2024. Jan. 11th

Computing w_{ij} and θ_i is very troublesome.

↓ We like to find $\boxed{\text{a program}}$ to compute them.

$$E(x_1, x_2, \ldots, x_N) = \underbrace{-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} x_i x_j + \sum_{i=1}^{N}\theta_i x_i + C}_{\text{The standard form of the energy.}}$$

(1) At first we compute C

$$E(0, 0, \ldots, 0) = C$$

C is computed just by substituting 0 to all ~~w_{ij} and θ_i, $i=1\sim N, j=1\sim N$~~ $x_i, i=1\sim N$.

(2) Then we compute θ_i

$$\theta_i = E(\underset{x_1}{0}, \underset{x_2}{0}, \ldots, \underset{x_{i-1}}{0}, \underset{x_i}{1}, \underset{x_{i+1}}{0}, \ldots, \underset{x_N}{0}) - C$$

Only x_i is 1, other $x_j, j \neq i = 0$

?

(3) Finally we can compute w_{ij}

$$E(\underset{x_1}{0},\underset{x_2}{0},\cdots,\underset{x_{i-1}}{0},\underset{x_i}{1},\underset{x_{i+1}}{0},\cdots,\underset{x_{j-1}}{0},\underset{x_j}{1},\underset{x_{j+1}}{0},\cdots,\underset{x_N}{0}) \quad — \text{(A)}$$

Only x_i and x_j take 1

and other $x_k, k \neq i, j$ take 0.

Substitute 1 to x_i and x_j

Substitute 0 to $x_{k \neq i,j}$

No self connection

$w_{kk} = 0$

symmetric connection

$w_{ij} = w_{ji}$

$$= -\frac{1}{2}\Big(\underbrace{w_{12}\cdot \underset{0}{x_1}\cdot \underset{0}{x_2}}_{=0} + \underbrace{w_{13}\cdot \underset{0}{x_1}\cdot \underset{0}{x_3}}_{=0} + \cdots \underbrace{w_{1i}\cdot \underset{0}{x_1}\, \underset{1}{x_i}}_{=0}$$

$$+ \underbrace{w_{ij}\, \underset{1}{x_i}\, \underset{1}{x_j}}_{=w_{ij}} + \underbrace{w_{ij+1}\, \underset{1}{x_i}\, \underset{0}{x_{j+1}}}_{=0} + \underbrace{\cdots}_{=0} \quad 0$$

$$+ \underbrace{w_{ji}\, \underset{=1}{x_j}\, \underset{=1}{x_i}}_{= w_{ji}} + \cdots \underbrace{\cdot\ \cdot}_{=0}\ \underbrace{\cdot}_{=0} + \underbrace{w_{ij}\, x_N x_{N-1}}_{=0} + \underbrace{w_{ij}\, x_N x_N}_{=0}$$

$$+ \Big(\underbrace{\theta_1 x_1}_{=0} + \underbrace{\theta_2 x_2}_{=0} + \cdots + \underbrace{\theta_{i-1} x_{i-1}}_{=0} + \underbrace{\theta_i x_i}_{=\theta_i} + \cdots + \underbrace{\theta_{j-1} x_{j-1}}_{=0} + \underbrace{\theta_j x_j}_{\theta_j} + \underbrace{\theta_{j+1} x_{j+1}}_{=0} + \cdots + \underbrace{\theta_N x_N}_{=0}\Big)$$

$$+ C$$

$$= -\frac{1}{2}\left(w_{ij} + w_{ji}\right) + \theta_i + \theta_j + C$$

symmetric $w_{ij} = w_{ji}$

$$= -w_{ij} + \theta_i + \theta_j + C$$

$$E(0,0,\dots 0,\underbrace{1}_{=x_i},0,\dots,0,\underbrace{1}_{=x_j},0\dots 0)$$

$$= -w_{ij} + \theta_i + \theta_j + C$$

We can compute w_{ij} by

$$w_{ij} = \theta_i + \theta_j + C - E(0,0,\dots 0,\underbrace{1}_{=x_i},0,\dots 0,\underbrace{1}_{=x_j},0,\dots 0)$$

From (1), (2) and (3), we can make a program to compute C, θ_i and w_{ij} $\begin{array}{l} i=1\sim N \\ j=1\sim N \end{array}$

(1) $C = E(0,0,\dots,0)$

(2) $\theta_i = E(0,0,\dots 0,\underbrace{1}_{=x_i},0,\dots 0) - C$ $x_i=1$, $x_k=0$ $k\neq i$

