ASSG 2 COMP 2121

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1. **Set Theory** [19 Marks]

   (a) (5 marks) Find all sets X, Y, Z that satisfy the following condition: $X = \{1, |Y|, |Z|\}$, $Y = \{2, |X|, |Z|\}$, and $Z = \{1, 2, |X|, |Y|\}$.

   (b) (6 marks) For arbitrary finite sets A, B and C, prove that $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B)$.

   (c) (8 marks) Suppose that $A, B_1, B_2, \ldots, B_n$ are finite sets where n is a positive integer. Prove that

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i).$$

a. $X = \{1, |Y|, |Z|\}$

$Y = \{2, |X|, |Z|\}$

$Z = \{1, 2, |X|, |Y|\}$

excluding the unknown variable value, we know that $1 \leq |X| \leq 3$ $\quad 1 \leq |Y| \leq 3$ $\quad 2 \leq |Z| \leq 4$

knowing that $|Z|$ is at least two and $|Z|$ is part of $|X|$, then $2 \leq |X| \leq 3$, so $|X|$ is either 2 or 3

if $|X| = 2$

$Y = \{2, |Z|\}$, this means $|Y|$ is either 1 or 2

$Z = \{1, 2, |Y|\}$, if $|Y|$ is either 1 or 2, $|Z|$ is 2

assume that $|X| = 2$, $|Z| = 2$, then

$Y = \{2\}$

$X = \{1, 2\}$

$Z = \{1, 2\}$

if $|X| = 3$

$|Y| \neq |Z| \neq 1$

$Y = \{2, 3, |Z|\}$ $\rightarrow$ $|Y|$ is either 2 or 3

$|Z| = \{1, 2, 3, |Y|\}$ $\rightarrow$ regardless of $|Y|$ value, $|Z|$ is 3 then $|Y|$ is 2

assume $|X| = 3$, $|Z| = 3$, $|Y| = 2$

$X = \{1, 2, 3\}$

$Y = \{2, 3\}$

$Z = \{1, 2, 3\}$

there are only 2 possible solutions

$X = \{1, 2\}$, $Y = \{2\}$, $Z = \{1, 2\}$

$X = \{1, 2, 3\}$, $Y = \{2, 3\}$, $Z = \{1, 2, 3\}$

b. arbitary set A, B, C

$(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B)$

let $x \in (A \setminus C) \setminus (B \setminus C)$

$\forall x \ (x \in (A \setminus C) \wedge x \notin (B \setminus C))$ difference relation

$\forall x \ (x \in A \wedge x \notin C \wedge (x \notin B \vee x \in C))$ difference relation & De Morgan's Law

$\forall x \ ((x \in A \wedge x \notin C \wedge x \notin B) \vee (x \in A \wedge x \notin C \wedge x \in C))$ distributive law

$\forall x \ (((x \in A \vee x \notin B) \wedge x \notin C) \vee \emptyset)$ associative law & domination law

$\forall x \ (x \in (A \setminus B) \wedge x \notin C)$ difference relation

as $(A \setminus B) \wedge (x \in \&)$ must be a subset of $(A \setminus B)$, it's proven that $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B)$

c. $A, B_1, B_2, \ldots, B_n$ are finite sets, $n \in \mathbb{Z}^+$

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i)$$

i. LHS $\subseteq$ RHS

$A \cup \left(\bigcap_{i=1}^{n} B_i \right)$ means $x \in A \vee x \in \left(\bigcap_{i=1}^{n} B_i \right)$

$x \in A \cup x \in \left(\bigcap_{i=1}^{n} B_i \right)$

$\equiv x \in A \cup ((x \in B_1) \cap (x \in B_2) \cap (x \in B_3) \ldots \cap (x \in B_n))$ generelised intersection

$\equiv (x \in A \cup x \in B_1) \cap (x \in A \cup x \in B_2) \ldots \cap (x \in A \cup x \in B_n)$ distributive law

$\equiv x \in ((A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n))$ distributive law

$\equiv x \in \bigcap_{i=1}^{n} (A \cup B_i)$ generelised intersection

$\equiv \bigcap_{i=1}^{n} (A \cup B_i)$ proven that LHS $\subseteq$ RHS

ii. RHS $\subseteq$ LHS

$\bigcap_{i=1}^{n} (A \cup B_i)$

$x \in \bigcap_{i=1}^{n} (A \cup B_i)$

$\equiv x \in ((A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n))$ generelised intersection

