UNIT II – FOURIER SERIES

Definition:

Periodic function:

A function $f(x)$ is said to have a period T for all x, $f(x+T) = f(x)$, where T is a +ve constant. The least value of $T>0$ is called the period of $f(x)$.

e.g: $\sin x$, $\cos x$ are periodic function with period 2π.

Fourier series:

If $f(x)$ is a periodic function and satisfies Dirichlet conditions, then it can be represented by an infinite series called Fourier series.

Uses of Fourier series:

Fourier series are particularly suitable for expansion of periodic functions. we come across many periodic functions in voltage, current flex density; applied force, potential and electiomagnetic force in electricity, hence

Fourier series are very useful in electrical engineering problems.

Deduction: Sum of the Fourier Series

Continuous	Discontinuous	
	End point	Middle point
Substitute the value directly	Average values at end points	$\frac{L.H.S + R.H.S}{2}$

Problems:

① Sum of the Fourier series for

$$f(x) = \begin{cases} x^2, & -\pi \le x \le 0 \\ 0, & 0 \le x \le \pi \end{cases} \quad \text{at } x = \frac{\pi}{2}, \frac{-\pi}{2}.$$

Sol:

$x = \frac{\pi}{2}$ is a continuous point at $(0, \pi)$

$\therefore f(x) = 0.$

$x = -\frac{\pi}{2}$ is a continuous point at $(-\pi, 0)$

$f(x) = x^2$

$\Rightarrow f\left(\frac{-\pi}{2}\right) = \left(\frac{-\pi}{2}\right)^2 = \frac{\pi^2}{4}.$

$\therefore f(x) = \frac{\pi^2}{4}.$

(2) Sum the Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases} \quad \text{at } x=0.$$

Sol:

$x=0$ is a discontinuous and end point.

$$\text{Sum of the fourier series} = \frac{f(0)+f(2)}{2}$$

$$= \frac{0+2}{2} = \frac{2}{2}$$

$$\left.\text{Sum of the fourier series}\right\} = 1.$$

(3) Sum of the fourier series for

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ x^2, & \pi < x < 2\pi \end{cases} \quad \text{at } x=\pi.$$

Sol:

$x=\pi$ is a discontinuous and middle point.

$$\text{Sum of the fourier series} = \frac{f(\pi-)+f(\pi+)}{2}$$

$$= \frac{\pi+\pi^2}{2}.$$

Dirichlet condition:

i) $f(x)$ is periodic, single valued and finite

ii) $f(x)$ has a finite no. of finite discontinuous.

iii) $f(x)$ has no infinite discontinuous

iv) $f(x)$ has a finite no. of maxima and minima

Formula for fourier series in $(0, 2\pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\, dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx\, dx.$$

Problems:

① Expand $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2\, dx$$

$$= \frac{1}{\pi} \left(\frac{x^3}{3}\right)_0^{2\pi}$$

$$= \frac{8\pi^{\cancel{3}2}}{3\cancel{\pi}}$$

$$= \frac{8\pi^2}{3}.$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\, dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2 \cos nx\, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx\, dx.$$

Here $u = x^2$ $\quad \int dv = \int \cos nx$

$u_1 = 2x \quad V = \frac{\sin nx}{n}$

$u_2 = 2 \quad V_1 = -\frac{\cos nx}{n^2}$

$V_2 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2 \sin 2n\pi}{n} + \frac{4\pi\cos 2n\pi}{n^2} - \frac{2\sin 2n\pi}{n^3} - \frac{2\cdot 0\cos 2n\pi}{n^2}\right] \quad \left(\because \begin{array}{l}\sin 2n\pi = 0\\ \cos 2n\pi = 1.\end{array}\right.$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$= \frac{4}{n^2}.$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$u = x^2 \qquad \int dv = \int \sin nx$

