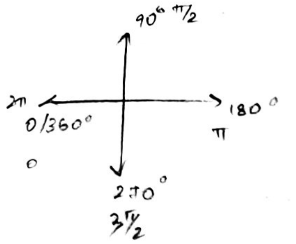
17/02

MODULE - 02



(0 → 2π) one cycle
↳ is a period

PERIODIC FUNCFUNCTIONS:

A function is said to have a period T or to be periodic with period T, if for all t, $f(t+T) = f(t)$, where T is a positive constant.

The least value of $T > 0$ is called the principle period or the [functional] fundamental period or simply the period of $f(t)$.

Example:

The function $\sin x$ has period 2π since $\sin(x+2\pi) = \sin x$

PIECEWISE CONTINUOUS FUNCTION:

A function is called piecewise continuous on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval [i.e. the subinterval without its endpoints and has a finite limits at the endpoints.

DIRICHLET'S CONDITIONS :

* $f(x)$ defined and single valued except possibly at a finite number of points in interval $(c, c+2l)$.

* $f(x)$ is periodic in the interval $(c, c+2l)$

* $f(x)$, $f'(x)$ are piecewise continuous in the interval $(c, c+2l)$.

* $f(x)$ has no or finite number of maxima or minima in the interval $(c, c+2l)$

DEFINITION :

FORMULA : [Fourier Series]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{l}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

Fourier coefficients :

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left[\frac{n\pi x}{l}\right]dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left[\frac{n\pi x}{l}\right]dx$$

eg:
$f(x) = \cos x$

$f(x) = x^2$

$f(-x) = (-x)^2 \Rightarrow x^2$

$f(x) = f(-x)$

Even function :

$b_n = 0$

eg:
$f(x) = \sin x, \tan x$

$f(x) = x^3$

$f(-x) = (-x)^3 \Rightarrow -x^3$

$f(-x) = -f(+x)$

$f(-x) = -f(x)$

Odd function

$a_0 = a_n = 0$

$$\int_{-a}^{a} = 2\int_{0}^{a}$$

$\left.\begin{matrix} 0, 2l \\ -l, l \end{matrix}\right]$ full limits

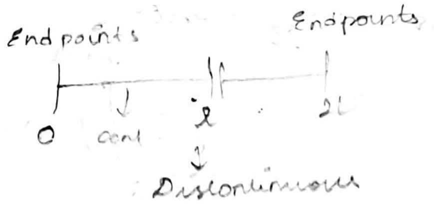
$0, l \rightarrow$ positive (half)

Bernoulli's Equation :

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$$

$$\int u\,dv = uv - \int v\,du$$

Formula :



CONTINUOUS POINTS

$f(x)$ at $x = a$ $\therefore$ $f(a)$

ENDPOINTS

$\{f(x)\}$ at $x = \underset{a}{\text{endpoint}}$ $\Rightarrow \dfrac{f(a)+f(b)}{2}$

DISCONTINUOUS POINTS

$$f(x) = \begin{cases} g(x) & \text{for } a < x < b \\ h(x) & \text{for } b < x < c \end{cases}$$

$\{f(x)\}$ at $x = b$ $\Rightarrow \left\{\dfrac{g(b) + h(b)}{2}\right\}$

Discontinuous

Qn :1
Find the fourier series of period $2l$ for the function $f(x) = x(2l-x)$ in $(0-2l)$.
Deduce the sum of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots$

Soln:

Gn: $f(x) = x(2l-x) \quad , 0 \le x \le 2l$

$$(c, c+2l) = (0, 2l)$$

$$\boxed{c = 0 \quad , l = l}$$

To find fourier series, find fourier coefficients

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x)\,dx$$

$$a_0 = \frac{1}{l} \int_{0}^{2l} x(2l-x)\,dx \Rightarrow \frac{1}{l} \int_{0}^{2l} [2lx - x^2]\,dx$$

$$a_0 = \frac{1}{l} \int_{0}^{2l} -x^2\,dx + 2lx\,dx$$

$$= \frac{1}{l} \left[2l \int_{0}^{2l} x\,dx + \int_{0}^{2l} -x^2\,dx \right]$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \frac{1}{l} \left[2l \left[\frac{x^2}{2} \right]_0^{2l} + \left[-\frac{x^3}{3} \right]_0^{2l} \right]$$

$$= \frac{1}{l} \left[2l \left[\frac{x^2}{2} \right]_0^{2l} - \left[\frac{x^3}{3} \right]_0^{2l} \right]$$

$$= \frac{1}{l}\left[2l\left[\frac{(2l)^2}{2} - \frac{0}{2}\right] - \left[\frac{(2l)^3}{3} - \frac{0}{3}\right]\right]$$

$$= \frac{1}{l}\left[2l\left(\frac{4l^2}{2}\right) - \left(\frac{8l^3}{3}\right)\right]$$

$$= \frac{1}{l}\left[l(4l^2) - \left(\frac{8l^3}{3}\right)\right] \Rightarrow \frac{1}{l}\left[4l^3 - \frac{8l^3}{3}\right]$$

$$= \frac{1}{l} \times l^3\left[\frac{12-8}{3}\right] \Rightarrow \frac{1}{l} \times l^3\left(\frac{4}{3}\right) \Rightarrow \frac{4l^2}{3}$$

$$a_0 = \frac{4l^2}{3}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left[\frac{n\pi x}{l}\right]dx$$

$$\therefore a_n = \frac{1}{l}\int_0^{2l} x(2l-x) \times \cos\left[\frac{n\pi x}{l}\right]dx$$

$$= \frac{1}{l}\int_0^{2l}(2lx - x^2)\left(\cos\left(\frac{n\pi x}{l}\right)\right)dx$$

FORMULA: $\int \cos(ax)\,dx = \frac{1}{a}(\mathrm{si}$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 \ldots$$

$u = (2lx - x^2)$

$v = \cos\left(\frac{n\pi x}{l}\right)$

$u' = 2l - 2x$

$v_1 = \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = -2$

$u''' = 0$

$$v_2 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi \cancel{x}}{l}\right)^2} \quad , \quad v_3 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi \cancel{x}}{l}\right)^3}$$

$$a_n = \frac{1}{l}\left[\frac{(2lx - x^2)}{\left(\frac{n\pi \cancel{x}}{l}\right)} \sin\left(\frac{n\pi x}{l}\right) + \frac{(2l - 2x)}{\left(\frac{n\pi \cancel{x}}{l}\right)^2} \cos\left(\frac{n\pi x}{l}\right) + \frac{(-2)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$a_n = \frac{1}{l}\left[\left[\frac{(2lx - x^2)}{\left(\frac{n\pi}{l}\right)} \sin\left(\frac{n\pi x}{l}\right)\right] + \frac{(2l - 2x)}{\left(\frac{n\pi}{l}\right)^2} \cos\left(\frac{n\pi x}{l}\right) + \frac{2\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$\sin 0 = 0$ $\qquad$ $\cos 0 = 1$

