CHAPTER 11

EXTREME OF FUNCTION OF SEVERAL VARIABLES

Maximum value and Minimum Value

Let f(x, y) be a function of two variables. Then we say f has maximum value at the point (a, b) if

$$f(x, y) < f(a, b)$$

for all (x, y) lies in some open disk containing (a, b).

Let f(x, y) be a function of two variables. Then we say f has minimum value at the point (a, b) if

$$f(x, y) > f(a, b)$$

for all (x, y) lies in some open disk containing (a, b).

Extreme point: The point (a, b) is said to be extreme point of given function f(x, y) if the function f(x, y) has maximum or minimum value at (a, b).

Stationary point (or critical point): The point, at which the first differential function is zero, is known as the stationary point of the function.

Saddle point: A point (a, b) is called the saddle point of the function f(x, y) if $f_{xx}f_{yy} - (f_{xy})^2$ is negative where the suffixes indicates the derivative of the function with respect to the variable in the suffix.

Condition to have extreme value for f(x, y) at (a, b)

Let f(x, y) be a function of two variables and (a, b) be its stationary point.

(a) Then f(x, y) has maximum value at (a, b) if it satisfies,

(i) $f_{xx} < 0$ and (ii) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$.

(b) and f(x, y) has minimum value at (a, b) if it satisfies

(i) $f_{xx} > 0$ and (ii) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$.

Condition to have extreme value for f(x, y, z) at (a, b, c)

Let f(x, y, z) be a function of two variables and (a, b, c) be its stationary point.

(a) Then f(x, y, z) has maximum value at (a, b, c) if it satisfies,

(i) $f_{xx} < 0$ (ii) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$ (iii) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$

(b) and f(x, y, z) has minimum value at (a, b, c) if it satisfies

(i) $f_{xx} > 0$ (ii) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$ (iii) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} > 0$

Method of Lagrange Multipliers

Suppose f(x, y, z) and g(x, y, z) are differentiable functions.

To find the local maximum and minima of f subject to the constraint g(x, y, z) = 0, we find a scalar value λ such that

$$\nabla f = \lambda \nabla g \text{ such that } g(x, y, z) = 0$$

Where $\nabla = \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$

Here, the scalar value λ is known as Lagrange's multipliers.

Exercise 11.1

1. Test the following surface for maxima, minima and saddle points. Find the functional values at these points.

(a) $z = x^2 + xy + y^2 + 3x - 3y + 4$

Solution: Given that

$$z = x^2 + xy + y^2 + 3x - 3y + 4$$

Then,

$$z_x = 2x + y + 3 \quad \text{and} \quad z_y = x + 2y - 3$$

$$z_{xx} = 2, \qquad z_{yy} = 2$$

Also, $z_{xy} = 1$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow \quad 2x + y + 3 = 0 \qquad \Rightarrow \quad x + 2y - 3 = 0$$

Solving these equations we get,

$$x = -3 \text{ and } y = 3.$$

Now,

$$z_{xx} = 2 > 0.$$

and, $$\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$$

Therefore, z is minimum at (–3, 3) and minimum value is,

$$z = 9 - 9 + 9 - 9 - 9 + 4 = -5.$$

(b) $z = 5xy - 7x^2 + 3x - 6y + 2$

Solution: Given function is,

$$z = 5xy - 7x^2 + 3x - 6y + 2$$

Then,

$$z_x = 5y - 14x + 3 \quad \text{and} \quad z_y = 5x - 6$$

$$z_{xx} = -14 \qquad z_{yy} = 0$$

Also, $z_{xy} = 5$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow \quad 5x - 14x + 3 = 0 \qquad \Rightarrow 5x - 6 = 0$$

Solving these equations we get

$$x = \frac{6}{5} \quad \text{and} \quad y = \frac{69}{5}$$

Now,

$$z_{xx} = -14 < 0.$$

$$\text{and} \quad \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -14 & 5 \\ 5 & 0 \end{vmatrix} = 0 - 25 = -25 < 0.$$

This shows that the function is saddle at $\left(\frac{6}{5}, \frac{69}{5}\right)$.

And value of z at the point is

$$z = \frac{2070}{25} - \frac{252}{25} + \frac{18}{5} - \frac{414}{5} + 2 = \frac{2070 - 252 + 90 - 2070 + 50}{25} = -\frac{112}{25}.$$

(c) $z = x^2 + xy + 3x + 2y + 5$

Solution: Given function is,

$$z = x^2 + xy + 3x + 2y + 5$$

Then,

$$z_x = 2x + y + 3 \quad \text{and} \quad z_y = x + 2$$

$$z_{xx} = 2 \qquad z_{yy} = 0$$

Also, $z_{xy} = 1$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow \quad 2x + y + 3 = 0 \qquad \Rightarrow \quad x + 2 = 0$$

Solving these equations we get,

$$x = -2, y = 1.$$

Now,

$$z_{xx} = 2 > 0$$

$$\text{and} \quad \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0.$$

This shows that the function is saddle at (–2, 1).

And, the value of z at the point is,

$$z = 4 - 2 - 6 + 2 + 5 = 3.$$

(d) $z = 2xy - 5x^2 - 2y^2 + 4x - 4$

Solution: Given function is

$$z = 2xy - 5x^2 - 2y^2 + 4x - 4$$

Then,

$$z_x = 2y - 10x + 4 \quad \text{and} \quad z_y = 2x - 4y$$

$$z_{xx} = -10 \qquad z_{yy} = -4$$

Also, $z_{xy} = 2$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow \quad 2y - 10x + 4 = 0 \qquad \Rightarrow \quad 2x - 4y = 0$$

Solving these equations we get,

$x = \frac{4}{9}$ and $y = \frac{2}{9}$.

Now, at $(x, y) = \left(\frac{4}{9}, \frac{2}{9}\right)$,

$$z_{xx} = -10 < 0.$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -10 & 2 \\ 2 & -4 \end{vmatrix} = 40 - 4 = 36 > 0.$

This shows that z is maximum at $\left(\frac{4}{9}, \frac{2}{9}\right)$. And maximum value is.

$$z = \frac{16}{81} - \frac{80}{81} - \frac{8}{81} + \frac{16}{9} - 4 = \frac{16 - 80 - 8 + 144 - 324}{81} = -\frac{252}{81} = -\frac{28}{81}.$$

(e) $z = x^2 + xy + y^2 + 3y + 3$

Solution: Given function is

$$z = x^2 + xy + y^2 + 3y + 3$$

Then,

$z_x = 2x + y$ and $z_y = x + 2y + 3$

$z_{xx} = 2$ $\quad z_{yy} = 2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 2x + y = 0$ $\qquad \Rightarrow \quad x + 2y + 3 = 0.$

Solving these equations we get,

$x = 1$ and $y = -2$.

Now, at $(x, y) = (1, -2)$,

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

This shows that z is minimum at $(1, -2)$. And minimum value at the point is,

$$z = 1 - 2 + 4 - 6 + 3 = 0.$$

(f) $z = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Solution: Given function is

$$z = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

Then,

$z_x = 4x + 3y - 5$ and $z_y = 3x + 8y + 2$

$z_{xx} = 4$ $\quad z_{yy} = 8$

Also $z_{xy} = 3$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 4x + 3y - 5 = 0$ $\qquad \Rightarrow \quad 3x + 8y + 2 = 0$

Solving these equations we get,

$x = 2$ and $y = -1$.

Now, at $(x, y) = (2, -1)$,

$z_{xx} = 4 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & 8 \end{vmatrix} = 32 - 9 = 23 > 0.$

This shows that z is minimum at (x, y) = (2, −1). And minimum value at the point is,

$$z = 8 - 6 + 4 - 10 - 2 = -6.$$

) $z = x^2 - 4xy + 4y^2 - 5x + 2y$

Solution: Given function is

$$z = x^2 - 4xy + 4y^2 - 5x + 2y$$

Then,

$z_x = 2x - 4y - 5$ and $z_y = -4x + 8y + 2$

$z_{xx} = 2$ $\quad z_{yy} = 8$

Also, $z_{xy} = -4$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \; 2x - 4y - 5 = 0$ $\quad \Rightarrow \; -4x + 8y + 2 = 0$

Solving these equations we get,

$$x = \frac{1}{6} \text{ and } y = \frac{4}{3}$$

Now, at the point $(x, y) = \left(\frac{1}{6}, \frac{4}{3}\right)$,

$z_{xx} = 2 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 16 - 16 = 0.$

This shows that the z gives no information at $(x, y) = \left(\frac{1}{6}, \frac{4}{3}\right)$.

