FOURIER SERIES

Periodic function

A function f(x) is said to be periodic, if there exists a nonzero positive number p such that f(x) = f(x + p) for all x, where p is said to be period of f(x).

Even and odd functions:

A function f(x) is said to be even if f(–x) = f(x) for all x.

A function f(x) is said to be odd if f(–x) = – f(x) for all x.

Some results on odd and even functions:

(a) Product of two odd functions is even. [2011 Fall – Short] [2003 Fall – Short]

Proof: Let f(x) and g(x) are odd functions for all x.

Then $f(-x) = -f(x)$ and $g(-x) = -g(x)$.

Now,

$$f(-x)\,.\,g(-x) = \{-f(x)\}\,\{-g(x)\} = f(x)\,g(x).$$

This shows that the product of two odd functions is an even function.

(b) Product of odd and even functions is odd.

Proof: Let f(x) is an odd and g(x) is an even function for all x. Then,

$$f(-x) = -f(x) \text{ and } g(-x) = g(x)$$

Now,

$$f(-x).\,g(-x) = \{-f(x)\}.\,g(x) = -\{f(x).\,g(x)\}$$

This shows that the product of odd and even functions is an odd function.

(c) Product of two even functions is even.

Proof: Let f(x) and g(x) are even functions for all x. Then,

$$f(-x) = f(x) \text{ and } g(-x) = g(x)$$

Now,

$$f(-x)\,.\,g(-x) = f(x).\,g(x)$$

This shows that the product of two even functions is even.

(d) If f(x) is an even function then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Proof: Let f(x) is an even function. Then f(–x) = f(x), for all x.

Now,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

...e x = – t for first integral on the right,

$$= \int_{a}^{0} f(-t)\,(-dt) + \int_{a}^{0} f(x)\,dx$$

$$= \int_0^a f(-t)\,dt + \int_0^a f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \qquad [\because f(-x) = f(x)]$$

$$= 2\int_0^a f(x)\,dx$$

Thus, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ whenever $f(x)$ is an even.

(e) **If $f(x)$ is an odd function then $\int_{-a}^{a} f(x)\,dx = 0$.**

Proof: Let $f(x)$ is an odd function. Then $f(-x) = -f(x)$ for all x.
Then by (d),

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

Thus, $\int_{-a}^{a} f(x)\,dx = 0$ whenever $f(x)$ is an odd function.

Fourier series

Let $f(x)$ be a periodic function of period 2π and is integrable over a finite interval. Then a trigonometric series,

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots S(1)$$

is called a Fourier series of $f(x)$ if the series (1) converges uniformly to $f(x)$. And, the coefficients are called Euler's coefficients.

Determination of Euler's coefficients for 2π periodic function:

Let the Fourier series of $f(x)$ with period 2π is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots\ldots(1)$$

Integrating both sides of (1) over $(-\pi, \pi)$ then

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= a_0 \cdot 2\pi + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right] \quad \ldots\ldots (2)$$

Since,

$$\int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0 \qquad [\because \sin n\pi = 0]$$

And

$$\int_{-\pi}^{\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} = -\frac{1}{n}[\cos n\pi - \cos(-n\pi)]$$

$$= -\frac{1}{n}[\cos n\pi - \cos n\pi] = 0$$

Therefore (2) becomes,

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 \;\Rightarrow\; a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \ldots\ldots\ldots (3)$$

Next, multiplying both sides of (1) by $\cos mx$ and then taking integration over $(-\pi, \pi)$ then

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0 \int_{-\pi}^{\pi} \cos mx\,dx +$$

$$\sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \cos mx \sin nx\,dx\right] \quad \ldots (4)$$

Since,

$$\int_{-\pi}^{\pi} \cos mx\,dx = \left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi} = 0$$

And,

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} \qquad \text{for } \sin n\pi = 0$$

$$= \begin{bmatrix} 0 & \text{for } n \neq m \\ \text{in 0/0 form} & \text{for } n = m \end{bmatrix}$$

So for $n = m$,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\left[\frac{\cos (n-m)x.\, x}{1}\right]_{-\pi}^{\pi} \quad [\because \text{applying L-hopital rule for } n = m]$$

$$= \frac{1}{2}.\, 2\pi = \pi.$$

Also,

$$\int_{-\pi}^{\pi} \cos mx \sin x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin (n+m)x + \sin(n-m)x] \, dx$$

$$= \frac{1}{2}\left[\frac{-\cos (n+m)x}{n+m} - \frac{\cos (n-m)x}{n-m}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2(n+m)}[\cos(n+m)\pi - \cos((n+m)(-\pi))] - \frac{1}{2(n-m)}[\cos(n-m)\pi - \cos((n-m)(-\pi))]$$

$$= -\frac{1}{2(n-m)}[\cos(n+m)\pi - \cos(n+m)\pi] - \frac{1}{2(n-m)}[\cos(n-m)\pi - \cos(n-m)\pi] = 0.$$

Thus, (4) gives us,

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_0 0 + \sum_{n=1}^{\infty} (a_n.\, \pi + b_n.0) \quad \text{at } n = m$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n.\, \pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{for } n = 1, 2, 3, \ldots\ldots \quad \ldots..(5)$$

Similarly, if we multiply (1) by sin mx and then taking integration over $(-\pi, \pi)$ then we get,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad \text{for } n = 1, 2, 3, \ldots.. \quad \ldots\ldots.(6)$$

Thus, the value in (3), (5) and (6) are required Euler's coefficients of (1).

Fourier sine and cosine series of period 2π.

We have the Fourier series of a function f(x) of period 2π is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \ldots\ldots.(1)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin x \, dx$

First we assume that f(x) is an odd function. So, f(x) cosnx is odd and f(x) sinx is even. Then

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} f(x) \sin x \, dx = 2\int_{0}^{\pi} f(x) \sin x \, dx$$

Also, $\int_{-\pi}^{\pi} f(x)\, dx = 0.$

Thus, we observe, $a_n = 0$, $a_0 = 0$.

and, $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$

Therefore (1) becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots..(2)$$

where, $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$

Here (2) is known as Fourier sine series with value of b_n.

Next, we assume that f(x) is even. Then f(x) cos nx is even and f(x) sin nx is odd.

Then,

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\int_{0}^{\pi} f(x)\, dx; \quad \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 2\int_{0}^{\pi} f(x) \cos nx \, dx$$

and $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$

This gives,

$$a_0 = \frac{2}{2\pi}\int_{0}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx; \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx$$

and $b_n = \frac{1}{\pi}.\, 0 = 0.$

Therefore, (1) becomes,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots\ldots.(3)$$

where,

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx \text{ and } a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx$$

Here, (3) is known as Fourier cosine series with value of a_0 and a_n.

Fourier series of a function f(x) with period 2l

Consider a periodic function f(x) with period $2l$, is defined in the interval $-l < x < l$.

Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \qquad \text{.........(1)}$$

Integrate (1) term–by–term on the interval $-l \le x \le l$ then,

$$\int_{-l}^{l} f(x)\,dx = a_0\int_{-l}^{l} dx + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= 2l\,a_0$$

because,

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx = \left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right]_{-l}^{l} = 0$$

and $$\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)dx = \frac{-l}{n\pi}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} = 0$$

Thus,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \qquad \text{...........(2)}$$

Next, multiply both sides of (1) by $\cos\left(\frac{n\pi x}{l}\right)$ and integrate term–by–term on the integral $[-l, l]$. We obtain,

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = a_0\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)dx +$$

$$\sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{2\pi x}{l}\right)dx\right] \qquad \text{...(*)}$$

Since we have,

$$\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = 0 \qquad \text{for all m and n.}$$

and, $$\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = 0 \qquad \text{for all m and n.}$$

And for m = n,

$$\int_{-l}^{l}\cos^2\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left[1 + \cos\left(\frac{2n\pi x}{l}\right)\right]dx$$

$$= \frac{1}{2}\int_{-l}^{l} dx + \frac{1}{2}\int_{-l}^{l}\cos\left(\frac{2n\pi x}{l}\right)dx = \frac{1}{2}\cdot 2l + 0 = l.$$

Also,

$$\int_{-l}^{l}\cos\left(\frac{m\pi x}{l}\right)dx = 0$$

Therefore (*) becomes,

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = 0 + l.a_n + 0 \qquad \text{for } n = m$$

$$\Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \qquad \text{........(3)}$$

Again if we multiply both sides of (i) by $\sin\left(\frac{m\pi x}{l}\right)$ and then integrate term–by–term on the interval $[-l, l]$. So that, as above, we obtain,

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad \text{......(4)}$$

Thus, (1) is Fourier series with the value of coefficients given in (2), (3) and (4).

Note: If f(x) is periodic with period 2π, is defined on $[-\pi, \pi]$ then above relation deduces to

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin x)$$

with $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx; \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad \text{for } n = 1, 2, 3, \text{.......}$$

Fourier sine and cosine series with period 2l

We have, the Fourier series of f(x) with period $2l$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \qquad \text{.....(1)}$$

with $$a_0 = \frac{1}{2l}\int_{-\pi}^{\pi} f(x)\,dx; \qquad a_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\text{and } b_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Suppose that f(x) is an even function. So, f(x) $\cos\left(\frac{n\pi x}{l}\right)$ is even f(x) $\sin\left(\frac{n\pi x}{l}\right)$ is odd. So that,

$$a_0 = \frac{2}{2l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx \quad \text{......(2)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \text{.....(3)}$$

$$b_n = 0 \quad \text{........(3)}$$

Thus, (1) becomes,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

with $a_0 = \frac{1}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

which is known as Fourier cosine series with period 2l.

Next, we suppose that f(x) is an odd function. Then f(x) $\cos\left(\frac{n\pi x}{l}\right)$ is odd and f(x) $\sin\left(\frac{n\pi x}{l}\right)$ function. So that,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\cdot 0 = 0; \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\cdot 0 = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Therefore (1) becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

with $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$

we called Fourier sine series with period 2l.

EXERCISE 3.1

1. **Find the smallest period p of the following functions: cosx, sinx, cos2x, sin2x, cosπx, sinπx, cos2πx, sin2πx.**

[2013 Fall Q. No. 7(b)]

Solution:

(i) Let f(x) = cosx.

Then $f(x+p) = \cos(x+p) = \cos x \cos p - \sin x \sin p$

Suppose that given function is of period p. So,

$f(x) = f(x+p) \Rightarrow \cos x = \cos x \cos p - \sin x \sin p$

Comparing the both sides then

$\cos p = 1 = \cos 2\pi$, $\quad \sin p = 0 = \sin 2\pi$

$\Rightarrow p = 2\pi$, $\quad \Rightarrow p = 2\pi$

This shows that the given function is of period $p = 2\pi$.

(ii) – (vii) Process as above.

(viii) Let $f(x) = \sin 2\pi x$.

Then $f(x+p) = \sin 2\pi(x+p) = \sin 2\pi x \cos 2\pi p + \cos 2\pi x \sin 2\pi p$

Suppose that the given function is of period p. So,

$f(x) = f(x+p) \Rightarrow \sin 2\pi x = \sin 2\pi x \cos 2\pi p + \cos 2\pi x \sin 2\pi p$

Comparing the like terms from both sides. Then,

$\cos 2\pi p = 1 = \cos 2\pi$ and $\sin 2\pi p = 0 = \sin 2\pi$

$\Rightarrow 2\pi p = 2\pi$ $\quad \Rightarrow 2\pi p = 2\pi$

$\Rightarrow p = 1$ $\quad \Rightarrow p = 1$

This shows that the given function is of period p = 1.

2. **Are the following functions odd, even or neither odd nor even?**

(i) $|x^3|$, xcosnx, x^2 cosnx, coshx, sinhx, sinx + cosx, x|x|

(ii) $\frac{e^x + e^{-x}}{2}$ |sinx|, $2 - 3x^4 + \sin^2 x$, sinh2x, $\sqrt{1+x+x^2} - \sqrt{1-x+x^2}$, x^{2n}, x^{2n+1}, sinx + cosx, $\log\left(\frac{1-x}{1+x}\right)$.

Solution: (i)

(a) Let $f(x) = |x^3|$.

Then, $f(-x) = |(-x)^3| = |-x^3| = |x^3| = f(x)$.

$\Rightarrow f(x) = f(-x)$.

So, $f(x) = |x^3|$ is an even function.

(b) Let f(x) = x cosnx.

So, $f(-x) = (-x)\cos n(-x) = -x \cos nx$ $\quad [\because \cos(-\theta) = \cos\theta]$

$\Rightarrow f(x) = -f(-x)$.

So, f(x) is an odd function.

(c) Let $f(x) = x^2\cos nx$.

So, $f(-x) = (-x)^2 \cos n(-x) = x^2\cos nx$ $[\because \cos(-\theta) = \cos\theta]$

$\Rightarrow f(x) = f(-x)$.

So, f(x) is an even function.

(d) Let $f(x) = \cosh x$.

So, $f(-x) = \cosh(-x) = \cosh x = f(x)$

$\Rightarrow f(x) = f(-x)$.

So, f(x) is an even function.

(e) Let $f(x) = \sinh x$.

So, $f(-x) = \sinh(-x) = -\sinh x = -f(x)$.

$\Rightarrow f(x) = -f(-x)$.

Therefore, f(x) is an odd function.

(f) Let, $f(x) = \sin x + \cos x$.

So, $f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x = -(\sin x - \cos x)$

$\Rightarrow f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$.

So, f(x) is neither odd nor even.

(g) Let, $f(x) = x|x|$.

So, $f(-x) = (-x)|-x| = -x|x|$.

$\Rightarrow f(x) = -f(-x)$.

Therefore, f(x) is an odd function.

(ii) (a) Let, $f(x) = \dfrac{e^x + e^{-x}}{2} = \cos hx$, which is an even function.

(b) Let, $f(x) = |\sin x|$.

So, $f(-x) = |\sin(-x)| = |-\sin x| = \sin x = f(x)$.

$\Rightarrow f(x) = f(-x)$.

So, f(x) is an even function.

(c) Let, $f(x) = 2 - 3x^4 + \sin^2 x$.

So, $f(-x) = 2 - 3(-x)^4 + \sin^2(-x) = 2 - 3x^4 + \sin^2 x = f(x)$.

$\Rightarrow f(x) = f(-x)$.

So, f(x) is an even function.

(d) Let $f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$

So, $f(-x) = \sqrt{1 + (-x) + (-x)^2} - \sqrt{1 - (-x) + (-x)^2}$

$= \sqrt{1 - x + x^2} - \sqrt{1 + x + x^2}$

$= -[\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}] = f(x)$.

$\Rightarrow f(x) = -f(-x)$.

So, f(x) is an odd function.

(e) Let $f(x) = x^{2n} = (x^2)^n$.

So, $f(-x) = ((-x)^2)^n = (x^2)^n = x^{2n} = f(x)$.

$\Rightarrow f(x) = f(-x)$.

