1	8	27

The nature of the curve is as follows—



$y = x^3$

(F) Plotting of Square Root Function

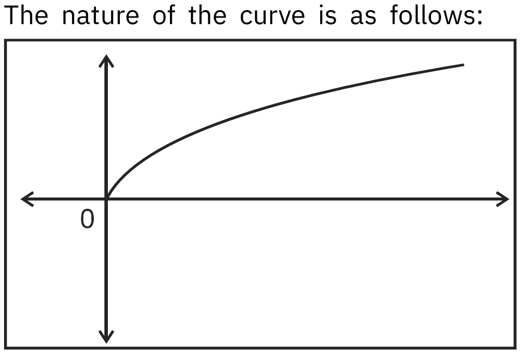
Consider the function $y = \sqrt{x}$.

Since square root of x is taken, x cannot be negative. As x is always positive or zero, the value of y will always be positive, the graph of this function will always lie in the first quadrant.

Different values of x & y can be tabulated as below:

x	0	1	4	9	16	25
y	0	1	2	3	4	5

The nature of the curve is as follows:



$y = \sqrt{x}$

(G) Plotting of Reciprocal Function

Consider the function $y = \frac{1}{x}$.

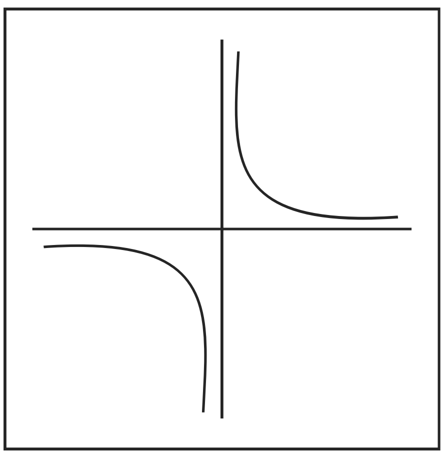
If x is positive, y will be positive and if x is negative, y will be negative.

As x,y cannot be zero, the graph will not touch any of the axes.

Different values of x and y can be tabulated as follows:

x	–10	–4	–1	1	4	10
y	–0.1	–0.25	–1	1	0.25	0.1

The nature of the curve is as follows—

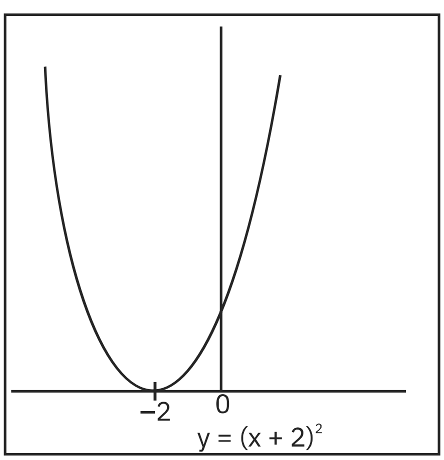


y = 1/x

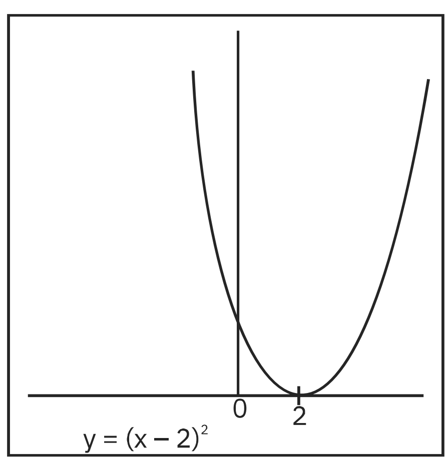
Shifting of the curve Upward, Downward, Left and Right

The curve $y = (x + k)^2$ is the curve of $y = x^2$ shifted towards left by k units. Similarly, the curve $y = (x - k)^2$ is the curve of y shifted towards right by k units.

The curves $y = (x + z)^2$ and $y = (x - z)^2$ are as shown below:



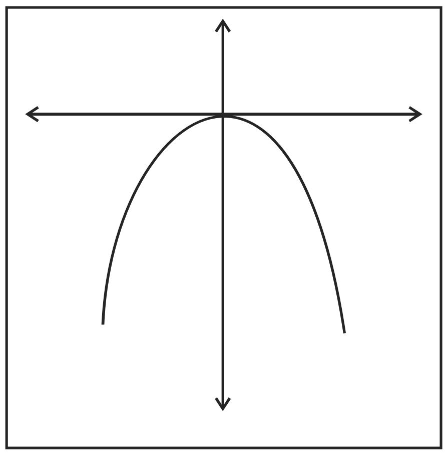
$y = (x + 2)^2$



$y = (x - 2)^2$

Now, consider $y = -x^2$.

In the graph of $y = -x^2$, y will be negative for all values of x and the shape will be as follows:



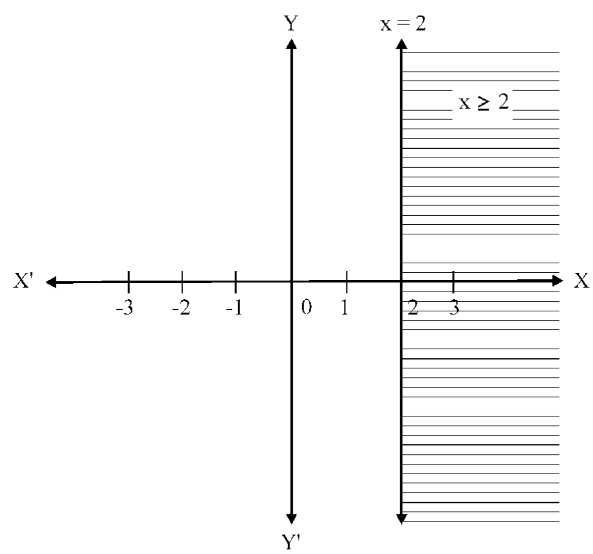
Linear Inequalities Represented on the Cartesian Plane

Most of the solution sets of an inequation are infinite sets and hence easier to visualize graphically.

Linear inequality in one variable

The graph of $x \geq 2$ is constructed by considering the graph of x = 2. The line x = 2 divides the plane into two regions called half planes.

As seen in the figure every abscissa to the right hand side of the graph x = 2 is greater than 2 and every abscissa to the left of graph x = 2 is less than 2. The plane to the right of the graph x = 2 is $x \geq 2$ (including the line) and that to the left of the graph x = 2 is x < 2 (excluding the line)



Some important points

1. If the inequality is ‘strictly less than’ i.e. ‘<’ or ‘strictly greater than’ i.e. ‘>’, then the points on the line would not be included in the region.
2. The inequalities x > 0 and y > 0 represent the entire first quadrant.
3. The inequalities x < 0 and y > 0 represent the entire second quadrant.
4. The inequalities x < 0 and y < 0 represent the entire third quadrant.
5. The inequalities x > 0 and y < 0 represent the entire fourth quadrant.

Linear inequality in two variables

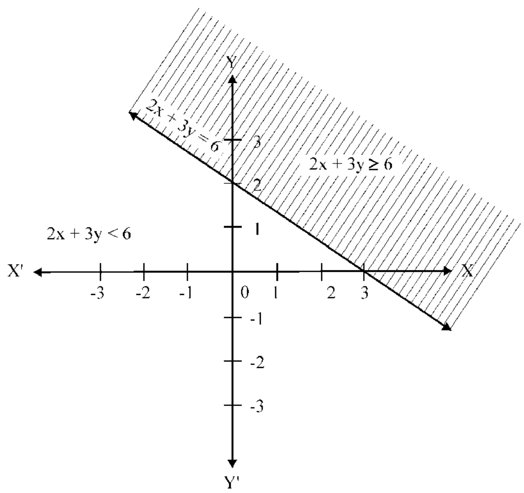
Plot the graph of $2x + 3y \geq 6$.

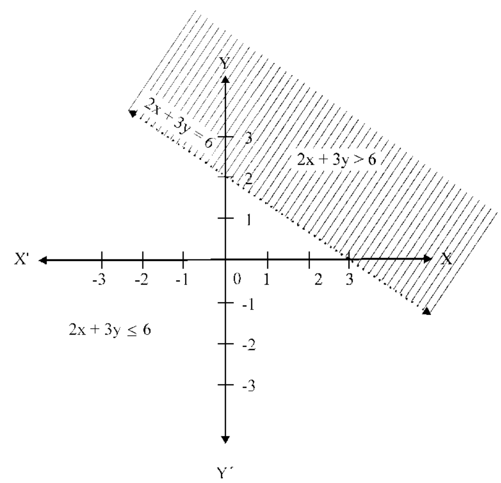
The graph of the straight line $2x + 3y = 6$ divides the plane into two planes, one of these planes contains the origin.

If $x = 0$ and $y = 0$ then $2x + 3y = 0$. Since 0 is not greater than or equal to 6, the origin will lie on the graph of $2x + 3y < 6$.

$\therefore$ The plane without the origin is the graph of $2x + 3y \geq 6$ and the one with the origin is the graph of $2x + 3y < 6$.

$2x + 3y > 6$ will be denoted by the same graph but the line $2x + 3y = 6$ will be dotted to indicate that the line is not included.





Some important points

1. Always write the equation in the form $ax + by + c = 0$ making the coefficient of x positive.
2. If the line passes through the origin, check for any other point, which is not on the line to decide the required plane.
3. The union of a half plane and its boundary is called a closed half plane.

Solution set of two or more inequalities

Example

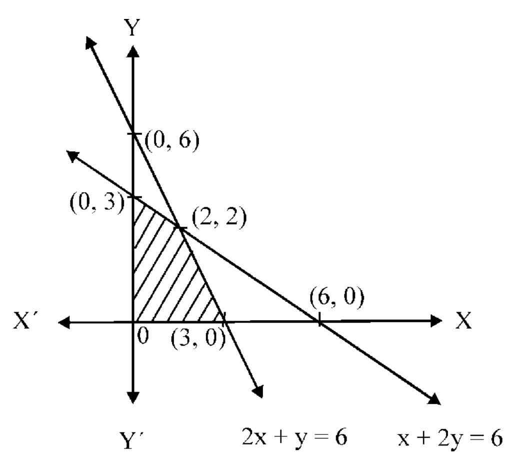
(i) Find the region bounded by $x + 2y \leq 6$, $2x + y \leq 6$,

$x \geq 0$, $y \geq 0$.

$x + 2y = 6$, 0 is less than 6. Hence, origin side of $x + 2y = 6$ is $x + 2y \leq 6$.

$2x + y = 6$, 0 is less than 6. Hence, origin side of $2x + y = 6$ is $2x + y \leq 6$.

$x \geq 0$ and $y \geq 0$ determine the first quadrant. Required region is the shaded part as shown in the figure.



(ii) Find the region bounded by $y - 2x \leq 4$ and $x + y \geq -2$

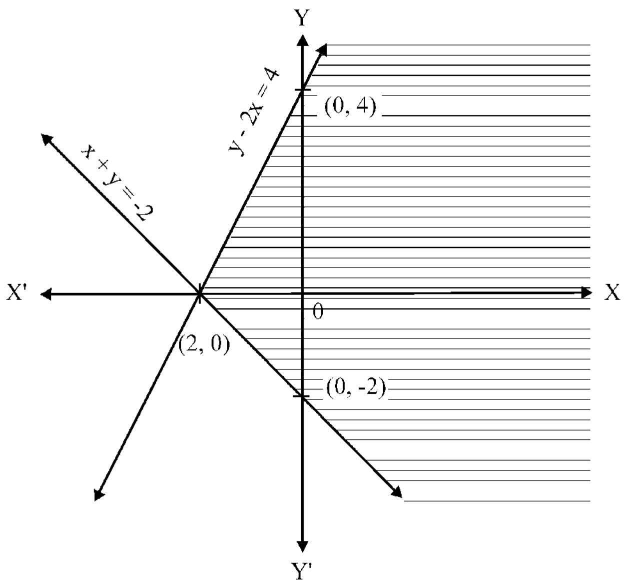
Zero is less than 4.

$\therefore$ The origin side of the line $y - 2x = 4$ is $y - 2x \leq 4$.

Zero is greater than -2.

$\therefore$ The origin side of the line $x + y = -2$ is $x + y \geq -2$.

The shaded region is an infinite region represented by $y - 2x \leq 4$ and $x + y \geq 2$

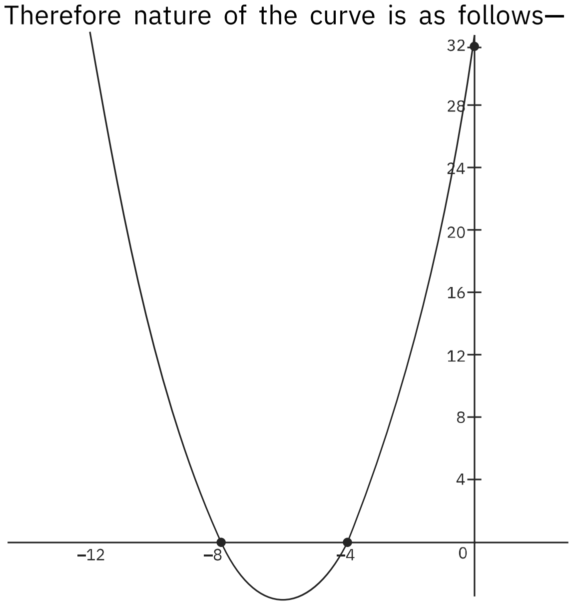


Graphical Approach to Quadratic Inequalities

Once we plot the nature of the quadratic curve, problems related to inequalities can be solved as shown below—

Q : Solve $x^2 + 12x + 32 > 0$

A : $x^2 + 12x + 32 = (x + 8)(x + 4)$ Therefore, the curve interesects x-axis at $x = -8$ and $x = -4$.
Putting $x = 0$, $x^2 + 12x + 32 = 32 > 0$
Therefore nature of the curve is as follows—



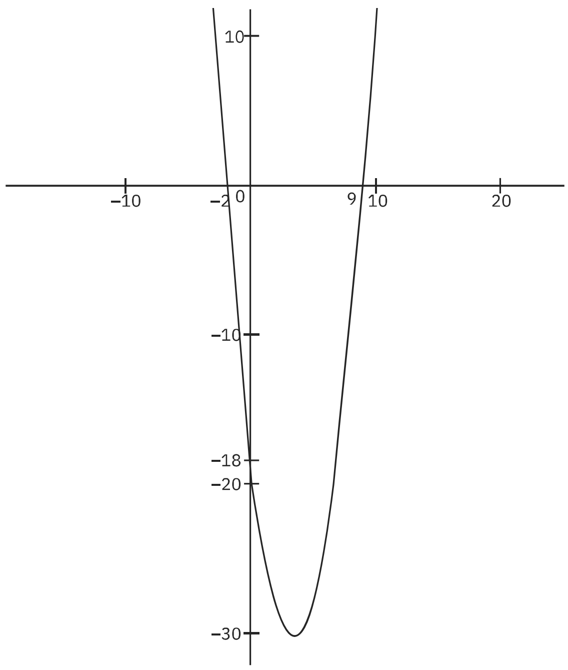
As we can see from the curve, $x^2 + 12x + 32 > 0$ for $x < -8$ and $x > -4$

Q : Solve $x^2 - 7x - 18 < 0$

A : $x^2 - 7x - 18 = (x - 9)(x + 2)$. Therefore the curve intersects x-axis at $x = 9$ and $x = -2$.

Putting $x = 0$, $x^2 - 7x - 18 = -18 < 0$

Therefore, the nature of the curve is as follows—



It can be seen that $x^2 - 7x - 18 < 0$ for $-2 < x < 9$

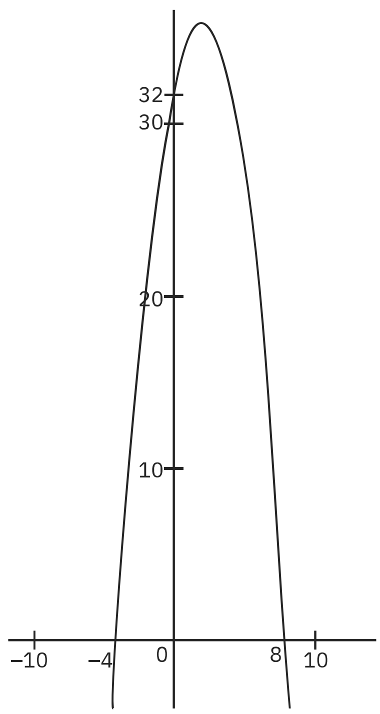
Q : Solve $-x^2 + 4x + 32 \leq 0$

A : $-x^2 + 4x + 32 = -(x^2 - 4x - 32) = -(x - 8)(x + 4)$.

Therefore, the curve interescts x-axis at $x = 8$ and $x = -4$.

Putting $x = 0$, $-x(x - 8)(x + 4) = 32 > 0$

Therefore, nature of the curve is as follows—



It can be seen that $-x^2 + 4x + 32 \leq 0$ for $x \leq -4$ and $x \geq 8$

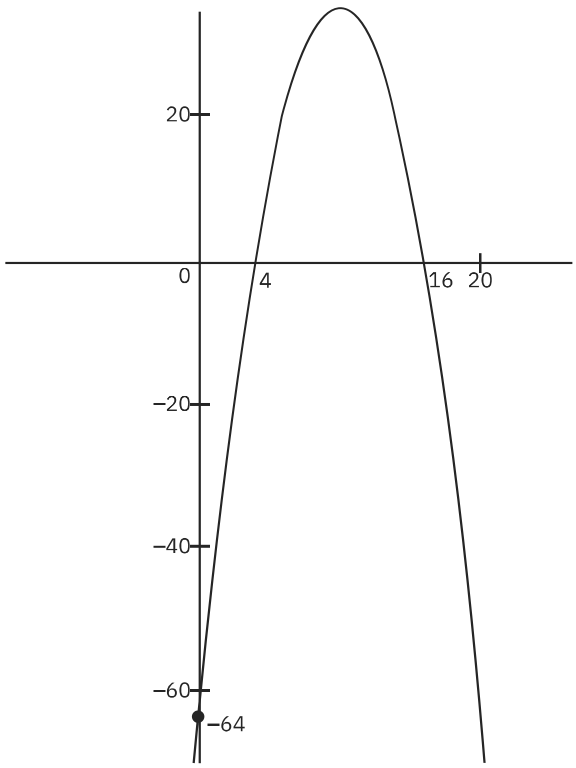
Q : Solve $-x^2 + 20x - 64 \geq 0$

A : $-x^2 + 20x - 64 = -(x^2 - 20x + 64 = -(x - 16)(x - 4)$.

Therefore, the curve interescts x-axis at $x = 16$ and $x = 4$.

Putting $x = 0$, $-(x - 16)(x - 4) = -64 < 0$

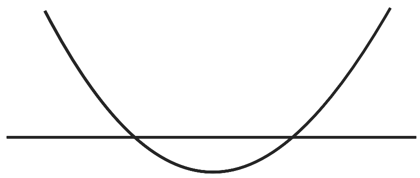
Therefore, nature of the curve is as follows—



It can be seen that $-(x - 16)(x - 4) \geq 0$ for $4 < x \leq 16$.

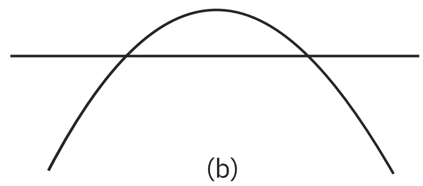
Maxima and Minima of Quadratic Functions

As seen in the previous section, the nature of quadratics curve is either of the following two—



(a)

or



(b)

The curve as shown in (a) has a minimum value (minima) while there is no upper limit on the value it can take. Similarly, the curve as shown in (b) has a maximum value (maxima) while there is no lower limit on the value it can take.

Method for Calculating Maxima/Minima of Quadratic Functions

1) Determine whether $x^2 - 4x - 12$ has maxima or minima and calculate its value—
For such problems, express the given expression in the form $(x + a)^2 + b$, where a and b are constants.
$x^2 - 4x - 12 = (x^2 - 4x + 4) - 16 = (x - 2)^2 - 16$.
Since $(x - 2)^2$ is a square, the minimum value taken by $(x - 2)^2$ is zero and it occurs at $x = 2$
$\therefore$ minimum value of the expression: $0 - 16 = -16$ or minima = 16 at $x = 2$

2) Determine whether $x^2 + 6x + 5$ has maxima or minima and calculate its value $x^2 + 6x + 5 = 9x^2 + 6x + 9) - 4 = (x + 3)^2 - 4$.
Since $(x + 3)^2$ is a square, the minimum value taken by $(x + 3)$ is zero and it occurs at $x = -3$.
$\therefore$ minimum value of the expression: $0 - 4 = -4$ or minima $= -4$ at $x = -3$.

3) Determine wheter $x^2 + 10x + 38$ has maxima has maxima or minima and calculate its value.
$x^2 + 10x + 38 = (x^2 + 10x + 25) + 13 = (x + 5)^2 + 13$.
Since $(x + 5)^2$ is a square, the minimum value taken by $(x + 5)^2$ is zero and it occurs at $x = -5$
$\therefore$ minimum value of the expression: $0 + 13 =$ or minima $= 13$ at $x = -5$.

4) Determine whether $-x^2 + 4x + 21$ has maxima or minima and calculate its value—
$-x^2 + 4x + 21 = -(x^2 - 4x - 21) = -[(x^2 - 4x + 4) - 25]$
$= [(x - 2)^2 - 25] = -(x - 2)^2 + 25$
The minimum value $(x - 2)^2$ can take is zero.
$\therefore$ The maximum value $-(x - 2)^2$ can take is zero.
$\therefore$ The maximum value $-(x - 2)^2 + 25$ can take is $0 + 25 = 25$ and it occurs at $x = 2$

5) Determine whether $-x^2 + 8x - 15$ has maxima or minima and calculate its value—
$-x^2 + 8x - 15 = -(x^2 - 8x + 15)$
$= -[(x^2 - 8x + 16) - 1]$
$= -[(x - 4)^2 - 1]$
$= -(x - 4)^2 + 1$
The minimum value $(x - 4)^2$ can take is zero.
$\therefore$ The maximum value $-(x - 4)^2$ can take is zero.
$\therefore$ The maximum value $-(x - 4)^2 + 1$ can take is 1
$\therefore$ Maximum value of the expression = 1 or maxima = 1 at $x = 4$.

Maxima and Minima of Modulus Functions

As we have seen earlier, the graph of the modulus function with positive coefficient is of the shape of 'v' while the graph of the modulus function with negative coefficient is of the shape of inverted 'v'. Therefore the modulus function with positive coefficient has a minima while the modulus function with negative coefficient has a maxima.

Therefore functions of the form $|2x + 4|$, $2|3x + 5|$ etc have minima

While the functions of the form $-|2x + 4|$, $-2|3x + 5|$ etc have maxima

Method for Calculating Maxima or Minima of Modulus Functions

1) Consider function $y = |x + 2| + |x - 4|$

The function has a minima because both $|x + 2| + |x - 4|$ have positive coefficients.

Step 1: Since Graphs of $|x + 2|$ and $|x - 4|$ touch the x axis at $x = -2$ and $x = 4$ respectively, the minima will occur either at $x = -2$ or 4.

Step 2: The minimum value of the function occurs at one of the roots.
At $x = -2$, $y = 6$
At $x = 4$, $y = 6$

Therefore minimum value of the function is 6.

2) Consider function $y = -|x + 3| - |x - 4|$

The function has maxima because both $-|x + 3|$ and $-|x - 4|$ have negative coefficients.

Step 1: Since Graphs of $-|x + 3|$ and $|x - 4|$ touch the x axis at $x = -3$ and $x = 4$ respectively, the maxima will occur either at $x = -3$ or $+4$.

Step 2: At $x = -3$, $y = -7$

At $x = 4$, $y = -7$

Therefore maxmum value of the function is -7.

3) Consider function $y = |2x + 4| + |3x + 5|$

The functions has minima because both $|2x + 4|$ and $|3x + 5|$ have positive coefficients.

Step 1: Since Graphs of $|2x + 4|$ and $|3x + 5|$ touch the x axis at $x = -2$ and $x = -\frac{5}{3}$ respectively, the minima will occur either at $x = -2$ or $-\frac{5}{3}$.

Step 2: At $x = -2$, $y = 1$

At $x = -\frac{5}{3}$, $y = \frac{2}{3}$

Therefore minimum value of function is $\frac{2}{3}$.

4) Consider function $y = |3x + 2| + |2x + 3|$

Step 1: Since Graphs of $|3x + 2|$ and $|2x + 3|$ touch the x axis at $x = -\frac{2}{3}$ and $x = -\frac{3}{2}$ respectively, the minima will occur either at $x = -\frac{2}{3}$ or $x = -\frac{3}{2}$.

Step 2: At $x = -\frac{2}{3}$, $y = \frac{5}{3}$

At $x = -\frac{3}{2}$, $y = \frac{5}{2}$

Therefore minima of the function is $\frac{5}{3}$

5) Consider function $y = |2x - 4| - |x - 1|$
In this problem, one of the modulus functions has positive coefficient while the other has negative coefficient. Slope of $|2x - 4|$ is 2 while slope of $|x - 1|$. Since the term with positive coefficient has greater slope, the function has a minima.

Step 1: Since Graphs of $|2x - 4|$ and $|x - 1|$ touch the x axis at $x = 2$ and $x = 1$ respectively, the minima will occur at $x = 2$ or 1.

Step 2: At $x = 2$, $y = -1$

At $x = 1$, $y = 2$

Therefore minimum value of y is -1.

6) Consider function $y = |x + 1| - |3x + 1|$
Slope of $|x + 1|$ is 1 while slope of $|3x + 1|$ is 3. Since the slope of the term with negative coefficient is more the function has a maxima,

Step 1: Since Graphs of $|x + 1|$ and $|3x + 1|$ touch the x axis at $x = -1$ and $x = -\frac{1}{3}$ respectively, the maxima will occur either at $x = -1$ or $-\frac{1}{3}$.

Step 2: At $x = -1$, $y = -2$

At $x = -\frac{1}{3}$, $y = \frac{2}{3}$

Therefore maximum value of y is $\frac{2}{3}$.

CLASS EXERCISE

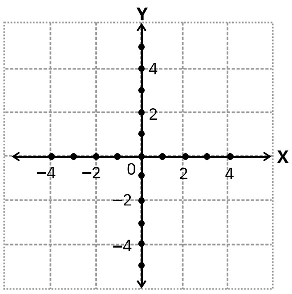
Teaser

Yesterday, my colleague Rounak was telling me about the new social networking site Facepalm. He explained how two people can link to each other and become “friends”. He proudly proclaimed that he already had 500 “friends” on the site. He also boasted that no two of those “friends” have the same number of “friends”, but none of them had as many as he does. I believe that he was not telling the truth. Why?