$\sin \frac{\pi}{2} = 1$ $\qquad$ $\cos(\pi/2) = 0$

$\sin \pi = 0$ $\qquad$ $\cos \pi = -1$

$\sin(n\pi) = 0$ $\qquad$ $\cos(n\pi) = (-1)^n$

$\sin(2n\pi) = 0$ $\qquad$ $\cos(2n\pi) = 1$

$$a_n = \frac{1}{l}\left[\left[0 + \left[\frac{-2l}{(n\pi/l)^2}\right] + 0\right] - \left[0 + \frac{2l}{(n\pi/l)^2} + 0\right]\right]$$

$$a_n = \frac{1}{l}\left[\frac{-2l^3}{n^2\pi^2} - \frac{2l^3}{n^2\pi^2}\right] \Rightarrow \frac{1}{l}\left[\frac{-4l^3}{n^2\pi^2}\right] \Rightarrow \frac{-4l^3}{n^2\pi^2}$$

$$\boxed{a_n = \frac{-4l^2}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} x(2l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} (2lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

u

$2lx - x^2$

$u' = 2l - 2x$

$u'' = -2$

$u''' = 0$

$dv \Rightarrow \sin\left(\frac{n\pi x}{l}\right)dx$

$v = \sin\left(\frac{n\pi x}{l}\right)$

$v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^3}$

$$\cancel{b_n = \frac{1}{l}\left[\left[\frac{(2lx - x^2)\left(\sin\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)}\right] - \left[\frac{(2l-2x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{(n\pi/l)}\right]\right.}$$

$$b_n = \frac{1}{l}\left[\left[\frac{(2lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)}\right] - \left[\frac{(2l-2x)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right.$$

$$\left. + \left[\frac{(-2)\left(\cos\left(\frac{n\pi x}{l}\right)\right)}{(n\pi/l)^3}\right]\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left[-\frac{\left[(2lx - x^2)(\cos(n\pi x/l))\right]}{\frac{n\pi}{l}}\right] + \left[\frac{(2l-2x)\left(\sin\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right.$$

$$\left. \frac{-2\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left[0 + 0 \quad \frac{-2}{(n\pi/l)^3}\right] - \left[0 + 0 \quad \frac{-2}{(n\pi/l)^3}\right]\right]$$

$$= \frac{1}{l}\left[\frac{-2}{\left(\frac{n\pi}{l}\right)^3} + \frac{2}{(n\pi/l)^3}\right] \Rightarrow 0$$

$$\boxed{b_n = 0}$$

Fourier series Expansion :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore f(x) = \frac{\left(\frac{4l^2}{3}\right)}{2} + \sum_{n=1}^{\infty} \left[\frac{-4l^2}{n^2\pi^2}\right] \cos\left[\frac{n\pi x}{l}\right] + 0$$

$$f(x) = \frac{2l^2}{3} - \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \left[\left[\frac{1}{n^2}\right] \cos\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = \frac{2l^2}{3} - \frac{4l^2}{\pi^2} \left[\frac{\cos\left(\frac{1\pi x}{l}\right)}{1^2} + \frac{\cos\left(\frac{2\pi x}{l}\right)}{2^2} + \frac{\cos\left(\frac{3\pi x}{l}\right)}{3^2}\right.$$

Since l is a continuous point between 0 to $2l$ interval :

formula : $f(x) = f(a) \quad (x = a)$

$\therefore \; x = l$

$f(x) = f(l)$

$$\therefore \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\left(\frac{\pi \cancel{l}}{\cancel{l}}\right)}{1^2} + \frac{\cos\left(\frac{2\pi \cancel{l}}{\cancel{l}}\right)}{2^2} + \frac{\cos\left(\frac{3\pi l}{l}\right)}{3^2}\right.$$

$f(l) = l(2l - \cancel{l})$

$$\frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2}\right.$$

$$l(2l - l) \Rightarrow \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$2l^2 - l^2 \Rightarrow \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} \frac{-1}{3^2} + \cdots\right]$$

$$l^2 - \frac{2l^2}{3} = -\frac{4l^2}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$l^2 - \frac{2l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

Hence Deduced

$\frac{\pi^2}{12}$

3. Find the fourier series expansion with period of $f(x)$ with period $2l$ of $f(x)$ defined by $f(x) = (l-x)^2$ in $(0, 2l)$. Hence deduced that $1/1^2 + 1/2^2 + 1/3^2 + \dots = \pi^2/6$

Soln:

$$f(x) = (1-x)^2, \quad 0 \le x \le 2l$$

$$(c, c+2l) \Rightarrow (0, 2l)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_0 = \frac{1}{l}\int_0^{2l} (1-x)^2\,dx$$

$$= \frac{1}{l}\int_0^{2l} (x^2 + 1^2 - 2x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\,dx + \int_0^{2l} 1\,dx - \int_0^{2l} 2x\,dx$$

$$= \frac{1}{l}\left[\left[\frac{x^3}{3}\right]_0^{2l} + [x]_0^{2l} - \left[2 \times x^2/2\right]_0^{2l}\right]$$

$$= \frac{1}{l}\left[\left(\frac{x^3}{3}\right)_0^{2l} + (x)_0^{2l} + (x^2)_0^{2l}\right]$$

$$= \frac{1}{l}\left[\frac{8l^3}{3} + 2l + 4l^2\right]$$

$$\Rightarrow \frac{1}{l}\Bigg[$$

$$a_0 = \frac{1}{l}\int_0^{2l} (l-x)^2\, dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l} \Rightarrow \frac{1}{l}\left[\frac{(l-2l)^3}{-3} - \left[\frac{(l-0)^3}{-3}\right]\right]$$

$$= \frac{1}{l}\left[\frac{-l^3}{-3} + \frac{l^3}{3}\right] \Rightarrow \left[\frac{l^3}{3} + \frac{l^3}{3}\right] \times \frac{1}{l}$$

$$\boxed{a_0 \Rightarrow \frac{2l^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l}(l-x)^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$uv \cdot u'v_1$

$\int uv\,dx = uv_1 - u'v_2$

$u = (l-x)^2 \quad 2(l-x)(-1)$, $dv = \cos\left(\frac{n\pi x}{l}\right)$

$u' = \cancel{2(l-x)}(-1)$

$\Rightarrow -\frac{2l-2x}{0-1}$

$v_1 = \dfrac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$u'' \Rightarrow -\cancel{2l} + 2$

$v_2 = \dfrac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$u''' \Rightarrow 0$

$v_3 = \dfrac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^3}$

$$\therefore \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}\right) - \left[\frac{(-2l+2x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{(n\pi/l)^2}\right] + \left(\frac{2\times(-\sin(n\pi x/l))}{(n\pi/l)^3}\right)\right]_0^{2l}$$

$$\Rightarrow \frac{1}{l}\left[\frac{(l-x)^2}{\left(\frac{n\pi}{l}\right)}\left(\sin\left(\frac{n\pi x}{l}\right)\right) + \left[\frac{(2x-2l)}{(n\pi/l)^2}\cos\left(\frac{n\pi x}{l}\right)\right] - \frac{2\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$4l - 2l$

$2n\pi$

$$\Rightarrow \frac{1}{l}\left[0 + \left(\frac{2l}{(n\pi/l)^2}\right) + 0\right] - \left[0 + \left(\frac{-2l}{(n\pi/l)^2}\right) + 0\right]$$

$$= \frac{1}{l}\left[\frac{2l}{\left(n\pi/l\right)^2} + \frac{2l}{\left(n\pi/l\right)^2}\right]$$

$$= \frac{1}{l}\left[4l^3/(n\pi)^2\right]$$

$$a_n \Rightarrow \frac{4l^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

$$b_n = \frac{1}{l}\int_0^{2l}(l-x)^2\sin\left(\frac{n\pi x}{l}\right)dx.$$

$u = (l-x)^2$ $\quad v = \sin\left(\frac{n\pi x}{l}\right)$, $v_1 = \dfrac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u' = 2(l-x)(-1)$