So, f(x) is an even function.

(f) Let $f(x) = x^{2n+1} = x^{2n}.x$

So, $f(-x) = (-x)^{2n}.(-x) = x^{2n}(-x) = -x^{2n}.x = -x^{2n+1} = -f(x)$.

$\Rightarrow f(x) = -f(-x)$.

So, f(x) is an odd function.

(g) done in (i).

(h) Let, $f(x) = \log\left(\dfrac{1-x}{1+x}\right)$.

So, $f(-x) = \log\left(\dfrac{1-(-x)}{1+(-x)}\right) = \log\left(\dfrac{1+x}{1-x}\right) = -\log\left(\dfrac{1+x}{1-x}\right)^{-1} = -\log\left(\dfrac{1-x}{1+x}\right)$

$\Rightarrow f(x) = -f(-x)$.

So, f(x) is an odd function.

3. **Are the following functions f(x), which are assumed to be periodic, of period 2π, even, odd or neither even nor odd?**

(i) $f(x) = x^2$ for $0 < x < 2\pi$. **(ii) $f(x) = e^{-|x|}$ for $-\pi < x < \pi$**

(iii) $f(x) = x^3$ for $\dfrac{-\pi}{2} < x < \dfrac{3\pi}{2}$.

Solution: (i) Let $f(x) = x^2$ for $0 < x < 2\pi$

So, $f(-x) = (-x)^2 = x^2$ for $0 < (-x) < 2\pi$

$\Rightarrow f(-x) = x^2$ for $0 > x > -2\pi$

Thus, $f(x) \neq f(-x)$ for $0 < x < 2\pi$

Therefore, f(x) is neither even nor odd.

(ii) Let $f(x) = e^{-|x|}$ for $-\pi < x < \pi$

So, $f(-x) = e^{-|-x|}$ for $-\pi < (-x) < \pi$

$\Rightarrow f(-x) = e^{-|x|}$ for $\pi > x > -\pi$

Thus, $f(x) = f(-x)$ for $\pi > x > -\pi$.

So, f(x) is an even function.

(iii) Let, $f(x) = x^3$ for $-\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

So, $f(-x) = (-x)^3$ for $-\dfrac{\pi}{2} < (-x) < \dfrac{3\pi}{2}$

$\Rightarrow f(-x) = -x^3$ for $\dfrac{\pi}{2} > x > \left(-\dfrac{3\pi}{2}\right)$

This shows that $f(x) \neq f(-x)$ for $-\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.

Also, $f(x) = -f(-x)$ but not for $-\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.

So, f(x) is neither even nor odd for $-\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.

4. **Find Fourier series of the functions**

(i) – (ii) See figure from book. **(iii) $f(x) = x$ $(-\pi < x < \pi)$.**

(iv) $f(x) = x^3 \ (-\pi < x < \pi)$.

(v) $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$

(vi) $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$.

(vii) $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$.

Solution:

(i) Here, $f(x) = \begin{cases} 1 & \text{for } -\pi/2 \le x \le \pi/2 \\ 0 & \text{for } -\pi \le x < -\pi/2,\ \pi/2 < x \le \pi \end{cases}$

Here, period of f(x) is, $2p = \pi - (-\pi)$ [$\because$ upper limit–lower limit]

$= 2\pi$

So, the function f(x) is 2π periodic.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{.... (i)}$$

with the value of a_0, a_n, b_n.

Here,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2\pi}[x]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2} = \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx = 0, \text{ being sine function is odd.}$$

Therefore (1) becomes,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\cos(nx)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \dots\right]$$

(ii) Here, $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$

Similar as (i)

(iii) Here, $f(x) = x \ (-\pi < x < \pi)$

Then the Fourier series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x) \quad \text{.... (i)}$$

with the value of a_0, a_n, b_n.

Here,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{4\pi}[\pi^2 - \pi^2] = 0.$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos n\pi}{n^2}\right] = 0.$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} + 0 - \pi\frac{\cos n\pi}{n}\right]$$

$$= -\frac{2}{n}\left[\frac{\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n = -\frac{2}{n}(-1)^n$$

Substituting the value of a_0, a_n, b_n in equation (i) we get,

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right).$$

(iv) Given, $f(x) = x^3$ for $-\pi < x < \pi$

Put $x = -x$, $f(-x) = (-x)^3$ for $-\pi < -x < \pi$

$= -x^3$ for $\pi > x > -\pi$

$= -x^3$ for $-\pi < x < \pi$

$= -f(x)$

This shows that f(x) is odd.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{......(1)}$$

with the value of a_0, a_n, b_n.

Since f(x) is odd. So, $a_0 = 0$ and $a_n = 0$. And,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3} - 0\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi}{n^3}\cos n\pi\right] = 2\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]\cos n\pi.$$

Thus (1) becomes,

$$f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]\cos n\pi \sin nx$$

$$= 2\sum_{n=1}^{\infty}\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right](-1)^n \sin nx$$

$$= 2\left[\left(\frac{\pi^2}{1} - \frac{6}{1}\right)\sin n - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2x + \left(\frac{\pi^2}{3} - \frac{6}{3^3}\right)\sin 3x \ldots\right].$$

(v) Given that, $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi \end{cases}$

Here,

$$f(-x) = \begin{cases} 1 & \text{if } -\pi < -x < 0 \\ -1 & \text{if } 0 < -x < \pi \end{cases}$$

$$= \begin{cases} 1 & \text{if } \pi > x > 0 \\ -1 & \text{if } 0 > x > -\pi \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases} = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases} = -f(x).$$

Thus, $f(-x) = -f(x)$. This shows that f(x) is odd function.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots\ldots(1)$$

with the value of a_0, a_n, b_n.

Since f(x) is odd. So, $a_0 = 0$ and $a_n = 0$. And,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx$$

$$= -\frac{2}{\pi}\int_0^{\pi} \sin nx dx$$

$$= -\frac{2}{\pi}\left[\frac{\cos nx}{-n}\right]_0^{\pi} = -\frac{2}{\pi}\left[\frac{\sin \pi}{-n} + \frac{1}{n}\right] = \frac{2}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right].$$

Thus (1) becomes,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right]\sin nx$$

$$= -\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right).$$

(vi) Similar as (v).

(vii) Given, $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

Put $x = -x$,

$$f(-x) = \begin{cases} -x & \text{if } -\pi/2 < -x < \pi/2 \\ 0 & \text{if } \pi/2 < -x < 3\pi/2 \end{cases}$$

$$= \begin{cases} -x & \text{if } \pi/2 > x > -\pi/2 \\ 0 & \text{if } -\pi/2 > x > -3\pi/2 \end{cases}$$

$$= \begin{cases} -x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } -3\pi/2 < x < \pi/2 \end{cases}$$

$$\neq f(x)$$

So, f(x) is neither even nor odd.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots\ldots(1)$$

with the value of a_0, a_n, b_n.

Here,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} f(x)\, dx + \int_{\pi/2}^{3\pi/2} f(x)\, dx\right] = \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} x\, dx + \int_{\pi/2}^{3\pi/2} dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2} = \frac{1}{4\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{4}\right] = 0.$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{3\pi/2} f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + 0\right]$$

$$=\frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - \left(\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left[\frac{\pi}{2n}\sin\frac{\pi}{2} - \frac{\pi}{2n}\sin n\frac{\pi}{2}\right] = 0.$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(\frac{\cos nx}{-n}\right) - \left(\frac{\sin nx}{-n^2}\right)\right]_{-\pi/2}^{\pi/2}$$

$$=\frac{1}{\pi}\left[\frac{\sin n\frac{\pi}{2}}{n^2} + \frac{\sin n\frac{\pi}{2}}{n^2}\right] = \frac{2}{\pi n^2}\left[\sin n\frac{\pi}{2}\right]$$

Thus (1) becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[\sin n\frac{\pi}{2}\right]\sin nx$$

$$=\frac{2}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$$

5. Find the fourier series of the periodic function f(x), of period p = 2l

(i) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$

(ii) $f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 & \text{if } 0 < x < 2 \end{cases}$

(iii) $f(x) = 2x$ if $-1 < x < 1$

(iv) $f(x) = 3x^2$ if $-1 < x < 1$

(v) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$

(vi) $f(x) = \pi\sin\pi x$ if $0 < x < 1$.

Solution: (i) Given that,

$$f(x) = \begin{cases} -1 & \text{for } -1 < x < 0 \\ 1 & \text{for } 0 < x < 1 \end{cases}$$

Clearly f(x) is of 2l–periodic function.

The Fourier series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x) \quad \ldots\ldots(i)$$

with $a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx$, $a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx$ and $b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx$

ere,

$$a_0 = \frac{1}{2}\left\{\int_{-1}^{0}(-1)\,dx + \int_{0}^{1} 1\,dx\right\} = \frac{1}{2}\left\{[-x]_{-1}^{0} + [x]_{0}^{1}\right\} = \frac{1}{2}(0+1+1-0) = 1.$$

and,

$$a_n = -\int_{-1}^{0}\cos n\pi x\,dx + \int_{0}^{1}\cos n\pi x\,dx$$

$$=-\left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin n\pi x}{n\pi}\right]_{0}^{1} = 0 \qquad [\because \sin n\pi = 0 = \sin 0]$$

Also,

$$b_n = -\int_{-1}^{0}\sin n\pi x\,dx + \int_{0}^{1}\sin n\pi x\,dx$$

$$=-\left[-\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[-\frac{\cos n\pi x}{n\pi}\right]_{0}^{1}$$

$$=\frac{1}{n\pi}[1 - \cos n\pi - \cos n\pi + 1] = \frac{2}{n\pi}[1 - \cos n\pi]$$

Now (i) becomes,

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{2}{n\pi}(1 - \cos n\pi)\sin n\pi x \quad \ldots\ldots(ii)$$

Since, $1 - \cos n\pi = 1 - 1 = 0$ for n is even

and $1 - \cos n\pi = 1 + 1 = 2$ for n is odd.

Therefore (ii) becomes,

$$f(x) = 1 + \sum_{n\text{-odd}}\frac{4}{n\pi}\sin n\pi x$$

$$\Rightarrow f(x) = 1 + \frac{4}{\pi}\left[\sin\pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \ldots\right]$$

This is the required Fourier of f(x).

(ii) Similar to (i).

(iii) Given function is.

$f(x) = 2x$ for $-1 < x < 1$.

Clearly f(x) is 2–periodic function.

Put $x = -x$, $f(-x) = -2x$ if $-1 < -x < 1$

$= -2x$ if $1 > x > -1$

$= -2x$ if $-1 < x < 1$

$= -f(x)$

This shows that f(x) is odd. So, its Fourier series becomes as a Fourier sine series.

And the function f(x) on period $2L = 1 - (-1) = 2$. Therefore, $L = 1$.

And the Fourier series of f(x) is,

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{L}x \quad \ldots\ldots(i)$$

where, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$.

Here, $b_n = 2\int_0^1 2x \sin n\pi x\, dx$

$$= 4\int_0^1 x \sin n\pi x\, dx = 4\left[x\frac{\cos n\pi x}{-n\pi} - \frac{\sin n\pi x}{-(n\pi)^2}\right]_0^1$$

$$= 4\left[-\frac{\cos n\pi}{n\pi}\right] = -\frac{4}{n\pi}(-1)^n.$$

Therefore (i) becomes,

$$f(x) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin n\pi x$$

$$= \frac{4}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots\right).$$

This is the required Fourier of f(x).

(iv) Given function is, $f(x) = 3x^2$ if $-1 < x < 1$.

Put $x = -x$, $f(-x) = 3(-x)^2$ if $-1 < -x < 1$

$= 3x^2$ if $-1 < x < 1$

$= f(x)$.

This shows that f(x) is even. So, its Fourier series becomes as a Fourier cosine series. And the function f(x) on period $2L = 1 - (-1) = 2$. Therefore, $L = 1$.

Therefore the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \dots\dots\dots (1)$$

where, $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$ and $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx.$

Since f(x) is even so,

$$a_0 = \int_0^1 3x^2\, dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1.$$

$$a_n = 2\int_0^1 3x^2 \cos n\pi x\, dx$$

$$= 6\left[x^2\frac{\sin n\pi x}{n\pi} - 2x\frac{\cos n\pi x}{-(n\pi)^2} + 2\frac{\sin n\pi x}{-(n\pi)^3}\right]_0^1$$

$$= 6\left[\frac{2\cos n\pi}{(n\pi)^2}\right] = \frac{12\cos n\pi}{(n\pi)^2}.$$

ow (1) becomes,

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{12\cos n\pi}{(n\pi)^2}\cos n\pi x$$

$$= 1 - \frac{12}{\pi^2}\left(\cos \pi x - \frac{\cos 2\pi x}{4} + \frac{\cos 3\pi x}{9} - \dots\right).$$

[2004 Spring Q. No. 6(a)]

(v) Similar to (ii).

(vi) Given function is, $f(x) = \pi \sin\pi x$ for $0 < x < 1$.

Clearly f(x) is a periodic function with 1–period.

The Fourier series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(2n\pi x) + b_n \sin(2n\pi x)] \quad \dots\dots(i)$$

with $a_0 = \int_0^1 f(x)\, dx$, $a_n = 2\int_0^1 f(x)\cos(2n\pi x)\, dx$

and, $b_n = 2\int_0^1 f(x)\sin(2n\pi x)\, dx$

Here,

$$a_0 = \int_0^1 \pi \sin\pi x\, dx = \pi\left[\frac{-\cos \pi x}{\pi}\right]_0^1 = 1 - \cos\pi = 1 + 1 = 2.$$

and,

$$a_n = 2\int_0^1 \pi \sin\pi x \cos(2n\pi x)\, dx$$

$$= \pi\int_0^1 [\sin(2n\pi + \pi)x - \sin(2n\pi - \pi)x]\, dx$$

$$= -\pi\left[\frac{\cos(2n\pi + \pi)x}{(2n+1)\pi} - \frac{\cos(2n\pi - \pi)x}{(2n-1)\pi}\right]_0^1$$

$$= -\pi\left[\frac{\cos(2n\pi + \pi)}{(2n+1)\pi} - \frac{\cos(2n\pi - \pi)}{(2n-1)\pi} - \frac{1}{(2n+1)\pi} + \frac{1}{(2n-1)\pi}\right]$$

$$= -\pi\left[\frac{\cos\pi}{(2n+1)\pi} - \frac{\cos\pi}{(2n-1)\pi} - \frac{1}{(2n+1)\pi} + \frac{1}{(2n-1)\pi}\right] \quad [\because \cos 2n\pi = 1]$$

$$= -\pi\left[\frac{-1}{(2n+1)\pi} + \frac{1}{(2n-1)\pi} - \frac{1}{(2n+1)\pi} + \frac{1}{(2n-1)\pi}\right] \quad [\because \cos\pi = -1]$$

$$= -\frac{2\pi}{\pi}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= -2\left(\frac{2}{4n^2 - 1}\right) = -\frac{4}{4n^2 - 1}.$$

Also,

$$b_n = 2\int_0^1 \pi \sin\pi x \sin(2n\,\pi x)\, dx$$

$$= \pi\left[\frac{\sin(2n\pi-\pi)x}{2n\pi-\pi} - \frac{\sin(2n\pi+\pi)x}{2n\pi+\pi}\right]_0^1$$

$= 0$ [∵ $\sin n\pi = \sin(2n-1)\pi = 0$ for n is integer]

Now, (i) becomes,

$$f(x) = 2 + \sum_{n=1}^{\infty}\left(-\frac{4}{4n^2-1}\right)\cos(2n\pi x)$$

$$= 2 - 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\cos(2n\pi x)$$

$$= 2 - 4\left[\frac{\cos 2\pi x}{1.3} + \frac{\cos 4\pi x}{3.5} + \frac{\cos 6\pi x}{5.7} + \ldots\ldots\right]$$

This is the required Fourier series for f(x).

