HALF RANGE COSINE SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\,.dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,.dx$$

HALF RANGE SINE SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\,.dx$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\,.dx.$$

Obtain the sine series for the function $f(x) = \pi - x$ in the interval $(0, \pi)$ and hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \pi/4$.

for sine (a_0, a_n) ✗.

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx$$

$u = \pi - x$ $\quad v = \sin nx$ (i)

$u' = -1$ $\quad v_1 = \frac{-\cos nx}{n}$ (ii)

$u'' = 0$ $\quad v_2 = \frac{-\sin nx}{n^2}$ (i)

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[+\cancel{\pi}\left(\frac{1}{n}\right)\right]$$

$$= \frac{2}{n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left[\frac{n\pi x}{l}\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\pi x}{\pi}\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)$$

considering $x = \pi/2$.

$f(x) = \pi - x$

$f(\pi/2) = \pi - \pi/2 \Rightarrow \pi/2$

$x = \pi/2$

$$f(x) = 2\left[\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right]$$

$$\pi/2 = 2\left[\frac{1}{1}\sin\pi/2 + \frac{1}{2}\sin(\pi) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin(2\pi) + \ldots\right]$$

$$\frac{\pi}{2} = 2\left[1 + 0 + \frac{1}{3}(-1) + 0 + \frac{1}{5}(1) + \ldots\right]$$

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5}\right]\ldots$$

$$\frac{\pi}{2}\times\frac{1}{2} = 1 - \frac{1}{3} + \frac{1}{5}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

(i) Find the fourier half range cosine series.

(ii) half range sine series for the function $f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$

(i)
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{2}{l} \int_0^l f(x) \cdot dx$$

$$= \frac{\cancel{2}}{\cancel{2}} \left[\int_0^1 x \cdot dx + \int_1^2 2-x \cdot dx \right]$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2}[1-0] + \left[4 - \frac{4}{2} - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[4-2 - \left(\frac{4-1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[2 - \left(\frac{3}{2}\right)\right]$$

$$= \frac{1}{2} + \left[\frac{4-3}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1.$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{2} \left[\int_0^1 x \cos \frac{n\pi x}{l} \, dx + \int_1^2 2-x \cos \frac{n\pi x}{l} \, dx \right]$$

$$= \left[\int_0^1 x \cdot \cos \frac{n\pi x}{2} \cdot dx + \int_1^2 2-x \cos \frac{n\pi x}{2} \cdot dx \right].$$

$u = 2-x$ $\quad v = \cos \frac{n\pi x}{2}$

$u' = -1$ $\quad v_1 = \dfrac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$

$u'' = 0$ $\quad v_2 = \dfrac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$

$u = x$ $\quad v = \cos \frac{n\pi x}{2}$

$u' = 1$ $\quad v_1 = \dfrac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$

$u'' = 0$ $\quad v_2 = \dfrac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$

$$a_n = \left[\left\{ x \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right\}_0^1 + \left\{ (2-x) \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right\}_1^2 \right]$$

$$= \left[\cancel{\frac{2}{n\pi} \sin \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \right] + \left[-\frac{4}{n^2\pi^2} \cos n\pi - \cancel{\frac{2}{n\pi} \sin \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} \right]$$

$$= \frac{8}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \left[(-1)^n + 1 \right]$$

$$= \begin{cases} 0 & \text{if n is odd} \\ \frac{8}{n^2\pi^2} \left[(-1)^n - 1 \right] & \text{if n is even} \end{cases}$$

$$\frac{8(-1)^n}{n^2\pi^2} \quad \frac{-8}{n^2\pi^2} \text{ if n even}$$

$$\frac{8}{n^2\pi^2} \left((-1)^{\frac{n}{2}} - 1 \right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$= \frac{1}{2} + \frac{8}{\pi^2} \sum_{n=2,4,6\ldots}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{2}$$

$$= \frac{1}{2}$$

(49)

(ii) $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx$$

$$= \int_0^1 x \sin \frac{n\pi x}{2}\, dx + \int_1^2 (2-x) \sin \frac{n\pi x}{2}\, dx$$

