$$= e^x \left[x - \frac{2D}{D^2} \right] \frac{\sin x}{D^2}$$

$f(D) = D^2$

$f'(D) = 2D$

$f''(D) = 2$

$$= e^x \left[x \left(\frac{\sin x}{D^2} \right) - \frac{2D \sin x}{D^4} \right]$$

$$= e^x \left[x \frac{\cos x}{D} - \frac{2 \cos x}{D^2} \right]$$

$$= e^x \left[x (-\sin x) - 2 \cos x \right]$$

$$= -e^x \left[x \sin x + 2 \cos x \right]$$

$$y = C_1 e^x + x C_2 e^x - e^x \left[x \sin x + 2 \cos x \right]$$

24/11 EULER'S (OR) CAUCHY'S HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATION WITH VARIABLE CO-EFFICIENT.

Here $xD = \theta$

$x^2 D^2 = \theta(\theta - 1)$

$$\boxed{D = \frac{d}{dx} \qquad \theta = \frac{d}{dz}}$$

$x^3 D^3 = \theta(\theta-1)(\theta-2)$

$x = e^z$

$\log x = z \log e$

$\log x = z$

Solve $(x^2 D^2 + x D + 1) y = \sin(2 \log x) \sin(\log x$

The given equation is an Cauchy or Euler's differentia equation.

$\therefore$ we take

$xD = \theta$, $x^2D^2 = \theta(\theta-1) = \theta^2 - \theta$, $x = e^z$, $\log x = z$

$\Rightarrow [\theta(\theta-1) + \theta + 1]y = \sin 2z \sin z$

$[\theta^2 - \theta + \theta + 1]y = \sin 2z \sin z$

$[\theta^2 + 1]y = \sin 2z \sin z$

$m^2 + 1 = 0$

$m = \pm i$

$C.F = e^{0x}[C_1 \cos x + C_2 \sin x]$

$C.F = C_1 \cos z + C_2 \sin z$

$$P.I = \frac{\sin 2z \sin z}{\theta^2 + 1}$$

$$= \frac{\frac{1}{2}[\cos(2z - z) - \cos(2z + z)]}{\theta^2 + 1}$$

$$= \frac{\frac{1}{2}[\cos z - \cos 3z]}{\theta^2 + 1}$$

$$= \frac{1}{2}\left[\frac{\cos z}{\theta^2+1} - \frac{\cos 3z}{\theta^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{z \cos z}{2\theta} - \frac{\cos 3z}{-9+1}\right]$$

$a = 3$

$\theta^2 = -(3)^2$

$= -9$

$$= \frac{1}{2}\left[\frac{z}{2}\int \cos z + \frac{\cos 3z}{8}\right]$$

$$= \frac{1}{2}\left[\frac{z}{2}\sin z + \frac{\cos 3z}{8}\right]$$

$$= \frac{z}{4}\sin z + \frac{\cos 3z}{16}$$

$$= \frac{z}{4}\sin \log x + \frac{\cos(3 \log x)}{16}$$

$$y = C_1 \cos \log x + C_2 \sin \log x + \frac{\log x}{4}\sin \log x + \frac{\cos(3 \log x)}{16}$$

$(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

The given equation is an Cauchy or Euler's differential equation

∴ we take $xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x = e^z$, $\log x = z$

$$[\theta(\theta-1) - \theta + 1]y = \left(\frac{z}{e^z}\right)^2$$

$$[\theta^2 - \theta - \theta + 1]y = z^2e^{-2z}$$

$$[\theta^2 - 2\theta + 1]y = z^2e^{-2z}$$

$(m-1)^2 = 0$

$m = 1, 1$

$C.F = c_1e^z + zc_2e^z$.

$$P.I = \frac{z^2e^{-2z}}{(\theta-1)^2}$$

$$= e^{-2z}\left[\frac{z^2}{(\theta-1)^2}\right]$$

$$= e^{-2z}\left[\frac{z^2}{(\theta-2-1)^2}\right]$$

$$= e^{-2z}\left[\frac{z^2}{(\theta-3)^2}\right]$$

$$= \frac{e^{-2z}}{(-3)^2}\left[\frac{z^2}{9}\right]$$

$a = -2$.

$D = D + a$.

$\theta(z^2) = 2z$

$\theta^2(z^2) = 2$

$\theta^3(z^2) = 0$.

27/01/25

Solve $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$

$$[x^2D^2 + 4xD + 2]y = x^2 + \frac{1}{x^2}$$

Substituting $xD = \theta$, $x^2D^2 = \theta(\theta-1)$ $\quad x = e^z \quad \log x = z$

$$[\theta(\theta-1) + 4(\theta) + 2]y = e^{2z} + \frac{1}{e^{2z}}$$

$$[\theta^2 - \theta + 4\theta + 2]y = e^{2z} + e^{-2z}$$

$$[\theta^2 + 3\theta + 2]y = e^{2z} + e^{-2z}$$

$m^2 + 3m + 2 = 0$

$m = -1, -2$

Roots are real and distinct.