6. State whether the given function is even or odd. Find its Fourier series.

(i) $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$ and show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}$.

[2004 Fall Q. No. 6(b)] [2002 Q. No. 6(b)]

Solution: Given, $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

Clearly f(x) is 2π periodic function.

Put $x = -x$, $f(-x) = \begin{cases} k & \text{if } -\pi/2 < -x < \pi/2 \\ 0 & \text{if } \pi/2 < -x < 3\pi/2 \end{cases}$

$= \begin{cases} k & \text{if } \pi/2 > x > -\pi/2 \\ 0 & \text{if } -\pi/2 > x > -3\pi/2 \end{cases}$

$= \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } -3\pi/2 < x < -\pi/2 \end{cases}$

$\neq f(x)$

This shows that f(x) is neither even nor odd.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots\ldots\text{(i)}$$

ith $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

e,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} dx + \int_{\pi/2}^{3\pi/2} 0\,dx\right]$$

$$= \frac{k}{2\pi}[x]_{-\pi/2}^{\pi/2} = \frac{k}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{k}{2}$$

And, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} k\cos nx\,dx + 0\right]$$

$$= \frac{k}{\pi}\left[\frac{\sin n\pi}{n}\right]_{-\pi/2}^{\pi/2} = \frac{k}{n\pi}\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right] = \frac{2}{n\pi}\sin n\frac{\pi}{2}.$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi/2}^{\pi/2} k\sin nx\,dx + 0\right]$$

$$= \frac{k}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = -\frac{k}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right] = 0$$

Now, (i) becomes,

$$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty}\frac{2k}{n\pi}\sin\frac{n\pi}{2}\cos nx$$

$$= \frac{k}{2} + \frac{2k}{\pi}\cos x - \frac{2k}{3\pi}\cos 3 + \frac{2k}{5\pi}\cos 5x \quad \ldots\ldots\text{(ii)}$$

This is required Fourier series for f(x)

Next, put $x = 0$ in (ii) then,

$$f(0) = \frac{k}{2} + \frac{2k}{\pi} - \frac{2k}{3\pi} + \frac{2k}{5\pi} \ldots\ldots$$

$$\Rightarrow k = \frac{k}{2} + \frac{2k}{\pi} - \frac{2k}{3\pi} + \frac{2k}{5\pi} \ldots\ldots$$

$$\Rightarrow \frac{k}{2} = \frac{2k}{\pi} - \frac{2k}{3\pi} + \frac{2k}{5\pi} \ldots\ldots$$

$$\Rightarrow \frac{k}{2} = \frac{2k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots$$

This completes the proof.

6. State whether the given function is even or odd. Find its Fourier series.

(ii) $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$ **[2014 Spring Q. No. 4(b)]**

Solution: Given function is

...2π period.

$$f(-x)=\begin{cases}-x & \text{for } -\pi/2<-x<\pi/2\\ \pi+x & \text{for } \pi/2<-x<3\pi/2\end{cases}$$

$$=\begin{cases}-x & \text{for } \pi/2>x>-\pi/2\\ \pi+x & \text{for } -\pi/2>x>-3\pi/2\end{cases}$$

$$\neq f(x)$$

This shows that f(x) is neither even nor odd.

Now, Fourier series of f(x) is

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad \text{........(i)}$$

with $\quad a_0=\frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(x)\cos nx\,dx$

and $\quad b_n=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(x)\sin x\,dx$

Here,

$$a_0=\frac{1}{2\pi}\left\{\int_{-\pi/2}^{\pi/2}x\,dx+\int_{\pi/2}^{3\pi/2}(\pi-x)\,dx\right\}$$

$$=\frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2}+\left[\pi x-\frac{x^2}{2}\right]_{\pi/2}^{3\pi/2}\right\}$$

$$=\frac{1}{2\pi}\left[0+\frac{3\pi^2}{2}-\frac{\pi^2}{2}-\frac{9\pi^2}{8}+\frac{\pi^2}{8}\right]=\frac{\pi}{16}[12-4-9+1]=0$$

and,

$$a_n=\frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2}x\cos nx\,dx+\int_{\pi/2}^{3\pi/2}(\pi-x)\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]_{-\pi/2}^{\pi/2}+\left[(\pi-x)\frac{\sin nx}{n}-\frac{\cos nx}{n^2}\right]_{\pi/2}^{3\pi/2}\right\}$$

$$=\frac{1}{\pi}\left\{\left[\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)-\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)-\frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]\right.$$

$$\left.+\left[\frac{-\pi}{2^n}\sin\left(\frac{2n\pi}{2}\right)-\frac{\pi}{2n}\sin\left(\frac{2\pi}{2}\right)-\frac{1}{n^2}\cos\left(\frac{3n\pi}{2}\right)+\frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]\right\}$$

$$=\frac{1}{\pi}\left\{\left(\frac{-\pi}{2n}\right)\left[\sin\left(\frac{3n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right]-\frac{1}{n^2}\left[\cos\left(\frac{3n\pi}{2}\right)-\cos\left(\frac{n\pi}{2}\right)\right]\right\}$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{2n}\left[-\sin\left(\frac{\pi n}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right]-\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{2}\right)-\cos\left(\frac{n\pi}{2}\right)\right]\right]$$

$$\left[\because \sin\left(\frac{3n\pi}{2}\right)=\sin\left(2\pi-\frac{\pi}{2}\right)n=-\sin\left(\frac{\pi n}{2}\right)\right.$$
and similar for cos.$\left.\right]$

$$=0$$

Also,

$$b_n=\frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2}x\sin nx\,dx+\int_{\pi/2}^{3\pi/2}(\pi-x)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right]_{-\pi/2}^{\pi/2}+\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-\frac{\sin nx}{n^2}\right]_{\pi/2}^{3\pi/2}\right\}$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\left(\frac{nx}{2}\right)-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)+\right.$$
$$\left.\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)-\frac{1}{n^2}\left[\sin\left(\frac{3n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right]\right]$$

$$=\frac{1}{\pi n^2}\left[2\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\left[\because \sin\left(\frac{3n\pi}{2}\right)=\sin\left(2\pi-\frac{\pi}{2}\right)n=-\sin\left(\frac{\pi n}{2}\right)\right]$$

$$=\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right).$$

Therefore (i) becomes,

$$f(x)=\sum_{n=1}^{\infty}\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx \quad \text{.......(ii)}$$

Since $\sin n\pi=0$. So, $\sin\left(\frac{n\pi}{2}\right)=0$ for n is even.

Then (ii) can be written as

$$f(x)=\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right)\sin x+\frac{1}{3^2}\sin\left(\frac{3\pi}{2}\right)\sin 3x+\frac{1}{5^2}\sin\left(\frac{5\pi}{2}\right)\sin 5x+\frac{1}{7^2}\sin 7x+\dots\right]$$

$$\Rightarrow f(x)=\frac{4}{\pi}\left[\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\dots\right].$$

This is the required Fourier series for f(x).

6. **State whether the given function is even or odd. Find its Fourier series.**

(iii) $f(x)=\frac{x^2}{2}$ if $-\pi<x<\pi$ and show that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots=\frac{\pi^2}{6}$ and $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\dots=\frac{\pi^2}{12}$.

Solution: Given function is, $f(x)=\frac{x^2}{2}$ for $-\pi<x<\pi$.

Clearly f(x) is 2π–periodic function.

$= f(x).$

This shows that f(x) is an even function.

So, the Fourier series of f(x) is same to the Fourier cosine series of f(x).

Now, the Fourier cosine series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{.......(i)}$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$

Here,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{x^2}{2}\right)dx = \frac{1}{4\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{12\pi}(\pi^3+\pi^3) = \frac{\pi^2}{6}.$$

And,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{2\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{2\pi n^2}(\pi\cos n\pi + \pi\cos n\pi) \qquad [\because \sin n\pi = 0]$$

$$= \frac{2\cos n\pi}{n^2}$$

Then (i) becomes,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2\cos n\pi}{n^2}\cos nx$$

$$\Rightarrow f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx \qquad \text{.......(ii)}$$

This is required Fourier series (Fourier cosine series) for f(x).

In particular if we take x = 0 then we get f(0) = 0. So (ii) gives,

$$f(0) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{(-2)^n}{n^2} \qquad [\because \cos 0 = 1]$$

$$\Rightarrow -\frac{\pi^2}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 2\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots\right]$$

$$\Rightarrow \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots = \frac{\pi^2}{12}$$

And if we take $x = \pi$ then $f(\pi) = \frac{\pi^2}{2}$. So, (ii) gives,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$\Rightarrow \frac{2\pi^2}{6} = 2\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad [\because \cos n\pi = (-1)^n \text{ and } (-1)^{2n} = 1]$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots$$

7. **Find the Fourier cosine series as well as Fourier sine series of the following function:**

(i) $f(x) = x$ if $0 < x < L$ **[2005 Fall Q. No. 5(b)] [2003 Fall Q. No. 5(b)]**

(ii) $f(x) = \pi - x$ if $0 < x < \pi$. **[2010 Spring Q. No. 4(b)]**

(iii) $f(x) = e^x$ if $0 < x < L$ **[2010 Fall Q. No. 6(b)] [2009 Spring Q. No. 6(b)]**

Solution: (i) Let $f(x) = x$ for $0 < x < l$.

Clearly f(x) is half range periodic function on $(0, l)$. Then the Fourier cosine series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \text{........(i)}$$

with $a_0 = \frac{1}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$

Here,

$$a_0 = \frac{1}{l}\int_0^l x\,dx = \frac{1}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{1}{2l}(l^2) = \frac{l}{2}.$$

and,

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[x\,\frac{\sin(n\pi x/l)}{n\pi/l} - (1)\left(\frac{-\cos(n\pi x/l)}{(n\pi/l)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right] \qquad [\because \sin 0 = 0]$$

Then (i) becomes,

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2}{l}\left[\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right]\cos\left(\frac{n\pi x}{l}\right) \qquad \text{......(ii)}$$

Since, $\sin n\pi = 0$ for n = 1, 2, 3, ...

and $\cos n\pi - 1 = \begin{cases} -2 & \text{for n is odd} \\ 0 & \text{for n is even}\end{cases}$

Then (ii) becomes,

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n\text{-odd}}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \quad f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\left(\frac{\pi x}{l}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{l}\right) + ...\right)$$

This is required Fourier cosine series for f(x).

Also, the Fourier sine series of f(x) is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad ....(iii)$$

with $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$

Here, $b_n = \frac{2}{l}\int x \sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[x\left(\frac{-\cos(n\pi x/l)}{n\pi/l}\right) - (1)\left(\frac{-\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_0^l$

$$= \frac{2}{l}\left[-\frac{l^2}{n\pi}(\cos n\pi) + \frac{l^2}{n^2\pi^2}\sin n\pi\right].$$

Then (iii) becomes,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{l}\left[-\frac{l^2}{n\pi}(\cos \pi) + \frac{l^2}{n^2\pi^2}\sin n\pi\right]\sin\left(\frac{n\pi x}{l}\right) \qquad ......(iv)$$

Since $\sin n\pi = 0$ for n = 1, 2, 3, ... And, $\cos n\pi = \begin{cases} -1 & \text{for n is odd} \\ 1 & \text{for n is even}\end{cases}$

Therefore (iv) becomes,

$$f(x) = \frac{2l}{\pi}\left[-\sum_{n-even}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right) + \sum_{n-odd}\frac{1}{n}\sin\frac{n\pi x}{l}\right]$$

$$\Rightarrow \quad f(x) = \frac{2l}{\pi}\left[\sin\left(\frac{x\pi}{l}\right) - \frac{1}{2}\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) - \frac{1}{4}\sin\left(\frac{4\pi x}{l}\right) + ...\right]$$

This is required Fourier sine series for f(x) on (0, l).

(ii) Given function is, $f(x) = \pi - x$ for $0 < x < \pi$.

Clearly f(x) is half range periodic function on $(0, \pi)$.

Now the Fourier cosine series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad .....(i)$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$, and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}.$$

And,

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx = \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n\pi}(1 - \cos n\pi).$$

Then (i) becomes,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}(1 - \cos n\pi)\cos nx \qquad ......(ii)$$

Since, $1 - \cos n\pi = 1 - (-1)^n = \begin{cases} 0 & \text{if n is even} \\ 2 & \text{if n is odd}\end{cases}$

Therefore (ii) becomes,

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n-odd}\frac{\cos nx}{n^2}$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

This is the Fourier cosine series for f(x).

Next, the Fourier sine series of f(x) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ......(iii)$$

with $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

because f(x) is half range periodic function on $(0, \pi)$.

Here,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\pi - x)\sin nx\,dx = \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{n} = \frac{2}{n}.$$

Now (iii) becomes,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}\sin nx = 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]$$

This is required Fourier sine series for f(x).

(iii) Let $f(x) = e^x$ for $0 < x < l$.

Clearly f(x) is a half range periodic function on (0, l).

Now, the Fourier cosine series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad ....(i)$$

with $a_0 = \frac{1}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$.

Here, $a_0 = \frac{1}{l}\int_0^l e^x\,dx = \frac{1}{l}[e^x]_0^l = \frac{e^l-1}{l}$.

and $a_n = \frac{2}{l}\int_0^l e^x\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\left[\frac{e^x}{1+(n\pi/l)^2}\right]\left[1.\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c\right]$$

$$= \frac{2}{l}\left[\frac{l^2 e^l}{l^2+n^2\pi^2}\left(\cos n\pi + \frac{n\pi}{l}\sin n\pi\right) - \frac{l^2}{l^2+n^2\pi^2}\right]$$

Then (i) becomes,

$$f(x) = \left(\frac{e^l-1}{l}\right) + 2l\sum_{n=1}^{\infty}\left(\frac{1}{l^2+n^2\pi^2}\right)\left[\left(e^l\cos n\pi + \frac{n\pi}{l}\sin n\pi\right) - 1\right]\cos\left(\frac{n\pi x}{l}\right) \quad \text{.......(ii)}$$

Since, $\sin n\pi = 0$ for n = 1, 2, 3, ... and $\cos n\pi = \begin{cases} 1 & \text{for n is even} \\ -1 & \text{for n is odd} \end{cases}$

Then (ii) becomes,

$$f(x) = \left(\frac{e^l-1}{l}\right) + 2l\left[\sum_{n-even}\left(\frac{e^l-1}{l^2+n^2\pi^2}\right)\cos\left(\frac{n\pi x}{l}\right) - \sum_{n-odd}\left(\frac{e^l-1}{l^2+n^2\pi^2}\right)\cos\left(\frac{n\pi x}{l}\right)\right]$$

This is required Fourier cosine series for f(x).

