Laplace transform of the derivative.

suppose $f(t)$ be a continuous function defined for all $t \geq 0$ satisfies $|f(t)| < m.e^{\gamma t}$ for some s and γ and has a derivative $f'(t)$ which is piecewise continious on every finite interval in the range of $t \geq 0$ then

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0) \quad s > \gamma$$

soln

let $f(t)$ be a piecewise continious function defined for all $t \geq 0$ and it's derivative $f'(t)$ is also continous.

By defination

$$L f'(t) = \int_0^\infty e^{-st} f'(t)\, dt$$

Using integration by parts

$$L f'(t) = \left[e^{-st} \int f'(t)\,dt\right]_0^\infty - \int_0^\infty \left[\frac{d}{dt}(e^{-st}) \int f'(t)\, dt\right] dt$$

$$L f'(t) = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty e^{-st} (-s) \cdot f(t)\, dt$$

$$= 0 - e^{0} f(0) + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\,L f(t)$$

$$\therefore L f'(t) = s\,L f(t) - f(0)$$

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Similarly

$Lf''(t) = s^2 Lf(t) - sf(0) - f'(0)$

$Lf^n(t) = s^n f(t) - s^{(n-1)} f(0) - s^{n-2} f(0)$

$- s^{n-3} f''(0) - f^{n-1}(0)$

$Lf''(t) = [e^{-st} f'(t)]_0^\infty + s\int_0^\infty$

$= 0 \cdot f'(0) + s\{sLf(t) \cdot f(0)$

$= -f'(0) + s^2 Lf(t) \cdot sf(0)$

$Lf''(t) = s^2 Lf(t) - f'(0) - sf$

Note:

$Ly' = s \cdot L(y) - y(0)$

$Ly'' = s^2 L(y) - sy(0) - y'(0)$

$Ly''' = s^3 L(y) - s^2 y(0) - y''(0) - sy'(0)$

find the laplace transform of $f(t) = t^2$ using derivative.

soln

Given function is

$f(t) = t^2 \longrightarrow (i)$

$f'(t) = 2t$

$f''(t) = 2$

we have

$Lf''(t) = s^2 Lf(t) - sf(0) - f'(0)$

$L2 = s^2 Lt^2 - s \cdot 0 - 0$

$L2 = s^2 Lt^2$

$\frac{2}{s} = s^2 Lt^2$

$Lt^2 = \frac{2!}{s^3}$

$f(t) = e^{at}$

$f'(t) = a \cdot e^{at}$

we have

$L f'(t) = s L f(t) - f(0)$

$L a \cdot e^{at} = s \cdot L e^{at} - 1$

$a L e^{at} = s \cdot L e^{at} - 1$

$1 = s L e^{at} - a L e^{at}$

$1 = (s-a) L e^{at}$

$L e^{at} = \dfrac{1}{s-a}.$

$f(t) = t \cdot \sin at$

$f'(t) = t a \cos at + \sin at$

$f''(t) = ta(-a\sin at) + a\cos at + a\cos at$

$= 2a\cos at - a^2 t \sin at$

we have

$L f''(t) = s^2 L f(t) - s \cdot f(0) - f'(0)$

$L(2a\cos at - a^2 t \sin at) = s^2 L t \sin at - s \cdot 0 - (0+0)$

$L 2a\cos at - a^2 t \sin at = s^2 L t \sin at$

$2a L \cos at - a^2 L t \sin at = s^2 L t \sin at$

$2a \cdot \dfrac{s}{s^2+a^2} = L t \sin at (s^2+a^2)$

$L t \sin at = \dfrac{2as}{(s^2+a^2)^2}$

iv) $f(t) = t\cos at$

$f'(t) = t(-\sin at)a + \cos at$

$f''(t) = -at\cos at \cdot a - a\sin at - \sin at\, a$

$= -a^2 t\cos at - 2a\sin at$

we have

$Lf''(t) = s^2 Lf(t) - s \cdot f(0) - f'(0)$

$L(-a^2 t\cos at - 2a\sin at) = s^2 L t\cos at - s \cdot 0 - (0+1)$

$-a^2 L t\cos at - 2aL\sin at = s^2 L t\cos at - 1$

$-a^2 L t\cos at - 2a \cdot \frac{a}{s^2+a^2} = s^2 L t\cos at - 1$

$1 - \frac{2a^2}{s^2+a^2} = (s^2+a^2) L t\cos at$

$\frac{s^2+a^2-2a^2}{(s^2+a^2)^2} = L t\cos at$

$L t\cos at = \frac{s^2-a^2}{(s^2+a^2)^2}$

) $L t\sinh at$

let $f(t) = t\sinh at$

$f'(t) = ta\cosh at + \sinh at$

$f''(t) = 2a\cosh at + a^2 t\sinh at$

we have

$Lf''(t) = s^2 Lf(t) - s \cdot f(0) - f'(0)$

$L(2a\cosh at + a^2 t\sinh at) = s^2 L t\sinh at - s \cdot 0 - 0$

$2aL\cosh at + a^2 L t\sinh at = s^2 L t\sinh at$

$\frac{2a \cdot s}{(s^2-a^2)} = (s^2-a^2) L t\sinh at$

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$L\, t\sinh at = \dfrac{2as}{(s^2-a^2)^2}$.

vii) $L\, t\cosh at$

$f(t) = t\cosh at$

$f'(t) = t\sinh at \cdot a + \cosh at$

$f''(t) = a^2 t\cosh at \cdot a + a\sinh at + \sinh at \cdot a$

$= a^2 t\cosh at + 2a\sinh at$

we have $=$ $a\sinh at + a^2 t\cosh at + a\sinh at$

$L f''(t) = s^2 L f(t) - s\cdot f(0) - f'(0)$

$L(a^2 t\cosh at + 2a\sinh at) = s^2 L\, t\cosh at - s\cdot 0 \cdot 0$

$a^2 \cdot L\, t\cosh at + 2a\, L\sinh at = s^2 L\, t\cosh at - 1$

$2a \cdot \dfrac{a}{s^2-a^2} + 1 = (s^2-a^2)\, L\, t\cosh t$

$\dfrac{2a^2 + s^2 - a^2}{(s^2-a^2)} = (s^2-a^2)\, L\, t\cosh t)$

$L\, t\cosh t = \dfrac{s^2+a^2}{(s^2-a^2)^2}$ ans.

$$L^{-1}\left[\frac{1}{(s^2+w^2)^2}\right] = \frac{1}{2w^3}(\sin wt - wt\cos wt)$$

Soln here

$= L\frac{1}{2w^2}(\sin wt - wt\cos wt)$

$= \frac{1}{2w^2}\left\{L\sin wt - L\,wt\cos wt\right\}$

$= \frac{1}{2w^3}\left\{\frac{w}{s^2+w^2} - w\,L\,t\cos wt\right\}$

$= \frac{1}{2w^3}\left\{\frac{w}{s^2+w^2} - \frac{w(s^2-w^2)}{(s^2+w^2)^2}\right\}$

$= \frac{w}{2w^3}\left\{\frac{s^2+w^2-s^2+w^2}{(s^2+w^2)^2}\right\}$

$\cdot \frac{1}{2w^2}\left(\frac{2w^2}{(s^2+w^2)^2}\right)$

$= \frac{1}{(s^2+w^2)^2}$

$\therefore L\frac{1}{2w^2}(\sin wt - wt\cos wt) = \frac{1}{(s^2+w^2)^2}$

$$L^{-1}\left(\frac{1}{(s^2+w^2)^2}\right) = \frac{1}{2w^2}(\sin wt - wt\cos wt)$$

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vi) $L^{-1}\left[\frac{s^2}{(s^2+\omega^2)^2}\right] = \frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

soln

$L\,\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

$= \frac{1}{2\omega}\left\{L\sin\omega t + \omega L\, t\cos\omega t\right\}$

$= \frac{1}{2\omega}\left\{\frac{\omega}{s^2+\omega^2} + \frac{\omega(s^2-\omega^2)}{(s^2+\omega^2)^2}\right\}$

$= \frac{\omega}{2\omega}\left\{\frac{1}{s^2+\omega^2} + \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\}$

$= \frac{1}{2}\left\{\frac{s^2+\omega^2+s^2-\omega^2}{(s^2+\omega^2)^2}\right\}$

$= \frac{1}{2}\,\frac{2s^2}{(s^2+\omega^2)^2}$

$L\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t) \quad \frac{s^2}{(s^2+\omega^2)^2}$

$L^{-1}\frac{s^2}{(s^2+\omega^2)^2} = \frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$

vii) $L^{-1}\left(\frac{s}{(s^2+\omega^2)^2}\right) - \frac{t\sin\omega t}{2\omega}$

soln

$L\,\frac{t\sin\omega t}{2\omega}$

$= \frac{1}{2\omega}L\,t\sin\omega t$

$= \frac{1}{2\omega}\left(\frac{2\omega s}{(s^2+a^2)^2}\right)$

$$L^{-1}\frac{t\sin\omega t}{2\omega} = \frac{s}{(s^2+\omega^2)^2}$$

then,

$$L^{-1}\frac{s}{(s^2+\omega^2)^2} = \frac{t\sin\omega t}{2\omega}$$

Note:
laplace transform of $\int_0^t F(\tau)\,d\tau$

$$L\{\int_0^t f(\tau)\,d\tau\} = \frac{1}{s}L\{f(\tau)\}$$

$$= L^{-1}[1/s\,F(s)] = \int_0^t f(\tau)\,d\tau.$$

) Find $f(t)$, if $L\{f(t)\}$ equals the following.

i) $\frac{1}{4s+s^3}$

let $F(s) = \frac{1}{4s+s^3}$

$$F(s) = \frac{1}{s(s^2+2^2)} \quad \text{—(i)}$$

Suppose $\frac{1}{s(s^2+2^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$$1 = A(s^2+4) + (Bs+C)s$$

$$1 = As^2 + 4A + Bs^2 + Cs$$

equating the coeff of s^2, s and constant term

coeff of s^2; $\quad 0 = A+B \Rightarrow A = -B$ —(i)

coeff of s $\quad C = 0$

coff of constant $\quad 1 = 4A$

$A = 1/4$

$\therefore B = \frac{-1}{4}$.

$= \frac{1}{4s} + \frac{-1/4 B}{s^2+4}$

$= \frac{1}{4s} - \frac{B}{4(s^2+4)}$

from (i)

$F(s) = \frac{1}{4s} - \frac{s}{4(s^2+4)}$

taking inverse laplace transformation on both sides

$L^{-1}F(s) = L^{-1}\left[\frac{1}{4s}\right] - \frac{s}{4(s^2+4)}\Big]$

$= \frac{1}{4} L^{-1}\frac{1}{s} - \frac{1}{4} L^{-1}\frac{s}{s^2+2^2}$

$f(t) = \frac{1}{4}\{1 - \cos 2t\}$

$f(t) = \frac{(1-\cos 2t)}{4}$

$f(t) = \frac{\sin^2 t}{2}$

v) $\frac{9}{s^2}\frac{(s+1)}{(s^2+9)}$

soln Given function is

$F(s) = \frac{9(s+1)}{s^2(s^2+9)}$ —(i)

let

$$\frac{9(s+1)}{s^2(s^2+9)} = \frac{A\cancel{s}+B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$9s+9 = (As+B)(s^2+9) + (Cs+D)s^2$$

$$9s+9 = As^3 + 9As + Bs^2 + 9B + Cs^3 + s^2 D$$

now, comparing the coeff 3, 2, 1 and constant term

$$0 = A + C$$

$$A = -C \rightarrow (ii)$$

$$B \cancel{+C} + D = 0$$

$$B = -D \quad — (iii)$$

$$9 = 9A$$

$$A = 1$$

$$C = -1$$

$$9 = 9B$$

$$B = 1$$

$$\therefore A = 1, B = 1, C = -1, D = -1$$

$$\therefore \frac{9s+9}{s^2(s^2+9)} = \frac{s+1}{s^2} + \frac{-1s-1}{s^2+9}$$

$$= \frac{s+1}{s^2} - \frac{s+1}{s^2+9}$$

$$= \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+9} - \frac{1}{s^2+9}$$

now, from (i)

$$F(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+9} - \frac{1}{s^2+9}$$

taking inverse laplace transform on both sides.

$$L^{-1}F(s) = L^{-1}\left(\frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+9} - \frac{1}{s^2+9}\right)$$

$$f(t) = L^{-1}\frac{1}{s} + L^{-1}\frac{1}{s^2} - L\frac{s}{s^2+9} - \frac{1}{3}L^{-1}\frac{3}{s^2+3^2}$$

$$= 1 + t - \cos 3t - \frac{1}{3}\sin 3t$$

i) $\frac{1}{s^2+s}$

$\frac{1}{s(s+1)}$

$Lf(t) = \frac{1}{s\ \cancel{(s}\ s+1)}$

$f(t) = L^{-1}\frac{1}{s(s+1)}$

$F(s) = \frac{1}{s+1}$

$f(t) = L^{-1}\frac{1}{s+1}$

$= e^{-t}$

$f(t) = e^{-t}$.

$f(\tau) = e^{-\tau}$.

then,

$f(t) = \int_0^t e^{-\tau} d\tau$

$= \left(-e^{-\tau}\right)_0^t$

$= -e^{-t} + 1$

$= 1 - e^{-t}$ ans.

iii) $\frac{1}{s}\frac{(s-a)}{s+a}$

given function is

$F(s) = \frac{s-a}{s(s+a)}$ —(1)

consider

$\frac{s-a}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$

$s-a = A(s+a) + Bs$

put $s=0$

$-a = aA + 0 = A = -1$

put $s = -a$

$-2a = 0 + B(-a)$

$B = 2$

$\therefore \frac{s-a}{s(s+a)} = \frac{-1}{s} + \frac{2}{s+a}$.

taking inverse laplace

$L^{-1}F(s) = L^{-1}\left(-1/s + 2/s+a\right)$

$= L^{-1}-1/s + 2L^{-1}\, 1/s+a$

$= -1 + 2e^{-at}$

$= 2e^{-at} - 1$ ans

iv) $\frac{8}{s^4 - 4s^2}$

soln

given function is

$$F(s) = \frac{8}{s^4 - 4s^2} = \frac{8}{s^2(s^2-4)}$$

consider,

$$\frac{8}{s^2(s^2-4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2-4}$$

$$8 = (As+B)(s^2-4) + (Cs+D)s^2$$

$$8 = As^3 - 4As + Bs^2 - 4B + Cs^3 + Ds^2$$

$$8 = (A+C)s^3 + (B+D)s^2 - 4As - 4B$$

comparing

$A+C=0$ — (1)

$B+D=0$ — (1)

$-4A = 0 \Rightarrow A=0$

$-4B = 8 \Rightarrow B = -2$

$C=0$

$D=2$

then,

$$\frac{8}{s^2(s^2-4)} = \frac{-2}{s^2} + \frac{2}{s^2-4}$$

$$F(s) = \frac{-2}{s^2} + \frac{2}{s^2-4}$$

taking inverse laplace

$$L^{-1}F(s) = L^{-1}\, -2/s^2 + 2/s^2-4$$

$$f(t) = -L^{-1}\, 2/s^2 + 2L^{-1}\, 1/s^2-4$$

$$f(t) = -2t + \sinh 2t$$

$$f(t) = \sinh 2t - 2t$$

ii)
$$\frac{1(s+1)}{s^2(s^2+1)}$$

Given function is

$$F(s) = \frac{s+1}{s^2(s^2+1)}$$

consider

$$\frac{s+1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$s+1 = (As+B)(s^2+1) + (Cs+D)s^2$$

$$= As^3+As+Bs^2+B+Cs^3+Ds^2$$

$$s+1 = (A+C)s^3+(B+D)s^2+As+B.$$

$A+C=0$

$B+D=0$

$A=1$

$B=1$

$C=-1$

$D=-1$

then,

$$\frac{s+1}{s^2(s^2+1)} = \frac{s+1}{s^2} + \frac{-s-1}{s^2+1}$$

taking inverse laplace transform

$$L^{-1}F(s) = L^{-1}\left(\frac{s}{s^2}+\frac{1}{s^2}\right) + L^{-1}\left(\frac{-s}{s^2+1}-\frac{1}{s^2+1}\right)$$

$$= L^{-1}\frac{1}{s} + L^{-1}\frac{1}{s^2} - L^{-1}\frac{s}{s^2+1^2} - L^{-1}\frac{1}{s^2+1}$$

$$= 1+t-\cos t-\sin t$$

ans.

