

Definition of matrix

A rectangular arrangement of numbers (may real or complex), that are enclosed by a bracket [] or (), is called a matrix. It is denoted by capital alphabet A, B, C, ... etc. For example,

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}, B = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \text{ etc.}$$

Order of matrix

Let A be a matrix having m^{th} row and n^{th} column then the order of A is $m \times n$ and is denoted by $(A)_{m \times n}$ or, $A = (a_{ij})_{m \times n}$

Statement: In general, the matrices are not commutative in multiplication.

Proof: Let,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}$$

So, AB is defined.

However, $B = (b_{ij})_{n \times p}$ and $A = (a_{ij})_{m \times n}$. So, BA is not defined if $p \neq m$.

This proves that $AB \neq BA$ for $m \neq p$.

If $m = p$, that is if the matrices are square matrices, then $AB \neq BA$ in general but $AB = BA$ as a special case.

For this, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

Then,

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Thus, $AB \neq BA$.

And, if we let, $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

Then,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This shows that $AB = BA$.

Statement: If $A = 0$ or $B = 0$ then $AB = 0$ but converse may not hold.

Proof: Let $A = 0$ or $B = 0$ then trivially $AB = 0$.

But the converse may not hold.

For this, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $A \neq 0, B \neq 0$.

However,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Definition of transpose of a matrix

The transpose of a matrix A is the matrix obtained by interchanging the rows into columns or the columns into rows. It is denoted by A' or A^T or A^t .

That is, if $A = (a_{ij})_{m \times n}$ then $A^T = (a_{ji})_{n \times m}$

For example If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Statement: Show that $(A + B)^T = A^T + B^T$.

Proof: Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ji})_{n \times m}$. Then, $A + B$ is possible.

Let $A + B = C = (C_{ij})_{m \times n}$ where, $C_{ij} = a_{ij} + b_{ji}$

Then,

$$\begin{aligned} C^T &= ((C_{ij})_{m \times n})^T = (C_{ij})_{n \times m} = (a_{ij} + b_{ji})_{n \times m} = (a_{ij})_{n \times m} + (b_{ji})_{n \times m} \\ &\quad = ((a_{ij})_{m \times n})^T + ((b_{ji})_{m \times n})^T = A^T + B^T \\ \Rightarrow (A + B)^T &= A^T + B^T. \end{aligned}$$

Statement: If A and B are matrices of specified order, then $(AB)^T = B^T A^T$.

Proof: Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ji})_{n \times p}$. Then AB is defined.

Also, $A^T = (a_{ji})_{n \times m}$ and $B^T = (b_{ji})_{p \times n}$. Then $B^T A^T$ is defined.

Then we wish to show $(AB)^T = B^T A^T$.

Let $AB = C = (c_{ij})_{m \times p}$

Here,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{r=1}^n a_{ir}b_{rj}$$

$$\begin{aligned} \text{Then, } C^T &= (c_{ij})^T = c_{ji} = \sum_{r=1}^n a_{jr}b_{ri} = \sum_{r=1}^n (a_{rj})^T(b_{ri})^T = \sum_{r=1}^n (b_{ri})^T(a_{rj})^T \\ &\quad = B^T A^T \\ \Rightarrow C^T &= B^T A^T \\ \Rightarrow (AB)^T &= B^T A^T \end{aligned}$$

Definition of symmetric matrix

A square matrix A is called symmetric if $A = A^T$. That means a matrix $A = (a_{ij})_{n \times n}$ is symmetric if $a_{ij} = a_{ji}$ for all i, j .

For example,

If $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = A$. So, A is a symmetric matrix.

Definition of skew-symmetric matrix

A square matrix A is called skew-symmetric matrix if $A^T = -A$. That is, if $A = (a_{ij})_{n \times n}$ then A is skew-symmetric if $a_{ij} = -a_{ji}$ for all i, j .

For example: Let $A = \begin{bmatrix} a & -b & -c \\ b & d & -e \\ c & e & f \end{bmatrix}$. Then $A^T = -A$, so, A is skew-symmetric matrix.

Note: For a skew-symmetric matrix, the leading diagonal elements should be zero.
But it is not sufficient.

Proof: Let $A = (a_{ij})_{n \times n}$ is skew-symmetric matrix. Then, $a_{ij} = -a_{ji}$ for all i, j .

For leading diagonal element, we should have $i = j$.

So, $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$, for all i .

This shows that, the leading diagonal elements are zero.

But a matrix,

$$B = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$$

is not skew-symmetric however it has 0 as leading diagonal elements.

Definition of orthogonal matrix

A square matrix A is said to be orthogonal if $A^T A = I = AA^T$.

For example, let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

So, A is an orthogonal matrix.

Statement: If A and B are any two orthogonal matrices of same order then AB is also orthogonal.

Proof: Let A and B are orthogonal matrices of same order. Then AB is defined.

Being A is orthogonal, $AA^T = I = A^T A$.

Also, being B is orthogonal, $BB^T = I = B^T B$.

Now,

$$(AB)(AB)^T = (AB)(B^T A^T) = A(BB^T)A^T = AIA^T = AA^T = I$$

Also,

$$(AB)^T(AB) = (B^T A^T)(AB) = B^T(A^T A)B = B^T I B = B^T B = I$$

Thus, $(AB)(AB)^T = I = (AB)^T(AB)$.

This shows that AB is orthogonal.

Note: AB may replace by BA in above statement.

Definition of complex matrix:

A matrix is said to be complex matrix if any (may all) element is complex number.

For example,

$$A = \begin{bmatrix} 1 & 5+i \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2+i & 4+i \\ 3-i & i \end{bmatrix}, C = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

are complex matrix.

Definition of conjugate (or complex conjugate) matrix:

If $A = (a_{ij})_{n \times n}$ is a matrix then its conjugate (or complex conjugate) matrix is obtained by replacing each element of A by its complex conjugate. It is denoted by

A. *For example,*

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For example,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Definition of minor of a matrix

The minor of an element in a determinant is the value determined by the determinant obtained by deleting the row and column in which the element of the determinant lies. It is denoted by M_{ij} for minor of a_{ij} . For an instance, let the given determinant be

$$\begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix}$$

Then

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{11} & b_{23} \\ a_{32} & b_{33} \end{vmatrix}$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} a_{11} & b_{13} \\ a_{21} & b_{23} \end{vmatrix}$$

are minors of the given determinant.

Definition of cofactor of a matrix

The cofactor of the element a_{ij} in the determinants A, is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij} .

Definition of adjoint of a matrix

Let $A = (a_{ij})_{n \times n}$ be a square matrix. Then the adjoint matrix of A is the transpose of cofactor matrix of A. It is denoted by $\text{Adj}(A)$ or $\text{adj}(A)$. For an instance, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } \text{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

where A_{ij} be cofactor of a_{ij} .

Statement: If Λ is $n \times n$ square matrix. Then $\text{Adj}(\Lambda) \cdot \Lambda = \Lambda \cdot \text{Adj}(\Lambda) = |\Lambda|I$ where I is $n \times n$ unit matrix.

Proof: Let $A = (a_{ij})_{n \times n}$ be a square matrix. Also, A_{ij} be cofactor of a_{ij} .

Then,

$$\begin{aligned} \text{Adj}(A) &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix}^T \\ &= \begin{bmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & A_{3n} & \dots & A_{nn} \end{bmatrix} \end{aligned}$$

Now,

$$A \cdot \text{Adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1}^n a_{1r}A_{1r} & \sum_{r=1}^n a_{1r}A_{2r} & \sum_{r=1}^n a_{1r}A_{nr} \\ \sum_{r=1}^n a_{2r}A_{1r} & \sum_{r=1}^n a_{2r}A_{2r} & \sum_{r=1}^n a_{2r}A_{nr} \\ \dots & \dots & \dots \\ \sum_{r=1}^n a_{nr}A_{1r} & \sum_{r=1}^n a_{nr}A_{2r} & \sum_{r=1}^n a_{nr}A_{nr} \end{bmatrix}$$

We know that $\sum_{r=1}^n a_{ir} A_{ij} = \begin{cases} |A| & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$

Therefore,

$$A \cdot \text{Adj}(A) = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} = |A|I$$

Similarly, $\text{Adj}(A) \cdot A = |A|I$

Thus, $A \cdot \text{Adj}(A) = \text{Adj}(A) \cdot A = |A|I$

Singular and non-singular matrix

A square matrix A is called singular if $|A|=0$. And, the matrix A is non-singular if $|A| \neq 0$.

Definition of inverse of a matrix

Let A be a given square matrix. Also, B be another square matrix having order as the matrix A has. Moreover, if $AB = BA = I$ then we called B is inverse (or reciprocal) matrix of A and vice-versa.

Note: (i) If B is inverse of A then A is known as invertible.

(ii) We have,

$$\text{Adj}(A) \cdot A = |A|I = A \cdot \text{Adj}(A)$$

$$\Rightarrow \frac{\text{Adj}(A) \cdot A}{|A|} = I = \frac{A \cdot \text{Adj}(A)}{|A|} \Rightarrow A^{-1} = \frac{\text{Adj}(A)}{|A|} \text{ for } |A| \neq 0.$$

Statement: The necessary and sufficient condition for a square matrix A to possess inverse is that A is non-singular.

Proof: Necessary condition

Suppose that a square matrix A to possess inverse of A then we wish to show A is non-singular i.e. $|A| \neq 0$.

Let B is an inverse of A . So, $AB = I = BA$.

$$\text{Therefore, } |AB| = |I| = |BA| \Rightarrow |A||B| = 1 = |B||A|$$

This shows that neither $|A| \neq 0$ nor $|B| \neq 0$

This proves that A is singular.

Sufficient condition

Suppose that A is non-singular. So, $|A| \neq 0$.

Define, $B = \frac{\text{Adj}(A)}{|A|}$ then we wish to show B is inverse of A .

Here,

$$AB = \frac{A \cdot \text{Adj}(A)}{|A|} = \frac{|A|I}{|A|} = I \quad \text{and} \quad BA = \frac{\text{Adj}(A) \cdot A}{|A|} = \frac{|A|I}{|A|} = I$$

Thus, $AB = BA = I$. This proves that B is inverse of A . Hence, A possess inverse.

Note: Since $A^{-1} = \frac{\text{Adj}(A)}{|A|}$ that will exist only if $|A| \neq 0$.

So, the necessary condition for existence of A^{-1} is $|A| \neq 0$

Theorem: Let A and B are two matrices of specified order, then $(AB)^{-1} = B^{-1}A^{-1}$ where $|A| \neq 0, |B| \neq 0$.

OR The inverse of the product is the product of their inverse in reverse order.
OR If A and B are any two non-singular matrices of specified order and AB is also non-singular then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Let A and B are any two non-singular matrices of specified order. Then AB is defined. Also, let AB is non-singular.

So, $|A| \neq 0, |B| \neq 0, |AB| \neq 0$. Therefore, A^{-1}, B^{-1} and $(AB)^{-1}$ exist.

Also, $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$.

Now,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} \quad [\because \text{matrix is associative}] \\ = AIA^{-1} = AA^{-1} = I$$

$$\text{and} \quad (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B \quad [\because \text{matrix is associative}] \\ = B^{-1}IB = B^{-1}B = I$$

$$\text{Thus, } (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

This proves that AB is inverse of $B^{-1}A^{-1}$. That is $(AB)^{-1} = B^{-1}A^{-1}$

Discuss coefficient matrix and augmented matrix

Consider a system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

Then the matrix form of coefficient of variables as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

is coefficient matrix of given system of linear equations. And, the matrix form of coefficient of variables with results of the equations, formed as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ a_{21} & a_{22} & a_{23} : b_2 \end{bmatrix}$$

is the augmented matrix of given system of linear equations.

Discuss Echelon and canonical form of a matrix

The echelon form is the triangular matrix of the coefficient of the variables in system of linear equations.

The canonical form is the diagonal matrix of the coefficient of the variables in system of linear equations.

Definition of linear dependence and independence

A set of vectors $\{x_1, x_2, \dots, x_n\}$ is called linearly dependent if there exist real numbers c_1, c_2, \dots, c_n such that

2. If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$. Show that $AB \neq BA$.

Solution: Given that,

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

Then,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 2 + 3 \times 1 + 0 \times (-1) & 1 \times 3 + 3 \times 2 + 0 \times 1 & 1 \times 4 + 3 \times 3 + 0 \times 2 \\ (-1) \times 2 + 2 \times 1 + 1 \times (-1) & (-1) \times 3 + 2 \times 2 + 1 \times 1 & (-1) \times 4 + 2 \times 3 + 1 \times 2 \\ 0 \times 2 + 0 \times 1 + 2 \times (-1) & 0 \times 3 + 0 \times 2 + 2 \times 1 & 0 \times 4 + 0 \times 3 + 2 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}. \end{aligned}$$

And,

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 1 + 3 \times (-1) + 4 \times 0 & 2 \times 3 + 3 \times 2 + 4 \times 1 & 2 \times 4 + 3 \times 1 + 4 \times 0 \\ 1 \times 1 + 2 \times (-1) + 3 \times 0 & 1 \times 3 + 2 \times 2 + 3 \times 1 & 1 \times 4 + 2 \times 1 + 3 \times 0 \\ -1 \times 1 + 1 \times (-1) + 2 \times 0 & -1 \times 3 + 1 \times 2 + 1 \times 0 & -1 \times 4 + 1 \times 1 + 2 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}. \end{aligned}$$

$$\text{Thus, } BA = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix} = BA.$$

$$\Rightarrow AB \neq BA.$$

3. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$. Find AB and explain why BA is not defined.

Solution: Given that,

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot (-1) + 2 \cdot 2 & 0 \cdot (-2) + 1 \cdot 0 + 2 \cdot (-1) \\ 1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 2 & 1 \cdot (-2) + 2 \cdot 0 + 3 \cdot (-1) \\ 2 \cdot 1 + 3 \cdot (-1) + 4 \cdot 2 & 2 \cdot (-2) + 3 \cdot 0 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

And,

$$BA = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Here the first row of first matrix has 2 values but the first column of second matrix has 3 values. So, we cannot operate the system of multiplication between matrices. Thus, BA is undefined.

Note: Remember that the multiplication between 2-matrices is possible only if number of column(s) in first matrix is same as the number of row(s) in second matrix.

4. If $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$. Verify $AB = BA = I$.

Solution: Given that,

$$A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

Now,

$$\begin{aligned} AB &= \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \times 1 + 3 \times 2 + (-1) \times 3 & -2 \times 3 + 3 \times 2 + (-1) \times 0 & -2 \times (-1) + 3 \times 1 + (-1) \times (-1) \\ -1 \times 1 + 2 \times 2 + (-1) \times 3 & -1 \times 3 + 2 \times 2 + (-1) \times 1 & -1 \times (-1) + 2 \times 1 + (-1) \times (-1) \\ -6 \times 1 + 9 \times 2 + (-4) \times 3 & -6 \times 3 + 9 \times 2 + (-4) \times 0 & -6 \times (-1) + 9 \times 1 + (-4) \times (-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$

Also,

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \times -2 + 3 \times -1 + (-1) \times -6 & -2 \times 3 + 3 \times 2 + (-1) \times 0 & -2 \times (-1) + 3 \times 1 + (-1) \times (-1) \\ -4 \times -2 + 6 \times -1 + 0 \times -6 & -4 \times 3 + 6 \times 2 + 0 \times 1 & -4 \times (-1) + 6 \times 1 + 0 \times (-1) \\ -6 \times -2 + 9 \times -1 + 0 \times -6 & -6 \times 3 + 9 \times 2 + 0 \times 0 & -6 \times (-1) + 9 \times 1 + 0 \times (-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$

Thus, $AB = BA = I$.

5. Find $AB - BA$, where $A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$.

Solution: Given that,

$$A = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 9 \times 7 & 2 \times 5 + 9 \times 2 \\ 4 \times 1 + 3 \times 7 & 4 \times 5 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} 65 & 28 \\ 25 & 26 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 5 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 5 \times 4 & 2 \times 9 + 5 \times 3 \\ 7 \times 1 + 2 \times 4 & 7 \times 9 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 22 & 24 \\ 14 & 69 \end{bmatrix}$$

Now,

$$AB - BA = \begin{bmatrix} 65 & 28 \\ 25 & 26 \end{bmatrix} - \begin{bmatrix} 22 & 24 \\ 14 & 69 \end{bmatrix} = \begin{bmatrix} 43 & 4 \\ 3 & -43 \end{bmatrix}.$$

6. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find AB or BA whichever exists.

Solution: Given that,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Here, matrix A has 4-columns but matrix B has only 3-rows. So, AB does not exist being (number of column of A) = 4 \neq 3 = (number of rows of B).

And, B has 3-columns and A has 3-rows. Thus, (number of columns of B) = (number of rows of A). So, BA is defined.

Now,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2+2+0 & 4+0+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 12+4+5 \\ 1+0+3 & 2+0+1 & 4+0+5 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}. \end{aligned}$$

7. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$. Show that $(AB)C = A(BC)$ and $A(B+C) = AB + AC$.

Solution: Given that,

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 1+6 \\ -4+6 & -2+9 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$B+C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

So that,

$$(AB)C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

Thus,

$$(AB)C = A(BC)$$

Also,

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

And,

$$AB+AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 12 & -12 \end{bmatrix} = \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-12 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

Thus,

$$A(B+C) = AB+AC$$

8. If $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $(2I-A)(10I-A) = I$.

Solution: Given that,

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$2I-A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}$$

and,

$$10I-A = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

Now,

$$(2I-A)(10I-A) = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9 & 0+0 \\ -24+24 & 9+0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 9I$$

9. If $A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A-B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. Find AB .

Solution: Given that,

$$A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \text{ and } A-B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Then,

$$(2A) = [(A+B)+(A-B)] = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

And,

$$A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - A = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2+0 & -2+0 \\ -2+1 & -2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$$

10. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 7I = 0$, where I is a unit matrix of size 2×2 .

Solution: Let, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then,

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Now,

$$\begin{aligned} A^2 - 5A + 7I &= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \end{aligned}$$

Thus, $A^2 - 5A + 7I = 0$.

11. Let $A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \\ -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$. Show that AB and AC does not exist.

Solution: Given that,

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -3 \\ -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$$

Since the matrix A has 2-columns but both B and C have 3-rows. Thus, (number of column of A) \neq (number of row of B or C). So, AB and AC are not defined. Therefore, AB and AC does not exist.

12. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ where $i^2 = -1$, then show that $(A + B)^2 = A^2 + B^2$.

Solution: Let,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{where } i^2 = -1$$

Then,

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

So,

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0+(1-i)(1+i) & 0+0 \\ 0+0 & (1-i)(1+i)+0 \end{bmatrix} \\ &= \begin{bmatrix} 1-i^2 & 0 \\ 0 & 1-i^2 \end{bmatrix} = \begin{bmatrix} 1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

And, $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Also, $B^2 = B \cdot B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0-i^2 & 0+0 \\ 0+0 & -i^2+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So that,

$$A^2 + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = (A+B)^2$$

Thus, $(A+B)^2 = A^2 + B^2$

Note: In general, $(A+B)^2$ may not be equal to $A^2 + B^2$. For this, check the condition with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$ Show that $A^3 - A^2 - 18A - 30I = 0$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\begin{aligned} A^2 = A \cdot A &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8+3 \\ 2-2+12 & 4+1+4 & 6-4+4 \\ 3+2+3 & 6-1+1 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \end{aligned}$$

And,

$$\begin{aligned} A^3 = A \cdot A^2 &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 14+24+24 & 3+18+18 & 14+12+42 \\ 28-12+32 & 6-9+24 & 28-6+56 \\ 42+12+8 & 9+9+6 & 42+6+14 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} \end{aligned}$$

Now,

$$\begin{aligned} A^3 - A^2 - 18A - 30I &= \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - 30 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 62-14-18-30 & 39-3-36-0 & 68-14-54-0 \\ 48-12-36-0 & 21-9+18-30 & 78-6-72-0 \\ 62-8-54-0 & 24-6-18-0 & 6214-18-30 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

Thus, $A^3 - A^2 - 18A - 30I = 0$.

14. If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Show that $A^3 - 4A^2 - 3A + 11I = 0$, where I is the unit matrix of order 3.

Solution: Process as 13.

EXERCISE - 1.2

1. If $A = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}$, $D = [4 \ 3 \ 0]$.

Calculate the following products or give reasons why they are not defined.

- (i) BA , $A^T B$, AB
- (ii) C^2 , $C^T C$, CC^T
- (iii) $A^T D$, $A^T D^T$, DA , AD
- (iv) BB^T , $B^T B$, BB^T , B

Solution: Let,

$$A = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, \quad D = [4 \ 3 \ 0].$$

Then, the transpose of the matrices be

$$A^T = [1 \ 4 \ 3], \quad B^T = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}, \quad D^T = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

(i) (a) BA – Since (number of column in B) \neq (number of rows in A). So, BA is undefined.

(b) $A^T B$ – Here, A^T has 3-columns and B has 3-rows. So, $A^T B$ is defined. And,

$$\begin{aligned} A^T B &= [1 \ 4 \ 3] \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \\ &= [2+0+0 \quad -3+8+3] = [2 \ 8]. \end{aligned}$$

(c) AB – Here, A has 1-column and B has 3-rows. So, (number of column in A) \neq (number of rows in B). Therefore, AB is not defined.

(ii) (a) $C^2 = C \cdot C$ – Since both C are 3×3 square matrices. So, C^2 is defined.

Here,

$$C^2 = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+36+4 & 24+0+6 & 8+18-2 \\ 24+0+6 & 36+0+9 & 12+0-3 \\ 8+18-2 & 12+0-3 & 4+9+1 \end{bmatrix} = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$$

(b) $C^T C, CC^T$ - Since $C^T = C$. So, $C^T C = CC^T = CC = C^2$.
Therefore, see (a) for $C^T C = CC^T$.

(iii) (a) $A^T D$ - Here, A^T has 3-columns and D has 1-rows.
So, (number of columns in A^T) \neq (number of rows in D).

Therefore, $A^T D$ is undefined.

(b) $A^T D^T$ - Here, (number of columns in A) $=$ (number of rows in D^T).
So, $A^T D^T$ is defined. And,

$$A^T D^T = [1 \ 4 \ 3] \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = (4 + 12 + 0) = (16).$$

(c) DA - Here D has 3-columns and A has 3-rows. So, DA is defined. And,

$$DA = [4 \ 3 \ 0] \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = (4 + 12 + 0) = (16)$$

(d) AD - Here A has 1-column and D has 1-row. So, AD is defined. And,

$$AD = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} [4 \ 3 \ 0] = \begin{bmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{bmatrix}$$

(iv) (a) BB^T - Here B has 2-columns and B^T has 2-rows. So, BB^T is defined. And,

$$\begin{aligned} BB^T &= \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4+9 & 0-6 & 0-3 \\ 0-6 & 0+4 & 0+2 \\ 0-3 & 0+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 13 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix}. \end{aligned}$$

(b) $B^T B$ - Here B^T has 3-columns and B has 3-rows. So, $B^T B$ is defined. And,

$$\begin{aligned} B^T B &= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4+0+0 & -6+0+0 \\ -6+0+0 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \end{aligned}$$

(c) $BB^T B$ - Since we know $BB^T B = (BB^T)B = B(B^T B)$.

Since, B has 2-columns and $B^T B$ has 2-rows.

So, $B(B^T B) = (BB^T)B$ is defined.

