$$= \int_{-3}^{1} [xy + y]_{2x}^{3-x^2} dx = \int_{-3}^{1} [x(3-x^2) + (3-x^2) - (2x^2 + 2x)]\, dx$$

$$= \int_{-3}^{1} (3x - x^3 + 3 - x^2 - 2x^2 - 2x)\, dx = \int_{-3}^{1} (3 + x - 3x^2 - x^3)\, dx$$

$$= \left[3x + \frac{x^2}{2} - x^3 - \frac{x^4}{4}\right]_{-3}^{1} = \left(3 + \frac{1}{2} - 1 - \frac{1}{4}\right) - \left(-9 + \frac{9}{2} + 27 - \frac{81}{4}\right)$$

$$= \frac{8+2-1}{4} - \frac{72+18-81}{4} = 0$$

Thus, the volume of the solid is 0 cubic units.

<u>2018 Spring 3 (a)</u>

Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}$ **dy dx by changing polar integral.**

<u>2017 Fall Q.No. 3 (b), 2018 Spring Q.No. 3(b)</u>

Find the volume of the solid whose base is the region in xy-plane that is bounded by the parabola $y = 3 - x^2$, $y = 2x$ while the top is bounded by the plane $z = x + 1$.

<u>2015 Spring Q.No. 3 (a)</u>

Find the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

SHORT QUESTIONS

2003 Fall: Evaluate: $\int_0^{\log 2} \int_{e^y}^{2}$ **dx dy.**

Solution: Let,

$$I = \int_0^{\log 2} \int_{e^y}^{2} dx\, dy = \int_0^{\log 2} [x]_{e^y}^{2} dy = \int_0^{\log 2} (2 - e^y)\, dy$$

$$= [2y - e^y]_0^{\log 2} = [2.\log(2) - e^{\log(2)}] - [0 - e^0]$$

$$= 2\log(2) - 2 + 1 = 2\log(2) - 1.$$

Thus, $I = 2\log(2) - 1$

2004 Spring: Convert the given integral to polar form $\int_0^3 \int_0^{x\sqrt{3}} \frac{y\, dy\, dx}{\sqrt{x^2+y^2}}$.

Solution: See the solution of 2008 Spring.

2009 Spring: Change Cartesian integral $\int_0^2 \int_0^x$ **y dydx, to equivalent polar integral.**

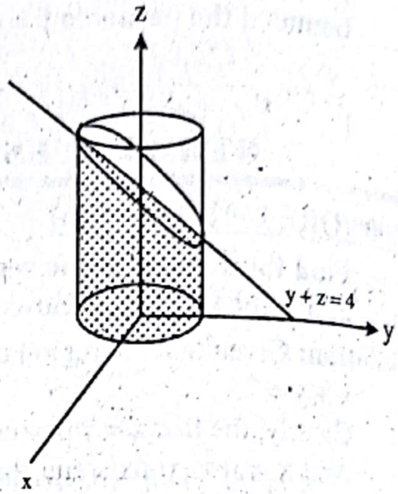
Solution: See the required part of Exercise 9.3 Q. No. 3.

□□□

2002 Q. No. 3(b) OR

Find the volume bounded by circle cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$ and $z = 0$.

Solution: Given that the solid is bounded by a circle cylinder $x^2 + y^2 = 4$, on the top by the plane $y + z = 4$ and by $z = 0$.



Clearly the plane $y + z = 4$ is parallel to x–axis and makes intercept 4 on y–axis and z–axis.

Then the volume generated by the solid is obtained by integrating z over the circle $x^2 + y^2 = 4$ that has radius 2.

Now, volume of the solid is

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

Since 4 is even and y is an odd function. So,

$$V = 16\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} dy\,dx - 0 = 16\int_{0}^{2} [y]_0^{\sqrt{4-x^2}}\,dx = 16\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 16\,[0 + 2\sin^{-1}(1)] = 16\cdot\left(2\cdot\frac{p}{2}\right) = 16p$$

Thus, the volume of the solid is 16p cubic units.

2003 Fall Q. No. 3(b)

Find the volume of the solid whose base is the region in xy–plane that is bounded by the parabola $y = 3 - x^2$, $y = 2x$ while the top is bounded by the plane $z = x + 1$.

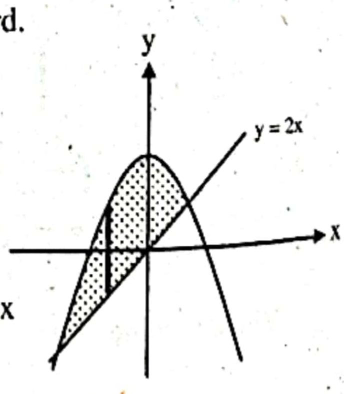
Solution: Given that the base of solid is bounded by the parabola $y = 3 - x^2$ and the line

$y = 2x$. The solid is bounded on the top by the plane $z = x + 1$.

Since, the parabola $y = 3 - x^2 \Rightarrow x^2 = -(y - 3)$ has vertex at (0, 3) and equation of line of symmetry is $x = 0$. So, it has down open ward.

Solving $y = 3 - x^2$, $y = 2x$ we get $x = -3$ and $x = 1$.

On these bases the sketch of the base of solid is shown in figure.



Now, for the volume of solid, we integrate the plane z over the region of base of solid. So,

$$V = \int_{-3}^{1}\int_{2x}^{3-x^2} (z)\,dy\,dx = \int_{-3}^{1}\int_{2x}^{3-x^2} (x+1)\,dy\,dx$$

Similar Question for Practice:

2006 Fall Q. No. 3(b)

Find the volume under the parabolic cylinder $z = x^2$ above the region bounded the parabola $y = 6 - x^2$ and the line $y = x$ in xy–plane.

OTHER QUESTIONS FROM FINAL EXAM

2000(OR); 2002 Q. No. 3(b)

Find the volume of the region that lies under the parabolied $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$.

Solution: Given that the region of integration is enclosed by the lines $x = y$, $x = 0$ and $x + y = 2$.

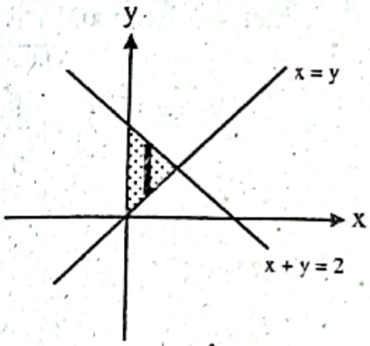
Clearly, the line $x = y$ passes through (0, 0) and (1, 1).

And $x = 0$ is y–axis and the line $x + y = 2$ passes through the point s(2, 0) and (0, 2).

On these bases, the region is sketch as in the figure.

For the volume of the region under the parabolid $z = x^2 + y^2$ and the region shown in figure, we integrate z over the region taking vertical strip,

$$V = \int_0^1 \int_x^{2-x} (z)\, dy\, dx$$



$$= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx$$

$$= \int_0^1 \left[x^2(2-x) - x^3 + \frac{(2-x)^3}{3} - \frac{x^3}{3}\right] dx$$

$$= \int_0^1 \left[2x^2 - x^3 - x^3 - \frac{8 - x^3 - 12x + 6x^2}{3} - \frac{x^3}{3}\right] dx$$

$$= \int_0^1 \left[2x^2 - 2x^3 - \frac{8}{3} + 4x - 2x^2\right] dx = \int_0^1 \left(-2x^3 + 4x - \frac{8}{3}\right) dx$$

$$= \left[-\frac{2x^4}{4} + 2x^2 - \frac{8x}{3}\right]_0^1 = -\frac{2}{4} + 2 - \frac{8}{3}$$

$$= -\frac{1}{2} + 2 - \frac{8}{3}$$

$$= \frac{-3 + 12 - 16}{6} = -\frac{7}{6}$$

Thus, volume of the parabolid under the given boundies is $\frac{7}{6}$ cubic units.

$$= \frac{32a^2}{3} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\Rightarrow \quad A = \frac{16a^2}{3}$$

Thus, area of the region bounded by $x^2 = 4ay$ and $y^2 = 4ax$ is $\frac{16a^2}{3}$ sq. units.

Similar Question for Practice;

2006 Fall Q. No. 3(b) OR

Find the area bounded by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$ by using double integration.

Determining Volume by using Double Integral

1999(OR); 2001(OR); 2004 Fall Q. No. 3(b)

Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy plane.

Solution: Given that the solid is bounded above the cylinder $z = x^2$, below by $y = 2 - x^2$ and is the xy–plane, the solid is bounded by the line $x = y$.

Clearly, the parabola $y = 2 - x^2 \Rightarrow x^2 = -(y-2)$ has vertex at (0, 2) and line of symmetry is $x = 0$. So, the parabola has down openward. Also, the line $x = y$ passes through the point (0, 0) and (1, 1).

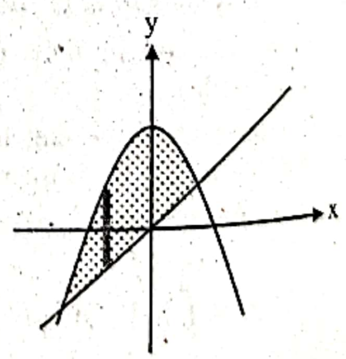
On these bases the sketch base of the solid in xy–plane is as shown in figure.

For the volume of the solid, we integrate z over the region in xy–plane, taking vertical strip. Solving the curves $x = y$ and $y = 2 - x^2$ we get the point of contacts are (–2, –2) and (1, 1).

Now, volume of the solid is

$$V = \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy \, dx$$

$$= \int_{-2}^{1} x^2 [y]_x^{2-x^2} dx = \int_{-2}^{1} x^2 (2 - x^2 - x) \, dx$$



$$= \int_{-2}^{1} (2x^2 - x^4 - x^3) \, dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4}\right]_{-2}^{1}$$

$$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(\frac{-16}{3} + \frac{32}{5} - \frac{16}{4}\right) = \frac{18}{3} - \frac{33}{5} - \frac{1}{4} + 4$$

$$= 6 - \frac{132 + 5}{20} + 4 = 10 - \frac{137}{20} = \frac{200 - 137}{20} = \frac{63}{20}$$

Thus, volume of the solid is $\frac{63}{20}$ cubic units.

2002 Q. No. 3(b)

Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$ by using double integral.

Solution: Given that the region is bounded by $y = 4x - x^2$ and $y = x$. Since, the curve $y = 4x - x^2 \Rightarrow (x-2)^2 = -(y-2)$ which is a parabola having vertex at (2, 2) and equation of line of symmetry be $x - 2 = 0 \Rightarrow x = 2$. So, the parabola is down open ward.

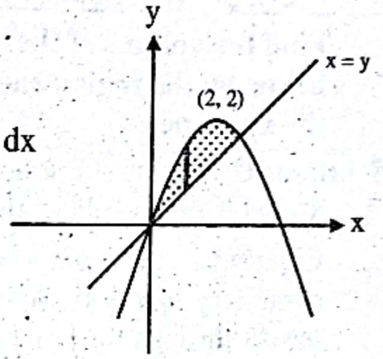
And the line $x = y$ passes through the points (0, 0) and (1, 1).

The sketch of the region is as shown in figure.

Clearly, the point of contact of the curve and the line is (0, 0) and (2, 2).

Now, the area of the region determined by the given curve and the line is

$$\text{Area} = \int_0^2 \int_x^{4x-x^2} dy\, dx = \int_0^2 [y]_x^{4x-x^2} dx$$



$$\Rightarrow A = \int_0^2 (4x - x^2 - x)\, dx = \int_0^2 (3x - x^2)\, dx$$

$$\Rightarrow A = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 = 6 - \frac{8}{3} = \frac{10}{3}$$

Thus, the area of the region is $\frac{10}{3}$ square units.

2002 (II) Q. No. 3(b)

Using the double integrals find the area of the region, $y^2 = 4ax$ and the parabola $x^2 = 4ay$.

Solution: Given that the required region is bounded the curves $y^2 = 4ax$ and $x^2 = 4ay$. Clearly the curve $y^2 = 4ax$ is a parabola having vertex at (0, 0) and equation of line of symmetry is $y = 0$. So, it has right open ward.

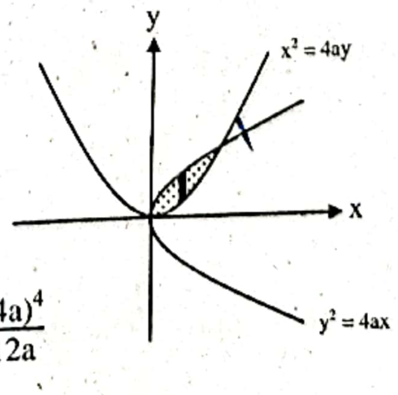
And the curve $x^2 = 4ay$ is a parabola having vertex at (0, 0) and the equation of line of symmetry is $x = 0$. So, it has up open ward.

Moreover, solving these curves, the point of intersection between them is (0, 0) and (4a, 4a).

On these bases, the sketch of the region of integration is as in the figure in which the shaded portion is the region of integration.

Now, for the area of the region, taking vertical strip we get,

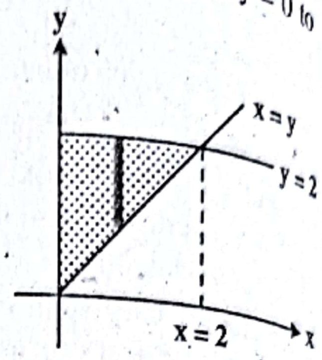
$$A = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$



$$= \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}\frac{x^{3/2}}{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^4}{12a}$$

Now, reversing the order of integration, region has horizontal strip as in figure 2 in which x varies from x = 0 to x = y. Also, the strip moves from y = 0 to y = 2. Then (1) becomes,



(fig. 1)

$$I = \int_0^2 \int_0^y y^2 \sin xy \, dx \, dy$$

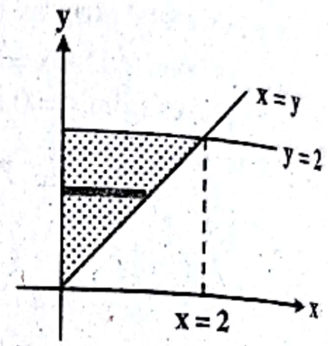
$$= \int_0^2 y^2 \left[-\frac{\cos xy}{y}\right]_0^y dy$$

$$= -\int_0^2 y\,[\cos(y^2) - 1]\, dy$$

Put $y^2 = t$ then $2y\,dy = dt$.

Also $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow t = 4$.

So,



(fig. 2)

$$I = \frac{1}{2}\int_0^4 [\cos t - 1]\, dt$$

$$= \frac{1}{2}[\sin t - t]_0^4 = \frac{\sin 4 - 4}{2}$$

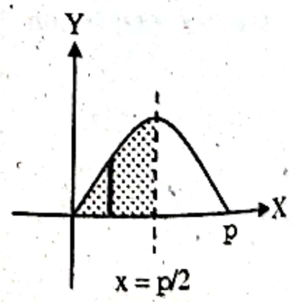
Determining Area by using Double Integral

1999; 2001 Q. No. 3(b)

Using the double integration find the area of the region bounded curves y = sinx and the line x = 0 and $x = \frac{p}{2}$.

Solution: Given that the region is bounded by curves y = sinx, x = 0 and $x = \frac{p}{2}$.

Clearly the region is the shaded portion in the figure that is bounded by $0 \le y \le \sin x$ and $0 \le x \le \frac{p}{2}$.



Now,

$$I = \int_0^{p/2} \int_0^{\sin x} dy\, dx$$

$$= \int_0^{p/2} [y]_0^{\sin x} dx$$

$$= \int_0^{p/2} \sin x\, dx = [-\cos x]_0^{p/2} = 1 - 0 = 1.$$

Thus, area of the region is 1 sq. unit.

$$I = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx \quad ....(1)$$

Here, the region of integration of (1) is R: $0 \le y \le 4 - x^2, 0 \le x \le 2$.
Since, $y = 0$ is a straight line.
and $y = 4 - x^2 \Rightarrow x^2 = -(y - 4)$ is a parabola having vertex at (0, 4) and down–open ward.
Also, both $x = 0$, $x = 2$ are straight line. Thus, the region of integration of (i) is the shaded portion that has vertical strip as shown in figure (1).

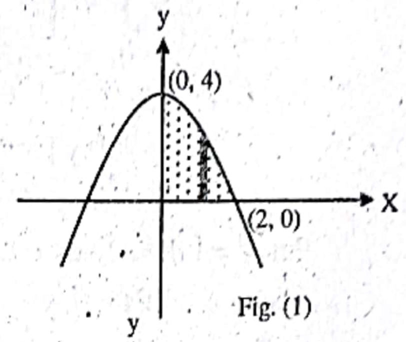


Fig. (1)

Now, reversing the order of integration we take the horizontal strip as in figure (2) for which the strip is bounded by $x = 0$ and $x = \sqrt{4-y}$. And, the strip moves from $y = 0$ to $y = 4$.
Therefore, (1) becomes,

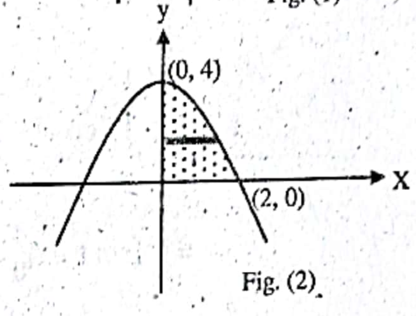


Fig. (2)

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \left(\frac{x\,e^{2y}}{4-y}\right) dx\,dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2}\int_0^4 \frac{e^{2y}}{4-y}(4-y)\,dy = \frac{1}{2}\int_0^4 e^{2y}\,dy = \frac{1}{2}\left[\frac{e^{2y}}{2}\right]_0^4 = \frac{e^8-1}{4}$$

Thus, $I = \dfrac{e^8-1}{4}$

[2012 Fall Q. No. 3(a)]; 2008 Fall Q. No. 3(a)

Sketch the region of integration and evaluate the integral by reversing the order of integration $\int_0^2 \int_x^2 y^2 \sin xy\, dydx$.

Solution: Given that,

$$I = \int_0^2 \int_x^2 y^2 \sin xy\,dy\,dx$$

Here the region of integration is bounded by $y = x$, and by $y = 2$.
Since the line $y = x$ passes through the points (0, 0) and (1, 1). And the line $y = 2$ is a straight line that is parallel to x–axis.
Next, the line $x = 0$ is y–axis. And the line $x = 2$ is a straight line that is parallel to y–axis.
On the basis of these boundaries the sketch of figure is shown as in fig–1.
Clearly, the required region generated by the integral (1) is the shaded portion that has vertical strip as shown in figure 1.

$$= \frac{1}{2}\left[4x - \frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left(16 \frac{16}{2}\right)$$

$$= \frac{1}{2}\left(\frac{16}{2}\right)$$

$$= 4$$

Thus, I = 4 sq. units.

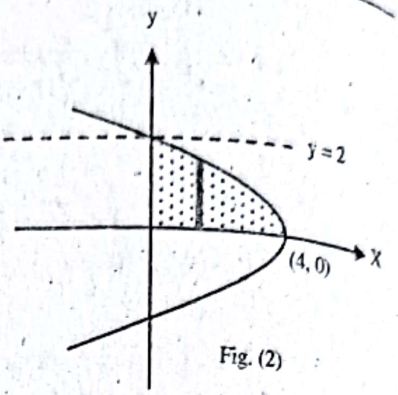
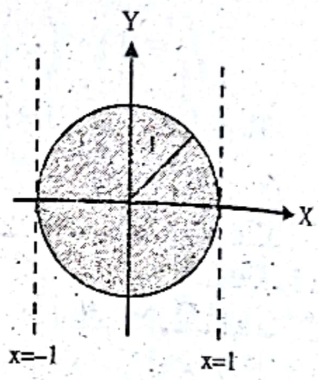


Fig. (2)

2004 Fall Q. No. 3(a)

Change the following integral into polar form and evaluate:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dy\,dx}{(1+x^2+y^2)^2}$$



Solution: Given integral is

$$I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{(1+x^2+y^2)^2}$$

Here, the region of integration be R: $-1 < x < 1$, $-\sqrt{1-x^2} < y < \sqrt{1-x^2}$.

Clearly, $(\sqrt{1-x^2})^2 = y^2 \Rightarrow x^2 + y^2 = 1$.

This shows that the region is a circle with radius r = 1.

So, $0 < r < 1$ and $0 < q < 2p$.

Set, $x = r\cos q$, $y = r\sin q$. Then $x^2 + y^2 = r^2$. Also, $dx\,dy = r\,dr\,dq$.

Now,

$$I = \int_0^{2p}\int_0^1 \left(\frac{2}{(1+r^2)^2}\right) r\,dr\,dq$$

Put $1 + r^2 = t$ then $2r\,dr = dt$. And, $r = 0 \Rightarrow t = 1$ and $r = 1 \Rightarrow t = 2$.

Therefore,

$$I = \int_0^{2p}\int_1^2 \left(\frac{1}{t^2}\right) dt\,dq = \int_0^{2p}\left[\frac{-1}{t}\right]_1^2 dq = -\int_0^{2p}\left[\frac{1}{2} - 1\right] dq$$

$$= \frac{1}{2}\int_0^{2p} dq = \frac{1}{2}\cdot 2p = p$$

Thus, I = p.