$u = x \quad v = \sin \frac{n\pi x}{2}$

$u' = 1 \quad v_1 = \frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}}$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$

$$= \left[\left\{ x \left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) + \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right\}_0^1 + \left\{ (2-x) \left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right\}_1^2 \right]$$

$$= -\frac{2}{n\pi} \cancel{\cos \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{2}{n\pi} \cos \frac{n\pi}{2}} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$= \begin{cases} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$f(x) = \frac{8}{\pi^2} \left[\frac{1}{1^2} \sin \frac{\pi x}{2} - \frac{1}{3^2} \sin \frac{3\pi x}{2} + \frac{1}{5^2} \sin \frac{5\pi x}{2} - \dots \infty \right]$$

Find the half-range cosine series of $f(x) = \sin x$ in $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^\pi \sin x\, dx$$

$$= \frac{2}{\pi} \left[-\cos x \right]_0^\pi$$

$= -\frac{2}{\pi}\left[\cos x\right]_0^{\pi}$

$= -\frac{2}{\pi}[-1-1]$

$a_0 = \frac{+4}{\pi}$

$\cos A \sin B = \frac{1}{2}\, \sin(A+B) - \sin(A-B)$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \cos \frac{n\pi x}{l}\, dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \cdot dx$

$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\pi} \sin(n+1)x - \sin(n-1)x \quad dx$

$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$

$= \frac{1}{\pi}\left[\frac{1}{n+1}\left(1-(-1)^{n+1}\right) + \frac{1}{n-1}\left((-1)^{n-1}-1\right)\right]$

$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{\pi}\left(\frac{2}{n+1}\right) - \frac{2}{n-1} & \text{if } n \text{ is even} \end{cases}$

$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{\pi}\left[\frac{2(n-1)-2(n+1)}{n^2-1^2}\right] & \text{if } n \text{ is even} \end{cases}$

$= \begin{cases} 0 & \text{if odd} \\ \frac{1}{\pi}\left[\frac{2n-2-2n-2}{n^2-1}\right] & \text{if } n \text{ is even} \end{cases}$

$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \quad \text{if } n \neq 1 \\ \frac{-4}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$

$\sin 2x = 2\sin x \cos x$

$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$

$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \cdot dx$

$$a_1 = \frac{1}{\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-1+1\right]$$

$$a_1 = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\sin x = \frac{4/\pi}{2} + \sum_{n=2}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

$$\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=2,4,6..}^{\infty} \frac{1}{n^2-1} \cos nx\right]$$

X

(A+B) -

-B)

A/3 ONE DIMENSIONAL WAVE EQUATION.

The one dimensional wave equation is where

$a^2 = \frac{T}{m}$ $l \to$ string length.

The three possible solution of one dimensional wave equation are

$y(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda at} + De^{-\lambda at})$

$\boxed{y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)}$

$y(x,t) = (Ax + B)(Ct + D)$

where A, B, C and D are arbitrary constants.

Here we use the second solution as solution for the periodic function with problems on vibrations of string.

Now, to solve the one dimensional wave equation the solution is equ (2) where we need to find the constants A, B, C and D with the following initial and boundary conditions.

⇒ If 0 initial velocity and y. is given as $f(x)$ the conditions are

$y(0,t) = 0 \quad \forall t \geq 0$
$y(l,t) = 0 \quad \forall t \geq 0$ } boundary conditions

$\left(\frac{dy}{dt}\right)_{t=0} = 0 \quad 0 \leq x < l$
$y(x,0) = f(x) \quad 0 \leq x < l$ } Initial conditions

→ If the initial velocity is given as f(x) the

$$y(0,t) = 0 \quad \forall t \geq 0$$
$$y(l,t) = 0 \quad \forall t \geq 0$$

$$y(x,0) = 0 \qquad 0 \leq x < l$$

$$\left(\frac{dy}{dt}\right)_{t=0} = f(x) \qquad 0 \leq x < l.$$

$$\left(\frac{dy}{dt}\right)_{t=0} = \frac{dy(x,0)}{dt}$$

WITH ZERO INITIAL VELOCITY :

1. The tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y=f(x)$. If it is released from rest from this position, find the displacement at any time and at any distance from the end $x=0$.

where $f(x) = kx(l-x)$.