(3) $w_{ij} = \theta_i + \theta_j + C - E(0,0,\dots,0,\underset{x_i=1}{1},0,\dots 0,\underset{x_j=1}{1},0,\dots 0)$ $\begin{array}{l} x_k=0 \\ k\neq i,j \end{array}$

$$
\begin{aligned}
C = E(0,0,\cdots,0) &= (0-2\times0+0-0-1)^2 \\
&\quad +(-0-0+0+2\times0-2)^2 \\
&\quad +(2\times0+0-0+0-2)^2 \\
&\quad +(-2\times0+3\times0-2\times0-0+5)^2 \\
&= (-1)^2+(-2)^2+(-2)^2+(5)^2 \\
&= \cancel{35}\ 34
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= E(1,0,\cdots,0) - 34 \\
&= (1-2\times0+0-0-1)^2 \\
&\quad +(-1-0+0+2\times0-2)^2 \\
&\quad +(2\times1+0-0+0-2)^2 \\
&\quad +(-2\times1+3\times0-2\times0-0+5)^2 \\
&\quad -34 \\
&= (0)^2+(-3)^2+(0)^2+(3)^2-34 = -16
\end{aligned}
$$

Task 4-1

Make a program to compute C, θ_i, $i=1\sim N$,
w_{ij}, $i=1\sim N$, $j=1\sim N$

for the energy function of Task 3.

$$E(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_3 - x_4 - 1)^2 + (-x_1 - x_2 + x_3 + 2x_4 - 2)^2 + (2x_1 + x_2 - x_3 + x_4 - 2)^2 + (-2x_1 + 3x_2 - 2x_3 - x_4 + 5)^2$$

For this function, we already computed w_{ij}, θ_i, and C by our hand.

Please check if you can get the same results.

Now, You got a program to compute w_{ij}, θ_i and C, $i = 1 \sim N$, $j = 1 \sim N$

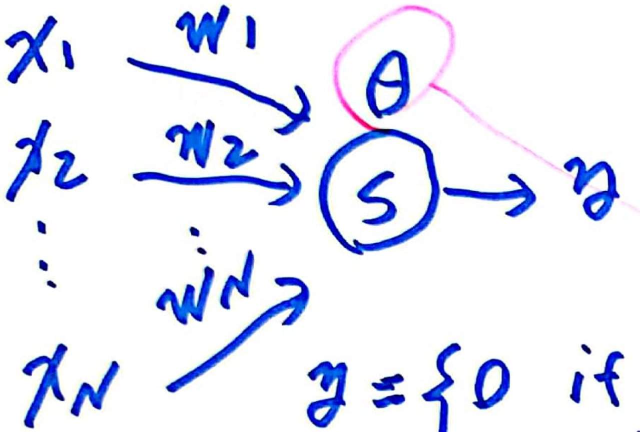
You can apply RNN to more complex problems.

I like to explain some tips to make your program simple.

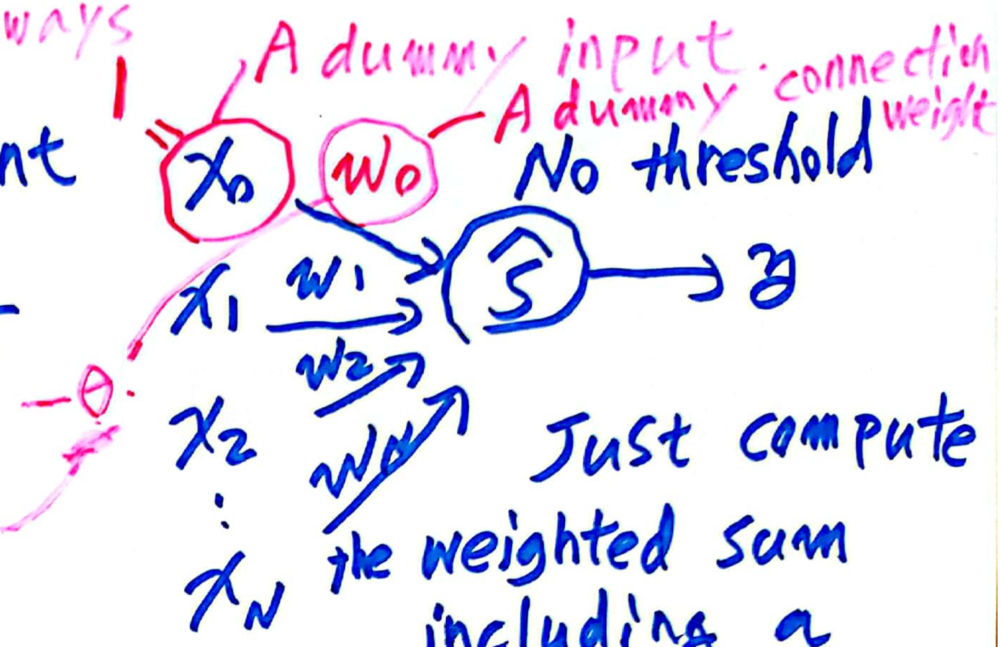
So far, we used w_{ij} and θ_i separately.
So we have to make different procedures for w_{ij} and θ_i.

