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MODULE - 3

DISCRETE AND INTEGRAL TRANSFORM

Laplace Transform :

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \qquad t \rightarrow \text{positive}$$

→ piecewise continuous function

A function $f(x)$ is said to be exponential order when $\lim_{t\to\infty} e^{-st} f(x) = 0$

The function t^2 is of exponential order.

$$\lim_{t\to\infty} e^{-st}\, t^2 = \lim_{t\to\infty} \frac{t^2}{e^{st}} = \lim_{t\to\infty} \frac{2t}{s e^{st}}$$

$\boxed{\text{L-hospitals rule}}$

$$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = \frac{2}{\infty} = 0 \text{ (finite)}$$

t^2 is of exponential order.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

exists if

Should be continuous or piecewise continuous in the closed interval $[a, b]$

Should be of exponential order.

LAPLACE TRANSFORM		INVERSE LAPLACE TRANSFORM	
$f(t)$	$L[f(t)]$	$F(s)$	$L^{-1}[F(s)]$
k, a constant	$L[k] = \frac{k}{s}$ $s>0$, k is constant	$\frac{k}{s}, s\neq 0$ k constant	$L^{-1}\left[\frac{k}{s}\right] = k$
e^{at}	$L[e^{at}] = \frac{1}{s-a}$	$\frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
e^{-at}	$L[e^{-at}] = \frac{1}{s+a}$	$\frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
t^n	$L[t^n] = \begin{cases} \frac{n!}{s^{n+1}} & n=0,1,2,3 \\ \frac{\Gamma(n+1)}{s^{n+1}} & n=\frac{1}{2},\frac{3}{2} \end{cases}$	$\frac{1}{s^n}$	$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ $n=1,2,3$
a^t	$L[a^t] = \frac{1}{s-\log a}$	$\frac{1}{(s-a)^n}$	$L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)!}$
$\sqrt{t}$	$L[t^{1/2}] = \frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
$\frac{1}{\sqrt{t}}$	$L\left[\frac{1}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}$	$\frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
$\sin at$	$L[\sin at] = \frac{a}{s^2+a^2}$	$\frac{1}{s^2-a^2}$	$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
$\cos at$	$L[\cos at] = \frac{s}{s^2+a^2}$	$\frac{s}{s^2-a^2}$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
$\sinh at$	$L[\sinh at] = \frac{a}{s^2-a^2}$	$\frac{1}{(s-a)^2+b^2}$	$L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$
$\cosh at$	$L[\cosh at] = \frac{a}{s^2-a^2}$	$\frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

Prove that $L[e^{at}] = \frac{1}{s-a}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[f(t)] = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{-(s-a)}\right] - \left[\frac{e^{0}}{-(s-a)}\right]$$

$$= \frac{0}{-(s-a)} - \left[\frac{1}{-(s-a)}\right]$$

$$= \frac{1}{s-a}$$

$$L[e^{at}] = \frac{1}{s-a}$$

Find the laplace transform of $\cos^3 2t$.

$$\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$$

$$\cos^3 A = \frac{\cos 3A + 3\cos A}{4}$$

$$L[\cos^3 A] = \frac{1}{4} L\left[\cos 3(2t) + 3\cos(2t)\right]$$

$$= \frac{1}{4}\left[L(\cos 6t) + 3L(\cos 2t)\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + 3\cdot\frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3\cdot s}{s^2+4}\right]$$

$$= \frac{s}{4}\left[\frac{1}{s^2+36} + \frac{3}{s^2+4}\right]$$

$$= \frac{s}{4}\left[\frac{s^2+4+3s^2+108}{(s^2+36)(s^2+4)}\right]$$

$$= \frac{s}{4}\cdot\left[\frac{4s^2+112}{(s^2+36)(s^2+4)}\right]$$

$$L[\cos^3 2t] = \frac{s^3+28s}{(s^2+36)(s^2+4)}$$

Find the laplace transform of $\sin 2t \cos 3t$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos 3t \sin 2t = \frac{1}{2}\left[\sin 5t - \sin t\right]$$

$$L\left[\cos 3t \sin 2t\right] = \frac{1}{2} L\left[\sin(5t) - \sin(t)\right]$$

$$\boxed{L\left[\sin at\right] = \frac{a}{s^2+a^2}}$$

$$= \frac{1}{2}\left[L(\sin 5t) - L(\sin t)\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{5(s^2+1^2) - (s^2+25)}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{5s^2+5 - s^2 - 25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$L\left[\cos 3t \sin 2t\right] = \frac{2s^2-10}{(s^2+25)(s^2+1)}$$

Find the laplace transform of $\sin^2 2t$.

