The Step Function

$Ku(t) = 0 \ @ \ t < 0$

$Ku(t) = K \ @ \ t > 0$

$Ku(t-\alpha) = 0 \ @ \ t < \alpha$

$Ku(t-\alpha) = K \ @ \ t > \alpha$

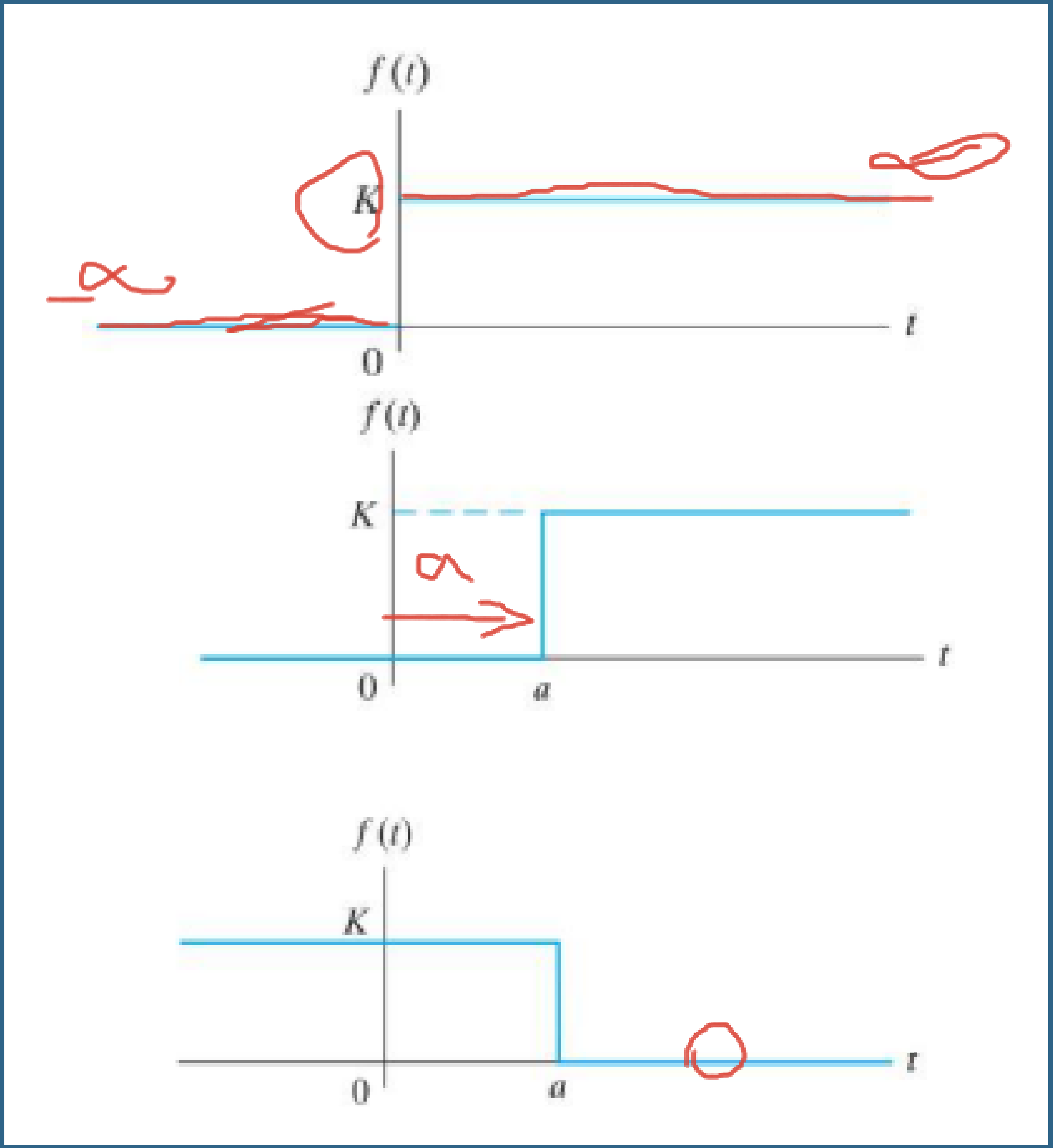
$t - \alpha > 0 \rightarrow t > \alpha$