vi) $\frac{1}{s^4-2s^3}$

$\frac{1}{s^3(s-2)}$

consider

$$\frac{1}{s^3(s-2)} = \frac{As^2+Bs+C}{s^3} + \frac{D}{s-2}$$

$$\frac{1}{s^3(s-2)} = \frac{(s-2)(As^2+Bs+C)+Ds^3}{s^3(s-2)}$$

$$1 = As^3+Bs^2+Cs-2As^2-2Bs-2C+Ds^3$$

$$= (A+D)s^3 + s^2(B-2A) + s(C-2B)s - 2C$$

$A+D=0$

$B-2A=0$

$C-2B=0$

$-2C=1$

$2C=-1$

$\cancel{B=0}$ $C=-1/2$

then,

$B=-1/4$

$A=-1/8$

$D=1/8$

then,

$$\frac{1}{s^3(s-2)} \quad \frac{-1/8 s^2 + -1/4 s - 1/2}{s^3} + \frac{1}{8(s-2)}$$

$$= \frac{-s^2-2s-4}{8s^3} + \frac{1}{8(s-2)}$$

taking inverse laplace transform

$$L^{-1}F(s) = L^{-1}\left[\frac{-s^2}{8s^3} - \frac{2s}{8s^3} - \frac{4}{8s^3}\right] + L^{-1}\frac{1}{8(s-2)}$$

$$= \frac{-1}{8}L^{-1}\frac{1}{s} - \frac{1}{4}L^{-1}\frac{1}{s^2} - 0\,L^{-1}\frac{4!}{s^3} + \frac{1}{8}L^{-1}\frac{1}{s-2}$$

$$= -\frac{1\times 1}{8} - \frac{1}{4}t - 2t^2 + \frac{1}{8}e^{2t}$$

$$= \frac{1}{8}\left(e^{2t} - 1 - 2t - 2t^2\right) \text{ ans}$$

vii)
$$\frac{1}{s^2+4s}$$

$$\frac{1}{s(s+4)}$$

Consider

$$\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$1 = A(s+4) + Bs$$

put $s = 0$

$A = 1/4$

put $s = -4$

$B = -1/4$

then

$$\frac{1}{s(s+4)} = \frac{1}{4s} - \frac{1}{4(s+4)}$$

taking laplace inverse transform

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$$L^{-1}F(s) = L^{-1}\frac{1}{4s} - L^{-1}\frac{1}{4(s+4)}$$

$$= \frac{1}{4}L^{-1}\frac{1}{s} - \frac{1}{4}L^{-1}\frac{1}{s+4}$$

$$= \frac{1}{4}\cdot 1 - \frac{1}{4}e^{-4t}$$

$$= \frac{1}{4}(1 - e^{-4t}$$

$$\frac{1}{s(s^2+\omega^2)}$$

$$F(s) = \frac{1}{s(s^2+\omega^2)}$$

consider

$$\frac{1}{s(s^2+\omega^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega^2}$$

$$1 = A(s^2+\omega^2) + s(Bs+c)$$

$$1 = As^2 + A\omega^2 + Bs^2 + cs$$

$$A+B = 0$$

$$c = 0$$

$$A = 1/\omega^2$$

$$B = -1/\omega^2$$

$$\frac{1}{s(s^2+\omega^2)} = \frac{1}{\omega^2 s} + \frac{-1}{\omega^2(s^2+\omega^2)}$$

taking inverse laplace transform

$$L^{-1}F(s) = \frac{1}{\omega^2}L^{-1}1/s - \frac{1}{\omega^2}\int\frac{s}{s^2+\omega^2}$$

$$= \frac{1}{\omega^2} - \frac{1}{\omega^2}\cos\omega t$$

$$= \frac{1}{\omega^2}(1 - \cos\omega t)$$

ix) $\frac{1}{s^3-s}$

$F(s) = \frac{1}{s(s^2-1)}$

consider

$\frac{1}{s(s^2-1)} = \frac{A}{s} + \frac{Bs+C}{s^2-1}$

$1 = As^2 - A + Bs^2 + Cs$

then,

$(A+B) = 0$

$C = 0$

$-A = 1$

$A = -1$

$B = 1$

$\frac{1}{s(s^2-1)} = \frac{-1}{s} + \frac{s}{s^2-1}$

taking laplace inverse transform

$L^{-1}F(s) = L^{-1} -1/s + L^{-1}\frac{s}{s^2-1^2}$

$= -1 + \cosh t$

$= \cosh t - 1$. ans

3) solve the following initial value problems using the laplace transformation.

i) $4y'' + \pi^2 y = 0$, $y(0) = 2$, $y'(0) = 0$.

soln

$4y'' + \pi^2 y = 0$

taking laplace transform, we get

$L\,4y'' + \pi^2 y = 0$

$4Ly'' + \pi^2 Ly = 0$

$4(s^2 Ly - sy'(0) - y'(0)) + \pi^2 Ly = 0$

$4(s^2 Ly - 2s - 0) + \pi^2 Ly$

$(4s^2 + \pi^2) Ly = 2s$

$Ly = \dfrac{2s}{4s^2 + \pi^2}$

taking inverse laplace

$y = \dfrac{2}{4} L^{-1} \dfrac{2s}{s^2 + (\pi/2)^2}$

$= \dfrac{1}{2} L^{-1} \dfrac{s}{s^2 + (\pi/2)^2}$

$= \dfrac{1}{2} \cos\dfrac{\pi}{2} t.$

ii) $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y'(0) = 8$

taking laplace transform

$L\,y'' + 2y' - 8y = 0$

$s^2 Ly - sy(0) - y'(0) + 2(sLy - y(0)) - 8Ly = 0$

$s^2 Ly - s - 8 + 2sLy - 2 - 8Ly = 0$

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$(s^2+2s-8)Ly = s+8$

$Ly = \frac{s+10}{s^2+2s-8}$

$Ly = \frac{s+10}{s^2+2s-8} = \frac{s+10}{(s-2)(s+4)}$

then,

$s+10 = A(s+4) + B(s-2)$

$s+10 = As + 4A + Bs - 2B$

$A+B = 1$

$4A - 2B = 10$

put $s=-4$ then $B=-1$

put $s=2$ then $A=2$.

then,

$Ly = \frac{2}{s-2} + \frac{-1}{s+4}$

$y = 2L^{-1}\frac{1}{s-2} - L^{-1}\frac{1}{s-(-4)}$

$= 2e^{2t} - e^{-4t}$.

$y'' - ky' = 0 \quad y(0) = 2 \quad y'(0) = k.$

soln

$y'' - ky' = 0$

taking laplace transform we get

$Ly'' - Lky' = 0$

$s^2Ly - sy(0) - y'(0) - ksLy + ky(0) = 0$

$s^2Ly - 2s - k - ksLy + 2k = 0$

$(s^2 - ks)Ly = \cancel{k+2s+2}\ 2s-k$

Ans

$$Ly = \frac{2s-k}{s(s-k)}$$

consider

$$\frac{2s-k}{s(s-k)} = \frac{A}{s} + \frac{B}{s-k}$$

$$2s-k = A(s-k) + Bs$$

put $s = k$

$$k = Bk$$

$$B = 1$$

put $s = 0$

$$-k = -kA$$

$$A = 1$$

then,

$$\frac{2s-k}{s(s-k)} = \frac{1}{s} + \frac{1}{s-k}$$

taking inverse laplace transform

$$= L^{-1}\frac{1}{s} + L^{-1}\frac{1}{s-k}$$

$$y = 1 + e^{kt}.$$

iv) $y'' + \omega^2 y = 0$ $\quad y(0) = A$ $\quad y'(0) = B$ (ω real, not zer

soln

$$y'' + \omega^2 y = 0$$

taking laplace transform

$$Ly'' + \omega^2 Ly = 0$$

$$s^2 Ly - s\,y(0) - y'(0) + \omega^2 Ly = 0$$

$$s^2 Ly - As - B + \omega^2 Ly = 0$$

$$Ly = \frac{As+B}{(s^2+\omega^2)} \longrightarrow (i)$$

$$Ly = \frac{As}{s^2+\omega^2} + \frac{B}{s^2+\omega^2}$$

$$y = AL^{-1}\frac{s}{s^2+\omega^2} + BL^{-1}\frac{1}{s^2+\omega^2}$$

$$= A\cos\omega t + \frac{B}{\omega}L^{-1}\frac{\omega}{s^2+\omega^2}$$

$$= A\cos\omega t + \frac{B}{\omega}\sin\omega t$$

$y' + 3y = 10\sin t$, $y(0) = 0$

taking laplace we get

$$Ly' + L3y = 10L\sin t$$

$$sLy + y(0) + L3y = \frac{10}{s^2+1^2}$$

$$sLy + 3Ly = \frac{10}{(s^2+1)}$$

$$Ly = \frac{10}{(s^2+1)(s+3)}$$

then

$$\frac{10}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$$

$$10 = (As+B)(s+3) + C(s^2+1)$$

$$10 = As^2 + 3As + Bs + 3B + Cs^2 + C$$

then

$$A + C = 0$$

$$3A + B = 0$$

$$3B + C = 10$$

$$\frac{10}{(s^2+1)(s+3)} =$$

vi) $y' + 0.2y = 0.01t \qquad y(0) = -0.25$

sol^n

$y' + 0.2y = 0.01t$

taking laplace

$Ly' + 0.2y = 0.01t$

$sLy - y(0) - 0.2Ly = 0.01\,Lt$

$sLy + 0.25 - 0.2Ly = \frac{0.01}{s^2}$

$Ly(s-0.2) = \frac{0.01}{s^2} \cancel{+} - 0.25$

$$Ly = \frac{0.01 - 0.25s^2}{(s-0.2)s^2}$$

$$Ly = \frac{0.01 - 0.25s^2}{(s-0.2)s^2} \quad \text{——— (i)}$$

consider

$$\frac{0.01-0.25s^2}{s^2(s+0.2)} = \frac{As+B}{s^2} + \frac{C}{s+0.2}$$

$$0.01-0.25s^2 = (As+B)(s+0.2) + Cs^2$$

$$0.01-0.25s^2 = As^2 + 0.2As + Bs + 0.2B + Cs^2$$

$A + C = -0.25$

$0.2A + B = 0$

$0.2B = 0.01$

$B = 0.01/0.2$

$= 0.05$

then

$A = -0.25$

$C = 0$

again.

$$\frac{0.01-0.25s^2}{s^2(s+0.2)} = \frac{-0.25s+0.05}{s^2} + 0$$

$$= \frac{-0.25\cancel{s}}{s^{\cancel{2}}} + \frac{0.05}{s^2}$$

$$= \frac{-0.25}{s} + \frac{0.05}{s^2}$$

now taking laplace inverse transform

$$y = L^{-1}\, \frac{-0.25}{s} + \frac{0.05}{s^2}$$

$$= -0.25\, L^{-1} 1/s + 0.05\, L^{-1} 1/s^2$$

$$= -0.25 \cdot 1 + 0.05t$$

$$= 0.05t - 0.25$$

$y'' + ay' - 2a^2 y = 0 \quad y(0) = 6,\ y'(0) = 0$

soln

$y'' + ay' - 2a^2 y = 0$

taking laplace tranform

$Ly'' + aLy' - 2a^2 Ly = 0$

$s^2 Ly - sy(0) - y'(0) + asLy - ay(0) - 2a^2 Ly = 0$

$s^2 Ly - 6s - 0 + asLy - 6a - 2a^2 Ly = 0$

$(s^2 + as - 2a^2) Ly = 6a + 6s$

$$Ly = \frac{6a + 6s}{(s^2 + as - 2a^2)}$$

$$Ly = \frac{6a + 6s}{(s+2a)(s-a)} \quad -(1)$$

consider,

$$\frac{6a+6s}{(s+2a)(s-a)} = \frac{A}{s+2a} + \frac{B}{s-a}.$$

$6a + 6s = (s-a)A + B(s+2a)$

put $s = a$

$12a = B3a$

$B = 4$

put $s = -2a$

$-6a = -3aA$

$2 = A$

then,

$$\frac{6a+6s}{(s+2a)(s-a)} = \frac{2}{s+2a} + \frac{4}{s-a}$$

$$Ly = \frac{2}{s+2a} + \frac{4}{s-a}$$

taking inverse laplace transform.

$$y = 2L^{-1}\frac{1}{s+2a} + 4L^{-1}\frac{1}{s-a}.$$

$$y = 2e^{-2at} + 4e^{at}$$ ans.

viii) $y'' - 4y' + 3y = 6t - 8 \quad y(0) = 0 \quad y'(0) = 0$

soln

$y'' - 4y' + 3y = 6t - 8$

taking laplace transform

$$Ly'' - 4Ly' + 3Ly = L6t - 8L8$$

$$s^2Ly - sy(0) - y'(0) - 4(sLy - y(0)) + 3Ly = \frac{6}{s^2} - \frac{8}{s}$$

$$s^2Ly - 4sLy + 3Ly = \frac{6-8s}{s^2}$$

$$Ly(s^2 - 4s + 3) = \frac{6-8s}{s^2}$$

$$Ly = \frac{6-8s}{s^2(s^2-4s+3)} = \frac{6-8s}{s^2(s-3)(s-1)}$$

consider

$$\frac{6-8s}{s^2(s-3)(s-1)} = \frac{As+B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1}$$

$$6 - 8s = (As+B)(s-3)(s-1) + Cs^2(s-1) + D(s-3)s^2$$

now,

put $s=1$

$-2 = -2D$

$D = 1$

put $s=0$

$6 = (A\cdot 0 + B)\cdot(-3)(-1) + 0 + 0$

$B = 2$

put $s=3$

$-18 = 0 + C\cdot 9\cdot 2 + 0$

$C = -1$

eqn coff of s,

$-4B + 3A = -8$

$-4\cdot 2 + 3A = 8$

$A = 0$

then

$$\frac{6-8s}{s^2(s-3)(s-1)} = \frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3}$$

then

$$Ly = \frac{2}{s^2} + \frac{1}{s-1} + \frac{1}{s-3}$$

$$y = 2L^{-1}\,1/s^2 + L^{-1}\,1/s-1 + L^{-1}\,1/s-3$$

$$y = 2t + e^{at} + e^{3t}.$$ ans.

$y'' + 2y' - 3y = 6e^{-2t}$ $y(0) = 2$ $y'(0) = -14$.

Given.

$y'' + 2y' - 3y = 6e^{-2t}$ $y(0) = 2$ and $y'(0) = -14$

taking laplace transform

$L(y'' + 2y' - 3y) = L(6e^{-2t})$

$Ly'' + 2Ly' - 3Ly = 6Le^{-2t}$

$s^2Ly - sy(0) - y'(0) + 2\{sLy - y(0)\} - 3Ly = \frac{6}{s+2}$

$s^2Ly - 2s + 14 + 2sLy - 2.2 - 3Ly = \frac{6}{s+2}$

$(s^2 + 2s - 3)Ly = 2s - 10 + \frac{6}{s+2}$

$(s^2 + 2s - 3)Ly = \frac{(2s-10)(s+2) + 6}{s+2}$

$Ly = \frac{2s^2 + 4s - 10s - 20 + 6}{(s+2)(s^2+2s-3)}$

$Ly = \frac{2s^2 - 6s - 14}{(s+2)(s+3)(s-1)}$ —— (iii)

let.