$u'' = 2$ $\quad v_2 = \dfrac{-\sin\left(\frac{n\pi x}{l}\right)}{\left[\frac{n\pi}{l}\right]^2}$, $v_3 = \dfrac{+\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$u''' = 0$

$$= \frac{1}{l}\left[\left[\frac{(l-x)^2}{(n\pi/l)}\left[-\cos\left(\frac{n\pi x}{l}\right)\right]\right] - \left[\frac{(2x-2l)}{(n\pi/l)^2}\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\right]\right]_0^{2l}$$

$$+\left[\frac{2\left(\cos\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left[-\frac{((l-x)^2)}{(n\pi/l)}\left[\cos\left(\frac{n\pi x}{l}\right)\right] + \left[\frac{(2x-2l)}{(n\pi/l)^2}\sin\left(\frac{n\pi x}{l}\right)\right] + 2\,\frac{\left[\cos\left(\frac{n\pi x}{l}\right)\right]}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left[-\frac{(l^2 + \cancel{4l^2} - \cancel{4l^2})}{(n\pi/l)} \times 1\right] + [0] + \left[\frac{2}{\left(\frac{n\pi}{l}\right)^3}\right] - \left[\left(-\frac{(l)^2}{(n\pi/l)}\right) + (0) + \frac{2}{(n\pi/l)^3}\right]\right]$$

$$= \frac{1}{l}\left[\left[-\frac{\cancel{(l^2)}}{\cancel{(n\pi/l)}} + \frac{2l^3}{(n\pi)^3}\right] - \left[\frac{-\cancel{(l^2}}{\cancel{(n\pi/l)}} + \cancel{\frac{2l^3}{(n\pi)^3}}\right]\right]$$

$$= \frac{1}{l}[0] \Rightarrow \boxed{b_n \Rightarrow 0}$$

$\therefore$ $\boxed{x = 0}$ $\Rightarrow$ Endpoints :

$\therefore$ $\frac{f(a)+f(b)}{2}$ $\Rightarrow f(0) \Rightarrow l - 0 \Rightarrow l$

22/02/25

Find the fourier series exponsion of,

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(1-x), & 1 \le x < 2 \end{cases}$$ and hence deduce

i) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ (or) $\sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$

Soln:

Limits $[c, c+2l] = (0, 2)$

$\boxed{c=0}$, $c+2l = 2 \Rightarrow \boxed{l=1}$

$$\therefore a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_0 = \frac{1}{l}\int_0^2 f(x)\,dx \Rightarrow \frac{1}{1}\int_0^2 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx \Rightarrow \pi\int_0^1 x\,dx + \pi\int (2-x)$$

$$a_0 = \left[\pi\left(\frac{x^2}{2}\right)_0^1\right] + \left[\pi\left[2x - \frac{x^2}{2}\right]_1^2\right]$$

$$= \pi\left[\frac{1}{2} - 0\right] + \pi\left[\left[4 - \frac{4}{2}\right] - \left[2 - \frac{1}{2}\right]\right]$$

$$= \pi\left[\frac{1}{2}\right] + \pi\left[\frac{8-4}{2} - \left(\frac{4-1}{2}\right)\right] = \frac{\pi}{2} + \pi\left[\frac{1}{2}\right]$$

$$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_{0}^{1} \pi x\cos\left(\frac{n\pi x}{l}\right)dx + \int \pi\int_{1}^{2} 2-x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 1\int_{0}^{1} \pi x\cos(n\pi x)\,dx + \pi\int_{1}^{2} 2-x\cos(n\pi x)\,dx$$

$u = x$	$v = \cos(n\pi x)$
$u' = 1$	$v_1 = \frac{\sin(n\pi x)}{(n\pi)}$
$u'' = 0$	$v_2 = \frac{-\cos(n\pi x)}{(n\pi)^2}$

$u = 2-x$	$v = \cos(n\pi x)$
$u' = -1$	$v_1 = \frac{\sin(n\pi x)}{n\pi}$
$u'' = 0$	$v_2 = \frac{-\cos(n\pi x)}{(n\pi)^2}$

$$a_n = \left[[\pi]\left[x\left(\sin\left[\frac{n\pi x}{n\pi}\right]\right) - \left[\frac{1(-\cos(n\pi x))}{(n\pi)^2}\right]\right]\right]_0^1$$

$$+(\pi)\left[(2-x)\frac{\sin(n\pi x)}{n\pi} - \left[-1\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]\right]_1^2$$

$$a_n = \pi\left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}\right]_0^1$$

$$+ (\pi)\left[\frac{(2-x)\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{(n\pi)^2}\right]_1^2$$

$$= \pi\left[0 + \frac{(-1)^n}{(n\pi)^2}\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$\Rightarrow \frac{2\pi(-1)^n}{(n^2\pi^2)} - \frac{2\pi}{n^2\pi^2} \Rightarrow \frac{2}{n^2\pi^2}(-1)^n - 1$$

$$a_n = \begin{cases} -4/n^2\pi^2 & \to n \text{ is odd} \\ 0 & \to \text{is even} \end{cases}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\,\cancel{dx}\left(\sin\left(\frac{n\pi x}{l}\right)dx\right)$$

$$= 1\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x(\sin(n\pi x))\,dx + \pi\int_1^2 2-x \times \sin(n\pi x)$$

$$= \pi \int_0^1 x(\sin(n\pi x))\,dx + \pi \int_1^2 (2-x)\sin(n\pi x)\,dx$$

$$= \pi \left[x \left[-\cos(n\pi x)\right] \times \frac{1}{n\pi} \right]_0^1 - \left[1\left(-\sin(n\pi x)\right) \times \frac{1}{(n\pi)^2} \right]$$

$$+ \pi \left[\left[2-x \left(-\cos(n\pi x)\right) \times \frac{1}{n\pi} \right] - \left[\frac{-1}{(n\pi)^2}\left(-\sin(n\pi x)\right) \right] \right]_1^2$$

$$= \pi \left[\left[-\frac{1}{n\pi}(-1)^n \right] - 0 \right] - [0] \Big] + \pi \left[(0-0) - \left[\frac{-1}{(n\pi)}(-1)^n \right] \right]$$

$$= \pi \left[-\frac{1}{n\pi}(-1)^n \right] + \pi \left[(-1)^n \left(\frac{1}{n\pi} \right) \right]$$

$$\boxed{b_n = 0}$$

Fourier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) +$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{-4}{n^2\pi^2} \left(\cos(n\pi x) \right) \right] + 0$$

$$= \frac{\pi}{2} + \left[\frac{-4}{\pi^2}\right] \sum_{n=1}^{\infty} \left[\cos(n\pi x) \times \frac{1}{n^2} \right]$$

$$\Rightarrow \frac{\pi}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos(n\pi x) \right]$$

24/10 $f(x) = x\cos x \quad , 0 \leq x \leq 2\pi$

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{l}\right]dx + \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]dx$$

$$a_0 = \frac{1}{l}\int f(x)\,dx.$$

$(0 \, , 2\pi) \Rightarrow c, \, c+2l$

$c = 0$

$c + \cancel{2}l = \cancel{2}\pi$

$\boxed{l = \pi}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos x\,dx$$

$udv = uv' + vu'$

$x (\sin$

$$= \frac{1}{2\pi}\left[x\sin x - 1[(-\cos x)^2]\right]^2$$

$$= \frac{1}{2\pi}\left[0 + 1 - 0 - 1\right] \Rightarrow 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n+1)x + \cos(n-1)x\right]dx$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$= \left[\frac{1}{2\pi}\left\{(x)\frac{\sin(n+1)x}{n+1} + \frac{\cos(n+1)(x)}{(n+1)^2}\right]_0 + \right.$$

$$\left.\left\{\frac{x}{n-1}\sin(n+1)x\right) + \frac{\cos(n-1)x}{(n-1)^2}\Bigg]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(0 + \frac{1}{(n+1)^2}\right) - \left(0 + \frac{1}{(n-1)^2}\right)\right]$$

$$\boxed{a_n = 0}$$

for $n \geq 1$

NOTE :
$f(x) = \sin(kx)$ or $\cos(kx)$

Calculating $\Rightarrow a_0, a_n, b_n$ & additionally a_K, b_K.