[2013 Fall Q. No. 3(a)]; 2006 Fall Q. No. 3(a)

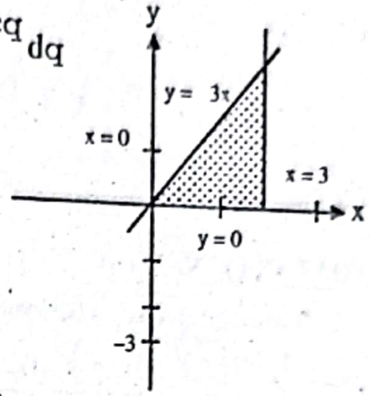
Sketch the region of integration and evaluate by interchanging the order of integration of the double integral: $\displaystyle\int_{x=0}^{2}\int_{y=0}^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx.$

Solution: Given integral is

$$I = \int_0^{p/3} \int_0^{3\sec q} \frac{r \sin q\, r\, dr\, dq}{\sqrt{(r\cos q)^2 + (r \sin q)^2}}$$

$$= \int_0^{p/3} \int_0^{3\sec q} \frac{r^2 \sin q\, dr\, dq}{r} = \int_0^{p/3} \sin q\, [r]_0^{3\sec q}\, dq$$

$$= \int_0^{p/3} \sin q\, 3 \sec q.\, dq = 3 \int_0^{p/3} \tan q.\, dq$$



$$= 3\,[\log(\sec q)]_0^{p/3}$$

$$= 3\left[\log\left(\sec \frac{p}{3}\right) - \log(\sec 0)\right] = 3 \log(2).$$

Thus, $I = 3 \log(2) = \log(8)$.

Determining the value of Double Integral after Reversing the Order of Integration

1999; 2001 Q. No. 3(a)

Change the order of integration and evaluate $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$.

Solution: Given integral is

$$I = \int_0^2 \int_0^{4-y^2} y\, dx\, dy \quad \text{.........(i)}$$

Here, the region of integration is R: $0 < x < 4 - y^2$, $0 < y < 2$.

Since, $x = 4 - y^2 \Rightarrow y^2 = -(x - 4)$ which is a parabola having vertex at (4, 0) and open–left–ward.

Thus, the integral (1) has region of shaded portion as shown in figure (1), that has horizontal strip.

Now, reversing the order of integration, we take the vertical strip as in figure (2) for which y varies from $y = 0$ to the curve $y = \sqrt{4 - x}$. And, the strip moves from $x = 0$ to $x = 4$.

Then,

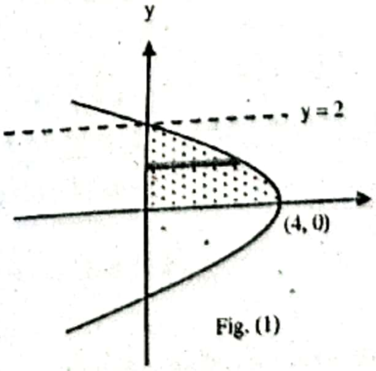


Fig. (1)

$$I = \int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx$$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx = \int_0^4 \left(\frac{4-x}{2}\right) dx$$

Now,

$$|I| = \left| \int_0^2 [x^2y + 2y^2]_{2x}^{x^2} dx \right| = \left| \int_0^2 [(x^2.x^2 + 2x^4) - (x^2.2x + 8x^2)] dx \right|$$

$$= \left| \int_0^2 (3x^4 - 2x^3 - 8x^2) dx \right|$$

$$= \left| \left[\frac{3x^5}{5} - \frac{2x^4}{4} - \frac{8x^3}{3} \right]_0^2 \right|$$

$$= \left| \frac{3 \times 32}{5} - \frac{2 \times 16}{4} - \frac{8 \times 8}{3} \right|$$

$$= \left| \frac{64}{5} - 8 - \frac{64}{3} \right| = \left| 8 \left[\frac{8}{5} - 1 - \frac{8}{3} \right] \right|$$

$$= \left| 8 \left(\frac{24 - 15 - 40}{15} \right) \right| = \frac{248}{15}$$

Thus, $I = \frac{248}{15}$

Determining the value of D.I. after changing the Cartesian form to Polar form
2008 Spring Q. No. 3(a)

Evaluate the double integral $\int_0^3 \int_0^{x\sqrt{3}} \frac{y\,dy\,dx}{\sqrt{x^2 + y^2}}$, by changing Cartesian integral to equivalent polar integral.

Solution: Given integral is,

$$I = \int_0^3 \int_0^{x\sqrt{3}} \frac{y\,dy\,dx}{\sqrt{x^2 + y^2}}$$

Here, the variables x varies from $x = 0$ to $x = 3$ and the variable y varies from $y = 0$ to $y = x\sqrt{3}$.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r \sin q$.

And, to find r, $x = 0$, $\quad x = 3$

$\Rightarrow \quad r\cos q = 0 \quad \Rightarrow r\cos q = 3$

$\Rightarrow \quad r = 0 \quad \Rightarrow r = 3\sec q$

Also, to find q,

$y = 0 \quad y = x\sqrt{3}$

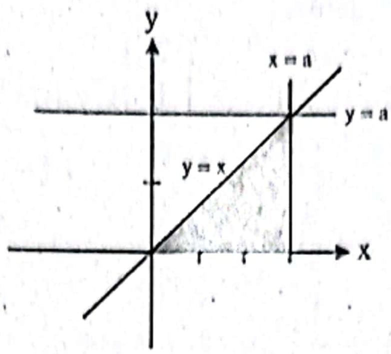
$\Rightarrow \quad r\sin q = 0 \quad \Rightarrow r\sin q = \sqrt{3}\, r\cos q$

$\Rightarrow \quad q = 0 \quad \Rightarrow \tan q = \sqrt{3} = \tan\frac{p}{3}$

$\Rightarrow \quad q = \frac{p}{3}$.

Now, the above integration change to,

$$I = \int_0^{p/4}\int_0^{a\sec q} \frac{(r\cos q)^2\, r\,dr\,dq}{\sqrt{(r\cos q)^2 + (r\sin q)^2}}$$



$$= \int_0^{p/4}\int_0^{a\sec q} \frac{r^2\cos^2 q\, dr\, dq}{r}$$

$$= \int_0^{p/4} \cos^2 q \left[\frac{r^3}{3}\right]_0^{a\sec q} dq$$

$$= \int_0^{p/4} \cos^2 q \times \frac{a^3\sec^3 q}{3}\, dq = \frac{a^3}{3}\int_0^{p/4} \sec q\, dq$$

$$= \frac{a^3}{3}[\log(\sec q + \tan q]_0^{p/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sec\frac{p}{4} + \tan\frac{p}{4}\right) - \log(\sec 0 + \tan 0)\right] = \frac{a^3}{3}[\log(\sqrt{2}+1)].$$

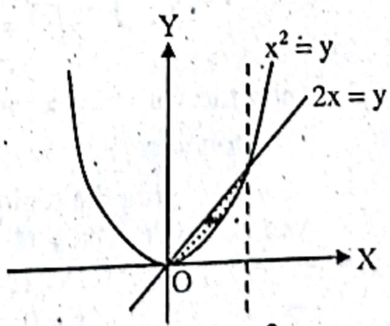
Thus, $I = \frac{a^3}{3}[\log(\sqrt{2}+1)]$.

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

<u>Determining the value of Double Integral</u>

<u>2007 Fall Q. No. 3(a)</u>

Let R be the region in the xy plane bounded by the curves $y = x^2$ and $y = 2x$, evaluate: $\iint_R (x^2 + 4y)dA$.



Solution: Given that,

$$I = \iint_R (x^2 + 4y)\, 2A \qquad \text{............(1)}$$

That is bounded by $y = x^2$ and $y = 2x$.

Solving the curves we get $x = 0, 2$. Then (1) can be written as

$$I = \int_0^2\int_{2x}^{x^2} (x^2 + 4y)\, dy\, dx \qquad \text{........(2)}$$

Here, region of integration is R: $2x \le y \le x^2$, $0 \le x \le 2$.

Clearly, $y = 2x$ is a straight line and $y = x^2$ is a parabola having vertex at (0, 0) and with up open ward.

Clearly, the integral (2) has region of shaded portion in the figure.

$$I = \int_{p/4}^{p/2} \int_{0}^{4a \cot q \operatorname{cosec} q} \left(\frac{r^2(\cos^2 q - \sin^2 q)}{r^2}\right) r\, dr\, dq$$

$$= \int_{p/4}^{p/2} \int_{0}^{4a \cot q \operatorname{cosec} q} \cos 2q\, r\, dr\, dq = \int_{p/4}^{p/2} \cos 2q \left[\frac{r^2}{2}\right]_0^{4a \cot q \operatorname{cosec} q} dq$$

$$= \frac{1}{2} \int_{p/4}^{p/2} \cos 2q\,(16a^2 \cot^2 q \operatorname{cosec}^2 q - 0)\, dq$$

$$= 8a^2 \int_{p/4}^{p/2} (\cos^2 q - \sin^2 q) \cot^2 q \operatorname{cosec}^2 q\, dq$$

$$= 8a^2 \int_{p/4}^{p/2} (\tan^2 q - 1) \cot^2 q\, dq = 8a^2 \int_{p/4}^{p/2} (1 - \tan^2 q)\, dq$$

$$= 8a^2 \int_{p/4}^{p/2} (2 - \sec^2 q)\, dq = 8a^2 [2q - \tan q]_{p/4}^{p/2} = 8a^2 \left[\frac{p}{2} - 1\right].$$

Thus, $I = 8a^2 \left[\frac{p}{2} - 1\right]$.

(10) $\displaystyle\int_0^a \int_y^a \frac{x^2\, dx\, dy}{\sqrt{x^2 + y^2}}$

Solution: Given integral is,

$$I = \int_0^a \int_y^a \frac{x^2\, dx\, dy}{\sqrt{x^2 + y^2}}$$

Here, the variables y varies from $y = 0$ to $y = a$ and the variable x varies from $x = y$ to $x = a$.

Now, changing the region to polar we substitute $x = r \cos q$ and $y = r \sin q$.

Then, to find r,

$x = 0,$	$x = a$
$\Rightarrow r \cos q = 0$	$\Rightarrow r \cos q = a$
$\Rightarrow r = 0$	$\Rightarrow r = a \sec q.$

And, to find q,

$y = 0$	$y = x$
$\Rightarrow r \sin q = 0$	$\Rightarrow r \sin q = r \cos q$
$\Rightarrow q = 0$	$\Rightarrow \tan q = 1 = \tan 45°$
	$\Rightarrow q = \frac{p}{4}$

So that the above integral changes to

Then to find r,

$x = 0, \quad x = a$

$\Rightarrow \quad r\cos q = 0, \qquad \Rightarrow r\cos q = a$

$\Rightarrow \quad r = 0, \qquad \Rightarrow r = a\sec q$

So, $0 \le r \le a\sec q$.

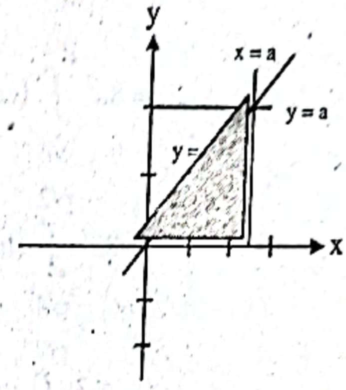
And, to find q,

$y = 0, \quad y = x$

$\Rightarrow \quad r\sin q = 0, \qquad \Rightarrow r\sin q = r\cos q$

$\Rightarrow \quad q = 0, \qquad \Rightarrow \tan q = 1 = \tan\frac{p}{4} \Rightarrow q = \frac{p}{4}$.

So that, the above integral changes to,



$$I = \int_0^{p/4}\int_0^{a\sec q} \frac{r\cos q.\ r\,dr.\,dq}{r^2}$$

$$= \int_0^{p/4} \cos q\,[r]_0^{a\sec q}\,dq = \int_0^{p/4} \cos q \times a\sec q.\ dq$$

$$= a[q]_0^{p/4} = \frac{pa}{4}.$$

Thus, $I = \frac{pa}{4}$.

(9) $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$ **[2015 Fall Q. No. 3(a)]**

Solution: Given integral is,

$$I = \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$$

Here, the variables x varies from $x = \frac{y^2}{4a}$ to $x = y$ and the variable y varies from $y = 0$ to $y = 4a$.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

Then, to find r,

$x = 0, \quad x = 4a$

$\Rightarrow \quad r\cos q = 0 \qquad \Rightarrow r\cos q = 4a$

$\Rightarrow \quad r = 0 \qquad \Rightarrow r = 4a\sec q$

And, to find q,

$y^2/4a = x \quad y = x$

$\Rightarrow \quad r^2\sin^2 q = 4ar\cos q \qquad \Rightarrow r\sin q = r\cos q$

$\Rightarrow \quad r\sin^2 q = 4a\cos q \qquad \Rightarrow \tan q = 1 = \tan\frac{p}{4}$

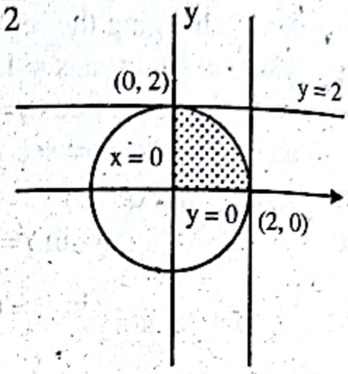
$\Rightarrow \quad r = \frac{4a\cos q}{\sin^2 q} \qquad \Rightarrow q = \frac{p}{4}$

$\Rightarrow r = 4a\cot q\,\mathrm{cosec}\,q$

The above integration is changes to

Here, the variables x varies from $x = 0$ to $x = \sqrt{4-y^2}$ and the variable y varies from $y = 0$ to $y = 2$.
Thus, the region of integration is from the line $x = 0$ to the circle $x^2 + y^2 = 2^2$.
Also, the region moves from origin to the length 2 toward y–axis.
Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.
Clearly the circle has radius $r = 0$ to $r = 2$. So, the region of integration is, $0 \le r \le 2$.
And, to find q,

$y = 0$	$y = 2$	
$\Rightarrow$	$r\sin q = 0$	$\Rightarrow r\sin q = 2$
$\Rightarrow$	$\sin q = 0$	$\Rightarrow 2\sin q = 2$ [Being $r = 2$]
$\Rightarrow$	$q = 0$	$\Rightarrow q = \frac{p}{2}$



The above integral change to

$$I = \int_0^{p/2}\int_0^2 \cos(r^2\cos^2 q + r^2\sin^2 q)\, r\, dr.\, dq$$

$$= \int_0^{p/2}\int_0^2 \cos r^2 r\, dr.\, dq$$

Put $r^2 = t$ then $\frac{dt}{dr} = 2r \Rightarrow \frac{dt}{2} = r\,dr$. Also, $r = 0 \Rightarrow t = 0$ and $r = 2 \Rightarrow t = 4$.

Then,

$$I = \int_0^{p/2}\int_0^4 \cos t \frac{dt}{2} dq = \frac{1}{2}\int_0^{p/2} [\sin t]_0^4\, dq = \frac{\sin 4}{2}\int_0^{p/2} dq = \frac{p}{4}\sin 4.$$

Thus, $I = \frac{p}{4}\sin 4$.

(8) $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

Solution: Given integral is,

$$I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} \quad \text{... (i)}$$

Here, the region be $y \le x \le a$, $0 \le y \le a$.
Now, reversing the order of integration, for which x varies from $x = 0$ to $x = a$.
Also, the strip moves from $y = 0$ to $y = x$. Therefore, after changing the order of integration of (i), it becomes,

$$I = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2} \quad \text{... (ii)}$$

Here, the variables y varies from $y = 0$ to $y = x$ and the variable x varies from $x = 0$ to $x = a$.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

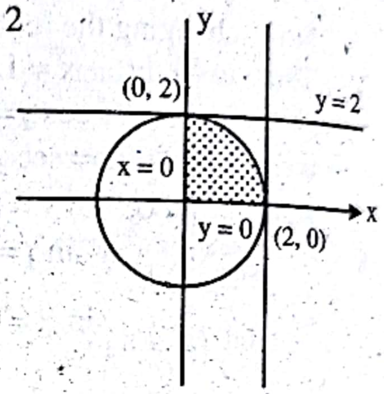
Here, the variables x varies from $x = 0$ to $x = \sqrt{4 - y^2}$ and the variable y varies from $y = 0$ to $y = 2$.

Thus, the region of integration is from the line $x = 0$ to the circle $x^2 + y^2 = 2^2$. Also, the region moves from origin to the length 2 toward y–axis.

Now, changing the region to polar we substitute $x = r \cos q$ and $y = r \sin q$. Clearly the circle has radius $r = 0$ to $r = 2$. So, the region of integration is, $0 \le r \le 2$.

And, to find q,

$y = 0$	$y = 2$	
$\Rightarrow$	$r \sin q = 0$	$\Rightarrow r \sin q = 2$
$\Rightarrow$	$\sin q = 0$	$\Rightarrow 2 \sin q = 2$ [Being $r = 2$]
$\Rightarrow$	$q = 0$	$\Rightarrow q = \frac{p}{2}$



The above integral change to

$$I = \int_0^{p/2} \int_0^2 \cos(r^2 \cos^2 q + r^2 \sin^2 q)\, r\, dr.\, dq$$

$$= \int_0^{p/2} \int_0^2 \cos r^2 r\, dr.\, dq$$

Put $r^2 = t$ then $\frac{dt}{dr} = 2r \Rightarrow \frac{dt}{2} = r\, dr$. Also, $r = 0 \Rightarrow t = 0$ and $r = 2 \Rightarrow t = 4$.

Then,

$$I = \int_0^{p/2} \int_0^4 \cos t \frac{dt}{2} dq = \frac{1}{2} \int_0^{p/2} [\sin t]_0^4\, dq = \frac{\sin 4}{2} \int_0^{p/2} dq = \frac{p}{4} \sin 4.$$

Thus, $I = \frac{p}{4} \sin 4$.

(8) $\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$

Solution: Given integral is,

$$I = \int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2} \quad \text{... (i)}$$

Here, the region be $y \le x \le a$, $0 \le y \le a$.

Now, reversing the order of integration, for which x varies from $x = 0$ to $x = a$. Also, the strip moves from $y = 0$ to $y = x$. Therefore, after changing the order of integration of (i), it becomes,

$$I = \int_0^a \int_0^x \frac{x\, dy\, dx}{x^2 + y^2} \quad \text{... (ii)}$$

Here, the variables y varies from $y = 0$ to $y = x$ and the variable, x varies from $x = 0$ to $x = a$.

Now, changing the region to polar we substitute $x = r \cos q$ and $y = r \sin q$.

Thus, $I = \frac{p}{2}(1 - e^{-a^2})$.

(6) $\int_1^2 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}}$.

Solution: Given integral is,

$$I = \int_1^2 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}}$$

Here, the variables x varies from x = 1 to x = 2 and the variable y varies from y = 0 to y = x.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

Then, to find r, $x = 1$, $\quad x = 2$

$\Rightarrow r\cos q = 1 \quad \Rightarrow r\cos q = 2$

$\Rightarrow r = \sec q \quad \Rightarrow r = 2\sec q$

And to find q, $y = 0 \quad y = x$

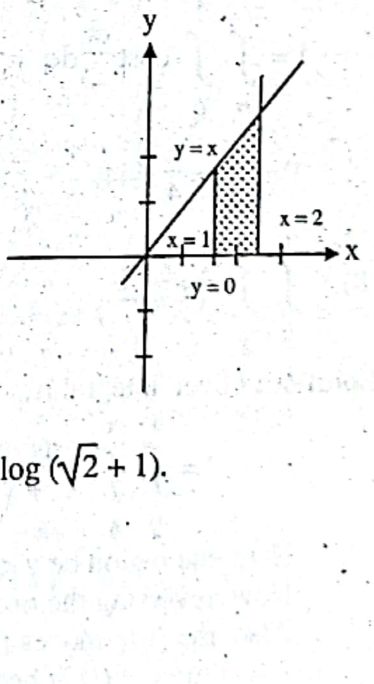
$\Rightarrow r\sin q = 0 \quad \Rightarrow r\sin q = r\cos q$

$\Rightarrow \sin q = 0 \quad \Rightarrow \tan q = 1 = \tan\frac{p}{4}$

$\Rightarrow q = 0 \quad \Rightarrow q = \frac{p}{4}$

Thus, $0 \le q \le \frac{p}{4}$.

So that, the above integral changes to



$$I = \int_0^{p/4} \int_{\sec q}^{2\sec q} \frac{r\,dr.\,dq}{\sqrt{r^2\cos^2 q + r^2\sin^2 q}}$$

$$= \int_0^{p/4} [r]_{\sec q}^{2\sec q}\, dq = \int_0^{p/4} \sec q.\, dq$$

$$= [\log(\sec q + \tan q)]_0^{p/4}$$

$$= \log\left(\sec\frac{p}{4} + \tan\frac{p}{4}\right) - \log(\sec q + \tan q) = \log(\sqrt{2} + 1).$$

Thus, $I = \log(\sqrt{2} + 1)$.