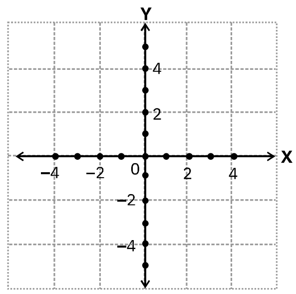
Plotting of Equations

1) Plot the following:

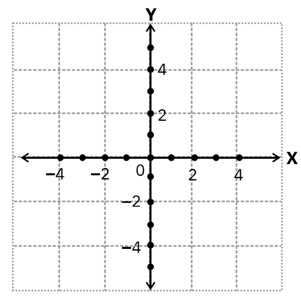
a) $y = x + 3$



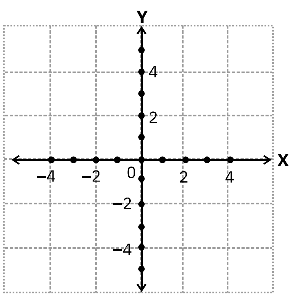
b) $y = 2x$



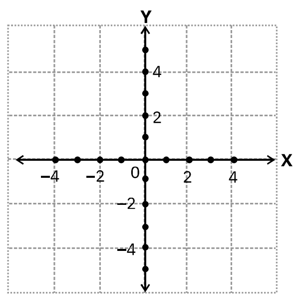
c) $y = -x$



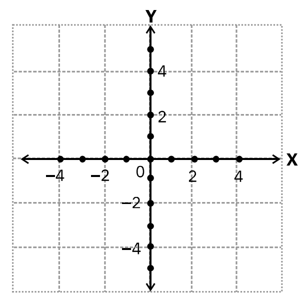
d) $y = x^2$



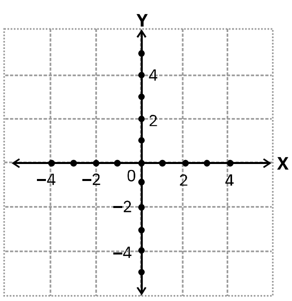
e) $y = \frac{1}{2}x^2$



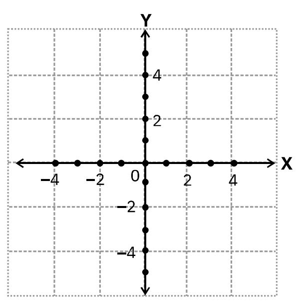
f) $y = -2x^2$



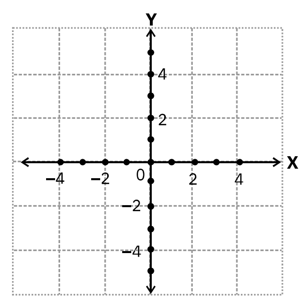
g) $y = x^2 - 2$



h) $y = \frac{x^2 - 9}{3}$



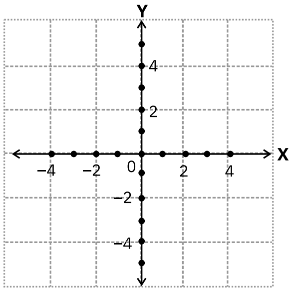
i) $y = |x^2 - 4|$



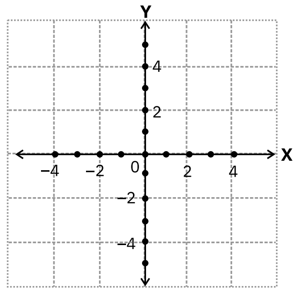
If $y = ax^2 + bx + c$,

- 'a' indicates the steepness of the curve. If 'a' is positive, the curve will face upwards ($\cup$), while if 'a' is negative, it will face downwards ($\cap$). The higher the value of $|a|$, the steeper the curve.
- 'c' indicates the value of y when $x = 0$ i.e. the y-intercept of the curve
- The roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ give the x-intercepts. If $b^2 - 4ac < 0$, then the curve will not intersect the X-axis.

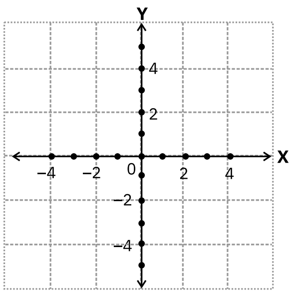
j) $y < 3x$



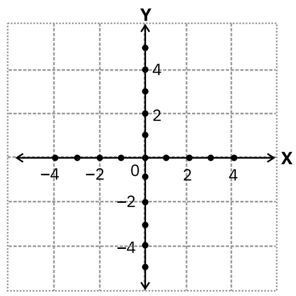
k) $y \leq \frac{3}{5}x - 3$



l) $y < 3$

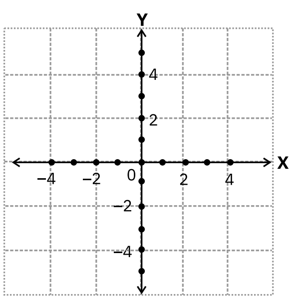


* m) $0 \leq y \leq x \leq 3y \leq 6$

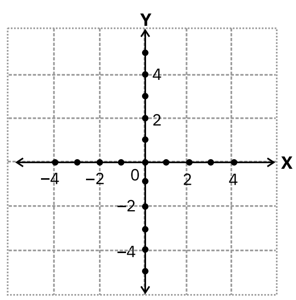


2) Plot the following:

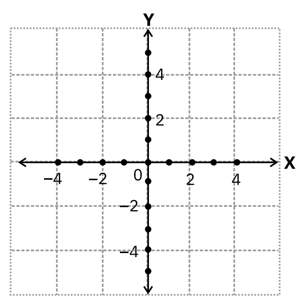
a) $|x| = 2$



b) $y = 3 - |x - 1|$

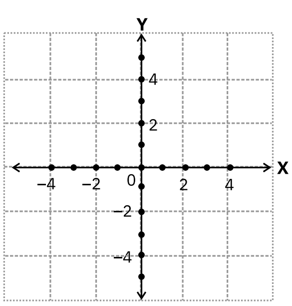


c) $x + 13 = 11 - y$

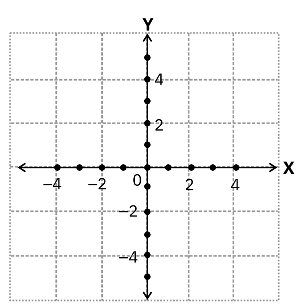


3) How many points with integer coordinates lie within or on the boundary of the region formed by the following?

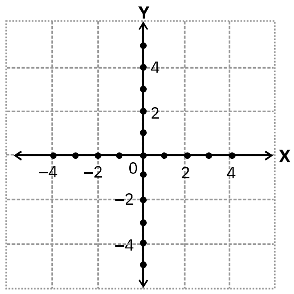
a) $0 \leq x \leq 4,\ 0 \leq y \leq x$



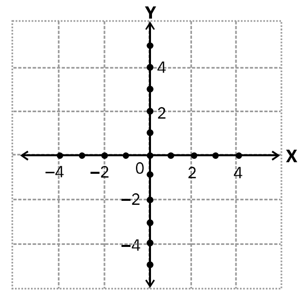
b) $-3 \leq x \leq 3,\ -2 \leq y \leq 2$



c) $|x| + |y| = 3$



d) Circle centred at origin with radius 5



Maximum or Minimum Values

4) Find whether each of the following modulus expressions has a maximum or a minimum. In each case, also identify the x-value at which it occurs, and the value of the expression at that point..

a) $|2x + 3|$ b) $2 + |x - 3|$ c) $5 - |x + 2|$ d) $3 - |3x - 3|$

A modulus function $|ax + b|$ will have a minimum at $x = -\frac{b}{a}$

A modulus function $-|ax + b|$ will have a maximum $x = -\frac{b}{a}$

The maximum or minimum value will be obtained by substituting $x = -\frac{b}{a}$ into the original expression

5) Identify which of the following quadratics has a maximum. Also, identify the coordinates of the extremum point (i.e. maximum or minimum) for each expression.

a) $x^2 + 5x - 6$ b) $4x^2 - 4x + 1$ c) $x^2 + 2x + 3$ d) $10 - 3x - x^2$

A quadratic expression $ax^2 + bx + c$ will have a maximum if a is negative and a minimum if a is positive

The maximum/minimum value of a quadratic expression $ax^2 + bx + c$ will occur at $x = -\frac{b}{2a}$

The maximum/minimum value of a quadratic expression $ax^2 + bx + c$ will be $y = -\left(\frac{b^2 - 4ac}{4a}\right) = \frac{\Delta}{4a}$

Challengers

1) If we define an expression $P(x) = x^2$ and an expression $Q(x) = |4x|$, then for how many integer values is $P(x) - Q(x)$ a negative quantity?

1) 2 2) 4 3) 6 4) 8

2) Plot the region on the XY plane satisfying the inequality: $|x| + |y| + |x + y| < 4$

How many points with integer co-ordinates lie in the interior of this region?

1) 4 2) 5 3) 6
4) 7 5) 8

3) How many points with integer coordinates lie within or on the boundary of the region formed by $(x - 3)^2 + (y + 2)^2 = 25$?

1) 79 2) 81 3) 83 4) 85

4) What is the maximum value of $25 + 2x^2 - x^4$?

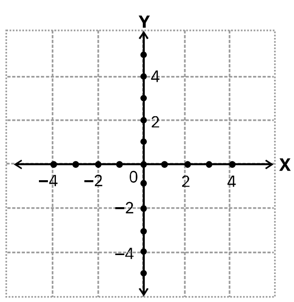
1) 26 2) 28 3) 30 4) 32

5) How many points with non-negative integer coordinates lie in the region $y \le 12 - x^2$?

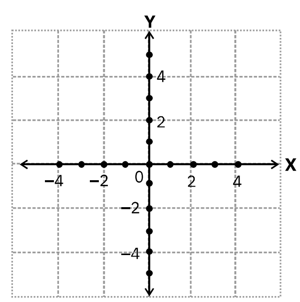
1) 34 2) 36 3) 38 4) 40

*6) Plot the following equations:

a) $y = |x + 1| + |x - 1|$

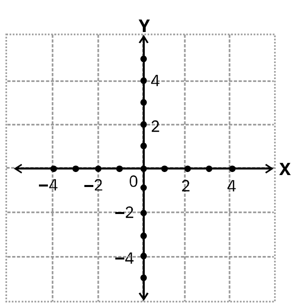


b) $y = 1 - \left|\dfrac{(x-2)}{2}\right|$

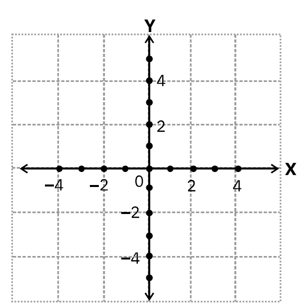


*7) Plot the following regions:

a) $|y| = 6 - 2|x|$



b) $|x + y| = 4$



PRACTICE EXERCISE

DIRECTIONS for questions 1 and 2: Solve as directed.

1. Solve the following inequalities graphically:

   (a) $-x^2 - 6x + 7 \leq 0$ (b) $x^2 + 8x + 16 \geq 0$

2. Mention whether the following functions have maxima or minima and calculate maxima/minima:

   (a) $4x^2 + 20x - 34$ (b) $-x^2 + 8x + 30$

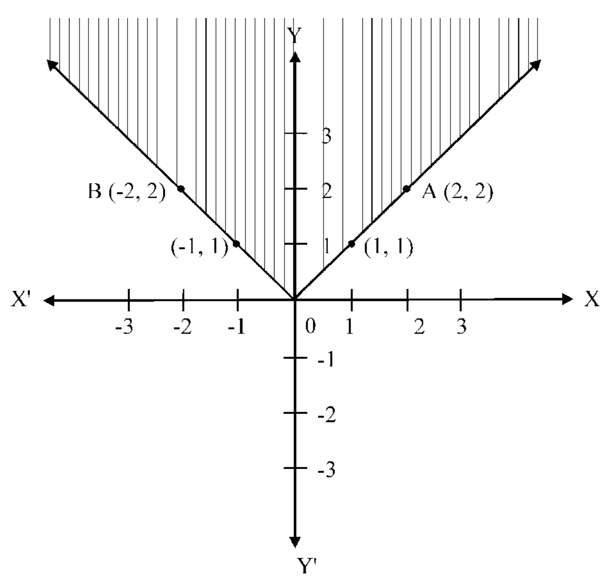
   (c) $-x^2 + 11x - 18$ (d) $x^2 + 7x + 3$

   (e) $|2x + 3| - |3x - 4|$ (f) $|x + 4| + |x + 6|$

   (g) $|3x + 4| - |2x + 3|$

DIRECTIONS for questions 3 to 8: Choose the correct alternative.

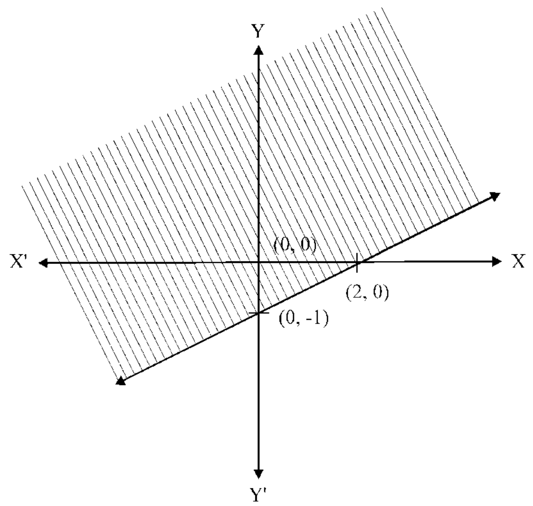
3.



The shaded region is represented by the inequation:

1) $y \geq x$ 2) $y \geq -x$ 3) $y \geq | x |$ 4) $y \leq | x |$

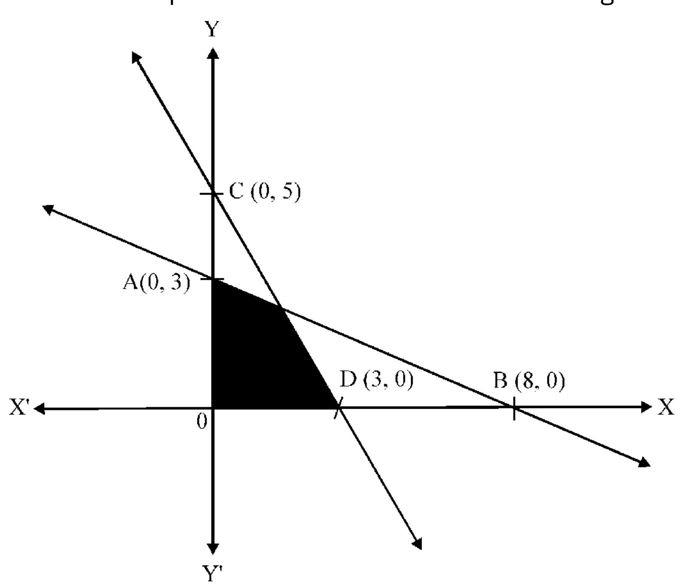
4.



The shaded region is represented by the inequality:

1) $x - 2y \leq 2$ 2) $x - 2y \geq 2$ 3) $2x - y \leq 2$ 4) $2x - y \geq 2$

5. Find the inequations that bound the shaded region.



1) $x \geq 0$, $y \geq 0$, $8x + 3y \geq 24$, $3y + 5x \geq 15$
2) $x \geq 0$, $y \geq 0$, $3x + 8y \geq 24$, $5x + 3y \geq 15$
3) $x \geq 0$, $y \geq 0$, $8x + 3y \leq 24$, $3y + 5x \leq 15$
4) $x \geq 0$, $y \geq 0$, $3x + 8y \leq 24$, $5x + 3y \leq 15$

6. Find the area of the region bounded by the equations $|x + y| \leq 5$ and $|x - y| \leq 3$.

1) 15 sq.units 2) 30 sq.units 3) 20 sq.units 4) 25 sq.units

7. Find the area of the region bounded by $y - 2 \geq 0$, $x - y \geq 1$ and $x + 2y \leq 10$.

1) 1.5 sq. units 2) 2 sq. units 3) 3 sq. units 4) 4 sq. units

8. What is the area bounded by the curves $|x + y| = 1$, $|x| = 1$, $|y| = 1$?

1) 4 2) 3 3) 2 4) 1

DIRECTIONS for questions 9 and 10: Solve as directed.

9. Find the area (in square units) of the region bounded by the curves $y = |x + 1| + |1 - x|$ and $y = 4$.

10. Find the number of points with integer coordinates

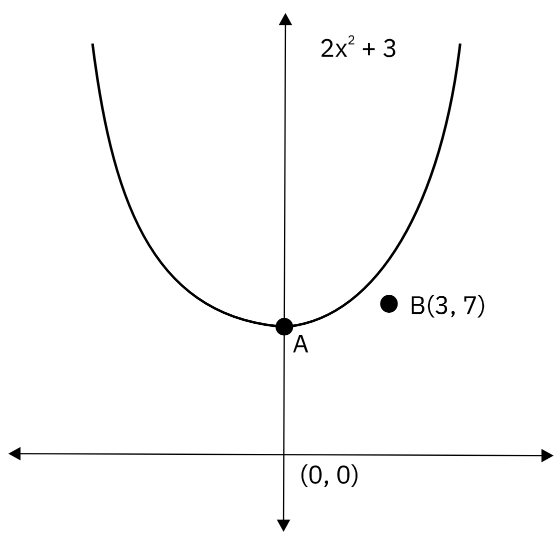
a) which fall within the region bounded by $|2x + y| = 6$ and $|2x - y| = 6$.

b) which satisfy both $y \le 5 - |2x|$ and $y \ge |x| - 4$

c) which satisfy both $4y \ge x^2 - 24$ and $4y \le 24 - x^2$

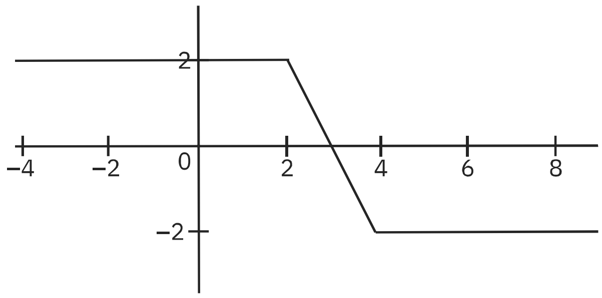
DIRECTIONS for questions 11 to 17: Choose the correct alternative.

11. The curve $y = 2x^2 + 3$ is shifted horizontally and vertically such that point A now corresponds to point B (3, 7). What will be the equation of the shifted curve?



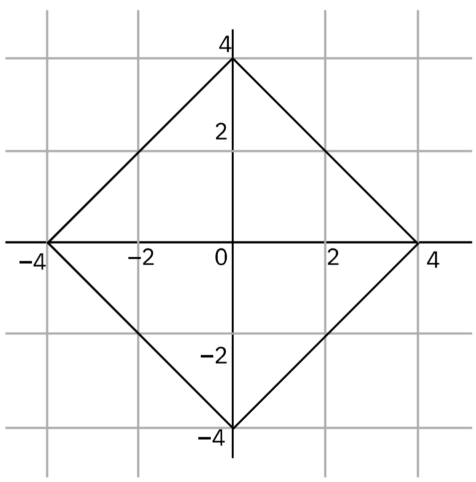
1) $y = 2x^2 - 12x + 25$
2) $y = 2x^2 - 12x + 18$
3) $y = 2x^2 - 12x + 11$
4) $y = 2x^2 + 12x + 25$

12. What is the equation of the following graph?



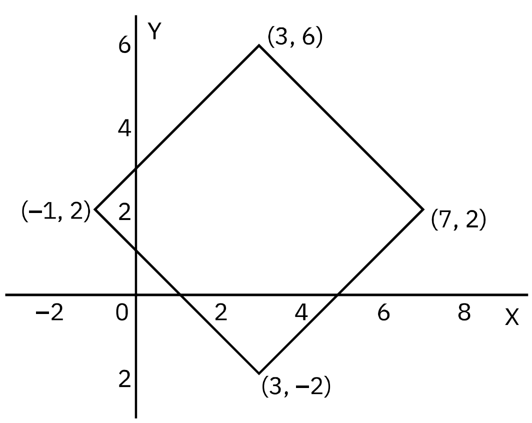
1) $y = |x - 4| - |x - 2|$
2) $y = |x + 4| - |x - 2|$
3) $y = |x - 4| - |x + 4|$
4) $y = |x + 4| - |x + 2|$

13. What is the equation of the graph described below?



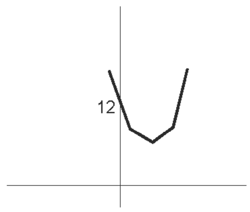
1) $|x| + |y| = 4$ 2) $|x + y| = 4$ 3) $|x| + |y| = 2$ 4) $|x + y| = 2$

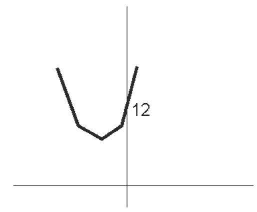
14. Which of the following equations describe the given graph?

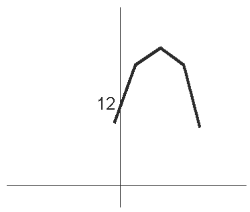


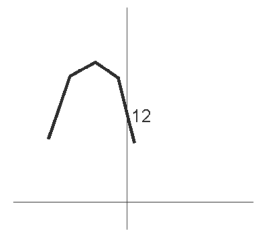
1) $|x + 3| + |y + 2| = 4$
2) $|x + 3| + |y + 2| = 16$
3) $|x - 3| + |y - 2| = 4$
4) $|x - 3| + |y - 2| = 16$

15. Which of the following graphs best describes the function f(x) = |x − 2| + |x − 4| + |x − 6|?

1) 

2) 

3) 

4) 

16. What is the maximum possible distance between a point that satisfies the equations $2 \leq x \leq 5$, $y \geq 1$ and $5x + 8y \leq 40$ and another point that satisfies the equations $2 \leq x \leq 5$, $y \leq 8$ and $5x + 8y \geq 40$?

1) $\sqrt{58}$ 2) $\sqrt{13}$ 3) $2\sqrt{13}$ 4) $\sqrt{41}$

17. Consider the function f(x) = | x − 2 | − | x − 3 |. Which of the following is true about the function?

1) A part of the graph of the function lies on the line y = 2x − 5
2) There exists a value of x for which f(-x) = − f(x)
3) $|f(x)| = 1$ if $|x| \geq 3$
4) All of the above are true.

DIRECTIONS for question 18: Solve as directed.

18. What is the minimum value of the function y = maximum (|4 − 3x|, |6 − 2x|)?

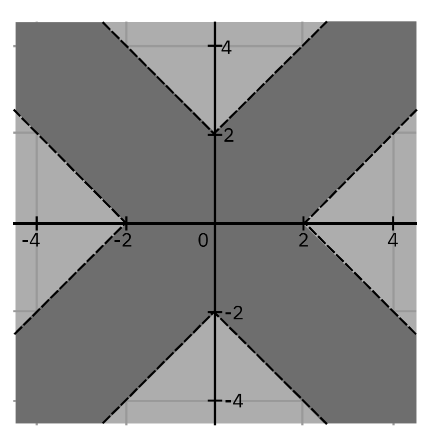
DIRECTIONS for questions 19 and 20: Choose the correct alternative.

19. The minimum value of the function $f(x) = a^2x^2 + (2ab + a)x + b^2 + b + 1$ is _____.

1) $\frac{1}{4}$ 2) $\frac{1}{2}$ 3) $\frac{4}{3}$ 4) $\frac{3}{4}$

20. Which of the following sets of equations represent the dark shaded area in the graph below?

1) |y| > |x| − 2 and |y| < |x| + 2
2) |y| > |x| − 2 and |y| > |x| + 2
3) |y| < |x| − 2 and |y| < |x| + 2
4) |y| < |x| − 2 and |y| > |x| + 2



QA-2.5 | FUNCTIONS AND GRAPHS — THEORY

Introduction

By definition, a function is a special case of a relation between the members of two sets.

The following table gives the values of y on a certain graph for a given value of x.

x	1	2	4	7	10	12	15
y	2	4	8	14	20	24	30

What are the observations that can be made from the table above? We can say that in the first column, value of x is 1 and value of y is 2 which is double the value of x. This relationship is observed across all the columns in the above table i.e., $y = 2x$.

In the above the example, x is the independent variable and y is the dependent variable. We can say that y is function of x. A function can be defined as follows:

"It is a **relationship** between two variables (**Independent** & **Dependent**) such that, for each value of the Independent variable there is **one and only one** value of the Dependent variable"

We normally write functions as $f(x)$ and read this as "function f of x". We can also use other letters for functions like $g(x)$ and $h(x)$.

Example

Area of the circle is the function of its radius. This can be written as $A = f(r) = \pi r^2$

Now consider following table:

x	−7	−4	−3	−2	2	3	4
y	49	16	9	4	4	9	16

Here, the relationship between y and x is $y = x^2$

It can be seen that at $x = -2$, $y = 4$

at $x = 2$, $y = 4$

Thus, at $y = 4$, there exist two values of x, but for every x there is only one value of y.

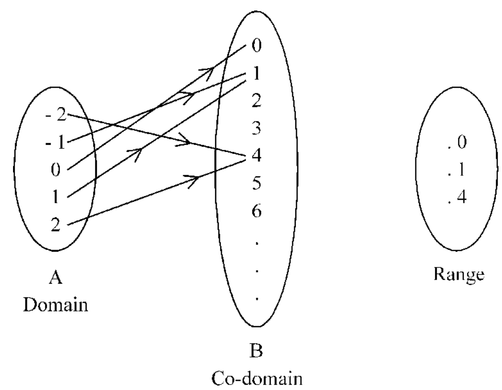
If all the values of x are denoted by a set A and all the values of y are denoted by a set B, for every value of set A, one and only one value of set B can be associated. Then we say that there exists a function from set A to set B.

In other words, if a general element of set A is denoted by x, and the corresponding element of set B is denoted by y, then we say that y is a function of x if, for every $x \in A$, one and only one value of $y \in B$ can be determined.

