task1

a)

From the definition of conditional probability and the nature of the assumptions in this exercise, we know that

$$\begin{aligned}
\mathrm{p}(\boldsymbol{\eta}, \boldsymbol{u}, \kappa_u, \kappa_v|\boldsymbol{y}) &\propto \mathrm{p}(\boldsymbol{y}\,|\boldsymbol{\eta}, \boldsymbol{u}, \kappa_u, \kappa_v)\mathrm{p}(\boldsymbol{\eta}\,|\boldsymbol{u}, \kappa_u, \kappa_v)\mathrm{p}(\boldsymbol{u}\,|\kappa_u, \kappa_v)\mathrm{p}(\kappa_u\,|\kappa_v)\mathrm{p}(\kappa_v) \\
&\propto \mathrm{p}(\boldsymbol{y}\,|\boldsymbol{\eta})\mathrm{p}(\boldsymbol{\eta}\,|\boldsymbol{u}, \kappa_v)\mathrm{p}(\boldsymbol{u}\,|\kappa_u)\mathrm{p}(\kappa_u)\mathrm{p}(\kappa_v).
\end{aligned}$$

By inserting the corresponding probabilities, this becomes

$$\begin{aligned}
\mathrm{p} &\propto \left(\prod_{i=1}^{n} (E_i e^{\eta_i})^{y_i}\, e^{E_i e^{\eta_i}}\right) |\kappa_v \mathrm{I}|^{\frac{1}{2}}\, e^{-\frac{\kappa_v}{2}(\boldsymbol{\eta}-\boldsymbol{u})^T(\boldsymbol{\eta}-\boldsymbol{u})} \kappa_u^{(n-1)/2} e^{-\frac{\kappa_u}{2}\boldsymbol{u}^T\mathbf{R}\boldsymbol{u}} \kappa_u^{\alpha_u-1} e^{-\beta_u\kappa_u} \kappa_v^{\alpha_v-1} e^{-\beta_v\kappa_v} \\
&\propto \kappa_u^{\frac{n-1}{2}+\alpha_u-1} \kappa_v^{\frac{n}{2}+\alpha_v-1} \exp\left\{-\beta_u\kappa_u - \beta_v\kappa_v - \frac{\kappa_v}{2}\left(\boldsymbol{\eta}-\boldsymbol{u}\right)^T\left(\boldsymbol{\eta}-\boldsymbol{u}\right) - \frac{\kappa_u}{2}\boldsymbol{u}^T\mathbf{R}\boldsymbol{u} + \sum_i \left(y_i\eta_i - E_i e^{\eta_i}\right)\right\}.
\end{aligned}$$

b)

The sum over e^{η_i} in the posterior means that the full conditional of η_i is difficult to sample from. We therefore want to approximate the distribution of $\boldsymbol{\eta}$ it with a multivariate normal in order to use Metropolis-Hastings steps for it. We define the function

$$f(\eta_i) = y_i\eta_i - E_i e^{\eta_i},$$

which has derivatives

$$\begin{aligned}
f'(\eta_i) &= y_i - E_i e^{\eta_i} \\
f''(\eta_i) &= -E_i e^{\eta_i}.
\end{aligned}$$

This yeilds the following Taylor series expansion of f around z_i,

$$\begin{aligned}
\tilde{f}(\eta_i) &= y_i z_i - E_i e^{z_i} + (y_i - E_i e^{z_i})(\eta_i - z_i) + \frac{1}{2}(-E_i e^{z_i})(\eta_i - z_i)^2 \\
&= a(z_i) + b(z_i)\eta_i - \frac{1}{2}c(z_i)\eta_i^2,
\end{aligned}$$

where $a(z_i) = E_i e^{z_i}(z_i - z_i^2/2 - 1)$, $b(z_i) = y_i + E_i e^{z_i}(z_i - 1)$ and $c(z_i) = E_i e^{z_i}$.

c)

