ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY
Near Anjugramam Junction, Kanyakumari Main Road, Palkulam, Variyoor P.O - 629401
Kanyakumari Dist, Tamilnadu., E-mail : admin@rcet.org.in, Website : www.rcet.org.in

ENGINEERING MATHEMATICS -II

SUBJECT CODE: MA8251

(Regulation 2017)
Common to all branches of B.E

UNIT-V

LAPLACE TRANSFORM

UNIT-V

LAPLACE TRANSFORM

5.1 INTRODUCTION

A transformation is an operation which converts a mathematical expression to a different but equivalent form. The well known transformation logarithms reduce multiplication and division to a simpler process of addition subtraction.

The Laplace transform is a powerful mathematical technique which solves linear equations with given initial conditions by using algebra methods. The Laplace transform can also be used to solve systems of differential equations, Partial differential equations and integral equations. In this chapter, we will discuss about the definition, properties of Laplace transform and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.2 LAPLACE TRANSFORM

Let $f(t)$ be a function of t defined for all $t \geq 0$ .then the Laplace transform of $f(t)$, denoted by $L[\,f(t)]$ is defined by

$$L[\,f(t)] = \int_0^\infty e^{-st}\,f(t)dt$$

Provided that the integral exists, "s" is a parameter which may be real or complex. Clearly $L[\,f(t)]$is a function of s and is briefly written as $F(s)$ $(i.e.)$ $L[\,f(t)] = F(s)$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st}f(t)$ is a finite quantity, where $s > 0$(exists).

Example: 5. 1 Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution:

$$\lim_{t\to\infty} e^{-st}\,e^{t^3} = \lim_{t\to\infty} e^{-st+t^3} = \lim_{t\to\infty} e^{t^3-st}$$

$$= e^{\infty} = \infty, \text{ not a finite quantity.}$$

Hence $f(t) = e^{t^3}$is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$

ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: 5.2 Prove that Laplace transform of e^{t^2} does not exist.

Solution:

$$\lim_{t\to\infty} e^{-st}\, e^{t^2} = \lim_{t\to\infty} e^{-st+t^2} = \lim_{t\to\infty} e^{t^2-st}$$

$$= e^{\infty} = \infty \text{ ,not a finite quantity.}$$

$\therefore e^{t^2}$ is not of exponential order.

Hence Laplace transform of e^{t^2} does not exist.

5.3 PROPERTIES OF LAPLACE TRANSFORM

Property: 1 Linear property

$\boldsymbol{L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]}$ **, where a and b are constants.**

Proof:

$$L[af(t) \pm bg(t)] = \int_0^\infty [af(t) \pm bg(t)]\, e^{-st} dt$$

$$= a\int_0^\infty f(t) e^{-st} dt \pm b\int_0^\infty g(t)\, e^{-st} dt$$

$$\boxed{L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]}$$

Property: 2 Change of scale property.

If $\boldsymbol{L[f(t)] = F(s)}$, **then** $\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$; $\boldsymbol{a} > 0$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt \cdots\cdots (2)$$

Put at= x $ie., t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$(2) \Rightarrow L[f(at)] = \int_0^\infty e^{\frac{-sx}{a}} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{\frac{-sx}{a}} f(x) dx$$

Replace x by t, $L[f(at)] = \frac{1}{a} \int_0^\infty e^{\frac{-st}{a}} f(t) dt$

$$\boxed{\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)} \text{ ; } \boldsymbol{a} > 0}$$

Property: 3 First shifting property.

If $\boldsymbol{L[f(t)] = F(s)}$, **then i)** $\boldsymbol{L[e^{-at} f(t)] = F(s+a)}$

ii) $\boldsymbol{L[e^{at} f(t)] = F(s-a)}$

Proof:

(i) $L[e^{-at} f(t)] = F(s+a)$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st}\, f(t)\, dt = F(s) \cdots (1)$$

By the definition of Laplace transform, we have

$$L[e^{-at} f(at)] = \int_0^\infty e^{-st}\, e^{-at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, f(t)\, dt$$

$$= F(s+a) \quad \text{by (1)}$$

(ii) $L[e^{at} f(at)] = \int_0^\infty e^{-st}\, e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t}\, f(t)\, dt$$

$$= F(s-a) \quad \text{by (1)}$$

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st}\, f'(t)\, dt = \int_0^\infty u dv$$

$$= [uv]_0^\infty - \int u dv$$

$$= [e^{-st}\, f(t)]_0^\infty - \int_0^\infty f(t)\,(-s)e^{-st}\, dt$$

$$= 0 - f(0) + sL[f(t)]$$

$$= sL[f(t)] - f(0)$$

$$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)}$$

$u = e^{-st}$

$\therefore du = -se^{-st}dt$

$dv = f'(t)dt$

$\therefore v = \int f'(t)dt$

$= f(t)$

Property: 5 Laplace transform of derivative of order n

$$\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1}\, f(0) - s^{n-2} f'(0) \cdots - s^{n-3}\, f''(0) - \cdots f^{n-1}\,(0)}$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \cdots\cdots (1)$

$$L[f^n(t)] = L[[f'(t)]']$$

$$= sL[f'(t)] - f'(0)$$

$$= s[sL[f(t)] - f(0)] - f'(0)$$

$$= s^2 L[f(t)] - sf(0) - f'(0)$$

Similarly, $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

In general, $\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1}\, f(0) - s^{n-2} f'(0) \cdots - s^{n-3}\, f''(0) - \cdots f^{n-1}\,(0)}$

Laplace transform of integrals

Theorem: 1 If$L[f(t)] = F(s)$, then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)dt$

$\therefore g'(t) = f(t)$

And $g(0) = \int_0^0 f(t)dt = 0$

Now$L[g'(t)] = L[f(t)]$

$sL[g(t)] - g(0) = L[f(t)]$

$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$

$L[g(t)] = \frac{L[f(t)]}{s}$

$\therefore \boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}}$

Theorem: 2 If$L[f(t)] = F(s)$, then $L[tf(t)] = -\frac{d}{ds}F(s)$

Proof:

Given $L[f(t)] = F(s)$

$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$

Differentiating (1) with respect to s, we get

$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$

$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\, dt = \frac{d}{ds}F(s)$

$\int_0^\infty (-t)e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$

$-\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$

$-L[tf(t)] = \frac{d}{ds}F(s)$

$\therefore \boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$

Note: In general $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

Example: 5.3 If $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$ then find $L[f(2t)]$.

Solution:

Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$L[f(2t)] = \frac{1}{2}F\left(\frac{s}{2}\right)$

$= \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$

$= \frac{1}{2}\frac{\left[\frac{s^2}{4} - \frac{s}{2}+4\right]}{(s+1)^2\left(\frac{s-2}{2}\right)}$

$= \frac{s^2-2s+1}{4(s+1)^2(s-2)}$

Laplace transform of some Standard functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$\therefore \boldsymbol{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$ $\qquad \because \int_0^\infty e^{-u} u^n\, du$

Let $st = u \cdots\cdots (1)$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

When $t \to 0 (1) => u \to 0$

,

$t \to \infty, (1) => u \to \infty$

Note: If n is an integer, then $\Gamma(n+1) = n!$

$\therefore L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer

If $n = 0$, then $L[1] = \frac{1}{s}$

If $n = 1$, then $L[t] = \frac{1}{s^2}$

Similarly $L[t^2] = \frac{2!}{s^3}$

$L[t^3] = \frac{3!}{s^4}$

Result: 2 Prove that $\boldsymbol{L(e^{at}) = \frac{1}{s-a}}$, $\boldsymbol{s > a}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s-a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $\boldsymbol{L(e^{-at}) = \frac{1}{s+a}}$, $\boldsymbol{s > a}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s+a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[sinat] = \frac{a}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[sinat] = \int_0^\infty e^{-st}\, sinat\, dt$$

$\therefore L[sinat] = \frac{a}{s^2+a^2}, s > |a|$ $\left[\because \int_0^\infty e^{-at}\, sinbt\, dt = \frac{b}{a^2+b^2}\right]$

Result: 5 Prove that $L[cosat] = \frac{s}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[cosat] = \int_0^\infty e^{-st}\, cosat\, dt$$

$\therefore L[cosat] = \frac{s}{s^2+a^2}, s > |a|$ $\because \int_0^\infty e^{-at}\, cosbt\, dt = \frac{a}{a^2+b^2}$

Result: 6 Prove that $L[sinhat] = \frac{a}{s^2-a^2}$ $, s > |a|$

Proof:

We have $L[sinhat] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$\therefore L[sinhat] = \frac{a}{s^2-a^2}$ $, s > |a|$

Result: 7 Prove that $L[coshat] = \frac{s}{s^2-a^2}$ $, s > |a|$

Proof:

We have $L[coshat] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$\therefore L[coshat] = \frac{s}{s^2-a^2}$ $, s > |a|$

Example: 5.4 Find $L\left[t^{\frac{1}{2}}\right]$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = \frac{1}{2}$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

Example: 5.5 Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\frac{1}{2}$

$$\therefore L\left[t^{-\frac{1}{2}}\right] = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$ $L[t] = \frac{1}{s^2}$ $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ **if n is not an integer** $L[t^n] = \frac{n!}{s^{n+1}}$ **if n is an integer** $L(e^{at}) = \frac{1}{s-a}$ $L(e^{at}) = \frac{1}{s+a}$	$L[sinat] = \frac{a}{s^2+a^2}$ $L[cosat] = \frac{s}{s^2+a^2}$ $L[coshat] = \frac{s}{s^2-a^2}$ $L[sinhat] = \frac{a}{s^2-a^2}$

Problems using Linear property

Example: 5.6 Find the Laplace transform for the following

i.	$3t^2 + 2t + 1$	v.	$sin\sqrt{2}\,t$	ix.	$sin^2 t$
ii.	$(t+2)^3$	vi.	$sin(at+b)$	x.	$cos^2 2t$
iii.	a^t	vii.	$cos^3 2t$	xi.	$cos5tcos4t$

iv. e^{2t+3}	**viii.** sin^3t	

Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$L[f(t)] = L[3t^2 + 2t + 1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= 3L[t^2] + 2L[t] + L[1]$$
$$= 3\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t+2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$L[f(t)] = L[t^3 + 3t^2(2) + 3t2^2 + 2^3]$$
$$= L[t^3] + L[6t^2] + L[12t] + L[8]$$
$$= L[t^3] + 6L[t^2] + 12L[t] + 8L[1]$$
$$= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}$$

(iii) Given $f(t) = a^t$

$$L[f(t)] = L[a^t] = L[e^{tloga}]$$
$$L[a^t] = \frac{1}{s - loga}$$

(iv)Given $f(t) = e^{2t+3}$

$$L[f(t)] = L[e^{2t+3}] = L[e^{2t}.e^3]$$
$$= e^3L[e^{2t}]$$
$$= e^3\left[\frac{1}{s-2}\right]$$
$$\therefore L[e^{2t+3}] = e^3\left[\frac{1}{s-2}\right]$$