$C.F = C_1e^{-x} + C_2e^{-2x}$

$a = 2$

$C.F = C_1e^{-z} + C_2e^{-2z}$.

$$P.I = \frac{e^{2z} + e^{-2z}}{\theta^2 + 3\theta + 2}.$$

$$= \frac{e^{2z}}{\theta^2+3\theta+2} + \frac{e^{-2z}}{\theta^2+3\theta+2}$$

$$= \frac{e^{2z}}{4+6+2} + \frac{z.\ e^{-2z}}{2\theta+3}$$

$$= \frac{e^{2z}}{12} + \frac{z.\ e^{-2z}}{-4+3}$$

$$= \frac{e^{2z}}{12} + z.e^{-2z}$$

$$= \frac{e^{2z}}{12} + \frac{z.}{e^{2z}}$$

$$= \frac{1}{12}x^2 + \frac{1}{x^2}\log x$$

LEGENDRE'S DIFFERENTIAL EQUATION

$$ax + b = e^{z}.$$

$$z = \log(ax+b)$$

Then,

$$(ax+b)D = a\theta$$

$$(ax+b)^2 D^2 = a^2 \theta(\theta-1)$$

$$\theta = \frac{d}{dz}.$$

Solve $(5+2x)^2 y'' - 6(5+2x)y' + 8y = 0$

$a = 2 \qquad b = 5$

Let $\quad 2x+5 = e^{z}$

$$x = \frac{e^{z}-5}{2}.$$

$$\log(2x+5) = z.$$

$(2x+5)D = 2\theta \qquad (2x+5)^2 D^2 = 4\theta(\theta-1)$

$$= 4\theta^2 - 4\theta.$$

$$[4\theta^2 - 4\theta - 6(2\theta) + 8]y = 0$$

$$[4\theta^2 - 16\theta + 8]y = 0$$

$$m^2 - 4m + 2 = 0 \qquad \frac{4 \pm \sqrt{16-8}}{2},$$

$$m = 2 \pm \sqrt{2} \qquad \frac{4 \pm \sqrt{8}}{2}$$

Roots are real and unequal $\qquad 2 \pm \sqrt{2}$

C.F $= C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z}$ $\qquad$ P.I $= 0$

$$y = C_1 e^{2+\sqrt{2}z} + C_2 e^{2-\sqrt{2}z}$$

$y = C_1 (2x+5)^{2+\sqrt{2}} +$

$C_2 \quad (2x+5)^{2-\sqrt{2}}$

$e^z = ax+b$

$e^{2z} = (ax+b)^2$

$e^{(2+\sqrt{2})z} = (ax+b)^{2+\sqrt{2}}$

Solve $\quad (x+1)^2 y''(x) + (x+1) y'(x) + y(x) = 4 \cos(\log(x+1))$

$x+1 \rightarrow a=1 \qquad b=1.$

$(x+1) D = \theta$

$e^z = x+1$

$(x+1)^2 D^2 = \theta^2 - \theta$

$x = e^z - 1$

$z = \log(x+1)$

$[\theta^2 + 1] y = 4 \cos z.$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

Roots are imaginary.

$C.F = C_1 \cos z + C_2 \sin z.$

$P.I = \dfrac{4 \cos z}{D'^2 + 1}$

$a = 1$

$D^2 = -(a^2)$

$= \dfrac{4 \cos z}{-1+1}$

$= \dfrac{z}{2D'} 4 \cos z.$

$= \dfrac{z}{2} 4 \sin z$

$= 2z \sin z$

$y = C.F + P.I$

$= C_1 \cos z + C_2 \sin z + 2z \sin z$

$= C_1 \cos(\log(x+1)) + B \sin(\log(x+1)) + 2 \log x \sin(\log(x+1))$

2011

$((2x-1)^2D^2 - 4(2x-1)D + 8)y = 8x$

$2x-1 = e^z$ $\quad$ $2x-1\,d = 2\theta$

$x = \frac{e^z+1}{2}$ $\quad$ $(2x-1)^2d^2 = 2^2(\theta)(\theta-1) = 4\theta^2 - 4\theta$