The function f, taking $x \in A$ to $y \in B$, is denoted as $f : A \rightarrow B$ and we write $f(x) = y$.
A is called the **domain** of the function of f and B is called the **co-domain** of the function f.
Since $x \in A$ is associated with unique $y \in B$, y is called the image of x under f or the value of the function f at x, commonly written as $y = f(x)$; $x \in A$. Also, x is called the pre-image of y.
The set $\{f(x)/x \in A\}$ is called the **range** of f.

Example

$A \equiv \{-2, -1, 0, 1, 2\}$ and B is the set of whole numbers.
For every $x \in A$, $f(x) \in B$ and $f(x) = x^2$
Here, A is the domain and B i.e., the set of whole numbers is the co-domain but the range of $f \equiv \{(-2)^2, (-1)^2, 0^2, 1^2, 2^2\}$.



f(a) is the value of the function f, when x takes the value a, i.e., when x is replaced by a. The elements of the co-domain which is equal to f(x) form the range.
when $x = -2$, $f(-2) = (-2)^2 = 4$
when $x = -1$, $f(-1) = (-1)^2 = 1$
when $x = 0$, $f(0) = (0)^2 = 0$
when $x = 1$, $f(1) = (1)^2 = 1$
when $x = 2$, $f(2) = (2)^2 = 4$

About Functions

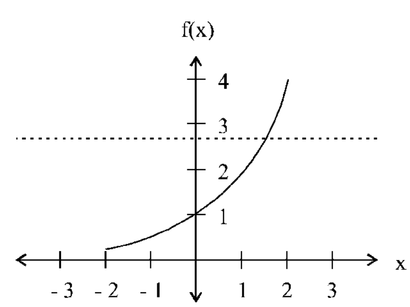
1. Each element of the set A must be associated with a unique element of set B.
2. Two or more elements of the set A may be associated with the same element of the set B.
3. There may be some elements of the set B, i.e., in the co-domain, which are not assigned to any element of the set A.
4. Range is a subset of co-domain.

Classification of Functions

1. On the basis of correspondence

One-one function

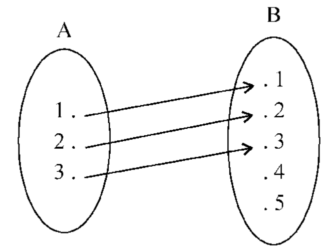
A function f is called a one-one function if no two different elements in A have the same image in B i.e., each element of A has one and only one image in B and each element of B has one and only one pre-image in A. Thus, the function is a one-one correspondence.



Graphically, a function is one-one if and only if no line parallel to the x-axis meets the graph of the function in more than one point.

Example

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ and $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

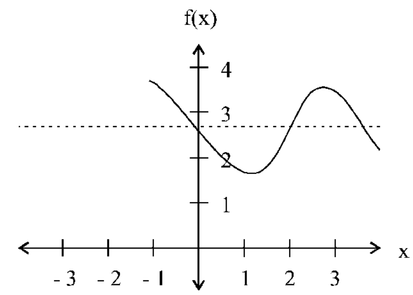


Note: The dots represent any point in the set.

Many-one function

A function f : A → B is called a many-one function if at least two elements in A have the same image in B i.e., at least one element of B has more than one pre-image in A.

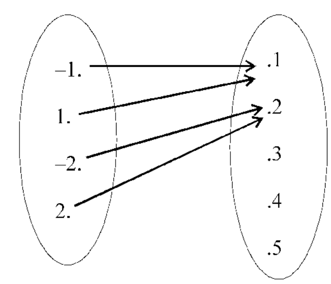
Graphically, a function is many-one if and only if a line parallel to the x-axis meets the graph of the function in more than one point.



Note: One-many function does not exist.

Example

$f(x_1) = f(x_2)$ does not imply $x_1 = x_2$;
or $x_1 \neq x_2$ but $f(x_1) = f(x_2)$ for some x_1 and $x_2 \in A$

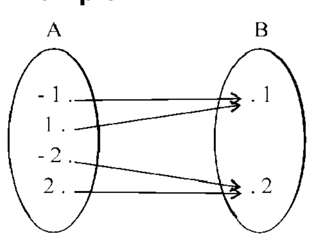


Onto function

A function f : A → B is called onto function if for every element y of B there is atleast one element in A so that f(x) = y

In an onto function, range ≡ co-domain.

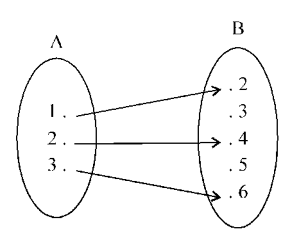
Example



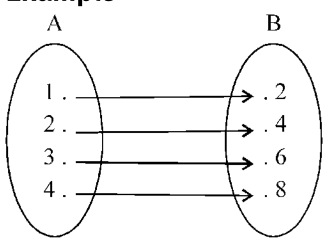
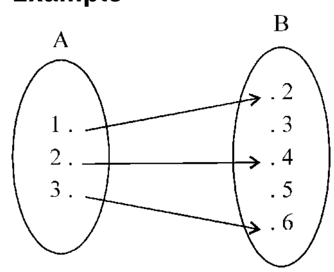
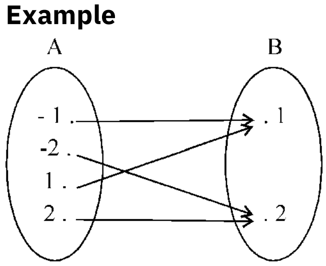
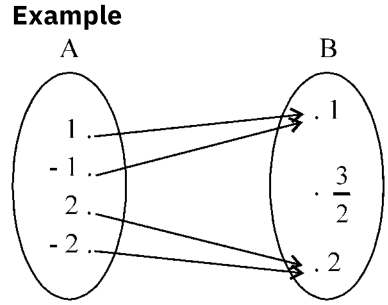
Into function

A function f : A → B is called an into function if there is atleast one element of the set B which has no pre-image in the set A.

In an into function, range is a proper subset of co-domain.



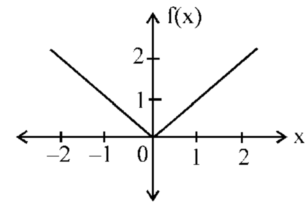
Based on the correspondence between elements of two sets, functions can be classified as the following types:

One-one-onto function	**One-one-into function**
If a function f : A $\rightarrow$ B is both a one-one function and an onto function then it is one-one onto function. i.e. (i) Range $\equiv$ Co-domain and (ii) $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ **Example** 	A function f : A $\rightarrow$ B is a one-one into function if it is both one-one and into function. i.e., (i) Range $\in$ Co-domain (ii) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ **Example** 
Many-one-onto function	**Many-one-into function**
If a function f : A $\rightarrow$ B is both many-one and onto function then it is a many-one onto function. i.e., (i) Range $\equiv$ Co-domain and (ii) $x_1 \neq x_2$ but $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$ **Example** 	A function f : A $\rightarrow$ B which is both many-one and into function is called a many-one into function. i.e., (i) Range $\in$ Co-domain (ii) $x_1 \neq x_2$ but $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$ **Example** 

2. On the basis of Symmetry

Even function

A function f: A $\rightarrow$ B is called an even function if $f(-x) = f(x)$ for all $x \in A$.



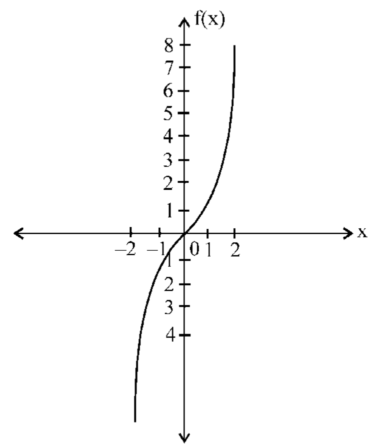
Example

Consider the graph of the function $f(x) = |x|$.

Note $f(-2) = |-2| = 2 = |2| = f(2)$
$f(-1) = |-1| = 1 = |1| = f(1)$
As can be seen,
the graph of an even function is symmetric about the function axis.

Odd function

A function f: A $\rightarrow$ B is called an odd function if $f(-x) = -f(x)$ for all $x \in A$.



Example

$f(x) = x^3$
$f(-x) = (-x)^3 = -x^3 \quad \therefore\ f(-x) = -f(x)$
In particular $f(1) = 1^3 = 1$
$f(-1) = (-1)^3 = -1$

The graph of an odd function is a double reflection, first in the function axis and then in the other axis.

> **Note:** There are functions, which are neither even nor odd.
> **Example**
> $f(x) = x^3 + x^2 + x + 5.$

3. On the basis of Application

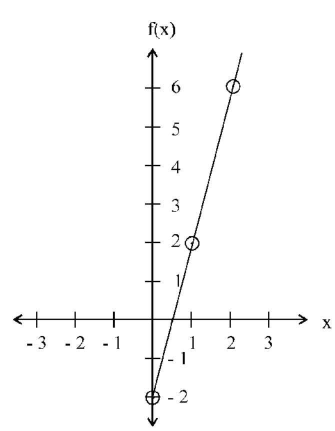
Linear function

A function f : A $\rightarrow$ B of the form $f(x) = ax + b$, where a and b are real numbers is called a linear function.

The graph of a linear function is always a straight line.

Example

Graph of $f(x) = 4x - 2$ is as shown below:



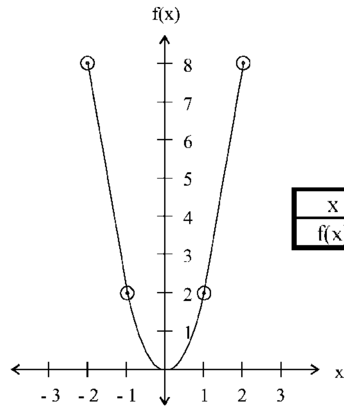
x	1	0	2
f(x)	2	-2	6

Quadratic function

A function f : A $\rightarrow$ B is called a quadratic function if it is of the form $y = ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$. The graph of such a function is a curve called a parabola.

Example

Graph of $f(x) = 2x^2$ is as shown below:
A quadratic function where b = 0 is an even function



x	-2	1	-1	2
f(x)	8	2	2	8

Modulus function

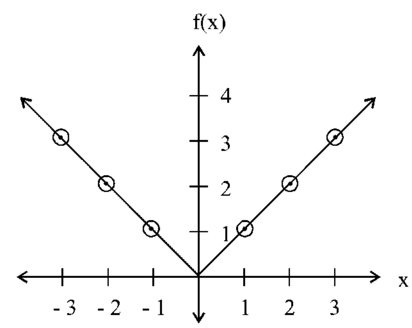
A function f : A $\rightarrow$ B is called a modulus function if $f(x) = P|Qx + R| + S$. A special application is $f(x) = |x|$, i.e., f(x) takes only the magnitude of x.

$f(x) = -x$ if $x < 0$
$\quad = x$ if $x \geq 0$

A modulus function is also an even function.

Example

The graph of a modulus function, $f(x) = |x|$ is as shown:
$f(1) = f(-1) = 1$, $f(2) = f(-2) = 2$, $f(3) = f(-3) = 3$

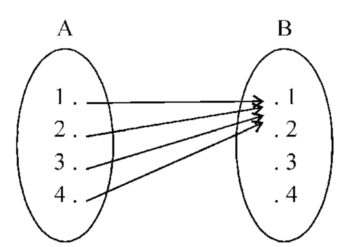


Constant function

A function f : A $\rightarrow$ B is a constant function if each element of the domain is associated with a single element of the co-domain i.e., the range of the function is a singleton set, i.e., f(x) = some constant. Constant function is also an even function.

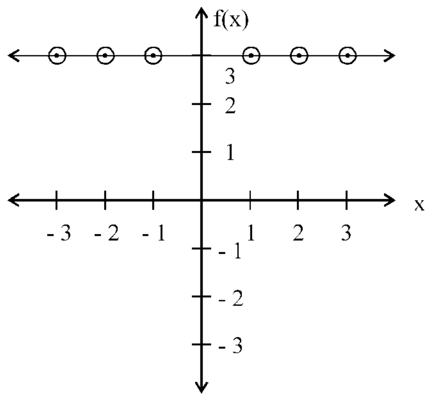
Example

The constant function f: A $\rightarrow$ B given by f(x) = 1.
Graphically, a constant function will be a straight line parallel to the variable axis (i.e., x-axis in general) or which is the same, perpendicular to the function axis (i.e., f(x) axis).



Example

f(x) = 3 (i.e., y = 3 if we consider the y-axis to be the function axis.)



x	1	2	3	–1	–2	–3
f(x)	3	3	3	3	3	3

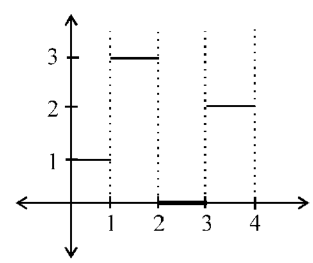
Step function

A function f : A $\rightarrow$ B is called step function, if it is a collection of constant functions over various parts of its domain;

Example

which is given by f: [0, 4] $\rightarrow$ {0, 1, 2, 3} as

f(x) = 0 if $2 \le x < 3$
= 1 if $0 \le x < 1$
= 2 if $3 \le x \le 4$
= 3 if $1 \le x < 2$



Example

A simpler and more common example of a step function is f(x) = [x]; i.e., the greatest integer less than or equal to x e.g., If f(x) = [x], then

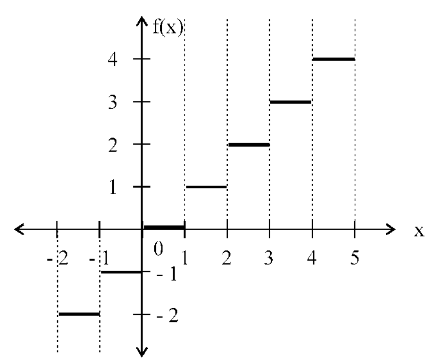
f(1.35) = 1, f(2.89) = 2, f(3) = 3, f(–1.29) = –2 etc.

Its graphical representation is as shown below:

f(x) = 0, where $0 \le x < 1$

f(x) = 1, where $1 \le x < 2$

f(x) = –1, where $-1 \le x < 0$

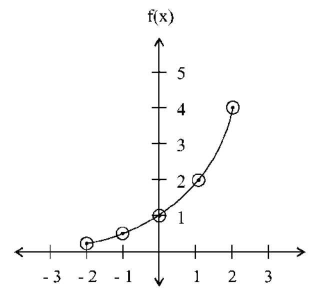


Exponential function

A function f : A $\rightarrow$ B of the form $f(x) = a^x$, where $a > 1$ and $x \in R$ is called an exponential function.

Example

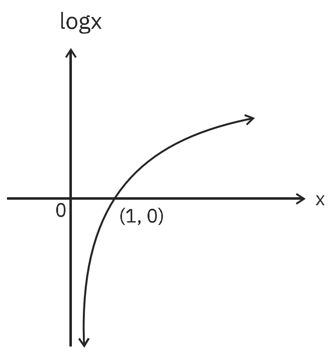
The graph of the exponential function $f(x) = 2^x$ is as shown below:



Note: The exponential functions a^x will never be negative for any value of x. The least value that it can have is 0 when x tends to negative infinity.

Logarithmic Function

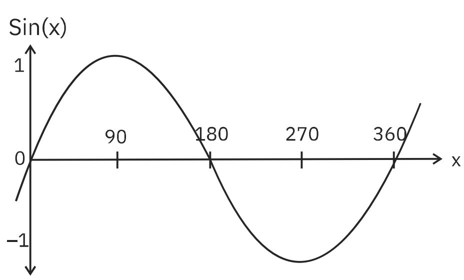
A function of the form f(x) = Log x, where x > 0, is called a 'Logarithmic Function'. The graph is as shown below:



- The Graph of 'Log Function' is only in 1^{st} and 4^{th} Quadrant as the 'Log' of negative values and '0' is not defined.

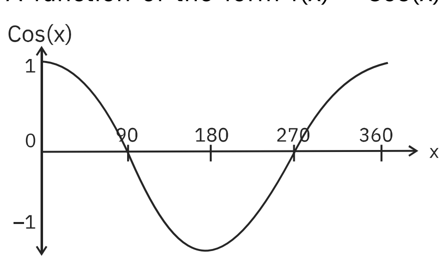
Sine & Cosine Function

A function of the form f(x) = sin(x) is called a 'Sine Function'. The graph is as shown below:



Sine function is an **Odd Function**.

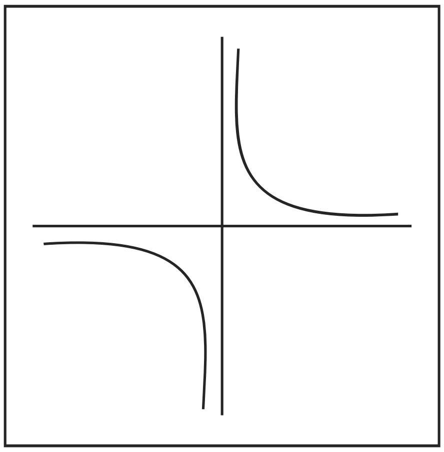
A function of the form f(x) = Cos(x) is called a 'Cosine Function'. The graph is as shown below:



Cosine function is an **Even Function**.

Reciprocal Function

A function of the form f(x) = 1/x, where $x \in R$ (except when x = 0) is called a 'Reciprocal Function'. The graph of the Reciprocal function is shown below:



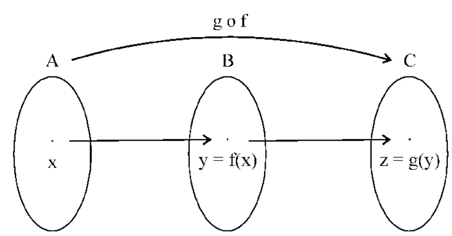
- It is an Odd Function.
- Its domain is the set of Real Numbers, except zero.

Composite and Inverse Functions

Composite function

If $f : A \rightarrow B$ is a function and $g : B \rightarrow C$ is another, then the composite of the functions f and g denoted by (gof) is a function defined from A to C.

$(gof) : A \rightarrow C$ is $(gof)(x) = g[f(x)], x \in A$



Let there be an element $x \in A$, then its image $y = f(x) \in B$ and B is the domain of the function g and $z = g(y) \in C$ is the range of g.

Example

If $f(x) = x^2$ and g(x) = 3x, find the value of gof for x = 1, x = 2 and x = 3.

$f(1) = 1^2 = 1$

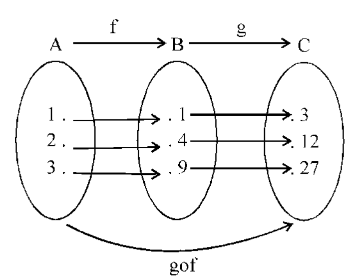
$(gof)(1) = g[f(1)] = g(1) = 3 \times 1 = 3$

$f(2) = 2^2 = 4$

$(gof)(2) = g[f(2)] = g(4) = 3 \times 4 = 12$

$f(3) = 3^2 = 9$

$(gof)(3) = g[f(3)] = g(9) = 3 \times 9 = 27$



Alternatively,

$(gof)(x) = g[f(x)] = 3[x^2] = 3x^2$

$\therefore$ (gof) (1) $= 3 \times 1^2 = 3$

(gof) (2) $= 3 \times 2^2 = 12$

(gof) (3) $= 3 \times 3^2 = 27$

Inverse function

A one-one onto function $f : A \to B$ has an inverse function f^{-1}. For each $y \in B$, $f^{-1}(y) \in A$ and is unique.

$f^{-1} : B \to A$ is a function defined by

$f^{-1}(y) = x$ if and only if $f(x) = y$.

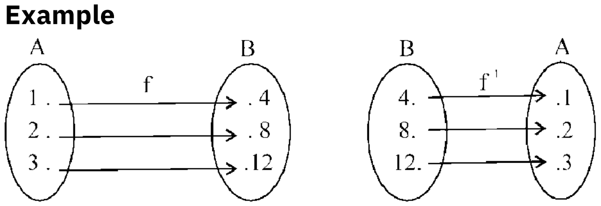
f^{-1} is the inverse of function of f.

Example

Consider $f : A \to B$ given by $f(x) = 4x$
where $A \equiv \{1, 2, 3\}$ and $B \equiv \{4, 8, 12\}$
Note, $f(1) = 4$, $f(2) = 8$, $f(3) = 12$.
Now, consider $f^{-1} : B \to A$ given by $f^{-1}(x) = \frac{x}{4}$
where $A \equiv \{1, 2, 3\}$ and $B \equiv \{4, 8, 12\}$.
Note, $f^{-1}(4) = 1$, $f^{-1}(8) = 2$, $f^{-1}(12) = 3$.
Thus, f^{-1} is a function which takes the images under f (i.e., f(x)) back to the respective pre-images (i.e., x).

Inverse functions should not be confused by interpreting inverse of f(x) as $f\left(\frac{1}{x}\right)$.

Example



Concept Builder 1

1. What will be the domain of the function $f(x) = \sin x$?
2. Is Range a Proper Subset of Co-domain for $f(x) = x$?
3. A line drawn parallel to Y-axis cuts the graph of f(x) at two different points. Is f(x) a function?
4. If a function f(x) is an Even function and another function g(x) is an Odd function then '$f(x) \times g(x)$' will be an Even function. True or False?
5. For $f(x) = x^2 + 2x + 2$, Is $f\left(\frac{1}{2}\right) + f(3) = f\left(\frac{1}{2} + 3\right)$
6. For the given $f(x) = |x|$, the graph of $f(x) + 2$ will shift towards the right. True or False
7. If f(x) is a Least Integer Function, then find $f\left(-\frac{1}{2}\right)$
8. If $f(x) = 2^x$, then the range of f(x) will be:
9. Is the 'Greatest Integer Function' an Odd function?

Answer Key

1. $\in \times R$	2. No	3. No	4. False
5. Yes	6. False	7. 0	8. R^+
9. No			

SOLVED EXAMPLES

Q : Is $f(x) = 2x^2 - 4$ an even function?

A : $f(x) = 2x^2 - 4$

$f(-x) = 2(-x)^2 - 4 = 2x^2 - 4$

$\therefore$ $f(-x) = f(x)$; Hence, $f(x) = 2x^2 - 4$ is an even function.

Q : Which of the following is an Odd Function?

(a) $|x^2| - 4x$ (b) $x + \dfrac{1}{x}$ (c) $2^{2x} + 3^{-x}$ (d) $x^2 + 4$

A : (a) $f(x) = |x^2| - 4x.$

$f(-x) = |(-x)^2| - 4(-x)$

$= |x^2| + 4x \neq -f(x)$ So, it is not an Odd Function

(b) $f(x) = x + \dfrac{1}{x}.$

$f(-x) = -x - \dfrac{1}{x}$

$= -(x + 1/x) = -f(x)$ So, it is an Odd Function

(c) $f(x) = x^2 + 4.$

$f(-x) = (-x)^2 + 4$

$= x^2 + 4 \neq -f(x)$ So, it is not an Odd Function

Q : Find f(4) if $f(x) = x^4 - 3x^3 + 6x^2 - 10x + 16.$

A : $f(x) = x^4 - 3x^3 + 6x^2 - 10x + 16$

$= (4)^4 - 3(4)^3 + 6(4)^2 - 10(4) + 16$

$= 256 - 192 + 96 - 40 + 16 = 136.$

Q : Find f(–1.5), if $f(z) = 2^{z-2}$

A : $f(z) = 2^{z-2}$

$$f(-1.5) = 2^{-1.5-2} = 2^{-3.5} = 2-\frac{7}{2} = \frac{1}{2^{\frac{7}{2}}} = \frac{1}{\sqrt{128}} = \frac{1}{8\sqrt{2}}$$

Q : $f(t) = 2t^2 + \dfrac{2}{t^2} + \dfrac{5}{t} + 5t$. Find $f\left(\dfrac{1}{t}\right)$.

A : $f(t) = 2t^2 + \dfrac{2}{t^2} + \dfrac{5}{t} + 5t$

$$f\left(\frac{1}{t}\right) = 2\left(\frac{1}{t}\right)^2 + \frac{2}{\left(\frac{1}{t}\right)^2} + \frac{5}{\left(\frac{1}{t}\right)} + 5\left(\frac{1}{t}\right) = \frac{2}{t^2} + 2t^2 + 5t + \frac{5}{t} = f(t)$$

Q : If $f(x) = 2x$ and $g(x) = x + 1$. Find (gof)(x).

A : $(gof)(x) = g[f(x)] = g(2x) = (2x) + 1 = 2x + 1$

Q : If $f(x, y) = 3x^2 - 2xy - y^2 + 4$, find $f(1,-1)$.

A : $f(x, y) = 3x^2 - 2xy - y^2 + 4$

$f(1,-1) = 3(1)^2 - 2(1)(-1) - (-1)^2 + 4 = 3 + 2 - 1 + 4 = 8$

Q : If $f(x) = 3x - 5$ and $f[g(x)] = 2x$, then find $g(x)$.

A : $f(x) = 3x - 5$

$f[g(x)] = 3g(x) - 5 = 2x \quad \therefore\ 3g(x) = 2x + 5$

$g(x) = \dfrac{2x+5}{3}$

Q : If $f(x) = ax^2 + bx + 2$. $f(1) = 3$ and $f(4) = 42$, find 'b'.

A : $f(x) = ax^2 + bx + 2$

$f(1) = a(1)^2 + b(1) + 2 = 3$

$\therefore\ a + b + 2 = 3$

$\therefore\ a + b = 1 \qquad \ldots$ (i)

$f(4) = a(4)^2 + b(4) + 2 = 42$

$\therefore\ 16a + 4b + 2 = 42$

$16a + 4b = 40 \qquad \ldots$ (ii)

Solving (i) and (ii) simultaneously, $b = -2$

Data for next 2 examples

$f(x) = 2x^2 + 1$ if $x \geq 1$

$= \dfrac{2}{x} + 1$ if $0 < x < 1$

$= 3|x|$ if $x \leq 0$

Q : Which of the following is true?

(i) $f\left(\dfrac{1}{2}\right) < f(1)$ (ii) $f(-2) = f(2)$ (iii) $f(-3) = f(2)$

A : $f\left(\dfrac{1}{2}\right) = \dfrac{2}{\frac{1}{2}} + 1 = 5$

$f(1) = 2(1)^2 + 1 = 3 \quad \therefore$ (i) is not true.

$f(2) = 2(2)^2 + 1 = 9$

$f(-2) = 3|-2| = 6 \quad \therefore$ (ii) is not true.

$f(-3) = 3|-3| = 9 \quad \therefore\ f(-3) = f(2)$ is true. $\therefore$ (iii) is true.