As $\mathrm{p}(\theta_i|\boldsymbol{\theta}_{\setminus i}, \boldsymbol{y}) \propto \mathrm{p}(\boldsymbol{\theta}|\boldsymbol{y})$, we can quite easily find the full conditionals from the posterior. Using this, we see that

$$\mathrm{p}(\kappa_u|\boldsymbol{y}, \kappa_v, \boldsymbol{\eta}, \boldsymbol{u}) \propto \kappa_u^{(n-1)/2+\alpha_u-1} e^{-(\beta_u + \frac{1}{2}\boldsymbol{u}^T\mathbf{R}\boldsymbol{u})\kappa_u}.$$

We recognise this as the core of a gamma distribution which means that the full conditional density of κ_u is gamma$(\frac{n-1}{2}+\alpha_u, \beta_u+\frac{1}{2}\boldsymbol{u}^T\mathbf{R}\boldsymbol{u})$. By the same reasoning, we see that gamma$(\frac{n}{2}+\alpha_v, \beta_v+\frac{1}{2}\left(\boldsymbol{\eta}-\boldsymbol{u}\right)^T\left(\boldsymbol{\eta}-\boldsymbol{u}\right))$ is the full conditional density of κ_v. Similarly

$$\begin{aligned}
\mathrm{p}(\boldsymbol{u}|\boldsymbol{y},\boldsymbol{\eta}.\kappa_u,\kappa_v) &\propto \exp\left\{-\frac{\kappa_v}{2}\left(\boldsymbol{\eta}-\boldsymbol{u}\right)^T\left(\boldsymbol{\eta}-\boldsymbol{u}\right)-\frac{\kappa_u}{2}\boldsymbol{u}^T\mathbf{R}\boldsymbol{u}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{u}^T(\kappa_u\mathbf{R}+\kappa_v\mathrm{I})\boldsymbol{u}+\kappa_v\boldsymbol{u}^T\boldsymbol{\eta}\right\}.
\end{aligned}$$

We recognise this as the canonical form of a multivariate normal distribution. All these distributions are easy to sample from, and can be used in the Gibbs algorithm directly.

The full conditional distribution for $\boldsymbol{\eta}$ however takes the form

$$\mathrm{p}(\boldsymbol{\eta}|\boldsymbol{y},\boldsymbol{u},\kappa_u,\kappa_v) \propto \exp\left\{-\frac{\kappa_v}{2}\left(\boldsymbol{\eta}-\boldsymbol{u}\right)^T\left(\boldsymbol{\eta}-\boldsymbol{u}\right)+\sum_i f(\eta_i)\right\}.$$

This does not correspond to any standard distribution, but by applying the approximation $\tilde{f}(\eta_i)$, we get

$$\begin{aligned}
\mathrm{q}(\boldsymbol{\eta}|\boldsymbol{z},\boldsymbol{y},\boldsymbol{u},\kappa_u,\kappa_v) &\propto \exp\left\{-\frac{\kappa_v}{2}\boldsymbol{\eta}^T\boldsymbol{\eta}+\kappa_v\boldsymbol{\eta}^T\boldsymbol{u}-\frac{1}{2}\boldsymbol{\eta}^T\mathrm{diag}(c(\boldsymbol{z}))\boldsymbol{\eta}+\boldsymbol{\eta}^Tb(\boldsymbol{z})\right\} \\
&= \exp\left\{-\frac{1}{2}\boldsymbol{\eta}^T\Big(\kappa_v\mathbf{I}+\mathrm{diag}(c(\boldsymbol{z}))\Big)\boldsymbol{\eta}+\boldsymbol{\eta}^T\left(\kappa_u\boldsymbol{u}+b(\boldsymbol{z})\right)\right\},
\end{aligned}$$

where $\boldsymbol{z}=[z_1,...,z_n]^T$ is the point around which we Taylor expand f, $b(\boldsymbol{z})=[b(z_1),...,b(z_n)]^T$ and $c(\boldsymbol{z})=[c(z_1),...,c(z_n)]^T$. This is the canonical form of a multivariate normal distribution, and can be used for Metropolis-Hastings steps for $\boldsymbol{\eta}$.