School of Computer Science and Technology
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Exercise Sheet 1 for
Design and Analysis of Algorithms
Autumn 2022

Due 1 Oct 2022 at 23:59

Exercise 1

Suppose you have an unfair dice such that if you roll it, the probability that each odd number (i.e., $1, 3, 5$) appears on the top is $\frac{1}{12}$, and the probability that each even number (i.e., $2, 4, 6$) appears on the top is $\frac{1}{4}$.
(1) Suppose that you roll this dice exactly once.

- What is the expected value of the number X on top of the dice?
- What is the variance of X?

(2) Suppose that you roll this dice n times. Let Y denote the sum of the numbers that are on the top of the dice throughout all the n rolls.

- What is the expected value of Y?
- What is the variance of Y?
- Give an upper bound of $\Pr[Y > 4n]$ by using Markov inequality.
- Give an upper bound of $\Pr[Y > 4n]$ by using Chebyshev's inequality.
- Give an upper bound of $\Pr[Y > 4n]$ by using Chernoff Bound.

Solution:

- $E(X) = (1 + 3 + 5)/12 + (2 + 4 + 6)/4 = 3.75$
- $Var(X) = E(X^2) - (E(X)^2) = (1 + 9 + 25)/12 + (4 + 16 + 36)/4 - 3.75^2 = 2.85$
- Because of the Y is the sum of n independent variables, $E(Y) = nE(X) = 3.75n$
- $Var(Y) = nVar(X) = 2.85n$
- $Pr(Y > 4n) \le Pr(Y \ge 4n) = Pr(Y \ge tE(Y)) \le \frac{1}{t}$, set $3.75nt = 4n$, $t = \frac{16}{15}$, so we can get $Pr(Y > 4n) \le \frac{15}{16} = 0.94$.
- $Pr(Y > 4n) \le Pr(|Y - 3.75n| \ge 0.25n) = Pr(|Y - E(Y)| \ge t\sqrt{Var(Y)}) \le \frac{1}{t^2}$, set $t\sqrt{2.85n} = 0.25n$ $\frac{1}{t^2} = 2.85/(0.25^2 n) = 45.6/n$, so we can get $Pr(Y > 4n) \le \frac{45.6}{n}$
- $Pr(Y > 4n) \le Pr(|Y - 3.75n| \ge 0.25n) = Pr(|Y - E(Y)| \ge \delta E(Y)) \le 2\exp(\frac{-E(Y)\cdot\delta^2}{3})$, set $3.75n\delta = 0.25n$, $\delta = 0.07$, so we can get $Pr(Y > 4n) \le 2\exp(\frac{-3.75n\cdot 0.0049}{3}) = 2\exp(-0.0030625n)$

Exercise 2 *(An application of Markov's inequality: turning a Las Vegas algorithm to a Monte Carlo algorithm)*

Let $\mathcal{A}$ be a randomized algorithm for some decision problem P (e.g., to decide if a graph is connected or not). Suppose that for any given instance of P of size n, $\mathcal{A}$ runs in *expected* $T(n)$ time and always outputs the correct answer.
Use $\mathcal{A}$ to give a new randomized algorithm NewAlg for the problem P such that NewAlg *always* runs in $100T(n)$ time and

- if the input is a 'Yes' instance[1], then it will be accepted with probability at least $\frac{5}{6}$;
- if the input is a 'No' instance, then it will always be rejected.

Describe your algorithm and justify its correctness and running time.

Hint: You may use Markov's inequality.

Solution:

Consider the algorithm (NewAlg): run the $\mathcal{A}$ for t steps, and if it has not stopped yet, just abort it.

If the input is a 'No' instance, then it will always be rejected obviously.
If the input is a 'Yes' instance, consider the probability of accepted a answer is at least: if the running time of $\mathcal{A}$ is T, we can use Markov's inequality to get $Pr(T < t) = 1 - Pr(T \geq t) \geq 1 - \frac{E\mathcal{A}}{t} = 1 - \frac{T(n)}{t}$,
$\because Pr(T < t) \geq \frac{5}{6} \therefore 1 - \frac{T(n)}{t} = \frac{5}{6} \therefore t = 6T(n)$. So we just run the $\mathcal{A}$ at least for $6T(n)$ steps which is smaller than $100T(n)$, we can ensure the accepted probability is at least $\frac{5}{6}$.

Exercise 3
Take R to be the IQ of a random person you pull off the street. What is the probability that some's IQ is at least 200 given that the average IQ is 100 and the variance of R is 100?

Solution:

- Using Markov's inequality: $Pr(IQ \geq 200) = Pr(IQ \geq tE(IQ)) \leq \frac{1}{t}$, set $tE(IQ) = 200$, $t = 2$, so we can get $Pr(IQ \geq 200) \leq \frac{1}{2}$
- Using Chebyshev inequality: $Pr(IQ \geq 200) \leq Pr(|IQ - 100| \geq 100) = Pr(|IQ - E(IQ)| \geq t\sqrt{Var(IQ)}) \leq \frac{1}{t^2}$, set $t\sqrt{Var(IQ)} = 100$, $t = 10$, so we can get $Pr(IQ \geq 200) \leq \frac{1}{100}$

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[1] An instance is a 'Yes'-instance if the correct answer to the Problem is 'Yes'. Otherwise (the correct answer is 'No'), it is a 'No'-instance.