Q : Find $f[f(x)]$ if $x < 0$ and x is an integer.

A : $f(x) = 3|x|$ as $x < 0$

Now, $3|x| > 3$

$\therefore\ f[f(x)] = 2[3|x|]^2 + 1 = 2 \times 9x^2 + 1 = 18x^2 + 1$

Q : Find $f^{-1}(x)$ for $f(x) = 2 + 2/x$?

A : Consider $f(x) = y = 2 + 2/x$

So, $x = 2/(y-2)$

Since, $x = f^{-1}(y)$,

$f^{-1}(y) = 2 / (y-2)$

replacing y by x,

$\mathbf{f^{-1}(x) = 2 / (x-2)}$

Data for next 3 examples

The functions f and g are defined for natural numbers as:

$f(x) = 1 + x^2$, if x is even

$= (1 + x)^2$, otherwise

$g(x) = (x - 1)^2$, if x is prime

$= x^2 - 1$, otherwise

Q : Find f[g(x)], if x = 3.

A : $g(x) = (3 - 1)^2 = 4$

$f[g(x)] = f(4) = 1 + 4^2 = 17$

Q : For which number is $f[g(x)] = 1 + [(x - 1)^2]^2$ true.

A : $g(x) = (x - 1)^2$ which implies that x is prime.

$f[g(x)] = 1+ [(x - 1)^2]^2$ i.e., $f(x) = 1 + x^2$ is applied i.e., $(x - 1)^2$ is even.

These two conditions are true for all primes other than 2.

$\therefore$ $f[g(x)] = 1 + [(x - 1)^2]^2$ is applicable for all prime numbers except 2.

Q : Which of the following is not true?

(i) $f(2) + g(1) = f(2)$ (ii) $f(5) = g(7)$ (iii) $\frac{g(3)}{f(3)} = 4$

A : $g(1) = 1^2 - 1 = 0$

$\therefore$ $f(2) + g(1) = f(2)$ $\therefore$ (i) is true.

$f(5) = (1 + 5)^2 = 6^2$

$g(7) = (7 - 1)^2 = 6^2$ $\therefore$ (ii) is true.

$g(3) = (3 - 1)^2 = 2^2 = 4$

$f(3) = (1 + 3)^2 = 4^2 = 16$

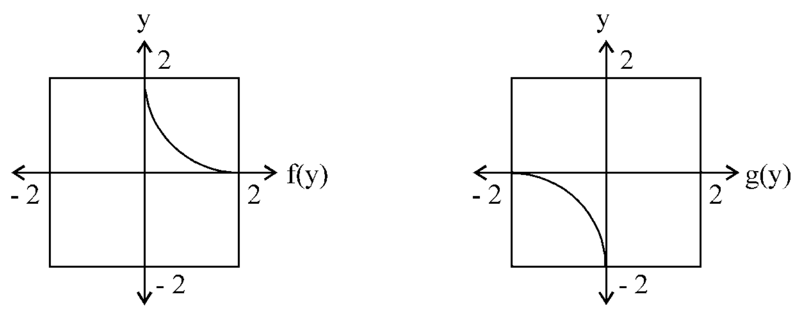
$\therefore$ $\frac{g(3)}{f(3)} = \frac{4}{16} = \frac{1}{4}$ $\therefore$ (iii) is not true.

We say that two functions f and g are

1. Even with respect to each other if g(–x) = f(x) for every x in the domain.
2. Odd with respect to each other if g(–x) = –f(x) every x in the domain.

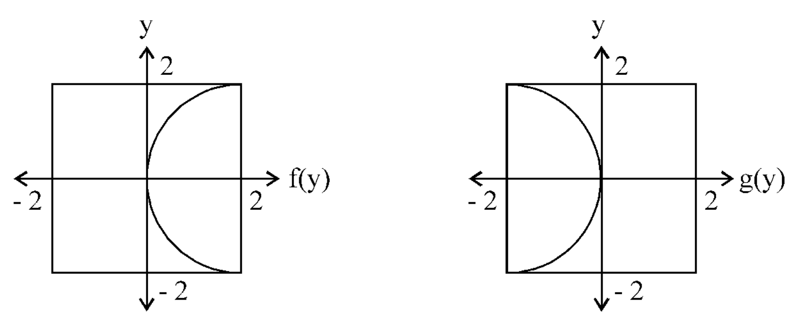
Directions for next 2 examples: Determine whether the graphs given below are even or odd or otherwise with respect to each other.

Q :



A : For positive y, e.g., y = a, we get f(y) as positive and for y = –a, g(y) as negative.
The nature of f(y) and g(y) is the same but they are in the first and third quadrants, respectively. Thus, f(–a) = –g(a).
∴ The functions f(y) and g(y) are odd with respect to each other.
Alternatively,
If we take reflection of the graph of f(y) with respect to the **Origin**, we get the graph g(y). So they are **Odd** with respect to each other.

Q :



A : f(y) and g(y) by themselves are even functions. f(y) and g(y) are curves of a parabola, where g(y) = $-y^2$ and f(y) = y^2.
∴ g(–y) = –f(y) and f(–y) = –g(y).
Thus, g(y) and f(y) are odd with respect to each other.
Alternatively,
As the graph g(y) is a double reflection of the function f(y), first about the function axis f(x) and then about the other axis, i.e., y, it represents an odd function.

CLASS EXERCISE

Teaser

Nick wants to travel to Hogsmeade from Hogwarts. There are trains leaving Hogwarts for Hogsmeade every hour starting 8:00 am. Similarly there are trains from Hogsmeade to Hogwarts every hour starting 8:00 am. The journey in both the directions takes exactly 4 hours. If Nick boards a train at Hogsmeade at 2 pm, how many trains travelling in the opposite direction will he meet on the way (excluding the endpoints)?

Functions and Graphs

For questions 1 to 6:

Consider the following functions defined from R to R (R denotes the set to real numbers)

$f(x) = x$ $g(x) = 3x$ $h(x) = x^2$ $k(x) = 2x - 5$

1. Find:

   1) k (10) 2) f [g (5)] 3) h o g (-5) – g o h (-5)

   4) k o g o h (x^2) 5) g^{-1} (4.5) 6) h o k $^{-1}$(3)

2. What is the range of:

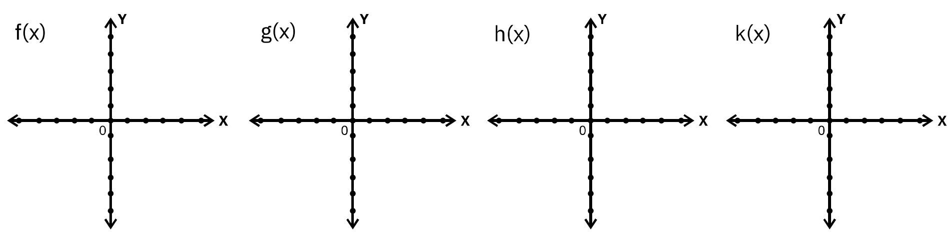
   1) h(x)

   2) k(x) if the domain of k(x) is reduced to the set (2,3), i.e. $\{x \in R: 2 < x < 3\}$

3. Are the following functions even / odd / neither:

   1) h o g (x) 2) g o f (x) 3) k(x) 4) $m(x) = x^3$

4. Draw rough graphs of the functions f(x), g(x), h(x) and k(x) defined above:



5. * Is $h^{-1}(y)$ a well-defined function on the set of real numbers?

6. *Which of the following functions are inverses of each other on the set of all real numbers?

   1) f(x) and $f^*(x) = \frac{1}{x}$ 2) h(x) and $h^*(x) = \sqrt{x}$

   3) k(x) and $k^*(x) = \frac{x+5}{2}$ 4) All the above

7. * Suppose a function f: A $\rightarrow$ B is defined where

   A: set of all real numbers and
   B: set of all non-negative real numbers.
   Indicate whether the following functions are into functions into or onto functions.

   1) $f(x) = x^2$ 2) $f(x) = 10^x$

   3) $f(x) = |x|$ 4) $f(x) = \frac{1}{|x|}$ $(x \neq 0)$

8. Draw rough graphs of the following functions:

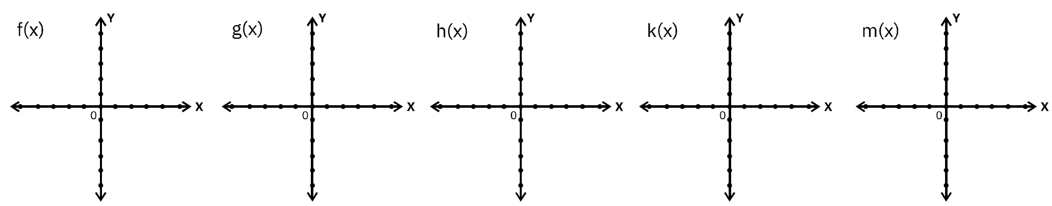
1) $f(x) = \frac{1}{x}$

2) $*g(x) = 2^x$

3) $h(x) = |x - 2| - 2$

4) $k(x) = 2$ for $x < 2$
$= 3$ for $2 \leq x < 3$
$= 4$ for $3 \leq x < 4$
$= 5$ for $x \geq 4$

5) $m(x) = (x - 3)(x + 4)$



Suppose operation # is defined on two real numbers such that
1) If a # b = b # a, the operation # is said to be commutative
2) If m (a # b) = ma # mb, where m is a constant, the operation # is said to be distributive
3) If (a # b) # c = a # (b # c), the operation # is said to be associative.

9. Consider the relation $a \# b = \frac{a}{b} - \frac{b}{a}$. Which is the following is true?
1) It is commutative but not distributive
2) It is distributive but not commutative
3) It is both commutative and distributive
4) It is neither commutative not distributive

Q-10-12

Suppose operations # and @ are defined for real numbers as follows

a # b = ab if both a and b are positive
= 1 otherwise

a @ b = $(a + b)^{a+b}$ if ab is positive
= 1 otherwise

10. $\frac{3\#4}{3@-4}$ =

1) $\frac{1}{12}$ 2) 12 3) 144 4) $\frac{1}{144}$

11. $\frac{[(1\#-1)\#-2]}{[(-2)@(1)]}$ =

1) 8 2) 1 3) 2 4) 4

12. Suppose $\left[\frac{(x\# - y)}{(-x@y)}\right] = 1$, what can be said about x and y?

1) Only one of x and y is positive while the other one is negative.
2) Both x and y are either positive or negative
3) Both x and y are positive but cannot be negative
4) Both x and y are negative but cannot be positive

13. Let f(x) be a function satisfying f(x) f(y) = f(xy) for all real x, y. If f(2) = 4, then what is the value of $\left(\frac{1}{2}\right)$?

1) 0 2) $\frac{1}{4}$ 3) $\frac{1}{2}$
4) 1 5) Cannot be determined

14. Which of the following represents the domain of the function $f(x) = \frac{\sqrt{x^2 - 4}}{x^2 - x - 12}$ correctly?

1) All real numbers except x = –3 and 4
2) All real numbers except x = 1, 2, –3 and 4
3) | x| > 2
4) All |x| $\geq$ 2 except x = –3 and 4

Higher Order Curves

15. Consider the polynomial P = (x – 1)(x + 2)(x – 4)(x + 5)

a) How many roots does P have?
b) What are the roots of P?
c) At a very large positive value of x, is P positive or negative?

If the function is of the form $f(x) = (x - \alpha)^{\text{even number}} \times (x - \beta)^{\text{odd number}}$, the graph of the polynomial just touches the x-axis at x = α and intersects the x-axis at x = β.
In general, a function with degree n can touch/intersect x-axis in maximum n points.

16. *Draw rough graphs of the following functions

1. $f(x) = |(x - 1)^2 (x - 2)|$
2. $f(x) = |(x - 1)(x - 2)^2 (x - 3)^3 (x - 4)|$

In a graph of an absolute value of f(x), the part of the graph where f(x)>0 (i.e. where the graph is above x-axis) remains unchanged. However the part of the graph where f(x)<0 (i.e. where graph is below x-axis) is chopped off and the mirror image of that part appears above x-axis.

Points of intersection of two curves

17. Find the points of intersections of the following pair of functions
    1. $y = x^2 - 7x + 10$ and $y = 4$
    2. $y = x^2 - 7x + 10$ and $y = x - 4$
    3. $y = 2x^2 - 7x + 5$ and $y = x^2 - 3x + 2$
    4. *$y = x^2 - 7x + 10$ and $x = 4$
    5. *$y = 5x^2 - 28x + 15$ and $y = 3x^2 - 19x + 6$

> Graphs of a line and a function representing a quadratic function can intersect in either 0/1/2 points depending on the value of discriminant obtained. If $\Delta > 0$, the line intersects the quadratic curve in two points, if $\Delta = 0$, the line touches the quadratic function in one point. If $\Delta < 0$, the line does not intersect the quadratic function.

6. $y = 2x^2 + 10x + 29$ and $y = x^2 + 4x + 5$
7. $y = x^2 + 6x + 14$ and $y = x^2 + 4x + 10$

> Two quadratic functions can intersect in zero/one or two points.

8. $y = x^3 + 6x^2 + 6x - 9$ and $y = x^3 + 2x^2 + 10x - 10$

> In general, graphs of a function of power m and power n (m>n) will intersect in at most m points.

Common roots of two curves

18. Calculate the common roots between the following pairs of equations
    1. $y = x^3 + 6x^2 + 11x + 6$ and $y = x^3 + 7x^2 + 11x + 5$
    2. $y = x^3 + 6x^2 + 11x + 6$ and $y = 2x^3 + 6x^2 + 10x + 6$
    3. *$y = 2x^3 - 6x^2 - 2x + 6$ and $y = x^3 + 2x^2 - x - 2$

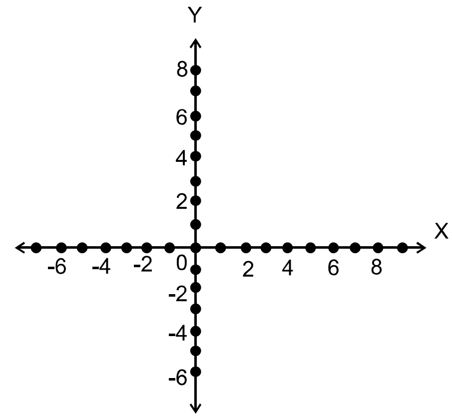
Challengers

1. Consider the function f(x) = max (x + 3, 7 − x).

   This means that for every value of x, y will take the larger value out of (x + 3) and (7 − x), i.e. the 'maximum of these two values.

   Then:

   1) Find the values of f(3) and f(-2)
   2) For which value/s of x is f(x) = 8?
   3) Find the minimum value that f(x) can take
   4) Plot the graph of f(x)



2. If f(x) is a well defined function on the set of all positive real numbers such that f(xy) = f(x) + f(y) for any positive real numbers x and y then which of the following is certainly true:

   1) f(0) = 0 2) f(1) = 1 3) f(1) = 0 4) None of these

Q-3-4

Suppose three functions are defined as follows

f(x) = 2x + 4

g(x) = $\frac{x-4}{2}$

h(x) = x − 4

3. f o g o f o g o f o g o f o g (2000) =

   1) 2000 2) 4004 3) 998 4) 1996

4. h o f o h o f o h (2004) =

   1) 250 2) 16000 3) 4000 4) 8000

Directions for question 5: Answer the questions on the basis of the information given below.

$f_1(x)$ = x $0 \le x \le 1$

= 1 $x \ge 1$

= 0 Otherwise

$f_2(x) = f_1(-x)$ for all (x)

$f_3(x) = -f_2(x)$ for all (x)

$f_4(x) = f_3(-x)$ for all (x)

5. How many of the following products are necessarily zero for every x:

   $f_1(x)f_2(x)$, $f_2(x)f_3(x)$, $f_2(x)f_4(x)$?

   1) 0 2) 1 3) 2 4) 3

6. A function f(x) satisfies f(1) = 7200 and f(1) + f(2) + ... + f(n) = n^2. f(n) for all positive integers n > 1. Which of the following is the value of f(8)?

   1) 200 2) 400 3) 800 4) 900

PRACTICE EXERCISE - 1

DIRECTIONS for questions 1 to 4: Choose the correct alternative.

1. The functions f(x) and g(x) are related as f(g(x)) = xg(f(f(x))), where $f(x) = \frac{x}{x-1}$. What could be the functional form of g(x)?

   1) $\frac{x}{x+1}$ 2) $\frac{x+1}{x}$ 3) $\frac{x}{x-1}$ 4) $\frac{x-1}{x}$

2. If a function is defined as:
   $f(x, y) = f(2x - y, x - 2y) + x + y$ for $x - y < 0$
   $= f(y - 2x, 2y - x) - x - y$ for $x - y > 0$
   then, find the value of f(–9, 0) – f(1, 0).

   1) –9 2) 9 3) 16 4) –16

3. The number f(n); $n \in N$, satisfies the recurrence f(n + 2) = f(n) + 2 for $n \in N$, with f(1) = 1 and f(2) = 3. What is the ratio of f(45) to f(14)?

   1) 1 2) 2 3) 3 4) Data insufficient

4. If $f(x) = ax^2 - bx + 7$ and f(2) = 5 and f(4) = 11 find the value of a + b.

   1) 1 2) 3 3) 2 4) 4

DIRECTIONS for questions 5 and 6: Solve as directed.

5. Suppose a function f(x) is defined as follows: $3f(x) + 4f\left(\frac{3x+9}{x-3}\right) = 7x$, What is the value of f(6)?

6. Suppose a function on natural numbers x and y is defined as f(x × y) = f(x) × f(y). If f(2) = 2, what is the value of f(512) – f(1)?

DIRECTIONS for questions 7 and 8: Refer to the data below and answer the questions that follows.

A function f(x) is said to be even if f(-x) = f(x), and odd if f(-x) = –f(x). Thus, for example, the function given by $f(x) = x^2$ is even, while the function given by $f(x) = x^3$ is odd. Using this definition, answer the following questions.

7. The function given by $f(x)^3 = |x|^3$ is

   1) even 2) odd 3) neither 4) both

8. The sum of two odd functions

   1) is always an even function
   2) is always an odd function
   3) is sometimes odd and sometimes even
   4) may be neither odd nor even

Direction for questions 9 and 10: Refer to the data below and answer the questions that follows.

A, S, M and D are functions of x and y, and they are defined as follows.

$A(x, y) = x + y$

$S(x, y) = x - y$

$M(x, y) = xy$

$D(x, y) = \frac{x}{y}$, $y \neq 0$

9. What is the value of M(M(A(M(x, y), S(y, x)), x), A(y, x)) for x = 2, y = 3?

   1) 60   2) 140   3) 25   4) 70

(Past CAT question)

10. What is the value of S[M(D(A(a, b), 2), D(A(a, b), 2)), M(D(S(a, b), 2), D(S(a, b), 2))]?

    1) $a^2 + b^2$ 2) ab   3) $a^2 - b^2$ 4) b

(Past CAT question)

DIRECTIONS for questions 11 to 16: Choose the correct alternative.

11. If $f(x) = x^2 + 6x + 8$, how many real values of ‘x’ satisfy f(f(x)) = 0 ?

    1) 0   2) 2   3) 3   4) 4

12. P and Q are real numbers.

    $f(P, Q) = |P|\ |Q|$.

    $g(P, Q) = -f(P, Q)$

    $h(P, Q) = -g(P, Q)$

    Which of the following is not true?

    1) h(P, Q) = f(P, Q) for any value of P and Q.
    2) g(P, Q) = f(P, Q) for P > 0 and Q > 0.
    3) g(P, Q) × f(P, Q) = g(P, Q) × h(P, Q).
    4) None of these.

13. A function can sometimes reflect on itself, i.e. if y = f(x), then x = f(y). Both of them retain the same structure and form. Which of the following functions has this property?

    1) $\frac{2x+3}{3x+4}$
    2) $\frac{2x+3}{3x-2}$
    3) $\frac{3x+4}{4x-5}$
    4) None of the above.

14. If y = f(x) and $f(x) = \frac{1-x}{1+x}$, which of the following is true?

    1) $f(2x) = f(x) - 1$ 2) $x = f(2y) - 1$ 3) $f\left(\frac{1}{x}\right) = f(x)$ 4) $x = f(y)$

(Past CAT question)

15. For two positive integers a and b define the function h (a, b) as the greatest common factor (gcf) of a, b. Let A be a set of n positive integers G(A), the gcf of the elements of set A is computed by repeatedly using the function h. The minimum number of times h is required to be used to compute G is:

1) $\frac{n}{2}$ 2) n − 1 3) n 4) None of these

(Past CAT question)

16. There are two whole numbers – x and y. A function of x and y is defined such that:

$f(0,y) = y + 1,$

$f(x + 1,0) = f(x, 1)$

$f(x + 1,y + 1) = f(x, f(x + 1,y))$

What is the value of f (1,2)?

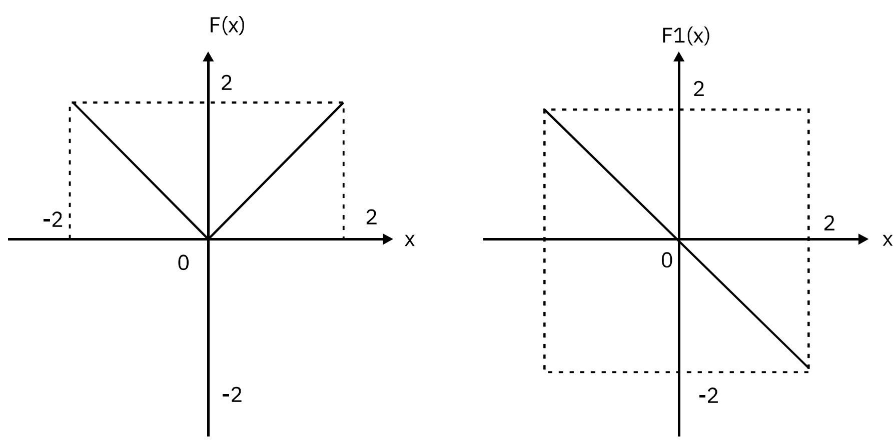
1) 2 2) 4 3) 3 4) Cannot be determined

(Past CAT question)

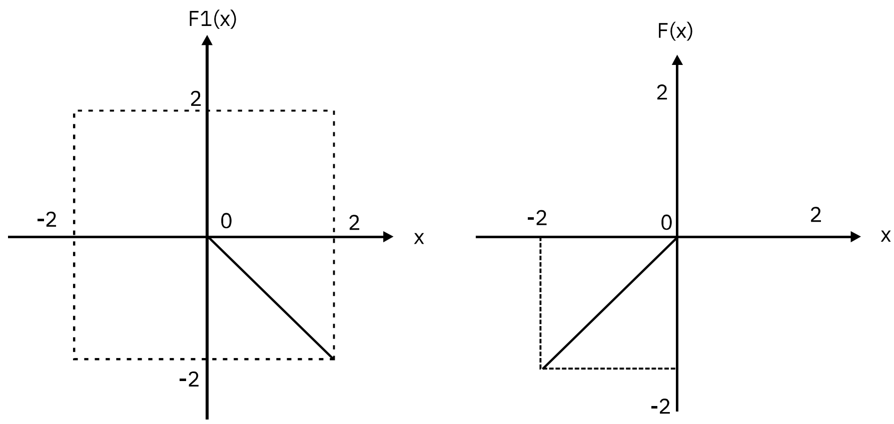
Directions for questions 17 to 20: Choose the answer as:

1) if F1(x) = -F(x)
2) if F1(x) = F(-x)
3) if F1(x) = -F(-x)
4) if None of the above is true

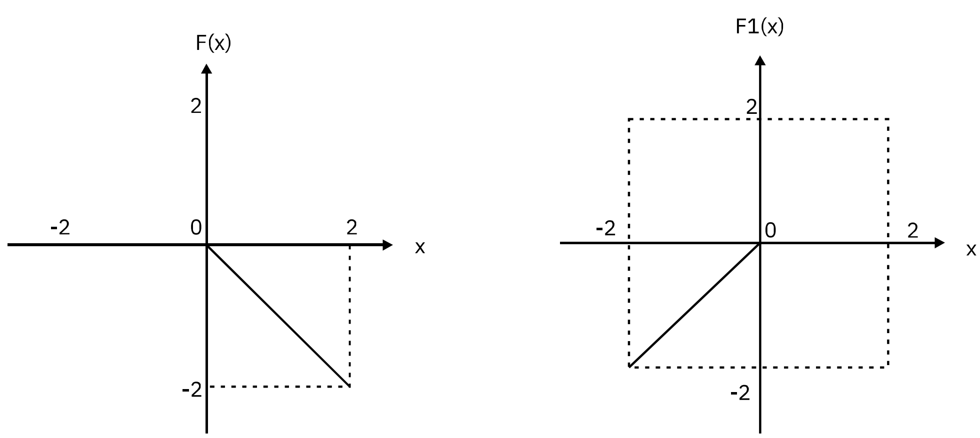
17.



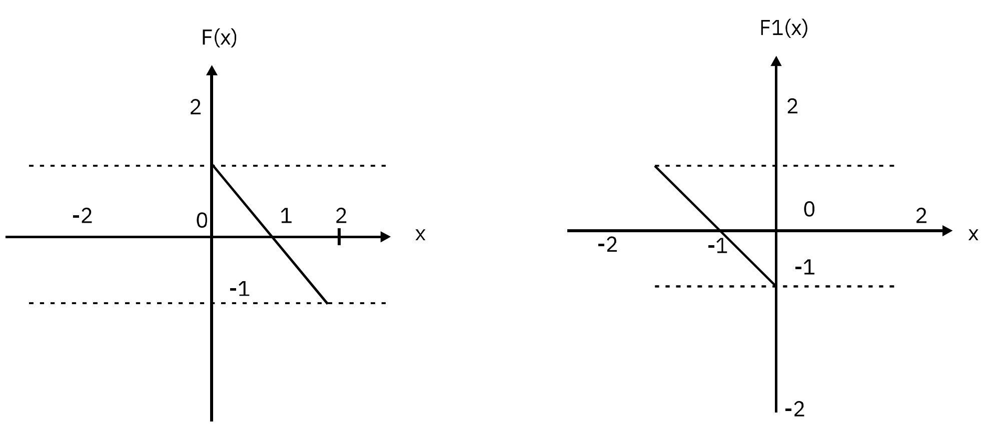
18.


F1(x)
2
-2
0
2
x
-2
F(x)
2
-2
0
2
x
-2

19.


F(x)
2
-2
0
2
x
-2
F1(x)
2
-2
0
2
x
-2

20.