$\alpha - t > 0$

$\alpha > t$

$Ku(\alpha - t) = K, \ t < \alpha$

$Ku(\alpha - t) = 0, \ t > \alpha$



Using a Step Function to Represent Finite Duration

① $2t$

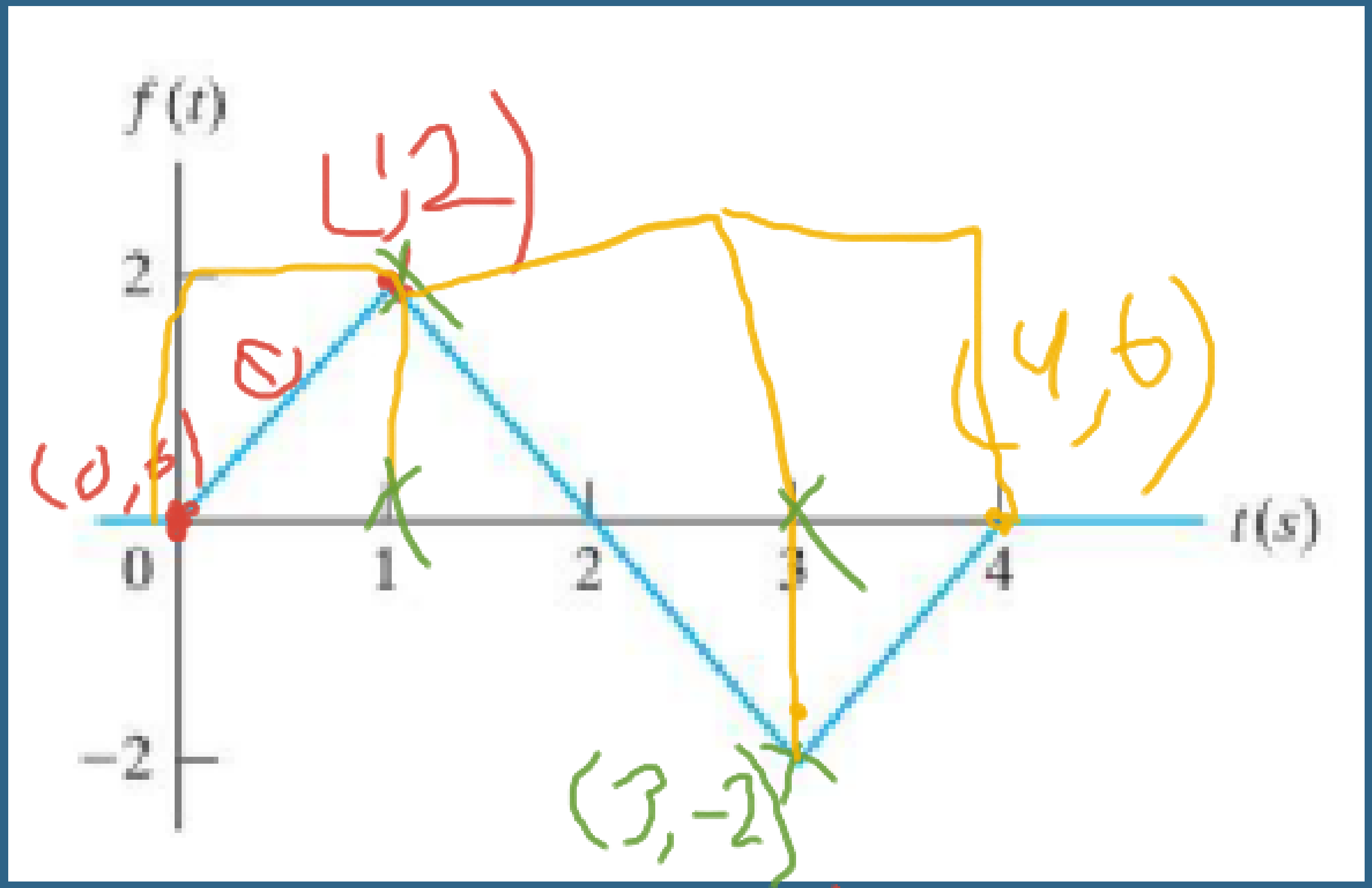
② $-2t+4$

③ $2t-8$