But by using a tip (trick), θ_i can be dealt as a kind of w_{ij}.

↳ We can use a common procedure for w_{ij} and θ_i.



Equivalent



$$y = \begin{cases} 0 & \text{if } \mathrm{ran}() > p \\ 1 & \text{else} \end{cases}$$

$$p = \mathrm{sigmoid}\,(S - \theta) = \hat{S}$$

$$S = \sum_{i=1}^{N} w_i x_i$$

the same S

Just compute the weighted sum including a dummy input and a dummy connection weight.

$$\hat{S} = \sum_{i=0}^{N} w_i x_i$$

— start from the dummy.

$$= \underbrace{w_0}_{= -\theta} \underbrace{x_0}_{= 1} + \underbrace{\sum_{i=1}^{N} w_i x_i}_{S}$$

$$= (S - \theta)$$

The energy function can be simple if you use the dummy input and the dummy connection.

$$E = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\, x_i x_j + \sum_{i=1}^{N} \theta_i x_i + C$$

A dummy neuron

$$= -\frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N} w_{ij}\, x_i x_j + C$$

$w_{00} = 0$
$w_{i0} = w_{0i} = -\theta_i$) symmetric condition
$x_0 = 1$

A dummy neuron for a dummy input.
$x_0 = 1$

Dummy connections for thresholds

Now we are ready for difficult optimization problems.

Today : Eight Queen Problem

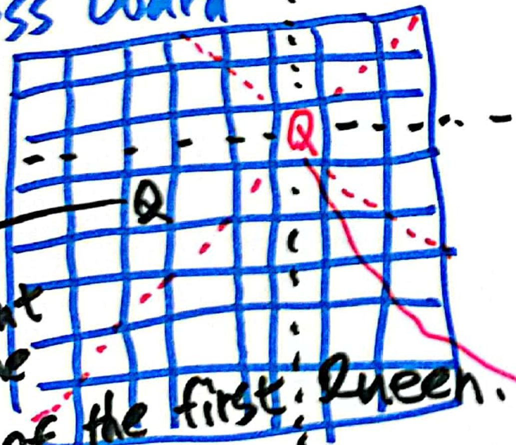
Tommorrow : Travelling Salesman Problem.

Eight Queen Problem (Common Problem for Programming Exercise)

Queen of Chess

Like this, we have to put Eight Queens on the chess board so that they do not conflict with each other.

Chess Board



Put Eight Queens on this chess Bord avoiding the confliction among them.

The second Queen must be put avoiding the influence of the first Queen.

This Queen has its influence along the horizontal, vertical and diagonal lines.

We need to find the location of Eight Queens.

The location is represented using neurons.

$$
\begin{array}{c|ccccc}
 & j=1 & j=2 & j=3 & \cdots & j=8 \\
\hline
i=1 & x_{11} & x_{12} & x_{13} & \cdots & x_{18} \\
i=2 & x_{21} & x_{22} & x_{23} & \cdots & x_{28} \\
\vdots & \vdots & \vdots & & & \vdots \\
i=8 & x_{81} & x_{82} & x_{83} & \cdots & x_{88}
\end{array}
\quad \Big\} \; 8
$$

$\underbrace{\qquad\qquad\qquad\qquad}_{8}$

We use $8 \times 8 = 64$ neurons.

↓

Represent 8×8 locations (cells) on the chess Board.

x_{ij} ⇒ Using these two indexes, we can specify the position of the cell.

$\underline{i}\ \underline{j}$ = Two indexes to represent position (i,j)

If a Queen is on a cell represented by x_{ij}, then x_{ij} takes the value 1, and if there is no Queen on the cell then x_{ij} takes the value 0.

The idea of "the winner takes all" can be used for Eight Queen Problem.

Along the horizontal (column) line (row) (vertical) (diagonal)

only one Queen can exist.

↳ only one neuron x_{ij} can take 1, and other neurons on the same horizontal (vertical) (diagonal) line must take 0.

same as "the winner takes all".

For i-th horizontal line, only one x_{ij} takes 1 and other $x_{ik} = 0$ $_{k \neq j}$

↳ The energy function should be

$$(x_{i1} + x_{i2} + \cdots + x_{i8} - 1)^2$$

This energy function ~~has~~ takes its minimum 0

when only one of $x_{i1}, x_{i2}, \cdots x_{i8}$ takes 1

and others take 0.

For example:

$$(\overset{x_{i1}}{0} + \overset{x_{i2}}{0} + \cdots \overset{x_{ij-1}}{0} + \underset{x_{ij}}{1} + \underset{x_{ij+1}}{0} \cdots + \underset{x_{i8}}{0} - 1)^2 = \underline{\underline{0}}$$

0 is the minimum because the squared value is always 0 or positive.

