Laplace Transforms

Definition of Laplace Transforms

Let $f(t)$ be a function of defined for all positive values of t. Then the Laplace transforms of $f(t)$, denoted by $L[f(t)]$ and is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

where is a parameter provided that the integral exists.
Note: The parameter may be a real or complex number.

In the following link you can see a video that explains Definition with problems

https://drive.google.com/open?id=1WZZpWExYYpem7vg5PT1u46QCwxIWwmSm

Laplace Transforms

Introduction

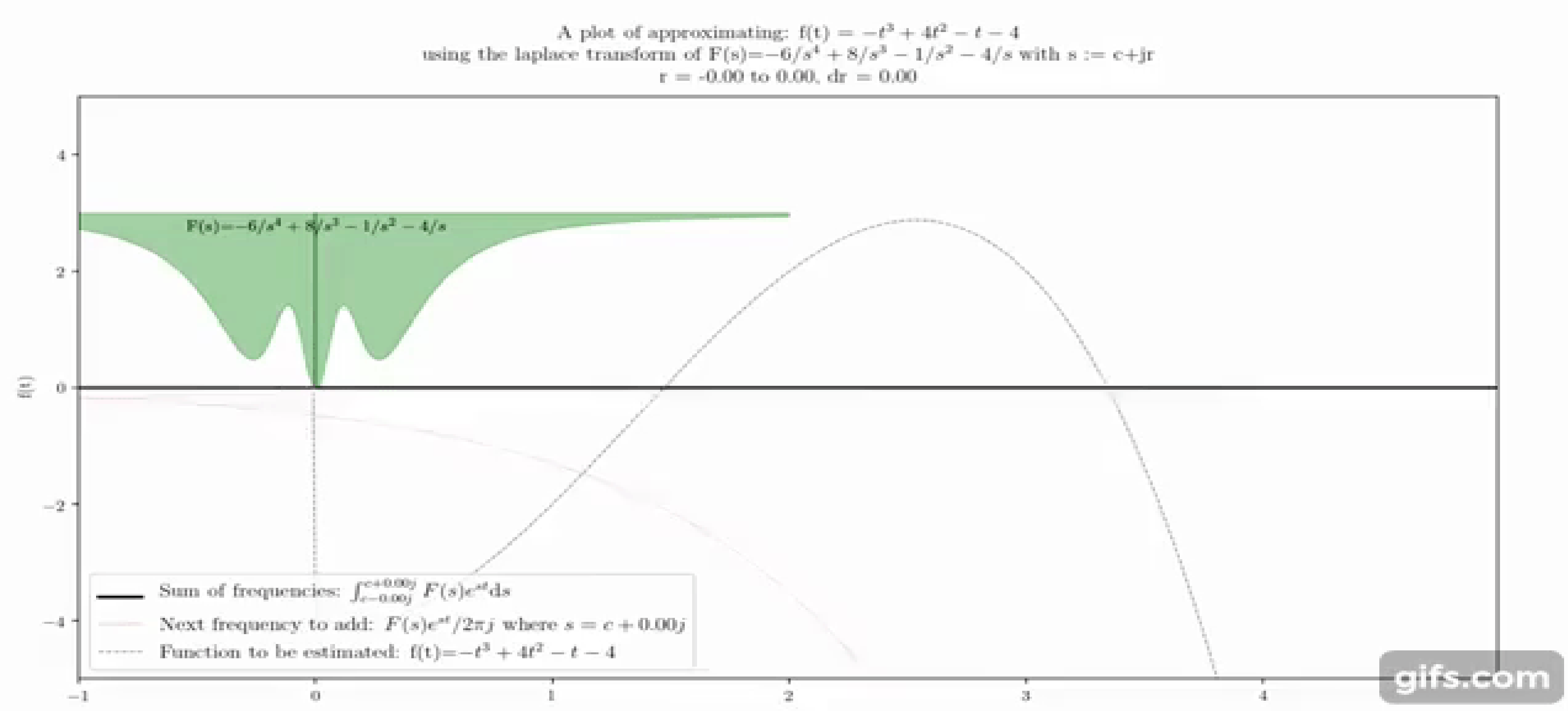
The knowledge of Laplace transforms has in recent years becomes an essential part of mathematical back ground required of engineering and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

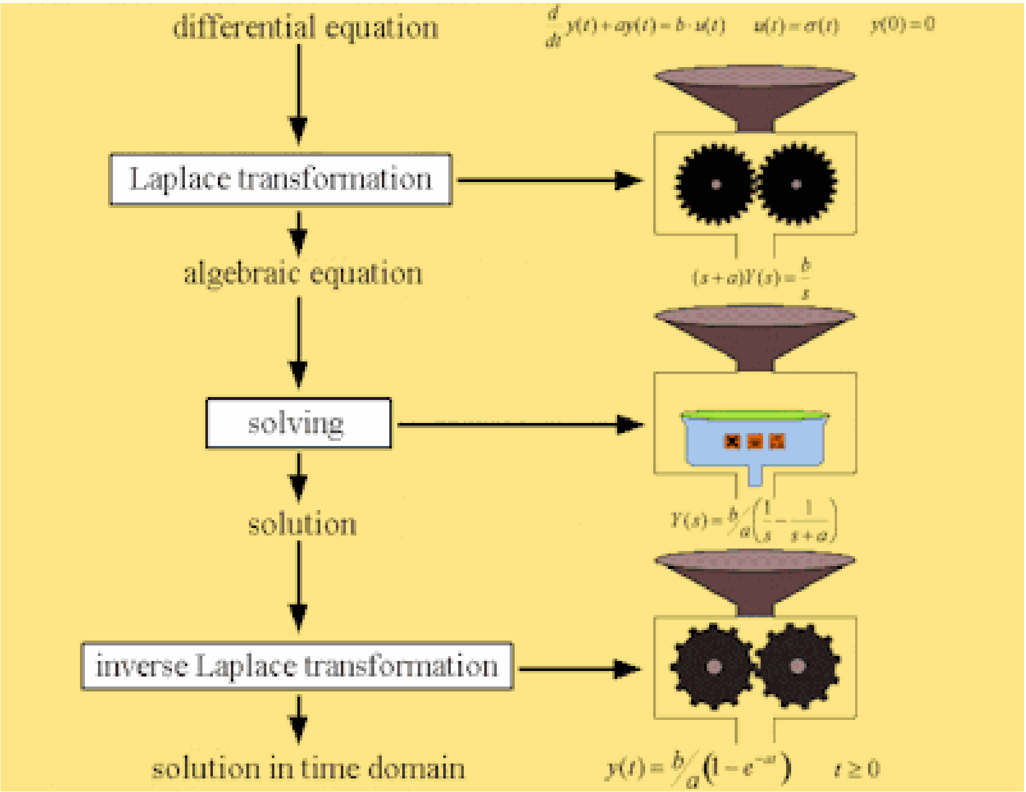
This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850-1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916-17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.

In the following link you can see a video that discusses the relationship to the transfer function and the Laplace Transform.
https://drive.google.com/open?id=1hM5sXtmLRIIUJuCg3h0b-XZi0Taoisj9

Additional references explains you about
Laplace Transform Explained and Visualized Intuitively
https://drive.google.com/open?id=10-ojploUiEhzo7opbzFIBXZuPKWCFIMH

The Laplace Transform - A Graphical Approach
https://drive.google.com/open?id=1oCbivfAgeBbuT6HD2WWATUJ39farami





differential equation
$\frac{d}{dt}y(t) + ay(t) = b \cdot u(t)$
$u(t) = \sigma(t)$
$y(0) = 0$
Laplace transformation
algebraic equation
$(s+a)Y(s) = \frac{b}{s}$
solving
solution
$Y(s) = b/a\left(\frac{1}{s} - \frac{1}{s+a}\right)$
inverse Laplace transformation
solution in time domain
$y(t) = b/a\left(1 - e^{-at}\right)$
$t \geq 0$

Laplace Transforms

Applications

✓The Laplace transformation is used to find the solution of linear differential equations-Ordinary as well as partial.

✓It helps in solving the differential equation with boundary values without finding the general solution and then finding the values of the arbitrary constants.

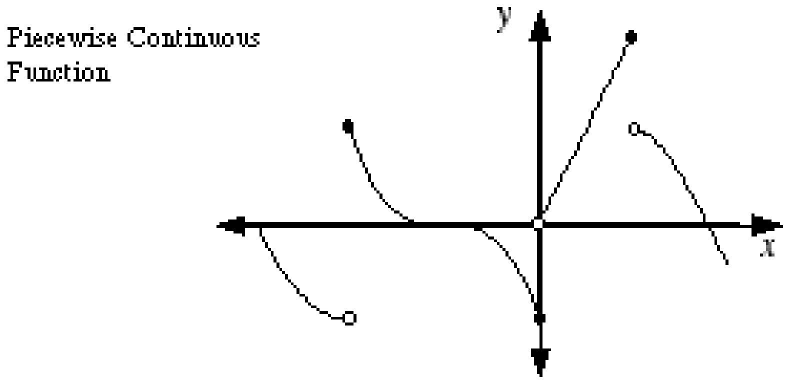
In the following link you can see a video that explains 1.What does the Laplace Transform really tell us? A visual explanation https://drive.google.com/open?id=1Ucep9U9jgeSWP65wQP8SDi-XQxmCC3nC
Additional reference explains you about 1. Quick Review of Laplace Transform https://drive.google.com/open?id=1JNO8K5bTtNiss4dTCBh01t0QKMQxUqLb 2. Laplace Transforms and Electric Circuits https://drive.google.com/open?id=1pOYtW6KFBQYwFgSI-AdebxumbfzN019J

Conditions for existence

Definitions

Piecewise Continuous function

A function $f(t)$ is said to be piecewise continuous in any interval [a,b] if it is defined on that interval and is such that the interval can be broken up into a finite number of sub-intervals in each of which $f(t)$ is continuous.



Function of Exponential order

A function $f(t)$ is said to be of exponential order if $\lim_{t\to\infty} e^{-st} f(t) = 0.$

Example: The function t^2 is of exponential order.

For,

$$\lim_{t\to\infty} e^{-st} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{st}} = \lim_{t\to\infty} \frac{2t}{se^{st}} = \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = \frac{2}{\infty} = 0 \;(\,finite)$$

$$\therefore t^2 \text{ is of exponential order.}$$

Conditions for existence

Existence of Laplace transform

$$L\left[f(t)\right]=\int_0^\infty e^{-st} f(t)dt$$

exists, if

1. Should be continuous or piecewise continuous in the closed interval [a,b].

2. Should be of exponential order.

In the following link you can see a video that explains the Conditions for existence of Laplace Transforms
https://drive.google.com/open?id=1rCmzYs0tRdZ7c2SLpA1R3wrlIlQPZ5mW

Additional reference explains you about the Conditions for existence from introduction https://drive.google.com/open?id=1IY1mpuRXc1tQp-YOiHOTNkyd9YVQPXpU

Transform of standard functions

- Constant
- Exponential function
- Trigonometric function
- Polynomial function t^n
- Hyperbolic function

LAPLACE TRANSFORMS AND INVERSE LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

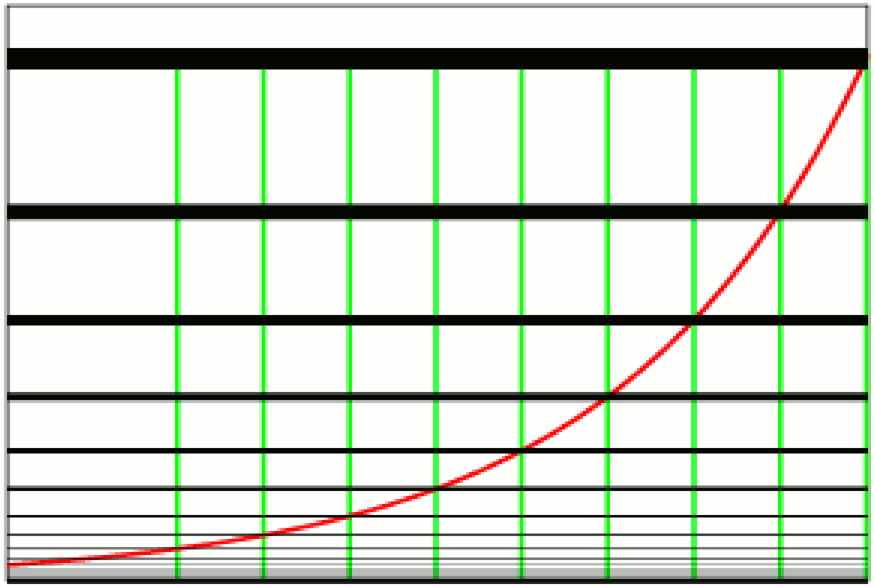
Sl.No	Laplace Transforms		Inverse Laplace Transforms	
	$f(t)$	$L[f(t)]$	$F(S)$	$L^{-1}[F(S)]$
1	k, a constant	$L[k]=\frac{k}{s}$, s > 0 k is a constant	$\frac{k}{s}$, s ≠ 0 k is a constant	$L^{-1}\left[\frac{k}{s}\right]=k$
2	e^{at}	$L\left[e^{at}\right]=\frac{1}{s-a}$	$\frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
3	e^{-at}	$L\left[e^{-at}\right]=\frac{1}{s+a}$	$\frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$
4	t^n	$L\left[t^n\right]=\begin{cases}\frac{n!}{s^{n+1}}, \text{ n } = 0,1,2,3\ldots \\ \frac{\Gamma(n+1)}{s^{n+1}}, \text{ n } = \frac{1}{2},\frac{3}{2},\ldots\end{cases}$	$\frac{1}{s^n}$	$L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$ n = 1,2,3...
5	a^t	$L\left[a^t\right]=\frac{1}{s-\log a}$	$\frac{1}{(s-a)^n}$	$L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}t^{n-1}}{(n-1)!}$
6	$\sqrt{t}$	$L\left[t^{\frac{1}{2}}\right]=\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$
7	$\frac{1}{\sqrt{t}}$	$L\left[\frac{1}{\sqrt{t}}\right]=\frac{\sqrt{\pi}}{\sqrt{s}}$	$\frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