$u_1 = 2x \qquad v = -\frac{\cos nx}{n}$

$u_2 = 2 \qquad v_1 = -\frac{\sin nx}{n^2} \qquad v_2 = \frac{\cos nx}{n^3}.$

$$b_n = \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2\cos 2n\pi}{n} + \frac{4\pi\sin 2n\pi}{n^2} + \frac{2\cos 2n\pi}{n^3}\right.$$

$$b_n = \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^2} - \frac{2}{n^3}\right]$$

$$b_n = -\frac{4\pi}{n}.$$

$$\therefore f(x) = \frac{8\pi^2}{3.2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx.$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx.$$

Deduction:

Put $x=0$ [end points discontinuous]

$$\frac{f(0)+f(2\pi)}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{0+4\pi^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{6\pi^2 - 4\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\frac{2\pi^2}{3.4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(2) Expand $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{\pi} \left[-\frac{(\pi - x)^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[+\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \frac{2\pi^3}{3}$$

$$= \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$u = (\pi - x)^2$ $\qquad \int dv = \int \cos nx$

$u_1 = -2(\pi - x)$ $\qquad v = \frac{\sin nx}{n}$

$$a_n = \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - \frac{2(\pi-x)\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2(\pi-2\pi)\cos 2n\pi}{n^2} + \frac{2\pi\cos 2n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{4\pi\cos 2n\pi}{n^2}\right) = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right)$$

$$= \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2 \sin nx\,dx.$$

$u = (\pi-x)^2$ $\quad \int dv = \int \sin nx$

$u_1 = -2(\pi-x)$ $\quad v = -\frac{\cos nx}{n}$

$u_2 = 2.$ $\quad v_1 = -\frac{\sin nx}{n^2}$; $v_2 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\left[-\frac{(\pi-x)^2\cos nx}{n} - \frac{2(\pi-x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(\pi-2\pi)^2\cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} + \frac{\pi^2\cos 0}{n} - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$b_n = 0.$

$$\therefore \quad f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

Deduction:

Put $x=0$ (end point & discontinuous)

$$\frac{f(0) + f(2\pi)}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{\pi^2 + \pi^2}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\pi^2 - \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{4\pi^2}{3 \cdot 4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Problems on $(0, 2\pi)$:

1. Expand $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

sol:

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{\cancel{2}} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left(\pi^2\right)$$

$$a_0 = \pi.$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$u = x$ $\quad \int dv = \int \cos nx\,dx$ $\quad u = (2\pi - x)$

$u_1 = 1$ $\quad V = \frac{\sin nx}{n}$ $\quad u_1 = -1.$

$u_2 = 0$ $\quad V_1 = \frac{-\cos nx}{n^2}$ $\quad u_2 = 0$

$$a_n = \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{2(-1)^n - 2}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx.$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$u = x$ $\quad \int dv = \int \sin nx \quad$ $u = (2\pi - x)$

$u_1 = 1$ $\quad v = -\frac{\cos nx}{n} \quad$ $u_1 = -1.$

$u_2 = 0$ $\quad v_1 = -\frac{\sin nx}{n^2} \quad$ $u_2 = 0$

$$b_n = \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi} + \left(-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$= 0.$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi} \cos nx.$$

Deduction:

Put $x=0$.

$$\pi = \frac{\pi}{2} + 4\sum_{n=odd}^{\infty} \frac{1}{\pi n^2}$$

$$\pi - \frac{\pi}{2} = \frac{4}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{2} \times \frac{\pi}{4} = \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

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(2) F.S of $f(x) = x \sin x$ & hence deduce that

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - + \cdots$$

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x\,dx$$

$u = x \qquad \int dv = \int \sin x$

$u_1 = 1 \qquad v = -\cos x$

$$a_0 = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\cos 2\pi\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-2\cancel{\pi}\right]$$

$$a_0 = -2.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$u = x$ $\quad dv = \int \sin(n+1)x\,dx$ $\quad \int dv = \int \sin(n-1)x\,dx$

$u_1 = 1$ $\quad v = -\frac{\cos(n+1)x}{n+1}$ $\quad v = -\frac{\cos(n-1)x}{n-1}$