The energy function for the horizontal lines $i = 1 \sim 8$

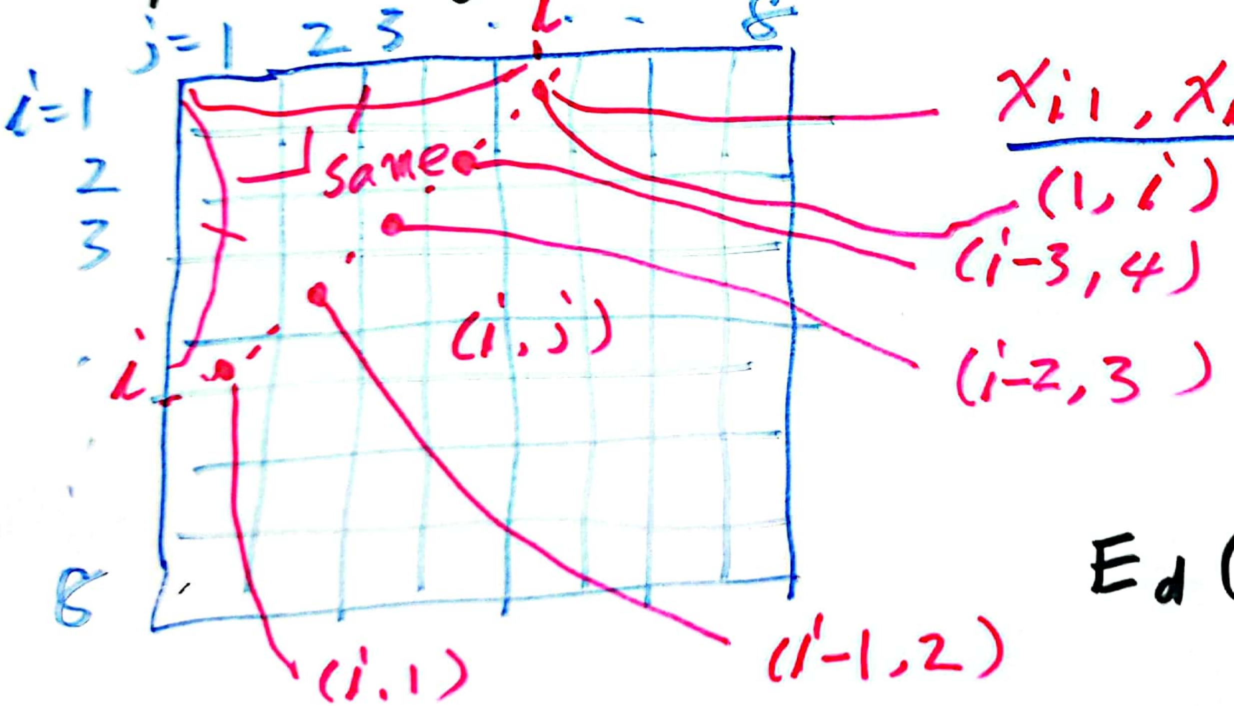
$$E_h(\underbrace{x_{11}, \cdots, x_{88}}_{64}) = \sum_{i=1}^{8} \left(\sum_{j=1}^{8} x_{ij} - 1 \right)^2$$

↳ To avoid horizontal conflicts among eight queens, this energy function must take 0 (minimal)

For vertical conflicts, we can use the following energy function.

$$E_v(\underbrace{x_{11}, \cdots, x_{88}}_{64}) = \sum_{j=1}^{8}\left(\sum_{i=1}^{8} x_{ij} - 1\right)^2$$

For diagonal conflics.



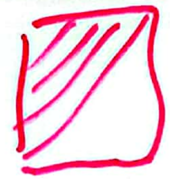
$x_{i1}, x_{i-1,2}, x_{i-2,3}, \cdots, x_{1,i}$

↓ Among these neurons

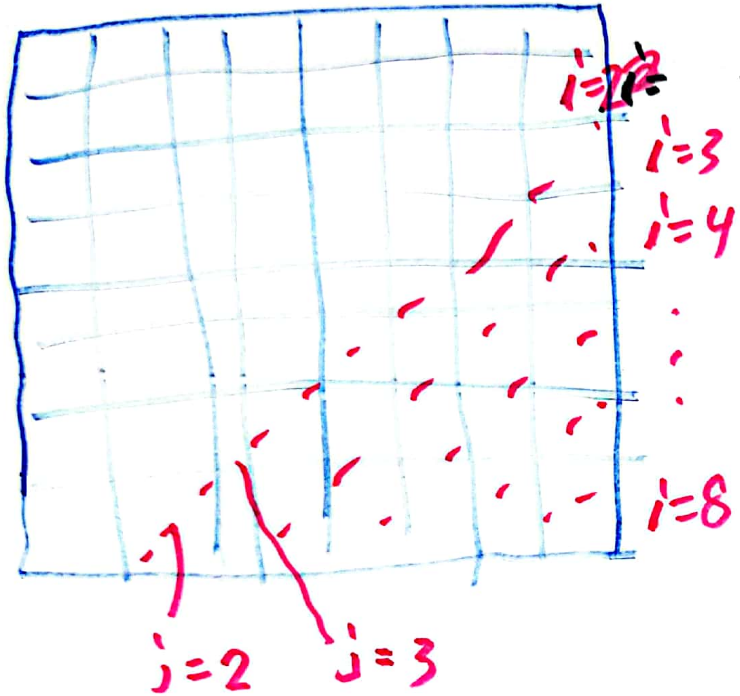
Only one neuron can take 1 and other neurons take.