(v) $L\left[sin\sqrt{2}t\right] = \frac{\sqrt{2}}{s^2+2}$

(vi)Given $f(t) = \sin(at + b) = sinatcosb + cosatsinb$

$$L[f(t)] = L[\sin(at + b)]$$
$$= L[sinatcosb + cosatsinb]$$
$$= cosb\ L[\mathrm{sin}at] + sinb\ L[\mathrm{cos}at]$$
$$L[\sin(at + b)] = cosb\frac{s}{s^2+a^2} + sinb\ \frac{s}{s^2+a^2}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3cos2t + cos6t]$

$$\boxed{\because\ \cos^3\theta = \frac{3cos\theta + cos3\theta}{4}}$$

$$L[f(t)] = \frac{1}{4}L[3cos2t + cos6t]$$
$$= \frac{1}{4}[3L(cos2t) + L(cos6t)]$$
$$= \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

$$L[\cos^3 2t] = \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3sint - sin3t]$

$$L[f(t)] = \frac{1}{4}L[3sint - sin3t]$$

$$= \frac{1}{4}[3L(sint) - L(sin3t)]$$

$$= \frac{1}{4}\left[3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-cos2t}{2}$

$$L[f(t)] = L\left[\frac{1-cos2t}{2}\right]$$

$$= \frac{1}{2}[L(1) - L(cos2t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

(x) Given $f(t) = \cos^2 2t = \frac{1+cos4t}{2}$

$$L[f(t)] = L\left[\frac{1+cos4t}{2}\right]$$

$$= \frac{1}{2}[L(1) + L(cos4t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

(xi) Given $f(t) = cos5tcos4t$

$$L[f(t)] = L[cos5tcos4t]$$

$$= \frac{1}{2}[L(cos9t) + L(cost)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+81} + \frac{s}{s^2+1}\right]$$

Problems using First Shifting theorem

$$\boxed{\begin{aligned} L[e^{-at}f(t)] &= L[f(t)]_{s\to s+a} \\ L[e^{at}f(t)] &= L[f(t)]_{s\to s-a} \end{aligned}}$$

Example: 5.7 Find the Laplace transform for the following:

i. te^{-3t}	**vii.** $t^2 2^t$
ii. $t^3 e^{2t}$	**viii.** $t^3 2^{-t}$
iii. $e^{4t}sin2t$	**ix.** $e^{-2t}sin3tcos2t$
iv. $e^{-5t}cos3t$	**x.** $e^{-3t}cos4tcos2t$
v. $sinh2tcos3t$	**xi.** $e^{4t}cos3tsin2t$

vi. cosh3tsin2t	

(i) $\boldsymbol{te^{-3t}}$

$$L[te^{-3t}] = L[t]_{s\to s+3}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s+3} \qquad \because L(t) = \frac{1}{s^2}$$

$$\therefore L[te^{-3t}] = \frac{1}{(s+3)^2}$$

(ii) $\boldsymbol{t^3e^{2t}}$

$$L[t^3e^{2t}] = L[t^3]_{s\to s-2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s-2} \qquad \because L(t) = \frac{3!}{s^{3+1}}$$

$$\therefore L[t^3e^{2t}] = \frac{6}{(s-2)^4}$$

(iii) $\boldsymbol{e^{4t}sin2t}$

$$L[e^{4t}sin2t] = L[sin2t]_{s\to s-4}$$

$$= \left(\frac{2}{s^2+2^2}\right)_{s\to s-4}$$

$$= \frac{2}{(s-4)^2+4}$$

$$= \frac{2}{s^2-8s+16+4}$$

$$\therefore\ L[e^{4t}sin2t] = \frac{2}{s^2-8s+20}$$

(iv) $\boldsymbol{L[e^{-5t}cos3t]}$

$$L[e^{-5t}cos3t] = L[cos3t]_{s\to s+5}$$

$$= \left(\frac{s}{s^2+3^2}\right)_{s\to s+5}$$

$$= \frac{s+5}{(s+5)^2+9}$$

$$= \frac{s+5}{s^2+10s+25+9}$$

$$\therefore\ L[e^{-5t}cos3t] = \frac{s+5}{s^2+10s+34}$$

(v) $\boldsymbol{L[sinh2tcos3t]}$

$$L[sinh2tcos3t] = L\left[\left(\frac{e^{2t}-e^{-2t}}{2}\right)cos3t\right]$$

$$= \frac{1}{2}[L(e^{2t}cos3t) - L(e^{-2t}cos3t)]$$

$$= \frac{1}{2}[L(cos3t)_{s\to s-2} - L(cos3t)_{s\to s+2}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+3^2}\right)_{s\to s-2} - \left(\frac{s}{s^2+3^2}\right)_{s\to s+2}\right]$$

$$\therefore L[sinh2tcos3t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+9} - \frac{s+2}{(s+2)^2+9}\right]$$

(vi) $\boldsymbol{L[cosh3tsin2t]}$

$$L[cosh3tsin2t] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right)sin2t\right]$$

$$= \frac{1}{2}[L(e^{3t}sin2t) + L(e^{-3t}sin2t)]$$

$$= \frac{1}{2}[L(sin2t)_{s\to s-3} + L(sin2t)_{s\to s+3}]$$

$$= \frac{1}{2}\left[\left(\frac{2}{s^2+2^2}\right)_{s\to s-3} + \left(\frac{2}{s^2+2^2}\right)_{s\to s+3}\right]$$

$$\therefore\ L[cosh3tsin2t] = \frac{1}{2}\left[\frac{2}{(s-3)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

(vii) $\boldsymbol{t^2 2^t}$

$$L[t^2 2^t] = L\left[t^2 e^{log2^t}\right]$$

$$= L[t^2 e^{tlog2}] = L[t^2]_{s\to s-log2}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to s-log2}$$

$$= \frac{2}{(s-log2)^3}$$

$$\therefore L[t^2 2^t] = \frac{2}{(s-log2)^3}$$

(viii) $\boldsymbol{t^3 2^{-t}}$

$$L[t^3 2^{-t}] = L\left[t^3 e^{log2^{-t}}\right]$$

$$= L[t^3 e^{-tlog2}] = L[t^3]_{s\to s+log2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s+log2}$$

$$= \frac{6}{(s+log2)^4}$$

$$\therefore L[t^3 2^{-t}] = \frac{6}{(s+log2)^4}$$

(ix) $\boldsymbol{L[e^{-2t}sin3tcos2t]}$

$$L[e^{-2t}sin3tcos2t] = L[sin3tcos2t]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin(3t+2t) + \sin(3t-2t)]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin 5t + \sin t]_{s\to s+2}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) + L(\sin t)]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right]$$

$$\therefore L[e^{-2t}sin3tcos2t] == \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right]$$

(x) $\boldsymbol{L[e^{-3t}cos4tcos2t]}$

$$L[e^{-3t}cos4tcos2t] = L[cos4tcos2t]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos(4t+2t) + \cos(4t-2t)]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos 6\mathrm{t} + \cos 2t]_{s\to s+3}$$

$$= \frac{1}{2}[\mathrm{L}(\cos 6\,t) + L(\cos 2\,t)]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}\right]$$

$$\therefore L[e^{-3t}cos4tcos2t] = \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}\right]$$

(xi) $\boldsymbol{L[e^{4t}cos3tsin2t]}$

$$L[e^{4t}cos3tsin2t] = L[cos3tsin2t]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin(3t+2t) - \sin(3t-2t)]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin 5t - \sin t]_{s\to s-4}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) - L(\sin t)]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}\right]$$

$$\therefore L[e^{4t}cos3tsin2t] = \frac{1}{2}\left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}\right]$$

Exercise: 5.1

Find the Laplace transform for the following

1. $\cos^2 3t$ **Ans:** $\frac{1}{4}\left[\frac{3s}{s^2+9} + \frac{s}{s^2+81}\right]$

2. $sin3tcos4t$ **Ans:** $\frac{1}{4}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right]$

3. te^{2t} **Ans:** $\frac{1}{(s-2)^2}$

4. t^4e^{-3t} **Ans:** $\frac{4!}{(s-3)^5}$

5. $e^{4t}sin2t$ **Ans:** $\frac{2}{(s-4)^2+4}$

6. $e^{-5t}cos3t$ **Ans:** $\frac{s+5}{(s+5)^2+9}$

7. $t^3 3^t$ **Ans:** $\frac{3!}{(s-log3)^4}$

8. $t^5 4^{-t}$ **Ans:** $\frac{5!}{(s+log4)^6}$

9. $e^{-2t}sin3tcos2t$ **Ans:** $\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}$

10. $e^{-3t}cos4tcos2t$ **Ans:** $\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}$

11. $sinhtsin4t$ **Ans:** $\frac{4}{(s-1)^2+16} - \frac{4}{(s+1)^2+16}$

12. $cosh2tcos2t$ **Ans:** $\frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} - \frac{s+2}{(s+2)^2+4}\right]$

5.4 LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$L[tf(t)] = \frac{-d}{ds}L[f(t)]$$

Example: 5.8 Find the Laplace transform for $tsin4t$

Solution:

$$L[tsin4t] = \frac{-d}{ds}L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: 5.9 Find $L[tsin^2t]$

Solution:

$$L[tsin^2t] = \frac{-d}{ds}L[\sin^2 t] = \frac{-d}{ds}L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{4}{s(s^2+4)}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{1}{s(s^2+4)}\right]$$

$$= -2\left[\frac{0-(3s^2+4)}{(s^3+4s)^2}\right]$$

$$\therefore L[tsin^2t] = \frac{2(3s^2+4)}{(s^3+4s)^2}$$

Example: 5.10 Find $L[tcos^2 2t]$

Solution:

$$L[cos^2 2t] = \frac{-d}{ds}L[\cos^2 2t] = \frac{-d}{ds}L\left[\frac{(1+cos4t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) + L(cos4t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+16)1-s.2s}{(s^2+16)^2}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{s^2+16-2s^2}{(s^2+16)^2}\right]$$

$$\therefore L[cos^2 2t] = \frac{1}{2}\left[\frac{1}{s^2} - \frac{16-s^2}{(s^2+16)^2}\right]$$

Example: 5.11 Find the Laplace transform for $tsinh2t$

Solution:

$$L[sinh2t] = \frac{-d}{ds}L[sinh2t]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2-4}\right]$$

$$= \frac{-[(s^2-4)0-2(2s)]}{(s^2-4)^2}$$

$$\therefore L[tsinh2t] = \frac{4s}{(s^2-4)^2}$$

Example: 5.12 Find the Laplace transform for $f(t) = sinat - atcosat$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L(tcosat)}$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds}L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: 5.13 Find the Laplace transform for the following

(i) $te^{-3t}sin2t$ (ii) $te^{-t}cosat$ 9iii) $tsinhtcos2t$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore \quad L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4\,)^2}$$

(ii) $L[te^{-t}cosat] = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore\ L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2\,)^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^tcos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4\,)^2} + \frac{4-(s+1)^2}{((s+1)^2+4\,)^2}\right]$$