$z = \log(2x-1)$

$ax+b \Rightarrow a=2 \quad b=-1$

$[4\theta^2 - 4\theta - 8\theta + 8]y = 8 \cdot \frac{e^z+1}{2}$

$[4\theta^2 - 12\theta + 8]y = 4e^z + 4$

$\div$ by 4

$m^2 - 3m + 2 = 0$

$m = 1, 2$

$CF = C_1e^z + C_2e^{2z}$

$PI = \frac{e^z + e^{0z}}{\theta^2 - 3\theta + 2}$

$= \frac{e^z}{\theta^2 - 3\theta + 2} + \frac{e^{0z}}{\theta^2 - 3\theta + 2}$

$= \frac{e^z}{0} + \frac{e^{0z}}{0 - 0 + 2}$

$= \frac{z \cdot e^z}{2\theta - 3} + \frac{e^{0z}}{2}$

$= z \cdot \frac{e^z}{-1} + \frac{1}{2}$

$P.I = -z \cdot e^z + \frac{1}{2}$

$y = C_1e^z + C_2e^{2z} - z \cdot e^z + \frac{1}{2}$

$= C_1(2x-1) + C_2(2x-1)^2 - [\log(2x-1)\,(2x-1)] + \frac{1}{2}$

METHOD OF VARIATION OF PARAMETERS

[RHS = $\tan x$ or $\sec x$].

Step : 1
Let $y = Af(x) + Bg(x)$ be the solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Step : 2

Assume $y = A(x)f(x) + B(x)g(x)$ solution of the given equation.

Step 3 :

Wronskion

Find $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$. and check whether $W \neq 0$.

Step 4 :

Compute $A(x) = -\int \frac{g(x)R}{W} + C_1$

$B(x) = \int \frac{f(x)R}{W} + C_2$.

Step 5 :

Substituting $A(x)$ and $B(x)$ in $y = A(x)f(x) + B(x)f(x)$ we get the soln.

Solve $y'' - 2y' + 2y = e^x \tan x$

Consider

$y'' - 2y' + 2y = 0$

$m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$m = 1 \pm i$

C.F $= e^x [C_1 \cos x + C_2 \sin x]$.

$y = C_1 e^x \cos x + C_2 e^x \sin x$

$f(x) = e^x \cos x$ $\qquad g(x) = e^x \sin x$.

$f'(x) = e^x(-\sin x) + \cos x . e^x$

$= -e^x \sin x + e^x \cos x$.

$g'(x) = e^x \cos x + \sin x \, e^x$

$= e^x \cos x + e^x \sin x$.

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x \end{vmatrix}$$

$= e^x \cos x (e^x \cos x + e^x \sin x) -$

$(-e^x \sin x + e^x \cos x) \, e^x \sin x$.

$= e^{2x} \cos^2 x + \cancel{e^x \cos x \, e^x \sin x} + e^{2x} \sin^2 x - \cancel{e^x \cos x \, e^x \sin x}$

$= e^{2x} [\cos^2 x + \sin^2 x] \qquad = e^{2x}$

$$A(x) = -\int \frac{g(x) . R}{W} + C_1$$

$$= -\int \frac{e^x \sin x . e^x \tan x}{e^{2x}} + C_1$$

$$= \int \sin x . \frac{\sin x}{\cos x} + C_1$$

$$= \int \frac{\sin^2 x}{\cos x} + C_1$$

$$= \int \frac{1 + \cos^2 x}{\cos x} + C_1$$

$= \int \frac{1}{\cos x} \mp \int \cos x + C_1$

$= \int \sec x + \sin x + C_1$

$A(x) = -\log(\sec x + \tan x) + \sin x + C_1$

$$B(x) = \int \frac{f(x) \cdot R}{W} + C_2$$

$$= \int \frac{\cancel{e^x \cos x} \cdot e^x \tan x}{\cancel{e^{2x}}} + C_2$$

$$= \int \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} + C_2$$

$y = \underset{\smile}{A(x)}\, e^x \cos x + B(x)\, e^x \sin x$

$$= \int \sin x + C_2$$

$$= -\cos x + C_2$$

$$y = (-\log(\sec x + \tan x) + \sin x + C_1)$$

$$e^x \cos x + (-\cos x + C_2)\, e^x \sin x$$

27/1

Solve $\quad y'' - 4y' + 4y = \frac{e^{2x}}{x}.$

$$m^2 - 4m + 4 = 0.$$

$$\begin{array}{c} 4 \\ -2 \; -2 \end{array}$$

$$m = 2, 2.$$

$$C.F = C_1 e^{2x} + x C_2 e^{2x}$$

$$y = A(x)\, e^{2x} + B(x)\, e^{2x} x$$

$f(x) = e^{2x} \qquad g(x) = x \cdot e^{2x}$

$f'(x) = 2e^{2x}. \qquad g'(x) = x \cdot 2e^{2x} + e^{2x}$

$$W = \begin{vmatrix} e^{2x} & x \cdot e^{2x} \\ 2e^{2x} & x \cdot 2e^{2x} + e^{2x} \end{vmatrix}$$

