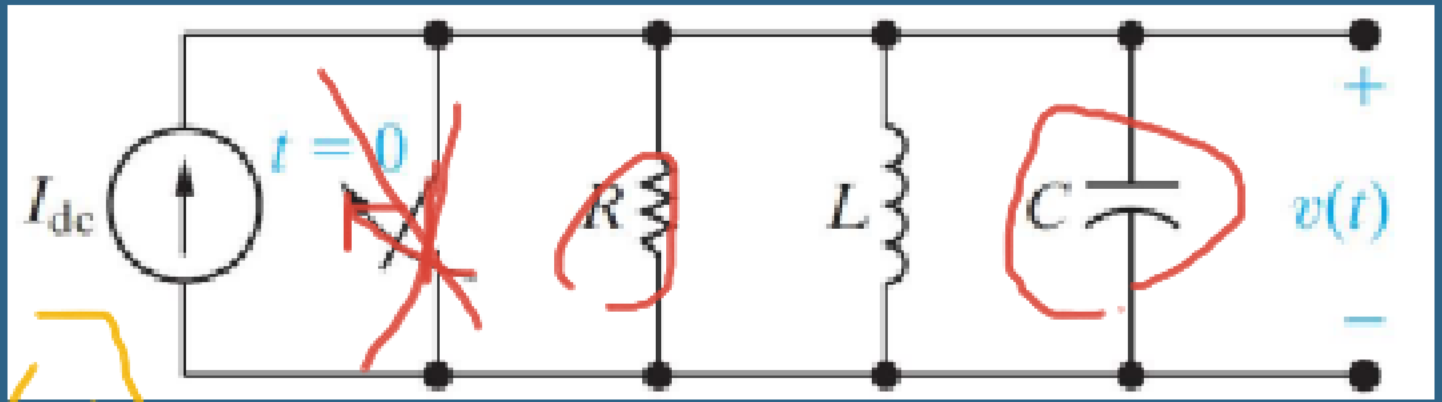
Laplace transform for RLC Step Response

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$



$$C[sV(s) - \overset{0}{v(0^-)}]$$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C \cdot s\,V(s) = I_{dc}\left(\frac{1}{s}\right)$$

$$V(s)\left[\frac{1}{R} + \frac{1}{Ls} + Cs\right] = I_{dc}\left(\frac{1}{s}\right)$$

$$V(s) = \frac{I_{dc}/s}{\frac{1}{R} + \frac{1}{Ls} + Cs} \quad \times \frac{s}{c}$$

$$\frac{I_{dc}/C}{s^2 + s/RC + 1/LC}$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

Inverse Laplace Transform

==> The expression V(s) is a rational function of s.

==> This means V(s) is a ratio of two polynomials in s where only integer powers of s appear in the polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}.$$

==> For linear circuits with constant component values, the s-domain expression for the unknown voltages and currents are always rational functions of s.

$$\frac{x^3 + x^2 - 7}{x^5 + x + 1}$$

Inverse Laplace Transform of Polynomials

$F(s)$	$f(t)\ (t > 0^-)$	Type
1	$\delta(t)$	(impulse)
$\frac{1}{s}$	$u(t)$	(step)
$\frac{1}{s^2}$	t	(ramp)
$\frac{1}{s+a}$	e^{-at}	(exponential)
$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	(sine)
$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	(cosine)
$\frac{1}{(s+a)^2}$	te^{-at}	(damped ramp)
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \sin \omega t$	(damped sine)
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$	(damped cosine)

Get the response in the frequency domain

Partial fraction

$$\frac{s+6}{\cancel{s}\cancel{(s+3)}(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{s+1} + \frac{K_4}{(s+1)^2}$$

$$\Rightarrow K_1 = \frac{0+6}{(0+3)(0+1)^2} = \frac{6}{3} = 2$$

$$\Rightarrow K_2 = \frac{-3+6}{-3(-3+1)^2} = \frac{3}{-12} = -\frac{1}{4}$$

$$\frac{2}{s} \rightarrow 2u(t) \qquad \frac{-1/4}{s+3} = -\frac{1}{4}e^{-3t}u(t)$$

How to get K values in partial fraction expansion

$$F(s) = \frac{96(s+5)(s+12)}{s\cancel{(s+8)}\cancel{(s+6)}} \equiv \frac{K_1}{s} + \frac{K_2}{(s+8)} + \frac{K_3}{(s+6)}.$$

$K_1 = 120.$

$K_2 = -72.$

$K_3 = 48$

$$K_1 \rightarrow \frac{(5)(12)96}{(8)(6)} = \boxed{120}$$

$$K_2 \rightarrow \frac{96(-8+5)(-8+12)}{(-8)(-8+6)} = \checkmark$$

$$K_3 \rightarrow \frac{96(-6+5)(-6+12)}{-6(-6+8)} = \boxed{48}$$

$s+6=0$

$\boxed{s=-6}$

Convert to time domain

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} - \frac{72}{(s+8)} + \frac{48}{(s+6)}.$$

$$(120 - 72e^{-8t} + 48e^{-6t})u(t) \ \#$$

Different Cases for Partial Fraction Expansion

Distinct real roots $\dfrac{N(s)}{(s+p_1)(s+p_2)(s+p_3)}$ $= \dfrac{K_1}{s+p_1} + \dfrac{K_2}{s+p_2} + \dfrac{K_3}{s+p_3}$

Distinct complex roots $\dfrac{A(s+z_1)}{(s+\alpha-j\beta)(s+\alpha+j\beta)}$ $= \dfrac{K_1}{(s+\alpha-j\beta)} + \dfrac{K_2}{(s+\alpha+j\beta)}$

$K_1 = |K| e^{j\theta}$ $K_2 = |K| e^{-j\theta}$

Repeated real roots $\dfrac{N(s)}{(s+p)^r}$ $= \dfrac{K_1}{s+p} + \dfrac{K_2}{(s+p)^2} + \cdots + \dfrac{K_r}{(s+p)^r}$

$\dfrac{N(s)}{(s+p)^2} = \dfrac{K_1}{s+p} + \dfrac{K_2}{(s+p)^2}$

What if ~~Polynomial~~ of numerator >= denominator

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}.$$

$$s^2 + 4s + 10$$

$$s^2 + 9s + 20 \,\big)\, s^4 + 13s^3 + 66s^2 + 200s + 300$$

$$s^4 + 9s^3 + 20s^2$$

$$4s^3 + 46s^2 + 200s + 300$$

$$4s^3 + 36s^2 + 80s$$

$$10s^2 + 120s + 300$$

$$10s^2 + 90s + 200$$

$$30s + 100$$

$$= s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$R_e$$

Inverse Laplace Transform of Different Polynomials

Pair Number	Nature of Roots	$F(s)$	$f(t)$
1	Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$
4	Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$

$$\mathscr{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\}=\frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t).$$

$$\mathscr{L}^{-1}\left\{\frac{|K|\angle\theta}{(s+\alpha-j\beta)^r}+\frac{|K|\angle-\theta}{(s+\alpha+j\beta)^r}\right\}$$
$$=\left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t).$$