1/01/25

MODULE - 1

DIFFERENTIAL EQUATION

ORDINARY DIFFERENTIAL EQUATIONS :

Linear differential Equation with constant coefficient :

Linear differential equations of n^{th} order with constant co-efficient are of the form,

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X \quad \hookrightarrow (1)$$

Here $k_1, k_2, \dots k_n$ are constants

Let $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ... then

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n y = X \quad \hookrightarrow (2)$$

$$\boxed{f(D)y = X}$$

Complementary function: (CF)

For $f(D)y = F[x]$

Auxillary eqn. is $f(m) = 0$

$\Rightarrow m = ?$

Case : 1 Roots are real & equal

$$C.F = C_1 e^{m_1 x} + x C_2 e^{m_1 x} + x^2 C_3 e^{m_1 x} + \dots + x^{n-1} C_4 e^{mx}$$

$$CF = (c_1 + x c_2 + x^2 c_3 + \dots + x^{n-1} c_n) e^{m_1 x}$$

Case 2 : Roots are real & distinct

$$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

Special case : $(m_1 = m_2 = m_3 \neq m_4 \neq m_5)$

$$CF = (c_1 + x c_2 + x^2 c_3) e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x}$$

Case 3 : Roots are imaginary.

$$m = \alpha \pm i \beta$$

$$C.F = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$$

Problem - 1

Solve : $(y'' + 5y' + 6y) = 0$

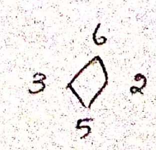
$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \quad [\text{or}] \quad (D^2 + 5D + 6)y = 0$$

It is of the form $[f(D)]y = F[x]$

$$f(D) = D^2 + 5D + 6$$

$$f(x) = 0$$

Auxillary eqn. is $f(m) = 0$



$$m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$\boxed{m = -3, -2}$

$\cancel{ax^2 + bx + c = 0}$ $\therefore$ Roots are real & distinct

$\cancel{x = -b}$

$$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{-2x} + C_2 e^{-3x}$$

$PF = 0$ as $F(x) = 0$

$$y = C.S = C.F + P.I$$

$$= C_1 e^{-2x} + C_2 e^{-3x} + 0$$

$$y = C_1 e^{-2x} + C_2 e^{-3x}$$

Problem - 2

Solve $(D^2 + n^2) y = 0$

$$D^2 y + n^2 y = 0$$

$$\frac{d^2 y}{dx^2} + n^2 y = 0$$

It is of the form $[f(D)] y = F(x)$

$$f(D) = D^2 + n^2$$

$$f(x) = 0$$

Auxillary eqn. is $f(m) = 0$

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm in$$

$\therefore$ $m = \alpha \pm i\beta$ $\alpha = 0$ $\beta = n$

$\therefore$ The roots are imaginary

$CF = A\cos nx + B\sin nx$

$P.I = 0$

$y = CF + P.I$

$= A\cos nx + B\sin nx + 0$

$y = A\cos nx + B\sin nx$

$CF = e^{\alpha x}[A\cos(\beta x) + B\sin(\beta x)]$

$= e^{0}[A\cos nx + B\sin nx]$

$CF = A\cos nx + B\sin nx$

~~sol~~

Problem -3

solve $(D^2 + 3D - 40)y = 0$

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 40y = 0$ $[f(D)]y = 0$

$\Downarrow$ CF $\Downarrow$ P.F

$f(D) = D^2 + 3D - 40$

$f(x) = 0$

Auxillary eqn,

$f(m) = 0$

$m^2 + 3m - 40 = 0$

$(m+8)(m-5) = 0$

$m = -8, 5$

-40

8 -5

3

Roots are real & distinct,

$$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{-8x} + C_2 e^{5x}$$

$$PF = 0$$

$$y = CF + PF = C.S$$

$$= C_1 e^{-8x} + C_2 e^{5x} + 0$$

$$y = C_1 e^{-8x} + C_2 e^{5x}$$

Particular Integral (P.I) :

$$P.I = \frac{F(x)}{f(D)}$$

Replace $D = a$

Type : I : $\cancel{f(D)} = e^{ax}$

If $f(a) = 0$ &

$$P.I = \frac{F[x]}{f(a)} = \frac{x\ F[x]}{f'(a)}$$

replace $D = a$

Problem - 4 :

Solve $(D^2 + 3D + 2)y = e^x$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$$