Problems using the formula

$$\boldsymbol{L[t^2f(t)] = \frac{d^2}{ds^2}L[f(t)]}$$

Example: 5.14 Find the Laplace transform for (i) t^2sint (ii) t^2cos2t

Solution:

(i) $L[t^2sint] = \frac{d^2}{ds^2}L[sint]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$$

$$= \frac{-2\left[(s^2+1)^2(1)-s(2)(s^2+1)(2s)\right]}{(s^2+1)^4}$$

$$= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4}$$

$$= \frac{-2[1-3s^2]}{(s^2+1)^3}$$

$$\therefore L[t^2sint] = \frac{6s^2-2}{(s^2+1)^3}$$

(ii) $L[t^2cos2t] = \frac{d^2}{ds^2}L[cos2t]$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$= \frac{\left[(s^2+4)^2(-2s)-(4-s^2)2(s^2+4)(2s)\right]}{(s^2+4)^4}$$

$$= \frac{2s(s^2+4)[(s^2+4)(-1)-(4-s^2)2]}{(s^2+4)^4}$$

$$= \frac{2s[s^2-12]}{(s^2+4)^3}$$

$\therefore L[t^2\cos 2t] = \frac{2s[s^2-12]}{(s^2+4)^3}$

Example: 5.15 Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \left[\frac{\left[(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= (s^2+1)\left[\frac{[(s^2+1)(-2s)-4s(1-s^2)]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+2}$$

$$= \left[\frac{2s^3-6s}{(s^2+1)^3}\right]_{s\to s+2}$$

$\therefore \quad L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$

(ii)$L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]_{s\to s-4} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= -6\left[\frac{\left[(s^2+9)^2(1)-(s)2(s^2+9)(2s)\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6(s^2+9)\left[\frac{[(s^2+9)-4s^2]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$$

$$\therefore\ L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Exercise: 5.2

Find the Laplace transform for the following

1. $tsinat$ **Ans:** $\frac{2as}{(s^2+a^2)^2}$
2. $tcosat$ **Ans:** $\frac{s^2-a^2}{(s^2+a^2)^2}$
3. $te^{-4t}sin3t$ **Ans:** $\frac{6(s+4)}{(s+4)^2+9}$
4. $tcos2tsin6t$ **Ans:** $\frac{8s}{(s^2+64)^2} - \frac{4s}{(s^2+16)^2}$
5. $te^{-2t}cos2t$ **Ans:** $\frac{(s-2)^2-4}{((s+4)^2+4)^2}$

Problems using the formula

$$\boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds}$$

This formula is valid if $\lim\limits_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = logs$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int \frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: 5.16 Find $\boldsymbol{L\left[\frac{cosat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

$\therefore$ Laplace transform does not exists.

Example: 5.17 Find $\boldsymbol{L\left[\frac{sinat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{sinat}{t} = \frac{sina(0)}{0} = \frac{0}{0}$$

$$= \lim_{t\to 0} acosat \qquad \text{(by applying L−Hospital rule)}$$

$\lim_{t\to 0} acosat = acos0 = a$, finite quantity.

Hence Laplace transform exists

$$L\left[\frac{sinat}{t}\right] = \int_s^\infty L[(sinat)]ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2} ds$$

$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{a}\right]$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{a}\right]$$

$$\therefore L\left[\frac{sinat}{t}\right] = cot^{-1}\frac{s}{a}$$

Example: 5.18 Find $L\left[\frac{sin^3 t}{t}\right]$

Solution:

$$\frac{sin^3 t}{t} = \frac{3sint - sin3t}{4t}$$

$$\lim_{t\to 0}\frac{sin^3 t}{t} = \lim_{t\to 0}\frac{3sint - sin3t}{4t}$$

$$= \frac{0-0}{0} = \frac{0}{0} \qquad \text{(by applying L}-\text{Hospital rule)}$$

$$= \lim_{t\to 0}\frac{3sint - sin3t}{4t} = 0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3 t}{t}\right] = L\left[\frac{3sint - sin3t}{4t}\right]$$

$$= \frac{1}{4}\int_s^\infty L[(3sint - sin3t)]ds$$

$$= \frac{1}{4}\int_s^\infty \left(3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right) ds$$

$$= \frac{1}{4}\left[3\tan^{-1} s - \tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{1}{4}\left[3(\tan^{-1}\infty - \tan^{-1} s) - \left(\tan^{-1}\infty - \tan^{-1}\frac{s}{3}\right)\right]$$

$$= \frac{1}{4}\left[(\frac{\pi}{2} - \tan^{-1} s) - \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)\ \right]$$

$$= \frac{1}{4}\left[\cot^{-1} s - cot^{-1}\frac{s}{3}\right]$$

Example: 5.19 Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{sin2tcos3t}{t}\right] = L\left[\frac{sin2tcos3t}{t}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1}s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - (\tan^{-1}\infty - \tan^{-1}s)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1}s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1}s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Example: 5.20 Find the Laplace transform for $\frac{e^{-at}-e^{-bt}}{t}$

Solution:

$$\lim_{t\to 0}\frac{e^{-at}-e^{-bt}}{t} = \lim_{t\to 0}\frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L– Hospital rule)}$$

$$= \lim_{t\to 0}\frac{-ae^{-at}+be^{-bt}}{1}$$

$$= -a+b = b-a = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L[e^{-at}-e^{-bt}]ds$$

$$= \int_s^\infty [L(e^{-at}) - L(e^{-bt})]ds$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[log\frac{s\left(1+\frac{a}{s}\right)}{s\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$$= log1 - log\frac{s+a}{s+b} = 0 - log\frac{s+a}{s+b} \qquad \because log1 = 0$$

$$= log\frac{s+a}{s+b}$$

Example: 5.21 Find the Laplace transform of $\frac{1-cost}{t}$

Solution:

$$\lim_{t\to 0}\frac{1-cost}{t} = \frac{0}{0} \qquad \lim_{t\to 0}\frac{sint}{1} = \frac{0}{1} = 0 \qquad \text{(use L– Hospital rule)}$$

$L\left[\frac{1-cost}{t}\right]$ exists.

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[(1-cost)]ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+1}\right]_s^\infty$$

$$= \left[log\, \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= 0 - log\, \frac{s}{\sqrt{s^2+1}}$$

$$= log\, \frac{\sqrt{s^2+1}}{s}$$

Example: 5.22 Find the Laplace transform for $\frac{cosat - cosbt}{t}$

Solution:

$$\lim_{t\to 0} \frac{cosat - cosbt}{t} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L– Hospital rule)}$$

$$= \lim_{t\to 0} \frac{-asinat + bsinbt}{1} = 0 = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty [L(cosat) - L(cosbt)]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[log\, \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\, \frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\, \frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log1 - log\, \frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\left[log\, \frac{s^2+a^2}{s^2+b^2}\right] \qquad [\because log1 = 0]$$

$$= \frac{1}{2}\left[log\, \frac{s^2+b^2}{s^2+a^2}\right]$$

Example: 5.23 Find the Laplace transform of $\frac{sin^2 t}{t}$

Solution:

$$\frac{sin^2 t}{t} = \frac{1 - cos2t}{2t}$$

$$\lim_{t\to 0} \frac{\mathbf{1 - cos2t}}{\mathbf{2t}} = \frac{0}{0}$$

$$\lim_{t\to 0} \frac{\mathbf{2sin2t}}{\mathbf{2}} = \frac{0}{1} = 0 \qquad \text{(use L– Hospital rule)}$$

Laplace transform exists.

$$L\left[\frac{sin^2t}{t}\right] = L\left[\frac{1-cos2t}{2t}\right] = \frac{1}{2}\int_s^\infty L[(1-cos2t)]ds$$
$$= \frac{1}{2}\int_s^\infty [L(1) - L(cos2t)]ds$$
$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$
$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$
$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$
$$= \frac{1}{2}\left[log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$
$$= \frac{1}{2}\left[0 - log\frac{s}{\sqrt{s^2+4}}\right]$$
$$= \frac{1}{2}log\frac{\sqrt{s^2+4}}{s}$$

Example: 5.24 Find the Laplace transform for $\frac{sin2tsin5t}{t}$

Solution:

$$L\left[\frac{sin2tsin5t}{t}\right] = \int_s^\infty L[sin2tsin5t]ds$$
$$= \int_s^\infty \frac{1}{2}[L(\cos(-3t) - L(cos7t)]ds$$
$$= \frac{1}{2}\int_s^\infty [L(\cos(3t) - L(cos7t)]ds \qquad [\because \cos(-\theta) = cos\theta]$$
$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+49}\right)ds$$
$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+9) - \frac{1}{2}\log(s^2+49)\right]_s^\infty$$
$$= \frac{1}{4}\left[log\frac{s^2+9}{s^2+49}\right]_s^\infty$$
$$= \frac{1}{4}\left[log\frac{s^2\left(1+\frac{9}{s^2}\right)}{s^2\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$
$$= \frac{1}{4}\left[log\frac{\left(1+\frac{9}{s^2}\right)}{\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$
$$= \frac{1}{4}\left[log1 - log\frac{s^2+9}{s^2+49}\right] = -\frac{1}{4}\left[log\frac{s^2+9}{s^2+49}\right] \qquad [\because log1 = 0]$$
$$= \frac{1}{4}\left[log\frac{s^2+49}{s^2+9}\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: 5.25 Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ (ii) $\int_0^t cos2tdt$ (iii) $\int_0^t tsin3tdt$ (iv) $t\int_0^t costdt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}\left[L[sin3t]\right]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[t\int_0^t costdt\right] = \frac{-d}{ds}L\left[\int_0^t costdt\right]$

$$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{2s}{(s^2+1)^2}$$

Example: 5.26 Find the Laplace transform for $e^{-t}\int_0^t tcos4tdt$

Solution:

$$L\left[e^{-t}\int_0^t tcos4tdt\right] = L\left[\int_0^t tcos4tdt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(cos4t)\right]_{s\to s+1}$$

$$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\ \frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t tcos4tdt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16\ )^2}\right]$$

Example: 5.27 Find the Laplace transform of $e^{-t}\int_0^t \frac{sint}{t}dt$

Solution:

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = L\left[\int_0^t \frac{sint}{t}dt\right]_{s\to s+1}$$

$$= \left[\frac{1}{s} L\left(\frac{sint}{t}\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty L(sint)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}[\tan^{-1} s]_s^\infty\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1} s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\frac{\pi}{2} - \tan^{-1} s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Exercise: 5.3

Find the Laplace transform of

1. $\frac{sint}{t}$ **Ans:** $\cot^{-1}\frac{s}{2}$
2. $e^{-2t}\frac{sint}{t}$ **Ans:** $cot^{-1}(s+2)$
3. $\frac{sinat-sinbt}{t}$ **Ans:** $\cot^{-1}\frac{s}{a} - \cot^{-1}\frac{s}{b}$
4. $\frac{e^{-at}-cosbt}{t}$ **Ans:** $log\frac{\sqrt{s^2+b^2}}{s+a}$
5. $\frac{1-e^{-t}}{t}$ **Ans:** $log\frac{s+1}{s}$
6. $e^{-t}\int_0^t \frac{sint}{t}dt$ **Ans:** $\frac{1}{s+1}cot^{-1}(s+1)$
7. $e^{-t}\int_0^t tcostdt$ **Ans:** $\frac{1}{s+1}\left[\frac{s^2+2s}{(s^2+2s+2)^2}\right]$
8. $e^{-t}\int_0^t te^{-t}sintdt$ **Ans:** $\frac{1}{s}\left[\frac{2(s+1)}{s^2+2s+2}\right]$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^\infty f(t)e^{-at}dt = [L[f(t)]]_{s=a}$