F(x)
2
-2
0
1
2
x
-1
F1(x)
2
0
2
-2
-1
x
-1
-2

PRACTICE EXERCISE - 2

DIRECTIONS for questions 1 and 2: Choose the correct alternative.

1. A function, f(x), is defined as follows:

$f(x) = x + x^2 + x^3$ if $x > 0$

$= g(x)$ if $x \leq 0$

$g(x) = 2x + 30$

Find f{g[f(g(x))]}, if x = −14.

1) 58 2) 0 3) 198534 4) 2069

2. The function $f : N \to N$, satisfies the recurrence $f(n + 2) = 2f(n) + 1$, for $n \in I$, with $f(1) = 1$ and $f(2) = 3$. What is (approximately) the ratio of f(41) to f(21)?

1) 128 2) 1024 3) 8192 4) 65536

DIRECTIONS for questions 3 to 5: Refer to the data below and answer the questions that follows. Graphs of some functions are given. Mark option:

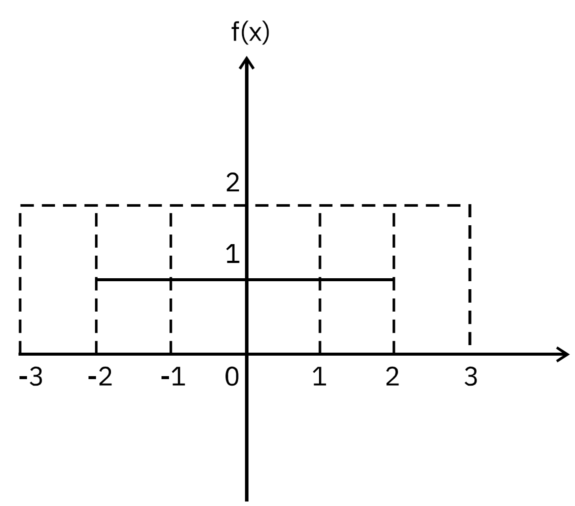
1. If f(x) = 3f(−x)

2. If f(x) = f(−x)

3. If f(x) = −f(−x)

4. If 3f(x) = 6f(−x), for x ≥ 0

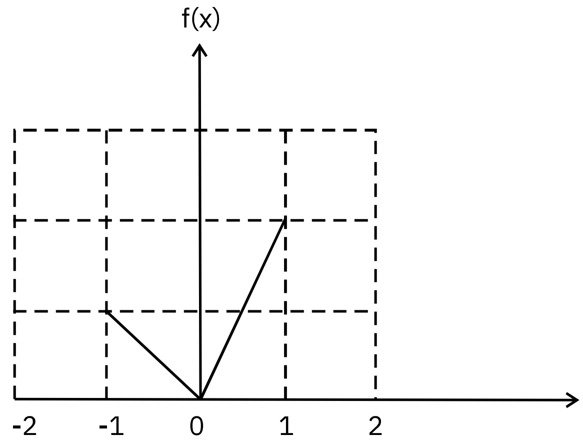
3.



1) 1 2) 2 3) 3 4) 4

(Past CAT question)

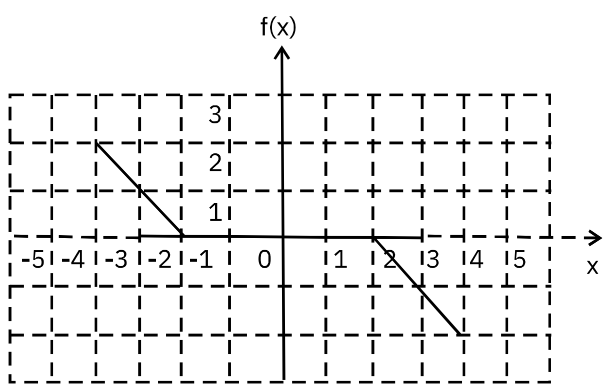
4.



1) 1 2) 2 3) 3 4) 4

(Past CAT question)

5.



1) 1 2) 2 3) 3 4) 4

(Past CAT question)

DIRECTIONS for question 6: Solve as directed.

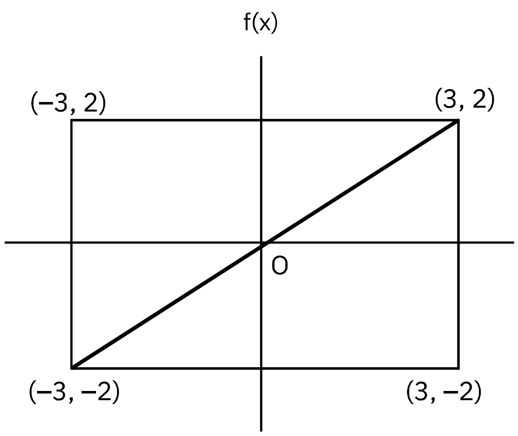
6. If we define a function f(x) on natural numbers as f(x + 1) = f(x) + x and f(1) = 1, what is the sum f(1) + f(2) + f(3) + ... + f(30)?

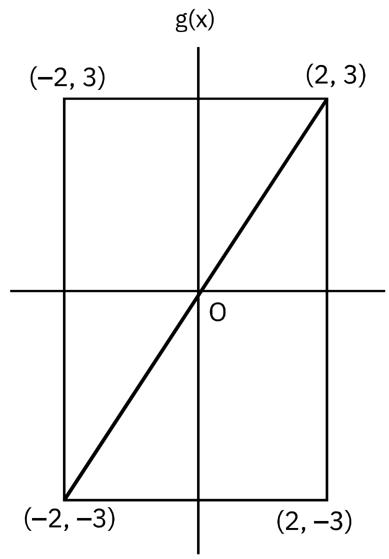
DIRECTIONS for questions 7 to 9: Choose the correct alternative.

7. If a function f(x) is defined as $f(x) = \frac{1}{1+x}$ and $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$ and so on, what is the product of $f(x) \times f^2(x) \times \ldots \times f^{10}(x)$?

1) $\frac{1}{134+89x}$ 2) $\frac{1}{55+34x}$ 3) $\frac{1}{89+55x}$ 4) $\frac{1}{89+34x}$

8. The graph of functions f(x) and g(x) are shown below.





Which of the following is true?

1) $f(x) = g(x)$ 2) $f(-x) = g(-x)$ 3) $f(x) = -g(x)$ 4) None of these

9. If a function f(x) satisfies the equation $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, $x \neq 0$, then f(x) equals

1) $x^2 - 2$ for $x \neq 0$
2) $x^2 - 2$ for all satisfying $|x| \geq 2$
3) $x^2 - 2$ for all satisfying $|x| < 2$
4) None of these

DIRECTIONS for questions 10 and 11: Refer to the data below and answer the questions that follows.

A function f(x, y) is defined such that
$f(x, y) = (x + y)^{0.5}$ (the positive root) if $(x + y)^{0.5}$ is real
$f(x, y) = (x + y)^2$ otherwise
$g(x, y) = (x + y)^2$ if $(x + y)^{0.5}$ is real
$g(x, y) = -(x + y)$ otherwise

10. Which expression yields positive values for non zero and real values of x and y?

1) $f(x,y) - g(x,y)$ 2) $f(x,y) - [g(x,y)]^2$ 3) $g(x,y) - [f(x,y)]^2$ 4) $f(x,y) + g(x,y)$

(Past CAT question)

11. When is $f(x,y) > g(x,y)$?

1) $y \geq x$
2) Both x & y are less than −1
3) Both x & y are greater than 0
4) Both x & y are less than 0

(Past CAT question)

DIRECTIONS for questions 12 to 14: Choose the correct alternative.

12. f is function defined on natural numbers such that

$2f(n) \times f(2n + 1) = f(2n) \times [2f(n) + 1]$ and $8f(n) > f(2n) > 4f(n)$. Find the value of f(12).

1) $6^3f(1) + 108$ 2) $6f(1) + 9$ 3) $6^2f(1) + 6$ 4) $f(1) + 108$

13. A function f(x) is defined as $f(x) = ax^2 + bx + |c|$, $(a \neq 0)$. If it is known that $f(1) = f(-2)$, under what conditions will the function have real roots?

1) $a = b$; $a < 4c$ 2) $a = b$; $a > 4c$
3) $a = b$; $\frac{-b}{4} < c < \frac{b}{4}$ 4) $a \neq b$; $\frac{-b}{4} < c < \frac{b}{4}$

14. If $f(x) = |x^2 - 7|$, find $f(x) - f(f(x)) + f(f(f(x))) - f(f(f(f(x)))) + \ldots$ up to 20 terms, at $x = 2$.

1) 10 2) 20 3) 30 4) Cannot be determined

DIRECTIONS for questions 15 and 16: Refer to the data below and answer the questions that follows.

For real number x, let $f(x) = \frac{1}{1+x}$, if x is non-negative
$= 1 + x$, if x is negative
$f^n(x) = f(f^{n-1}(x))$, n = 2, 3, ...

15. What is the value of product, $f(2)\ f^2(2)\ f^3(2)\ f^4(2)\ f^5(2)$?

1) $\frac{1}{3}$ 2) 3 3) $\frac{1}{18}$ 4) None of these

(Past CAT question)

16. If r is an integer ≥ 2, then what is the value of $f^{r-1}(-r) + f^r(-r) + f^{r+1}(-r)$?

1) −1 2) 0 3) 1 4) None of these

(Past CAT question)

DIRECTIONS for questions 17 to 20: Choose the correct alternative.

17. $f(x) = ax^3 - bx^2 + cx + 7$, if $f(1) = 12$, $f(2) = 27$, what is the value of $(2a - c)$?

1) 1 2) 0 3) −1 4) Cannot be determined

18. A function defined on a two-dimensional plane is called a 'pseudometric', if it satisfies the following properties:
i) $f((x_1, x_2), (y_1, y_2)) > 0$
ii) $f((x_1, x_2), (y_1, y_2)) = 0$ if and only if $(x_1, x_2) = (y_1, y_2)$.
Which of the following is not a pseudometric?

1) $f((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$
2) $f((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
3) $f((x_1, x_2), (y_1, y_2)) = \text{maximum } \{|x_1 - y_1|, |x_2 - y_2|\}$
4) $f((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 - (x_2 - y_2)^2$

19. If $f(x) = -3x^3 + \frac{5}{x^2} + 5x^2 - \frac{3}{x^3}$ which of the following is/are true?

I. $f(x) = f(-x)$ II. $f\left(\frac{1}{x}\right) = f(x)$

1) Only I 2) Only II 3) Both I and II 4) Neither I nor II

20. Find x if $f(x) = 1 + x - x^2$ and $f(x + 1) = f(x + 2)$

1) 0 2) 1 3) − 1 4) 2

QA-2.6 | INEQUALITIES — THEORY

Definition of an inequation

A mathematical statement, which states that two expressions are not equal is called an **inequation** and is denoted by $\mathbf{x \neq y}$.

A mathematical statement, which states that one expression is greater than the other in value is called an **inequality**.

Symbols of inequality

$x > y$ for x is greater than y.
$x < y$ for x is less than y.
$x \geq y$ for x is greater than or equal to y.
$x \leq y$ for x is less than or equal to y.

Any quantity x is said to be greater than another quantity y when '$x - y$' is positive. Thus -2 is greater than -3 as $-2 - (-3) = 1$ is positive.

Any quantity x is said to be less than y when '$x - y$' is negative. Thus, -5 is less than -2 as $-5 - (-2) = -3$ is negative.

Unconditional Inequality

An unconditional inequality is one that holds for all values of the variables.

Example

$3x^2 + 2 > x - 2$ which is true for all values of x.

Conditional Inequality

A conditional inequality is true only for certain values of the variable.

Example

$3x + 2 > 8 \quad \therefore 3x > 6 \quad \therefore x > 2$

Thus, the inequation $3x + 2 > 8$ is true only for those values of x which are greater than 2.

The set of values, which satisfies the given statement is called the **solution set or truth set**.

Example

Solution set of $-2 < x < 7$, where x is an integer is $(-1, 0, 1, 2, 3, 4, 5, 6)$

Rules of Inequalities

1. An inequality will still hold after each side has been increased or diminished by the same number.
   i.e., if $a > b$ and c is any number
   $a + c > b + c$
   $a - c > b - c$
2. An inequality will still hold after each side has been multiplied or divided by the **same positive quantity**. i.e., if $a > b$ and $c > 0$

$ac > bc$

$\frac{a}{c} > \frac{b}{c}$

3. If the sides of an inequality are multiplied or divided by the same negative quantity, the sign of the inequality must be reversed. i.e., if $a > b$ and $c < 0$, then $ac < bc$

Example

$-2 < 4$

Multiplying both sides of the inequality by -1, we get LHS = 2 and RHS = -4.

$\because\ 2 > -4$

Therefore on multiplying both sides of the inequality by the same negative value, the sign of the inequality gets reversed.

4. In an inequality, any term may be transposed from one side to the other if its sign is changed.

i.e., if $a - c > b$, then $a > b + c$ or $-c > b - a$

Summary of Rules

1. If $a > b$ then $a + c > b + c$ where $c \in R$
2. If $a > b$ then
   (i) $ac > bc$ if $c > 0$
   (ii) $ac < bc$ if $c < 0$
3. If $a > b$ then
   (i) $\frac{a}{c} > \frac{b}{c}$ if $c > 0$
   (ii) $\frac{a}{c} < \frac{b}{c}$ if $c < 0$

Linear Inequalities

1. Linear Inequalities with Addition and Subtraction

To solve an inequality in one variable, we use the method used for solving a linear equation. Transpose all the terms containing the unknown variable to the left hand side of the inequality and the remaining to the right hand side.

Example

$2x + 3 < 4x + 5$

$2x - 4x < 5 - 3$

i.e., $-2x < 2$

$\therefore\ x > -1$

2. Linear Inequalities with Multiplication and Division

To solve an inequality with multiplication/division transpose all the terms to the left hand side of the inequality and 0 to the right hand side.

Example

$\frac{3x-2}{x+1} < 1$

$\frac{3x-2}{x+1} - 1 < 0$

$\frac{3x-2-(x+1)}{x+1} < 0$

$\frac{2x-3}{x+1} < 0$

This means that the above fraction is negative. This will happen only if both the numerator and the denominator have a different sign. We have to consider 2 cases.

Case 1: $2x - 3 < 0$ and $x + 1 > 0$

$x < \frac{3}{2}$ and $x > -1$

i.e., $-1 < x < \frac{3}{2}$

Case 2: $2x - 3 > 0$ and $x + 1 < 0$

$x > \frac{3}{2}$ and $x < -1$

Both inequalities cannot occur simultaneously

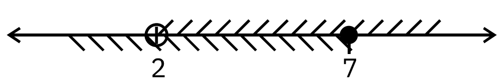
So the required solution is $-1 < x < \frac{3}{2}$

Union and intersection of the solution

Intersection

1) $x > 2$ and $x \leq 7$

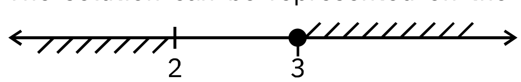
The solution can be represented on the number line as follows:



The common region lies between 2 & 7 (execluding 2 and including 7) i.e. $2 < x \leq 7$

2) $x \geq 3$ and $x < 2$

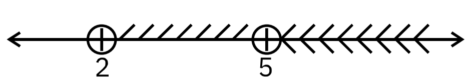
The solution can be represented on the number line as follows:



There is no common region. Hence, no solution.

3) $x > 2$ and $x \geq 5$

The solution can be represented on the number line as follows:



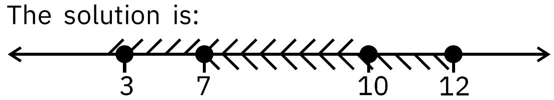
The solution is:

$x \geq 5$ i.e. $x \in [5, \infty)$

4) $3 \leq x \leq 10$ and $7 \leq x \leq 12$

The solution can be represented on the number line as follows:

The solution is:



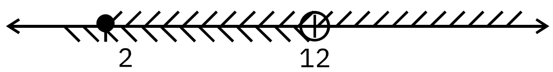
$7 \leq x \leq 10$ i.e. $x \in [7, 10]$

5) $5x - 3 \geq 7$ and $x - 4 < 8$

The solution can be represented on the number line as follows:

$5x - 3 \geq 7$ i.e. $x \geq 2$

$x - 4 < 8$ i.e. $x < 12$



The solution is $2 \leq x < 12$ i.e. $x \in [2, 12)$

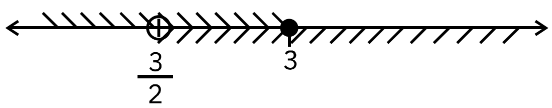
6) $7 - x \geq 4$ and $5 - 2x < 2$

$7 - x \geq 4$ i.e. $x \leq 3$

$5 - 2x < 2$ i.e. $2x > 3$ i.e. $x > \frac{3}{2}$

The solution can be represented on the number line as follows:

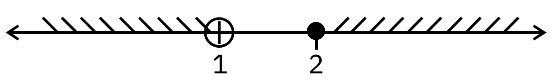
The solution is:



The solution is $\frac{3}{2} < x \leq 3$ i.e. $x \in \left(\frac{3}{2}, 3\right]$

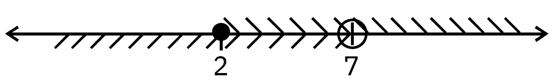
Union

1) $x \geq 2$ or $x < 1$



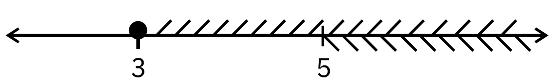
The solution is $x \in (-\infty, 1) \cup [2, \infty)$

2) $x \geq 2$ or $x < 7$



The solution is $x \in (-\infty, \infty)$

3) $x \geq 3$ or $x > 5$



The solution is $x \in [3, \infty]$

Note: When an inequality is such that the unknown variable (say x) has an inequality sign on both sides, then we solve both the inequations individually to get the range of the values of x.

Quadratic Inequalities

A **quadratic inequality** is defined as an inequality of the form:
$\mathbf{ax^2 + bx + c \ \forall \ 0}$ $(a \neq 0)$ where the symbol $\forall$ represents any of the inequalities $>$, $<$, $\geq$ or $\leq$

Solving a Quadratic inequality

To solve a quadratic inequality, we first find the factors of the quadratic polynomial $ax^2 + bx + c$ using any of the methods discussed in the chapter on quadratic equations. Express factors of the quadratic polynomial as $(x - \alpha)$ and $(x - \beta)$, i.e.,
$ax^2 + bx + c = (x - \alpha)(x - \beta)$ where $\alpha < \beta$
Now depending on the sign of the inequality in the polynomial $ax^2 + bx + c \ \forall \ 0$ $(a \neq 0)$ we can derive the following rules

RULE 1
If $ax^2 + bx + c < 0$
$\Rightarrow (x - \alpha)(x - \beta) < 0$
$\Rightarrow \alpha < x < \beta$ i.e., x lies between α and β
This is an illustrated with an example below

Example
$x^2 + 8x + 7 < 0$
$\Rightarrow (x + 1)(x + 7) < 0$
For the product of the above 2 terms i.e., $(x + 1)$ and $(x + 7)$ to be negative one of the terms is positive and the other is negative. Since we do not know the value of x, we have to consider 2 cases

Case 1
$x + 1 < 0$ and $x + 7 > 0$
$\Rightarrow x < -1$ and $x > -7$
$\Rightarrow -7 < x < -1$
$\therefore -7 < x < -1.$

Case 2
$x + 1 > 0$ and $x + 7 < 0$
$\Rightarrow x > -1$ and $x < -7$
Since both inequalities cannot occur simultaneously, only Case 1 is possible.
$\therefore -7 < x < -1.$
Alternatively,
$(x + 1)(x + 7) < 0$
$\Rightarrow [x - (-1)][x - (-7)] < 0$
Rearranging the terms,
$\Rightarrow [x - (-7)][x - (-1)] < 0$
Comparing this with $(x - \alpha)(x - \beta) < 0$ such that $\alpha < \beta$ we get,
$-7 < x < -1.$

RULE 2
If $ax^2 + bx + c > 0$
$\Rightarrow (x - \alpha)(x - \beta) > 0$ where $\alpha < \beta$
$\Rightarrow x < \alpha$ or $x > \beta$

Example
$\Rightarrow x^2 + 7x + 12 > 0$

$\Rightarrow$ (x + 3) (x + 4) > 0

For the product of the 2 terms to be positive either both terms are positive or both terms are negative. So we again consider 2 cases.

Case 1

x + 3 > 0 and x + 4 > 0

$\Rightarrow$ x > −3 and x > −4

$\Rightarrow$ x > −3

Case 2

x + 3 < 0 and x + 4 < 0

$\Rightarrow$ x < −3 and x < −4

$\Rightarrow$ x < −4

i.e., either x < −4 or x > −3

Alternatively,

(x + 3) (x + 4) > 0

$\Rightarrow$ [x − (−3)] [x − (−4)] > 0

Rearranging the terms such that $\alpha < \beta$

$\Rightarrow$ [x − (−4)] [x − (−3)] > 0

$\Rightarrow$ x < −4 or x > −3

RULE 3

If $ax^2 + bx + c > 0$ such that roots of the quadratic expression are equal i.e., $\alpha = \beta$ then the solution of the inequality is such that $x < \alpha$ or $x > \alpha$ i.e., $x \neq \alpha$

Example

Solve $x^2 - 10x + 25 > 0$

$y = x^2 - 10x + 25 > 0$

$\Rightarrow (x - 5)^2 > 0$

y becomes positve for all real values of x except at x = 5

$\therefore$ the solution is $x \neq 5$

Shortcut Method for Quadratic Inequality

1. If ab > 0

   then a > 0 and b > 0

   or a < 0 and b < 0

   Example

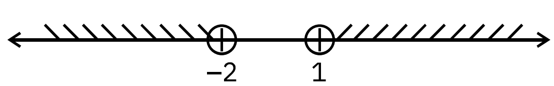
   If (x − 1)(x + 2) > 0, find x.

   (x − 1)(x + 2) > 0 (x − 1)(x + 2) > 0

   $\therefore$ x − 1 > 0 and x + 2 > 0 $\therefore$ x − 1 < 0 and x + 2 < 0

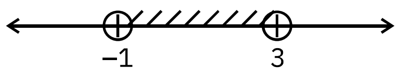
   $\therefore$ x > 1 and x > −2 $\therefore$ x < 1 and x < − 2

   $\therefore$ x > 1 $\therefore$ x < − 2



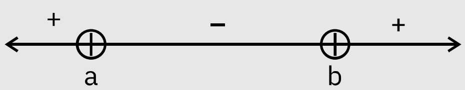
   The solution is x < −2 and x > 1.

2. If $ab < 0$
then $a < 0$ and $b > 0$
or $a > 0$ and $b < 0$
eg if $(x - 3)(x + 1) < 0$, find x.
$(x - 3)(x + 1) < 0$
$\therefore\ x - 3 < 0$ and $x + 1 > 0$
$\therefore\ x < 3$ and $x > -1$



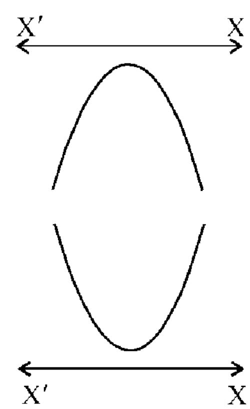
The solution is $-1 < x < 3$

Note: The value of the expression $(x - a)(x - b)$ is negative if $a < x < b$ (provided $a < b$), the value of the expression $(x - a)(x - b)$ is positive, if $x < a$ and $x > b$ (provided $a < b$)



Understanding of Quadratic Inequalities by Graphical Representation

For the quadratic polynomial of the form $ax^2 + bx + c$, the discriminant $\Delta = b^2 - 4ac$
Let $y = ax^2 + bx + c$ where a, b, c are real and $a \neq 0$ then y represents a parabola whose axis is parallel to y-axis. For some values of x, y may be positive, negative or zero. Also if $a > 0$ then parabola opens upwards and for $a < 0$ the parabola opens downwards. This gives the following cases.



Case 1: $\Delta < 0$

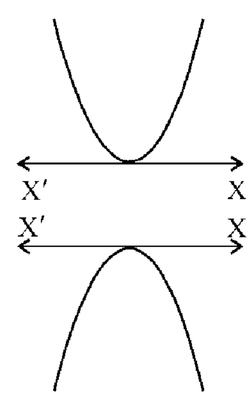
i) If $a < 0$, y is negative for all real values of x.

ii) If $a > 0$, then $ax^2 + bx + c > 0$ for all x.

y will always be positive for all real values of x.

(i.e.,) if $\Delta < 0$ then the values of the quadratic expression takes the same sign as the coefficient of x^2.

In other words, if $\Delta < 0$ then all the real values of x are solutions of the inequalities $ax^2 + bx + c > 0$ and $ax^2 + bx + c \geq 0$ for $a > 0$ and have no solution if $a < 0$.
Also, if $\Delta < 0$ all real values of x are solutions of inequalities $ax^2 + bx + c < 0$ and $ax^2 + bx + c \leq 0$ if $a < 0$ and these inequalities will not have any solution for $a > 0$.



Case 2: $\Delta = 0$

i) When $a > 0$

y will be positive for all real values of x except at the vertex where = 0

ii) When $a < 0$

y is negative for all real values of x except at the vertex where $y = 0$.

If the discriminant of a quadratic expression is equal to zero, then the value of the quadratic expression takes the same sign as that of the coefficient of x^2 (except when $x = -\frac{b}{2a}$, at which the value of the quadratic expression becomes 0).

For $\Delta = 0$,

(i) The inequality $ax^2 + bx + c > 0$ has a solution any $x \neq -\left(\frac{b}{2a}\right)$ if $a > 0$ and has no solution if $a < 0$

(ii) The inequality $ax^2 + bx + c < 0$ has a solution any $x \neq -\left(\frac{b}{2a}\right)$ if $a < 0$ and has no solution if $a > 0$

(iii) The inequality $ax^2 + bx + c \geq 0$ has as solution any x if $a > 0$ and has unique solution

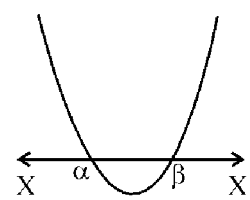
$x = -\frac{b}{2a}$ if $a < 0$.

(iv) The inequality $ax^2 + bx + c \leq 0$ has as solution any x if $a < 0$ and has unique solution $x = -\frac{b}{2a}$ for $a > 0$.