$\frac{2s^2 - 6s - 14}{(s+2)(s+3)(s-1)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+3}$

$2s^2 - 6s - 14 = A(s+2)(s+3) + B(s-1)(s+3) + C(s-1)(s+2)$

put $s = 1$ $-18 = 12A = -3/2$

$s = -2$ $6 = -3B = -2$

$s = -3$ $22 = 4C = 11/2$

$$\frac{2s^2-6s-14}{(s+2)(s+3)(s-1)} = \frac{-3}{2(s-1)} - \frac{2}{(s+2)} + \frac{11}{2(s+3)}$$

now, from (ii)

$$Ly = \frac{11}{2(s+3)} - \frac{3}{2(s-1)} - \frac{2}{s+2}$$

$$y = \frac{11}{2} L^{-1} \frac{1}{s+3} - \frac{3}{2} L^{-1} \frac{1}{s-1} - 2 L^{-1} \frac{1}{s+2}$$

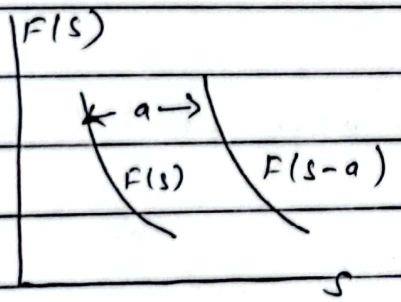
$$= \frac{11}{2} e^{-3t} - \frac{3}{2} e^{t} - 2e^{-2t}.$$

first shifting theorem for laplace transformatio

statement If $Lf(t) = F(s)$ then,

$$L\{e^{at} f(t)\} = F(s-a)$$

i.e replacing s by $s-a$ in the transform o the corresponds to multiplying the original function by e^{at}.



proof:-

let $f(t)$ be the continious function define for all $t>0$

then

$$Lf(t) = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$L e^{at} f(t) = \int_0^{\infty} e^{at} e^{-st} f(t)\, dt$$

$$= \int_0^{\infty} e^{(a-s)t} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a) \quad \text{pd}$$

Similarly

we can prove.

$L e^{-at} f(t) = F(s+a)$

Note

$$L e^{at} f(t) = \left[L f(t) \right]_{s \to s-a}.$$

1) $L e^{at} 1 = (L 1)_{s \to s-a}$

$$= \frac{1}{s}$$

$$= \frac{1}{s-a}.$$

2) $L e^{at} t = (L t)_{s \to s-a}$

$$= \left(\frac{1}{s^2} \right)_{s-a}$$

$$= \frac{1}{(s-a)^2}$$

3) $L\, e^{at}\sin\omega t = \left[(L\sin\omega t)\right]_{s\to s-a}$

$$= \left[\frac{\omega}{s^2+\omega^2}\right]_{s\to s-a}$$

$$= \frac{\omega}{(s-a)^2+\omega^2}$$

4) $L\, e^{at}\cos\omega t = \frac{(s-a)}{(s-a)^2+\omega^2}$

5) $L\, e^{at}\sinh\omega t = \frac{\omega}{(s-a)^2-\omega^2}$

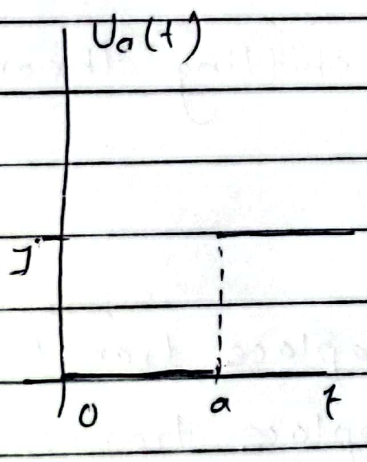
6) $L\, e^{at}\cosh\omega t = \frac{(s-a)}{(s-a)^2-\omega^2}$

Unit step function.

The unit step function is denoted by $U_a(t)$ and is defined by

$$U_a(t) = \left.\begin{array}{ll} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{array}\right\} \text{ when } a \geq 0.$$

$U_a(t)$ is also written as $u(t-a)$



laplace transformation of $U_a(t)$

soln let $f(t) = U_a(t)$

then

$$L\{U_a(t)\} = \int_0^\infty e^{-st}\, U_a(t)\, dt$$

$$= \int_0^a e^{-st}\, U_a(t)\, dt + \int_a^\infty e^{-st}\, U_a(t)\, dt$$

$$= 0 + \int_0^\infty e^{-st}\, U_a(t)\, dt$$

$$= \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \left[\frac{e^{-s\infty}}{-s}\right] - \left[\frac{e^{-sa}}{-s}\right]$$

$$= \frac{e^{-sa}}{s}$$

$$= \frac{e^{-as}}{s} \Rightarrow L^{-1}\left(\frac{e^{-as}}{s}\right) = U_a(t).$$

state and proof second shifting theorem for laplace transform

statement:

If $F(s) = $ · the laplace transform of $f(\cdot)$

then $e^{-as} F(s)$ be the laplace transform of $\overline{F(t)}$

where,

$\overline{F(t)} = 0$ for $t < a$

$f(t-a) :$ if $t > a$

if $F(s) = L f(t)$, then,

$e^{-as} F(s) = L\{\overline{F(t)}\}$.

$e^{-as} F(s) = L\, F(t-a)\, u_a(t)$

$L^{-1} e^{-as} F(s) = F(t-a)\, u_a(t)$

i.e

$$\begin{cases} e^{-as} F(s) = L f(t-a)\, U_a(t) \\ L^{-1} e^{-a} F(s) = h(t-a) U_a(t) \quad t > 0 \end{cases}$$

proof:

let $f(\tau)$ be the given function defined for all $\tau \geq 0$ then by def n.

$$F(\tau) : \int_0^{\infty} e^{-s\tau} f(\tau)\, d\tau$$

$$e^{-as} F(s) : \int_0^{\infty} e^{-as} e^{-s\tau} F(\tau)\, d\tau$$

$$= \int_0^{\infty} e^{-(a+\tau)s} F(\tau)\, d\tau$$

put $a + \tau = t \Rightarrow \tau = t - a$

$d\tau = dt$

$$e^{-as} F(s) = \int_0^{\infty} e^{-st} f(t-a)\, dt$$

$$\int_0^{\infty} e^{-st} f(t-a)\, U_a(t)\, dt$$

$$e^{-as} F(s) = L f(t-a) \cdot U_a(t)\}$$

$L^{-1} e^{-as} F(s) = f(t-a)\, U_a(t)$ proved #

CHAPTER 10

PARTIAL DIFFERENTIATION

Partial derivative:

Let $z = f(x, y)$ be a function of two variables x and y. Then the partial derivative of f w. r. t. x is defined as,

$$f_x = \frac{\partial f}{\partial x} = \lim_{n \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

And, the partial derivative of f w. r. to y is defined as,

$$f_y = \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Total derivative:

Let u be a function of x and y. Let $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous. Then the total derivative of u is defined as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Derivative of composite function:

Let $u = f(x, y)$ be a function. Also, let $x = \phi(t)$ and $y = \psi(t)$. Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Homogeneous function:

A function $f(x, y)$ is said to be homogeneous of degree n if it can be expressed in the form,

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right) \quad \text{or} \quad f(x, y) = y^n \psi\left(\frac{y}{x}\right)$$

Note: A function $f(x, y, z)$ is said to be homogeneous of degree n if it can be expressed as,

$$f = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad f = y^n \psi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad f = z^n r\left(\frac{x}{z}, \frac{y}{z}\right).$$

Homogeneous Function:

A function $f(x_1, x_2, \ldots, x_n)$ is called a homogeneous function of degree n if for all values of parameter t,

$$f(tx_1, tx_2, \ldots, tx_n) = t^n f(x_1, x_2, \ldots, x_n).$$

Euler's theorem on homogeneous function of two variables

[1999, 2001 Q. No. 2(b)] [2000 Q. No. 2(b) OR]

Statement: If $u = f(x, y)$ be a homogeneous function of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: Let f(x, y) be a homogeneous function of degree n.

So, $f(tx, ty) = t^n f(x, y)$, where t is independent of x, y for all values of t.

Set, tx = p, ty = q. Then,

$$x = \frac{\partial p}{\partial t}, \quad y = \frac{\partial q}{\partial t}$$

Here,

$$f(p, q) = f(tx, ty) = t^n f(x, y)$$

$$\Rightarrow f(p, q) = t^n f(x, y) \quad \text{......... (i)}$$

Differentiating (i) w. r. to t then,

$$\frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} = nt^{n-1} f(x, y).$$

$$\Rightarrow x\frac{\partial f}{\partial p} + y\frac{\partial f}{\partial q} = n\,t^{n-1} f(x, y)$$

$$\Rightarrow x\frac{\partial f(p, q)}{\partial p} + y\frac{\partial f(p, q)}{\partial q} = nt^{n-1} f(x, y)$$

Set t = 1 then p = 1.x = x, q = 1.y = y. Then,

$$x\frac{\partial}{\partial x} f(x, y) + y\frac{\partial}{\partial y} f(x, y) = n\,f(x, y)$$

$$\Rightarrow x \cdot \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Euler's Theorem on Homogeneous Function of three Variables

[2008 Fall Q. No. 2(a) OR] [2009 Spring Q. No. 2(a)]

Statement: If f(x, y, z) be a homogeneous function of degree n then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

Proof: Let f(x, y, z) be a homogeneous function of degree n.

So, $f(tx, ty, tz) = t^n f(x, y, z)$, t is independent of x, y and z for all values of t.

Set, tx = p, ty = q and tz = r. Then,

$$x = \frac{\partial p}{\partial t}, \quad y = \frac{\partial q}{\partial t}, \quad \text{and } z = \frac{\partial r}{\partial t}$$

Here,

$$f(p, q, r) = f(tx, ty, tz) = t^n f(x, y, z)$$

$$\Rightarrow f(p, q, r) = t^n f(x, y, z) \quad \text{......... (i)}$$

Differentiating (i) w. r. to t then,

$$\frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial t} = nt^{n-1} f(x, y, z).$$

$$\Rightarrow x\frac{\partial f}{\partial p} + y\frac{\partial f}{\partial q} + z\frac{\partial f}{\partial r} = n\,t^{n-1} f(x, y, z)$$

$$\Rightarrow x\frac{\partial f(p, q, r)}{\partial p} + y\frac{\partial f(p, q, r)}{\partial q} + z\frac{\partial f(p, q, r)}{\partial r} = nt^{n-1} f(x, y, z)$$

Set t = 1 then p = 1.x = x, q = 1.y = y, r = 1.z = z. Then,

$$x\frac{\partial}{\partial x} f(x, y, z) + y\frac{\partial}{\partial y} f(x, y, z) + z\frac{\partial}{\partial z} f(x, y, z) = n\,f(x, y, z)$$

$$\Rightarrow \quad x \cdot \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

Mixed Derivative Theorem For partial Differentiation

If f(x, y) be a function of two variables and the second order partial differentiation of exist then $f_{xy} = f_{yx}$

i.e., $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Exercise 10.1

1. Find the first partial derivatives of F.

(i) $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1$

Solution: Given that, $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1.$

Then, $\frac{\partial f}{\partial x} = 8x^3y^3 - y^2$ and $\frac{\partial f}{\partial y} = 6x^4y^2 - 2xy + 3.$

(ii) $f(x, y) = xe^y + y\sin x$

Solution: Given that, $f(x, y) = x\,e^y + y\sin x$

Then, $\frac{\partial f}{\partial x} = e^y + y\cos x$ and $\frac{\partial f}{\partial y} = xe^y + \sin x.$

(iii) $f(x, y) = x\cos\left(\frac{x}{y}\right)$

Solution: Given that, $f(x, y) = x\cos\left(\frac{x}{y}\right)$

Then, $$\frac{\partial f}{\partial x} = x\frac{\partial\left\{\cos\left(\frac{x}{y}\right)\right\}}{\partial\left(\frac{x}{y}\right)} \times \frac{\partial\left(\frac{x}{y}\right)}{\partial x} + \cos\left(\frac{x}{y}\right).1$$

$$= -\frac{x}{y}\sin\left(\frac{x}{y}\right) + \cos\left(\frac{x}{y}\right) = \cos\left(\frac{x}{y}\right) - \frac{x}{y}\sin\left(\frac{x}{y}\right).$$

and $$\frac{\partial f}{\partial y} = x\frac{\partial\left\{\cos\left(\frac{x}{y}\right)\right\}}{\partial\left(\frac{x}{y}\right)} \times \frac{\partial\left(\frac{x}{y}\right)}{\partial y} = \frac{x^2}{y^2}\left(\frac{x}{y}\right).$$

(iv) $f(x, y, z) = 3x^2z + xy^2$

Solution: Given that, $f(x, y, z) = 3x^2z + xy^2$

Then, $\frac{\partial f}{\partial x} = 6xz + y^2$ $\quad \frac{\partial f}{\partial y} = 2xy$ and $\frac{\partial f}{\partial z} = 3x^2$

(v) $f(r, s, t) = r^2 e^{2s}\cos t$

Solution: Given that, $f(r, s, t) = r^2 e^{2s}\cos t$

Then, $\frac{\partial f}{\partial r} = 2re^{2s}\cos t$ $\quad \frac{\partial f}{\partial s} = 2r^2e^{2s}\cos t$ and $\frac{\partial f}{\partial t} = -r^2e^{2s}\sin t.$

(vi) $f(x, y, z) = xe^z - ye^x + ze^{-y}$

Solution: Given that, $f(x, y, z) = xe^z - ye^x + ze^{-y}$

Then, $\frac{\partial f}{\partial x} = e^z - ye^x$ $\quad \frac{\partial f}{\partial y} = -e^x - ze^{-y}$ and $\quad \frac{\partial f}{\partial z} = xe^z + e^{-y}$

2. Verify that $u_{xy} = u_{yx}$

(i) $u = xy^4 - 2x^2y^3 - 4x^2 + 3x$

Solution: Given that, $u = xy^4 - 2x^2y^3 - 4x^2 + 3x.$

Differentiating,

$u_x = y^4 - 4xy^3 - 8x + 3$ and $u_y = 4xy^3 - 6x^2y^2$

$(u_x)_y = 4y^3 - 12xy^2$ $\quad (u_y)_x = 4y^3 - 12xy^2$

This shows that, $u_{xy} = u_{yx}$.

(ii) $u = x^3e^{-2y} + y^{-2}\cos x$

Solution: Given that, $u = x^3e^{-2y} + y^{-2}\cos x$

Differentiating,

$u_x = 3x^2e^{-2y} - y^{-2}\sin x$ and $u_y = -2x^3e^{-2y} - 2y^{-3}\cos x$

$u_{xy} = -6x^2e^{-2y} + 2y^{-3}\sin x$ $\quad u_{yx} = -6x^2e^{-2y} + 2y^{-3}\sin x$

This shows that, $u_{xy} = u_{yx}$

(iii) $u = \frac{x^2}{x + y}$

Solution: Given that, $u = \frac{x^2}{x + y}$

Differentiating,

$$u_x = \frac{(x + y)\,2x - x^2}{(x + y)^2} = \frac{2x^2 + 2xy - x^2}{(x + y)^2} = \frac{x^2 + 2xy}{(x + y)^2}$$

$$u_{xy} = \frac{(x + y)^2\,2x - (x^2 + 2xy)\,2\,(x + y).1}{(x + y)^4}$$

$$= \frac{(x + y)^2\,2x - (x^2 + 2xy)\,(2x + 2y)}{(x + y)^4}$$

$$= \frac{2x^3 + 4x^2y - 2x^3 - 6x^2y - 2xy^2 - 2xy^2}{(x + y)^4} = -\frac{(2x^2y + 2xy^2)}{(x + 4)^4} = -\frac{2xy}{(x + y)^3}$$

And,

$$u_y = \frac{(x + y) \times 0 - x^2.1}{(x + y)^2} = \frac{-x^2}{(x + y)^2}$$

$$u_{yx} = \frac{-\{(x + y)^2\,.\,2x - x^2\,2(x + y).1}{\{(x + y)^2\}^2}$$

$$= -\frac{(2x^3 + 4x^2y + 2xy^2 - 2x^3 - 2x^2y)}{(x + y)^4} = -\frac{(2x^2y + 2xy^2)}{(x + y)^4} = -\frac{2xy}{(x + 4)^4}$$

This shows that, $u_{xy} = u_{yx}$.