8	$\sin at$	$L[\sin at] = \dfrac{a}{s^2 + a^2}$	$\dfrac{1}{s^2 - a^2}$	$L^{-1}\left[\dfrac{1}{s^2 - a^2}\right] = \dfrac{1}{a}\sinh at$
9	$\cos at$	$L[\cos at] = \dfrac{s}{s^2 + a^2}$	$\dfrac{s}{s^2 - a^2}$	$L^{-1}\left[\dfrac{s}{s^2 - a^2}\right] = \cosh at$
10	$\sinh at$	$L[\sinh at] = \dfrac{a}{s^2 - a^2}$	$\dfrac{1}{(s-a)^2 + b^2}$	$L^{-1}\left[\dfrac{1}{(s-a)^2 + b^2}\right] = \dfrac{1}{b} e^{at} \sin bt$
11	$\cosh at$	$L[\cosh at] = \dfrac{s}{s^2 - a^2}$	$\dfrac{s-a}{(s-a)^2 + b^2}$	$L^{-1}\left[\dfrac{s-a}{(s-a)^2 + b^2}\right] = e^{at} \cos bt$

Transform of standard functions

Exponential function

Prove that $L\left[e^{at}\right]=\dfrac{1}{s-a}$



Proof

$$L[f(t)]=\int_0^\infty e^{-st} f(t)dt$$

$$L\left[e^{at}\right]=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-(s-a)t}dt$$

$$=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty \qquad \left[\because \int e^{-at}dt=\frac{e^{-at}}{-a}\right]$$

$$=\left[\frac{e^{-(s-a)\infty}}{-(s-a)}\right]-\left[\frac{e^{-(s-a)0}}{-(s-a)}\right]$$

$$=\left[\frac{0}{-(s-a)}\right]-\left[\frac{1}{-(s-a)}\right] \qquad \left[\because e^{-\infty}=0, e^{0}=1\right]$$

$$L\left[e^{at}\right]=\frac{1}{s-a}$$

Transform of standard functions

Exponential and constant function

$$L\left[e^{at}\right] = \frac{1}{s-a}$$

$$L[k] = \frac{k}{s}, \text{s} > 0 \text{ k is a constant}$$

In the following link you can see a video that explains Laplace Transform of Exponential function and Constant
https://drive.google.com/open?id=1f8BqxWfUWRS5fWAHj5iSeike9b7lydW9

Problem 1:
https://drive.google.com/open?id=1-aSr6K8kAOUJ-IASvTKLsX2oAhJSw8jA

Problem 2:
https://drive.google.com/open?id=1BbWUw03xumMRLX9iQM8ZUjQR-bJDA30t

Transform of standard functions

Exponential and constant function

Problems for practice

1 $L[e^{-at}]$

2 $L[e^{5t}]$

3 $L[5]$

4.3. Transform of standard functions

Trigonometric function

Prove that $L[\cos at] = \dfrac{s}{s^2+a^2}$

Proof

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt$$

$$= Real\ Part\ of \int_0^\infty e^{-st} e^{iat}\ dt$$

$$= Real\ Part\ of\ L\left[e^{iat}\right]$$

$$= Real\ Part\ of\ \frac{1}{s\text{-}ia} \qquad \left[\because L\left(e^{at}\right) = \frac{1}{s-a}\right]$$

$$= Real\ Part\ of\ \frac{1}{s\text{-}ia} \times \frac{s+ai}{s+ai}$$

Transform of standard functions

Trigonometric function

Prove that $L[\cos at] = \dfrac{s}{s^2 + a^2}$

Proof

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at \, dt$$

$$= Real\ Part\ of \int_0^\infty e^{-st} e^{iat} \, dt$$

$$= Real\ Part\ of\ L\left[e^{iat}\right]$$

$$= Real\ Part\ of\ \frac{1}{s\text{-}ia} \qquad \left[\because L\left(e^{at}\right) = \frac{1}{s-a}\right]$$

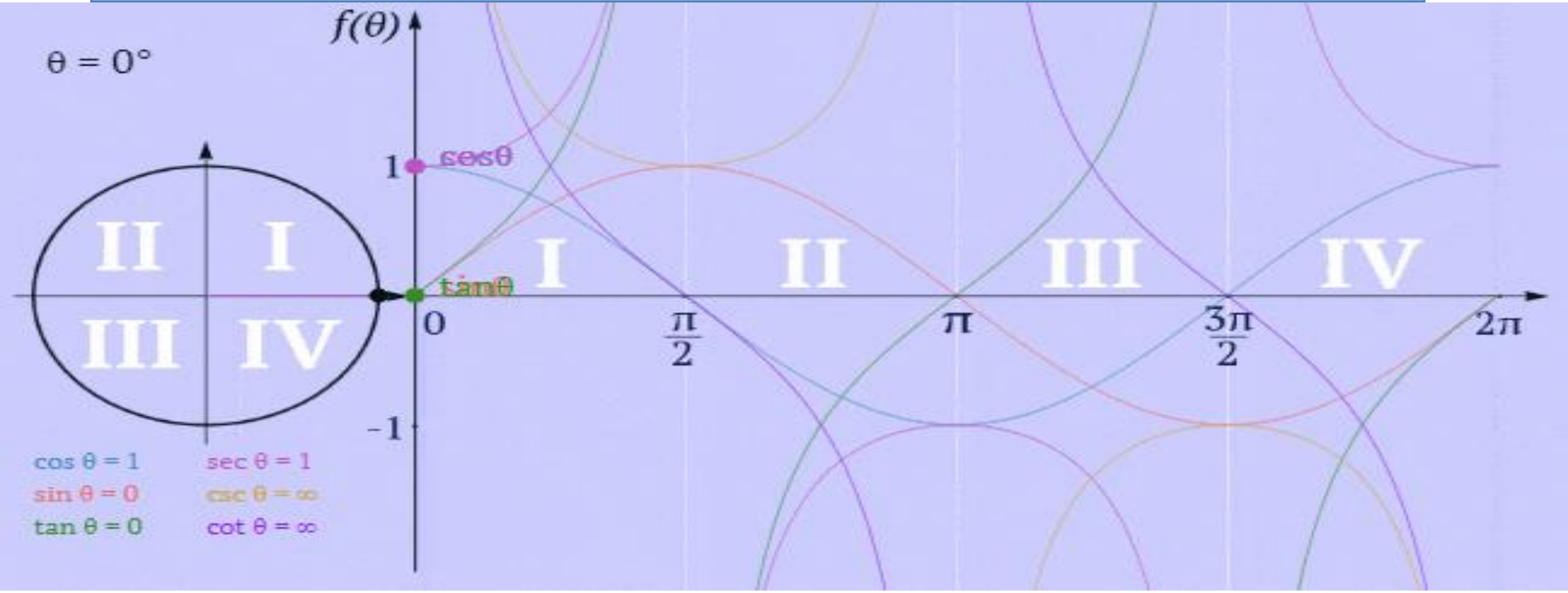
$$= Real\ Part\ of\ \frac{1}{s\text{-}ia} \times \frac{s+ai}{s+ai}$$

$$= \text{Real Part of } \frac{s + ai}{s^2 - (ai)^2}$$

$$= \text{Real Part of } \frac{s + ai}{s^2 + a^2}$$

$$= \text{Real Part of } \left[\frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2} \right]$$

$$L[\cos at] = \frac{s}{s^2 + a^2}$$



Transform of standard functions

Trigonometric function

Problems

1. Find the Laplace Transform of $\sin 2t \cos 3t$.

Solution:

We know that $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$\cos 3t \sin 2t = \frac{1}{2}\left[\sin(5t) - \sin(t)\right]$$

$$L\left[\cos 3t \sin 2t\right] = \frac{1}{2} L\left[\sin(5t) - \sin(t)\right]$$

$$= \frac{1}{2}\left[L\left(\sin 5t\right) - L\left(\sin t\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right] \qquad \left[\because L(\sin at) = \frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{5(s^2+1)-(s^2+25)}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$L\left[\cos 3t \sin 2t\right] = \frac{2s^2-10}{(s^2+25)(s^2+1)}$$

2. Find the Laplace Transform of $\sin^2 2t$.

Solution:

We know that, $\sin^2 A = \dfrac{1-\cos 2A}{2}$

$$L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2} L[1-\cos 4t]$$

$$= \frac{1}{2}[L(1) - L(\cos 4t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right] \qquad \left[\because L[1] = \frac{1}{s},\quad L[\cos at] = \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$$

$$L[\sin^2 2t] = \frac{8}{s(s^2+16)}$$

3. Find the Laplace Transform of $\cos^3 2t$

Solution:

We know that, $\cos^3 A = \dfrac{\cos 3A + 3\cos A}{4}$

$$L[\cos^3 2t] = \frac{1}{4} L\left[\cos 3(2t) + 3\cos(2t)\right]$$

$$= \frac{1}{4}\left[L\left(\cos 6t\right) + 3L\left(\cos 2t\right)\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + 3.\frac{s}{s^2+2^2}\right] \qquad \left[\because L\left[\cos at\right] = \frac{s}{s^2+a^2}\right]$$

$$= \frac{s}{4}\left[\frac{1}{s^2+36} + \frac{3}{s^2+4}\right]$$

$$= \frac{s}{4}\left[\frac{s^2+4+3s^2+108}{(s^2+36)(s^2+4)}\right]$$

$$= \frac{s}{4}\left[\frac{4s^2+112}{(s^2+36)(s^2+4)}\right]$$

$$L[\cos^3 2t] = \left[\frac{s^3+28s}{(s^2+36)(s^2+4)}\right]$$

Problems for practice

Find the Laplace transform of the following functions

1. $\sin at$
2. $\cos^2 2t$
3. $\sin^3 t$
4. $\sin\frac{t}{2}$

Transform of standard functions

Trigonometric function

$$L[\sin at] = \frac{a}{s^2 + a^2} \qquad L[\cos at] = \frac{s}{s^2 + a^2}$$

In the following link you can see a video that explains the Laplace transform of a trigonometric function
https://drive.google.com/open?id=11kBQkgJiVFPSNFzthjdmHK-vUTL14Euw

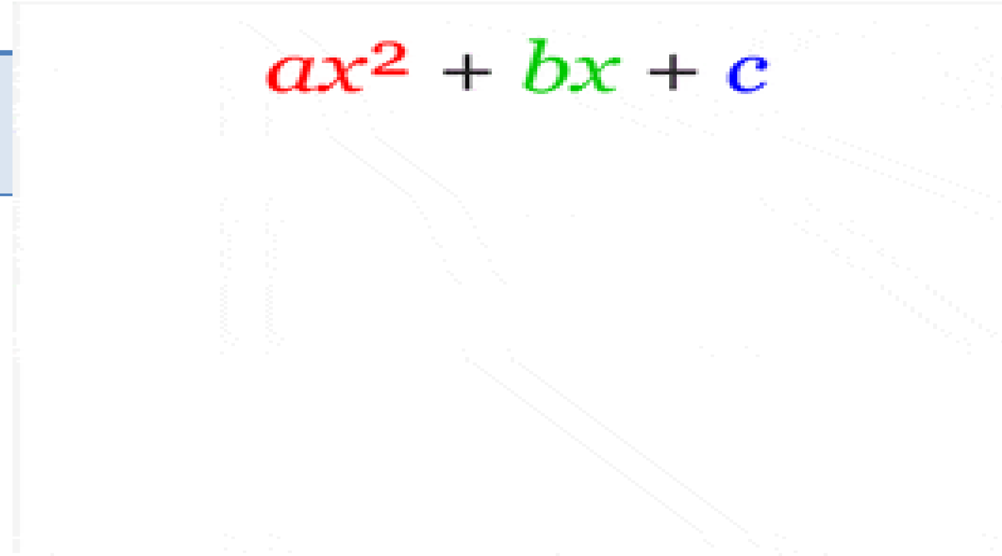
Problems:

1:https://drive.google.com/open?id=1fRsetvNUktP5UX1d88pRTSfFFPpL7o5K
2:https://drive.google.com/open?id=1suD_V0QaizXVZEIQ4xVC3AbBogW5Qgu8
3:https://drive.google.com/open?id=1RETN9UisHeMZ0Jm5CUxC_l-z5iO-vV3v

Transform of standard functions

Polynomial function t^n

$$L\left[t^n\right] = \begin{cases} \dfrac{n!}{s^{n+1}}, \mathrm{n} = 0,1,2,3\ldots \\ \dfrac{\Gamma(n+1)}{s^{n+1}}, \mathrm{n} = \dfrac{1}{2}, \dfrac{3}{2}, \ldots \end{cases}$$



In the following link you can see a video that explains the derivation of t^n
https://drive.google.com/open?id=124nTZBeLKO76wuJobaU_BOKmu085dJE-

Problem 1:
https://drive.google.com/open?id=176GwV_N5SaLP9tDumGDP_ieW5eYPQX9Z

Transform of standard functions

Hyperbolic function

Prove that . $L[\cosh at] = \dfrac{s}{s^2 - a^2}$

Proof:

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right] \qquad \left[\because \cosh at = \frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L\left[e^{at}\right] + L\left[e^{-at}\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} \qquad \left[\because L\left[e^{at}\right] = \frac{1}{s-a}, L\left[e^{-at}\right] = \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right]$$

$$L[\cosh at] = \left[\frac{s}{s^2 - a^2}\right]$$

Transform of standard functions

Hyperbolic function

$$L[\sinh at] = \frac{a}{s^2 - a^2} \qquad L[\cosh at] = \frac{s}{s^2 - a^2}$$

In the following link you can see a video that explains

1. Laplace transform of sinh (at)
https://drive.google.com/open?id=1IFI-PcCgQ6-3j6m1sFc72cDtUqzSgt8I

2. Laplace transform of cosh (at)
https://drive.google.com/open?id=1mZ3odwY_WDhOq2bZtt0g2waQu4k9xq3H

Problem for practice $L[sinhat]$

Transform of standard functions

Problems

Problems in Laplace transforms of constants, trigonometric functions, hyperbolic functions, exponential function, Polynomial function (t^n)

In the following link you can see a video of examples for

1. Part-1
   https://drive.google.com/open?id=1KWwNCaSJPPP9PHoPGaY9k0YqjpaXggrp

2. Part-2
   https://drive.google.com/open?id=1VZ_Rc_dlnsvDCn-xFegcag-qhUS9POIs

Basic properties

- Linearity Property
- First shifting property
- Second shifting property
- Change of scale property
- Derivatives of Transforms
- Integrals of Transforms

Basic properties

Linearity Property

If $L\left[f(t)\right] = F(s) \,\&\, L\left[g(t)\right] = G(s)$,

then $L\left[af(t) \pm bg(t)\right] = aF(s) \pm bG(s),\ a \,\&\, b$ are constants.