$$= e^{2x}(x \cdot 2e^{2x} + e^{2x}) - 2x e^{4x}$$

$$= \cancel{2xe^{4x}} + e^{4x} - \cancel{2xe^{4x}}$$

$$W = e^{4x}$$

$$A(x) = -\int \frac{g(x)\, R}{W} + C_1$$

$$= -\int \frac{x e^{2x} \cdot \frac{e^{2x}}{x}}{e^{4x}} + C_1$$

$$= -\int 1 + C_1$$

$$= -x + C_1$$

$$B(x) = \int \frac{f(x)\, R}{W} + C_2$$

$$= \int \frac{e^{2x} \cdot \frac{e^{2x}}{x}}{e^{4x}} + C_2$$

$$= \int \frac{1}{x} + C_2$$

$$= \log x + C_2 .$$

$$y = e^{2x}(-x + C_1) + xe^{2x}(\log x + C_2)$$

$$= C_1 e^{2x} - xe^{2x} + x\log x \cdot e^{2x} + C_2 x e^{2x}$$

$$= C_1 e^{2x} + C_2\, x e^{2x} + xe^{2x}[\log x - 1]$$

Solve $\quad x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = \sin(\log x)$

$$[x^2D^2 - 4xD + 6]y = \sin(\log x).$$

$xD = \theta$

$x^2D^2 = \theta(\theta - 1) = \theta^2 - \theta$

$x = e^z$

$z = \log x.$

$[\theta^2 - \theta - 4\theta + 6]y = \sin(\log x)$.

$[\theta^2 - 5\theta + 6]y = \sin z$.

$m^2 - 5m + 6 = 0$.

$m = 2, 3$

$CF : c_1 e^{2z} + c_2 e^{3z}$.

$y = A(x) e^{2z} + B(x) e^{3z}$

$f(x) = e^{2z}$ $\quad g(x) = e^{3z}$

$f'(x) = 2 \cdot e^{2z}$ $\quad g'(x) = 3e^{3z}$

$$W = \begin{vmatrix} e^{2z} & e^{3z} \\ 2 \cdot e^{2z} & 3e^{3z} \end{vmatrix}$$

$= 3e^{5x} - 2e^{5x}$

$W = e^{5x}$

$A(x) = -\int \frac{g(x) \cdot R}{W} + c_1$

$= -\int \frac{\cancel{e^{3x}} \cdot \sin z}{e^{5z}} + c_1$

$= -\int \frac{\sin z}{e^{2z}} + c_1$

$= -\int e^{-2z} \sin z + c_1$ $\quad a = -2$, $b = 1$

$= -\frac{e^{-2x}}{5}[-2\sin x - \cos x] + c_1$

$= \frac{e^{-2x}}{5}[2\sin x + \cos x] + c_1$

$B(x) = \int \frac{f(x) \cdot R}{W} + c_2$

$= \int \frac{e^{2z} \cdot \sin z}{e^{5z}} + c_2$

$= \int e^{-3z} \sin z + c_2$ $\quad a = -3$, $b = 1$

$= \frac{e^{-3z}}{10}[-3\sin z - \cos z] + c_2$

30/1

PARTIAL DIFFERENTIAL EQUATION

LAGRANGE'S LINEAR EQUATIONS

The equation of the form

$$P.\frac{dz}{dx} + Q\frac{dz}{dy} = R \text{ (or) } \rightarrow (1)$$

$$pP + qQ = R. \rightarrow (2)$$ with P, Q, R being the functions x, y and z & the first order partial derivatives $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$.

Now,

the subsidary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

METHOD OF GROUPING :

Taking two pairs from the subsidiary equation and separate one variable in left side and the another in right side, by direct integration, we solve both the pairs and get

$u(x,y) = c_1$, $v(y,z) = c_2$ then

the general solution of equation 1 is

$$\phi(u,v) = 0.$$

Solve $pyz + qzx = xy$.

Comparing with $pP + qQ = R$

$P = yz \quad Q = zx \quad R = xy$.

Subsidiary Equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}.$$

Take $\frac{dx}{yz} = \frac{dy}{zx}$.

$x \cdot dx = y \cdot dy$

$\int x \cdot dx = \int y \cdot dy$

$\frac{x^2}{2} = \frac{y^2}{2} + \frac{C_1}{2}$

$\div$ by 2

$x^2 = y^2 + C_1$

$x^2 - y^2 = C_1$

$u = C_1$

$u = x^2 - y^2$

Take $\frac{dy}{zx} = \frac{dz}{xy}$.

$y.dy = z.dz$.