(h) $z = x^2 - y^2 - 2x + 4y + 6$

Solution

Given function is

$$z = x^2 - y^2 - 2x + 4y + 6$$

Then,

$z_x = 2x - 2$ and $z_y = -2y + 4$

$z_{xx} = 2$ $\quad z_{yy} = -2$

Also $z_{xy} = 0$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \; 2x - 2 = 0$ $\quad \Rightarrow \; -2y + 4 = 0$

Solving these equations we get,

$$x = 1 \text{ and } y = 2.$$

Now, at the point (x, y) = (1, 2),

$z_{xx} = 2 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4 < 0$

This shows that z is saddle at (x, y) = (1, 2). And the value of z at the point is

$$z = 1 - 4 - 2 + 8 + 6 = 9.$$

(i) $z = x^2 + 2xy$

Solution: Given function is

$$z = x^2 + 2xy$$

Then,

$z_x = 2x + 2y$ and $z_y = 2x$

$z_{xx} = 2$ $z_{yy} = 0$

Also, $z_{xy} = 2$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 2x + 2y = 0$ $\Rightarrow 2x = 0$

Solving these equations we get,

$x = 0$ and $y = 0$.

Now, at the point (x, y) = (0, 0),

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = 0 - 4 = -4$

This shows that z is saddle at (x, y) = (0, 0). And the value of z at the point is

$$z = 0 + 0 = 0$$

(j) $z = x^2 + xy + y^2 + x - 4y + 5$

Solution: Given function is

$$z = x^2 + xy + y^2 + x - 4y + 5$$

Then,

$z_x = 2x + y + 1$ and $z_y = x + 2y - 4$

$z_{xx} = 2$ $z_{yy} = 2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 2x + y + 1 = 0$ $\Rightarrow x + 2y - 4 = 0$

Solving these equations we get,

$x = -2$ and $y = 3$.

Now, at the point (x, y) = (–2, 3),

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$

This shows that z is minimum at (x, y) = (–2, 3). And minimum value is

$$z = 4 - 6 + 9 - 2 - 12 + 5 = -2$$

(k) $z = 3x^2 - xy + 2y^2 - 8x + 9y + 10$

Solution: Given function is

$$z = 3x^2 - xy + 2y^2 - 8x + 9y + 10$$

Then,

$$z_x = 6x - y - 8 \quad \text{and} \quad z_y = -x + 4y + 9$$

$$z_{xx} = 6 \qquad\qquad z_{yy} = 4$$

Also, $z_{xy} = -1$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow 6x - y - 8 = 0 \qquad \Rightarrow -x + 4y + 9 = 0$$

Solving these equations we get,

$$x = 1 \text{ and } y = -2$$

Now, at the point $(x, y) = (1, -2)$,

$$z_{xx} = 6 > 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 24 - 1 = 23 > 0$

This shows that z is minimum at (1, –2). And minimum value at the point is

$$z = 3 + 2 + 8 - 8 - 18 + 10 = -3.$$

(*l*) $z = x^3 - y^3 - 2xy + 6$

Solution: Given function is

$$z = x^3 - y^3 - 2xy + 6$$

Then $z_x = 3x^2 - 2y$ and $z_y = -3y^2 - 2x$

$$z_{xx} = 6x \qquad\qquad z_{yy} = -6y$$

Also, $z_{xy} = -2$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$

$$\Rightarrow 3x^2 - 2y = 0 \qquad \Rightarrow -3y^2 - 2x = 0$$

Solving these equations, we get,

$$x = 0, \frac{2}{3} \text{ and } y = 0, \frac{2}{3}$$

Now, at the point $(x, y) = (0, 0)$,

$$z_{xx} = 6.0 = 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0.$

This shows that z is saddle at $(x, y) = (0, 0)$.

And value of z at the point is

$$z = 0 - 0 - 0 + 6 = 6.$$

Next, at the point $(x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)$,

$$z_{xx} = 6 \times \frac{2}{3} = 4 > 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ -2 & -4 \end{vmatrix} = -16 - 4 = -20 < 0.$

This shows that z is saddle at $(x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

And value of z at the point is

$$z = \frac{8}{27} - \frac{8}{27} - \frac{8}{9} + 6 = \frac{-8+54}{9} = \frac{46}{9}.$$

(m) $z = 6x^2 - 2x^3 + 3y^2 + 6xy$

Solution: Given function is

$$z = 6x^2 - 2x^3 + 3y^2 + 6xy$$

Then,

$z_x = 12x - 6x^2 + 6y$ and $z_y = 6y + 6x$

$z_{xx} = 12 - 12x$ $\quad z_{yy} = 6$

Also, $z_{xy} = 6$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 12x - 6x^2 + 6y = 0 \quad \Rightarrow \quad 6y + 6x = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, -1$.

Now, at the point $(x, y) = (0, 0)$,

$$z_{xx} = 12 - 0 = 12 > 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 12 & 6 \\ 6 & 6 \end{vmatrix} = 72 - 36 = 36 > 0$

This shows that z is minimum at (0, 0). And, minimum value at the point is

$z = 0$.

Next, at the point $(1, -1)$,

$$z_{xx} = 12 - 12 = 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 6 \\ 6 & 6 \end{vmatrix} = 0 - 36 = -36 < 0.$

This shows that z is saddle at $(1, -1)$.

And value of z at the point is

$$z = 6 - 2 + 3 - 6 = 1.$$

(n) $z = x^3 + y^3 - 3xy + 15$

Solution: Given function is

$$z = x^3 + y^3 - 3xy + 15$$

Then, $z_x = 3x^2 - 3y$ and $z_y = 3y^2 - 3x$

$z_{xx} = 6x$ $\quad z_{yy} = 6y$

Also, $z_{xy} = -3$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 3x^2 - 3y = 0 \quad \Rightarrow 3y^2 - 3x = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, 1$.

Now, at point (x, y) = (0, 0),

$$z_{xx} = 6.0 = 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0.$

This shows that z is saddle at (0, 0). And value of z at the point is

$$z = 0 + 0 - 0 + 15 = 15.$$

Next, at point (x, y) = (1, 1),

$$z_{xx} = 6.1 = 6 > 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 = 27 > 0.$

This shows that z is minimum at (x, y) = (1, 1). And minimum value of z at the point is,

$$z = 1 + 1 - 3 + 15 = 14.$$

(o) $z = 4xy - x^4 - y^4$

Solution: Given function is

$$z = 4xy - x^4 - y^4$$

Then

$z_x = 4y - 4x^3$ and $z_y = 4x - 4y^3$

$z_{xx} = -12x^2$ $\quad$ $z_{yy} = -12y^2$

Also, $z_{xy} = 4$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 4y - 4x^3 = 0$ $\quad$ $\Rightarrow 4x - 4y^3 = 0$

Solving these equations we get

$$x = 1, -1, 0 \text{ and } y = 1, -1, 0$$

Now, at point (x, y) = (0, 0),

$z_{xx} = -12 . 0 = 0$ and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0.$

This shows that z is saddle at (0, 0).