Case 3: $\Delta > 0$

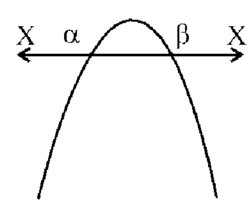
i) When $a > 0$

Let α, β be the two real roots of $y = 0$ ($\alpha < \beta$), then y will be positive for all real values of x which are lower than α or higher than β; $y = 0$ when $x = \alpha$ or β. When x lies between α and β then y will be negative.



ii) When $a < 0$

Let α,β ($\alpha < \beta$) be the two real roots of $y = 0$. Then y will be negative for all real values of x that are lower than α or higher than β. and $y = 0$ when x equals to either α or β. When x lies between α and β then y will be positive.



If α, β ($\alpha < \beta$) are the roots of the quadratic expression then it can be said

(i) For $a > 0$, $ax^2 + bx + c$ is positive for all values of x outside the interval $[\alpha, \beta]$ and is negative for all x within the interval (α, β).

Besides, for values $x = \alpha$ or $x = \beta$ the value of the quadratic expression becomes zero.

(ii) For $a < 0$, $ax^2 + bx + c$ is negative for all values of x outside the interval $[\alpha, \beta]$ and is positive for x in the interval (α, β). Besides for $x = \alpha$ or β the value of the quadratic expression becomes zero.

Applications of Rules of Inequalities

1. For positive numbers, if $a > x$, $b > y$, $c > z$ then $a + b + c +\ldots > x + y + z + \ldots$ and $abc\ldots > xyz\ldots$

2. If x is positive and $a < b$, then $\frac{a+x}{b+x} > \frac{a}{b}$

   If x is positive and $a > b$, then $\frac{a+x}{b+x} < \frac{a}{b}$

3. If x is positive and a > b > x, then, $\frac{a-x}{b-x} > \frac{a}{b}$

   If x is positive and x < a < b, then, $\frac{a-x}{b-x} < \frac{a}{b}$

4. $\frac{a+c+e+\ldots}{b+d+f+\ldots}$ is less than the greatest and greater than the least of the fractions $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$

5. If the sum of two positive quantities is given, their product is greatest when they are equal; and if the product of two positive quantities is given, their sum is least when they are equal.

6. If a > b and a, b $\geq$ 0 then $a^n > b^n$, and $\frac{1}{a^n} < \frac{1}{b^n}$, or $a^{-n} < b^{-n}$; for any positive integer n.

7. The square of every real quantity is positive and therefore must be greater than zero.
   i.e., for $a \neq b$, $(a-b)^2 > 0$; $a^2 + b^2 > 2ab$;
   Similarly, if x > 0, y > 0 then, $\frac{x+y}{2} \geq \sqrt{xy}$
   Hence, the arithmetic mean of two positive quantities is greater than or equal to their geometric mean.

8. If a, b, c...k are n unequal quantities, then, $\left(\frac{a+b+c+d\ldots+k}{n}\right)^n > a \times b \times c \times d \ldots \times k$
   i.e., $\frac{a+b+c+\ldots+k}{n} > (a \times b \times c \times \ldots k)^{1/n}$

Note:
The arithmetic mean of any number of positive quantities is greater than their geometric mean.

9. If a and b are positive and unequal, $\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$ except when m is a positive proper fraction.

   If m is a positive integer or any negative quantity $\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$

   If m is positive and less than 1, $\frac{a^m+b^m}{2} < \left(\frac{a+b}{2}\right)^m$

   If there are n positive quantities a, b, c... k, then:

   $\left(\frac{a^m+b^m+c^m+\ldots+k^m}{n}\right) > \left(\frac{a+b+c+\ldots+k}{n}\right)^m$ unless m is a positive proper fraction.

10. $(n!)^2 > n^n$, for n > 2.

11. For any positive integer n, $2 \leq \left(1+\frac{1}{n}\right)^n \leq 3$

12. $a^2 + b^2 + c^2 \geq bc + ca + ab$

13. $a^2b + b^2c + c^2a \geq 3abc$

14. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$

15. $a^4 + b^4 + c^4 + d^4 \geq 4abcd$

16. If a, b, c are positive and not all equal, then, $(a + b + c)(ab + bc + ca) > 9abc$ and, $(b + c)(c + a)(a + b) > 8abc$.

Concept Builder 1

1. $5x + 7 < -8$,
2. $2(3x - 1) \leq 5(x + 4)$
3. $\frac{x+1}{x-1} < 5$
4. $x^2 - 3x + 2 < 0$
5. $2(x - 2) + 4 < 6 - (1 - 5x)$
6. $x^2 - 6x + 9 > 0$
7. $x^2 - 7x + 12 > 0$
8. $(x + 2)(3x - 4) < 0$
9. $(2x - 1)(x + 1) > 0$

Answer Key

1. $x < -3$
2. $x \leq 22$
3. $x > \frac{3}{2}$, $x < 1$
4. $1 < x < 2$
5. $x > -\frac{5}{3}$
6. $x \neq 3$
7. $x < 3$, $x > 4$
8. $-2 < x < \frac{4}{3}$
9. $x < -1$ and $x > \frac{1}{2}$

SOLVED EXAMPLES

Q : Which of the two numbers $(1.000001)^{1000000}$ and 2 is greater?

A : $(1.000001)^{1000000} = \left(1+\frac{1}{1000000}\right)^{1000000}$ which is greater than 2.

Q : Which of the two numbers 1000^{1000} and 1001^{999} is greater?

A : $\frac{1001^{999}}{1000^{1000}} = \left(\frac{1001}{1000}\right)^{1000} \cdot \frac{1}{1001} = \left(1+\frac{1}{1000}\right)^{1000} \cdot \frac{1}{1001}$

$2 \leq \left(1+\frac{1}{1000}\right)^{1000} \leq 3$

$\therefore \left(1+\frac{1}{1000}\right)^{1000} \times \frac{1}{1001} < 1 \quad \therefore \frac{1001^{999}}{1000^{1000}} < 1 \quad \therefore 1000^{1000} > 1001^{999}$

Q : Solve $(-2x + 3) \leq 6$

A : $-2x + 3 \leq 6 \quad \therefore -2x \leq 3 \quad \therefore 2x \geq -3 \qquad \therefore x \geq -\frac{3}{2}$

Q : If w satisfies both the following inequalities and w is an integer, what values can we have?

(i) $5(w + 10) - 4w > 0$ (ii) $8 + 7w < 3(2w + 1)$

A : From (i): $5(w + 10) - 4w > 0$

$w + 50 > 0 \quad \therefore w > -50$

From (ii): $8 + 7w < 6w + 3$

$w < -5$

$\therefore$ From (i) and (ii), w lies between −50 and −5. i.e., $-50 < w < -5$.

Q : Between what values of x, is the expression $19x - 2x^2 - 35$ positive?

A : Let y denote the given expression

$y = -(2x^2 - 19x + 35) = -(2x - 5)(x - 7)$

$= (2x - 5)(7 - x) = 2\left(x-\frac{5}{2}\right)(7 - x)$

(Refer property of quadratic inequalities)

For y to be positive,

$\left(x-\frac{5}{2}\right)(x - 7) < 0 \therefore \frac{5}{2} < x < 7$

Q : Find the range of value of x if $x^3 - 7x^2 + 16x - 10$ is positive.

A : $x^3 - 7x^2 + 16x - 10$ has a factor $x - 1$. Using synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 16 & -10 \\ & & 1 & -6 & 10 \\ \hline & 1 & -6 & 10 & \boxed{0} \end{array}$$

The second factor is $x^2 - 6x + 10$

$\therefore\ x^3 - 7x^2 + 16x - 10 = (x - 1)(x^2 - 6x + 10) = (x - 1)[(x - 3)^2 + 1]$

$[(x - 3)^2 + 1]$ is always positive.

$\therefore\ x^3 - 7x^2 + 16x - 10$ will be positive if $x > 1$.

Inequalities of Higher Order Curves

Wavy Curve Method

In order to solve the inequalities of the form

$$f(x) = \frac{(x-a_1)^{n1}.(x-a_2)^{n2}.....(x-a_k)^{nk}}{(x-b_1)^{m1}(x-b_2)^{m2}.....(x-b_3)^{mp}} > 0$$

where n_1, n_2, ... , nk, m_1, m_2, ..., mp are real numbers and a_1, a_2, ..., ak, b_1, b_2, ... bp are any real numbers such that $a_i \neq b_j$

where i = 1, 2, 3, ... k and j = 1, 2, 3, ..., p.

We will begin with simple inequalities to understand the concept.

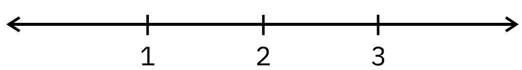
Example

$(x - 1)(x - 2)(x - 3) < 0$

Equate the L.H.S. to zero and get the values of x.

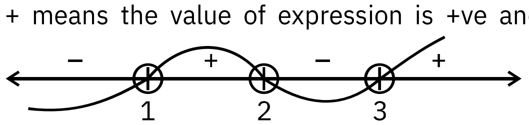
If $(x - 1)(x - 2)(x - 3) = 0$ then x = 1, 2, 3.

These are the points where the value of the expression will be zero. Draw a number line and mark these points. The 3-points will give you 4 intervals as shown below:



Now, starting from rightmost interval mark the intervals alternately + and –.

+ means the value of expression is +ve and – means the value of expression is –ve in that interval.



$\therefore -\infty < x < 1$ and $2 < x < 3$ is the required region.

It can be written as $x \in (-\infty, 1) \cup (2, 3)$

For equation $(x - 1)(x - 2)(x - 3) > 0$, $x \in (1, 2) \cup (3, \infty)$

Example

$(x + 3)^4 (x - 2)^5 < 0$

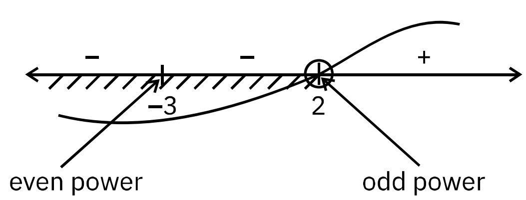
Equate LHS to zero.

If $(x + 3)^4 (x - 2)^5 = 0$, then x = –3, 2.

The powers of the breakets can be interpreted as follows:

While moving from rightmost interval to left, if the power of the corrosponding bracket is even, do not change the sign & if the power is odd, change the sign.

This can be done as shown below:



The sign will change at $x = 2$ but will not change at $x = -3$

The solution is $x \in (-\infty, +2)$

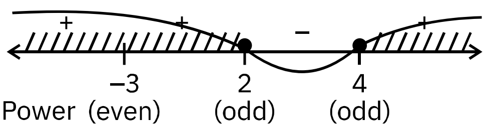
Example

$(x + 3)^4 (x - 2)^5 (x - 4) \geq 0$

Equate LHS to zero.

If $(x + 3)^4 (x - 2)^5 (x - 4) = 0$ then $x = -3, 2, 4.$

Using Wavy Curve Method, we get,



The sign will change at $x = 2$ & $x = 4$ but not at $x = -3$.

The solution is $x \in (-\infty, 2] \cup [4, \infty]$ i.e., $x \in R - (2, 4)$

Concept Builder 2

Solve the following for x.

1. $(x + 3)(x - 5) < 0$
2. $(x - 1)(x + 4) \geq 0$
3. $5x - 3 \leq 2$ and $3x + 5 \geq 2$
4. $(x + 3)(x - 1)(x + 2) \leq 0$
5. $(2x + 1)(x - 4)(x + 5) > 0$
6. $(x + 7)^2 (x - 3)(x + 4)^3 \leq 0$
7. $(x + 2)^{17} (x - 4)^{24} \geq 0$
8. $(3x - 2)(x + 2)^7 (x - 1)^{10} \leq 0$

Answer Key

1. $-3 < x < 5$
2. $x \leq -4, x \geq 1$
3. $-1 \leq x \leq 1$
4. $x \leq -3$ or $-2 \leq x \leq 1$
5. $-5 < x < -\frac{1}{2}$ or $x > 4$
6. $-4 \leq x \leq 3$
7. $x \geq -2$
8. $-2 \leq x \leq \frac{2}{3}$

SOLVED EXAMPLES

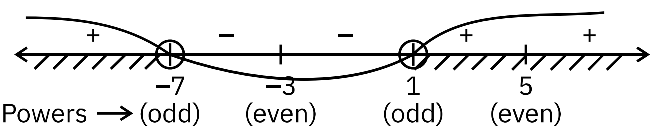
Q : Solve for x: $(x + 7)^7 (x - 5)^4 (x + 3)^2 (x - 1)^3 > 0$

A : Equate LHS = 0

If $(x + 7)^7 (x - 5)^4 (x + 3)^2 (x - 1)^3 = 0$

$\therefore$ x = −7, −3, 1, 5.

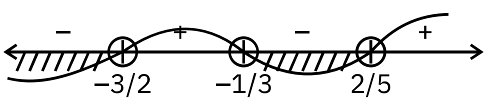
Draw the number line



The solution is $(-\infty, -7) \cup (1, \infty)$

Q : $(3x + 1)(3 + 2x)(5x - 2) < 0$

A : Equate LHS to zero. Then $x = -\frac{3}{2}, -\frac{1}{3}, \frac{2}{5}$



The solution is $x \in \left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{1}{3}, \frac{2}{5}\right)$

Q : $(3 - 2x)(1 - 5x)(-2 - x) > 0$

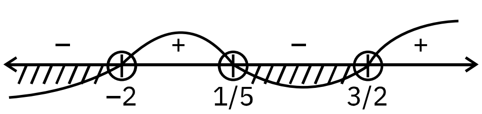
A : $(3 - 2x)(1 - 5x)(-2 - x) > 0$

$\therefore (-1)(2x - 3)(-1)(5x - 1)(-1)(x + 2) > 0$

$\therefore (-1)(-1)(-1)[(2x - 3)(5x - 1)(x + 2)] > 0$

$\therefore (2x - 3)(5x - 1)(x + 2) < 0$

Equate LHS to zero. Then $x = -2, \frac{1}{5}, \frac{3}{2}$



The solution is $x \in (-\infty, -2) \cup \left(\frac{1}{5}, \frac{3}{2}\right)$

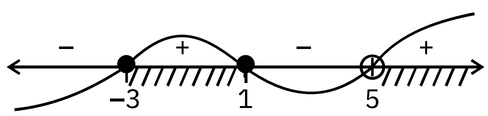
Q : $\dfrac{(x-1)(x+3)}{(x-5)} \geq 0$

A : $\dfrac{(x-1)(x+3)}{(x-5)} \geq 0$

$\dfrac{(x-1)(x+3)(x-5)}{(x-5)^2} \geq 0$

Equate LHS to zero.

x = 1, −3, 5.



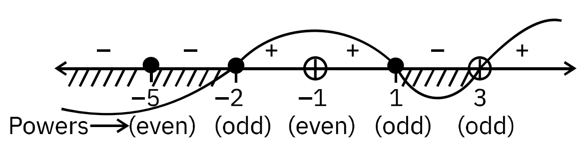
The solution is: $x \in [-3, 1] \cup (5, \infty)$

Note: As $(x - 5)$ is the denominator, x cannot be 5.

Q : $\dfrac{(x+2)^3(x-1)^5(x+5)^2}{(x+1)^4(x-3)} \leq 0$

A : The break points can be obtained directly by equating numerator & denominator of the LHS to 0.

$\therefore$ x = −2, 1, −5, −1, 3



The solution is

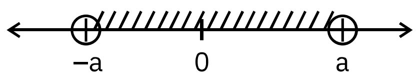
x $(-\infty, -2] \cup [1, 3)$

Note: At x = 3 & x = −1 the graph will be discontinuous as the denominator is $(x+1)^4(x-3)$. $\therefore$ x cannot take these values.

At x = −5, x = −2 and x = 1, the graph cuts the x axis. Hence, the value of the expression on LHS is zero.

Inequalities with Modulus Sign

If an inequality is given as $|x| < a$, it can be interpreted as $x < a$ and $-x < a$ i.e. $x < a$ and $x > -a$. Hence the solution is $-a < x < a$. (a and –a are not inclusive)



In other words, the solution lies between the interval stretched upto a distance 'a' units away from the centre, in both the sides.
For $|x| < a$ the solution is: $-a < x < a$

Example

i) $|x| < 10$ means $-10 < x < 10$

ii) $|x + 5| < 10$ means $-10 < x + 5 < 10 \Rightarrow -5 - 10 < x < -5 + 10$

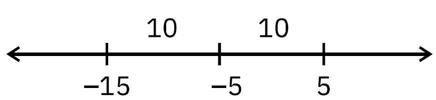
$\therefore -15 < x < 5$

Short Cut Method

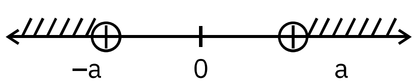
Here $x + 5$ is equated to 0. $\therefore x + 5 = 0$

$\therefore x = -5$. Now the origin will be –5.

Subtract and add 10 to get the minimum and maximum limit of the solution interval.



For $|x| > a$, the solution is



$x < -a$ and $x > a$

Example

$|x + 1| > 3$
As $x + 1 = 0$, $x = -1$
Take –1 as the origin.
$-1 - 3 > x$ and $x > -1 + 3$
i.e. $-4 > x$ and $x > 2$

Some important identities are as follows:

Absolute value and inequalities

1. $|x| = x$; if $x \geq 0$ and
   $-x$; if $x < 0$
   Where x is any real number
2. $|x|$ is always positive.
3. $|x| \geq x$
4. $|x - y| = |y - x|$
5. $|x.y| = |x| \, . \, |y|$
6. $|x + y| \leq |x| + |y|$
7. $|x + y| \geq |x| - |y|$
8. $|x - y| \geq |x| - |y|$
9. If $x > y$ then $|x - y| = x - y$ and if $x < y$ then $|x - y| = y - x$.
10. $|x| \leq a \Rightarrow -a \leq x \leq a$
11. $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$
12. $-|x| \leq x \leq |x|$

Concept Builder 3

Solve the following for x.

1. $|x - 5| < 3$
2. $|2x + 3| > -5$
3. $|x - 3| \leq 7$
4. $|7x - 2| \leq 5$
5. $|3(4x - 6)| \geq 0$
6. $|x + 7| < -1$
7. $|2x + 1| \geq 3$
8. $|5x - 1| \geq 7$

Answer Key

1. $2 < x < 8$
2. x can take any value
3. $-4 \leq x \leq 10$
4. $\frac{-3}{7} \leq x \leq 1$
5. x can take any value
6. No solution
7. $x \leq -2$, $x \geq 1$
8. $x \geq \frac{8}{5}$, $x \leq -\frac{6}{5}$

SOLVED EXAMPLES

Q : $|x^2 - 5| < 7$, then find the value of x.

A : $\therefore\ |x^2 - 5| < 7.$

$\therefore\ -2 < x^2 < 12$

$-2 < x^2$ is always true.

As $x^2 < 12$,

$x^2 - 12 < 0$

$(x + 2\sqrt{3})\ (x - 2\sqrt{3}) < 0$

$-2\sqrt{3} < x < 2\sqrt{3}$

Hence the solution is $x \in (-2\sqrt{3},\ 2\sqrt{3})$

Q : If $\left|\frac{2x-1}{x+1}\right| < 3$, then find the value of x.

A : $\left|\frac{2x-1}{x+1}\right| < 3,$

$\therefore\ \left|\frac{2x-1}{x+1}\right| - 3 < 0$

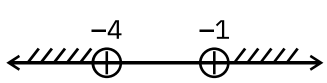
When $\frac{2x-1}{x+1} - 3 < 0$

$\therefore\ \frac{2x-1-3x-3}{x+1} < 0$

$\therefore\ \frac{-x-4}{x+1} < 0$

$\therefore\ \frac{x+4}{x+1} > 0$

$x < -4$ or $x > -1$



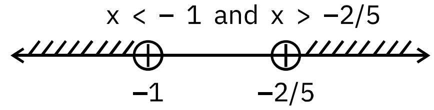
When $-\frac{2x-1}{x+1} - 3 < 0$

$\therefore\ \frac{-2x+1-3x-3}{x+1} < 0$

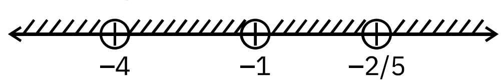
$\therefore\ \frac{-5x-2}{x+1} < 0$

$\therefore\ \frac{5x+2}{x+1} > 0$

$x < -1$ and $x > -2/5$



Combining the two cases we can conclude



x can be any real number except -1, -4 and $-\frac{2}{5}$.

Q : If $\left|\frac{x^2-9}{x+5}\right| > 0$, then find the value of x.

A : The modulus value is always positive.

$\therefore\ \left|\frac{x^2-9}{x+5}\right| > 0$ is true for all real values.

But $x + 5 \neq 0$ i.e. $x \neq -5$

Also $x^2 - 9 \neq 0$ i.e. $x \neq 3$ & $x \neq -3$

$\therefore$ The solution is $x \in R - \{-5, 3, -3\}$

Q : If $\dfrac{|x+3|+x}{x+2} > 1$, then find the value of x.

A : $\dfrac{|x+3|+x}{x+2} > 1$

$\therefore \dfrac{|x+3|+x}{x+2} - 1 > 0$

$\therefore \dfrac{|x+3|-2}{x+2} > 0$

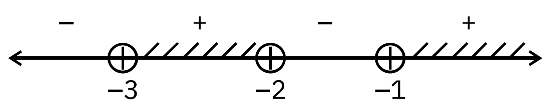
Case I: When x + 3 is positive,

$|x + 3| = x + 3$ & $x > -3$

Again, $\dfrac{x+3-2}{x+2} > 0$

$\therefore \dfrac{x+1}{x+2} > 0$

The solution is



Hence, $x \in (-3, -2) \cup (-1, \infty)$

Case II: When x + 3 is negative

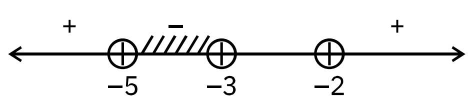
$|x + 3| = -(x + 3)$ and $x < -3$

Again, $\dfrac{-(x+3)-2}{x+2} > 0$

$\therefore \dfrac{-x-5}{x+2} > 0$

$\therefore \dfrac{x+5}{x+2} < 0$

The solution is



Hence, $x \in (-5, -3)$

Case III: When x + 3 = 0

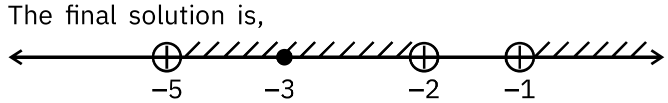
$|x + 3| = 0 \quad \therefore x = -3$

It gives $\dfrac{|-3+3|-3}{-3+2}$

$= \dfrac{0-3}{-1} = \dfrac{-3}{-1} = 3 > 0$

Hence, $x = -3$

The final solution is,



$x \in (-5, -2) \cup (-1, \infty)$

CLASS EXERCISE

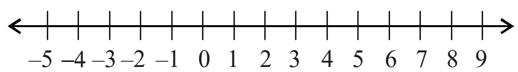
Teaser

A magician is performing for 10 children at a party. He places ten coins on a table, 5 with a Head showing and the other 5 with a Tail. Then he moves away and asks nine of the children to go to the table one by one and turn exactly 1 coin each (chosen at random) upside down. He tells the tenth child to cover one of the coins. Looking at the remaining 9 coins, he sees 2 heads and 7 tails. Is the hidden coin more likely to show a head or a tail?

Linear Inequalities

- If a > b and c is any number
  $\Rightarrow a + c > b + c$
  $\Rightarrow a - c > b - c$
- If a > b and c > 0
  $\Rightarrow ac > bc$
  $\Rightarrow \frac{a}{c} > \frac{b}{c}$
- If a > b and c < 0, then ac < bc
- If a − c > b, then a > b + c or −c > b − a.

1. Plot the values of x on a number line for the following cases.



a) $x = 3$
b) $x < 4$
c) $x > 5$
d) $x \leq 4$
e) $-2 < x \leq 5$
f) $x \in (-2, 4]$
g) $x \in (-\infty, 3) \cup [6, \infty)$

2. Find the values/ range of values of x that satisfy the given inequalities:

a) $2x + 3 < 11$
b) $3x - 7 \geq 11$
c) $17 - 4x > 7$
d) $1 < 2x+9 \leq 7$
e) $-5 \leq 28 - 3x < 37$

3. Which of the following integral values can x have if $2x + 8 < \frac{20}{3}$?

1) 0 2) 1 3) −2 4) 2

Linear Inequalities with Modulus Sign

- $|x| < a \Rightarrow -a < x < a$
- $|x| > a \Rightarrow x > a$ or $x < -a$

4. Find the values/ range of values of x that satisfy the given inequalities:

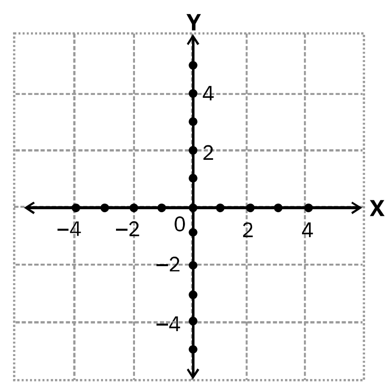
a) $|x| < 5$
b) $|x - 2| > 5$
c) $\left|\frac{x}{2} + 5\right| \leq$ 7d)
d) $1 < |4x - 3| < 13$
e) $|2x + 3| > -5$

Quadratic Inequalities

- If $x^2 < a$, where a is a positive number
  $\Rightarrow -\sqrt{a} < x < \sqrt{a}$ ($\sqrt{a}$ is the positive square root of a)
- If $x^2 > a$, where a is a positive number
  $\Rightarrow x > \sqrt{a}$ or $x < -\sqrt{a}$ ($\sqrt{a}$ is the positive square root of a)

5. Find the values/range of values of x that satisfy the given inequalities:
   a) $x^2 < 16$
   b) $x^2 \geq 49$
   c) $(x - 2)^2 < 36$
   d) $(2x + 3)^2 < 81$
   e) $(3x - 2)^2 > 25$

6. Plot the graph for $y = x^2 - x - 2$
   a) Find the range of x for which the corresponding values of y is negative.
   b) Find the range of x for which the corresponding values of y is positive.