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2\pi} x\cos x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left[\frac{1+\cos(2x)}{2}\right]dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\,dx + \int_0^{2\pi} x\cos(2x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2}\right)_0^{2\pi} + \left[\left(\frac{x}{2}(\sin(2x))\right] + \frac{\cos(2x)}{4}\right.\right.$$

$$= \frac{1}{2\pi}\left[\left[\frac{4\pi^2}{2} - 0\right] + \left[0 + \frac{1}{4} - (0 + 1/4)\right]\right]$$

$$= \frac{1}{2\pi} \times \frac{4\pi^2}{2} \Rightarrow \frac{4\pi^2}{4\pi} \Rightarrow \boxed{\pi}$$

$$\boxed{a_1 = \pi}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_0^{2\pi} x\cos x\,(\sin nx)\, dx$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right]$$

$$= \frac{1}{l}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(\sin(n+1)x)\,dx - \int_0^{2\pi} \sin(n-1)x\,dx$$

$$b_n = -1/n+1 \quad -1/n-1$$

$$b_n = \frac{-n+1-n-1}{(n^2-1)} \Rightarrow -\frac{2n}{n^2-1} \quad (n \neq 1)$$

$$b_1 = \frac{1}{l}\int_{c}^{c+2l} f(x)\, \sin\left[\frac{\pi x}{l}\right] dx$$

$$= \frac{1}{l}\int_{0}^{2\pi} x\cos x \; \sin x \, dx$$

$$b_1 = 1/2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{\pi}\right) + \sum_{n=2}^{\infty} a_n \cos(\pi x) + b_1 \sin\left(\frac{\pi x}{\pi}\right)$$

$$+ \sum_{n=2}^{\infty} b_n \sin[\pi x]$$

$$= \pi(\cos x) + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\left[\frac{-2n}{n^2-1}\right]\sin(nx)$$

---

hw:

$$f(x) = \begin{cases} x & \text{in } 0 < x < 2 \\ 0 & \text{in } 2 < x < 4 \end{cases}$$

② $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 3 \end{cases}$

$c, c+2l \Rightarrow (0, 3)$

$0, \; 0 + 2l = 3$

$2l = 3$

$$\boxed{l = 3/2}$$

$f(x) = x^2$ when $-\pi < x < \pi$ in a fourier series of 2π. Hence deduce.

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ to $\infty = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$ to $\infty = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ to $\infty = \frac{\pi^2}{8}$

__Soln:__

__given:__ $f(x) = x^2$, $(-\pi \leq x \leq \pi)$

$\therefore (c, c+2l) \Rightarrow (-\pi, \pi)$

$\boxed{c = -\pi,}$

$-\pi + 2l = \pi \Rightarrow 2l = 2\pi$

$\boxed{l = \pi}$

$f(x) = x^2$,

$f(-x) = (-x)^2$

$f(-x) = x^2$

$f(x) = f(-x)$

$\therefore f(x)$ is an even function.

Hence $\boxed{b_n = 0}$

$$\int_{-a}^{a} = 2\int_{0}^{a}$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi} \times 2\int_{0}^{\pi} x^2\,dx \Rightarrow \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \Rightarrow \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] \Rightarrow \boxed{\frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left[\frac{n\pi x}{l}\right]dx$$

$$= \frac{1}{l}[2]\int_0^{\pi} x^2\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$u = x^2$

$u' = 2x$

$u'' = 2$

$u''' = 0$.

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi}\left[x^2\left(\frac{\sin(nx)}{n}\right) - \left[2x\left[\frac{-\cos(nx)}{n^2}\right]\right] + \left[2\left(\frac{-\sin(nx)}{n^3}\right)\right]\right]$$

$\cos(nx) = v$

$v_1 = \frac{\sin(nx)}{n}$

$v_2 = \frac{-\cos(nx)}{n^2}$

$v_3 = \frac{-\sin(nx)}{n^3}$

$$= \frac{2}{\pi}\int_0^{\pi}\left[\frac{x^2}{n}\left[\sin(nx)\right] + \frac{2x}{n^2}\left[\cos(nx)\right] - \frac{2}{n^3}\left[\sin(nx)\right]\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}\left[(-1)^n\right] - \left[0 + 0 + 0\right]\right] \Rightarrow \frac{4\pi}{\pi n^2}(-1)^n$$

$$a_n = 4/n^2\,(-1)^n$$

Fourier Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{l}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2}(-1)^n\right] \cos\left[\frac{n\pi x}{\pi}\right] + 0$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2}(-1)^n\right] \cos(nx)$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \left[\frac{1}{n^2}(-1)^n\right] \cos(nx).$$

To Deduce :

1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$ to $\frac{\pi^2}{6}$

Let $\boxed{x = \pi}$ where 'π' is an endpoint

$$f(x) = \frac{f(a) + f(b)}{2} \Rightarrow \frac{f(-\pi) + f(\pi)}{2}$$

$$f(-\pi) = (-\pi)^2, \quad f(\pi) = \pi^2$$

$$f(x) = \frac{\pi^2 + \pi^2}{2} \Rightarrow \frac{2\pi^2}{2} \Rightarrow \pi^2$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}(-1)^n\right] \cos(nx)$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2}(-1)^1[\cos(x)]\right] + \left[\frac{1}{2^2}(-1)^2[\cos 2x]\right]$$

$$+ \left[\frac{1}{3^2}(-1)^3[\cos(3x)]\right] \cdots \quad \left[\begin{matrix}\text{since} \\ x = \pi\end{matrix}\right]$$

$$\frac{3\pi^2 - \pi^2}{3} = 4\left[\left[\frac{1}{1^2}(-1)^1[\cos\pi]\right] + \left[\frac{1}{2^2}(-1)^2[\cos 2\pi]\right] + \left[\frac{1}{3^2}(-1)^3[\cos 3\pi]\right] \dots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{-1}{1^2}(-1) + \frac{1}{2^2}(1)(1) + \frac{1}{3^2}(-1)(-1)\right]$$

$$\frac{2\pi^2}{4\times 3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \infty \rightarrow (1)$$

ii] To Deduce to,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \infty \text{ to } = \frac{\pi^2}{12}$$

Let $x = 0$, a continuous point.

$$\therefore f(x) = f(0) \Rightarrow [0]^2 \Rightarrow 0.$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}(-1)^n\left[\cos\left[\frac{n\pi x}{l}\right]\right]\right]$$

$$\frac{-\pi^2}{3} = 4\left[\frac{1}{1^2}(-1)^1[\cos(0)] + \frac{1}{2^2}(-1)^2[\cos 0] + 1/3^2 (-1)^3[\cos 0]\right]$$

$$-\frac{\pi^2}{12} = \left[-\left[\frac{1}{1^2}\right] + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$-\frac{\pi^2}{12} = -\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \infty \rightarrow (2)$$

To Deduce to,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \infty \text{ to } \frac{\pi^2}{8}$$

Let's, add (1) & (2)

$$\frac{\pi^{2\times 2}}{6\times 2} + \frac{\pi^2}{12} = \left[\frac{1}{1^2} + \cancel{\frac{1}{2^2}} + \frac{1}{3^2} + \dots + \infty\right] +$$

$$\left[\frac{1}{1^2} - \cancel{\frac{1}{2^2}} + \frac{1}{3^2} - \dots \infty\right]$$

2 | 6, 12
3 | 3, 6
2 | 1, 2
 | 1, 1

$$\frac{2\pi^2 + \pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \infty\right]$$

$$\frac{3\pi^2}{12}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \infty$$

Hence Proved

Find the fourier series of

$f(x) = |\cos x|$ in $(-\pi, \pi)$

Soln:

Given: $(c, c+2l) = (-\pi, \pi)$

$\boxed{c = -\pi}$, $-\pi + 2l = \pi$

$\boxed{l = \pi}$

$f(x) = |\cos x|$

$f(-x) = \cos(-x)$

$f(-x) = \cos x$

$\therefore f(x) = f(-x)$,

it is an even function,

$\boxed{b_n = 0}$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\,dx$$

$\rightarrow 0 \rightarrow 90 \; (+,+) \Rightarrow \cos(+ve)$ [1st]