The wave equation is $\dfrac{d^2y}{dt^2} = a^2\dfrac{d^2y}{dx^2}$ → ①

the solution of the wave equation ① is given by

$$y(x,t) = [A\cos\lambda x + B\sin\lambda x][C\cos\lambda at + D\sin\lambda at] \rightarrow ②$$

The problem is given with zero initial velocity and $f(x) = kx(l-x)$ hence the boundary conditions are,

$$y(0,t) = 0 \quad \text{for } t \geq 0$$
$$y(l,t) = 0 \quad \text{for } t \geq 0$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\partial y(x,0)}{\partial t} = 0 \quad 0 \le x < l$$

$$y(x,0) = f(x) \quad 0 \le x < l$$

Now apply boundary condition (i) in (2).

put $x = 0$ $y(0,t) = 0$

$$(A \cos 0 + B \sin 0)\,(C \cos(\lambda at) + D \sin(\lambda at)) = 0$$

$$A\,(C \cos(\lambda at) + D \sin(\lambda at)) = 0$$

$\Rightarrow A = 0$ & $C \cos(\lambda at) + D \sin(\lambda at) \neq 0$.

(2) $\Rightarrow$

$$y(x,t) = B \sin(\lambda x)\left[C \cos(\lambda at) + D \sin(\lambda at)\right] \rightarrow (3)$$

Apply (ii) in (3).

$y(l,t) = 0$

$\Rightarrow x = l$.

$$B \sin(\lambda l)\left[C \cos(\lambda at) + D \sin(\lambda at)\right]$$

$B \neq 0$. $\sin(\lambda l) = 0$. $C \cos(\lambda at) + D \sin(\lambda at) \neq 0$

$$\lambda l = \sin^{-1}(0)$$

$$\lambda l = n\pi$$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

$l \Rightarrow$ string length (known parameter)

Sub λ in (3)

$$y(x,t) = B \sin\frac{n\pi x}{l}\left[C \cos\left(\frac{n\pi at}{l}\right) + D \sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow (4)$$

Apply (iii) in ④.

Diff ④ partially w.r.t to t.

$$\frac{dy}{dt}(x,t) = B \sin \frac{n\pi x}{l}\left[-C \frac{an\pi}{l} \sin\left(\frac{n\pi at}{l}\right) + D\frac{n\pi a}{l}\cos\left(\frac{n\pi at}{l}\right)\right]$$

$t = 0$.

$$B \sin \frac{n\pi x}{l}\left[D\frac{n\pi a}{l}\right] = 0 .$$

$\left(\frac{a\pi n}{l}\right)$ known constant.

$B \sin \frac{n\pi x}{l} \neq 0$. $\quad D\left(\frac{an\pi}{l}\right) = 0. \quad \Rightarrow D = 0$.

$$④ \Rightarrow \quad y(x,t) = B \sin \frac{n\pi x}{l}\left[C \cos\left(\frac{an\pi t}{l}\right)\right]$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{an\pi t}{l}\right) \rightarrow ⑤.$$

Apply (iv) (i.e) $t = 0$.

$$y(x,0) = f(x)$$

$$f(x) = kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

$u = lx - x^2$

$u' = l - 2x$

$u'' = -2$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$v = \sin\frac{n\pi x}{l}$

$$= \frac{2k}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right) dx.$$

$v_1 = \dfrac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$v_2 = \dfrac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$v_3 = \dfrac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + (l-2x)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) - 2\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{2l^3}{(n\pi)^3}\cos n\pi + \frac{2l^3}{(n\pi)^3}\right]$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2k}{l} \times \frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} \dfrac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}.$$

T|3

A tightly stretched string with fixed end points $x=0$ is initially in a position given by

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}.$$

If it is released from rest from this position find the displacement y at any distance x from one end at any time t.