$\equiv x \in (A \cup (\bigcap_{i=1}^{n} B_n))$ distributive law & generelised union

$\equiv A \cup (\bigcap_{i=1}^{n} B_n)$ proven that RHS $\subseteq$ LHS

$\therefore$ as $A \cup (\bigcap_{i=1}^{n} B_n) \subseteq \bigcap_{i=1}^{n} (A \cup B_i)$ and $\bigcap_{i=1}^{n} (A \cup B_i) \subseteq A \cup (\bigcap_{i=1}^{n} B_i)$, $A \cup (\bigcap_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} (A \cup B_i)$

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2. **Properties of Relations** [18 Marks]

(a) (6 marks) List all the symmetric, antisymmetric and asymmetric relations on the set $A = \{\Phi, \{\Phi\}\}$. Conjecture a formula for the numbers of antisymmetric and asymmetric relations on a set of m elements. Substantiate your conjecture with brief (informal) reasoning.

(b) (6 marks) Consider the homogeneous relation R on set $A = \{3, 4, 5, 8, 10, 24, 25, 27, 48, 50, 81, 100, 125, 128\}$ defined by aRb if and only if a and b have exactly the same prime factors. Determine whether it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric. Then state whether it is (a) an equivalence relation and (b) a partial order. If it is an equivalence relation, determine its equivalence classes.

(c) (6 marks) Let $A := \{a, b, c, d\}$. Find the smallest relation S on A that includes the relation $R = \{(a, b), (b, a), (b, c), (c, d), (d, a)\}$ as a subset and is both reflexive and transitive.

a. $A = \{\varnothing, \{\varnothing\}\} = \{\{\}, \{\{\}\}\}$

$|A| = 2$

There are $2^{2^2} = 16$ possible relations

symmetric relations of R

- $\{(\varnothing, \varnothing), (\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \varnothing), (\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing)\}$
- $\{(\varnothing, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing)\}$
- $\{(\varnothing, \varnothing)\}$ & $\{(\{\varnothing\}, \{\varnothing\})\}$
- $\{\}$

total: 8

anti symmetric relations of R

- $\{(\varnothing, \varnothing), (\varnothing, \{\varnothing\}), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \varnothing), (\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \varnothing), (\varnothing, \{\varnothing\})\}$
- $\{(\varnothing, \varnothing), (\{\varnothing\}, \varnothing)\}$
- $\{(\varnothing, \{\varnothing\}), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$
- $\{(\varnothing, \{\varnothing\})\}$ & $\{(\{\varnothing\}, \varnothing)\}$
- $\{\}$
- $\{(\varnothing, \varnothing)\}$
- $\{(\{\varnothing\}, \{\varnothing\})\}$

total: 12

asymmetric relations of $R \rightarrow ((a, b) \in R \wedge (b, a) \notin R)$

- $\{(\varnothing, \{\varnothing\})\}$
- $\{(\{\varnothing\}, \varnothing)\}$
- $\{\}$

total: 3

<u>symmetric relation</u>

$$\begin{bmatrix} n & \cdot & \cdot & \cdot \\ \vdots & n_2 & \cdot & \cdot \\ \vdots & \vdots & \ddots & \cdot \\ \cdot & \cdot & \cdot & n_m \end{bmatrix}$$

diagonal relations((a,b) where a=b) = m pairs

other relations at upper triangle: $\frac{m^2 - m}{2}$ pairs

other relations

a_i, b_j	1	0
a_j, b_i	1	0

a diagonal relation has 2 possible value 0 / 1

another relation at the upper triangle has 2 possible value (a_i, b_j) 0 / 1, (a_j, b_i) will have the same value

number of symmetric relation for m element $2^m \cdot 2^{\frac{m^2-m}{2}} = 2^{m + \frac{m^2-m}{2}} = 2^{\frac{m^2+m}{2}}$