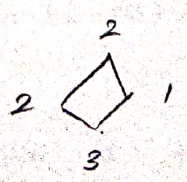
Auxillary eqn ⇒

$$[f(D)]y = e^x)$$

$\cancel{f(m) = e^x}$

$$f(m) = 0$$

$$m^2 + 3m + 2 = 0$$



$$(m+2)(m+1) = 0$$

$$m = -2 \quad m = -1$$

Roots are real & distinct,

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{-2x} + c_2 e^{-x}$$

$e^{ax} = e^x$

$a = 1$

$$P.I = \frac{F[x]}{f(D)} = \frac{e^x}{D^2+3D+2}$$

$$P.I = \frac{F[x]}{f(a)} = \frac{e^x}{a^2+3a+2} = \frac{e^x}{1+3+2} = \frac{e^x}{6}$$

$$P.I = \frac{e^x}{6}$$

$$y = CF + PI$$

$$= c_1 e^{-2x} + c_2 e^{-x} + \frac{e^x}{6}$$

Problem - 5 :

Solve $(D-2)^2 y = e^{2x}$

$$(D^2 - 4D + 4)y = e^{2x}$$

Auxillary eqn :

$$f(m) = 0$$

$$m^2 - 4m + 4 = 0$$

$$m = 2,2 \quad (m-2)(m-2) = 0$$

4, −2, −2, −4

The roots are real & equal

$$C.F = (C_1 + x\, C_2) e^{m_1 x}$$

$$= (C_1 + x\, C_2) e^{2x}$$

$$p.I = \frac{F[x]}{f(D)} = \frac{e^{2x}}{(D-2)^2}$$

$e^{ax} = e^{2x}$

$a = 2$

$$\frac{F[x]}{f(a)} = \frac{e^{2x}}{(a-2)^2} = \frac{e^{2x}}{0}$$

So, $$PI = \frac{F[x]}{f(a)} = \frac{x\,F[x]}{f'(a)}$$

$$= \frac{x e^{2x}}{2(a-2)} = \frac{x e^{2x}}{2(2-2)} = \frac{x e^{2x}}{2(0)}$$

$$= \frac{x e^{2x}}{0}$$

a diff means $\Rightarrow 1$

$a' = 1$

So, $$PI = \frac{x F(x)}{f''(a)} = \frac{x(x e^{2x})}{2(1)}$$

$$= \frac{x^2 e^{2x}}{2}$$

$y = CF + PI$

$$= (c_1 + x c_2) e^{2x} + \frac{x^2 e^{2x}}{2}$$

Type : 2

$F(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{F[x]}{f(D)}$$ replace $D^2 = -(a^2)$

Practice problem:

① solve $(D^2 + 5D + 4)\, y = e^x$

$f(D)\, y = f(x)$

$f(D) = D^2 + 5D + 4 \qquad f(x) = e^x$

Auxillary eqn.

$f(m) = 0$

$m^2 + 5m + 4 = 0$

$(m + 4)(m + 1) = 0$

$m = -1, -4$

4 4 1 5

Roots are real & distinct,

$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F = c_1 e^{-x} + c_2 e^{-4x}$

$P.I = \frac{F[x]}{f(D)} \qquad e^{ax} = e^x$

$P.I = \frac{e^x}{D^2 + 5D + 4} \qquad a = 1$

$D = a$

$= \frac{e^x}{1 + 5 + 4} = \frac{e^x}{10}$

$y = C.F + P.I$

$y = c_1 e^{-x} + c_2 e^{-4x} + \frac{e^x}{10}$

10/01/25 Type 2 :

Problem 6 :

solve : $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 3x$

$(D^2 + 3D + 2)y = \sin 3x$

$f[(D)y] = f(x)$

$f(D) = D^2 + 3D + 2$ $f(x) = \sin 3x$

Auxiliary eqn :

$f(m) = 0$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -1, -2$

Roots are real & distinct,

$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F = c_1 e^{-x} + c_2 e^{-2x}$

$P.I = \frac{F[x]}{f(D)}$ $F(x) = \sin(ax+b) = \sin 3x$

$\boxed{a = 3}$

$D^2 = -(a)^2$

$= -(3)^2$

$D^2 = -9$

$= \frac{\sin 3x}{D^2 + 3D + 2} = \frac{\sin 3x}{-9 + 3D + 2}$

$= \frac{\sin 3x}{3D - 7} \times \frac{(3D+7)}{(3D+7)}$

$= \frac{(3D+7)\sin 3x}{9D^2 - 49} = \frac{3D\sin 3x + 7\sin 3x}{-81 - 49}$

$= \frac{3D \sin(3x) + 7\sin 3x}{-130}$ $\quad \frac{d(\sin(3x))}{dx} = 3\cos 3x$