(iv) $u = y^2\,e^{x^2} + \frac{1}{x^2y^3}$

Solution: Here, $u = y^2\,e^{x^2} + \frac{1}{x^2y^3}$

So, $u_x = 2xe^{x^2}y^2 + \frac{-2x^{-3}}{y^3}$ and $u_y = 2y\,e^{x^2} + \frac{-3y^{-4}}{x^2}$

Also,

$$u_{xy} = 4xy\,e^{x^2} + 6x^{-3}y^{-4} \quad \text{and} \quad u_{yx} = 4xye^{x^2} + 6x^{-3}y^{-4}$$

Thus, $u_{xy} = u_{yx}$.

(v) $u = \sqrt{x^2 + y^2 + z^2}$

Solution: Given that, $u = \sqrt{x^2 + y^2 + z^2}$

Differentiating,

$$u_x = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$u_{xy} = \frac{\sqrt{x^2+y^2+z^2} \times 0 - x\dfrac{1}{2\sqrt{x^2+y^2+z^2}} \times 2y}{(x^2+y^2+z^2)} = -\frac{xy}{(x^2+y^2+z^2)^{3/2}}$$

And,

$$u_y = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2y = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$u_{yx} = \frac{\sqrt{x^2+y^2+z^2} \times 0 - y\dfrac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x}{(x^2+y^2+z^2)} = -\frac{xy}{(x^2+y^2+z^2)^{3/2}}$$

This shows that, $u_{xy} = u_{yx}$.

(vi) $u = 3x^2y^3z + 2xy^4z^2 - yz$

Solution: Given that, $u = 3x^2y^3z + 2xy^4z^2 - yz$

Differentiating,

$$u_x = 6xy^3z + 2y^4z^2 \quad \text{and} \quad u_y = 9x^2y^2z + 8xy^3z^2 - z$$

$$u_{xy} = 18xy^2z + 8y^3z^2 \qquad u_{yx} = 18xy^2z + 8y^3z^2$$

This shows that, $u_{xy} = u_{yx}$.

(vii) $u = \sin^{-1}\left(\frac{y}{x}\right)$

Solution: Given that, $u = \sin^{-1}\left(\frac{y}{x}\right)$

Differentiating,

$$u_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \times -\frac{y}{x^2} = -\frac{y}{x^2\dfrac{\sqrt{x^2-y^2}}{x}} = -\frac{y}{x\sqrt{x^2-y^2}} = -\frac{y}{\sqrt{x^4-x^2y^2}}$$

$$u_{xy} = -\frac{\left\{x\sqrt{x^2-y^2}\,.\,1 - y\dfrac{1}{2\sqrt{x^4-x^2y^2}} \times -2x^2y\right\}}{(x^4-x^2y^2)}$$

$$= -\frac{x^2(x^2-y^2)+x^2y^2}{(x^4-x^2y^2)^{3/2}} = -\frac{x^4}{(x^4-x^2y^2)^{3/2}} = \frac{x}{(x^2-y^2)^{3/2}}$$

and,

$$u_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \times \frac{1}{x} = \frac{x}{\sqrt{x^2-y^2}} \times \frac{1}{x} = \frac{1}{\sqrt{x^2-y^2}}$$

$$u_{yx} = \frac{\sqrt{x^2-y^2}\,.\,d(1) - \frac{1}{2\sqrt{x^2-y^2}} \times 2x}{(x^2-y^2)} = -\frac{x}{(x^2-y^2)^{3/2}}$$

This shows that, $u_{xy} = u_{yx}$.

(viii) $u = \log(x)\tan^{-1}(x^2+y^2)$

Solution: Given that, $u = \log(x)\tan^{-1}(x^2+y^2)$

Differentiating,

$$u_x = \tan^{-1}(x^2+y^2)\frac{1}{x} + \log x\,\frac{1}{1+(x^2+y^2)^2} \times 2x$$

$$= \frac{\tan^{-1}(x^2+y^2)}{x} + \frac{2x\log x}{1+(x^2+y^2)^2}$$

$$u_{xy} = \frac{x\,\frac{1}{1+(x^2+y^2)^2} \times 2y - \tan^{-1}(x^2+y^2) \times 0}{x^2} + \frac{\{1+(x^2+y^2)^2\}\,\{(2\log x + 2)\}\quad 2x\log x \times 2(x^2+y^2) \times 2y}{\{1+(x^2+y^2)^2\}^2}$$

$$= \frac{2xy}{x^2\{1+(x^2+y^2)^2\}} + \frac{2(1+\log x)\,\{1+(x^2+y^2)^2\} - 8xy\log x\,(x^2+y^2)}{\{1+(x^2+y^2)^2\}^2}$$

and,

$$u_y = \log x\,\frac{1}{1+(x^2+y^2)^2} \times 2y = \frac{2y\log x}{1+(x^2+y^2)^2}$$

$$u_{yx} = \frac{\{1+(x^2+y^2)^2\}\,2\frac{y}{x} - 2y\log x\,2(x^2+y^2) \times 2x}{\{1+(x^2+y^2)^2\}^2}$$

$$= \frac{\frac{2y}{x}\{1+(x^2+y^2)^2\} - 8xy\log x\,(x^2+y^2)}{\{1+(x^2+y^2)^2\}^2}$$

This shows that, $u_{xy} = u_{yx}$.

(ix) $u = ax^2 + 2hxy + by^2$

Solution: Given that, $u = ax^2 + 2hxy + by^2$

Differentiating,

$u_x = 2ax + 2hy$ and $u_y = 2hx + 2by$

$u_{xy} = 2h$ $\qquad$ $u_{yx} = 2h$

This shows that, $u_{xy} = u_{yx}$.

(x) $u = \tan^{-1}\left(\frac{y}{x}\right)$

Solution: Given that, $u = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiating,

$$u_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2} = -\frac{y}{x^2} \times \frac{x^2}{(x^2+y^2)} = -\frac{y}{(x^2+y^2)}$$

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$$u_{xy} = -\frac{[(x^2+y^2).1 - y.2y]}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

And,

$$u_y = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$u_{yx} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xi) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

Solution: Given that, $u = \log\left(\frac{x^2+y^2}{xy}\right)$.

Differentiating,

$$u_x = \frac{1}{\frac{x^2+y^2}{xy}} \times \frac{xy.2x - (x^2+y^2).y}{x^2y^2}$$

$$= \frac{2x^2y - x^2y - y^3}{xy(x^2+y^2)} = \frac{x^2y-y^3}{xy(x^2+y^2)} = \frac{y(x^2-y^2)}{xy(x^2+y^2)} = \frac{(x^2-y^2)}{(x^3+xy^2)}$$

$$u_{xy} = \frac{(x^3+xy^2)\times -2y - (x^2-y^2)\times 2xy}{(x^3+xy^2)^2} = \frac{-2x^3y - 2xy^3 - 2x^3y + 2xy^3}{(x^3+xy^2)^2}$$

$$= -\frac{4x^3y}{x^2(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

And,

$$u_y = \frac{1}{\frac{x^2+y^2}{xy}} \times \frac{xy.2y - (x^2+y^2).x}{x^2y^2} = \frac{2xy^2 - xy^2}{xy(x^2+y^2)} = \frac{xy^2 - x^3}{xy(x^2+y^2)}$$

$$= \frac{x(y^2-x^2)}{xy(x^2+y^2)} = \frac{y^2-x^2}{y(x^2+y^2)}$$

$$u_{yx} = \frac{y(x^2+y^2)\times -2x - (y^2-x^2)\,2xy}{\{y(x^2+y^2)\}^2}$$

$$= \frac{-2x^3y - 2xy^3 - 2xy^3 + 2x^3y}{y^2(x^2+y^2)^2} = \frac{-4xy^3}{y^2(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xii) $u = e^{ax}.\sin by$

Solution: Given that, $u = e^{ax}.\sin by$

Differentiating,

$u_x = ae^{ax}\sin by$ and $u_y = be^{ax}\cos by$

$u_{xy} = abe^{ax}\cos by$ $\quad$ $u_{yx} = abe^{ax}.\cos by$

This shows that, $u_{xy} = u_{yx}$.

(xiii) $u = \log(x \sin y + y \sin x)$

Solution: Given that, $u = \log(x \sin y + y \sin x)$

Differentiating,

$$u_x = \frac{1}{(x \sin y + y \sin x)}(\sin y + y\cos x)$$

$$u_{xy} = \frac{(x \sin y + y \sin x)(\cos y + \cos x) - (\sin y + y\cos x)(\cos y + \sin x)}{(x\sin y + y\sin x)^2}$$

$$= \frac{(x\sin y \cos y + x\sin y \cos x + y\sin x \cos y + y\sin x.\cos x) - (x\sin y.\cos y + \sin x.\sin y + xy\cos x.\cos y + y\sin x.\cos x)}{(x\sin y + y\sin x)^2}$$

$$= \frac{x\sin y \cos y + x\sin y \cos x + y\sin x \cos y + y\sin x \cos x - x\sin y \cos y - \sin x \sin y - xy\cos x \cos y - y\sin x.\cos x}{(x\sin y + y\sin x)^2}$$

$$= \frac{x\sin y \cos x + y\sin x \cos y - \sin x \sin y - xy\cos x \cos y}{(x\sin y + y\sin x)^2}$$

And,

$$u_y = \frac{1}{(x\sin y + y\sin x)}(x\cos y + \sin x)$$

$$u_{yx} = \frac{(x\sin y + y\sin x)(\cos y + \cos x) - (x\cos y + \sin x)(\sin y + y\cos x)}{(x\sin y + y\sin x)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xiv) $u = e^{x^2} + xy + y^2$

Solution: Given that, $u = e^{x^2} + xy + y^2$

Differentiating,

$u_x = 2xe^{x^2} + y$ and $u_y = x + 2y$

$u_{xy} = 1$ $u_{yx} = 1$

This shows that, $u_{xy} = u_{yx}$.

3. **If $u = x^2 + y^2 + z^2$ show that $xu_x + yu_y + zu_z = 2u$.**

Solution: Given that, $u(x, y, z) = x^2 + y^2 + z^2$

Set, $x = tx$, $y = ty$ and $z = tz$

Then $u(tx, ty, tz) = x^2 + y^2 + z^2 = t^2(x^2 + y^2 + z^2) = t^2\, u(x, y, z)$

This means u is homogeneous of degree 2. Then by Euler's theorem,

$$xu_x + yu_y + zu_z = 2u.$$

4. **If $u = x^2y + y^2z + z^2x$ show that $u_x + u_y + u_z = (x + y + z)^2$.**

Solution: Given that, $u = x^2y + y^2z + z^2x$

Differentiating,

$u_x = 2xy + z^2$ $u_y = x^2 + 2yz$ $u_z = y^2 + 2zx$

Now,

$$\begin{aligned} u_x + u_y + u_z &= 2xy + z^2 + x^2 + 2yz + y^2 + 2zx \\ &= (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) \\ &= (x + y + z)^2 \end{aligned}$$

Thus, $u_x + u_y + u_z = (x + y + z)^2$.

5. **If $F(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$ show that $x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} + z\dfrac{\partial F}{\partial z} = 0$.**

[2004 Fall Q. No. 2(b)]

Solution: Given that, $F(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$

Set, $x = tx$, $y = ty$ and $z = tz$

Then $\quad F(tx, ty, tz) = e^{tx/ty} + e^{ty/tz} + e^{tz/tx}$

$$= e^{x/y} + e^{y/z} + e^{z/x}$$

$$= F(x, y, z)$$

This means F is homogeneous of degree 0. Then by Euler's theorem,

$$x.\frac{\partial F}{\partial x} + y.\frac{\partial F}{\partial y} + z.\frac{\partial F}{\partial z} = 0.F = 0.$$

6. If $V = (\sqrt{x^2+y^2+z^2})$ show that $V_{xx} + V_{yy} + V_{zz} = \frac{2}{V}$.[2013 Spring Q.No. 2. (a)]

Solution: Given that, $V = (\sqrt{x^2+y^2+z^2})$

Differentiating,

$$V_x = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$V_{xx} = \frac{(x^2+y^2+z^2)^{1/2}.1 - x\frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x}{(x^2+y^2+z^2)} = \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2}{\sqrt{x^2+y^2+z^2}^{3/2}}$$

Since V is symmetrical in x, y and z. So, similarly,

$$V_{yy} = \frac{x^2+z^2}{\sqrt{(x^2+y^2+z^2)^{3/2}}} \quad \text{And} \quad V_{zz} = \frac{x^2+y^2}{\sqrt{(x^2+y^2+z^2)^{3/2}}}$$

Now,

$$V_{xx} + V_{yy} + V_{zz} = \frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2}{V}$$

Thus, $V_{xx} + V_{yy} + V_{zz} = \frac{2}{V}$.

(7) If $u = \log(\sqrt{x^2+y^2+z^2})$ show that: $(x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y2} + \frac{\partial^2 u}{\partial z^2}\right] = 1.$

[2007 Fall Q. No. 2(b)]

Solution: Given that, $u = \log(\sqrt{x^2+y^2+z^2})$

Differentiating,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \times \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{(x^2+y^2+z^2)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2).1 - x(2x)}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Since V is symmetrical in x, y and z. So, similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2} \quad \text{And} \quad \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

Now,

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$=(x^2+y^2+z^2)\left(\frac{y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2}\right)$$

$$=\frac{(x^2+y^2+z^2)^2}{(x^2+y^2+z^2)^2}=1.$$

Thus, $(x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y2}+\frac{\partial^2 u}{\partial z^2}\right]=1.$

(8) If $V = F(x, y, z)$, show that $x^2 V_{xx} = y^2 V_{yy} = z^2 V_{zz}$.

Solution: Given that, $V = F(x\ y\ z)$

Differentiating V w. r. t. x,

$$V_x = F'(x\ y\ z)\ yz \qquad \text{And} \qquad V_{xx} = F''(x\ y\ z)\ y^2.\ z^2$$

Multiplying by x^2 both sides,

$$x^2 V_{xx} = F''(xyz)\ x^2y^2z^2 \qquad ....(i)$$

Next, differentiating V w. r. t. y,

$$V_y = F'(xyz).\ xz \qquad \text{and} \qquad V_{yy} = F''(xyz).\ x^2 z^2$$

Multiplying by y^2

$$y^2 V_{yy} = F''(xyz)\ x^2y^2z^2 \qquad ....(ii)$$

Also, differentiating V w. r. t. z,

$$V_z = F'(xyz).\ xy \qquad \text{and} \qquad V_{zz} = F''(xyz).\ x^2y^2$$

Multiplying by z^2

$$z^2 V_{zz} = F''(xyz)\ x^2y^2z^2 \qquad ....(iii)$$

Now, from (i), (ii) & (iii), we get,

$$x^2V_{xx} = y^2 V_{yy} = z^2 V_{zz}.$$

(9) If $x = r\cos\theta$, $y = r\sin\theta$, show that

(i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$ **(ii)** $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x.\ \partial y}\right)^2.$

Solution: Let, $x = r\cos\theta$ and $y = r\sin\theta$.