(iii) $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

Example: 5.28 If $L[f(t)] = \frac{s+2}{s^2+4}$, then find the value of $\int_0^\infty f(t)dt$

Solution:

Given $L[f(t)] = \frac{s+2}{s^2+4}$

We know that $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

$$= \left[\frac{s+2}{s^2+4}\right]_{s=0} = \frac{2}{4}$$

$$\int_0^\infty f(t)dt = \frac{1}{2}$$

Example: 5.29 If $L[f(t)] = \frac{5s+4}{s^2-9}$, then find the value of $\int_0^\infty e^{-2t}f(t)dt$

Solution:

Given $L[f(t)] = \frac{5s+4}{s^2-9}$

We know that $\int_0^\infty e^{-2t}f(t)dt = \left[L[f(t)]\right]_{s=2}$

$$= \left[\frac{5s+4}{s^2-9}\right]_{s=2} = \frac{14}{-5}$$

$$\therefore \int_0^\infty e^{-2t}f(t)dt = \frac{-14}{5}$$

Example: 5.30 Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}cos2tdt$ **(ii)** $\int_0^\infty t^2e^{-t}sintdt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right)dt$

Solution:

(i) $\int_0^\infty te^{-2t}cos2tdt = L[tcos2t]_{s=2} = \left[\frac{-d}{ds}L(cos2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2e^{-t}sintdt = L[t^2sint]_{s=1} = \frac{d^2}{ds^2}L[sint]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)^2(1)-s.2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)[(s^2+1)-4s^2)\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t}-e^{-2t}]ds]_{s=0}$

$$= \int_s^\infty [[L(e^{-t}) - L(e^{-2t})]\square s]_{s=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds\right]_{s=0}$$

$$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$$

$$= \left\{\left[log\,\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$$

$$= \left\{log\,\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}\Bigg|_s^\infty\right\}_{s=0}$$

$$= \left[0 - log\,\frac{s+1}{s+2}\right]_{s=0} \qquad \because log1 = 0$$

$$= \left[log\,\frac{s+2}{s+1}\right]_{s=0} = log2$$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt$

$$\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt = L\left[\frac{1-cost}{t}\right]_{S=1} = \int_s^\infty [L[(1-cost)]ds]_{S=1}$$

$$= \int_s^\infty [[L(1) - L(cost)]ds]_{S=1}$$

$$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds\right]_{S=1}$$

$$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[\log s - \log\sqrt{s^2+1}\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[log\,\frac{s}{\sqrt{s^2+1}}\right]_s^\infty\right\}_{S=1}$$

$$= \left[0 - log\,\frac{s}{\sqrt{s^2+1}}\right]_{S=1}$$

$$= \left[log\,\frac{\sqrt{s^2+1}}{s}\right]_{S=1}$$

$$= log\sqrt{2}$$

(v) $\int_0^\infty \left(\frac{e^{-at} - cosbt}{t}\right) dt$

$$\int_0^\infty \left(\frac{e^{-at} - cosbt}{t}\right) dt = L\left[\frac{e^{-at} - cosbt}{t}\right]_{S=0} = \int_s^\infty [L[(e^{-at} - cosbt)]ds]_{S=0}$$

$$= \int_s^\infty [[L(e^{-at}) - L(cosbt)]ds]_{S=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2}\right) ds\right]_{S=0}$$

$$= \left\{\left[\log(s+a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[\log(s+a) - \log\sqrt{s^2+b^2}\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[log\,\frac{s+a}{\sqrt{s^2+b^2}}\right]_s^\infty\right\}_{S=0}$$

$$= \left[0 - log\frac{s+a}{\sqrt{s^2+b^2}}\right]_{s=0}$$

$$= \left[log\frac{\sqrt{s^2+b^2}}{s+a}\right]_{s=0}$$

$$= log\frac{\sqrt{b^2}}{a}$$

$$= log\frac{b}{a}$$

Exercise: 5.4

Find the values of the following integrals using Laplace transforms

1. $\int_0^\infty te^{-2t}costdt$ **Ans:** $\frac{3}{25}$
2. $\int_0^\infty te^{-3t}sintdt$ **Ans:** $\frac{13}{250}$
3. $\int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right)dt$ **Ans:** $log\frac{b}{a}$
4. $\int_0^\infty e^{-2t}\frac{\sin^2 t}{t}dt$ **Ans:** $\frac{1}{4}log2$
5. $\int_0^\infty \left(\frac{cosat-cosbt}{t}\right)dt$ **Ans:** $log\frac{a}{b}$

Laplace transform of Piecewise continuous functions

$$\int_0^\infty \boldsymbol{f(t)e^{-st}dt = L[f(t)]}$$

Example: 5.31 Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{e^{-t}; 0 < t < \pi} \\ \boldsymbol{0\,; t > \pi} \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\pi e^{-st}e^{-t}dt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-(s+1)t}dt$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\pi = \frac{e^{-(s+1)\pi-e^0}}{-(s+1)}$$

$$\therefore L[f(t)] = \frac{1-e^{-(s+1)\pi}}{-(s+1)}$$

Example: 5.32 Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{sint; 0 < t < \pi} \\ \boldsymbol{0\,; t > \pi} \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} e^{-st}sintdt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-st}sintdt$$

$$= \left[\frac{\boldsymbol{e^{-st}}}{\boldsymbol{(-s)^2+1}}(-ssint - cost)\right]_{\boldsymbol{0}}^{\boldsymbol{\pi}} = \frac{e^{-s\pi}}{s^2+1}[-ssin\pi - cos\pi] - \frac{e^0}{s^2+1}[-ssin0 - cos0]$$

$$= \frac{e^{-s\pi}}{s^2+1}(0+1) - \frac{1}{s^2+1}(-1) = \frac{e^{-s\pi}+1}{s^2+1}$$

$\therefore L[f(t)] = \frac{e^{-s\pi}+1}{s^2+1}$

Example: 5.33 Find the Laplace transform of $f(t) = \begin{cases} t; 0 < t < 1 \\ 0\,; t > 1 \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^1 e^{-st}tdt + \int_1^\infty e^{-st}0dt$$

$$= \int_0^1 te^{-st}dt$$

$$= \left[t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{(-s)^2}\right]_0^1 = \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - 0 + \frac{1}{s^2}$$

$$\therefore L[f(t)] = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

Exercise: 5.5

1. Find the Laplace transform of $f(t) = \begin{cases} 0; 0 < t < 2 \\ 3\,; t > 2 \end{cases}$ **Ans:** $\frac{3e^{-2s}}{s}$

2. Find the Laplace transform of $f(t) = \begin{cases} e^t; 0 < t < 1 \\ 0\,; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-(s-1)}}{s-1}$

3. Find the Laplace transform of $f(t) = \begin{cases} 1; 0 < t < 1 \\ 0\,; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-s}}{s}$

Unit step function

The unit step function $U(t-a)$ is defined as $U(t-a) = \begin{cases} 0; t < a \\ 0\,; t > a \end{cases}$

Example: 5.34 Find the Laplace transform of unit step functions.

Solution:

$$L[U(t-a)] = \int_0^\infty U(t-a)e^{-st}dt$$

$$= \int_0^a 0dt + \int_a^\infty (1)e^{-st}dt = \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s}$$

$$L[U(t-a)] = \frac{e^{-sa}}{s}$$

Second Shifting theorem

Statement: If $L[f(t)] = F(s)$, then $L[f(t-a)U(t-a)] = e^{-as}F(s)$

Proof:

$$U(t-a)f(t-a) = \begin{cases} 0; t < a \\ f(t-a)\,; t > a \end{cases}$$

By the definition of Laplace transform,

$$L[U(t-a)f(t-a)] = \int_0^\infty U(t-a)f(t-a)e^{-st}dt$$

$$= \int_0^a 0dt + \int_a^\infty f(t-a)e^{-st}dt$$

$$L[U(t-a)f(t-a)] = \int_0^\infty e^{-s(a+x)}f(x)dx$$

$$= \int_0^\infty e^{-sa} e^{-sx} f(x) dx$$

$$= e^{-sa} \int_0^\infty e^{-sx} f(x) dx$$

Let $t - a = x \cdots (1)$

$t = a + x$

$dt = dx$

When $t = a, (1) => x = 0$

When $t = \infty, (1) => x = \infty$

Replace x by t

$$L[U(t-a)f(t-a)] = e^{-sa} \int_0^\infty e^{-st} f(t) dt$$

$$= e^{-sa} L[f(t)] = e^{-sa} F(s)$$

$$L[U(t-a)f(t-a)] = e^{-sa} F(s)$$

5.5 PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T) = f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t + 2\pi) = \sin(t + 4\pi) \cdots$

$\therefore$ $sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$\boxed{L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt}$$

Problems on Laplace transform of Periodic function

Example: 5.35 Find the Laplace transform of $f(t) = \begin{cases} \sin\omega t; 0 < t < \frac{\pi}{\omega} \\ 0; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Solution:

The given function is a periodic function with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} \sin\omega t e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} (0) dt \right]$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} \sin\omega t e^{-st} dt$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{(-s)^2 + \omega^2} (-s\sin\omega t - \omega\cos\omega t) \right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left\{ \frac{e^{\frac{-s\pi}{\omega}}}{s^2 + \omega^2} [-s\sin\pi - \omega\cos\pi] + \frac{\omega}{s^2 + \omega^2} \right\}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{\frac{-s\pi}{\omega}} \omega + \omega}{s^2 + \omega^2} \right]$$

$$= \frac{1}{1^2 - \left(e^{\frac{-\pi s}{\omega}}\right)^2} \left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}} + 1\right)}{s^2 + \omega^2} \right]$$

$$= \frac{1}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

Example: 5.36 Find the Laplace transform of $f(t) = \begin{cases} E; 0 \le t \le a \\ -E\,; a \le t \le 2a \end{cases}$ **given that** $f(t+2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[E\int_0^a e^{-st}dt - E\int_a^{2a} e^{-st}dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$

$$= \frac{E}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: 5.37 Find the Laplace transform of $f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1\,; \frac{a}{2} \le t \le a \end{cases}$ **given that** $f(t+a) = f(t)$.