(7) $\int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2)\,dx\,dy$

[2016 Spring Q.NO. 3(a), 2014 Spring Q. No. 3(a)]

Solution: Given integral is,

$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2)\,dx\,dy$$

$$= 3\left[\log\left(\sec\frac{p}{3} + \tan\frac{p}{3}\right) - \log(\sec 0 + \tan 0)\right] = 3\,[\log(2+\sqrt{3})].$$

Thus, $I = 3\,[\log(2+\sqrt{3})]$.

(5) $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx$ **[2003 Fall Q. No. 3(a)]**

Solution: Given integral is,

$$I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx$$

Here, the variable y varies from $y = 0$ to $y = \sqrt{a^2-x^2}$ and the variable x varies from $x = -a$ to $x = a$.

Thus, the region of integration is the half circle that is in only the positive region of y.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

Here, $y = \sqrt{a^2-x^2}$, so $r \neq 0$.

Clearly the circle has radius $r = 0$ to $r = a$. So, the region of integration is, $0 \le r \le a$.

And to find q,

$y = 0$ $\qquad\qquad y = \sqrt{a^2-x^2}$

$\Rightarrow \quad r\sin q = 0 \qquad\qquad \Rightarrow \quad r\sin q = \sqrt{a^2 - r^2\cos^2 q}$

$\Rightarrow \quad \sin q = 0 \qquad\qquad \Rightarrow \quad \sin q = \sin q \quad$ [being $r = a$]

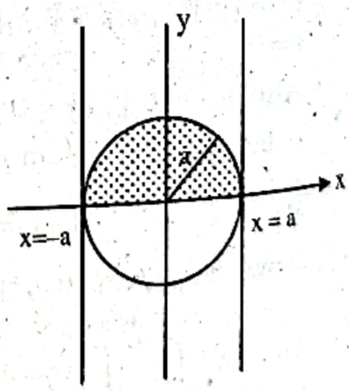
$\Rightarrow \quad q = 0, p$

Also, $dx\,dy = r\,dq\,dr$.

So that the above integral changes to,

$$I = \int_0^p\int_0^a e^{-(r^2\cos^2 q + r^2\sin^2 q)}\, r\,dr.\,dq$$

$$= \int_0^p\int_0^a e^{-r^2}\, r\,dr\,dq$$



Put, $t = r^2$ then $\dfrac{dt}{dr} = 2r \Rightarrow \dfrac{dt}{2} = r\,dr$.

Also, $r = 0 \Rightarrow t = 0$ and $r = a \Rightarrow t = a^2$. Then,

$$I = \int_0^p\int_0^{a^2} e^{-t}\frac{dt}{2}\,dq = \frac{1}{2}\int_0^p \left[\frac{e^{-t}}{-1}\right]_0^{a^2} dq = \frac{1}{-2}\int_0^p (e^{-a^2}-1)\,dq$$

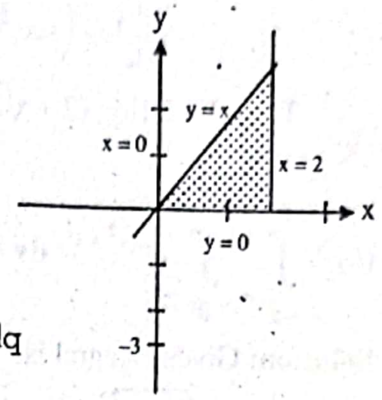
$$= \frac{1}{2}(1-e^{-a^2})\int_0^p dq = \frac{1}{2}(1-e^{-a^2})\,[q]_0^p$$

$$= \frac{p}{2}(1-e^{-a^2}).$$

$$= \int_0^{p/4} \sin q \left[\frac{r^3}{3}\right]_0^{2\sec q} dq$$

$$= \int_0^{p/4} \sin q \frac{(2\sec q)^3}{3} dq$$

$$= \frac{8}{3}\int_0^{p/4} \sin q.\ \sec^3 q\, dq = \frac{8}{3}\int_0^{p/4} \tan q.\ \sec^2 q\, dq$$



Put, $\tan q = t$ then $\sec^2 q.\ dq = dt$. Also, $q = 0 \Rightarrow t = 0$ and $q = \frac{p}{4} \Rightarrow t = 1$. Then,

$$I = \frac{8}{3}\int_0^1 t\, dt = \frac{8}{3}\left[\frac{t^2}{2}\right]_0^1 = \frac{4}{3}.$$

(4) $\displaystyle\int_0^3 \int_0^{x\sqrt{3}} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$,

Solution: Given integral is,

$$I = \int_0^3 \int_0^{x\sqrt{3}} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$$

Here, the variables x varies from $x = 0$ to $x = 3$ and the variable y varies from $y = 0$ to $y = x\sqrt{3}$.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

And, to find r, $\quad x = 0, \qquad x = 3$

$r\cos q = 0 \qquad r\cos q = 3$

$r = 0 \qquad r = 3\sec q$

Also, to find q, $\quad y = 0 \qquad y = x\sqrt{3}$

$r\sin q = 0 \qquad r\sin q = \sqrt{3}\, r\cos q$

$q = 0 \qquad \tan q = \sqrt{3} = \tan\frac{p}{3} \Rightarrow q = \frac{p}{3}.$

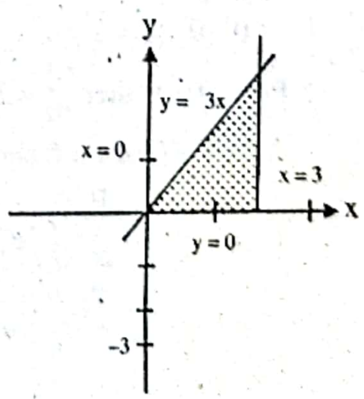
Now, the above integration change to,

$$I = \int_0^{p/3}\int_0^{3\sec q} \frac{r\, dr\, dq}{\sqrt{(r\cos q)^2 + (r\sin q)^2}}$$

$$= \int_0^{p/3}\int_0^{3\sec q} \frac{r\, dr\, dq}{r} = \int_0^{p/3} [r]_0^{3\sec q} dq$$

$$= \int_0^{p/3} 3\sec q.\ dq = 3\int_0^{p/3} \sec q.\ dq$$

$$= 3\,[\log(\sec q + \tan q)]_0^{p/3}$$



Here $x = \sqrt{a^2 - y^2}$, so $r \neq 0$.

Clearly the circle has radius $r = 0$ to $r = a$. So, the region of integration is, $0 \le r \le a$.

And, to find q, the region is bounded by the line $x = y$. So, $q = \frac{p}{4}$.

Thus the angular form moves from $q = 0$ to $q = \frac{p}{4}$.

Also, $dx\, dy = r\, dr\, dq$.

Then the above integral changes to

$$I = \int_0^{p/4}\int_0^{a} r\cos q.\, r dr.\, dq = \int_0^{p/4}\int_0^{a} r^2 \cos q.\, dr.\, dq$$

$$= \int_0^{p/4} \cos q \left[\frac{r^3}{3}\right]_0^a dq = \int_0^{p/4} \frac{a^3}{3}\cos q.\, dq = \frac{a^3}{3}[\sin q]_0^{p/4} = \frac{a^3}{3\sqrt{2}}$$

Thus, $I = \frac{a^3}{3\sqrt{2}} = \frac{a^3\sqrt{2}}{6}$.

(3) $\displaystyle\int_0^2\int_0^x y\, dy\, dx$

Solution: Given integral is,

$$I = \int_0^2\int_0^x y\, dy\, dx.$$

Here, the variables x varies from $x = 0$ to $x = 2$ and the variable y varies from $y = 0$ to $y = x$.

Thus, in the region of integrating limits for y are $y = 0$ to the line $y = x$. Also, the region moves from origin to the length 2 units toward x–axis.

Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

Then, to find r, $\quad x = 0, \qquad x = 2$

$r\cos q = 0, \qquad r\cos q = 2$

$r = 0, \qquad r = 2\sec q$

Therefore, $0 \le r \le 2\sec q$.

And, to find q, $\quad y = 0 \qquad y = x$

$r\sin q = 0 \qquad r\sin q = r\cos q$

$q = 0 \qquad \tan q = 1 = \tan\frac{p}{4} \Rightarrow q = \frac{p}{4}$

So, the above integral change to,

$$I = \int_0^{p/4}\int_0^{p/4} r\sin q\, r dr\, dq = \int_0^{p/4}\int_0^{2\sec q} r^2 \sin q\, dr dq$$

Exercise 9.3

Change the Cartesian integral into an equivalent polar integral and evaluate the polar integral.

(1). $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

Solution: Given integral is,

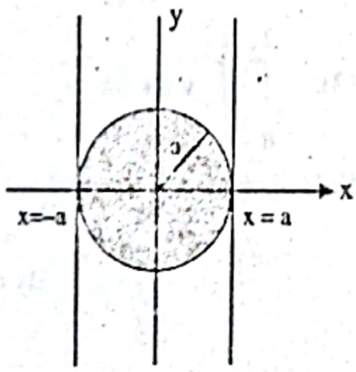
$$I = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

Here, the region of integration is bounded by $y = \sqrt{a^2-x^2}$ to $y = -\sqrt{a^2-x^2}$, $x = -a$ to $x = a$. That is the region of integration is the circle $x^2 + y^2 = a^2$ as shown in figure. For the radical strip in the region, r varies from $r = 0$ to $r = a$. In order to cover the region such type of radical strip q varies from $q = 0$ to $q = 2p$. Then the above integral changes

$$I = \int_{0}^{2p}\int_{0}^{a} r\,dr\,dq = \int_{0}^{2p}\left[\frac{r^2}{2}\right]_0^a dq$$

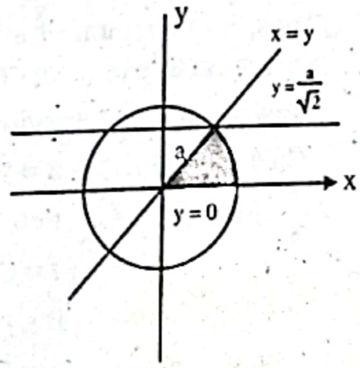
$$= \int_{0}^{2p}\frac{a^2}{2}dq = \frac{a^2}{2}[q]_0^{2p}$$

$$= \frac{a^2}{2}2p = a^2p$$



Thus $I = a^2p$.

(2) $\displaystyle\int_{0}^{a/\sqrt{2}}\int_{y}^{\sqrt{a^2-y^2}} x\,dx\,dy$



Solution: Given integral is,

$$I = \int_{0}^{a/\sqrt{2}}\int_{y}^{\sqrt{a^2-y^2}} x\,dx\,dy$$

Here, the variables x varies from $x = y$ to $x = \sqrt{a^2-y^2}$ and the variable y varies from $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

Thus, the region of integration is from the line $x = y$ to the circle $x^2 + y^2 = a^2$.

Also, the region moves from origin to the length $\frac{a}{\sqrt{2}}$ toward y–axis.

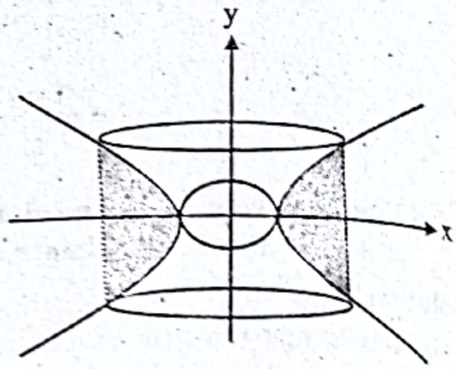
Now, changing the region to polar we substitute $x = r\cos q$ and $y = r\sin q$.

$$= 8\int_{q=0}^{p/2}\int_{0}^{\sqrt{3}} t^2\, dt\, dq$$

$$= \int_{q=0}^{p/2} dq \left[\frac{t^3}{3}\right]_1^{\sqrt{3}}$$

$$= 8\frac{p}{2}\left[\frac{3\sqrt{3}}{3}\right]$$

$$= 4\sqrt{3}p.$$



16. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ intercepted between the parabolic $x^2 + y^2 = 2az$ and the xy plane.

Solution

Given that the solid is bounded by the cylinder $x^2 + y^2 - 2ax = 0$ intercepted between the parabolic $x^2 + y^2 = 2az$ and the xy plane.

In xy–plane, $z = 0 \Rightarrow x^2 + y^2 = 0$.

Also given cylinder is,

$$x^2 + y^2 - 2ax = 0$$

In polar form,

$$r^2 = 2ar\cos q$$

$$\Rightarrow r = 2a\cos q$$

And given region of integration in xy plane is a circle with $r > 0$, so q varies on the region from $q = \frac{-p}{2}$ to $q = \frac{p}{2}$.

Therefore,

$$\text{Volume} = 2\iint z\, dy\, dx = \int_{-p/2}^{p/2}\int_{0}^{2a\cos q} \frac{r^2}{2a}\, r\, dr\, dq$$

$$= \frac{1}{2a}\int_{-p/2}^{p/2} \left[\frac{r^4}{4}\right]_0^{2a\cos q} dq = \frac{1}{2a}\int_{-p/2}^{p/2} \frac{16a^4\cos^4 q}{4}\, dq$$

$$= 2a^3\int_{-p/2}^{p/2} \cos^4 q\, dq$$

$$= 4a^3\int_{0}^{p/2} \cos^4 q\, dq \qquad \text{[Being } \cos^4 q \text{ is even]}$$

$$= 4a^3\,\frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)}$$

$$= \frac{3}{4}pa^3$$

Thus, the volume of the solid is $\frac{3}{4}pa^3$ cubic units.

$$= \frac{2}{3}\left[\frac{t^3}{3}\right]_0^3 + \frac{2}{9}\left[9x - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{27-0}{3}\right] + \frac{2}{9}\left[27 - \frac{27}{3}\right] = 6 + 4 = 10.$$

14. **Find the volume bounded by the parabolic $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.**

Solution

Given that the solid is bounded by the parabolic $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

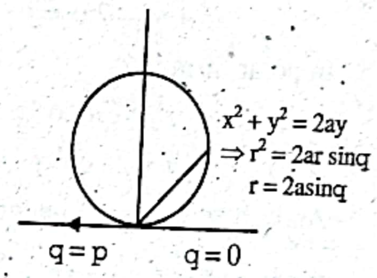
And, $x^2 + y^2 = 2ay$

$\Rightarrow x^2 + y^2 - 2ay + a^2 = a^2$

$\Rightarrow (x-0)^2 + (y-a)^2 = a^2.$

From the above equation it is clear that the centre of circle lies in (0, a) and radius is a. For required volume, we integrate $z = \left(\frac{x^2+y^2}{a}\right)$ over the circle $x^2 + y^2 = 2ay$. For this y varies from $y = 0$ to $y = 2a$ and x varies from $x = 0$ to $x = \sqrt{2ay - y^2}$. Then,

$$\text{Volume} = \int_{q=0}^{p} \int_{r=0}^{2a\sin q} \frac{r^2}{a} \cdot r\, dr\, dq$$



$$= \frac{1}{a}\int_{q=0}^{p} \left[\frac{r^4}{4}\right]_0^{2a\sin q} dq$$

$$= \frac{1}{a}\int_{q=0}^{p} 4a^4 \sin^4 q\, dq = 4a^3\, 2\int_0^{p/2} \sin 4q\, dq$$

$$= 8a^3 \frac{\Gamma\frac{4+1}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{4+0+2}{2}} = \frac{8a^3 \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2\Gamma 3} = \frac{3pa^3}{2}$$

15. **Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.**

Solution

Given that the solid is bounded by the cylinder $x^2 + y^2 = 4$, the hyperboloid $x^2 + y^2 - z^2 = 1$.

In xy–plane, $z = 0 \Rightarrow x^2 + y^2 = 1$. That is, $r^2 = 1$.

Therefore, the region of integration in xy plane in which r varies from $r = 1$ to $r = 2$ and q varies from $q = 0$ to $q = 2p$.

$$\text{Total volume} = 8\int_{q=0}^{p/2}\int_{r=1}^{2} \sqrt{(r^2-1)}\, r\, dr\, dq$$

$$= -4\sqrt{2a}\left\{-\frac{4a^2\sqrt{2a}}{3} + \frac{4a^2\sqrt{2a}}{5}\right\}$$

$$= \frac{16a^2 \times 2a}{3} - \frac{16a^2 \times 2a}{5} = \frac{32a^3}{3} - \frac{32a^3}{5}$$

$$= \frac{32a^3(5-3)}{15} = \frac{64a^3}{15}.$$

Thus, the volume determine by the cylinder is $\frac{64a^3}{15}$ cubic units.

12. Find the volume bounded by the plane $z = 0$, surface $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.

Solution

The solid is bounded by $z = 0$, $z = x^2 + y^2 + 2$, $x^2 + y^2 = 4$.

Here, the solid has volume in xy–plane. Clearly, the solid has four symmetrical parts. And the circle $x^2 + y^2 = 4$ has radius 2 and it moves form $q = 0$ to $q = 2p$.

Also, $x = r\cos q$, $y = r\sin q$ and $dx\, dy = r\,dr\,dq$

Now, volume of the solid is

$$V = \iint z\, dx\, dy = \int_0^{2p}\int_0^{2} (r^2 + 2)\, r\,dr\,dq = \int_0^{2p}\left[\frac{r^4}{4} + r^2\right]_0^2 dq$$

$$= \int_0^{2p}\left[\frac{16}{4} + 4\right] dq = 8\int_0^{2p} dq = 8\,[q]_0^{2p} = 16p$$

Thus, volume of the solid is 16p.

13. Find the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.

Solution

Given that the solid is bounded by, $z = x + y$, $4x^2 + 9y^2 = 36$.

Here, the solid has volume in xy–plane. Clearly, the solid has four symmetrical parts.

In the first quadrant of the ellipse, y varies from $y = 0$ to $y = \frac{2}{3}\sqrt{9 - x^2}$ and then x varies from $x = 0$ to $x = 3$.

Now, volume of the solid is

$$V = \iint z\, dx\, dy = \int_0^3 \int_0^{\frac{2}{3}\sqrt{9-x^2}} (x + y)\, dy\, dx = \int_0^3 \left[xy + \frac{y^2}{2}\right]_0^{\frac{2}{3}\sqrt{9-x^2}} dx$$

$$= \int_0^3 \left[\frac{2}{3}x\sqrt{9 - x^2} + \frac{2}{9}(9 - x^2)\right] dx$$

$$= -\frac{2}{3}\int_3^0 t^2\, dt + \frac{2}{9}\int_0^3 (9 - x^2)\, dx$$

$$= 4\int_{x=0}^{a} \left[x^2y + \frac{y^3}{3}\right]_0^a dx$$

$$= 4\int_{x=0}^{a} \left[ax^2 + \frac{a^3}{3}\right] dx = 4\left[a\frac{x^3}{3} + \frac{a^3x}{3}\right]_0^a = 4\left[\frac{a^4}{3} + \frac{a^4x}{3}\right] = \frac{8a^4}{3}$$

Thus, the volume determine by the cylinder is $\frac{8a^4}{3}$, cubic units.

11. **Proved that the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{64a^3}{15}$.**

2016 Fall Q. No. 3(b); 2011 Fall Q. No. 3(a)

Find the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

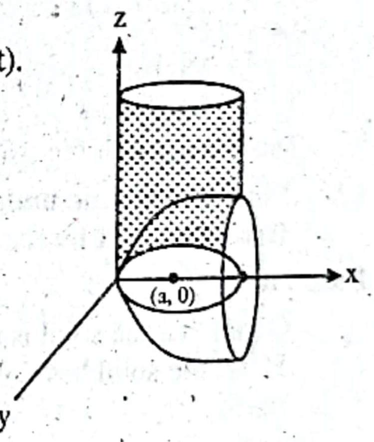
Solution

Given that the solid is enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.
To find the required volume z is to be integrated over the circle $x^2 + y^2 = 2ax$ in xy–plane.

Also, $x^2 + y^2 = 2ax \Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$

$\Rightarrow (x-a)^2 + (y-0)^2 = a^2$

From the above equation, it is clear that radius of circle is a and centre lies at (a, 0).
Now, taking vertical strip,



Volume = $\iint z\, dy\, dx$ = 2 (volume in the first quadrant).

$$= 2\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \sqrt{2ax}\, dy\, dx$$

$$= 2\int_0^{2a} \sqrt{2ax}\,[y]_0^{\sqrt{2ax-x^2}}\, dx$$

$$= 2\int_0^{2a} \sqrt{2ax}.\sqrt{2ax - x^2}\, dx$$

$$= 2\int_0^{2a} \sqrt{x}\sqrt{2a}\sqrt{x}\sqrt{2a-x}\, dx = 2\sqrt{2a}\int_0^{2a} x\sqrt{2a-x}\, dx$$

Put, $t = \sqrt{2a-x}$ then $2t\frac{dt}{dx} = -1 \Rightarrow 2t\,dt = -dx$. Also, $x = 0 \Rightarrow t = \sqrt{2a}$ and $x = 2a \Rightarrow t = 0$. Then,

$$= -2\sqrt{2a}\int_{\sqrt{2a}}^{0} (2a - t^2)t.\,2t dt = -4\sqrt{2a}\int_{\sqrt{2a}}^{0} (2at^2 - t^4)\, dt$$

$$= -4\sqrt{2a}\left[2a\frac{t^3}{3} - \frac{t^5}{5}\right]_{\sqrt{2a}}^{0}$$

$$= -4\sqrt{2a}\left[-\frac{2a}{3}(\sqrt{2a})^3 + \frac{(\sqrt{2a})^5}{5}\right]$$

$$= \left[\frac{3}{2} \cdot 2p - \frac{1}{3}(\sin 2p - \cos 2p)\right] - \left[0 - \frac{1}{3}(0 - 1)\right]$$

$$= 3p + \frac{1}{3} - \frac{1}{3} = 3p$$

Thus, the volume of the solid is 3p cubic units.