The wave equation is $\dfrac{d^2y}{dt^2} = a^2 \dfrac{d^2y}{dx^2}$. $\rightarrow$ (1)

The solution of the wave equation (1) is given by

$$y(x,t) = \left[A\cos(\lambda x) + B\sin(\lambda x)\right]\left[C\cos(\lambda a t) + D\sin(\lambda a t)\right] \rightarrow (2)$$

The boundary conditions are

$y(0,t) = 0 \qquad t \geq 0.$

$y(l,t) = 0 \qquad t \geq 0$

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\partial y(x,0)}{\partial t} = 0 \qquad 0 \leq x < l$

$y(x,0) = f(x) \qquad 0 \leq x < l.$

Using (i) in ②

put $x = 0$ $y(0,t) = 0$

$$y(0,t) = [A\cos 0 + B\sin 0][C\cos(\lambda at) + D\sin(\lambda at)] = 0.$$

$$A[C\cos(\lambda at) + D\sin(\lambda at)] = 0$$

$A = 0 \qquad C\cos(\lambda at) + D\sin(\lambda at) \neq 0.$

② $\Rightarrow$

$$y(x,t) = B\sin(\lambda x)[C\cos(\lambda at) + D\sin(\lambda at)] \longrightarrow ③$$

Using (ii) in ③

$y(l,t) = 0$

$x = l$

$$B(\sin \lambda l)[C\cos(\lambda at) + D\sin(\lambda at)] = 0$$

If $B = 0$ then we get a trivial solution.

$\sin \lambda l = 0 \qquad C\cos(\lambda at) + D\sin(\lambda at) \neq 0.$

$\lambda l = \sin^{-1}(0)$

$\lambda l = n\pi$

$\lambda = \frac{n\pi}{l}$

Sub λ in ③

$$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow ④$$

Apply (iii) in (A)

Diff A partially w.r.t 't'

$$\frac{dy}{dt}(x,t) = B \sin\left(\frac{n\pi x}{l}\right)\left[-C \frac{an\pi}{l} \sin\left(\frac{n\pi at}{l}\right) + D \frac{n\pi a}{l} \cos\left(\frac{n\pi at}{l}\right)\right]$$

$t = 0$

$$B \sin\left(\frac{n\pi x}{l}\right)\left[D \frac{n\pi a}{l}\right] = 0.$$

$D = 0 \qquad B \sin\left(\frac{n\pi x}{l}\right) \neq 0.$

(A) $\Rightarrow$

$$y(x,t) = B \sin \frac{n\pi x}{l}\left[C \cos\left(\frac{n\pi at}{l}\right)\right]$$

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos\left(\frac{n\pi at}{l}\right) \rightarrow (5)$$

App (iv) to (5)

$y(x,0) = f(x). \quad t = 0.$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$\boxed{\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}}$$

$$b_1 \sin \frac{n\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \cdots$$

$$= y_0\left[\frac{3 \sin \frac{\pi x}{l}}{4} - \frac{\sin \frac{3\pi x}{l}}{4}\right]$$

$$b_1 \sin \frac{n\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \cdots$$

$$= \frac{3}{4} y_0 \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

By comparing.

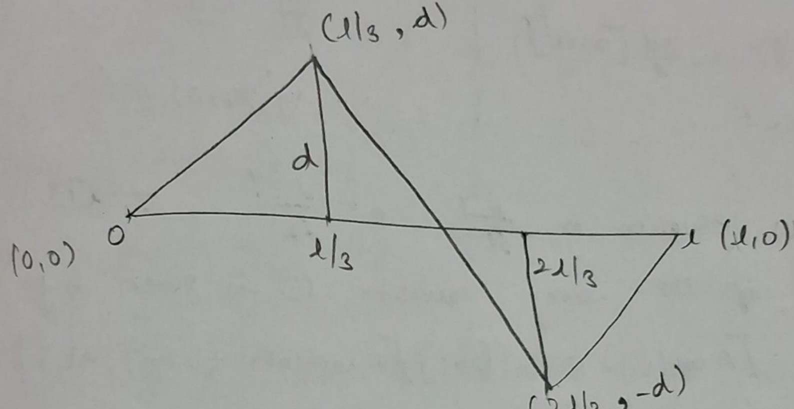
$$b_1 = \frac{3y_0}{4} \qquad b_2 = 0 \qquad b_3 = -\frac{y_0}{4} \qquad b_4 = 0.$$