$\int y\,dy = \int z.dz$

$\frac{y^2}{2} = \frac{z^2}{2} + \frac{C_2}{2}$

$y^2 = z^2 + C_2$

$C_2 = y^2 - z^2$

$v = C_2$

$v = y^2 - z^2$.

The general solution of Lagrange's Linear equation is $\phi(u, v) = 0$.

$\phi(x^2 - y^2, y^2 - z^2) = 0$.

Solve $p\tan x + q\tan y = \tan z$

$P = \tan x \quad Q = \tan y \quad R = \tan z.$

S.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}.$$

Take $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$\cot x\, dx = \cot y . dy.$

$\int \cot x . dx = \int \cot y\, dy.$

$\log(\sin x) = \log(\sin y) + \log c_1$

$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$

$\frac{\sin x}{\sin y} = c_1$

$u = \frac{\sin x}{\sin y}.$

Take $\frac{dy}{\tan y} = \frac{dz}{\tan z}.$

$\cot y\, dy = \cot z\, dz$

$\int \cot y . dy = \int \cot z . dz.$

$\log(\sin y) = \log(\sin z) + \log C_2$

$\log\left(\frac{\sin y}{\sin z}\right) = \log C_2$

$C_2 = \frac{\sin y}{\sin z}.$

$v = \frac{\sin y}{\sin z}.$

The general solution of Lagrange's Linear equation is $\phi(u, v) = 0$

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$$

Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

$P = x^{1/2}$ $\quad Q = y^{1/2}$ $\quad R = z^{1/2}$

S.E: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^{1/2}} = \frac{dy}{y^{1/2}} = \frac{dz}{z^{1/2}}$$

Take

$$\frac{dx}{x^{1/2}} = \frac{dy}{y^{1/2}}$$

$$x^{-1/2}\,dx = y^{-1/2}\,dy$$

$$\int x^{-1/2}\,dx = \int y^{-1/2}\,dy$$

$$\frac{x^{-1/2+1}}{-1/2+1} = \frac{y^{-1/2+1}}{-1/2+1} + 2C_1$$

$$\frac{x^{1/2}}{1/2} = \frac{y^{1/2}}{1/2} + 2C_1$$

$$2\sqrt{x} = 2\sqrt{y} + 2C_1$$

$$\sqrt{x} = \sqrt{y} + C_1$$

$$\sqrt{x} - \sqrt{y} = C_1$$

$$u = \sqrt{x} - \sqrt{y}$$

Take

$$\frac{dy}{y^{1/2}} = \frac{dz}{z^{1/2}}$$

$$y^{-1/2}\,dy = z^{-1/2}\,dz$$

$$\int y^{-1/2}\,dy = \int z^{-1/2}\,dz$$

$$\frac{y^{-1/2+1}}{-1/2+1} = \frac{z^{-1/2+1}}{-1/2+1} + 2C_2$$

$$\frac{y^{1/2}}{1/2} = \frac{z^{1/2}}{1/2} + 2C_2$$

$$2\sqrt{y} = 2\sqrt{z} + 2C_2$$

$$\sqrt{y} = \sqrt{z} + C_2$$

$$\sqrt{y} - \sqrt{z} = C_2$$

$$v = \sqrt{y} - \sqrt{z}$$

The general solution of Lagrange's Linear equation is $\phi(u,v) = 0$

$$\phi(\sqrt{x} - \sqrt{y},\ \sqrt{y} - \sqrt{z}) = 0$$

Solve $xp + yq = x$.

$P = x \quad Q = y \quad R = x$

S.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}.$$

Take

$\dfrac{dx}{x} = \dfrac{dy}{y}$.

$\int \dfrac{dx}{x} = \int \dfrac{dy}{y}$.

$\log x - \log y = \log c_1$

$\log\left(\dfrac{x}{y}\right) = \log c_1$

$\dfrac{x}{y} = c_1$

$u = \dfrac{x}{y}$.

Take $\dfrac{dx}{\cancel{x}} = \dfrac{dz}{\cancel{x}}$

$\int 1\,dx = \int 1\,dz$.

$x = z + c_2$

$x - z = c_2$

$v = x - z$.

The general solution of Lagrange's linear equation is

$\phi(u, v) = 0$

$\phi\left(\dfrac{x}{y}, x - z\right) = 0$.

METHOD OF MULTIPLIERS

Here choose any 3 multiplier l, m, n which may be constants or functions of x, y and z such that in the subsidiary equation can be written as,

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}}$$

where $lP + mQ + nR = 0$.

Hence, $l\,dx + m\,dy + n\,dz = 0$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = k \text{ (say)}$$

$$l\,dx + m\,dy + n\,dz = k\,(lP + mQ + nR)$$

$$l\,dx + m\,dy + n\,dz = 0.$$

By direct integration we get,

$$u(x, y, z) = c_1$$

Again choose another set of multiplier l', m' and n' which is not a combination of l, m, n.