$V_1 = -\frac{\sin(n+1)x}{(n+1)^2}$ $\quad V_1 = -\frac{\sin(n-1)x}{(n-1)^2}$

$$a_n = \frac{1}{2\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} - \left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 2(n+1)\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$$= \frac{\cancel{2\pi}}{\cancel{2\pi}}\left[\frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1}\right]$$

$$= \frac{2}{n^2-1} \quad \because (n \neq 1)$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \frac{\sin 2x}{2} \, dx$$

$u = x \qquad dv = \int \sin 2x \, dx$

$u_1 = 1 \qquad v = -\frac{\cos 2x}{2}$

$u_2 = 0 \qquad v_1 = -\frac{\sin 2x}{4}$

$$a_1 = \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi \cos 2\pi}{2}\right]$$

$$= \frac{1}{2}$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(n-1)x - \cos(n+1)x\right)dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x \, dx - \int_0^{2\pi} x\cos(n+1)x \, dx\right]$$

$$= \frac{1}{2\pi}\left[\left(x\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right)_0^{2\pi} - \left(\frac{x\sin(n+1)x}{} + \frac{\cos(n+1)x}{(n+1)^2}\right)_0^{2\pi}\right]$$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \frac{1}{(n+1)^2} - \cancel{\frac{1}{(n-1)^2}} + \frac{1}{(n+1)^2}\right]$$

$$b_n = 0.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}}\right]$$

$$= \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2}\right] = \frac{1}{2\pi}\cdot\frac{4\pi^2}{2}$$

$$b_1 = \pi.$$

$$\therefore\ f(x) = -\frac{2}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx - \frac{1}{2}\cos x + \pi\sin x$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx.$$

Deduction:

put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = -1 + \pi + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} + 1 - \pi = 2\sum_{n=2}^{\infty}\frac{1}{(n+1)(n-1)}\cos\frac{n\pi}{2}$$

$$\left(1-\frac{\pi}{2}\right)\cdot\frac{1}{2} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} \cos\frac{n\pi}{2}$$

$$\frac{1}{2}-\frac{\pi}{4} = \frac{1}{1\cdot 3}\cos\pi + \frac{1}{4\cdot 2}\cos\frac{3\pi}{2} + \frac{1}{5\cdot 3}\cos 2\pi + \cdots$$

$$\frac{1}{2}-\frac{\pi}{4} = \frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots$$

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi-2}{4}.$$

Formula for fourier series in $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\frac{n\pi x}{l} dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\frac{n\pi x}{l} dx.$$

Problems based on $(0, 2l)$.

1. Expand $f(x) = \begin{cases} l-x, & 0 < x \le l \\ 0, & l \le x \le 2l \end{cases}$ &

hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Sol:

$$a_0 = \frac{2}{2l} \int_0^l (l-x)\, dx$$

$$= \frac{1}{l} \left(lx - \frac{x^2}{2}\right)_0^l$$

$$= \frac{1}{l}\left(l^2 - \frac{l^2}{2}\right)$$

$$= \frac{l^2}{2l}$$

$$= \frac{l}{2}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx.$$

$u = l-x \qquad \int dv = \int \cos\frac{n\pi x}{l}\,dx$

$u_1 = -1 \qquad v = \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u_2 = 0$

$$v_1 = \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$$

$$a_n = \frac{1}{l}\left[(l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{1}{l}\left[-\frac{\cos\frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{\cos 0}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{1}{l}\left[-\frac{(-1)^n\cdot l^2}{n^2\pi^2} + \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{l^2}{l\cdot n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \frac{2l}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{2l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$u = l - x \qquad \int dv = \int \sin\frac{n\pi x}{l}\,dx$$

$$u_1 = -1 \qquad v = -\cos\frac{n\pi x}{l}\Big/ n\pi/l$$

$$v_1 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$$

$$b_n = \frac{1}{l}\left[-(l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{1}{l}\left[l\,\frac{\cos 0}{\frac{n\pi}{l}}\right] = \frac{1}{l}\left[l \cdot \frac{l}{n\pi}\right]$$

$$b_n = \frac{l}{n\pi}$$

$$f(x) = \frac{l}{4} + \sum_{n=odd}^{\infty} \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l}$$

Deduction:

Put $x = l$ (continuous)

$$0 = \frac{l}{4} + \sum_{n=odd}^{\infty}\frac{2l}{n^2\pi^2}\cos n\pi$$

$$-\frac{l}{4} = \frac{2l}{\pi^2}\sum_{n=odd}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{1}{4}\cdot\frac{\pi^2}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Put $x = \frac{l}{2}$ (continuous)

$$l - \frac{l}{2} = \frac{l}{4} + \sum_{n=odd}^{\infty} \frac{2l}{n^2\pi^2} \cos\frac{n\pi l}{2l} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin\frac{n\pi . l}{2l}$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2} \sum_{n=odd}^{\infty} \frac{\cos n\pi}{2} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi}{2}$$

$$\frac{l}{4} = \sum_{n=1}^{\infty} \frac{l}{n\pi} \frac{\sin n\pi}{2}$$

$$\frac{l}{4} \cdot \frac{\pi}{l} = \frac{\sin \pi/2}{1} + \frac{\sin 2\pi/2}{2} + \frac{\sin \frac{3\pi}{2}}{3} + \frac{\sin \frac{4\pi}{2}}{4}$$

$$\frac{\pi}{4} = 1 + 0 + \frac{\sin(\pi + \frac{\pi}{2})}{3} + 0 + \frac{\sin(2\pi + \frac{\pi}{2})}{5} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{\sin\frac{\pi}{2}}{5} + \cdots$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

(2) $f(x) = \begin{cases} x, & 0 \le x \le 3 \\ 6 - x, & 3 \le x \le 6 \end{cases}$

sol:

$$2l = 6 \Rightarrow l = \frac{6}{2} = 3$$

$$a_0 = \frac{2}{6-0} \int_0^6 f(x)\, dx.$$

$$a_0 = \frac{1}{3}\left[\int_0^3 x\,dx + \int_3^6 (6-x)\,dx\right]$$

$$= \frac{1}{3}\left[\left(\frac{x^2}{2}\right)_0^3 + \left(6x - \frac{x^2}{2}\right)_3^6\right]$$

$$= \frac{1}{3}\left[\frac{9}{2} + 36 - \cancel{\frac{36}{2}}^{18} - 18 + \frac{9}{2}\right]$$

$$= \frac{1}{3}\left[9\right]$$

$$a_0 = 3.$$

$$a_n = \frac{2}{6}\left[\int_0^3 x\cos\frac{n\pi x}{3}dx + \int_3^6 (6-x)\cos\frac{n\pi x}{3}dx\right]$$

$$= \frac{1}{3} \qquad \int dv = \int \cos\frac{n\pi x}{3}\,dx$$

$u = x \qquad v = \dfrac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} \qquad u = 6-x$

$u_1 = 1 \qquad\qquad\qquad\qquad u_1 = -1$

$\qquad\qquad v_1 = -\dfrac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \qquad u_2 = 0$

$u_2 = 0$

$$a_n = \frac{1}{3}\left[\left(x\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right)_0^3 + \left((6-x)\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}} - \frac{\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right)_3^6\right]$$

$$= \frac{1}{3}\left[\frac{(-1)^n 9}{n^2\pi^2} - \frac{9}{n^2\pi^2} - \frac{9}{n^2\pi^2} + \frac{(-1)^n 9}{n^2\pi^2}\right]$$

$$a_n = \frac{2 \cdot 9 \cdot 3}{\cancel{7} \cdot n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{6}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-12}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\, \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{6}\left[\int_0^3 x \sin\frac{n\pi x}{3}\, dx + \int_3^6 (6-x) \sin\frac{n\pi x}{3}\, dx\right]$$

$u = x \qquad \int dv = \int \sin\frac{n\pi x}{3}\, dx \qquad u = 6 - x$

$u_1 = 1 \qquad v = \frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} \qquad u_1 = -1$