Proof $$L\left[f(t)\right] = \int_0^{\infty} \mathrm{e}^{-st}\ \mathrm{f(t)}\ \mathrm{dt}$$

$$L\left[af(t) \pm bg(t)\right] = \int_0^{\infty} \left[af(t) \pm bg(t)\right] e^{-st}\, dt$$

$$= a\int_0^{\infty} e^{-st}\, f(t)dt \pm b\int_0^{\infty} e^{-st}\, g(t)dt$$

$$= aL\left[f(t)\right] \pm bL\left[g(t)\right]$$

$$L\left[af(t) \pm bg(t)\right] = aF(s) \pm bG(s)$$

Basic properties

Linearity Property-Problem

1. Find the Laplace Transform of $e^{-2t}\cosh^3 2t.$

Solution:

We know that, $\cosh A = \dfrac{e^{A} + e^{-A}}{2}$

$$\cosh^3 2t = \left[\frac{e^{2t} + e^{-2t}}{2}\right]^3$$

$$= \frac{1}{8}\left[(e^{2t})^3 + (e^{-2t})^3 + 3(e^{2t})^2(e^{-2t}) + 3(e^{2t})(e^{-2t})^2\right]$$

$$\cosh^3 2t = \frac{1}{8}\left[e^{6t} + e^{-6t} + 3e^{2t} + 3e^{-2t}\right]$$

$$L\left[e^{-2t}\, cosh^3\, 2t\right] = \frac{1}{8} L\left[e^{-2t}\left[e^{6t} + e^{-6t} + 3e^{2t} + 3e^{-2t}\right]\right]$$

$$=\frac{1}{8}L\left[e^{6t}+e^{-6t}+3e^{2t}+3e^{-2t}\right]_{s\to s+2}$$

$$=\frac{1}{8}\left[L(e^{6t})+L(e^{-6t})+3L(e^{2t})+3L(e^{-2t})\right]_{s\to s+2}$$

$$=\frac{1}{8}\left[\frac{1}{s-6}+\frac{1}{s+6}+3\frac{1}{s-2}+3\frac{1}{s+2}\right]_{s\to s+2}$$

$$=\frac{1}{8}\left[\frac{1}{s+2-6}+\frac{1}{s+2+6}+3\frac{1}{s+2-2}+3\frac{1}{s+2+2}\right]$$

$$L\left[e^{-2t}\cosh^3 2t\right]=\frac{1}{8}\left[\frac{1}{s-4}+\frac{1}{s+8}+\frac{3}{s}+\frac{3}{s+4}\right]$$

Basic properties

Linearity Property

$$L\left[af(t) \pm bg(t)\right] = aF(s) \pm bG(s)$$

Where $$L\left[f(t)\right] = F(s) \;\&\; L\left[g(t)\right] = G(s)$$

In the following link you can see a video that explains Linearity Property of Laplace transform (Existence not necessary)

https://drive.google.com/open?id=11TeyKTWL_igK3NDJuvpfQdwt7Sb5PTDS

Problem for practice

Find the Laplace transform of the following functions $\sinh^2 2t$

Basic properties

First shifting property

If $L[f(t)] = F(s)$ then

$$(i) L\left[e^{-at} f(t)\right] = F(s+a)$$

$$(ii) L\left[e^{at} f(t)\right] = F(s-a)$$

Proof of (i)

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$L\left[e^{-at} f(t)\right] = \int_0^\infty e^{-st} [e^{-at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s+a)} f(t) dt$$

$$L\left[e^{-at} f(t)\right] = F(s+a)$$

$$\because L\left[e^{-at} f(t)\right] = F(s+a)$$

$$= F[s]_{s \to (s+a)}$$

Proof of (ii)

$$L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt = F(s)$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)] \, dt$$

$$= \int_0^\infty e^{-(s-a)} f(t) dt$$

$$L[e^{at} f(t)] = F(s-a)$$

$$\therefore L[e^{at} f(t)] = F(s-a)$$

$$= F[s]_{s \to (s-a)}$$

Basic properties

First shifting property-Problems

1. Find the Laplace Transform of $\left(t^3+3e^{2t}-5\sin 3t\right)e^{-t}$

Solution:

$$L\left[\left(t^3+3e^{2t}-5\sin 3t\right)e^{-t}\right]$$

$$=\left[L[t^3]+3L[e^{2t}]-5L[\sin 3t]\right]_{s\to s+1}$$

$$=\left[\frac{3!}{s^4}+3\frac{1}{s-2}-5\frac{3}{s^2+9}\right]_{s\to s+1}$$

$$L\left[e^{at}\right]=\frac{1}{s-a}$$

$$L[t^n]=\frac{n!}{s^{n+1}},$$

$$=\frac{6}{(s+1)^4}+\frac{3}{s-1}-\frac{15}{(s+1)^2+9}$$

$$\left[\because L\left[\sin at\right]=\frac{a}{s^2+a^2}\right]$$

$$L\left[e^{-t}\left(t^3+3e^{2t}-5\sin 3t\right)\right]=\frac{6}{(s+1)^4}+\frac{3}{s-1}-\frac{15}{s^2+2s+10}$$

2. Find the Laplace Transform of $\left(1+te^{-t}\right)^3$

Solution:

$$L\left(1+te^{-t}\right)^3 = L\left(1^3+3(1)^2(te^{-t})+3(1)(te^{-t})^2+(te^{-t})^3\right)$$

$$= L[1]+3L(te^{-t})+3L(t^2e^{-2t})+L(t^3e^{-3t})$$

$$= \frac{1}{s}+3\left[L\left(t\right)\right]_{s\to s+1}+3\left[L\left(t^2\right)\right]_{s\to s+2}+\left[L\left(t^3\right)\right]_{s\to s+3}$$

$$= \frac{1}{s}+3\left[\frac{1}{s^2}\right]_{s\to s+1}+3\left[\frac{2!}{s^3}\right]_{s\to s+2}+\left[\frac{3!}{s^4}\right]_{s\to s+3}$$

$$L\left(1+te^{-t}\right)^3 = \frac{1}{s}+\left[\frac{3}{(s+1)^2}\right]+\left[\frac{6}{(s+2)^3}\right]+\left[\frac{6}{(s+3)^4}\right]$$

$$(a+b)^3 = a^3+3ab^2+3a^2b+b^3$$

3. Find the Laplace Transform of $\cosh at \cos at$.

Solution:

We know that $\cosh t = \dfrac{e^{t} + e^{-t}}{2}$

$$L[\cosh at \cos at] = L\left[\left(\frac{e^{at} + e^{-at}}{2}\right)\cos at\right]$$

$$= \frac{1}{2}\left[L\left(e^{at} + e^{-at}\right)\cos at\right]$$

$$= \frac{1}{2}\left[L\left(e^{at} \cos at\right) + L\left(e^{-at} \cos at\right)\right]$$

$$= \frac{1}{2}\left[L(\cos at)_{s \to s-a} + L(\cos at)_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2 + a^2}\right)_{s \to s-a} + \left(\frac{s}{s^2 + a^2}\right)_{s \to s+a}\right]$$

$$L[\cosh at \cos at] = \frac{1}{2}\left[\left(\frac{s-a}{(s-a)^2 + a^2}\right) + \left(\frac{s+a}{(s+a)^2 + a^2}\right)\right]$$

4. Find the Laplace Transform of $e^{3t}\sin 2t\sin t$

Solution:

We know that $$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

$$\sin 2t\sin t=\frac{1}{2}\left[\cos(t)-\cos(3t)\right]$$

$$L\left[\sin 2t\sin t\right]=\frac{1}{2}L\left[\cos(t)-\cos(3t)\right]$$

$$L\left[e^{3t}\sin 2t\sin t\right]=\frac{1}{2}L\left[e^{3t}\left[\cos(t)-\cos(3t)\right]\right]$$

$$=\frac{1}{2}\left[L\left(e^{3t}\cos t\right)-L\left(e^{3t}\cos 3t\right)\right]$$

$$=\frac{1}{2}\left[L\left(\cos t\right)_{s\to s-3}-L\left(\cos 3t\right)_{s\to s-3}\right]$$

$$=\frac{1}{2}\left[\left(\frac{s}{s^2+1^2}\right)_{s\to s-3}-\left(\frac{s}{s^2+3^2}\right)_{s\to s-3}\right]$$

$$L\left[e^{3t}\sin 2t\sin t\right]=\frac{1}{2}\left[\left(\frac{s-3}{(s-3)^2+1^2}\right)-\left(\frac{s-3}{(s-3)^2+3^2}\right)\right]$$

Basic properties

First shifting property-Problem for practice

Find the Laplace Transform of the following functions

1. $e^{t}\sin^{3}2t$

2. $e^{-t}\sin 2t\cos 3t$

3. $t\cosh^{3}t$

4. $\sinh\dfrac{t}{2}\sin\dfrac{\sqrt{3}}{2}t$

Basic properties

Second shifting property

If $L\left[f(t)\right]=F(s)\ \&\ g(t)=\begin{cases} f(t-a), & t>a \\ 0, & 0\le t<a \end{cases}$

then $L\left[g(t)\right]=e^{-as}F(s)$

In the following link you can see a video that explains Second shifting property

https://drive.google.com/open?id=1sOns45WJhyit43nz60Su9DMaQzXTLUSy

1. Find the Laplace Transform of $f(t)=\begin{cases}\cos\left(t-\frac{2\pi}{3}\right), & t>\frac{2\pi}{3}\\ 0, & t<\frac{2\pi}{3}\end{cases}$

Solution:

Here $g(t)=\cos t$ and $a=\frac{2\pi}{3}$

$$\therefore L(\cos t)=\frac{s}{s^2+1}$$

by second shifting theorem

$$L\left(\cos\left(t-\frac{2\pi}{3}\right)\right)=e^{-\frac{2\pi s}{3}}L(\cos t)=e^{\frac{-2\pi s}{3}}\frac{s}{s^2+1}$$

Basic properties

Derivatives and Integrals of Transforms

$$\text{i) } L[t\, f(t)] = (-1)\frac{d}{ds}F(s) \qquad ii) L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) = (-1)^n F^n(s)$$

$$iii) L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} L[f(t)]ds = \int_s^{\infty} F(s)ds$$

Where

$$L[f(t)] = F(s) = \overline{f(s)}$$

In the following link you can see a video that explains First shifting Property, Derivatives and Integrals of Laplace Transforms
https://drive.google.com/open?id=1zUUUgYQ4ffKmQLgxXro0FYBuR2okWj3J

Additional reference explains you about the First shifting Property
https://drive.google.com/open?id=1gWnAfLedb8SVysFFmZFA-oa2BDOyA_CI

Basic properties

Problems based on Derivatives of Transforms

1. Find The Laplace transform of $t \sin at$.

Solution:

$$L\left[t\sin at\right]=-\frac{d}{ds}L\left[\sin at\right]=-\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]=\frac{2as}{\left(s^2+a^2\right)}$$

2. Find The Laplace transform of $L\left[t\cos\frac{t}{a}\right]$

Solution:

$$L\left[t\cos\frac{t}{a}\right] = -\frac{d}{ds}L[\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+\left(\frac{1}{a}\right)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+\left(\frac{1}{a}\right)^2}\right] = -\left[\frac{s^2+\left(\frac{1}{a}\right)^2-2s^2}{s^2+\left(\frac{1}{a}\right)^2}\right] = \left[\frac{a^4s^2-a^2}{\left(a^2s^2+1\right)^2}\right]$$

3. Find The Laplace transform of $L[t\sin at]^2$

Solution:

$$L[t\sin at]^2 = L\left[t^2\sin^2 at\right] = \frac{d^2}{ds^2}L\left[\sin^2 at\right] = \frac{d^2}{ds^2}L\left[\frac{1-\cos 2at}{2}\right]$$

$$= \frac{1}{2}\frac{d^2}{ds^2}\left\{L[1] - L[\cos 2at]\right\} = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} + \frac{s^2-4a^2}{\left(s^2+4a^2\right)^2}\right] = \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s\left(12a^2-s^2\right)}{\left(s^2+4a^2\right)^2}\right]$$

4. Find The Laplace transform of $L\left[t\,cosh\,t\,cos\,t\right]$

Solution:

$$L\left[t\cosh t\cos t\right]=-\frac{d}{ds}L\left[\cosh t\cos t\right]=-\frac{d}{ds}L\left[\frac{e^{t}+e^{-t}}{2}*\cos t\right]$$

$$=-\frac{1}{2}\frac{d}{ds}L\left[e^{t}\cos t+e^{-t}\cos t\right]=-\frac{1}{2}\frac{d}{ds}\left[\left(\frac{s}{s^{2}+1}\right)_{s\to s-1}+\left(\frac{s}{s^{2}+1}\right)_{s\to s+1}\right]$$

$$=-\frac{1}{2}\frac{d}{ds}\left[\frac{s-1}{(s-1)^{2}+1}+\frac{s+1}{(s+1)^{2}+1}\right]=-\frac{1}{2}\left[\frac{(s-1)^{2}-1}{\left((s-1)^{2}+1\right)^{2}}+\frac{(s+1)^{2}-1}{\left((s+1)^{2}+1\right)^{2}}\right]$$

$$=\frac{1}{2}\left[\frac{s^{2}-2s}{\left((s-1)^{2}+1\right)^{2}}+\frac{s^{2}+2s}{\left((s+1)^{2}+1\right)^{2}}\right]$$