Now,

$$\begin{aligned} BB^T B &= B(B^T B) = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 8+18 & -12-42 \\ 0-12 & 0+2 \\ 0-6 & 0+14 \end{bmatrix} = \begin{bmatrix} 24 & -54 \\ -12 & 28 \\ -6 & 14 \end{bmatrix}. \end{aligned}$$

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Verify that $(AB)^T = B^T A^T$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Now,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -3 \\ 5 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix} \end{aligned}$$

And,

$$\begin{aligned} B^T A^T &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+4+5 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 9 \\ 3 & 6 & 0 \end{bmatrix} \end{aligned}$$

Thus,

$$(AB)^T = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 9 \\ -3 & 6 & 0 \end{bmatrix} = B^T A^T$$

This verifies $(AB)^T = B^T A^T$.

3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$, show that $(A^2)^T = (A^T)^2$.

Solution: Let, $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}$
Now,

$$\begin{aligned} A^2 &= A \cdot A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 4+15 & 6-21 \\ 10-35 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix}. \end{aligned}$$

And,

$$\begin{aligned} (A^T)^2 &= A^T \cdot A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 4+15 & 10-35 \\ 6-21 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix}. \end{aligned}$$

Thus,

$$(A^2)^T = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix} = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix} = (A^T)^2$$

$$= \begin{bmatrix} 16+36+4 & 24+0+6 & 8+18-2 \\ 24+0+6 & 36+0+9 & 12+0-3 \\ 8+18-2 & 12+0-3 & 4+9+1 \end{bmatrix} = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$$

(b) $C^T C, CC^T$ - Since $C^T = C$, So, $C^T C = CC^T = CC = C^T$.

Therefore, see (a) for $C^T C = CC^T$.

(iii) (a) $A^T D$ - Here, A^T has 3-columns and D has 1-rows.

So, (number of columns in A^T) \neq (number of rows in D).

Therefore, $A^T D$ is undefined.

(b) $A^T D^T$ - Here, (number of columns in A) $=$ (number of rows in D^T).

So, $A^T D^T$ is defined. And,

$$A^T D^T = [1 \ 4 \ 3] \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = (4 + 12 + 0) = (16).$$

(c) DA - Here D has 3-columns and A has 3-rows. So, DA is defined. And,

$$DA = [4 \ 3 \ 0] \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = (4 + 12 + 0) = (16)$$

(d) AD - Here A has 1-column and D has 1-row. So, AD is defined. And,

$$AD = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} [4 \ 3 \ 0] = \begin{bmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{bmatrix}$$

(iv) (a) BB^T - Here B has 2-columns and B^T has 2-rows. So, BB^T is defined. And,

$$\begin{aligned} BB^T &= \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4+9 & 0-6 & 0-3 \\ 0-6 & 0+4 & 0+2 \\ 0-3 & 0+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 13 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix}. \end{aligned}$$

(b) $B^T B$ - Here B^T has 3-columns and B has 3-rows. So, $B^T B$ is defined. And,

$$\begin{aligned} B^T B &= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4+0+0 & -6+0+0 \\ -6+0+0 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \end{aligned}$$

(c) $BB^T B$ - Since we know $BB^T B = (BB^T)B = B(B^T B)$.

Since, B has 2-columns and $B^T B$ has 2-rows.

So, $B(B^T B) = (BB^T)B$ is defined.

Now,

$$\begin{aligned} BB^T B &= B(B^T B) = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 8+18 & -12-42 \\ 0-12 & 0+2 \\ 0-6 & 0+14 \end{bmatrix} = \begin{bmatrix} 24 & -54 \\ -12 & 28 \\ -6 & 14 \end{bmatrix}. \end{aligned}$$

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Verify that $(AB)^T = B^T A^T$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Now,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -3 \\ 5 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix} \end{aligned}$$

And,

$$\begin{aligned} B^T A^T &= \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+4+5 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 9 \\ 3 & 6 & 0 \end{bmatrix} \end{aligned}$$

Thus,

$$(AB)^T = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix} = B^T A^T$$

This verifies $(AB)^T = B^T A^T$.

3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$, show that $(A^2)^T = (A^T)^2$.

Solution: Let, $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}$

Now,

$$\begin{aligned} A^2 &= A \cdot A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 4+15 & 6-21 \\ 10-35 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix}. \end{aligned}$$

And,

$$\begin{aligned} (A^T)^2 &= A^T \cdot A^T = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 4+15 & 10-35 \\ 6-21 & 15+49 \end{bmatrix} = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix}. \end{aligned}$$

Thus,

$$(A^2)^T = \begin{bmatrix} 19 & 15 \\ -25 & 64 \end{bmatrix} = \begin{bmatrix} 19 & -25 \\ 15 & 64 \end{bmatrix} = (A^T)^2$$

4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Verify $(AB)^T = B^T A^T$.

Solution: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

So,

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Here, A has 2-columns and B has 2-rows. So, AB is defined. And, B^T has 2-columns and A^T has 2-rows. So, $B^T A^T$ is defined.

Now,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2+6 & 1+4 & 3+2 \\ 6+12 & 3+8 & 9+4 \\ 10+18 & 5+12 & 15+6 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 18 & 11 & 13 \\ 28 & 17 & 21 \end{bmatrix}. \end{aligned}$$

So,

$$(AB)^T = \begin{bmatrix} 8 & 18 & 28 \\ 5 & 11 & 17 \\ 5 & 13 & 21 \end{bmatrix}.$$

And,

$$\begin{aligned} B^T A^T &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2+6 & 6+12 & 10+18 \\ 1+4 & 3+8 & 5+12 \\ 3+2 & 9+4 & 15+6 \end{bmatrix} = \begin{bmatrix} 8 & 18 & 28 \\ 5 & 11 & 17 \\ 5 & 13 & 21 \end{bmatrix}. \end{aligned}$$

Thus, $(AB)^T = B^T A^T$.

Find A^* , where $A = \begin{bmatrix} 1+i & 2 & 3i \\ 2+i & i & 3 \end{bmatrix}$.

ution: Given that,

$$A = \begin{bmatrix} 1+i & 2 & 3i \\ 2+i & i & 3 \end{bmatrix}$$

So, $\bar{A} = \begin{bmatrix} 1-i & 2 & -3i \\ 2-i & -i & 3 \end{bmatrix}$ being \bar{A} is a complex conjugate of A.

Now,

$$A^* = (\bar{A})^T = \begin{bmatrix} 1-i & 2 & -3i \\ 2-i & -i & 3 \end{bmatrix}^T = \begin{bmatrix} 1-i & 2-i \\ 2 & -i \\ -3i & 3 \end{bmatrix}.$$

ow that the matrix $\begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix}$ is Hermitian.

Let, the given matrix is

$$A = \begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix}$$

Then the complex conjugate \bar{A} of A is,

$$\bar{A} = \begin{bmatrix} 7 & 7+4i & 1-i \\ 7-4i & 2 & 2+i \\ 1+i & 2-i & 1 \end{bmatrix}$$

So,

$$A^* = (\bar{A})^T = \begin{bmatrix} 7 & 7+4i & 1-i \\ 7-4i & 2 & 2+i \\ 1+i & 2-i & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & 7-4i & 1+i \\ 7+4i & 2 & 2-i \\ 1-i & 2+i & 1 \end{bmatrix} = A$$

Thus,

$$A^* = (\bar{A})^T = A$$

So, the given matrix A is Hermitian.

7. Show that the square matrix $A = \begin{bmatrix} i & 2+i & 3-i \\ -2-i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix}$ is skew Hermitian matrix.

Solution: Let

$$A = \begin{bmatrix} i & 2+i & 3-i \\ -2-i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix}$$

Then the complex conjugate \bar{A} of A is,

$$\bar{A} = \begin{bmatrix} -i & 2-i & 3+i \\ -2-i & -2i & 2 \\ -3+i & -2 & i \end{bmatrix}$$

Then,

$$(\bar{A})^T = \begin{bmatrix} -i & -2-i & -3+i \\ 2-i & -2i & -2 \\ 3+i & 2 & i \end{bmatrix}^T = -\begin{bmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{bmatrix} = -A$$

Thus,

$$A^* = (\bar{A})^T = -A$$

Therefore, A is skew-Hermitian matrix.

8. If $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ then show that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric matrix.

Solution: Let, $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Then, $A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Now,

$$A + A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}.$$

Here $a_{ij} = a_{ji}$ for $a_{ij} \in (A + A^T)$. So $(A + A^T)$ is symmetric.

And,

$$A - A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

Here, $a_{ij} = -a_{ji}$ for $a_{ij} \in (A - A^T)$. So, $A - A^T$ is skew-symmetric.

Prove that the following.

- (i) The sum of two symmetric matrix is symmetric.

Solution: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

are two symmetric matrices.

Thus, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$.
Now,

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix}$$

Since, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. So, $a_{12} + b_{12} = a_{21} + b_{21}$, $a_{n1} + b_{n1} = a_{1n} + b_{1n}$ and so on.
This shows that $A + B$ is a symmetric matrix.

- (ii) The sum of two skew-symmetric matrix is skew-symmetric.

Solution: Let,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

are two skew-symmetric matrices.

Therefore, $a_{ij} = -a_{ji} \quad \forall i \neq j$ and $a_{ii} = a_{ji} \quad \forall i = j$.Also, $b_{ij} = -b_{ji} \quad \forall i \neq j$ and $b_{ii} = b_{ji} \quad \forall i = j$.

Now,

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{pmatrix}$$

Here, $a_{ij} + b_{ij} = (-a_{ji}) + (-b_{ji}) = -(a_{ji} + b_{ji}) \quad \forall i \neq j$.Also, $a_{ii} + b_{ii} = a_{ii} + b_{ii} \quad \forall i = j$.This proves that $A + B$ is skew-symmetric matrix.

- (i) If
- A
- is a square matrix then
- $A + A^T$
- is symmetric and
- $A - A^T$
- skew-symmetric.

Solution: Let,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

be a square matrix. Then, its transpose matrix be

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Now,

$$A + A^T = \begin{pmatrix} a_{11} + a_{11} & a_{12} + a_{21} & \dots & a_{1n} + a_{n1} \\ a_{21} + a_{12} & a_{22} + a_{22} & \dots & a_{2n} + a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{n1} + a_{1n} & a_{n2} + a_{2n} & \dots & a_{nn} + a_{nn} \end{pmatrix}$$

This matrix has element $a_{ij} + b_{ji}$ on upper triangular form and $a_{ji} + a_{ij}$ on the lower triangular form.Since, $a_{ij} + a_{ji} = a_{ji} + a_{ij} \quad \forall i, j$

Thus, the element on the upper triangular is equal to the term on respective lower triangular position.

So, $A + A^T$ is symmetric matrix.

Next,

$$A - A^T = \begin{pmatrix} a_{11} - a_{11} & a_{12} - a_{21} & \dots & a_{1n} - a_{n1} \\ a_{21} - a_{12} & a_{22} - a_{22} & \dots & a_{2n} - a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{n1} - a_{1n} & a_{n2} - a_{2n} & \dots & a_{nn} - a_{nn} \end{pmatrix}$$

Here, the element in the upper triangular position has the form $a_{ij} - a_{ji}$ and in the lower triangular position has the form $a_{ji} - a_{ij}$, $\forall i$ and $\forall j$.Since, $a_{ij} - a_{ji} = -(a_{ji} - a_{ij})$.This shows that every element in upper triangular has same value but in negative form, in the respective lower triangular position. So, $A - A^T$ is skew-symmetric.

- (iv) Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrix.

Solution: Let A be a square matrix. Then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric matrices.Moreover, $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix.Here, $A = \frac{1}{2}[(A + A^T) + (A - A^T)]$.This shows that A can be expressed as the sum of its symmetric and skew-symmetric form.**EXERCISE 1.3**

1. Without expanding, show that each of the following determinants vanishes.

$$(i) \begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Applying $C_1 = C_1 - C_2 - C_3$ then,

$$= \begin{vmatrix} 36 & 1 & 6 \\ 24 & 7 & 4 \\ 12 & 3 & 2 \end{vmatrix}$$

Taking the common value 6 from first column. Then,

$$= 6 \begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}$$

Here, $C_1 = C_3$. So, the property of determinant tells us that the value of the determinant is 0. So,

$$\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix} = 6 \begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix} = 6 \times 0 = 0$$

$$(ii) \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ and $C_3 = C_3 - C_1$. Then,

$$= \begin{vmatrix} 265 & 21 & 46 \\ 240 & 27 & 42 \\ 219 & 17 & 38 \end{vmatrix}$$

$$(iii) \begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$, $R_3 = R_3 - R_4$. Then

$$= \begin{vmatrix} -3 & -5 & 1 & 0 \\ 18 & 14 & 4 & 7 \\ 1 & 1 & 1 & 1 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Again applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$, $C_3 = C_3 - C_4$. Then,

$$= \begin{vmatrix} 2 & -6 & 1 & 0 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix}$$

Again applying $R_1 = R_1 - R_4$ then,

$$= \begin{vmatrix} 0 & 0 & 0 & 2 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix}$$

Again applying $R_1 = R_1 - 2R_3$ then

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 4 & 10 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -1 & 2 \end{vmatrix} = 0.$$

being the first row has zero value in each position.

$$(iv) \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Applying $C_3 = C_3 + C_2$ then,

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = (a+b+c) \times 0 = 0$$

because if two columns have same value then the value of determinant is zero.

$$(v) \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$ then,

$$= \begin{vmatrix} 0 & a-b & a^2-b^2+ac-bc \\ 0 & b-c & b^2-c^2+ab-ac \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & (a-b)(a+b+c) \\ 0 & b-c & (b-c)(a+b+c) \\ 1 & c & c^2-ab \end{vmatrix}$$

Taking-out the common value $a-b$ from R_1 and $(b-c)$ from R_2 then

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & c & c^2-ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ then

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & a+b+c \\ 1 & c & c^2-ab \end{vmatrix} = 0, \text{ being each value in } R_1 \text{ is zero.}$$

$$(vi) \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applying $C_3 = C_3 + C_2$ then,

$$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Taking-out the common value $ab+bc+ca$ from C_3 then

$$= (ab+bc+ca) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} = (ab+bc+ca) \times 0 = 0$$

because if two columns of a determinant has same value then the determinant determines zero value.

$$(vii) \begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$$

Solution: Here,

$$\begin{aligned} & \left| \begin{array}{ccc} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{array} \right| \\ & = \frac{abc}{abc} \left| \begin{array}{ccc} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{array} \right| \end{aligned}$$

Applying $R_1 = aR_1$, $R_2 = bR_2$ and $R_3 = cR_3$ then

$$= \frac{1}{abc} \left| \begin{array}{ccc} 1 & a^2 & abc \\ 1 & b^2 & abc \\ 1 & c^2 & abc \end{array} \right|$$

Taking-out the common value abc from C_3 . Then,

$$= \frac{abc}{abc} \left| \begin{array}{ccc} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{array} \right| = 1 \times 0 = 0$$

because if the determinant has same value in two columns the value of determinant zero.

$$(viii) \left| \begin{array}{cccc} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + acd \\ 1 & c & c^2 & c^3 + abd \\ 1 & d & d^2 & d^3 + abc \end{array} \right|$$

[2004 Fall Q.No. 1(a)]

Solution: Here,

$$\begin{aligned} & \left| \begin{array}{cccc} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + acd \\ 1 & c & c^2 & c^3 + abd \\ 1 & d & d^2 & d^3 + abc \end{array} \right| \\ & = \left| \begin{array}{cccc} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{array} \right| + \left| \begin{array}{cccc} 1 & a & a^2 & bcd \\ 1 & b & b^2 & acd \\ 1 & c & c^2 & abd \\ 1 & d & d^2 & abc \end{array} \right| \end{aligned}$$

Applying for second matrix as $R_1 = aR_1$, $R_2 = bR_2$, $R_3 = cR_3$ and $R_4 = dR_4$. Then,

$$= \left| \begin{array}{cccc} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{array} \right| + \frac{1}{abcd} \left| \begin{array}{cccc} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{array} \right|$$

Taking-out the common value $abcd$ from second matrix. Then,

$$\begin{aligned} & \left| \begin{array}{cccc} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{array} \right| + \frac{abcd}{abcd} \left| \begin{array}{cccc} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{array} \right| \\ & = \left| \begin{array}{cccc} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{array} \right| + (-1)^3 \left| \begin{array}{cccc} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{array} \right| = 0 + 0 \end{aligned}$$

C_3 cross three parallel columns in second matrix.

$= 0$; adding 1st and 2nd matrices.

$$\begin{pmatrix} x & bc & b+c \\ x^2 & ac & a+c \\ x^3 & ab & a+b \end{pmatrix}$$

Solution: Here,

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ac & a+c \\ a^2b^2 & ab & a+b \end{vmatrix}$$

Multiply R_1 by a^2 , R_2 by b^2 and R_3 by c^2 and then taking out the column value $a^2b^2c^2$ from C_3 then,

$$= \frac{a^2b^2c^2}{a^2b^2c^2} \left| \begin{array}{ccc} 1 & a^2bc & a^2(b+c) \\ 1 & ab^2c & b^2(a+c) \\ 1 & abc^2 & c^2(a+b) \end{array} \right|$$

Taking out the common value abc from C_2 then,

$$= abc \left| \begin{array}{ccc} 1 & a & (a^2b + a^2c) \\ 0 & b-a & (ab^2 + b^2c) \\ 0 & c-a & (ac^2 + bc^2) \end{array} \right|$$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$ then,

$$= abc \left| \begin{array}{ccc} 1 & a & (a^2b + a^2c) \\ 0 & b-a & (ab^2 + b^2c - a^2b + a^2c) \\ 0 & c-a & (ac^2 + bc^2 - a^2b - a^2c) \end{array} \right|$$

$$= \left| \begin{array}{ccc} 1 & a & a^2b + a^2c \\ 0 & b-a & ab^2 - a^2b + b^2c - abc + abc - ca^2 \\ 0 & c-a & ac^2 - a^2c + bc^2 - abc + abc - a^2b \end{array} \right|$$

$$= \left| \begin{array}{ccc} 1 & a & a^2b + a^2c \\ 0 & b-a & (b-a)(ab + bc + ca) \\ 1 & c-a & (c-a)(ab + bc + ca) \end{array} \right|$$

Taking the common factor $(b-a)$ from R_2 and $(c-a)$ from R_3 then,

$$= abc(b-a)(c-a) \left| \begin{array}{ccc} 1 & a & a^2b + a^2c \\ 0 & 1 & ab + bc + ca \\ 0 & 1 & ab + bc + ca \end{array} \right|$$

Again, applying $R_3 \rightarrow R_3 - R_2$ then,

$$= abc(b-a)(c-a) \left| \begin{array}{ccc} 1 & a & a^2b + a^2c \\ 0 & 1 & ab + bc + ca \\ 0 & 0 & 0 \end{array} \right|$$

$\equiv 0$, being all values in R_3 are zero.

$$(x) \left| \begin{array}{ccc} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{array} \right|$$

Solution: Here,

$$\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$$

Applying $R_3 = R_3 - R_2$ and $R_2 = R_2 - R_1$ then,

$$= \left| \begin{array}{ccc} x+a & x+2a & x+3a \\ a & a & a \\ 2a & 2a & 2a \end{array} \right|$$

Taking-out the common value a from R_2 and $2a$ from R_3 then

$$= 2a^2 \left| \begin{array}{ccc} x+a & x+2a & x+3a \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right| = 2a^2 \times 0 = 0, \text{ being } R_2 = R_3.$$

2. Show that:

$$(i) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-z)(x-y)(z-x)$$

Solution: Here,

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_1 - R_2$ and $R_2 \leftrightarrow R_2 - R_3$, then

$$= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Taking out the common value $x-y$ from R_1 and $y-z$ from R_2 then,

$$= (x-y)(y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

Now, expanding from C_1 then,

$$= (x-y)(y-z) \{(y+z)-(x+y)\}$$

$$= (x-y)(y-z)(z-x)$$

$$(ii) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution: Here,

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3$ then

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking out the common value $a+b+c$ from R_1 then

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$ then

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking the common value $(a+b+c)$ from C_1 and C_2 then,

$$= (a+b+c)^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding the determinant from R_1 then,

$$= (a+b+c)^3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (a+b+c)^3$$

$$(iii) \begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$$

Solution: Here,

$$\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix}$$

Applying $R_1 = R_1 - R_2 - R_3$ then

$$= \begin{vmatrix} 0 & -2c & -2c \\ a & a+c & c \\ b & c & b+c \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} 0 & 0 & -2c \\ a & a & c \\ b & -b & b+c \end{vmatrix}$$

Now, expanding from R_1 then

$$= -2c \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2c(-ab - ab) = 4abc.$$

$$(iv) \begin{vmatrix} b+c & a & b \\ a+c & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$$

Solution: Here,

$$\begin{vmatrix} b+c & a & b \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3$ then,

$$= \begin{vmatrix} 2(a+b+c) & a+b+c & a+b+c \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Taking out the common value $(a+b+c)$ from R_1 then

$$= (a+b+c) \begin{vmatrix} 2 & 1 & 1 \\ a+c & c & a \\ a+b & b & c \end{vmatrix}$$

Applying $C_1 = C_1 - C_2 - C_3$ and $C_2 = C_2 - C_3$ then

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & c-a & a \\ a-c & b-c & c \end{vmatrix}$$

Now, expanding from R_1 then

$$= (a+b+c)(a-c)^2$$

$$v) \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = x^3(4+x)$$

Solution: Here,

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ then,

$$= \begin{vmatrix} 4+x & 4+x & 4+x & 4+x \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Taking-out the common value $4+x$ from R_1 then,

$$= (4+x) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

Again applying $C_2 = C_2 - C_1$, $C_3 = C_3 - C_1$, $C_4 = C_4 - C_1$ then

$$= (4+x) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x & 0 & 0 \\ 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix}$$

Expanding the determinant from C_1 then

$$= (4+x)(x^3)$$

$$= (4+x)x^3$$

$$(vi) \begin{vmatrix} a & x & x & x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix} = (a+3x)(a-x)^3$$

Solution: Here,

$$\begin{vmatrix} a & x & x & x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ then

$$= \begin{vmatrix} a+3x & a+3x & a+3x & a+3x \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Taking-out the common value $(a+3x)$ from R_1 then

$$= (a+3x) \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & x & x \\ x & x & a & x \\ x & x & x & a \end{vmatrix}$$

Applying $C_2 = C_2 - C_1$, $C_3 = C_3 - C_1$, $C_4 = C_4 - C_1$, then

$$= (a+3x) \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & a-x & 0 & 0 \\ x & 0 & a-x & 0 \\ x & 0 & 0 & a-x \end{vmatrix}$$

Now, expanding the determinant from R_1 then

$$= (a+3x)(a-x)^3$$

$$(vii) \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = 1+a+b+c+d$$

Solution: Here,

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

Applying $C_1 = C_1 + C_2 + C_3 + C_4$ then

$$= \begin{vmatrix} 1+a+b+c+d & b & c & d \\ 1+a+b+c+d & 1+b & c & d \\ 1+a+b+c+d & b & 1+c & d \\ 1+a+b+c+d & b & c & 1+d \end{vmatrix}$$

Taking-out the common value $1+a+b+c+d$ then,

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix}$$

Applying $R_2 = R_2 - R_1$, $R_3 = R_3 - R_1$, $R_4 = R_4 - R_1$ then

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Expanding the determinant from C_1 then

$$= (1+a+b+c+d)$$

$$(viii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution: Here,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$ and $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

Taking-out the common value $(a-b)$ from C_1 and $(b-c)$ from C_2 then

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Now, expanding from R_1 then

$$\begin{aligned} &= (a-b)(b-c) \begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix} \\ &= (a-b)(b-c) [(b^2+bc+c^2) - (a^2+ab+b^2)] \\ &= (a-b)(b-c)(c-a)(a+b+c) \end{aligned}$$

$$(ix) \begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ y & z & 0 \end{vmatrix} = (x-2y+z)^2$$