$u_2 = 0 \qquad v_1 = \frac{-\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2} \qquad u_2 = 0$

$$b_n = \frac{1}{3}\left\{\left[-\frac{x\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} + \frac{\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right]_0^3 + \left[(6-x)\frac{\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}} - \frac{\sin\frac{\pi x n}{3}}{\left(\frac{n\pi}{3}\right)^2}\right]_3^6\right\}$$

$$= \frac{1}{3}\left[\frac{-3\,3(-1)^n}{\frac{n\pi}{3}} + 3\,\frac{(-1)^n \cdot 3}{n\pi}\right]$$

$$= \frac{1}{3}\left[0\right]$$

$$b_n = 0.$$

$$\therefore f(x) = \frac{3}{2} + \sum_{n=odd}^{\infty} \frac{-12}{n^2\pi^2} \cos\frac{n\pi x}{3}$$

Even function:

If $f(x)$ is an even function, then

$f(x) = f(-x) \Rightarrow b_n = 0.$

Ex: $\cos x,\ |x|,\ x^2.$

$f(x) = x^2 \Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x) = f(-x).$

$\therefore f(x)$ is an even function $\Rightarrow b_n = 0.$

Odd function:

If $f(x)$ is an odd function, then

$-f(x) = +f(-x).$

Ex: $f(x) = x^3,\ \sin x,\ x^3,\ x\cos x.$

$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Problems based on $(-\pi, \pi)$ & $(-l, l)$

First check whether the function is odd or even.

If the function is even using the fourier formula in the interval $(-\pi, \pi)$.

$f(x) = a_0 + \sum^{\infty} a_n \cos nx$ Here $b_n = 0.$

If the function is odd using the fourier formula in the interval $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx \, dx$

Here $a_0 = 0$ & $a_n = 0$.

If the function is even using the fourier formula in the interval $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_0 = \frac{2}{b-a} \int_{-l}^{l} f(x) \, dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{n\pi x}{l} \, dx.$$

If the function is odd using the function fourier formula in the interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{b-a} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{8}$$

Sol:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi} & , \ -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi} & , \ 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi} & , \ 0 \le x \le \pi \\ 1 + \frac{2x}{\pi} & , \ -\pi \le x \le 0 \end{cases}$$

$f(-x) = f(x)$.

$\therefore$ $f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} \left(1 \overset{-}{\cancel{+}} \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x \overset{-}{\cancel{+}} \frac{2x^2}{2\pi} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi \overset{-}{\cancel{+}} \frac{\pi^2}{\pi} \right] = \cancel{\frac{2}{\pi} \cdot 2\pi}$$

$a_0 = 0$.

$$a_n = \frac{2}{\ldots} \int^b f(x) \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

Here $u = 1-\frac{2x}{\pi}$ $\qquad \int dv = \int \cos nx\,dx$

$u_1 = -\frac{2}{\pi}$ $\qquad v = \frac{\sin nx}{n}$

$u_2 = 0$ $\qquad v_1 = -\frac{\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2\cos nx}{\pi n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - \frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$a_n = \frac{4}{\pi n^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \frac{8}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} \frac{8}{\pi n^2}\cos nx.$$

$$1 = \frac{8}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos nx.$$

$$\frac{\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{6^2} + \cdots$$

$f(x) = |\cos x|$.

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{8}.$$

Sol:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$f(-x) = f(x)$.

$\therefore f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x - \frac{2x^2}{2\pi} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi - \frac{\pi^2}{\pi} \right] = \cancel{\frac{2}{\pi} \cdot 2\pi}$$

$a_0 = 0$.

$$a_n = \frac{2}{\quad} \int^b f(x) \cos nx\,dx$$

$f(x)$ is even. $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x|\,dx = \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\left\{\int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{(\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right\}$$

$$= \frac{2}{\pi}\{2\} = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_{0}^{\pi/2} \cos x \cos nx\,dx - \int_{\pi/2}^{\pi} \cos x \cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(n+1)\pi/2}{n+1} + \frac{2\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin n\pi/2\cos\pi/2 + \cos n\pi/2\sin\pi/2}{n+1} + \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\frac{\cos n\pi/2}{n^2-1}\left[n-1-n-1\right]$$