Then, $x^2 + y^2 = r^2$

Differentiating partially w. r. t. 'x' then

$$2x = 2r.\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.$$

Again differentiating partially w. r. t. x, then,

$$\frac{\partial^2 r}{\partial x^2}=\frac{r.1-x.\frac{\partial r}{\partial x}}{r^2}=\frac{r-x.\frac{x}{r}}{r^2} \qquad \left[\text{using } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{r^2-x^2}{r^3}=\frac{r^2(1 \square \cos^2\theta)}{r^3} \qquad [\text{Being } x = r\cos\theta]$$

$$=\frac{r^2(\sin^2\theta)}{r^3}=\frac{y^2}{r^3} \qquad [\text{Being } y = r\sin\theta]$$

Thus, $\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

Also, differentiating $\frac{\partial r}{\partial x}$ partially w. r. t. y, then,

$$\frac{\partial^2 r}{\partial x . \partial y} = \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = \left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} = -x . \frac{1}{r^2} \times \frac{y}{r} = \frac{-xy}{r^3}$$

Now,

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

$\Rightarrow \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{1}{r}\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right]$

$\Rightarrow \frac{x^2 + y^2}{r^3} = \frac{x^2 + y^2}{r^3}$

This proves (i).

(ii) $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x . \partial y}\right)^2$

$\Rightarrow \frac{y^2}{r^3} . \frac{x^2}{r^3} = \left(-\frac{xy}{r^3}\right)^2$

$\Rightarrow \frac{x^2y^2}{r^6} = \frac{x^2y^2}{r^6}$

This proves (ii).

(10) If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution: Given that, $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

Set, $x = tx$, $y = ty$ and $z = tz$

Then $u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx} = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = u(x, y, z)$

This means u is homogeneous of degree 0. Then by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.u = 0.$$

(11) If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$. Show that: $u_x + u_y + u_z = 0$.

Solution: Let, $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = x^2(y - z) - y^2(x - z) + z^2(x - y)$

$= x^2y - x^2z - xy^2 + y^2z + xz^2 - yz^2$

Differentiating we get,

$u_x = 2xy - 2xz - y^2 + z^2$

$u_y = x^2 - 2xy + 2yz - z^2$

and, $u_z = -x^2 + y^2 + 2xz - 2yz$

Now,

$u_x + u_y + u_z = 2xy - 2xz - y^2 + z^2 + x^2 - 2xy + 2yz - z^2 - x^2 + y^2 + 2xz - 2yz$
$= 0.$

Thus, $u_x + u_y + u_z = 0$.

(12) If $u = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$, $t = \frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = \log(e^x + e^y)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{e^x + e^y} e^x = \frac{e^x}{e^x + e^y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(e^x + e^y)\,e^x - e^x . e^x}{(e^x + e^y)^2} = \frac{e^{2x} + e^{xy} - e^{2x}}{(e^x + e^y)^2} = \frac{e^{xy}}{(e^x + e^y)^2}$$

And,

$$\frac{\partial u}{\partial y} = \frac{1}{e^x + e^y} e^y = \frac{e^y}{e^x + e^y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(e^x + e^y)\,e^y - e^y . e^y}{(e^x + e^y)^2} = \frac{e^{xy}}{(e^x + e^y)^2}$$

Also, $$\frac{\partial^2 u}{\partial x . \partial y} = \frac{(e^x + e^y)\,0 - e^y . e^x}{(e^x + e^y)^2} = -\frac{e^{xy}}{(e^x + e^y)^2}$$

Now,

$$rt - s^2 = \frac{\partial^2 u}{\partial x^2} . \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x . \partial y}\right)^2 = \frac{e^{xy}}{(e^x + e^y)^2} . \frac{e^{xy}}{(e^x + e^y)^2} - \left(\frac{e^{xy}}{(e^x + e^y)^2}\right)^2$$

$$= \frac{e^{x^2 . y^2}}{\{(e^x + e^y)\}^2} - \frac{e^{x^2 . y^2}}{\{(e^x + e^y)\}^2} = 0$$

Thus, $rt - s^2 = 0$.

(13) If $u = \tan^{-1}\frac{(xy)}{\sqrt{1 + x^2 + y^2}}$. Then show that $\frac{\partial^2 u}{\partial x . \partial y} = (1 + x^2 + y^2)^{-3/2}$.

Solution: Let, $u = \tan^{-1}\frac{(xy)}{\sqrt{1 + x^2 + y^2}}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{(xy)}{\sqrt{1 + x^2 + y^2}}\right)^2} \times \frac{(1 + x^2 + y^2)^{1/2} . y - xy\frac{1}{2\sqrt{1 + x^2 + y^2}} \times 2x}{(1 + x^2 + y^2)}$$

$$= \frac{(1 + x^2 + y^2)}{1 + x^2 + y^2 + x^2y^2} \times \frac{(1 + x^2 + y^2)y - x^2y}{(1 + x^2 + y^2)^{3/2}}$$

$$= \frac{(1 + x^2 + y^2)\,(y + x^2y + y^3 - x^2y)}{(1 + x^2 + y^2 + x^2y^2)\,(1 + x^2 + y^2)^{3/2}}$$

$$= \frac{(y + y^3)}{(1 + x^2 + y^2 + x^2y^2)\,(1 + x^2 + y^2)^{1/2}}$$

$$= \frac{y(1 + y^2)}{(1 + y^2)\,(1 + x^2)\,(1 + x^2 + y^2)^{1/2}} = \frac{y}{(1 + x^2)\,(1 + x^2 + y^2)^{1/2}}$$

And, $$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2)}\left\{\frac{(1 + x^2 + y^2)^{1/2} . 1 - y\frac{1}{2\sqrt{1 + x^2 + y^2}} \times 2y}{(1 + x^2 + y^2)}\right\}$$

$$= \frac{1}{(1+x^2)}\left\{\frac{1+x^2+y^2-y^2}{(1+x^2+y^2)^{3/2}}\right\} = \frac{(1+x^2)}{(1+x^2)(1+x^2+y^2)^{3/2}}$$

$$= (1+x^2+y^2)^{-3/2}$$

Thus, $\frac{\partial^2 u}{\partial x.\,\partial y} = (1+x^2+y^2)^{-3/2}$.

(14) If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$.

Solution: Let, $u = e^{xyz}$

Differentiating we get,

$$\frac{\partial u}{\partial z} = e^{xyz}.\,xy$$

And, $\frac{\partial^2 u}{\partial y \partial z} = xy\,(e^{xyz}.\,xz) + e^{xyz}.\,x = x^2yz\,e^{xyz} + z\,e^{xyz}$

Also, $\frac{\partial^2 u}{\partial x \partial y \partial z} = yz\,(2x\,e^{xyz} + x^2\,e^{xyz}\,yz) + e^{xyz}.xyz + e^{xyz}.1$

$$= 2xyz\,e^{xyz} + x^2y^2z^2\,e^{xyz} + xyze^{xyz} + e^{xyz}$$

$$= e^{xyz}\,(2xyz + x^2y^2z^2 + xyz + 1) = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$$

Thus, $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$.

(15) If $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, show that: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \left(\frac{1}{x^2+y^2}\right)2x + \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2+y^2} + \frac{-y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).\,2 - (2x-y).\,2x}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} = \frac{2y^2+2xy-2x^2}{(x^2+y^2)^2}$$

Also,

$$\frac{\partial u}{\partial y} = \left(\frac{1}{x^2+y^2}\right)2y + \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\frac{1}{x} = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+2y}{x^2+y^2}$$

And,

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2).\,2 - (x+2y).\,2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-2xy-4y^2}{(x^2+y^2)^2} = \frac{2x^2-2xy-2y^2}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2 + 2xy - 2x^2 + 2x^2 - 2xy - 2y^2}{(x^2+y^2)^2} = 0.$$

Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

(16) If $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Solution: Let, $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \times \frac{(x^2-y^2)\,2y - 2xy\,.\,2x}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2} \times \frac{2x^2y - 2y^3 - 4x^2y}{(x^2-y^2)^2} = \frac{-2y^3 - 2x^2y}{(x^2-y^2)^2+4x^2y^2}$$

$$= \frac{-2y\,(x^2+y^2)}{(x^2+y^2)^2} = \frac{-2y}{x^2+y^2}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = -\left(\frac{(x^2+y^2)\,.\,0 - 2y\,.\,2x}{(x^2+y^2)^2}\right) = \frac{4xy}{(x^2+y^2)^2}$$

Again,

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \times \frac{(x^2-y^2)\,2y - 2x - 2xy.\,x - 2y}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2+y^2)^2} \times \frac{2x^3 - 2xy^2 + 4xy^2}{(x^2-y^2)^2} = \frac{2x^3 + 2xy^2}{(x^2+y^2)^2} = \frac{2x\,(x^2+y^2)}{(x^2+y^2)^2} = \frac{2x}{x^2+y^2}$$

Also,

$$\frac{\partial^2 u}{\partial y^2} = -\left(\frac{(x^2+y^2)\,.\,0 - 2x\,.\,2y}{(x^2+y^2)^2}\right) = -\frac{4xy}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2} = 0$$

Thus, this proves the statement.

(17) If $u = \tan(y + ax) - (y - ax)^{3/2}$ show that $\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = \tan(y + ax) - (y - ax)^{3/2}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \sec^2(y+ax) \times a - \frac{3}{2}(y - ax)^{1/2}(-a)$$

$$= \frac{2a\sec^2(y+ax) + 3a(y-ax)^2}{2}$$

And, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[2a.\sec(y+ax).\sec(y+ax)\tan(y+ax).a + 3a.\frac{1}{2\sqrt{y-ax}} \times -a\right]$

Also,

$$\frac{\partial f}{\partial y} = \sec^2(y+ax).\,1 - \frac{3}{2}(y-ax)^{1/2}.\,1$$

And,

$$\frac{\partial^2 u}{\partial y^2} = 2\sec(y+ax).\sec(y+ax).\tan(y+ax).\,1 - \frac{3}{2}\times\frac{1}{2}(y-ax)^{-1/2}$$

$$= 2\sec^2(y+ax).\tan(y+ax) - \frac{3}{4}(y-ax)^{-1/2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} = a^2\left[2\sec^2(y+ax).\tan(y+ax) - \frac{3}{4}(y-ax)^{-1/2}\right]$$

$$= a^2\frac{\partial^2 u}{\partial y^2}$$

This proves the requirement.

(18) If $u = \log(x^2+y^2+z^2)$ show that $x\frac{\partial^2 u}{\partial x.\,\partial z} = y\frac{\partial^2 u}{\partial z.\,\partial x} = z\frac{\partial^2 u}{\partial x.\,\partial y}$.

Solution: Let, $u = \log(x^2+y^2+z^2)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{(x^2+y^2+z^2)}\times 2x = \frac{2x}{(x^2+y^2+z^2)}$$

And,

$$\frac{\partial^2 u}{\partial x.\,\partial y} = \frac{(x^2+y^2+z^2).0 - 2x.2y}{(x^2+y^2+z^2)^2} = \frac{-4xy}{(x^2+y^2+z^2)^2}$$

Then,

$$z\frac{\partial^2 u}{\partial x.\,\partial y} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \qquad \text{.......(i)}$$

Also,

$$\frac{\partial^2 u}{\partial x.\,\partial z} = \frac{(x^2+y^2+z^2) - 2x.2z}{(x^2+y^2+z^2)^2} = \frac{-4xz}{(x^2+y^2+z^2)^2}$$

Then,

$$y\frac{\partial^2 u}{\partial x.\,\partial z} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \qquad \text{.......(ii)}$$

Next,

$$\frac{\partial u}{\partial y} = \frac{1}{(x^2+y^2+z^2)}\times 2y - \frac{2y}{(x^2+y^2+z^2)}$$

And,

$$\frac{\partial^2 u}{\partial y.\,\partial z} = \frac{(x^2+y^2+z^2).0 - 2y.2z}{(x^2+y^2+z^2)^2} = \frac{-4yz}{(x^2+y^2+z^2)^2}$$

Then,

$$x\frac{\partial^2 u}{\partial y.\,\partial z} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \qquad \text{.......(iii)}$$

Now, from (i), (ii) and (iii), we observe,

$$x\frac{\partial^2 u}{\partial y.\,\partial z} = y\frac{\partial^2 u}{\partial x.\,\partial z} = z\frac{\partial^2 u}{\partial x.\,\partial y}.$$

(19) If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2\left(\tan^{-1}\frac{x}{y}\right)$, show that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution: Let, $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2\left(\tan^{-1}\frac{x}{y}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = 2x\,.\tan^{-1}\left(\frac{y}{x}\right) + x^2\frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right) - y^2\frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{y}$$

$$= 2x\,.\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

And, $$\frac{\partial^2 u}{\partial x\,\partial y} = \left(2x\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\frac{1}{x}\right) - \left(\frac{(x^2+y^2)\,x^2 - x^2y.\,2y}{(x^2+y^2)^2}\right) - \left(\frac{(x^2+y^2)\,3.y^2 - y^3\,(2y)}{(x^2+y^2)^2}\right)$$

$$= \frac{2x^2}{x^2+y^2} - \frac{x^4 + x^2y^2 - 2x^2y^2}{(x^2+y^2)^2} - \frac{3x^2y^2 + 3y^4 - 2y^4}{(x^2+y^2)^2}$$

$$= \frac{2x^2\,(x^2+y^2) - x^4 + x^2y^2 - 3x^2y^2 - y^4}{(x^2+y^2)^2}$$

$$= \frac{2x^4 + 2x^2y^2 - x^4 - 2x^2y^2 - y^4}{(x^2+y^2)^2}$$

$$= \frac{x^4 - y^4}{(x^2+y^2)^2} = \frac{(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^2} = \frac{x^2-y^2}{x^2+y^2}$$

Thus, $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2-y^2}{x^2+y^2}$.

(20) If $u = \sqrt{x^2+y^2+z^2}$, show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$ (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$

Solution: Let, $u = \sqrt{x^2+y^2+z^2}$

Differentiating w. r. t. x we get,

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)^{1/2}\,.\,1 - x\,.\frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x}{(x^2+y^2+z^2)}$$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}}$$

Similarly, differentiating w. r. t. y and z then we get,

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

(i)

Now,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1.$$

Thus, $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1.$

(ii)

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{2}{\sqrt{x^2+y^2+z^2}}=\frac{2}{u}$$

Thus, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$.

(21) If $u=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}$, show that $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$.

Solution: Let,

$$u=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}$$

Then,

$$\frac{\partial u}{\partial t}=-\frac{1}{2}t^{-3/2}\,e^{\left(-\frac{x^2}{4a^2t}\right)}+\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{x^2}{4a^2t^2}\right)$$

And,

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)$$

Also,

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\sqrt{t}}\,e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)^2+\frac{1}{\sqrt{t}}\,e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)$$

So,

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{x^2}{4a^2t^2}\right)-\frac{1}{2}t^{-3/2}\,e^{\left(-\frac{x^2}{4a^2t}\right)}=\frac{\partial u}{\partial t}$$

$$\Rightarrow\ \frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

(22) If $u = (1 - 2xy + y^2)^{-1/2}$, show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$

Solution: Let, $u = (1 - 2xy + y^2)^{-1/2}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \times (-2y) = y(1 - 2xy + y^2)^{-3/2}$$

Then,

$$x.\frac{\partial u}{\partial x} = xy\,(1 - 2xy + y^2)^{-3/2} \quad ....(i)$$

Next,

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \times (-2x + 2y)$$

Then,

$$y\frac{\partial u}{\partial y} = (xy - y^2)\,(1 - 2xy + y^2)^{-3/2} \quad ....(ii)$$

Now,

$$x.\frac{\partial u}{\partial x} - y.\frac{\partial u}{\partial y} = \frac{xy}{(1 - 2xy + y^2)^{3/2}} - \frac{xy - y^2}{(1 - 2xy + y^2)^{3/2}}$$

$$= \frac{xy - xy + y^2}{(1 - 2xy + y^2)^{3/2}}$$

$$= y^2\,((1 - 2xy + y^2)^{-1/2})^3 = y^2 \times u^3$$

Thus, $x.\frac{\partial u}{\partial x} - y.\frac{\partial u}{\partial y} = y^2 \times u^3$.

(23) If $u = e^x\,(x \cos y - y \sin y)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = e^x\,(x \cos y - y \sin y)$.