5. Find The Laplace transform of $L\left[t^{2}e^{-t}\cos t\right]$

Solution:

$$L\left[t^{2}\cos t\right]=\frac{d^{2}}{ds^{2}}L\left[\cos t\right]=\frac{d^{2}}{ds^{2}}\left[\frac{s}{s^{2}+1}\right]=\frac{d}{ds}\left[\frac{1-s^{2}}{\left(s^{2}+1\right)^{2}}\right]=\left[\frac{2s^{2}-6s}{\left(s^{2}+1\right)^{3}}\right]$$

$$L\left[e^{-t}t^{2}\cos t\right]=\left[\frac{2s^{2}-6s}{\left(s^{2}+1\right)^{3}}\right]_{s\to s=1}=\left[\frac{2\left(s+1\right)^{2}-6\left(s+1\right)}{\left(\left(s+1\right)^{2}+1\right)^{3}}\right]$$

Basic properties

Problems based on Derivatives of Transforms

$$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^n(s)$$

where $L\left[f(t)\right] = F(s) = \overline{f(s)}$

In the following link you can see a video that explains problems in Derivatives of Laplace Transforms

Problem 1:
https://drive.google.com/open?id=17sj0itIHueTuNu3B4jufXjAszvub0lGR
(*Variable x is used for the variable t in this video)*

Problem 2:
https://drive.google.com/open?id=1NLsNh0w4t_iv7dyVmtXAGWuXqGjusE8u

Problem 3:
https://drive.google.com/open?id=1mznR8YMpBGzjHruvRW42DSUJbymCD1Ua

Basic properties

Problem for practice

Find the Laplace Transform of the following functions

1. $t \sin at$

2. $t \cosh t \cos t$

3. $t^2 e^{-t} \cos t$

Basic properties

Problems based on Integrals of Transforms

1. Find The Laplace transform of $\frac{\sin at}{t}$.

Solution:

$$L\left[\frac{\sin at}{t}\right] = \int_s^{\infty} L[\sin at]\,ds = \int_s^{\infty} \frac{a}{s^2 + a^2}\,ds$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

2. Find The Laplace transform of $L\left[\frac{e^{at}-\cos bt}{t}\right]$

Solution:

$$L\left[\frac{e^{at}-\cos bt}{t}\right]=\int_s^\infty L\left[e^{at}-\cos bt\right]ds=\int_s^\infty\left[\frac{1}{s-a}-\frac{s}{s^2+b^2}\right]ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty$$

$$=\left[\frac{1}{2}\log\left(\frac{(s-a)^2}{s^2+b^2}\right)\right]_s^\infty=\left[\frac{1}{2}\log\left(\frac{s^2+b^2}{(s-a)^2}\right)\right]$$

3. Find The Laplace transform of $L\left[\frac{2\sin 2t \sin t}{t}\right]$

Solution:

$$L\left[\frac{2\sin 2t \sin t}{t}\right] = \int_s^\infty L[2\sin 2t \sin t]\,ds$$

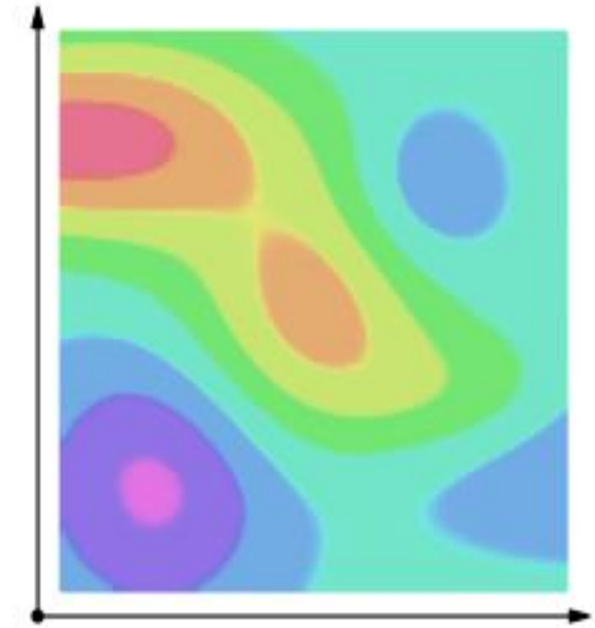
$$= \int_s^\infty L[\cos t - \cos 3t]\,ds = \int_s^\infty \left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right)ds$$

$$= \left[\frac{1}{2} log\left(\frac{s^2+9}{s^2+1}\right)\right]_s^\infty = 0 - \frac{1}{2} log\left(\frac{s^2+1}{s^2+9}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

Basic properties

Problems in Integrals of Transforms



$$L\left[\frac{f(t)}{t}\right]=\int_s^{\infty} L[f(t)]ds=\int_s^{\infty} F(s)ds$$

where $$L[f(t)]=F(s)=\overline{f(s)}$$

In the following link you can see a video that explains problems in Integrals of Laplace Transforms

Problem 1:
https://drive.google.com/open?id=1YJ9cLOXfo5hQZ4sLCjbRbf6qaFiHXSf-

Problem 2:
https://drive.google.com/open?id=1MIxwWue0jTp7q__tKjpngV3X6z7fvPAT

Problem 3:
https://drive.google.com/open?id=18bmArYO0QcSts6lQOF89qDtXLmPMEwNr

Problem 4:
https://drive.google.com/open?id=19vd33RjJhKwRARL37sKMM1MkTUcZ10Oo

Basic properties

Change of scale property(Time Scaling)

If $L\left[f(t)\right] = F(s)$, then $L\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\left[f(t)\right] = \int_0^{\infty} e^{-st}\, f(t)\; dt$$

$$L\left[f(at)\right] = \int_0^{\infty} e^{-st}\, f(at)\; dt$$

$$put\;\; at = x \Rightarrow t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$$

$$L\left[f(at)\right] = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \; \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x) \; dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t) \; dt \qquad \left[\because x \text{ is a dummy variable}\right]$$

$$L\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

1. If $L[f(t)] = \log\frac{s+3}{s+1}$ **find** $L[f(2t)]$

Solution :

Given $L[f(t)] = \log\frac{s+3}{s+1}$

by Change of scale property $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

$$L[f(2t)] = \frac{1}{2}\log\frac{\frac{s}{2}+3}{\frac{s}{2}+1} \Rightarrow \frac{1}{2}\log\frac{s+6}{s+2}$$

Problem for practice

Use Change of scale property(Time Scaling) in Laplace transform evaluate the following

1. if $L[f(t)] = \dfrac{8(s-3)}{(s^2-6s+25)^2}$ Find $L[f(2t)]$

2. if $L[f(t)] = \dfrac{2}{s^2}e^{-s}$ Find $L[f(3t)]$

Basic properties

Change of scale property(Time Scaling)

If $$L\left[f(t)\right] = F(s),$$

Then $$L\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

In the following link you can see a video that explains Change of scale property
https://drive.google.com/open?id=1m-m8HkmJQUXxLCBWeqq9F4bdz84E-qoO

Basic properties

Problems

This problems illustrates how to solve Laplace Transform in combination with product of 3 functions using First shifting property and Derivatives of Transforms

In the following link you can see a video of the problem

Problem 1:
https://drive.google.com/open?id=1m2ScqKQp93wCElnMIVSsFe_F5XPIcV_9
Problem 2:
https://drive.google.com/open?id=1m-UzFK7QFQPdDMp81V8xOroq8TeBaAeN

Additional NPTEL Reference Tutorial on Laplace Transform of standard functions and properties (Problems 1 to 4 only)
https://drive.google.com/open?id=1ZPtHitfDrmG2uyqmr8WiLRWswgKJ691q

Basic properties

Evaluating integrals using Laplace Transforms

$$\int_0^{\infty} e^{-at}\, f(t)\, dt = L\left[f(t)\right]_{s=a}$$

In the following link you can see a video of the problem

Problem 1:
https://drive.google.com/open?id=14_FHz16bacOAolv_QSiK2DCtBlsqZKpq

1. Using Laplace Transform evaluate $\int_0^\infty e^{-3t}\sin 4t\,dt.$

Solution:

We know that, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$ $\quad L[\sin at] = \frac{a}{s^2+a^2}$

$$\int_0^\infty e^{-3t} \sin 4t\, dt = \left[\int_0^\infty e^{-st} \sin 4t\, dt\right]_{s=3}$$

$$= [L(\sin 4t)]_{s=3}$$

$$= \left[\frac{4}{s^2+16}\right]_{s=3}$$

$$= \left[\frac{4}{3^2+16}\right]$$

$$\int_0^\infty e^{-3t} \sin 4t\, dt = \frac{4}{25}$$

2. Using Laplace Transform evaluate $\int_0^\infty e^{-3t} t^2 dt.$

Solution:

We know that, $L[f(t)] = \int_0^\infty e^{-st} f(t) dt.$

$$L[t^n] = \frac{n!}{s^{n+1}}$$

$$\int_0^\infty e^{-3t} t^2 dt = L[t^2]_{s=3} = \left[\frac{2!}{s^{2+1}}\right]_{s=3} = \left[\frac{2}{s^3}\right]_{s=3} = \left[\frac{2}{3^3}\right] = \frac{2}{27}$$

3. Using Laplace Transform evaluate $\int_0^\infty te^{-2t}\sin 3t\,dt$.

Solution:

We know that, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$f(t) = \sin 3t$

$$L[tf(t)] = -\frac{d}{ds}L[f(t)]$$

$$\int_0^\infty e^{-2t} t\sin 3t\,dt = \left[\int_0^\infty e^{-st} t\sin 3t\,dt\right]_{s=2}$$

$$= L[t\sin 3t]_{s=2}$$

$$= -\frac{d}{ds}L[\sin 3t]_{s=2}$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]_{s=2} \qquad \because L[\sin at] = \frac{a}{s^2+a^2}$$

$$= -\left[\frac{-3}{(s^2+9)^2}(2s)\right]_{s=2} \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$= \left[\frac{6s}{(s^2+9)^2}\right]_{s=2}$$

$$= \frac{6(2)}{(2^2+9)^2} = \frac{12}{13^2}$$

$$\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}$$

4. Using Laplace Transform evaluate $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}\,dt.$

Solution:

We know that, $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}\,dt = \left[\int_0^\infty e^{-st}\,\frac{\sin\sqrt{3}t}{t}\,dt\right]_{s=1}$$

$$= \left[L\left(\frac{\sin\sqrt{3}t}{t}\right)\right]_{s=1} \qquad \mathrm{f(t) = \sin\sqrt{3}t}$$

$$= \left[\int_s^\infty L\left[\sin\sqrt{3}t\right]ds\right]_{s=1} \qquad \because L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$$

$$= \left[\int_s^\infty \left[\frac{\sqrt{3}}{s^2 + (\sqrt{3})^2} \right] ds \right]_{s=1} \qquad \because L[\sin at] = \frac{a}{s^2 + a^2}$$

$$= \left[\sqrt{3} \left(\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{s}{\sqrt{3}} \right) \right)_s^\infty \right]_{s=1} \qquad \because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right)$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\sqrt{3}} \right) \right]_{s=1}$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\sqrt{3}} \right) \right]_{s=1} \qquad \because \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\sqrt{3}} \right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}} \right) \qquad \because \cot\left(\frac{\pi}{3} \right) = \frac{1}{\sqrt{3}}$$

$$\int_0^\infty \frac{e^{-t} \sin \sqrt{3}t}{t} \, dt = \frac{\pi}{3}$$

Problem for practice

Use Laplace transform evaluate the following integrals

1. $$\int_0^\infty e^{-2t}\cos t\,dt$$

2. $$\int_0^\infty t\,e^{-2t}\sin 3t\,dt$$

3. $$\int_0^\infty \frac{\cos at-\sin at}{t}\,dt$$

4. $$\int_0^\infty \frac{\sin^2 t}{te^t}\,dt$$

Transforms of derivatives and integrals

Definition and Problems

$$L[y(t)] = L(y)$$

$$L[y'(t)] = s\,L|y(t)| - y(0)$$

$$L[y''(t)] = s^2\,L[y(t)] - s\,y(0) - y'(0)$$

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

In the following links you can see Transforms of derivatives
https://drive.google.com/open?id=1TDpx_w_VC1t5ogE6rI2eHgGDrYI6DsDQ

Transforms of integrals with problems
https://drive.google.com/open?id=167PgiU5nnQEMfS6QwgSOaz6t3eoYueaC

Additional Reference for Problems based on Change of scale property, Second shifting property and followed by Transforms of derivatives
https://drive.google.com/open?id=1UUzMihVKnIKp5yjvK0vg9ELnHlsVUCl4

1. If $L\left[\sin\sqrt{t}\right]=\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$ then Find The Laplace transform of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Solution: Here $f(t)=\sin\sqrt{t} \Rightarrow f'(t)=\dfrac{1}{2}\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ **and f(0)=0**

we know that $L\left[f'(t)\right]=sF(s)-f(0)$

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right]=sL\left[\sin\sqrt{t}\right]-f(0)$$

$$=s\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}-0 \quad =\frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}e^{-\frac{1}{4s}}$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]=\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4s}}$$

2. Using Laplace Transform evaluate $\int_0^t t \cos t \ dt.$

Solution:

We know that, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

Here $f(t) = t\cos t$

$$L[f(t)] = -\frac{d}{ds}L(\cos t) \quad = -\frac{d}{ds}L(\cos t) = -\frac{d}{ds}\frac{s}{s^2+1}$$

$$= -\frac{1-s^2}{(1+s^2)^2} \quad = \frac{s^2-1}{(1+s^2)^2}$$

$$\therefore \ L\left[\int_0^t t \cos t \ dt.\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}\frac{s^2-1}{(s^2+1)^2}$$

3. Find the Laplace Transform of $\int_0^t te^{-t}\sin t\,dt.$

Solution:

We know that, $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$ $\quad L[tf(t)] = -\frac{d}{ds}L[f(t)]$

$$L\left[\int_0^t te^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[te^{-t}\sin t\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}L\left(e^{-t}\sin t\right)\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}L(\sin t)_{s\to s+1}\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)_{s\to s+1}\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right)\right]$$

$$= -\frac{1}{s}\left[-\frac{1}{\left(s^2+2s+2\right)^2}\left(2s+2\right)\right] \qquad \left[\because \frac{d}{dx}\left(\frac{1}{ax+b}\right) = -\frac{1}{(ax+b)^2}(a)\right]$$

$$L\left[\int_0^t te^{-t}\sin t\,dt\right] = \frac{1}{s}\left[\frac{2(s+1)}{\left(s^2+2s+2\right)^2}\right]$$