- If $ax^2 + bx + c < 0$
  $\Rightarrow (x - \alpha)(x - \beta) < 0$, where $\alpha < \beta$
  $\Rightarrow \alpha < x < \beta$ i.e., x lies between α and β
- If $ax^2 + bx + c > 0$
  $(x - \alpha)(x - \beta) > 0$, where $\alpha < \beta$
  $x < \alpha$ or $x > \beta$

7. What is the value of m which satisfies $3m^2 - 21m + 30 < 0$?
   1) $m < 2$ or $m > 5$ 2) $m > 2$ 3) $2 < m < 5$ 4) $-5 < m < -2$

(Past CAT question)

8. Which of the following values can x have so that $2x^2 - 11x + 12$ is positive?
   1) -3 2) 2 3) 3 4) $\frac{7}{2}$

9. Find the common range of x for the equations $x^2 + x - 6 < 0$ and $x^2 - x - \frac{15}{4} < 0$.
   1) $-\frac{3}{2} < x < \frac{5}{2}$ 2) $-3 < x < \frac{5}{2}$ 3) $-\frac{3}{2} < x < 2$ 4) $-3 < x < -\frac{3}{2}$

10. If $\frac{x^2-2}{-x^2-3\sqrt{2}} \le \frac{x+10}{-x^2-3\sqrt{2}}$. Find the range of the values of x:

1) $-3 \le x \le 4$
2) $x \ge 4$ or $x \le -3$
3) $x = 0$
4) $x \ge 3$

Inequalities of Higher Order Curves

11. Consider the polynomial $P = (x-1)(x+2)(x-4)(x+5)$

What is the solution set for the inequality $P \le 0$?

1) $x \in [-5, -2] \cup [1, 4]$
2) $x \in [-2, 1] \cup (4, \infty)$
3) $x \in [-5, -2] \cup [4, \infty)$
4) None of these

12. Solve: $(x-4)^4 (x-3)^3 (x+2)^2 (x+1) \le 0$

1) $x \in [-2, -1] \cup [3, 4]$
2) $x \in (-\infty, 2] \cup [-1, 4]$
3) $x \in (-2) \cup [-1, 3] \cup (4)$
4) None of these

13. Find the number of positive integer solutions for $(x+3)^3 (x-1)^5 (x-4)^7 < 0$

14. Solve the inequality $\frac{(x-1)^2 (x+1)^3}{(x-2)} \le 0$.

1) $-2 \le x < 1$
2) $-1 \le x < 2$
3) $-1 \le x \le -2$
4) $-2 \le x \le 1$

Challengers

1. If $6 \ge x \ge -2$ and $4 \ge y \ge -4$, find the limits for $\frac{y}{x}$, where x and y are non zero integers.

1) $\frac{y}{x} \ge 2$, $\frac{y}{x} \le \frac{2}{3}$
2) $\frac{y}{x} \ge \frac{-2}{3}$, $\frac{y}{x} \le 2$
3) $\frac{y}{x} \ge \frac{-2}{3}$, $\frac{y}{x} \le \frac{1}{4}$
4) $\frac{y}{x} \ge -4$, $\frac{y}{x} \le 4$

2. The number of distinct integer solutions of the inequation $x^2 - 8|x| + 15 \le 0$ is:

1) 4
2) 5
3) 6
4) 8

3. The number of integral values that do not satisfy the inequation $|x+1| + |x-4| \ge 7$

1) 5
2) 6
3) 7
4) 8

4. Find the range of values of x that satisfy the equation $\frac{3x^2-5x-22}{x^2-3x+10} < 2$

1) $-6 < x < 7$
2) $-7 < x < 6$
3) $-7 < x < -6$
4) $6 < x < 7$

5. If $\frac{x+3}{|x+2|+1} \geq 1$, then what is the range of the values of x?

1) $x \geq -2$ 2) $x \leq -2$ 3) $-2 \leq x \leq 3$ 4) $1 \leq x \leq 2$

(Past CAT question)

6. Which of the following range of values do not satisfy the inequation $x^2 + 4x + 6 \geq |(x^2 + 3x)|$?

1) $-6 \leq x \leq -2$ and $x \geq -3/2$
2) $x \in [-6, \infty) - (-2, -3/2)$
3) $-6 \leq x \leq -3$
4) $-2 < x < -3/2$

7. The solution set for the inequation $7P - 3 < |4P + 3| < 3P - 1$ is:

1) $R - \left(-\frac{6}{11}, -\frac{2}{7}\right)$
2) $R - [-4, \infty)$
3) $R - \left[-4, -\frac{6}{11}\right]$
4) None of these

PRACTICE EXERCISE

DIRECTIONS for questions 1 to 20: Choose the correct alternative.

1. The number of integers n satisfying $-n + 2 \geq 0$ and $2n \geq 4$ is

   1) 0   2) 1   3) 2   4) 3

   (Past CAT question)

2. Given that $-2 \leq x \leq 2$, $-2 \leq y \leq 1.5$ and $-4 \leq z \leq -1$ and $U = \frac{x+y}{z}$, then which of the following is necessarily true?

   1) $0 \leq U \leq 1$ 2) $-3.5 \leq U \leq 4$
   3) $3.5 \leq U \leq 4$ 4) $-1 \leq U \leq 1$

3. What values of x satisfy $X^{\frac{2}{3}} + X^{\frac{1}{3}} - 2 \leq 0$?

   1) $-8 \leq x \leq 1$ 2) $-1 \leq x \leq 8$ 3) $1 < x < 8$ 4) $1 \leq x \leq 8$

   (Past CAT question)

4. If $\frac{x-4}{-x^2-6} \leq \frac{2}{-x^2-6}$ then:

   1) $x \leq 2$ 2) $x \geq 2$ 3) $x \leq 6$ 4) $x \geq 6$

5. If $0 < x < 5$ and $1 < y < 2$, then which of the following is true?

   1) $x + y < 0$ 2) $-3 < 2x - 3y < 4$
   3) $-6 < 2x - 3y < 7$ 4) $-3 < 3x - y < 2$

6. Which of the following values of x do not satisfy the inequality $(x^2 - 3x + 2 > 0)$ at all?

   1) $1 \leq x \leq 2$ 2) $-2 \leq x \leq -1$ 3) $0 \leq x \leq 2$ 4) $-2 \leq x \leq 0$

   (Past CAT question)

7. Find the range of values satisfying $(x - 2011)^{2011} (x - 2014)^{2014} (x - 2017)^{2017} (x - 2020)^{2020} < 0$.

   1) $(2011, 2014) \cup (2017, 2020)$ 2) $(2011, 2017)$
   3) $(2011, 2014) \cup (2014, 2017)$ 4) $[2011, 2017] - (2014)$

8. The solution set for the inequation $5m + 2 < |2m + 3| < 4 - m$ is:

   1) $\left(-\frac{1}{3}, 7\right)$ 2) $\left(\frac{1}{3}, 7\right)$ 3) $\left(-7, \frac{1}{3}\right)$ 4) $\left(-7, -\frac{1}{3}\right)$

9. If $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$ then which of the following is true?

1) $x \in (-3, -2) \cup \left(-\frac{2}{3}, \frac{1}{2}\right)$
2) $x \in \left(-1, -\frac{2}{3}\right) \cup \left(-\frac{1}{2}, \infty\right)$
3) $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$
4) $x \in (-3, -1) \cup (2, 5)$

10. What real values of x satisfy $|x + 3| + |3x - 1| > -3$?

1) $\frac{1}{3} < x < 3$
2) $-3 < x < 3$
3) $-\frac{1}{3} < x < \frac{1}{3}$
4) All real numbers

11. Find the range of the values of x if $\frac{2x^2+9x+7}{(x-3)(x^2-9)} \le 0$

1) $(-\infty, -7/2] \cup (-3, -1]$
2) $[-7/2, -3) \cup [-1, 3)$
3) $[-7/2, -3) \cup (-3, 3)$
4) $(-\infty, -3) \cup [-1, 3)$

12. Find the range of the values of x if $\frac{(x-7)^{2019}(x-5)^{2011}}{(x-2)^{1983}(x-3)^{1992}} \ge 0$

1) $(-\infty, 2) \cup [3, 5] \cup [7, \infty)$
2) $(2, 5] \cup [7, \infty] - (3)$
3) $[2, 3) \cup [5, 7] - (2)$
4) $(-\infty, 3) \cup [5, 7]$

13. The number of integral values of x that satisfy the inequation $|x + 3| + |x - 2| \le 9$ is

1) 5
2) 8
3) 10
4) 12

14. $(x + 1)(x - 3)(x + 5).......(x - 99) < 0$ has how many integer solutions?

1) 72
2) 75
3) 64
4) 87

15. If $\frac{|x-1|+3}{x-4} < 1$, then what is the range of the values of x?

1) $(-\infty, 4)$
2) $(-\infty, -4) \cup (1, 4)$
3) $(4, \infty)$
4) $(-1, 3) \cup (4, \infty)$

16. Which of the following represents the solution for the inequality $x^3 + 8x^2 - 5x - 84 > 0$?

1) $x < -3$ or $4 < x < 7$
2) $x < 3$ or $4 < x < 7$
3) $x > 7$ or $-7 < x < -4$
4) $x > 3$ or $-7 < x < -4$

17. If $-3 \le a \le 1$, $-2 \le b \le 2$, $-1 \le c \le 4$ and $1 \le d \le 3$, which of the following is true?

1) $-9 \le ab + cd \le 12$
2) $-6 \le ab + cd \le 18$
3) $-9 \le ab + cd \le 18$
4) $-6 \le ab + cd \le 12$

18. How many pairs of (x,y) with both x and y having integer values satisfy the inequality $|2x + 3y| + |2x - 3y| < 12$?

1) 24 2) 18 3) 15 4) None of these

19. If x satisfies the inequality $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

1) $0 \leq x \leq 4$ 2) $x \leq 0$ or $x \geq 4$
3) $x \leq -2$ or $x \geq 3$ 4) None of the above

20. Which of the following range of values does not satisfy the inequality?
$|2x^2 - x - 1| \leq x^2 + 1$

1) $\left(-\frac{1}{2}, 0\right]$ 2) $\left[\frac{1}{3}, 1\right)$ 3) $\left[-1, -\frac{1}{2}\right)$ 4) $\left(0, \frac{1}{3}\right)$

QA-2.7 | SEQUENCES & SERIES — THEORY

Definition of a sequence

A sequence is a succession of terms arranged in a definite order according to some rule.

Example

(i) Sequence of even numbers
2, 4, 6, 8, 10, 12, ...

(ii) Sequence of prime numbers from 1 to 30
2, 3, 5, 7, 11, 13, 17, 19, 23, 29

The numbers in a sequence are called the terms of the sequence. A sequence, which has a last term, is called a finite sequence. A sequence which does not have a last term i.e. has infinite terms is called an infinite sequence.

The first term of a sequence is denoted by a_1, second term by a_2 and so on. Thus, a_n represents the n^{th} term of the sequence, where n is a positive integer.

The n^{th} term of the sequence is considered 'the rule' for forming the sequence and may be denoted by f(n) or T_n. The n^{th} term of the sequence is also called the general term of the sequence.

Example

The n^{th} term of a sequence is given by $T_n = 2n + 1$. Find the first, second, third and tenth term of the sequence.

First term $(n = 1) \equiv a_1 = 2 \times 1 + 1 = 3$

Second term $(n = 2) \equiv a_2 = 2 \times 2 + 1 = 5$

Third term $(n = 3) \equiv a_3 = 2 \times 3 + 1 = 7$

Tenth term $(n = 10) \equiv a_{10} = 2 \times 10 + 1 = 21$

> **Note:** The function f(n) i.e., T_n may be specified by an algebraic, trigonometrical or any other analytical formula.

Definition of a series

The indicated sum of the terms of a sequence is called a series.

In case of a finite sequence $a_1, a_2, a_3, \ldots, a_n$ the corresponding series is $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i-1}^{n} a_i$

The Greek letter Σ (Sigma) is the summation sign whose upper and lower limit i.e., i = 1 and n indicate the range of the variable 'i' over which the sum is calculated

Example

$$\sum_{i-1}^{3} a_i = a_1 + a_2 + a_3$$

The sum of the first n terms of a sequence is denoted by $S_{n.}$ i.e., $S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

$S_{n-1} = T_1 + T_2 + T_3 ..... + T_{n-1}$

$S_n = T_1 + T_2 + T_3 ...... + T_{n-1} + T_n$

Hence,$T_n = S_n - S_{n-1}$

Find S_5 for $T_n = 3n - 2$.

$T_1 = 1, T_2 = 4, T_3 = 7, T_4 = 10, T_5 = 13$

$\therefore S_5 = 1 + 4 + 7 + 10 + 13 = 35$

Sequences following certain patterns are more often called progression. Arithmetic progression, Geometric progression and Harmonic progression are three important types of progression.

Types of sequences

1. Arithmetic progression (A.P.)

A sequence is called an arithmetic progression if the difference between any term and its previous term is a constant. If the first term is 'a' and the common difference i.e., constant is 'd', then the progression takes the form a, a + d, a + 2d, a + 3d ....

The n^{th} term of the arithmetic progression is given by $T_n = a + (n - 1)d$ and the sum of the n terms of the arithmetic progression is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$

Example

Find the S_n and T_n for the Arithmetic progression 6, 10, 14, 18,....

$a = 6, d = 10 - 6 = 4$

$T_n = a + (n - 1) d = 6 + (n - 1) 4 = 6 + 4n - 4 = 2 + 4n = 2(1 + 2n)$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{n}{2}[2 \times 6 + (n - 1)4] = \frac{n}{2}[12 + 4n - 4] = 2n(2 + n)$

About Arithmetic progressions

1. An arithmetic progression can be continually increasing or decreasing by the same number.
2. $d = T_n - T_{n-1}$

3. $S_n = \frac{n}{2}[T_1 + T_n]$

4. When three terms are in Arithmetic progression, the middle term is called arithmetic mean between the other two. i.e., if a, b and c are in A.P then $b = \frac{a+c}{2}$.

5. Three numbers in A.P. are taken as a – d, a , a + d. Four numbers in A.P. are taken as a – 3d, a – d, a + d, a + 3d.

Inserting 'n' Numbers between 2 given numbers

If we insert n numbers $a_1, a_2, a_3 \ldots a_n$ between two numbers P and Q, such that P, $a_1, a_2, a_3, \ldots a_n$, Q are in Arithmetic Progression, then there will be a total of 'n + 2' numbers in the new sequence

Let 'd' be the common difference between the terms of the above Arithmetic Progression.

Also, P is the 1st term and 'Q' is the $(n + 2)^{th}$ term in the progression $\therefore$ Q = P + (n + 1)d

$d = \frac{Q-P}{n+1} \Rightarrow a = p + \frac{Q-P}{n+1}$, $a_2 = P + \frac{2(Q-P)}{n+1}$ and so on

Hence,

Example

If 10 numbers are inserted between 5 and 38 such that the sum of they form an AP, then find the sum of the numbers

Let the numbers be $a_1, a_2, a_3 \ldots a_{10}$.

Then, 5, $a_1, a_2, a_3 \ldots a_{10}$, 38 are in AP and 38 is the 12^{th} term in the arithmetic progression.

Hence, 38 = 5 + 11d $\therefore$ d = 3

So, the inserted numbers are 8, 11, 14 ... 35

Sum of all numbers = $\frac{12}{2}[2 \times 5 + (12 - 1)3]_n$

$\Rightarrow 6[10 + 33] \Rightarrow 258$

2. Geometric progression (G.P.)

A sequence is called a geometric progression if the ratio of any term to the preceding term is a constant, called common ratio.

If the first term is 'a' and the common ratio is 'r', then the sequence takes the form a, ar, ar^2, ar^3, ...

Example

Sequence	Common ratio
(i) 1, 2, 4, 8, 16,...	2
(ii) $\frac{16}{27}, -\frac{8}{9}, \frac{4}{3}, -2, \ldots$	$-\frac{3}{2}$

The n^{th} term of the geometric progression is given by $T_n = ar^{n-1}$
The sum of the first n terms of the geometric progression is given by

$S_n = a\left[\frac{(1-r^n)}{1-r}\right]$; **for r < 1 and** $S_n = a\left[\frac{(r^n-1)}{r-1}\right]$; **for r > 1**

For r = 1, S_n from the above formulae is indeterminate.

Example

Find S_6 and T_5 for the G.P. 1, 3, 9, 27, ...

$a = 1;\ r = \frac{3}{1} = 3$

$T_n = a\ r^{n-1}$
$T_5 = 1 \times 3^{5-1} = 3^4 = 81$

$S_n = a\left[\frac{r^n-1}{r-1}\right]$

$S_6 = 1 \times \left[\frac{3^6-1}{3-1}\right] = \frac{729-1}{2} = 364$

About Geometric progressions

1. If three quantities a, b and c are in G.P., then b is the geometric mean of a and c
   $\therefore\ b = \sqrt{ac}$
2. Three numbers in G.P. can be taken as $\frac{a}{r}$, a, ar. Four numbers in G.P. can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3.

Inserting 'n' Numbers between 2 given numbers

If we insert n numbers $a_1, a_2, a_3 \ldots a_n$ between two numbers P and Q, such that P, $a_1, a_2, a_3, \ldots a_n$, Q are in Geometric Progression.

Let 'r' be the common ratio between the terms of the above Geometric Progression.

$\because$ 'Q' is the $(n + 2)^{th}$ term in the G.P. $\therefore\ Q = P.r^{n+1}$

$\mathbf{r} = \left(\frac{Q}{P}\right)^{\frac{1}{n+1}}$

Hence, $\mathbf{a_1 = Pr}$, $\mathbf{a_2 = Pr^2}$**........** $\mathbf{a_n = Pr^n}$

Example

Insert three numbers between 3 and 48 such that they form a G.P.

Let the numbers be a_1, a_2, a_3, then 3, a_1, a_2, a_3, 48 will be in a G.P.

If the common ratio is 'r', then $48 = 3\times r^4$.

$\therefore\ r = 2$

So, the three numbers are: 6, 12, 24

Infinite Geometric series

Sum of an infinite geometric series, where | r | < 1 is given by $S_\infty = \frac{a}{1-r}$
For | r | > 1, sum of an infinite G.P. tends to infinity.

The symbol ∞ indicate infinite.

Example

What will be the sum of all the terms of the sequence: 2, $\frac{1}{4}$, $\frac{1}{32}$, $\frac{1}{256}$ ... ∞

The given sequence is an Infinite Geometric Progression with the first term as '2' and the Common Ratio as '$\frac{1}{8}$'.

So, the sum of all the terms $(S_\infty) = \frac{2}{1-\frac{1}{8}} = \frac{16}{7}$

3. Harmonic progression (H.P.)

A sequence formed by the terms a_1, a_2, a_3, a_n,... for which the reciprocals of the terms, $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \frac{1}{a_n}, \ldots$ form an arithmetic progression is called a harmonic progression.

The n^{th} term of an harmonic progression is given by,

$T_n = \frac{1}{a+(n-1)d}$ **where, a and d are the first term and common difference of the corresponding Arithmetic progression.**

Example

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$, ... form an harmonic progression since the reciprocal of the term, 2, 4, 6, 8,... forms an Arithmetic progression

About Harmonic Progressions

1. There is no general formula for the sum of any number of terms in harmonic progression. Questions based on harmonic progression are generally solved by converting them into an arithmetic progression by taking the reciprocal of the terms.
2. The harmonic mean of a and b is $\frac{2ab}{a-b}$. The concept of harmonic mean is used to find the average speed of a body over a particular distance.

Some important Formulae

(i) Sum of the first n natural numbers is given by $\sum_{r-1}^{n} r = \frac{n(n+1)}{2}$

(ii) Sum of the first n even natural numbers = n(n+1)

(iii) Sum of the first n odd natural numbers = n^2

(iv) Sum of the squares of the first n natural numbers is given by $\sum_{r-1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(v) Sum of the cubes of the first n natural numbers is given by $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)^2}{2}\right]$

Miscellaneous Types of Sequences

Combination of an A.P. and G.P.

It is a sequence of the form a, (a + d)r, (a + 2d)r^2 . . . . where $T_n = [a + (n - 1)d]r^{n-1}$ ....

Example

Find out the sum of is the terms of the sequence: $\frac{3}{4}, \frac{5}{4^2}, \frac{7}{4^3}, \frac{9}{4^4}, ...\infty$

In the above sequence, 3, 5, 7, ... are in A.P. and $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, ...$ are in G.P.

Let, $\mathbf{S} = \frac{3}{4} + \frac{5}{4^2} + \frac{7}{4^3} + \frac{9}{4^4} + ...\infty$ ... (i)

Multiplying both sides of equation (i) by the common ratio i.e., $\frac{1}{4}$.

$\Rightarrow \frac{1}{4}\mathbf{S} = \frac{3}{4^2} + \frac{5}{4^3} + \frac{7}{4^4} + \frac{9}{4^5} ...\infty$ ...(ii)

Subtracting (ii) from (i),

$\Rightarrow \frac{3}{4}\mathbf{S} = \frac{3}{4} + \frac{5-3}{4^2} + \frac{7-5}{4^3} + \frac{9-7}{4^4} ......\infty$

$\Rightarrow \frac{3}{4}\mathbf{S} = \frac{3}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \frac{2}{4^4} + ...\infty \quad \Rightarrow \frac{3}{4}S = \frac{3}{4} + 2\left(\frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + ...\infty\right)$

$\Rightarrow \frac{3}{4}S = \frac{3}{4} + 2\left(\frac{\frac{1}{4^2}}{1-\frac{1}{4}}\right) \quad \Rightarrow \frac{3}{4}S = \frac{3}{4} + 2\left[\frac{1}{12}\right]$

$\Rightarrow \frac{3}{4}S = \frac{3}{4} + \frac{1}{6} \Rightarrow \frac{3}{4}S = \frac{11}{12} \Rightarrow S = \frac{11}{9}$

Sequences where Second Level Differences are Constant

The second level difference is the difference between the difference of the terms in a sequence. If this difference is constant, then the n^{th} term in this sequence (T_n) can be expressed in quadratic form i.e.

$T_n = an^2 + bn + c$

This is illustrated with an example below

Example

Find the 50^{th} term and sum of first 50 terms of the sequence: 4, 6, 9, 13, 18...

Observing the terms of the sequence, we see that the second level differences are constant, as shown below:

4	6	9	13	18...
2	3	4	5...	
1	1	1...		

Putting n = 1 in the term $T_n = an^2 + bn + c$, we get

$T_1 = a + b + c = 4$ ... (i)

$T_2 = 4a + 2b + c = 6$ ... (ii)

$T_3 = 9a + 3b + c = 9$ ... (iii)

Subtracting (i) from (ii) and (i) from (iii) we get $3a + b = 2$ and $5a + b = 3$

So, $a = \frac{1}{2}$, $b = \frac{1}{2}$ and $c = 3$ $\Rightarrow$ So, $T_n = \frac{1}{2}n^2 + \frac{1}{2}n + 3 = \frac{1}{2}(n^2 + n + 6)$

So, the 50^{th} term $= T_{50} = \frac{1}{2}(50^2 + 50 + 6) = 1278$

Now, the sum of '50' terms $S_n = \sum_{n=1}^{50} \frac{1}{2}(n^2 + n + 6)$ $\quad \therefore S_n = \frac{1}{2}\left(\sum_{1}^{50} n^2 + \sum_{1}^{50} n + \sum_{1}^{50} 6\right)$

$$\therefore S_n = \frac{1}{2}\left[\frac{50(50+1)(2\times 50+1)}{6} + \frac{50(50+1)}{2} + 50(6)\right] = \frac{1}{2}(42925 + 1275 + 300) = 22250$$

Concept Builder

Which type of progression (i.e., A.P, G.P, H.P or A.G.P) do the following sequences represent?

1. $\frac{3}{4}, 2, \frac{13}{4}, \frac{9}{2}, \ldots$
2. $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \frac{4}{30}, \ldots$
3. 6, 18, 54, 162, ...
4. $\frac{3}{5}, \frac{5}{10}, \frac{7}{20}, \frac{9}{40}, \ldots$

Answer the following questions.

5. What is the Arithmetic mean of A.P. : 3, 7, 11, 15, 19, 23
6. What is the Geometric mean of G.P.: 3, 6, 12, 24, 48
7. Find the sum of the terms of AP whose 1^{st} term, last term & common difference are 7, 100, and 3, respectively.
8. In a G.P of Real Numbers, the 2^{nd} term is '9' & the sixth term is $\frac{1}{9}$, then find the 4^{th} term.
9. The sum of the first 25 terms of an A.P. is '0'. Which term of this A.P. will be '0'?
10. Find the sum of the first 50 natural numbers.

Answer Key

1. A.P. 2. H.P. 3. G.P. 4. A.G.P 5. 13
6. 12 7. 1712 8. 1 9. 13^{th} 10. 1275

Solved Examples

Q : Find the sum of 30 terms of the series 5 + 11 + 17 + ...

A : The terms 5, 11, 17, ... form an Arithmetic Progression.

as $d = 11 - 5 = 17 - 11 = 6$; also, $a = 5$

$\therefore S_{30} = \frac{n}{2}[2a + (n - 1)d] = \frac{30}{2}[2 \times 5 + 29 \times 6]$

$= 15\ [10 + 174] = 2760$

Q : If $S_n = n(n + 8)$, find T_1 and T_n

A : $S_1 = T_1 = 1(1 + 8) = 9$

$S_{n-1} = (n - 1)\ (n - 1 + 8)$

$= (n - 1)\ (n + 7)$

$= n^2 + 6n - 7$

$T_n = S_n - S_{n-1} = (n^2 + 8n) - (n^2 + 6n - 7) = 2n + 7$

Q : The sum of the first 31 terms of an A.P is '0'. Which term of this A.P. will surely be a non-negative integer?

A : In an A.P. with sum of all terms '0', Sum of the first & last term, second and second-last term, third and third-last term and so on, will also be'0'. But whether these terms are positive or negative and whether they are integers or not, cannot be determined.

Since, the number of terms in the given A.P. is 31 (i.e. odd), the middle term (i.e. 16^{th} term) has to be '0'.

So, the 16^{th} term will surely be a non-negative integer.

Q : If the 4^{th} and the 9^{th} terms of a geometric progression are $\frac{1}{3}$ and 81 respectively, find the first term.

A : $T_n = a\ r^{n-1}$

$T_4 = ar^3;\ \frac{1}{3} = ar^3$ ... (i)

$T_9 = ar^8;\ 81 = ar^8$ ... (ii)

Dividing (ii) by (i); $81 \times 3 = r^5\ \therefore r = 3$

Substituting value of r in equation (i)

$\frac{1}{3} = a(3)^3 \quad \therefore a = \frac{1}{81}$. Hence, the first term is $\frac{1}{81}$.

Q : How many terms of the series 1 + 5 + 9 + ... must be taken in order, so that the sum is 190?