$\rightarrow (90-180)$ [2nd] $\Rightarrow \cos(-ve)$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right.$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[(0-1) - [(-1)-0]\right] \Rightarrow \frac{2}{\pi}(-1+1) \Rightarrow 0$$

$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$

$= \frac{2}{\pi}\left[(1-0) - (0-(1))\right] \Rightarrow \frac{2}{\pi}(1+1)$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left[\frac{n\pi x}{l}\right]dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x\left[\cos\left(\frac{n\pi x}{l}\right)\right]dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\,(\cos nx) + \int_0^{+\pi} \cos x\,(\cos nx)\,dx\right.$

$= \left[\right]$

26/02

1. Find the fourier series of $f(x) = x$ in interval $(-4, 4)$

Soln:

Given: $(c, c+2l) \Rightarrow (-4, 4)$

$c = -4, \quad -4+2l = 4$

$2l = 8$

$\boxed{l = 4}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{l}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \Rightarrow \frac{1}{4}\int_{-4}^{4} x\,dx$$

$$a_0 = \frac{1}{4}\left[x^2/2\right]_{-4}^{4} \Rightarrow \frac{1}{8}\left[16-16\right] \Rightarrow 0$$

$a_0 = 0$

$\therefore f(x) = x$

$f(-x) = -x$

$f(x) \neq f(-x)$

So odd function, $(\therefore a_0, a_n \Rightarrow 0)$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\left[\frac{n\pi x}{l}\right] dx$$

$$= \frac{1}{4}\int_{-4}^{4} x \sin\left[\frac{n\pi x}{4}\right] dx$$

$$= \frac{1}{4}\left[x \frac{\left[-\cos\left(\frac{n\pi x}{4}\right)\right]}{\left(\frac{n\pi x}{4}\right)} - \left[1 \frac{\left(-\sin\left(\frac{n\pi x}{4}\right)\right)}{\left(\frac{n\pi x}{4}\right)^2} \right] \right]_{-4}$$

$u = x$

$u' = 1$

$u'' = 0$

$\sin\left(\frac{n\pi x}{4}\right) = v$

$\int v_1 = \frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi x}{4}\right)}$

$-\sin$

$\frac{-4(4)}{n\pi}(-1)^n - \left[\frac{-4(-4)}{n\pi}\left[(-1)^n(-1)\right]\right]$

$$\frac{1}{4}\left[\frac{-x\left(\cos\left(\frac{n\pi x}{4}\right)\right)}{\left(\frac{n\pi x}{4}\right)} + \frac{\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi x}{4}\right)^2} \right]_{-4}^{4}$$

$\frac{16}{4} / 64$

$$= \frac{1}{4}\left[\left[\frac{-16}{n\pi}(-1)^n + 0\right] + \left[\frac{64}{(n\pi)^2}(-1)^n\right]\right]$$

$$= \frac{1}{4}\left[\frac{-16}{n\pi}(-1)^n + \frac{64}{(n\pi)^2}(-1)^2\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{n\pi x}{l}\right] + \sum_{n=1}^{\infty} b_n\left[\sin\left(\frac{n\pi x}{l}\right)\right]$$

$\int^{l}$

Q] $f(x) = x^2 + x$ in $(-2, 2)$ also find,

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Soln:

$f(x) = x^2 + x$

$f(-x) = (-x)^2 + (-x) \Rightarrow x^2 - x$

$f(x) \neq f(-x)$

~~Q~~ $f(x) \neq -f(x)$

$\therefore$ Neither odd nor even.

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx$$

$(c, c+2l) \Rightarrow (-2, 2)$

$\boxed{c = -2}$, $-2 + 2l = 2$, $\boxed{l = 2}$

$$a_0 = \frac{1}{2} \int_{-2}^{2} (x^2 + x)\,dx$$

$$a_0 = \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-2}^{2}$$

$$= \frac{1}{2} \left[\underbrace{\left(\frac{8}{3} - \frac{(-8)}{3} \right)}_{\int x^2 dx \text{ even}} + \underbrace{\left(\frac{4}{2} - 4/2 \right)}_{\int x\,dx \text{ odd} \Rightarrow 0} \right]$$

$\boxed{a_0 = 0}$

$$= \frac{1}{2} \left[\frac{16}{3} \right] \Rightarrow \frac{8}{3}$$

$a_0 = 8/3$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left[\frac{n\pi x}{l}\right] dx$$

$u = x^2 + x$

$u' = 2x + 1$

$u'' = 2$

$$a_n = \frac{1}{2}\int_{-2}^{2} (x^2+x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$v = \cos\left(\frac{n\pi x}{2}\right)$

$v_1 = \dfrac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}$

$v_2 = \dfrac{\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)}$

$$a_n = \frac{1}{2}\left[\int_{-2}^{2} x^2\cos\left(\frac{n\pi x}{2}\right) dx + \int_{-2}^{2} x\cos\left(\frac{n\pi x}{2}\right) dx\right]$$ odd $= 0$.

$v_3 = -\dfrac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^3}$

$u = x^2$

$u' = 2x$

$u'' = 2$

$u''' = 0$

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2\cos\left(\frac{n\pi x}{2}\right) dx + 0$$

$$a_n = \frac{1}{2}\left[\frac{x^2\left(\sin\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)} - \left[\frac{2x\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)^2}\right] + \left[\frac{2\left(-\sin\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)^3}\right]\right]_{-2}^{2}$$

$$a_n = \frac{1}{2}\left[\left[\frac{2x^2}{n\pi}\left(\sin\left(\frac{n\pi \overset{2}{x}}{2}\right)\right)^{0}\right] + \left[\frac{4x(2)}{(n\pi)^2}\cos\left(n\pi \overset{(-2)}{x}/2\right)\right] \left[\frac{8}{(n\pi)^3}\left(\sin\left(\frac{n\pi x}{2}\right)^{0}\right)\right]\right]_{-2}^{2}$$

$$a_n = \frac{1}{2}\left[\frac{4(2)}{(n\pi)^2}[-1]^n - \left[\frac{+8(2)}{(n\pi)^2}\left[-(-1)^n\right]\right]\right]$$

$$a_n = \frac{1}{2}\left[\frac{8}{(n\pi)^2}(-1)^n + \left[\frac{16}{(n\pi)^2}(-1)^n\right]\right]$$

$$a_n = \frac{8}{2(n\pi)^2}\left[(-1)^n + 2(-1)^n\right]$$

$$a_n = \frac{24}{(n\pi)^2}(-1)^n \times \frac{1}{2} \Rightarrow \boxed{\frac{12}{(n\pi)^2}(-1)^n}$$

HALF

FOR

Half Range Sine & Cosine Series:

Half Range Cosine Series ::

Interval $(0, l)$

Formula : $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\dfrac{n\pi x}{l}\right]$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left[\frac{n\pi x}{l}\right]dx.$$

FIND $\to a_0, a_n, a_1$

$a_1 \to$ coeff of x

Half Range Sine Series:

Formula : $\sum_{n=1}^{\infty} b_n \sin\left[\dfrac{n\pi x}{l}\right]$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left[\frac{n\pi x}{l}\right]dx$$

FIND

b_n, b_1

coeff of x.