$2t\left[u(t)-u(t-1)\right]$

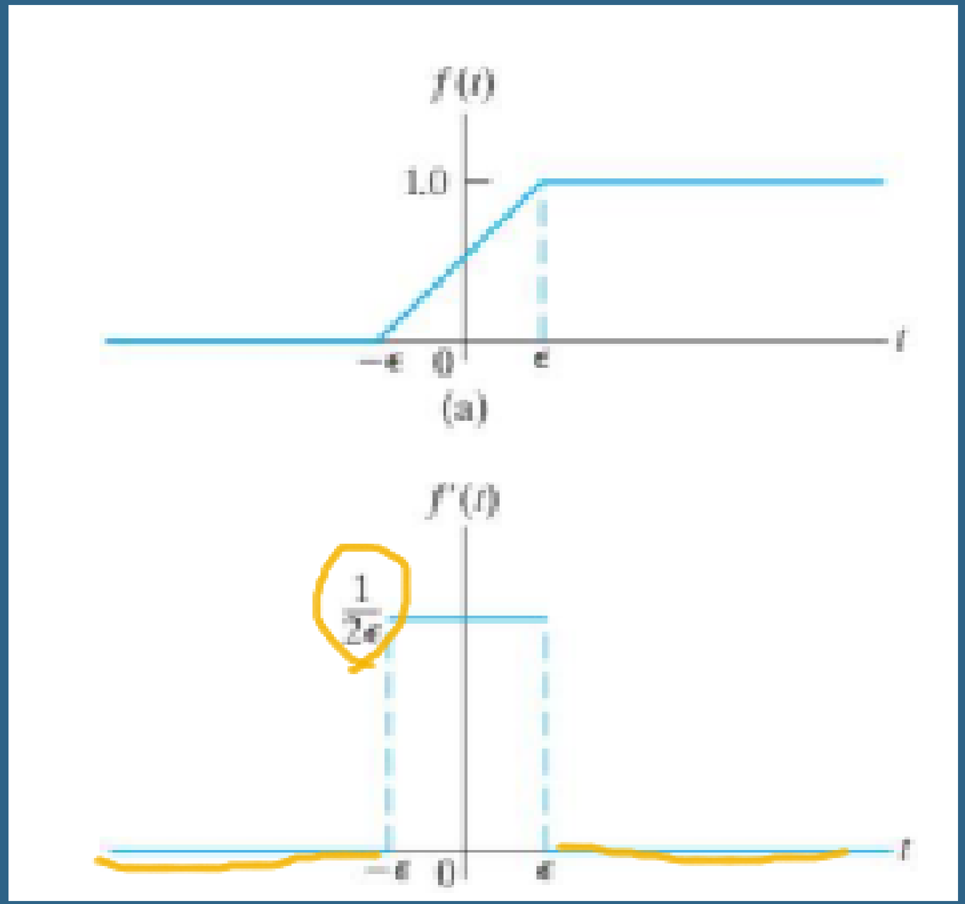
$(-2t+4)\left[u(t-1)-u(t-3)\right]$

$(2t-8)\left[u(t-3)-u(t-4)\right]$

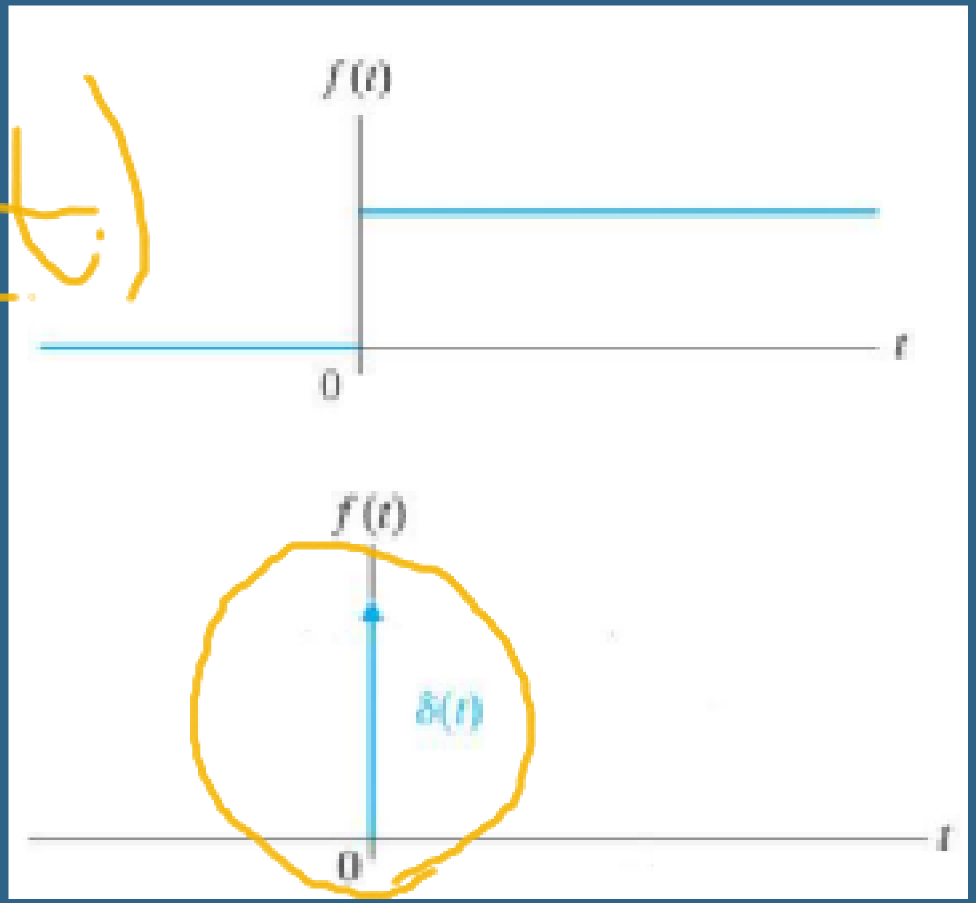


u+ → u

The impulse function



u(t)