And, value of z at (0, 0) is

$$z = 0.$$

Next, at point (x, y) = (1, 1)

$$z_{xx} = -12 < 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0$

This shows that z is maximum at (1, 1). And, the maximum value of z at (1, 1) is

$$z = 4 - 1 - 1 = 2$$

Next, at point (x, y) = (–1, –1)

$$z_{xx} = -12 < 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -14 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0$

This shows that z is maximum at (–1, –1). And maximum value of z at (–1, –1) is,

$$z = 4 - 1 - 1 = 2.$$

(p) $u = 16 - (x + 2)^2 - (y - 2)^2$

Solution: Given function is

$$u = 16 - (x + 2)^2 - (y - 2)^2$$

Then,

$u_x = -2(x + 2)$ and $u_y = -2(y - 2)$

$u_{xx} = -2$ $\qquad u_{yy} = -2$

Also, $u_{xy} = 0$

For extreme point, set,

$u_x = 0$ and $u_y = 0$

$\Rightarrow -2(x + 2) = 0$ $\qquad \Rightarrow -2(y - 2) = 0$

Solving these equations we get,

$$x = -2 \text{ and } y = 2.$$

Now, at point (x, y) = (–2, 2),

$$u_{xx} = -2 < 0$$

$$\text{and, } \begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0.$$

This shows that u is maximum at (x, y) = (–2, 2). And, maximum value of u at (–2, 2) is

$$u = 16 - (-2 + 2)^2 - (2 - 2)^2 = 16.$$

(q) $z = x^3 - x^2 - y^2 + xy$

Solution: Given function is

$$z = x^3 - x^2 - y^2 + xy$$

Then,

$z_x = 3x^2 - 2x + y$ and $z_y = -2y + x$

$z_{xx} = 6x - 2$ $\qquad z_{yy} = -2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 3x^2 - 2x + y = 0$ $\qquad \Rightarrow -2y + x = 0$

Solving these equations we get,

$$x = 0, \frac{1}{2} \text{ and } y = 0, \frac{1}{4}$$

Now, at point (x, y) = (0, 0),

$$z_{xx} = 0 - 2 = -2 < 0$$

$$\text{and, } \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0.$$

This shows that z is maximum at (x, y) = (0, 0). And, maximum value of z at (0, 0) is,

$$z = 0.$$

2. **Find the extreme and stationary points of f.**

(i) $f(x, y) = -x^2 - 4x - y^2 + 2y - 1$

Solution: Given function is

$$f(x, y) = -x^2 - 4x - y^2 + 2y - 1$$

Then,

$$f_x = -2x - 4 \quad \text{and} \quad f_y = -2y + 2$$
$$f_{xx} = -2 \qquad f_{yy} = -2$$

Also, $f_{xy} = 0$

For extreme point, set,

$$f_x = 0 \quad \text{and} \quad f_y = 0$$
$$\Rightarrow -2x - 4 = 0 \qquad \Rightarrow -2y + 2 = 0$$

Solving these equations we get,

$$x = -2 \text{ and } y = 1.$$

Now, at point $(x, y) = (-2, 1)$,

$$f_{xx} = -2 < 0$$

$$\text{and, } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0.$$

This shows that f is maximum at $(-2, 1)$ and maximum value at $(-2, 1)$ is,

$$f(-2, 1) = -4 + 8 - 1 + 2 - 1 = 4.$$

(ii) $f(x, y) = x^2 + 4y^2 - x + 2y$

Solution: Given function is

$$f(x, y) = x^2 + 4y^2 - x + 2y$$

Then

$$f_x = 2x - 1 \quad \text{and} \quad f_y = 8y + 2$$
$$f_{xx} = 2 \qquad f_{yy} = 8$$

Also, $f_{xy} = 0$

For extreme point, set,

$$f_x = 0 \quad \text{and} \quad f_y = 0$$
$$\Rightarrow 2x - 1 = 0 \qquad \Rightarrow 8y + 2 = 0$$

Solving these equations we get,

$$x = \frac{1}{2} \text{ and } y = -\frac{1}{4}$$

Now, at point $(x, y) = \left(\frac{1}{2}, -\frac{1}{4}\right)$,

$$f_{xx} = 2 > 0$$

$$\text{and, } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16 > 0$$

This shows that f(x, y) is minimum at $(x, y) = \left(\frac{1}{2}, -\frac{1}{4}\right)$ and minimum value of f is,

$$f\left(\frac{1}{2}, -\frac{1}{4}\right) = \frac{1}{4} + \frac{4}{16} - \frac{1}{2} - \frac{2}{4} = \frac{1 + 1 - 2 - 2}{4} = \frac{-2}{4} = -\frac{1}{2}$$

(iii) $f(x, y) = x^2 + 2xy + 3y^2$

Solution: Given function is

$$f(x, y) = x^2 + 2xy + 3y^2$$

Then,

$f_x = 2x + 2y$ and $f_y = 2x + 6y$

$f_{xx} = 2$ $\quad f_{yy} = 6$

Also, $f_{xy} = 2$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x + 2y = 0$ $\quad \Rightarrow 2x + 6y = 0$

Solving these equations we get,

$x = 0$ and $y = 0$.

Now, at point $(x, y) = (0, 0)$

$$f_{xx} = 2 > 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = 12 - 4 = 8 > 0$

This shows that f is minimum at (0, 0). And its minimum value at the point is,

$$f(0, 0) = 0.$$

(iv) $f(x, y) = x^3 + 3xy^2 - y^3$

Solution: Given function is

$$f(x, y) = x^3 + 3xy^2 - y^3$$

Then,

$f_x = 3x^2 + 3y^2$ and $f_y = 6xy - 3y^2$

$f_{xx} = 6x$ $\quad f_{yy} = 6x - 6y$

Also, $f_{xy} = 6y$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2 + 3y^2 = 0$ $\quad \Rightarrow 6xy - 3y^2 = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, -1$.

Now, at point $(x, y) = (0, 0)$,

$$f_{xx} = 6.0 = 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$

This shows that f gives no information at (0, 0).

(v) $f(x, y, z) = 2x^2 + xy + 4y^2 + xy + z^2 + 2$

Solution: Given function is

$$f(x, y, z) = 2x^2 + xy + 4y^2 + xy + z^2 + 2$$

Then,

$f_x = 4x + y + z,$ $\quad f_y = x + 8y$ and $f_z = x + 2z$

$f_{xx} = 4$ $\quad f_{yy} = 8$ $\quad f_{zz} = 2$

$f_{xy} = 1$ $\quad f_{yz} = 0$ $\quad f_{zx} = 1$

For extreme point, set,

$f_x = 0,$ $\quad f_y = 0$ and $f_z = 0$

$\Rightarrow 4x + y + z = 0$ $\quad \Rightarrow x + 8y = 0$ $\quad x + 2z = 0$

Solving these equations we get,

$x = 0, y = 0$ and $z = 0$.

Now, at point $(x, y, z) = (0, 0, 0)$,

(a) $f_{xx} = 4 > 0$

(b) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 32 - 1 = 31 > 0$

(c) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4(16 - 0) - 1(2 - 0) + 1(0 - 8)$

$= 64 - 2 - 8 = 54 > 0.$

This shows that f is minimum at (0, 0, 0) and minimum value is

$f(0, 0, 0) = 0 + 0 + 0 + 0 + 0 + 2 = 2.$

(vi) $f(x, y, z) = 35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$

Solution: Given function is

$$f(x, y, z) = 35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$$

Then,

$f_x = -2(2x + 3)$, $f_y = -2(y - 4)$ and $f_z = -2(z + 1)$

$f_{xx} = -4$, $f_{yy} = -2$, $f_z = -2$

$f_{xy} = 0$, $f_{yz} = 0$, $f_{zx} = 0$

For extreme point, set,

$f_x = 0$, $f_y = 0$ and $f_z = 0$

$\Rightarrow -2(2x + 3) = 0$, $\Rightarrow -2(y - 4) = 0$, $\Rightarrow -2(z + 1) = 0$

Solving these equations we get,

$x = -\frac{3}{2}, y = 4, z = -1.$

Now, at the point,

(a) $f_{xx} = -4 < 0$

(b) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 8 > 0$

(c) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -16 < 0$

This shows that f is maximum at the point. And maximum value is,

$$f\left(-\frac{3}{2}, 4, -1\right) = 35 - (-3 + 3)^2 - (4 - 4)^2 - (-1 + 1)^2$$

$$= 35.$$

(vii) – (ix) Similar to (v) and (vi)

3. Similar to Q. No. 2.

4. Find the extreme value of $f = 48 - (x - 5)^2 - 3(y - 4)^2$ such that $x + 3y = 9$.

Solution: Let

$$f(x, y) = 48 - (x - 5)^2 - 3(y - 4)^2$$
$$= -x^2 - 3y^2 + 10x + 24y - 25 \quad .....(1)$$

Then $\nabla f = (-2x + 10)\vec{i} + (-6y + 24)\vec{j}$ ....(2)

Also, let

$$g(x, y) = x + 3y - 9 = 0$$

Then $\nabla g = \vec{i} + 3\vec{j}$

By method of Lagrange's multiplier's, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g(x, y) = 0$$

i.e. $(-2x + 10)\vec{i} + (-6y + 24)\vec{j} = \lambda(\vec{i} + 3\vec{j})$

This gives

$$-2x + 10 = \lambda \quad \text{and} \quad -6y + 24 = 3\lambda$$

$$\Rightarrow \quad x = \frac{10 - x}{2}, y = \frac{24 - 3\lambda}{6}$$

Since we have,

$$g(x, y) = 0 \Rightarrow x + 3y - 9 = 0$$

$$\Rightarrow \left(\frac{10 - \lambda}{2}\right) + \frac{3}{6}(24 - 3\lambda) - 9 = 0$$

$$\Rightarrow 10 - \lambda + 24 - 3\lambda - 18 = 0$$

$$\Rightarrow -4\lambda + 16 = 0$$

$$\Rightarrow \lambda = 4$$

Therefore, $x = 3$, $y = 2$.