$$E_d(x_{11}, \cdots, x_{88}) = \sum_{i=1}^{8} \cdots$$

$$E_d(x_{11}, \dots x_{88}) = \sum_{i=1}^{8} \left(\sum_{k=1}^{i} x_{i-k+1,k} - 1 \right)^2$$

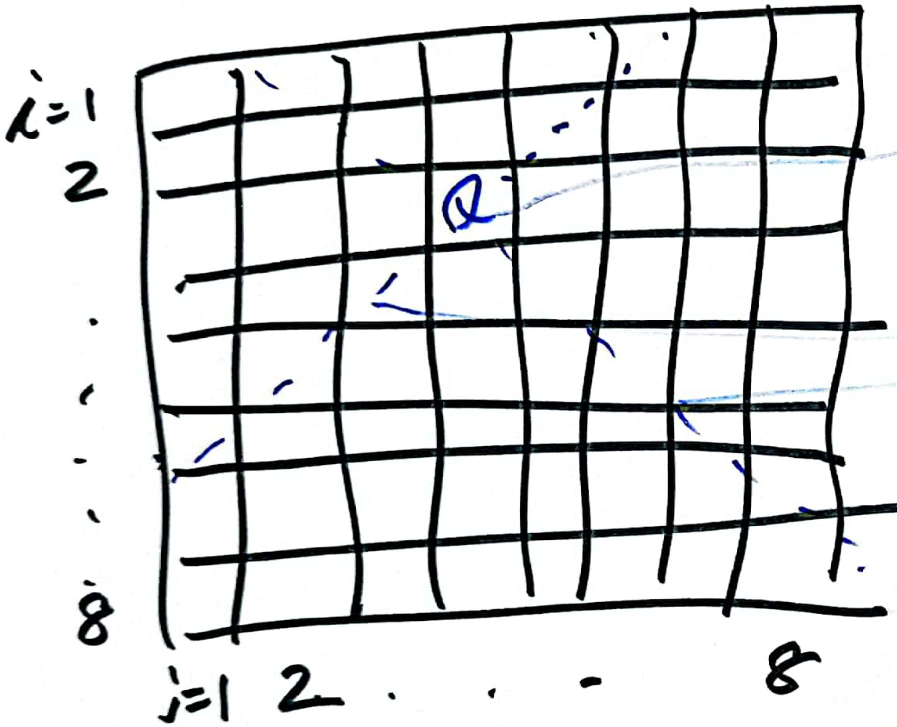


$$+ \sum_{i=2}^{8} \Big($$



2024. Jan. 12th

For diagonal conflicts among Queens, we use the following energy function.



If there is a Queen here, there should be no other Queens on these diagonal lines.

$$E_d = \sum_{i=1}^{8} \sum_{j=1}^{8} x_{i,j} \cdot \left(\sum_{d=-8}^{8} x_{i+d\, j+d} \right) + \sum_{i=1}^{8} \sum_{j=1}^{8} x_{i,j} \cdot \left(\sum_{d=-8}^{8} x_{i+d\, j-d} \right)$$

$1 \times 0 = 0$

Energy can be small, then

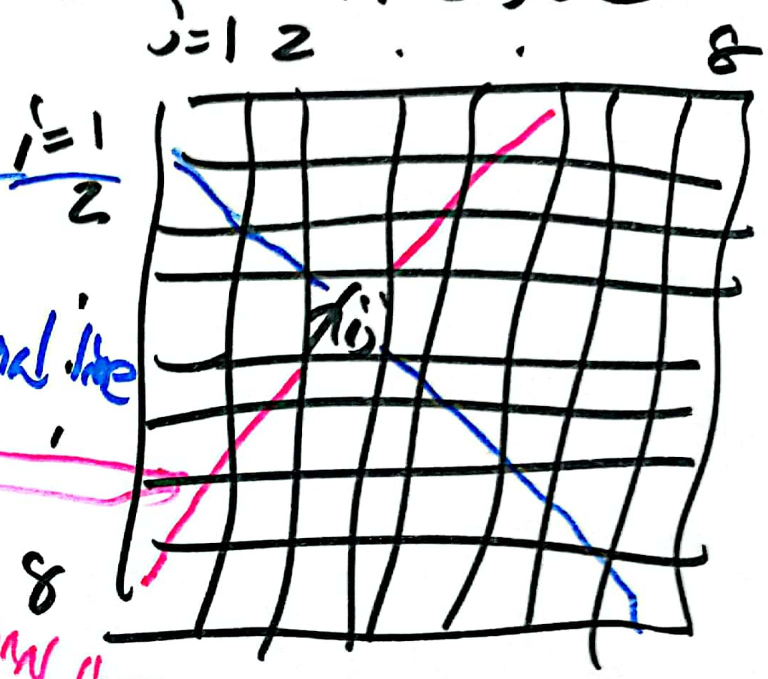
If there is a Queen on (i, j), it takes 1, these should take 0.

