Office Hours

9/18/24

↳ $\boxed{MLR}$

Pfs about: $y \sim N(\_\_, \_\_)$

$\hat{\beta} \sim N(\_\_, \_\_)$

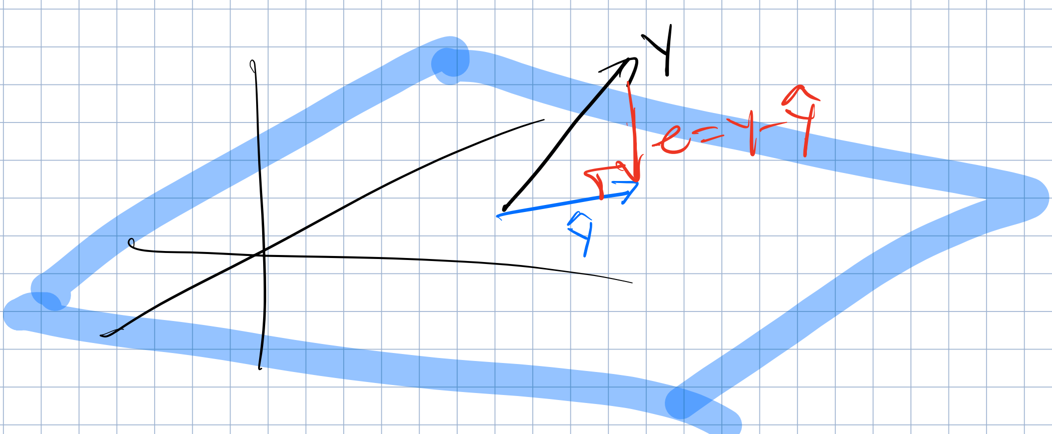
$\hat{y} \sim N(\_\_, \_\_)$

$e \sim N \ \_\_ \ \_\_$

SSE

Ideas about Project

↳ Hat matrix H < ① sym, ② idemp ⟹ projection



6.23

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i \qquad i = 1, \ldots, n$$

$$\begin{cases} Y_1 = \beta_1 X_{11} + \beta_2 X_{12} + \varepsilon_1 \\ \vdots \\ Y_n = \beta_1 X_{n1} + \beta_2 X_{n2} + \varepsilon_n \end{cases}$$

Vectors & Matrices:

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad \underset{n \times 2}{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ \vdots & \vdots \\ X_{n1} & X_{n2} \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\Rightarrow \quad Y = X\beta + \varepsilon$$

with these defs

$$\underline{Q(\beta)} = \|\varepsilon\|^2 = \|Y - X\beta\|^2$$

$$= (Y - X\beta)^T (Y - X\beta)$$

$$= Y^T Y - \underline{Y^T X \beta} - \underline{(X\beta)^T Y} + (X\beta)^T (X\beta)$$

$$= Y^T Y - \underline{2\beta^T X^T Y} + \beta^T X^T X \beta$$

b/c scaler!

$$\underbrace{(\underline{Y^T X \beta})}_{\underbrace{1\times n \; n\times 2 \; 2\times 1}_{1\times 1}} = \underline{(Y^T X \beta)^T} = \underline{\beta^T X^T Y}$$

$\Longrightarrow$ <u>SLR</u>

$$\boxed{\hat{\beta} = (\underline{X^T X})^{-1} \underline{X^T Y}}$$ is the OLS est

$$\|$$

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \bar{Y} - \hat{\beta}_1 \bar{X} \\ \frac{SSXY}{SSX} \end{pmatrix} \leftarrow \text{SLR}$$

