Jumps in Real-time Financial Markets:

A New Nonparametric Test and Jump Dynamics

Suzanne S. Lee and Per A. Mykland*

Abstract

This paper introduces a new nonparametric jump test for continuous-time asset pricing models. It distinguishes jump arrival times and realized jump sizes in asset prices as precisely as at intra-day levels. We demonstrate the likelihood of misclassification of jumps in discrete data becomes negligible when we use high-frequency returns. We explore real-time jump dynamics using intra-day U.S. individual equity prices through the test, and find empirical evidence that jump arrivals are associated with both pre-scheduled earnings announcements and unscheduled real-time news release.

*Lee is with the Georgia Institute of Technology, Finance Area. Mykland is with the University of Chicago, Department of Statistics. We would like to thank Federico Bandi, George Constantinides, Pietro Veronesi, and Ruey Tsay for suggestions and comments. Financial support for this research from the National Science Foundation (DMS-02-04-639), Oscar Mayer Dissertation Fellowships, and Financial Mathematics Program at the University of Chicago is gratefully acknowledged. Any comments are welcome. Please send any correspondence to: Suzanne S. Lee, phone: 404.822.1552, fax: 404.894.6030, email: suzanne.lee@mgt.gatech.edu.

Financial market evolution is often interrupted by various incidences such as market crashes, corporate defaults or announcements, central bank announcements, and international events. These market uncertainties generate significant discontinuities in financial variables. Empirical evidences of such discontinuities, so called "*Jumps*" in financial markets, have been well documented in recent literature. A number of studies have proved the substantial impact of jumps on financial management from portfolio and risk management to option or bond pricing and hedging: see Merton (1976), Bakshi, Cao, and Chen (1997), Bates (2000), Liu *et al.* (2003), Eraker *et al.* (2003), Naik and Lee (1990), Duffie *et al.* (2000), and Piazzesi (2003). It turns out to be important to incorporate jumps into continuous-time asset pricing models because they provide reasonable explanations for market phenomena such as the excess kurtosis and skewness of return distributions and the implied volatility smile in option markets. Furthermore, it allows us for the management of a different kind of risk associated with their different hedging demand, in addition to the usual diffusive risk to which market participants are usually exposed. In order for investors to manage the two types of risk differently, it is critical to distinguish jumps from diffusion for their applications.

Despite its attractiveness, disentangling jumps in continuous-time models is difficult since we can only observe realized data at discrete times. It becomes a challenge to make an econometric inference for continuous-time jump diffusion models because some of discrete data may be from the diffusion part, but still yield exactly the same discreteness in data as that which the jump part can provide. Incorrectly identified jumps will affect jump intensity and jump distribution estimation, which eventually mislead investors to make unwanted financial decisions. In addition, classifying the arrival times and sizes of realized jumps can be necessary in terms of their relevance to the application. A thorough investigation on the jump arrival dynamics can help us characterize and forecast future return distributions as well as deepen our understanding on different stylized facts of individual equity options such as flatter implied volatilities of equity options than index options: see Bakshi, Kapadia, and Madan (2003) and Mahue and McCurdy (2004).

The purpose of this paper is to propose a new testing device that can identify arrival dates and times in a day of realized jumps and their sizes to provide a model-free tool for our investigation on the dynamic behavior of jumps as precisely as at intra-day levels. Overall characteristics of return distributions can be found as a by-product since both signs and sizes of jumps can be obtained whenever we detect jumps. The importance of detecting sizes and arrivals lies in dynamic hedging applications among many others: see Naik and Lee (1990), and Bertsimas *et al.* (2001). The impact of jumps becomes more crucial under a larger variance of jumps, especially as jumps increase in size, since hedging error is more likely to go beyond one's tolerance level. In other words, the size of jumps determines the degrees of market incompleteness. Arrival time detection becomes important in a dynamic re-balance of a derivative hedging portfolio, as discussed in Collin-Dufresne and Hugonnier (2001) on the temporal resolution of uncertainty.

Our test is nonparametric, which makes the results robust to model misspecification. There are a few other existing nonparametric approaches to jumps in the literature. Aït-Sahalia (2002) suggests a diffusion criterion based on transition density to test the presence of jumps. Bandi and Nguyen (2003) and Johannes (2004) provide consistent nonparametric estimators for jump diffusion models based on the kernel estimation method. Barndorff-Nielsen and Shephard (2006) suggest tests to indicate the presence of jumps over a certain time interval by comparing realized power and bipower variation. Because their methods are using two integrated quantities and the integrations contain all possible jumps in the interval, they cannot distinguish how many jumps were present within the interval, when in the interval a jump occurs, and how large the realized jump sizes are, although it can still tell its presence. We examine by simulation that our test performs equally or better than tests by Barndorff-Nielsen and Shephard (2006). Tauchen and Zhou (2005) used the test of Barndorff-Nielse and Shephard (2006) to find realized jump sizes by strictly assuming there is at most one jump in a day. According to our empirical evidence found by our test, however, it is possible that there can be more than one jump a day. There is increased interest in this problem recently by other research groups. A wavelet approach by Fan and Wang (2005) and a test based on a variance swap replicating strategy by Jiang and Ooment (2005) are also under development.

A number of different parametric inference methodologies have been developed and applied in empirical studies for jumps in continuous-time asset pricing models using time series and cross sectional data. They include Implied State Generalized Method of Moments (IS-GMM): see Pan (2002), Maximum Likelihood Estimation: see Shaumburg (2001), simulation-based Efficient Method of Moment (EMM): see Andersen *et al* (2002) and Chernov *et al*. (2003), Baysian approach: Eraker *et al*. (2003), volatility estimation and different types of jumps under Levy processes: see Aït-Sahalia (2003) and Aït-Sahalia and Jacod (2005) . All, however, run the risk of incorrect specification for functionals in their chosen models, which is not the case with our nonparametric test.

Another merit of our new test is that it is robust to nonstationarity of the processes, which is a common feature of financial variables.

We conduct an empirical study on real-time jump dynamics in the U.S. individual equity market. It is based on high-frequency returns from 4 different individual stock prices transacted in New York Stock Exchange (NYSE) over the period of September 1 to November 30, 2005. Our empirical observation through our new test indicates that most of jumps in equity prices arrive with news events. Specifically, we connect jump arrival times we detected with real-time news release from Factiva. We find that accumulated real-time news about a company from early morning before the market starts for a day tend to create jump in the company's stock prices around market opening time. Pre-scheduled news announcement of corporate earnings turns out to be always associated with a jump around the opening time. In addition to the pre-scheduled news events, majority of jumps were connected with unscheduled news and the magnitudes of jump size with unscheduled news are comparable to those with pre-scheduled ones. We believe the outcome of this type of study using high-frequency time series information on underlying assets can suggests a foundation for individual equity option pricing models associated with sound fundamentals. Price of jump risks can be better examined with cross-sectional option data after setting up models with more precise jump dynamics though our test.

The rest of the paper is organized as follows. Section 1 sets up a theoretical model framework for financial variables to test jumps. It describes the intuition behind our new test, develops the test statistic, and derives its asymptotic distribution to provide a benchmark for tests. We discuss in Section 2 how rejection region for the test can be determined and the likelihood of misclassifications. Section 3 investigates finite sample performance of the test by simulation. Section 4 presents empirical evidence of real-time jump dynamics in equity prices and the association with real-time news. Finally, we conclude in Section 5. All the proofs are in Section 6.

1 A Theoretical Model for the Test and Its Asymptotic Theory

We employ a one-dimensional asset return process with a fixed complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$ where $\{\mathcal{F}_t : t \in [0, T]\}$ is a right-continuous information filtration for market participants, and $\mathcal{P}$ is a data-generating measure. Let the continuously compounded return be written as $d \log S(t)$

for $t \geq 0$, where $S(t)$ is the asset price at t under $\mathcal{P}$. We are interested in testing jumps in the asset returns as follows. The null hypothesis of no jumps in the market is represented as

$$d \log S(t) = \mu(t)dt + \sigma(t)dW(t) \tag{1}$$

where $W(t)$ is a $\mathcal{F}_t$-adapted standard Brownian Motion. The drift $\mu(t)$ and spot volatility $\sigma(t)$ are $\mathcal{F}_t$-measurable functions, such that the underlying process is a diffusion which has continuous sample paths. Its alternative hypothesis is given by

$$d \log S(t) = \mu(t)dt + \sigma(t)dW(t) + Y(t)dJ(t) \tag{2}$$

where $dJ(t)$ is a jump-counting process with intensity $\lambda(t)$, and $Y(t)$ is the jump size whose mean is $\mu_y(t)$ and standard deviation is $\sigma_y(t)$, which are also $\mathcal{F}_t$-measurable functions. We assume $W(t)$ and $J(t)$ are independent. Observation of $S(t)$, equivalently $\log S(t)$, only occurs at discrete times $0 = t_0 < t_1 < ... < t_n = T$. For simplicity, this paper assumes observation times are equally spaced: $\Delta t = t_i - t_{i-1}$. This simplified assumption can easily be generalized to non-equidistant cases by letting $\max_i(t_i - t_{i-1}) \to 0$. We also impose the following necessary assumption on price processes throughout this paper.

