28/03

Z - TRANSFORM

$$\boxed{Z\{f(n)\} = f(z) = \sum_{n=1}^{\infty} f(n) z^{-n}}$$ [one-sided]

1. Find $Z[a^n]$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$Z\{a^n\} = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + x^3 \cdots}$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

Find Z

Z

Z [

x by z and ÷

z to bring

terms of

2. Find $z\left[\frac{1}{n}\right]$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$z\left[\frac{1}{n}\right] = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n z^n}$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= -\log\left(1-\frac{1}{z}\right) = \log\left(1-\frac{1}{z}\right)^{-1}$$

$$= \log\left(\frac{z-1}{z}\right)^{-1}$$

$$z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$$

Find $z\left[\frac{1}{n+1}\right]$

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+1) z^n}$$

$$= 1 + \frac{\frac{1}{z}}{2} + \frac{\left(\frac{1}{z}\right)^2}{3} + \frac{\left(\frac{1}{z}\right)^3}{4} + \cdots$$

x by z and ÷ by z to bring in terms of project

$$= z\left(\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right)$$

$$= z \log\left(\frac{z}{z-1}\right)$$

Find $z\left[\frac{a^n}{n!}\right]$

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

$$z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{a}{z}\right)^n}{n!}$$

$$= 1 + \frac{\frac{a}{z}}{1!} + \frac{\left(\frac{a}{z}\right)^2}{2!} + \frac{\left(\frac{a}{z}\right)^3}{3!} + \cdots$$

$$z\left[\frac{a^n}{n!}\right] = e^{a/z}$$

Find $Z[r^n \cos n\theta]$ and $Z[r^n \sin n\theta]$

We know $z[a^n] = \frac{z}{z-a}$

put $a = re^{i\theta}$

$$z\left[(re^{i\theta})^n\right] = \frac{z}{z - re^{i\theta}}$$

$$z\left[r^n(\cos n\theta + i\sin n\theta)\right] = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z - r\cos\theta) - ir\sin\theta} \times \frac{(z - r\cos\theta) + ir\sin\theta}{(z - r\cos\theta) + ir\sin\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 + r^2\cos^2\theta - 2zr\cos\theta + r^2\sin^2\theta}$$

$$Z[r^n \cos n\theta + i r^n \sin n\theta] = \frac{z(z - r\cos\theta) + i z r \sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating real parts

$$Z[r^n \cos n\theta] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

Equating imaginary parts

$$Z[r^n \sin n\theta] = \frac{z r \sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Find $Z[t]$

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \qquad \boxed{t = nT}$$

$$Z[t] = \sum_{n=0}^{\infty} nT z^{-n} = T \sum_{n=0}^{\infty} n z^{-n}$$

$$= T\left[z^{-1} + 2z^{-2} + 3z^{-3} + \cdots\right]$$

$$\boxed{(1-z)^{-2} = 1 + 2z + 3z^2 + \cdots}$$

$$= \frac{T}{z}\left[1 + 2z^{-1} + 3z^{-2} + \cdots\right]$$

$$= \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{T}{z}\left(\frac{z-1}{z}\right)^{-2}$$

$$= \frac{T}{z} \frac{z^2}{(z-1)^2} = \frac{Tz}{(z-1)^2}$$

$$Z[t] = \frac{Tz}{(z-1)^2}$$

1/4

Z-TRANSFORM OF UNIT IMPULSE FUNCTION $\delta(n)$

$$\boxed{\begin{aligned} &* \ \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \\ &* \ Z(\delta(n)) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 \\ &* \ Z(\delta(n)) = 1 \end{aligned}}$$