9. **Find the volume bounded by the xy–plane; the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.**

[2019 Fall Q.No. 3(a), 2009 Spring; 2010 Spring Q. No. 3(b)]

Solution:

We have to generate the volume of the solid bounded by xy–plane, parabolic $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.

For this, we integrate z over the circle $x^2 + y^2 = 4$ of radius $r = 2$.

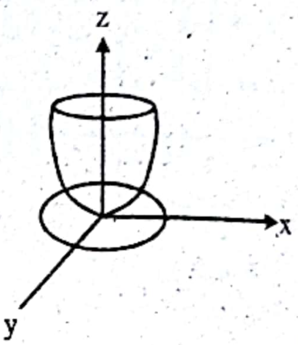
We change the integration in polar form as,

$$x = r\cos q, \quad y = r\sin q$$

Then, $r = 0$ to 2 and $q = 0$ to $q = 2p$.

Moreover, $dxdy = r dr\, dq$

Now, volume of the solid is



$$V = \iint z\, dx\, dy = \int_0^{2p}\int_0^{2} \frac{r^2}{2} \cdot r\, dr\, dq$$

$$= \frac{1}{2}\int_0^{2p}\int_0^{2} r^3\, dr\, dq = \frac{1}{2}\int_0^{2p} \left[\frac{r^4}{4}\right]_0^2 dq$$

$$= \frac{1}{8}\int_0^{2p} (16)\, dq = 2\int_0^{2p} dq = 2\,[q]_0^{2p} = 2 \cdot 2p = 4p.$$

Thus, volume of the solid is 4p cubic units.

10. **Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.**

Solution

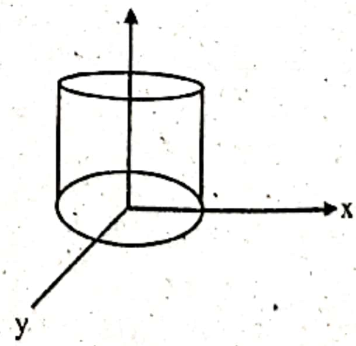
We have to determine the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, $y = -a$, $y = a$.

Clearly, the given solid is uniform cylinder that has four symmetrical parts.

So, volume of the solid be

$$V = 4\int_{x=0}^{a}\int_{y=0}^{a} z\, dy\, dx$$

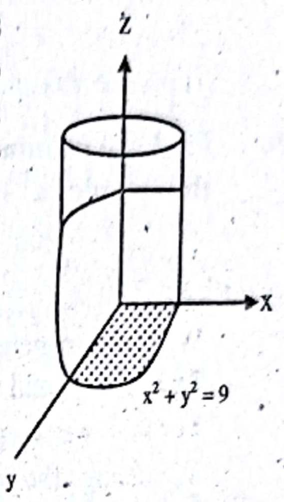
$$= 4\int_{x=0}^{a}\int_{y=0}^{a} (x^2 + y^2)\, dy\, dx$$



Find the volume V of the solid that lies in the first octant and is bounded by the three co-ordinate planes and the cylinder $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$.

tion:

Given that the solid lies is the first octant and it is bounded by $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. Here, these two cylinders have common portion whose volume is required. We need the volume in the first octant. For required volume, we integrate $z = \sqrt{9 - y^2}$ over the circle $x^2 + y^2 = 9$ in first quadrant. For this x varies from y = 0 to y = 3 and y varies from x = 0 to $x = \sqrt{9 - y^2}$. Then,



$$V = \int_0^3 \int_0^{\sqrt{9-y^2}} z\,dx\,dy = \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-y^2}\,dx\,dy$$

$$= \int_0^3 \sqrt{9-y^2} \int_0^{\sqrt{9-y^2}} dx\,dy$$

$$= \int_0^3 \sqrt{9-y^2}\,[x]_0^{\sqrt{9-y^2}}\,dy$$

$$= \int_0^3 (9-y^2)\,dy = \left[9y - \frac{y^3}{3}\right]_0^3 = 27 - \frac{27}{3} = 18$$

Thus, volume of the solid be 18 cubic units.

8. **Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.** **[2008 Spring Q. No. 3(b)]**

Solution

Here, we have to determine the volume of the solid bounded by xy-plane, $x^2 + y^2 = 1$ and $x + y + z = 3$. For this we integrate the plane $z = 3 - x - y$ over $x^2 + y^2 = 1$ which is a circle with radius r = 1.

Set $x = r\cos q$, $y = r\sin q$ and the radius of circle is 1. So, r = 1. Moreover, the region is circle. So, the angle q varies from 0 to 2p in the circle.

Also, $z = 3 - x - y = 3 - r\cos q - r\sin q$.

And, $dx\,dy = r\,dr\,dq$

Now, volume of the solid is

$$V = \iint z\,dx\,dy = \int_0^{2p}\int_0^1 z\,r\,dr\,dq = \int_0^{2p}\int_0^1 (3r - r^2\cos q - r^2\sin q)\,dr\,dq$$

$$= \int_0^{2p} \left[\frac{3r^2}{2} - \frac{r^3}{3}(\cos q + \sin q)\right]_0^1 dq$$

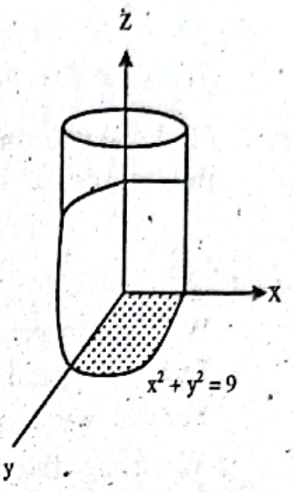
$$= \int_0^{2p} \left[\frac{3}{2} - \frac{1}{3}(\cos q + \sin q)\right] dq$$

$$= \left[\frac{3}{2}q - \frac{1}{3}(\sin q - \cos q)\right]_0^{2p}$$

7. **Find the volume V of the solid that lies in the first octant and is bounded by the three co-ordinate planes and the cylinder $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$.**

Solution:

Given that the solid lies is the first octant and it is bounded by $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. Here, these two cylinders have common portion whose volume is required. We need the volume in the first octant. For required volume, we integrate $z = \sqrt{9 - y^2}$ over the circle $x^2 + y^2 = 9$ in first quadrant. For this x varies from y = 0 to y = 3 and y varies from x = 0 to $x = \sqrt{9 - y^2}$. Then,



$$V = \int_0^3 \int_0^{\sqrt{9-y^2}} z\, dx\, dy = \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-y^2}\, dx\, dy$$

$$= \int_0^3 \sqrt{9-y^2} \int_0^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_0^3 \sqrt{9-y^2}\, [x]_0^{\sqrt{9-y^2}}\, dy$$

$$= \int_0^3 (9 - y^2)\, dy = \left[9y - \frac{y^3}{3}\right]_0^3 = 27 - \frac{27}{3} = 18$$

Thus, volume of the solid be 18 cubic units.

Find the volume bounded by the xy–plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. **[2008 Spring Q. No. 3(b)]**

Solution

Here, we have to determine the volume of the solid bounded by xy–plane, $x^2 + y^2 = 1$ and $x + y + z = 3$. For this we integrate the plane $z = 3 - x - y$ over $x^2 + y^2 = 1$ which is a circle with radius r = 1.

Set $x = r\cos q$, $y = r\sin q$ and the radius of circle is 1. So, r = 1. Moreover, the region is circle. So, the angle q varies from 0 to 2p in the circle.

Also, $z = 3 - x - y = 3 - r\cos q - r\sin q$.

And, $dx\, dy = r\, dr\, dq$

Now, volume of the solid is

$$V = \iint z\, dx\, dy = \int_0^{2p} \int_0^1 z\, r\, dr\, dq = \int_0^{2p} \int_0^1 (3r - r^2\cos q - r^2 \sin q)\, dr\, dq$$

$$= \int_0^{2p} \left[\frac{3r^2}{2} - \frac{r^3}{3}(\cos q + \sin q)\right]_0^1 dq$$

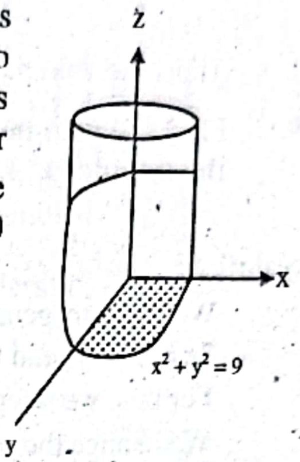
$$= \int_0^{2p} \left[\frac{3}{2} - \frac{1}{3}(\cos q + \sin q)\right] dq$$

$$= \left[\frac{3}{2}q - \frac{1}{3}(\sin q - \cos q)\right]_0^{2p}$$

7. **Find the volume V of the solid that lies in the first octant and is bounded by the three co–ordinate planes and the cylinder $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$.**

Solution:

Given that the solid lies is the first octant and it is bounded by $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. Here, these two cylinders have common portion whose volume is required. We need the volume in the first octant. For required volume, we integrate $z = \sqrt{9 - y^2}$ over the circle $x^2 + y^2 = 9$ in first quadrant. For this x varies from y = 0 to y = 3 and y varies from x = 0 to $x = \sqrt{9 - y^2}$. Then,



$$V = \int_0^3 \int_0^{\sqrt{9-y^2}} z\, dx\, dy = \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-y^2}\, dx\, dy$$

$$= \int_0^3 \sqrt{9-y^2} \int_0^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_0^3 \sqrt{9-y^2}\, [x]_0^{\sqrt{9-y^2}}\, dy$$

$$= \int_0^3 (9 - y^2)\, dy = \left[9y - \frac{y^3}{3}\right]_0^3 = 27 - \frac{27}{3} = 18$$

Thus, volume of the solid be 18 cubic units.

8. **Find the volume bounded by the xy–plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.** **[2008 Spring Q. No. 3(b)]**

Solution

Here, we have to determine the volume of the solid bounded by xy–plane, $x^2 + y^2 = 1$ and $x + y + z = 3$. For this we integrate the plane $z = 3 - x - y$ over $x^2 + y^2 = 1$ which is a circle with radius r = 1.

Set $x = r\cos q$, $y = r\sin q$ and the radius of circle is 1. So, r = 1. Moreover, the region is circle. So, the angle q varies from 0 to 2p in the circle.

Also, $z = 3 - x - y = 3 - r\cos q - r\sin q$.

And, $dx\, dy = r\, dr\, dq$

Now, volume of the solid is

$$V = \iint z\, dx\, dy = \int_0^{2p} \int_0^1 z\, r\, dr\, dq = \int_0^{2p} \int_0^1 (3r - r^2\cos q - r^2\sin q)\, dr\, dq$$

$$= \int_0^{2p} \left[\frac{3r^2}{2} - \frac{r^3}{3}(\cos q + \sin q)\right]_0^1 dq$$

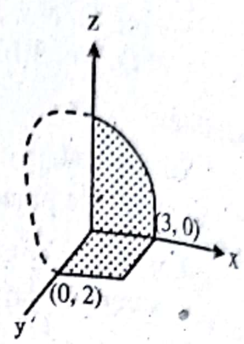
$$= \int_0^{2p} \left[\frac{3}{2} - \frac{1}{3}(\cos q + \sin q)\right] dq$$

$$= \left[\frac{3}{2}q - \frac{1}{3}(\sin q - \cos q)\right]_0^{2p}$$

5. Find the volume of the solid in the first octant bounded by the co-ordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

Solution

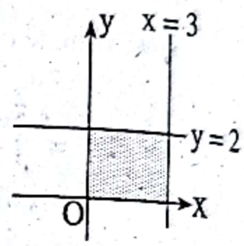
Given that the volume of the solid is restricted in the first octant bounded by coordinate planes, the plane $x = 3$ and the parabolic cylinder $z = 4 - y^2$.



Here, when $z = 0$ then $y = \pm 2$ that projected in xy-plane. The region of integration in xy plane is as shown in the figure. Now, volume of the solid is

$$V = \int_{x=0}^{3}\int_{y=0}^{2} z\,dy\,dx = \int_{x=0}^{3}\int_{y=0}^{2} (4 - y^2)\,dy\,dx$$

$$= \int_{x=0}^{3} \left[4y - \frac{y^3}{3}\right]_0^2 dx$$



Base of Solid

$$= \int_{x=0}^{3} \left(8 - \frac{8}{3}\right) dx$$

$$= \frac{16}{3}\int_{x=0}^{3} dx = \frac{16}{3}[x]_0^3 = 16$$

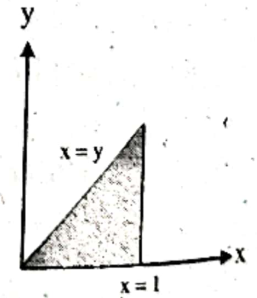
Thus, volume of the solid is 16 cubic units.

6. Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the line $y = x$, $x = 1$ and top lies in the plane $z = f(x, y) = 3 - x - y$.

Solution

Given that the prism has base as the triangle in xy-plane that bounded by x-axis, $y = x$ and $x = 1$. And the top of the prism is bounded by $z = 3 - x - y$.
Thus, the limits of the prism for y are $y = 0$ and $y = x$.
And, the limits for x are $x = 0$ to $x = 1$.
Now, the volume generated by prism be,

$$V = \int_{x=0}^{1}\int_{y=0}^{x} z\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{x} (3 - x - y)\,dy\,dx$$



base of prism

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1.$$

Thus, volume of the prism is 1 cubic units.

4. **Find the volume in the first octant bounded by the co-ordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$.**

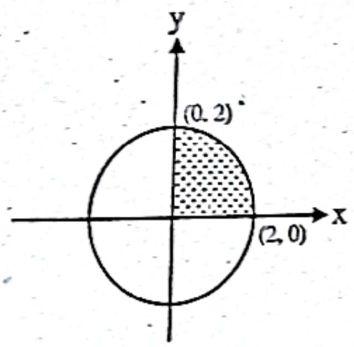
[2019 Fall 3(b), 2017 Spring 3(b), 2016 Fall 3(b), 2015 Fall Q. No. 3(b), 2014 Spring Q. No. 3(b), 2013 Spring Q. No. 3(b), 2011 Spring Q. No. 3(b), 2008 Fall Q. No. 3(b), 2004 Spring Q. No. 3(b)]

Solution:

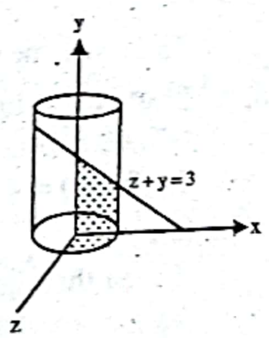
Given that, we have observe the volume in the first octant bounded by the coordinate planes, cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$.

From the figure it is clear that $z = 3 - y$ is to be integrated over the first quadrant of the circle $x^2 + y^2 = 4$.

For the base, limits for y are $y = 0$, and $y = \sqrt{4 - x^2}$ and limits for x are $x = 0$ and $x = 2$.



base of cylinder



Now, volume of the solid is

$$V = \int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} z\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} (3 - y)\, dy\, dx$$

$$= \int_{x=0}^{2} \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{x=0}^{2} \left(3\sqrt{4-x^2} - \frac{4-x^2}{2}\right) dx$$

$$= \left[3\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{2}\left(4x - \frac{x^3}{3}\right)\right]_0^2$$

$$= 3\,[0 + 2\sin^{-1}(1)] - \frac{1}{2}\left(8 - \frac{8}{3}\right) - 3[0 + 2\sin^{-1}0] + 0$$

$$= 6\sin^{-1}(1) - \frac{8}{3} \qquad [\because \sin^{-1} 0 = 0]$$

$$= 6 \cdot \frac{p}{2} - \frac{8}{3}$$

$$= 3p - \frac{8}{3}$$

Thus, volume of the solid is $3p - \frac{8}{3}$ cubic units.

$$= \left(\frac{-32}{5} + 4 - \frac{16}{3} + 8 + 16\right) - \left(\frac{-1}{5} + \frac{1}{4} + \frac{2}{3} + 2 - 8\right)$$

$$= \frac{-96-80}{15} + 28 - \frac{-3+10}{15} + \frac{1}{4} + 6 = \frac{-61}{5} + \frac{1}{4} + 34$$

$$= \frac{423}{20}$$

Thus, the volume of the solid is $\frac{423}{20}$.

3. **Find the volume of the solid whose base is the region in the xy plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.**

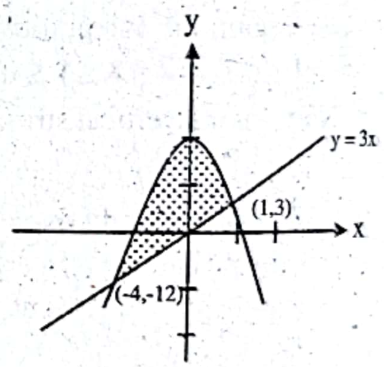
[2018 Fall Q.No 3(b), 2012 Fall Q. No. 3(b), 2014 Fall Q. No. 3(b) [2006 Spring; 2007 Fall; 2009 Fall; 2011 Fall Q. No. 3(b)]

Solution:

Given that the parabola $y = 4 - x^2$ and the line $y = 3x$ made a solid in xy–plane which is bounded in the top by a plane $z = x + 4$.

By solving $y = 4 - x^2$ and $y = 3x$, we get the base has common points as $(1, 3)$ and $(-4, -12)$. The region of integration is as shown in the figure.

Now, integrate the plane z over the region of base then,



base of solid

$$V = \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} (z)\, dy\, dx$$

$$= \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} (x+4)\, dy\, dx$$

$$= \int_{x=-4}^{1} (x+4)\, [y]_{3x}^{4-x^2}\, dx$$

$$= \int_{x=-4}^{1} (x+4)(4 - x^2 - 3x)\, dx = \int_{x=-4}^{1} (-8x - x^3 - 7x^2 + 16)\, dx$$

$$= \left[-4x^2 - \frac{x^4}{4} - \frac{7x^3}{3} + 16x\right]_{-4}^{1}$$

$$= \left(-4 - \frac{1}{4} - \frac{7}{3} + 16\right) - \left(-64 - 64 + \frac{448}{3} - 64\right)$$

$$= \left(12 - \frac{31}{12}\right) - \left(-192 + \frac{448}{3}\right) = \frac{144 - 31 + 2304 - 1792}{12} = \frac{625}{12}$$

Thus, volume of the solid is $\frac{625}{12}$ cubic units.

$$=\frac{1}{4}\int_0^4 x^3\left[\frac{\sqrt{x}}{2}-\frac{x^2}{16}\right]dx=\int_0^4\left(\frac{x^{7/2}}{2}-\frac{x^5}{16}\right)dx$$

$$=\left[\frac{1}{2}\times\frac{2}{9}x^{9/2}-\frac{1}{16}\times\frac{1}{6}x^6\right]_0^4=\left[\frac{1}{9}2^{2\times9/2}-\frac{1}{96}4^6\right]$$

$$=\frac{512}{9}-\frac{4096}{96}$$

$$=\frac{512}{9}-\frac{256}{6}=\frac{1024-768}{18}=\frac{256}{18}=\frac{128}{9}$$

Thus, the volume of the solid is cubic units.

(v) $\mathbf{z=x^2+4,\ y=4-x^2,\ x+y=2,\ z=0}$

Solution: Given curves are $z=x^2+4$, $y=4-x^2$, $x+y=2$, $z=0$.

In xy plane, $z=0 \Rightarrow x^2+4=0$.

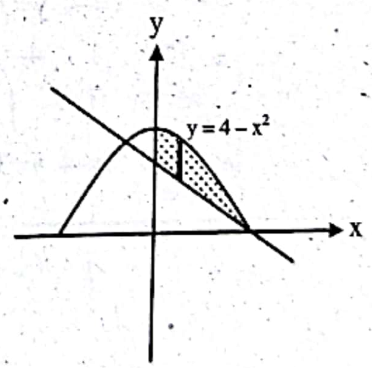
And, the region is bounded by the curve $y=4-x^2 \Rightarrow x^2=-(y-4)$. This is a parabola having vertex at (0, 4), line of symmetry $x=0$ and down open–ward.

Also, the region is bounded by the line $x+y=2$ passes through the points (0, 2) and (2, 0).

On these bases the base of the region is shown as in figure.