Differentiating we get,

$$\frac{\partial u}{\partial x} = \cos y\,(e^x + xe^x) - e^x y \sin y$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \cos y(e^x + e^x + xe^x) - e^x y \sin y$$

$$= 2e^x \cos y + xe^x \cos y - e^x y \sin y \quad .... (i)$$

Also,

$$\frac{\partial u}{\partial y} = -\,xe^x \sin y - e^x\,(\sin y + y\cos y)$$

And, $$\frac{\partial^2 u}{\partial y^2} = -\,xe^x \cos y - e^x(\cos y + \cos y - y\sin y)$$

$$= -xe^x \cos y - 2e^x \cos y + e^x y \sin y \quad ....(ii)$$

From (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^x\cos y + xe + x\cos y - e^x y\sin y - xe^x \cos y - 2e^x\cos y + e^x y\sin y = 0$$

Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(24) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$. Show that: $\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

Solution: Let, $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz) = \frac{3x^2-3yz}{(x^3+y^3+z^3-3xyz)}$$

And,

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz) = \frac{3y^2-3xz}{(x^3+y^3+z^3-3xyz)}$$

Also,

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy) = \frac{3z^2-3xy}{(x^3+y^3+z^3-3xyz)}$$

Now,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2-3yz-3xz+3z^2-3xy}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2-z^2-xy-xz-yz)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2-xy-xz-yz)}{(x+y+z)(x^2+y^2+z^2-xy-xz-yz)} = \frac{3}{(x+y+z)}.$$

Then, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}$.

(25) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$.

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Solution: Let, $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3+z^3-3xyz)} \times (3x^2-3yz) = \frac{3x^2-3yz}{(x^3+y^3+y^3-3xyz)}$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^3+y^3+z^3-3xyz)(6x)-(3x^2-3yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{6x^4+6xy^3+6xz^3-18x^2yz-(9x^4-9x^2yz-9x^2yz+9y^2z^2)}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{6x^4+xy^3+6xz^3-18x^2yz-9x^4+18x^2yz-9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{-3x^4+6xy^3+6xz^3-9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

Similarly,

$$\frac{\partial u}{\partial x} = \frac{3y^2-3xz}{(x^3+y^3+z^3-3xyz)} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{6x^3y-3y^4+6yz^3-9x^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

Also, $$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{(x^3+y^3+z^3-3xyz)} \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = \frac{6x^3z+6y^3z-3z^4-9x^2y^2}{(x^3+y^3+z^3-3xyz)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{-3x^4+6xy^3+6xz^3-9y^2z^2+6x^3y-3y^4+6yz^3-9x^2z^2+6x^3z+6y^3z-3z^4-9x^2y^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{-3(x^4+y^4+z^4)-9(x^2y^2+y^2z^2+x^2z^2)+6(xy^3+xz^3+x^3y+yz^3+x^4z+y^3z)}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{-3}{(x+y+z)^2}$$

Thus, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{-3}{(x+y+z)^2}$

(26) **If $x = r\cos\theta$, $y = r\sin\theta$ so that $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ prove that:**

(i) $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$ **(ii)** $\frac{1}{r}\frac{\partial r}{\partial \theta}=r\cdot\frac{\partial \theta}{\partial r}$ **(iii)** $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0.$

Solution: Let, $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$.

(i) Since $r^2 = x^2 + y^2$. So, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$

Also, $x = r\cos\theta$. So, $\frac{\partial x}{\partial r}=\cos\theta=\frac{x}{r}$

Thus, $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$.

(ii) Since we know that,

$$\frac{\partial r}{\partial \theta}=\frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

Also, $\frac{\partial r}{\partial \theta}=\frac{\partial \theta}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial \theta}{\partial y}\cdot\frac{\partial y}{\partial r}$

Since, $r^2 = x^2 + y^2$.

So, $2r\cdot\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$ and $2r\cdot\frac{\partial r}{\partial y}=2y \Rightarrow \frac{\partial r}{\partial y}=\frac{y}{r}$

Also, $x = r\cos\theta$ and $y = r\sin\theta$

So, $\frac{\partial x}{\partial \theta}=r(-\sin\theta)=-y$ So, $\frac{\partial y}{\partial \theta}=r\cos\theta=x$

Moreover, we have, $\theta=\tan^{-1}\left(\frac{y}{x}\right)$. So,

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=\frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}=-\frac{y}{r^2}$$

and

$$\frac{\partial \theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{1}{x}\right)=\frac{x^2}{x^2+y^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{r^2}$$

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Now,

$$\frac{\partial r}{\partial \theta} = \frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{x}{r}(-y) + \frac{y}{r}\cdot x = 0$$

$$\frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial y}{\partial r} = \left(-\frac{y}{r^2}\right)\left(\frac{x}{r}\right) + \frac{x}{r^2}\frac{y}{r} \qquad [\because \text{using (i)}]$$

$$= 0$$

This shows that the proof part is trivial.

(iii) Since we have,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

From (ii) $\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2}$ and $\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$

Then, $\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$

Thus, $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$

(27) If $z = f(x + ay) + (x - ay)$, prove that: $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$

Solution: Let, $z = f(x + ay) + (x - ay)$

Differentiating we get,

$\frac{\partial z}{\partial y} = f'(x + ay) . a + (-a)$ and, $\frac{\partial^2 z}{\partial y^2} = f''(x + ay) . a^2$

Also, $\frac{\partial z}{\partial x} = f'(x + ay) . 1 + 1$ and $\frac{\partial^2 z}{\partial x^2} = f''(x + ay)$

Now, $\frac{\partial^2 z}{\partial y^2} = f''(x + ay).a^2 = a^2\frac{\partial^2 z}{\partial x^2}$

(28) If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Solution: Let, $z = \frac{x^2+y^2}{x+y}$

Differentiating we get,

$$\frac{\partial z}{\partial x} = \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

And, $$\frac{\partial z}{\partial y} = \frac{(x+y).2y - (x^2+y^2).1}{(x+y)^2} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now,

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \left\{\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy+x^2}{(x+y)^2}\right\}^2 = \left\{1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right\}$$

$$\Rightarrow \left\{\frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2}\right\}^2 =$$

$$4\left(\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right)$$

$$\Rightarrow \left\{\frac{2(x^2-y^2)}{(x+y)^2}\right\}^2 = 4\left(\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right)$$

$$\Rightarrow \left\{\frac{2(x+y)(x-y)}{(x+y)^2}\right\}^2 = 4\frac{(x-y)^2}{(x+y)^2}$$

$$\Rightarrow \frac{4(x-y)^2}{(x+y)^2} = 4\frac{(x-y)^2}{(x+y)^2}$$

This proves that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$.

Exercise 10.2

1. Verify Euler's theorem for the following functions:

(i) $u = ax^2 + 2hxy + by^2$ **[2007 Fall; 2008 Fall –Short]**

Solution: Let, $u = ax^2 + 2hxy + by^2$

Set x as tx and y as ty then

$$u(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2 = t^2(ax^2 + 2hxy + by^2) = t^2u.$$

This shows that u is homogeneous function of degree (n) = 2.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 2hx + 2by$$

So, $$x\frac{\partial u}{\partial x} = 2ax^2 + 2hxy \quad \text{and} \quad y\frac{\partial u}{\partial y} = 2hxy + 2by^2$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2) = 2u$$

Hence, u verifies the Euler's theorem.

(ii) $u = (x^2 + y^2)^{1/3}$

Solution: Let, $u = (x^2 + y^2)^{1/3}$

Set x as tx and y as ty then

$$u(tx, ty) = \{(tx)^2 + (ty)^2\}^{1/3} = t^{2/3}(x + y^2)^{1/3} = t^{2/3} . u$$

This shows that u is homogeneous function of degree (n) $= \frac{2}{3}$.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{1}{3}(x^2+y^2)^{-2/3} \times 2x \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{3}(x^2+y^2)^{-2/3} \times 2y$$

$$= \frac{2x}{3}(x^2+y^2)^{-2/3} \qquad = \frac{2y}{3}(x^2+y^2)^{-2/3}$$

So,

$$x\frac{\partial u}{\partial x} = \frac{2x^2}{3}(x^2+y^2)^{-2/3} \quad \text{and} \quad y\frac{\partial u}{\partial y} = \frac{2y^2}{3}(x^2+y^2)^{-2/3}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2}{3}(x^2+y^2)^{-2/3} + \frac{2y^2}{3}(x^2+y^2)^{-2/3}$$

$$= \frac{2}{3}(x^2+y^2)^{-2/3}(x^2+y^2) = \frac{2}{3}(x^2+y^2)^{1/3} = \frac{2}{3}u$$

Hence, u verifies the Euler's theorem.

(iii) $u = x^n \tan^{-1}\left(\frac{y}{x}\right)$

Solution: Let, $u = x^n \tan^{-1}\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = (tx)^n \tan^{-1}\left(\frac{ty}{tx}\right) = t^n x^n \tan^{-1}\left(\frac{y}{x}\right) = t^n.u$$

This shows that u is homogeneous function of degree (n) = n.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = nx^{n-1}\tan^{-1}\left(\frac{y}{x}\right) + x^n \frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right)$$

And $\frac{\partial u}{\partial y} = x^n \times \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = nx^{n-1}\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^n y}{x^2+y^2} = \frac{x^{n+1}}{x^2+y^2}$

So,

$$x\frac{\partial u}{\partial x} = nx^n \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^{n+1}}{x^2+y^2} \quad \text{And} \quad y\frac{\partial u}{\partial y} = \frac{yx^{n+1}}{x^2+y^2}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^{n+1}y}{x^2+y^2} + \frac{yx^{n+1}}{x^2+y^2}$$

$$= nx^n \tan^{-1}\left(\frac{y}{x}\right) = n.u$$

Hence, u verifies the Euler's theorem.

(iv) $u = x\,f\left(\frac{y}{x}\right)$

Solution: Let, $u = x\,f\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = tx\,f\left(\frac{ty}{tx}\right) = t\,x\,f\left(\frac{y}{x}\right) = tu$$

This shows that u is homogeneous function of degree (n) = 1.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = f\left(\frac{y}{x}\right) + x\,.\,f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$$

And $$\frac{\partial u}{\partial y} = x\,f'\left(\frac{y}{x}\right).\frac{1}{x} = f'\left(\frac{y}{x}\right)$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,f\left(\frac{y}{x}\right) - y\,f'\left(\frac{y}{x}\right) + y\,f'\left(\frac{y}{x}\right)$$

$$= 1\,.\,x\,f\left(\frac{y}{x}\right) = 1\,.\,u$$

Hence, u verifies the Euler's theorem.

(v) $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ **[2008 Spring Q. No. 2(b) OR]**

Solution: Let, $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Set x as tx and y as ty then

$$u(tx, ty) = \frac{(tx)^{1/4} + (ty)^{1/4}}{(tx)^{1/5} + (ty)^{1/5}}$$

$$= \frac{t^{1/4}\,(x^{1/4} + y^{1/4})}{t^{1/5}\,(x^{1/5} + y^{1/5})} = t^{1/4 - 1/5}\frac{(x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})} = t^{1/20}\,.\,u$$

This shows that u is homogeneous function of degree (n) = $\frac{1}{20}$.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{u}{20}$.

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5} + y^{1/5})\frac{1}{4}x^{-3/4} - (x^{1/4} + y^{1/4})\frac{1}{5}x^{-4/5}}{(x^{1/5} + y^{1/5})^2}$$

And $$\frac{\partial u}{\partial y} = \frac{(x^{1/5} + y^{1/5})\frac{1}{4}x^{-3/4} - (x^{1/4} + y^{1/4})\frac{1}{5}x^{-4/5}}{(x^{1/5} + y^{1/5})^2}$$

Then,

$$x\frac{\partial u}{\partial x}=\frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

And $$y\frac{\partial u}{\partial x}=\frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=\frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})+\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$=\frac{\frac{1}{4}(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})-\frac{1}{5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$=\frac{(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})\left(\frac{1}{4}-\frac{1}{5}\right)}{(x^{1/5}+y^{1/5})^2}=\frac{1}{20}\cdot\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}=\frac{1}{20}\cdot u$$

Hence, u verifies the Euler's theorem.

(vii) $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Solution: Let, $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty)=\sin^{-1}\left(\frac{tx}{ty}\right)+\tan^{-1}\left(\frac{ty}{tx}\right)$$

$$=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=t^0u$$

This shows that u is homogeneous function of degree (n) = 0.

Then we wish to show $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.u=0.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\times\frac{1}{y}+\frac{1}{1+\frac{y^2}{x^2}}\times\left(-\frac{y}{x^2}\right)=\left(\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}\right)$$

And $$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\times\left(-\frac{x}{y^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\times\frac{1}{x}=\left(\frac{-x}{y\sqrt{y^2-x^2}}-\frac{x}{x^2+y^2}\right)$$

Then,

$$x\frac{\partial u}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}\quad\text{And}\quad y\frac{\partial u}{\partial y}=-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$
$$=0.\ u=0.$$

Hence, u verifies the Euler's theorem.

(viii) $u = x^3+y^3+z^3-3xyz$

Solution: Let, $u = x^3+y^3+z^3-3xyz$

Set x as tx, y as ty and z as tz then

$$u(tx, ty, tz) = (tx)^3+(ty)^3+(tz)^3-3tx.\ ty.\ tz$$
$$= t^3(x^3+y^3+z^3-3xyz) = t^3.\ u$$

This shows that u is homogeneous function of degree (n) = 3.

Then we wish to show $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3u.$

Here differentiating u partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial u}{\partial x}=3x^2-3yz,\qquad \frac{\partial u}{\partial y}=3y^2-3xz \qquad\text{and}\qquad \frac{\partial u}{\partial z}=3z^2-3xy$$

Then,

$$x\frac{\partial u}{\partial x}=3x^3-3xyz,\qquad y\frac{\partial u}{\partial y}=3y^3-3xyz\quad\text{and}\qquad z\frac{\partial u}{\partial z}=3z^3-3xyz$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3x^3-3xyz+3y^3-3xyz+3z^3-3xyz$$
$$=3(x^3+y^3+z^3-3xyz)=3u$$

Hence, u verifies the Euler's theorem.

(2) **If $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$**

[1999, 2001 Q. No. 2(b) OR] [2008 Fall Q. No. 2(b)]

Solution: Let, $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let, $f=\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Set x as tx and y as ty then

$$f(tx, ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=\frac{t(x+y)}{\sqrt{t}\,(\sqrt{t}+\sqrt{y})}=t^{1/2}.\ f$$

This shows that f is homogeneous function of degree (n) = $\frac{1}{2}$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{2}.f$$
$$\Rightarrow\quad x\frac{\partial \cos u}{\partial x}+y\frac{\partial \cos u}{\partial y}=\frac{1}{2}\cos u$$

$$\Rightarrow \quad x\ (-\sin u)\frac{\partial u}{\partial x} + y\ (-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow \quad -\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\cos u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{1}{2} = -\frac{1}{2}\cot u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

(3) If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, $x \neq y$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

Solution: Let, $u = \tan^{-1}\dfrac{x^3+y^3}{x-y} \Rightarrow \tan u = \dfrac{x^3+y^3}{x-y}$ for $x \neq y$

Let, $$f = \tan u = \frac{x^3+y^3}{x-y}$$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(tx)^3 + (ty)^3}{(tx - ty)} = \frac{t^3(x^3+y^3)}{t(x-y)} = \frac{t^2(x^3+y^3)}{x-y} = t^2 f$$

This shows that f is homogeneous function of degree (n) = 2.

Then by Euler's theorem

$$x\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial y} = 2f$$

$$\Rightarrow \quad x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \quad \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

(4) If, prove that $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 1$.

Solution: Let, $$v = \log\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow e^v = \frac{x^2+y^2}{x+y}$$

Let, $$f = e^v = \frac{x^2+y^2}{x+y}$$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{tx+ty} = \frac{t^2(x^2+y^2)}{t(x+y)} = t.f$$

This shows that f is homogeneous function of degree (n) = 1.