Assumption 1

$$\textbf{A1.1} \;\; sup_i \;\; sup_{t_i \leq u \leq t_{i+1}} |\mu(u) - \mu(t_i)| = O_p(\sqrt{\Delta t})$$

$$\textbf{A1.2} \;\; sup_i \;\; sup_{t_i \leq u \leq t_{i+1}} |\sigma(u) - \sigma(t_i)| = O_p(\sqrt{\Delta t})$$

Following Pollard (2002), we use O_p notation throughout this paper to mean that for random vectors $\{X_n\}$ and non-negative random variable $\{d_n\}$, $X_n = O_p(d_n)$ if for each $\epsilon > 0$, there exists a finite constant M_ϵ such that $P(|X_n| > M_\epsilon d_n) < \epsilon$ eventually. One can interpret **Assumption**

1 as the drift and diffusion coefficient functions do not dramatically changes over short time interval. Furthermore, it allows the drift and diffusion coefficients to depend on the process itself. Therefore, this assumption is general enough to cover most of the models that incorporate jumps in continuous-time asset price processes in finance literature. It also satisfies the stochastic volatility plus finite activity jump semi-martingale class in Barndorff-Nielsen and Shephard (2004) and the reference therein.

1.1 Intuition and Definition of the Nonparametric Jump Test

In this subsection, we address the basic intuition behind our new testing technique and define the jump test statistic, T. We focus our discussion on a single test at time t_i. We do not assume there was a jump or not before or after t_i. Generalization to global test to determine whether a diffusion model (1) is rejected is straightforward by a multiple test (single tests over available times). As long as there is at least one rejection in any single test, we can simply conclude that an appropriate model is a jump diffusion model (2). Global test is interesting in itself and is also part of our goal. Given the literature emphasizing the importance, our main purpose of this paper is not only to find out whether there are jumps in markets but also to advance our knowledge on how these jumps evolve over time. Multiple test can provide us the information on jump arrival dynamics.

Imagine that asset prices evolve over time continuously. There was a jump arrived in a market at some time, say t_i. If the horizon is short, we would expect the realized asset return to be much higher than usual returns due only to purely continuous random innovations. How about the situation when the volatility at that time was also high? Even if there was no jump arrived, if the volatility is high and if we can only observe prices in discrete times, the realized return we observe may also be as high as the return due to jump. To distinguish those two cases, it is natural to standardize the return by a measure that explains the local variation only from the continuous part of the process. We call this measure as *local volatility*[1] in this paper. This intuition is incorporated into our test. It compares a realized return at any given time to a consistently estimated local volatility using corresponding local movement of returns. More specifically, the ratio of realized return to estimated local volatility creates the test statistic for jumps.

Now how do we estimate *local volatility*? A commonly used nonparametric estimator for variance in literature is the *realized power (quadratic) variation* defined as the sum of squared returns

$$\text{plim}_{n\to\infty}\sum_{i=2}^{n}(\log S(t_i) - \log S(t_{i-1}))^2.$$

Using high frequency returns within some period just before our testing time, it suggests a variance estimate over that period. However, this well-known variance estimator is unfortunately inconsistent under the presence of jumps in a return process. Alternatively, a slightly modified version called the *realized bipower variation* defined as the sum of products of consecutive absolute returns

$$\text{plim}_{n\to\infty}\sum_{i=3}^{n}|\log S(t_i) - \log S(t_{i-1})||\log S(t_{i-1}) - \log S(t_{i-2})|$$

has been suggested and is shown to be a consistent estimator for the integrated volatility, even when there are jumps in return processes: see Barndorff-Nielsen and Shephard (2003) and Aït-Sahalia (2004). Despite the intuition that jumps in a process may have impact on the volatility estimation, it remains consistent, no matter how large or small the jump sizes are mixed with the diffusive part of pricing models. Our test is based on this interesting insight. Even if highly volatile market environment makes the distinguishing of jumps harder, infrequent Poisson jumps will be detected by our procedure pretty accurately as long as we use high frequency observations.

[1]Note that in the literature on implied binomial tree for option pricing, the term "local volatility" is used in a different manner

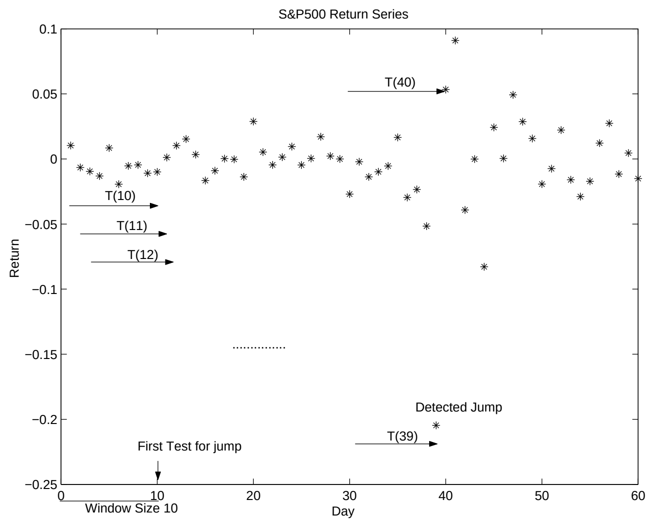

Figure 1: Formation of our new test with a window size $K = 10$

From now on, we describe the formulation of the statistic, and provide its mathematical definition. Suppose we have a fixed time horizon T, and n is the number of observation in $[0, T]$. $\Delta t = \frac{T}{n}$. Consider a local movement of the process within a window size K. With realized returns in the window consisting of previous K observations just before a testing time t_i, the time-varying local volatility is estimated based on the realized bipower variation. We, then, compare this estimated local volatility to the next realized return by taking ratio to determine whether there was a jump and how large the jump size was. For example, if $\Delta t = 5$ minutes, $t_i = 10:05$ AM and $K = 10$, then we test jump by the relative magnitude of realized return from $10:00$ AM to $10:05$ AM compared to local volatility estimated with 5 minute returns from $9:10$ AM to $10:00$ AM. Figure 1 also illustrates the construction of the test. Mathematical notation of the test statistics is as follows.

Definition 1 *The statistic $T(i)$ which tests at time t_i whether there was a jump from t_{i-1} to t_i*

is defined as

$$T(i) = \frac{\log S(t_i) - \log S(t_{i-1})}{\sqrt{\frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |\log S(t_j) - \log S(t_{j-1})||\log S(t_{j-1}) - \log S(t_{j-2})|}}$$

Notice that the realized bipower variation is used in the local volatility estimation for the denominator of the statistic. This makes our technique immune to the presence of jumps in previous times, especially those of which are considered for local volatility estimation. We choose the window size K in such a way that the effect of jumps on the volatility estimation disappears. In the following subsections, we show that T asymptotically follows a normal distribution if there is no jump at testing times, and suggest a criterion to distinguish arrival of jumps based on this asymptotic distribution.

We state our results ignoring the drift. For our analysis using high-frequency data, the drift (of order dt) is mathematically negligible compared to the diffusion (of order $\sqrt{dt}$) and the jump component (of order 1). In fact, the drift estimates have higher standard errors so that it makes the precision of variance estimates decrease if included in variance estimation. We study a simplified version of the model without the drift term, i.e., $\mu = 0$. We also show that the main result we present as follows continues to hold with the nonzero drift term in Appendix 5.1. A modified statistic T_μ for the nonzero drift case is defined, and a corresponding theorem is presented therein as well.

1.2 Under the Absence of Jump at Testing time t_i

Suppose our realized return from time t_{i-1} to t_i is from a scalar diffusion process without a drift as

$$d \log S(t) = \sigma(t) dW(t).$$

The asymptotic null distribution of the jump test statistic T is provided in the following Theorem 1.

Theorem 1 *Let* $T(i)$ *be as in* **Definition 1** *under the null at time* t_i *and* $K = O_p(\Delta t^{\alpha})$ *where* $-1 < \alpha < -0.5$*. Suppose* **Assumption 1** *is satisfied. Then as* $\Delta t \to 0$*,*

$$sup_i |T(i) - \widehat{T}(i)| = O_p(\Delta t^{\frac{3}{2} - \delta + \alpha})$$

where δ *is in* $0 < \delta < \frac{3}{2} + \alpha$*, and*

$$\widehat{T}(i) = \frac{U_i}{c}.$$

Here $U_i = \frac{1}{\sqrt{\Delta t}}(W_{t_i} - W_{t_{i-1}})$ *and* $c = E|U_i| = \sqrt{2}/\sqrt{\pi} \approx 0.7979$*.*

Proof of Theorem 1: *See Appendix 5.2.*

Theorem 1 states our test statistic $T(i)$ approximately follows the same distribution as $\widehat{T}(i)$. And we find $\widehat{T}(i)$ follows a normal distribution with mean 0 and variance $\frac{1}{c^2}$ because U_i is a standard normal random variable. In case we absolutely know a priori the absence of jumps in the whole price process, the usage of realized quadratic variation for estimating *local volatility* would yield the same asymptotic distribution. As discussed in our intuition, however, we do not condition on the absence of jump in earlier or later times. Therefore, using quadratic variation does not suffice in this case. As also can be noticed, $T(i)$ is asymptotically independent and normally distributed over time; hence, one can easily find a joint asymptotic null distribution of the test statistics for various periods if interested.