Eg

$\sin(mx)$,

$\sin(10x)$

1] Obtain sine series for $f(x) = \pi - x$ in the interval $(0, \pi)$ and hence prove that,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \pi/4.$$

$(0, l) \Rightarrow (0, \pi)$

$\boxed{l = \pi}$

Soln : let $\boxed{l = \pi}$

FORMULA :

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left[\frac{n\pi x}{l}\right] dx$$

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x) \sin\left[\frac{n\cancel{\pi} x}{\cancel{\pi}}\right] dx$$

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x) \sin(nx)\, dx$$

$u = (\pi - x)$

$u' = -1$

$u'' = 0$

$v = \sin(nx)$

$v_1 = \frac{-\cos(nx)}{n}$

$v_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{2}{\pi}\left[\left[\frac{(\pi - x)(-\cos(nx))}{n}\right] - \left[\frac{-1}{n^2}(-\sin(nx))\right]\right]_0^\pi$$

$$b_n = \frac{2}{\pi}\left[\left(-\left[\frac{\pi - x}{n}\cos(nx)\right] - \left[\frac{1}{n^2}(\sin(nx))\right]\right]\right]_0^\pi$$

$$b_n = \frac{-2}{\pi}\left[\frac{(\pi - x)}{n}\cos(nx) + \frac{1}{n^2}(\sin(nx))\right]_0^\pi$$

$$b_n = \frac{-2}{\pi}\left[(0 + 0) - \left(\pi/n + 0\right)\right] \Rightarrow \frac{-2}{\pi}\left[0 - \pi/n\right] \Rightarrow \boxed{2/n}$$

$\frac{2}{\cancel{\pi}}(\cancel{\pi}/n)$

$\boxed{b_n = 2/n}$

$\therefore$ Fourier sine series or half range sine series or sine series for given $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$\therefore \cancel{f(x) = \frac{2}{n} \sum_{n=1}^{\infty} b_n}$$

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin\left[\frac{n\cancel{\pi} x}{\cancel{\pi}}\right]$$

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

Considering $x = \pi/2$ $[\sin(n\pi) = 0]$

$\therefore$ $f(x) = \pi - x$

$f(\pi/2) = \pi - \pi/2 \Rightarrow \pi/2$

$\boxed{\pi/2 = x}$

360
+90
450

90
5

$$f(x) = 2\left[\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right]$$

90
5

$$\pi/2 = 2\left[\frac{1}{1}\sin\pi/2 + \frac{1}{2}\sin\left[2\times\pi/2\right] + \frac{1}{3}\sin\left[3\pi/2\right] + \frac{1}{4}\sin\left[\frac{4\pi}{2}\right] + \frac{1}{5}\sin\left[5\pi/2\right]\right] \ldots$$

$$\frac{\pi}{2} = 2\left[1 + 0 + (1/3(-1)) + 0 + \frac{1}{5}(1) \ldots\right]$$

$$\pi/2 = 2\left[1 - 1/3 + 1/5 \ldots\right] \Rightarrow \frac{\pi}{2} \times \frac{1}{2} = \left[1 - 1/3 + 1/5\right]$$

$$\Rightarrow \pi/4 = (1 - 1/3 + 1/5)$$

2] i] Find the cosine series and ii] half range cosine series, for the function.

$$f(x) = \begin{cases} x & , 0 < x < 1 \\ (2-x) & , 1 < x < 2 \end{cases}.$$

i] Cosine

ii] Sine.

i] COSINE SERIES :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^2 f(x)\,dx \Rightarrow \frac{2}{l}\int_0^1 x\,dx + \frac{2}{l}\int_1^2 (2-x)\,dx$$

$$= \frac{2}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{2}{2}\left[2x - \frac{x^2}{2}\right]_1^2$$

$$\Rightarrow 2[x]_1^2 - \left[\frac{x^2}{2}\right]_1^2$$

$$\Rightarrow \left[2[2] - 2[1]\right] - \left[\frac{4}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{2} + \left[\left[(2)(2) - \frac{(2)^2}{2}\right] - \left[2(1) - \frac{1^2}{2}\right]\right]$$

$$(4-2) - (3/2)$$

$$2 - 3/2 \Rightarrow \frac{4-3}{2} \Rightarrow 1/2$$

$$= \frac{1}{2} + \left[\left[4 - \frac{4}{2}\right] - \left[2 - \frac{1}{2}\right]\right] \Rightarrow \frac{1}{2} + \left[\frac{2}{2} - \frac{1}{2}\right] \Rightarrow \frac{1}{2} + \frac{1}{2}$$

$$\boxed{a_0 \Rightarrow 1}$$

$$a_n = \frac{2}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\left[x\left(\frac{\sin\frac{n\pi x}{2}}{(n\pi/2)}\right) - 1\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]_0^1$$

$u = x$

$u' = 1$

$u'' = 0$

$v = \cos\left(\frac{n\pi x}{2}\right)$

$v_1 = \frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}$

$v_2 = \frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}$

$$+$$

$$\left[\frac{(2-x)\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)} - 1\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right]\right]_1^2$$

$u = 2-x$

$u' = -1$

$u'' = 0$.

$$\left[\frac{2x}{n\pi}\left(\sin\left(\frac{n\pi x}{2}\right)\right) + \frac{4}{(n\pi)^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]\right]_0^1$$

$$+$$

$$\left[\frac{(2-x)2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right] + \left[\frac{4}{(n\pi)^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]\right]\right]_1^2$$

$$\Rightarrow \left[\frac{2}{n\pi} + 0 - \left(0 + \frac{4}{(n\pi)^2}\right)\right] + \left[0 + \frac{4}{(n\pi)^2}(-1)^n - \left(\frac{2(2-1)}{n\pi} + 0\right)\right]$$

$$= \left[\cancel{\frac{2}{n\pi}} - \frac{4}{(n\pi)^2} + \frac{4}{(n\pi)^2}(-1)^n - \cancel{\frac{2}{n\pi}}\right] \Rightarrow \frac{-4}{(n\pi)^2}\left[1 + (-1)^n\right]$$

COSINE SERIES:

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n^2}(1+(-1)^n)\cos\left[\frac{n\pi x}{2}\right]\right)\right]$$

ii) SINE SERIES: $(0, l) \Rightarrow (0, 2) \quad \therefore \boxed{l = 2}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left[\frac{n\pi x}{l}\right]dx \Rightarrow \frac{2}{2}\int_0^2 f(x)\sin\left[\frac{n\pi x}{2}\right]dx$$

$$\Rightarrow \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = x$
$u' = 1$
$u'' = 0$

$\sin\left(\frac{n\pi x}{2}\right) = v$

$v_1 = \dfrac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}$

$v_2 = \dfrac{-\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}$

$u = 2 - x$
$u' = 0 - 1 \Rightarrow -1$
$u'' = 0$

$$\Rightarrow \left[\frac{x\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)} - \left[1\left(\frac{-\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]\right]_0^1 +$$

$$\left[(2-x)\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right] - \left[-1\left(\frac{-\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)\right]\right]_1^2$$

$$\Rightarrow \left[-\frac{2x}{n\pi}\left(\cos\left(\frac{n\pi x}{2}\right)\right) + \left[\frac{4}{(n\pi)^2}\left[\sin\left(\frac{n\pi x}{2}\right)\right]\right]\right]_0^1$$

$$+\left[-\left[\frac{2(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right] - \left[\frac{4}{(n\pi)^2}\left(\sin\left(\frac{n\pi x}{2}\right)\right)\right]\right]_1^2$$

$\sin\left(\frac{n\pi}{2}\right) \Rightarrow ?$

$$\Rightarrow \left[\left[0 + \frac{4}{(n\pi)^2}\right] - \left[0 + 0\right]\right] + \left[0 - 0 - \left(0 - \frac{4}{(n\pi)^2}\right)\right]$$

$$\Rightarrow \frac{4}{(n\pi)^2} + \frac{4}{(n\pi)^2} \Rightarrow \frac{8}{(n\pi)^2}$$

$$\boxed{b_n = 8/(n\pi)^2}$$

$$\therefore \quad f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left[\frac{n\pi x}{2}\right]$$

ONE DIMENSIONAL WAVE EQUATION

One dimensional wave equation is

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}} \rightarrow (1) \quad \text{, where}$$

$$\boxed{a^2 = T/m}$$

The three possible solns of 1d wave equations :