Z-TRANSFORM OF UNIT STEP SEQUENCE

$$* \ u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$* \ Z(k) = \sum_{n=0}^{\infty} k z^{-n} = k + \frac{k}{z} + \frac{k}{z^2} + \dots \infty$$

$$= k \frac{1}{1 - \frac{1}{z}}$$

$$= \frac{kz}{z-1}$$

Where the region of convergence is $\left|\frac{1}{z}\right| < 1$ (or) $|z| > 1$

In particular

$$Z\{u(n)\} = Z(1) = \frac{z}{z-1} \text{ if } |z| > 1 \text{ and}$$

$$Z\{u(n)\} = \frac{z}{z-1}$$

LINEAR PROPERTY:

$$Z[af(n) + bg(n)] = a\,Z[f(n)] + b\,Z[g(n)]$$

FREQUENCY SHIFTING PROPERTY:

$$Z[a^n f(n)] = \left[F\left(\frac{z}{a}\right)\right]$$

$$Z[a^n f(t)] = \left[F\left(\frac{z}{a}\right)\right]$$

DIFFERENTIATION IN Z-DOMAIN:

$$Z[n f(n)] = -z \frac{d}{dz}[F(z)]$$

$$Z[n f(t)] = -z \frac{d}{dz}[F(z)]$$

4 →

$$Z[a^{-n} f(n)] = [F(az)]$$

TIME SHIFTING:

$$Z[f(n+1)] = z\,F(z) - z\,f(0)$$

6 →

$$Z[f(n+2)] = z^2\left(F(z) - f(0) - f(1)z^{-1}\right)$$

INITIAL VALUE THEOREM:

$Z[f(n)] = F(z)$ then

$$f(0) = \lim_{z\to\infty} F(z)$$

FINAL VALUE THEORY:

$Z[f(n)] = F(z)$ then $\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1)F(z)$

Find the $Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right]$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right] = Z\left[\frac{1-\cos\frac{n\pi}{2}}{2}\right]$

$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$= \frac{1}{2}\left[Z(1) - Z\left(\cos\frac{n\pi}{2}\right)\right]$

$Z(a) = \frac{z}{z-a}$

$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$

2(ii) If $F(z) = \frac{10z}{(z-1)(z-2)}$, find $f(0)$.

Initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{10z}{(z-1)(z-2)}$$

$$= \lim_{z\to\infty} \frac{10z}{z^2\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)}$$

$$= 0.$$

If $F(z) = \frac{5z}{(z-2)(z-3)}$ find $\lim_{t\to\infty} f(t)$

Final value theorem $\lim_{t\to\infty} = \lim_{z\to 1} (z-1)F(z)$

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0$$

Find $Z[e^{-at} t]$

$$Z(t^k) = -Tz \frac{d}{dz}\left[Z(t^{k-1})\right]$$ $\boxed{\text{sub } k=1}$

$$Z(t) = -Tz \frac{d}{dz}\left[Z(1)\right]$$

$$= -Tz \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$\frac{u}{v} = \frac{vu'-uv'}{v^2}$$

$$= -Tz\left[\frac{(z-1) - z(1-0)}{(z-1)^2}\right]$$

$$= -Tz\left[\frac{z-1-z}{(z-1)^2}\right]$$

$$= -Tz\left[\frac{-1}{(z-1)^2}\right]$$

$$= \frac{Tz}{(z-1)^2}$$

$$z[e^{-at}t] = [z(t)]_{z \to ze^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}}$$

$$= \frac{Tze^{aT}}{(ze^{aT}-1)^2}$$

INVERSE Z-TRANSFORMS.

$F(z) = z(f(n)$ then inverse z-transform is defined as

$$z^{-1}(F(z)) = f(n)$$

$z(a^n) = \frac{z}{z-a}$ $\quad z^{-1}\left(\frac{z}{z-a}\right) = a^n$

$z(n) = \frac{z}{(z-1)^2}$ $\quad z^{-1}\left(\frac{z}{(z-1)^2}\right) = n$

$z(a^n n) = \frac{az}{(z-a)^2}$ $\quad z^{-1}\left(\frac{az}{(z-a)^2}\right) = a^n n$