$= -\frac{1}{130}\left[3(3\cos(3x)) + 7\sin 3x\right]$

$P.I = \frac{9\cos 3x + 7\sin 3x}{-130}$

$y = CF + P.I$

$y = C_1 e^{-x} + C_2 e^{-2x} - \frac{9\cos 3x + 7\sin 3x}{130}$

problem - 7 :

$(D^2+9)y = \cos 3x$

given: $(D^2+9)y = \cos 3x$

$(D^2+3^2)y = \cos 3x$

$f(D) = D^2+3^2$

$f(x) = \cos 3x$

Auxilarly eqn, $f(m) = 0$

$m^2+3^2=0$

$m^2 = -3^2$

$m = \pm 3i$

$\alpha = 0 \quad \beta = 3$

Roots are imaginary.

$C.F = e^{\alpha x}\,[C_1 \cos(\beta x) + C_2 \sin(\beta x)]$

$= e^{0}\,[C_1 \cos 3x + C_2 \sin 3x]$

$C.F = C_1 \cos 3x + C_2 \sin 3x$

$\lambda^{n} = n\lambda^{n-1}$

$P.I = \frac{F[x]}{f(D)} = \frac{\cos 3x}{D^2 + 3^2}$

$\cos 3x = (\cos ax + b)$

$\boxed{a = 3}$

$D^2 = -(a)^2$
$= -(3)^2$
$= -9$

$= \frac{\cos 3x}{-9 + 9}$

$= \frac{\cos 3x}{0}$

$P.I = x\,\frac{F(x)}{f'(D)} = x\,\frac{\cos 3x}{2D}$

$\int \cos(ax)\,dx = -\frac{\sin(ax)}{a}$

$= \frac{x}{2} \times \frac{\cos 3x}{D}$

$= \frac{x}{2}\,[\int \cos 3x\,dx]$

$= \frac{x}{2}\left(-\frac{\sin 3x}{3}\right)$

$= -\frac{x \sin 3x}{6}$

$y = CF + PI$

$= C_1 \cos 3x + C_2 \sin 3x - \frac{x}{6}\sin(3x)$

solve $(D^2+1)^2 y = 2\sin x \cos 3x$

Auxillary eqn:

$(m^2+1)^2 = 0$

$m^2 = -1^2$

$m = \pm i$

$CF = (Ax+B)\cos x + (Cx+D)\sin x$

$$P.I = \frac{2\sin x \cos 3x}{(D^2+1)^2} = \frac{F(X)}{f(D)}$$

$$= \frac{1}{(D^2+1)^2}(\sin 4x - \sin 2x)$$

$\sin 4x = \sin ax$

$a = 4$

$$= \frac{\sin 4x}{(D^2+1)^2} - \frac{\sin 2x}{(D^2+1)^2}$$

$\sin 2x = \sin ax$

$a = 2$

$$= \frac{\sin 4x}{(-16+1)^2} - \frac{\sin 2x}{(-4+1)^2}$$

$-(2)^2$

$-4+1$

$-4+1$

$$= \frac{\sin 4x}{(-15)^2} - \frac{\sin 2x}{(-3)^2}$$

$$= \frac{\sin 4x}{225} - \frac{\sin 2x}{9}$$

$$y = (Ax+B)\cos x + (Cx+D)\sin x + \frac{\sin 4x}{225} - \frac{\sin 2x}{9}$$

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Binomial series

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3$

Type -3 :

$f(x)$ is polynomial

$$P.I = \frac{F(x)}{f(D)}$$

$$= f^{-1}(D)\, F(x)$$

while taking $f^{-1}(D)$, $f(D)$ to be written in the form of $(1 \pm x)^n$ and take the inverse.