The region of integration in xy–plane bounded by parabola and lines be, R: $-1\le x\le 2,\ 2-x\le y\le(4-x^2)$.

Now, taking vertical strip,



$$\text{Volume}=\iint z\,dy\,dx=\int_{-1}^{2}\int_{2-x}^{(4-x^2)}(x^2+4)\,dy\,dx$$

$$=\int_{-1}^{2}(x^2+4)\,[y]_{2-x}^{4-x^2}\,dx$$

$$=\int_{-1}^{2}(x^2+4)(4-x^2-2+x)\,dx$$

$$=\int_{-1}^{2}(4x^2-x^4-2x^2+x^3+16-4x^2-8+4x)\,dx$$

$$=\int_{-1}^{2}(-x^4+x^3-2x^2+4x+8)\,dx$$

$$=\left[\frac{x^5}{5}+\frac{x^4}{4}-\frac{2x^3}{3}+\frac{4x^2}{2}+8x\right]_{-1}^{2}$$

$$\text{Volume} = -\int_4^0 \left\{\frac{u^{3/2}}{3} + u^{1/2}(4-u)^2\right\} du = -\int_4^0 \left\{\frac{u^{3/2}}{3} + u^{1/2}(16-8u+u^2)^2\right\} du$$

$$= -\int_4^0 \left(\frac{u^{3/2}}{3} + 16u^{1/2} - 8u^{3/2} + u^{5/2}\right) du$$

$$= -\frac{1}{3}\int_4^0 (48u^{1/2} - 23u^{3/2} + 3u^{5/2})\, du$$

$$= -\frac{1}{3}\left[48 \times \frac{2}{3}u^{3/2} - 23 \times \frac{2}{5}u^{5/2} + 3 \times \frac{2}{7}u^{7/2}\right]_4^0$$

$$= \frac{1}{3}\left[32 \times u^{3/2} - \frac{46}{5}4^{5/2} + \frac{6}{7}4^{7/2}\right]$$

$$= \frac{1}{3}\left[32 \times 2^{2\times 3/2} - \frac{46}{5}2^{2\times 5/2} + \frac{6}{7}2^{2\times 7/2}\right] = \frac{1}{3}\left[32 \times 8 - \frac{46}{5}32 + \frac{6}{7} + 128\right]$$

$$= \frac{1}{3}\left[\frac{8960 - 10304 + 8340}{35}\right] = \frac{2496}{105} = \frac{832}{35}.$$

Thus, the volume the solid is $\frac{832}{35}$.

(iv) $z = x^3, x = 4y^2, 16y = x^2, z = 0$

Solution: Given curves are $z = x^3$, $x = 4y^2$, $16y = x^2$, $z = 0$.

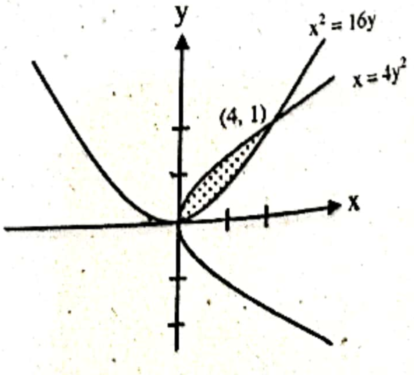
In xy plane $z = 0$.

And, the region is bounded by the curve $x = 4y^2$ this is a parabola having vertex at (0, 0), line of symmetry $y = 0$ and right openward.

Also, the region is bounded by the curve $16y = x^2$ this is a parabola having vertex at (0, 0), line of symmetry $x = 0$ and up openward. Solving $x = 4y^2$ and $x^2 = 16y$ we get the point of contacts are (0, 0) and (4, 1).

On this basis the base of the region is shown as in figure.

The region of integration in xy plane bounded by parabolas be, R: $0 \le x \le 4$, $\frac{x^2}{16} \le y \le \sqrt{\frac{x}{4}}$.



Now, taking vertical strip,

$$\text{Volume} = \iint z\, dy\, dx = \int_0^4 \int_{x^2/16}^{\sqrt{x/4}} x^3\, dy\, dx$$

$$= \int_0^4 x^3\, [y]_{x^2/16}^{\sqrt{x/4}}\, dx$$

$$= \int_0^4 \left\{4\frac{(4-y)}{2} - \left(\frac{4-y}{2}\right)^2 - y\left(\frac{4-y}{2}\right)\right\} dy$$

$$= \int_0^4 \left\{8 - 2y - \frac{(16-8y+y^2)}{4} - \frac{(4y-y^2)}{2}\right\} dy$$

$$= \frac{1}{4}\int_0^4 (32 - 8y - 16 + 8y - y^2 - 8y + 2y^2)\, dy$$

$$= \frac{1}{4}\int_0^4 (y^2 - 8y + 16)\, dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3} - \frac{8y^2}{2} + 16y\right]_0^4 = \frac{1}{4}\left[\frac{64}{3} - 64 + 64\right] = \frac{16}{3}$$

Thus the volume of the solid is $\frac{16}{3}$ cubic units.

(iii) $z = x^2 + y^2, y = 4 - x^2, x = 0, y = 0, z = 0.$

Solution: Given curves are $z = x^2 + y^2$, $y = 4 - x^2$, $x = 0$, $y = 0$, $z = 0$.

In xy–plane $z = 0$.

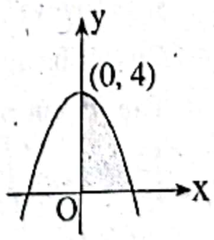
And, the region is bounded by the curve $y = 4 - x^2$ this is a parabola having vertex at (0, 4), line of symmetry $x = 0$ and down openward.

Also, the region is bounded by $x = 0$ and $y = 0$.

On these bases the base of the region is shown as in figure.

The region of integration in the first octant xy plane bounded by above parabola and lines be, R: $0 \le y \le 4, 0 \le x \le \sqrt{4-y}$.

Now, taking horizontal component



base of solid

$$\text{Volume} = \iint_z dx\,dy = \int_0^4 \int_0^{\sqrt{4-y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2x\right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \left\{\frac{(4-y)^{3/2}}{3} + \sqrt{(4-y)}\, y^2\right\} dy$$

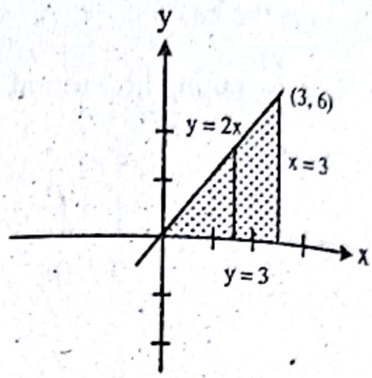
Put, $u = 4 - y$ then $\frac{du}{dy} = -1 \Rightarrow -du = dy$. Also, $y = 0 \Rightarrow u = 4$; $y = 4 \Rightarrow u = 0$.

Then the above part becomes,

The base of the figure is shown as in figure.

The region of integration in the first octant xy plane is bounded by above line by $y = 2x$ and below by $y = 0$ and between $x = 0$ to $x = 3$.

Now, taking vertical strip,



$$\text{Volume} = \int_0^3 \int_0^{2x} \sqrt{9-x^2}\, dy\, dx$$

$$= \int_0^3 \sqrt{9-x^2}\, [y]_0^{2x}\, dx$$

$$= \int_0^3 2x\sqrt{9-x^2}\, dx$$

Put, $u = 9 - x^2$ then $\frac{du}{dx} = -2x \Rightarrow -du = 2x\, dx$. Also, $x = 0 \Rightarrow u = 9$ and $x = 3 \Rightarrow u = 0$. then,

$$\text{Volume} = \int_9^0 -\sqrt{u}\, du = -\left[\frac{2}{3} \times u^{3/2}\right]_9^0 = \frac{2}{3} \times 3^3 = 18.$$

Thus the volume of the solid is 18 cubic units.

(ii) $2x + y + z = 4, x = 0, y = 0, z = 0$.

Solution: Given curves are $2x + y + z = 4$, $x = 0$, $y = 0$, $z = 0$

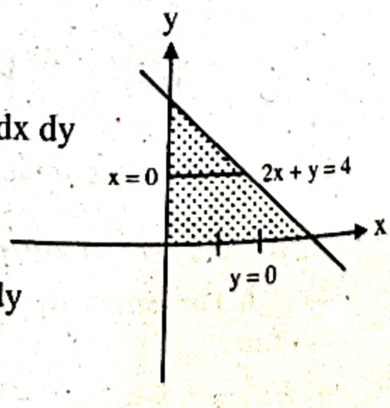
In xy plane, $z = 0$. Then $2x + y = 4$ which is passing through the point (2, 0) and (0, 4).

Also, the line $y = 0$, $x = 0$ are the axes.

The base of the figure is shown as in figure.

The region of integration in the first octant xy–plane bounded by above line and curves be, R: $0 \le y \le 4, 0 \le x \le \frac{4-y}{2}$.

Now, taking horizontal strip,



$$\text{Volume} = \iint dx\, dy = \int_0^4 \int_0^{(4-y)/2} (4 - 2x - y)\, dx\, dy$$

$$= \int_0^4 [4x - x^2 - yx]_0^{(4-y)/2}\, dy$$

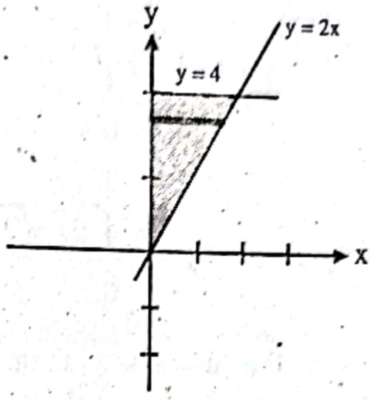
$$= \int_0^4 [4x - x^2 - yx]_0^{(4-y)/2}\, dy$$

Since the line y = 2x passes through the points (0, 0) and (1, 2). And the line y = 4 is a straight line that is parallel to x–axis. Also, the region is bounded by y–axis.

On the bases of these boundaries the region is as in the figure.

Now, taking horizontal strip, in the region find that x varies from $x = 0$ to $x = \frac{y}{2}$,

$$\text{Area} = \int_0^4 \int_0^{y/2} dx\,dy = \int_0^4 [x]_0^{y/2}\,dy$$



$$= \int_0^4 \frac{y}{2}\,dy$$

$$= \left[\frac{y^2}{4}\right]_0^4 = \frac{16}{4} = 4.$$

Thus the area of the region is 4.

(vi) $\mathbf{x = y^2, x = 2y - y^2.}$

Solution: Here the region of integration is bounded by $x = y^2$, and $x = 2y - y^2$.

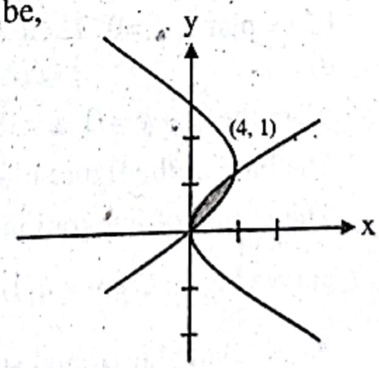
Since the curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward.

Also, the curve $x = 2y - y^2 \Rightarrow (y-1)^2 = -(x-1)$ is a parabola that has vertex at (1, 1) and has line of symmetry $y = 1$. So, the parabola is left openward.

On these bases the region of integration is as shown in figure.

Now, the area of the region bounded by the curves be,

$$\text{Area} = \int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 [x]_{y^2}^{2y-y^2}\,dy$$



$$= \int_0^1 (2y - y^2 - y^2)\,dy$$

$$= \left[\frac{2y^2}{2} - \frac{2y^3}{3}\right]_0^1 = \left(\frac{2}{2} - \frac{2}{3}\right) = \frac{1}{3}$$

Thus, the area of the region is $\frac{1}{3}$.

(vii) – (viii) Similar as above, left for practice.

2. **Sketch the solid in the first octant bounded by the curves and find its volume.**

(i) $\mathbf{x^2 + z^2 = 9, y = 2x, y = 0, z = 0}$

Solution: Given curves are $x^2 + z^2 = 9$, $y = 2x$, $y = 0$, $z = 0$.

In xy plane, $z = 0$. And, $x^2 = 9 \Rightarrow x = \pm 3$.

Also, the line $y = 2x$ passes through (0, 0) and (1, 2).

$$= \int_0^1 \int_x^{3x} dy\, dx + \int_1^2 \int_x^{3x} dy\, dx$$

$$= \int_0^1 [y]_x^{3x} dx + \int_1^2 [y]_x^{3x} dx$$

$$= \int_0^1 (3x - x)\, dx + \int_1^2 (3x - x)\, dx$$

$$= \int_0^1 2x\, dx + \int_1^2 (2x)\, dx = [x^2]_0^1 + [x^2]_1^2 = 1 + (2 - 1) = 2.$$

Thus, area of the region is 2.

(iv) $y = e^x, y = \sin x, x = -p, x = p$

Solution: Here,

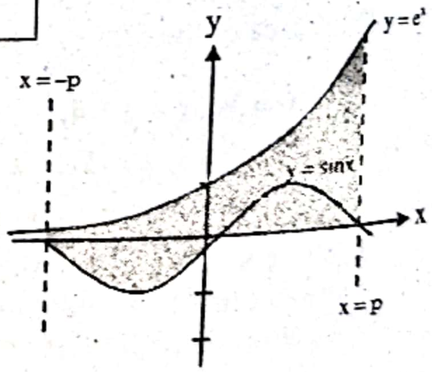
For, $y = e^x$

x	0	1	2	-1
y	1	e	e^2	1/e

For, $y = \sin x$

x	0	p/2	p	-p	-p/2
y	0	1	0	0	-1

Taking vertical strip



$$\text{Area} = \int_{-p}^{p} \int_{\sin x}^{e^x} dy\, dx$$

$$= \int_{-p}^{p} [y]_{\sin x}^{e^x} dx$$

$$= \int_{-p}^{p} (e^x - \sin x)\, dx = [e^x + \cos x]_{-p}^{p} = e^x - 1 - e^{-p} + 1 = (e^p - e^{-p}).$$

Thus, area of the region is $(e^p - e^{-p})$.

(v) The y–axis, the line y = 2x, and the line y = 4.

Solution: Given that the required region is bounded by y–axis, the line y = 2x, and the line y = 4.

$$= \int_0^1 \int_x^{3x} dy\, dx + \int_1^2 \int_x^{3x} dy\, dx$$

$$= \int_0^1 [y]_x^{3x} dx + \int_1^2 [y]_x^{3x} dx$$

$$= \int_0^1 (3x - x)\, dx + \int_1^2 (3x - x)\, dx$$

$$= \int_0^1 2x\, dx + \int_1^2 (2x)\, dx = [x^2]_0^1 + [x^2]_1^2 = 1 + (2 - 1) = 2.$$

Thus, area of the region is 2.

(iv) $y = e^x, y = \sin x, x = -p, x = p$

Solution: Here,

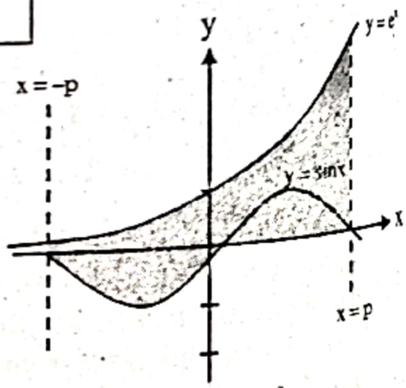
For, $y = e^x$

x	0	1	2	–1
y	1	e	e^2	1/e

For, $y = \sin x$

x	0	p/2	p	–p	–p/2
y	0	1	0	0	–1

Taking vertical strip



$$\text{Area} = \int_{-p}^{p} \int_{\sin x}^{e^x} dy\, dx$$

$$= \int_{-p}^{p} [y]_{\sin x}^{e^x} dx$$

$$= \int_{-p}^{p} (e^x - \sin x)\, dx = [e^x + \cos x]_{-p}^{p} = e^x - 1 - e^{-p} + 1 = (e^p - e^{-p}).$$

Thus, area of the region is $(e^p - e^{-p})$.

(v) **The y–axis, the line y = 2x, and the line y = 4.**

Solution: Given that the required region is bounded by y–axis, the line y = 2x, and the line y = 4.

$$= \left[-\frac{1}{x}+\frac{x^3}{3}\right]_1^2 = \left(-\frac{1}{2}+\frac{8}{3}+1-\frac{1}{3}\right) = \left(\frac{-3+16+6-2}{6}\right) = \frac{17}{6}.$$

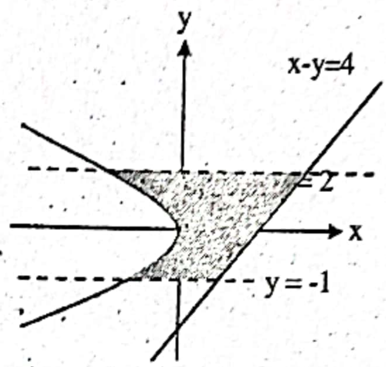
(ii) $y^2 = -x, x - y = 4, y = -1, y = 2$

Solution: Here, $y^2 = -x$, $x - y = 4$, $y = -1$, $y = 2$.

The curve $y^2 = -x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is left openward. And, the line $x - y = 4$ passes through he point (4, 0) and (0, –4)

Also, the line $y = -1$ and $y = 2$ are parallel o the x–axis.

On he bases the region of integration is as shown in the figure.



Here the required region is the shaded part in the corresponding figure.

Taking horizontal strip

$$\text{Area} = \int_{-1}^{2}\int_{-y^2}^{y+4} dx\,dy$$

$$= \int_{-1}^{2} [x]_{-y^2}^{y+4} dy = \int_{-1}^{2} (y+4+y^2)\,dy = \left[\frac{y^2}{2}+4y+\frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{4}{2}+8+\frac{8}{3}\right) - \left(\frac{1}{2}-4-\frac{1}{3}\right)$$

$$= \left(\frac{12+48+16}{6} - \frac{3-24-2}{6}\right) = \frac{76+23}{6} = \frac{99}{6} = \frac{33}{2}$$

Thus, area of the region is $\frac{33}{2}$.

(iii) $y = x, y = 3x, x + y = 4.$

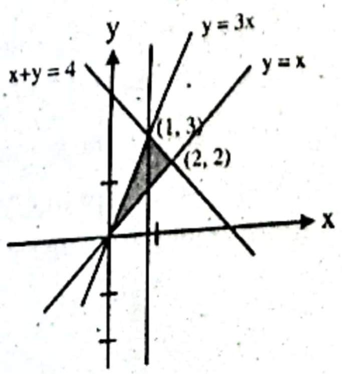
Solution: Here, $y = x$, $y = 3x$, $x + y = 4$.

Here the region of integration is bounded by $y = x$, and by $y = 3x$ and $x + y = 4$.

The line $y = x$ passes through the points (0, 0) and (1, 1). And the line $y = 3x$ is a straight line that passes through the point (0, 0) and (1, 3). Also, the line $x + y = 4$ passes through the points (4, 0) and (0, 4).

On the basis of these boundaries, the region is as in the figure.

Divide the region by $x = 1$. For the region left of $x = 1$, limits for y are $y = x$ to $y = 3x$ and then x varies from $x = 0$ to $x = 1$. Again for the region on the right of $x = 1$, limits for y are $y = x$ to $y = 3x$ and x varies from $x = 1$ to $x = 2$.



Therefore,

$$\text{Area} = \int_{R1}\int + \int_{R2}\int$$

$$= \left[\frac{x^2}{2}\log 2x^2 - \int \frac{1}{2x^2} \times 4x \times \frac{x^2}{2}\,dx - x^2 + \frac{px^2}{2}\right]_0^{a/\sqrt{2}} + [\log(x^2 + \sqrt{a^2 - x^2})\,(\frac{x}{2}$$
$$\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}) - \int \frac{1}{x^2 + \sqrt{a^2 - x^2}}\,(2x - \frac{2x}{\sqrt{a^2 - x^2}})\,(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}$$
$$\sin^{-1}\frac{x}{a})\,dx - 2\frac{x}{2}\sqrt{a^2 - x^2} - \frac{2a^2}{2}\sin^{-1}\frac{x}{a} + \tan^{-1}\frac{\sqrt{a^2 - x^2}}{x} \times 2\frac{x^2}{2} -$$
$$\int \frac{1}{1 + \left(\frac{a^2 - x^2}{x^2}\right)} \cdot \frac{2x^2}{2}\,dx]_{a/\sqrt{2}}^{a}$$

$$= \left[\frac{x^2}{2}\log 2x^2 - \frac{x^2}{2} - x^2 + \frac{px^2}{2}\right]_0^{a/\sqrt{2}} +$$
$$\left[\int\left(\frac{2x}{x^2 + \sqrt{a^2 - x^2}} - \frac{2x}{x^2\sqrt{a^2 - x^2} + (a^2 - x^2)}\right)\right] + x^2\tan^{-1}\frac{\sqrt{a^2 - x^2}}{x}$$
$$\left. - \int \frac{x^2}{a^2} \times x^2\,dx\right]_{a/\sqrt{2}}^{a}$$

$$= \frac{pa^2}{4}\left(\log a - \frac{1}{2}\right).$$

Exercise 9.2

1. Sketch the region bounded by the graph of the equation and find its area using one or more double integral.

(i) $y = \frac{1}{x^2}, y = -x^2, x = 1, x = 2.$

Solution: Here,

For, $y = \frac{1}{x^2}$.