Thus, the extreme value of f is

$$f(3, 2) = 48 - (3 - 5)^2 - 3(2 - 4)^2$$
$$= 48 - 4 - 12$$
$$= 32$$

5. **Find the extreme value of $f = x^2 + y^2 + z^2$ such that $ax + by + cz = p$.**

[1999 Q. No. 2(a); 2001 Q. No. 2(a)]

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2$$

Then $\nabla f = 2x\vec{i} + 3y\vec{j} + 2z\vec{k}$

Also, let

$$g(x, y, z) = ax + by + cz - p = 0.$$

Then $\nabla g = a\vec{i} + b\vec{j} + c\vec{k}$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g(x, y, z) = 0$$

i.e. $2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(a\vec{i} + b\vec{j} + c\vec{k})$

This gives

$$2x = a\lambda,\ 2y = b\lambda,\ 2z = c\lambda$$

$$\Rightarrow \quad x = \frac{a\lambda}{2}, y = \frac{b\lambda}{2}, z = \frac{c\lambda}{2}$$

Since we have

$g(x, y, z) = 0$ i.e. $ax + by + cz - p = 0$

$$\Rightarrow \lambda(a^2 + b^2 + c^2) - 2p = 0$$

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$$\Rightarrow \quad \lambda = \frac{2p}{a^2 + b^2 + c^2}$$

Therefore,

$$x = \frac{ap}{a^2 + b^2 + c^2},\ y = \frac{bp}{a^2 + b^2 + c^2},\ z = \frac{cp}{a^2 + b^2 + c^2}$$

Thus, the extreme value of f is

$$f = \frac{(a^2 + b^2 + c^2)\, p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

6. **Find the maximum value of $f = xyz$, given $x + y + z = 24$.**

Solution: Let

$$f = xyz$$

Then $\quad \nabla f = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

Also, let

$$g = x + y + z - 24 = 0$$

Then $\quad \nabla g = \vec{i} + \vec{j} + \vec{k}$

By method of Lagrange's multiplier, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g(x, y, z) = 0.$$

i.e. $\quad yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k})$

This gives,

$$yz = \lambda,\ zx = \lambda,\ xy = \lambda$$

$$\Rightarrow \quad y = \frac{\lambda}{z},\ x = \frac{\lambda}{z},\ xy = \frac{1}{A}$$

That is

$$\frac{\lambda^2}{z^2} = \lambda \quad \Rightarrow \lambda = z^2$$

Then, $x = y = z$.

Since we have

$$g(x, y, z) = 0 \quad \text{i.e. } x + y + z - 24 = 0$$

$$\Rightarrow \quad 3x - 24 = 0$$

$$\Rightarrow \quad x = 8.$$

So, $x = y = z = 8$.

Now, the extreme value of f such that $g(x, y, z) = 0$ is,

$$f(8, 8, 8) = (8)(8)(8) = 512.$$

7. **Find extreme value of $f = x^2 + y^2 + z^2$ such that $x + z = 1$ and $2y + z = 2$.**

[2014 Fall Q.No. 2(b), 2004 Spring Q. No. 2(a)]

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2$$

Then $\quad \nabla f = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$.

Also, let

$$g(x, y, z) = x + z - 1 = 0, \qquad h(x, y, z) = 2y + z - 2 = 0.$$

Then $\quad \nabla g = \vec{i} + \vec{k}, \qquad \nabla h = 2\vec{j} - \vec{k}$

By method of Lagrange's multipliers, for some scalars λ and μ,

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$\text{i.e.}\quad 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} + \vec{k}) + \mu(2\vec{j} + \vec{k})$$
$$= \lambda\vec{i} + 2\mu\vec{j} + (\lambda + \mu)\vec{k}$$

This gives
$$2x = \lambda,\ 2y = 2\mu,\ 2z = \lambda + \mu.$$
$$\Rightarrow\quad x = \frac{\lambda}{2},\ y = \mu.$$

Therefore,
$$2z = \lambda + \mu = 2x - y$$
$$\Rightarrow\quad y = 2x - 2z.$$

Since we have
$$g = x + z - 1 = 0$$
$$\Rightarrow\quad x = 1 - z$$

and
$$h = 2y + z - 2 = 0$$
$$\Rightarrow\quad 4x - 4z + z - 2 = 0$$
$$\Rightarrow\quad 4x - 3z - 2 = 0$$
$$\Rightarrow\quad 4(1 - z) - 3z - 2 = 0 \qquad (\because x = 1 - z)$$
$$\Rightarrow\quad 2 - 7z = 0$$
$$\Rightarrow\quad z = \frac{7}{3}$$

Then
$$x = 1 - z = \frac{-4}{3}\ \text{and}\ y = 2x - 2z = -5$$

Now, the extreme value of f such that g(x, y, z) = 0, h(x, y, z) = 0 is,
$$f\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{9}{9} = 1.$$

8. Find the minimum value of $f = x^2 + xy + y^2 + 3z^2$ such that $x + 2y + 4z = 60$.
[2019 Fall 2(b) OR]

Solution: Let
$$f = x^2 + xy + y^2 + 3z^2$$

Then
$$\nabla f = (2x + y)\vec{i} + (x + 2y)\vec{j} + 6z\vec{k}$$

Also, let
$$g = x + 2y + 4z - 60 = 0$$

Then
$$\nabla g = \vec{i} + 2\vec{j} + 4\vec{k}$$

By method of Lagrange's multipliers, for some scalar λ,
$$\nabla f = \lambda \nabla g \qquad \text{such that } g(x, y, z) = 0.$$
$$\text{i.e.}\quad (2x + y)\vec{i} + (x + 2y)\vec{j} + 6z\vec{k} = \lambda(\vec{i} + 2\vec{j} + 4\vec{k})$$

This gives
$$2x + y = \lambda,\ x + 2y = 2\lambda,\ 6z = 4\lambda.$$

Solving we get

$$x = 0, y = \lambda, z = \frac{2\lambda}{3}$$

Since we have

$$g(x, y, z) = 0 \quad \text{i.e.} \quad x + 2y + 4z - 60 = 0$$

$$\Rightarrow \quad 0 + 2\lambda + \frac{8\lambda}{3} - 60 = 0$$

$$\Rightarrow \quad 14\lambda = 180$$

$$\Rightarrow \quad \lambda = \frac{90}{7}$$

Therefore,

$$x = 0, y = \frac{90}{7}, z = \frac{60}{7}.$$

Now, the extreme value of f such that g(x, y, z) = 0 is

$$f\left(0, \frac{90}{7}, \frac{60}{7}\right) = 0 + 0 + \left(\frac{90}{7}\right)^2 + 3\left(\frac{60}{7}\right)^2 = \frac{2700}{7}$$

9. **Find the minimum value of $f = x^2 + y^2 + z^2$ such that $x+y+z = 1$ and $xyz = 1$.**
[2017 Spring Q.No. 2(b), 2013 Fall Q.No.2 (b)]

Solution: Given that,

$$f(x, y, z) = x^2 + y^2 + z^2$$

Then

$$f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

Also, let

$$g(x, y, z) = x + y + z - 1 = 0, \quad h(x, y, z) = xyz - 1 = 0$$

Then

$$\nabla g = \vec{i} + \vec{j} + \vec{k}, \quad \nabla h = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

By method of Lagrange's multipliers, for some scalars λ and μ,

$$\nabla f = \lambda\nabla g + \mu\nabla h$$

$$\text{i.e.} \quad 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k}) + \mu(yz\vec{i} + zx\vec{j} + xy\vec{k})$$

This gives,

$$2x = \lambda + yz\mu, \quad 2y = \lambda + zx\mu, 2z = \lambda + xy\mu$$

$$\Rightarrow \quad 2x^2 + \lambda x + xyz\mu, 2y^2 = \mu y + xyz\mu, 2z^2 = \lambda z + xyz\,\mu.$$

This implies

$$2x^2 - \lambda x = 2y^2 - \lambda y = 2z^2 - \lambda z.$$

Here,

$$2x^2 - \lambda x = 2y^2 - \lambda y$$

$$\Rightarrow \quad 2(x^2 - y^2) = \lambda(x - y)$$

$$\Rightarrow \quad 2(x + y) = \lambda.$$

Similarly, $2(y + z) = \lambda$ and $2(z + x) = \lambda$.

