

# TMA4300Ex2

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# Loading libraries
library(ggplot2)

# Set seed so that the task can be reproduced
set.seed(42)
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## Exercise 1: Derivations

a)

From the definition of conditional probability and the nature of the assumptions in this exercise, we know that

$$\begin{aligned} p(\boldsymbol{\eta}, \mathbf{u}, \kappa_u, \kappa_v | \mathbf{y}) &\propto p(\mathbf{y} | \boldsymbol{\eta}, \mathbf{u}, \kappa_u, \kappa_v) p(\boldsymbol{\eta} | \mathbf{u}, \kappa_u, \kappa_v) p(\mathbf{u} | \kappa_u, \kappa_v) p(\kappa_u | \kappa_v) p(\kappa_v) \\ &\propto p(\mathbf{y} | \boldsymbol{\eta}) p(\boldsymbol{\eta} | \mathbf{u}, \kappa_v) p(\mathbf{u} | \kappa_u) p(\kappa_u) p(\kappa_v). \end{aligned}$$

By inserting the corresponding probabilities, this becomes

$$\begin{aligned} p &\propto \left( \prod_{i=1}^n (E_i e^{\eta_i})^{y_i} e^{E_i e^{\eta_i}} \right) |\kappa_v \mathbf{I}|^{\frac{1}{2}} e^{-\frac{\kappa_v}{2} (\boldsymbol{\eta} - \mathbf{u})^T (\boldsymbol{\eta} - \mathbf{u})} \kappa_u^{(n-1)/2} e^{-\frac{\kappa_u}{2} \mathbf{u}^T \mathbf{R} \mathbf{u}} \kappa_u^{\alpha_u - 1} e^{-\beta_u \kappa_u} \kappa_v^{\alpha_v - 1} e^{-\beta_v \kappa_v} \\ &\propto \kappa_u^{\frac{n-1}{2} + \alpha_u - 1} \kappa_v^{\frac{n}{2} + \alpha_v - 1} \exp \left\{ -\beta_u \kappa_u - \beta_v \kappa_v - \frac{\kappa_v}{2} (\boldsymbol{\eta} - \mathbf{u})^T (\boldsymbol{\eta} - \mathbf{u}) - \frac{\kappa_u}{2} \mathbf{u}^T \mathbf{R} \mathbf{u} + \sum_i (y_i \eta_i - E_i e^{\eta_i}) \right\}. \end{aligned}$$

b)

The sum over  $e^{\eta_i}$  in the posterior means that the full conditional of  $\eta_i$  is difficult to sample from. We therefore want to approximate the distribution of  $\boldsymbol{\eta}$  with a multivariate normal in order to use Metropolis-Hastings steps for it. We define the function

$$f(\eta_i) = y_i \eta_i - E_i e^{\eta_i},$$

which has derivatives

$$\begin{aligned} f'(\eta_i) &= y_i - E_i e^{\eta_i} \\ f''(\eta_i) &= -E_i e^{\eta_i}. \end{aligned}$$

This yields the following Taylor series expansion of  $f$  around  $z_i$ ,

$$\begin{aligned}\tilde{f}(\eta_i) &= y_i z_i - E_i e^{z_i} + (y_i - E_i e^{z_i})(\eta_i - z_i) + \frac{1}{2}(-E_i e^{z_i})(\eta_i - z_i)^2 \\ &= a(z_i) + b(z_i)\eta_i - \frac{1}{2}c(z_i)\eta_i^2,\end{aligned}$$

where  $a(z_i) = E_i e^{z_i}(z_i - z_i^2/2 - 1)$ ,  $b(z_i) = y_i + E_i e^{z_i}(z_i - 1)$  and  $c(z_i) = E_i e^{z_i}$ .