1.3 Under the Presence of Jump at Testing time t_i

We show in this subsection how the jump test react to the arrival of jump and discuss the choice of window size. The realized return from time t_{i-1} to t_i is now from a scalar jump diffusion, which describes the arrival of jumps in addition to the diffusive market innovation as

$$d \log S(t) = \sigma(t) dW(t) + Y(t) dJ(t).$$

Theorem 2 demonstrates how our test behaves differently when jump arrives. It specifically shows that as the sampling interval Δt goes to 0, the test statistic becomes so large that we can reject the hypothesis that there is no jump arrived.

Theorem 2 *Let* $T(i)$ *be as in* **Definition 1** *under the alternative at time* t_i*. If* $K = O_p(\Delta t^{\alpha})$ *where* $-1 < \alpha < -0.5$*, then*

$$T(i) \approx \frac{U_i}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}} I_{\tau \in (t_{i-1}, t_i]}$$

and $T(i) \to \infty$ *as* Δt *goes to 0.*

Proof of Theorem 2: *See Appendix 5.3.*

The benefit of the bipower variation as a *local volatility* estimator in the denominator of the statistic is that the presence of jumps in earlier times doesn't affect the consistency of volatility estimation. Therefore, our test is robust to earlier jumps in detecting current jumps. This does not imply that we make no use of earlier jumps. If one would like to learn about earlier jump arrival, he can do the same single test at that earlier time. Several single tests over time become a multiple test, which can provide us more information on jump dynamics.

In order to retain the benefit of bipower variation, the window size K has to be large enough, but obviously smaller than the number of observations n, so that the effect of jumps in return on estimating local volatility disappears. The condition $K = O_p(\Delta t^{\alpha})$ with $-1 < \alpha < -0.5$ satisfies this requirement. Therefore, the choice of sampling frequency Δt will determine the window size. For example, if daily data are used in the analysis, $\Delta t = \frac{1}{252}$, and $K = \beta \Delta t^{\alpha}$ with $\beta = 1$, integers between 15.87 and 252 are within the required range. The simulation study in subsection 3.4 finds that if K is within the range, increasing K only elevates the computational burden without large marginal contribution. Hence, the smallest integer K that satisfies the necessary condition would be an optimal choice of K. In this example, $K^{opt} = 16$, i.e. the local volatility estimation is based on observations of about three weeks just prior to the test if we use daily data.

1.4 Selection of Rejection Region

In this subsection, we address rejection region for our proposed test. In effect, we demonstrate the suggested rejection region allows us to be able to distinguish jumps more precisely at higher frequency of observations.

As studied in Theorem 1 and 2, our test statistics present completely different limiting behavior depending on the existence of jumps at testing times. If there is no jump at testing time, our test statistic follows approximately a normal distribution. But if there is, it becomes very large. To determine reasonable rejection region, we raise a question of how large our test statistics can be when there is no jump. Hence, we first study the asymptotic behavior or distribution of maximums of our test statistics under the null hypothesis. Such a distribution then guide us to choose the relevant threshold for the test to distinguish the presence of jumps at a testing time. Following **Lemma 1** states the limiting distribution of the maximums as follows.

Lemma 1 *If the conditions for $T(i)$, K, and c are as in* **Theorem 1** *under the null at time t_i,* *then as* $\Delta t \to 0$,

$$\frac{max|T(i)| - C_n}{S_n} \to \xi$$

where ξ *has a cumulative distribution function* $P(\xi \leq x) = exp(-e^{-x})$,

$$C_n = \frac{(2\log n)^{1/2}}{c} - \frac{\log \pi + \log(\log n)}{2c(2\log n)^{1/2}} \text{ and } S_n = \frac{1}{c(2\log n)^{1/2}}.$$

where n is the number of observations.

Proof of Lemma 1: *See Appendix 5.4.*

In short, the main idea in selecting rejection region is that if our observed test statistics are not even within the usual region of maximums, it is unlikely that the realized return is from a diffusion process. To apply this result to select rejection region, for instance, we can set a significance level of 1%. Then the threshold for $\frac{|T(i)|-C_n}{S_n}$ is β^* such that $P(\xi \leq \beta^*) = exp(-e^{-\beta^*}) = 0.99$. Equivalently, $\beta^* = -\log(-\log(0.99)) = 4.6001$. Therefore, if $\frac{|T(i)|-C_n}{S_n} > 4.6001$, then we reject the hypothesis of no jump at t_i.

2 Misclassifications

In what follows, we define 4 different types of misclassification that can occur in our study. Then we demonstrate that the probability of such misclassifications becomes negligible at higher frequencies of observations. For a single testing time, say t_i, there can be 2 kinds of misclassifications. The first kind is when there is a jump in interval $(t_{i-1}, t_i]$, but the test fails to declare the existence. We call it as *failure to detect actual jumps(FTD_i)* at t_i. The second kind is when there is no jump in $(t_{i-1}, t_i]$, but the test wrongly declare there is one. We call this as *spurious detection of jumps(SD_i)* at t_i. It is usually the case that we do this test n times with n number of observations. Extension of these concepts to *global* is straightforward. In words, if there are some jumps over the whole time horizon, $[0, T]$, but the test fails to detect any one of the presence, we call it as *global failure to detect jumps(GFTD)*. Finally, if there are some returns that are not due to jumps but the procedure wrongly declares any one of them as due to jump, then we call it as *global spurious detection of jumps(GSD)*. We will use following mathematical notations whenever necessary to explain the above situations. We let A_n be jump times among n observations and B_n be times at which the test declares the presence of a jump. We use J_i (J is for jumps) to denote the event that there is a jump in $(t_{i-1}, t_i]$. Note that $J_i = \{i \in A_n\}$. D_i (D is for declaring jumps) denotes the event that our test declares a jump in $(t_{i-1}, t_i]$. In this case, $D_i = \{i \in B_n\}$. Then following statements holds.

$$\textit{failure to detect actual jump at } t_i \textit{ (local property)}\ (FTD_i) = J_i \bigcap D_i^C$$

$$\textit{spurious detection of jump at } t_i \textit{ (local property)}\ (SD_i) = J_i^C \bigcap D_i$$

$$\textit{failure to detect actual jumps (global property)}\ (GFTD) = \bigcup_{i=1}^{n} \left(J_i \bigcap D_i^C \right)$$

$$\textit{spurious detection of jumps (global property)}\ (GSD) = \bigcup_{i=1}^{n} \left(J_i^C \bigcap D_i \right)$$

With all the new notations, we now generalize the above example of using a fixed significance level to any significance level α_n that approaches to 0. Alternatively, β_n approaches to ∞. In Theorem 3 and Corollary 1, we explicitly show that the conditional and unconditional probability of *global failure to detect actual jumps* approach to 0 respectively. They specifically show how fast these probabilities converge to 0.

Theorem 3 *Let β_n be the $(1-\alpha_n)^{th}$ percentile of the limiting distribution of ξ in* **Lemma 1** *where α_n is the significance level of test. Suppose there are N jumps in $[0, T]$. Then, the probability of* *failing to detect actual jumps (global property) is*

$$P(GFTD|N\ jumps) = \frac{2}{\sqrt{2\pi}} y_n N + o(y_n N).$$

where $y_n = (\beta_n S_n + C_n)c\sigma\sqrt{\Delta t}$. *Therefore, as long as* $\beta_n \to \infty$ *slower than* $\sqrt{n \log n}$,

$$P(GFTD|N\ jumps) \longrightarrow 0$$

Proof of Theorem 3: *See Appendix 5.5.*

Corollary 1 *If the jump intensity is* λ *and time horizon is from 0 to T, then*

$$E[P(GFTD)] = \frac{2}{\sqrt{2\pi}} y_n \lambda T + o(y_n \lambda T).$$

Theorem 4 also presents a generalized likelihood that we spuriously detect jumps approaches to 0 quickly. The corresponding convergence rate is provided as the significance level α_n become close to 0 or equivalently, the rejection threshold β goes to ∞.

Theorem 4 *Let* β_n *be as in* **Theorem 3**. *Again, suppose there are N jumps in* $[0, T]$. *Then, as* $\Delta t \to 0$, *the probability of spurious detection of jumps (global property)*

$$P(GSD|N jumps) = exp(-\beta_n) + o(exp(-\beta_n)).$$

Therefore, as $\beta_n \to \infty$,

$$P(GSD|N\ Jumps) \longrightarrow 0$$

Proof of Theorem 4: *See Appendix 5.6.*

Immediate consequence of our finding in previous two theorems is on the accuracy of stochastic jump intensity estimators based on our test. If both of the probabilities in Theorem 3 and 4 becomes negligible, then the likelihood of global misclassification is also negligible as stated in the following Theorem 5.

Theorem 5 *If* $\widehat{\Lambda}(T)$ *is the estimator of the number of jumps in* $[0,T]$ *using our test, (which is formally a cumulative jump intensity estimator) and* $\Lambda^{actual}(T)$ *is the number of actually realized jumps in* $[0,T]$*, then the probability of global misclassification is*

$$P(\widehat{\Lambda} \neq \Lambda^{actual}|N\ Jumps) = P(GFTD\ or\ GSD|N\ jumps) = \frac{2}{\sqrt{2\pi}}y_n N + exp(-\beta_n) + o(exp(-\beta_n)).$$

It can be minimized at $\beta_n^* = -\log\left(\frac{\sigma\sqrt{T}N}{\sqrt{2n\log n}}\right)$*. Moreover, the overall optimal convergence rate is* $\frac{\sigma\sqrt{T}N}{\sqrt{2n\log n}}$.