Then by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1.f$$

$$\Rightarrow\quad x\frac{\partial e^v}{\partial x}+y\frac{\partial e^v}{\partial y}=1.e^v$$

$$\Rightarrow\quad x\,e^v\frac{\partial v}{\partial x}+ye^v\frac{\partial v}{\partial y}=e^v$$

$$\Rightarrow\quad e^v\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=e^v$$

$$\Rightarrow\quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1.$$

(5) If $u=\sin^{-1}\left[\dfrac{x^2+y^2}{x+y}\right]$ show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u$.

Solution: Let, $u=\sin^{-1}\left[\dfrac{x^2+y^2}{x+y}\right]$ $\Rightarrow$ $\sin u=\dfrac{x^2+y^2}{x+y}$

Let $f=\sin u=\dfrac{x^2+y^2}{x+y}$

Set x as tx and y as ty then

$$f(tx,xy)=\frac{(tx)^2+(ty)^2}{(tx+ty)}=\frac{t^2(x^2+y^2)}{t(x+y)}=t\left(\frac{x^2+y^2}{x+y}\right)=t.f$$

This shows that f is homogeneous function of degree (n) = 1.

Then by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1.f$$

$$\Rightarrow\quad x\frac{\partial \cos u}{\partial x}+y\frac{\partial \cos u}{\partial y}=1.f$$

$$\Rightarrow\quad x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\Rightarrow\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

(6) If $u=\sin^{-1}\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}+3\tan u=0$.

Solution: Let, $u=\sin^{-1}\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ $\Rightarrow$ $\sin u=\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$

Let, $f=\sin u=\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$.

Set x as tx, y as ty and z as tz then

$$f(tx,ty,tz)=\frac{tx+2ty+3tz}{\sqrt{(tx)^8+(ty)^8+(tz)^8}}=\frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+z^8}}=t^{-3}f$$

This shows that f is the homogeneous function of degree (n) = –3

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=-3.f$$

$$\Rightarrow \quad x\frac{\partial \sin u}{\partial x}+y\frac{\partial \sin u}{\partial y}+z\frac{\partial \sin u}{\partial z}=-3\sin u$$

$$\Rightarrow \quad x.\cos u\frac{\partial u}{\partial x}+y.\cos u\frac{\partial u}{\partial y}+z\cos u.\frac{\partial u}{\partial z}=-3\sin u$$

$$\Rightarrow \quad \cos u\left(x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=-3\sin u$$

$$\Rightarrow \quad x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0.$$

(7) If $u=\log\left[\frac{x^4+y^4}{x+y}\right]$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$.

Solution: Let, $u=\log\left[\frac{x^4+y^4}{x+y}\right]$. $\Rightarrow e^u=\frac{x^4+y^4}{x+y}$.

Let $$f=e^u=\frac{x^4+y^4}{x+y}$$

Set x as tx and y as ty then

$$f(tx,ty)=\frac{(tx)^4+(ty)^4}{tx+ty}=\frac{t^4(x^4+y^4)}{t(x+y)}=t^3.f$$

This shows that f is the homogeneous function of degree (n) = 3.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f.$$

$$\Rightarrow \quad x\frac{\partial e^u}{\partial x}+y\frac{\partial e^u}{\partial y}=3e^u.$$

$$\Rightarrow \quad x\,e^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

(8) $u=\sqrt{x^2-y^2}\sin^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$.

Solution: Let, $u(x,y)=\sqrt{x^2-y^2}\sin^{-1}\left(\frac{y}{x}\right)$.

Set x as tx and y as ty then

$$u(tx,ty)=\sqrt{(tx)^2-(ty)^2}\sin^{-1}\left(\frac{ty}{tx}\right)=t\sqrt{x^2-y^2}\sin^{-1}\left(\frac{y}{x}\right)=t^1.u$$

This shows that u is the homogeneous function of degree (n) = 1.

Then by Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=u.$$

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(9) If $u = \sqrt{y^2 - x^2}\sin^{-1}\left(\frac{x}{y}\right) + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$ show that $x\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = u$.

Solution: Let, $u(x, y) = \sqrt{y^2 - x^2}\sin^{-1}\left(\frac{x}{y}\right) + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

Set x as tx and y as ty then

$$u(tx, ty) = \sqrt{(ty)^2 - (tx)^2}\sin^{-1}\left(\frac{ty}{tx}\right) + \frac{(tx)^2 + (ty)^2}{\sqrt{(tx)^2 + (ty)^2}}$$

$$= t\sqrt{y^2 - x^2}\sin^{-1}\left(\frac{x}{y}\right) + \frac{t^2(x^2 - y^2)}{t\sqrt{x^2 + y^2}}$$

$$= t\left(\sqrt{y^2 - x^2}\sin^{-1}\left(\frac{y}{x}\right) + \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right) = t^1.u$$

This shows that u is the homogeneous function of degree (n) = 1.

So according to Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u.$$

(10) If $u = \cos\left[\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right]$ **show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution: Let, $u = \cos\left[\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right] \Rightarrow \cos^{-1}u = \frac{xy + yz + zx}{x^2 + y^2 + z^2}$

Let, $f = \cos^{-1}u = \frac{xy + yz + zx}{x^2 + y^2 + z^2}$.

Set x as tx, y as ty and z as tz then

$$f(tx, ty) = \frac{tx.\,ty + ty.\,tz + tz.\,tx}{(tx)^2 + (ty)^2 + (tz)^2} = \frac{t^2(xy + yz + zx)}{t^2(x^2 + y^2 + z^2)} = t^0.\,u$$

This shows that f is the homogeneous function of degree (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial x} = 0.f.$$

$$\Rightarrow \quad x\frac{\partial \cos^{-1}u}{\partial x} + y\frac{\partial \cos^{-1}u}{\partial y} + z\frac{\partial \cos^{-1}u}{\partial x} = n.\cos^{-1}u\,.$$

$$\Rightarrow \quad x\left(-\frac{1}{\sqrt{1 - u^2}}\right)\frac{\partial u}{\partial x} + y\left(-\frac{1}{\sqrt{1 - u^2}}\right)\frac{\partial u}{\partial y} + z\left(-\frac{1}{\sqrt{1 - u^2}}\right)\frac{\partial u}{\partial z} = 0$$

$$\Rightarrow \quad -\frac{1}{\sqrt{1 - u^2}}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 0$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

(11) If $\sin u = \dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$.

Solution: Let, $$\sin u = \frac{x^2y^2}{x+y}$$

Let, $$f = \sin u = \frac{x^2y^2}{x+y}$$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(tx)^2(ty)^2}{tx+ty} = \frac{t^4x^2y^2}{t(x+y)} = t^3\frac{(x^2y^2)}{x+y} = t^3.u$$

This shows that f is the homogeneous function of degree (n) = 3.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

$$\Rightarrow \quad x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 3\sin u$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$$

$$\Rightarrow \quad x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u.$$

(12) If $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u\cos 2u}{4\cos^3 u}$.

[2018 Spring Q.No. 2(a)]

Solution: Let, $$u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right] \Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let, $$f = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = t^{1/2}u$$

This shows that f is the homogeneous function of degree (n) $= \frac{1}{2}$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f$$

$$\Rightarrow \quad x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = n.\sin u$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \qquad \text{........ (i)}$$

Differentiating above equation w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial y\partial x}=\frac{1}{2}\sec^2 u.\frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial y\partial x}=\frac{1}{2}x\sec^2 u.\frac{\partial u}{\partial x} \qquad \text{........ (ii)}$$

Differentiating (i) w. r. t. 'y' then

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u.\frac{\partial u}{\partial y}$$

Multiplying by y

$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=\frac{1}{2}y\sec^2 u\frac{\partial u}{\partial y} \qquad \text{......(iii)}$$

Adding (ii) and (iii) then,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{1}{2}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\tan u=\frac{1}{2}\sec^2 u\frac{1}{2}\tan u \qquad \text{[using (i)]}$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u-1\right)$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1}{2\cos^2 u}-1\right)$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{1-2\cos^2 u}{2\cos^2 u}\right)$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

(13) If $u=(x^2+y^2)^{1/3}$, show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{2u}{9}$.

Solution: Let, $\quad u=(x^2+y^2)^{1/3}$

Set x as tx and y as ty then

$$u(tx, ty)=\{(tx)^2+(ty)^2\}^{1/3}=t^{2\times 1/3}(x^2+y^2)^{1/3}=t^{2/3}u$$

This shows that u is the homogeneous function of degree (n) $=\frac{2}{3}$.

Then by Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2}{3}u \qquad \text{.........(i)}$$

Differentiating (i) partially w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial y}{\partial x\partial y}=\frac{2}{3}\frac{\partial u}{\partial x}$$

Multiplying by 'x'

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 y}{\partial x\partial y}=\frac{2}{3}x\frac{\partial u}{\partial x} \qquad \text{.........(ii)}$$

Again differentiating (i) partially w. r. t. 'y' then,

$$x\frac{\partial^2 y}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{2}{3}\frac{\partial u}{\partial y}$$

Multiplying by 'y'

$$xy\frac{\partial^2 y}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=\frac{2}{3}\frac{\partial u}{\partial y} \qquad \text{.........(iii)}$$

Adding (ii) and (iii) then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2}{3}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{2}{3}u=\frac{2}{3}\times\frac{2}{3}u$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{4}{9}u-\frac{2}{3}u$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{4u-6u}{9}$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{2u}{9}.$$

(14) If $u=\tan^{-1}\left(\frac{y^2}{x}\right)$ show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u.\sin^2 u$

Solution: Let, $\tan u=\frac{y^2}{x}$

Let $f=\tan u=\frac{y^2}{x}$

Set x as tx and y as ty then

$$f(tx, ty)=\frac{(ty)^2}{tx}=\frac{t^2y^2}{tx}=t\left(\frac{y^2}{x}\right)=t^1u$$

Here u is the homogeneous function of degree (n) = 1.
Then by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1.f.$$

$$\Rightarrow\quad x\frac{\partial \tan u}{\partial x}+y\frac{\partial \tan u}{\partial y}=1.\tan u.$$

$$\Rightarrow\quad x\sec^2u\frac{\partial u}{\partial x}+y\sec^2u\frac{\partial u}{\partial y}=1\tan u$$

$$\Rightarrow\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}\times\cos^2u=\frac{1}{2}\sin 2u \qquad \text{........ (i)}$$

Differentiating (i) partially w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{2}{2}\cos 2u\frac{\partial u}{\partial x}$$

Multiplying by x, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=x\cos 2u\frac{\partial u}{\partial x} \qquad \text{....... (ii)}$$

Again, differentiating (i) partially w. r. t. 'y' then,

$$x^2\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{2}{2}\cos 2u\frac{\partial u}{\partial y}$$

Multiplying by y

$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=y\cos 2u\frac{\partial u}{\partial y} \quad .... (iii)$$

Adding (ii) and (iii)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\sin^2 u=\cos 2u\,.\frac{1}{2}\sin 2u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sin^2(\cos 2u-1)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sin 2u(\cos 2u-1)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}[1-\sin^2 u-1]=-\frac{1}{2}\sin 2u.\,2\sin^2 u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u.\,\sin^2 u.$$

(15) Find $\frac{dw}{dt}$ where (i) $w=x^3-y^3,\ x=\frac{1}{t+1},\ y=\frac{1}{t+1}$

(ii) $w=r^2-s\tan v,\ r=\sin^2 t,\ s=\cos t,\ v=4t.$

Solution:

(i) Let, $w=x^3-y^3,\ x=\frac{1}{t+1},\ y=\frac{t}{t+1}$

Differentiating w partially,

$$\frac{\partial w}{\partial x}=3x^2,\quad \frac{\partial w}{\partial y}=-3y^2,\quad \frac{dx}{dt}=-\frac{1}{(t+1)^2},\quad \frac{dy}{dt}=\frac{1}{(t+1)^2}$$

Now,

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}=-\frac{1}{(t+1)^2}(3x^2-3y^2)$$

$$=\frac{-3}{(t+1)^2}(x^2-y^2)=\frac{-3(1+t^2)}{(t+1)^4}.$$

(ii) Let, $w=r^2-s\tan v,\ r=\sin^2 t,\ s=\cos t,\ v=4t.$

Differentiating w partially w. r. t. 'r', 's' and 'v' then,

$$\frac{\partial w}{\partial r}=2r,\quad \frac{dr}{dt}=2\sin t\cos t,\quad \frac{\partial w}{\partial s}=-\tan v$$

$$\frac{ds}{dt}=-\sin t,\quad \frac{\partial w}{\partial v}=s\sec^2 v,\quad \frac{dv}{dt}=4$$

We have,

$$\frac{dw}{dt}=\frac{\partial w}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial w}{\partial s}\cdot\frac{ds}{dt}+\frac{\partial w}{\partial v}\cdot\frac{dv}{dt}$$

$$= 2r.\ 2\sin t.\ \cos t + (-\tan v)(-\sin t) + (-\sec^2 v).4$$

$$= 4r \sin t.\ \cos t + \tan v.\ \sin t - 4s \sec^2 v$$

$$= 4 \sin^2 t.\ \sin t \cos t + \tan 4t.\ \sin t - 4\cos t.\sec^2 4t$$

$$\Rightarrow \frac{dw}{dt} = 4\sin^3 t.\ \cos t + \tan 4t.\sin t - 4\ \cos t.\ \sec^2 4t.$$

(16) Find $\frac{dz}{dx}$ it $z = (y + x)\, e^{xy}$, $y = \frac{1}{x^2}$.

Solution: Let, $z = (y + x)\, e^{xy}$, $y = \frac{1}{x^2}$.

Differentiating z partially w. r. t. 'x' then,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \quad ....(i)$$

Since,

$$z = (y + x)\, e^{xy},\ y = \frac{1}{x^2}.$$

Then,

$$\frac{\partial z}{\partial x} = (0 + 1)\, e^{xy} + (y + x)\, y.e^{xy} = e^{xy}(1 + xy + y^2)$$

And, $$\frac{\partial z}{\partial y} = (1 + 0)\, e^{xy} + (y + x)\, xe^{xy} = e^{xy}(1 + xy + x^2)$$

Also, $$\frac{dy}{dx} = -\frac{2}{x^3}$$

Now, (i) becomes,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} = e^{xy}(1 + xy + y^2) + e^{xy}(1 + xy + x^2) \times \left(-\frac{2}{x^3}\right)$$

$$= e^{xy}\left(1 + xy + y^2 - \frac{2}{x^3}(1 + xy + x^2)\right)$$

$$= e^{1/x}\left(1 + x.\frac{1}{x^2} + \left(\frac{1}{x^2}\right)^2 - \frac{2}{x^3} - \frac{2}{x^2.x^2} - \frac{2}{x}\right)$$

$$= \left(1 + \frac{1}{x} + \frac{1}{x^4} - \frac{2}{x^3} - \frac{2}{x^4} - \frac{2}{x}\right) e^{1/x}$$

$$\Rightarrow \frac{dz}{dx} = \left(1 - \frac{1}{x} - \frac{2}{x^3} - \frac{1}{x^4}\right) e^{1/x}$$

(17) If $z = f(x, y)$ and if $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

[2000 Q. No. 2(b)]

Solution: Let, $z = f(x, y)$ and if $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

So, $$\frac{\partial x}{\partial u} = e^u,\quad \frac{\partial x}{\partial v} = -e^{-v},\quad \frac{\partial y}{\partial u} = -e^{-u},\quad \frac{\partial y}{\partial v} = -e^v$$

Now,

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)-\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)$$

$$=\left(\frac{\partial z}{\partial x}e^{u}+\frac{\partial z}{\partial y}\times(-e^{-u})\right)-\left(\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}\times(-e^{v})\right)$$

$$=\frac{\partial z}{\partial x}(e^{u}+e^{-v})-\frac{\partial z}{\partial y}(e^{-u}-e^{-v})$$

$$=\frac{\partial z}{\partial x}\,x-\frac{\partial z}{\partial y}\,y$$

Thus, $\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$

(18) Find $\dfrac{dy}{dx}$ is the following cases using $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y}$.