A : $a = 1,\ d = 4$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$190 = \frac{n}{2}[2 \times 1 + (n - 1)4]$

$380 = 2n + 4n^2 - 4n$

$2n^2 - n - 190 = 0$

$(2n + 19)(n - 10) = 0$

$\therefore\ n = -\frac{19}{2}$ or $n = 10$

n cannot be $-\frac{19}{2}$; $\therefore\ n = 10$

Q : Find the geometrical progression whose sum to infinity is $\frac{9}{2}$ and the second term of which is -2.

A : $ar = -2\ \therefore\ a = \frac{-2}{r}$

$\frac{a}{1-r} = \frac{9}{2}$

$\frac{-2}{r(1-r)} = \frac{9}{2}$

$-4 = 9r - 9r^2$

$9r^2 - 9r - 4 = 0$

$(3r + 1)(3r - 4) = 0$

$\therefore\ r = -\frac{1}{3}$ or $r = \frac{4}{3}$

If $r = -\frac{1}{3}$, $a = -\frac{2}{r} = 6$ and the series is $6, -2, \frac{2}{3}, \ldots\ldots.$

The value of $r = \frac{4}{3}$ is inadmissible, for r must be numerically less than unity.

Q : The number of bacteria in a culture triples in every 15 minutes. Find the number of bacteria in the culture after 105 minutes, if there were 10,000 bacteria initially.

A : $a = 10{,}000;\ r = 3;\ n = 8$

$T_n = a(r^{n-1}) = 10{,}000(3^{8-1})$

$= 10{,}000(2187) = 2187 \times 10^4$ bacteria.

Q : Find the value of K so that 8K + 3, 6K − 3 and 2K + 6 are in Arithmetic progression

A : If 8K + 3, 6K − 3 and 2K + 6 are in Arithmetic progression then $(6K - 3) - (8K + 3) = (2K + 6) - (6K - 3)$ ... [common difference]

$-2K - 6 = -4K + 9 \qquad \therefore\ K = \frac{15}{2} = 7.5$

Q : Find the sum of the first 20 terms of the sequence 5, 5.5, 5.55, 5.555...

A : $S_{20} = 5 + 5.5 + 5.55 + \ldots + T_{20}$

$= \frac{5}{9}[(10 - 1) + (10 - 0.1) + (1\ 0 - 0.01) + \ldots + (10 - 10^{-19})]$

$= \frac{5}{9}[10 + 10 + 10 + \ldots \text{ upto 20 terms}]$

$- \frac{5}{9}\left[1 + \frac{1}{10} + \frac{1}{100} + \ldots\text{upto 20 terms}\right]$

$= \frac{5}{9}\left[10 \times 20 - 1\left[\frac{1 - \frac{1}{(10)^{20}}}{1 - \frac{1}{10}}\right]\right]$

$= \frac{5}{9}\left[200 - \frac{10}{9}\left[1 - \frac{1}{10^{20}}\right]\right]$

$= \frac{5}{81}\left[1790 + \frac{1}{10^{19}}\right] = \frac{5}{81} \times 1790 \ldots \left(\frac{1}{10^{19}} \approx 0\right) = 110.5$

Q : Find the sum of the first 10 terms of the sequence whose n^{th} term is $2n - 8$.

A : $T_n = 2n - 8$

$a = T_1 = 2 \times 1 - 8 = -6$

$T_2 = 2 \times 2 - 8 = -4$

$T_3 = 2 \times 3 - 8 = -2$

$-6, -4, -2$ form an A.P. $\therefore$ $d = T_2 - T_1 = -4 - (-6) = 2$

$S_{10} = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{10}{2}[2 \times (-6) + 9 \times 2]$

$= 5\ [-12 + 18] = 30$

Q : What will be the 27^{th} term in the sequence: $(3 \times 5) + (5 \times 8) + (7 \times 11) + \ldots$

A : By observation, we can make out that each term in this sequence is multiplication of two expressions i.e. (2n+1) & (3n+2), where n = 1, 2, 3,...

Now, the n^{th} term of the sequence $T_n = (2n+1) \times (3n+2)$

$\therefore$ 27^{th} term can be find out by putting n = 27.

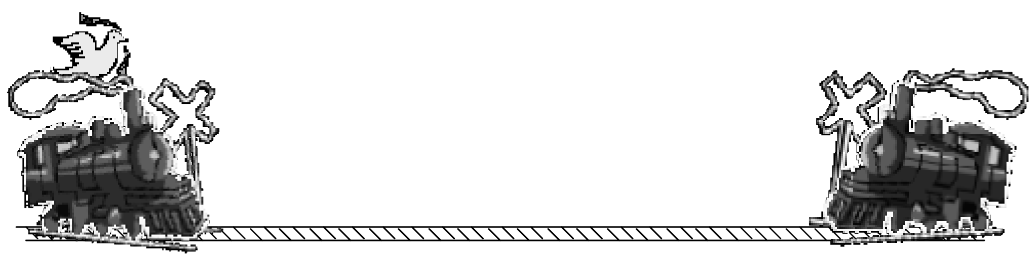
$T_{27} = (2 \times 27 + 1) \times (3 \times 27 + 2) = 4565$

Note: In the above question, you will find that the second level difference is constant. So, you can also use the method explained earlier to derive the n^{th} term of the sequence i.e $T_n = an^2 + bn + c$)

CLASS EXERCISE

Teaser

Two trains A and B, travelling at speeds of 15 kmph are approaching each other on the same railway track. A bird C is sitting on the engine of train A. When the trains are 120 km apart, the bird starts flying at 25 kmph towards B. The moment it meets B, it turns back and flies towards A. It keeps this pattern up until the two trains collide. What is the total distance the bird will cover in the process?



Sequences, Series and Progressions: Theory

1. The Maharaja of Mysore was extremely fond of the game of chess. Two artisans Dodda and Chikka designed for him a large chessboard with king-size chessmen. The Maharaja was pleased and decided to award them with grains of gold. He asked them how many grains they would like to have. Dodda said he would like 100 golden grains on the first square of the board and in every consecutive square, 50 grains more than in the previous. Chikka said that he wanted only one grain in the first square but in every consecutive square he wanted twice as many grains as in the previous.

   a) How many grains will Dodda get in the 10^{th} square?
   b) In how many squares will Chikka get more grains than Dodda?
   c) How many grains will Chikka get in the last square?
   d) How many grains will Dodda get in all?

2. Consider the series 1, 4, 7, 10, 13...

   a) What will be the 100^{th} number (term) in the series?
   b) Will the number 172 be there in this series? If yes, what will be its position?
   c) What will be the sum of the first 100 terms in the series?
   d) What will be the difference between 100^{th} and 102^{nd} term?

3. Consider the series 2, 6, 18, 54...

a) What will be the 50th term in the series?

1) 2×3^{49} 2) 2×3^{50} 3) 2×3^{51} 4) 3^{50}

b) Will the number 1008 be there in this series? If yes, what will be its position?

1) No 2) Yes, 336^{th} 3) Yes, 168^{th} 4) Yes, 56^{th}

c) What will be the sum of first 10 terms of this series? (Note: $3^{10} = 59049$)

1) 59048 2) 29524 3) 118098 4) 19682

4. What will be the sum of infinite terms of the series: 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, , ...?

For an Arithmetic Progression (A.P.) with 1^{st} term (t_1) = a and common difference 'd',
- n^{th} term (t_n) = $a + (n - 1)d$
- Sum of n terms (S_n) = $\frac{n}{2}$

For a Geometric Progression (G.P.) with 1^{st} term (t_1) = a and common ratio 'r',
- n^{th} term (t_n) = ar^{n-1}
- Sum of n terms (S_n) = $\frac{a(1-r^n)}{1-r}$
- If r < 1, Sum of infinite terms (S_∞) = $\frac{a}{1-r}$

5. Find the sum of $\frac{3}{4} + \frac{15}{16} + \frac{63}{64}$ ..... up to 'n' terms

6. Find the next term in the following series:
   a) 4, 7, 13, 25, 49, 97, _____
   b) 5, 8, 13, 21, 34, 55, _____
   c) 3, 6, 10, 15, 21, 28, _____

7. Find the expression for the n^{th} term (Tn) of the following series:
   a) 15, 22, 29, 36, 43...
   b) 5, 7, 11, 19, 35,
   c) 3, 6, 10, 15, 21, 28...

8. Find the sum of the first ten terms of the following series:
   a) 11, 12, 13, 14, 15....
   b) 36, 49, 64, 81, 100...
   c) 3, 6, 10, 15, 21, 28...

> The sum of the first n natural numbers is given by $\sum_1^n n = \frac{n \times (n+1)}{2}$
>
> The sum of the first n perfect squares is given by $\sum_1^n n^2 = \frac{n \times (n+1) \times (2n+1)}{6}$
>
> The sum of the first n perfect cubes is given by $\sum_1^n n^3 = \frac{n^2 \times (n+1)^2}{4} = \left(\sum_1^n n\right)^2$

9. * Find the sum of $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$

10. * In a bacteria culture the number of bacteria doubles after every minute. If a dish is full of bacteria exactly after 10 minutes when was it half full?

11. * Marbles are arranged in the shape of a regular tetrahedron such that the marble at the top rests on 3 marbles, which further rest on 6 marbles and so on.
    a) If there are 120 marbles in all, how many layers of marbles are there?
    b) If there are 120 marbles in the last row, how many layers of marbles are there?

Sequences, Series and Progressions: Examples

1. A certain sequence is defined by the recursive relation $T_{n+1} = 2T_n + 3$
   a) If $T_1 = 2$, find T_5
   b) If $T_5 = 125$, find T_1
   c) If $T_1 = 3$ and $T_{11} = 6141$, find $T_2 + T_3 + T_4 + \ldots + T_{10}$

2. If in an Arithmetic Progression with general term a_n, it is known that $a_7 + a_9 = a_3 + a_4 + a_{10}$, then what is the first term of the AP?

3. A group of 630 children is arranged in rows for a group photograph session. Each row contains three fewer children than the row in front of it. What number of rows is not possible?
1) 3 2) 4 3) 5 4) 6 5) 7

4. A tennis ball is initially dropped from a height of 180 m. After striking the ground, it rebounds (3/5)th of the height from which it has fallen. The total distance that the ball travels before it comes to rest is
1) 540 m 2) 600 m 3) 720 m 4) 900 m

5. If the sum $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$ .... to n terms is $\frac{3069}{512}$, find the value of n.
1) 5 2) 10 3) 15 4) 20

6. Find the last digit of the 200-digit number 122333444455555.....

7. Calculate sum of $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \ldots + \frac{1}{99\times100}$

8. $\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{100}+\sqrt{99}}$ equals

9. The inverse of the sum of the following series up to n terms $\frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \ldots$. can be written as
1) $\frac{(n-1)^2}{n^2+2n}$ 2) $\frac{n^2+2n}{(n-1)^2}$ 3) $\frac{n^2+2n}{(n+1)^2}$ 4) $\frac{(n+1)^2}{n^2+2n}$

10 The sum of the series $1^2 - 2^2 + 3^2 - 4^2 + \ldots + 2001^2 - 2002^2 + 2003^2$ is
1) 2007006 2) 1005004 3) 200506 4) None of the above

11 * A child was asked to add the first few natural numbers (i.e. 1 + 2 + 3 + ...) so long as his patience permitted. When he stopped, he gave the sum as 575. When the teacher declared the result wrong, the child discovered that he had missed one number in the sequence during addition. The number he missed was
1) 10 2) 18 3) 20 4) None of the above

12. * If the sum of the first 5 terms of an Arithmetic Progression is equal to the sum of the first 10 terms, then which term of the AP must necessarily be zero?
1) 6^{th} 2) 8^{th} 3) 10^{th} 4) 9^{th}

13. * A square is drawn by joining the midpoints of the sides of a given square. A third square is drawn inside the second square in the same way and this process is continued indefinitely. If a side of the first square is 8 cm, the sum of the areas of all the squares such formed (in sq. cm) is
1) 128 2) 120 3) 96 4) None of these

14. * In an Arithmetic Progression, the 8th, 12th and 17th terms are in Geometric Progression. What is the ratio of the first and tenth terms?

15. * What is the sum of the series $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ .... to 10 terms?

1) $\frac{175}{132}$ 2) $\frac{36}{55}$ 3) $\frac{175}{264}$ 4) $\frac{209}{312}$

16. * Two men X and Y started working for a certain company at similar jobs on January 1, 1950. X asked for an initial monthly salary of Rs. 300 with an annual increment of Rs. 30 on the monthly salary. Y asked for an initial monthly salary of Rs. 200 with a rise of Rs.15 on a monthly salary every 6 months. Assume that the arrangements remained unaltered till December 31, 1959. Salary is paid on the last day of the month. What is the total amount paid to them as salary during the period?

1) Rs. 93,300 2) Rs. 93,200 3) Rs. 93,100 4) None of the above

17. * The infinite sum $1+\frac{4}{7}+\frac{9}{7^2}+\frac{16}{7^3}+\frac{25}{7^4}$ ... equals

1) $\frac{27}{14}$ 2) $\frac{21}{13}$ 3) $\frac{49}{27}$ 4) $\frac{256}{147}$

Challengers

1. Consider a sequence where nth term, $t_n = \frac{n}{n+2}$, n = 1, 2, ...

The value of $t_3 \times t_4 \times t_5 \times$ .... $\times t_{53}$ equals:

1) $\frac{2}{495}$ 2) $\frac{2}{477}$ 3) $\frac{12}{55}$

4) $\frac{1}{1485}$ 5) $\frac{1}{2970}$

2. If $a_1 = 1$, $a_{n+1} = 2a_n + 5$, n = 1, 2, ... then a_{100} is equal to

1) $5 \times 2^{99} - 6$ 2) $5 \times 2^{99} + 6$ 3) $6 \times 2^{99} + 5$ 4) $6 \times 2^{99} - 5$

3. Given that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....=\frac{\pi}{6}$, find the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+....$

1) $\frac{\pi}{8}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{10}$ 4) $\frac{\pi}{12}$

4. There are 8436 steel balls, each with a radius of 1 centimeter, staked in a pile with 1 ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth and so on. The number of horizontal layers in the pile is

1) 34 2) 38 3) 36 4) 32

5. There is a series of terms such that $T_1 = 1$, $T_2 = 5 + 5^2$, $T_3 = 5^3 + 5^4 + 5^5$ and so on. Find T_{10}.

1) $\frac{5^{54}(5^{10}-1)}{4}$ 2) $\frac{5^{45}(5^{9}-1)}{4}$ 3) $\frac{5^{45}(5^{10}-1)}{4}$ 4) $\frac{5^{44}(5^{10}-1)}{4}$

PRACTICE EXERCISE - 1

DIRECTIONS for questions 1 to 4: Choose the correct alternative.

1. Five numbers are in geometric progression such that their product is 1024. Find the third number.

   1) 1 2) 4 3) 16 4) 2

2. A ball dropped from a height of 24 m rebounds two-third of the distance it falls. How much distance will it travel before coming to rest?.

   1) 240 m 2) 60 m
   3) 120 m 4) Cannot be determined

3. The three angles of a triangle are in GP. What type of triangle is it, if one of its interior angle is 60°?

   1) Equilateral triangle 2) Isosceles triangle
   3) Scalene triangle 4) Data insufficient

4. a, b, c are in Arithmetic Progression with common difference = d. Which of the following is certainly TRUE (select the best option):

   1) At least one of a, b, c, d is divisible by 3
   2) At most one of a, b, c, d is divisible by 3
   3) Either all or none of a, b, c, d is divisible by 3
   4) None of the above

DIRECTIONS for question 5: Solve as directed.

5. The sum of first 10 terms of an AP is equal to the sum of first 15 terms.

   a) If the sum of first ‘k’ terms is 0, find the value of ‘k’.
   b) If the 25th term is 10, find the first term and the common difference of the A.P.

DIRECTIONS for questions 6 to 15: Choose the correct alternative.

6. Find the sum to infinity of a decreasing GP with the common ratio x such that $|x| < 1$; $x \neq 0$. The ratio of the fourth term to the second term is $\frac{1}{16}$ and the ratio of third term to the square of the second term is $\frac{1}{9}$.

   1) 6 2) 12 3) 18 4) None of these

7. A series has terms of the form a^b where a is the base and b is the exponent. The bases in the series increase as a GP, while the exponent increase as an AP. Another series is constructed from this series as follows.

   If T_1, T_2, T_3, ... T_r, ... are the terms of the given series.

   The new series is $\frac{T_2}{T_1}, \frac{T_3}{T_2}, \frac{T_4}{T_3}$, ... $\overline{r\,1}$, ... etc.

   Thus, the new series will be a:

   1) AP 2) GP 3) HP 4) None of these

8. For a generalized fibonacci sequence, which of the following statement/s is/are true? Fibonacci sequence is 1, 1, 2, 3, 5, ...

   I. $\sum_{k=1}^{n} a_k = a_{n+2} - a_2$

   II. $a_n^2 = (a_{n-1} \times a_{n+1}) + a_2$
   (where a_n is the n^{th} term of the fibonacci sequence)

   1) Only I 2) Only II 3) Both I and II 4) None of these

9. The sum of the reciprocal of the product of first three natural numbers, the reciprocal of the product of three consecutive natural numbers starting with 2, the reciprocal of the product of next three consecutive natural numbers starting with 3 and so on till the reciprocal of the product of $(n - 2)^{th}$, $(n - 1)^{th}$ and n^{th} natural numbers will be:

   1) $\frac{n^2 - n - 2}{4(n-1)n}$ 2) $\frac{n-1}{(n-2)n}$ 3) $\frac{4n}{n(n-1)}$ 4) $\frac{n(n+1)}{2}$

10. A sequence, $\{a_n\}$, is defined recursively as $a_1 = 2$; $a_{n+1} = a_n + 2n$ $(n \geq 1)$. Find a_{100}.

   1) 9900 2) 9902 3) 9904 4) 9906

11. Sum of the series: $1 + 2 + \frac{7}{4} + \frac{5}{4} + \frac{13}{16} + \frac{16}{32} + \frac{19}{64} + \ldots\ldots$ is:

   1) 8 2) 10 3) 9 4) ∞

12. If the product of z positive real numbers is unity, then their sum is necessarily:

   1) a multiple of z 2) equal to $z + \frac{1}{z}$
   3) never less than z 4) a positive integer

13. Consider three numbers in GP such that the middle number is 36. If A = Sum of the three numbers and B = Sum of the products of two numbers taken at a time, then find the relation between A and B.

   1) $B = 36^2A$ 2) $B = 6A$ 3) $B = 36A$ 4) $A = 36B$

14. A body dropped from an airplane falls 10 meters in the first second of its motion, 15 meters in the second, 20 meters in the third, 25 meters in the fourth and so on. Find the distance travelled by the body if it hits the earth in 120 seconds.

   1) 73.8 km 2) 40 km 3) 36.9 km 4) 1428 km

15. The price of an X-ray machine is Rs.3,00,000. If the machine is bought in monthly installments in a period of 3 years with Rs.500 as the first installment and increasing the value of the installment by Rs.500 every succeeding month, find the extra amount paid by the buyer in the installment scheme.

   1) Rs.3000 2) Rs.3300 3) Rs.30000 4) Rs.33000

DIRECTIONS for questions 16 and 17: Solve as directed.

16. The sum of n terms of a sequence is $4n^2 + 7n$. What is the sum of the 6^{th}, 7^{th} and 8^{th} terms of the sequence?

17. The k^{th} term of an AP is given by formula $T_k = 2016 - 23k$. Find the smallest value of n for which S_n, the sum of first n terms, is negative.

DIRECTIONS for questions 18 to 20: Choose the correct alternative.

18. The fourth term of an arithmetic progression is 8. What is the sum of the first 7 terms of the arithmetic progression?

    1) 7 2) 64 3) 56 4) Indeterminate

(Past CAT question)

19. If $\frac{1}{y+z}, \frac{1}{z+x}$ and $\frac{1}{x+y}$ are in A.P., which of the following terms are also in A.P.?

    1) x, y, z 2) x^2, y^2, z^2 3) $\frac{1}{y-z}, \frac{1}{z-x}, \frac{1}{x-y}$ 4) None of these

(Past CAT question)

20. What is the sum of the following series:

    $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{100\times 101}$?

    1) $\frac{99}{100}$ 2) $\frac{1}{100}$ 3) $\frac{100}{101}$ 4) $\frac{101}{102}$

(Past CAT question)

PRACTICE EXERCISE - 2

DIRECTIONS for questions 1 to 3: Solve as directed.

1. Find the sum $1^2 + 3^2 + 5^2 + \ldots + 99^2$.

2. If the sum of three distinct natural numbers in Geometric Progression (having common ratio r, where r is a natural number) is 57, what will be the sum of their squares?

3. The sum of the first 'n' terms of an arithmetic progression is given by the expression $2n^2 - 15n$. What is the sum of the 2^{nd} term, 4^{th} term, 6^{th} term … 18^{th} term and the 20^{th} term?

DIRECTIONS for questions 4 to 18: Choose the correct alternative.

4. The three interior angles of a quadrilateral are in AP, such that the difference between the largest and the second largest angle is equal to the smallest angle considering the same three angles. Also the fourth angle is equal to one of the three angles that are in AP. At least one of the angles of the quadrilateral is 90°. The least angle of the quadrilateral can be:

   1) 40° 2) 45° 3) 51.3° 4) Data insufficient

5. A ball dropped from a certain height bounces $\left(\frac{6}{7}\right)^{th}$ its original height every time it hits the ground. What is the ratio of the total distance travelled by the ball in the downward direction (towards the ground) to the total distance travelled by the ball in the upward direction (away from the ground) from the moment it is dropped till it comes to a complete halt? Assume that the ball bounces at the same point on the ground each time.

   1) 7 : 6 2) 13 : 7 3) 7 : 1 4) Data insufficient

6. Three integers a, b, c are such that 2a + 3b − 4c, b + 2c − 3a and 2a + 2c − b are in arithmetic progression. The integers a + b, b + c, a + c are also in arithmetic progression. Then a, b, c are in the ratio

   1) 3 : 2 : 4 2) 2 : 3 : 1 3) 3 : 1 : 5 4) Cannot be determined

7. What is the sum of the series given below?

   $$\frac{2}{1^3} + \frac{6}{1^3+2^3} + \frac{12}{1^3+2^3+3^3} + \frac{20}{1^3+2^3+3^3+4^3} + \ldots \text{ upto } \infty.$$

   1) ∞ 2) 2 3) 4 4) 1

8. If $f(x) = \sum_{k=-n}^{n} x^k$ for non-zero x, then the minimum value f(x) can attain is:

   1) 2^n 2) $2(n + 1)$ 3) $2n + 1$ 4) 2^{n+1}

9. $T = \{T_n \mid T_n = \sum_{i=0}^{n} i \text{ for } n \epsilon W\}$ and

   $A_i = \{T_{i-1} + k + 1 \mid 0 \le k \le i-1\}$ for $i \in N$ and $T_{i-1} \in T$ and where N and W are sets of natural numbers and whole numbers, respectively. Which of the following represents the sum of the elements of set A_n?

   1) $2T_{n-1} + T_n$ 2) $nT_{n-1} + T_n$ 3) $\frac{n}{2}(T_{n-1} + T_n)$ 4) Cannot be determined

10. If $x = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \ldots + \frac{1}{100}$, what can be said about x ?

    1) $\frac{1}{3} < x < \frac{1}{2}$ 2) $x > \frac{3}{2}$ 3) $\frac{1}{2} < x < 1$ 4) Cannot be determined

11. Find the value of $\dfrac{1\times 50 + 2\times 49 + 3\times 48 + \ldots + 50\times 1}{1\times 1 + 2\times 2 + 3\times 3 + \ldots + 50\times 50}$.

1) $\dfrac{52}{101}$ 2) $\dfrac{53}{101}$ 3) $\dfrac{152}{101}$ 4) $\dfrac{153}{101}$

12. The lowest platelet count in human blood is 50,000. The leukemia reduces the platelet count in the blood and a radiation dose is given to a patient whose platelet count is less than 10000. For a patient suffering from leukemia ALL L1 type the following platelet count was recorded every fortnight over a cycle of 2 months 51000, 44500, 38250, 32125 ..... And the trend is continued in the same manner. When will the radiation dose be given to the patient?

1) 3^{rd} month 2) After 3 and a half months
3) End of 4^{th} month 4) 4 and a half months

13. Find the value of 'x' if

$x = \dfrac{5}{3} + \dfrac{17}{15} + \dfrac{37}{35} + \dfrac{65}{63} + \dfrac{101}{99} + \ldots + \dfrac{2501}{2499}$

1) $\dfrac{1325}{51}$ 2) $\dfrac{1326}{51}$ 3) $\dfrac{1274}{51}$ 4) $\dfrac{1276}{51}$

14. The ratio of the 9^{th} and the 12^{th} terms of an AP is 20:27. What will be the ratio of the 3^{rd} and the 6^{th} terms of the same AP?

1) 6 : 13 2) 25 : 39 3) 25 : 46 4) 9 : 23

15. Consider all APs that can be formed using natural numbers up to 100 such that the first term of the AP is 1 and the last term is 100. Which of the following cannot be the total number of terms in that AP?

1) 4 2) 10 3) 11 4) 34

16. Each term a_k of the sequence $a_1, a_2, \ldots, a_{50}$ is defined as k less than the sum of the other 49 terms of the sequence. Find the value of S_{50}, the sum of all the 50 terms of the sequence.

1) $\dfrac{1225}{48}$ 2) $\dfrac{425}{16}$ 3) 1275 4) $\dfrac{1275}{16}$

17. What is the approximate sum up to 10 terms of the following series?

$\dfrac{39}{4} + \dfrac{131}{8} + \dfrac{403}{16} + \ldots$

1) 546 2) 551 3) 601 4) 646

18. If 'p' times the $(p - 1)^{th}$ term of an AP is equal to 'q' times the $(q - 1)^{th}$ term of the AP, what will be the $(p + q - 1)^{th}$ term of the AP?

1) 0 2) 1 3) 2 4) Data insufficient

DIRECTIONS for questions 19 and 20: Solve as directed.

19. Kulbushan takes every 3^{rd} number counting from 1001 up to 4000 (1003, 1006, 1009, ..., 3997, 4000) and writes them down in sequence. What will be the 251st digit in this sequence?

20. Evaluate the following expression:

$$\frac{1^2 + 3^2 + 4^2 + 5^2 + 7^2 + 8^2 + 9^2 + 11^2 + 12^2 + 13^2 + 15^2 + 16^2 + \ldots + 49^2}{1249}$$

(Past CAT question)