And, the Fourier sine series of f(x) is,

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right) \quad \text{.......(iii)}$$

with $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Here, $b_n = \frac{2}{l}\int_0^l e^x\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\left[\frac{l^2 e^x}{l^2+n^2\pi^2}\left\{\sin\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right\}\right]_0^l$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c\right]$$

$$= \frac{2l}{l^2+n^2\pi^2}\left[e^l\left(\sin n\pi - \frac{n\pi}{l}\cos n\pi\right) + \frac{n\pi}{l}\right]$$

Then (iii) becomes,

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2l}{l^2+n^2\pi^2}\right)\left[e^l\left(\sin n\pi - \frac{n\pi}{l}\cos n\pi\right) + \frac{n\pi}{l}\right]\sin\left(\frac{n\pi x}{l}\right) \quad \text{.....(iv)}$$

Since, $\sin n\pi = 0$ for n = 1, 2, 3, ... and $\cos n\pi = (-1)^n$ for n = 1, 2, 3, ...

Therefore (iv) becomes,

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2l}{l^2+n^2\pi^2}\right)[1-(-1)^n e^l]\frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2n\pi}{l^2+n^2\pi^2}[1-(-1)^n e^l]\sin\left(\frac{n\pi x}{l}\right)$$

This is required Fourier sine series of f(x).

8. Find the Fourier expansion of the following function in the interval $0 \le x \le 2\pi$.

(i) $f(x) = x^2$.

Solution: Let, $f(x) = x^2$ for $0 \le x \le 2\pi$.

Clearly the function is 2π–periodic function.

Now, the Fourier series of f(x) on [0, 2π] is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx] \quad \text{.....(i)}$$

with $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Here, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^3}{6\pi} = \frac{4\pi^2}{3}$.

and $a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$

$= \frac{4\pi}{\pi n^2}$ for n = 1, 2, 3, ...

$= \frac{4}{n^2}$ for n = 1, 2, 3, ...

[∵ $\sin 2n\pi = 0$, $\cos 2n\pi = 1$ for n = 1, 2, 3, ...]

Also, $b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$

$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left(-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right)$ for n = 1, 2, 3, ...

$$= \frac{-4\pi}{n} \quad \text{for } n = 1, 2, 3, \ldots\ldots$$

Then (i) becomes,

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right)$$

This is required Fourier series of f(x) on $[0, 2\pi]$.

(ii) Similar to Q.5 (ii).

(iii) $f(x) = \begin{cases} x & \text{for } 0 \le x < \pi \\ 2\pi - x & \text{for } \pi \le x < 2\pi \end{cases}$

Solution: Given function is,

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi \\ 2\pi - x & \text{for } \pi \le x < 2\pi \end{cases}$$

Clearly, the function f(x) is 2π–periodic.

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots\ldots(i)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

Since the Fourier series is periodic so we may change the period with equal length. That means we may change the period $(-\pi, \pi)$ with the period $(0, 2\pi)$. Here,

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\left[2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\left[\frac{4\pi^2 - 3\pi^2}{2}\right] = \frac{\pi}{4} + \frac{1}{2\pi}\cdot\frac{\pi^2}{2} = \frac{\pi + \pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

And,

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x\cos x\,dx + \int_{\pi}^{2\pi}(2\pi - 3)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{\sin nx}{n} - \frac{\cos nx}{-n^2}\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\sin nx}{n} + \frac{\cos nx}{-n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos\pi}{n^2} - \frac{1}{n^2} + \frac{\cos 2n\pi}{-n^2} + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} + \frac{2\cos\pi x}{n^2}\right]$$

Also,

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{-\cos nx}{n} - \frac{\sin nx}{-n^2}\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\cos nx}{-n} + \frac{\sin nx}{-n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \pi\frac{\cos n\pi}{n}\right]$$

$$= 0.$$

Now, (i) becomes,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2\cos n\pi - \cos 2\pi n - 1}{\pi n^2}\right)\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos x + \sum_{n=2}^{\infty}\left(\frac{2\cos n\pi - \cos 2\pi n - 1}{\pi n^2}\right)\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

8. Find the Fourier expansion of the following function in the interval $0 \le x \le 2\pi$.

(iv) $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le x \le \pi \\ -1 & \text{for } \pi < x \le 2\pi \end{cases}$ (v) $f(x) = x$ for $0 \le x \le 2\pi$.

Solution: (iv) Similar to Q.4 (v). (v) Similar to Q.7 (i).

9. Find the Fourier expansion of f(x):

(i) $f(x) = \cosh x$ for $-\pi \le x \le \pi$.

Solution: Given function is

$$f(x) = \cosh x \qquad \text{for } -\pi \le x \le \pi$$

Clearly, the function f(x) is 2π–periodic.

Here,

$$\begin{aligned} f(-x) &= \cosh(-x) && \text{for } -\pi \le -x \le \pi \\ &= \cosh x && \text{for } \pi \ge x \ge -\pi \\ &= \cosh x && \text{for } -\pi \le x \le \pi \\ &= f(x) \end{aligned}$$

This shows that the given function f(x) is even. So the Fourier series for f(x) is same as the Fourier cosine series of f(x).

Now, the Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots\ldots(i)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Since, $b_n = 0$, being f(x) is even.

Here, $a_0 = \frac{2}{2\pi}\int_0^{\pi}\cosh x\,dx = \frac{1}{\pi}[\sinh x]_0^{\pi} = \frac{\sinh\pi}{\pi}$.

and, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cosh x\cos nx\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^x + e^{-x})\cos nx\,dx \quad \left[\because \cosh x = \frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^x}{(1+n^2)}(\cos nx + n\sin nx) + \frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c\right]$$

$$= \frac{1}{2\pi(1+n^2)}[\cos n\pi\,(e^{\pi} - e^{-\pi}) - \cos n\pi\,(e^{-\pi} - e^{\pi})]$$

for n = 1, 2, 3, ...

$$= \frac{2}{2\pi(1+n^2)}\cdot\cos n\pi(e^{\pi} - e^{-\pi}) \quad \text{for } n = 1, 2, 3, \ldots$$

$$= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)} \quad \text{for } n = 1, 2, 3, \ldots\ldots$$

Then (i) becomes,

$$f(x) = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{e^{\pi} - e^{-\pi}}{1+n^2}\right)\cos nx$$

$$\Rightarrow f(x) = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n\sinh\pi}{\pi(1+n^2)}\right)\cos nx \quad \left[\because \sinh\pi = \frac{e^{\pi} - e^{-\pi}}{2}\right]$$

$$\Rightarrow f(x) = \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{1+n^2}\right]$$

This is required Fourier series for f(x).

i) $f(x) = \begin{cases} 0 & \text{for } \pi < x < 0 \\ \pi & \text{for } 0 < x < \pi \end{cases}$

lution: Let, $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \pi & \text{for } 0 < x < \pi \end{cases}$

$$\Rightarrow f(x) = \pi\begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$$

Clearly f(x) is 2π–periodic function.

Also, let $g(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$

Then g(x) is also 2π–periodic function.

Now, the Fourier series of g(x) is

$$g(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \ldots\ldots\text{(i)}$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx$.

Here, $a_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$

And, $a_n = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$ for n = 1, 2, 3, ...

Also, $b_n = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{1 - \cos n\pi}{n\pi} = \frac{1 - (-1)^n}{n\pi}$

for n = 1, 2, 3, ...

Then (i) becomes,

$$g(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1 - (-1)^n}{n\pi}\right)\sin nx$$

$$\Rightarrow g(x) = \frac{1}{2} + \sum_{n\text{–odd}}\frac{2\sin nx}{n\pi}$$

So, the Fourier series of f(x) is,

$$f(x) = \pi.g(x) = \frac{\pi}{2} + 2\sum_{n\text{–odd}}\left(\frac{\sin nx}{n}\right)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + 2\sum_{n=0}^{\infty}\left(\frac{\sin(2n+1)x}{2n+1}\right)$$

This is required Fourier series of f(x).

(iii) $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$ and hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}$.

Solution: Let, $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$

Then by (ii), the Fourier series of f(x) is,

$$f(x) = \frac{1}{2} + \sum_{n-odd}\left(\frac{2\sin nx}{n\pi}\right)$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\ldots\right]$$

Set $x = \frac{\pi}{2}$ then, $f\left(\frac{\pi}{2}\right) = 1$. So,

$$1 = \frac{1}{2} + \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right]$$

$$\Rightarrow \frac{1}{2} = \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$$

(iv) $f(x) = \begin{cases} -1/2 & \text{for } -\pi < x < 0 \\ 1/2 & \text{for } 0 < x < \pi \end{cases}$

Solution: Let, $f(x) = \begin{cases} -1/2 & \text{for } -\pi < x < 0 \\ 1/2 & \text{for } 0 < x < \pi \end{cases}$

$$\Rightarrow f(x) = -\frac{1}{2}\begin{cases} 1 & \text{for } -\pi < x < 0 \\ -1 & \text{for } 0 < x < \pi \end{cases}$$

Then by Q. No. 4(v), the Fourier series of f(x) is

$$f(x) = -\frac{1}{2}\left[-\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)\right].$$

10. Expand the following functions in both a Fourier cosine series and a Fourier sine series on the interval (0, π).

(i) $f(x) = x$ for $0 < x < \pi$.

Solution: Let, $f(x) = x$ for $0 < x < \pi$. **[2008 Spring Q. No. 6(b)]**

See Q.7 (i) with replacing l by π.

(ii) $f(x) = \sin x$ for $0 < x < \pi$.

Solution: Let $f(x) = \sin x$ for $0 < x < \pi$.

Clearly f(x) is half range 2π-periodic function.

The Fourier cosine series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots\ldots(ii)$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

Here, $a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{1-\cos\pi}{\pi} = \frac{2}{\pi}$

And, $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2-1} + (-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2-1} + \frac{2(-1)^n}{n^2-1}\right]$$

$$= \frac{-2}{\pi(n^2-1)}[1 + (-1)^n]$$

$$= \begin{cases} -4/\pi(n^2-1) & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$$

Therefore (ii) becomes,

$$f(x) = \frac{2}{\pi} + \sum_{n-even}\left[-\frac{4\cos nx}{\pi(n^2-1)}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n-even}\left[\frac{\cos nx}{(n-1)(1+n)}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \ldots\right]$$

This is required Fourier cosine series of f(x).

And, the Fourier sine series of f(x) is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots(i)$$

with $b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

Here, $b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\frac{\sin(n-1)\pi}{n-1}\quad \left[\text{in } \frac{0}{0} \text{ fom at } n = 1, \text{ is } 0 \text{ for } n = 2, 3, \ldots\right]$$

$$= \frac{1}{\pi}\begin{cases}\dfrac{\cos(n-1).\pi}{1} & \text{at } n = 1\\ 0 & \text{for } n = 2, 3, \ldots\end{cases}$$

$$= \begin{cases}1 & \text{at } n = 1\\ 0 & \text{for } n = 2, 3, \ldots\end{cases}$$

Then (i) becomes, $f(x) = \sin x$.

(iii) $f(x) = l - x$ for $0 < x < l$. **[2013 Spring Q. No. 6(b)]**

Solution: Let $f(x) = l - x$ for $0 < x < l$.

Clearly $f(x)$ is half range $2l$–periodic function.

Now, the Fourier cosine series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \ldots\ldots(i)$$

with $a_0 = \frac{1}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$.

Here, $$a_0 = \frac{2}{l}\int_0^l (l - x)\,dx = \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{1}{l}\left(l^2 - \frac{l^2}{2}\right) = \frac{l^2}{2l} = \frac{l}{2}.$$

and, $$a_n = \frac{1}{l}\int_0^l (l - x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[(l-x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (-1)\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\times\frac{l^2}{n^2\pi^2}[\cos n\pi - 1]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1]$$

$$= \begin{cases}\dfrac{-4l}{n^2\pi^2} & \text{for n–odd}\\ 0 & \text{for n–even}\end{cases}$$

Then (i) becomes,

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n\text{–odd}}\frac{\cos\left(\frac{n\pi x}{l}\right)}{n^2}$$

This is required Fourier cosine series of $f(x)$.

1. Expand the function $f(x) = x^2$ for $0 \le x \le \pi$ in a Fourier cosine series and deduce that

(i) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{\pi^2}{6}$ **(ii)** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Solution: Given function is, $f(x) = x^2$ for $0 \le x \le \pi$

Here, $f(-x) = (-x)^2$ for $0 \le -x \le \pi$
$= x^2$ for $0 \ge x \ge -\pi$
$\ne f(x)$.

So, $f(x)$ is neither odd nor even.

Clearly $f(x)$ is period on half range period $(0, \pi)$.

Now, the Fourier cosine series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots\ldots(i)$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

Here, $$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}.$$

and,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4\pi}{\pi x^2}\text{Cos } n\pi$$

$$= \frac{4}{n^2}\cos n\pi$$

Then (i) becomes,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nx$$

$$\Rightarrow f(x) = \frac{2\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\cos nx \qquad \ldots\ldots(ii)$$

This is the Fourier cosine series of $f(x) = x^2$ for $0 \le x \le \pi$.

(i) In particular, if we take $x = \pi$ then $f(\pi) = \pi^2$. So, (ii) gives,

$$\pi^2 = \frac{2\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\cos n\pi$$

$$\Rightarrow \frac{6\pi^2 - 2\pi^2}{6} = 2.\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}(-1)^n \qquad [\because \cos n\pi = (-1)^n]$$

$$\Rightarrow \frac{4\pi^2}{6} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad [\because (-1)^n(-1)^n = (-1)^{9n} = 1]$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2.$$

(ii) In particular if we take x = 0 then f(0) = 0. So, (ii) gives,

$$0 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.4 \qquad [\because \cos 0 = 1]$$

$$\Rightarrow \frac{2\pi^2}{6} = -4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

12. Show that in the range $0 < x < \pi$, the function sinx can be represented by sinx $= \frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots\ldots\right)$.

Solution: Here we have to show,

$$\sin x = \frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots\right) \qquad \ldots\ldots\text{(i)}$$

for $0 < x < \pi$.

If we take, $f(x) = \sin x$ for $0 < x < \pi$.