Proof of Theorem 5: *See Appendix 5.7*

3 Monte Carlo Simulation

In this subsection, we study the effectiveness of the test through Monte Carlo simulation. Our asymptotic argument in previous sections requires the sampling interval Δt converge to 0. This idealistic requirement cannot be perfectly met in real applications. This subsection investigates the finite sample performance of this test. The main result from this simulation study shows that as we increase the frequency of observation, the precision of our test increases. For the series generation, we used the Euler-Maruyama Stochastic Differential Equation (SDE) discretization scheme (Kloeden and Platen (1992)), an explicit order 0.5 strong and order 1.0 weak scheme. We discard the burn-in period – first part of the whole series – to avoid the starting value effect. We use the notation $\Delta t = \frac{1}{252 \times nobs}$ with *nobs* as the number of observations per day throughout.

3.1 Constant Volatility

We first consider the simplest model in the class with a fixed volatility. Table I presents the probability of *spurious detection of jumps at* t_i (SD_i). We simulate two constant volatility diffusion processes with fixed spot volatilities at realistic annualized values of 30% and 60% respectively. A thousand series of returns over 1 year are simulated at several different frequencies from 1 to 288 observations per day – at highest 5-minute observations. The significance level for this study is 5%. Table I shows that increasing the frequency of observation reduces the probabilities of *spurious detection of jumps* (SD_i).

Table II lists the probability of success to detect actual jump, that is, 1-the probability of *failure to detect actual jump at* t_i (FTD_i). A thousand simulated tests at different frequencies, again from 1 to 96 observations per day, are performed. Arrivals of six different jump sizes are assumed at three times to 10% of the given volatility level of 30%. We chose different jump sizes to show that it is harder to detect smaller-sized jumps at low frequency. However, we show that using our technique, as we increase the frequency, we obtain very high detecting power (above 98%) even for very small sized jumps. For instance, from Table II, we can see that when the relative magnitude of jumps are 10% of volatility, econometricians are less likely to tell the difference between price changes due to volatility part and those due to jump part with lower frequency such as daily. According to our study, only 2% of times, they can detect the presence of jumps. On the other hand, at frequencies as high as 30-minutes, we can distinguish the difference more than 95% of the times.

nobs	$\sigma = 0.3$	(SE)	$\sigma = 0.6$	(SE)	SV	(SE)
1	1.3305e-03	(7.4050e-05)	1.3305e-03	(7.6239e-05)	3.9110e-03	(1.3319e-04)
2	5.7380e-04	(3.4901e-05)	5.3222e-04	(3.3570e-05)	2.3306e-03	(7.7336e-05)
4	2.0696e-04	(1.4460e-05)	2.1209e-04	(1.4790e-05)	1.3289e-03	(4.3670e-05)
12	5.2879e-05	(4.3701e-06)	5.5911e-05	(4.2684e-06)	4.8131e-04	(1.6731e-05)
24	2.1775e-05	(1.9032e-06)	2.5126e-05	(2.0353e-06)	2.7688e-04	(1.0062e-05)
48	8.8436e-06	(8.3749e-07)	8.6768e-06	(8.3965e-07)	1.4467e-04	(7.2952e-06)
96	3.4947e-06	(3.7430e-07)	4.1876e-06	(4.2736e-07)	8.9449e-05	(4.2818e-06)
288	9.6810e-07	(1.1824e-07)	9.8630e-07	(2.2875e-07)	1.2986e-05	(1.2463e-06)

Table I: The mean and standard error (in parenthesis) of $P(SD_i)$, the probability of rejecting a spurious jump. The significance level α is 5%. The null model is a diffusion process with fixed volatilities, σ at 30% and 60% or with stochastic volatility(SV). *nobs* denotes the number of observations per day.

3.2 Stochastic Volatility

We examine how the test performs differently for stochastic volatility. Following the empirical study on realized variance of foreign currency exchange rates in Barndorff-Nielsen and Shephard (2004), we assume the spot volatility to be a sum of two uncorrelated Cox, Ingersoll and Ross (1985) square root processes. Specifically, the spot volatility process is modeled as a sum of separate solutions of two different stochastic differential equations,

$$d\sigma_s^2(t) = -\theta_s\{\sigma_s^2(t) - \kappa_s\}dt + \omega_s\sigma_s(t)dB(\theta_s t)$$

where B denotes a Brownian Motion, $\theta_s > 0$, and $\kappa_s \geq \omega_s^2/2$ for $s = 1, 2$. For this simulation, we use estimates calibrated by Barndorff-Nielsen and Shephard (2002) with exchange rate data in order for our study to mimic the real markets. The values are from

$$E(\sigma_s^2) = p_s 0.509, Var(\sigma_s^2) = p_s 0.461, \text{ for } s = 1, 2$$

	Constant Volatility σ at 30%					
Jump size	3σ	2σ	1σ	0.5σ	0.25σ	0.1σ
$nobs = 1$	0.9920	0.9880	0.9810	0.9270	0.4690	0.0260
	(0.0028)	(0.0034)	(0.0043)	(0.0082)	(0.0158)	(0.0050)
$nobs = 4$	0.9860	0.9780	0.9820	0.9700	0.9050	0.1520
	(0.0037)	(0.0046)	(0.0042)	(0.0054)	(0.0093)	(0.0114)
$nobs = 24$	0.9950	0.9860	0.9890	0.9890	0.9770	0.8880
	(0.0022)	(0.0037)	(0.0033)	(0.0033)	(0.0047)	(0.0100)
$nobs = 96$	0.9980	0.9970	0.9960	0.9920	0.9970	0.9820
	(0.0014)	(0.0017)	(0.0020)	(0.0028)	(0.0017)	(0.0042)
	Stochastic Volatility					
Jump size	$3\widetilde{\sigma(t)}$	$2\widetilde{\sigma(t)}$	$1\widetilde{\sigma(t)}$	$0.5\widetilde{\sigma(t)}$	$0.25\widetilde{\sigma(t)}$	$0.1\widetilde{\sigma(t)}$
$nobs = 1$	0.9470	0.9330	0.8540	0.5720	0.2500	0.0320
	(0.0071)	(0.0079)	(0.0112)	(0.0157)	(0.0137)	(0.0056)
$nobs = 4$	0.9770	0.9690	0.9410	0.8480	0.5320	0.1400
	(0.0047)	(0.0055)	(0.0075)	(0.0114)	(0.0158)	(0.0110)
$nobs = 24$	0.9870	0.9860	0.9830	0.9610	0.8770	0.5260
	(0.0036)	(0.0037)	(0.0041)	(0.0061)	(0.0104)	(0.0158)
$nobs = 96$	0.9970	0.9990	0.9980	0.9920	0.9610	0.8100
	(0.0017)	(0.0010)	(0.0014)	(0.0028)	(0.0061)	(0.0130)

Table II: The mean and standard error (in parenthesis) of $[1 - P(FTD_i)]$. The significance level α is 5%. The null model is a diffusion process with a fixed volatility σ at 30% and stochastic volatility. The jump size are determined compared to volatility. For stochastic volatility, the jump size depends on the mean of volatility $\widetilde{\sigma(t)} = E[\sigma(t)]$. $nobs$ denotes the number of observations per day.

with $p_1 = 0.218, p_2 = 0.782, \theta_1 = 0.0429,$ and $\theta_2 = 3.74$. We assume no correlation between two Brownian motions in volatility and the random terms in return process, which leaves us with no leverage effect in this simulation study. The jump size and the Poisson jump counting process are set to be the same as in the case of Constant Volatility. We include the result when the volatility is stochastic in Table I in order to directly compare with constant volatility case. It confirms our intuition that if the volatility moves over time, it would be more difficult to disentangle jumps. At every frequency, the corresponding success probability for stochastic volatility is greater than those with fixed volatility. We find the same result from the comparison in Table II: the probability of success to detect a jump at some given time decreases under stochastic volatility. This shows that stochastic volatility reduces the precision of jump detection. However, this study does not alter our conclusion in the previous subsection, namely, if we increase the frequency of observation, we can still improve our ability to detect jumps. Indeed, for the case of $nobs = 96$, the success rate for stochastic volatility is as high as for constant volatility.

3.3 Global Misclassification

In this subsection, we examine the global likelihood of misclassification by either *spurious detection of jumps (GSD)* or *failure to detect actual jumps (GFTD)* as studies in Theorem 5. This tells us how accurately we can locate actual jump arrival times. We simulate five hundred different series of 1 year observations at different frequencies from 1 to 96 observations per day – daily to 15 minute returns. We consider 5 different jump sizes from 3 time to 25% of volatility level. As discussed in Theorem 5, we find the likelihood becomes negligible at higher frequencies. Table III presents the probability of how likely the number of jumps counted by our test is not equal to the actual number of jumps.

