Second shifting property.

$$L[f(t)] = F(s) \;\&\; g(t) = \begin{cases} f(t-a) & t > a \\ 0 & 0 \le t < a \end{cases}$$

$$L[g(t)] = e^{-as}F(s)$$

Find the Laplace Transform of $f(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$

$$g(t) = \cos t \qquad a = \frac{2\pi}{3}$$

$$L[\cos t] = \frac{s}{s^2+1}$$

By second shifting theorem.

$$L\left(\cos\left(t - \frac{2\pi}{3}\right)\right) = e^{-\frac{2\pi s}{3}} L(\cos t) = e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$$

Derivatives and Integrals of Transforms.

$$L[tf(t)] = (-1)\frac{d}{ds}F(s)$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^n(s)$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds = \int_s^\infty F(s).ds$$

where

$$L[f(t)] = F(s) = \overline{f(s)}$$

Find the laplace transform of $t\sin at$.

$$L[t\sin at] = -\frac{d}{ds} L[\sin at]$$

$$\boxed{d\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}}$$

$$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\left[\frac{-a(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{2as}{(s^2+a^2)^2}$$

Find the laplace transform of $L\left[t\cos\frac{t}{a}\right]$

$$L\left[t\cos\frac{t}{a}\right] = -\frac{d}{ds} L\left[\cos\frac{t}{a}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+\left(\frac{1}{a}\right)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+\frac{1}{a^2}}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{\frac{s^2a^2+1}{a^2}}\right]$$

$$= -\frac{d}{ds}\left[\frac{sa^2}{s^2a^2+1}\right]$$

$$= -\left[\frac{a^2(s^2a^2+1) - sa^2(2a^2s)}{(s^2a^2+1)^2}\right]$$

$$= \frac{-a^2(s^2a^2+1) + sa^2.(2sa^2)}{(1+s^2a^2)^2}$$

$$= \frac{-a^4s^2 - a^2 + 2s^2a^4}{(1+s^2a^2)^2} = \frac{s^2a^4 - a^2}{(1+s^2a^2)^2}$$

Find the Laplace transform of $L[t\sin at]^2$.

$$L[t\sin at]^2 = L[t^2\sin^2 at]$$

$$= \frac{d^2}{ds^2} L[\sin^2 at]$$

$$= \frac{d^2}{ds^2} L\left[\frac{1-\cos 2at}{2}\right]$$

$$= \frac{1}{2}\frac{d^2}{ds^2}\left\{L[1] - L[\cos 2at]\right\}$$

$$\frac{u}{v}$$

$$= \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} - \frac{(s^2+4a^2) - s(2s)}{(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} - \frac{s^2 - 2s^2 + 4a^2}{(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} + \frac{s^2-4a^2}{(s^2+4a^2)^2}\right]$$

$$= \frac{1}{2}\left[2\frac{1}{s^3} + \frac{2s(s^2+4a^2)^2 - (s^2-4a^2)2(s^2+4a^2)(2s)}{(s^2+4a^2)^4}\right]$$

s^{-2}

$2s^{-3}$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s(s^2+4a^2) - 4s(s^2-4a^2)}{(s^2+4a^2)^3}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s\left[s^2+4a^2 - 2(s^2-4a^2)\right]}{(s^2+4a^2)^3}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + 2s\left[\frac{s^2+4a^2-2s^2+8a^2}{(s^2+4a^2)^3}\right]\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + 2s\left[\frac{-s^2+12a^2}{(s^2+4a^2)^3}\right]\right]$$

(HW)

Find the laplace transform of $L[t\cosh t \cos t]$

$$L[t\cosh t \cos t] = -\frac{d}{ds} L[\cosh t \cos t]$$

$$= -\frac{d}{ds} L\left[\frac{e^{t}+e^{-t}}{2} * \cos t\right]$$

$$= -\frac{1}{2}\frac{d}{ds} L\left[e^{t}\cos t + e^{-t}\cos t\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left\{\left(\frac{s}{s^2+1}\right)_{s\to s-1} + \left(\frac{s}{s^2+1}\right)_{s\to s+1}\right\}$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s-1}{(s-1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= -\frac{1}{2}\left[\frac{(s-1)^2+1-(s-1)2(s-1)}{((s-1)^2+1)^2} + \frac{(s+1)^2+1-(s+1)\,2(s+1)}{((s+1)^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{(s-1)^2-1}{((s-1)^2+1)^2} + \frac{(s+1)^2-1}{((s+1)^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{s^2-2s+1-1}{((s-1)^2+1)^2} + \frac{s^2+2s+1-1}{((s+1)^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{s^2-2s}{((s-1)^2+1)^2} + \frac{s^2+2s}{((s+1)^2+1)^2}\right]$$

Find the laplace transform of $L[t^2 e^{-t} \cos t]$

$$L[t^2 e^{-t} \cos t] = \frac{d^2}{ds^2} L[\cos t] = \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]$$

$$= \frac{d}{ds}\left[\frac{1(s^2+1) - s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+1 - 2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \left[\frac{-2s(s^2+1)^2 - \cancel{2}(s^2+1)(2s)(1-s^2)}{(s^2+1)^4}\right]$$

$$= \left[\frac{-2s(s^2+1) - 4s(1-s^2)}{(s^2+1)^3}\right]$$

$$= \frac{-2s^3 - 2s - 4s + 4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$L[e^{-t} t^2 \cos t] = \left[\frac{2s^3 - 6s}{(s^2+1)^3}\right]_{s \to s+1}$$

$$= \left[\frac{2(s+1)^3 - 6(s+1)}{((s+1)^2+1)^3}\right]$$

Find the Laplace transform of $\frac{\sin at}{t}$.

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at] \cdot ds$$

$$\int_s^\infty \frac{a}{s^2+a^2} = \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \int_s^\infty \frac{a}{s^2+a^2} \cdot ds$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$90 - \tan^{-1}(x)$$
$$= \cot^{-1}(x)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$