<u>anti symmetric relation</u>

a_i, b_j	1	0	0
a_j, b_i	0	1	0

a diagonal relation has 2 possible value

an other relation at the upper triangle has 3 possible outcome

number of anti symmetric relation : $2^m \cdot 3^{\frac{m^2-m}{2}}$

asymmetric relation

all diagonal relations have to be 0, a diagonal relation has 1 possible value

an other relation at the upper triangle has 3 option, like anti symmetric

number of asymmetric relations $= 1^m \times 3^{\frac{m^2-m}{2}} = 3^{\frac{m^2-m}{2}}$

b. $A = \{3, 4, 5, 8, 10, 24, 25, 27, 48, 50, 81, 100, 125, 128\}$

aRb if and only if a and b has the same prime factor

$R = \{(3,3), (3,27), (3,81), (4,4), (4,8), (4,128), (5,5), (5,25), (5,125), (8,4), (8,8), (8,128),$
$(10,10), (10,50), (10,100), (24,24), (24,48), (25,5), (25,25), (25,125), \ldots, (128,128)\}$

(i) it is reflexive as for $\forall a(a \in A \mid (a,a) \in R)$, same number will have the same factor

(ii) a symmetric relation has $\forall a \forall b((a,b) \in R \longrightarrow (b,a) \in R)$ property

when $(a,b) \in R$ means a's prime factor(s) = b's prime factor(s), this also mean, b's prime factor(s) = a's

which implies $(b,a) \in R$. Hence, it is symmetric

(iii) a transitive relation has $\forall a \forall b \forall c\ (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$ property

when $(a,b) \in R$, b should be the first element in one other relation.

$(a,b) \in R$ means a's prime factor(s) = b's prime factor(s)
$(b,c) \in R$ means b's prime factor(s) = c's prime factor(s)
} we can conclude a's prime factor(s) = c's

Hence $(a,c) \in R$, it is transitive

Additionally, as R is symmetric & reflexive to the very least there are (a,b), (b,a), and (a,a)

(iv) an anti symmetric has $\forall a \forall b\ ((a,b) \in R \wedge (b,a) \in R) \longrightarrow (a=b)$

however this relation clearly have lots of counterexample, $(4,8) \in R \wedge (8,4) \in R$, but $4 \neq 8$

Hence, it is not anti symmetric

So R is an equivalence relation as it is reflexive symmetric, transitive, but not anti symmetric

equivalence classes:

$[3]_R = \{b \in A \mid b \bmod 3 = 0\} = \{3, 27, 81\}$
$[4]_R = \{b \in A \mid b \bmod 2 = 0\} = \{4, 8, 128\}$
$[5]_R = \{b \in A \mid b \bmod 5 = 0\} = \{5, 25, 125\}$
$[24]_R = \{b \in A \mid b \bmod 6 = 0\} = \{24, 48\}$
$[10]_R = \{b \in A \mid b \bmod 10 = 0\} = \{10, 50, 100\}$

c. $A = \{a, b, c, d\}$

$R = \{(a,b), (b,a), (b,c), (c,d), (d,a)\}$

R + reflexive pairs : S_1

$S_1 = \{(a,b), (b,a), (b,c), (c,d), (d,a), (a,a), (b,b), (c,c), (d,d)\}$

(a,b) & (b,c) are not transitive, we need to add (a,c)

(b,c) & (c,d) are not transitive, we need to add (b,d)

(c,d) & (d,a) are not transitive, we need to add (c,a)

(d,a) & (a,b) are not transitive, we need to add (d,b)

(d,a) & (a,c) are not transitive, we need to add (d,c)

(a,c) & (c,d) are not transitive, we need to add (a,d)

(c,a) & (a,b) are not transitive, we need to add (c,b)

S_1 + additional transitivity pairs : S

$S = \{(a,b), (b,a), (b,c), (c,d), (d,a), (a,a), (b,b), (c,c), (d,d), (a,c), (a,d), (b,d), (c,a),$
$(c,b), (d,b), (d,c)\}$

$= A \times A$

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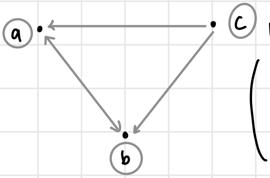
3. **Representing Relations** [10 **Marks**]