(5) $\Rightarrow$

$$y(x,t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + b_3' \sin \frac{3\pi at}{l} \cos \frac{3\pi at}{l} + \cdots$$

$$y(x,t) = \frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi at}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi at}{l}\right)$$

The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string always remains at rest.



$$y(x,0) = f(x) = \begin{cases} \text{Equ. of AB}, & 0 < x < \frac{l}{3} \\ \text{Equ. of BC} & \frac{l}{3} < x < \frac{2l}{3} \\ \text{Equ. of CD} & \frac{2l}{3} < x < l \end{cases}$$

Equ. of AB $(0,0)$ $(l/3, d)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{d-0} = \frac{x-0}{\frac{l}{3}-0}$$

$$y = d\left(\frac{x}{l/3}\right)$$

$$y = \frac{3dx}{l} \qquad 0 < x < \frac{l}{3}$$

Equ of BC $(l/3, d)$ $(2l/3, -d)$

$$\frac{y-d}{-d-d} = \frac{x-l/3}{\frac{2l}{3}-l/3}$$

$$\frac{y-d}{-2d} = \frac{3x-l}{l}$$

$$yl - dl = -6dx + 2dl$$

$$yl = -6dx + 3dl$$

$$y = \frac{-6dx}{l} + 3d$$

$$y = \frac{3d}{l}\left[l - 2x\right]$$

$$\frac{l}{3} < x < \frac{2l}{3}$$

Eq. of CD $\left(\frac{2l}{3}, -d\right)$ $(l, 0)$

$$\frac{y+d}{d} = \frac{3x-2l}{3l-2l}$$

$$\frac{y+d}{d} = \frac{3x-2l}{l}$$

$$l(y+d) = d(3x-2l)$$

$$yl + dl = 3xd - 2dl$$

$$yl = 3dx - 3dl$$

$$y = \frac{3d}{l}[x-l]$$

$$\frac{2l}{3} < x < l$$

$$y(x,0) = f(x) = \begin{cases} \frac{3dx}{l} & 0 < x < \frac{l}{3} \\ \frac{3d}{l}(l-2x) & \frac{l}{3} < x < \frac{2l}{3} \\ \frac{3d}{l}(x-l) & \frac{2l}{3} < x < l \end{cases}$$

Boundary conditions are

$y(0,t) = 0 \qquad t > 0$

$y(l,t) = 0 \qquad t > 0$

$\frac{\partial y}{\partial t}(x,0) = 0 \qquad 0 \le x < l$

$y(x,0) = f(x) \qquad 0 \le x < l$

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$

The solution of the wave equation (1) is given by

$$y(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)][C\cos(\lambda at) + D\sin(\lambda at)] \longrightarrow (2)$$

Using (i) in (2)

Put $x = 0$ $\qquad y(0,t) = 0$

$$y(0,t) = [A\cos 0 + B\sin 0][C\cos(\lambda at) + D\sin(\lambda at)] = 0$$

$$A[C\cos(\lambda at) + D\sin(\lambda at)] = 0 \qquad A = 0 \qquad (C\cos(\lambda at) + D\sin(\lambda at) \neq 0)$$

(2) $\Rightarrow$

$$y(x,t) = B\sin(\lambda x)[C\cos(\lambda at) + D\sin(\lambda at)] \longrightarrow (3)$$

Using (ii) in (3)

$y(l,t) = 0$

$x = l$

$$B(\sin \lambda l)[C\cos(\lambda at) + D\sin(\lambda at)] = 0$$ [If $B = 0$ then we get trivial soln]

$$\sin \lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

(3) $\Rightarrow$ $$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow (4)$$

$= \begin{cases} \frac{3dx}{l} & 0 < x < \frac{l}{3} \\ \frac{3d}{l}(l-2x) & \frac{l}{3} < x < \frac{2l}{3} \\ \frac{3d}{l}(x-l) & \frac{2l}{3} < x < l \end{cases}$

are

$t > 0$

$t > 0$

$0 \le x < l$

$0 \le x < l$

$] \to (2)$

$\sin(\lambda at) \ne 0$

get

$\to (4)$

Apply (iii) in (4)