Probability of Global Misclassification					
Jump Size	3σ	2σ	1σ	0.5σ	0.25σ
$nobs = 1$	0.2140	0.2120	0.1640	0.3540	0.9860
	(0.0184)	(0.0183)	(0.0166)	(0.0214)	(0.0053)
$nobs = 2$	0.1252	0.1312	0.1531	0.1173	0.7714
	(0.0148)	(0.0151)	(0.0161)	(0.0144)	(0.0188)
$nobs = 4$	0.0477	0.0407	0.0407	0.0506	0.0755
	(0.0095)	(0.0088)	(0.0088)	(0.0098)	(0.0118)
$nobs = 12$	0.0036	0.0046	0.0023	0.0063	0.0050
	(0.0027)	(0.0030)	(0.0021)	(0.0035)	(0.0031)
$nobs = 24$	0.0012	0.0012	0.0005	0.0007	0.0008
	(0.0015)	(0.0015)	(0.0010)	(0.0011)	(0.0013)
$nobs = 48$	0.0001	0.0001	0.0006	0.0004	0.0002
	(0.0003)	(0.0004)	(0.0011)	(0.0009)	(0.0006)
$nobs = 96$	0.0001	0.0001	0.0001	0.0002	0.0000
	(0.0001)	(0.0002)	(0.0005)	(0.0006)	(0.0001)

Table III: The mean and standard error (in parenthesis) of probability of global misclassification, $P\left(\Lambda(T) \neq \Lambda^{actual}(T)\right)$. The significance level α is 5%. The null model is a diffusion process with stochastic volatility and y is relative jump size compared to volatility.

3.4 Comparison with Other Jump Tests

Most comparable tests to ours are those introduced by Barndorff-Nielsen and Shephard (BNS)(2006) and Jiang and Oomen (JO)(2005), both of which are also nonparametric, which makes test results robust to model specification. We spend this subsection to explain difference between the other tests and ours. BNS(2006) takes the differences (or ratios) between the realized quadratic variation and bipower variation during a certain time interval to distinguish the presence of jumps in that interval. JO(2005)'s swap variance test takes a similar approach to BNS (2006)'s. Instead of using bipower variation, JO(2005) uses cumulative delta-hedged gain or loss of a variance swap replicating strategy. They both also provide asymptotic null distributions for their jump test statistics. As stated in BNS(2006) after simulation study under alternative hypothesis, one of the features of their test is that they cannot distinguish two jumps with low variance and one jump with high variance a day in terms of rejection rate. The reason for this problem is because their tests depends on integrated quantities. JO (2005)'s swap variance test share this feature because it also depends on an integrated quantity in their test. For instance, two jumps with same jump sizes, say 10% would yield the similar impact on their tests as one jump with a jump size of 20%. Following example can illustrate the difference more clearly. Suppose there are two jumps in a day and an analyst chooses one day as the interval for the test of BNS(2006). Then the presence of jump in that day could be recognized but not "how many" jumps were there within that day, whether the jump was negative or positive, and at which time of the day those jumps occur. Those issues, however, can be resolved by ours.

Not only can our test do more jobs as explained above but also it outperforms tests by BNS(2006) on the same job. We only do a comparative study with BNS (2006) since except the convergence rate, the same result is expected from JO (2005). For the rest of this subsection, we report a simulation study to compare the power of our new test to that of the linear test by BNS(2006). We choose their linear test because they show the linear test performs better than the adjusted ratio test in their simulation study in BNS(2006), despite the little difference. We presume comparison with their linear test and presenting our test outperforms is sufficient.

We design this simulation by introducing one or two jumps a day to the diffusion process with a constant volatility. We consider 3000 separate simulated series of process for a day and each jump arrives randomly as a Poisson process. The number of jump was set as either one or two. If there is a given number of jumps for some time, (one day in this study), then the Poisson jump arrival time is uniformly distributed: see Ross (1995). Hence, we select randomly the arrival times from uniform distribution. σ is set at 30% as before. The variance of jump size distribution is chosen at 10%,5%,1% and 0.5% respectively of σ^2. At higher variances of jump sizes, the difference between two tests are not large. Table IV shows the cases with lower variances that present better performance of our test. In conclusion, our test perform equally or better in the all cases of jump sizes, the number of jumps, and frequencies.

3.5 Optimal Window Size K

The optimal choice for window size is studied by simulation in this subsection. We show in plots the relationship between Mean Squared Error and the choice of window sizes. In particular, four cases are plotted in Figure 2 to show optimal choice of window size. The upper left panel shows Mean Squared Error as a function of window size K using daily data with the horizon $T = 1$ year. The remaining three use 6 hourly data with the horizon $T = 1, 3,$ and 7 years. It turns out that it is optimal for K to be the smallest integer in the condition set, $K = O_p(\Delta t^{\alpha})$ with $-1 < \alpha < -0.5$, because it gives the lowest Mean Squared Error in our simulation. Increasing

	Number of jumps per day $N = 1$				Number of Jumps per day $N = 2$			
10%	Linear test (BNS(2004))		Our test		Linear test (BNS(2004))		Our test	
nobs	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)
48	0.8520	(0.0112)	0.8717	(0.0107)	0.9690	(0.0055)	0.9820	(0.0042)
72	0.8819	(0.0102)	0.8967	(0.0097)	0.9860	(0.0037)	0.9879	(0.0036)
96	0.8987	(0.0091)	0.9220	(0.0085)	0.9860	(0.0037)	0.9890	(0.0033)
288	0.9360	(0.0075)	0.9620	(0.0060)	0.9930	(0.0026)	0.9990	(0.0010)
5%	Linear test (BNS(2004))		Our test		Linear test (BNS(2004))		Our test	
nobs	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)
48	0.8040	(0.0123)	0.8480	(0.0114)	0.9570	(0.0064)	0.9730	(0.0051)
72	0.8248	(0.0120)	0.8509	(0.0113)	0.9690	(0.0055)	0.9770	(0.0047)
96	0.8347	(0.0117)	0.8740	(0.0105)	0.9720	(0.0052)	0.9810	(0.0043)
288	0.8900	(0.0096)	0.9340	(0.0079)	0.9890	(0.0033)	0.9960	(0.0020)
1%	Linear test (BNS(2004))		Our test		Linear test (BNS(2004))		Our test	
nobs	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)
48	0.5650	(0.0157)	0.6310	(0.0153)	0.8160	(0.0123)	0.8670	(0.0107)
72	0.6640	(0.0149)	0.7130	(0.0140)	0.8677	(0.0107)	0.9039	(0.0093)
96	0.6470	(0.0151)	0.7330	(0.0143)	0.8770	(0.0104)	0.9260	(0.0083)
288	0.7610	(0.0135)	0.8470	(0.0114)	0.9490	(0.0070)	0.9740	(0.0050)
0.5%	Linear test (BNS(2004))		Our new test		Linear test (BNS(2004))		Our new test	
nobs	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)	$1 - FTD$	(SE)
48	0.4603	(0.0158)	0.5375	(0.0158)	0.6857	(0.0147)	0.7668	(0.0134)
72	0.4940	(0.0158)	0.6127	(0.0155)	0.7570	(0.0136)	0.8230	(0.0121)
96	0.5493	(0.0157)	0.6590	(0.0150)	0.8070	(0.0125)	0.8740	(0.0105)
288	0.6753	(0.0145)	0.7990	(0.0127)	0.9130	(0.0089)	0.9570	(0.0064)

Table IV: The probabilities of rejecting the null hypothesis (standard errors in parenthesis) with the linear test by Barndorff-Nielsen and Shephard (2006) and our new test given that there is one or two jumps a day. The variance of jumps are 10%,5%,1% and 0.5% respectively of σ^2.