Solution: Here,

$$\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$$

Applying $R_1 = R_1 - 2R_2 + R_3$, then

$$= \begin{vmatrix} 0 & 0 & 0 & x - 2y + z \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$$

Expanding from R_1 , we get,

$$= -(x - 2y + z) \begin{vmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ x & y & z \end{vmatrix}$$

Applying $C_1 = C_1 - 2C_2 + C_3$, then

$$= -(x - 2y + z) \begin{vmatrix} 0 & 6 & 7 \\ 0 & 7 & 8 \\ x - 2y + z & y & z \end{vmatrix}$$

Expanding from C_1 , then

$$\begin{aligned} &= -(x - 2y + z)(x - 2y + z)(48 - 49) \\ &= (x - 2y + z)^2 \end{aligned}$$

$$(x) \begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix} = -(a - b)^4$$

Solution: Here,

$$\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$, Then

$$= \begin{vmatrix} a-b & b & a & a \\ a-b & b & b & b \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Taking the common value $(a - b)$ from C_1 , then

$$= (a - b) \begin{vmatrix} 1 & b & a & a \\ 1 & b & b & b \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Again applying $R_2 = R_2 - R_1$, Then

$$= (a - b) \begin{vmatrix} 1 & b & a & a \\ 0 & 0 & b-a & b-a \\ 0 & b & b & a \\ 0 & a & b & a \end{vmatrix}$$

Expanding from C_1 , then

$$= (a - b) \begin{vmatrix} 0 & b-a & b-a \\ b & b & a \\ a & b & a \end{vmatrix}$$

Taking the common value $(b - a)$ from R_1 , then

$$= (a - b)(b - a) \begin{vmatrix} 0 & 1 & 1 \\ b & b & a \\ a & b & a \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$. Then

$$= -(a - b)(b - a) \begin{vmatrix} 0 & 0 & 1 \\ b & b-a & a \\ a & b-a & a \end{vmatrix}$$

Expanding from R_1 then

$$= -(a - b)^2 [b(b - a) - a(b - a)]$$

$$= -(a - b)^2 (b - a)^2$$

$$= -(a - b)^4$$

$$(xi) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (b - c)(c - a)(a - b)$$

Solution: Here,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ and $R_2 = R_2 - R_3$, then

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ bc-ca & ca-ab & ab \end{vmatrix}$$

Taking common value $(a - b)$ from C_1 and $(b - c)$ from C_2 then

$$\begin{aligned} &= (a - b)(b - c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ c & a & ab \end{vmatrix} = (a - b)(b - c) \begin{vmatrix} 1 & 1 \\ c & a \end{vmatrix} \\ &= (a - b)(b - c)(c - a) \end{aligned}$$

$$(xii) \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx)$$

[2006 Spring Q.No. 1(a) OR]

Solution: Here,

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$, $R_2 = R_2 - R_3$ then

$$= \begin{vmatrix} x-y & x^2-y^2 & yz-zx \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix}$$

Taking-out the common value $x - y$ from R_1 and $y - z$ from R_2 , then

$$= (x-y)(y-z) \begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 = R_1 - R_2$ then,

$$= (x-y)(y-z) \begin{vmatrix} 0 & x-z & x-z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Taking-out the common value $x-z$ from R_1 then

$$= (x-y)(y-z)(x-z) \begin{vmatrix} 0 & 1 & 1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $C_2 = C_2 - C_3$ then

$$= (x-y)(y-z)(x-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & x+y+z & -x \\ z & z^2-xy & xy \end{vmatrix}$$

Expanding from R_1 then

$$= (x-y)(y-z)(x-z) \begin{vmatrix} 1 & x+y+z \\ z & z^2-xy \end{vmatrix}$$

$$= (x-y)(y-z)(x-z)(z^2-xy-xz-yz-z^2)$$

$$= (x-y)(y-z)(x-z)(-1)(xy+yz+zx)$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$(xiii) \begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3 \quad [2007 Fall Q.No. 1(a) OR]$$

Solution: Here,

$$\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix}$$

Multiplying R_1 by c , R_2 by a and R_3 by b then

$$= \frac{1}{abc} \begin{vmatrix} c(a+b)^2 & c^2a & bc^2 \\ ca^2 & a(b+c)^2 & a^2b \\ b^2c & ab^2 & b(c+a)^2 \end{vmatrix}$$

Taking-out the common value c from C_1 , a from C_2 and b from C_3 then

$$= \frac{cab}{abc} \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

Applying $C_1 = C_1 - C_3$ and $C_2 = C_2 - C_3$ then

$$= \begin{vmatrix} (a+b)^2 - c^2 & 0 & c^2 \\ 0 & (b+c)^2 & a^2 \\ b^2 - (c+a)^2 & b^2 - (c+a)^2 & (c+a)^2 \end{vmatrix}$$

Taking the common value $(a+b+c)$ from C_1 and C_2 then

$$= (a+b+c)^2 \begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ b-c-a & b-c-a & (c+a)^2 \end{vmatrix}$$

Applying $R_3 = R_3 - R_1 - R_2$ then

$$= (a+b+c)^2 \begin{vmatrix} a+b-c & 0 & c^2 \\ 0 & b+c-a & a^2 \\ -2a & -2c & 2ca \end{vmatrix}$$

Now, expanding we get,

$$\begin{aligned} &= (a+b+c)^2 \left\{ (a+b-c) \begin{vmatrix} b+c-a & a^2 \\ -2c & 2ca \end{vmatrix} + c^2 \begin{vmatrix} 0 & b+c-a \\ -2a & -2ca \end{vmatrix} \right\} \\ &= (a+b+c)^2 [(a+b+c)[2abc + 2c^2a - 2ca^2 + 2ca^2] + c^2(2ab + 2ac - 2a^2)] \\ &= (a+b+c)^2 (2a^3bc + 2a^2c^2 + 2ab^2c + 2abc^2 - 2abc^2 - 2ac^3 + 2abc^2 + 2ac^3 - 2a^2c^2) \\ &= 2abc(a+b+c)^2(a+b+c) \\ &= 2abc(a+b+c)^3 \end{aligned}$$

$$(xiv) \begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2 + d^2$$

[2014 Spring Q. No. 1(a) OR]

[2011 Spring Q.No. 1(a)] [2010 Fall Q.No. 1(a) OR] [2005 Fall Q.No. 1(a)]

Solution: Here,

$$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix}$$

Multiplying R_1 by a , R_2 by b , R_3 by c and R_4 by d then,

$$= \frac{1}{abcd} \begin{vmatrix} a(a^2+1) & a^2b & a^2c & a^2d \\ ab^2 & b(b^2+1) & b^2c & b^2d \\ ac^2 & bc^2 & c(c^2+1) & c^2d \\ ad^2 & bd^2 & d^2c & d(d^2+1) \end{vmatrix}$$

Taking-out the common value a from C_1 , b from C_2 , c from C_3 , d from C_4 then

$$= \frac{abcd}{abcd} \begin{vmatrix} a^2+1 & a^2 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 & b^2 \\ c^2 & c^2 & c^2+1 & c^2 \\ d^2 & d^2 & d^2 & d^2+1 \end{vmatrix}$$

Applying $R_1 = R_1 + R_2 + R_3 + R_4$ and the taking-out the common value $1 + a^2 + b^2 + c^2 + d^2$ from R_1 . So that,

$$= (1 + a^2 + b^2 + c^2 + d^2) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 & b^2 \\ c^2 & c^2 & c^2+1 & c^2 \\ d^2 & d^2 & d^2 & d^2+1 \end{vmatrix}$$

Applying $C_1 = C_1 - C_2$, $C_2 = C_2 - C_3$, $C_3 = C_3 - C_4$. Then,

$$= (1 + a^2 + b^2 + c^2 + d^2) \begin{vmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & b^2 \\ 0 & -1 & 1 & c^2 \\ 0 & 0 & -1 & d^2+1 \end{vmatrix}$$

Now, expanding the determinant from R_1 then

$$= (-1)(1 + a^2 + b^2 + c^2 + d^2) \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

Again applying $C_2 = C_2 + C_1$ then

$$= (-1)(1 + a^2 + b^2 + c^2 + d^2) \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding the determinant from R_1 we get,

$$\begin{aligned} &= (-1)^2 (1 + a^2 + b^2 + c^2 + d^2) (1 - 0) \\ &= (1 + a^2 + b^2 + c^2 + d^2) \end{aligned}$$

$$(xv) \begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + d^2 + \lambda)$$

[2008 Spring Q.No. 1(b) OR]

Solution: Here,

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

Multiplying R₁ by a, R₂ by b, R₃ by c, R₄ by d then

$$= \frac{1}{abcd} \begin{vmatrix} a(a^2+\lambda) & a^2b & a^2c & a^2d \\ ab^2 & b(b^2+\lambda) & b^2c & b^2d \\ ac^2 & bc^2 & c(c^2+\lambda) & c^2d \\ ad^2 & bd^2 & cd^2 & d(d^2+\lambda) \end{vmatrix}$$

Then taking-out the common value a from C₁, b from C₂, c from C₃ and d from C₄ then

$$= \frac{abcd}{abcd} \begin{vmatrix} a^2+\lambda & a^2 & a^2 & a^2 \\ b^2 & b^2+\lambda & b^2 & b^2 \\ c^2 & c^2 & c^2+\lambda & c^2 \\ d^2 & d^2 & d^2 & d^2+\lambda \end{vmatrix}$$

Now, applying R₁ = R₁ + R₂ + R₃ + R₄ and the taking-out the common a² + b² + c² + d² + λ from R₁. So that,

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^2 & b^2+\lambda & b^2 & b^2 \\ c^2 & c^2 & c^2+\lambda & c^2 \\ d^2 & d^2 & d^2 & d^2+\lambda \end{vmatrix}$$

Applying C₂ = C₂ - C₁, C₃ = C₃ - C₁, C₄ = C₄ - C₁ then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^2 & \lambda & 0 & 0 \\ c^2 & 0 & \lambda & 0 \\ d^2 & 0 & 0 & \lambda \end{vmatrix}$$

Expanding the determinant from R₁ then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

Expanding from C₁ then

$$= (a^2 + b^2 + c^2 + d^2 + \lambda) \lambda^3$$

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

Now: Here,

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Taking-out the common value a from C₁, b from C₂, c from C₃ and d from C₄ then

$$= abcd \begin{vmatrix} (1/a)+1 & 1/b & 1/c & 1/d \\ 1/a & (1/b)+1 & 1/c & 1/d \\ 1/a & 1/b & (1/c)+1 & 1/d \\ 1/a & 1/b & 1/c & 1+(1/d) \end{vmatrix}$$

Then applying C₁ = C₁ + C₂ + C₃ + C₄ and the taking-out the common value $\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$ from C₁ then

$$= (abcd) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 1 & 1+(1/b) & 1/c & 1/d \\ 1 & 1/b & 1+(1/c) & 1/d \\ 1 & 1/b & 1/c & 1+(1/d) \end{vmatrix}$$

Again applying R₂ = R₂ - R₁, R₃ = R₃ - R₁, R₄ = R₄ - R₁ then

$$= (abcd) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Now, expanding the determinant from C₁ then

$$\begin{aligned} &= (abcd) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) (1) \\ &= abcd \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \end{aligned}$$

$$(xvii) \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$$

Solution: Here,

$$\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying R₁ = R₁ + R₂ then

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Taking-out the common value (a + b + c) from R₁ then

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying C₂ = C₂ - C₁ and C₃ = C₃ - C₁ then

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b+c & b-a & b-c \\ a^2 & b^2-a^2 & c^2-b^2 \end{vmatrix}$$

Taking-out the common value $(b-a)$ from C_2 and $(b-c)$ from C_3 then

$$= (a+b+c)(b-a)(b-c) \begin{vmatrix} 1 & 0 & 0 \\ b+c & 1 & 1 \\ a^2 & b+a & c+b \end{vmatrix}$$

Applying $C_1 = C_3 - C_2$ then

$$= (a+b+c)(b-a)(b-c) \begin{vmatrix} 1 & 0 & 0 \\ b+c & 1 & 0 \\ a^2 & b+a & c-a \end{vmatrix}$$

Expanding from R_1 then

$$= (a+b+c)(b-a)(b-c)(c-a)$$

$$= -(a-b)(b-c)(c-a)(a+b+c)$$

$$(xviii) \begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

Taking-out the common value

$$\Rightarrow abc \begin{vmatrix} 1 & bc & 1 \\ b & ca & 1 \\ c & ab & 1 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Multiplying R_1 by a , R_2 by b and R_3 by c on the left determinants then

$$\Rightarrow \begin{vmatrix} a^2 & abc & a \\ b^2 & abc & b \\ c^2 & abc & c \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Taking-out the common value on the left determinant then,

$$\Rightarrow abc \begin{vmatrix} a^2 & 1 & a \\ b^2 & 1 & b \\ c^2 & 1 & c \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow (-1)abc \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow (-1)^2 abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

This completes the requirement.

$$(xix) \begin{vmatrix} bcd & 1 & a^2 & a^3 \\ acd & 1 & b^2 & b^3 \\ abd & 1 & c^2 & c^3 \\ abc & 1 & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

Solution: Here,

$$\begin{vmatrix} bcd & 1 & a & a^2 \\ acd & 1 & b & b^2 \\ abd & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix}$$

Multiplying R_1 by a , R_2 by b , R_3 by c and R_4 by d and then taking out the common value $abcd$ from C_1 . So that,

$$= \frac{abcd}{abcd} \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

$$(xx) \begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2 \quad [2012 Fall Q.No. 1(a)]$$

Solution: Here,

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

Multiplying R_1 by a , R_2 by b , R_3 by c and then applying $R_1 \rightarrow R_1 - R_2 - R_3$ then,

$$= \frac{1}{abc} \begin{vmatrix} 0 & -2bc^2 & -2b^2c \\ ab^2 & bc^2+a^2b & b^2c \\ ac^2 & bc^2 & a^2c+b^2c \end{vmatrix}$$

Thing out common value $-2bc$ from R_1 then,

$$= \frac{-2bc}{abc} \begin{vmatrix} 0 & c & b \\ ab^2 & bc^2+a^2b & b^2c \\ ac^2 & bc^2 & a^2c+b^2c \end{vmatrix}$$

Taking out the common value 'a' from C_1 , 'b' from C_2 and 'c' from C_3 then,

$$\begin{aligned} &= -2bc \left\{ 0 - b \begin{vmatrix} c & b \\ bc^2+a^2b & b^2c \end{vmatrix} + c \begin{vmatrix} c & b \\ c^2+a^2 & bc \end{vmatrix} \right\} \\ &= -2bc \{ 0 - b(a^2c + b^2c - b^2c) + c(bc^2 - bc^2 - a^2b) \} \\ &= -2bc \{ -b(a^2c) + c(-a^2b) \} \\ &= 2bc(a^2bc + a^2bc) \\ &= 2bc \times 2a^2bc \\ &= 4a^2b^2c^2 \end{aligned}$$

$$(xi) \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution: Here,

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Taking-out the common value a from C_1 , b from C_2 , c from C_3 then,

$$= abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Applying $C_1 = C_1 + C_3$ and $C_2 = C_2 - C_3$, then,

$$\begin{aligned}
 &= (b-a)(c-a)(bc^2 + ac^2 - b^2c - ab^2) \\
 &= (b-a)(c-a)[(c-b)bc + a(c^2 - b^2)] \\
 &= (b-a)(c-a)(c-b)(bc + a(b+c)) \\
 &= (a-b)(b-c)(c-a)(bc + ab + ca) \\
 &= (a-b)(b-c)(c-a)(ab + bc + ca).
 \end{aligned}$$

3. Solve the following equations

$$(i) \begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Solution: Given that,

$$\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Applying $C_1 = C_1 + C_3 - 2C_2$ then,

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ -3 & 4x+3 & 6x+3 \\ -3 & 6x+4 & 8x+4 \end{vmatrix} = 0$$

Again, applying $R_3 = R_3 - R_1$ then

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ -3 & 4x+3 & 6x+3 \\ -3 & 4x+3 & 5x+3 \end{vmatrix} = 0$$

Again, applying $R_2 = R_2 - R_1$ then

$$\Rightarrow \begin{vmatrix} 0 & 2x+1 & 3x+1 \\ 0 & 0 & x \\ -3 & 4x+3 & 5x+3 \end{vmatrix} = 0$$

Now, expanding the determinant from C_1 then,

$$\Rightarrow (-3) \begin{vmatrix} 2x+1 & 3x+1 \\ 0 & x \end{vmatrix} = 0$$

$$\Rightarrow (-3)(-x)(2x+1) = 0$$

$$\Rightarrow x(2x+1) = 0 \quad [\because 3 \neq 0]$$

$$\Rightarrow x = 0, \frac{1}{2}$$

$$(ii) \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$$

Solution: Here,

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

Solution: Here,

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 2R_1$, then,

$$\Rightarrow \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 2x+2 & 3x+1 & 4x-1 \\ 1 & -4 & 0 \end{vmatrix} = 0$$

Again, applying $C_2 = C_2 + 4C_1$ then

$$\Rightarrow \begin{vmatrix} 4x & 22x+2 & 8x+1 \\ 2x+2 & 11x+9 & 4x-1 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Again, applying $R_1 = R_1 - 2R_2$ then

$$\Rightarrow \begin{vmatrix} -4 & -16 & 3 \\ 2x+2 & 11x+9 & 4x-1 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Now, expanding the determinant from R_3 then,

$$\Rightarrow \begin{vmatrix} -16 & 3 \\ 11x+9 & 4x-1 \end{vmatrix} = 0$$

$$\Rightarrow -64x + 16 - 33x - 27 = 0$$

$$\Rightarrow 97x + 11 = 0$$

$$\Rightarrow x = \frac{-11}{97}$$

$$4. \text{ If } a+b+c=0, \text{ solve } \begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$$

[2011 Fall Q.No. 1(a) OR] [2010 Spring Q.No. 1(a) OR]

Solution: Let, $a+b+c=0$

Here,

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$ and then taking out the common value $a+b+c-x$ from C_1 . Then,

$$\Rightarrow (a+b+c-x) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$$\Rightarrow (-x) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0 \quad [\because a+b+c=0]$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ then,

$$\Rightarrow (-x) \begin{vmatrix} 1 & c & b \\ 0 & b-c-x & a-b \\ 0 & a-x & c-b-x \end{vmatrix} = 0$$

$$\Rightarrow (-x) [(b-c-x)(c-b-x) - (a-b)(a-c)] = 0$$

$$\Rightarrow (-x) [-(b-c-x)(b-c+x) + (a-b)(c-a)] = 0$$

$$\Rightarrow (-x) [-(b-c)^2 + x^2 + (a-b)(c-a)] = 0$$

$$\Rightarrow x[x^2 + (a-b)(c-a) - (b-c)^2] = 0$$

$$\Rightarrow x[x^2 + ac - bc - a^2 + ab - b^2 + 2bc - c^2] = 0$$

$$\Rightarrow x[x^2 + ac + bc + ab - a^2 - b^2 - c^2] = 0$$

$$\Rightarrow x[x^2 - (a+b+c)^2 + 3(ac+bc+ab)] = 0$$

$$\Rightarrow x[x^2 + 3(ac + bc + ab)] = 0 \quad [\because a + b + c = 0]$$

This gives,

$$x = 0, \text{ or, } x = \pm \sqrt{3ab + bc + ac}.$$

5. Show that:

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$$

Solution: Here,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 &= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \\ &= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \end{aligned}$$

[∴ interchanging C_2 and C_3] Solution: Here,

$$\begin{aligned} &= \begin{vmatrix} a & b & c \\ -b & c & a \\ -c & a & b \end{vmatrix} \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \\ &= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -bc+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+h^2 & -bc+a^2+hc & -c^2+ab+ab \end{vmatrix} \\ &= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \end{aligned}$$

Next, we have

$$\begin{aligned} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 &= [a(bc-a^2) - b(b^2-ca) + c(ab-c^2)]^2 \\ &= [abc - a^3 - b^3 + abc + abc - c^3]^2 \\ &= (-1)^2 (a^3 + b^3 + c^3 - 3abc)^2 \\ &= (a^3 + b^3 + c^3 - 3abc)^2 \end{aligned}$$

6. Show that:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} \text{ as the product of two determinants and equals to } 4a^2b^2c^2.$$

$$\text{Solution: Here, } \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out the common value 'a' from C_1 , 'b' from C_2 and 'c' from C_3 . Then,

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 - C_3$, then,

$$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$

Again applying $R_2 \rightarrow R_2 - R_3$ then,

$$= abc \begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix}$$

$$= 2ab^2c (ac - c^2 + ac + bc^2)$$

$$= 4a^2b^2c^2$$

$$= \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

7. Show that the determinant

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} \text{ is a perfect square and the value}$$

$$\begin{aligned} &\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+0+ca \\ 0+0+ab & c^2+0+a^2 & 0+0+bc \\ 0+0+ca & 0+0+bc & b^2+a^2+0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 \end{aligned}$$

This shows that the given determinant is a perfect square a determinant. Moreover,

$$\begin{aligned} &\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = [0 - c(0 - ab) + b(ac - 0)]^2 \\ &= (abc + abc)^2 \\ &= 4a^2b^2c^2 \end{aligned}$$

EXERCISE 1.4

1. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$. Show that its adjoint is itself.

Solution: Let

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Here,

$$\text{Cofactor of } -4 = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0 - 4 = -4$$

$$\text{Cofactor of } -3 = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(3 - 4) = 1$$

$$\text{Cofactor of } -3 = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = (4 - 0) = 4$$

$$\text{Cofactor of } 4 = (a_{23}) = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1 - 2) = 1$$

$$\text{Cofactor of } 1 = (a_{31}) = (-1)^{3+1} \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = (8 - 5) = 3$$

$$\text{Cofactor of } 1 = (a_{32}) = (-1)^{3+2} \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = -(4 - 15) = 11$$

$$\text{Cofactor of } 2 = (a_{33}) = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = (1 - 6) = -5$$

Thus, the adjoint matrix of A is

$\text{Adj.}(A) = \text{Transpose of cofactor matrix of } A$

$$= \begin{bmatrix} -2 & -2 & 2 \\ 1 & -3 & 1 \\ 3 & 11 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

And,

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 1(2 - 4) - 2(6 - 4) + 5(3 - 1) \\ = -2 - 4 + 10 = 4.$$

Here, $|A| = 4 \neq 0$. So, A^{-1} exists.

Now,

$$A^{-1} = \frac{\text{Adj.}(A)}{|A|} = \frac{1}{4} \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

Note: If $|A| = 0$ then A^{-1} is not exist.

$$(ii) \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$(v) \begin{bmatrix} 3 & -1 & 5 \\ 2 & 6 & 4 \\ 5 & 5 & 9 \end{bmatrix}$$

$$(viii) \begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 1 & 4 & 9 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$(vii) \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Solution: Process as (i).

$$4. \text{ If } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}. \text{ Verify that } (AB)^{-1} = B^{-1} A^{-1}.$$

Solution: Let,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Here,

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{vmatrix} = 1(18 - 12) - 1(2 - 8) + 1(3 - 18) \\ = 6 + 6 - 15 = -3$$

Thus, $|A| = -3 \neq 0$. So A^{-1} exists.

Also,

$$|B| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 1(9 - 1) - 2(6 + 1) + 0 \\ = 8 - 14 = -6$$

Thus, $|B| = -6 \neq 0$. So B^{-1} exists.