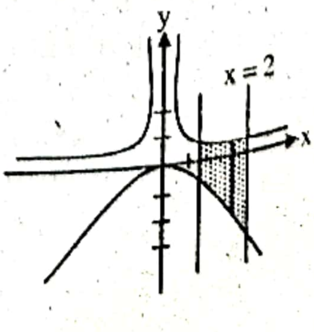
x	0	1	2	-1	-2	+ ½
u	¥	1	¼	1	¼	4

The curve $x^2 = -y$ is a parabola that has vertex at (0, 0) and has line of symmetry $x = 0$. So, the parabola is down openward. And, line $x = 1$ and $x = 2$ are parallel o the y–axis.

On he bases the region of integration is as shown in the figure.

Here the required region is the shaded part in the corresponding figure.

Now, taking vertical strip,



$$\text{Area} = \int_1^2 \int_{-x^2}^{1/x^2} dy\,dx$$

$$= \int_1^2 \int [y]_{-x^2}^{1/x^2}\,dx = \int_1^2 \int \left(\frac{1}{x^2} + x^2\right) dx$$

$$= \int_0^{a/\sqrt{2}} \int [\log (x^2+y^2) \int dy - \int \left\{ \frac{d \log(x^2+y^2)}{dy} \int dy \right\} dy]_0^x \, dx +$$

$$\int_{a/\sqrt{2}}^{a} [\log (x^2+y^2) \int dy - \int \left\{ \frac{d \log(x^2+y^2)}{dy} \int dy \right\} dy]_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \int_0^{a/\sqrt{2}} \int [\log (x^2+y^2) \times y - \frac{1}{x^2+y^2} \times 2y \times y dy]_0^x \, dx +$$

$$\int_{a/\sqrt{2}}^{a} \int [\log (x^2+y^2) \times y - \int \left\{ \frac{1}{x^2+y^2} \times 2y . y \, dy \right\}_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^{a/\sqrt{2}} \int \left[y \log (x^2+y^2) - 2 \int \left(\frac{x^2+y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2} \right) dy \right]_0^x dx +$$

$$\int_{a/\sqrt{2}}^{a} \left[y \log (x^2+y^2) - 2 \int \left(\frac{x^2+y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2} \right) dy \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^{a/\sqrt{2}} \int \left[y \log (x^2+y^2) - 2y + \frac{2x^2}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx +$$

$$\int_{a/\sqrt{2}}^{a} \int \left[y \log (x^2+y^2) - 2y + \frac{2x^2}{x} \tan^{-1} \frac{y}{x} \right]_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^{a/\sqrt{2}} \int \left(x \log 2x^2 - 2x + 2x \times \frac{p}{2} \right) dx +$$

$$\int_{a/\sqrt{2}}^{a} \sqrt{a^2-x^2} \log (x^2 + \sqrt{a^2-x^2}) - 2\sqrt{a^2-x^2} + 2x \tan^{-1} \frac{\sqrt{a^2-x^2}}{x}$$

$$= \left[\log 2x^2 \int x - \int \left(\frac{d \log 2x^2}{dx} \int x \, dx \right) . dx - \frac{2x^2}{2} + p \frac{x^2}{2} \right]_0^{a/\sqrt{2}} +$$

$$[\log (x^2 + \sqrt{a^2-x^2}) \int \sqrt{a^2-x^2} dx - \int \frac{d}{dx} \log (x^2 + \sqrt{a^2-x^2} \int \sqrt{a^2-x^2} -$$

$$2 \int \sqrt{a^2-x^2} \, dx + \tan^{-1} \frac{\sqrt{a^2-x^2}}{x} \int 2x \, dx - \int \frac{d}{dx} \tan^{-1} \sqrt{a^2-x^2} \int 2x . \, dx]_{a/\sqrt{2}}^{a}$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{x=ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy = \left[\left(\frac{a^3}{3} + a\right)\frac{y^4}{4} - \frac{a^3}{3}\cdot\frac{y^7}{7} - a.\frac{y^5}{5}\right]_0^1$$

$$= \left(\frac{a^3}{3} + a\right)\cdot\frac{1}{4} - \frac{a^3}{21} - \frac{a}{5} = a^3\left(\frac{1}{12} - \frac{1}{21}\right) + a\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= a^3\left(\frac{21-12}{252}\right) + a\left(\frac{5-4}{20}\right) = a^3\left(\frac{9}{252}\right) + \frac{a}{20} = \frac{a^3}{28} + \frac{a}{20}$$

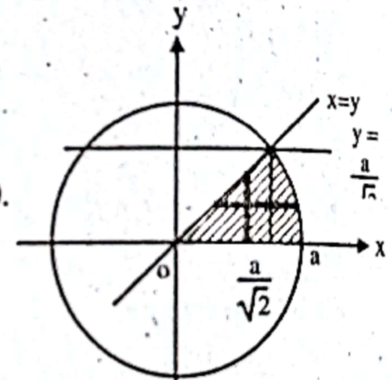
Thus, $I = \frac{a^3}{28} + \frac{a}{20}$

(xiii) $\displaystyle\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\, dy$ **for $a > 0$.** **[2017 Spring Q.No. 3(b)]**

Solution

Given integral is

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\, dy \text{ for } a > 0.$$



Here, the region of integration is

$$0 \le y \le \frac{a}{\sqrt{2}},\ y \le x \le \sqrt{a^2-y^2}$$

Clearly, the region generated by I is the shaded part in the figure that has horizontal strip in which the strip is bounded by $x = y$ and $x = \sqrt{a^2-y^2}$ and it moves from $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

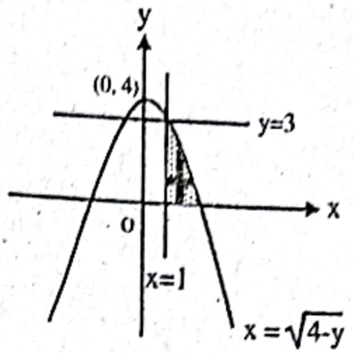
Now, changing the order of integration, the region has vertical strip. From figure it is clearly that the strip is bounded by $y = 0$ and $y = x$ to till when the strip moves form $x = 0$ to $x = \frac{a}{\sqrt{2}}$ and the strip is bounded by $y = 0$ and $y = \sqrt{a^2-x^2}$ when the strip moves from $x = \frac{a}{\sqrt{2}}$ to $x = a$.

Then,

$$I = \int_0^{a/\sqrt{2}} \int_0^x \log(x^2+y^2)\, dy\, dx + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} \log(x^2+y^2)\, dy\, dx$$

$$= \int_0^{a/\sqrt{2}} \int_0^x \log(x^2+y^2)\, dy\, dx + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} \log(x^2+y^2)\, dy\, dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{1}{2}(4-x^2)^2 \right] dx$$



$$= \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2 \right) dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3} \right]_1^2$$

$$= \left(\frac{16}{2} - \frac{16}{4} + 16 - \frac{32}{10} - \frac{32}{3} \right) - \left(\frac{4}{2} - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3} \right)$$

$$= 16\left(\frac{1}{2} - \frac{1}{4} + 1 + \frac{2}{10} - \frac{2}{3} \right) - \left(2 - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3} \right)$$

$$= 16\left(\frac{30-15+60+12-40}{60} \right) - \left(\frac{120-15+480+6-80}{6} \right)$$

$$= \frac{1}{60}[16(102-55) - (606-95)] = \frac{1}{60}(752-511) = \frac{241}{60}$$

Thus, $I = \frac{241}{60}$

(xii) $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$

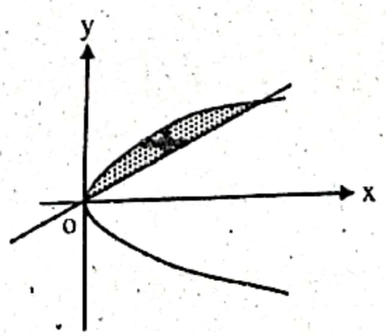
Solution: Given integral is

$$I = \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$$

Here, the region of integration of I is, $0 \le x \le a, \frac{x}{a} \le y \le \sqrt{\frac{x}{a}}$. The line $x = 0$ is y–axis and $x = a$ is parallel to y–axis. Solving the curves $\frac{x}{a} = y$ and $y = \sqrt{\frac{x}{a}}$ we get the point of contacts are (0, 0) and (a, 1).

Clearly, the region generated by I, is the shaded portion in the figure that has vertical strip in which the strip bounded by $y = \frac{x}{a}$ and $y = \sqrt{\frac{x}{a}}$ and it moves from $x = 0$ to $x = a$.

Now, changing the order of integration, the region has horizontal strip that is bounded by $x = ay$ and $x = ay^2$ and the strip moves from $y = 0$ to $y = 1$.



Then,

$$I = \int_0^1 \int_{ay^2}^{ay} (x^2 + y^2)\, dx\, dy$$

Clearly, the region generated by I is the shaded portion in the figure that has vertical strip in which y varies from $y = 0$ to $y = x$ that moves from $x = 0$ to $x \to \infty$.

Now, changing the order of integration, the region has horizontal strip in which x varies from $x = y$ to $x \to \infty$ that moves from $y = 0$ to $y \to \infty$. Then,

$$I = \int_0^\infty \int_y^\infty x\, e^{-x^2/y}\, dy\, dx$$

Put $\frac{x^2}{y} = t$ then $\frac{2x}{y} dx = dt$. And $x = y \Rightarrow t = y$, $x \to \infty \Rightarrow t \to \infty$. So that,

$$I = \int_0^\infty \int_y^\infty e^{-t}\, y \frac{dt}{2} dy = \int_0^\infty \frac{y}{2} \int_y^\infty e^{-t}\, dt\, dy$$

$$= \int_0^\infty \frac{y}{2} \left[\frac{e^{-t}}{-1}\right]_y^\infty dy = \int_0^\infty \frac{y}{2}\left(\frac{0 - e^{-y}}{-1}\right) dy = \frac{1}{2}\int_0^\infty y\, e^{-y}\, dy$$

$$\Rightarrow \quad I = \frac{1}{2}\left[y\left(\frac{e^{-y}}{-1}\right) - (1)\frac{e^{-y}}{(-1)^2}\right]_0^\infty \quad [\because \text{Applying integrating by parts}]$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

Thus, $I = \frac{1}{2}$

(xi). $\int_0^3 \int_1^{\sqrt{4-y}} (x + y)\, dx\, dy$

Solution: Given integral is

$$I = \int_0^3 \int_1^{\sqrt{4-y}} (x + y)\, dx\, dy$$

Here, the region of integration is $0 \le y \le 3$, $1 \le x \le \sqrt{4 - y}$.

Since, $x = \sqrt{4 - y} \Rightarrow x^2 = -(y - 4)$ which is a parabola having vertex at (0, 4) and down-open ward.

Clearly, the region determined by I is the shaded portion has horizontal strip in which the strip is bounded by $x = 1$ and $x = \sqrt{4 - y}$ and it moves from $y = 0$ to $y = 3$.

Now, changing the order of integration of I, the region takes vertical strip that is bounded by $y = 0$ and $y = 4 - x^2$ and it moves from $x = 1$ to $x = 2$.

Then,

$$I = \int_1^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx = \int_1^2 \left[xy + \frac{y^2}{2}\right]_{y=0}^{4-x^2} dx$$

$$= \frac{1}{a}\int_{-a}^{0} \frac{a^2 - y^2}{y^2 - a^2}\,dy = -\frac{1}{a}\int_{-a}^{0} dy = -\frac{1}{a}[y]_{-a}^{0} = -\frac{1}{a}(0 + a) = -1$$

Thus, $I = -1$.

(ix) $\displaystyle\int_0^{¥}\int_x^{¥} \left(\frac{e^{-y}}{y}\right) dy\,dx$

[2017 Fall Q.No. 3 (a), 2016 Fall Q. No. 3(a), 2009 Fall Q. No. 3(a)]

Solution: Given integral is

$$I = \int_0^{¥}\int_x^{¥} \left(\frac{e^{-y}}{y}\right) dy\,dx$$

Here, region of integration is $0 \le x < ¥$, $x \le y < ¥$.

Clearly, the region generated by I is the shaded portion in the figure that has vertical strip that moves from $x = 0$ to x fi ¥.

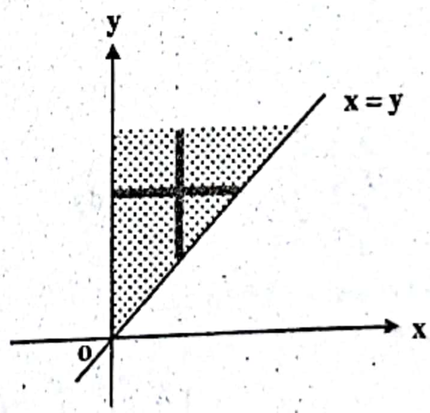
Now, changing the order of integration of I, the region takes horizontal strip in which x varies from $x = 0$ to $x = y$ that moves from $y = 0$ to y fi ¥. Then,

$$I = \int_0^{¥}\int_0^{y} \left(\frac{e^{-y}}{y}\right) dx\,dy$$

$$= \int_0^{¥} \left(\frac{e^{-y}}{y}\right) \int_0^{y} dx\,dy$$

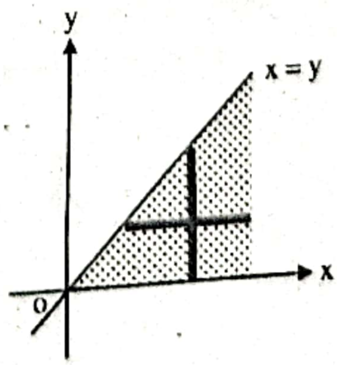
$$= \int_0^{¥} \left(\frac{e^{-y}}{y}\right) [x]_0^{y}\,dy$$

$$= \int_0^{¥} \frac{e^{-y}}{y}\cdot y\,dy = \int_0^{¥} e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^{¥} = \left(\frac{e^{-¥} - e^{0}}{-1}\right) = \left(\frac{0-1}{-1}\right) = 1$$



Thus, $I = 1$.

(x) $\displaystyle\int_0^{¥}\int_0^{x} x\,e^{-x^2/y}\,dy\,dx$



Solution: Given integral is

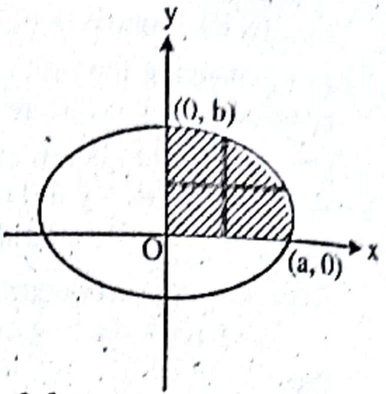
$$I = \int_0^{¥}\int_0^{x} x\,e^{-x^2/y}\,dy\,dx$$

Here, the region of integration is $0 \le y \le x$, $0 \le x < ¥$.

Here, the region of integration be $0 \le y \le b, 0 \le x \le \frac{a\sqrt{b^2 - y^2}}{b}$.

Clearly the required region is the shaded portion that has horizontal strip.
Now, by changing the order of integration, the region takes vertical strip that is bounded by $y = 0$ to $y = \frac{b\sqrt{a^2 - x^2}}{a}$ and then x varies from $x = 0$ to $x = a$.

Then,



$$I = \int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} xy\, dy\, dx$$

$$= \int_0^a x \left[\frac{y^2}{2}\right]_0^{\frac{b\sqrt{a^2-x^2}}{a}} dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{b^2(a^2-x^2)}{a^2}\right] dx \; \frac{1}{2a^2}\int_0^a (a^2b^2x - b^2x^3)\, dx$$

$$= \frac{a^2b^2}{2a^2}\left[\frac{x^2}{2}\right]_0^a - \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{a^4b^2}{4a^2} - \frac{a^4b^2}{8a^2} = \frac{a^2b^2}{4} - \frac{a^2b^2}{8} = \frac{a^2b^2}{8}$$

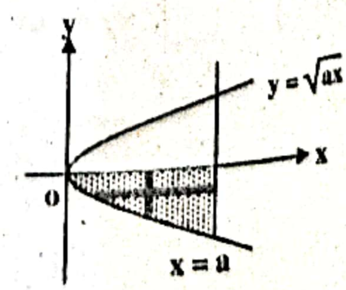
Thus, $I = \frac{a^2b^2}{8}$

(viii) $\displaystyle\int_0^a \int_{\sqrt{ax}}^0 \frac{y^2\, dy\, dx}{y^4 - a^2y^2}$

Solution: Given integral is

$$I = \int_0^a \int_{\sqrt{ax}}^0 \frac{y^2\, dy\, dx}{y^4 - a^2y^2} = \int_0^a \int_{\sqrt{ax}}^0 \frac{dy\, dx}{\sqrt{y^2 - a^2}}$$

Here, the region of integration is, $0 \le x \le a, \sqrt{ax} \le y \le 0$.
Clearly, the region generated by I, is the shaded portion in the figure that has vertical strip.
Now, changing the order of integration of I, the region takes horizontal strip in which x varies from $x = \frac{y^2}{a}$ to $x = a$. For this, the strip moves from $y = 0$ to $y = -a$. Then,



$$I = \int_{-a}^0 \int_{y^2/a}^a \frac{dy\, dx}{y^2 - a^2} = \int_{-a}^0 \left(\frac{1}{y^2 - a^2}\right)[x]_{y^2/a}^a\, dy$$

$$= \int_{-a}^0 \left(\frac{1}{y^2-a^2}\right)\left(a - \frac{y^2}{a}\right) dy$$

Solution: Given integral is

$$I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$$

Here, the region of integration is $\frac{x^2}{a} \le y \le 2a - x;\ 0 \le x \le a$.

Clearly, the required region (shaded part in the figure) has vertical strip.

Now, changing the order of integration, the region takes horizontal strip that is bounded by three different curves. Divide the region of integration by the line $y = a$. For region below the line $y = a$ and limits for x are $x = 0$ to $x = \sqrt{ay}$ and then y varies from $y = 0$ to $y = a$. For the region above the line $y = a$, limits for x are $x = 0$ to $x = 2a - y$ and then for y are $y = a$ to $y = 2a$.

Since, the strip is bounded by $x = 0$ and $x = \sqrt{ay}$ for $y = 0$ to $y = a$ and by $x = 0$, $x = 2a - y$ for $y = a$ to $y = 2a$.

Then,

$$I = \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy + \int_0^{2a} \int_0^{(2a-y)} xy\, dx\, dy$$

$$= \int_0^a a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_0^{2a} y \left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a \frac{a}{2} y^2\, dy + \frac{1}{2}\int_a^{2a} y\,(4a^2 + y^2 - 4ay)\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{4a^2}{2}\left[\frac{y^2}{2}\right]_a^{2a} + \frac{1}{2}\left[\frac{y^4}{4}\right]_a^{2a} - \frac{4a}{2}\left[\frac{y^3}{3}\right]_a^{2a}$$

$$= \frac{a^4}{6} + a^2(4a^2 - a^2) + \frac{1}{8}(16a^4 - a^4) - \frac{4a}{6}(8a^3 - a^3)$$

$$= \frac{a^4}{6} + 3a^4 + \frac{15a^4}{8} - \frac{28a^4}{6} = \frac{8a^4 + 144a^4 + 90a^4 - 224a^4}{48} = \frac{18a^4}{48} = \frac{3a^4}{8}$$

Thus, $I = \frac{3a^4}{8}$.

(vii) $\displaystyle\int_0^b \int_0^{\left(\frac{a\sqrt{b^2-y^2}}{b}\right)} xy\, dx\, dy$

Solution: Given integral is

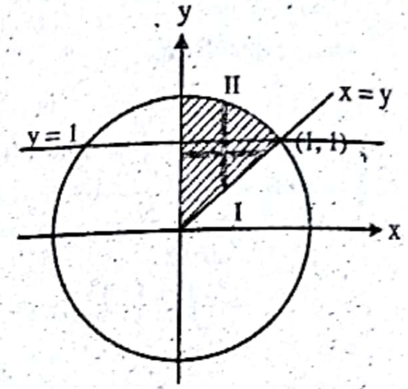
$$I = \int_0^b \int_0^{\left(\frac{a\sqrt{b^2-y^2}}{b}\right)} xy\, dx\, dy$$

The shaded region is region of integration. Divide the region by the line $x = 1$. For the region below $x = 1$, limits for x are $x = 0$ to $x = y$ and then y varies from $y = 0$ to $y = 1$. And, x varies from $x = 0$ to $x = \sqrt{2-y^2}$ for $y = 1$ to $y = \sqrt{2}$. Therefore,

$$I = \int_0^1 \int_0^y \frac{x\, dydx}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\, dxdy}{\sqrt{x^2+y^2}}.$$

Put $x^2 + y^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = y^2$, and $x = y \Rightarrow t = 2y^2$.