Then the right part of (i) indicates the Fourier cosine series of f(x) for half range $(0, \pi)$ whenever f(x) is periodic on $(0, \pi)$ because the series includes only the cosine terms.

Since, the Fourier cosine series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots\ldots\text{(i)}$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{1}{\pi}(1 - \cos\pi) = \frac{1}{\pi}(1+1) = \frac{2}{\pi}.$$

and,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx \qquad [\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi \,.\, \cos\pi}{n-1} - \frac{\cos n\pi \,.\, \cos\pi}{n+1} + \frac{-2}{n^2-1}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos n\pi - \frac{2}{n^2-1}\right] \qquad [\because \cos\pi = -1]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2-1}\cos n\pi - \frac{2}{n^2-1}\right]$$

$$= \frac{-2}{\pi(n^2-1)}[\cos n\pi + 1]$$

Therefore (i) becomes,

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-2}{\pi(n^2-1)}[\cos n\pi + 1]\cos nx$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-2}{\pi(n^2-1)}[(-1)^n + 1]\cos nx \qquad \ldots\ldots\text{(ii)}$$

Here, $(-1)^n - 1 = \begin{cases} 0 & \text{if n is odd} \\ +2 & \text{if n is even} \end{cases}$

Then (ii) becomes,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\ldots\right]$$

$$\Rightarrow \sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{3} + \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots\ldots\right]$$

13. A function f(x) is defined as follows $f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$

Show that the Fourier sine series for f(x) in $0 \le x \le \pi$ is given by

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$$

Solution: Given that, $f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi/2 \\ \pi - x & \text{for } \pi/2 \le x \le \pi \end{cases}$ ...... (i)

Clearly f(x) is 2π–periodic function in which the half range is given as in (i).

Now, Fourier sine series of f(x) in $0 \le x \le \pi$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\text{(ii)}$$

with $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Here,

$$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x \sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \right.$$

$$\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$=\frac{2}{\pi}\left\{-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)+\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$=\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

Then (ii) becomes,

$$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin x \qquad \text{..........(iii)}$$

Since $\sin\left(\frac{n\pi}{2}\right)=0$ if n is even. So, (iii) gives,

$$f(x)=\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx$$

$$=\frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right)\sin x+\frac{1}{3^2}\sin\left(\frac{3\pi}{2}\right)\sin 3x+\frac{1}{5^2}\sin\left(\frac{5\pi}{2}\right)\sin 5x+\frac{1}{7^2}\sin\left(\frac{7\pi}{2}\right)\sin 7x+\ldots\right]$$

$$=\frac{4}{\pi}\left[\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\ldots\right]$$

14. A function f(x) is defined by $f() = \pi x - x^2$ for $0 \le x \le \pi$. Show that f(x) can be represented by the Fourier cosine series by $f(x)=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}$ valid for the interval $0 \le x \le \pi$.

Solution: Given function is, $f(x)=\pi x-x^2$ for $0\le x\le\pi$.

Clearly, f(x) is half range periodic function with period 2π. So, the Fourier cosine series of f(x) is,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx \qquad \text{.... (i)}$$

with $a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\,dx$ and $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$.

Here,

$$a_0=\frac{1}{\pi}\int_0^{\pi}(\pi x-x^2)\,dx=\frac{1}{\pi}\left[\frac{\pi x^2}{2}-\frac{x^3}{3}\right]_0^{\pi}=\frac{1}{\pi}\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)=\frac{\pi^3}{6\pi}=\frac{\pi^2}{6}.$$

And,

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\frac{\sin nx}{n}-(\pi-2x)\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{n^2}\cos n\pi-\frac{\pi}{n^2}\right]$$

$$=-\frac{2}{n^2}[\cos n\pi+1].$$

Then (i) becomes,

$$f(x)=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{2}{n^2}(\cos n\pi+1)\cos nx$$

$$=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{2}{n^2}[(-1)^n+1]\cos nx$$

$$=\frac{\pi^2}{6}-4\left[\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\ldots\right]=\frac{\pi^2}{6}-4\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2}$$

$$=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}$$

Thus, $f(x)=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}$.

15. A function f(x) is defined by $f(x)=\pi x-x^2$ for $0\le x\le\pi$. Show that the fourier sine series valid for the interval $0\le x\le\pi$, is given by

$$f(x)=\frac{8}{\pi}-\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3} \text{ and then deduce } \frac{\pi^2}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}-\ldots\ldots\ldots$$

Solution: Similar to Q. 14 and find b_n.

For the last part set value of x as in Q. 6.

16. A periodic function f(x) of period 2π is defined for $-\pi\le x\le\pi$ as follows:

$$f(x)=\begin{cases}2\cos x & \text{for } |x|<\pi/2\\ 0 & \text{otherwise}\end{cases}$$

. Show that the fourier series of f(x) is given by

$$f(x)=\frac{4}{\pi}\left[\frac{1}{2}+\frac{\pi}{4}\cos x+\frac{1}{1.3}\cos 2x-\frac{1}{3.5}\cos 4+\frac{1}{5.7}\cos 6x-\ldots\ldots\right].$$

Solution: Given that, $f(x)=\begin{cases}2\cos x & \text{for } |x|<\pi/2\\ 0 & \text{for otherwise}\end{cases}$

And f(x) is 2π–periodic function.

Here,

$$f(-x)=\begin{cases}2\cos(-x) & \text{for } |-x|<\pi/2\\ 0 & \text{for otherwise}\end{cases}$$

$$=\begin{cases}2\cos x & \text{for } |x|<\pi/2\\ 0 & \text{for otherwise}\end{cases}$$

$$=f(x)$$

This shows that f(x) is an even function. So, $b_n=0$ in Fourier series of f(x).

Now, Fourier series of f(x) over $-\pi\le x\le\pi$ is,

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad \text{......(i)}$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Here, $b_n = 0$, being f(x) is an even function.

And,

$$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 2\cos x\,dx = \frac{2}{2\pi}[\sin x]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left(\sin\frac{\pi}{2}+\sin\frac{\pi}{2}\right) = \frac{2}{\pi}\sin\frac{\pi}{2} = \frac{2}{\pi}$$

Also,

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} [\cos(n+1)x + \cos(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1}\right)\sin(n+1)\frac{\pi}{2} + \left(\frac{1}{n-1}\right)\sin(n-1)\frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1}\right)\cos\left(\frac{n\pi}{2}\right) - \left(\frac{1}{n-1}\right)\cos\left(\frac{n\pi}{2}\right)\right] \quad \left[\because \begin{array}{l}\cos \pi/2 = 0\\ \sin \pi/2 = 1\end{array}\right]$$

$$= \frac{1}{\pi}\cos\left(\frac{n\pi}{2}\right)\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{-2}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right)$$

Therefore, (i) becomes,

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2-1}\right)\cos\left(\frac{n\pi}{2}\right)\cos nx \quad .....(ii)$$

Since, $\cos\left(\frac{n\pi}{2}\right) = 0$ for n is odd.

So,
$$f(x) = \frac{2}{\pi}\left[1 - \sum_{n=\text{even}}^{\infty}\left(\frac{1}{n^2-1}\right)\cos\left(\frac{n\pi}{2}\right)\cos nx\right]$$

$$= \frac{2}{\pi}\left[1 - \sum_{n-\text{even}}\left(\frac{1}{(n-1)(n+1)}\cos\left(\frac{n\pi}{2}\right)\cos nx\right)\right]$$

$$= \frac{2}{\pi}\left[1 - \left(\frac{1}{1.3}\right)\cos\pi\cos 2x + \frac{1}{3.5}\cos 2\pi\cos 4x + \frac{1}{5.7}\cos 3\pi\cos 6x + \frac{1}{7.9}\cos 4\pi\cos 8x + ...\right]$$

$$= \frac{2}{\pi}\left[1 + \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x + ...\right]$$

17. Find the Fourier expansion of the following functions:

(i) $f(x) = -x$ for $-l \le x < l$

(ii) $f(x) = \begin{cases} 1 & \text{for } -l \le x < 0 \\ 0 & \text{for } 0 \le x < l \end{cases}$

(iii) $f(x) = \begin{cases} x & \text{for } -1 < x \le 0 \\ x+2 & \text{for } 0 < x \le 1 \end{cases}$

and hence deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

Solution: Process as Q.5 with given period.

EXERCISE 3.2

1. Find the Fourier series of the following periodic functions

(i) $f(x) = \begin{cases} -x & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}$ [2014 Spring Q. No. 4(a)]

Solution: Let, $f(x) = \begin{cases} -x & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}$

Clearly, f(x) is 4–periodic function. So, $l = 2$.

Now, Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)\right] \quad ......(i)$$

with $a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx$, $a_n = \frac{1}{2}\int_{-1}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$ and $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

Since, $f(-x) = \begin{cases} x & \text{for } -2 < -x < 0 \\ -x & \text{for } 0 < -x < 2 \end{cases}$

$$\Rightarrow f(-x) = \begin{cases} x & \text{for } 2 > x > 0 \\ -x & \text{for } 0 > x > -2 \end{cases}$$

$$= f(x)$$

So, f(x) is even function. So, $b_n = 0$.

Here,

$$a_0 = \frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx + \int_{0}^{2} x\,dx\right] = \frac{1}{4}\left\{\left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{4}\left(\frac{4}{2}+\frac{4}{2}\right) = \frac{1}{4}.4 = 1.$$

and,

$$a_n = \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[(-x)\left(\frac{\sin(n\pi x/2)}{n\pi/2}\right) - (-1)\left(\frac{-\cos(n\pi x/2)}{(n\pi/2)^2}\right)\right]_{-2}^{0} + \left[x\left(\frac{\sin(n\pi x/2)}{n\pi/2}\right) - (1)\left(\frac{-\cos(n\pi x/2)}{(n\pi/2)^2}\right)\right]_{0}^{2}$$

$$= \frac{1}{2}\times\frac{4}{n^2\pi^2}[(\cos n\pi - 1) + (\cos n\pi - 1)] \quad \text{for } n = 1, 2, 3, ...$$

$$=\frac{4}{n^2\pi^2}[(-1)^n-1]$$

Therefore, (i) becomes,

$$f(x)=1+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$=1+\frac{4}{\pi^2}\sum_{n-odd}\left(-\frac{2}{n^2}\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$=1-\frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right)+\frac{1}{9}\cos\left(\frac{3\pi x}{2}\right)+\frac{1}{25}\cos\left(\frac{5\pi x}{2}\right)+.....\right]$$

This is required Fourier series of f(x).

(ii) $f(x)=\begin{cases}1 & \text{if } -1<x<0\\ 0 & \text{if } 0<x<1\end{cases}$.

Solution: Let, $f(x)=\begin{cases}1 & \text{for } -1<x<0\\ 0 & \text{for } 0<x<1\end{cases}$

Clearly f(x) is 2–periodic function.

Now, the Fourier series of f(x) is,

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x) \quad ....(i)$$

With $a_0=\frac{1}{2}\int_{-1}^{1}f(x)\,dx$, $a_n=\int_{-1}^{1}f(x)\cos(n\pi x)\,dx$, $b_n=\int_{-1}^{1}f(x)\sin(n\pi x)\,dx$

Here, $a_0=\frac{1}{2}\left[\int_{-1}^{0}dx+\int_{0}^{1}0\,.\,dx\right]=\frac{1}{2}[x]_{-1}^{0}=\frac{1}{2}$

and, $a_n=\int_{-1}^{0}\cos(n\pi x)\,dx+\int_{0}^{1}0\,.\cos(n\pi x)\,dx$

$$=\left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0}+0$$

$=0$ for n = 1, 2, 3, ... [∵ $\sin n\pi=0$ for n = 1, 2,...]

Also,

$$b_n=\int_{-1}^{0}\sin(n\pi x)\,dx+\int_{0}^{1}\sin(n\pi x)\,dx$$

$$=\left[-\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0}+0=\frac{1}{n\pi}(\cos n\pi-1)=\frac{(-1)^n-1}{n\pi}.$$

Then (i) becomes,

$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n\pi}\right)\sin(n\pi x)$$

$$\Rightarrow f(x)=\frac{1}{2}-\frac{2}{\pi}\sum_{n-odd}\frac{\sin n\pi x}{n} \qquad [\because (-1)^n-1=0 \text{ for n is even}]$$

$$\Rightarrow f(x)=\frac{1}{2}-\frac{2}{\pi}\left[\sin(\pi x)+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}+....\right].$$

This is the required Fourier series for f(x).

(iii) $f(x)=|\cos x|$ if $-\pi<x<\pi$.

Solution: Let $f(x)=|\cos x|$ for $-\pi<x<\pi$.

Clearly f(x) is 2π – periodic function.

Here,

$f(-x)=|\cos(-x)|$ for $-\pi<-x<\pi$

$=|\cos x|$ for $\pi>x>-\pi$

$=f(x)$.

This shows that f(x) is even function. So, the Fourier series is same as to Fourier cosine series of f(x).

Now, Fourier cosine series of f(x) is,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx \quad ....(i)$$

with $a_0=\frac{1}{\pi}\int_{0}^{\pi}f(x)\,dx$ and $a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx$

Here,

$$a_0=\frac{1}{\pi}\int_{0}^{\pi}|\cos x|\,dx=\frac{1}{\pi}\int_{0}^{\pi}\cos x\,dx=\frac{1}{\pi}[\sin x]_0^{\pi}=0.$$

And, $a_n=\frac{2}{\pi}\int_{0}^{\pi}|\cos x|\cos nx\,dx$

$$=\frac{2}{\pi}\int_{0}^{\pi}\cos nx\cos x\,dx=\frac{1}{\pi}\int_{0}^{\pi}[\cos(n+1)x+\cos(n-1)x]\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\begin{cases}\frac{\sin(n-1)\pi}{n-1} & \text{at } n=1\\ 1 & \\ 0 & \text{for } n=2,3,......\end{cases}$$

[∵ $\sin n\pi=0$ for n = 1, 2, 3 ....]

At n = 1, a_n has 0/0 form. So, applying L'hopital rule then,

$$a_n=\frac{1}{\pi}\cdot\frac{\cos(n-1)\pi\,.\,\pi}{1} \quad \text{at } n=1$$

$\Rightarrow a_1=1$ and $a_n=0$ for n = 2, 3, ...

Then (i) becomes,

$$f(x)=\cos x.$$

(iv) $f(x) = \pi - 2|x|$ if $-\pi < x < \pi$.

Solution: Let, $f(x) = \pi - 2|x|$ for $-\pi < x < \pi$.

Clearly $f(x)$ is 2π-periodic

Here,

$$f(x) = \begin{cases} \pi + 2x & \text{for } -\pi < x < 0 \\ \pi - 2x & \text{for } 0 < x < \pi \end{cases}$$

So, $$f(-x) = \begin{cases} \pi - 2x & \text{for } -\pi < -x < 0 \\ \pi + 2x & \text{for } 0 < -x < \pi \end{cases}$$

$$= \begin{cases} \pi - 2x & \text{for } \pi > x > 0 \\ \pi + 2x & \text{for } 0 > x > -\pi \end{cases}$$

$= f(x)$.