The range of summation needs to be adjusted for each (i, j)

$$E_d = \sum_{d=-8}^{8} x_{i+d\; j+d} + \sum_{\hat{d}=-8}^{8} x_{i+\hat{d}\; j-\hat{d}}$$

reprents the blue diagonal line

represent the red diagonal line



This summantion is conducted only for $1 \leqq i+d \leqq 8$ and $1 \leqq j+d \leqq 8$

This summantion is conducted only for $1 \leqq i+d \leqq 8$ and $1 \leqq j-d \leqq 8$

$E_d = 0$;

When you conduct $\sum_{d=-8}^{8} \underline{x_{i+d\ j+d}}$

Sum this value only when these conditions are satisfied.

For $d = 1 \sim 8$

$\{$ if $i+d \leqq 8$

if $i+d \geqq 1$

if $j+d \leqq 8$

if $j+d \geqq 1$

$E_d = E_d + x_{i+d\ j+d}$ $\}$

For $\hat{d} = 1 \sim 8$

$\{$ if $(i+\hat{d} \leqq 8)$ AND $(i+\hat{d} \geqq 1)$ AND $(j-\hat{d} \leqq 8)$

AND $(j-\hat{d} \geqq 1)$

$E_d = E_d + x_{i+\hat{d}\ j-\hat{d}}$ $\}$

The energy function of Eight Queen Problem

$$E = E_h + E_v + E_d$$

$$E_h = \sum_{i=1}^{8}\left(\sum_{j=1}^{8} x_{ij} - 1\right)^2$$

$$E_v = \sum_{j=1}^{8}\left(\sum_{i=1}^{8} x_{ij} - 1\right)^2$$

maybe $-2x_{ij}$

Top-right to bottom left

$$E_d = \sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij} * \left(\sum_{d=-8}^{8} x_{i+d\, j+d} + \sum_{\hat{d}=-8}^{8} x_{i+\hat{d}\, j-\hat{d}}\right)$$

~~only for d~~

only for d which is within this range

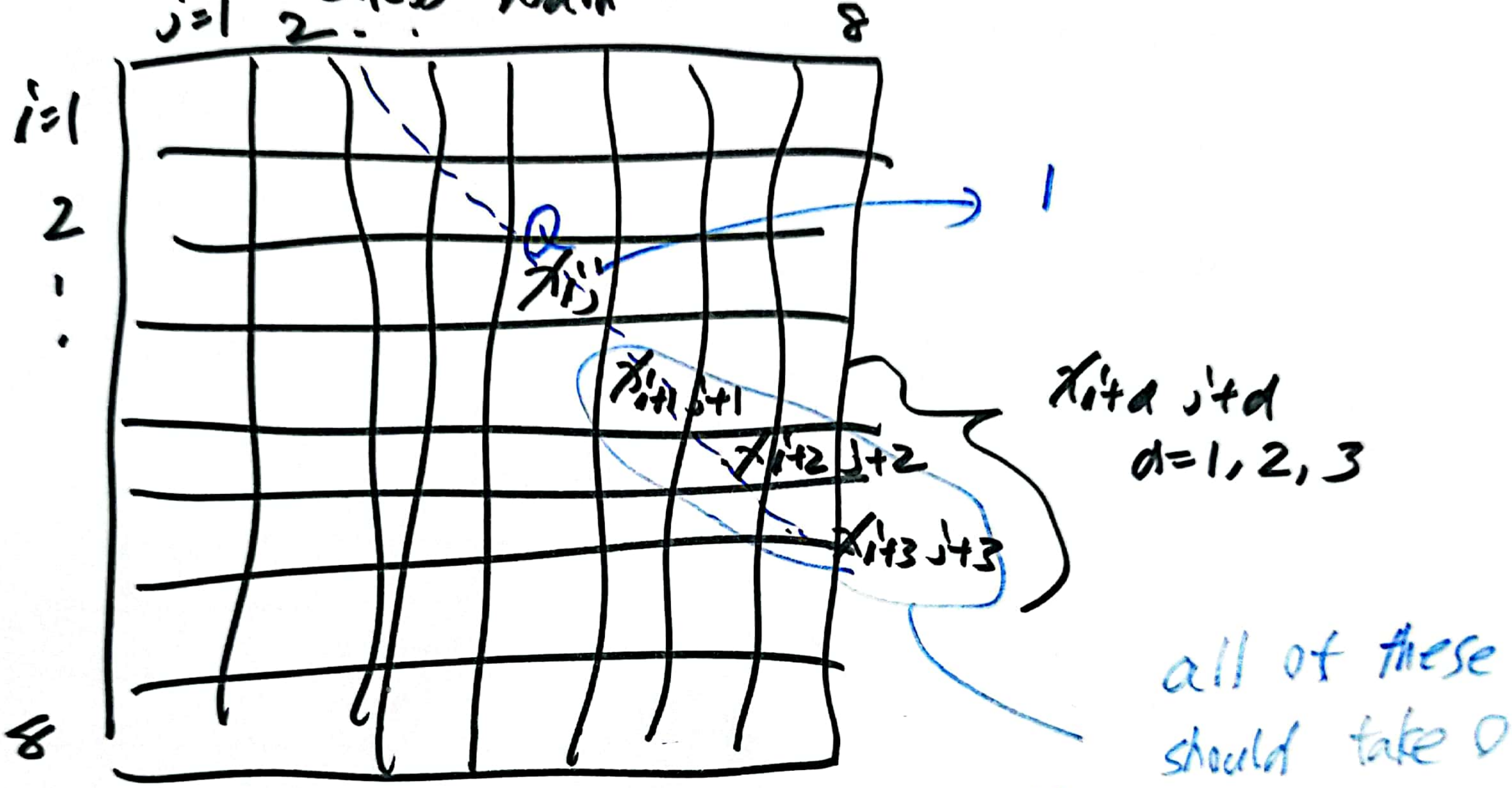
$$1 \leq i+d \leq 8$$
$$1 \leq j+d \leq 8$$