Given the directed graphs representing two relations, how can the directed graph of the union, intersection, difference and composition of these relations be found? Illustrate this with the relations R_1 and R_2 on a set A represented by the matrices

$$M_{R_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{R_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

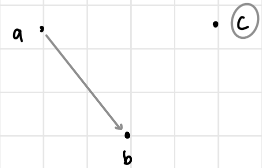
3. union



$M_{R_1} \vee M_{R_2}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

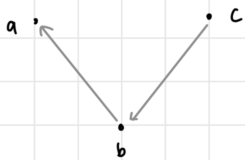
intersection



$M_{R_1} \wedge M_{R_2}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

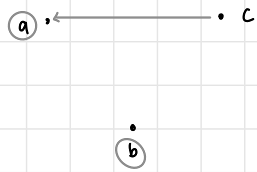
difference ($M_{R_1} - M_{R_2}$)



$M_{R_1} - M_{R_2}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

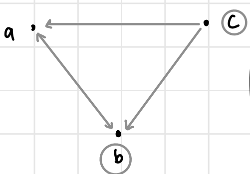
difference ($M_{R_2} - M_{R_1}$)



$M_{R_2} - M_{R_1}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

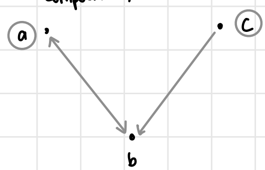
composition ($M_{R_2} \circ M_{R_1}$)



$M_{R_2} \circ M_{R_1}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

composition ($M_{R_1} \circ M_{R_2}$)



$M_{R_1} \circ M_{R_2}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\underset{M_{R_1}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \cdot \underset{M_{R_2}}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

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4. **Functions [18 Marks]**

(a) (6 marks) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined as follows. For all $x \in \mathbb{R}$, $f(x) := x + |x|$ and $g(x) = |x| - x$ where $|\cdot|$ denotes the absolute value function. Find the functions $f \circ g$ and $g \circ f$.

(b) (6 marks) Let R be a homogeneous relation on a finite set A. Suppose R is an Equivalence Relation as well as a Function, is it necessarily the case that R is the Identity Relation defined by $I(a) = a$ for all $a \in A$? Prove your answer.

(c) (6 marks) Identify whether the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{x}{x^2+1}$ for all $x \in \mathbb{R}$ is (a) Injective and (b) Surjective. Explain your answer.

a. $f(x) = x + |x|$

$g(x) = |x| - x$

$f \circ g = |x| - x + \underline{||x| - x|}$ → regardless of x value $|x| - x$ will be positive / 0

$= |x| - x + |x| - x$

$= 2|x| - 2x$

$g \circ f = |x + |x|| - (x + |x|)$

$= x + |x| - x - |x|$

$= 0$

b. equivalence relation:

- reflexive : $\forall a \, ((a,a) \in R)$
- symmetric : $\forall a \forall b \, ((a,b) \in R \to (b,a) \in R)$
- transitive : $\forall a \forall b \forall c \, (((a,b) \in R \land (b,c) \in R) \to (a,c) \in R)$

transitive suggests that the function is surjective

homogeneous relation suggests that the function is one to one

Hence the function's invertible

let $(a,b) \in R$, but from the fact that it's reflective we know that $(a,a) \in R$

and due to it's homogeneous property $b = a$. This means every element is related to itself and only itself

$\therefore$ this implies that R is the identity function on A for every $a \in A$

c. $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \frac{x}{x^2+1}$

c.a. injective

injective means one to one relation

suppose $f(x_1) = f(x_2)$

$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$x_1(x_2^2+1) = x_2(x_1^2+1)$$

$$x_1 x_2^2 + x_1 = x_1^2 x_2 + x_2$$

$x_1 x_2 (x_2 - x_1) = x_2 - x_1$

$(x_2 - x_1)(x_1 x_2 - 1) = 0$

so it's either $x_2 = x_1$ or $x_1 x_2 = 1$

there's 2 possible outcomes, this function is not injective

c.b. surjective

Surjective means that codomain is equal to range

$$y = \frac{x}{x^2+1}$$

$y(x^2+1) = x$

$x^2 y + y - x = 0$

discriminant $= (-1)^2 - 4(y)(y)$

$= 1 - 4y^2$

since discriminant needs to be positive for any real y, real roots will always exist if

$1 - 4y^2 \geq 0$

$y^2 \leq \frac{1}{4}$

$-\frac{1}{2} \leq y \leq \frac{1}{2}$

so the range is $(-\frac{1}{2}, \frac{1}{2}) \neq$ codomain

$\therefore$ it's not surjective

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5. **Special Functions [19 Marks]**

(a) (6 marks) Solve for $x \in \mathbb{R}$ the equation $\lceil 7x + 3\lfloor x \rfloor \rceil = 17$.