(*) $y(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda at} + De^{-at})$

(*) $y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$

(*) $y(x,t) = (Ax+B)(Ct+D)$

where, A, B, C, D $\rightarrow$ arbitrary constants

→ We use (2) solution for the periodic function, which deals with the problems on vibration of string

⇒ Now, to solve 1D wave eqn, the soln (2) where we need to find the constant A, B, C, D. with the following initial & boundary conditions

If zero initial velocity, the condition and position is given as $f(x)$

*) $y(0,t) = 0 \quad , \forall t \geq 0$
*) $y(l,t) = 0 \quad , \forall t \geq 0$

} boundary condition (limits)

*) $\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0 \quad , 0 \leq x < l$

*) $y(x,0) = f(x) \quad , 0 \leq x < l$

} initial condition ($t=0$)

If the initial velocity is given a $f(x)$ then conditions are,

$$y(0, t) = 0 \quad , \quad \forall t \geq 0$$

$$y(l, t) = 0 \quad , \quad \forall t \geq 0$$

$$y(x, 0) = 0 \quad , \quad 0 \leq x < l$$

$$\left[\frac{\partial y}{\partial t}\right]_{t=0} = f(x) \quad , \quad 0 \leq x < l$$

PROBLEMS :

ZERO INITIAL VELOCITY

i] A tightly stretched string with fixed endpoints $x=0$, $x=l$ is initially in a position given by $y=f(x)$, if released from rest from this position, find the displacement at any time at any distance from the end $x=0$.

Soln: Where $f(x)$

The wave eqn : $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ → ①

the soln of wave eqn ① is given by,

$$y(x,t) = \left[(A\cos(\lambda x) + B\sin(\lambda x))(C\cos(\lambda at) + D\sin(\lambda at))\right]$$

Given ⇒ zero initial velocity,

$$\therefore \boxed{f(x) = kx(l-x)}$$

since the boundary conditions are,

i] $y(0, t) = 0$ for $t \geq 0$

ii] $y(l, t) = 0$ for $t \geq 0$ } → BOUNDARY

~~iii] $[\partial y/\partial t]_{t=0} = 0$ for~~

iii) $\left[\frac{\partial y}{\partial t}\right]_{t=0} = \frac{\partial y(x,0)}{\partial t} = 0$ for $0 \le x \le l$

iv) $y(x,0) = f(x)$, $0 \le x \le l$ initial cond.

Applying the boundary condition (i) in the solution ② eqn

i) $y(0,t) = 0$ in

$\rightarrow y(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)](C\cos(\lambda at) + D\sin(\lambda at))$

Put $x = 0$,

$\therefore$

$\Rightarrow [A\cos(0) + B\sin(0)][C\cos(\lambda at) + D\sin(\lambda at)] = 0$

$\Rightarrow [1(A) + 0][C\cos(\lambda at) + D\sin(\lambda at)] = 0$

$\Rightarrow A[C\cos(\lambda at) + D\sin(\lambda at)] = 0$

$\boxed{A = 0}$ and $\boxed{C\cos(\lambda at) + D\sin(\lambda at) \neq 0}$

$y(x,t) = B\sin(\lambda x)[C\cos(\lambda at) + D\sin(\lambda at)] \rightarrow ③$

Now apply (ii) in ③, (i.e) Put $\boxed{x = l}$

$y(l,t) = 0$

$B\sin(\lambda l)[C\cos(\lambda at) + D\sin(\lambda at)] = 0$

$B \neq 0$, $\sin(\lambda l) = 0$, $C\cos(\lambda at) + D\sin(\lambda at) \neq 0$

$$\boxed{\begin{aligned}\lambda l &= \sin^{-1}(0)\\ \lambda l &= n\pi\end{aligned}}$$

$\therefore \boxed{\lambda = \frac{n\pi}{l}}$ in eqn

$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(a\left(\frac{n\pi}{l}\right)t\right) + D\sin\left[\frac{a(n\pi)}{l}t\right]\right] \rightarrow ③$ eqn becomes $\rightarrow ④$

Diff. (4) p.w.r.to. 't'

$$\frac{\partial y(x,t)}{\partial t} = B\sin\left[\frac{n\pi x}{l}\right]\left[-\frac{Can\pi}{l}\sin\left[\frac{an\pi t}{l}\right]\right] + \left[\frac{Dan\pi}{l}\cos\left(\frac{an\pi t}{l}\right)\right]$$

Apply (iii) $\Rightarrow$ (i.e) $\boxed{t=0}$

$$B\sin\left[\frac{n\pi x}{l}\right]\left[0 + \frac{Dan\pi}{l}\right] = 0$$

$$B \neq 0 \quad , \quad \sin\left[\frac{n\pi x}{l}\right] \neq 0 \quad , \quad \frac{Dan\pi}{l} = 0 \; , \; (D=0)$$

(4) $\Rightarrow$ $y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[c\cos\left(\frac{an\pi t}{l}\right)\right]$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{an\pi t}{l}\right)$$

Apply (iv) $\Rightarrow$ (i.e) $\boxed{t=0}$

$$y(x,0) = f(x)$$

$$Kx(l-x) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left[\frac{n\pi x}{l}\right]dx.$$

$u = Kx(l-x)$

$u = Kxl - x^2$

$\Rightarrow K[xl - x^2]$

$\boxed{u' \Rightarrow Kl - 2xK}$

$\boxed{u'' = -2K}$

$$b_n = \frac{2}{l}\int_0^l Kx(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\left[\frac{(Kx(l-x))}{(n\pi/l)}\left[-\cos\left(\frac{n\pi x}{l}\right)\right] - \left[Kl - 2xK\right] - \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right.$$

$$+ \left[-2K\left(\cos\left(n\pi x/l\right)\right)/(n\pi/l)^3\right)$$

$$= \frac{2k}{l}\left[\frac{x(l-x)}{(n\pi/l)}\left[-\cos\left(\frac{n\pi x}{l}\right)\right] + \left[\frac{(l-2x)}{(n\pi/l)^2}\right]\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$-\left[2/(n\pi/l)^3\right]\cos(n\pi x/l)\Big]\Big]_0^l$$

$$= \frac{2k}{l}\left[\frac{l^2x - lx^2}{n\pi}\left[-\cos\left(\frac{n\pi x}{l}\right)\right] + \left[\frac{l^2(l-2x)}{(n\pi)^2}\right]\left[\sin\left(\frac{n\pi x}{l}\right)\right]\right.$$

$$\left.-\frac{2l^3}{(n\pi)^3}\left[\cos\left(\frac{n\pi x}{l}\right)\right]\right]_0^l$$

$$= \frac{2k}{l}\left[\left(0+0-\frac{2l^3}{(n\pi)^3}(-1)^n\right) - \left(0+0-\frac{2l^3}{(n\pi)^3}\right)\right]$$

$$= \frac{2k}{l}\left[-\frac{2l^3}{(n\pi)^3}(-1)^n + \frac{(2l^3)}{(n\pi)^3}\right] \Rightarrow$$

$$\Rightarrow \frac{2k}{l}\left[\frac{2l^3}{(n\pi)^3}\left[1-(-1)^n\right]\right]$$

$$\boxed{b_n \Rightarrow \frac{4kl^2}{(n\pi)^3}\left[1-(-1)^n\right]}$$

$n \to$ odd $\qquad\qquad$ $n \to$ even

$$\frac{4kl^2}{(n\pi)^3}[1+1] \to \frac{8kl^2}{(n\pi)^3}, \quad \frac{4kl^2}{(n\pi)^3}(1-1) = 0$$

Sub B_n in 5

$$y(x,t) = \sum_{n=1,3,5}^{\infty}\left[\frac{8kl^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)\right] \ldots \to 6$$

(6) $y(x,0) = f(x) = \begin{cases} \text{Equ of } AB, & 0 < x < l/3 \\ \text{" of } BC, & l/3 < x < 2l/3 \\ \text{" of } AC, & 2l/3 < x < l \end{cases}$