$$a_n = \frac{-4}{\pi}\frac{\cos n\pi/2}{n^2-1}.$$

$$a_1 = \frac{2}{\ }\left\{\int^{\pi/2}\cos x\cos x\,dx - \int^{\pi}\cos x\cos x\,dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx$$

$$a_1 = \frac{2}{\pi}\left[\left(\frac{1}{2}x+\frac{\sin 2x}{4}\right)_0^{\pi/2} - \left(\frac{1}{2}x+\frac{\sin 2x}{4}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2}+\frac{\sin 2\pi/2}{4} - \frac{1}{2}\pi - \frac{\sin 2\pi}{4} + \frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin 2\cdot\pi/2}{4}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}-\frac{\pi}{2}+\frac{\pi}{4}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{4}-\frac{\pi}{2}\right]$$

$$a_1 = 0.$$

$$f(x) = \frac{4}{\pi\cdot 2} + \sum_{n=2}^{\infty}\frac{-4}{\pi}\cdot\frac{\cos n\pi/2}{n^2-1}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos n\pi/2}{n^2-1}.$$

Problems based on odd function:

1) Determine the fourier series for the functi

$$f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$$ & hence dedu

Given $f(x) = \begin{cases} x-1 & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$

$f(-x) = \begin{cases} -x-1 & -\pi < -x < 0 \\ 1-x & 0 < -x < \pi \end{cases}$

$= -\begin{cases} x+1 & 0 < x < \pi \\ x-1 & -\pi < x < 0 \end{cases}$

$f(-x) = -f(x)$

$f(x)$ is an odd function $\Rightarrow a_0 = 0,\ a_n = 0$

$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi} \int_0^{\pi} x+1 \ \sin nx \, dx$

$= \frac{2}{\pi} \left[-(x+1)\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$

$= \frac{2}{\pi} \left[-(\pi+1) \frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{1}{n} \right]$

$= \frac{2}{\pi} \left[-(\pi+1) \frac{(-1)^n}{n} + \frac{1}{n} \right]$

$b_n = \frac{2}{n\pi} \left[1 - (1+\pi)(-1)^n \right]$

2. Prove that $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} - \frac{\sin 3x}{3^3} + \dots$ in the interval $(-\pi, \pi)$.

Sol:

Let $f(x) = \frac{x(\pi^2 - x^2)}{12}$

$$f(-x) = (-x)\left(\frac{\pi^2 - (-x)^2}{12}\right)$$

$$= -x\left(\frac{\pi^2 - x^2}{12}\right)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Let the required fourier series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x(\pi^2 - x^2)}{12} \sin nx \, dx.$$

$u = x\pi^2 - x^3$ $\int dv = \int \sin nx$

$u_1 = \pi^2 - 3x^2$ $v = \frac{-\cos nx}{n}$

$u_2 = -6x$ $v_1 = \frac{-\sin nx}{n^2}$

$u_3 = -6$ $v_2 = \frac{\cos nx}{n^3}$

$$= \frac{2}{12\pi}\int_0^{\pi} (\pi^2 x - x^3) \sin nx \, dx$$

$$= \frac{1}{6\pi}\left[-(\pi^2 x - x^3)\frac{\cos nx}{n} + (\pi^2 - 3x^2)\frac{\sin nx}{n^2} - 6x\frac{\cos nx}{n^3} + 6\frac{\sin nx}{n^4}\right]_0^{\pi}$$

$$b_n = -\frac{\cos n\pi}{n^3}$$

$$= -\frac{(-1)^n}{n^3}$$

$$\therefore f(x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx.$$

$$\frac{x(\pi^2-x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots$$

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③ $f(x) = x^2+x$ in $(-\pi,\pi)$ & $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = x+x^2$$

$$f(-x) = -x+(-x)^2$$

$$= -x+x^2$$

$$f(x) \neq f(-x)$$

$$f(x) = -f(x)$$

$\therefore$ $f(x)$ is neither even nor odd.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x+x^2\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}+\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x+x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} + (1+2x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos n(-\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{n^2\pi}\left[\cancel{1}+2\pi-\cancel{1}+2\pi\right]$$

$$= \frac{(-1)^n 4\pi}{n^2\pi}$$

$$a_n = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x+x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + (1+2x)\frac{\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} + (-\pi+\pi^2)\frac{\cos n(-\pi)}{n} - \frac{2\cos n(-\pi)}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} - \frac{\pi^2 \cancel{(-1)^n}}{\cancel{n}} - \frac{\pi(-1)^n}{n} + \frac{\pi^2 \cancel{(-1)^n}}{\cancel{n}}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-2\cancel{\pi}(-1)^n}{n}\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ -\frac{2}{n} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\cancel{2}\pi^2}{3\cdot\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