4. Find the Laplace Transform of $\int_0^t \frac{e^{-t}\sin t}{t}\,dt.$

Solution:

We know that $$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds$$

$$L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right] = \frac{1}{s}L\left[\frac{e^{-t}\sin t}{t}\right]$$

$$= \frac{1}{s}\left[\int_s^\infty L\left[e^{-t}\sin t\right]ds\right]$$

$$= \frac{1}{s}\left[\int_s^\infty L[\sin t]_{s\to s+1}\,ds\right]$$

$$= \frac{1}{s}\left[\int_s^{\infty}\left[\frac{1}{s^2+1}\right]_{s\to s+1} ds\right]$$

$$= \frac{1}{s}\left[\int_s^{\infty}\left[\frac{1}{(s+1)^2+1}\right] ds\right]$$

$$= \frac{1}{s}\left[\tan^{-1}(s+1)\right]_s^{\infty} \qquad \Theta \int \frac{1}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{s}\left[\tan^{-1}(\infty) - \tan^{-1}(s+1)\right]$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]$$

$$L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right] = \frac{1}{s}\cot^{-1}(s+1)$$

5. Find the Laplace Transform of $e^{-t}\int_0^t t\cos t\,dt.$

Solution:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = L\left[\int_0^t t\cos t\,dt\right]_{s\to s+1} \qquad \left[\because L\left[e^{-at}f(t)\right] = L[f(t)]_{s\to s+a}\right]$$

$$= \left[\frac{1}{s}L[t\cos t]\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left[\frac{-d}{ds}L(\cos t)\right]\right]_{s\to s+1} \qquad \left[\because L[tf(t)] = -\frac{d}{ds}L[f(t)]\right]$$

$$= \left[\frac{1}{s}\left[\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)\right]\right]_{s\to s+1}$$

$$= \left[-\frac{1}{s}\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]\right]_{s\to s+1}$$

$$= \left[-\frac{1}{s} \left[\frac{s^2+1-2s^2}{(s^2+1)^2} \right] \right]_{s\to s+1}$$

$$= \left[-\frac{1}{s} \left[\frac{1-s^2}{(s^2+1)^2} \right] \right]_{s\to s+1}$$

$$= \left[\frac{1}{s} \left[\frac{s^2-1}{(s^2+1)^2} \right] \right]_{s\to s+1}$$

$$= \frac{1}{s+1} \left[\frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2} \right]$$

$$= \frac{1}{s+1} \left[\frac{s^2+2s+1-1}{\left[s^2+2s++1\right]^2} \right]$$

$$L\left[e^{-t} \int_0^t t\cos t\, dt \right] = \frac{s^2+2s}{(s+1)\left[s^2+2s+2\right]^2}$$

Problem for practice

Find the Laplace transform of the following functions

1. L $\left(e^{-t} \int_0^t t \cos t \, dt \right)$

2. L $\left(e^{-t} \int_0^t \frac{\sin t}{t} \, dt \right)$

3. L $\left(\int_0^t \frac{\sin t}{t} \, dt \right)$

4. L $\left(t \int_0^t e^{-4t} \sin 3t \, dt \right)$

Transform of standard functions

$$L[k]=\frac{k}{s}, \mathrm{s}>0 \text{ k is a constant}$$

$$L\left[e^{at}\right]=\frac{1}{s-a}$$

$$\mathrm{L}\left[\mathrm{e}^{-\mathrm{at}}\right]=\frac{1}{\mathrm{s}+\mathrm{a}}$$

$$L\left[t^{n}\right]=\begin{cases}\dfrac{n!}{s^{n+1}}, \mathrm{n}=0,1,2,3\ldots \\ \dfrac{\Gamma(n+1)}{s^{n+1}}, \mathrm{n}=\dfrac{1}{2}, \dfrac{3}{2}, \ldots\end{cases}$$

$$\mathrm{L}[\cos \mathrm{at}]=\frac{\mathrm{s}}{\mathrm{s}^{2}+\mathrm{a}^{2}}$$

$$L[\sin at]=\frac{a}{s^{2}+a^{2}}$$

$$L[\cosh at]=\frac{s}{s^{2}-a^{2}}$$

$$L[\sinh at]=\frac{a}{s^{2}-a^{2}}$$

Properties

i) $L[af(t) \pm bg(t)] = aF(s) \pm bG(s)$

$$\text{(ii)}L\left[e^{-at}f(t)\right] = F(s+a) \qquad (ii)L\left[e^{at}f(t)\right] = F(s-a)$$

$$\text{(iii) } L[f(t)] = F(s)\ \&\ g(t) = \begin{cases} f(t-a), & t > a \\ 0, & 0 \le t < a \end{cases} \Rightarrow L[g(t)] = e^{-as}F(s)$$

$$\text{(iv) } L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\text{v) } L[t\,f(t)] = (-1)\frac{d}{ds}F(s)$$

$$\text{vi) } L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds = \int_s^\infty F(s)ds$$

$$\text{(vii) } \int_0^\infty e^{-at}\ f(t)\,dt = L\,[f(t)]_{s=a} \qquad \text{(viii) } L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}L[\,f(t)]$$

Laplace Transform of special functions

Unit step function

The function f(t) $= \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$ where $a \geq 0$ is called **Heavyside's unit step function** and is

denoted by

$$U_a(t)\, or \quad U(t-a)$$

and

$$L[U_a(t)] = \frac{1}{s}$$

Unit impulse function

The function f(t) $= \begin{cases} \frac{1}{h}, a-\frac{h}{2} \le t \le a+\frac{h}{2} \\ 0, \text{ otherwise} \end{cases}$ is called Unit impulse function or Direc Delta

function and is denoted by $\delta_a(t)\, or \quad \delta(t-a)$

and $L[\delta_a(t)] = e^{-as}$

1. Find the Laplace transforms of $(t-1)^2 U_1(t)$

Solution:

$$L[(t-1)^2 U_1(t)] = \int_0^{\infty} e^{-st}\,(t-1)^2 U_1(t)\,dt$$

$$= \int_0^{1} e^{-st}\,(t-1)^2[0]\,dt + \int_1^{\infty} e^{-st}\,(t-1)^2[1]\,dt$$

$$= 0 + \left[(t-1)^2 \frac{e^{-st}}{-s} - 2(t-1)\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3}\right]_1^{\infty}$$

$$= \left[(0-0+0) - (0-0+2\frac{e^{-st}}{(-s)^3}\right]$$

$$= \frac{2e^{-st}}{s^3}$$

2. Find the Laplace transforms of $\sin t U_{\pi}(t)$

Solution:

$$L[sint(U_{\pi}(t)] = \int_{0}^{\infty} e^{-st} sint\, U_{\pi}(t)dt$$

$$= \int_{0}^{\pi} e^{-st} sint\, [0]dt + \int_{\pi}^{\infty} e^{-st} sint\, [1]dt$$

$$= 0 + \left[\frac{e^{-st}}{s^2+1}(-s\, sint - cost)\right]_{\pi}^{\infty}$$

$$= \left[0 - \frac{e^{-s\pi}}{s^2+1}(-s\, sin\pi - cos\pi)\right]$$

$$= -\frac{e^{-s\pi}}{s^2+1}$$

3. **Find the Laplace transforms of** $e^{-\pi t}\delta(t-a)$

Solution:

$$L[e^{-\pi t}\delta(t-a)] = L[\delta(t-a)]_{S\to S+\pi}$$

$$= [e^{-as}]_{S\to S+\pi}$$

$$= e^{-a(S+\pi)}$$

4. **Find the Laplace transforms of** $\dfrac{\delta(t-\pi)}{t}$

Solution:

$$L\left[\frac{\delta(t-\pi)}{t}\right] = \int_{s}^{\infty} L[\delta(t-\pi)]ds$$

$$= \int_{s}^{\infty} e^{-S\pi}\, ds$$

$$= \left[\frac{e^{-S\pi}}{-\pi}\right]_{s}^{\infty}$$

$$= 0 - \frac{e^{-S\pi}}{-\pi}$$

$$= \frac{e^{-S\pi}}{\pi}$$

Basic properties

Initial Value Theorem

Final Value Theorem

Initial Value Theorem

If $L[f(t)] = F(s)$, **then** $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\,F(s)$

1. Verify initial value theorem for the function $1 + e^{-2t}$

Solution: Given $f(t) = 1 + e^{-2t}$

$$L[f(t)] = F(s) = L[1 + e^{-2t}] = L(1) + L(e^{-2t}) \Rightarrow F(s) = \frac{1}{s} + \frac{1}{s+2}$$

By the initial value theorem

L.H.S

$$\lim_{t\to 0}[f(t)] = \lim_{t\to 0}(1 + e^{-2t}) = 2$$

R.H.S

$$\lim_{s\to\infty}[s\,F(s)] = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{1}{s+2}\right] = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{1}{s(1+2/s)}\right] = 2$$

Final Value Theorem

If $L[f(t)] = F(s)$, **then** $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

2. Verify initial and final value theorem for the function $1 + e^{-t}(\sin t + \cos t)$

Solution:

$$\text{Given } f(t) = 1 + e^{-t}(\sin t + \cos t)$$

$$L[f(t)] = F(s) = L[1 + e^{-t}(\sin t + \cos t]$$

$$= L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$F(s) = \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} \quad = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

By initial Value Theorem I.V.T $\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\,F(s)]$

L.H.S $\lim_{t\to 0}[f(t)] = \lim_{t\to 0}(1+e^{-t}(\sin t+\cos t)) = 2$

R.H.S $\lim_{s\to\infty}[s\,F(s)] = \lim_{s\to\infty} s\left[\frac{1}{s}+\frac{s+2}{(s^2+1)^2+1}\right] = \lim_{s\to\infty} s\left[\frac{1}{s}+\frac{s(1+\frac{1}{s})}{s^2((1+\frac{1}{s})^2+\frac{1}{s^2})}\right]$

$$=\lim_{s\to\infty} s\left(\frac{1}{s}\right)\left[1+\frac{(1+\frac{1}{s})}{((1+\frac{1}{s})^2+\frac{1}{s^2})}\right] = 2$$

R.H.S=L.H.S

By Final Value Theorem

$$\underline{\text{F.V.T}}\ \lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

$$\text{L.H.S}\quad \lim_{t\to\infty}[f(t)] = \lim_{t\to\infty}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1$$

$$\text{R.H.S}\ \ \lim_{s\to 0}[sF(s)] = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= \lim_{s\to 0}\left[1 + \frac{s}{(s+1)^2+1} + \frac{s(s+1)}{(s+1)^2+1}\right] = 1$$

L.H.S=R.H.S

3. If $L\left(e^{-t}\cos^2 t\right)=\phi(s)$ **find**

$$i)\lim_{s\to 0}\left[s\,\phi(s)\right]\quad ii)\lim_{s\to\infty}\left[s\,\phi(s)\right]$$

Solution:

$$Given\quad f(t)=e^{-t}\cos^2 t$$

$$i)\lim_{s\to 0}\left[s\,\phi(s)\right]=\lim_{t\to\infty}\left[f(t)\right]$$

$$=\lim_{t\to\infty}\left[e^{-t}\cos^2 t\right]$$

$$=0$$

$$ii)\lim_{s\to\infty}\left[s\,\phi(s)\right]=\lim_{t\to 0}\left[f(t)\right]$$

$$=\lim_{t\to 0}\left[e^{-t}\cos^2 t\right]=1$$

4. If $L[f(t)] = \dfrac{1}{s(s+1)(s+2)}$ **find** $i)\lim_{t\to\infty}[f(t)]\ ii)\lim_{t\to 0}[f(t)]$

Solution:

$$Given\, F(s) = \frac{1}{s(s+1)(s+2)}$$

$$i)\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\,F(s)]$$

$$= \lim_{s\to 0}\left[s\left(\frac{1}{s(s+1)(s+2)}\right)\right]$$

$$= 1/2$$

$$ii)\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\,F(s)] = \lim_{s\to\infty} s\left(\frac{1}{s(s+1)(s+2)}\right) = 0$$

Basic properties

Problem for practice

Verify Initial and Final value Theorem for the following functions

1. $f(t) = 3e^{-2t}$

2. $f(t) = 1 - e^{-at}$

3. If $L[f(t)] = \dfrac{1}{s(s+a)}$, find $f(0), f(\infty)$

Transform of periodic functions

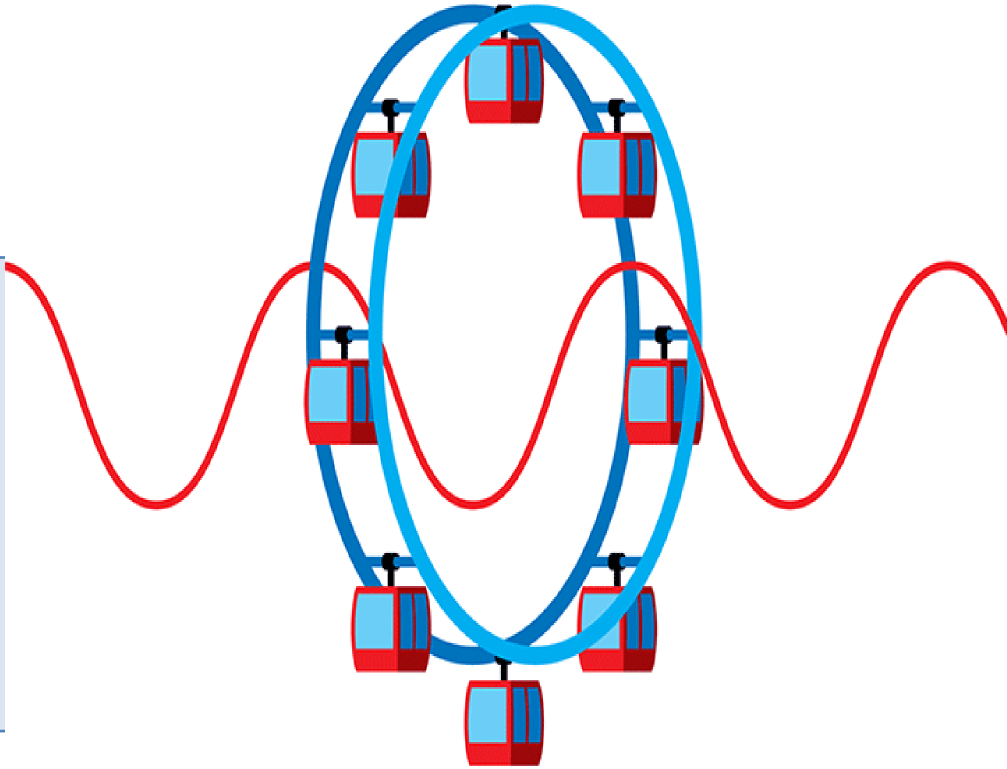
Definition of Periodic function with examples