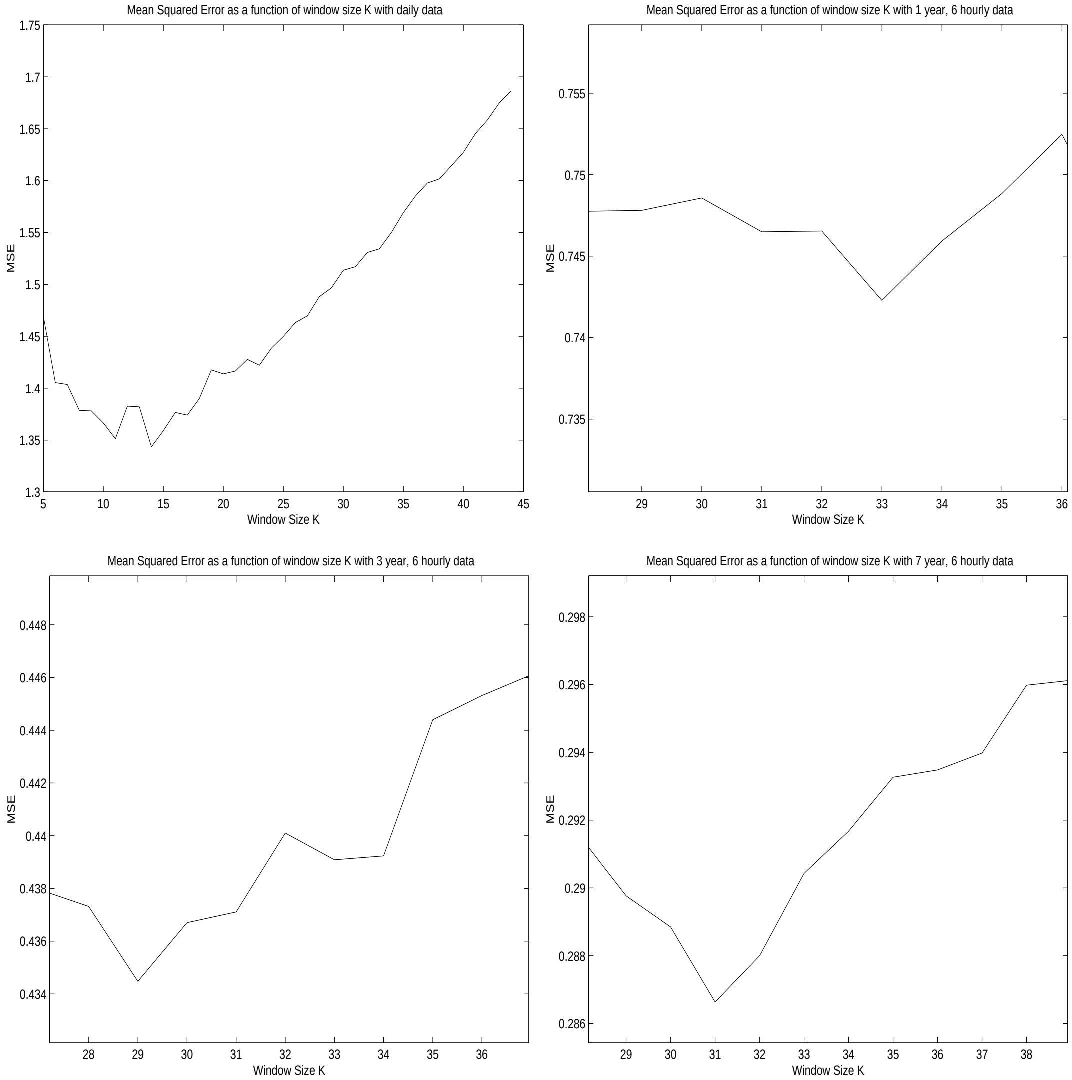

Figure 2: Graphs of Mean Squared Total Error with daily and 6 hourly data as a function of window size K

window size would not always increase the efficiency of test, especially for the intensity estimation, because it lowers the number of multiple tests if we have a fixed number of observations n.

4 Empirical Applications: Dynamics of Jump Arrivals

One of the exclusive advantages of our new test is it facilitates the study on jump dynamics as high as at intra-day levels. Not only can it guide researchers to develop correctly improved models for many empirical studies that include jumps but also allow investigation of more interesting unknown phenomenon in real-time financial markets.

Improvement in continuous-time option pricing models with time-varying intensity has been already stressed out in a number of previous theoretical and empirical studies. See Bates (2000, 2002), Duffie, *et al* (2000), Andersen, Benzoni, and Lund (2002), Pan (2002), Chernov *et al* (2003), Dubinsky and Johannes (2005) among others. In some of these studies, volatility levels, jump sizes, or earnings announcements were chosen to explain time-varying jump intensities in linear and nonlinear fashion. As reported earlier in this paper, only few extremely large and obvious jumps can be detected and estimated using daily observations which all of aforementioned empirical studies employed. In-dept investigation on jump dynamics using a high-frequency time series of underlying securities on which options depend is first warranted with our more advanced econometric technique. As a next step, they can use cross-sectional option data to pin down the market price of jump size or intensity risks based on the pre-determined model.

In this paper, we conduct an empirical study to explore real-time jump dynamics in individual U.S. stock prices. From the Trade and Quote (TAQ) database, we collected ultra high-frequency stock prices and generate returns by taking differences of log prices. We multiplied all returns by 100 to make them presented in percentages. The time span used is 3 months from September

1 to November 30, 2005, which is most recent available and never investigated in earlier studies. We chose 4 different U.S. major companies including Wal Mart (WMT), IBM (IBM), Coca Cola (KO), and General Electric (GE). Using stock prices transacted in New York Stock Exchange (NYSE), 15 minute returns were used for results in Table V. The significance level for all series is 5%. The outcome of the tests are each arrival date, time, jump size, and mean and variance of the detected jump size distribution. We do not assume that there is at most one jump per day but we assume when jump occur, the jump size dominates the return. We find that most of the jumps in equity prices under consideration arrive around market opening time. Except one or two jumps, they were associated with new events. We searched for real-time business news and information around jump arrival times through Factiva. Resources for Factiva include the Wall Street Journal, the Financial Times, Dow Jones and Reuters newswires. During 3 months, there was one scheduled corporate news event of each company, which is the third-quarter earnings announcement. Around earnings announcement times, we find similar evidence as Dubinsky and Johannes (2005) who sets a equity option pricing model with jump events conditional on deterministic earnings announcement date (EAD). In addition to pre-scheduled announcements, however, we find that jump times are indeed with more of unscheduled news events as well as scheduled events. We find evidence that sizes of jumps that come with earnings announcements are not necessarily largest in our sample, which suggests that scheduled events are not sufficient for individual equity pricing model with jumps.

This analysis can deepen our understanding on jump dynamics of individual equity and we can also exploit profitable trading strategy based on the evidence of dynamics.

Wal Mart (WMT)					IBM (IBM)				
Date	Time	Size(%)	News	S	Date	Time	Size(%)	News	S
Sep 26	9:30am	1.29	Law Suit	N	Sep 13	9:45am	-0.76	Announcement	N
Oct 06	9:31am	1.12	EAD	Y	Sep 19	9:30am	-0.74	Deal News	N
Oct 10	9:30am	1.53	New Launch	N	Sep 21	9:30am	-0.96	Option Market	N
Oct 14	9:30am	0.93	Law Suit	N	Sep 27	9:45am	1.03	Good News	N
Oct 19	10:45am	0.84	Investment	N	Oct 07	9:45am	1.04	Good News	N
Oct 31	9:30am	1.33	Sales Up	N	Oct 10	9:30am	0.98	New Product	N
Nov 15	9:48am	-1.25	Announcement	N	Oct 11	9:30am	1.24	Expansion	N
Nov 18	9:30am	1.11	Announcement	N	Oct 18	9:30am	1.99	EAD	Y
					Oct 19	9:30am	-1.29	Rival EAD	Y
					Nov 18	9:30am	1.30	Announcement	N
$\mu_y = 0.8625$		$\sigma_y = 0.8817$			$\mu_y = 0.3830$		$\sigma_y = 1.1801$		

General Electric (GE)					Coca Cola (KO)				
Date	Time	Size(%)	News	S	Date	Time	Size(%)	News	S
Sep 15	9:45am	0.70	Good News	N	Sep 13	9:45am	-1.59	Bad News	N
Sep 21	9:30am	-0.95	Bad News	N	Sep 21	9:30am	0.96	Announcement	Y
Oct 06	9:31am	2.03	Announcement	N	Oct 20	9:30am	2.60	EAD	Y
Oct 07	9:30am	0.92	Good News	N	Nov 18	9:30am	1.42	Announcement	N
Oct 14	9:30am	1.11	EAD	Y	Nov 18	9:45am	-0.83	Announcement	N
Nov 18	9:30am	2.20	Announcement	N	Nov 29	9:30am	-0.88	Announcement	N
$\mu_y = 1.0017$		$\sigma_y = 1.1324$			$\mu_y = 0.28$		$\sigma_y = 1.6260$		

Table V: Jump dates, Jump times, and Jump observed sizes, and Associated News Events. μ_y and σ_y are mean and standard deviation of observed jump sizes. The result is based on transaction prices from NYSE during 3 months from September 1 to November 30, 2005. The significance level of each test is set at 5%. S denotes whether it was scheduled or not.

5 Concluding Remarks

A new nonparametric test is introduced to detect realizations of jump arrival and characterize their sizes and overall dynamic jump and return distributions. Careful Monte Carlo simulation experiments show that an intra-day time series data increase the precision of jump tests: hence, decrease the likelihood of global misclassification of jumps. Using our new test, we perform an empirical study and find an evidence of the association between jump arrivals and both pre-scheduled earnings announcements and real-time news release in U.S. equity markets. Our results out of the test are robust to model specification and non-stationarity of the series applied. Our test assumes the Poisson type of jumps in pricing models. One potential extension is to develop a nonparametric test for jumps in a pure jump processes such as infinite activity Levy process. Another extension of our work is to develop a modified test in the presence of market microstructure noise, which is under development.

6 Appendix

6.1 The Nonzero Drift

The main conclusion of Theorem 1 is not altered under an extension to the nonzero drift case. Suppose now we have the nonzero drift coefficient $\mu(t)$ for the return process, that is,

$$d \log S(t) = \mu(t)dt + \sigma dW(t).$$

A modified version of Definition 1 for this case is as follows.

Definition 1.1 *The statistic $T_\mu(t_i)$ which tests at time t_i whether there was a jump from t_{i-1} to*

t_i is defined as

$$T_\mu(t_i) = \frac{\log S(t_i) - \log S(t_{i-1}) - \widehat{m}_i}{\sqrt{\frac{1}{K-2}\sum_{j=i-K+2}^{i-1} |\log S(t_j) - \log S(t_{j-1})||\log S(t_{j-1}) - \log S(t_{j-2})|}}$$

where

$$\widehat{m}_i = \frac{1}{K-1}\sum_{j=i-K+1}^{i-1} (\log S(t_j) - \log S(t_{j-1}))$$

Then we can extend Theorem 1 to the case of nonzero drift as below.