(b) (6 marks) Solve for $x \in \mathbb{R}$ the equation $(6\log_5 x)^2 + \log_5 x^{11} - 35 = 0$.

(c) (7 marks) Compute $\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$ (give a closed form expression for the sum). Check your answer for the case $n = 10$.

a. $\lceil 7x + 3\lfloor x \rfloor \rceil = 17$

$\lfloor x \rfloor = n \Rightarrow n \le x < n+1 \;, \; n \in \mathbb{Z}$

$\lceil 7x + 3n \rceil = 17$

$16 < 7x + 3n \le 17$

$\frac{16-3n}{7} < x \le \frac{17-3n}{7} \quad \ldots ①$

$n \le x < n+1 \quad \ldots ②$

$\frac{16-3n}{7} < n+1$ $\qquad\qquad$ $n < \frac{17-3n}{7}$

$16 - 3n < 7n + 7$ $\qquad\qquad$ $7n < 17 - 3n$

$10n > 9$ $\qquad\qquad$ $10n < 17$

$n > \frac{9}{10}$ $\qquad\qquad$ $n < \frac{17}{10}$

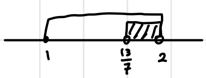
$\frac{9}{10} < n < \frac{17}{10}$

as $n \in \mathbb{Z}$, $n = 1$

Substituting $n=1$ to ① $\qquad\qquad$ Subs $n=1$ to ②

$\frac{16-3}{7} < x \le \frac{17-3}{7}$ $\qquad\qquad$ $1 \le x < 2$

$\frac{13}{7} < x \le 2$



$\therefore$ from the intersection, we get that $\frac{13}{7} < x < 2$

b. $(6\log_5 x)^2 + \log_5 x^{11} - 35 = 0$

$(6\log_5 x)^2 + 11\log_5 x - 35 = 0$

let $\log_5 x = a$

$36a^2 + 11a - 35 = 0$

$a = \dfrac{-11 \pm \sqrt{121 + 5040}}{72}$

$\log_5 x = \dfrac{-11 + \sqrt{5161}}{72}$ $\qquad$ $\log_5 x = \dfrac{-11 - \sqrt{5161}}{72}$

$x = 5^{\frac{-11+\sqrt{5161}}{72}}$ or $x = 5^{\frac{-11-\sqrt{5161}}{72}}$

c. $\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$

$\lfloor\sqrt{0}\rfloor + \lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \lfloor\sqrt{4}\rfloor + \lfloor\sqrt{5}\rfloor + \lfloor\sqrt{6}\rfloor + \lfloor\sqrt{7}\rfloor + \lfloor\sqrt{8}\rfloor + \lfloor\sqrt{9}\rfloor + \lfloor\sqrt{10}\rfloor + \lfloor\sqrt{11}\rfloor + \lfloor\sqrt{12}\rfloor + \lfloor\sqrt{13}\rfloor + \lfloor\sqrt{14}\rfloor + \lfloor\sqrt{15}\rfloor + \dots + \lfloor\sqrt{n}\rfloor$

$0 + 1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + \dots + \lfloor\sqrt{n}\rfloor$

0 - 3: 0, 1, 1, 1 $\quad (3 = 2.1 + 1)$

4 - 8 : 2, 2, 2, 2, 2 $\quad (5 = 2 \cdot 2 + 1)$

9 - 15 : 3, 3, 3, 3, 3, 3, 3 $\quad (7 = 2.3 + 1)$

and so on

for every integer x, $(x+1)^2 - 1 - x^2 = 2x + 1$

$\sum_{k=0}^{n} \sqrt{k}$, $x^2 \le k < (x+1)^2$

can be rewritten as

$\sum_{k=0}^{\lfloor\sqrt{n}\rfloor} k(2k+1) - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor + 1)^2 - n)$