A function $f(x)$ is said to be periodic iff $f(x+T)=f(x)$ is true for some value of p and every value of *x*. The smallest positive value of T for which this equation is true for every value of *x* will be called the period of the function. The Laplace Transformation of a periodic function $f(t)$ with period p given by

$$L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)dt$$

In the following links you can see Definition of Periodic function with examples

https://drive.google.com/open?id=1_LIh69QtY8_JBwytru1nTIzq72V9YWgX



Transform of periodic functions

Periodic function-Problems

1. Find the Laplace transform of $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$ with f(t + 2a) = f(t).

Solution:

$$L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) dt$$

Since $f(t)$ is periodic function with period T=2a we get

$$L\{f(t)\} = \frac{1}{1-\ e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st} (2a-t) dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[-t\frac{e^{-st}}{s}-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a+\left[-(2a-t)\frac{e^{-st}}{s}+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[\left(-a\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\left(\frac{e^{-2as}}{s^2}\right)-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$=\frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]=\frac{1}{1-e^{-2as}}\left[\frac{\left(1-e^{-as}\right)^2}{s^2}\right]$$

$$=\frac{1}{\left(1^2-\left(e^{-as}\right)^2\right)}\frac{\left(1-e^{-as}\right)^2}{s^2}=\frac{1}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)}\frac{\left(1-e^{-as}\right)^2}{s^2}$$

$$=\frac{1}{s^2}\frac{\left(1-e^{-as}\right)}{\left(1+e^{-as}\right)}=\frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

2. Find the Laplace transform of square wave function of period ‘a’ defined as $f(t)=\begin{cases} k, & 0<t<a \\ -k, & a<t<2a \end{cases}$

Solution:

The function is periodic in the interval (0, 2a) with period 2a

$$L\{f(t)\}=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} k\,dt+\int_a^{2a} e^{-st}(-k)\,dt\right]$$

$$=\frac{k}{1-e^{-2as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a}-\left[\frac{e^{-st}}{-s}\right]_a^{2a}\right\}$$

$$=\frac{k}{1-e^{-2as}}\left\{\left[\frac{e^{-as}-1}{-s}\right]-\left[\frac{e^{-2as}-e^{-as}}{-s}\right]\right\}$$

$$=\frac{k}{\left(1-e^{-2as}\right)s}\left[-e^{-as}+1+e^{-2as}-e^{-as}\right]$$

$$=\frac{k}{\left(1-e^{-2as}\right)s}\left[1-2e^{-as}+e^{-2as}\right]$$

$$=\frac{k}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)s}\left(1-e^{-as}\right)^{2}$$

$$=\frac{k\left(1-e^{-as}\right)}{\left(1+e^{-as}\right)s} \qquad =\frac{k}{s}\left[\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right]$$

$$=\frac{k}{s}tanh\left(\frac{sb}{2}\right)$$

3. Find the Laplace transform of the Half wave rectifier function .

$$f(t)=\begin{cases} E\sin\omega t, 0<t<\dfrac{\pi}{\omega} \\ 0, \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega} \end{cases}$$

Solution:

This function is a periodic function with period $\dfrac{2\pi}{\omega}$ in the interval $\left(0, \dfrac{2\pi}{\omega}\right)$.

$$L\{f(t)\}=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} E\sin\omega t\,dt+0\right]$$

$$=\frac{E}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} \sin\omega t\,dt$$

$$=\frac{E}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right]_0^{\frac{\pi}{\omega}}$$

$$=\frac{E}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-\frac{s\pi}{w}}\omega+\omega}{s^2+\omega^2}\right]$$

$$=\frac{E\omega\left(1+e^{\frac{-\pi s}{\omega}}\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

$$=\frac{E\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

4. Find the Laplace transform of the full sine wave rectifier function f(t), defined as

$$f(t) = |\sin \omega t|, t \geq 0 \text{ and } f\left(t + \frac{\pi}{\omega}\right) = f(t).$$

Solution:

$|\sin \omega t|$ is a periodic function with period π.

$$L\{f(t)\} = \frac{1}{1 - e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt$$

$$= \frac{1}{1 - e^{-\frac{\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2 + \omega^2} \left(-s \sin \omega t - \omega \cos \omega t \right) \right]_0^{\pi/\omega}$$

$$=\frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left(\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right)$$

$$=\frac{\omega}{1-e^{-\frac{\pi s}{\omega}}}\left(\frac{e^{\frac{-\pi s}{\omega}}+1}{s^2+\omega^2}\right)$$

$$=\frac{\omega}{s^2+\omega^2}\left(\frac{1+e^{\frac{-\pi s}{\omega}}}{1-e^{-\frac{\pi s}{\omega}}}\right)=\frac{\omega}{s^2+\omega^2}\coth\left(\frac{\pi s}{2\omega}\right)$$

Transform of periodic functions

Periodic function-Problems

$$L\left[f(t)\right]=\frac{1}{1-e^{-Ts}}\int_{0}^{T}e^{-st}f(t)dt$$

Video Link for Problems

1. https://drive.google.com/open?id=1sWDKMj9x1N0L8anm_5YL5pJ1x-5v9anM

2.https://drive.google.com/open?id=1zdYu3bNSHgRTQi9m4lTIszRvu3OYHxi3

3. https://drive.google.com/open?id=1nSQPmmNr-C2NemfLTH-o77MIs-SJ0AS-

Problems for practice

1. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t & 0 < t \le 2 \\ 4 - t & 2 \le t \le 4 \end{cases}$$

2. Prove the Laplace transform of a half-wave rectifier function

$$f(t) = \begin{cases} a \sin \omega t & 0 \le t \le \pi / \omega \\ 0 & \pi / \omega \le t \le 2\pi / \omega \end{cases}$$

3. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

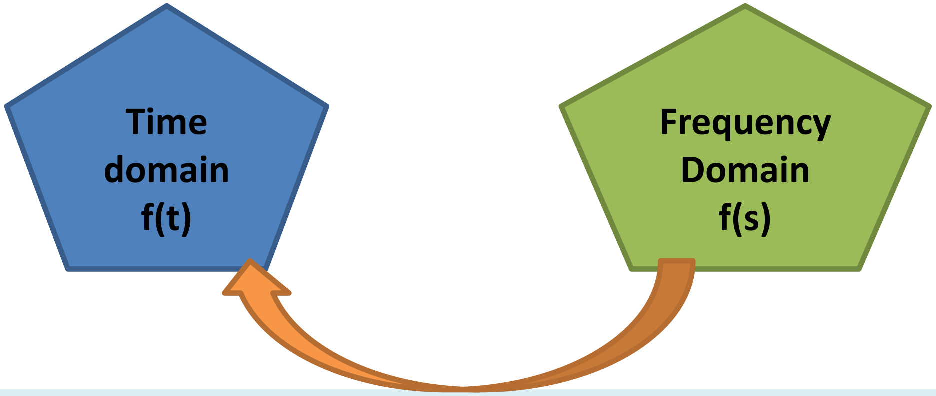
4.Find the Laplace transform of

$$f(t) = \begin{cases} \sin \omega t & 0 < t < \pi / \omega \\ 0 & \pi / \omega < t < 2\pi / \omega \end{cases}$$

Inverse Laplace Transforms

Definition and Problems

Inverse Laplace can **convert** any variable domain back to time domain or any basic domain like from frequency domain back to time domain. These properties allow them to be **used** for solving and analysing linear dynamical systems and optimisation purposes.



In the following links you can see the definition and formulas of Inverse Laplace Transforms
https://drive.google.com/open?id=1muQovXJQobvMfpE2lM4wejADupOjqnl6

Inverse Laplace Transforms

Problems

In the following links you can see the Problems on Inverse Laplace Transforms

i. https://drive.google.com/open?id=1l6Wj_BzbKITTkPOrODrt-u5JCuoRQCIF
ii. https://drive.google.com/open?id=1Lwwz5hHTifops7INbqifdEw6vsBXAg0v
iii. https://drive.google.com/open?id=1lPi9TZj5llR8N8rb0RhW-GJWn_VSN-0-

Inverse Laplace transforms

$$L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{n!}$$

$$L^{-1}[F(s+a)] = e^{-at}f(t)$$

$$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$$

$$L^{-1}[F(s-a)] = e^{at}f(t)$$

$$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$$

Problems based on inverse Laplace transforms

Example: No: 1

Find $L^{-1}\left[\dfrac{1}{s^2-25}\right]$

$$L^{-1}\left[\frac{1}{s^2-25}\right] = L^{-1}\left[\frac{1}{5}\,\frac{5}{s^2-25}\right]$$

$$= \frac{1}{5}L^{-1}\left[\frac{5^2}{s^2-5^2}\right]$$

$$= \frac{1}{5}\sinh 5t$$

Problems based on inverse Laplace transforms

Example: No: 2

$$\text{Find } L^{-1}\left[\frac{1}{(s-2)^2+1}\right]$$

$$L^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{2t}\sin t$$

Problems based on inverse Laplace transforms

Example: No: 3

Find $L^{-1}\left[\dfrac{s+2}{(s+2)^2-36}\right]$

$$L^{-1}\left[\frac{s+2}{(s+2)^2-36}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2-6^2}\right]$$

$$= e^{-2t}\cosh 6t$$

Problems based on inverse Laplace transforms

Example: No: 4

Find $L^{-1}\left[\frac{2s^2-4s+5}{s^3}\right]$

$$L^{-1}\left[\frac{2s^2-4s+5}{s^3}\right] = L^{-1}\left[\frac{2}{s}-\frac{4}{s^2}+\frac{5}{s^3}\right]$$

$$= L^{-1}\left[\frac{2}{s}\right]-L^{-1}\left[\frac{4}{s^2}\right]+L^{-1}\left[\frac{5}{s^3}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right]-4L^{-1}\left[\frac{1}{s^2}\right]+5L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2(1)-4(t)+\frac{5}{2}t^2$$

$$= 2-4t+\frac{5}{2}t^2$$

Problems based on inverse Laplace transforms

Example: No: 5

Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{s-3}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2-5}{(s+2)^2+3^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - 5e^{-2t}L^{-1}\left[\frac{1}{3}\frac{3}{s^2+3^2}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{5}{3}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

Problems based on inverse Laplace transforms

Problems for practice

1. Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

2. Find $L^{-1}\left[\frac{2s}{s^2+4s+13}\right]$

3. Find $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

4. Find $L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$

5. Find $L^{-1}\left[\frac{1}{2(s-1)^2+32}\right]$

Inverse Laplace transforms

Properties

$$L^{-1}\left[sF(s)\right] = \frac{d}{dt} L^{-1}\left[F(s)\right] = \frac{d}{dt} f(t)$$

Inverse Laplace transforms

1. Find $L^{-1}\left[\frac{s}{(s+3)^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[s\frac{1}{(s+3)^2}\right]$$

$$L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)] = \frac{d}{dt}f(t)$$

$$= \frac{d}{dt}L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= \frac{d}{dt}\left(e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]\right)$$

$$= \frac{d}{dt}\left(e^{-3t}t\right) = e^{-3t}(1) + t(-3)e^{-3t}$$

$$= e^{-3t}(1-3t)$$

Inverse Laplace transforms

Patrice Problem

1.Find $L^{-1}\left[\dfrac{3s}{2s+9}\right]$

2.Find $L^{-1}\left[\dfrac{s}{(s-2)^2}\right]$

3.Find $L^{-1}\left[\dfrac{s}{(s+2)^2+4}\right]$

Inverse Laplace transforms

Properties

$$L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}\left[f(t)\right]dt=\int_0^t f(t)\,dt$$

Inverse Laplace transforms

1.Find $L^{-1}\left[\dfrac{1}{s\left(s^2-2s+5\right)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}[f(t)]\,dt=\int_0^t f(t)\,dt$$

$$L^{-1}\left[\frac{1}{s\left(s^2-2s+5\right)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2-2s+5}\right]dt$$

$$=\int_0^t L^{-1}\left[\frac{1}{s^2-2s+1+4}\right]dt \quad =\int_0^t L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right]dt$$

$$=\int_0^t e^t L^{-1}\left[\frac{1}{s^2+2^2}\right]dt \quad =\frac{1}{2}\int_0^t e^t \sin 2t\,dt$$

$$=\frac{1}{2}\left[\frac{e^t}{1+2^2}\left(\sin 2t-2\cos 2t\right)\right]_0^t$$

$$=\frac{1}{2}\left[\frac{e^t}{1+2^2}\left(\sin 2t-2\cos 2t\right)-\frac{e^0}{1+2^2}(0-2)\right] \quad =\frac{1}{10}\left[e^t\left(\sin 2t-2\cos 2t\right)+2\right]$$

Problems based on inverse Laplace transforms

Problems for practice

1.Find $L^{-1}\left[\dfrac{1}{s^2(s+2)}\right]$

2.Find $L^{-1}\left[\dfrac{s^2+3}{s(s^2+9)}\right]$

Inverse Laplace transforms

Properties

$$L^{-1}[F'(s)] = -tL^{-1}[F(s)] = -t\,f(t)$$

$$L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)] = -\frac{1}{t}\frac{d}{ds}L^{-1}[F(s)]$$