$z(a^{n-1}n) = \frac{z}{(z-a)^2}$ $\quad z^{-1}\left(\frac{z}{(z-a)^2}\right) = a^{n-1}n$

$z(a^{n-1}) = \frac{1}{z-a}$ $\quad z^{-1}\left(\frac{1}{z-a}\right) = a^{n-1}$

PARTIAL FRACTION :

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Power of numerator should be less than the denominator

Find the inverse Z-transform of $\frac{z^2}{z^2-3z+2}$

$$F(z) = \frac{z^2}{(z^2-3z+2)}$$

$$= \frac{z \cdot z}{z^2-3z+2}$$

$$\frac{F(z)}{z} = \frac{z}{z^2-3z+2} = \frac{z}{(z-1)(z-2)}$$

$$\frac{F(z)}{z} = \frac{z}{(z-1)(z-2)} \longrightarrow (1)$$

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\frac{z}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$z = A(z-2) + B(z-1)$$

when $z=2$

$2 = B$

when $z=1$

$1 = -A$

$A = -1$

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2} \longrightarrow 2$$

$$\frac{F(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-2}$$

$$F(z) = \frac{-z}{z-1} + \frac{2}{z-2}$$

$$z^{-1}(F(z)) = z^{-1}\left\{\frac{-z}{z-1} + \frac{2z}{z-2}\right\}$$

By lineal property.

$$z^{-1}(F(z)) = z^{-1}\left\{\frac{-z}{z-1}\right\} + z^{-1}\left\{\frac{2z}{z-2}\right\}$$

$$= -1\, z^{-1}\left\{\frac{z}{(z-1)}\right\} + 2\, z^{-1}\left\{\frac{z}{z-2}\right\}$$

$$z^{-1}\left\{\frac{z^2}{z^2-3z+2}\right\} = -1(1)^n + 2(2)^n = -1 + 2^{n+1}$$

$$\boxed{z^{-1}\left\{\frac{z}{z-a}\right\} = a^n}$$

Find $z^{-1}\left\{\frac{4z^3}{(2z-1)^2(z-1)}\right\}$

$$F(z) = \frac{4z^3}{(2z-1)^2(z-1)}$$

$$\frac{F(z)}{z} = \frac{4z^2}{(2z-1)^2(z-1)} \longrightarrow 1$$

$$\frac{4z^2}{(2z-1)^2(z-1)} = \frac{A}{2z-1} + \frac{B}{(2z-1)^2} + \frac{C}{z-1}$$

$$4z^2 = A(2z-1)(z-1) + B(z-1) + C(2z-1)^2$$

When $z = 1$

$$4 = C$$

When $z = \frac{1}{2}$

$$1 = -\frac{1}{2}B$$

$$B = -2$$

Eq. co.eff z^2

$$4 = 2A + 4C$$

$$4 = 2A + 16$$

$$4 - 16 = 2A$$

$$-12 = 2A$$

$$A = -6.$$

$$\frac{4z^2}{(2z-1)^2(z-1)} = \frac{-6}{(2z-1)} - \frac{2}{(2z-1)^2} + \frac{4}{z-1}$$

$$\frac{F(z)}{z} = -\frac{6}{2z-1} - \frac{2}{(2z-1)^2} + \frac{4}{z-1}$$

$$F(z) = \frac{-6z}{2z-1} - \frac{2z}{(2z-1)^2} + \frac{4z}{z-1}$$

$$F(z) = \frac{-6z}{2(z-\frac{1}{2})} - \frac{2z}{2(z-\frac{1}{2})^2} + \frac{4z}{z-1}$$