This shows that $f(x)$ is an even function. So, the Fourier series of $f(x)$ is same as the Fourier cosine series of $f(x)$.

Now, Fourier cosine series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{.......(i)}$$

with $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx$ and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\,dx$

Here, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(\pi - 2x)\,dx = \dfrac{1}{\pi}[\pi x - x^2]_0^{\pi} = \dfrac{1}{\pi}(\pi^2 - \pi^2) = 0.$

And,

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos nx\,dx = \frac{2}{\pi}\left[(\pi - 2x)\frac{\sin nx}{n} - (-2)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{4(1 - \cos n\pi)}{\pi n^2} \quad \text{for } n = 1, 2, 3, ..$$

$[\because \sin n\pi = 0 \text{ for } n = 1, 2, 3, \text{........}]$

$$= \frac{4(1 - (-1)^n)}{\pi n^2}$$

$$= \begin{cases} 8/\pi n^2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

Then (i) becomes,

$$f(x) = \frac{8}{\pi}\sum_{n\text{-odd}} \frac{\cos nx}{n^2}$$

$$\Rightarrow f(x) = \frac{8}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \text{.....}\right).$$

This is required Fourier series for $f(x)$.

(v) $f(x) = x^2$ if $0 < x < 2$.

Solution: Let $f(x) = x^2$ for $0 < x < 2$.

Clearly $f(x)$ is 2-periodic function. So, $l = 1$.

Now, Fourier series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\pi x) + b_n\sin(n\pi x)] \quad \text{..........(i)}$$

with $a_0 = \dfrac{1}{2}\displaystyle\int_0^2 f(x)\,dx$, $a_n = \displaystyle\int_0^2 f(x)\cos(n\pi x)\,dx$, $b_n = \displaystyle\int_0^2 f(x)\sin(n\pi x)\,dx$

Here,

$$a_0 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

and, $a_n = \displaystyle\int_0^2 x^2\cos(n\pi x)\,dx$

$$= \left[x^2\frac{\sin(n\pi x)}{n\pi} - 2x\left(-\frac{\cos(n\pi x)}{(n\pi)^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \frac{4\cos 2n\pi}{n^2\pi^2} \quad \text{for } n = 1, 2, 3, \text{....} \quad [\because \sin n\pi = 0 \text{ for } n = 1, 2, \text{......}]$$

$$= \frac{4}{n^2\pi^2} \quad [\because \cos 2n\pi = 1 \quad \text{for } n = 1, 2, 3, \text{.......}]$$

Also,

$$b_n = \int_0^2 x^2\sin n\pi x\,dx$$

$$= \left[x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\frac{-4}{n\pi} + \frac{2}{n^3\pi^3}(1 - 1)\right] \quad \text{for } n = 1, 2, 3, \text{........}$$

$[\because \cos 2n\pi = 1,\ \sin n\pi = 0, \quad \text{for } n = 1, 2, \text{.......}]$

$$= \frac{-4}{n\pi}$$

Therefore (i) becomes,

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{4\cos n\pi x}{n^2\pi^2} - \frac{4\sin n\pi x}{n\pi}\right).$$

$$= \frac{4}{3} + \frac{4}{x^2}\left(\cos\pi x + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \text{......}\right) - \frac{4}{\pi}\left(\sin\pi x + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \text{.....}\right)$$

This is required Fourier series of $f(x)$.

2. Find Fourier series of $f(x) = x - x^2$ if $-\pi < x < \pi$ and show that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \text{......} = \dfrac{\pi^2}{12}$.

Solution: Let $f(x) = x - x^2$ for $-\pi < x < \pi$.

Clearly $f(x)$ is 2π – periodic function.

Now, Fourier series of f(x) is.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad ......(i)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Here,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right) = \frac{-2\pi^3}{6\pi} = \frac{\pi^2}{3}.$$

and,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{\cos n\pi}{n^2}(1 - 2\pi - 1 - 2\pi) \quad \text{for } n = 1, 2, 3, ...$$

$[\because \sin n\pi = 0$ for $n = 1, 2, ...]$

$$= \frac{-4\cos n\pi}{n^2} = \frac{-4(-1)^n}{n^2}$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi + \pi^2}{n}\cos n\pi - \frac{\pi + \pi^2}{n}\cos n\pi - \frac{2}{n^3}(\cos n\pi - \cos n\pi)\right]$$

for $n = 1, 2, 3, ...$ $[\because \sin n\pi = 0$ for $n = 1, 2, 3, ...]$

$$= \frac{1}{\pi}\left(\frac{-\pi + \pi^2 - \pi - \pi^2}{n}\right)(-1)^n$$

$$= \frac{-2(-1)^n}{n}$$

Therefore (i) becomes,

$$f(x) = \frac{-\pi^2}{3} - 2\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\cos nx + \frac{(-1)^n}{n}\sin nx\right) \quad ......(ii)$$

This is required Fourier series of f(x).

Set $x = 0$ then $f(0) = 0$. So, (ii) gives,

$$0 = -(-\pi^2/3) - 2\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}$$

3. **Find Fourier series of $f(x) = e^{-x}$ if $0 < x < 2\pi$.**

Solution: Let $f(x) = e^{-x}$ for $0 < x < 2\pi$.

Clearly f(x) is 2π-periodic function.

Now, Fourier series of f(x) on $(0, 2\pi)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad ......(i)$$

with $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Here, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{2\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{1 - e^{-2\pi}}{2\pi}.$

and, $a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1 + n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$

$$= \frac{1}{\pi(1 + n^2)}(-e^{-2\pi} + 1) \quad [\because \cos 2n\pi = 1]$$

for $n = 1, 2, 3, ...$

Also, $b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1 + n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$

$$= \frac{n\,(1 - e^{-2\pi})}{\pi\,(1 + n^2)} \quad \text{for } n = 1, 2, 3, ...$$

Then (i) becomes,

$$f(x) = \left(\frac{1 - e^{-2\pi}}{\pi}\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx + n\sin nx}{1 + n^2}\right)\right]$$

This is required Fourier series of f(x) on $(0, 2\pi)$.

4. Find Fourier series of $f(x) = |x|$ if $-\pi < x < \pi$ and show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} .... = \frac{\pi^2}{8}$. [2010 Fall Q. No. 6(a)]

Solution: Let $f(x) = |x|$ for $-\pi < x < \pi$

Clearly f(x) is 2π-periodic function.

Here,

$f(-x) = |-x|$ for $-\pi < -x < \pi$

$= |x|$ for $\pi > x > -\pi$

$= f(x).$

This shows that f(x) is even. Therefore, the Fourier series of f(x) is same as the Fourier cosine series of f(x). For Fourier cosine series, see Q.10 (i), Exercise 3.1

By first part we have,

$$f(x) = \frac{\pi}{2} - \left[\frac{4}{\pi}\sum_{n-odd}\frac{1}{n^2}\cos(nx) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right] \quad .....(i)$$

Set $x = 0$ then $f(x) = 0$. So that (i) reduces to

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n-odd}\frac{1}{n^2} \Rightarrow \sum_{n-odd}\frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\Rightarrow -1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

5. **Expand the function $f(x) = x \sin x$ if $-\pi \le x \le \pi$ by using Fourier series.**

Solution: Let, $f(x) = x \sin x \quad$ for $-\pi \le x \le \pi$.

Clearly, $f(x)$ is 2π-periodic function.

Here,

$$f(-x) = (-x)\sin(-x) \quad \text{for } -\pi \le -x \le \pi$$
$$= x \sin x \quad \text{for } \pi \ge x \ge -\pi$$
$$= f(x)$$

This shows that $f(x)$ is even. Therefore, the Fourier series of $f(x)$ is same as the Fourier cosine series of $f(x)$.

Now, Fourier cosine series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad ....(i)$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

Here, $a_0 = \frac{1}{\pi}\int_0^{\pi} x \sin x\,dx = \frac{1}{\pi}[x(-\cos x) - 1(-\sin x)]_0^{\pi} = \frac{1}{\pi}\cdot\pi = 1$

And, $a_n = \frac{1}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\} - \frac{\sin(n-1)\pi}{(n-1)^2}\right] \quad \text{for } n = 1, 2, 3, ...$$

$$[\because \sin n\pi = 0 \text{ for } n = 1, 2, ...]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1}\right\}\right] \quad \text{for } n = 2, 3, ...$$

$$= (-1)^{n+1}\left(\frac{1}{n-1} - \frac{1}{n-1}\right)$$

$$= (-1)^{n+1}\left(\frac{-2}{n^2-1}\right) \quad ....(ii)$$

Also,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{-\pi}{2\pi} = -\frac{1}{2}$$

Therefore (i) becomes,

$$f(x) = 1 - \frac{\cos x}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx.$$

This is required Fourier series of $f(x)$.

6. **Obtain the Fourier series of $f(x)$ where $f(x) = \begin{cases} 1 + 2x/\pi & \text{for } -1 \le x \le 0 \\ 1 - 2x/\pi & \text{for } 0 \le x \le 1 \end{cases}$ and show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$.**

Solution: Let, $f(x) = \begin{cases} 1 + 2x/\pi & \text{for } -1 \le x \le 0 \\ 1 - 2x/\pi & \text{for } 0 \le x \le 1 \end{cases}$

Clearly $f(x)$ is 2-periodic function.

Here,

$$f(-x) = \begin{cases} 1 - 2x/\pi & \text{for } -1 \le -x \le 0 \\ 1 + 2x/\pi & \text{for } 0 \le -x \le 1 \end{cases}$$

$$= \begin{cases} 1 - 2x/\pi & \text{for } 1 \ge x \ge 0 \\ 1 + 2x/\pi & \text{for } 0 \ge x \ge -1 \end{cases}$$

$$= f(x)$$

This shows that $f(x)$ is an even function. So, the Fourier series of $f(x)$ is same as the Fourier cosine series of $f(x)$.

Now, Fourier cosine series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad ......(i)$$

with $a_0 = \frac{1}{2}\int_0^1 f(x)\,dx$ and $a_n = \int_0^1 f(x)\cos(n\pi x)\,dx$.

Here, $a_0 = \int_0^1\left(1 + \frac{2x}{\pi}\right)dx = \left[x + \frac{x^2}{\pi}\right]_0^1 = \left(1 + \frac{1}{\pi}\right)$.

And,

$$a_n = 2\int_0^1\left(1 + \frac{2x}{\pi}\right)\cos(n\pi x)\,dx$$

$$= 2\left[\left(1 + \frac{2x}{\pi}\right)\left(\frac{\sin n\pi x}{n\pi}\right) - \frac{2}{\pi}\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^3}(\cos n\pi - 1) \quad \text{for } n = 1, 2, 3, ... \quad [\because \sin n\pi = 0 \text{ for } n = 1, 2, 3, ...]$$

$$= \frac{4}{n^2\pi^3}[(-1)^n - 1]$$

$$= \begin{cases} -8/n^2\pi^3 & \text{for n is odd} \\ 0 & \text{for n is even} \end{cases}$$

Therefore, (i) becomes,

$$f(x) = \left(1 + \frac{1}{\pi}\right) - \frac{8}{\pi^3} \sum_{n-odd} \left(\frac{\cos n\pi x}{n^2}\right) \quad ......(ii)$$

This is required Fourier series of f(x) on [–1, 1].

Set x = 0 then f(0) = 1. Then (ii) gives,

$$1 = 1 + \frac{1}{\pi} - \frac{8}{\pi^3} \sum_{n-odd} \left(\frac{1}{n^2}\right)$$

$$\Rightarrow \sum_{n-odd} \left(\frac{1}{n^2}\right) = \frac{\pi^3}{8\pi} = \frac{\pi^2}{8}$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$$

7. Find the Fourier series of $f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$ **and show that** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Solution: First part: See from exercise 3.1, Q.8 (iii).

From the first part we have,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$$

At x = 0, we get f(0) = 0. Then

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos 0}{(2n+1)^2} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{4} \times \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

8. Find the fourier series of $f(x) = \begin{cases} -k & \text{when } -\pi \le x \le 0 \\ k & \text{when } 0 < x \le \pi \end{cases}$ **and show that** $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

Solution: Given that, $f(x) = \begin{cases} -k & \text{for } -\pi \le x \le 0 \\ k & \text{for } 0 \le x \le \pi \end{cases}$

$$= k \begin{cases} -1 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 \le x \le \pi \end{cases}$$

Set, $g(x) = \begin{cases} -1 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 \le x \le \pi \end{cases}$

Then process as similar to Q.4 (v), Exercise 3.1 and then multiply the result by k to each term that is the required Fourier series for f(x).

Then result is,

$$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right].$$

If we take $x = \frac{\pi}{2}$ then $f\left(\frac{\pi}{2}\right) = k$. So that,

$$k = \frac{4k}{\pi}\left[\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + ...\right].$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ......$$

Find Fourier series of $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \pi \\ 2 & \text{if } \pi < x \le 2\pi \end{cases}$ **and show that** $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

Solution: Given that, $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 2 & \text{for } \pi \le x \le 2\pi \end{cases}$

Clearly, f(x) is of 2π–periodic function.

The Fourier series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad ....(i)$$

with, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$.

Here,

$$a_0 = \frac{1}{2\pi}\left\{\int_0^{\pi} dx + 2\int_{\pi}^{2\pi} dx\right\} = \frac{1}{2\pi}\left\{[x]_0^{\pi} + 2\,[x]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{2\pi}(\pi + 4\pi - 2\pi) = \frac{3}{2}.$$

And,

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} \cos nx\,dx + 2\int_{\pi}^{2\pi} \cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin x}{n}\right]_0^{\pi} + 2\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}.0 = 0 \quad [\because \sin n\pi = 0 = \sin 2n\pi]$$

Also,

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} \sin nx\,dx + 2\int_{\pi}^{2\pi} \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\cos nx}{n}\right]_0^{\pi} + 2\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{n\pi}[1 - \cos n\pi + 2\cos n\pi - 2\cos 2n\pi]$$

$$= \frac{1}{n\pi}(1 + \cos n\pi - 2\cos 2n\pi)$$

Now (i) becomes,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1 + \cos n\pi - 2\cos 2n\pi)\sin nx$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(\cos n\pi - 1)\sin x \quad ....(ii) \quad [\because \cos 2n\pi = 1]$$

Since, $\cos n\pi - 1 = 1 - 1 = 0$ for n is even
$\cos n\pi - 1 = -1 - 1 = -2$ for n is odd.

So, (ii) becomes,

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\sin x + \frac{\sin 3}{3} + \frac{\sin 5x}{5} + ...\right]$$

This is the required Fourier series for f(x).