Problems based on inverse Laplace transforms

Example: No: 6

$$\text{Find } L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(s^2+a^2\right)-\log\left(s^2+b^2\right)\right]\right]$$

$$=-\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right\}$$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = -\frac{2}{t}L^{-1}\left[\frac{s}{s^2+a^2}\right] + \frac{2}{t}L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= -\frac{2}{t}\cos at + \frac{2}{t}\cos bt$$

$$= \frac{2}{t}\left(\cos bt - \cos at\right)$$

Problems based on inverse Laplace transforms

Example: No: 7

Find $L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right]$

$$L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(s^2+1\right)-\log s-\log\left(s+1\right)\right]\right]$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right\}$$

Problems based on inverse Laplace transforms

$$L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right] = -\frac{2}{t}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{t}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{t}L^{-1}\left[\frac{1}{s+1}\right]$$

$$= -\frac{2}{t}\cos t + \frac{1}{t}(1) + e^{-t}$$

$$= \frac{1}{t}\left(1 + e^{-t} - 2\cos t\right)$$

Example: No: 8

Find $L^{-1}\left[\cot^{-1}(s)\right]$

Solution:

$$L^{-1}\left[\cot^{-1}(s)\right]=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(s)\right]=\frac{-1}{t}L^{-1}\left[\frac{-1}{1+s^2}\right]=\frac{1}{t}\sin t$$

Example: No: 9

Find inverse Laplace transform of $\tan^{-1}\left(\frac{s}{a}\right)$

Solution:

$$L^{-1}\left[\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{a}{a^2+s^2}\right] = -\frac{1}{t}\sin at$$

Problems based on inverse Laplace transforms

Problems for practice

1.Find $L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]$

2.Find $L^{-1}\left[\tan^{-1}\frac{1}{s}\right]$

3.Find $L^{-1}\left[\cot^{-1}\frac{s}{a}\right]$

Inverse Laplace transforms using Partial fraction method

Case 1: Factors are linear and Distinct

$$F(s) = \frac{p(s)}{(s+a)(s+b)} \quad \Rightarrow F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case 2: Factors are linear and Repeated

$$F(s) = \frac{p(s)}{(s+a)(s+b)^2} \quad \Rightarrow F(s) = \frac{A}{s+a} + \frac{B}{s+b} + \frac{c}{(s+b)^2}$$

Case 3: Factors are quadratic and Distinct

$$F(s) = \frac{p(s)}{(s^2+as+b)(s^2+cs+db)}$$

$$F(s) = \frac{As+B}{(s^2+as+b)} + \frac{Cs+D}{(s^2+cs+d)}$$

Problems based on inverse Laplace transforms using partial fraction method

Example: No: 1

Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)}$

Solution:

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

Put s=-2 we get B=-1

Put s=-1 we get A=1

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] = e^{-t} + e^{-2t}$$

Problems based on inverse Laplace transforms using partial fraction method

Example: No: 2

Find the inverse Laplace transform of $\dfrac{s+2}{s(s+1)(s+3)}$

Solution: let $F(s)=\dfrac{s+2}{s(s+1)(s+3)}$

By partial fraction expansion $F(s)=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+3}$

$$\Rightarrow s+2=A(s+1)(s+3)+Bs(s+3)+Cs(s+1)\cdots(1)$$

Put s=0 in equ (1) $A=\dfrac{2}{3}$

Put s=-1 in equ (1) $B=-\dfrac{1}{2}$

Put s=-3 in equ (1) $C=-\dfrac{1}{6}$

$$\therefore\ F(s)=\frac{2}{3}\frac{1}{s}-\frac{1}{2}\frac{1}{s+1}-\frac{1}{6}\frac{1}{s+3}$$

Problems based on inverse Laplace transforms

$$L^{-1}\left[F(s)\right] = \frac{2}{3}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

$$L^{-1}\left[\frac{s+2}{s(s+1)(s+3)}\right] = \frac{2}{3} - \frac{1}{2}e^{-2} - \frac{1}{6}e^{-3t}$$

Problems based on inverse Laplace transforms using partial fraction method

Example: No: 3

Find the inverse Laplace transform of $\dfrac{2s}{s^4+4}$

Solution: let $F(s)=\dfrac{2s}{s^4+4}$

$$F(s)=\frac{2s}{s^4+4}=\frac{2s}{\left(s^2\right)^2+2^2}=\frac{2s}{\left(s^2\right)^2+4+4s^2-4s^2}$$

$$=\frac{2s}{\left(s^2+2\right)^2+-\left(2s\right)^2}=\frac{2s}{\left(s^2+2+2s\right)\left(s^2+2-2s\right)}$$

By partial fraction expansion

$$\frac{2s}{\left(s^2+2+2s\right)\left(s^2+2-2s\right)}=\frac{As+B}{s^2+2+2s}+\frac{Cs+D}{s^2+2-2s}$$

Problems based on inverse Laplace transforms

$$2s = (As + B)(s^2 + 2 - 2s) + (Cs + D)(s^2 + 2 + 2s) \cdots\cdots (1)$$

Put s=0 in equ(1) $B + D = 0 \Rightarrow B = -D$

Put s=1 in eqqu (1) $2 = A + B + (C + D)(5)$

Compare the coefficient of S^3 $A + C = 0 \Rightarrow A = -C$

compare the coefficient of S^2 $-2A + B + 2C + D = 0$

$$\Rightarrow -2A - 2A = 0 \quad \Rightarrow -4A = 0 \Rightarrow A = 0 \text{ and } C = 0$$

$$\therefore 2 = A + B + (C + D)(5) \quad \Rightarrow 2 = B + 5D \Rightarrow 2 = -D + 5D$$

$$\Rightarrow 2 = 4D \Rightarrow D = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\frac{2s}{(s^2 + 2 + 2s)(s^2 + 2 - 2s)} = \frac{-\frac{1}{2}}{s^2 + 2 + 2s} + \frac{\frac{1}{2}}{s^2 + 2 - 2s}$$

Problems based on inverse Laplace transforms

$$L^{-1}\left[\frac{2s}{(s^2+2+2s)(s^2+2-2s)}\right]=-\frac{1}{2}L^{-1}\left[\frac{1}{s^2+2+2s}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+2-2s}\right]$$

$$=-\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$=-\frac{1}{2}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{1}{2}e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=-\frac{1}{2}e^{-t}\sin t+\frac{1}{2}e^{t}\sin t$$

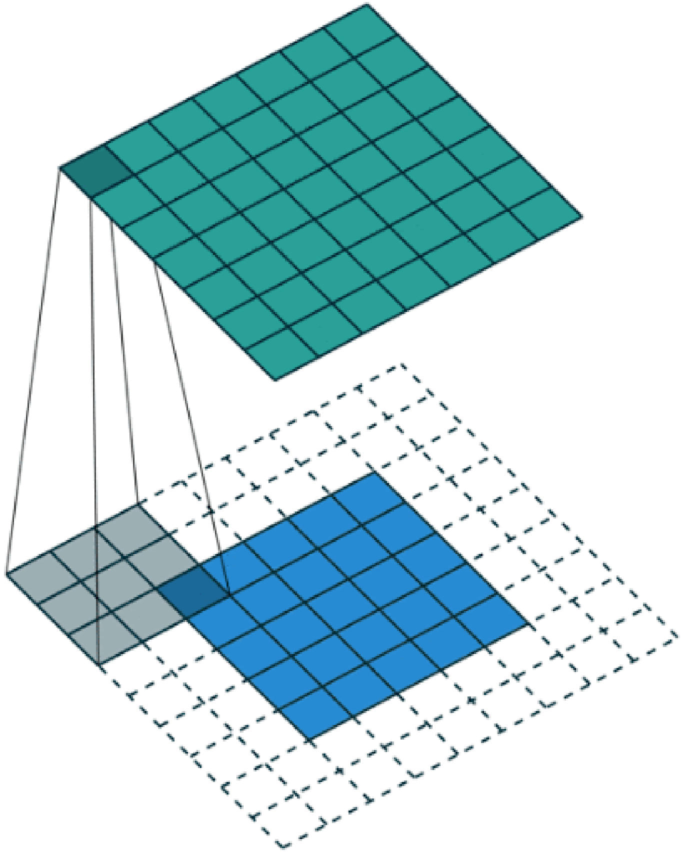
Convolution theorem

Statement and Problems

$$L\left[f(t)*g(t)\right]=F(s)\,G(s)$$

$$L^{-1}[F(s)G(s)]=f(t)*g(t)$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$



In the following links you can see the formulation of convolution of 2 functions
https://drive.google.com/open?id=1RhCesi0PZyiSjPokq8sUTqmfxiH43JpP

In the following links you can see the example for convolution of 2 functions
https://drive.google.com/open?id=1RWNBW2DnasHO7thR7uJ-MIY3Sr2NFF2q

Convolution theorem

Problems in Convolution of 2 functions

1. Find the value of $e^t * \sin t$

   Solution:

$$Given\ f(t) = e^t, g(t) = \sin t$$

$$f(u) = e^u, g(t-u) = \sin(t-u)$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$e^t * \sin t = \int_0^t e^u \sin(t-u)\, du$$

$$\int e^{ax}\sin(bx+c)dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c)-b\cos(bx+c)\right]$$

$$=\frac{e^{u}}{1+1}\left[1.\sin(-u+t)-(-1)\cos(-u+t)\right]_0^t$$

$$=\left[\frac{e^{t}}{2}(1.\sin(-t+t)+\cos(-t+t))-\frac{1}{2}(\sin(0+t)+\cos(0+t))\right]$$

$$=\frac{1}{2}\left[e^{t}-(\sin t+\cos t)\right]$$

2. Find the value of $1 * \cos t$

Solution:

$$\text{Given } f(t) = 1, g(t) = \cos t$$

$$f(u) = 1, g(t-u) = \cos(t-u)$$

$$\text{U sin g Convolution theroem}$$

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$1 * \cos t = \int_0^t 1.\cos(t-u) du$$

$$= \int_0^t [\cos t \cos u + \sin t \sin u] du$$

$$= \cos t \int_0^t \cos u \, du + \sin t \int_0^t \sin u \, du$$

$$= \cos t [\sin u]_0^t + \sin t [-\cos u]_0^t \quad = \sin t$$

Convolution theorem

Problems in Inverse of Laplace Transforms using convolution Theorem.

$$L\left[f(t)*g(t)\right]=F(s)\,G(s)$$

$$L^{-1}[F(s)G(s)]=f(t)*g(t)$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$

In the following links you can see problems to find Inverse of Laplace Transforms using convolution Theorem.

1. https://drive.google.com/open?id=15FoB7o7uaaTmNacAFvYOXJaTYJ-hLC0
2. https://drive.google.com/open?id=1OXst8OTnxX9s_deiX30q8hNEMPvpAEV
3. https://drive.google.com/open?id=1X_yDkbDo2ndoKzaGoqOXIYWJdLZgHkg
4. https://drive.google.com/open?id=1mhm2ypOOtenuqGIXfTdq9MyTXPs9sU7

1. Using convolution theorem find the inverse Laplace transform of

$$\frac{1}{(s+a)(s+b)}$$

Solution:

$$e^{-at} * e^{-bt} = \int_0^t e^{-au}\, e^{-b(t-u)}\, du = \int_0^t e^{-au-bt+bu}\, du$$

$$= e^{-bt} \int_0^t e^{-u(a-b)} du$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)} \right]_0^t$$

$$= \frac{e^{-bt}}{-(a-b)} \left[e^{-(a-b)t} - 1 \right]$$

$$= \frac{e^{-bt}}{-(a-b)} \left[e^{-at}\, e^{bt} - 1 \right]$$

$$= \frac{1}{(a-b)} \left[e^{-bt} - e^{-at} \right]$$

2. Using convolution theorem find the inverse Laplace transform of

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Solution:

$$Given\,F(s) = \frac{s}{(s^2+a^2)}$$

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2+a^2}\right)$$

$$f(t) = \cos at$$

$$f(u) = \cos au$$

$$Given\,G(s) = \frac{s}{(s^2+b^2)}$$

$$g(t) = L^{-1}(G(s)) = L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$g(t) = \cos bt$$

$$g(t-u) = \cos b(t-u)$$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$\cos at * \cos bt = \int_0^t \cos au\,\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t cos\left(au + bt - bu\right) + cos\left(au - bt + bu\right)du$$

$$= \frac{1}{2}\int_0^t cos(\left(a - b\right) u + bt) + cos((a + b) u - bt)du$$

$$= \frac{1}{2}\left[\frac{sin\left(a - b\right) u + bt}{\left(a - b\right)} + \frac{sin\left(a + b\right) u - bt}{(a + b)}\right]$$

$$= \frac{a \sin at - b \sin bt}{a^2 - b^2}$$

3. Using convolution theorem find the inverse Laplace transform of

$$\frac{1}{s^2\left(s^2+25\right)}$$

Solution:

$$Given\,F(s)=\frac{1}{s^2}$$

$$f(t)=L^{-1}(F(s))=L^{-1}\left(\frac{1}{s^2}\right)$$

$$f(t)=t$$

$$f(u)=u$$

$$Given\,G(s)=\frac{1}{\left(s^2+25\right)}$$

$$g(t)=L^{-1}(G(s))=L^{-1}\left(\frac{1}{\left(s^2+25\right)}\right)$$

$$g(t)=\frac{\sin 5t}{5}$$

$$g(t-u)=\frac{\sin 5(t-u)}{5}$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$

$$t * \frac{\sin 5t}{5} = \int_0^t u \ \frac{\sin 5(t-u)}{5} du$$

$$= \frac{1}{5}\int_0^t u \ \sin\left(5t - 5u\right) du$$

$$= \frac{1}{5}\int_0^t u\left[\sin 5t \cos 5u - \cos 5t \sin 5u\right] du$$

$$= \ \frac{1}{5}\left[\sin 5t \int_0^t u \cos 5u \, du - \cos 5t \int_0^t u \sin 5u \, du \ \right]$$

$$= \frac{1}{125}\left[5t - \sin 5t\right]$$

4. Using convolution theorem find the inverse Laplace transform of

$$\frac{1}{(s-2)(s+2)^2}$$

Solution:

$$Given\, F(s)=\frac{1}{(s-2)}$$

$$Given\, G(s)=\frac{1}{(s+2)^2}$$

$$f(t)=L^{-1}(F(s))=L^{-1}\left(\frac{1}{(s-2)}\right)$$

$$g(t)=L^{-1}(G(s))=L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$f(t)=e^{2t}$$

$$g(t)=e^{-2t}\,L^{-1}\left[\frac{1}{s^2}\right]=e^{-2t}\,t$$

$$f(u)=e^{2u}$$

$$g(t-u)=e^{-2(t-u)}(t-u)$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)du$$

$$e^{2t} * e^{-2t}\, t = \int_0^t e^{2u}(t-u)e^{-2(t-u)}\, du$$

$$= \int_0^t (t-u)e^{-2t}\, e^{4u}\, du$$

$$= e^{-2t} \int_0^t e^{4u}\,(t-u)du$$

$$= \frac{1}{16}\left[e^{2t} - e^{-2t} - 4t\,e^{-2t}\right]$$

5. Using convolution theorem find the inverse Laplace transform of

$$\frac{s}{\left(s^2+a^2\right)^2}$$

Solution:

$$\cos at * \frac{\sin at}{a} = \frac{1}{a}\int_0^t \cos au \ \sin a(t-u)\,du \quad = \frac{1}{a}\int_0^t \cos au \sin(at-au)\,du$$

$$= \frac{1}{a}\int_0^t \sin(at-au)\cos au\,du$$

$$= \frac{1}{2a}\int_0^t \sin(at) - \sin(at-2au)\,du$$

$$= \frac{1}{2a}\left[\sin(at)u - \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t.\sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a}\right] \quad = \frac{1}{2a}t\sin at$$