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$L\left[\sin^2 2t\right] = L\left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2} L\left[1-\cos 4t\right]$$

$$= \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$L(1) = \frac{1}{s}$

$L[\cos at] = \frac{s}{s^2+a^2}$

$$= \frac{1}{2}\left[\frac{s^2+16 - s^2}{s(s^2+16)}\right]$$

$$= \frac{1}{2}\left[\frac{16}{s(s^2+16)}\right]$$

$$= \frac{8}{s(s^2+16)}$$

Find the laplace transform of $\sin t/2$.

$$L\left[\sin \tfrac{1}{2}t\right] = \frac{\frac{1}{2}}{s^2 + \left(\frac{1}{2}\right)^2}$$

$L[\sin at] = \frac{a}{s^2+a^2}$

$$= \frac{\frac{1}{2}}{s^2 + \frac{1}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{4s^2+1}{4}}$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{4}^{2}}{4s^2+1}$$

$$L\left[\sin \tfrac{1}{2}t\right] = \frac{2}{4s^2+1}$$

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Polynomial function t^n.

$$L[t^n] = \begin{cases} \dfrac{n!}{s^{n+1}} & n = 0, 1, 2, 3 \\ \dfrac{\Gamma(n+1)}{s^{n+1}} & , n = 1, 2, 3 \end{cases}$$

$$\boxed{\begin{aligned} \Gamma\tfrac{1}{2} &= \sqrt{\pi} \\ \Gamma\tfrac{5}{2} &= \tfrac{3}{2}\Gamma\tfrac{3}{2} \\ \Gamma\tfrac{3}{2} &= \tfrac{1}{2}\Gamma\tfrac{1}{2} \end{aligned}}$$

Prove that $L[\cosh at] = \dfrac{s}{s^2-a^2}$

$$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$$

$$\boxed{L[\cosh at] = \frac{e^{at}+e^{-at}}{2}}$$

$$= \frac{1}{2}\left\{L[e^{at}] + L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$\boxed{\begin{aligned} L[\sinh at] &= \frac{a}{s^2-a^2} \\ L[\cosh at] &= \frac{s}{s^2-a^2} \end{aligned}}$$

$$= \frac{1}{2}\left\{\frac{s+a+s-a}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\}$$

$$L[\cosh at] = \frac{s}{s^2-a^2}$$

Linear Property:

$$L[f(t)] = F(s)$$

$$L[g(t)] = G(s)$$

$$L[af(t) \pm bg(t)] = aF(s) \pm bG(s)$$

Find the Laplace Transform of $e^{-2t} \cosh^3 2t$

$$\cosh^3 2t = \left[\frac{e^{2t} + e^{-2t}}{2}\right]^3$$

$$\boxed{\cosh A = \frac{e^{A} + e^{-A}}{2}}$$

$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3b^2$$

$$= \frac{1}{8}\Big[(e^{2t})^3 + (e^{-2t})^3 +$$

$$3(e^{2t})^2(e^{-2t}) + 3(e^{2t})(e^{-2t})^2\Big]$$

$$\cosh^3 2t = \frac{1}{8}\left[e^{6t} + e^{-6t} + 3e^{2t} + 3e^{-2t}\right]$$

$$L\left[e^{-2t} \cosh^3 2t\right] = \frac{1}{8} L\left[e^{-2t}\left[e^{6t} + e^{-6t} + 3e^{2t} + 3e^{-2t}\right]\right]$$

$$= \frac{1}{8} L\left[e^{4t} + e^{-8t} + 3 + 3e^{-4t}\right]$$

$$= \frac{1}{8}\left\{L[e^{4t}] + L[e^{-8t}] + L[3] + 3L[e^{-4t}]\right\}$$

$$\boxed{\begin{aligned} L[e^{at}] &= \frac{1}{s-a} \\ L[e^{-at}] &= \frac{1}{s+a} \end{aligned}}$$

$$= \frac{1}{8}\left\{\frac{1}{s-4} + \frac{1}{s+8} + \frac{3}{s} + \frac{3}{s+4}\right]$$

First shifting property:

If $L[f(t)] = F(s)$ then

$$L\left[e^{-at} f(t)\right] = F(s+a) \Rightarrow L\left[e^{-at} f(t)\right] = F(s)_{s \to s+a}$$

$$L\left[e^{at} f(t)\right] = F(s-a)$$

Find Laplace transform of $(t^3+3e^{2t}-5\sin 3t)e^{-t}$.

$$\boxed{L\left[e^{-at}f(t)\right]=F(s)_{s\to(s+a)}}$$

$$L\left[e^{at}\right]=\frac{1}{s-a}$$

$$L\left[t^n\right]=\frac{n!}{s^{n+1}}$$

$$L\left[\sin at\right]=\frac{a}{s^2+a^2}$$

$$L\left[(t^3+3e^{2t}-5\sin 3t)e^{-t}\right]$$

$$=\left[L[t^3]+3L[e^{2t}]-5L[\sin 3t]\right]_{s\to s+1}$$

$$=\left[\frac{3!}{s^4}+3\,\frac{1}{s-2}-5\,\frac{3}{s^2+9}\right]_{s\to s+1}$$

$$=\frac{6}{(s+1)^4}+\frac{3}{(s-1)}-\frac{15}{(s+1)^2+9}$$

$$L\left[(t^3+3e^{2t}-5\sin 3t)e^{-t}\right]=\frac{6}{(s+1)^4}+\frac{3}{(s-1)}-\frac{15}{s^2+2s+10}$$