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & \cdots & & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

$$= \begin{pmatrix} n & \Sigma_i x_i \\ \Sigma_i x_i & \Sigma_i x_i^2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$(X^T X)^{-1} = \frac{1}{n \Sigma_i x_i^2 - (n\bar{x})^2} \begin{pmatrix} \Sigma_i x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} n\bar{y} \\ \Sigma_i x_i y_i \end{pmatrix}$$

$$\hat{\beta} = \frac{1}{n \Sigma_i x_i^2 - (n\bar{x})^2} \begin{pmatrix} \Sigma_i x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix} \begin{pmatrix} n\bar{y} \\ \Sigma_i x_i y_i \end{pmatrix}$$

$$= \frac{1}{n\sum_i x_i^2 - (n\bar{x})^2} \begin{pmatrix} \sum_i x_i^2 (n\bar{y}) - n\bar{x}(\sum_i x_i y_i) \\ -n^2 \bar{x}\,\bar{y} + n\sum_i x_i y_i \end{pmatrix} \quad \uparrow \bar{y} - \hat{\beta}_1 \bar{x}$$

$$= \begin{pmatrix} \cdots \\ \frac{\cancel{n}(\sum_i x_i y_i - n\bar{x}\bar{y})}{n(\sum_i x_i^2 - n\bar{x}^2)} \end{pmatrix} = \begin{pmatrix} \bar{y} - \hat{\beta}_1 \bar{x} \\ \frac{SSXY}{SSX} \end{pmatrix} \hat{\beta}_1$$

$$\frac{\sum_i x_i^2 (n\bar{y}) - (n\bar{x})(\sum_i x_i y_i)}{n(\sum_i x_i^2 - n\bar{x}^2)} - \frac{n\bar{x}^2\bar{y} + n\bar{x}^2\bar{y}}{n(\sum_i x_i^2 - n\bar{x}^2)}$$

$$= \frac{(\sum_i x_i^2 n)\bar{y} - n\bar{x}^2\bar{y} - n\bar{x}(\sum_i x_i y_i) + (n\bar{x})(\bar{x}\bar{y})}{n(\sum_i x_i^2 - n\bar{x}^2)}$$

$$= \frac{\cancel{n(\sum_i x_i^2 - n\bar{x}^2)}\,\bar{y}}{\cancel{n(\sum_i x_i^2 - n\bar{x}^2)}} - \frac{\cancel{n}\bar{x}\left(\sum_i x_i y_i - n\bar{x}\bar{y}\right)}{\cancel{n}(\sum_i x_i^2 - n\bar{x}^2)}$$

$$- \frac{SSXY\ \bar{x}}{SSX}$$

$$= \bar{y} - \left(\frac{SSXY}{SSX}\right)\bar{x} = \bar{y} - \hat{\beta}_1 \bar{x} = \hat{\beta}_0$$

6.23b

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$$

$$y_i \sim N(\beta_1 x_{i1} + \beta_2 x_{i2},\ \sigma^2)$$

$$\mathcal{L}(\beta) = \prod_{i=1}^{n} f_{Y_i}(y_i)$$

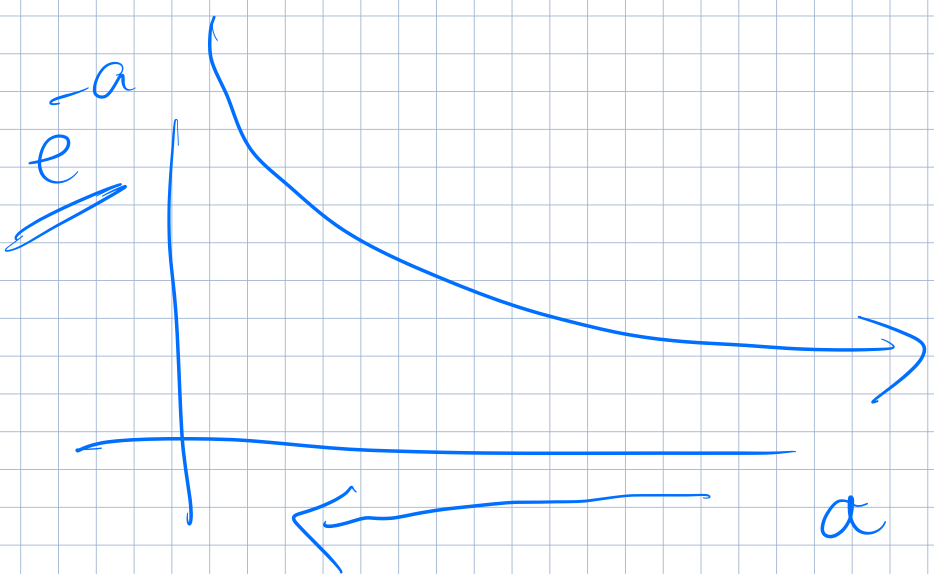
$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) e^{-\frac{(y_i - \beta_1 x_{i1} - \beta_2 x_{i2})^2}{2\sigma^2}}$$