Solution:

The given function is a periodic function with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} (1)e^{-st}dt + \int_{\frac{a}{2}}^a (-1)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt - \int_{\frac{a}{2}}^a e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{2a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because tanhx = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Example: 5.38 Find the Laplace transform of $f(t) = \begin{cases} t; 0 \le t \le a \\ 2a - t\,; a \le t \le 2a \end{cases}$ **given that** $f(t + 2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^a - \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Exercise: 5.6

1. Find the Laplace transform of

$f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1\,; \frac{a}{2} \le t \le a \end{cases}$ given that $f(t + a) = f(t)$. **Ans:** $\frac{k}{s}\tanh\left(\frac{as}{2}\right)$

2. Find the Laplace transform of

$f(t) = \begin{cases} t; 0 \le t \le a \\ 2a - t; a \le t \le 2a \end{cases}$ given that $f(t+2a) = f(t)$. **Ans:** $\frac{1}{s^2}\tanh\left(\frac{s\pi}{2}\right)$

3. Find the Laplace transform of

$f(t) = \begin{cases} \frac{t}{a}; 0 \le t \le a \\ \frac{2a-t}{a}; a \le t \le 2a \end{cases}$ given that $f(t+2a) = f(t)$. **Ans:** $\frac{1}{as^2}\tanh\left(\frac{sa}{2}\right)$

4. Find the Laplace transform of

$f(t) = \begin{cases} sint; 0 < t < \pi \\ 0; \pi < t < 2\pi \end{cases}$ $f(t+2\pi) = f(t)$ **Ans:** $\frac{1}{(1-e^{-\pi s})(s^2+1)}$

5.6 INVERSE LAPLACE TRANSFORM

Definition

If the Laplace transform of a function $f(t)$ is $F(s)$ *ie.*, $L[f(t)] = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically $f(t) = L^{-1}[F(s)]$, where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ *if n is an integer*	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

$L[sinhat] = \frac{a}{s^2 - a^2}$	$L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{sinhat}{a}$
$L[cosat] = \frac{s}{s^2 - a^2}$	$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = coshat$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ **and** $L[g(t)] = G(s)$ **,then** $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

$\because f(t) = L^{-1}[F(s)]$

$\because g(t) = L^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}\left[F(s+a) = e^{-at}L^{-1}[F(s)]\right]$

(ii) $L^{-1}\left[F(s-a) = e^{at}L^{-1}[F(s)]\right]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}\big[F[s+a]\big]$

$L^{-1}\big[F[s+a]\big] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ **and** $f(0) = 0$**, then** $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Result: 4 Division by s.

$$\boldsymbol{L^{-1}}\left[\frac{F(s)}{s}\right] = \int_{\boldsymbol{0}}^{\boldsymbol{t}} L^{-1}[F(s)]dt$$

Proof:

We know that $L\left[\int_0^t f\,(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$$\int_0^t f\,(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

$$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

Result: 5 Inverse Laplace transform of derivative

$$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$$tf(t) = -L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$f(t) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Result: 6 Inverse Laplace transform of integral

$$\boldsymbol{L^{-1}[F(s)] = t}L^{-1}\left[\int_s^\infty F(s)ds\right]$$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$$= \int_s^\infty F(s)\,ds$$

Operating L^{-1} on both sides, we get

$$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$f(t) = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

Problems under inverse Laplace transform of elementary functions

Example: 5.39 Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$$= \frac{1}{2} e^{-\frac{-3t}{2}}$$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$$= \frac{1}{4} L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$$

$$= \frac{1}{4}\frac{1}{3/2} sin\frac{3}{2}t$$

$$= \frac{1}{6} sin\frac{3}{2}t$$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$$

$$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3cos6t + \frac{5sin6t}{6}$$

Inverse Laplace transform using First shifting theorem

$$\boldsymbol{L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]}$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i)	$\frac{1}{(s+2)^2}$	**(ii)**	$\frac{1}{(s-3)^4}$	**(iii)**	$\frac{1}{(s+3)^2+9}$	**(iv)**	$\frac{1}{s^2-2s+2}$
(v)	$\frac{1}{s^2-4s+13}$	**(vi)**	$\frac{s+2}{(s+2)^2+25}$	**(vii)**	$\frac{s+2}{s^2+4s+20}$	**(viii)**	$\frac{s}{(s+3)^2}$
(ix)	$\frac{s}{(s-4)^3}$	**(x)**	$\frac{s}{s^2-2s+2}$	**(xi)**	$\frac{2s+3}{s^2+6s+25}$	**(xii)**	$\frac{s}{s^2+6s-7}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}\right] = e^{-2t}\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-3t}\frac{sin3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^{t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{t}sint$

(v) $L^{-1}\left[\frac{1}{s^2-4s+13}\right] = L^{-1}\left[\frac{1}{(s-2)^2+9}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{2t}\frac{sin3t}{3}$

(vi) $L^{-1}\left[\frac{s+2}{(s+2)^2+25}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2+25}\right] = e^{-2t}cos5t$

(vii) $L^{-1}\left[\frac{s+2}{s^2+4s+20}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+16}\right]$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+16}\right] = e^{-2t}cos4t$$

(viii) $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right]$

$$= L^{-1}\left[\frac{s+3}{(s+3)^2}\right] - L^{-1}\left[\frac{3}{(s+3)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+3}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= e^{-3t} - 3e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-3t} - 3e^{-3t}t$$

(ix) $L^{-1}\left[\frac{s}{(s-4)^3}\right] = L^{-1}\left[\frac{s-4+4}{(s-4)^3}\right]$

$$= L^{-1}\left[\frac{s-4}{(s-4)^3}\right] + L^{-1}\left[\frac{4}{(s-4)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s-4)^2}\right] + 4L^{-1}\left[\frac{1}{(s-4)^3}\right]$$

$$= e^{4t}L^{-1}\left[\frac{1}{s^2}\right] + 4e^{4t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{4t}t + 4e^{4t}\frac{t^2}{2!}$$

$$= e^{4t}t + 2e^{4t}t^2$$

(x) $L^{-1}\left[\frac{s}{s^2-2s+2}\right] = L^{-1}\left[\frac{s}{(s-1)^2+1}\right] = L^{-1}\left[\frac{s-1+1}{(s-1)^2+1}\right]$

$$= L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] + L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= e^{t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{s}{s^2-2s+2}\right] = e^{t}cost + e^{t}sint$$

(xi) $L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = L^{-1}\left[\frac{2s+3}{(s+3)^2+16}\right] = L^{-1}\left[\frac{2(s+3-3)+3}{(s+3)^2+16}\right]$

$$= L^{-1}\left[\frac{2(s+3)-6+3}{(s+3)^2+16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{2s-3}{s^2+16}\right]$$

$$= e^{-3t}\left[2L^{-1}\left[\frac{s}{s^2+16}\right] - 3L^{-1}\left[\frac{1}{s^2+16}\right]\right]$$

$$L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = e^{-3t}\left(2cos4t - \frac{3sin4t}{4}\right)$$

(xii) $L^{-1}\left[\frac{s}{s^2+6s-7}\right] = L^{-1}\left[\frac{s}{(s+3)^2-16}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2-16}\right]$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2-16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s}{s^2-16}\right] - 3e^{-3t}L^{-1}\left[\frac{1}{s^2-16}\right]$$

$$L^{-1}\left[\frac{s}{s^2+6s-7}\right] = e^{-3t}\left[cosh4t - \frac{3sinh4t}{4}\right]$$

Exercise: 5.7

Find the inverse Laplace transform for the following:

1. $\frac{2s-3}{s^2+5^2}$ **Ans:** $2cos5t - \frac{3sin5t}{5}$
2. $\frac{3s+5}{s^2+16}$ **Ans:** $3cos4t + \frac{5sin4t}{4}$
3. $\frac{1}{4s^2+9}$ **Ans:** $\frac{1}{6}sin\frac{3}{2}t$
4. $\frac{1}{(s+4)^5}$ **Ans:** $e^{-4t}\frac{t^4}{4!}$
5. $\frac{1}{s^2-4s+13}$ **Ans:** $\frac{e^{2t}}{3}sin3t$

Inverse using the formula

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $cot^{-1}\left(\frac{s}{a}\right)$ **(ii)** $tan^{-1}\left(\frac{a}{s}\right)$ **(iii)** $cot^{-1}as$

(iv) $tan^{-1}(s+a)$ **(v)** $log\left(\frac{s+a}{s+b}\right)$ **(vi)** $cot^{-1}\left(\frac{2}{s+1}\right)$ **(vii)** $tan^{-1}\left(\frac{2}{s^2}\right)$

Solution:

(i) $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t}sinat$$

(ii) $L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{1}{t}sinat$$

(iii) $L^{-1}[cot^{-1}as] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right] = \frac{1}{t}L^{-1}\left[\frac{a}{a^2\left(s^2+\frac{1}{a^2}\right)}\right]$$

$$= \frac{1}{at}L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right] = \frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$$

$$L^{-1}[cot^{-1}as] = \frac{1}{t}sin\frac{t}{a}$$

(iv) $L^{-1}[tan^{-1}(s+a)] = e^{-at}L^{-1}[tan^{-1}s]$

$$= e^{-at}\left[\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(tan^{-1}s)\right]\right]$$

$$= e^{-at}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= \frac{-1}{t}e^{-at}L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-e^{-at}}{t}sint$$

(v) $L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(log\left(\frac{s+a}{s+b}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s+a)-\log(s+b))\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$

$$= \frac{-1}{t}\left[e^{-at}-e^{-bt}\right]$$

$$L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right] = \frac{-1}{t}\left[e^{-at}-e^{-bt}\right]$$

(vi) $L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right] = e^{-t}L^{-1}\left[cot^{-1}\left(\frac{2}{s}\right)\right]$

$$= e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{2}{s}\right)\right)\right]$$

$$= e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{-1}{1+\frac{4}{s^2}}\left(\frac{-2}{s^2}\right)\right] = -\frac{e^{-t}}{t}L^{-1}\left[\frac{1}{\frac{s^2+4}{s^2}}\left(\frac{2}{s^2}\right)\right]$$

$$= -\frac{e^{-t}}{t}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right] = -\frac{e^{-t}}{t}sin2t$$

(vii) $L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{2}{s^2}\right)\right)\right]$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right] = \frac{4}{t}L^{-1}\left[\frac{1}{\frac{s^4+4}{s^4}}\left(\frac{1}{s^3}\right)\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2)^2+2^2}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right] \qquad \boxed{\because a^2+b^2 = (a+b)^2-2ab}$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{-4s}\left(\frac{1}{s^2+2+2s}-\frac{1}{s^2+2-2s}\right)\right] \qquad \because \left\{\begin{array}{c}\frac{1}{(s^2+ax+b)(s^2+ax+c)} \\ = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b}-\frac{1}{s^2+ax+c}\right]\end{array}\right\}$$

$$= \frac{-1}{t}L^{-1}\left[\left(\frac{1}{s^2+2s+2}-\frac{1}{s^2-2s+2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}-\frac{1}{(s-1)^2+1}\right]$$

$$= \frac{-1}{t}\left(e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$= \frac{-1}{t}(e^{-t}sint - e^{t}sint)$$

$$= \frac{sint}{t}(e^{-t} - e^{t})$$

$$= \frac{sint}{t}2sinht$$

$$L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{2sintsinht}{t}$$

Inverse using the formula

$$\boldsymbol{L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]}$$

Example: 5.42 Find $\boldsymbol{L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]}$