6. Using convolution theorem find the inverse Laplace transform of

$$\frac{4}{\left(s^2+2s+5\right)^2}$$

Solution:

$$Given\, F(s) = \frac{2}{(s^2+2s+5)}$$

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{2}{(s^2+2s+5)}\right) = L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= e^{-t} L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$f(t) = e^{-t}\sin 2t \Rightarrow f(u) = e^{-u}\sin 2u$$

$$Given\,G(s)=\frac{2}{(s^2+2s+5)}$$

$$g(t)=L^{-1}(G(s))=L^{-1}\left(\frac{2}{(s^2+2s+5)}\right)=L^{-1}\left[\frac{2}{(s+1)^2+4}\right]=e^{-t}\,L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$g(t)=e^{-t}\sin 2t\Rightarrow g(t-u)=e^{-(t-u)}\sin 2(t-u)$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$

$$e^{-t}\sin 2t*e^{-t}\sin 2t=\int_0^t e^{-u}\sin 2u\;e^{-(t-u)}\sin 2(t-u)\,du$$

$$= e^{-t}\int_0^t \sin 2u\ \sin 2\left(t-u\right)du$$

$$= \frac{1}{2}e^{-t}\int_0^t \cos\left(2t-2u-2u\right)-\cos(\,2t-2u+2u\,)du$$

$$= \frac{e^{-t}}{2}\int_0^t \cos(\,2t-4u\,)-\cos(\,2t\,)du$$

$$= \frac{e^{-t}}{2}\left[\frac{\sin\left(2t-4u\right)}{-4}-\cos\left(2t\right)u\right]_0^t$$

$$= \frac{e^{-t}}{4}\left[\sin 2t-2t\cos 2t\right]$$

7. Using convolution theorem find the inverse Laplace transform of

$$\frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}$$

Solution: $$Given\, F(s)=\frac{s}{(s^2+1)}$$

$$f(t)=L^{-1}(F(s))=L^{-1}\left(\frac{s}{(s^2+1)}\right)$$

$$f(t)=\cos t \Rightarrow f(u)=\cos u$$

$$Given\, G(s)=\frac{s+1}{(s^2+2s+2)}=\frac{s+1}{(s+1)^2+1}$$

$$g(t)=L^{-1}(G(s))=L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)=e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right]=e^{-t}\cos t$$

$$g(t)=e^{-t}\cos t \Rightarrow g(t-u)=e^{-(t-u)}\cos(t-u)$$

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$

$$cos\,t * e^{-t}\,cos\,t = \int_0^t cos\,u\; e^{-(t-u)}\,cos(t-u)\,du$$

$$= e^{-t}\int_0^t e^{u}\,cos\,u\;cos(t-u)\,du$$

$$= \frac{1}{2}e^{-t}\int_0^t e^{u}\,cos(u+t-u) - cos(u-t+u)\,du$$

$$= \frac{e^{-t}}{2}\int_0^t e^{u}(cos\,t + cos(2u-t)\,du$$

$$= \frac{e^{-t}}{2}\left[\int_0^t e^{u}\,cos\,t\,du + \int_0^t e^{u}\,cos(2u-t)\,du\right]$$

$$= \frac{e^{-t}}{2}\left[cos\,t\left[e^{u}\right]_0^t - \left[\frac{e^{u}}{1+4}\left(2\,cos(2u-t) - t\,sin(2u-t)\right)\right]_0^t\right]$$

$$= \frac{e^{-t}}{2}\left[\left(e^{t}-1\right)cos\,t + \frac{e^{t}}{5}\left(2\,cos\,t - t\,sin\,t\right) - \frac{1}{5}\left(2\,cos\,t + t\,sin\,t\right)\right]$$

Problems for practice

Find the inverse Laplace transform of the following functions

1. $$\frac{1}{(s+1)(s+2)}$$

2. $$\frac{1}{s(s^2+4)}$$

3. $$\frac{1}{s^3(s+5)}$$

4. $$\frac{s}{(s^2+1)(s^2+4)}$$

5. $$\frac{1}{s^2(s+1)^2}$$

Solution of linear ordinary differential equation of second order with constant Coefficient using Laplace Transform techniques

We know that

$$L[y''(t)] = s^2\, L[y(t)] - s\, y(0) - y'(0)$$

$$L[y'(t)] = s\, L|y(t)| - y(0)$$

$$L[y(t)] = L(y)$$

1. Solve the equation $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=2e^{-3t}, y(0)=1 and\ y'(0)=-2$

by Laplace transforms.

Solution:

$$L[y''(t)]+6L[y'(t)]+9L(y)=2L\left(e^{-3t}\right)$$

$$s^2L[y(t)]-s\,y(0)-y'(0)+6[s\,L(y(t))-y(0)]+9\,L[y(t)]=\frac{2}{s+3}$$

$$s^2L(y(t))-s.1+2+6s\,L[y(t)]-6+9\,L[y(t)]=\frac{2}{s+3}$$

$$L[y(t)]\left(s^2+6s+9\right)=\frac{2}{s+3}+4+s$$

$$y(t) = L^{-1}\left[\frac{2}{(s+3)^3}\right] + L^{-1}\left[\frac{4+s}{(s+3)^2}\right]$$

$$L[y(t)] = \frac{2}{(s+3)^3} + \frac{4+s}{(s+3)^2}$$

$$y(t) = 2L^{-1}\left[\frac{1}{(s+3)^3}\right] + L^{-1}\left[\frac{s+3+1}{(s+3)^2}\right]$$

$$= 2e^{-3t}L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{s+3}{(s+3)^2}\right] + L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= 2e^{-3t}\frac{t^2}{2} + L^{-1}\left[\frac{1}{(s+3)}\right] + e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= t^2e^{-3t} + e^{-3t} + te^{-3t} = e^{-3t}(t^2 + t + 1)$$

2. Solve the equation $y'' + 2y' - 3y = \sin t,\, y = 0,\, y' = 0 \; when \;\; t = 0$
by Laplace transforms.

Solution:

$$L[y''(t)] + L[y'(t)] - 3L(y) = L(\sin t)$$

$$s^2\, L[y(t)] - s\, y(0) - y'(0) + 2[s\, L(y(t)) - y(0)] - 3\, L[y(t)] = \frac{1}{s^2 + 1}$$

$$s^2\, L(y(t)) - s(0) - 0 + 2s\, L[y(t)] - 0 - 3\, L[y(t)] = \frac{1}{s^2 + 1}$$

$$L[y(t)]\left(s^2 + 2s - 3\right) \quad = \frac{1}{s^2 + 1}$$

$$L[y(t)] = \qquad \frac{1}{(s^2 + 1)\left(s^2 + 2s - 3\right)}$$

$$L[y(t)] = \frac{1}{(s^2 + 1)(s - 1)(s + 3)}$$

$$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{(s^2+1)} + \frac{C}{(s-1)} + \frac{D}{(s+3)}$$

$$A = -1/10, B = -1/5, C = 1/8, D = -1/40$$

$$y(t) = L^{-1}\left[\frac{\frac{-1}{10}s - \frac{1}{5}}{(s^2+1)} + \frac{\frac{1}{8}}{(s-1)} + \frac{\frac{-1}{40}}{s+3}\right]$$

$$= -\frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{8}e^{-t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t + -\frac{1}{40}e^{-3t}$$

3. Solve the equation $\left(D^2-2D+1\right)x=e^t, x=2\, and\ Dx=-1 at\ \ t=0$

by Laplace transforms.

Solution:

$$\left(x''(t)-2x'(t)+x\right)=e^t, x(0)=2, x'(0)=-1$$

$$L[x''(t)]-2L[x'(t)]+L(x(t))=L\left[e^t\right]$$

$$s^2 L[x(t)]-s\,x(0)-x'(0)-2[s\,L(x(t))-x(0)]+L[x(t)]=\frac{1}{s-1}$$

$$s^2 L(x(t))-s(2)-(-1)-2s\,L[x(t)]+4+L[x(t)]=\frac{1}{s-1}$$

$$L[x(t)]\left(s^2-2s+1\right)+5-2s \quad =\frac{1}{s-1}$$

$$L[x(t)]\left(s^2-2s+1\right)=\frac{1}{s-1}-5+2s$$

$$L[x(t)]=\frac{2s^2-7s+6}{(s-1)\left(s^2-2s+1\right)}$$

$$L[x(t)]=\frac{2s^2-7s+6}{(s-1)^3}$$

$$x(t)=L^{-1}\left[\frac{2s^2-7s+6}{(s-1)^3}\right]$$

$$x(t)=2L^{-1}\left[\frac{s^2}{(s-1)^3}\right]-7L^{-1}\left[\frac{s}{(s-1)^3}\right]+6L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$=2\frac{d^2}{dt^2}L^{-1}\left[\frac{1}{(s-1)^3}\right]-7\frac{d}{dt}L^{-1}\left[\frac{1}{(s-1)^3}\right]+6e^t L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2\frac{d^2}{dt^2}e^t L^{-1}\left[\frac{1}{s^3}\right] - 7\frac{d}{dt}e^t L^{-1}\left[\frac{1}{s^3}\right] + 6e^t L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2\frac{d^2}{dt^2}e^t\frac{t^2}{2} - 7\frac{d}{dt}e^t\frac{t^2}{2} + 6e^t\frac{t^2}{2}$$

$$= \left[e^t 2t + t^2 e^t + 2te^t + 2e^t\right] - \frac{7}{2}\left[t^2 e^t + 2te^t\right] + 3t^2 e^t$$

$$= e^t\left[2t + t^2 + 2t + 2 - \frac{7}{2}t^2 - 7t + 3t^2\right]$$

$$= e^t\left[\frac{1}{2}t^2 - 3t + 2\right]$$

4. Solve the equation $y'' + y' - 2y = 3\cos 3t - 11\sin 3t,\ y(0) = 0,\ y'(0) = 6$

by Laplace transforms.

Solution:

$$L[y''(t)] + L[y'(t)] - 2L(y) = 3L(\cos 3t) - 11L(\sin t)$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) + [s\,L(y(t)) - y(0)] - 2\,L[y(t)] = \frac{3s}{s^2+9} - \frac{33}{s^2+9}$$

$$s^2 L(y(t)) - s(0) - 6 + s\,L[y(t)] - 0 - 2\,L[y(t)] = \frac{3s}{s^2+9} - \frac{33}{s^2+9}$$

$$L[y(t)]\left(s^2 + s - 2\right) = \frac{3s-33}{s^2+9} + 6$$

$$L[y(t)] = \frac{3s - 33 + 6\left(s^2+9\right)}{(s^2+9)\left(s^2+s-2\right)}$$

$$L[y(t)] = \frac{6s^2 + 3s + 21}{(s^2+9)\left(s^2+s-2\right)}$$

$$y(t) = L^{-1}\left[\frac{6s^2+3s+21}{(s^2+9)(s^2+s-2)}\right]$$

$$\frac{6s^2+3s+21}{(s^2+9)(s^2+s-2)} = \frac{6s^2+3s+21}{(s^2+9)(s+2)(s-1)}$$

$$\frac{6s^2+3s+21}{(s^2+9)(s+2)(s-1)} = \frac{As+B}{(s^2+9)} + \frac{C}{(s+2)} + \frac{D}{(s-1)}$$

$$A = 0, B = 3, C = -1, D = 1$$

$$y(t) = L^{-1}\left[\frac{3}{(s^2+9)} - \frac{1}{(s+2)} + \frac{1}{s-1}\right] = \sin 3t - e^{-2t} + e^{t}$$

5. Solve the equation $(D^2+9)y = 6\cos 3t, y=1, Dy=0 \, when \, t=0$

by Laplace transforms.

Solution:

$$L[y''(t)] + 9L(y) = 6\,L(cos\,3t)$$

$$s^2\,L[y(t)] - s\,y(0) - y'(0) + 6\,L[y(t)] = \frac{6s}{s^2+9}$$

$$s^2\,L(y(t)) - s(1) - 0 + 6L[y(t)] = \frac{6s}{s^2+9}$$

$$L[y(t)]\left(s^2+9\right) = \frac{6s}{(s^2+9)^2} + s$$

$$L[y(t)] = \frac{s^3 + 6s + 9s}{(s^2+9)^2}$$

$$y(t) = L^{-1}\left[\frac{s^3 + 9s}{(s^2+9)^2}\right] + L^{-1}\left[\frac{6s}{(s^2+9)^2}\right]$$

$$= L^{-1}\left[\frac{s(s^2+9)}{(s^2+9)^2}\right] + L^{-1}\left[\frac{6s}{(s^2+9)^2}\right]$$

$$y(t) = L^{-1}\left[\frac{s}{(s^2+9)}\right] + L^{-1}\left[\frac{6s}{(s^2+9)^2}\right] = cos\,3t + t\,sin\,3t$$