Therefore, $x + y = y + z = z + x.$

This gives

$$x = y = z.$$

Given that

$$g = 0 \quad \text{i.e. } 3x = 1 \Rightarrow x = \frac{1}{3}$$

and $$h = 0 \quad \text{i.e. } x^3 = -1 \Rightarrow x = -1.$$

So, the critical points are $(x, y, z) = (1/3, 1/3, 1/3)$ and $(x, y, z) = (-1, -1, -1)$.

Here $$f(1/3, 1/3, 1/3) = (1/3)^2 + (1/3)^2 + (1/3)^2 = 1/3.$$

and $$f(-1, -1, -1) = (-1)^2 + (-1)^2 + (-1)^2 = 3.$$

10. Find the extreme value for the function $f(x, y) = x^2 + y^2$ under the condition $x + 4y = 2$.

Solution: Given that,

$$f(x, y) = x^2 + y^2.$$

Then $$\nabla f = 2x\vec{i} + 2y\vec{j}$$

Also, let

$$g(x, y) = x + 4y - 2 = 0.$$

Then $$\nabla g = \vec{i} + 4\vec{j}$$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g(x, y) = 0.$$

$$\text{i.e. } 2x\vec{i} + 2y\vec{j} = \lambda(\vec{i} + 4\vec{j}).$$

This gives, $$2x = \lambda,\ 2y = 4\lambda.$$

Since we have,

$$g(x, y) = 0 \quad \text{i.e. } x + 4y - 2 = 0$$

$$\Rightarrow \frac{\lambda}{2} + 8\lambda - 2 = 0$$

$$\Rightarrow 17\lambda = 4$$

$$\Rightarrow \lambda = \frac{4}{17}$$

Therefore,

$$x = \frac{2}{17} \text{ and } y = \frac{8}{17}.$$

Now, the extreme value of f such that $g(x, y) = 0$ is

$$f\left(\frac{2}{17}, \frac{8}{17}\right) = \left(\frac{2}{17}\right)^2 + \left(\frac{8}{17}\right)^2 = \frac{68}{17 \times 17} = \frac{4}{17}.$$

11. Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ such the $x + y + z = 3a^2$.
[2012 Fall, Q.No. 2(b), 2004 Fall, 2007 Fall, 2008 Fall, 2008 Spring Q. No. 2(a)]

Solution: Given that,

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Then $$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

Also, let

$$g(x, y, z) = x + y + z - 3a^2 = 0$$

Then $\nabla g = \vec{i} + \vec{j} + \vec{k}$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \text{ such that } g = 0.$$

i.e. $2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k})$

This gives,

$$2x = \lambda,\ 2y = \lambda,\ 2z = \lambda.$$

Therefore, $x = y = z.$

Since we have

$$g = 0 \quad \text{i.e.} \quad x + y + z - 3a^2 = 0$$
$$\Rightarrow \quad 3z = 3a^2$$
$$\Rightarrow \quad z = a^2.$$

So, $x = y = z = a^2.$

Now, the extreme value of f such that g = 0 is

$$f(a^2, a^2, a^2) = 3a^2.$$

Since we know

$$x^2 + y^2 + z^2 \geq x + y + z = 3a^2.$$

So, $3a^2$ is the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$.

12. If the sum of three positive number is 8, what is the maximum value of their product.

Solution: Let the numbers are x, y, z.

Given that the sum of these numbers is 8. So,

$$x + y + z = 8 \quad \text{......(1)}$$

And we have find the minimum value of product of x, y, z that is of xyz.

Let

$$f(x, y, z) = xyz \quad \text{....(2)}$$

Then $\nabla f = yz\vec{i} + zx\vec{j} + xy\vec{k}$

Also, let

$$g(x, y, z) = x + y + z - 8 = 0 \quad \text{....(3)}$$

Then $\nabla g = \vec{i} + \vec{j} + \vec{k}$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0.$$

i.e. $yz\vec{i} + zx\vec{j} + xy\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k})$

This gives,

$$yz = \lambda,\ zx = \lambda,\ xy = \lambda$$
$$\Rightarrow \quad y = \frac{\lambda}{z},\ x = \frac{\lambda}{z}$$

So,

$$xy = \lambda \Rightarrow \frac{\lambda^2}{z^2} = \lambda \Rightarrow z^2 = \lambda.$$

Therefore,

$$x = y = z.$$

Since

$$g = 0 \quad \text{i.e.} \quad x + y + z - 8 = 0.$$

$$\Rightarrow \quad 3x = 8.$$

$$\Rightarrow \quad x = \frac{3}{3}$$

So, $x = y = z = \frac{8}{3}$.

Now the number are $x = \frac{8}{3}, y = \frac{8}{3}, z = \frac{8}{3}$. And the value of the product of these numbers is

$$f = \left(\frac{8}{3}\right)\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = \frac{512}{27}.$$

13. Find the minimum values of $x^2 + y^2 + z^2$ where
(i) $x + y + z = 3a$ (ii) $xyz = a^3$.

Solution:

Let $\quad f(x, y, z) = x^2 + y^2 + z^2$

Then $\quad \nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

(i) Also let **[2013 Spring Q.No. 2(b)]**

$$g(x, y, z) = x + y + z - 3a = 0.$$

Then $\quad \nabla g = \vec{i} + \vec{j} + \vec{k}$.

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0.$$

$$\text{i.e.} \quad 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k})$$

This gives

$$2x = \lambda, 2y = \lambda, 2z = \lambda.$$

Since we have

$$g = 0. \quad \text{i.e.} \quad x + y + z - 3a = 0$$

$$\Rightarrow \quad 3\lambda - 6a = 0.$$

$$\Rightarrow \quad \lambda = 2a.$$

Therefore, $x = y = z = a$.

Now, the extreme value of f such that $g = 0$ is

$$f(a, a, a) = 3a^2.$$

(ii) Also, let

$$g(x, y, z) = xyz - a^3 = 0.$$

Then $\quad \nabla g = yz\vec{i} + zx\vec{j} + xy\vec{k}$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0.$$

$$\text{i.e.} \quad 2x\vec{i} + 2y\vec{j} + 2\vec{k} = \lambda(yz\vec{i} + zx\vec{j} + xy\vec{k})$$

This gives

$$2x = \lambda yz,\ 2y = \lambda zx,\ 2z = \lambda xy$$

$$\Rightarrow \quad 2x^2 = \lambda xyz = \lambda a^3,\ 2y^2 = \lambda xyz = \lambda a^3,\ 2z^2 = \lambda xyz = a^3\lambda$$

That is

$$2x^2 = 2y^2 = 2z^2$$

$$\Rightarrow \quad x^2 = y^2 = z^2.$$

Since we have

$$g = 0 \quad \text{i.e.} \quad xyz - a^3 = 0$$

$$\Rightarrow \quad x^2y^2z^2 = a^6$$

$$\Rightarrow \quad x^6 = a^6$$

Therefore $x^2 = y^2 = z^2 = a^2$.