If $\cancel{x=0}$, $x=\pi$

$$\pi+\pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos n\pi$$

$$\pi+\pi^2 - \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

If $x=\pi$ (discontinuous & end point)

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

$$\frac{\cancel{-\pi}+\pi^2+\cancel{\pi}+\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\cancel{2}\pi^2}{3\cdot\cancel{4}2} = \cancel{4}\cdot \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Half range Sine Series in the interval $(0,\pi)$ $(0,l)$

Formula : $(0,\pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

Formula: $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{n\pi x}{l}\,dx$$

Problems:

(1) $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Sol:

$$f(x) = x(\pi - x)$$
$$= x\pi - x^2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

$u = x\pi - x^2$ $dv = \sin nx$

$u_1 = \pi - 2x$ $v = -\frac{\cos nx}{n}$

$u_2 = -2$ $v_1 = -\frac{\sin nx}{n^2}$

$v_2 = \frac{\cos nx}{n^3}$

$$= \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\frac{\cos nx}{n} + (\pi - 2x)\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$b_n = \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= \begin{cases} \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin nx$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{\sin n\pi/2}{(2n-1)^3}$$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

1) Obtain the sine series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{l}{2} \\ l-x, & \frac{l}{2} \le x \le l \end{cases}$$

Sol:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\ } \int^{b} f(x) \sin nx \, dx$$

$u = x \quad dv = \sin\frac{n\pi x}{l} \quad u = l - x$

$u_1 = 1 \quad v = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \quad u_1 = -1$

$u_2 = 0 \quad v_1 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \quad u_2 = 0$

$$b_n = \frac{2}{l}\left[\left(-x\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)_0^{l/2} + \left(-(l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)_{l/2}^{l}\right]$$

$$\cancel{b_n = \frac{2}{l}\left[-\frac{l}{2}\frac{1}{\frac{n\pi}{l}} + \left(\frac{l}{2}\right)\frac{1}{\frac{n\pi}{l}}\right]}$$

$$b_n = \frac{2}{l}\left[-\frac{l^2}{2}\frac{\cos\frac{n\pi}{2}}{n\pi} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2\sin n\pi}{n^2\pi^2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left(\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

(3) $f(x) = (2-x)$ in $(0, 2)$.

Sol:

$$b_n = \frac{2}{2}\int_0^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$u = 2-x \qquad dv = \sin\frac{n\pi x}{2}$

$$v = -\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$$

$u_1 = -1 \qquad v_1 = -\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$

$$b_n = \left[-(2-x)\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= 2\,\frac{2}{n\pi}$$

$$= \frac{4}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\frac{n x\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[1-(-1)^n\right] \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi} \cos nx$$

---

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0,\pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\pi - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

Half range cosine series in the interval $(0,\pi), (0,l)$

Formula: $(0,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

Formula: $(0,l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

1. Find the half range cosine series of $f(x)=x$ in $(0,\pi)$.

Sol:

$$f(x) = x.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[1-(-1)^n\right]\cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd} \frac{-4}{n^2\pi}\cos nx$$

2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0,\pi)$.