Diff 4 partially wrt 't'

$$\frac{\partial y}{\partial t}(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\,\frac{an\pi}{l}\sin\left(\frac{n\pi at}{l}\right) + D\,\frac{n\pi a}{l}\cos\left(\frac{n\pi at}{l}\right)\right]$$

$t=0$ $\quad B\sin\left(\frac{n\pi x}{l}\right)\left[D\,\frac{n\pi a}{l}\right] = 0$

$D = 0 \qquad B\sin\left(\frac{n\pi x}{l}\right) \ne 0.$

(4) $\Rightarrow$

$$y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right)\right]$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \to (5)$$

App (iv) to (5)

$y(x,0) = f(x) \qquad t = 0$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/3}\frac{3dx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3}\frac{3d}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l}\frac{3d}{l}(x-l)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{6d}{l^2}\left[\int_0^{l/3} x\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{2l/3}^{l}(x-l)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{6d}{l^2}\left\{\left[-x\cdot\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/3} + \left[-(l-2x)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - 2\,\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/3}^{2l/3} + \left[-(x-l)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_{2l/3}^{l}\right\}$$

$u = x \quad v = \sin\frac{n\pi x}{l}$

$u' = 1 \quad v_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$u = l - 2x$

$u' = -2$

$v = \sin\frac{n\pi x}{l}$

$v_1 = \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$u = x - l$

$u' = 1$

$v = \sin\frac{n\pi x}{l}$

$v_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \qquad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$$= \frac{6d}{l^2}\left\{\left[-\frac{l}{3}\,\frac{\cos\left(\frac{n\pi l}{3l}\right)}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi l}{3l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] +\right.$$

$$\left[-\left(l-\frac{2l}{3}\right)\frac{\cos\left(\frac{2n\pi l}{3l}\right)}{\frac{n\pi}{l}} - 2\,\frac{\sin\left(\frac{2n\pi l}{3l}\right)}{\left(\frac{n\pi}{l}\right)^2} + \left(\frac{l}{3}-\frac{2l}{3}\right)\left(\frac{\cos\frac{n\pi l}{3l}}{\frac{n\pi}{l}}\right)\right.$$

$$\left. + 2\,\frac{\sin\frac{n\pi l}{3l}}{\left(\frac{n\pi}{l}\right)^2}\right] +$$

$$\left.\left[\left(\frac{2l}{3}-l\right)\frac{\cos\left(\frac{2n\pi l}{3l}\right)}{\frac{n\pi}{l}} - \frac{\sin\left(\frac{2n\pi l}{3l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]\right\}.$$

$$= \frac{6d}{l^2}\left\{\left[-\frac{l}{3}\times\frac{l}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\right] +\right.$$

$$\left[\frac{l}{3}\times\frac{l}{n\pi}\cos\left(\frac{2n\pi}{3}\right) - \frac{2l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) - \frac{l\times l}{3n\pi}\cos\left(\frac{n\pi}{3}\right) +\right.$$

$$\left.\left.\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\right] + \left[-\frac{l}{3}\times\frac{l}{n\pi}\cos\left(\frac{2n\pi}{3}\right) - \frac{l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\right]\right.$$

$$= \frac{6d}{l^2}\left\{\left[-\frac{l^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\left(\frac{2n\pi}{3}\right)\right.\right.$$

$$-\frac{2l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) - \frac{l^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$\left. - \frac{l^2}{3n\pi},\cos\left(\frac{2n\pi}{3}\right) - \frac{l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\right]$$

A string is stretched and fastened to two points l apart. Motion is started by displaing the string into the form of the curve $y = kx(l-x)$ and then releasing it from this position at time $t = 0$. Find the displacement function $y(x,t)$.

$$f(x) = kx(l-x) \qquad = k(xl - x^2)$$

$$f(x) = k(xl - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}$$

$u = lx - x^2$

$u' = l - 2x$

$u'' = -2$

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \cdot dx$$

$v = \sin \frac{n\pi x}{l}$

$$= \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$v_1 = \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}$$

$\left(\sin \frac{n\pi x}{l} \right)$

$-\sin \frac{n\pi x}{l}$

PROBLEMS WITH INITIAL VELOCITY:

If a string of length is initially at rest in its equilibrium position and each of its point is given velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$ $0 \le x \le l$. Determine the displacement.