Eqn of AB $(0,0)$ & $(l/3, d)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{d-0} = \frac{x-0}{l/3-0} \Rightarrow \frac{y}{d} = \frac{x}{l/3}$$

$$\boxed{y = \frac{3dx}{l}}$$

Eqn of BC $(l/3, d)$ & $(2l/3, -d)$

$$\frac{y-d}{-2d} = \frac{3x-l}{2l-l} \Rightarrow y = \frac{3d}{l}(l-2x),$$

$$l/3 < x < 2l/3$$

Eqn of CD $(2l/3, -d), (l, 0)$

$$\frac{y+d}{d} = \frac{3x-2l}{3l-2l} \Rightarrow y = \frac{3d}{l}(x-l),$$

$$2l/3 < x < l.$$

The initial position of the string is,

$$y(x,0) = f(x) = \begin{cases} \frac{3dx}{l}, & 0 < x < l/3 \\ \frac{3d}{l}(l-2x), & l/3 < x < 2l/3 \\ 3d/l\,(x-l), & 2l/3 < x < 3l/3 \end{cases}$$

BOUNDARY CONDITIONS:

i] $y(0,t) = 0, \quad \forall t > 0$

ii] $y(l,t) = 0$

iii] $\frac{\partial y(x,0)}{\partial t} = 0$

iv] $y(x,0) = \begin{cases} \frac{3dx}{l}, & 0 < x < l/3 \\ 3d/l\,(l-2x), & l/3 < x < 2l/3 \\ \frac{3d}{l}(x-l), & 2l/3 < x < l \end{cases}$

10) $y(x,0) = f(x) = \begin{cases} \text{Equ of } AB, & 0 < x < l/3 \\ \text{" of } BC, & l/3 < x < 2l/3 \\ \text{" of } AC, & 2l/3 < x < l \end{cases}$

Eqn of AB $(0,0)$ & $(l/3, d)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{d-0} = \frac{x-0}{l/3-0} \Rightarrow \frac{y}{d} = \frac{x}{l/3}$$

$$\boxed{y = \frac{3dx}{l}}$$

Eqn of BC $(l/3, d)$ & $(2l/3, -d)$

$$\frac{y-d}{-2d} = \frac{3x-l}{2l-l} \Rightarrow y = \frac{3d}{l}(l-2x),$$

$l/3 < x < 2l/3$

Eqn of CD $(2l/3, -d), (l, 0)$

$$\frac{y+d}{d} = \frac{3x-2l}{3l-2l} \Rightarrow y = \frac{3d}{l}(x-l),$$

$2l/3 < x < l.$

The initial position of the string is,

$$y(x,0) = f(x) = \begin{cases} \frac{3dx}{l}, & 0 < x < l/3 \\ \frac{3d}{l}(l-2x), & l/3 < x < 2l/3 \\ 3d/l\,(x-l), & 2l/3 < x < 3l/3 \end{cases}$$

BOUNDARY CONDITIONS:

i] $y(0,t) = 0$, $\forall t > 0$

ii] $y(l,t) = 0$

iii] $\frac{\partial y(x,0)}{\partial t} = 0$

iv] $y(x,0) = \begin{cases} \frac{3dx}{l}, & 0 < x < l/3 \\ 3d/l\,(l-2x), & l/3 < x < 2l/3 \\ \frac{3d}{l}(x-l), & 2l/3 < x < l \end{cases}$

PROBLEMS WITH INITIAL VELOCITY

i] If a string of length is initially at rest in its equilibrium position and each of its point is given velocity $v_0 \sin^3\left(\frac{\pi x}{l}\right)$, $0 \le x < l$. Displacement

Soln:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

BOUNDARY CONDITIONS:

i] $y(0,t) = 0$, for $t>0$

ii] $y(l,t) = 0$, for $t>0$

iii] $y(x,0) = 0$, for x in $(0,l)$

iv] $\frac{\partial y(x,0)}{\partial t} = v_0 \sin^3\left(\frac{\pi x}{l}\right)$, x in $(0,l)$

Apply [i] in soln ①

$y(x,t) = (A\cos(\lambda x) + B\sin(\lambda x))[C\cos(\lambda at) + D\sin(\lambda at)]$

$\boxed{x=0}$

$y(0,t) = (A\cos(0) + B\sin(0))[C\cos(\lambda at) + D\sin(\lambda at)]$

$y(0,t) = A[C\cos(\lambda at) + D\sin(\lambda at)] = 0$.

$\boxed{A=0}$, $\boxed{C\cos(\lambda at) + D\sin(\lambda at) \neq 0}$ in ①

$\therefore y(x,t) = B\sin(\lambda x)[C\cos(\lambda at) + D\sin(\lambda at)] \to$ ②

Apply (ii) in ② $\Rightarrow y(l,t) = B\sin(\lambda l)[C\cos(\lambda at) + D\sin(\lambda at)] = 0$.

$B=0$, $\sin(\lambda l) = 0$, $C\cos(\lambda at) + D\sin(\lambda at) \neq 0$

in ③.

$\sin^{-1}(0) = \lambda l$

$\lambda l = n\pi$

$\boxed{\lambda = n\pi/l}$

$y(x,t) = B\sin(\lambda x)[C\cos(an\pi t/l) + D\sin\left(\frac{n\pi at}{l}\right)] \to$ ④

Apply (iii) in ④. $y(x,0)$ $\boxed{t=0}$

$y(x,0) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{an\pi t}{l}\right) + D\left(\sin\left(\frac{an\pi t}{l}\right)\right)\right]$

$\therefore y(x,0) = B\sin\left(\frac{n\pi x}{l}\right)[C] = 0$ $\therefore \boxed{C=0}$

$\boxed{B\sin\left(\frac{n\pi x}{l}\right) \neq 0}$

in ④.

$\hookrightarrow y(x,t) = \left[B\sin\left(\frac{n\pi x}{l}\right) D\sin\left(\frac{n\pi at}{l}\right)\right] \Rightarrow \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \sin$

$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \left[\left[\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{a n\pi t}{l}\right)\right] \times \left[\frac{a n\pi}{l}\right]\right]$

$\left[\frac{\partial y}{\partial t}\right]_{t=0} = \sum_{n=1}^{\infty} c_n \left[\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{a n\pi t}{l}\right)\right]$

$\left(\frac{dy}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = f(x) \Rightarrow v_0 \sin^3\left(\frac{\pi x}{l}\right)$

$c_1 \sin\left(\frac{\pi x}{l}\right) + c_2 \sin\left(\frac{2\pi x}{l}\right) + c_3\left(\sin\left(\frac{3\pi x}{l}\right)\right) \dots = \frac{v_0(3)}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{v_0}{4}\left(\sin\left(\frac{3\pi x}{l}\right)\right)$

$\boxed{c_1 = \frac{3v_0}{4}, \quad c_3 = \frac{-v_0}{4}}$

$c_n = \frac{b_n a n\pi}{l} \quad \therefore \quad b_n = \frac{c_n l}{a n\pi}$

$b_1 = \frac{c_1 l}{a\pi} \Rightarrow \frac{3v_0 l}{4a\pi}, \quad b_3 \Rightarrow \frac{-v_0 l}{12 a\pi}$

$y(x,t) = b_1 \sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{a\pi t}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\sin\left(\frac{2a\pi t}{l}\right) + b_3 \sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3a\pi t}{l}\right) \dots$

$y(x,t) = \frac{3v_0 l}{4a\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{a\pi t}{l}\right) + \left[\frac{-v_0 l}{12 a\pi}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3a\pi t}{l}\right)\right]$

$= \frac{3v_0 l}{4a\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{a\pi t}{l}\right) - \frac{v_0 l}{12 a\pi}\left(\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi a t}{l}\right)\right)$

A string is stretched between two fixed points at a distance $2l$ apart