If we set $x = \frac{\pi}{2}$ then $f(\pi/2) = 1$. So,

$$1 = \frac{3}{2} - \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

10. Find the Fourier series of $f(x) = x - x^2$ if $-1 < x < 1$.
Solution: Similar to Q.2, Exercise 3.2.

11. Find the Fourier series of $f(x) = x^2 - 2$ if $-2 \le x \le 2$.
Solution: Similar to Q.2, Exercise 3.2.

12. Obtain the Fourier series of $f(x) = \begin{cases} \pi x & \text{if } 0 \le x \le 1 \\ \pi(2-x) & \text{if } 1 \le x \le 2 \end{cases}$

Solution: Given that, $f(x) = \begin{cases} \pi x & \text{for } 0 \le x \le 1 \\ \pi(2-x) & \text{for } 1 \le x \le 2 \end{cases}$

Clearly f(x) is 2–periodic function whose Fourier series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} f(x)dx, \quad a_n = \int_0^2 f(x)\cos n\pi x\, dx, \quad b_n = \int_0^2 f(x)\sin n\pi x\, dx$$

Here,

$$a_0 = \frac{1}{2}\left\{\int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi x^2}{2}\right]_0^1 + \left[2\pi x - \frac{\pi x^2}{2}\right]_1^2\right\} = \frac{1}{2}\left\{\frac{\pi}{2} + 4\pi - 2\pi - 2\pi + \frac{\pi}{2}\right\} = \frac{\pi}{2}.$$

and,

$$a_n = \left\{\pi\int_0^1 x\cos n\pi x\, dx + \pi\int_1^2 (2-x)\cos n\pi x\, dx\right\}$$

$$= \pi\left\{\left[x\,\frac{\sin n\pi x}{n\pi} - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(2-x)\frac{\sin n\pi x}{n\pi} - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2\right\}$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{\cos 2n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right]$$

$[\because \cos 2n\pi = 1]$

$$= \frac{2\pi}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1)$$

Also,

$$b_n = \pi\int_0^1 x\sin n\pi x\, dx + \pi\int_1^2 (2-x)\sin n\pi x\, dx$$

$$= \pi\left\{\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[(2-x)\frac{-\cos n\pi x}{n\pi} - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2\right\}$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi}\right]$$

$$= \pi \cdot 0 = 0$$

Now (i) becomes,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2} + (\cos\pi - 1)\cos n\pi x\right] \quad ......(ii)$$

Since, $\cos n\pi - 1 = 1 - 1 = 0$ if n is even
and $\cos n\pi - 1 = -1 - 1 = -2$ if n is odd.

Then (ii) becomes,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos n\pi}{1^2} + \frac{\cos 3n\pi}{3^2} + \frac{\cos 5n\pi}{5^2} + \frac{\cos 7n\pi}{7^2} + ...\right]$$

This is required Fourier series for f(x).

13. Find Fourier series of $f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$

Solution: Given that, $f(x) = \begin{cases} 0 & \text{for } -2 < x < -1 \\ k & \text{for } -1 < x < 1 \\ 0 & \text{for } 1 < x < 2. \end{cases}$

Clearly f(x) is 4–periodic function whose Fourier series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{2}\right) \quad .....(i)$$

with, $a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\, dx$, $a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx$, $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx$

Here,

$$a_0 = \frac{1}{4}\left(\int_{-2}^{-1} + \int_{-1}^{1} + \int_{1}^{2}\right) f(x)\, dx = \frac{1}{4}\int_{-1}^{1} dx = \frac{k}{4}[x]_{-1}^{1} = \frac{2k}{4} = \frac{k}{2}.$$

and,

$$a_n = \frac{1}{2}\int_{-1}^{1}\cos\left(\frac{n\pi x}{2}\right) dx = \frac{k}{2}\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1}$$

$$b_n = \frac{k}{2}\int_{-1}^{1} \sin\left(\frac{n\pi x}{2}\right) dx = \frac{k}{2}\left[\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1}$$

$$= \frac{k}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right] = 0.$$

Now (i) becomes,

$$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty} \frac{2k}{n\pi} \sin \cos\left(\frac{n\pi x}{2}\right) \quad \text{.......(ii)}$$

We know that, $\sin\left(\frac{n\pi}{2}\right) = 0$ for n is even

$\sin\left(\frac{n\pi}{2}\right) = 1$ for $n = 1, 5, 9, ...$

$\sin\left(\frac{n\pi}{2}\right) = -1$ for $n = 3, 7, 11, ...$

Therefore (ii) becomes,

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + ...\right]$$

This is required Fourier series for f(x).

14. Expand $f(x) = \pi x - x^2$ is $0 \le x \le \pi$ in a Fourier sine series.

Solution: See the solution part of Q.15, Exercise 3.1.

15. If $f(x) = \begin{cases} x & \text{if } 0 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < \pi \end{cases}$. Show that:

(i) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - ......\right)$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\right)$

Solution: (i) See the solution part of Q.13, Exercise 3.1.

(ii) Given that, $f(x) = \begin{cases} x & \text{for } 0 < x < \pi/2 \\ \pi - x & \text{for } \pi/2 < x < \pi \end{cases}$

Clearly f(x) is of half range 2π–periodic function. So, the Fourier cosine series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) \quad \text{....... (i)}$$

with $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$

Here,

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right\} = \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right)\right]$$

$$= \frac{\pi^2}{8\pi}[1 + 8 - 4 - 4 + 1]$$

$$= \frac{\pi}{8} \times 2 = \frac{\pi}{4}.$$

Also,

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos\left(\frac{n\pi x}{\pi}\right) dx + \int_{\pi/2}^{\pi} (\pi - x) \cos\left(\frac{n\pi x}{\pi}\right) dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{2\pi n^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1 - \cos n\pi + \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - \cos n\pi\right]$$

$$= \frac{4}{4\pi n^2}(\cos n\pi - 1)$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1).$$

Now (i) becomes,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos(2nx) \quad \text{.....(ii)}$$

Since $-\cos n\pi - 1 = -1 - 1 = -2$ for n is even

$-\cos n\pi - 1 = 1 - 1 = 0$ for n is odd.

$\cos\left(\frac{n\pi}{2}\right) = 0$ if n is odd and $\cos\left(\frac{n\pi}{2}\right) = 0$ if n is even.

Therefore (ii) becomes,

$$f(x) = \frac{\pi}{4} + \sum_{n\text{–odd}} \left(\frac{-2}{\pi n^2}\right)\cos 2nx$$

$$\Rightarrow n(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\right].$$

16. Find Fourier cosine series of $f(x) = x \sin x$ in $(0, \pi)$ and show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} .... = \frac{\pi - 2}{4}$.

Solution: Given that, $f(x) = x \sin x$ for $0 < x < \pi$.

Clearly f(x) is of π–periodic function. The Fourier cosine series of f(x) is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{.......(i)}$$

With $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{1}{\pi}[x(-\cos x) - (1)(-\sin x)]_0^{\pi} = -\frac{\pi\cos\pi}{\pi} = 1.$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left[x\frac{-\cos 2x}{2} - (1)\frac{-\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-1}{2}\right)\right] = \frac{-1}{2}.$$

and,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx \quad \text{for } n \ge 2.$$

$$= \frac{2}{\pi}\times\frac{1}{2}\int_0^{\pi} x[\sin(nx+x) - \sin(nx-x)]\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right) - x\left(\frac{-\cos(n-1)x}{n-1}\right) + (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{n+1} + \frac{\sin(n+1)\pi}{(n+1)^2} + \frac{\pi\cos(n-1)\pi}{n-1} - \frac{\sin(n-1)\pi}{(n-1)^2}\right].$$

Then (i) becomes,

$$f(x) = 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\left[\left(\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right) + \frac{1}{\pi}\left(\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1}\right)\right]\cos(nx)$$

$$= 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\left[\left(\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right)\right]\cos(nx)$$

$$\text{.........(ii)}$$

Since, $\frac{\sin(n-1)\pi}{n-1}$ has 0 value for $n = 2, 3, \ldots$

Similarly, $\frac{\sin(n+1)\pi}{n+1} = 0$ for every $n = 2, 3, \ldots\ldots$

Also, $\cos(n-1)\pi = (-1)^{n-1}$ and $\cos(n+1)\pi = (-1)^{n+1}$

Therefore,

$$\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} = \frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} \quad \text{for } n \ge 2.$$

$$= (-1)^{2n}\left(\frac{(-1)^{-1}}{n-1} - \frac{(-1)^{1}}{n+1}\right)$$

$$= \left(\frac{-1}{n-1} + \frac{1}{n+1}\right)$$

$$= \left(\frac{-2}{n^2-1}\right)$$

Now, (ii) becomes,

$$f(x) = 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\left(\frac{-2}{n^2-1}\right)\cos(nx) \quad \text{.......(iii)}$$

$$\Rightarrow f(x) = 1 - \frac{\cos x}{2} - 2\left(\frac{\cos 2x}{3} + \frac{\cos 3x}{8} + \ldots\right)$$

This is required Fourier cosine series for f(x).

In particular if we take $x = \frac{\pi}{2}$ then by (iii),

$$f(\pi/2) = 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\left(\frac{-2}{n^2-1}\right)\cos(nx)$$

$$\Rightarrow \frac{\pi}{2} = 1 + \sum_{n=2}^{\infty}\left(\frac{-2}{n^2-1}\right)\cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi-2}{2} = -2\sum_{n=2}^{\infty}\left[\frac{1}{(2n-1)(2n+1)}\right]\cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi-2}{4} = -\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \frac{1}{7.9} - \ldots\right]$$

$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots$$

17. Find Fourier sine series of $f(x) = e^x$ if $0 < x < 1$.

Solution: Given function is, $f(x) = e^x$ for $0 < x < 1$.

Clearly f(x) is 2–periodic function. The Fourier sine series of f(x) is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{.......(i)}$$

with $b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$.

Here,

$$b_n = 2\int_0^1 e^x \sin(n\pi x)\,dx$$

$$= 2\left[\frac{e^x}{1^2+n^2\pi^2}(\sin(n\pi x) - n\pi\cos(n\pi x))\right]_0^1$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c\right]$$

$$= 2\left(\frac{1}{1+n^2\pi^2}\right)[n\pi - (-1)^n e n\pi] \quad \left[\begin{array}{l}\because \cos n\pi = (-1)^n \\ \sin n\pi = 0\end{array}\right]$$

$$= \frac{4n\pi}{1+n^2\pi^2}[1 - e(-1)^n].$$

Now, (i) becomes,

$$f(x) = \sum_{n=1}^{\infty}\frac{4n\pi}{1+n^2\pi^2}[1-e(-1)^n]\sin(n\pi x).$$

18. Obtain the Fourier cosine series of $f(x) = \begin{cases} kx & \text{for } 0 \le x \le l/2 \\ k(l-x) & \text{for } l/2 \le x \le l \end{cases}$ and show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots = \frac{\pi^2}{8}$.

Solution: Given function is, $f(x) = \begin{cases} kx & \text{for } 0 \le x \le l/2 \\ k(l-x) & \text{for } l/2 \le x \le l \end{cases}$

Clearly, the function is of $2l$–periodic.

The Fourier cosine series of f(x) is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) \quad \text{.......(i)}$$

with $a_0 = \frac{1}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$.

Here,

$$a_0 = \frac{1}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right\}$$

$$= \frac{k}{l}\left\{\left[\frac{x^2}{2}\right]_0^{l/2} + \left[lx - \frac{x^2}{2}\right]_{l/2}^{l}\right\} = \frac{k}{l}\left[\frac{l^2}{8} + l^2 - \frac{l^2}{2} - \frac{l^2}{2} + \frac{l^2}{8}\right]$$

$$= \frac{l^2k}{8l}[1+8-4-4+1] = \frac{kl}{4}$$

and,

$$a_n = \frac{2k}{l}\left\{\int_0^{l/2} x\cos\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2k}{l}\left\{\left[x\left(\frac{\sin(n\pi x/l)}{2n\pi/l}\right) - (1)\left(\frac{-\cos(n\pi x/l)}{(n\pi/l)^2}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{\sin(n\pi x/l)}{n\pi/l}\right) - (-1)\left(\frac{-\cos(n\pi x/l)}{(n\pi/l)^2}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left[\left(\frac{l}{n\pi}\right)^2(\cos n\pi - 1) - \left(\frac{l}{n\pi}\right)^2(\cos n\pi - \cos n\pi)\right]$$

$$[\because \sin n\pi = 0 = \sin 0]$$

$$= \frac{kl}{n^2\pi^2}[(\cos n\pi - 1) + (\cos n\pi - 1)] \quad [\because \cos n\pi = 1]$$

$$= \frac{2kl}{n^2\pi^2}(\cos n\pi - 1).$$

Therefore (i) becomes,

$$f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty}\frac{2kl}{n^2\pi^2}(\cos n\pi - 1)\cos\left(\frac{n\pi x}{l}\right) \quad \text{........(ii)}$$

Since, $\cos n\pi - 1 = 1 - 1 = 0$ for n is even

$\cos n\pi - 1 = -1 - 1 = -2$ for n is odd.

Therefore (ii) becomes,

$$f(x) = \frac{kl}{4} - \frac{4kl}{\pi^2}\sum_{n\text{–odd}}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x) = \frac{kl}{4} - \frac{4kl}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{l}\right) + \ldots\right]$$

This is required Fourier cosine series of f(x).

And, at x = 0 we get, f(0) = 0. Then,

$$0 = \frac{kl}{4} - \frac{4kl}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots..\right]$$

$$\Rightarrow \frac{kl}{4}\times\frac{\pi^2}{4kl} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots..$$

$$\Rightarrow \frac{\pi^2}{16} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots..$$

OTHER IMPORTANT QUESTION FROM FINAL EXAM

FOURIER SERIES

2014 Fall Q. No. 2(a)

Define Fourier series representation of a periodic function f(x) with period 2π. Find the fourier series representation of the periodic function $f(x) = \frac{x^2}{2}$ for $-\pi < x < \pi$.

Solution: First Part: See the definition of the Fourier series.

Second Part: See the solution part of Exercise 3.1 Q. No. 6(iii).

2014 Fall Q. No. 6(b)

Find Fourier series of $f(x) = \begin{cases} k & \text{for } 0 \le x < \pi \\ 0 & \text{for } \pi \le x < 2\pi \end{cases}$

Solution: See solution of Exercise 3.1 Q. No. 8(ii) with multiplying by k.

2013 Fall Q. No. 3(a)

Find Fourier series of $F(x) = \begin{cases} 1 & \text{for } 0 \le x < \pi \\ 0 & \text{for } \pi \le x < 2\pi \end{cases}$.

Solution: Similar to 2014.

2012 Fall Q. No. 5(a); 2006 Spring Q.No. 5(a).

Find the Fourier series of the function, $f(x) = \begin{cases} x & 0 < x < 1 \\ 1 - x & 1 < x < 2 \end{cases}$.

Solution: Similar to the solution of Exercise 3.2, Q. No. 12.