For cofactor matrix of A,

$$\text{Cofactor of } a_{11} = (18 - 12) = 6$$

$$\text{Cofactor of } a_{12} = -(2 - 3) = 1$$

$$\text{Cofactor of } a_{13} = (4 - 9) = -5$$

$$\text{Cofactor of } a_{21} = -(2 - 8) = 6$$

$$\text{Cofactor of } a_{22} = (2 - 2) = 0$$

$$\text{Cofactor of } a_{23} = -(4 - 1) = -3$$

$$\text{Cofactor of } a_{31} = (3 - 18) = -15$$

$$\text{Cofactor of } a_{32} = -(3 - 2) = -1$$

$$\text{Cofactor of } a_{33} = (9 - 1) = 8$$

For cofactor matrix of B,

$$\text{Cofactor of } a_{11} = (9 - 1) = 8$$

$$\text{Cofactor of } a_{12} = -(6 + 1) = -7$$

$$\text{Cofactor of } a_{13} = (-2 - 3) = -5$$

$$\text{Cofactor of } a_{21} = -(6 - 0) = -6$$

$$\text{Cofactor of } a_{22} = (3 - 0) = 3$$

$$\text{Cofactor of } a_{23} = (-1 - 2) = 3$$

$$\text{Cofactor of } a_{31} = (-2 - 0) = -2$$

$$\text{Cofactor of } a_{32} = -(-1 - 0) = 1$$

$$\text{Cofactor of } a_{33} = (3 - 4) = -1$$

Thus, the adjoint matrix of A is

$$\text{Adj.}(A) = \text{Transpose of cofactor matrix of } A \\ = \begin{bmatrix} 6 & 1 & -5 \\ 6 & 0 & -3 \\ -15 & -1 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

And, the adjoint matrix of B is

$$\text{Adj.}(B) = \text{Transpose of cofactor matrix of } B \\ = \begin{bmatrix} 8 & -7 & -5 \\ -6 & 3 & 3 \\ -2 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 8 & 6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

Hence, the inverse matrix of A and B are,

$$A^{-1} = \frac{\text{Adj.}(A)}{|A|} = -\frac{1}{3} \begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

and

$$B^{-1} = \frac{\text{Adj.}(B)}{|B|} = -\frac{1}{6} \begin{bmatrix} 8 & 6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

Now,

$$B^{-1} A^{-1} = \frac{1}{18} \begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix} \\ = \frac{1}{18} \begin{bmatrix} 48 - 6 + 10 & 48 + 0 + 6 & -120 + 6 - 16 \\ -42 + 3 - 5 & -42 + 0 - 3 & 105 - 3 + 8 \\ -30 + 3 + 5 & -30 + 0 + 3 & 75 - 3 - 8 \end{bmatrix} \\ = \frac{1}{18} \begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 641 \end{bmatrix}, \dots\dots (1)$$

Next,

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \\ = \begin{bmatrix} 1+2+1 & 2+3-2 & 0-1+6 \\ 1+18+3 & 2+27-3 & 0-9+9 \\ 1+8+2 & 2+12-2 & 0-4+6 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{bmatrix}$$

Then,

$$|AB| = \begin{vmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{vmatrix} = 5(22 - 0) - 3(44 - 0) + 5(264 - 286) \\ = 260 - 132 - 110 = 18$$

Thus, $|AB| = 18 \neq 0$. So, $(AB)^{-1}$ exists.

Here for cofactor matrix of AB

$$\begin{aligned} \text{Cofactor of } 4 &= (52 - 0) = 52 \\ \text{Cofactor of } 3 &= -(44 - 0) = -44 \\ \text{Cofactor of } 5 &= (264 - 286) = -22 \\ \text{Cofactor of } 22 &= -(6 - 60) = 54 \\ \text{Cofactor of } 26 &= (10 - 55) = -45 \\ \text{Cofactor of } 0 &= -(60 - 33) = -27 \\ \text{Cofactor of } 11 &= (0 - 130) = -130 \\ \text{Cofactor of } 12 &= -(0 - 110) = 110 \\ \text{Cofactor of } 2 &= (130 - 66) = 64 \end{aligned}$$

Thus, the adjoint matrix of AB is

$\text{Adj.}(AB) = \text{Transpose of cofactor matrix of } AB$

$$= \begin{bmatrix} 52 & -44 & -22 \\ 54 & -45 & -27 \\ -130 & 110 & 64 \end{bmatrix}^T = \begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix}$$

Now,

$$(AB)^{-1} = \frac{\text{Adj.}(AB)}{|AB|} = \frac{1}{18} \begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix} \quad \dots \dots \dots (2)$$

Hence, by (1) and (2), we conclude that,

$$(AB)^{-1} = B^{-1}A^{-1}$$

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Process as Q. 4

6. Find A, where $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution: Since we know that $(A^{-1})^{-1} = A$.

$$\text{Given that, } A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

So, we wish to find the inverse of A^{-1}

Here,

$$|A^{-1}| = \begin{vmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{vmatrix} = 1(0 - 1) + 3(0 + 2) + 2(3 - 6) \\ = -1 + 6 - 6 = -1 \neq 0$$

So, inverse of A^{-1} exists

For cofactor of A^{-1} ,

$$\begin{aligned} \text{Cofactor of } a_{11} &= 0 - 1 = -1 \\ \text{Cofactor of } a_{13} &= 3 - 6 = -3 \\ \text{Cofactor of } a_{22} &= 0 - 4 = -4 \\ \text{Cofactor of } a_{31} &= 3 - 6 = -3 \\ \text{Cofactor of } a_{33} &= (3 - 9) = -6 \end{aligned}$$

Thus, the adjoint matrix of A is,

$\text{Adj.}(A^{-1}) = \text{Transpose of Cofactor matrix of } A^{-1}$

$$= \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}^T = (-1) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Hence the inverse of A^{-1} is,

$$A = (A^{-1})^{-1} = \frac{\text{Adj.}(A^{-1})}{|A^{-1}|} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

EXERCISE - 1.5

Solve the following system of linear equations by using Gauss elimination method:

$$6x + 4y = 2$$

$$3x - 5y = -34$$

Solution: Given system of linear equations be

$$6x + 4y = 2$$

$$3x - 5y = -34$$

The augmented matrix of given equations be

$$\left[\begin{array}{cc|c} 6 & 4 & 2 \\ 3 & -5 & -34 \end{array} \right]$$

Applying $R_2 = 2R_2 - R_1$ then

$$\left[\begin{array}{cc|c} 6 & 4 & 2 \\ 0 & -14 & -70 \end{array} \right]$$

Thus, from R_2 , we have

$$-14y = -70 \Rightarrow y = 5$$

and from R_2 , we have

$$\begin{aligned} 6x + 4y &= 2 \Rightarrow 6x = 2 - 4y \Rightarrow 6x = 2 - 20 \quad [\because y = 5] \\ &\Rightarrow 6x = -18 \\ &\Rightarrow x = -3 \end{aligned}$$

Hence $x = -3, y = 5$ be required solution.

$$3x - 0.5y = 0.6$$

$$1.5x + 4.5y = 6$$

Solution: Given system of linear equations be

$$3x - 0.5y = 0.6$$

$$1.5x + 4.5y = 6$$

The augment matrix of given equations be

$$\begin{bmatrix} 3 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6 \end{bmatrix}$$

Applying $R_2 = 2R_1 - R_1$, then,

$$\begin{bmatrix} 3 & -0.5 & 0.6 \\ 0 & 9.5 & 11.4 \end{bmatrix}$$

Thus, from R_2 we have,

$$9.5y = 11.4 \Rightarrow y = 1.2$$

and from R_1 , we have

$$\begin{aligned} 3x - 0.5y &= 0.6 \Rightarrow 3x - 0.6 = 0.6 \Rightarrow 3x = 1.2 \\ &\Rightarrow x = 0.4 \end{aligned}$$

Hence, $x = 0.4$, $y = 1.2$ are solution of given equations.

3. $x + y - z = 9$

$$8y + 6z = -6$$

$$-2x + 4y - 6z = 40$$

Solution: Given that,

$$x + y - z = 9$$

$$8y + 6z = -6$$

$$-2x + 4y - 6z = 40$$

The augment matrix of given system of equations be

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{bmatrix}$$

Applying $R_3 = R_3 + 2R_1$, then

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{bmatrix}$$

Again applying $R_3 = 4R_3 - 3R_2$, then

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -50 & 250 \end{bmatrix}$$

Thus, from R_3 , we have,

$$-50z = 250 \Rightarrow z = -5$$

and from R_2 ,

$$\begin{aligned} 8y + 6z &= -6 \Rightarrow 8y = -6 - 6z = -6 + 30 = 24 \\ &\Rightarrow y = 3 \end{aligned}$$

Also, from R_1 ,

$$x + y - z = 9 \Rightarrow x = 9 - y + z = 9 - 3 - 5 = 1$$

Hence, $x = 1$, $y = 3$, $z = -5$ are solutions of given equations.

$$13x + 12y = -6$$

$$-4x + 7y = -73$$

$$11x - 13y = 157$$

Solution: Given equations are

$$13x + 12y = -6$$

$$-4x + 7y = -73$$

$$11x - 13y = 157$$

The augmented matrix of given equation be

$$\begin{bmatrix} 13 & 12 & 0 & -6 \\ -4 & 7 & 0 & -73 \\ 11 & -13 & 0 & 157 \end{bmatrix}$$

Applying $R_2 = 13R_2 + 4R_1$ and $R_3 = 13R_3 - 11R_1$, then

$$\begin{bmatrix} 13 & 12 & 0 & -6 \\ 0 & 139 & 0 & -973 \\ 0 & -301 & 0 & 2107 \end{bmatrix}$$

Applying $R_2 = \frac{R_2}{139}$ and $R_3 = \frac{R_3}{-301}$ then

$$\begin{bmatrix} 13 & 12 & 0 & -6 \\ 0 & 1 & 0 & -7 \\ 0 & 1 & 0 & -7 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$\begin{bmatrix} 13 & 12 & 0 & -6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows the system is valid for only values (as required). From R_2 ,

$$y = -7$$

and from R_1 ,

$$\begin{aligned} 13x + 12y &= -6 \Rightarrow 13x = -6 - 12y = -6 + 84 = 78 \\ &\Rightarrow x = 6 \end{aligned}$$

Hence, $x = 6$, $y = -7$ are required solution for the given equations.

$$4y + 3z = 8$$

$$2x - z = 2$$

$$3x + 2y = 5$$

Solution: Given system of equations be

$$4y + 3z = 8$$

$$2x - z = 2$$

$$3x + 2y = 5$$

The augmented matrix of given system of equations be

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

Here the first value in R_1 is zero which we need non-zero. So, interchanging the R_1 and R_2 in the augmented matrix that has no significant change in the system. Therefore,

$$\begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \rightarrow 2R_1 - 3R_2$, then

$$\begin{bmatrix} 2 & 0 & -1 & : & 2 \\ 0 & 4 & 3 & : & 8 \\ 0 & 4 & 3 & : & 1 \end{bmatrix}$$

Again applying $R_1 \rightarrow R_1 - R_2$ then

$$\begin{bmatrix} 2 & 0 & -1 & : & 2 \\ 0 & 4 & 3 & : & 8 \\ 0 & 0 & 0 & : & -7 \end{bmatrix}$$

In R_3 , we see that $0 + 0 + 0 = -7$ which is not possible. So, the given system of equation has no solution.

6. $7x - 4y - 2z = -6$

$$16x + 2y + z = 3$$

Solution: Given system of equation be

$$7x - 4y - 2z = -6$$

$$16x + 2y + z = 3$$

The augmented matrix of given equation be

$$\begin{bmatrix} 7 & -4 & -2 & : & -6 \\ 16 & 2 & 1 & : & 3 \end{bmatrix}$$

Applying $R_2 \rightarrow 7R_2 - 16R_1$ then

$$\begin{bmatrix} 7 & -4 & -2 & : & -6 \\ 0 & 78 & 39 & : & 117 \end{bmatrix}$$

From R_2 , we have

$$78y + 39z = 117 \Rightarrow 2y + z = 3$$

$$\Rightarrow z = 3 - 2y$$

And from R_1 , we have

$$7x - 4y - 2z = -6 \Rightarrow 7x - 4y - 6 + 4y = -6$$

$$\Rightarrow 7x = 0 \Rightarrow x = 0$$

Thus, $x = 0, z = 3 - 2y$ are solution of given system. This implies that, the system has no unique solution.

7. $x - 2y + 3z = 11$

$$3x + y - z = 2$$

$$5x + 3y + 2z = 3$$

Solution: Given system of linear equations be,

$$x - 2y + 3z = 11$$

$$3x + y - z = 2$$

$$5x + 3y + 2z = 3$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 3 & 1 & -1 & : & 2 \\ 5 & 3 & 2 & : & 3 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 5R_1$, then

$$\begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 0 & 7 & -10 & : & -31 \\ 0 & 13 & -13 & : & -52 \end{bmatrix}$$

Applying $R_3 \rightarrow \frac{1}{13}R_3$, then

$$\begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 0 & 7 & -10 & : & -31 \\ 0 & 1 & -1 & : & -4 \end{bmatrix}$$

Again applying $R_3 \rightarrow 7R_3 - R_2$, then

$$\begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 0 & 7 & -10 & : & -31 \\ 0 & 0 & 3 & : & 3 \end{bmatrix}$$

Now, from R_3 , $3z = 3 \Rightarrow z = 1$.

And from R_2 , $7y - 10z = -31 \Rightarrow 7y = -31 + 10 = -21 \Rightarrow y = -3$.

Also, from R_1 , $x = 11 + 2y - 3z = 11 - 6 - 3 = 2$.

Thus, $x = 2, y = -3$ and $z = 1$ be the solution of gives system.

8. $2x + 3y + 4z = 20$

$$3x + 4y + 5z = 26$$

$$3x + 5y + 6z = 31$$

Solution: Given system of equations be

$$2x + 3y + 4z = 20$$

$$3x + 4y + 5z = 26$$

$$3x + 5y + 6z = 31$$

The augment matrix of given system of equations be

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 3 & 4 & 5 & : & 26 \\ 3 & 5 & 6 & : & 31 \end{bmatrix}$$

Applying $R_2 \rightarrow 2R_2 - 3R_1$ and $R_3 \rightarrow 2R_3 - 3R_1$ then

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 0 & -1 & -2 & : & -8 \\ 0 & 1 & 0 & : & 2 \end{bmatrix}$$

Again, applying $R_3 \rightarrow R_3 + R_2$ then,

$$\begin{bmatrix} 2 & 3 & 4 & : & 20 \\ 0 & -1 & -2 & : & -8 \\ 0 & 0 & -2 & : & -6 \end{bmatrix}$$

Here, from R_3 , we have

$$-2z = -6 \Rightarrow z = 3$$

and from R_2 , we have

$$-y - 2z = -8 \Rightarrow y = 8 - 2z = 8 - 6 = 2$$

Also, from R_1 , we have

$$2x + 3y + 4z = 20 \Rightarrow 2x + 6 + 12 = 20$$

$$\Rightarrow 2x = 18 \Rightarrow x = 9$$

Thus, $x = 9, y = 2, z = 3$ be solution of given system of equations.

9. $3x - y + z = -2$

$$x + 5y + 2z = 6$$

$$2x + 3y + z = 0$$

Solution: Given system of equations be

$$3x - y + z = -2$$

$$x + 5y + 2z = 6$$

$$2x + 3y + z = 0$$

The augmented matrix of given equations be

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -2 \\ 1 & 5 & 2 & 6 \\ 2 & 3 & 1 & 0 \end{array} \right]$$

Applying $R_2 = 3R_2 - R_1$ and $R_3 = 3R_3 - 2R_1$ then

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -2 \\ 0 & 16 & 5 & 20 \\ 0 & 11 & 1 & 4 \end{array} \right]$$

Again applying $R_3 = 16R_3 - 11R_2$ then

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & -2 \\ 0 & 16 & 5 & 20 \\ 0 & 0 & -39 & -156 \end{array} \right]$$

Thus, from R_3 , we have

$$-39z = -156 \Rightarrow z = 4$$

And, from R_2 , we have

$$16y + 5z = 20 \Rightarrow 16y + 20 = 20 \Rightarrow y = 0$$

Also, from R_1 , we have

$$\begin{aligned} 3x - y + z &= -2 \\ 3x - 0 + 4 &= -2 \\ 3x &= -6 \Rightarrow x = -2 \end{aligned}$$

Hence, $x = -2$, $y = 0$, $z = 4$ be solution of given system of equations.

10. $5x + 5y - 10z = 0$

$$2w - 3x - 3y + 6z = 2$$

$$4w + x + y - 2z = 4$$

Solution: Given system of linear equations be

$$5x + 5y - 10z = 0$$

$$2w - 3x - 3y + 6z = 2$$

$$4w + x + y - 2z = 4$$

The system has w with 0 as its coefficient is the first equation which is not valid in the process of solving equations by Gauss-elimination method.

So, interchanging the first and second equations then the above equation can be written as,

$$2w - 3x - 3y + 6z = 2$$

$$5x + 5y - 10z = 0$$

$$4w + x + y - 2z = 4$$

The augmented matrix of the system of equation be

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{array} \right]$$

Applying $R_3 = R_3 - 2R_1$ then

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{array} \right]$$

Again applying $R_3 = 5R_3 - 7R_2$ then

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Thus, R_3 is free. So, from R_2 , we have

$$5x + 5y - 10z = 0$$

$$\Rightarrow x + y - 2z = 0$$

And from R_1 we have,

$$2w - 3x - 3y + 6z = 2 \Rightarrow 2w - 3(x + y + 2z) = 2$$

$$\Rightarrow 2w - 3.0 = 2$$

$$\Rightarrow 2w = 2 \Rightarrow w = 1.$$

Thus, $w = 1$, $x + y = 2z$ be solution of given system. This shows that the system has no unique solution.

11. $10x + 4y - 2z = -4$

$$-3w - 17x + y + 2z = 2$$

$$w + x + y = 6$$

$$8w - 34x + 16y - 10z = 4$$

Solution: Given system of equations be

$$10x + 4y - 2z = -4$$

$$-3w - 17x + y + 2z = 2$$

$$w + x + y = 6$$

$$8w - 34x + 16y - 10z = 4$$

Here the first equation is free from first variable w in the system which is not acceptable. So, rearranging the equations as,

$$w + x + y = 6$$

$$10x + 4y - 2z = -4$$

$$-3w - 17x + y + 2z = 2$$

$$8w - 34x + 16y - 10z = 4$$

The augment matrix of the system be

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 8 & -34 & 16 & -10 & 4 \end{array} \right]$$

Applying $R_3 = R_3 + 3R_1$ and $R_4 = R_4 - 8R_1$ then

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 10 & 4 & -2 & -4 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & -42 & 8 & -10 & -44 \end{array} \right]$$

Again applying $R_3 = 5R_3 + 7R_2$, $R_4 = 5R_4 + 21R_2$ then,

$$\begin{vmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 48 & -4 & : & 72 \\ 0 & 0 & 124 & -92 & : & -304 \end{vmatrix}$$

Applying $R_3 = \frac{R_1}{4}$ and $R_4 = \frac{R_3}{4}$ then

$$\begin{vmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 12 & -1 & : & 18 \\ 0 & 0 & 31 & -23 & : & -76 \end{vmatrix}$$

Again applying $R_4 = 14R_3 - 31R_2$ then

$$\begin{vmatrix} 1 & 1 & 1 & 0 & : & 6 \\ 0 & 10 & 4 & -2 & : & -4 \\ 0 & 0 & 12 & -1 & : & 18 \\ 0 & 0 & 0 & -245 & : & -1470 \end{vmatrix}$$

Thus, from R_4 ,

$$-245z = -1470 \Rightarrow z = 6$$

And from R_3 ,

$$12y - z = 18 \Rightarrow y = \frac{1}{12}[18 + z] = \frac{1}{12}[18 + 6] = 2$$

And from R_2 ,

$$10x + 4y - 2z = -4 \Rightarrow 10x = -4 - 4y + 2z = -4 - 8 + 12 = 0 \\ \Rightarrow x = 0.$$

Also, from R_1 ,

$$w + x + y = 6 \Rightarrow w = 6 - x - y = 6 - 0 - 2 = 4.$$

Thus, $w = 4, x = 0, y = 2, z = 6$ are required solution of given system.

EXERCISE - 1.6

Solve by using Cramer's rule of the following system of linear equations:

$$1. \quad 5x - 3y = 37$$

$$-2x + 7y = -38$$

Solution: Given system of linear equations be,

$$5x - 3y = 37$$

$$-2x + 7y = -38$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 5 & -3 \\ -2 & 7 \end{vmatrix} = 35 - 6 = 29 \neq 0$$

So, the solution of the system is possible by the method.
Here,

$$D_1 = \begin{vmatrix} 37 & -3 \\ -38 & 7 \end{vmatrix} = 259 - 114 = 145$$

And,

$$D_2 = \begin{vmatrix} 5 & 37 \\ -2 & -38 \end{vmatrix} = -190 + 74 = -116$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{145}{29} = 5 \quad \text{and} \quad y = \frac{D_2}{D} = \frac{-116}{29} = -4.$$

Thus, $x = 5, y = -4$ are solution of given system of equations.

$$2. \quad 3x + 7y + 8z = -13$$

$$2x + 9z = -5$$

$$-4x + y - 26z = 2$$

Solution: Given system of liner equations be

$$3x + 7y + 8z = -13$$

$$2x + 9z = -5$$

$$-4x + y - 26z = 2$$

Here the determinant of the coefficients be

$$D = \begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix} = 3(-9) - 7(36 - 52) + 8(2) \\ = -27 + 112 + 16 = 101 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} -13 & 7 & 8 \\ -5 & 0 & 9 \\ 2 & 1 & -26 \end{vmatrix} = -13(1 - 9) - 7(130 - 18) + 8(-5 - 0) \\ = -707$$

And,

$$D_2 = \begin{vmatrix} 3 & -13 & 8 \\ 2 & -5 & 9 \\ -4 & 2 & -26 \end{vmatrix} = 3(130 - 18) + 13(-52 + 36) + 8(4 - 20) \\ = 336 - 208 - 128 = 0$$

Also,

$$D_3 = \begin{vmatrix} 3 & 7 & -13 \\ 2 & 0 & -5 \\ -4 & 1 & 2 \end{vmatrix} = 3(0 + 5) - 7(4 - 20) - 13(2 - 0) \\ = 15 + 112 - 26 = 101$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{-707}{101} = -7, \quad y = \frac{D_2}{D} = \frac{0}{101} = 0 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{101}{101} = 1$$

Thus, $x = -7, y = 0$ and $z = 1$ be the solution of the given system of equations.

$$3. \quad x + y + z = 0$$

$$2x + 5y + 3z = 1$$

$$-x + 2y + z = 2$$

Solution: Given system of linear equations

$$x + y + z = 0$$

$$2x + 5y + 3z = 1$$

$$-x + 2y + z = 2$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 1(5-6) - 1(2+3) + 1(4+5) \\ = -1 - 5 + 9 = 3 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 5 & 3 \\ 2 & 2 & 1 \end{vmatrix} = 0 - 1(1-6) + 1(2-10) = 0 + 5 - 8 = -3$$

and,

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = 1(1-6) - 0 + 1(4+1) = -5 - 0 + 5 = 0.$$

also,

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1(10-2) - 1(4+1) + 0 = 8 - 5 + 0 = 3$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-3}{3} = -1, \quad y = \frac{D_2}{D} = \frac{0}{3} = 0 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{3}{3} = 1$$

Thus, $x = -1, y = 0$ and $z = 1$ be the solution of the given system of equations.

4. $3x + y + 2z = 3$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

Solution: Given system of equations be,

$$3x + y + 2z = 3$$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(2+1) + 2(4+3) \\ = -3 - 3 + 14 = 8 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(-3+4) + 2(-6+12) \\ = -3 - 1 + 12 = 8.$$

and,

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4) - 3(2+1) + 2(8+3) \\ = 3 - 9 + 22 = 16.$$

also,

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-12+6) - 1(8+3) + 3(4+3) \\ = -18 - 11 + 21 = -8$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{8}{8} = 1, \quad y = \frac{D_2}{D} = \frac{16}{8} = 2 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{-8}{8} = -1.$$

Thus, $x = 1, y = 2$ and $z = -1$ be the solution of given equations.