Now, apply binomial series and expand, simplify

Problem - 1

Solve $(D^2+8)y = x^4 + 2x + 1$

Auxillary eqn :

$m^2 + 8 = 0$

$m^2 = -8$ $\quad m = \sqrt{-8}$

$m = \pm i\, 2\sqrt{2}$

$C.F = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$

$= c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x$

$$P.I = \frac{x^4 + 2x + 1}{D^2 + 8}$$

$$= \frac{x^4 + 2x + 1}{\dots} = \frac{1}{8}\left(\frac{D^2}{8} + 1\right)^{-1}(x^4 + 2x + 1)$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$\Rightarrow \frac{1}{8}\left[1 - \frac{D^2}{8} + \frac{D^4}{64} - \frac{D^6}{512} + \dots\right](x^4+2x+1)$

$D(x^4+2x+1) = 4x^3+2$

$D^2(x^4+2x+1) = 12x^2$

$D^3(x^4+2x+1) = 24x$

$D^4(x^4+2x+1) = 24$

$D^5(x^4+2x+1) = 0$

$D^6(x^4+2x+1) = 0$

$= \frac{1}{8}\left[(x^4+2x+1) - \frac{1}{8_2}(\cancel{12}^{3}x^2) + \frac{1}{\cancel{64}_8}(\cancel{24}^{3}) - \frac{1}{512}(0)\right]$

$= \frac{1}{8}\left[x^4+2x+1 - \frac{3}{2}x^2 + \frac{3}{8}\right]$

$= \frac{1}{8}\left[x^4+2x+\underbrace{1+\frac{3}{8}}_{\frac{11}{8}} - \frac{3}{2}x^2\right]$

$= \frac{x^4}{8} + \frac{\cancel{2}}{\cancel{8}_4}x + \frac{11}{64} - \frac{3}{16}x^2$

$P.I = \frac{x^4}{8} + \frac{x}{4} - \frac{3}{16}x^2 + \frac{11}{64}$

$y = C.F + P.I$

$= C_1\cos 2\sqrt{2}x + C_2\sin 2\sqrt{2}x + \frac{x^4}{8} + \frac{x}{4} - \frac{3}{16}x^2 + \frac{11}{64}$

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Type : 4 :

$P.I = \frac{e^{ax}F(x)}{f(D)}$ $\quad D = D+a$

$P.I = e^{ax}\left[\frac{F(x)}{f(D+a)}\right]$

simplify the specific type based on $F(x)$ and e^{ax} will come as a multiplier till the end.

problem - :

Solve $(D^2+6D+9)y = e^{-2x}x^3$

Soln auxillary eqn.

$m^2+6m+9=0$

$(m+3)^2=0$

$m=-3,-3$

$C.F = C_1 e^{m_1 x} + x C_2 e^{m_1 x} + x^2 C_3 e^{m_1 x} + \dots + x^{n-1} C_4 e^{m_1 x}$

$= C_1 e^{-3x} + x C_2 e^{-3x} + x^2 C_3 e^{-3x}$

$a=-2$

$D = D+a$

$= D-2$

$P.I = \dfrac{e^{-2x}x^3}{D^2+6D+9} = \dfrac{e^{-2x}x^3}{(D+3)^2}$

$= \dfrac{e^{-2x}x^3}{(D-2+3)^2} = \dfrac{e^{-2x}x^3}{(D+1)^2}$

It is as type -3

$= e^{-2x}\left[\dfrac{x^3}{(D+1)^2}\right]$

$= e^{-2x}(1+D)^{-2}x^3$ $\boxed{(1+x)^{-2} = 1-2x+3x^2-4x^3}$

$= e^{-2x}(1-2D+3D^2-4D^3)x^3$

$= e^{-2x}\left[x^3 - 2D(x^3) + 3D^2(x^3) - 4D^3(x^3)\right]$

$D(x^3) = 3x^2$

$D^2(x^3) = 6x$

$D^3(x^3) = 6$

$$= e^{-2x}\left[x^3 - 2(3x^2) + 3(6x) - 4(6)\right]$$

$$= e^{-2x}\left[x^3 - 6x^2 + 18x - 24\right]$$

$$y = C.F + P.I$$

$$y = C_1 e^{-3x} + x C_2 e^{-3x} + x^2 C_3 e^{-3x} + e^{-2x}\left[x^3 - 6x^2 + 18x - 24\right]$$