Top-left to bottom right

$$1 \leq i+\hat{d} \leq 8$$
$$1 \leq j-\hat{d} \leq 8$$

only for d which is within this range.

Chess board.

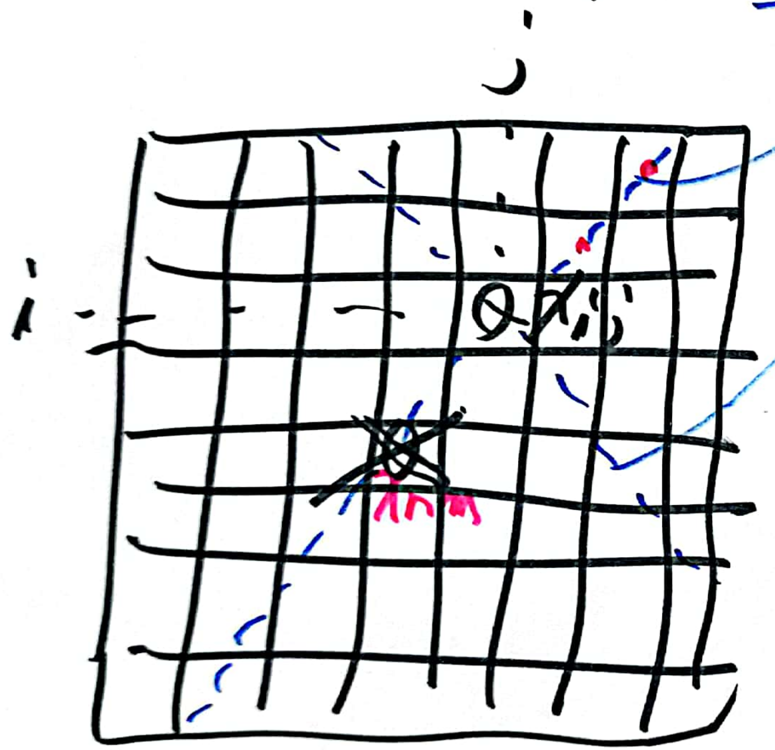


$$E_d = \dots + \underbrace{x_{ij}}_{1} * (\underbrace{x_{i+1\,j+1} + x_{i+2\,j+2} + \dots + x_{i+8\,j+8}})$$

By making these values 0, E_d takes a smaller value.

E_d : when $x_{ij} = 1$ (Queen is put on the cell (i, j))

there should be no Queen along the diagonal lines.



$x_{nm} = 0$ on these diagonal lines.

If you use the multiplicatin of $x_{ij} x_{nm}$

$x_{ij} * x_{nm}$

the energy can take a smaller value.

if $x_{nm} = 1$, then $x_{ij} * x_{nm} = 1$, the enery takes a larger value

when $x_{ij} = 1$, if $x_{nm} = 0$, This ~~been~~ becomes 0

Task 4-2

Update neurons for ~~Compute the energy value~~

~~for~~ the RNN with w_{ij}, θ_i and C

computed in Task 4-1

and find the solution of the simultaneous equation.