$= \sum_{k=0}^{\lfloor\sqrt{n}\rfloor} 2k^2 + k - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor + 1)^2 - n)$

$= 2\sum_{k=0}^{\lfloor\sqrt{n}\rfloor} k^2 + \sum_{k=0}^{\lfloor\sqrt{n}\rfloor} k - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor + 1)^2 - n)$

$= 2\left(\dfrac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)(2\lfloor\sqrt{n}\rfloor+1)}{6}\right) + \dfrac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)}{2} - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor + 1)^2 - n)$

$= \dfrac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)(4\lfloor\sqrt{n}\rfloor+5)}{6} - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor + 1)^2 - n)$

$n = 10$ $\qquad \dfrac{3 \quad 4 \,.\, 17}{6} - 3(16 - 10)$

$= 34 - 18$

$= 16$

$\lfloor\sqrt{0}\rfloor + \lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \lfloor\sqrt{4}\rfloor + \lfloor\sqrt{5}\rfloor + \lfloor\sqrt{6}\rfloor + \lfloor\sqrt{7}\rfloor + \lfloor\sqrt{8}\rfloor + \lfloor\sqrt{9}\rfloor$

$= 0 + 1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 3$

$= 16$

$$\frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)(4\lfloor\sqrt{n}\rfloor+5)}{6} - \lfloor\sqrt{n}\rfloor((\lfloor\sqrt{n}\rfloor+1)^2-n) \quad \text{is proven for } n:10$$

$$= \lfloor\sqrt{n}\rfloor \cdot \frac{(\lfloor\sqrt{n}\rfloor+1)(4\lfloor\sqrt{n}\rfloor+5) - 6(\lfloor\sqrt{n}\rfloor+1)^2 + 6n}{6}$$

$$= \lfloor\sqrt{n}\rfloor \cdot \frac{4\lfloor\sqrt{n}\rfloor^2 + 9\lfloor\sqrt{n}\rfloor + 5 - 6\lfloor\sqrt{n}\rfloor^2 - 12\lfloor\sqrt{n}\rfloor - 6 + 6n}{6}$$

$$= \lfloor\sqrt{n}\rfloor\left(\frac{-2\lfloor\sqrt{n}\rfloor^2 - 3\lfloor\sqrt{n}\rfloor - 1}{6} + n\right)$$

$$= \lfloor\sqrt{n}\rfloor\left(-\frac{(2\lfloor\sqrt{n}\rfloor+1)(\lfloor\sqrt{n}\rfloor+1)}{6} + n\right)$$

$$\therefore \sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \lfloor\sqrt{n}\rfloor\left(-\frac{(2\lfloor\sqrt{n}\rfloor+1)(\lfloor\sqrt{n}\rfloor+1)}{6} + n\right)$$

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6. **Growth of Functions and Relations [16 Marks]**

(a) (9 marks) Arrange the following functions on integers in increasing order of their growth rate. If $f(n)$ is $O(g(n))$ but $g(n)$ is not $O(f(n))$ then $f(n)$ must be below $g(n)$. If they are each big-O of each other, then they must be on the same level. All logs below have base 2.

$$f_1(n) = \log\log\log n \quad f_2(n) = \log n \cdot \log\log n, \quad f_3(n) = \log n^3, \quad f_4(n) = n^2 + n^3/100,$$
$$f_5(n) = n!, \quad f_6(n) = 2^n + 3^{\log n}, \quad f_7(n) = \left(n\log n + n^4\right)(n^3+1). \tag{1}$$

(b) (7 marks) Let R be the relation on the set of functions from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ such that (f, g) belongs to R if and only if f is $\Theta(g)$. Show that R is an equivalence relation. Describe the equivalence class containing $f(n) = 2n^3$ for this equivalence relation.