Now, the extreme value of f such that $g = 0$ is

$$f = 3a^2.$$

Here the arithmetic mean (A.M.) of x^2, y^2, z^2 is A.M. $= \dfrac{x^2+y^2+z^2}{3}$ is a^3 and the geometric mean (GM) of x^2, y^2, z^2 is GM $= (x^2y^2z^2)^{1/3} = (a^6)^{1/3} = a^2$.

Since AM $\geq$ GM therefore $x^2 + y^2 + z^2 \geq 3a^2$.

This shows the minimum value of f such that $g = 0$ is $3a^2$.

14. Find the extreme value for the function $x^2 + y^2$ under the condition $x + 4y = 2$.

Solution: Repeated Question, **See Q. No. 10.**

15. Show that the function $f(x, y) = y^2 + x^2y + x^4$ has a minimum value at (0, 0).

Solution: Given function is

$$f(x, y) = y^2 + x^2y + x^4 = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4}$$

This shows f(x, y) is always positive, so it has minimum at (0, 0).

16. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to $ax + by + cz = a + b + c$

Solution: Given function is

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Then $\quad \nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

Also, let

$$g(x, y, z) = ax + by + cz - a - b - c = 0$$

Then $\quad \nabla g = a\vec{i} + b\vec{j} + c\vec{k}$

By method of Lagrange's multipliers, for some scalar λ

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0$$

$$\text{i.e.} \quad 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(a\vec{i} + b\vec{j} + c\vec{k})$$

This gives

$$2x = \lambda a, \quad 2y = \lambda b, \quad 2z = \lambda.$$

This implies $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

We have

$$g = 0 \quad \text{i.e.} \quad ax + by + cz - a - b - c = 0$$

$$\Rightarrow \quad ax + \frac{b^2}{a}x + \frac{c^2 x}{a} = a + b + c$$

$$\Rightarrow \quad x(a^2 + b^2 + c^2) = a^2 + ab + ac$$

$$\Rightarrow \quad x = \frac{a^2 + ab + ac}{a^2 + b^2 + c^2}$$

Then

$$y = \frac{bx}{a} = \frac{ab + b^2 + bc}{a^2 + b^2 + c^2} \quad \text{and} \quad z = \frac{cx}{a} = \frac{ac + bc + c^2}{a^2 + b^2 + c^2}$$

Here,

$$(a^2 + ab + ac)^2 + (ab + b^2 + bc)^2 + (ac + bc + c^2)^2$$
$$= a^4 + a^2b^2 + a^2c^2 + 2a^3b + 2a^3c + 2a^2bc + a^2b^2 + b^4 + b^2c^2 + 2ab^3 + 2ab^2c + 2b^3c + a^2c^2 + b^2c^2 + c^4 + 2abc^2 + 2ac^3 + 2bc^3$$
$$= a^4 + b^3 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 + 2a^3b + 2a^3c + 2ab^3 + 2bc^3 + 2b^3c + 2ac^3 + 2a^2bc + 2b^2ac + 2abc^2$$
$$= (a^2 + b^2 + c^2)^2 + 2\{a^2(ab + ac + bc) + b^2(ab + bc + ac) + c^2(bc + ac + ab)\}$$
$$= (a^2 + b^2 + c^2)\{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\}$$
$$= (a^2 + b^2 + c^2)(a + b + c)^2$$
$$\geq 0.$$

So, f is minimum.

Now, the extreme value of f(x, y, z) such that g(x, y, z) = 0 is

$$f = \frac{(a^2 + ab + ac)^2 + (ab + b^2 + bc)^2 + (ac + bc + c^2)^2}{(a^2 + b^2 + c^2)^2}$$

$$= \frac{(a^2 + b^2 + c^2)(a + b + c)^2}{(a^2 + b^2 + c^2)^2} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}$$

(ii) Similar to above.

17. A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimension of the box requiring least material for its construction.

[2015 Fall Q.No. 2(b), 2009 Fall, 2009 Spring Q. No. 2(b)]

Solution: Let length of box = x, breadth of box = y and height of box = z.

Given that volume of the box = 32 cc.

Since we have, volume of a box, v = xyz.

So, $$xyz = 32 \Rightarrow z = \frac{32}{xy} \quad \text{........(i)}$$

Since the material to construct a box, is used in its surface.

And, we have the surface area of a open (at top) box is,

$$S = xy + 2yz + 2zx$$

Let

$$S(x, y, z) = xy + 2yz + 2zx$$

Then $$\nabla S = (y + 2z)\vec{i} + (x + 2z)\vec{j} + (2y + 2x)\vec{k}$$

Also, let
$$g(x, y, z) = xyz - 32 = 0$$

Then $$\nabla g = yz\vec{i} + 2zx\vec{j} + xy\vec{k}$$

By method of Lagrange's multipliers, for some scalar λ
$$\nabla S = \lambda \nabla g \quad \text{such that } g = 0$$

$$\text{i.e. } (y + 2z)\vec{i} + (x + 2z)\vec{j} + (2y + 2x)\vec{k} = \lambda(yz\vec{i} + xz\vec{j} + xy\vec{k})$$

This gives
$$y + 2z = \lambda yz,\ x + 2z = \lambda zx,\ 2(x + y) = \lambda xy$$

So,
$$xy + 2xz + \lambda xyz,\ xy + 2yz = \lambda xyz,\ 2(2x + 2y) + z = \lambda xyz.$$

This implies
$$xy + 2zx + xy + 2yz = 2xz + 2yz$$
$$\Rightarrow \quad x = y = 2z.$$

We have
$$g = 0 \quad \text{i.e.} \quad xyz = 32$$
$$\Rightarrow \quad 4z^3 = 32 \qquad (\because x = y = 2z)$$
$$\Rightarrow \quad z^3 = 8$$
$$\Rightarrow \quad z = 2.$$

So, $x = 4$, $y = 4$, $z = 2$.

Therefore the extreme value of S such that $g = 0$ is
$$S = 16 + 16 + 16 = 48.$$

Thus, the dimensions of the box are 4 cm, 4 cm and 2 cm.

18. Find the dimension of the rectangular box, open at the top of maximum capacity whose surface is 432 sq. **2015 Spring Q.No. 2 (a)**

Solution: Consider, length of box = x, breadth box = z and height box = y.

Given that, surface area (s) = 432 sq. cm.

Let $$v = xyz$$

Then $$\nabla v = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

Also, let
$$S = xz + 2xy + 2yz - 432 = 0$$

Then $$\nabla S = (z + 2y)\vec{i} + (2x + 2z)\vec{j} + (x + 2y)\vec{k}$$

By method of Lagrange's multipliers, for some scalar λ
$$\nabla v = \lambda \nabla S \quad \text{such that } S = 0$$

$$\text{i.e. } yz\vec{i} + zx\vec{j} + xy\vec{k} = \lambda(z + 2y)\vec{i} + (2x + 2z)\vec{j} + (x + 2y)\vec{k})$$

This gives,

$$yz = \lambda(z + 2y),\ zx = \lambda(2x + 2z),\ xy = \lambda(x + 2y)$$

So

$$xyz = \lambda(xz + 2xy),\ xyz = \lambda(2xy + 2yz),\ xyz = \lambda(xz + 2yz)$$

This implies,

$$xz + 2xy = 2xy + 2yz = xz + 2yz$$

$$\Rightarrow\ x = 2y = z.$$

We have

$$S = 0 \quad \text{i.e.}\quad xz + 2zy + yz - 432 = 0$$

$$\Rightarrow\ 4y^2 + 4y^2 + 4y^2 = 432$$

$$\Rightarrow\ 3y^2 = 108$$

$$\Rightarrow\ y^2 = 36$$

$$\Rightarrow\ y = \pm 6$$

Therefore $(x, y, z) = (12, 6, 12)$ or $(-12, -6, -12)$.

Now at $(x, y, z) = (12, 6, 12)$, $v = (12\,(6)\,(12)) = 864$

and at $(x, y, z) = (-12, -6, -12)$, $v = (-12)(-6)(-12) = -864$

Thus, the maximum capacity of the box is 864 cubic units at $(12, 6, 12)$.

(19) Prove that of all the rectangle parallelepiped of the same volume, the cube has the least surface. **[2010 Spring Q. No. 2(b)]**

Solution: Let, length = x, breadth = y height = z

Then the volume of the parallelepiped is, $(v) = xyz$

$$\Rightarrow\ z = \frac{v}{xy}$$

Since the all part of the parallelepiped is closed.