$$= (2\pi)^{-n/2} \sigma^{-n} e^{-\frac{\sum_i (y_i-\beta_1 x_{i1}-\beta_2 x_{i2})^2}{2\sigma^2}}$$

$$e^{a} e^{b} e^{c} = e^{a+b+c}$$

$$\mathcal{L}(\beta) = K e^{-\frac{Q(\beta)}{2\sigma^2}}$$

$$e^{-a}$$



maximizing $\mathcal{L}(\beta) \iff$ minimizing $Q(\beta)$

answer is OLS

$\hat{\beta} = (X^TX)^{-1}X^TY$!!

6.27

$$X(X^TX)^{-1}X^T$$
$6\times3 \quad \cdots \quad 3\times6$

$$I_6 - H$$
$6\times6 \quad 6\times6$

$$Y = \begin{pmatrix} 42 \\ 33 \\ 75 \\ 28 \\ 91 \\ 55 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 7 & 33 \\ 1 & 4 & 41 \\ 1 & 16 & 7 \\ 1 & 3 & 49 \\ 1 & 21 & 5 \\ 1 & 8 & 31 \end{pmatrix}$$

a. $\hat{\beta} = \underline{b} = (X^TX)^{-1}X^TY = \begin{pmatrix} \_ \\ \_ \\ \_ \end{pmatrix}$

np

pd

$$e = (I_6 - H)Y$$

$\boxed{r\ \beta_j\text{'s in } (n-r)}$ $\quad p\ \beta_j\text{'s}\ (n-p)$

$$F = \frac{(SSE_{Red} - SSE_{Full}) \,/\, \overbrace{(df_{SSE_{red}} - df_{SSE_{Full}})}^{(p-r)}}{SSE_F \,/\, \underbrace{df_{SSE_F}}_{(n-p)}}$$

Partial F-test

$$\boxed{H_0:\ \beta_r = \beta_{r+1} = \cdots = \beta_{p-1} = 0}$$

H_1: not H_0

↳ at least one of $\beta_r, \beta_{r+1}, \ldots, \beta_{p-1}$ is not actually zero.

Full: $Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_{r-1} X_{(r-1)i}$
$\qquad + \underbrace{\beta_r}_{0} X_{ri} + \cdots + \underbrace{\beta_{p-1}}_{0} X_{(p-1)i} + \varepsilon_i$

Red: $Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_{r-1} X_{(r-1)i} + \varepsilon_i$

$Y_i = \beta_0 + \varepsilon_i \leftarrow 1$

(AB)

$(A-B)(A-B)$

$(A-B)A - (A-B)B$

$A^2 - \underline{\underline{BA}} - \underline{\underline{AB}} + B^2 = A^2 - AB - BA + B^2$

Multivariate Normal $\leftarrow$ Cody Fix Notes

Sept 12

SSE's distribution

$Y \sim N(X\beta, \sigma^2 I_n)$

$SSE = e^T e$

$e = (I-H)Y$

$e = \underline{(I-H)}Y \sim N(\underline{(I-H)}X\beta, (I-H)\sigma^2 I (I-H)^T)$

$\sim N(0, \sigma^2 \underline{\underline{(I-H)}})$

$(I-H)X = IX - HX = X - HX$

$= X - X\cancel{(X^T X)^{-1}}\cancel{X^T} X$

$= 0$

if $AY \sim N(0, \Sigma)$ ↙ idempotent

$$\boxed{(Y^T A Y \sim \chi^2_{rank(A) = tr(A)})}$$