Taking

$$Z^{-1}(F(z)) = -3z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{1}{2}z^{-1}\left(\frac{z}{(z-1/2)^2}\right) +$$

$$4z^{-1}\left(\frac{z}{z-1}\right)$$

$$= -3\left(\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} n + 4(1)^n$$

$$= -3\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n n + 4$$

$$Z^{-1}\frac{z}{z-a} = (a^n)$$

$$Z^{-1}\frac{z}{(z-a)^2} = (na^{n-1})$$

$$= 4 - \left(\frac{1}{2}\right)^n (3+n)$$

CONVOLUTION THEOREM

$$\{f(n) * g(n)\} = \sum_{r=0}^{n} f(r)\, g(n-r)$$

Find the z-transform of $f(n) * g(n)$ given

$f(n) = \left(\frac{1}{2}\right)^n \quad g(n) = \cos n\pi$

$\bar{f}(z) = z\left[\left(\frac{1}{2}\right)^n\right] \qquad z[a^n] = \frac{z}{z-a}$

$$= \frac{z}{z-\frac{1}{2}}$$

$$= \frac{2z}{2z-1}$$

$\bar{g}(z) = z[\cos n\pi]$

$= z[(-1)^n] \quad = \frac{z}{z+1}$

By convolution

$z[f(n) * g(n)] = \bar{f}(z)\,\bar{g}(z)$

$$= \left(\frac{2z}{2z-1}\right)\left(\frac{z}{z+1}\right)$$

$$= \frac{2z^2}{2z^2-z+2z-1}$$

$$= \frac{2z^2}{2z^2+z-1}$$

INVERSE - Z TRANSFORM [CONVOLUTION THEOREM]

$Z[f(n) * g(n)] = F[z]\, G[z]$

$Z^{-1}[F[z]\, G[z]] = f(n) * g(n)$

$Z^{-1}[F(z)\, G(z)] = \sum_{r=0}^{n} f(m)\, g(n-r)$

$f(n) = z^{-1}[F(z)]$ & $g(n) = z^{-1}[G(z)]$

Find the value of $z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

$$z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = z^{-1}\left[\frac{8z^2}{2(z-\frac{1}{2})4(z-(\frac{-1}{4}))}\right]$$

$$\boxed{z^{-1}\left[\frac{z}{z-a}\right] = a^n}$$

$$= z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-(-\frac{1}{4}))}\right]$$

$$= z^{-1}\left[\frac{z}{z-1/2} \cdot \frac{z}{z-(-\frac{1}{4})}\right]$$

$$= \{f(n) * g(n)\}$$

$f(n) = z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] \Rightarrow f(n) = \left(\frac{1}{2}\right)^n$

$g(n) = z^{-1}\left[\frac{z}{z-(-\frac{1}{4})}\right] \Rightarrow g(n) = \left(\frac{-1}{4}\right)^n$

$$f(n) * g(n) = \sum_{n=0}^{n} f(k)\, g(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(-\frac{1}{4}\right)^{n-k}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(-\frac{1}{4}\right)^{-k}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} \frac{(1/2)^k}{(-1)^k \left(\frac{1}{2}\right)^{2k}}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} (-1)^k 2^k$$

$$= \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} (-2)^k$$

$$= \left(-\frac{1}{4}\right)^n \left[1 + (-2) + (-2)^2 + (-2)^3 + \dots (-2)^n\right]$$

$$= \left(-\frac{1}{4}\right)^n \left[1 + x + x^2 + x^3 \dots x^n\right]$$

When $x = -2$

$$= \left(-\frac{1}{4}\right)^n \frac{(-2)^{n+1} - 1}{-2-1}$$

$$\boxed{\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad \text{(or)} \quad \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}}$$

$$= -\frac{1}{3}\left[(-1)^n (2)^{-2n} \left[(-2)^{n+1} - 1\right]\right]$$

$$= -\frac{1}{3}\left[(-1)^n 2^{-2n} (-(-1))^n (2(2)^n) - \left(-\frac{1}{4}\right)^n\right]$$

$$= -\frac{1}{3}\left[-\underbrace{(-1)^{2n}}_{(1)} 2 2^{-n} - \left(-\frac{1}{4}\right)^n\right]$$

$$= -\frac{1}{3}\left[\frac{-2}{2^n} - \left(-\frac{1}{4}\right)^n\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n$$