Solve :

$$(D^2+4)y = 4e^{2x}\sin 3x$$

$$P.I = \frac{4e^{2x}\sin 3x}{D^2+4} \qquad D = D+2$$

$$= \frac{4e^{2x}\sin 3x}{(D+2)^2+4}$$

$$= 4e^{2x}\left[\frac{\sin 3x}{D^2+4D+8}\right]$$

$$= 4e^{2x}\left(\frac{\sin 3x}{\cancel{D}-9+4D+8}\right) = 4e^{2x}\left(\frac{\sin 3x}{\cancel{4D+1}\ 4D-1}\right)$$

$$= 4e^{2x}\left[\frac{\sin 3x\,(4D+1)}{(4D-1)(4D+1)}\right]$$

$$= 4e^{2x}\left[\frac{4D(\sin 3x) + \sin 3x}{16D^2 - 1}\right]$$

$$= 4e^{2x}\left[\frac{12\cos 3x + \sin 3x}{-145}\right]$$

$C.F =$

$y =$

Type-5 : $x\,[\sin(ax+b) \text{ or } \cos(ax+b)]$

$$P.I = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} \begin{array}{c}\sin(ax+b) \\ (\text{or}) \\ \cos(ax+b)\end{array}$$

Problem - 1

①. $(D^2 + D)\,y = x \cos x$

Aux. eqn :

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

real & distinct roots,

$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$= C_1 e^{0x} + C_2 e^{-x}$

$= C_1 + C_2 e^{-x}$

$$P.I = \frac{x \cos x}{D^2 + D} = \left(x - \frac{f'(D)}{f(D)}\right) \frac{\cos x}{f(D)}$$

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$$= \left[x - \left(\frac{2D+1}{D^2+D}\right)\right]\left(\frac{\cos x}{D^2+D}\right)$$

by type - 5

$f(D) = D^2 + D$

$f'(D) = 2D + 1$

$$= x\left(\frac{\cos x}{-1+D}\right) - \left(\frac{2D+1}{D^2+D}\right)\frac{\cos x}{(D^2+D)}$$

by type - 2

$\cos ax$

$a = 1$

$D^2 = -a^2 = -1$

$$= x\left(\frac{\cos x}{-1+D}\right) - \frac{(2D+1)(\cos x)}{(-1+D)^2}$$

$$= \frac{x(\cos x)(D+1)}{(D-1)(D+1)} - \frac{(2D+1)\cos x}{D^2+1-2D}$$

$$= x\left[\frac{\cos x(D+1)}{D^2-1}\right] - \frac{(2D+1)\cos x}{D^2+1-2D}$$

$D[\cos x] = -\sin x$

$$= x\left[\frac{-\sin x + \cos x}{-2} - \frac{(-2\sin x + \cos x)}{-2D}\right.$$

$$= \frac{x}{2}(\sin x - \cos x) + \frac{1}{2}\left[-2\int \sin x\,dx + \int \cos x\,dx\right]$$

$$PI = \frac{x}{2}(\sin x - \cos x) + \cos x + \frac{\sin x}{2}$$

$$y = CF + PI$$

$$= c_1 + c_2 e^{-x} + \frac{x}{2}(\sin x - \cos x) + \cos x + \frac{1}{2}\sin x$$

Q. Solve $(D^2-2D+1)y = xe^x \sin x$

Aux. eqn.

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

Roots are real & equal.

$$C.F = C_1 e^x + x C_2 e^x + x^2 C_3 e^x$$

by type - 4

$$P.I = \frac{x e^x \sin x}{(D-1)^2}$$

$e^{ax} \Rightarrow a = 1$

$D = D+1$

$$= e^x \left(\frac{x \sin x}{(D+1-1)^2} \right) = e^x \left(\frac{x \sin x}{D^2} \right)$$

by type - 5

$f(D) = D^2$

$f'(D) = 2D$

$$= e^x \left(x - \frac{f'(D)}{f(D)} \frac{\sin x}{f(D)} \right)$$

$$= e^x \left(x - \frac{2D}{D^2} \right) \frac{\sin x}{D^2}$$

type - 2

$e^{ax} = a = 1$

$D^2 = -a^2$

$= -1$

$$= e^x \left(\frac{x \sin x}{D^2} - \frac{2D \sin x}{D^4} \right)$$

$\Rightarrow -1$

$$= e^x \left(-x \sin x - 2D(\sin x) \right)$$

$$= e^x \left(-x \sin x - 2\cos x \right)$$

$$= -e^x (x \sin x + 2\cos x)$$

$$y = C_1 e^x + x C_2 e^x + x^2 C_3 e^x - e^x (x \sin x + 2\cos x)$$