Find the Laplace transform of $(1+te^{-t})^3$

$$L(1+te^{-t})^3=L\left[(1)^3+(te^{-t})^3+3(1)(te^{-t})+3(1)(te^{-t})^2\right]$$

$$=L[1]+\left[L(t^3)\right]_{s\to s+3}+3\left[L(t)\right]_{s\to s+1}+3\left[L(t^2)\right]_{s\to s+2}$$

$$=\frac{1}{s}+\left[\frac{3!}{s^4}\right]_{s\to s+3}+\left[\frac{1}{s^2}\right]_{s\to s+1}+3\left[\frac{2!}{s^3}\right]_{s\to s+2}$$

$$=\frac{1}{s}+\frac{6}{(s+3)^4}+\frac{1}{(s+1)^2}+\frac{6}{(s+2)^3}$$

Find the Laplace Transform of cosh at cos at.

$$\boxed{\cosh t = \frac{e^{t}+e^{-t}}{2}}$$

$L[\cosh at \cos at]$

$$= L\left[\frac{e^{at}+e^{-at}}{2} \cos at\right]$$

$$= \frac{1}{2}\left[L(e^{at}+e^{-at}) \cos at\right]$$

$$= \frac{1}{2}\left[L(e^{at}.\cos at) + L(e^{-at}. \cos at)\right]$$

$$= \frac{1}{2}\left[L(\cos at)_{s\to s-a} + L(\cos at)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+a^2}\right)_{s\to s-a} + \left(\frac{s}{s^2+a^2}\right)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

If need (based on marks)

$$= \frac{1}{2}\left[\frac{s-a}{s^2-2as+a^2+a^2} + \frac{s+a}{s^2+2as+a^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2-2as+2a^2} + \frac{s+a}{s^2+2as+2a^2}\right]$$

Find the laplace transform of $\sinh t/2 \sin \frac{\sqrt{3}}{2}t$.

$$\sinh t = \frac{e^{t}-e^{-t}}{2}$$

$$\sin at = \frac{a}{s^2+a^2}$$

$$L\left[\sinh \frac{1}{2}t \quad \sin \frac{\sqrt{3}}{2}t\right]$$

$$\sinh \frac{1}{2}t = \frac{e^{\frac{1}{2}t} - e^{-1/2 t}}{2}$$

$$L\left[\sinh t/2 \sin \frac{\sqrt{3}}{2}t\right] = L\left[\frac{e^{1/2 t}-e^{-1/2 t}}{2} \cdot \sin \frac{\sqrt{3}}{2}t\right]$$

$$= \left\{L\left[\frac{e^{1/2 t}}{2} \sin \frac{\sqrt{3}}{2}t\right] - L\left[\frac{e^{-1/2 t}}{2} \sin \frac{\sqrt{3}}{2}t\right]\right.$$

$$= \frac{1}{2}\left\{L\left[e^{1/2 t} \sin \frac{\sqrt{3}}{2}t\right] - L\left[e^{-1/2 t} \sin \frac{\sqrt{3}}{2}t\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right]_{s\to s-\frac{1}{2}} - \left[\frac{\sqrt{3}/2}{s^2+\frac{3}{4}}\right]_{s\to s+\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\frac{\sqrt{3}}{2}}{\frac{4s^2+3}{4}}\right]_{s\to s-1/2} - \left[\frac{\sqrt{3}/2}{\frac{4s^2+3}{4}}\right]_{s\to s+1/2}\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\sqrt{3}}{\cancel{2}} \times \frac{\cancel{4}}{4s^2+3}\right]_{s\to s-1/2} - \left[\frac{\sqrt{3}}{2} \times \frac{4}{4s^2+3}\right]_{s\to s+1/2}\right\}$$

$$= \frac{\sqrt{3}}{4(s-1/2)^2+3} - \frac{\sqrt{3}}{4(s+1/2)^2+3}$$

$$= \sqrt{3}\left[\frac{1}{4(s^2-s+\frac{1}{4})+3} - \frac{1}{4(s^2+s+\frac{1}{4})+3}\right]$$

$$= \sqrt{3}\left[\frac{1}{4\left(\frac{4s^2-4s+1}{4}\right)+3} - \frac{1}{4\left[\frac{4s^2+4s+1}{4}\right]+3}\right]$$

$$= \sqrt{3}\left[\frac{1}{4s^2-4s+1+3} - \frac{1}{4s^2+4s+1+3}\right]$$

$$= \sqrt{3}\left[\frac{1}{4s^2-4s+4} - \frac{1}{4s^2+4s-4}\right]$$

$$= \sqrt{3}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s-1}\right]$$