Theorem 1.1 *Let $T_\mu(t_i)$ be as in* **Definition 1.1** *under the null and $K = O_p(\Delta t^\alpha)$ where $-1 < \alpha < -0.5$. Suppose* **Assumption 1** *is satisfied. Then as $\Delta t \to 0$,*

$$sup_i|T_\mu(t_i) - \widehat{T}_\mu(t_i)| = O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})$$

where

$$\widehat{T}_\mu(t_i) = \frac{U_i - \bar{U}_{i-1}}{c}.$$

Here $\bar{U}_{i-1} = \frac{1}{K-1}\sum_{j=i-K+1}^{i-1} U_j$ and the rest of notations are the same as in **Theorem 1**.

Proof of Theorem 1.1: *See Appendix 5.2.*

The error rate is the same as the zero drift case since the error due to drift term is dominated by the error due to the diffusion part. $\widehat{T}_\mu(t_i)$ asymptotically follows a normal distribution with its mean 0 and variance $\frac{K}{c^2(K-1)} \to \frac{1}{c^2}$ since $K \to \infty$. Our choice of K makes the effect of jumps on $\widehat{m}_i$ vanish because of the property of the Poisson process for rare jumps, which says that there can be no more than a single jump in an infinitesimal time interval. Because this makes only a finite number, say L, of jumps in the window, the jump test statistic for the nonzero drift case becomes

$$T(t_i) \approx \frac{U_i - \bar{U}_{i-1}}{c} - \frac{L \times Y(\tau)}{c\sigma(K-1)\sqrt{\Delta t}} I_{\tau \in (t_{i-K}, t_{i-1}]}$$

The second term will disappear because of the condition $K\sqrt{\Delta t} \to \infty$. This shows that jumps in the window have asymptotically negligible effect on testing jumps from t_{i-1} to t_i with our choice of K.

6.2 Proof of Theorem 1 and Theorem 1.1 in Appendix 6.1

For $t_{i-K} < t < t_i$,

$$\log S(t) - \log S(t_{i-K}) = \int_{t_{i-K}}^{t} \mu(u)du + \int_{t_{i-K}}^{t} \sigma(u)dW(u).$$

Given the Assumption 1 imposed, we have with A1.1,

$$\int_{t_{i-1}}^{t_i} \mu(u)du - \mu(t_{i-1})\Delta t = O_p(\Delta t^{\frac{3}{2}})$$

$$\text{and } \int_{t_{i-K}}^{t_i} \mu(u)du - \mu(t_{i-K})K\Delta t = O_p(\Delta t^{\frac{3}{2}+\alpha})$$

uniformly in all i. This implies

$$sup_{i,t \le t_i} \left| \int_{t_{i-K}}^{t} \{\mu(u) - \mu(t_{i-K})\}du \right| = O_p(\Delta t^{\frac{3}{2}+\alpha})$$

Similarly to *lemma* 1 in Mykland and Zhang (2001), under the condition A1.2, we can apply Burkholder's Inequality (Protter (1995)) to get

$$sup_{i,t \le t_i} \left| \int_{t_{i-K}}^{t} \{\sigma(u) - \sigma(t_{i-K})\}dW(u) \right| = O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})$$

where δ can be any number in $0 < \delta < \frac{3}{2} + \alpha$. This result is also uniform in i for the $K = O_p(\Delta t^{\alpha})$ as specified. Therefore, over the window, for $t \in [t_{i-K}, t_i]$, $dlogS(t)$ can be approximated by $dlogS^i(t)$ such that

$$dlogS^i(t) = \mu(t_{i-K})dt + \sigma(t_{i-K})dW(t)$$

because

$$
\begin{aligned}
& |\left(\log S(t) - \log S(t_{i-K})\right) - \Big(\log S^i(t) - \log S^i(t_{i-K})\Big)| \\
&= |\int_{t_{i-K}}^{t} (\mu(u) - \mu(t_{i-K}))du + \int_{t_{i-K}}^{t} (\sigma(u) - \sigma(t_{i-K}))dW(u)| = O_p(\Delta t^{\frac{3}{2}-\delta+\alpha}).
\end{aligned}
$$

For all i, j and $t_j \in [t_{i-K}, t_i]$, the numerator is

$$
\begin{aligned}
& \log S(t_j) - \log S(t_{j-1}) - \widehat{m}_i \\
&= \log S^i(t_j) - \log S^i(t_{j-1}) - \frac{1}{K-1}\sum_{l=i-K+1}^{i-1} \Big(\log S^i(t_l) - \log S^i(t_{l-1})\Big) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha}) \\
&= \sigma(t_{i-K})W_{\Delta t} - \frac{1}{K-1}\sum_{l=i-K+1}^{i-1} \sigma(t_{i-K})W_{\Delta t} + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha}) \\
&= \sigma(t_{i-K})\sqrt{\Delta t}(U_j - \bar{U}_{i-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})
\end{aligned}
$$

where $U_j = \frac{1}{\sqrt{\Delta t}}(W_{t_j} - W_{t_{j-1}}) \sim iid$ Normal$(0,1)$, and $\bar{U}_{i-1} = \frac{1}{K-1}\sum_{j=i-K+1}^{i-1} U_j$.

For the denominator, we put a volatility estimator based on the realized bipower variation for integrated volatility estimator: see Barndorff-Nielsen and Shephard(2004). According to *Proposition* 2 in Barndorff-Nielsen and Shephard (2004), the impact of the drift term is negligible; hence, it does not affect the asymptotic limit behavior. Then, we are left to prove the following approximation of scaled volatility estimator.

$$
\begin{aligned}
\mathrm{plim}_{\Delta t\to 0} c^2\hat{\sigma}^2(t) &= \mathrm{plim}_{\Delta t\to 0}\frac{1}{(K-2)\Delta t}\sum_j |\log S(t_j) - \log S(t_{j-1})||\log S(t_{j-1}) - \log S(t_{j-2})| \\
&= c^2\sigma^2(t)
\end{aligned}
$$

It is due to

$$
\begin{aligned}
& \frac{1}{(K-2)\Delta t}\sum_{j=i-K+3}^{i-1} |\log S(t_j) - \log S(t_{j-1})||\log S(t_{j-1}) - \log S(t_{j-2})| \\
&= \frac{1}{(K-2)\Delta t}\sum_{j=i-K+3}^{i-1} |\log S^i(t_j) - \log S^i(t_{j-1}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})||\log S^i(t_{j-1}) - \log S^i(t_{j-2}) + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})|
\end{aligned}
$$

$$= \frac{1}{(K-2)\Delta t} \sum_{j=i-K+3}^{i-1} |\log S^i(t_j) - \log S^i(t_{j-1})||\log S^i(t_{j-1}) - \log S^i(t_{j-2})| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})$$

$$= \frac{1}{K-2} \sum_{j=i-K+3}^{i-1} \sigma^2(t_{i-K})|\sqrt{\Delta t} U_j||\sqrt{\Delta t} U_{j-1}| + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})$$

$$= \sigma^2(t_{i-K})c^2 + O_p(\Delta t^{\frac{3}{2}-\delta+\alpha})$$

where U_i's are iid $Normal(0,1)$ and $c = E(|U_i|) \approx 0.7979$. Then

$$T(t_i) = \frac{(U_i - \bar{U}_{i-1})}{c} + O_p(\Delta^{\frac{3}{2}-\delta+\alpha})$$

This proves Theorem 1 and Theorem 1.1.

Alternatively, for the non-zero drift case, we can use Girsanov's Theorem to suppose $\mu(t) = 0$ as in Zhang, Mykland and Ait-Sahalia (2005).

6.3 Proof of Theorem 2

When there is a possibility of rare Poisson jumps in the window, the scaled bipower variation, $c^2\hat{\sigma}^2(t)$ can be decomposed into two parts: one with jump terms and one without jump terms as follows.

$$c^2\hat{\sigma}^2(t) = \frac{1}{(K-2)\Delta t} \sum_{j=i-K+2}^{i-1} |\log S(t_j) - \log S(t_{j-1})||\log S(t_{j-1}) - \log S(t_{j-2})|$$

$$= \text{terms without jumps} + \frac{1}{(K-2)\Delta t} \sum_{\text{terms with jumps}} \sigma(t_j)|\Delta W_{t_j}||Jump|$$

$$= \text{terms without jumps} + \frac{1}{(K-2)\Delta t} O_p(\sqrt{\Delta t}) \sum_{\text{terms with jumps}} \sigma(t_j)|Jump|$$

$$= \text{terms without jumps} + \frac{1}{(K-2)\Delta t} O_p(\sqrt{\Delta t})$$

The order of the second term is due to the property of the Poisson jump process that allows finite number of jumps over the window. Since $\sigma(t_j)|Jump| = O_p(1)$, $\sum_{\text{terms with jumps}} \sigma(t_j)|Jump| = O_p(1)$. The effect of jump terms becoming negligible requires the second term to be $o_p(1)$ as Δt

goes to 0. The window size K that satisfies $K\sqrt{\Delta t} \to \infty$ and $K\Delta t \to 0$ as Δt goes to 0 will work. If we assume the window size to be $K = \beta \Delta t^{\alpha}$ where β is some constant, then the necessary condition for α is $-1 < \alpha < -0.5$. Accordingly,

$$lim_{\Delta t \to 0} \hat{\sigma}^2|_{alternative} = lim_{\Delta t \to 0} \hat{\sigma}^2|_{null} = \sigma^2(t).$$

Then putting the approximation for return above in the statistic yields

$$T(t_i) \approx \frac{\sigma_{t_{i-K}} \sqrt{\Delta t} U_i + Y(\tau) I_{\tau \in (t_{i-1}, t_i)}}{c\sigma(t_{i-K})\sqrt{\Delta t}} = \frac{U_i}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}} I_{\tau \in (t_{i-1}, t_i)}$$

6.4 Proof of Lemma 1

It follows from Aldous (1985) and the proof in Galambos (1978).