5. $x + y + z = 1$

$$2x + 3y + 2z = 2$$

$$3x + 3y + 4z = 1$$

Solution: Given system of equations is,

$$x + y + z = 1$$

$$2x + 3y + 2z = 2$$

$$3x + 3y + 4z = 1$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} = 1(12-6) - 1(8-6) + 1(6-9) \\ = 6 - 2 - 3 = 1 \neq 0$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 3 & 4 \end{vmatrix} = 1(12-6) - 1(8-2) + 1(6-3) \\ = 6 - 6 + 4 = 3.$$

and,

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 4 \end{vmatrix} = 1(8-2) - 1(8-6) + 1(2-6) \\ = 6 - 2 - 4 = 0.$$

also,

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 1 \end{vmatrix} = 1(3-6) - 1(2-6) + 1(6-9) \\ = -3 + 4 - 3 = -2.$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{4}{1} = 4, \quad y = \frac{D_2}{D} = \frac{0}{1} = 0 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{-2}{1} = -2.$$

Thus, $x = 4, y = 0$ and $z = -2$ is the solution set of given equations.

$$x + 2y + 3z = 6$$

$$2x + 4y + z = 7$$

$$3x + 3y + 9z = 14$$

Solution: Given system of equations be,

$$x + 2y + 3z = 6$$

$$2x + 4y + z = 7$$

$$3x + 3y + 9z = 14$$

Here, the determinant of the coefficients be

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 3 & 9 \end{vmatrix} = 1(36 - 2) - 2(18 - 3) + 3(4 - 12) \\ = 34 - 30 - 24 = -20 \neq 0.$$

So, the solution of the system is possible by the method.

Here,

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 3 & 9 \end{vmatrix} = 6(36 - 2) - 2(63 - 14) + 3(14 - 56) \\ = 204 - 98 - 126 = -20$$

and,

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = 1(63 - 14) - 6(18 - 3) + 3(28 - 21) \\ = 49 - 90 + 21 = -20$$

also,

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = 1(56 - 14) - 2(28 - 21) + 6(4 - 12) \\ = 42 - 14 - 48 = -20$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-20}{-20} = 1, \quad y = \frac{D_2}{D} = \frac{-20}{-20} = 1 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{-20}{-20} = 1$$

Thus, $x = 1, y = 1$ and $z = 1$ is the solution set of given equations.

7. $x + 3y + 6z = 2$

$$3x - y + 4z = 9$$

$$x - 4y + 2z = 7$$

Solution: Given system of equations is,

$$x + 3y + 6z = 2$$

$$3x - y + 4z = 9$$

$$x - 4y + 2z = 7$$

Then the determinant of coefficients of the equations be

$$D = \begin{vmatrix} 1 & 3 & 6 \\ 3 & -1 & 4 \\ 1 & -4 & 2 \end{vmatrix} = 1(-2 + 16) - 3(6 - 4) + 6(-12 + 1) \\ = 14 - 6 - 66 = -58 \neq 0$$

So, the solution of the system is possible by the method.
Here,

$$D_1 = \begin{vmatrix} 2 & 3 & 6 \\ 3 & -1 & 4 \\ 7 & -4 & 2 \end{vmatrix} = 2(-2 + 16) - 3(18 - 28) + 6(-36 + 7) \\ = 28 + 30 - 174 = -116$$

and,

$$= 28 + 30 - 174 = -116$$

$$D_2 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & 9 & 4 \\ 1 & 7 & 2 \end{vmatrix} = 1(18 - 28) - 2(6 - 4) + 6(21 - 9) = 58$$

also,

$$D_3 = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 9 \\ 1 & -4 & 7 \end{vmatrix} = 1(-7 + 36) - 3(21 - 9) + 2(-12 + 1) \\ = 29 - 36 - 22 = -29.$$

Now, by Cramer's rule, we have,

$$x = \frac{D_1}{D} = \frac{-116}{-58} = 2, \quad y = \frac{D_2}{D} = \frac{58}{-58} = -1 \quad \text{and}, \quad z = \frac{D_3}{D} = \frac{-29}{-58} = \frac{1}{2}$$

Thus, $x = 2, y = -1$ and $z = \frac{1}{2}$ is the solution set of given equations.

EXAMPLE 1.7

Are the following sets of vectors linearly independent or dependent?

(1, 0, 0), (1, 1, 0), (1, 1, 1).

ution: Given vectors are $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$ and $v_3 = (1, 1, 1)$.

And let a, b, c are any scalars.

$$\text{Now, } \begin{aligned} av_1 + bv_2 + cv_3 &= 0 \\ \Rightarrow a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1) &= 0 = (0, 0, 0) \\ \Rightarrow (a + b + c, b + c, c) &= (0, 0, 0) \end{aligned}$$

This gives,

$$a + b + c = 0, \quad b + c = 0, \quad c = 0.$$

Solving we get, $c = 0, b = 0, a = 0$.

This shows that the given vectors are linearly independent.

(-1, 5, 0), (16, 8, -3), (-64, 56, 9)

ition: Given vectors are $v_1 = (-1, 5, 0)$, $v_2 = (16, 8, -3)$ and $v_3 = (-64, 56, 9)$.

Let a, b, c are any three scalars.

$$\text{Now, } \begin{aligned} av_1 + bv_2 + cv_3 &= 0 \\ \Rightarrow a(-1, 5, 0) + b(16, 8, -3) + c(-64, 56, 9) &= 0 = (0, 0, 0) \\ \Rightarrow (-a + 16b - 54c, 5a + 8b + 56c, -3b + 9c) &= (0, 0, 0) \end{aligned}$$

This gives,

$$-a + 16b - 54c = 0, \quad 5a + 8b + 56c = 0, \quad -3b + 9c = 0.$$

Solving the equations, we get, $a = 0, b = 0, c = 0$.

This shows that the given vectors are linearly independent.

(2, -4), (1, 9), (3, 5)

ition: Given vectors are $v_1 = (2, -4)$, $v_2 = (1, 9)$, $v_3 = (3, 5)$.

Let a, b, c are any three scalars.

$$\text{Now, } \begin{aligned} av_1 + bv_2 + cv_3 &= 0 \\ \Rightarrow a(2, -4) + b(1, 9) + c(3, 5) &= 0 \\ \Rightarrow (2a + b + 3c, -4a + 9b + 5c) &= (0, 0) \end{aligned}$$

This gives,

$$2a + b + 3c = 0, \quad -4a + 9b + 5c = 0$$

Solving we get, $a = -c$ and $b = -c$

This shows that the given vectors are linearly dependent.

4. $(1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 8)$

Solution: Given vectors are $v_1 = (1, 9, 9, 8)$, $v_2 = (2, 0, 0, 3)$ and $v_3 = (2, 0, 8)$.

Let a, b, c are any scalars.

$$\text{Now, } av_1 + bv_2 + cv_3 = 0$$

$$\Rightarrow a(1, 9, 9, 8) + b(2, 0, 0, 3) + c(2, 0, 0, 8) = 0$$

$$\Rightarrow (a + 2b + 2c, 9a, 9a, 8a + 3b + 8c) = 0 = (0, 0, 0)$$

This gives,

$$a + 2b + 2c = 0, 9a = 0, 9a = 0, 8a + 3b + 8c = 0$$

Solving we get, $a = 0, b = 0, c = 0$.

This shows that the vectors are linearly independent.

- B. Show that the following sets of vectors are linearly independent:

1. $(1, 2), (1, 3)$

2. $(1, 1, 1), (1, -1, 0), (0, 1, 1)$

3. $(2, 3, 5), (4, 9, 11)$

4. $(1, 1, 2), (3, 1, 2), (0, 1, 4)$

5. $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

- C. Check dependent or independent of the following:

1. $(0, 1, 0), (0, 0, 1), (1, 1, 1)$

2. $(1, 0, 1), (1, 1, 0), (-1, 0, -1)$

3. $(1, 2, -1), (2, 3, 0), (0, 0, 0)$

4. $(2, 1, 1), (3, -2, 2), (-1, 2, -1)$

Solution: Process as in A.

EXERCISE 1.8

- A. Find rank of the following matrices.

$$1. \quad A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$$

$$\text{Solution: Let. } A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$$

Applying $C_2 = 2C_2 + C_1$ then

$$A \sim \begin{bmatrix} 8 & 0 \\ -2 & 0 \\ 6 & 0 \end{bmatrix}$$

Here, $\{C_1\}$ is linearly independent. So, rank of $A = 1$.

$$2. \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Solution: Let. } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly, the matrix is in echelon form. Here, $\{R_1, R_2\}$ be linearly independent. Again applying $R_3 = R_3 - R_1$ then rank of $A = 2$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{solution: Let, } A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Clearly A has $\{R_1\}$ as linearly independent. So, rank of A = 1.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\text{solution: Let, } A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Applying $R_2 = 3R_2 - 2R_1$ then

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$$

$$\text{ution: Let, } A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 0 & 8 & 7 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 8 & 7 \end{bmatrix}$$

Again applying $R_3 = R_3 + R_2$ then

$$A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -8 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

$$A = \begin{bmatrix} 8 & -3 & 7 \\ -20 & -17 & -15 \\ 11 & 2 & 9 \end{bmatrix} \quad 7. \quad \begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix}$$

tion: Process as Q. 5

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

tion: Let,

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ then

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as independent rows. So, rank of A = 2.

$$9. A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution: Let

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

Applying $R_3 = R_3 + R_1$ and $R_4 = 3R_4 - R_1$ then

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \end{bmatrix}$$

Applying, $R_1 = R_1 - R_2$, $R_4 = R_4 - R_2$ then

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, applying $R_1 = 2R_1 - R_2$ then

$$A = \begin{bmatrix} 3 & -3 & 0 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has $\{R_1, R_2\}$ as linearly independent rows. So, rank of A = 2.

$$10. A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Solution: Let,

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Applying $R_2 = 3R_2 - R_1$, $R_3 = 9R_3 - R_1$ then,

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 6 & 8 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Again applying $R_3 = R_3 + 2R_2$, $R_4 = R_4 - 2R_2$ then

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & -3 & 45 \end{bmatrix}$$

Again applying $R_4 = 4R_3 + R_1$ then

$$A \sim \begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & 0 & 153 \end{bmatrix}$$

This shows that A has $\{R_1, R_2, R_3, R_4\}$ as linearly independent rows. So, rank of A = 4.

$$11. A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$$

Solution: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$$

Here A is 2×4 matrix. So, maximum rank of A = 2.

Applying $R_2 = R_2 + 2R_1$ then

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 11 & 15 \end{bmatrix}$$

This shows that A has 2-independent rows R_1 and R_2 . So, rank of A = 2.

$$12. A = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Clearly A is of 3×3 matrix. So, maximum rank of A = 3.

Here, applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - 2R_1$ then

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Applying $R_3 = R_3 - R_2$ then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that A has 2-independent rows. So, rank of A = 2.

B. Find rank of AB, where

$$1. A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } B = [0 \ 3 \ 4]$$

Solution: Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } B = [0 \ 3 \ 4]$$

Here A is 3×1 and B is 1×3 matrix. So, AB is defined. And,

$$AB = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} [0 \ 3 \ 4] = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 = R_2 - 2R_1$ then

$$AB = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This shows that AB has only R_1 as independent row. So, rank of $(AB) = 1$.

$$2. A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Solution: Let,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Here A is 2×3 and B is 3×2 matrix. So, AB is defined. Then,

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{pmatrix} 1+4+0 & 4-2+0 \\ 3-4+10 & 12-4+10 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 9 & 14 \end{pmatrix}$$

Applying $C_2 = C_2 - 2C_1$, then

$$AB = \begin{bmatrix} 5 & -8 \\ 9 & 0 \end{bmatrix}$$

This shows that A has 2-independent rows. So, rank of $AB = 2$.

EXERCISE - 1.9

Check following system of linear equations is consistence or not, if consistence solve it.

$$1. x + y + z = -3$$

$$3x + y - 2z = -2$$

$$2x + 4y + 7z = 7$$

Solution: Given system of linear equations is

$$x + y + z = -3$$

$$3x + y - 2z = -2$$

$$2x + 4y + 7z = 7$$

The augmented matrix of given equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 3 & 1 & -2 & : & -2 \\ 2 & 4 & 7 & : & 7 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - 2R_1$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 2 & 5 & : & 11 \end{bmatrix}$$

Again applying $R_3 = R_3 + R_2$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 0 & 0 & : & 18 \end{bmatrix}$$

Here, R_3 shows that $0x + 0y + 0z = 18$ which is impossible. So, the given system inconsistency.

$$\begin{aligned} 3x - 4y &= 2 \\ -x + 3y &= 1 \end{aligned}$$

solution: Given system of linear equation is

$$\begin{aligned} 3x - 4y &= 2 \\ -x + 3y &= 1 \end{aligned}$$

The augmented matrix of given equation is

$$[A : B] = \begin{bmatrix} 3 & -4 & : & 2 \\ -1 & 3 & : & 1 \end{bmatrix}$$

Applying $R_2 = 3R_2 + R_1$ then

$$[A : B] = \begin{bmatrix} 3 & -4 & : & 2 \\ 0 & 5 & : & 5 \end{bmatrix}$$

This shows that the rank of coefficient matrix A is equal to the rank of augment matrix $[A : B]$. So, the given system is linear equations are consistence.

And, from last matrix we have,

$$3x - 4y = 2 \quad \text{and} \quad 0x + 5y = 5.$$

$$\Rightarrow 15x = 30 \quad \text{and} \quad \Rightarrow 5y = 5$$

$$\Rightarrow x = 2, y = 1$$

Thus, $x = 2, y = 1$ be solution of given equations.

$$x + y + z = 6$$

$$x + 7y + 3z = 10$$

$$x + 2y + 4z = 1$$

solution: Given system of linear equations be

$$x + y + z = 6$$

$$x + 7y + 3z = 10$$

$$x + 2y + 4z = 1$$

The augmented matrix of given equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 7 & 3 & : & 10 \\ 1 & 2 & 4 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 6 & 2 & : & 4 \\ 0 & 1 & 3 & : & -5 \end{bmatrix}$$

Again applying $R_3 = 6R_3 - R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 6 & 2 & : & 4 \\ 0 & 0 & 16 & : & -34 \end{bmatrix}$$

This shows that the rank of coefficient matrix (A) is equal to the rank of the augmented matrix $[A:B]$. So, the given system of linear equations is consistence.

And the last matrix gives us,

$$x + y + z = 6 \quad \dots \dots \text{(i)}$$

$$6y + 2z = 4 \quad \dots \dots \text{(ii)}$$

$$16z = -34 \quad \dots \dots \text{(iii)}$$

From (iii) we get,

$$z = -\frac{34}{16} \Rightarrow z = -\frac{17}{8}$$

From (ii), we get

$$6y + 2\left(-\frac{17}{8}\right) = 4 \Rightarrow 6y = 4 + \frac{17}{4} = \frac{33}{4} \Rightarrow y = \frac{11}{8}$$

And from (i) we get,

$$x = 6 - y - z = 6 - \frac{11}{8} + \frac{17}{8} \Rightarrow x = \frac{54}{8} = \frac{27}{4}$$

Thus, $x = \frac{27}{4}, y = \frac{11}{8}, z = -\frac{17}{8}$ be solution of given equations.

4. $x + y + z = 3$

$x + 2y + 3z = 4$

$2x + 3y + 4z = 9$

Solution: Given system of linear equations is

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$2x + 3y + 4z = 9$$

The augmented matrix of given system of equations is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 2 & 3 & 4 & : & 9 \end{bmatrix}$$

Applying $R_3 = R_1 - R_1 - R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 0 & 0 & 0 & : & 2 \end{bmatrix}$$

Here, R_3 shows that $0x + 0y + 0z = 2$ is impossible. So, the system of equations is inconsistent.

5. $x - y + 2z = 4$

$3x + y + 4z = 6$

$x + y + z = 1$

Solution: Given system of linear equations is

$$x - y + 2z = 4$$

$$3x + y + 4z = 6$$

$$x + y + z = 1$$

The augmented matrix of given system of equation is

$$[A : B] = \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 3 & 1 & 4 & : & 6 \\ 1 & 1 & 1 & : & 1 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$, and $R_3 = R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 1 & -1 & 2 & : & 4 \\ 0 & 4 & 0 & : & -2 \\ 0 & 0 & -1 & : & 0 \end{bmatrix}$$

Also, applying $R_1 = 4R_1 + R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 0 & 0 & : & -8 \\ 0 & 4 & 0 & : & 0 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix $[A : B]$).

So, the system is consistent. And, from last matrix we have,

$$x = -8,$$

$$4y = 0 \Rightarrow y = 0$$

$$\text{and } -z = -3 \Rightarrow z = 3$$

Thus, $x = -8, y = 0$ and $z = 3$ is the solution set of given system of linear equations.

$$2x + 5y + 6z = 13$$

$$3x + y - 4z = 0$$

$$x - 3y - 8z = -10$$

Solution: Given system of linear equations is

$$2x + 5y + 6z = 13$$

$$3x + y - 4z = 0$$

$$x - 3y - 8z = -10$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 3 & 1 & -4 & : & 0 \\ 1 & -3 & -8 & : & -10 \end{bmatrix}$$

Applying $R_2 = 2R_2 - 3R_1$ and $R_3 = 2R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 0 & -13 & -26 & : & -39 \\ 0 & -11 & -22 & : & -33 \end{bmatrix}$$

Taking out the common value -13 from R_2 and -11 from R_3 then,

$$[A : B] = 143 \begin{bmatrix} 2 & 5 & 6 & : & 13 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & 2 & : & 3 \end{bmatrix}$$

Again applying $R_3 = R_3 - R_2$ and $R_1 = R_1 - 3R_2$ then

$$[A : B] = 143 \begin{bmatrix} 2 & 2 & 0 & : & 4 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = 2 = (rank of augmented matrix $[A : B]$).

So, the given system is consistent. Then by the last matrix, we see

$$2x + 2y = 4 \quad \text{and} \quad y + 2z = 3$$

$$\Rightarrow x + y = 2, \quad \Rightarrow y + 2z = 3$$

That is, $x = 2 - y = 2 - 3 + 2z = 2z - 1$ and $y = 3 - 2z$.

Thus, $x = 2z - 1, y = 3 - 2z, z = z$ is the solution set of given system of linear equations.

$$x + y + z = 8$$

$$x - y + z = 6$$

$$3x + 5y - 7z = 14$$

$$10. \quad x + 2y - z = 3$$

$$3x - y + z = 1$$

$$2x - 2y + 3z = 2$$

Solution: Given system of linear equation is

$$x + 2y - z = 3$$

$$3x - y + z = 1$$

$$2x - 2y + 3z = 2$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 1 & : & 1 \\ 2 & -2 & 3 & : & 2 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - 2R_1$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 4 & : & -8 \\ 0 & -6 & 5 & : & -4 \end{bmatrix}$$

Again applying $R_3 = 7R_3 - 6R_2$ then

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 4 & : & -8 \\ 0 & 0 & 11 & : & 20 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent.

Now, from above matrix, we have,

$$11z = 20 \Rightarrow z = \frac{20}{11}$$

$$\text{And, } -7y + 4z = -8 \Rightarrow -7y = -8 - 4z \Rightarrow y = \frac{24}{11}$$

$$\text{Also, } x + 2y - z = 3 \Rightarrow x = 3 - 2y + z \Rightarrow x = \frac{5}{11}$$

Thus, $x = \frac{5}{11}$, $y = \frac{24}{11}$ and $z = \frac{20}{11}$ is solution set of given system of linear equations.

$$11. \quad 2x - 3y + 7z = 5$$

$$3x + y - 3z = 13$$

$$2x + 19y - 47z = 32$$

Solution: Given system of linear equations is

$$2x - 3y + 7z = 5$$

$$3x + y - 3z = 13$$

$$2x + 19y - 47z = 32$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

Applying $R_1 = 2R_2 - 3R_3$ and $R_3 = R_3 - R_1$ then

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 22 & -54 & : & 27 \end{bmatrix}$$

Again applying $R_1 = R_1 - 2R_2$ then

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 0 & 11 & -27 & : & 11 \\ 0 & 0 & 0 & : & 5 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) ≠ (rank of augmented matrix [A : B]).

So, the given system is inconsistent.

$$12. \quad 4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

Solution: Given system of linear equations is,

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

The augmented matrix of given system is

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 1 & 1 & -3 & : & -1 \end{bmatrix}$$

Applying $R_2 = 4R_2 - R_1$ then

$$[A : B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 0 & 6 & -18 & : & -12 \end{bmatrix}$$

This shows that

(rank of coefficient matrix A) = (rank of augmented matrix [A : B]).

So, the system is consistent. Now, from the last matrix we have,

$$4x - 2y + 6z = 8 \quad \text{and} \quad 6y - 18z = -12$$

$$\Rightarrow 2x - y + 3z = 4 \quad \text{and} \quad y - 3z = -2$$

$$\Rightarrow 2x = 4 + y - 3z \quad \text{and} \quad y = 3z - 2$$

$$\Rightarrow 2x = 4 + 3z - 2 - 3z \quad \text{and} \quad y = 3z - 2$$

$$\Rightarrow x = 1 \quad \text{and} \quad y = 3z - 2$$

Thus, $x = 1$, $y = 3z - 2$, $z = z$ is solution set of given system of linear equations.

EXERCISE - 1.10

Let $V = \mathbb{R}^2$ be a vector space. Show that $W = \{(x, y): x + 2y = 0\}$ is a vector subspace of V .

Solution: Here V is a vector space over a field \mathbb{F} of real numbers.

Here, $W = \{(x, y): x + 2y = 0\}$. Then W is a subset of V .

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ are in W and $a, b \in \mathbb{F}$.

Now,

$$\begin{aligned} au + bv &= a(x_1, y_1) + b(x_2, y_2) \\ &= (ax_1 + bx_2, ay_1 + by_2) \quad \dots \dots (1) \end{aligned}$$

Here,

$$\begin{aligned} (ax_1 + bx_2, ay_1 + by_2) &= (ax_1 + bx_2) + 2(ay_1 + by_2) \\ &= a(x_1 + 2y_1) + b(x_2 + 2y_2) = a \cdot 0 + b \cdot 0 = 0 \end{aligned}$$

So, $au + bv \in W$.

This means W is a vector subspace of V .

2. Let $V = \mathbb{R}^3$ be a vector space. Show that

(i) $W = \{(x, y, z) : x + 2y + z = 0\}$

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(ii) $W = \{(x, y, z) : 2x + y + 2z = 0\}$

are vector subspaces of V .

Solution: (i) Given $V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is a vector space over a field of real numbers.

Also given $W = \{(x, y, z) : x + 2y + z = 0\}$. Then W is a subset of V .

And, let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in W .

Then $x_1 + 2y_1 + z_1 = 0$ and $x_2 + 2y_2 + z_2 = 0$.

Now, for $a, b \in F$,

$$\begin{aligned} au + bv &= a(x_1, y_1, z_1) + b(x_2, y_2, z_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad \dots\dots\dots(1) \end{aligned}$$

Here,

$$\begin{aligned} (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) &= ax_1 + bx_2 + 2(ay_1 + by_2) + az_1 + bz_2 \\ &= a(x_1 + 2y_1 + z_1) + b(x_2 + 2y_2 + z_2) \\ &= a \cdot 0 + b \cdot 0 = 0 \end{aligned}$$

Then by (1), we have $au + bv \in W$.

This means W is a vector subspace of V .

(ii) Given $V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be a vector space over a field F .