So, the surface area of parallelepiped $(S) = 2(xy + yz + zx)$

Let

$$S = 2(xy + yz + zx)$$

Then

$$\nabla S = [2(y + z)\,\vec{i} + 2(x + z)\,\vec{j} + 2(y + x)\,\vec{k}$$

Also, let

$$V = xyz \quad \text{with given volume.}$$

Then

$$\nabla V = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla S = \lambda \nabla V.$$

$$\text{i.e. } 2(y + z)\,\vec{i} + 2(x + z)\,\vec{j} + 2(y + x)\,\vec{k} = \lambda(yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k})$$

This gives,

$$2(y + z) + \lambda yz,\ 2(x + z) = \lambda zx,\ 2(y + x) = \lambda xy$$

$$\Rightarrow\ 2x(y + z) = \lambda xyz,\ 2y(x + z) = \lambda xyz,\ 2z(x + y) = \lambda xyz.$$

Therefore,

$$x(y + z) = y(x + z) = z(x + y)$$

$$\Rightarrow\ xy + xz = xy + yz = zx + zy$$

$$\Rightarrow \quad x = y = z.$$

Given that

$$xyz = v(\text{fixed}) \quad \text{being } v \text{ is given volume}$$

$$\Rightarrow \quad x^3 = v$$

$$\Rightarrow \quad x = \sqrt[3]{v}$$

$$\Rightarrow \quad x = y = z = \sqrt[3]{v}$$

Therefore, the extreme value of S at $x = y = z = \sqrt[3]{v}$ is, $S = 6(v)^{2/3}$.

Therefore, the given parallelepiped is cube become $x = y = z$ and surface area is least.

(20) **Prove that of all the rectangular parallelepiped of given surface, cube has the maximum volume.**

Solution: Let, length of parallelepiped $(l) = x$

Breadth of parallelepiped $(b) = z$

Height of parallelepiped $(h) = y$

Since the all part of the parallelepiped is closed.

Surface area of parallelepiped $(s) = 2(xy + yz + zx)$

Let $\quad S = 2(xy + yz + zx)$ and $v = xyz$

Since S and V are symmetrical in x, y, z. So, the critical point for the volume is

$$x = y = z.$$

Thus, S has extremity at $x = y = z$ i.e. when the box is a cube.

Alternative method

Let

$$S = 2(xy + yz + zx) \text{ which is given}$$

Then $\quad \nabla S = 2(y + z)\vec{i} + 2(x + z)\vec{j} + 2(x + y)\vec{k}$

Also, let

$$v = xyz$$

Then $\quad \nabla v = yz\vec{i} + zx\vec{j} + xy\vec{k}$

By method of Lagrange's multiplier's for some scalar λ,

$$\nabla v = \lambda \nabla s.$$

i.e. $\quad yz\vec{i} + zx\vec{j} + xy\vec{k} = \lambda(2y + z)\vec{i} + 2(z+x)\vec{j} + 2(x + y)\vec{k}$

This gives

$$yz = 2\lambda(y+z),\ zx = 2\lambda(z+x),\ xy = 2\lambda(x+y)$$

$$\Rightarrow\ xyz = 2x\lambda(y+z),\ xyz = 2y\lambda(z+x),\ xyz = 2z\ \lambda(x+y)$$

This implies

$$2x(y+z) = 2y(z+x) = 2z(x+y)$$

$$\Rightarrow\ xy + xz = yz + xy = zx + yz$$

Therefore,

$$x = y = z.$$

This means the box has extreme volume when x = y = z i.e. when the box is a cube.

(21)

(a) Find the points on the ellipse $x^2 + 2y^2 = 1$, where $f(x, y) = xy$ has its extreme values.

Solution: Let $f(x, y) = xy$

Then $\nabla f = y\vec{i} + x\vec{j}$

Also, let

$$g(x, y) = x^2 + 2y^2 - 1 = 0$$

Then $\nabla g = 2x\vec{i} + 4y\vec{j}$

By method of Lagrange's multiplier's, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0.$$

i.e. $y\vec{i} + x\vec{j} = \lambda(2x\vec{i} + 4y\vec{j})$

This gives,

$$y = 2x\lambda,\quad x = 4y\lambda.$$

Then $x = 4(2x\lambda)\lambda \quad \Rightarrow \lambda^2 = \frac{1}{8} \ \Rightarrow \lambda = \pm\frac{1}{2}\sqrt{2}.$

Therefore,

$$y = \pm\frac{x}{\sqrt{2}}$$

We have

$$g = 0 \quad \text{i.e.}\quad x^2 + 2y^2 = 1$$

$$\Rightarrow\ x^2 = 2\left(\frac{x^2}{2}\right) = 1$$

$$\Rightarrow\ 2x^2 = 1.$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

Then $$y = \pm\left(\pm\frac{1}{2}\right) = \pm\frac{1}{2}$$

Therefore, the critical points are $(x, y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$

Here

$$\text{at}\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \quad f = \frac{1}{2\sqrt{2}} \qquad \text{at}\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right), \quad f = \frac{-1}{2\sqrt{2}}$$

$$\text{at}\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right), \quad f = \frac{-1}{2\sqrt{2}} \qquad \text{at}\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right), \quad f = \frac{1}{2\sqrt{2}}$$

Thus, the function f has maxima at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$.

And f has minima at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$.

(b) Find the maximum value of $f(x, y) = 9 - x^2 - y^2$ on the line $x + 3y = 12$.

Solution: Let

$$f(x, y) = 9 - x^2 - y^2$$

Then $$\nabla f = -2x\vec{i} - 2y\vec{j}$$

Also let

$$g = x + 3y - 12 = 0.$$

Then $$\nabla g = \vec{i} + 3\vec{j}$$

By method of Lagrange's multiplier's, for some scalar λ,

$$\nabla f = \lambda\nabla g \quad \text{such that } g = 0.$$

$$\text{i.e.} \quad -2x\vec{i} - 2y\vec{j} = \lambda(\vec{i} + 3\vec{j})$$

This gives, $$-2x = \lambda, \qquad -2y = 3\lambda$$

$$\Rightarrow \quad x = \frac{y}{3} \qquad \Rightarrow \quad y = 3x$$

We have

$$g = 0 \quad \text{i.e.} \quad x + 9x - 12 = 0$$

$$\Rightarrow \quad 10x = 12$$

$$\Rightarrow \quad x = \frac{6}{5}$$

Then $\qquad y = \frac{18}{5}$

Therefore the extreme value of f such that g = 0 is

$$f\left(\frac{6}{5}, \frac{18}{5}\right) = 9 - \frac{36}{25} - \frac{324}{25} = -\frac{135}{25}$$

(c) Find the extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

Solution: Let $\qquad f = x^2y$

Then $\qquad \nabla f = 2xy\vec{i} + x^2\vec{j}$

Also, let

$$g = x + y - 3 = 0$$

Then $\qquad \nabla g = \vec{i} + \vec{j}$

By method of Lagrange's multipliers, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0$$

$$\text{i.e. } 2xy\vec{i} + x^2\vec{j} = \lambda(\vec{i} + \vec{j})$$

This gives

$$2xy = \lambda,\ x^2 = \lambda \text{ i.e. } \qquad 2xy = x^2$$

$$\Rightarrow \quad 2y = x.$$

We have

$$g = 0 \quad \text{i.e.} \quad x + y - 3 = 0$$

$$\Rightarrow \quad 3y = 3$$

$$\Rightarrow \quad y = 1$$

Then, $\qquad x = 2$

Therefore the extreme value of f such that g = 0 is,

$$f(2, 1) = (4)(1) = 4.$$

(d) Find the minimum value of $x + y$ subject of $xy = 16$.