6.5 Proof of Theorem 3

We suppose there are N jumps from time $t = 0$ to $t = T$ and claim there is a jump if $|T(t_i)| > \beta_n S_n + C_n$. Fix a set of jump times as $A_n = \{i : \text{there is a jump in } (t_{i-1}, t_i]\}$. Then

$$P(\text{We correctly classify all N jumps}|\text{N jumps}) = P(\text{For all } i \in A_n, |T(t_i)| > \beta_n S_n + C_n)$$

$$\approx \prod_{i \in A_n} P(|T(t_i)| > \beta_n S_n + C_n) \approx \prod_{i \in A_n} P(|Y(t_i)| > (\beta_n S_n + C_n) c\sigma\sqrt{\Delta t})$$

$$= \prod_{i \in A_n} \Big(1 - F_{|Y|}(y_n)\Big) \sim \left(1 - \frac{2}{\sqrt{2\pi}} y_n + o(y_n^2)\right)^N = 1 - \frac{2}{\sqrt{2\pi}} y_n N + o(y_n^2 N)$$

6.6 Proof of Theorem 4

Let $A_n^C = \{0, 1, ..., n-1\} - A_n$ be a set of non-jump times. Then

$$P(\text{We incorrectly reject any non-jumps}|\text{N jumps})$$

$$= P(\text{for some } i \in A_n^C, |T(t_i)| > \beta_n S_n + C_n|\text{N jumps})$$

$$= P(max_{i\in A_n^C}|T(t_i)| > \beta_n S_n + C_n|\text{N jumps})$$

$$\approx P(max_{i\in A_n^C}|\widehat{T}(t_i)| > \beta_n S_n + C_n) = 1 - F_\xi(\beta_n) = exp(-\beta_n) + o(exp(-\beta_n)).$$

By L'Hopital's rule, we obtain the last step because

$$lim_{\beta_n\to\infty}\frac{1 - F_\xi(\beta_n)}{exp(-\beta_n)} = 1$$

6.7 Proof of Theorem 5

Prob(GMJ or $GMNJ$)=Prob(GMJ)+Prob($GMNJ$) and the results follows from Theorem 3 and 4. Minimum probability can be achieved at β_n^*, which can be obtained by taking the first derivative of probability with respect to β_n and setting it equal to 0 as

$$\frac{\partial P}{\partial \beta_n} = \frac{2}{\sqrt{2\pi}} S_n \sigma c\sqrt{\Delta t} N - exp(-\beta_n) = 0.$$

7 References

Aït-Sahalia, Y., 2002, "Telling from discrete data whether the underlying continuous-time model is a diffusion," *Journal of Finance* 57, 2075-2112.

Aït-Sahalia, Y., 2004, "Disentangling Diffusion from Jumps," *Journal of Financial Economics* 74, 487-528.

Aït-Sahalia, Y. and J. Jacod, 2005, "Fisher's Information for Discretely Sampled Levy Processes," Working Paper, Princeton University.

Aldous, D., 1989, "Probability Approximations via the Poisson Clumping Heuristic," Springer.

Andersen, T. G., L. Benzoni, and J. Lund, 2002, "An Empirical Investigation of Continuous-Time Equity Return Models," *Journal of Finance* 57, 1239-1284.

Andersen, T.G., T. Bollerslev, and F.X. Diebold, 2003, "Some Like it Smooth, and Some Like it Rough: Untangling Continuous and Jump Components in Measuring, Modeling, and Forecasting Asset Return Volatility," Working Paper, Department of Economics, Duke University.

Bakshi, G., C. Cao, and Z. Chen, 1997, "Empirical Performance of alternative Option Pricing Models," *Journal of Finance* 52, 2003-2049.

Bakshi, G., N. Kapadia, and D. Madan, 2003, "Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options," *The Review of Financial Studies*, 16, 101-143.

Bandi, F. M. and T. H. Nguyen, 2003, "On the functional estimation of jump-diffusion models," *Journal of Econometrics* 116, 293-328.

Barndorff-Nielsen, O. E., and N. Shephard, 2004, "Power and bipower variation with stochastic volatility and jumps," *Journal of Financial Econometrics* 2, 1-48.

Barndorff-Nielsen, O. E., and N. Shephard, 2004, "Econometrics of testing for jumps in financial economics using bipower variation," *Journal of Financial Econometrics* 4, 1-30.

Bates, D. S., 1996, "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsch Mark Options," *Review of Financial Studies* 9, 69-107.

Bates, D. S., 2000, "Post-'87 Crash Fears in the S&P 500 Futures Option Market," *Journal of Econometrics* 94, 181-238.

Bertsimas, D., L. Kogan, and A. Lo, 2001, "Hedging Derivative Securities and Incomplete Markets: An Epsilon-Arbitrage Approach," *Operations Research* 49, 372-397.

Black, Fischer and Myron S. Scholes, 1973, "The pricing of options and corporate liabilities," *Journal of Political Economy* 81, 637-654.

Chernov, M.,A.R. Gallant, E. Ghysels, and G. Tauchen, 2003, "Alternative Models for Stock Price Dynamics," *Journal of Econometrics*, 116, 225-257.

Collin-Dufresne, P. and J. Hugonnier, 2001, "Event Risk, Contingent Claims and the Temporal Resolution of Uncertainty," Working Paper, Carnegie Mellon University

Collin-Dufresne, P. and J. Hugonnier, 2002, "Pricing and Hedging in the Presence of Extraneous Risks," Working Paper, Carnegie Mellon University

Cox, J. T., J. E. Ingersoll, and S. A. Ross, 1985, "A theory of the term structure of interest rates," *Econometrica*, 53, 385-407.

Duffie, D., 1995, Dynamic Asset Pricing Theory. 2nd Edition. Princeton University Press.

Duffie, D., J. Pan, and K. Singleton, 2000, "Transform analysis and asset pricing for affine jump diffusions," *Econometrica* 68, 1343-1376.

Eraker, B., M. Johannes, N. Polson, 2003, "The Impact of Jumps in Volatility and Returns," *Journal of Finance* 53, 1269-1300.

Fan, J. and Y. Wang, 2005, "Multi-scale Jump and Volatility Analysis for High-Frequency Financial Data", Working Paper, Princeton University.

Galambos, J., 1978, The Asymptotic Theory of Extreme Order Statistics. John Wiley and Sons, Inc.

Huang, X. and G. Tauchen, 2005, “The Relative Contribution of Jumps to Total Price Variance,” *Journal of Financial Econometrics*, 3, 456-499.

Jacod, J. and A. N. Shriyaev, 1980, Limit Theorems for stochastic processes. Berlin New York: Springer.

Jiang, G. J. and R.C. Oomen, 2005, ”A New Test for Jumps in Asset Prices,” Working Paper, University of Arizona.

Johannes, M., 2004, “The statistical and Economic Role of Jumps in Interest Rates,” *Journal of Finance* 59, 227-260.

Kloeden, P. E. and E. Platen, 1992, Numerical Solution of Stochastic Differential Equations. Springer-Verlag.

Kou, S. G., 2002, “A Jump Diffusion Model for Option Pricing,” *Management Science* 48, 1086-1101.

Lee, S. Y., 2004, “Jumps in Efficient Prices under Market Microstructure Noise,” Working Paper, University of Chicago.

Liu, J., F. Longstaff, and J. Pan, 2003, “Dynamic Asset Allocation with Event Risk,” *Journal of Finance* 58, 231-259.

Maheu, J. M. and T. H. McCurdy, 2004, “New Arrival, Jump Dynamics, and Volatility Components for Individual Stock Returns,” *Journal of Finance* 59, 755-793.

Merton, R. C., 1976, “Option Pricing when Underlying Stock Returns are Discontinuous,” *Journal of Financial Economics* 3, 125-144.

Merton, R. C., 1990, Continuous-time Finance. Blackwell, Cambridge.

Mykland, P. and L. Zhang, 2006, "Anova for diffusions," *Forthcoming Annals of Statistics* 34.

Pan, J., 2002, "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journal of Financial Economics* 63, 3-50.

Piazzesi, M., 2003, "Bond yields and the Federal Reserve," *Journal of Political Economy* 113, 311-344.

Pollard, D., 2001, A User's Guide of Measure Theoretic Probability. Cambridge University Press.

Protter, Philip, 1995, Stochastic Integration and Differential Equations: A New Approach. Berlin New York: Springer.

Ross, S. M., 1995, Stochastic Processes, Wiley, Second Edition.

Schaumburg, E., 2001, "Maximum Likelihood Estimation of Jump Processes with Applications to Finance," Ph.D. Dissertation, Princeton University

Tauchen G. and H. Zhou, 2005, "Identifying Realized Jumps on Financial Markets," Working Paper, Duke University.

Tsay, Ruey S., 2001, A Manuscript, Chapter 10. Markov Chain Monte Carlo Methods with Applications.

Zhang, L., P. Mykland, and Y. Ait-Sahalia (2005), "A Tale of Two Time Scales: Determining Integrated Volatility with Noisy High-Frequency Data," *Journal of American Statistical Association* 472, 1394-1411.

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