Also given $W = \{(x, y, z) : 2x + y + 2z = 0\}$ then W is a subset of V .

And, let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in W .

Then, $2x_1 + y_1 + 2z_1 = 0$ and $2x_2 + y_2 + 2z_2 = 0$

Now, for $a, b \in F$,

$$\begin{aligned} au + bv &= a(x_1, y_1, z_1) + b(x_2, y_2, z_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad \dots\dots\dots(1) \end{aligned}$$

Here,

$$\begin{aligned} (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) &= 2(ax_1 + bx_2) + ay_1 + by_2 + 2(az_1 + bz_2) \\ &= a(2x_1 + y_1 + 2z_1) + b(2x_2 + y_2 + 2z_2) = a \cdot 0 + b \cdot 0 = 0 \end{aligned}$$

Thus, from (1), we have $au + bv \in W$.

This means W is a vector subspace of V .

3. Let $V = \mathbb{R}^3$ be a vector space. Let

(i) $W_1 = \{(0, 0, z) : z \in \mathbb{R}\}$ (ii) $W_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$

Show that W_1, W_2 and $W_1 \cap W_2$ are vector subspaces of V .

Solution: Given $V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be a vector space over a field of real numbers.

Also given $W_1 = \{(x, 0, z) : x, z \in \mathbb{R}\}$ and $W_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$.

Then W_1 and W_2 are subset of V .
Let $u_1 = (x_1, 0, z_1), v_1 = (x_2, 0, z_2)$ are in W_1 . And $u_2 = (0, y_3, z_3), v_2 = (0, y_4, z_4)$ are in W_2 .

Now, for $a, b \in F$,

$$\begin{aligned} au_1 + bv_1 &= a(x_1, 0, z_1) + b(x_2, 0, z_2) \\ &= (ax_1 + bx_2, 0, az_1 + bz_2) \in W_1 \end{aligned}$$

So W_1 is a vector subspace of V .

$$\begin{aligned} \text{And, } au_2 + bv_2 &= a(0, y_3, z_3) + b(0, y_4, z_4) \\ &= (0, ay_3 + by_4, az_3 + bz_4) \in W_2 \end{aligned}$$

Thus, W_2 is a vector subspace of V .

Also, Let $u = (0, 0, z_1)$ and $v = (0, 0, z_2)$ are in $W_1 \cap W_2$.

Then, $W_1 \cap W_2$ be a subset of V .

Now, for $a, b \in F$,

$$\begin{aligned} au + bv &= a(0, 0, z_1) + b(0, 0, z_2) \\ &= (0, 0, az_1 + bz_2) \end{aligned}$$

Since, $z_1, z_2 \in \mathbb{R}$ then $az_1 + bz_2 \in \mathbb{R}$. So, $au + bv \in W_1 \cap W_2$.

This means $W_1 \cap W_2$ be a vector subspace of V .

4. Let V = set of all 2×2 matrices, be a vector space. Let

(i) $W_1 = \left\{ \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in \mathbb{R} \right\}$ (ii) $W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in \mathbb{R} \right\}$

Show that W_1, W_2 are vector subspaces of V .

Solution: Given V = {set of all 2×2 matrices}, be a vector space over a field F .

(i) Given $W_1 = \left\{ \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in \mathbb{R} \right\}$. Then W_1 is a subset of V .

$$\text{Let } u_1 = \begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}. \text{ Then } u_1, u_2 \in W_1.$$

Now, for $a, b \in F$,

$$\begin{aligned} au_1 + bu_2 &= a \begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix} + b \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ab_1 + bb_2 \\ ac_1 + bc_2 & ad_1 + bd_2 \end{pmatrix} \end{aligned}$$

Since $b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$.

So, $ab_1 + bb_2, ac_1 + bc_2, ad_1 + bd_2 \in \mathbb{R}$ for $a, b \in \mathbb{R}$. So, $au_1 + bu_2 \in W_1$.

This means W_1 is a vector subspace of V .

(ii) Given $W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} : c, d \in \mathbb{R} \right\}$. Then W_2 is a subset of V .

$$\text{Let } v_1 = \begin{pmatrix} 0 & 0 \\ c_3 & d_3 \end{pmatrix} \text{ for } c_3, d_3 \in \mathbb{R}; \text{ and } v_2 = \begin{pmatrix} 0 & 0 \\ c_4 & d_4 \end{pmatrix} \text{ for } c_4, d_4 \in \mathbb{R}.$$

Then $v_1, v_2 \in W_2$.

Now, for $a, b \in F$,

$$av_1 + bv_2 = a \begin{pmatrix} 0 & 0 \\ c_3 & d_3 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ c_4 & d_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ ac_3 & ad_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ bc_4 & bd_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ac_3 + bc_4 & ad_1 + bd_1 \end{pmatrix}$$

Since, $c_3, d_3, c_4, d_4 \in R$. So, $ac_3 + bc_4, ad_1 + bd_1 \in R$.

So that, $av_1 + bv_2 \in W_2$.

This means W_2 is a vector subspace of V .

5. Check which of the following forms a basis of R^3 .

- (i) $(1, 1, 1), (1, 3, 2), (-1, 0, 1)$
- (ii) $(1, 2, 1), (2, 1, 0), (1, -1, 2)$
- (iii) $(1, 1, 0), (1, 0, 1)$
- (iv) $(1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)$

Solution: (i) Let $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space over a field F .

Then we wish to show $(1, 1, 1), (1, 3, 2), (-1, 0, 1)$ form a basis for V .

Now, for $a, b, c \in F$,

$$\begin{aligned} a(1, 1, 1) + b(1, 3, 2) + c(-1, 0, 1) &= 0 \\ \Rightarrow (a + b - c, a + 3b, a + 2b + c) &= 0 = (0, 0, 0) \end{aligned}$$

This gives $a + b - c = 0, a + 3b = 0, a + 2b + c = 0$

Solving these equations we get $a = 0, b = 0, c = 0$.

Thus, $(1, 1, 1), (1, 3, 2)$ and $(-1, 0, 1)$ are linearly independent.

Next, let $(x, y, z) \in V$ then we wish to show that (x, y, z) can be written as a linear combination of $(1, 1, 1), (1, 3, 2)$ and $(-1, 0, 1)$.

For this, let

$$\begin{aligned} (x, y, z) &= \alpha(1, 1, 1) + \beta(1, 3, 2) + \gamma(-1, 0, 1) \quad \dots \text{(i)} \\ &= (\alpha + \beta - \gamma, \alpha + 3\beta, \alpha + 2\beta + \gamma) \end{aligned}$$

This gives,

$$x = \alpha + \beta - \gamma, y = \alpha + 3\beta, z = \alpha + 2\beta + \gamma$$

Solving we get,

$$\alpha = x - y + z, \beta = \frac{1}{3}(2y - x - z) \text{ and } \gamma = -\frac{1}{3}(x + y - 2z)$$

Thus (i) becomes,

$$(x, y, z) = (x - y + z)(1, 1, 1) + \frac{1}{3}(2y - x - z)(1, 3, 2) - \frac{1}{3}(x + y - 2z)(-1, 0, 1)$$

This shows that the form $(1, 1, 1), (1, 3, 2), (-1, 0, 1)$ are in the linear combination (x, y, z) . So, the form generates R^3 . Hence, the form a basis $V = R^3$.

Solution: (ii) Given vectors are $(1, 2, 1), (2, 1, 0), (1, -1, 2)$

Let $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space over a field F .

Then we wish to show the forms a basis for V .

For this first we will show the forms are linearly independent.

Now, for $a, b, c \in F$,

$$\begin{aligned} a(1, 2, 1) + b(2, 1, 0) + c(1, -1, 2) &= 0 = (0, 0, 0) \\ \Rightarrow (a + 2b + c, 2a + b - c, a + 2c) &= (0, 0, 0) \end{aligned}$$

This gives,

$$a + 2b + c = 0, 2a + b - c = 0, a + 2c = 0$$

Solving we get, $a = 0, b = 0, c = 0$

This shows that the forms are linearly independent.

Next, let $(x, y, z) \in V$. Then we will show that (x, y, z) can be written as linear combination of the form $(1, 2, 1), (2, 1, 0), (1, -1, 2)$.

For this, let

$$\begin{aligned} (x, y, z) &= \alpha(1, 2, 1) + \beta(2, 1, 0) + \gamma(1, -1, 2) \\ &= (\alpha + 2\beta + \gamma, 2\alpha + \beta - \gamma, \alpha + 2\gamma) \end{aligned}$$

This gives,

$$x = \alpha + 2\beta + \gamma, y = 2\alpha + \beta - \gamma, z = \alpha + 2\gamma$$

Solving we get,

$$\alpha = \frac{1}{9}(4y - 2x + 3z), \beta = \frac{1}{9}(5x - y - 3z), \gamma = \frac{1}{9}(x - 2y + 3z)$$

This shows that (x, y, z) can be written as a linear combination of the forms $(1, 2, 1), (2, 1, 0), (1, -1, 2)$.

Therefore, the forms generates R^3 . Hence, the forms is a basis of $V = R^3$.

Solution: (iii) Given vectors are $(1, 1, 0), (1, 0, 1)$.

Let $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be a vector space over a field F .

Then we wish to show the vectors forms a basis for R^3 . For this first we will show the forms are linearly independent.

Now, for $a, b, c \in F$,

$$\begin{aligned} a(1, 1, 0) + b(1, 0, 1) &= 0 = (0, 0, 0) \\ \Rightarrow (a + b, a, b) &= (0, 0, 0) \end{aligned}$$

This gives, $a + b = 0, a = 0, b = 0$.

This shows that the forms are linearly independent.

Next, let $(x, y, z) \in V = R^3$. Then we will show (x, y, z) can be written as a linear combination of given forms.

Let

$$\begin{aligned} (x, y, z) &= \alpha(1, 1, 0) + \beta(1, 0, 1) \\ &= (\alpha + \beta, \alpha, \beta) \end{aligned}$$

This gives

$$\begin{aligned} x &= \alpha + \beta, y = \alpha, z = \beta \\ \Rightarrow x &= y + z, y = \alpha, z = \beta \end{aligned}$$

This shows that (x, y, z) can not be written as linear combination of given forms.

So, the forms $(1, 1, 0), (1, 0, 1)$ is not a basis of R^3 .

Solution: (iv) Given vectors are $(1, 1, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)$.

Then we wish to examine the vector forms a basis of R^3 or not.

For this, first we examine linearly independence of the forms.

Let, $a, b, c, d \in F$. Then,

$$\begin{aligned} a(1, 1, 0) + b(0, 1, 0) + c(0, 0, 1) + d(2, 3, 4) &= 0 = (0, 0, 0) \\ \Rightarrow (a + 2d, a + b + 3d, c + 4d) &= (0, 0, 0) \end{aligned}$$

This gives, $a + 2d = 0, a + b + 3d = 0, c + 4d = 0$

Solving we get, $a = -2d, b = -d, c = -4d$.

This shows that the forms are not linearly independent. So, the forms is not a basis of \mathbb{R}^3 .

6. Check the following transformations are linear or not.

(i) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $T(x, y) = x + y$. [2013 Spring Q. No. 7(a)]

Solution: Given that a map $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $T(x, y) = x + y$.

Let $u = (x_1, y_1), v = (x_2, y_2)$ then $u, v \in \mathbb{R}^2$. And let $a, b \in F$. Now,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= ax_1 + bx_2 + ay_1 + by_2 \\ &= a(x_1 + y_1) + b(x_2 + y_2) \\ &= aT(x_1, y_1) + bT(x_2, y_2) = aT(u) + bT(v) \end{aligned}$$

This shows that T is a linear transformation.

(ii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $T(x, y) = (x, y, xy)$

Solution: Given that a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by

$$T(x, y) = (x, y, xy)$$

Let $u = (x_1, y_1), v = (x_2, y_2)$ then $u, v \in \mathbb{R}^2$.

Then T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Now, for $a, b \in F$,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, ax_1y_1 + bx_2y_2) \quad \dots\dots(1) \end{aligned}$$

Next,

$$\begin{aligned} aT(u) + bT(v) &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= a(x_1, y_1, x_1y_1) + b(x_2, y_2, x_2y_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, ax_1y_1 + bx_2y_2) \quad \dots\dots(2) \end{aligned}$$

Thus from (1) and (2), we observe that

$$T(au + bv) \neq aT(u) + bT(v)$$

So, T is not a linear.

(iii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (x + 3, y)$ [2014 Spring Q. No. 7(c)]

Solution: Given that a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x + 3, y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u, v \in \mathbb{R}^2$.

Also, let $a, b \in F$

Then T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Now,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2 + 3, ay_1 + by_2) \quad \dots\dots(1) \end{aligned}$$

Next,

$$\begin{aligned} aT(u) + bT(v) &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= a(x_1 + 3, y_1) + b(x_2 + 3, y_2) \\ &= (ax_1 + bx_2 + 3(a+b), ay_1 + by_2) \quad \dots\dots(2) \end{aligned}$$

From (1) and (2), we observe that $T(au + bv) \neq aT(u) + bT(v)$. So, T is not linear transformation.

T : $\mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = x + 4$.

Solution: Given transformation $T: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $T(x) = x + 4$.

Let, $x, y \in \mathbb{R}$ and $a, b \in F$.

Then T is linear only if $T(ax + by) = aT(x) + bT(y)$.

Here,

$$T(ax + by) = ax + by + 4 \quad \dots\dots(1)$$

Next,

$$aT(x) + bT(y) = a(x + 4) + b(y + 4)$$

$$= ax + by + 4(a + b) \neq ax + by + 4 = T(ax + by)$$

This shows that T is not a linear transformation.

T : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (x, -2y)$.

Solution: Given transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x, -2y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u, v \in \mathbb{R}^2$.

The transformation T is linear only if $T(au + bv) = aT(u) + bT(v)$.

Here, for $a, b \in F$,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2, -2(ay_1 + by_2)) \\ &= (ax_1 + bx_2, -2ay_1 - 2by_2) \quad \dots\dots(1) \end{aligned}$$

Next,

$$aT(u) + bT(v) = aT(x_1, y_1) + bT(x_2, y_2)$$

$$= a(x_1, y_1, -2y_1) + b(x_2, y_2, -2y_2)$$

$$= (ax_1 + bx_2, -2ay_1 - 2by_2)$$

$$= T(au + bv) \quad [\because \text{using (1)}]$$

This shows that T is linear.

EXERCISE – 1.11

Find eigen value and eigen vector of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let λ and X are corresponding eigen value and eigen vector of A . The characteristics equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Thus, the eigen value of A are $\lambda = 1, \lambda = -1$.

And for the corresponding eigen vector is

$$\begin{aligned} AX = \lambda X &\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{at } \lambda = 1 \\ &\Rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

This gives,

$$\begin{aligned} x &= x \text{ and } -y = y \\ \Rightarrow x &= x \text{ (i.e. } x \text{ is free) and } y = 0 \end{aligned}$$

Thus, the required eigen vector with corresponding to $\lambda = 1$ is $(x, 0)$.

And, at $\lambda = -1$,

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= -1 \begin{bmatrix} x \\ y \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \end{aligned}$$

This gives,

$$\begin{aligned} x &= -x \text{ and } -y = y \\ \Rightarrow x &= 0 \text{ and } y = y \text{ (i.e. } y \text{ is free)} \end{aligned}$$

Thus the eigen vector with corresponding to $\lambda = -1$ is $(0, y)$.

Hence, the required eigen vector of A be $(x, 0)$ associated with $\lambda = 1$ and associated with $\lambda = -1$.

- (ii) $\begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$
 (v) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

Solution (ii) — (vi): process as (i).

(vii) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Let λ and X are corresponding eigen value and eigen vector of A.

The characteristic equation of A is,

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (2-\lambda)^2 - 1 = 0 \\ &\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \\ &\Rightarrow \lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \Rightarrow \lambda = 3, 1 \end{aligned}$$

Thus, the eigen value of A are $\lambda = 1, \lambda = 3$.

And, the corresponding eigen vector is

$$\begin{aligned} AX = \lambda X &\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

This gives $2x+y = \lambda x$ and $x+2y = \lambda y$

At $\lambda = 1$,

$$\begin{aligned} 2x+y &= x & \text{and} & x+2y = y \\ \Rightarrow x+y &= 0 & \text{and} & x+y = 0 \end{aligned}$$

Thus, at $\lambda = 1$, the corresponding eigen vector be $(x, -x)$.

And at $\lambda = 3$,

$$\begin{aligned} 2x+y &= 3x & \text{and} & x+2y = 3y \\ \Rightarrow x-y &= 0 & \text{and} & x-y = 0 \end{aligned}$$

This shows that at $\lambda = 3$, the corresponding eigen vector is (x, y) .

Hence, the required eigen vector of A is $(x, -x)$ associated with the eigen value $\lambda = 1$ and (x, x) associated with the eigen value $\lambda = 3$.

(viii) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Let λ and X are corresponding eigen value and eigen vector of A. The characteristic equation of A is,

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0 \\ &\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0 \\ &\Rightarrow \lambda^2 - 7\lambda + 6 = 0 \\ &\Rightarrow (\lambda-6)(\lambda-1) = 0 \Rightarrow \lambda = 1, 6 \end{aligned}$$

Thus, the eigen value of A are $\lambda = 1, \lambda = 6$.

And, the corresponding eigen vector is,

$$\begin{aligned} AX = \lambda X &\Rightarrow \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 5x+4y \\ x+2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

This gives, $5x+4y = \lambda x$ and $x+2y = \lambda y$

$$\begin{aligned} \text{At } \lambda = 1, \quad 5x+4y &= x & \text{and} & x+2y = y \\ \Rightarrow 4x+4y &= 0 & \text{and} & x+y = 0 \\ \Rightarrow x+y &= 0 & \text{and} & x+y = 0 \end{aligned}$$

Thus, at $\lambda = 1$, the corresponding Eigen vector is $(x, -x)$.

$$\begin{aligned} \text{At } \lambda = 6, \quad 5x+4y &= 6x & \text{and} & x+2y = 6y \\ \Rightarrow x-4y &= 0 & \text{and} & x-4y = 0 \end{aligned}$$

Thus, at $\lambda = 6$, the corresponding Eigen vector is $(4x, x)$.

2. Find eigen value as well as vector of the following:

$$(i) \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution: Let,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Let, λ and x are eigen value and eigen vector of A respectively. The characteristic equation of A is,

$$|Ax - \lambda x| = 0.$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -8-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-8-\lambda)(4-\lambda) = 0$$

$$\Rightarrow (3-\lambda)(8+\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda = 3, 4, -8$$

Thus, the eigen value of A are $\lambda = 3, \lambda = 4$ and $\lambda = -8$

And the corresponding eigen vector of A is,

$$Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} 3x \\ -8y \\ 4z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$$\Rightarrow 3x = \lambda x, -8y = \lambda y \text{ and } 4z = \lambda z$$

At $\lambda = 3$,

$$3x = 3x, -8y = 3y \quad \text{and} \quad 4z = 3z$$

These gives, $x = x, y = 0$ and $z = 0$.

Thus, the corresponding eigen vector to eigen value $\lambda = 3$ is $(x, 0, 0)$

And at $\lambda = 4$,

$$3x = 4x, -8y = 4y \text{ and } 4z = -8z$$

These gives $x = 0, y = y$ and $z = 0$.

Thus, the corresponding eigen vector to eigen value $\lambda = 3, 4, -8$ and corresponding Eigen vectors are $(x, 0, 0), (0, 0, z)$ and $(0, y, 0)$.

$$(ii) \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: process as (i)

$$v) \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

[2014 Fall Q. No. 1(b)]

Solution: Let,

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Let λ and x are eigen value and eigen vector of A , respectively.

The characteristic equation of A is,

$$|Ax - \lambda x| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda) \begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(-1-\lambda)(1-\lambda)-3] = 0$$

$$\Rightarrow (2-\lambda)[(1+\lambda)(1-\lambda)+3] = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda^2+3) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4) = 0$$

$$\Rightarrow \lambda = 2, -2$$

These are the eigen value of A .

And the corresponding eigen vector of A is,

$$Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x + y + 2z \\ -y + 3z \\ y + z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$$\Rightarrow 2x + y + 2z = \lambda x, -y + 3z = \lambda y, y + z = \lambda z$$

At $\lambda = 2$,

$$2x + y + 2z = 2x, \quad -y + 3z = 2y, \quad y + z = 2z$$

$$\Rightarrow 2x + y + 2z = 2x, \quad 3y - 3z = 0, \quad y - z = 0$$

$$\Rightarrow 2x + y + 2z = 2x \dots (i) \quad y - z = 0 \dots (ii) \quad y - z = 0 \dots (iii)$$

From (ii) and (iii) we have, $y = z$

And from (i), it is possible only if $y = z = 0$ then $2x = 2x$.

Thus, the corresponding vector is $(x, 0, 0)$.

Next, at $\lambda = -2$,

$$2x + y + 2z = -2x, \quad -y + 3z = -2y, \quad y + z = -2z$$

$$\Rightarrow 4x + y + 2z = 0 \quad y + 3z = 0 \quad y + 3z = 0$$

$$\Rightarrow 4x + y + 2z = 0 \dots (iv) \quad y + 3z = 0 \dots (v) \quad y + 3z = 0 \dots (vi)$$

From (v) $y = -3z$ then (iv) gives us,

$$4x - 3z + 2z = 0 \Rightarrow 4x = z$$

Thus, for $x = 1, z = 4$ and $y = -12$

Therefore, the corresponding eigen vector is, $(1, -12, 4)$.

Hence, the eigen value of A are $\lambda = 2$; and $\lambda = -2$ and their corresponding eigen vector are $(x, 0, 0)$ and $(x, -12y, 4z)$.

$$b) \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: Let, $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to corresponding matrix. The characteristic equation A is,

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (3-\lambda)^2(2-\lambda) - (2-\lambda) = 0 \\ &\Rightarrow (2-\lambda)\{(3-\lambda)^2 - 1\} = 0 \\ &\Rightarrow (2-\lambda)(3-\lambda-1)(3-\lambda+1) = 0 \\ &\Rightarrow (2-\lambda)(2-\lambda)(4-\lambda) = 0 \\ &\Rightarrow \lambda = 2, 2, 4 \end{aligned}$$

Thus, the eigen value are $\lambda = 2$ and $\lambda = 4$.

And, the corresponding eigen vector is,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$3x + y = \lambda x; \quad x + 3y = \lambda y; \quad 2z = \lambda z$$

At, $\lambda = 2$,

$$\begin{array}{l} x + y = 0 \\ x + y = 0 \\ 2z = 2z \end{array} \quad \begin{array}{l} -x + y = 0 \\ x - y = 0 \\ 2z = 0 \end{array}$$

This shows $x = -y, z = z$

At $\lambda = 4$

$$\begin{array}{l} x + y = 0 \\ x + y = 0 \\ 2z = 0 \end{array} \quad \begin{array}{l} x - y = 0 \\ x - y = 0 \\ 2z = 0 \end{array}$$

This shows $x = y, z = 0$

Thus, the eigen vector be $(x, -x, z)$ at $\lambda = 2$ and $(x, x, 0)$ at $\lambda = 4$.

(vii) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (x) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: Process as (vi)

(xi) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to the corresponding matrix A.

The characteristic equation of A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $R_2 \rightarrow R_2 + R_1$, then,

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $C_2 \rightarrow C_2 + C_3$, then,

$$\begin{aligned} \Rightarrow & \begin{vmatrix} 6-\lambda & 0 & 2 \\ 0 & 4-2\lambda & 2-\lambda \\ 2 & 2-\lambda & 3-\lambda \end{vmatrix} = 0 \\ \Rightarrow & (6-\lambda)\{(4-2\lambda)(3-\lambda) - (2-\lambda)^2\} = 0 \\ \Rightarrow & 2(6-\lambda)(2-\lambda)(3-\lambda-2+\lambda) = 0 \\ \Rightarrow & (6-\lambda)(2-\lambda) = 0 \quad [\because 2 \neq 0] \\ \Rightarrow & \lambda = 2, 6 \end{aligned}$$

Thus, the eigen value of A is 2 and 6.