Solution: Given that, $f = x + y$

Then $\qquad \nabla f = \vec{i} + \vec{j}$

Also, let $\qquad g(x, y) = xy - 16 = 0.$

Then $\qquad \nabla g = y\vec{i} + x\vec{j}$

By method of Lagrange's multiplier's, for some scalar λ,

$$\nabla f = \lambda \nabla g \quad \text{such that } g = 0$$

i.e. $\vec{i} + \vec{j} = \lambda(y\,\vec{i} + x\,\vec{j})$

This gives,

$1 = \lambda y,\ 1 = x\lambda$

$\Rightarrow\ x = y.$

We have,

$g = 0$ i.e. $x^2 = 16 \Rightarrow x = \pm 4$

Then $y = \pm 4.$

Therefore, the critical points are (4, 4) and (–4, 4)

Here

at (4, 4) $\quad f(4, 4) = 8$

at (–4, –4), $\quad f(-4, -4) = -8$

This means f has maxima 8 at (4, 4).

(e) **Find the maximum value of xy subject to x + y = 16.**

[2011 (Spring) Q.No. 2(b)]

Solution: Given that, $f = xy$

Then $\nabla f = y\,\vec{i} + x\,\vec{j}$

Also, let $g(x, y) = x + y - 16 = 0$

Then $\nabla g = \vec{i} + \vec{j}$

By method of Lagrange's multipliers, for some scalar λ

$\nabla f = \lambda \nabla g$ such that $g = 0$

i.e. $y\,\vec{i} + x\,\vec{j} = \lambda(\vec{i} + \vec{j})$

This gives

$y = \lambda,\ x = \lambda$

$\Rightarrow\ x = y$

We have

$g = 0 \Rightarrow 2x = 16 \Rightarrow x = 8.$

Thus, the critical point is (8, 8).

Therefore, the extrema of f(x, y) such that g(x, y) = 0 is,

$f(8, 8) = 64.$

(g) **The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.**

Solution: Let $T(x, y, z) = 400xyz^2$

Then $\nabla T = 400\,(yz^2\,\vec{i} + xz^2\,\vec{j} + 2xyz\,\vec{k})$

Also, let $S = x^2 + y^2 + z^2 - 1 = 0$

Then $\nabla S = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$

By method of Lagrange's multipliers, for some scalar λ,

$\nabla T = \lambda \nabla s$ such that $s = 0$

i.e. $400\,(yz^2\,\vec{i} + \vec{j} + 2xyz\,\vec{k}) = \lambda(2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k})$

This gives,

$400yz^2 = 2\lambda x,\ 400\,xz^2 = 2\lambda y,\ 800xyz = 2\lambda z.$

$\Rightarrow\ 400xyz^2 = 2\lambda x^2,\ 400xyz^2 = 2\lambda y^2,\ 400xyz^2 = \lambda z^2.$

This implies

$2x^2 = 2y^2 = z^2.$

Given that

$$S = x^2 + y^2 + z^2 - 1 = 0$$

$$x^2 + x^2 + 2x^2 = 1.$$

$\Rightarrow \quad 4x^2 = 1$

$\Rightarrow \quad x^2 = \frac{1}{4}.$

Therefore, $\quad x^2 = y^2 = \frac{1}{4}, z^2 = \frac{1}{2}.$

That is the critical points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$

Here

at $\left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$, $\quad T = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 50$

at $\left(\frac{-1}{2}, \pm\frac{-1}{2}, \pm\frac{1}{2}\right)$, $\quad T = 400\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) = 50$

at $\left(\frac{1}{2}, \pm\frac{-1}{2}, \pm\frac{1}{2}\right)$, $\quad T = 400\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) = -50$

at $\left(\frac{-1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$, $\quad T = 400\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -50$

Thus, the maximum temperature T such that s = 0 is,

$$T = 50 \text{ at } (x, y, z) = \left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \text{ or } \left(\frac{-1}{2}, \pm\frac{-1}{2}, \pm\frac{1}{2}\right).$$

(h) Show that $f(x, y) = y^2 + 2x^2y + 2x^4$ has minimum value at (0, 0).

[2017 Fall 2(b)]

Solution: Let, $\quad f(x, y) = y^2 + 2x^2y + 2x^4$

$$= (y + x^2)^2.$$

This means f(x, y) is always positive, so it has minimum value at (0, 0).

(i) similar to Q 2 (Local maxima or minima).

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

<u>**2000, 2002 (II) Q. No. 2(a), 2016 Spring 2(b)**</u>

Find the minimum value of $u = x^2 + y^2 + z^2$ when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Solution: We have, $f(x, y, z) = x^2 + y^2 + z^2$ and $\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

Let $f = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Using Lagrange's multiplier λ,

$\nabla u = \lambda \nabla f$

i.e., $(2x, 2y, 2z) = \lambda\left(\frac{-1}{x^2}, \frac{-1}{y^2}, \frac{-1}{z^2}\right)$

This gives, $2x = \frac{-\lambda}{x^2} \Rightarrow -2x^3 = \lambda \Rightarrow x = \left(\frac{-\lambda}{2}\right)^{\frac{1}{3}}$ and $y = z = \left(\frac{-\lambda}{2}\right)^{\frac{1}{3}}$

Thus, $x = y = z$

Therefore, (ii) gives,

$\frac{3}{x} = 1 \Rightarrow x = 3$

This means (i) has extreme value at (3, 3, 3) and extreme value is

$u = 9 + 9 + 9 = 27$

Since $u = x^2 + y^2 + z^2 \geq 0$ for each value of x, y, z. This means u has minima at the point of extremity.

hence u is minimum at (3, 3, 3) and minimum value is $u = 27$.

<u>2002 Q. No. 2(a)</u>

Find the minimum value of $x^2 + y^2 + z^2$ having given $lx + my + nz = k$.

Solution: See Exercise 11.1 Q. No. 5 with replacing a by l, b by m, c by n and p by k.

<u>2003 Fall Q. No. 2(a)</u>

Write down the necessary condition that f(x, y, z) to have maximum or minimum value. Find the minimum value of $u = x^2 + xy + y^2 + 3z^2$ subject is the condition $x + 2y + 4z = 60$.

Solution: First Part: See the condition.

Second Part: See Exercise 11.1 Q. No. 8.

<u>2006 Fall; 2011 Fall Q. No. 2(a), 2018 Fall Q.No. 2(a), 2018 Spring Q.No. 2(b)</u>

If the sum of the dimension of a rectangular swimming pool is given. Prove that the amount of water in the pool is maximum when it is a cube.

Solution: Let x, y and z be length, breadth and height of rectangular swimming pool.

Then we have $x + y + z = p$ (given) ....(1)

The amount of water $V = xyz$ ....(2)

We have to prove that the amount of water in the pool is maximum when it is a cube.

Then,

$$\nabla v = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)(xyz)$$

$$= yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$$

Also, let

$$g(x, y, z) = x + y + z - p = 0$$

Then

$$\nabla g = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)(x + y + z - p) = \vec{i} + \vec{j} + \vec{k}$$

By method of Lagrange's multiplier, for some scalar λ,

$\nabla f = \lambda \nabla g$ such that $g = 0$

i.e. $yz\vec{i} + zx\vec{j} + xy\vec{k} = \lambda(\vec{i} + \vec{j} + \vec{k})$

This gives,

$yz = \lambda$, $zx = \lambda$, $xy = \lambda$

$\Rightarrow x = y = z.$

This shows, the pool must be a cube when it is extreme.

We have

$x + y + z = p$

Therefore,

$3x = p \Rightarrow x = \frac{P}{3}$ $\qquad (\because x = y = z)$

So, $x = y = - z = \frac{P}{3}$

Clearly, $x \le p$, $y \le p$, $z \le p$ and $0 \le x$, $0 \le y$, $0 \le z$.

This means v will be extreme at either (0, 0, 0) or $\left(\frac{P}{3}, \frac{P}{3}, \frac{p}{3}\right)$. That means amount of water in the pool is maximum when it is a cube.

2006 Spring Q. No. 2(a)

What are the criteria of a function of two independent variables to l extreme values? Find the extreme value of $x^2 + y^2 + z^2$ w $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$

Solution: For criteria see the theoretical part of this chapter.

For the problem, see the solution of 2000.

2014 Spring Q. No. 2(b), 2019 Fall Q.No. 2(b)

If $f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$ when x and y are not simultaneously zero when x = y = 0 show that at (0, 0) $f_{xy} \ne f_{yx}$

2016 Fall Q. No. 2(b)

Find the minimum value of the function $x^2 + y^2 + z^2$ subject $xy + yz + zx = 3a^2$

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