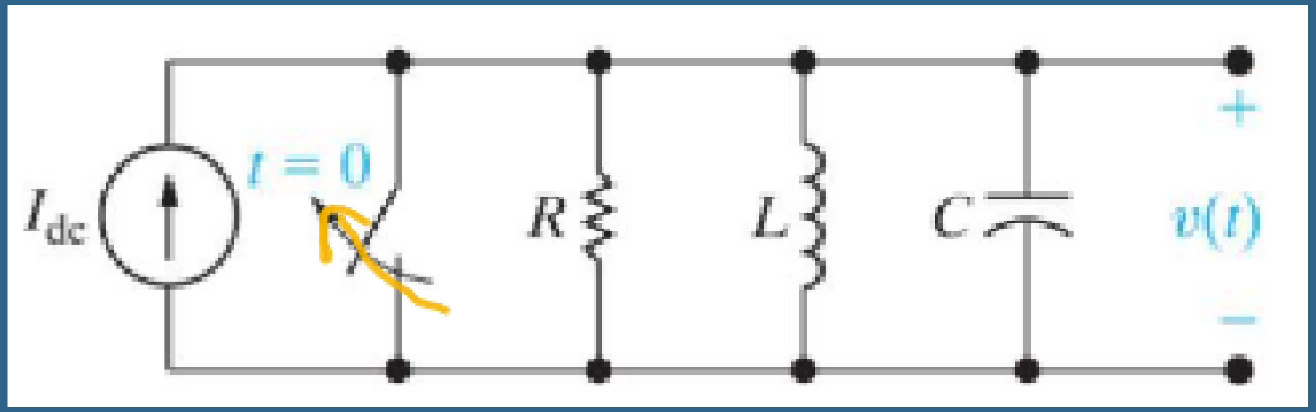
$$\int_{-\infty}^{\infty} K\delta(t)\,dt = K$$

$$\delta(t) = \frac{du(t)}{dt}$$

The Laplace transform

In circuit analysis, we use the Laplace transform to transform a set of integrodifferential equations in the time domain to a set of algebraic equations in the frequency domain.
We can therefore find the solution for an unknown quantity by solving a set of algebraic equations.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$



$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right)$$

Side Note:

What is the distinction between the Laplace transform and the Fourier series?

==> The Laplace transform converts a signal to a complex plane.

==>The Fourier transform transforms the same signal into the jw plane and is a subset of the Laplace transform in which the real part is 0.

Functional Transform is the Laplace transform of a specific function. $\sin(\omega t)$, e^{-at},

Operational transform defines a general mathematical property of the Laplace transform, such as finding the transform of the derivative of f(t).

Function Laplace transform

$\mathscr{L}\{u(t)\} =$

$$\int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

$\mathscr{L}\{\delta(t)\} =$

$$\int_0^\infty \delta(t) e^{-st} = e^{-s(0)} = \boxed{1}$$

$\mathscr{L}\{e^{-at}\}$

$$\int_0^\infty e^{-at} e^{-st} = \int_0^\infty e^{-(a+s)t} = \frac{1}{s+a}$$

$\mathscr{L}\{\sin \omega t\}$

$$= \int_0^\infty (\sin wt) e^{-st} = \left(\frac{e^{jwt} - e^{-jwt}}{2j}\right) e^{-st} = \frac{1}{2j}\left[\frac{1}{s-jw} - \frac{1}{s+jw}\right] = \frac{w}{s^2+w^2}$$

Laplace Transform of Most Common Functions

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0^-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Operational Laplace Transform

Addition and Subtraction

$$\mathscr{L}\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s)$$

Differentiation

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} \Rightarrow sF(s) - f(0^-)$$

$$\mathscr{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0) - \cdots$$

Integration

$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s}$$

Translation in the Time Domain

$\mathscr{L}\{f(t-a)u(t-a)\}$ $e^{-as}F(s)$

Translation in the Frequency Domain

$\mathscr{L}\{e^{-at}f(t)\}$ $F(s+a)$

$\mathscr{L}\{e^{-at}\cos\omega t\}$ $\dfrac{s+a}{(s+a)^2+w^2}$

Scale Changing

$\mathscr{L}\{f(at)\}$ $=\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Operational Table

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$ $- s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$