Solution:

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \cdots (1)$$

$$L^{-1}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s^2+a^2) - \log(s^2+b^2))\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s^2+a^2}2s - \frac{1}{s^2+b^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat - cosbt]$$

$$= \frac{2}{t}[cosbt - cosat]$$

Substituting in (1), we get

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt - cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt}$$

This formula is used when $F(s) = \frac{one\ term}{s(another\ term)}$

Example: 5.43 Find $\boldsymbol{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{-cosat}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[cosat]_0^t$$

$$= \frac{-1}{a^2}(cosat - cos0) = \frac{-1}{a^2}(cosat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-cosat}{a^2}$$

Example: 5.44 Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2-a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinhat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{coshat}{a}\right]_0^t$$

$$= \frac{1}{a^2}[coshat]_0^t$$

$$= \frac{1}{a^2}(coshat - cosh0) = \frac{1}{a^2}(coshat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{coshat-1}{a^2}$$

Example: 5.45 Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right]dt$$

$$= \int_0^t e^{-at}dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{-1}{a}(e^{-at} - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1-e^{-at}}{a}$$

Inverse using Partial Fraction

Example: 5.46 Find $L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right]$

Solution:

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$= \frac{A(s+2)(s-1)+Bs(s-1)+Cs(s+2)}{s(s+2)(s-1)}$$

$$A(s+2)(s-1) + Bs(s-1) + Cs(s+2) = s - 2 \cdots (1)$$

Put $s = 0$ in (1)	Put $s = -2$ in (1)	Put $s = 1$ in (1)
$A(2)(-1) = -2$	$B(-2)(-3) = -4$	$3C = -1$

$\Rightarrow A = 1$ $\qquad$ $\Rightarrow B = \frac{-4}{6} = \frac{-2}{3}$ $\qquad$ $\Rightarrow C = \frac{-1}{3}$

$$\therefore \frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{s+2} - \frac{1}{3(s-1)}$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = L^{-1}\left[\frac{1}{s}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right]$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = 1 - \frac{2}{3}e^{-2t} - \frac{1}{3}e^{t}$$

Example: 5.47 Find $L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right]$

Solution:

$$\frac{2s-3}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s - 3 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^2
		$A + B = 0$
$A = -1$	$C = 1$	$B = -A \Rightarrow B = 1$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right] + \left[\frac{1}{(s-2)^2}\right]$$

$$= -e^{t} + e^{2t} + e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$\therefore L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -e^{t} + e^{2t} + e^{2t}t$$

Example: 5.48 Find the inverse Laplace transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$= \frac{A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)}{(s-1)(s-2)^3}$$

$$A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) = 5s^2 - 15s - 11 \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^3
$A(-27) = 9$	$D(3) = -21$	$A + B = 0$
$A = \frac{9}{-27} \Rightarrow A = \frac{-1}{3}$	$D = \frac{-21}{3} = -7$	$B = -A \Rightarrow B = \frac{1}{3}$

Put $s = 0$ in (1), we get

$$-8A + 4B - 2C + D = -11$$

$$\frac{8}{3}+\frac{4}{3}-2C-7=-11$$

$$4-2C=7-11$$

$$-2C=-8 \Rightarrow C=4$$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1}{3(s+1)}+\frac{1}{3(s-2)}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}\frac{t^2}{2}$$

Example: 5.49 Find the inverse Laplace transform of $\frac{4s+5}{(s+1)(s^2+4)}$

Solution:

$$\frac{4s+5}{(s+1)(s^2+4)}=\frac{A}{s+1}+\frac{Bs+c}{s^2+4}$$

$$=\frac{A(s^2+4)+(Bs+c)(s+1)}{(s+1)(s^2+4)}$$

$$A(s^2+4)+(Bs+c)(s+1)=4s+5\cdots\cdots(1)$$

Put $s=-1$in (1)

$A(1+4)+0=4(-1)+5$

$A(5)=1 \Rightarrow A=\frac{1}{5}$

Equating coefficients of s^2term in (1)

$A+B=0$

$B=-A \Rightarrow B=\frac{-1}{5}$

Put $s=0$in (1)

$A(4)+C=5$

$C=5-4A=5-\frac{4}{5}$

$=\frac{25-4}{5}=\frac{21}{5}$

$$\therefore \frac{4s+5}{(s+1)(s^2+4)}=\frac{\frac{1}{5}}{s+1}+\frac{\frac{-1}{5}s+\frac{21}{5}}{s^2+4}$$

$$=\frac{1}{5(s+1)}-\frac{s}{5(s^2+4)}+\frac{21}{5}\frac{1}{(s^2+4)}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right]=\frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]+\frac{21}{5}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}cos2t+\frac{21}{5}\frac{sin2t}{2}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right]=\frac{1}{5}e^{-t}-\frac{1}{5}cos2t+\frac{21}{10}sin2t$$

Exercise: 5.8

Find the Inverse Laplace transforms using partial fraction for the following

1. $\frac{1}{(s+1)(s+3)}$ **Ans:** $\frac{1}{2}(e^{-t}-e^{-3t})$
2. $\frac{1}{s(s+1)(s+2)}$ **Ans:** $\frac{1}{2}(e^{-2t}-2e^{-t}+1)$
3. $\frac{54-3s-5}{(s+1)(s^2-3s+2)}$ **Ans:** $2e^{-t}+2\,e^{\frac{3t}{2}}cosh\frac{t}{2}2+8e^{\frac{3t}{2}}sinh\frac{t}{2}$

5. 7 INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If$L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to \infty$ on both sides, we have

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty}[e^{-st} f'(t)]dt + f(0)$$

$$= 0 + f(0) \qquad \because e^{-\infty} = 0$$

$$= f(0)$$

$$= \lim_{t\to 0} f(t)$$

$$\therefore \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$,then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to 0$ on both sides, we have

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to 0}[e^{-st} f'(t)]dt + f(0)$$

$$= \int_0^\infty f'(t)dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0)$$

$$= f(\infty)$$

$$= \lim_{t\to\infty} f(t)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Example: 5.50 Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

Given $f(t) = ae^{-bt}$

$F(s) = L[f(t)]$

$= L[ae^{-bt}]$

$= a\ \frac{1}{s+b}$

$sF(s) = \frac{as}{s+b}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt}$$

$$= a \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{as}{s+b}\right]$$

$$= \lim_{s\to\infty}\left[\frac{as}{s\left(1+\frac{b}{s}\right)}\right] == \lim_{s\to\infty}\left[\frac{a}{\left(1+\frac{b}{s}\right)}\right]$$

$$= a \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Example: 5.51Verify the initial value theorem and Final value theorem for the function

$\boldsymbol{f(t) = 1 + e^{-t}[sint + cost]}$.

Solution:

Given $f(t) = 1 + e^{-t}[sint + cost]$

$F(s) = L[f(t)]$

$= L\big[1 + e^{-t}[sint + cost]\big]$

$= L[1] + L\big[e^{-t}[sint + cost]\big]$

$= L[1] + L[sint + cost]_{s\to s+1}$

$= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$

$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$

$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$

$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\lim_{t\to 0} f(t) = \lim_{t\to 0}\big[1 + e^{-t}[sint + cost]\big]$

$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (1)$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$

$$= 1 + \lim_{s\to\infty}\left[\frac{1}{s\left(1+\frac{2}{s}+\frac{2}{s^2}\right)} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(1 + e^{-t}[sint + cost])$$

$$= 1 + 0 = 1 \cdots\cdots\cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$$

$$= 1 + 0 + 0 = 1 \cdots\cdots\cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: 5.52 Verify the initial value theorem and Final value theorem for the function

$$\boldsymbol{f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]}$$

Solution:

Given $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right] \cdots (1)$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^2}\right] dt = \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] dt$$

$$= \int_0^t e^{-2t} t \, dt$$

$$= \int_0^t t e^{-2t} \, dt$$

$$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \frac{(1)e^{-2t}}{(-2)^2}\right]_0^t$$

$$= -t\frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}$$

$$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$$sF(s) = \frac{1}{(s+2)^2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - \frac{1}{4} = 0$$

$$\therefore \lim_{t\to 0} f(t) = 0 \cdots (2)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+2)^2} = 0$$

$$\therefore \lim_{s\to\infty} sF(s) = 0 \cdots (3)$$

From (2) and (3), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - 0 = \frac{1}{4} \cdots (4)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4} \cdots (5)$$

From (4) and (5), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified

Example: 5.53 Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = e^{-t}(t+2)^2}$

Solution:

Given $f(t) = e^{-t}(t+2)^2$

$$= e^{-t}(t^2 + 4t + 4)$$

$$F(s) = L[f(t)]$$

$$= L[e^{-t}(t^2 + 4t + 4)]$$

$$= L[t^2 + 4t + 4]_{s\to s+1}$$

$$= [L(t^2) + 4L(t) + 4L(1)]_{s\to s+1}$$

$$= \left[\frac{2!}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}\right]_{s\to s+1}$$

$$= \frac{2}{(s+1)^3} + 4\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[e^{-t}(t^2 + 4t + 4)]$$

$$= 4 \cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s}{s^3\left(1+\frac{1}{s}\right)^3} + \frac{4s}{s^2\left(1+\frac{1}{s}\right)^2} + \frac{4s}{s\left(1+\frac{1}{s}\right)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2}{s^2\left(1+\frac{1}{s}\right)^3} + \frac{4}{s\left(1+\frac{1}{s}\right)^2} + \frac{4}{\left(1+\frac{1}{s}\right)}\right]$$

$$= 0 + 0 + 4$$

$= 4 \cdots (2)$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [e^{-t}(t^2 + 4t + 4)]$$

$$= 0 \cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= 0 \cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: 5.54 If $L[f(t)] = \frac{1}{s(s+1)}$, find the $\lim_{t\to 0} f(t)$ and $\lim_{t\to\infty} f(t)$ using initial and final value theorems.