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}}$$

where $l'P + m'Q + n'R = 0$. By integration

Hence, $l'dx + m'dy + n'dz = 0$. $v(x, y, z) = c_2$.

General solution is $\phi(u,v) = 0$.

Solve $(x(y-z))p + y(z-x)q = z(x-y)$

The eqn is of the form $Pp + Qq = R$

$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y)$

Hence the subsidiary eqn for Lagrange's linear PDE is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

∴ Grouping is impossible

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

when l, m, n are considered with value 1

$$lP + mQ + nR = 0$$

$$l\left[\frac{dx}{x(y-z)}\right] + m\left[\frac{dy}{y(z-x)}\right] + n\left[\frac{dz}{z(x-y)}\right]$$

$C_1 \Rightarrow$ any unknown value.

$$\frac{dx}{xy-xz} + \frac{dy}{yz-xy} + \frac{dz}{xz-yz} = 0$$

$$dx + dy + dz = 0$$

Integrating

$$\int dx + \int dy + \int dz = 0$$

$$x + y + z = c_1 \Rightarrow u(x,y,z) \rightarrow (1)$$

Now considering the multipliers as l', m', n' with value

$$l' = \frac{1}{x}, \quad m' = \frac{1}{y}, \quad n' = \frac{1}{z}$$

So that

$$l'P + m'Q + n'R = 0$$

$$l'\frac{dx}{x(y-z)} + m'\frac{dy}{y(z-x)} + n'\frac{dz}{z(x-y)} = 0$$

$$\Rightarrow \frac{\frac{dx}{x}}{\frac{1}{x}(x)(y-z)} + \frac{\frac{dy}{y}}{\frac{1}{y}(y(z-x)} + \frac{\frac{dz}{z}}{\frac{1}{z}(z)(x-y)} = 0$$

$$\Rightarrow \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0} = 0$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = C_2$$

$$\log(xyz) = C_2$$

$$xyz = \log C_2$$

$$v(xyz).$$

Sol. of PDE is $f(u,v) = 0$

$$f(x+y+z, xyz) = 0.$$

Solve $(mz-ny)p + (nx-lz)q = ly-mx$

The given eqn is in $Pp + Qq = R$

$P = mz-ny \quad Q = nx-lz \quad R = ly-mx$

Considering the multipliers l, m, n;

$$\Rightarrow \frac{l.dx}{l(mz-ny)} + \frac{m(dy)}{m(nx-lz)} + \frac{n\,dz}{n(ly-mx)}$$

$$\Rightarrow l(dx) + m(dy) + n(dz) = 0$$

$$l\int(dx) + m\int dy + n\int dz) = 0$$

$lx + my + nz = c_1$

$u(lx, my, nz) = c_1$

Considering l', m', n' as multipliers with value

x, y, z

$$\frac{l'dx}{P} + \frac{m'dy}{Q} + \frac{n'dz}{R} = 0$$

$$\Rightarrow \frac{x\,dx}{x(mz-ny)} + \frac{y\,dy}{y(nx-lz)} + \frac{z\,dz}{z(ly-mx)}$$

$$\Rightarrow \frac{x\,dx}{xmz-nyx} + \frac{y\,dy}{nyx-ylz} + \frac{z\,dz}{zly-mzx}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$x^2 + y^2 + z^2 = c_2 \Rightarrow v$$

$$v(x^2, y^2, z^2)$$

Solu of PDE is $f((lx + my + nz), (x^2 + y^2 + z^2)) = 0$.

3/1

PROBLEMS BASED ON USING BOTH METHOD OF GROUPING AND METHOD OF MULTIPLIERS

Solve $(x^2-y^2-z^2)p + 2xyq - 2xz = 0$

$$Pp + Qq = R$$

$P = x^2-y^2-z^2 \qquad Q = 2xy \qquad R = 2xz$

S.F $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}.$$

$$\frac{dy}{2\cancel{x}y} = \frac{dz}{2\cancel{x}z}.$$

$$\int \frac{dy}{2y} = \int \frac{dz}{2z}$$

$$\frac{1}{2}\log y = \frac{1}{\cancel{2}}\log z + \log C_1$$

$$\log y - \log z = \log C_1$$

$$\cancel{\log} C_1 = \cancel{\log}\left(\frac{y}{z}\right) \Rightarrow u.$$

$$u\left(\frac{y}{z}\right) = C_1$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2)+2xy^2+2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x^3+xy^2+xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dy}{y} = \frac{2(xdx + ydy + zdz)}{x^2+y^2+z^2}$$

$$\int \frac{dy}{y} = \int \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}$$

$$\log y = \log(x^2+y^2+z^2) + \log C_2$$

$$\log\left(\frac{y}{x^2+y^2+z^2}\right) = \log C_2.$$

$$\frac{y}{x^2+y^2+z^2} = C_2 = V.$$

$$\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0.$$

Solve $z(xp+yq) = y^2-x^2$

$$zxp + yzq = y^2-x^2.$$

$$Pp + Qq = R.$$

$P = zx \qquad Q = zy \qquad R = y^2-x^2$

S.E

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{zx} = \frac{dy}{zy} = \frac{dz}{y^2-x^2}$$

Take,

$$\frac{dx}{zx} = \frac{dy}{zy}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log C_1$$

$C_1 = \frac{x}{y}$.