a. $f_1(n) = \log^3 n$, $O(\log^3 n)$ which implies $|f(n)| \le C\,|\log^3 n|$ for $n > k$

can be seen from $C = 1$, $k = 1$

$f_2(n) = \log n \cdot \log^2 n$, $O(\log n \log^2 n)$ which implies $|f(n)| \le C\,|\log n \log^3 n|$ for $n > k$

can be seen from $C = 1$, $k = 1$

$f_3(n) = 3\log n$, $O(\log n)$ which implies $|f(n)| \le C\,|\log n|$ for $n > k$

can be seen from $C = 3$, $k = 1$

$f_4(n) = n^2 + \frac{n^3}{100}$, $O(n^3)$ which implies $|f(n)| \le C\,|n^3|$ for $n > k$

can be seen from $C = 1 + \frac{1}{100} = \frac{101}{100}$, $k = \frac{101}{99}$

$n^3 = n^2 + \frac{n^3}{100}$

$n = 1 + \frac{n}{100}$

$100(n-1) = n$

$n = \frac{100}{99} \rightarrow$ k value

$f_5(n) = n!$, $O(n^n)$ which implies $|f(n)| \le C\,|n^n|$ for $n > K$

as $n!$ in polynomial form is $(n)(n-1)(n-2)\ldots 1$, $O(n^n)$

can be seen from $C = 1$, $k = 1$

$f_6(n) = 2^n + 3^{\log n}$, $O(2^n)$ which implies $|f(n)| \le C\,|2^n|$ for $n > K$

can be seen from $C = 2$, $K = 1$

$2^n \cdot 2 \geq 2^n + 3^{\log n}$

$f_7(n) = n^4 \log n + n \log n + n^7 + n^4$, $O(n^7)$ which implies $|f(n)| \le C\,|n^7|$ for $n > k$

can be seen from $C = 4$, $k = 1$

by theory

$1 < \log n < \sqrt{n} < n < n\log n < n\sqrt{n} < n^2 < n^3 < \ldots < 2^n < 3^n < n^n$

so in order from the fastest growth

$f_5(n)$ as $f_{5n} = O(n^n)$

$f_6(n)$ as $f_{6n} = O(2^n)$

$f_7(n)$ as $f_{7n} = O(n^7)$

$f_4(n)$ as $f_{4n} = O(n^3)$

$f_2(n)$ as $f_{2n} = O(\log n \ \log^2 n)$

$f_3(n)$ as $f_{3n} = O(\log n)$

$f_1(n)$ as $f_{1n}: O(\log^3 n)$

proof $f_2(n) \geq f_3(n)$

$\log n . \log \log n \geq \log n^3$

$2^{\log n . \log\log n} \geq 2^{\log n^3}$

$n^{\log \log n} \geq n^3$

$\log \log n \geq 3$

as 3 is a constant

$\therefore$ order in increasing order of their growth rate $f_1(n), f_3(n), f_2(n), f_4(n), f_7(n), f_6(n), f_5(n)$

b. equivalence relation : reflecfire, symmetric, transitive

$(f,g) \in R \leftrightarrow f = \Theta g$; $f = O(g) \wedge f = \Omega(g)$

- reflextive : for any f function f is always Θf as it's itself
- symmetric : if $f = \Theta g$, then $g = \Theta f$. This imply that there exist c_1 & c_2 such that $c_1 . g \leq f \leq c_2 . g$ for all $n \geq n_0$. Hence, it's a symmetric.
- transitivity : if f is Θg $\wedge$ g is Θh $\rightarrow$ f is Θh

  there exist c_1, c_2, c_3, c_4 such that $c_1 . g \leq f \leq c_2 . g$ for $n \geq n_1$ $\wedge$ $c_3 h \leq g \leq c_4 h$ $n \geq n_2$

  if these inequalities combined, there will exist c_6, c_5 such that $c_5 . h \leq f \leq c_6 . h$ for all $n \geq n_3$

  Hence, it's transitive

$\therefore$ it's an equivalence relation

$f(n) = 2n^3$ as an equivalence class means that $f(n) = \Theta(g(n))$

so it'll have all $g(n)$ functions that have the same growth rate.

$f(n) = 2n^3 = \Theta(n^3)$

proof. $c_1 . n^3 \leq f(n) \leq c_2 . n^3$ for $n \geq 1$

$c_1 . n^3 \leq 2n^3$

is true for $c = 1$, $n \geq 1$

$c_2 . n^3 \geq 2n^3$

is true for $c = 2$, $n \geq 1$