Solution:

Given $L[f(t)] = \frac{1}{s(s+1)} \cdots (1)$

$ie., F(s) = \frac{1}{s(s+1)} \Rightarrow sF(s) = \frac{1}{(s+1)}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$= \lim_{s\to\infty} \frac{1}{(s+1)} = 0$$

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$= \lim_{s\to 0} \frac{1}{(s+1)} = 1$$

Exercise: 5.9

1. Verify the initial value theorem for the function $f(t) = e^{-t} sint$
2. Verify the initial value theorem for the function $f(t) = \sin^2 t$
3. Verify the initial value theorem for the function $f(t) = 1 + e^{-t} + t^2$
4. Verify the Final value theorem for the function $f(t) = 1 - e^{-at}$
5. Verify the Final value theorem for the function $f(t) = t^2 e^{-3t}$

5.8 CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by $f(t) * g(t) = \int_0^t f(u) g(t-u) du.$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$L[f(t) * g(t)] = \int_0^\infty [f(t) * g(t)]\, e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)due^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \cdots (1)$$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$L[f(t) * g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu$$

$$= \int_0^\infty f(u)\left[\int_u^\infty g(t-u)e^{-st}dt\right]du \cdots (2)$$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

$$(2) \Rightarrow L[f(t) * g(t)] = \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-s(u+x)}dx\right]du$$

$$= \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-su}e^{-sx}dx\right]du$$

$$= \int_0^\infty f(u)e^{-su}du \int_0^\infty g(x)e^{-sx}dx$$

$$= L[f(u)]L[g(x)]$$

$\therefore L[f(t) * g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems under Convolution theorem

Example: 5.55 Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{-au}\, e^{bu}du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{bt-at} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-bt+bt-at} - e^{-bt}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Example: 5.56 Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$$

$$= cosat * cosbt$$

$$= \int_0^t cosau\ cosb(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2}du$$

$$= \frac{1}{2}\int_0^t (\cos(au+bt-bu)+\cos(au-bt+bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos(a-b)\,u + bt + \cos(a+b)\,u - bt]du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b} + \frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)sinat+(a-b)sinat-(a+b)sinbt+(a-b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(asinat-bsinbt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinat-bsinbt}{a^2-b^2}$$

Example: 5.57 Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a}sinat * \frac{1}{b}sinbt$$

$$= \frac{1}{ab}\int_0^t sinau\ sinb(t-u)du$$

$$= \frac{1}{ab}\int_0^t \frac{\cos(au-bt+bu)-\cos(au+bt-bu)}{2}du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu)-\cos(au+bt-bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos[(\mathrm{a}+\mathrm{b})\mathrm{u}-\mathrm{bt}]-\cos[(\mathrm{a}-\mathrm{b})\mathrm{u}+\mathrm{bt}]]du$$

$$= \frac{1}{2ab}\left[\frac{sin[(a+b)u-bt]}{a+b}-\frac{sin[(a-b)u+bt]}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin(at-bt+bt)}{a-b}+\frac{sinbt}{a+b}+\frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{\sin at}{a+b}-\frac{\sin at}{a-b}-\frac{sinbt}{a+b}+\frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{(a-b)sinat-(a+b)sinat+(a-b)sinbt+(a+b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{-2bsinat+2asinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{2(asinbt-bsinat)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinbt-bsinat}{ab(a^2-b^2)}$$

Example: 5.58 Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\ \frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2}sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\ cos3(t-u)du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2}du$$

$$= \frac{1}{4}\int_0^t [\sin(3\mathrm{t}-u)+\sin(5\mathrm{u}-3\mathrm{t})]du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1}-\frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3\mathrm{t}-\mathrm{t})}{1}-\frac{\cos(5\mathrm{t}-3t)}{5}-\frac{cos3t}{1}+\frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t-\frac{\cos 2t}{5}-cos3t+\frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t-cos2t-5cos3t+cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t-4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t-cos3t}{5}$$

Example: 5.59 Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * cosat$$

$$= \frac{1}{a}\int_0^t sinau\ cosa(t-u)du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(au+at-au)+\sin(au-at+au)}{2} du$$

$$= \frac{1}{2a}\int_0^t [sinat + \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\int_0^t sinatdu + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat\int_0^t du + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat(u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos(2at-at)}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos at}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\ tsinat$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{tsinat}{2a}$$

Example: 5.60 Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * \frac{1}{a} sinat$$

$$= \frac{1}{a^2}\int_0^t sinau\ sina(t-u)du$$

$$= \frac{1}{a^2}\int_0^t \frac{\cos(au-at+au)-\cos(au+at-au)}{2} du$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - cosat]du$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du - \int_0^t cosat\,du\right]$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du - cosat\int_0^t du\right]$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin(2au-at)}{2a}\right)_0^t - cosat(u)_0^t\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - \frac{\sin(-at)}{2a} - tcosat\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} + \frac{sinat}{2a} - tcosat\right]$$

$$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a} - tcosat\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\frac{\sin at}{a} - tcosat\right]$$

Example: 5.61 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= cosat * cosat$$

$$= \int_0^t cosau\ cosa(t-u)du$$

$$= \int_0^t \frac{\cos(au+at-au)+\cos(au-at+au)}{2}du$$

$$= \frac{1}{2}\int_0^t [cosat + \cos(2au-at)]du$$

$$= \frac{1}{2}\left[\int_0^t cosatdu + \int_0^t \cos(2au-at)\,du\right]$$

$$= \frac{1}{2}\left[cosat\int_0^t du + \int_0^t \cos(2au-at)\,du\right]$$

$$= \frac{1}{2}\left[cosat(u)_0^t + \left(\frac{\sin(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2}\left[tcosat + \frac{\sin(2at-at)}{2a} + \frac{sinat}{2a}\right]$$

$$= \frac{1}{2}\left[tcosat + \frac{\sin at}{2a} + \frac{sinat}{2a}\right]$$

$$= \frac{1}{2}\left[tcosat + \frac{2\sin at}{2a}\right]$$

$$\therefore\ L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[tcosat + \frac{\sin at}{a}\right]$$

Example: 5.62 Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+2^2)}\ \frac{s}{(s^2+2^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+2^2)}\right] * L^{-1}\left[\frac{s}{(s^2+2^2)}\right]$$

$$= cos2t * cos2t$$

$$= \int_0^t cos2u\ cos2(t-u)du$$

$$= \int_0^t \frac{\cos(2u+2t-2u)+\cos(2u-2t+2u)}{2}du$$

$$= \frac{1}{2}\int_0^t [cos2t + \cos(4u-2t)]du$$

$$= \frac{1}{2}\left[\int_0^t cos2tdu + \int_0^t \cos(4u-2t)\,du\right]$$

$$= \frac{1}{2}\left[cos2t\int_0^t du + \int_0^t \cos(4u-2t)\,du\right]$$

$$= \frac{1}{2}\left[cos2t(u)_0^t + \left(\frac{\sin(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{\sin(4t-2t)}{4} - \frac{\sin(-2t)}{4}\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{\sin2t}{4} + \frac{sin2t}{4}\right]$$

$$= \frac{1}{2}\left[tcos2t + \frac{2\sin2t}{4}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[tcos2t + \frac{\sin2t}{2}\right]$$

Example: 5.63 Find $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{1}{s}\ \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 1 * \frac{\sin2t}{2}$$

$$= \frac{\sin2t}{2} * 1$$

$$= \int_0^t \frac{\sin2u\ (1)}{2}\,du$$

$$= \left[\frac{-cos2u}{4}\right]_0^t = \frac{1}{4}(-cos2t+1)$$

$$= \frac{1}{4}(1-cos2t)$$

Example: 5.64 Find the inverse Laplace transform $\frac{s+2}{(s^2+4s+13)^2}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{s^2+4s+13}\ \frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right] * e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}cos3t * \frac{e^{-2t}sin3t}{3}$$

$$= \int_0^t e^{-2u}cos3u\, e^{-2(t-u)}\frac{sin3(t-u)}{3}\,du$$

$$= \int_0^t e^{-2u}cos3u\, e^{-2t+2u}\frac{sin(3t-3u)}{3}\,du$$

$$= \frac{1}{3}\int_0^t e^{-2u-2t+2u}cos3u\,\sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{\sin(3u+3t-3u)-\sin(3u-3t+3u)}{2}\,du$$

$$= \frac{e^{-2t}}{6}\int_0^t [sin3t - \sin(6u - 3t)]du$$

$$= \frac{e^{-2t}}{6}\left[\int_0^t sin3tdu - \int_0^t \sin(6u - 3t)\, du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t\int_0^t du - \int_0^t \sin(6u - 3t)\, du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t(u)_0^t + \left(\frac{\cos(6u-3t)}{6}\right)_0^t\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos(6t-3t)}{6} - \frac{\cos(-3t)}{6}\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos 3t}{6} - \frac{cos3t}{6}\right]$$

$$= \frac{e^{-2t}}{6}\ tsin3t$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{6}\ tsin3t$$

Example: 5.65 Find the inverse Laplace transform $\frac{1}{(s+1)(s^2+4)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = L^{-1}\left[\frac{1}{s+1}\ \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= e^{-t} * cos2t$$

$$= \int_0^t e^{-(t-u)} cos2u\, du$$

$$= e^{-t}\int_0^t e^u cos2u\, du$$

$$\boxed{\because \int e^{at} cosbtdt = \frac{e^{at}}{a^2+b^2}(acosbt + bsinbt)}$$

$$= e^{-t}\left[\frac{e^u}{1^2+2^2}(cos2u + 2sin2u)\right]_0^t$$

$$= \frac{e^{-t}}{5}\left[e^t(cos2t + 2sin2t) - e^0(cos0 - 0)\right]$$

$$= \frac{e^{-t}}{5}\left[e^t(cos2t + 2sin2t) - 1\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = \frac{e^{-t}}{5}\left[e^t(cos2t + 2sin2t) - 1\right]$$

Exercise: 5.10

Find the inverse Laplace transforms using convolution theorem for the following

1. $\frac{1}{s(s^2+1)}$ **Ans:** $1 - cost$
2. $\frac{s}{(s^2+4)^2}$ **Ans:** $\frac{1}{8}\left[\frac{\sin 2t}{2} - tcos2t\right]$
3. $\frac{s^2}{(s^2+4)^2}$ **Ans:** $\frac{1}{2}\left[tcos2t + \frac{\sin 2t}{2}\right]$
4. $\frac{1}{(s+1)(s^2+1)}$ **Ans:** $\frac{1}{2}\left[e^{-t} + sint - cost\right]$
5. $\frac{1}{(s+1)(s^2+4)}$ **Ans:** $-\frac{1}{5}e^{-t} + \frac{1}{5}cos2t - \frac{1}{10}sin2t$

5.9 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$$

$$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$$

$$x(t) = 1 - 7e^{t} + 6e^{2t}$$

Example: 5.67 Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, with $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$$

Substituting $y(0) = 1 = y'(0)$.

$$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$$

$$= \frac{2+s^2+s-2s-2}{s+1}$$

$$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$$

$$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$$

$$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$$

Consider $\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$$

$$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$y(t) = \frac{13}{25} L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$

$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} te^{-t} + \frac{12}{25} e^{4t}$

Example: 5.68 Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$

$[s^2 L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$[s^2 L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1 + s(s+1) - 3(s+1)}{s+1}$

$= \frac{1 + s^2 + s - 3s - 3}{s+1}$

$(s-1)(s-2)\bar{y} = \frac{s^2 - 2s - 2}{s+1}$

$\bar{y} = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$

Consider $\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$

$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$

Put $s = -1$ in (1)	put $s = 1$ in (1)	put $s = 2$ in (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$
$A = \frac{1}{6}$	$B = \frac{3}{2}$	$C = \frac{-2}{3}$

$\therefore \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$

$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = sint$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = sint$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(sint)$

$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$

Substituting $y(0) = 0 = y'(0)$.