The wave equation is $\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}$

The boundary conditions are,

$y(0,t) = 0$ for every $t > 0$

$y(l,t) = 0$ for every $(t>0)$

$y(x,0) = 0$ for all x in $(0,l)$

$\frac{dy}{dt}(x,0) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$ $0 \le x \le l$.

$y(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)][C\cos(\lambda at) + D\sin(\lambda at)]$

(i) $y(0,t) = A$.

(ii) $\lambda = \frac{n\pi}{l}$

(iii) in (A)

$y(x,0) = 0$ $t=0$

(A) $\rightarrow$ $B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] = 0$

$B \ne 0$ $\sin\left(\frac{n\pi x}{l}\right) \ne 0$ $C+0$ $\boxed{C=0}$

(A) $\rightarrow$ $y(x,t) = B\sin\left(\frac{n\pi x}{l}\right) D \sin\left(\frac{an\pi t}{l}\right)$

$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{an\pi t}{l}\right)$ $\longrightarrow$ (5)

Diff w.r.t 't' (5)

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \left(\frac{an\pi}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{an\pi t}{l}\right)$$

$t=0$

$$\frac{dy}{dt}(x,0) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi a}{l}\right) \sin \frac{n\pi x}{l} = V_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = \frac{V_0}{4}\left[3 \sin \frac{n\pi}{l} - \sin \frac{3\pi x}{l}\right)$$

$$c_1 \frac{\pi a}{l} \sin \frac{\pi x}{l} + c_2 \frac{2\pi a}{l} \sin \frac{2\pi x}{l} + c_3 \frac{3\pi a}{l} \sin \frac{3\pi x}{l} + \cdots$$

$$= \frac{3V_0}{4} \sin \frac{n\pi}{l} - \frac{V_0}{4} \sin \frac{3\pi x}{4}$$

$$c_1 \frac{\pi a}{l} = \frac{3V_0}{4} \qquad c_2 = 0 \qquad c_3 \frac{3\pi a}{l} = \frac{-V_0}{4} \qquad c_4 = c_5 = \cdots = 0$$

$$y(x,t) = c_1 \sin \frac{\pi x}{l} \sin \frac{\pi a t}{l} + c_2 \sin \frac{2\pi x}{l} \sin \frac{2\pi a t}{l} +$$

$$c_3 \sin \frac{3\pi x}{l} \sin \frac{3\pi a t}{l} + c_4 \sin \frac{4\pi x}{l} \sin \frac{4\pi a t}{l} + \cdots$$

$$y(x,t) = \frac{3V_0 l}{4\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi a t}{l} - \frac{V_0 l}{12\pi a} \sin \frac{3\pi a}{l} \sin \frac{3\pi a t}{l}$$

8/3

Fourier $\rightarrow$ F

Inverse fourier $\rightarrow$

The

called

PARSEV

Find

W

FOURIER TRANSFORM

Fourier → $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Inverse Fourier → $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

The above two formulas are together called as Fourier Transform Pair.

PARSEVAL'S IDENTITY:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\boxed{\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}}$$

Find the fourier transforms of $f(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}$

We know that Fourier transform

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} 1 \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_a^b$$

$$= \frac{1}{\sqrt{2\pi} \times is} \left[e^{isb} - e^{isa} \right]$$

Find the fourier transforms of $f(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}$

We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$|x|<1$

$-1<x<1$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{isx}\, dx$$

$|x|>1$

$-\infty<x<-1$

$1<x<\infty$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi} \times is} \left[e^{is} - e^{-is} \right]$$

$$= \frac{1}{\sqrt{2\pi}\, is} \left[\cancel{\cos s} + i\sin s - \cos s + i\sin s \right]$$

$$= \frac{1}{\sqrt{2\pi}\, is} \left[2 i \sin s \right]$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin s}{s}.$$