And the corresponding eigen vector be,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$6x - 2y + 2z = \lambda x; \quad -2x + 3y - z = \lambda y; \quad 2x - y + 3z = \lambda z$$

At $\lambda = 2$,

$$6x - 2y + 2z = 0; \quad 2x - y + z = 0; \quad 2x - y + z = 0$$

This implies,

$$2x - y + z = 0 \quad [\because 2 \neq 0]; \quad 2x - y + z = 0; \quad 2x - y + z = 0.$$

(xii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (xiii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

[2013 Fall Q. No. 1(b)]

Solution: Process as (vi)

Solution: Let, $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Let λ and x are eigen value and eigen vector respectively to the matrix A.

The characteristic equation to A is,

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0 \\ &\Rightarrow \lambda = 2, 3, 5 \end{aligned}$$

Thus, the eigen value to A are $\lambda = 2, \lambda = 3, \lambda = 5$.

The corresponding eigen vector be,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$3x + y + 4z = \lambda x; \quad 2y = \lambda y; \quad 5z = \lambda z$$

At, $\lambda = 2$,

$$\begin{aligned} x + y + 4z &= 0 \\ 2y &= 2y \\ 3z &= 0. \end{aligned}$$

This shows that the corresponding vector is $(x, -x, 0)$.

Thus, the corresponding eigen vector are $(x, -x, 0)$ at $\lambda = 2$, $(x, 0, 0)$ at $\lambda = 3$ and $(0, x)$ at $\lambda = 5$.

3. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

Solution: Given matrix be,

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Let λ and x are eigen value and eigen vector of A , respectively.

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(1-\lambda)-6] - 4[3(1-\lambda)-14] + 1[9-7(2-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda + 2 - 6) - 4(3 - 3\lambda - 14) + (9 - 14 - 7\lambda) = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda + 12\lambda + 44 - 7\lambda - 5 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \quad \dots\dots\dots (1)$$

This is required characteristic equation of A .

To find the inverse of A , first we satisfy the Cayley-Hamilton condition. For this we should show the corresponding condition of (1).

$$A^3 - 4A^2 - 20A - 35I = 0 \quad \dots\dots\dots (2)$$

$$\text{Here, } A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\text{So, } A^2 = A \cdot A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\text{And, } A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 20+45+70 & 23+66+63 & 23+111+98 \\ 80+30+30 & 92+44+27 & 92+74+42 \\ 20+30+10 & 23+44+9 & 23+74+14 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\text{Now, } A^3 - 4A^2 - 20A - 35I$$

$$\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-68-0 \\ 60-40-20-0 & 76-36-40-0 & 111-56-20-35 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$$\Rightarrow A^3 - 4A^2 - 20A - 35I = 0 \quad \dots\dots\dots (3)$$

Now, multiplying (3) by A^{-1} then

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{35} (A^2 - 4A - 20I)$$

$$= \frac{1}{35} \left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{35} \begin{bmatrix} 20-4-20 & 23-12-0 & 23-28-0 \\ 15-16-0 & 22-8-20 & 37-12-0 \\ 10-4-0 & 9-8-0 & 14-4-20 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}.$$

4. Find the characteristics equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that

the equation is satisfied by A .

Solution: Given matrix is, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

Let, λ and x are eigen value and eigen vector of A , respectively.

Now, the characteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(-1-\lambda)^2 - 4] - 2[2(-1-\lambda) - 3] + 3[8 - 3(-1-\lambda)] = 0$$

$$\begin{aligned}
 & \Rightarrow (1-\lambda)(\lambda^2 + 2\lambda - 3) - 2(2\lambda - 5) + 3(3\lambda + 11) = 0 \\
 & \Rightarrow \lambda^3 + 2\lambda^2 - 3\lambda^2 - 2\lambda^2 + 3\lambda + 4\lambda + 10 + 9\lambda + 33 = 0 \\
 & \Rightarrow \lambda^3 + \lambda^2 - 18\lambda - 40 = 0 \quad \dots\dots\dots (i)
 \end{aligned}$$

The characteristics equation (i) will satisfy by A if we able to show
 $A^3 + A^2 - 18A - 40I = 0 \dots\dots\dots (ii)$

Here,

$$\begin{aligned}
 A^2 = AA &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8-3 \\ 2-2+3 & 4+1+4 & 6-4-4 \\ 3+8-3 & 6-1-1 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}
 \end{aligned}$$

And $A^3 = A^2 \cdot A = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

$$\begin{aligned}
 &= \begin{bmatrix} 14+24+6 & 3+18+12 & 8-4+42 \\ 28-12+8 & 6-9+16 & 16+2+56 \\ 42+12-2 & 9+9-4 & 24-2-14 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}
 \end{aligned}$$

Then,

$$\begin{aligned}
 A^3 + A^2 - 18A - 40I &= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 44+14-18-40 & 33+3-36-0 & 46+8-54-0 \\ 24+12-36-0 & 13+9+18-40 & 74-2-72-0 \\ 52+2-54-0 & 14+4-18-0 & 8+14+18-40 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.
 \end{aligned}$$

Thus, the equation (ii) is satisfied.

5. Using Cayley-Hamilton theorem find the inverse of

$$(i) \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \quad (ii) \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution: (i) Let, $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ The characteristics equation of A be,

$|A - \lambda I| = 0$

where, λ is eigen value of A

$\Rightarrow \begin{bmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 9 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 1 = 0 \quad \dots\dots\dots (1)$

This is required characteristics equation of A .The corresponding equation of A to (1) is,

$A^2 - 7A + I = 0 \quad \dots\dots\dots (2)$

This condition is the Cayley-Hamilton theorem.

Here,

$A^2 = AA = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4+9 & 6+15 \\ 6+15 & 9+25 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}$

Then,

$$\begin{aligned}
 A^2 - 7A + I &= \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} - 7 \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 13-14+1 & 21-21+0 \\ 21-21+0 & 34-35+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0
 \end{aligned}$$

Thus, (2) is verified. So, Cayley-Hamilton theorem is satisfied.

Now, multiplying the equation $A^2 - 7A + I = 0$ by A^{-1} then

$$\begin{aligned}
 A - 7I + A^{-1} &= 0 \\
 \Rightarrow A^{-1} &= -A + 7I = -\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -2+7 & -3 \\ -3 & -5+7 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}.
 \end{aligned}$$

$$(ii) \text{ Given matrix is, } A = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Let λ is eigen value of A . The characteristics equation of A is

$$\begin{aligned}
 |A - \lambda I| &= 0 \\
 \Rightarrow \begin{vmatrix} 7-\lambda & -1 & 3 \\ 6 & 1-\lambda & 4 \\ 2 & 4 & 8-\lambda \end{vmatrix} &= 0 \\
 \Rightarrow (7-\lambda)[(1-\lambda)(8-\lambda)-16] + 1[6(8-\lambda)-8] + 3[24-2(1-\lambda)] &= 0 \\
 \Rightarrow (7-\lambda)(8-9\lambda+\lambda^2-16) + (48-6\lambda-8) + 3(24-2+2\lambda) &= 0 \\
 \Rightarrow -56-63\lambda+7\lambda^2+8\lambda+9\lambda^2-\lambda^3-6\lambda+40+6\lambda+66 &= 0 \\
 \Rightarrow \lambda^3 - 16\lambda^2 + 55\lambda + 50 &= 0 \quad \dots\dots\dots (1)
 \end{aligned}$$

The corresponding value of A is,

$A^3 - 16A^2 + 55A - 50I = 0 \quad \dots\dots\dots (2)$

The condition (2) is Cayley-Hamilton condition for A .

Here,

$$\begin{aligned}
 A^2 = AA &= \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} \\
 &= \begin{bmatrix} 49-6+6 & -7-1+12 & 21-4+24 \\ 42+6+8 & -6+1+16 & 18+4+32 \\ 14+24+16 & -21+4+32 & 6+16+64 \end{bmatrix} = \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix}
 \end{aligned}$$

And,

$A^3 = AA^2 = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix}$

$$\begin{aligned}
 &= \begin{bmatrix} 343 - 56 + 162 & 28 - 11 + 102 & 287 - 54 + 258 \\ 294 + 56 + 216 & 24 + 11 + 136 & 246 + 54 + 344 \\ 98 + 224 + 432 & 8 + 44 + 272 & 82 + 216 + 688 \end{bmatrix} \\
 &= \begin{bmatrix} 449 & 99 & 491 \\ 296 & 171 & 644 \\ 754 & 324 & 986 \end{bmatrix}
 \end{aligned}$$

Then,

$$A^3 - 16A^2 - 55A + 52I$$

$$\begin{aligned}
 &= \begin{bmatrix} 449 & 99 & 491 \\ 296 & 171 & 644 \\ 754 & 324 & 986 \end{bmatrix} - 16 \begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} + 55 \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix} - 50 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 449 - 784 + 385 - 50 & 99 - 64 - 55 - 0 & 491 - 656 + 165 - 0 \\ 296 - 896 + 330 - 0 & 171 - 176 + 55 - 50 & 644 - 544 + 220 - 0 \\ 754 - 864 + 110 - 0 & 324 - 544 + 200 - 0 & 986 - 1376 + 440 - 50 \end{bmatrix} \\
 &= \begin{bmatrix} -336 & -100 & -170 \\ -100 & -20 & 100 \\ -100 & 0 & -400 \end{bmatrix}
 \end{aligned}$$

6. Diagonalise the following matrices and obtain the modal matrix in each case:

$$(i) \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \quad (ii) \begin{bmatrix} 9 & -1 & -9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$$

$$\text{Solution: (i)} \text{ Let, } A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Let λ be eigen value of A. Then the characteristic equation of A is,

$$\begin{aligned}
 IA - \lambda II = 0 &\Rightarrow \begin{bmatrix} -1 - \lambda & 1 & 2 \\ 0 & -2 - \lambda & 1 \\ 0 & 0 & -3 - \lambda \end{bmatrix} = 0 \\
 &\Rightarrow (-1 - \lambda)((-2 - \lambda)(-3 - \lambda) - 0) = 0 \\
 &\Rightarrow (1 + \lambda)(2 + \lambda)(3 + \lambda) = 0 \\
 &\Rightarrow \lambda = -1, -2, -3
 \end{aligned}$$

Thus, $\lambda = -1, \lambda = -2, \lambda = -3$ are eigen value of A.

And, the corresponding eigen vector of A is,

$$\begin{aligned}
 Ax = \lambda x &\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
 &\Rightarrow \begin{bmatrix} -x + y + 2z \\ -2y + z \\ -3z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}
 \end{aligned}$$

This gives,

$$-x + y + 2z = \lambda x, \quad -2y + z = \lambda y \quad \text{and} \quad -3z = \lambda z$$

At $z = -1$,

$$-x + y + 2z = -x, \quad -2y + z = -y \quad \text{and} \quad -3z = -z$$

$$\Rightarrow -x + y + 2z = -x, \quad y - z = 0 \quad \text{and} \quad 2z = 0$$

This implies $z = 0, y = 0$ and $x = x$.

Thus, the eigen vector $(x, 0, 0)$ is required vector corresponding to $\lambda = -1$.

If, $\lambda = -2$,

$$-x + y + 2z = -2x, \quad -2y + z = -2y \quad \text{and} \quad -3z = -2y$$

$$\Rightarrow x + y + 2z = 0, \quad 2y - z = 2y \quad \text{and} \quad z = 0$$

This implies $z = 0, y = y$ and $x = -y$.

Thus, the corresponding Eigen vector to $\lambda = -2$ is $(x, -x, 0)$.

Also, at $\lambda = -3$,

$$-x + y + 2z = -3x; \quad -2y + z = -3y \quad \text{and} \quad -3z = -3z$$

$$\Rightarrow 2x = y + 2z = 0; \quad y + z = 0 \quad \text{and} \quad z = z$$

$$\text{This implies } z = z, y = -z \quad \text{and} \quad x = \frac{1}{2}(-y - 2z) = \frac{1}{2}(z - 2z) \Rightarrow z = -2x.$$

Thus, the corresponding eigen vector to $\lambda = -3$ is $(-2x, 2x, -2x)$.

Hence, the modal matrix of A is,

$$B = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Next we wish to find out B^{-1} matrix.

For this,

$$\text{Cofactor of } a_{11} (1) = 2; \quad \text{Cofactor of } a_{12} (1) = 0; \quad \text{Cofactor of } a_{13} (-2) = 0$$

$$\text{Cofactor of } a_{21} (0) = -2; \quad \text{Cofactor of } a_{22} (-1) = -2; \quad \text{Cofactor of } a_{23} (0) = 0$$

$$\text{Cofactor of } a_{31} (0) = 0; \quad \text{Cofactor of } a_{32} (0) = -2; \quad \text{Cofactor of } a_{33} (-1) = -1$$

And,

$$|B| = \begin{vmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{vmatrix} = 1(2 - 0) = 2 \neq 0.$$

So, B^{-1} exists.

And,

$$B^{-1} = \frac{1}{|B|} \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & -2 \\ 0 & -2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & -2 \\ 0 & -2 & -1 \end{bmatrix}$$

We know the diagonal matrix of A is $B^{-1}AB$. That is,

$$\begin{aligned}
 B^{-1}AB &= \frac{1}{2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 + 0 + 0 & -1 - 1 + 0 & 2 + 2 + 4 \\ 0 + 0 + 0 & 0 + 2 + 0 & 0 - 4 + 2 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 - 6 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 8 \\ 0 & 2 & -2 \\ 0 & 0 & -6 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} -2 + 0 + 0 & -4 - 4 + 0 & 16 + 4 + 0 \\ 0 + 0 + 0 & 0 - 4 + 0 & 0 + 4 + 12 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 6 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} -2 & -8 & 20 \\ 0 & -4 & 16 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 10 \\ 0 & -2 & 8 \\ 0 & 0 & 3 \end{bmatrix}.
 \end{aligned}$$

Thus, the model matrix of A is $\begin{bmatrix} -1 & -4 & 10 \\ 0 & -2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$

(ii) Similar to (i).

OTHER IMPORTANT QUESTION FROM FINAL EXAM

2012 Fall Q.No. 1(a)

What is the Hermitian matrix? Show that the square matrix A is Hermitian if $A = \begin{pmatrix} i & 2+i & 3-i \\ -2+i & 2i & 2 \\ -3-i & -2 & -i \end{pmatrix}$.

Solution: See definition of Hermitian matrix.

For problem part, see solution of Exercise 1.2 - Q.7.

2010 Fall Q.No. 1(a); 2009 Fall Q.No. 1(a); 2007 Fall Q.No. 1(a)

Define transpose of the matrix. Show that $(AB)^T = B^T A^T$, where A and B are matrices of specific order.

OR - Define transpose of a matrix. If A and B are two matrices of order $m \times p$, $p \times n$ prove that $(AB)^T = B^T A^T$. [2009 Spring Q.No. 1(a)]

Solution: See definition of transpose of a matrix.

For problem part see the theoretical solution of $(AB)^T = B^T A^T$.

2008 Spring Q.No. 1(a)

Define inverse of a matrix. Let A and B be two matrices of specified order, show that $(AB)^{-1} = B^{-1}A^{-1}$ where A and B are non-singular matrix.

Solution: See definition of inverse of a matrix.

For problem part see the theoretical solution of $(AB)^{-1} = B^{-1}A^{-1}$.

2003 Spring Q.No. 1(a)

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}$, show that: $A(\text{Adj}A) = (\text{Adj}A)A$.

Solution: Let, $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix}$

Then the cofactor elements of A are:

Cofactor of $a_{11} = 27 - 20 = 7$	Cofactor of $a_{12} = -(18 - 16) = -2$
Cofactor of $a_{13} = 10 - 12 = -2$	Cofactor of $a_{21} = -(27 - 25) = -2$
Cofactor of $a_{22} = 9 - 20 = -11$	Cofactor of $a_{23} = -(5 - 12) = 7$
Cofactor of $a_{31} = 12 - 15 = -3$	Cofactor of $a_{32} = -(4 - 10) = 6$
Cofactor of $a_{33} = 3 - 6 = -3$	

Then the adjoint matrix of A is:

$\text{Adj}(A) = \text{Transpose of cofactor matrix of } A$

$$= \begin{bmatrix} 7 & -2 & -2 \\ -2 & -12 & 7 \\ -3 & 6 & -3 \end{bmatrix}^T = \begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -3 & 7 & -3 \end{bmatrix}$$

Now,

$$\begin{aligned} A[\text{Adj.}(A)] &= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix} \begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -3 & 7 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 7 - 6 - 10 & -2 - 36 + 35 & -3 + 18 - 15 \\ 14 - 6 - 8 & -4 - 36 + 28 & -6 + 18 - 12 \\ 28 - 10 - 18 & -8 - 60 + 63 & -12 + 30 - 27 \end{bmatrix} \\ &= \begin{bmatrix} -9 & -3 & 0 \\ 0 & -12 & 0 \\ 0 & -5 & -9 \end{bmatrix} \end{aligned}$$

Also,

$$\begin{aligned} (\text{Adj. } A)A &= \begin{bmatrix} 7 & -2 & -3 \\ -2 & -12 & 6 \\ -2 & 7 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 7 - 4 - 12 & -21 - 6 + 15 & 35 - 8 - 27 \\ -2 - 24 + 24 & -6 - 36 + 30 & -10 - 48 + 54 \\ -2 + 14 - 12 & -6 + 21 - 15 & -10 + 28 - 27 \end{bmatrix} \\ &= \begin{bmatrix} -9 & -12 & 0 \\ 0 & -9 & -9 \end{bmatrix} \end{aligned}$$

2001 Q.No. 1(a)

Define transpose of a matrix. Show that $(A + B)^T = A^T + B^T$.

Solution: See the definition of transpose of a matrix and see remark for problem.

USE OF PROPERTIES OF DETERMINANT

2009 Fall Q.No. 1(a) OR

Using the properties of determinant, show that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx)$$

Solution: See exercise 1.3 Q.2 (xii).

2002 Q.No. 1(a) OR

$$\text{Show that } \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Solution: We have,

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Apply, $C_2 - C_1, C_3 - C_1$ we get,

$$= \begin{vmatrix} (b+c)^2 & a^2 - (b+c)^2 & a^2 - (b+c)^2 \\ b^2 & (c+a)^2 - b^2 & b^2 - b^2 \\ c^2 - c^2 & (a+b)^2 - c^2 & (a+b)^2 - c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a-b-c)(a+b+c) \\ b^2 & (c+a-b)(a+b+c) & 0 \\ c^2 & 0 & (a+b-c)(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Apply $R_1 : R_1 - R_2 - R_3$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Expanding from R_1

$$\begin{aligned} &= (a+b+c)^2 [2bc(c+a-b)(a+b-c) + 2cb^2(a+b-c) - 2b(c+a-b)] \\ &= 2(a+b+c)^2 [bc(ac+bc-c^2+a^2+ab-ac-ab-b^2+bc) + b^3c - b^2c^2 + bc^3 + abc^2 - b^2c^2] \\ &= 2(a+b+c)^2 (ab^2 + b^2c^2 - bc^3 + a^2bc - abc^2 - b^3c + b^2c^2 + b^3c) \\ &= 2(a+b+c)^2 (abc^2 + a^2bc + b^2ac) \\ &= 2abc(a+b+c)^3. \end{aligned}$$

CONSISTENCY OF SYSTEM OF LINEAR EQUATIONS

2012 Fall Q.No. 1(b) OR

Check the consistency of the given system of equations and solve it:

$$5x + 3y + 7z = 4; 3x + 26y + 2z = 9; 7x + 2y + 10z = 5.$$

Solution: Given system of equations be,

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

The augmented matrix of the given system is,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

Applying $R_2 \rightarrow 5R_2 - 3R_1$ and $R_3 \rightarrow 5R_3 - 7R_1$ then,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Again applying $R_2 \rightarrow \frac{R_2}{11}$ then,

$$[A : C] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$, then

$$[A : C] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \dots \text{(i)}$$

Here, rank of the coefficient matrix (A) is 2 and rank of the augmented matrix ([A : C]) is also 2.

Thus, rank of A = rank of [A : C].

This shows that the given system is consistent.

Now, from R_2 in (i), $11y - z = 3$ and from R_1 in (i), $5x + 3y + 7z = 4$ Thus, $5x + 80y = 25, 11y - z = 3, z = 2$ be solution of given system.

2011 Spring Q.No. 1(b)

Check whether the following system of linear equation is consistent or not. If consistent solve it by using Gauss elimination method.

Solution: Given system of linear equations are

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

Rewrite the equations as,

$$x - y + z = 2$$

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

The augmented matrix of the system is,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1$ and $R_2 \rightarrow R_2 - 2R_1$ then,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

Again applying $R_3 \rightarrow R_3 - 2R_2$ then

$$[A : C] = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

Also, applying $R_3 \rightarrow -\frac{R_3}{2}$ then,

$$[A : C] = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \dots \text{(i)}$$

This shows that the rank of coefficient matrix (A) = 3 and rank of augmented matrix ([A : C]) = 3

That is rank of A = rank of [A : C]

Therefore, the given system is consistent

From R₃ of (i), $z = 3$.

$$R_2 \text{ of (i), } y + z = 5 \Rightarrow y = 5 - 3 = 2.$$

$$R_1 \text{ of (i), } x - y + z = 2 \Rightarrow x = 2 + 2 - 3 = 1.$$

Thus, $x = 1, y = 2, z = 3$ be solution of given system.

2010 Fall Q.No. 2(a)

Define consistency and inconsistency of a system of linear equations. Define consistency of given system of linear equation.

$$x + 2y + 3z = 1; 2x + 3y + 2z = 2; 2x + 3y + 4z = 1.$$

Solution: See the definition of consistency and inconsistency of a system of linear equations.

For the problem see exercise 1.9 Q. 9.

2010 Spring Q.No. 1(b)

Check whether the system of linear equations is consistent or not, if consistent solve it by using Gauss elimination method.

$$x + 6y + 2z = 0; 7x + 3y + z = 13; x + 2y + 3z = 20.$$

Solution: Given system of linear equations are,

$$x + 6y + 2z = 0$$

$$7x + 3y + z = 13$$

$$x + 2y + 3z = 20$$

The augmented matrix of given system be,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 7 & 3 & 1 & 13 \\ 1 & 2 & 3 & 20 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 7R_1$ and $R_3 - R_1$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & -39 & -13 & 13 \\ 0 & -4 & 1 & 20 \end{bmatrix}$$

Applying $R_2 \rightarrow \frac{-R_2}{39}$ and $R_3 \rightarrow \frac{-R_1}{4}$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 1 & -1/4 & -5 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$ then,

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & -7/12 & -14/3 \end{bmatrix}$$

Applying $R_3 \rightarrow \frac{-12R_3}{7}$ then

$$[A : C] = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 1 & 8 \end{bmatrix} \quad \dots \dots \dots \text{(i)}$$

This shows that rank of $(A) = 3 = \text{rank of } [A : C]$

That means the given system is consistent.

Now,

From R_3 of (i), $z = 8$

$$\text{From } R_2 \text{ of (i), } y + \frac{z}{3} = \frac{-1}{3} \Rightarrow y = \frac{-1}{3} - \frac{8}{3} = -3$$

$$\text{From } R_1 \text{ of (i), } x + 6y + 2z = 0 \Rightarrow x = 18 - 16 = 2$$

Thus, $x = 2, y = -3, z = 8$ be solution of given system.