$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$

$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$

$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$

$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$

$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$

Put $s = 1$ in (1)	Put $s = -3$ in (1)	equating the coefficients of s^2, we get
$8A = 0 + 1$	$B(-4)(10) = 1$	$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$
$A = \frac{1}{8}$	$B = \frac{-1}{40}$	$C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times3} \Rightarrow D = \frac{-1}{5}$

$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{\left(\frac{-1}{10}\right)s - \frac{1}{5}}{s^2+1}$

$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$

$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$

$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$

Example: 5.70 Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ $and y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$

$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$

$[s^2 - 3s + 2]\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$(s-1)(s-2)\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$\bar{y} = \frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2}$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2+B(s-1)(s-2)+C(s-1)}{(s-1)(s-2)^2}$

$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	equating the coefficients of s^2, we get
$A = 4 - 11$	$C = 4 + 0$	$A + B = -3 \Rightarrow -7 + B = -3$
$A = -7$	$C = 4$	$B = 4$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$= -7e^t + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = -7e^t + 4e^{2t} + 4e^{2t}t$$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with$y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2\ and y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$$

Substituting $y(0) = 2\ and y'(0) = -2$.

$$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$$

$$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$$

$$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$$

$$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$$

Put $s = 2$ in (1)	Put $s = 0\ in\ (1)$	equating the coefficients ofs^2, we get
$4A = 1 + 0$	$8A + 4B - 2C = 21$	$A + B = 2 \Rightarrow \frac{1}{4} + B = 2$
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 6e^{2t}\frac{sin2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 3e^{2t}sin2t$$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1\ at\ t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^tL^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^tt - 5e^tt$$

$$\therefore x = \frac{t^2e^t}{2} + 2e^t - 3e^tt$$

Example: 5.73 Solve the following differential equation using Laplace transform $(D^2 - 2D + 1)y = t^2e^t$ Given $y(0) = 2$ and $Dy(0) = 3$

Solution:

Given $(D^2 - 2D + 1)y = t^2e^t$ with $y(0) = 2$ and $Dy(0) = 3$

$ie., D^2y - 2Dy + y = t^2e^t$

$y'' - 2y' + y = t^2e^t$ With $y(0) = 2$ and $y'(0) = 3$

Apply Laplace transform on both sides, we get

$$L[y''(t)] - 2L[y'(t)] + L[y(t)] = L(t^2e^t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 2[sL[y(t)] - y(0)] + L[y(t)] = L[t^2]_{s\to s-1}$$

Substituting $y(0) = 2\ and y'(0) = 3$.

$$[s^2L[y(t)] - 2s - 3] - 2[sL[y(t)] - 2] + L[y(t)] = \left[\frac{2!}{s^3}\right]_{s\to s-1}$$

$$s^2L[y(t)] - 2s - 3 - 2sL[y(t)] + 4 + L[y(t)] = \frac{2}{(s-1)^3}$$

$$s^2L[y(t)] - 2sL[y(t)] + L[y(t)] = \frac{2}{(s-1)^3} + 2s - 1$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} - 2s\bar{y} + \bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$[s^2 - 2s + 1]\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$(s-1)^2\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$\bar{y} = \frac{2}{(s-1)^5} + \frac{2s}{(s-1)^2} - \frac{1}{(s-1)^2}$$

$$y(t) = L^{-1}\left[\frac{2}{(s-1)^5}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= 2e^tL^{-1}\left[\frac{1}{s^5}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= 2e^t\frac{t^4}{4!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - e^tt$$

$$= 2e^t\frac{t^4}{24} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tt - e^tt$$

$$\therefore x = \frac{t^4e^t}{12} + 2e^t + e^tt$$

Example: 5.74 Solve using Laplace transform $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$, given that $y(0) = 0$ and $y'(0) = 0$

Solution:

Given $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$ with $y(0) = 0$ and $y'(0) = 0$

$y'' + 6y' + 9y = 6t^2e^{-3t}$ With $y(0) = 0$ and $y'(0) = 0$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 6L(t^2e^{-3t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 6[sL[y(t)] - y(0)] + 9L[y(t)] = 6L[t^2]_{s\to s+3}$$

Substituting $y(0) = 0\ and y'(0) = 0$.

$$[s^2 L[y(t)] - 0 - 0] + 6[sL[y(t)] - 0] + 9L[y(t)] = 6\left[\frac{2!}{s^3}\right]_{s\to s+3}$$

$$s^2 L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

$$s^2 L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + 6s\bar{y} + 9\bar{y} = \frac{12}{(s+3)^3}$$

$$[s^2 + 6s + 9]\bar{y} = \frac{12}{(s+3)^3}$$

$$(s+3)^2\bar{y} = \frac{12}{(s+3)^3}$$

$$\bar{y} = \frac{12}{(s+3)^5}$$

$$y(t) = L^{-1}\left[\frac{12}{(s+3)^5}\right] = 12e^{-3t}L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 12e^{-3t}\frac{t^4}{4!}$$

$$\therefore y = \frac{t^4 e^{-3t}}{2}$$

Example: 5.75 Solve$\boldsymbol{\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}sint; x(0) = 0}$ and $\boldsymbol{x'(0) = 1}$

Solution:

Given $x'' + 2x' + 5x = e^{-t}sint; x(0) = 0; x'(0) = 1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + 2L[x'(t)] + 5L[x(t)] = L(e^{-t}sint)$$

$$[s^2 L[x(t)] - sx(0) - x'(0)] + 2[sL[x(t)] - x(0)] + 5L[x(t)] = L[sint]_{s\to s+1}$$

Substituting $x(0) = 0; x'(0) = 1$

$$[s^2 L[x(t)] - 0 - 1] + 2[sL[x(t)] - 0] + 5L[x(t)] = \left[\frac{1}{s^2+1}\right]_{s\to s+1}$$

$$s^2 L[x(t)] + 2sL[x(t)] + 5L[x(t)] - 1 = \frac{1}{(s+1)^2+1}$$

$$s^2 L[x(t)] + 2sL[x(t)] + 5L[x(t)] = \frac{1}{(s+1)^2+1} + 1$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} + 2s\bar{x} + 5\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{s^2+2s+2} + 1$$

$$\bar{x} = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{5-2}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$\boxed{\frac{1}{(s^2+ax+b)(s^2+ax+c)} = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c}\right]}$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{3(s^2+2s+2)} - \frac{1}{3(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$\bar{x} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$$

$$x(t) = \frac{1}{3}L^{-1}\left[\frac{1}{(s^2+2s+2)}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s^2+2s+5)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{3}e^{-t}sint + \frac{2}{3}e^{-t}\frac{sin2t}{2}$$

$$\therefore x = \frac{1}{3}e^{-t}[sint + sin2t]$$

Example: 5.76 Solve using Laplace transform $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $= 4$,$y' = -2$ when $t = 0$

Solution:

Given $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

$y'' + y' = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + L[y'(t)] = L(t^2) + L(2t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + [sL[y(t)] - y(0)] = \frac{2}{s^3} + 2\frac{1}{s^2}$$

Substituting $y(0) = 4\ and y'(0) = -2.$

$$[s^2L[y(t)] - 4s + 2] + [sL[y(t)] - 4] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2L[y(t)] + sL[y(t)] - 4s + 2 - 4 = \frac{2+2s}{s^3}$$

$$s^2L[y(t)] + sL[y(t)] = \frac{2(1+s)}{s^3} + 4s + 2$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + s\bar{y} = \frac{2(1+s)}{s^3} + 2(2s+1)$$

$$s(s^2 + s)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$s(s+1)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$\bar{y} = \frac{2(s+1)}{s^4(s+1)} + \frac{2(2s+1)}{s(s+1)}$$

$$= \frac{2}{s^4} + 2\left[\frac{s+(s+1)}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{1}{s+1} + \frac{1}{s}\right]$$

$$\bar{y} = \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}$$

$$y(t) = 2L^{-1}\left[\frac{2}{s^4}\right] + 2L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s}\right]$$

$$= 2\frac{t^3}{3!} + 2e^{-t} + 2(1)$$

$$\therefore y = \frac{t^3}{3} + 2e^{-t} + 2$$

Example: 5.77 Solve using Laplace transform $\boldsymbol{\frac{d^2x}{dt^2} + 9x = cos2t}$, if $(0) = 1$;$\boldsymbol{x\left(\frac{\pi}{2}\right) = -1}$

Solution:

Given $x'' + 9x = cos2t; x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

Since $x'(0)$ is not given assume $x'(0) = k$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + L[x(t)] = L(cos2t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 9L[x(t)] = L(cos2t)$$

Substituting $x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

$$[s^2L[x(t)] - s - k] + 9L[x(t)] = \frac{s}{s^2+4}$$

$$s^2L[x(t)] + 9L[x(t)] = \frac{s}{s^2+4} + s + k$$

$$[s^2 + 9]L[x(t)] = \frac{s}{s^2+4} + s + k$$

Put $L[x(t)] = \bar{x}$

$$[s^2 + 9]\bar{x} = \frac{s}{s^2+4} + s + k$$

$$\bar{x} = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{9-4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$\bar{x} = \frac{s}{5(s^2+4)} + \frac{(5s-s)}{5(s^2+9)} + \frac{k}{s^2+9}$$

$$= \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$x(t) = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}L^{-1}\left[\frac{s}{s^2+9}\right] + kL^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + k\frac{sin3t}{3} \cdots (1)$$

Given $x\left(\frac{\pi}{2}\right) = -1$

Put $t = \frac{\pi}{2}\ in\ (1)$

$$(1) \Rightarrow x\left(\frac{\pi}{2}\right) = \frac{1}{5}cos\frac{2\pi}{2} + \frac{4}{5}cos\frac{3\pi}{2} + k\frac{sin\frac{3\pi}{2}}{3}$$

$$-1 = \frac{1}{5}(-1) + 0 + \frac{k}{3}(-1)$$

$-\frac{k}{3} = \frac{1}{5} - 1 \Rightarrow -\frac{k}{3} = \frac{-4}{5} \Rightarrow k = \frac{12}{5}$

$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + \frac{12}{5}\frac{sin3t}{3}$

$\therefore x(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$

Exercise: 5.11

1. Solve using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$, given that $= 0$,$\frac{dy}{dt} = 2$ when $t = 0$

Ans:$-1 - \frac{1}{6}e^{-5t} + \frac{5}{6}e^{t}$

2. Using Laplace transform solve the differential equation $y'' + 5y' + 6y = 2$, with $y(0) = 0 = y'(0)$. Where $y' = \frac{dy}{dt}$ **Ans:**$y(t) = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}$

3. Using Laplace transform solve the differential equation $y'' + 4y' + 3y = e^{-t}$, with $y(0) = 1;\ y'(0) = 0.$ **Ans:**$y(t) = \frac{-1}{4}e^{-3t} - \frac{5}{4}e^{-t} + \frac{1}{2}te^{-t}$

4. Solve using Laplace transform $\frac{d^2y}{dt^2} + y = sint$ given $= 1$,$\frac{dy}{dt} = 0$ when $t = 0$

Ans:$y(t) = sint - tcost$

5. Solve using Laplace transform$\frac{d^2y}{dt^2} + 9y = cos2t$, if $y(0) = 1$;$y\left(\frac{\pi}{2}\right) = -1$

Ans:$y(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$