2011 Spring Q. No. 6(a); 2001 Q. No. 6(b)

Find the Fourier series representation of the periodic function $f(x) = |x|$ $-\pi < x < \pi$.

Solution: Process as in 2009 Fall Q. No. 6(a).

2010 Spring Q. No. 4(a)

Find the Fourier series of the periodic function $f(x) = x^2$ for $-\pi < x < \pi$.

Solution: Similar to the solution of Exercise 3.1, Q. No. 4(iv).

2013 Spring Q. No. 4(a); 2009 Fall Q. No. 6(a)

Write the Fourier coefficient of a function f(x). Find the Fourier expansion of the function $f(x) = x + |x|$ $(-\pi < x < \pi)$.

Solution: First Part: The Fourier series of f(x) for $2l$-period is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

with, $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$, $\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$,

and $\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$.

Here a_0, a_n and b_n are Fourier coefficients of f(x).

Second part: Given that, $f(x) = x + |x|$ for $-\pi < x < \pi$

Clearly f(x) is 2π-periodic function

The Fourier series of f(x) with 2π-period is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad .....(i)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

Here, $f(x) = x + |x|$ for $-\pi < x < \pi$

$$= \begin{cases} x - x & \text{for } -\pi < x < 0 \\ x + x & \text{for } 0 < x < \pi \end{cases}$$

$$= \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 2x & \text{for } 0 < x < \pi \end{cases}$$

So,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 2x\,dx = \frac{1}{2\pi}[x^2]_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}.$$

and,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(\cos n\pi - 1)\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{n}\cos n\pi\right) = -\frac{2}{n}\cos n\pi.$$

Now, (i) becomes,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}(\cos n\pi - 1)\cos nx - \frac{2}{n}\cos n\pi \sin nx\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}[(-1)^n - 1]\cos nx - \frac{2}{n}(-1)^n \sin nx\right] \quad .....(ii)$$

Since $(-1)^n - 1 = 0$ for n is even

$= -2$ for n is odd.

Then (ii) gives,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right)$$

This is the required Fourier series of f(x).

2009 Spring Q. No. 6(a)

Define periodic function. Find the Fourier series representation of the periodic function $f(x) = \frac{x^2}{2}$ for $-\pi \le x \le \pi$ and then show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$ and $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \frac{\pi^2}{12}$.

Solution: First Part: See the definition of periodic function.

Second Part: See the solution of Exercise 3.1, Q. No. 6(iii).

2008 Spring Q. No. 6(a)

Define periodic function with suitable example. Find the Fourier series of the periodic function $f(x) = x^2$ for $-\pi < x < \pi$. Using it shows that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$.

Solution: First Part: See the definition of periodic function.

Second Part: Similar to the solution of Exercise 3.1, Q. No. 6(iii).

2007 Fall Q. No. 6(a)

Define periodic function. Find the Flourier series of the function $f(x) = |x|$; $(-2 < x < 2)$, $p = 2L = 4$.

Solution: First Part: See the definition of periodic function.

Second Part: Similar to the solution of Exercise 3.2, Q. No. 4.

2007 Fall Q. No. 6(b)

Find the Fourier series of the following function: $f(x) = \begin{cases} 0, & -1 < x < 0 \\ -2x, & 0 \le x < 1 \end{cases}$.

Solution: Similar to second part of 2009 Spring Q.No. 6(a).

2006 Spring Q.No. 5(b)

Define an even and an odd function. Find the Fourier series of the function $f(x) = \frac{x^2}{2}$ for $-\pi < x < \pi$. Hence show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$.

Solution: First Part: See the definition of the odd and even function.

Second Part: See the solution part of Exercise 3.1 Q. No. 6(iii).

2005 Fall Q. No. 5(a)

Show that the product of two odd functions is an even function. Find the Fourier series of the following: $f(x) = \begin{cases} x & \text{if } \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$.

Solution: First Part: See the result for odd function.

Second Part: See the solution of Exercise 3.1 Q. No. 6(ii).

2004 Fall Q. No. 6(a)

Write the Fourier series of periodic function f(x) with period 2π and hence find the Fourier series of f(x) where $f(x) = x$ if $-\pi < x < \pi$.

Solution: First Part: See the definition of Fourier series with period 2π.

Second Part: See the solution of Exercise 3.1 Q. No. 4(iii).

2004 Spring Q. No. 6(b)

Find the Fourier series of the function having period 2π, $f(x) = \frac{x^2}{2}$ for $-\pi < x < \pi$ and hence show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \frac{\pi^2}{12}$.

Solution: See the solution part of 2009 Spring.

2003 Spring Q. No. 6(a)

Write the Fourier series of periodic function f(x) with period $p = 2l$ and hence find the Fourier series of $f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ -1 & \text{for } 0 < x < 1 \end{cases}$.

Solution: First Part: See the definition of Fourier series with $2l$ period.

Second Part: See the solution of Exercise 3.1 Q. No. 4(v).

2003 Spring Q. No. 6(b)

Find the Fourier of the function $f(x) = \frac{x^2}{2}$ if $-\pi < x < \pi$ with period 2π

Solution: See the solution part of Exercise 3.1 Q. No. 6(iii).

2002 Q. No. 6(a)

Define periodic function. Find the Fourier series of a function

$$f(x) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases} \quad \text{with period } p = 2\pi.$$

Solution: First Part: See the definition of periodic function.

Second Part: See the solution of Exercise 3.1 Q. No. 4(vi).

2002 Q. No. 6(a)

Write down the period of sinx. Find the Fourier series of the function:

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2, \end{cases} \quad p = 4$$

Solution: First Part: See the solution of Exercise 3.1 Q. No. 1.

Second Part: See the solution of Exercise 3.2 Q. No. 13.

2001 Q. No. 6(a)

What is the period cos nx? Find the Fourier series of the function f(t) of period T. $f(t) = \begin{cases} 0 & \text{for } -2 < t < 0 \\ 1 & \text{for } 0 < t < 2, \end{cases}$ $T = 4$

Solution: First Part: See the solution of Exercise 3.1 Q. No. 1.

Second Part: See the solution of Exercise 3.1 Q. No. 9(iii).

FOURIER COSINE AND SINE SERIES

2012 Fall Q. No. 5(b)

Find the Fourier cosine and sine series of the function,

$$f(x) = \begin{cases} x \text{ if } & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \pi - x \text{ if } & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

Solution: See the solution part of Exercise 3.2 Q. No. 4.

2011 Spring Q. No. 6(b); 2009 Fall Q. No. 6(b)

Find Fourier sine as well as cosine series representation of the half range function $f(x) = x^2$ for $0 < x < L$.

Solution: We have, $f(x) = x^2$ for $0 < x < L$.

Then its Fourier cosine series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad \text{.......(1)}$$

where,

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{L^2}{3}.$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^L x^2\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\left[x^2\frac{\sin\frac{n\pi}{L}x}{\frac{n\pi}{L}} - 2x\frac{\cos\frac{n\pi}{L}x}{-\left(\frac{n\pi}{L}\right)^2} + 2\frac{\sin\frac{n\pi}{L}x}{-\left(\frac{n\pi}{L}\right)^3}\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L^2\sin n\pi}{\frac{n\pi}{L}} + 2L\frac{\cos n\pi}{\left(\frac{n\pi}{L}\right)^2} - 2\frac{\sin n\pi}{\left(\frac{n\pi}{L}\right)^3}\right]$$

$$= \frac{2}{L}\frac{2L}{\left(\frac{n\pi}{L}\right)^2}\cos n\pi$$

$$= \frac{4L^2}{(n\pi)^2}\cos n\pi$$

Substituting these values in equation (1) we get

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2}{(n\pi)^2}\cos n\pi\cos\frac{n\pi}{L}x$$

$$\Rightarrow f(x) = \frac{L^2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{L^2}{n^2}\cos n\pi\cos\frac{n\pi}{L}x$$

This is the required fourier cosine series of the given function.

Again, we know fourier sine series of f(x) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad \text{.........(2)}$$

where,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^L x^2\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\left[x^2\frac{\cos\frac{n\pi}{L}x}{\frac{-n\pi}{L}} - 2x\frac{\sin\frac{n\pi}{L}x}{-\left(\frac{n\pi}{L}\right)^2} + 2\frac{\cos\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L$$

$$= \frac{2}{L}\left[\frac{-L^3}{n\pi}\cos n\pi + \frac{2}{\left(\frac{n\pi}{L}\right)^3}(\cos n\pi - 1)\right].$$

Substituting this value in equation (2), we get

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{L}\left\{\left(\frac{2L^3}{(n\pi)^3} - \frac{L^3}{n\pi}\right)\cos n\pi - \frac{2L^3}{(n\pi)^3}\right\}\sin\frac{n\pi}{L}x$$

This is the required Fourier sine series of the given function.

2003 Fall Q. No. 5(a)

What is the period of sin2x? Find the Fourier Sine series of the function $f(x) = x + |x|$ $(-\pi < x < \pi)$.

Solution: First Part: See the solution part of Exercise 3.1 Q. No. 1.

Second Part: See the solution of 2009 Fall.

2002 Q. No. 6(b)

Define even and odd functions. If $f(x) = x$, $0 < x < L$ then find (i) Fourier sine-series (ii) Fourier cosine-series.

Solution: First Part: See the definition of the odd and even function.

Second Part: See the solution of Exercise 3.1 Q. No. 7(i).

SHORT QUESTIONS

2014 Spring Q. No. 7(d): Find the smallest period of the function cos 2x.

Solution: Let p be the period of f(x). Then by definition,

$f(x + p) = f(x)$ for all x

$\Rightarrow \cos 2(x + p) = \cos 2x$

$\Rightarrow \cos 2x\cos 2p - \sin 2x\sin 2p = \cos 2x$

Comparing the coefficient we get,

$\cos 2p = 1 = \cos 2\pi$

$\Rightarrow p = \pi$

Thus required smallest period of $f(x) = \cos 2x$ is $p = \pi$.

2013 Spring Q. No. 7(c): Find the smallest period of the periodic function $f(x) = \sin\frac{n\pi}{L}x$.

Solution: Let p be the smallest period of f(x). Then,

$f(x + p) = f(x)$ for all x.

$$\Rightarrow \sin\frac{\pi n}{L}(x+p) = \sin\frac{n\pi}{L}x$$

$$\Rightarrow \sin\left(\frac{\pi nx}{L} + \frac{\pi np}{L}\right) = \sin\frac{n\pi}{L}x$$

$$\Rightarrow \sin\frac{\pi nx}{L}\cos\frac{\pi np}{L} + \cos\frac{\pi nx}{L}\sin\frac{\pi np}{L} = \sin\frac{\pi n}{L}x$$

Comparing coefficients, we get

$$\cos\frac{\pi np}{L} = 1 \text{ and } \sin\frac{n\pi p}{L} = 0$$

$$\Rightarrow \cos\frac{\pi np}{L} = \cos 2\pi \Rightarrow \frac{\pi np}{L} = 2\pi$$

$$\Rightarrow p = \frac{2L}{n}$$

Thus required smallest period of f(x) is $p = \frac{2L}{n}$

2012 Fall: Find the smallest period of the function $f(x) = \sin \pi x$.

Solution: See the solution of Exercise 3.1 Q. No. 1.

2011 Spring: Find the smallest period of the function cos2x.

Solution: See the solution of Exercise 3.1 Q. No. 1.

2010 Fall: Find the period of $f(x) = \cos\left(\frac{2\pi x}{k}\right)$.

Solution: Let p be the period of f(x). Then by definition,

$f(x + p) = f(x)$ for all x

$$\Rightarrow \cos\frac{2\pi}{k}(x+p) = \cos\frac{2\pi}{k}x$$

$$\Rightarrow \cos\left(\frac{2\pi}{k} + \frac{2\pi p}{k}\right) = \cos\frac{2\pi}{k}x$$

$$\Rightarrow \cos\frac{2\pi x}{k}\cos\frac{2\pi p}{k} - \sin\frac{2\pi p}{k}\sin\frac{2\pi}{k}x = \cos\frac{2\pi}{k}x$$

Comparing the coefficient we get,

$$\cos\frac{2\pi p}{k} = 1 \text{ and } \sin\frac{2\pi p}{k} = 0$$

$$\Rightarrow \frac{2\pi}{k}p = 2\pi$$

$$\Rightarrow p = k$$

Thus required smallest period of $f(x) = \cos\frac{2\pi}{k}x$ is $p = k$.

2009 Fall: Check even, odd or neither of $f(x) = x^3$ for $\frac{-\pi}{2} < x < \frac{3\pi}{2}$.

Solution: See the solution of Exercise 3.1 Q. No. 3(iii).

2008 Spring: Find the period of $f(x) = \cos 2x$.

Solution: See the solution of Exercise 3.1 Q. No. 3(iii).

2007 Fall: Find the smallest period of the periodic function $f(x) = \sin\left(\frac{2n\pi}{3}x\right)$.

Solution: Let p be the smallest period of f(x). Then,

$f(x + p) = f(x)$ for all x.

$$\Rightarrow \sin\frac{2\pi n}{k}(x+p) = \sin\frac{2\pi n}{k}x$$

$$\Rightarrow \sin\left(\frac{2\pi nx}{k} + \frac{2\pi np}{k}\right) = \sin\frac{2\pi nx}{k}$$

$$\Rightarrow \sin\frac{2\pi nx}{k}\cos\frac{2\pi np}{k} + \cos\frac{2\pi nx}{k}\sin\frac{2\pi np}{k} = \sin\frac{2\pi n}{k}x$$

Comparing coefficients, we get

$$\cos\frac{2\pi np}{k} = 1 \text{ and } \sin\frac{2n\pi p}{k} = 0$$

$$\Rightarrow \cos\frac{2\pi np}{k} = \cos 2\pi \Rightarrow \frac{2\pi np}{k} = 2\pi$$

$$\Rightarrow p = \frac{k}{n}$$

Thus required smallest period of f(x) is $p = \frac{k}{n}$

2006 Spring: Find the smallest period of $\sin 2\pi x$.

Solution: See the solution of Exercise 3.1 Q. No. 3(iii).

2004 Fall: When does f(x) is said to even or odd function? If $f(x) = x^3 \sin x$, examine whether f(x) is even function.

Solution: First part: See the properties of odd and even function.

Second Part: Given that, $f(x) = x^3 \sin x$

Here,

$$f(-x) = (-x)^3 \sin(-x) = (-x^3)(-\sin x) = x^3 \sin x = f(x).$$

This shows that f(x) is an even function.

2003 Spring: Define odd function. Examine whether $f(x) = x^3 \cos x$ is an even or odd function.

Solution: First part: See the properties of odd and even function.

Second Part: Given that, $f(x) = x^3 \cos x$

Here,

$$f(-x) = (-x)^3 \cos(-x) = (-x^3)(\cos x) = -x^3 \cos x = -f(x).$$

This shows that f(x) is an odd function.

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