Use the deterministic model and the probabilistic model.

What happens if you change the gain.

Check the decrease of the energy and if there is a local minimum.

Please compute the histogram of occurance of states.

$$S = \sum_{d=-8}^{8} x_{i+d\; j\underset{(-)}{+}d}$$

$$\underbrace{\qquad\qquad}_{\substack{1 \leqq i+d \leqq 8 \\ 1 \leqq j\underset{(-)}{+}d \leqq 8}}$$

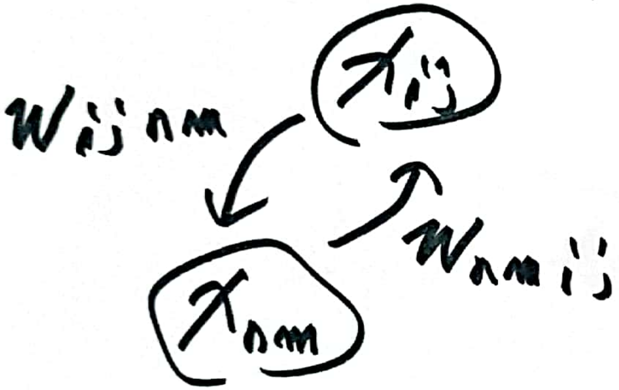
Please use such program as

$S = 0$

for $d = -8 \sim 8$

if $(1 \leqq i+d)$ AND $(i+d \leqq 8)$ AND $(1 \leqq j\underset{(-)}{+}d)$ AND $(j\underset{(-)}{+}d \leqq 8)$

then $S = S + x_{i+d\; j\underset{(-)}{+}d}$



Task 5-1

Compute W_{ijnm} and θ_{ij} and C for the energy of Eight Queen Problem.

$8 \times 8 \times 8 \times 8$ ⇒ many connection weights.

In your report just show a part of them.

Task 5-2

Use the deterministic model
and the probabilistic model,

check the decrease of the energy
for ~~deterministic~~ model,

Show the ~~histogram~~ of states

L there are two many states,

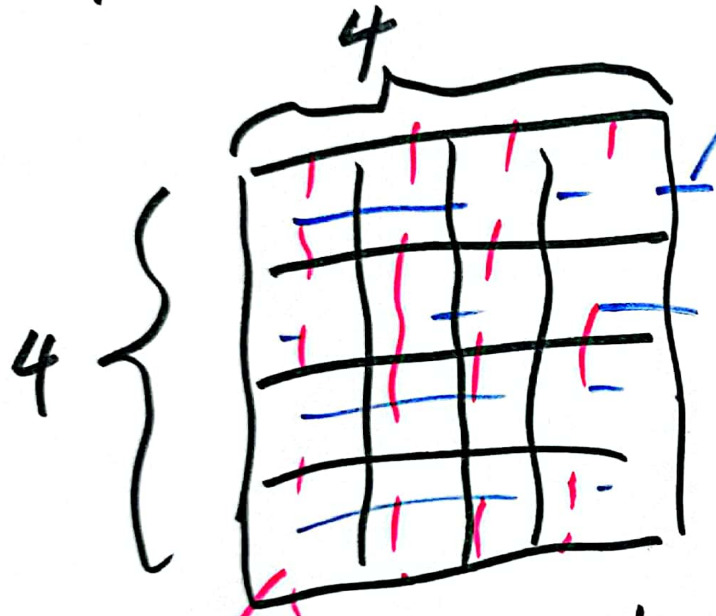
Just show the most frequently.

states which appears

In case you do not understand Eight Queen Problem,

please submit your report for Four Queen problem.

Horizontal conflicts.



4x4 cells

vertical conflics.

No diggonal conflicts.

put four Queens, on this 4x4 chess bord.

Avoid horizontal and vertical conflicts.

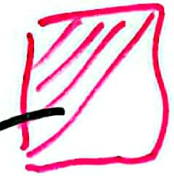
We do not consider about the diagonal conflicts.

Ignore the diagonal conflicts.

$$E_v = \sum_{j=1}^{4} \left(\sum_{i=1}^{4} x_{ij} - 1 \right)^2$$

$$E = E_h + E_v$$

$$E_h = \sum_{i=1}^{4} \left(\sum_{j=1}^{4} x_{ij} - 1 \right)^2$$

$$E_d(x_{11}, \ldots, x_{88}) = \sum_{i=1}^{8} \left(\sum_{k=1}^{i} x_{i-k+1,k} - 1 \right)^2 \rightarrow$$ 

$$+ \sum_{i=2}^{8} \left(\sum_{j=i}^{8} x_{8-j+1,j} - 1 \right)^2 \rightarrow$$