Sol:

$$f(x) = x\pi - x^2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\pi - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\cos nx\, dx$$

$u = x\pi - x^2$ $\quad\int dv = \int \cos nx\, dx$

$u_1 = \pi - 2x$ $\quad v = \frac{\sin nx}{n}$

$u_2 = -2$ $\quad v_1 = -\frac{\cos nx}{n^2}$

$u_3 = 0$ $\quad v_2 = -\frac{\sin nx}{n^3}$.

$$a_n = \frac{2}{\pi}\left[(x\pi - x^2)\frac{\sin nx}{n} + (\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= \frac{-2}{n^2}\left[1 + (-1)^n\right]$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=\text{even}}^{\infty} \frac{-4}{n^2}\cos nx$$

---

③ Find the half range cosine series of $f(x) = (x-1)^2,\ 0 < x < 1$.

$$a_0 = \frac{2}{l}\int_0^1 f(x)\,dx$$

$$= 2\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[0+\frac{1}{1}\right]$$

$$a_0 = \frac{2}{3}.$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 (x-1)^2 \cos\frac{n\pi x}{l}\,dx$$

$u = (x-1)^2$ $\qquad \int dv = \int \cos\frac{n\pi x}{l}.\,dx$

$u_1 = 2(x-1)$ $\qquad v = \frac{\sin\frac{n\pi x}{l}}{n\pi}$

$u_2 = 2$ $\qquad v_1 = -\frac{\cos n\pi x}{n^2\pi^2}$

$\qquad v_2 = -\frac{\sin n\pi x}{n^3\pi^3}$

$$a_n = 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} + 2(x-1)\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right.$$

$$= 2\left[+\frac{2}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}$$

Formula: Complex form $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{where } c_n = \frac{1}{b-a}\int_a^b f(x)\, e^{-inx}\, dx.$$

Formula: complex form $(-l, l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$\text{where } c_n = \frac{1}{\cancel{b-a}\ 2l}\int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx.$$

① Find the complex form $f(x) = e^{ax}, -\pi < x < \pi$ in the form $e^{ax} = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2} e^{inx}$ & hence prove that $\frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$.

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{b-a}\int_a^b f(x)\, e^{-inx}\, dx.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int e^{(a-in)x}\, dx$$

$$C_n = \frac{1}{2\pi}\left[\frac{e^{(a-in)\pi}}{a-in} - \frac{e^{(a-in)(-\pi)}}{a-in} \right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi} \right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i\sin n\pi) \right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n \right]$$

$$= \frac{(-1)^n}{2\pi(a-in)}\left(e^{a\pi} - e^{-a\pi} \right)$$

$$= \frac{(-1)^n}{\pi(a-in)} \frac{\sinh a\pi}{a+in} \cdot a+in$$

$$C_n = \frac{(-1)^n (a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)\sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)\sinh a\pi}{\pi(a^2+n^2)} e^{inx}.$$

Put $x=0$.

$$1 = \sum^{\infty} \frac{(-1)^n (a+in)\sinh a\pi}{}$$

Equating the real parts, we get

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{a^2+n^2}$$

$$\therefore \frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

2. $f(x) = e^{ax}$ in $(-l, l)$

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{inx\pi}{l}} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{ax} e^{-\frac{in\pi x}{l}} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(a - \frac{in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{a - \frac{in\pi}{l}}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{\frac{al - in\pi}{\cancel{l}}\cdot l}}{a - \frac{in\pi}{l}} - \frac{e^{\frac{al - in\pi}{\cancel{l}}\cdot(-l)}}{a - \frac{in\pi}{l}}\right]$$

$$= \frac{1}{\quad}\left[al - in\pi \quad -al \quad in\pi \right]$$

$$C_n = \frac{1}{2l\frac{(al-in\pi)}{l}}\left[e^{al}\cdot(-1)^n - e^{-al}(-1)^n\right]$$

$$= \frac{(-1)^n}{2(al-in\pi)}\left[e^{al} - e^{-al}\right]$$

$$C_n = \frac{(-1)^n}{(al-in\pi)}\sinh al$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh al}{(al-in\pi)} e^{+\frac{in\pi x}{l}}$$

---

③ Find the complex form of the fourier series of $f(x) = e^{-x}$ in $-1<x<1$.

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \qquad 2l=2,\ l=1.$$

$$C_n = \frac{1}{b-a}\int_a^b f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{-\frac{in\pi x}{1}}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-\left(1+\frac{in\pi}{1}\right)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{\ldots}\right]_{-1}^{1}$$