Also $x = \sqrt{2-y^2} \Rightarrow t = 2$.



Then,

$$I = \frac{1}{2}\int_0^1 \int_{y^2}^{2y^2} \frac{dt\, dy}{\sqrt{t}} + \frac{1}{2}\int_1^{\sqrt{2}} \int_{y^2}^{2} \frac{dt\, dy}{\sqrt{t}}$$

$$= \frac{1}{2}\int_0^1 \left[\frac{t^{1/2}}{1/2}\right]_{y^2}^{2y^2} dy + \frac{1}{2}\int_1^{\sqrt{2}} \left[\frac{t^{1/2}}{1/2}\right]_{y^2}^{2} dy$$

$$= \int_0^1 [(2y^2)^{1/2} - (y^2)^{1/2}]\, dy + \int_1^{\sqrt{2}} [(2)^{1/2} - (y^2)^{1/2}]\, dy$$

$$= \int_0^1 y(\sqrt{2}-1)\, dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\, dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \left[y\sqrt{2} - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= (\sqrt{2}-1)\frac{1}{2} + \left[\left(2-\frac{2}{2}\right) - \left(\sqrt{2}-\frac{1}{2}\right)\right]$$

$$= \frac{\sqrt{2}-1}{2} + 1 - \sqrt{2} + \frac{1}{2}$$

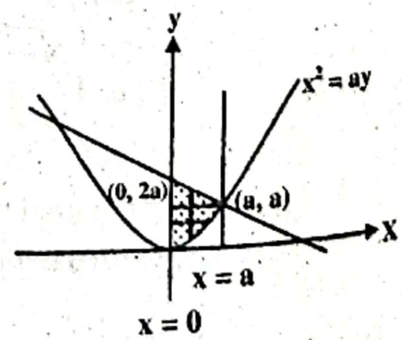
$$= \frac{\sqrt{2}-1+2-2\sqrt{2}+1}{2}$$

$$= \frac{2-\sqrt{2}}{2}$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)$$

Thus, $I = \left(1 - \frac{1}{\sqrt{2}}\right)$.

(vi) $\displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$

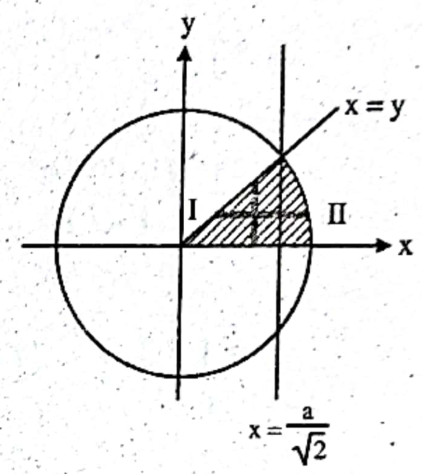


Here, the region of integration is, $y \le x \le \sqrt{a^2 - y^2}$. $0 \le y \le \frac{a}{\sqrt{2}}$.

Clearly, the region of integration is the shaded part over the given region that has horizontal strip.

Divide the region of integration in two parts by $x = \frac{a}{\sqrt{2}}$. For the region left of $x = \frac{a}{\sqrt{2}}$, y varies from y = 0 to y = x and x varies from x = 0 to $x = \frac{a}{\sqrt{2}}$. And for the next region y varies from y = 0 to $y = \sqrt{a^2 - x^2}$ and x varies from $x = \frac{a}{\sqrt{2}}$ to x = a. Therefore,

$$I = \int_{y}^{a/\sqrt{2}} \int_{0}^{x} x\,dy\,dx + \int_{a/\sqrt{2}}^{a} \int_{0}^{\sqrt{a^2-x^2}} x\,dy\,dx$$



$$= \int_{0}^{a/\sqrt{2}} x\,[y]_0^x\,dx + \int_{a/\sqrt{2}}^{a} [y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \int_{0}^{a/\sqrt{2}} x^2\,dx + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx$$

$$= \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3 + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx = \frac{a^3}{6\sqrt{2}} + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx$$

Put $a^2 - x^2 = t^2$ then $-2x\,dx = 2t\,dt$. And, $x = \frac{a}{\sqrt{2}} \Rightarrow t = \frac{a}{\sqrt{2}}$ and $x = a \Rightarrow t = 0$.

Then, $$I = \frac{a^3}{6\sqrt{2}} + \int_{0}^{a/\sqrt{2}} t^2\,dt = \frac{a^3}{6\sqrt{2}} + \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3 = \frac{2a^3}{6\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$$

Thus, $I = \frac{a^3}{3\sqrt{2}}$.

(v) $\int_{0}^{1} \int_{x}^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2+y^2}}$ **[2006 Spring Q. No. 3(a)]**

Solution: Given integral is

$$I = \int_{0}^{1} \int_{x}^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2+y^2}}$$

Here, region of integration is $x \le y \le \sqrt{2-x^2}$, $0 \le x \le 1$.

Clearly, the required region has vertical strip.

$$I = \int_{0}^{4}\int_{y}^{4} \frac{x\,dx\,dy}{x^2 + y^2} \qquad .... (1)$$

Here the region of integration is bounded on the left by x = y, and on the roght by x = 4.

The line x = y passes through the points (0, 0) and (1, 1). And the line x = 4 is a straight line that is parallel to y–axis.

Next, the line y = 0 is x–axis. And the line y = 4 is a straight line that is parallel to x–axis.

On the basis of these boundaries the sketch of region of integration is shown as in figure.

So, the region of integration generated by integral (1) is the shaded portion that has horizontal strip.

In order to change of order of integration we take vertical strip in the shaded region for which y varies from y = 0 to y = x and x moves from x = 0 to x = 4. Therefore,

$$I = \int_{0}^{4}\int_{0}^{x} \frac{x}{x^2 + y^2}\,dy\,dx$$

$$= \int_{0}^{4} x \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_{0}^{x} dx \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \int_{0}^{4} \tan^{-1}\left(\frac{x}{x}\right) dx \qquad [\because \tan^{-1} 0 = 0]$$

$$= \int_{0}^{4} \tan^{-1}(1)\,dx$$

$$= \int_{0}^{4} \frac{p}{4} dx \qquad \left[\because \tan\frac{p}{4} = 1 \Rightarrow \tan^{-1}(1) = \frac{p}{4}\right]$$

$$= \frac{p}{4}[x]_{0}^{4} = \frac{p}{4} \cdot 4 = p.$$

Thus, I = p.

(iv) $\displaystyle\int_{y}^{a\sqrt{2}} \int_{y}^{\sqrt{a^2 - y^2}} x\,dx\,dy$ **[2009 Spring Q. No. 3(a)]**

Solution: Given integral is

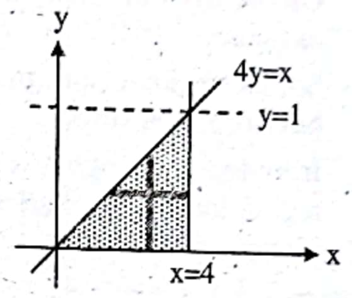
$$I = \int_{y}^{a\sqrt{2}} \int_{y}^{\sqrt{a^2 - y^2}} x\,dx\,dy$$

$$= \int_0^1 \frac{r^3}{2}\left[q - \frac{\sin 2q}{2}\right]_0^{p/2} dr = \int_0^1 \frac{r^3}{2}\frac{p}{2}\, dr = \frac{p}{4}\left[\frac{r^4}{4}\right]_0^1 = \frac{p}{16}$$

(ii) $\int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy$ **[2015 Spring Q.No. 3 (b), 2010 Spring Q. No. 3(a)]**

Solution: Given integral is

$$I = \int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy \quad \text{......... (1)}$$



Here the region of integration is bounded by $x = 4y$, and by $x = 4$.

The line $x = 4y$ passes through the points (0, 0) and (1, 4). And the line $x = 4$ is a straight line that is parallel to y–axis. They meet at (4, 1).

Next, the line $y = 0$ is x–axis. And the line $y = 1$ is a straight line that is parallel to x–axis.

On the basis of these boundaries, the sketch of region of integration is shown as in figure.

Clearly the region generated by (1) is the shaded portion in the corresponding figure that has horizontal strip.

In order to change of order of integration we take vertical strip in the shaded region for which y varies from $y = 0$ to $y = \frac{x}{4}$. and x moves from $x = 0$ to $x = 4$.

Then,

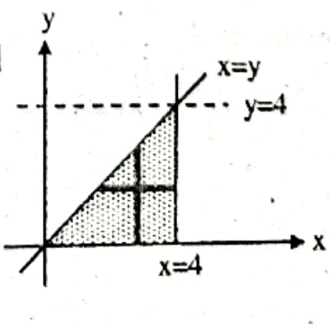
$$I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx = \int_0^4 e^{x^2}\, [y]_0^{x/4}\, dx = \frac{1}{4}\int_0^4 x\, e^{x^2}\, dx.$$

Put $x^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 4 \Rightarrow t = 16$. So that,

$$I = \frac{1}{4}\int_0^{16} e^t \frac{dt}{2} = \frac{1}{8}[e^t]_0^{16} = \frac{e^{16} - 1}{8} \qquad [\because e^0 = 1]$$

Thus, $I = \dfrac{e^{16} - 1}{8}$.

(iii) $\int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2}$ **[2018 Fall Q.No. 3(a)]**



Solution: Given integral is

$$I = \int_{y=1}^{2}\int_{x=1}^{2} f(x, y)\, dx\, dy$$

$$= \int_{y=1}^{2}\int_{x=1}^{2} \left(\frac{1}{xy}\right) dx\, dy$$

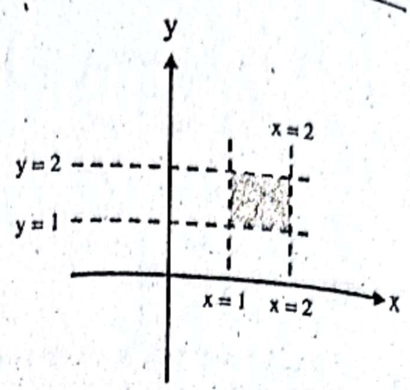
$$= \int_{y=1}^{2} \frac{1}{y} [\log x]_1^2\, dy$$

$$= \log (2) \int_{y=1}^{2} \frac{dy}{y} \qquad [\because \log(1) = 0]$$

$$= \log (2) [\log (y)]_1^2$$

$$= \log (2) \log (2) \qquad [\because \log (1) = 0]$$

$$= [\log (2)]^2$$



Thus, $I = [\log (2)]^2$

4. Evaluate, the following integrals by changing the order of integration.

(i) $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$

Solution: Given integral is

$$I = \int_0^1\int_0^{\sqrt{1-x^2}} y^2\, dy\, dx \qquad \text{......... (1)}$$

Here, the region of integral is R: $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$.

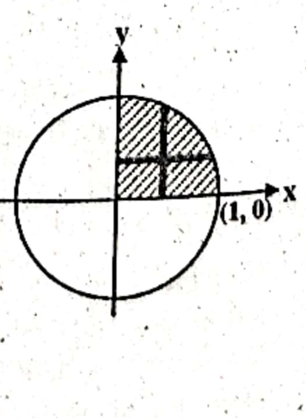
Since the region $y = \sqrt{1-x^2}$ is a circle having radius $r = 1$. And, $y = 0$ and $x = 0$ are the axes.

On the bases the region of integration is the shaded portion in the figure.

Put $x = r \cos q$, $y = r \sin q$. Then $dx\, dy = r\, dr\, dq$. Then by the figure $r = 0$, $r = 1$ and $q = 0$, $q = \frac{p}{2}$. Then (i) becomes,

$$I = \int_{r=0}^{1}\int_{q=0}^{p/2} r^2 \sin^2 q\, r\, dq\, dr$$

$$= \int_{r=0}^{1} r^3 \int_{q=0}^{p/2} \frac{1-\cos 2q}{2}\, dq\, dr$$



$$= \int_1^2 x \log\left(\frac{2x}{x}\right) dx = \log 2 \left[\frac{x^2}{2}\right]_1^2$$

$$= \log 2 \left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{2} \log 2.$$

Thus, $I = \frac{3}{2} \log 2.$

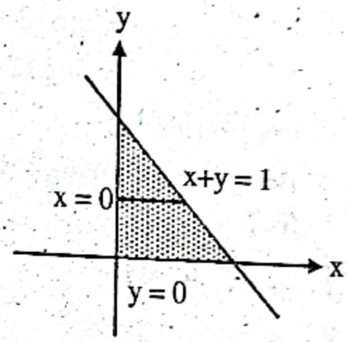
(ii) If $f(x, y) = y - \sqrt{x}$ over the triangular region cut from the first quadrant by the lines $x + y = 1$.

Solution: Given that the region is in the first quadrant bounded by the lines $x + y = 1$.

Here the region of integration is bounded by the line $x + y = 1$ that is passing through (1, 0) and (0, 1). Also, the region is bounded by the axes. On these basis the region of integration is as shown in figure.

Now, taking vertical strip then we observe that the strip is bounded below by $y = x$ and above by $y = 2x$. And the strip moves from $x = 1$ to $x = 2$. Therefore,

$$\int_0^1 \int_0^{1-y} (y - \sqrt{x})\, dx\, dy$$



$$= \int_0^1 \left\{ y\,[x]_0^{1-y} - \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{1-y} \right\} dy$$

$$= \int_0^1 \left(y(1-y) - \frac{2}{3}(1-y)^{\frac{3}{2}} \right) dy$$

$$= \int_0^1 \left[y - y^2 - \frac{2}{3}(1-y)^{\frac{3}{2}} \right] dy = \left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{2}{3}\frac{(1-y)^{5/2}}{5/2\,(-1)} \right]_0^1$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{4}{15}(1-y)^{5/2} \right]_0^1 = \frac{1}{2} - \frac{1}{3} - \frac{4}{15}$$

$$= \frac{15 - 10 - 8}{30} = \frac{-3}{30} = \frac{-1}{10}$$

Thus, $\int_0^1 \int_0^{1-y} (y - \sqrt{x})\, dx\, dy = \frac{-1}{10}.$

(iii) $f(x, y) = \frac{1}{xy}$ over the rectangle R: $1 \le x \le 2,\ 1 \le y \le 2$.

Solution: Given that, $f(x, y) = \frac{1}{xy}$ over the region R: $1 \le x \le 2,\ 1 \le y \le 2$.

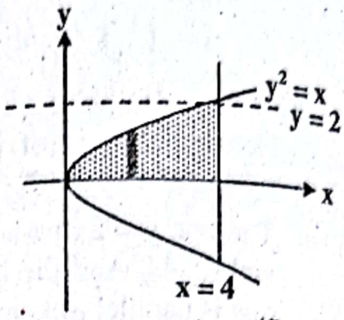
Here the region of integration is bounded below by $y = 1$, above by $y = 2$, on the left by $x = 1$ and on the right by $x = 2$. On these bases the region of integration is as shown in figure.

Now, taking vertical strip then we observe that the strip is bounded below by $y = x$ and above by $y = 2x$. And the strip moves from $x = 1$ to $x = 2$. Therefore, Also, the region is bounded below by $y = 0$ and above by $y = 2$.

On these bases the region of integration is as shown in figure.

Clearly, the region generated by (1) is the shaded portion that has horizontal strip, as shown in figure 1.

Now interchanging the order of integration of (1), the integral takes the vertical strip as in figure 2 in which y varies from $y = 0$ to $y = \sqrt{x}$, and y moves from $x = 0$ to $x = 4$.



(fig. 2)

Therefore (1) reduces to

$$I = \int_0^4 \int_0^{\sqrt{x}} y \cos(x^2)\, dy\, dx = \int_0^4 \cos(x^2) \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= \int_0^4 \cos(x^2) \cdot \frac{x}{2} dx$$

Put $x^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 4 \Rightarrow t = 6$. Then,

$$I = \int_0^6 \cos t \frac{dt}{4} = \frac{1}{4}[\sin t]_0^6 = \frac{\sin(6)}{4}$$

Thus, $I = \dfrac{\sin(6)}{4}$.

3. Integrate f(x, y) over the region R:

(i) $f(x, y) = \dfrac{x}{y}$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x, x = 1, x = 2$.

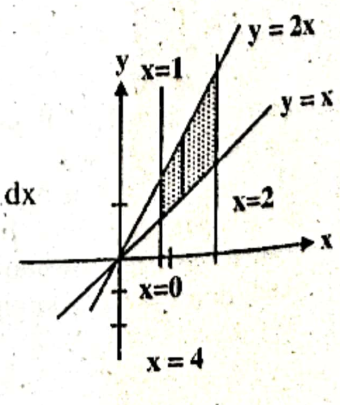
Solution: Given that the region of integration is in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$.

The line $y = 2x$ passes through (0, 0) and (2, 1). Also, the line $y = x$ passes through (0, 0) and (1, 1). On these basis the region of integration is as shown in figure.

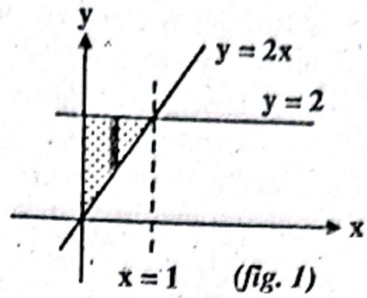
Now, taking vertical strip then we observe that the strip is bounded below by $y = x$ and above by $y = 2x$. And the strip moves from $x = 1$ to $x = 2$. Therefore,

$$I = \int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx = \int_1^2 x \int_x^{2x} \frac{1}{y}\, dy\, dx$$

$$= \int_1^2 x\, [\log y]_x^{2x}\, dx = \int_1^2 x\,(\log 2x - \log x)\, dx$$



$$I=\int_{0}^{1}\int_{2x}^{2} e^{y^2}\,dy\,dx \quad ....(1)$$



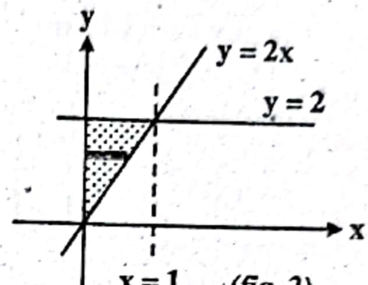
(fig. 1)

Here the region of integration is bounded by $y = 2x$, and by $y = 2$.

The line $y = 2x$ passes through the points (0, 0) and (2, 1). And the line $y = 2$ is a straight line that is parallel to x–axis.

Next, the line $x = 0$ is y–axis. And the line $x = 1$ is a straight line that is parallel to y–axis.

On the basis of these boundaries the sketch of figure is shown as in fig–1.



(fig. 2)

Clearly, the required region generated by the integral (1) is the shaded portion that has vertical strip as shown in figure 1.

Now, to reverse the order of integration, region has horizontal strip as in figure 2 in which x varies from $x = 0$ to $x = \frac{y}{2}$ and y moves from $y = 0$ to $y = 2$. Then (1) becomes,

$$I=\int_{0}^{2}\int_{0}^{y/2} e^{y^2}\,dx\,dy = \int_{0}^{2} e^{y^2}\cdot\frac{y}{2}\,dy$$

Put, $y^2 = t$ then $2y\,dy = dt$. Also, $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow t = 4$. Then,

$$I=\frac{1}{2}\int_{0}^{4} e^{t}\frac{dt}{2} = \frac{1}{4}\int_{0}^{4} e^{t}\,dt = \frac{1}{4}\left[e^{t}\right]_{0}^{4} = \frac{1}{4}(e^{4}-1).$$

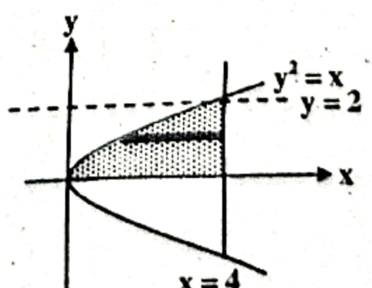
Thus, $I = \dfrac{(e^4-1)}{4}$

(vii) $\displaystyle\int_{0}^{2}\int_{y^2}^{4} y\cos(x^2)\,dx\,dy$ **[2011 Spring Q. No. 3(a)] [2011 Fall Q. No. 3(a)]**

[2014 Fall Q. No. 3(a)] [2013 Spring Q. No. 3(a)]

Solution: Given integral is

$$I=\int_{0}^{2}\int_{y^2}^{4} y\cos(x^2)\,dx\,dy \quad .... (1)$$



(fig. 1)

Here the region of integration is bounded by $x = y^2$, and by $x = 4$.

The curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry as $y = 0$. So, the parabola is right openward.