$u = \frac{x}{y}$.

$$\frac{dy}{zy} = \frac{x\,dx + y\,dy + z\,dz}{zx^2 + zy^2 + z(y^2 - x^2)}$$

$$\frac{dy}{zy} = \frac{x\,dx + y\,dy + z\,dz}{2zy^2}$$

$$2y\,dy = x\,dx + y\,dy + z\,dz$$

$$\int 2y\,dy = \int x\,dx + \int y\,dy + \int z\,dz$$

$$2\frac{y^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + \frac{c^2}{2}$$

$$x^2 - y^2 + z^2 = C_2$$

$$V = x^2 - y^2 + z^2$$

$$\phi\left(\frac{x}{y},\ x^2 - y^2 + z^2\right) = 0.$$

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION:

Partial derivatives are all of the same order (with degree 1 each).

The coefficients are constants.

Ex. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$.

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots +$$

$$a_n \frac{\partial^n z}{\partial y^n} = F(x,y).$$

$$\boxed{(a_0 D^n + a_1 D^{n-1}D' + \dots + a_n D'^n)z = F(x,y)}$$

$D = \frac{\partial}{\partial x}$ $\quad D' = \frac{\partial}{\partial y}$

$$f(D, D')z = F(x,y).$$

$$z = C.F + P.I$$

To find C.F

$\Rightarrow f(D, D')z = F(x,y)$

$f(m,1) = 0$ Replacing $D = m$ & $D' = 1$

$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0.$

$\Rightarrow$ m_1 m_2 m_2 ... m_n are the roots.

$\Rightarrow$ Case 1: Real and distinct.

$$Z = f_1(y+m_1x) + f_2(y+m_2x) + \cdots + f_n(y+m_nx)$$

$\Rightarrow$ Case 2: Roots are same.

$$Z = f_1(y+mx) + xf_2(y+mx) + \cdots + x^{n-1}f_n(y+mx).$$

Solve $(D^2 - 3DD' + 2D'^2)z = 0$

$m^2 - 3m + 2 = 0$

$m = 1, 2$

$$C.F = f_1(y+x) + f_2(y+2x)$$

RHS = 0 PI = 0

$$Z = f_1(y+x) + f_2(y+2x)$$

~~Solve~~ $\boxed{f(D,D')z = e^{ax+by}}$ $\Rightarrow$ TYPE : 1

To find P.I

$$P.I = \frac{e^{ax+by}}{f(D,D')}$$

Replace $D \to a$
$D' \to b.$

$$= \frac{1}{f(a,b)} e^{ax+by}$$ if $f(a,b) \neq 0.$

If $(a,b) = 0$ then

Replace $D \to a$, $D' \to b$

$$P.I = \frac{x}{f'(D,D')} e^{ax+by}$$

$$P.I = \frac{x}{f'(a,b)} e^{ax+by}$$

5|2 TYPE: 1

Solve $(D^2 - 3DD' + 4D'^2) z = 7$

[Find PI alone]

$$P.I = \frac{1}{D^2 - 3DD' + 4D'^2} 7e^{0x - 0y}$$

$$= \frac{x}{2D - 3D'} 7e^{0x - 0y}$$

$$= \frac{x^2}{2} 7e^{0x - 0y}$$

$$= \frac{7x^2}{2}$$

4
$\wedge$
-2 -2

Solve $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$C.F : z = f_1(y+2x) + x f_2(y+2x)$

$$P.I = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

$$= \frac{1}{2^2 - 4(2) + 4(1)^2} e^{2x+y}$$ $a=2 \quad b=1$

$$= \frac{1}{0} e^{2x+y}$$

$$= \frac{x}{2D - 4D'} e^{2x+y}.$$

$$= \frac{x}{2(2) - 4(1)} e^{2x+y}$$

$$= \frac{x}{0} e^{2x+y}$$

$$P.I = \frac{x^2}{2} e^{2x+y}.$$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

TYPE : 2

Solve $[D^2 - 3DD' + 2D'^2]z = 2\cosh(3x+4y)$.