**c)**

As  $p(\theta_i | \boldsymbol{\theta}_{\setminus i}, \mathbf{y}) \propto p(\boldsymbol{\theta} | \mathbf{y})$ , we can quite easily find the full conditionals from the posterior. Using this, we see that

$$p(\kappa_u | \mathbf{y}, \kappa_v, \boldsymbol{\eta}, \mathbf{u}) \propto \kappa_u^{(n-1)/2 + \alpha_u - 1} e^{-(\beta_u + \frac{1}{2} \mathbf{u}^T \mathbf{R} \mathbf{u}) \kappa_u}.$$

We recognise this as the core of a gamma distribution which means that the full conditional density of  $\kappa_u$  is  $\text{gamma}(\frac{n-1}{2} + \alpha_u, \beta_u + \frac{1}{2} \mathbf{u}^T \mathbf{R} \mathbf{u})$ . By the same reasoning, we see that  $\text{gamma}(\frac{n}{2} + \alpha_v, \beta_v + \frac{1}{2} (\boldsymbol{\eta} - \mathbf{u})^T (\boldsymbol{\eta} - \mathbf{u}))$  is the full conditional density of  $\kappa_v$ . Similarly

$$\begin{aligned}p(\mathbf{u} | \mathbf{y}, \boldsymbol{\eta}, \kappa_u, \kappa_v) &\propto \exp \left\{ -\frac{\kappa_v}{2} (\boldsymbol{\eta} - \mathbf{u})^T (\boldsymbol{\eta} - \mathbf{u}) - \frac{\kappa_u}{2} \mathbf{u}^T \mathbf{R} \mathbf{u} \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \mathbf{u}^T (\kappa_u \mathbf{R} + \kappa_v \mathbf{I}) \mathbf{u} + \kappa_v \mathbf{u}^T \boldsymbol{\eta} \right\}.\end{aligned}$$

We recognise this as the canonical form of a multivariate normal distribution. All these distributions are easy to sample from, and can be used in the Gibbs algorithm directly.

The full conditional distribution for  $\boldsymbol{\eta}$  however takes the form

$$p(\boldsymbol{\eta} | \mathbf{y}, \mathbf{u}, \kappa_u, \kappa_v) \propto \exp \left\{ -\frac{\kappa_v}{2} (\boldsymbol{\eta} - \mathbf{u})^T (\boldsymbol{\eta} - \mathbf{u}) + \sum_i f(\eta_i) \right\}.$$

This does not correspond to any standard distribution, but by applying the approximation  $\tilde{f}(\eta_i)$ , we get

$$\begin{aligned}q(\boldsymbol{\eta} | \mathbf{z}, \mathbf{y}, \mathbf{u}, \kappa_u, \kappa_v) &\propto \exp \left\{ -\frac{\kappa_v}{2} \boldsymbol{\eta}^T \boldsymbol{\eta} + \kappa_v \boldsymbol{\eta}^T \mathbf{u} - \frac{1}{2} \boldsymbol{\eta}^T \text{diag}(c(\mathbf{z})) \boldsymbol{\eta} + \boldsymbol{\eta}^T b(\mathbf{z}) \right\} \\ &= \exp \left\{ -\frac{1}{2} \boldsymbol{\eta}^T \left( \kappa_v \mathbf{I} + \text{diag}(c(\mathbf{z})) \right) \boldsymbol{\eta} + \boldsymbol{\eta}^T (\kappa_v \mathbf{u} + b(\mathbf{z})) \right\},\end{aligned}$$

where  $\mathbf{z} = [z_1, \dots, z_n]^T$  is the point around which we Taylor expand  $f$ ,  $b(\mathbf{z}) = [b(z_1), \dots, b(z_n)]^T$  and  $c(\mathbf{z}) = [c(z_1), \dots, c(z_n)]^T$ .  $q$  is the canonical form of a multivariate normal distribution, and can be used for Metropolis-Hastings steps for  $\boldsymbol{\eta}$ .

## Exercise 2: Implementation of the MCMC sampler