And, line $x = 4$ is a straight line that is parallel to y–axis.

$$I_2 = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Then (2) becomes,

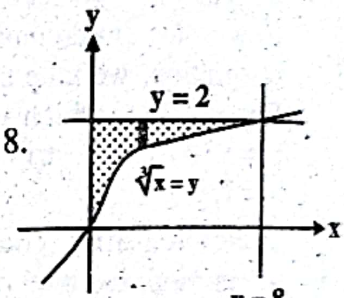
$$I = \frac{e-1}{2} - \frac{1}{2} = \frac{e-2}{2}$$

Thus, $I = \frac{e-2}{2}$

(v) $\int_0^8 \int_{x^{1/3}}^2 \left(\frac{1}{1+y^4}\right) dy\,dx$

Solution: Given integral is

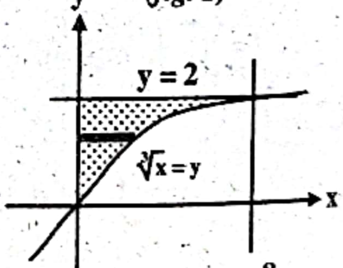
$$I = \int_0^8 \int_{x^{1/3}}^2 \left(\frac{1}{1+y^4}\right) dy\,dx \quad \text{........ (1)}$$



(fig. 1)

Here the region of integration is $\sqrt[3]{x} \le y \le 2, 0 \le x \le 8$. Clearly, the region generated by (1) the shaded portion that has vertical strip, as shown in figure 1.

Now, interchanging the order of integration, we get the region has horizontal strip as in figure 2 in which x varies from $x = 0$ to $x = y^3$ and y moves from $y = 0$, to $y = 2$. Thus, after interchanging the order of integration (1) deduces to



(fig. 2)

$$I = \int_0^2 \int_0^{y^3} \left(\frac{1}{1+y^4}\right) dx\,dy$$

$$= \int_0^2 \left(\frac{1}{1+y^4}\right) \int_0^{y^3} dx\,dy = \int_0^2 \frac{1}{1+y^4} \cdot y^3\,dy$$

Put $y^4 = t$ then $4y^3\,dy = dt$. Also, $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow y = 16$. Then,

$$I = \int_0^{16} \left(\frac{1}{1+t}\right) \cdot \frac{dt}{4} = \frac{1}{4}[\log(1+t)]_0^{16} = \frac{1}{4}\log(17) \quad [\because \log(1) = 0]$$

Thus, $I = \frac{1}{4}\log(17)$.

(vi) $\int_0^1 \int_{2x}^2 e^{y^2}\,dy\,dx$

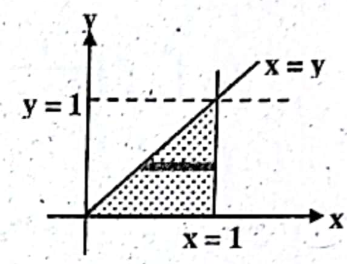
Solution: Given integral is

(iv) $\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$ [2002 Q. No. 3(a)]

Solution: Given integral is,

$$I = \int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy \quad \text{..........} (1)$$

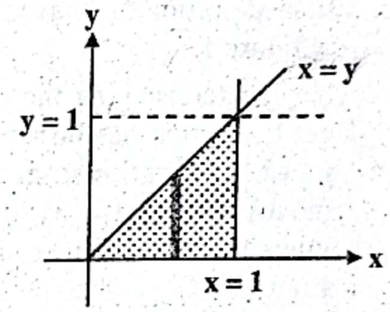
Here the region of integration is bounded by $x = y$, and by $x = 1$.
The line $y = x$ passes through the points (0, 0) and (1, 1). And the line $x = 1$ is a straight line that is parallel to y–axis.
Next, the line $y = 0$ is x–axis. And the line $y = 1$ is a straight line that is parallel to x–axis.
On the basis of these boundaries the sketch of figure is shown as in fig–1.
Clearly, the region generated by (1) is the shaded portion that has horizontal strip, as in figure 1.



(fig. 1)

Now, to interchange the order of integration, we take the vertical strip as in figure – 2 for which y varies from $y = 0$ to $y = x$. Also, the strip moves from $x = 0$ to $x = 1$.

Therefore, after changing the order of integration, the integral (1) becomes,



(fig. 2)

$$I = \int_0^1 \int_0^x x^2 e^x\, dy\, dx$$

$$= \int_0^1 x^2 \int_0^x e^{xy}\, dy\, dx$$

$$= \int_0^1 x^2 \left[\frac{e^{xy}}{x}\right]_0^x dx = \int_0^1 x^2 \left(\frac{e^{x^2} - 1}{x}\right) dx = \int_0^1 x e^{x^2}\, dx - \int_0^1 x \cdot dx$$

$$= I_1 - I_2$$

Here,

$$I_1 = \int_0^1 x e^{x^2}\, dx$$

Put, $x^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 1 \Rightarrow t = 1$. Then,

$$I_1 = \frac{1}{2}\int_0^1 e^t\, dt = \frac{1}{2}[e^t]_0^1 = \frac{e^1 - e^0}{2} = \frac{e-1}{2}$$

and

$$=\frac{1}{2}\int_{x=-2}^{2}\left[\frac{1}{2}(4-x^2)-0\right]dx$$

$$=\frac{1}{4}\int_{x=-2}^{2}(4-x^2)\,dx=\frac{1}{4}\left[4-\frac{x^3}{3}\right]_{-2}^{2}=\frac{1}{4}\left[\left(8-\frac{8}{3}-\left(-8+\frac{8}{3}\right)\right)\right]$$

$$=\frac{1}{4}\cdot\frac{32}{3}=\frac{8}{3}$$

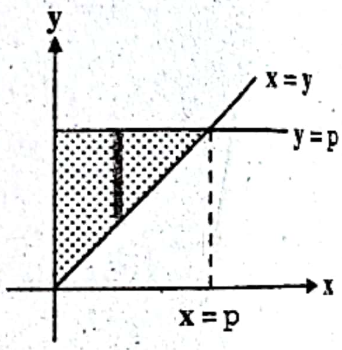
Thus, $I=\frac{8}{3}$.

(iii) $\int_0^p\int_x^p\left(\frac{\sin y}{y}\right)dy\,dx$ **[2017 Spring Q.No. 3(a), 2004 Spring Q. No. 3(a)]**

Solution: Given integral is

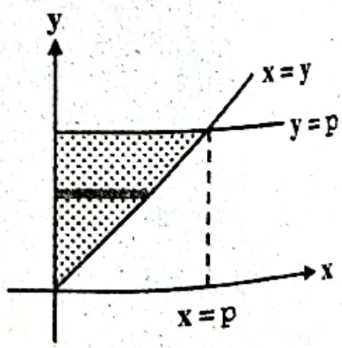
$$I=\int_0^p\int_x^p\left(\frac{\sin y}{y}\right)dy\,dx \quad \text{......... (1)}$$

Here the region of integration is bounded below by $y = x$, and above by $y = p$.
The line $y = x$ passes through the points (0, 0) and (1, 1). And the line $y = p$ is a straight line that is parallel to x–axis and meets $y = x$ at $(-p, p)$.
Next, the region is bounded on the left by line $x = 0$ is y–axis and on the right by the line $x = p$ is a straight line that is parallel to y–axis.
Then, the region generated by (1) is the shaded portion that has vertical strip, as shown in figure–1.



(fig. 1)

Now, to reverse the order of integration, we take the horizontal strip as in figure 2 for which x varies from $x = 0$ to $x = y$. Also, the strip moves from $y = 0$ to $y = p$. Therefore, after changing the order of integration of (1), it becomes,



(fig. 2)

$$I=\int_{y=0}^{p}\int_{x=0}^{y}\left(\frac{\sin y}{y}\right)dx\,dy$$

$$=\int_{y=0}^{p}\left(\frac{\sin y}{y}\right)\int_{x=0}^{y}dx\,dy=\int_{y=0}^{p}\left(\frac{\sin y}{y}\right)\cdot[x]_0^y\,dy$$

$$=\int_{y=0}^{p}\left(\frac{\sin y}{y}\cdot y\right)dy=\int_{y=0}^{p}\sin y\,dy=[-\cos y]_0^p=-[-1-1]=2$$

Thus, $I = 2$.

by x = 2. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_0^2 \int_1^{e^x} dy\,dx = \int_0^2 [y]_1^{e^x} dx$$

$$= \int_0^2 (e^x - 1)\,dx = [e^x - x]_0^2 = (e^2 - 2 - e^0 + 0) = e^2 - 3.$$

Thus, $I = \int_0^2 \int_1^{e^x} dy\,dx = e^2 - 3.$

(ii) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$

Solution: Given integral be

$$I = \int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy \qquad \ldots (1)$$

Here, the region is given by $-\sqrt{4-2y^2} \le x \le \sqrt{4-2y^2}$; $0 \le y \le \sqrt{2}$.

Since,

$$x = \sqrt{4-2y^2}$$

$$\Rightarrow x^2 = 4 - 2y^2$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1,$$

which is an ellipse having centre at (0, 0) and has vertex at ($\pm$ 2, 0).

Thus, the integral (1) has the region is the shaded portion as shown in the figure–1, that has horizontal strip.

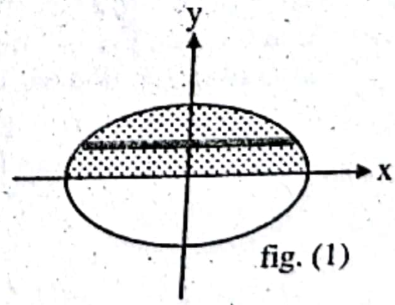


fig. (1)

Now, reversing the order of integration we take the vertical strip as in figure–2 for which y varies from y = 0 to the ellipse $y = \sqrt{\frac{1}{2}(4-x^2)}$. Also, the strip moves from x = – 2 to x = 2 (these are x-coordinates of vertices of the ellipse). Then,

$$I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{\frac{1}{2}(4-x^2)}} y\,dy\,dx$$

$$= \int_{x=-2}^{2} \left[\frac{y^2}{2}\right]_0^{\sqrt{\frac{1}{2}(4-x^2)}} dx$$

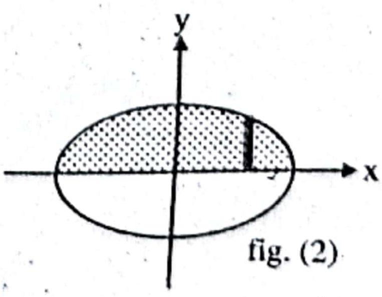
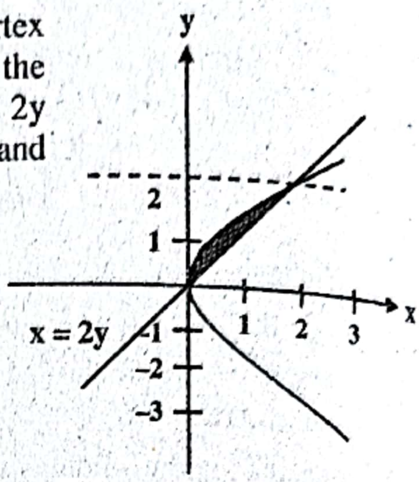


fig. (2)

$$I = \int_0^2 \int_{y^2}^{2y} (4x - y)\, dx\, dy$$

Here the region of integration is bounded on the left by $x = y^2$, and on the right by $x = 2y$.

Since the curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward. And, line $x = 2y$ passes through the points (0, 0) and (2, 1) and meets the parabola at (2, 1).



Also, the region is bounded on below by $y = 0$ and on above by $y = 2$.

On these bases the region of integration is as shown in figure.

Now,

$$I = \int_0^2 \int_{y^2}^{2y} (4x - y)\, dx\, dy = \int_0^2 \left[\frac{4x^2}{2} - xy\right]_{y^2}^{2y} dy$$

$$= \int_0^2 \left\{\left(4\left(\frac{4y^2}{2}\right) - 2y^2\right) - \left(\frac{4y^4}{2} - y^3\right)\right\} dy$$

$$= \int_0^2 (8y^2 - 2y^2 - 2y^4 + y^3)\, dy = \int_0^2 (6y^2 - 2y^4 + y^3)\, dy$$

$$= \left[6\left(\frac{y^3}{3}\right) - 2\left(\frac{y^5}{5}\right) + \frac{y^4}{4}\right]_0^2$$

$$= \left(16 - \frac{64}{5} + 4\right) = \frac{80 - 64 + 20}{5} = \frac{36}{5}.$$

Thus, $I = \int_0^2 \int_{y^2}^{2y} (4x - y)\, dx\, dy = \frac{36}{5}$.

(2) Evaluate, the integral by interchanging the equivalent integral obtained by reversing the order of integration if necessary.

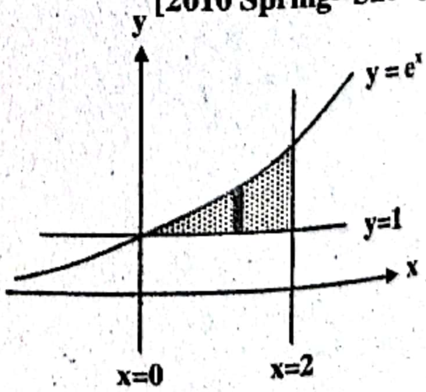
(i) $\int_0^2 \int_1^{e^x} dy\, dx$

[2010 Spring– Short]

Solution: Given integral be

$$I = \int_0^2 \int_1^{e^x} dy\, dx \quad \ldots (1)$$

Here, the region of integration is bounded below by $y = 1$, above by $y = e^x$, on the left by $x = 0$ and the right



Thus, $I = \int_{0}^{1}\int_{0}^{1/y} y e^{xy}\, dx\, dy = 9 - 9e.$

vii) $\int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^2y\, dy\, dx$

olution: Given integral is

$$I = \int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^2y\, dy\, dx$$

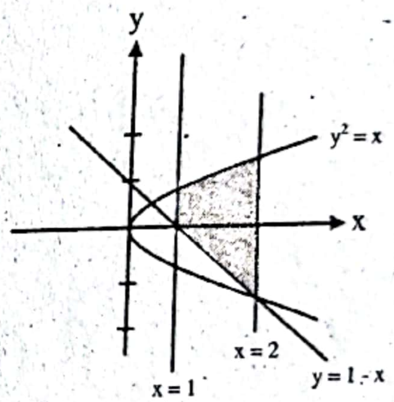
Here the region of integration is bounded by $(1 - x) \le y \le \sqrt{x}$, $1 \le x \le 2$.

Since the line $y = 1 - x$ passes through the points (0, 1) and (1, 0). And curve $y = \sqrt{x}$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward.

Also, the region is bounded on the left by $x = 1$ and on the right by $x = 2$.

On these bases the region of integration is as shown in figure.

Now,



$$I = \int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^2 y\, dy\, dx$$

$$= \int_{1}^{2} x^2 \left[\frac{y^2}{2}\right]_{1-x}^{\sqrt{x}} dx$$

$$= \int_{1}^{2} x^2 \left[\frac{x}{2} - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_{1}^{2} x^2 \left(\frac{x - 1 + 2x - x^2}{2}\right) dx = \frac{1}{2}\int_{1}^{2} x^2 (3x - 1 - x^2)\, dx$$

$$= \frac{1}{2}\int_{1}^{2} (3x^3 - x^2 - x^4)\, dx = \frac{1}{2}\left[3\left(\frac{x^4}{4}\right) - \frac{x^3}{3} - \frac{x^5}{5}\right]_1^2$$

$$= \frac{1}{2}\left[\left(\frac{48}{4} - \frac{8}{3} - \frac{32}{5}\right) - \left(\frac{3}{4} - \frac{1}{3} - \frac{1}{5}\right)\right] = \frac{163}{120}$$

Thus, $I = \int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^2 y\, dy\, dx = \frac{163}{120}.$

ii) $\int_{0}^{2}\int_{y^2}^{2y} (4x - y)\, dx\, dy$

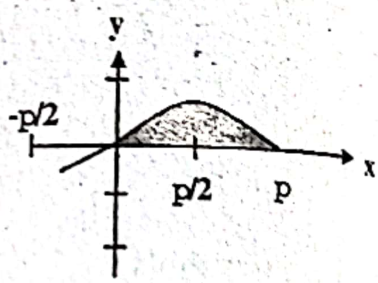
ution: Given integral is

(v) $\int_0^{p}\int_0^{\sin x} y\,dy\,dx$

Solution: Given integral is

$$I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx$$

Here the region of integration is bounded below by $y = 0$, above by $y = \sin x$, on the left by $x = 0$ and on the right by $x = p$. On these bases the region of integration is as shown in figure.



Now,

$$I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx = \int_0^{p}\left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

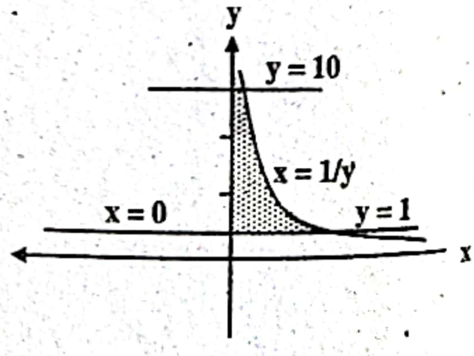
$$= \int_0^{p}\frac{\sin^2 x}{2}\,dx$$

$$= \frac{1}{2}\int_0^{p}\sin^2 x\,dx = \frac{1}{2}\int_0^{p}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{p} = \frac{1}{4}[p-0] = \frac{p}{4}$$

Thus, $I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx = \frac{p}{4}$.

(vi) $\int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy$



Solution: Given integral is

$$I = \int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy$$

Here the region of integration is bounded left by $x = 0$, right by $x = \frac{1}{y}$, on above by $y = 10$ and on below by $y = 1$. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy = \int_{10}^{1} y\left[\frac{e^{xy}}{y}\right]_0^{1/y} dy$$

$$= \int_{10}^{1}(e^1 - e^0)\,dy = (e-1)\int_{10}^{1} dy = (e-1)\,[y]_{10}^{1}$$

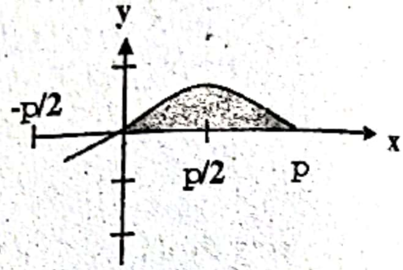
$$= (e-1)(-9) = 9 - 9e.$$

(v) $\int_0^{p}\int_0^{\sin x} y\,dy\,dx$

Solution: Given integral is

$$I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx$$

Here the region of integration is bounded below by y = 0, above by y = sin x, on the left by x = 0 and on the right by x = p. On these bases the region of integration is as shown in figure.



Now,

$$I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx = \int_0^{p}\left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

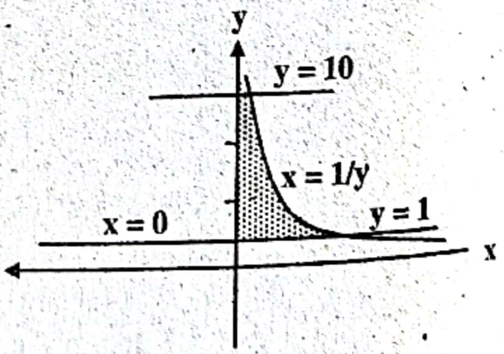
$$= \int_0^{p}\frac{\sin^2 x}{2}\,dx$$

$$= \frac{1}{2}\int_0^{p}\sin^2 x\,dx = \frac{1}{2}\int_0^{p}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{p} = \frac{1}{4}[p-0] = \frac{p}{4}$$

$$\text{Thus, } I = \int_0^{p}\int_0^{\sin x} y\,dy\,dx = \frac{p}{4}.$$

(vi) $\int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy$



Solution: Given integral is

$$I = \int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy$$

Here the region of integration is bounded left by x = 0, right by $x = \frac{1}{y}$, on above by y = 10 and on below by y = 1. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_{10}^{1}\int_0^{1/y} y\,e^{xy}\,dx\,dy = \int_{10}^{1} y\left[\frac{e^{xy}}{y}\right]_0^{1/y} dy$$

$$= \int_{10}^{1}(e^1 - e^0)\,dy = (e-1)\int_{10}^{1}dy = (e-1)\,[y]_{10}^{1}$$

$$= (e-1)(-9) = 9 - 9e.$$

CHAPTER 9

DOUBLE INTEGRAL

Fubini's Theorem – First Form

Let f(x, y) is continuous on the region R: a £ x £ b, c £ y £ d then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Note: This will possible only if all four limiting values are constant.

Fubini's Theorem – Second Form

Let f(x, y) is continuous on the region R: a £ x £ b, g_1 (x) £ y £ g_2 (x) then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy$$

Exercise 9.1

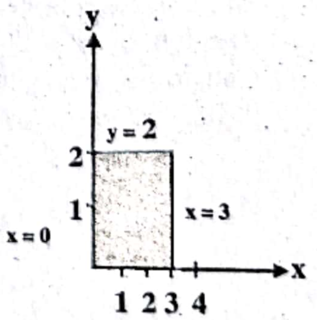
1. Evaluate the following integrates and sketch the region over which each integration takes place.

(i) $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$

Solution: Given integral is

$$I = \int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$$

Here the region of integration is bounded below by y = 0, above by y = 2, on the left by x = 0 and on the right by x = 3. On these bases the region of integration is as shown in figure.



Now,

$$I = \int_0^3 \int_0^2 (4 - y^2)\, dy\, dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_0^3 \left(8 - \frac{8}{3}\right) dx$$

$$= \frac{16}{3} \int_0^3 dx = \frac{16}{3}\, [x]_0^3 = \frac{16}{3}(3 - 0) = 16.$$