$m^2 - 3m + 2 = 0$

$m = 1, 2$

$C.F = f_1(y+x) + f_2(y+2x)$

$RHS = 2\cosh(3x+4y)$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}.$

$$= \cancel{2}\left[\frac{e^{3x+4y} + e^{-(3x+4y)}}{\cancel{2}}\right]$$

$$RHS. = e^{(3x+4y)} + e^{-(3x+4y)}$$

$$P.I = \frac{e^{(3x+4y)} + e^{-(3x+4y)}}{D^2 - 3DD' + 2D'^2}$$

$$P.I = \frac{e^{3x+4y}}{D^2 - 3DD' + 2D'^2} + \frac{e^{-3x-4y}}{D^2 - 3DD' + 2D'^2}$$ $a=3$, $b=4$

$$= \frac{e^{3x+4y}}{9 - 3(3)(4) + 2(4)^2} + \frac{e^{-3x-4y}}{(-3)^2 - 3(-3)(-4) + 2(-4)^2}$$ $a=-3$, $b=-4$

$$= \frac{e^{3x+4y}}{9-36+32} + \frac{e^{-(3x+4y)}}{9-36+32}$$ $\frac{-9}{1}$

$$= \frac{e^{3x+4y}}{5} + \frac{e^{-(3x+4y)}}{5}$$

$$= \frac{2\times 1}{2\;\;5}\left[(e^{3x+4y}) + (e^{(-3x+4y)})\right]$$ $\frac{e^{x}+e^{-x}}{2} = \cosh$

$$= \frac{2}{5}\cosh(3x+4y)$$

$$z = f_1(y+x) + f_2(y+2x) + \frac{2}{5}\cosh(3x+4y).$$

TYPE : 3

$$P\,I = \frac{1}{(D - m_1 D')(D - m_2 D')} F(x, y).$$

we $(D^2+DD'-6D'^2)z = y\cos x$

-6

$3 \quad -2$

$(m^2+m-6) = 0$

$m = -3, 2$

C.F : $f_1(y-3x) + f_2(y+2x)$.

P.I $= \dfrac{1}{(D-2D')} \, y$

TYPE : 4

$$f(D, D')z = x^m y^n$$

$$P.I = \frac{1}{f(D, D')} x^m y^n$$

$$P.I = \{f(D, D')\}^{-1} (x^m, y^n)$$

Expand by using binomial formulas.

numerator $D \rightarrow$ diff wrt x

numerator $D' \rightarrow$ diff wrt y.

Solve $D^2 + 2DD' + D'^2 = x^2 y$

$$(m^2 + 2m + 1) = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{x^2 y}{(D+D')^2}$$

$$= \frac{x^2 y}{D^2 (1+D')^2}$$

$$= \frac{1}{D^2} (1+D')^{-2} (x^2 y)$$

$$P.I = \frac{1}{D^2} \left(1 + \frac{D'}{D}\right) (x^2 y)$$

$$= \frac{1}{D^2} \left[1 - 2\frac{D'}{D} + 3\frac{D'^2}{D^2} + \cdots\right) (x^2 y)$$

$$= \frac{1}{D^2} \left(x^2 y - 2\frac{D'}{D} x^2 y + 3\frac{D'^2}{D^2} x^2 y + \cdot\right)$$

$$= \frac{1}{D^2}\left(x^2y - 2\frac{1}{D}x^2 + 3\frac{1}{D^2}(0) + \dots\right)$$

$$= \frac{1}{D^2}\left(x^2y - 2\frac{1}{D}x^2\right)$$

$$= \frac{x^2y}{D^2} - \frac{2}{D^3}x^2$$

$$= \frac{x^4y}{12} - 2\frac{x^5}{60}.$$

$$= \frac{x^4y}{12} - \frac{x^5}{30}.$$

$$Z = f_1(y-x) + xf_2(y-x) + \frac{x^4y}{12} - \frac{x^5}{30}.$$

TYPE-5

$f(D,D')z = \sin(ax+by)$ or $\cos(ax+by)$.

$P.I = \frac{1}{f(D,D')}\sin(ax+by)$ or $\cos(ax+by)$

Replacing $D^2 \to -a^2$, $D'^2 \to -b^2$, $DD' \to -ab$.

$P.I = \frac{1}{f(a,b)}\sin(ax+by)$ or $\cos(ax+by)$ $f(a,b)$.

If $f(a,b) = 0$

then, $PI = \frac{x}{f(D,D')}\sin(ax+by)$ or $\cos(ax+by)$

$D^2 \to -a^2$ $D'^2 \to -b^2$, $DD' \to -ab$.

$PI = \frac{x}{f(a,b)}\sin(ax+by)$ or $\cos(ax+by)$