SIMILAR QUESTIONS FOR PRACTICE2009 Spring Q.No. 2(b)

Define consistency and inconsistency of a system of linear equations. Solve the given system of equations by using Gauss elimination method.

$$x + y + z = 8; x - 2y + z = 6; 3x + 5y - 7z = 14.$$

2009 Fall Q.No. 1(b)

Solve the system of linear equations: $x + 3y + 6z = 2, 3x - y + 4z = 9, x - 4y + 2z = 7$, by Gauss elimination methods.

2008 Spring Q.No. 2(a)

Define consistency of the system of linear equations. Solve the given system of linear equations $x + 3y + 6z = 2, 3x - y + 4z = 9, x - 4y + 2z = 7$ by using Gauss elimination methods

2007 Fall Q.No. 1(b)

By using Gauss elimination method examine whether the following system of linear equations has solution or not, $2x - y + 3z = 9; x + y + z = 6; x - y + z = 2$.

2006 Spring Q.No. 2(a)

When is a set of simultaneous equations said to be consistent? Show that the set of simultaneous equations

$$7y + 3z = -12; 2x + 8y + z = 0; -5x + 2y - 9z = 26.$$

Is consistent and solve it using Gauss elimination method.

2005 Fall Q.No. 2(a)

Define consistency and inconsistency of a system of linear equations. Solve the given system of equations by using Gauss elimination method:

$$2x - y - z = 4; -x + 2y - z = -5; x - y + 2z = 1.$$

2004 Fall Q.No. 2(a) OR

Test for consistency and solve:

$$5x + 3y + 7z = 4; 3x + 26y + 2z = 9; 7x + 2y + 10z = 5.$$

2004 Spring Q.No. 1(b)

Test for consistency and then solve

$$x + y + z = 3; x + 2y + 3z = 4; x + 4y + 9z = 6.$$

2003 Fall Q.No. 1(b)

Solve by Gauss elimination method to solve the following system of linear equations: $x + y - z = 9; 8y + 6z = -6; -2x + 4y - 6 = 40$.

2003 Spring Q.No. 1(b); 2002 Q.No. 1(b)

When is a set of simultaneous equations said to be consistent? Show that the following system of equation is consistent and solve it:

$$x + y + z = 3; x + 2y + 3z = 4; 2x + 3y + 4z = 7.$$

2002 Q.No. 1(b)

Solve the following by Gauss elimination method

$$x + y + z = 6; x - y + z = 2; 2x - y + 3z = 9.$$

2001 Q.No. 1(b)

Test for consistency and solve the following equations:

$$x + y + z = 3; x + 2y + 3z = 4; x + 4y + 9z = 6.$$

2013 Fall Q. No. 1(a)

When does a system of equations said to be consistent? Test the consistency and solve $x + y + z = 6$; $2x - y + 3z = 11$; $4x + y + 5z = 23$.

2013 Spring Q. No. 1(a)

Define consistency of a system of linear equation. Check consistency of the system $x + y + z = 8$; $x - y + z = 6$; $3x + 5y - 7z = 14$ and solve it if it is consistent.

Solution: First Part: See definition of the consistency

Second Part: See Exercise 1.9 Q. No. 7.

2014 Fall Q. No. 1(a)

Define consistency of a system of equations. Check consistency of the equations $x_1 + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$. If it is consistency find its solution.

Solution: First Part: See definition of the consistency

Second Part: See 2012 Fall Q. No. 1(b) OR.

INVERSE MATRIX**2011 Fall Q.No. 1(a)**

When does the inverse of a matrix exist? Find A^{-1} if $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

Solution: For first part see note of definition of a inverse of a matrix.

$$\text{Problem part: Let, } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Here,

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 1(1-0) - 2(2+1) = 1-6 = -5 \neq 0$$

So, the A is non-singular. Therefore the inverse matrix of A exists.

Here, the cofactor element of A are,

$$\text{Cofactor of } a_{11} = 1 - 0 = 1$$

$$\text{Cofactor of } a_{12} = -(2+1) = -3$$

$$\text{Cofactor of } a_{13} = (0+1) = 1$$

$$\text{Cofactor of } a_{21} = -(2-0) = -2$$

$$\text{Cofactor of } a_{22} = (1-0) = 1$$

$$\text{Cofactor of } a_{23} = -(0+2) = -2$$

$$\text{Cofactor of } a_{31} = 2 - 0 = 2$$

$$\text{Cofactor of } a_{32} = -(1-0) = -1$$

$$\text{Cofactor of } a_{33} = 1 - 4 = -3$$

Therefore the adjoint matrix of A is,

$\text{Adj}(A) = \text{transpose of cofactor matrix of } A$

$$= \begin{bmatrix} 1 & -3 & 1 \\ -2 & 1 & -2 \\ 2 & -1 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & 2 \\ -3 & 1 & -1 \\ 1 & -2 & -3 \end{bmatrix}$$

thus, inverse matrix of A is,

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{-5} \begin{bmatrix} 1 & -2 & 2 \\ -3 & 1 & -1 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & \frac{-2}{5} \\ \frac{3}{5} & \frac{-1}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

SIMILAR QUESTIONS FOR PRACTICE**2010 Spring Q.No. 1(a)**

Define the inverse of a matrix. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

2007 Fall Q.No. 1(b)

Define inverse of a matrix. Given a matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -3 \\ 3 & 4 & 0 \end{bmatrix}$.

- i. Does A^{-1} exist?
ii. If A^{-1} exists, find A^{-1} .

2004 Spring Q.No. 1(a)

When does the inverse of a matrix exist? If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, find A^{-1} .

2003 Fall Q.No. 1(a)

What is the condition for matrix to have its inverse? If $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ find A^{-1} .

2002 Q.No. 1(a)

Define inverse of a matrix. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, Find A^{-1} .

2001Q. No. 1(b) OR

When does the inverse of a matrix exist? If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, find A^{-1} .

2002 Q.No. 1(a)

What do you mean by singular and non singular matrix? Write the necessary and sufficient condition for a square matrix A to have its inverse. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

2013 Fall Q. No. 4(a)

What is a matrix? When does the inverse of a matrix exists? If $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Find its inverse.

Solution: First Part: See definition of a matrix.

Second Part: The inverse of a matrix will exist if $|A| \neq 0$.

Third Part: See Exercise 1.4 Q. No. 3(ii).

2014 Spring Q. No. 1(a)

Prove that the inverse of a matrix if it exists, is unique. Find A^{-1} if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

1.4 & principle of work & energy & application

$$F_t = F_{\text{ext}}$$

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RANK

2010 Fall Q.No. 1(b); 2009 Spring Q.No. 2(a); 2004 Fall Q.No.
2002 Q.No. 2(a) OR; 2001 Q.No. 2(a)

Define rank of the matrix. Find the rank of the matrix $\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$.

Solution: For first part, see the definition of rank of a matrix.
For problem part, see Exercise 1.8 Q. 20.

2008 Spring Q.No. 1(b)

Define rank of the matrix. Find rank of the matrix BA where $A = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

Solution: Let $A = [1 \ 2 \ -1]$ and $B = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Then,

$$BA = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \times [1 \ 2 \ -1] = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

This matrix has only one column. So, rank of $BA = 1$, being (rank of matrix) \leq (number of non-zero column, number non-zero of row).

2006 Spring Q.No. 1(a)

Define a matrix with its order. Show that $(AB)^T = B^T A^T$, where A and B are matrices of specified order.

Solution: For first part, see the definition of a matrix and its order.
For problem part, see the part on transpose of a matrix.

2006 Spring Q.No. 1(b)

Define rank of a matrix and find the rank of the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Solution: Let

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Here, A is 3×4 matrix. So, rank of $A \leq 3$.

Now, applying $R_2 \rightarrow R_2 + 2R_1$ and $R_3 \rightarrow R_3 - 7R_1$, then;

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

Again applying $R_3 \rightarrow 2R_3 + R_2$ then,

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that A has only two non-zero columns. So, rank of $A = 2$.

SIMILAR QUESTIONS FOR PRACTICE

2005 Fall Q.No. 1(b)

Find the rank of the matrix: $\begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$

2003 Fall Q.No. 2(a)

Define rank of matrix. Find the rank of $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}$.

2003 Spring Q.No. 2(a) OR

What do you mean by the rank of a matrix? If $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & 2 \\ 6 & 9 & 3 & 3 \end{bmatrix}$, find the rank of A .

2002 O.No. 2(a)

Determine the rank of the matrix $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$.

BASIS; VECTOR SPACE

OR

2013 Fall Q.No. 4(a)

Prove that the set of vectors $(1, 1, 1), (1, 3, 2)$ and $(-1, 0, 1)$ forms a basis of \mathbb{R}^3 .

Solution: See the Exercise 1.10 Q. No. 5(i).

2012 Fall Q.No. 2(b); 2003 Spring Q.No. 2(a)

Define subspace of a vector space V over a field F . If $W = \{(x, y, z) : x + y + z = 0\}$ show that it is a subspace of vector space \mathbb{R}^3 over the field of real numbers R .

Solution: First Part: See the definition of vector subspace.

Second Part: Let, $W = \{(x, y, z) : x + y + z = 0\}$.

Then clearly W is subset of \mathbb{R}^3 being W is a three dimensional.

Let, $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in W .

Then W is subspace of \mathbb{R}^3 if $au + bv \in W$ for a and b are two constants.

Since, $u, v \in W$. So, $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$.

Now,

$$\begin{aligned} au + bv &= a(x_1, y_1, z_1) + b(x_2, y_2, z_2) \\ &= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \quad \dots \text{(i)} \end{aligned}$$

Here,

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2) \\ = a \cdot 0 + b \cdot 0 = 0.$$

This shows that $(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) \in W$.

$$\Rightarrow au + bv \in W$$

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This means W is a subspace in \mathbb{R}^3 .

2011 Spring Q.No. 2(a); 2010 Spring Q.No. 2(a)

Define vectors subspace of vector space and basis of vectors space. Check whether the following vectors form a basis of \mathbb{R}^3 or not: $(1, 2, 1), (2, 1, 0), (1, -1, 2)$.

Solution: First Part: See the definition of vector subspace and basis for a vector space.

Second Part: Given vectors are.

$$(1, -1, 1), (1, 1, -1), (-1, 1, 1), (0, 1, 0)$$

Then the vector are linear independent if for the constants c_1, c_2, c_3 and c_4

$$c_1(1, -1, 1) + c_2(1, 1, -1) + c_3(-1, 1, 1) + c_4(0, 1, 0) = 0$$

implies that $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$.

Here, number of vectors = 4 and number of dimension of vector is 3.

That is, no. of vectors \neq no. of dimension of vector.

So, the vectors are linearly dependent because the necessary condition for vectors to be linearly independent is the number of vectors is same as number of dimension of vector.

2011 Spring Q.No. 2(a) OR

Examine the following vectors for linear dependence: $(1, 2, 0), (0, 3, 1), (-1, 0, 1)$.

Solution: Given Vectors are $(1, 1, 0), (0, 3, 1)$ and $(-1, 0, 1)$.

Here, number of vectors = 3 = number of dimension of vector.

So, the vectors may be linearly independent being a necessary condition for linear independence for vectors is the number of vectors is same as number of dimension of vector.

Now, the vectors are linearly independent when for constants C_1, C_2, C_3

$$C_1(1, 2, 0) + C_2(0, 3, 1) + C_3(-1, 0, 1) = 0$$

Implies $C_1 = 0, C_2 = 0, C_3 = 0$.

Here,

$$\begin{aligned} C_1(1, 2, 0) + C_2(0, 3, 1) + C_3(-1, 0, 1) &= 0 = (0, 0, 0) \\ \Rightarrow (C_1 - C_3, 2C_1 + 3C_2, C_2 + C_3) &= (0, 0, 0) \end{aligned}$$

This gives,

$$C_1 - C_3 = 0, 2C_1 + 3C_2 = 0, C_2 + C_3 = 0.$$

Solving we get, $C_1 = 0, C_2 = 0, C_3 = 0$.

This shows that the given vectors are linearly independent.

2004 Spring Q.No. 2(a)

Show that the following vectors are linearly dependent.

$$x_1 = (1, -2, 3, 4), x_2 = (-2, 4, -1, -3), x_3 = (-1, 2, 7, 6).$$

Solution: Given vectors are,

$$x_1 = (1, -2, 3, 4), x_2 = (-2, 4, -1, -3), x_3 = (-1, 2, 7, 6).$$

$$\text{Here, } x_3 = 2x_2 + 3x_1.$$

So, the given vectors are linearly dependent.

2001 Fall Q.No. 2(b) OR; 2002 Q.No. 2(b) OR

What do you mean by linear transformation of vector space V to another vector space W , if $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $f(x, y, z) = (2x, 2y, 2z)$. Show f is linear.

solution: Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined as $f(x, y, z) = (2x, 2y, 2z)$.

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$. Then $u_1, v \in \mathbb{R}^3$

Let a and b are any two constants.

Now,

$$\begin{aligned} f(au + bv) &= f(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \\ &= \{2(ax_1 + bx_2), 2(ay_1 + by_2), 2(az_1 + bz_2)\} \\ &= \{a(2x_1, 2y_1, 2z_1) + b(2x_2, 2y_2, 2z_2)\} = a f(u) + b f(v). \end{aligned}$$

This shows that f is linear.

SIMILAR QUESTIONS FOR PRACTICE2011 Fall Q.No. 2(a)

Define the linearly dependent and independent vectors. Find whether the following sets of vectors are linearly dependent or independent: $(1, -1, 1), (1, 1, -1), (-1, 1, 1), (0, 1, 0)$.

2009 Fall Q.No. 2(a)

Define basis of a vector space over the field. Check the following vectors form a basis of \mathbb{R}^3 : $(1, 1, 1), (1, 2, 3), (-1, 0, 1)$.

2007 Fall Q.No. 2(a) OR

Define vector subspace of the vector space V over a field F . Let $V = \mathbb{R}^3$ be a vector space over a field R , show that $W = \{(x, y, z): x + 2y + z = 0\}$ is a subspace of V .

2004 Fall Q.No. 2(a)

Define linearly dependent and independent set of vectors in \mathbb{R}^3 and then show that the vectors $(1, 1, 1), (1, 2, 3)$ and $(1, 4, 9)$ in \mathbb{R}^3 are linearly independent.

2002 Q.No. 2(a)

Define linearly dependent and independent set of vectors and examine if the vectors $v_1 = (1, 2, 3), v_2 = (-1, 2, -3)$ and $v_3 = (4, -2, 1)$ are linearly dependent.

EIGEN VALUE EIGEN VECTOR2012 Fall Q.No. 2(a)

Find the eigen values and vectors of $A = \begin{pmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

solution: Let, $A = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Let λ and x are eigen value and eigen vector to A respectively. The characteristic equation to A is,

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 7 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -2 - \lambda \end{vmatrix} = 0 \Rightarrow (-\lambda)^2(-2 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, -2$$

Thus, the eigen values to are $\lambda = 0, \lambda = -2$.
And the corresponding eigen vector be,

$$AX = \lambda X \Rightarrow \begin{bmatrix} -\lambda & 2 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$2x = \lambda x, \quad 0 = \lambda y, \quad -2z = \lambda z.$$

$$\text{At } \lambda = 0, \quad x = 0, \quad y = 0, \quad z = 0.$$

$$\text{At } \lambda = -2, \quad x = 0, \quad y = 0, \quad z = 0.$$

Thus, the eigen vectors of A are $(0, y, 0)$ at $\lambda = 0$ and $(0, 0, z)$ at $\lambda = -2$.

2011 Spring Q.No. 2(b)

Find eigen-value and eigen vector of the following matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Solution: Let, } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let λ and x are eigen value and eigen vector to A respectively. The characteristic equation to A is,

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \\ &\Rightarrow -(1-\lambda)^2(1+\lambda) + 4(1+\lambda) = 0 \\ &\Rightarrow (1+\lambda)(4 - (1-\lambda)^2) = 0 \\ &\Rightarrow (1+\lambda)(3+2\lambda-\lambda^2) = 0 \\ &\Rightarrow (1+\lambda)(\lambda^2-2\lambda-3) = 0 \\ &\Rightarrow (1+\lambda)(\lambda-3)(\lambda+1) = 0 \quad \Rightarrow \lambda = -1, 3. \end{aligned}$$

Thus, the eigen value of A are $\lambda = -1, \lambda = 3$.

And the corresponding eigen vector be,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives,

$$x + 2y = \lambda x, 2x + y = \lambda y, x = -\lambda z$$

$$\text{At } \lambda = -1, \quad x + 2y = 0, 2x + y = 0, z = 0.$$

This shows $x = -y$ and $z = 0$. Thus, $(x, -x, 0)$ be a eigen vector of A at $\lambda = -1$. And at $\lambda = 3$,

$$-2x + 2y = 0, -x + y = 0, 4z = 0.$$

This shows $x = y, z = 0$. Thus, $(x, x, 0)$ be a eigen vector of A at $\lambda = 3$.

Hence, the eigen vector of A are $(x, -x, 0)$ at $\lambda = -1$ and $(x, x, 0)$ at $\lambda = 3$.

SIMILAR QUESTIONS FOR PRACTICE

2014 Spring Q. N. 1(b)

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

2013 Spring Q. No. 1(b)

Define eigen-value and eigen vector of the square matrix A. Find the eigen value and corresponding vectors of the square matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: First Part: See definition of eigen value and eigen vector.

Second Part: See Exercise J.11 Q. No. 2(xiii)

2011 Fall Q.No. 2(b)

Find eigen value and eigen vectors of the following matrix:

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

2010 Fall Q.No. 2(b)

Find the eigen value and eigen vector of the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2010 spring Q.No. 2(b)

Define eigen values and eigen vectors of a matrix. Find the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$.

2009 Fall Q.No. 2(b)

Determine the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

2009 Spring Q.No. 3(a)

Defined eigen value and eigen vectors of the square matrix. Find eigen value and vectors of the square matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$.

2008 Spring Q.No. 2(b)

Find the eigen value and eigen vector of the matrix: $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

2007 Fall Q.No. 2(a)

Define eigen value and eigen vectors of a square matrix A. Find Eigen value and their corresponding Eigen vector of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

2006 Spring Q.No. 2(b)

Find eigen value and eigen vector of the following matrix: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

2005 Fall Q.No. 2(b)

Define eigen value and eigen vector of a matrix, with its characteristic equation. Find Eigen value and Eigen vector of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2004 Fall Q.No. 2(b)

Define eigen values and eigen vector of a square matrix A. Determine eigen values and Eigen vectors of the matrix: $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

2004 Spring Q.No. 2(b)

Find the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

2003 Spring Q.No. 2(b)

Find the eigen values and eigen vectors of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

2003 Fall Q.No. 2(b)

Find eigen values of eigen vectors of the matrix: $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$.

2002 Q.No. 2(b)

Define characteristic equation of a square matrix A. Find the Eigen values.

Eigen vectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2001 Q.No. 2(b)

Find the eigen values and eigen vectors of the matrix: $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

2002 Q.No. 2(b)

Define eigen values and eigen vectors. Find eigen values and eigen vectors of a matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

SHORT QUESTIONS

2012 Fall: Check the linear dependence and independence of the vectors $(1, 1, 0)$, $(1, 1, 1)$.

Solution: Given vectors are, $x_1 = (1, 1, 0)$, $x_2 = (1, 1, 1)$.

Here, for any two scalars c_1 and c_2 ,

$$c_1x_1 + c_2x_2 = 0$$

$$\Rightarrow c_1(1, 1, 0) + c_2(1, 1, 1) = (0, 0, 0)$$

$$\Rightarrow (c_1 + c_2, c_1 + c_2, c_2) = (0, 0, 0)$$

$$\Rightarrow c_1 + c_2 = 0, c_1 + c_2 = 0, c_2 = 0,$$

$$\Rightarrow c_1 = 0 \text{ and } c_2 = 0.$$

This shows that the given vectors are linearly independent.

2010 Fall, 2007 Fall: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a transformation which is defined by $T(x, y) = (x + y, x - y)$, check the linearity of T.

Solution: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as $T(x, y) = (x + y, x - y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$. Then $u, v \in \mathbb{R}^2$

Also, let a and b are two constants.

Then T is linear if $T(au + bv) = aT(u) + bT(v)$.

Now,

$$\begin{aligned} T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2 + ay_1 + by_2, ax_1 + bx_2 - ay_1 - by_2) \\ &= (a(x_1 + y_1, x_1 - y_1) + b(x_2 + y_2, x_2 - y_2)) \\ &= aT(x_1, y_1) + bT(x_2, y_2) = aT(u) + bT(v). \end{aligned}$$

This shows that T is linear.

SIMILAR QUESTIONS FOR PRACTICE

2006 Spring: Show that the transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = |x + y|$ is not linear.

2008 Spring: Show that vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.

2011 Fall, 2010 Spring: Find the rank of $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution: Let, $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 4 \end{bmatrix}$.

Then the rank of A is min. non-zero {row, column} of A.

Here, applying $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - R_1$ then,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -7 \\ 0 & -1 & -1 \end{bmatrix}$$

Also, applying $R_3 \rightarrow 5R_3 - R_2$ then,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -7 \\ 0 & 0 & 2 \end{bmatrix}$$

This shows that rank of A = 3.

SIMILAR QUESTIONS FOR PRACTICE

2014 Fall Q. No. 7(a): Find the rank of $A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$.

2011 Spring: Find the rank of $A = \begin{bmatrix} 2 & 0 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 8 \end{bmatrix}$.

2005 Fall: Find the rank of $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

2002: Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 8 & -4 \end{bmatrix}$.

2009 Spring: Evaluate using Cramer's rule: $x + 2y = 5$ and $x - y = 2$.

Solution: Given system of equations are,

$$x + 2y = 5$$

$$x - y = 2$$

Then,

$$D = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0$$

So, the solution is possible by Cramer's rule

Therefore,

$$D_1 = \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = -5 - 4 = -9 \quad \text{and} \quad D_2 = \begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} = 2 - 5 = -3.$$

Now, by Cramer's rule,

$$x = \frac{D_1}{D} = \frac{-9}{-3} = 3, \quad y = \frac{D_2}{D} = \frac{-3}{-3} = 1$$

Thus, $x = 3, y = 1$ be solution of given system.

2011 Fall: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ find AB and BA .

Solution: Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$.

Then,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+0 \\ 0-1 & 0+3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+0 \\ -1+0 & 0+3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}.$$

2004 Spring: Evaluate, $\begin{vmatrix} a+b & a & b \\ b+c & b & c \\ c+a & c & a \end{vmatrix}$.

Solution: Here,

$$\Delta = \begin{vmatrix} a+b & a & b \\ b+c & b & c \\ c+a & c & a \end{vmatrix}$$

Applying $C_1 = C_1 - (C_2 + C_3)$ then

$$\Delta = \begin{vmatrix} 0 & a & b \\ 0 & b & c \\ 0 & c & a \end{vmatrix} = 0, \text{ being all values in } C_1 \text{ is zero.}$$

2002: Evaluate $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

Solution: Here,

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, then,

$$\begin{aligned} &= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0, \text{ being all values in } C_1 \text{ is zero.} \end{aligned}$$

2001: Evaluate $\begin{vmatrix} b+c & b & c \\ c+a & c & a \\ a+b & a & b \end{vmatrix}$.

Solution: Here,

$$\Delta = \begin{vmatrix} b+c & b & c \\ c+a & c & a \\ a+b & a & b \end{vmatrix}$$

Applying $C_1 = C_1 - (C_2 + C_3)$ then

$$\Delta = \begin{vmatrix} 0 & b & c \\ 0 & c & a \\ 0 & a & b \end{vmatrix} = 0, \text{ being all values in } C_1 \text{ is zero.}$$

2009 Fall, 2003 Fall: Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Solution: See Exercise 1.3 Q 2(i) with replacing x by a , y by b and z by c .