(i) $x^{2/3}+y^{2/3}=a^{2/3}$

Solution: Let, $f(x, y)=x^{2/3}+y^{2/3}-a^{2/3}=0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x=\frac{2}{3}x^{-1/3} \quad \text{and} \quad f_y=\frac{2}{3}y^{-1/3}$$

Now,

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=\frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}=-\left(\frac{y}{x}\right)^{1/3}.$$

(ii) $x^p y^q=(x+y)^{p+q}$

Solution: Let, $f(x, y)=x^py^q-(x+y)^{p+q}=0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x=y^q p.x^{p-1}-(p+q)(x+y)^{p+q-1}-(p+q)(x+y)^{p+q-1}$$
$$f_y=x^p.qy^{q-1}-(p+q)(x+y)^{p+1-1}=qx^py^{q-1}-(p+q)(x+y)^{p+q-1}$$

Now,

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=\frac{(p+q)(x+y)^{p+q-1}-py^qx^{p-1}}{qx^py^{q-1}-(p+q)(x+y)^{p+q-1}}$$

$$=\frac{(p+q)(x+y)^{p+q}(x+y)^{-1}-px^py^q.x^{-1}}{qx^py^qy^{-1}-(p+q)(x+y)^{p+q}(x+y)^{-1}}$$

$$=\frac{\frac{(p+q)x^py^q}{(x+y)}-\frac{px^py^q}{x}}{q\frac{x^py^q}{y}-\frac{(p+q)(x^py^q)}{(x+y)}}=\frac{\frac{p+q}{x+y}-\frac{p}{x}}{\frac{q}{y}-\frac{p+q}{x+y}}=\frac{y(qx-py)}{x(qx-py)}=\frac{y}{x}.$$

$$\frac{dy}{dx}=\frac{y}{x}.$$

(iii) $(\tan x)^y+(y)^{\tan x}=0$

Solution: Let, $f(x, y)=(\tan x)^y+(y)^{\tan x}$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x=y(\tan x)^{y-1}\sec^2x+y^{\tan x}.\log y.\sec^2x.$$

$$\left\{\begin{array}{l}\frac{d}{dx}a^x = a^x \log a\\ \frac{d}{dx}(x^n) = nx^{n-1}\end{array}\right.$$

$$f_y = (\tan x)^y \log \tan x + \tan x\, y^{\tan x - 1}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(\tan x)^{y-1}.\sec^2 x + y^{\tan x}.\log y \sec^2 x}{(\tan x)^y \log \tan x + \tan x\, y^{\tan x - 1}}$$

$$= -\frac{\sec^2 x\,[y(\tan x)^{y-1} + \log y.\, y^{\tan x}]}{(\tan x)^y \log \tan x + \tan x\, y^{\tan x - 1}}$$

(iv) $x^y = y^x$

Solution: Let, $f(x, y) = x^y - y^x = 0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\left\{\begin{array}{l}\frac{d}{dx}a^x = a^x \log x\\ \frac{d}{dx}(x^n) = nx^{n-1}\end{array}\right.$$

$$f_x = y\,x^{y-1} - y^x \log y$$

$$f_y = x^y \log x - xy^{x-1}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(yx^{y-1} - y^x \log y)}{x^y \log x - xy^{x-1}}.$$

(v) $x^y + y^x = a$

Solution: Let, $f(x, y) = x^y + y^x - a = 0.$

Differentiating f partially w. r. t. 'x' and 'y' then,

$f_x = yx^{y-1} + y^x \log y$ and $f_y = x^y \log x + xy^{x-1}$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(yx^{y-1} + y^x \log y)}{x^y \log x + xy^{x-1}}.$$

(19) If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

Solution: Let, $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$

So, $$\frac{dx}{dt} = 3,\quad \frac{dy}{dt} = 12t^2$$

Also, differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \times 1 = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}} \times (-1) = -\frac{1}{\sqrt{1-(x-y)^2}}$$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} \times 3 - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$= \frac{3 - 12t^2}{\sqrt{1 - 9t^2 + 24t^4 - 16t^6}}$$

$$= \frac{3 - 12t^2}{\sqrt{1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6}}$$

$$= \frac{3 - 13t^2}{\sqrt{1(1 - t^2) - 8t^2(1 - t^2) + 16t^4(1 - t^2)}}$$

$$= \frac{3 - 12t^2}{\sqrt{(1 - t^2)(1 - 8t^2 + 16t^4)}}$$

$$= \frac{3(1 - 4t^2)}{\sqrt{(1 - t^2)(1 - 4t^2)^2}} = \frac{3}{\sqrt{1 - t^2}}.$$

Thus, $\frac{du}{dt} = \frac{3}{\sqrt{1 - t^2}}$.

(20) If $u = x^2 + y^2$, $x = at^2$, $y = 2at$ then show that $\frac{du}{dt} = 4a^2t\,(t^2 + 2)$.

Solution: Let, $u = x^2 + y^2$, $x = at^2$, $y = 2at$

Differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial u}{\partial y} = 2y$$

Next, we have, $x = at^2$, $y = 2at$.

Differentiating partially w. r. t. 't' then,

$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 2x \times 2at + 2y \times 2a$$

$$= 2at^2 \times 2at + 2 \times 2at \times 2a = 4a^2t^3 + 8a^2t.$$

$$= 4a^2t\,(t^2 + 2).$$

Thus, $\frac{du}{dt} = 4a^2t\,(t^2 + 2)$.

(21) If $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$, show that $\frac{du}{dt} = 8e^{4t}$

Solution: Let, $u = x^2 + y^2 + z^2$

Differentiating u partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y \quad \& \quad \frac{\partial u}{\partial z} = 2z$$

Also, let, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$

Differentiating partially w. r. t. 't' then,

$$\frac{dx}{dt} = 2e^{2t}, \quad \frac{dy}{dt} = \cos 3t.\,2e^{2t} + e^{2t}\,3\,(-\sin 3t)$$

$$= 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$$

$$\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t.$$

Then, $\frac{\partial u}{\partial x}\cdot\frac{dx}{dt} = 2x.\,2e^{2t} = 4e^{2t}\,e^{2t} = 4e^{4t}$

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 2e^{2t}.\cos 3t\,(2e^{2t}\cos 3t - 3e^{2t}\sin 3t)$$
$$= 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t.\cos 3t.$$
$$\frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = 2e^{2t}\sin 3t\,(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$
$$= 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t.\cos 3t$$

Now,
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$
$$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t.\cos 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t.\cos 3t$$
$$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t) = 4e^{4t} + 4e^{4t} = 8e^{4t}.$$

(22) If $u = \sin\frac{x}{y}$, $x = e^t$, $y = t2$, show that $\frac{du}{dt} = \frac{e^t(t-2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$

Solution: Let, $u = \sin\frac{x}{y}$, $x = e^t$, $y = t^2$

Differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \cos\frac{x}{y}\cdot\frac{1}{y} = \frac{1}{y}\cos\frac{x}{y}, \qquad \frac{dx}{dt} = e^t$$

And $\frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}$ $\qquad \frac{dy}{dt} = 2t$

Now,
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = \frac{1}{y}\cos\frac{x}{y}.\,et + \left(-\frac{x}{y^2}\right)\cos\frac{x}{y}\times 2t$$
$$= \frac{e^t}{t^2}\cos\frac{e^t}{t^2} - \frac{e^t}{t^3}.\cos\frac{e^t}{t^2}\times 2t$$
$$= \cos\frac{e^t}{t^2}\left(\frac{e^t}{t^2} - \frac{2e^t}{t^3}\right) = \frac{e^t}{t^3}(t-2)\cos\left(\frac{e^t}{t^2}\right).$$

Thus, $\frac{du}{dt} = \frac{e^t}{t^3}(t-2)\cos\left(\frac{e^t}{t^2}\right)$.

(23) If $u = f(r, s)$, $r = x + y$, $s = x - y$, show that: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Solution: Let, $u = f(r, s)$, $r = x + y$, $s = x - y$.

Differentiating partially then,

$$\frac{\partial u}{\partial r} = f'(r, s)\,s \qquad \text{and} \qquad \frac{\partial u}{\partial s} = f'(r, s)\,r$$

Since, $r = x + y$, $s = x - y$.

Differentiating partially then,

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 1, \quad \frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = 1.$$

Now,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= f'(r, s).s.1 + f'(r, s).r + f'(r, s).s.1 + f'(r, s).r.(-1)$$

$$= f'(r, s)(s + r + s - r) = 2s\, f'(r, s) = 2\frac{\partial u}{\partial r}.$$

(24) If $z = e^{ax+by} f(ax - by)$, show that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$

Solution: Let, $z = e^{ax+by} f(ax - by)$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = e^{ax+by} \times a\, f(ax - by) + f'(ax - by)\, a\, e^{ax+by}$$

$$= a e^{ax+by} f(ax - by) + ae^{ax+by} f'(ax - by)$$

$$= ae^{ax+by} \{f(ax - by) + f'(ax - by)\}$$

And, $$\frac{\partial z}{\partial y} = e^{ax+by} \times b\, f(ax - by) + f'(ax - by) \times - be^{ax+by}$$

$$= be^{ax+by} \{f(ax - by) - f'(ax - by)\}$$

Now,

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = b[ae^{ax+by} \{f(ax-by) + f'(ax-by)\}] + a[be^{ax+by}\{f(ax-by) + f'(ax-by)\}]$$

$$= abe^{ax+by} \{f(ax - by) + f'(ax - by) + f(ax - by) - f'(ax - by)\}$$

$$= 2ab\, e^{ax+by} f(ax - by) = 2abz.$$

(25) If $u = f(r, s)$, $r = x + at$, $s = y + bt$ show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$

Solution: Let, $u = f(r, s)$, $r = x + at$, $s = y + bt$

Differentiating partially then,

$$\frac{\partial u}{\partial r} = f'(r, s), \quad \frac{\partial u}{\partial s} = f'(r, s), \quad \frac{\partial r}{\partial t} = a, \quad \frac{\partial s}{\partial t} = b,$$

$$\frac{\partial u}{\partial x} = f'(x + at, y + bt)(y + bt).1 = f'(r, s).s$$

$$\frac{\partial u}{\partial y} = f'(x + at, y + bt)(x + at).1 = f'(r, s).r$$

Now,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial t} = f'(r, s).s.a + f'(r, s).r.b.$$

$$= f'(r, s)[as + br]$$

And,

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a[f'(r, s).s] + b[f'(r, s).r]\, b$$

$$= f'(r, s)[as + br]$$

$$= \frac{\partial u}{\partial t}.$$

(26) If $z = x^2y$ and $x^2 + xy + y^2 = 1$ show that $\frac{dz}{dx} = 2xy - \frac{x^2(2x+y)}{(x+2y)}$

Solution: Let, $z = x^2y$.

Differentiating partially then,

$$\frac{\partial z}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial z}{\partial y} = x^2$$

And, $\quad f = x^2 + xy + y^2 - 1 = 0.$

Differentiating w. r. t. x then,

$$f_x = 2x + y \qquad \text{and} \qquad f_y = x + 2y$$

So, $\quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x+y}{x+2y}$

Now,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} = 2xy.1 - x^2\left(\frac{2x+y}{x+2y}\right)$$

(27) **If $x = e^r \cos\theta$, $y = e^r \sin\theta$ then**

(28) **If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.**

Solution: Let, $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$.

Since we have,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

Here, $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \text{and} \qquad \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r}$$

Also, $\quad \frac{\partial x}{\partial \theta} = -r\sin\theta = -y \quad \text{and} \quad \frac{\partial y}{\partial \theta} = r\cos\theta = x$

Then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{x}{r} + \frac{\partial w}{\partial y}\cdot\frac{y}{r} \qquad \text{and} \qquad \frac{\partial w}{\partial \theta} = -y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y}$$

So that,

$$\left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cdot\frac{x^2}{r^2} + \left(\frac{\partial w}{\partial y}\right)^2\cdot\frac{y^2}{r^2} + \frac{2xy}{r^2}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$$

$$\text{and } \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{y^2}{r^2}\cdot\left(\frac{\partial w}{\partial x}\right)^2 + \frac{x^2}{r^2}\cdot\left(\frac{\partial w}{\partial y}\right)^2 - \frac{2xy}{r^2}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$$

Now,

$$\begin{aligned}
\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 &= \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\left(\frac{y^2}{r^2} + \frac{x^2}{r^2}\right) \\
&= \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{x^2+y^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\cdot\left(\frac{x^2+y^2}{r^2}\right) \\
&= \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{r^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\cdot\left(\frac{r^2}{r^2}\right)
\end{aligned}$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \qquad [\because w = f].$$

Thus, $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$

(29) If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, show that $\frac{du}{dx} = e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x \sin 2x\right]$

Solution: Let, $u = xe^y z$, $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$,

Differentiating partially then,

$$\frac{\partial u}{\partial x} = e^y z, \qquad \frac{\partial u}{\partial y} = xe^y z, \qquad \frac{\partial u}{\partial z} = xe^y$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{a^2 - x^2}} = \frac{-x}{\sqrt{a^2 - x^2}} \quad \text{and} \quad \frac{dz}{dx} = 2 \sin x . \cos x = \sin 2x$$

Now,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx} = e^y z + xe^y z \times -\frac{x}{\sqrt{a^2 - x^2}} + xe^y . \sin 2x$$

$$= ey\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x \sin 2x\right]$$

(30) If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, show that $\frac{du}{dx} = \frac{2(b^2 - a^2)x}{b^2}\cos(x^2 + y^2)$

Solution: Let, $u = \sin(x^2 + y^2)$, and $a^2x^2 + b^2y^2 = c^2$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = 2x \cos(x^2 + y^2), \qquad \frac{\partial u}{\partial y} = 2y \cos(x^2 + y^2)$$

And, $f = a^2x^2 + b^2y^2 - c^2 = 0$

Differentiating partially then,

$$f_x = 2a^2x \qquad f_y = 2b^2y$$

Then, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2a^2x}{2b^2y} = -\frac{a^2x}{b^2y}$

Now,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x \cos(x^2 + y^2).1 + 2y \cos(x^2 + y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x \cos(x^2 + y^2) - 2 \cos(x^2 + y^2)\left(-\frac{a^2x}{b^2}\right)$$

$$= 2x \cos(x^2 + y^2)\left(1 - \frac{a^2}{b^2}\right) = 2x \cos(x^2 + y^2)\frac{(b^2 - a^2)}{b^2}.$$

(31) Find $\frac{dz}{dt}$ if $z = x \log y$, $x = t^2$, $y = e^t$

Solution: Let, $z = x \log y$, $x = t^2$, $y = e^t$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = \log y, \quad \frac{\partial z}{\partial y} = \frac{x}{y}, \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = e^t$$

Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \log(y)\, 2t + \frac{x}{y} e^t = \log(e^t)\,.\,2t + \frac{t^2}{e^t}\,.\,e^t = t\,.\,2t + t^2 = 2t^2 + t^2 = 3t^2.$$

Thus, $\frac{dz}{dt} = 3t^2$.

(32) If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}$

Solution: Let, $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$. ... (i)

Differentiating w. r. t. x,

$$\sqrt{1-y^2}\, 1 + x \times \frac{-2y}{2\sqrt{1-y^2}}\cdot\frac{dy}{dx} + \frac{dy}{dx}\cdot\sqrt{1-x^2} + y\,.\,\frac{-2x}{2\sqrt{1-x^2}} = 0$$

$$\Rightarrow \sqrt{1-y^2} - \frac{xy}{\sqrt{1-y^2}}\frac{dy}{dx} + \frac{dy}{dx}\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} = 0$$

$$\Rightarrow \frac{dy}{dx}\left[\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right] = \frac{-xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}}\right] = \frac{xy - \sqrt{1-y^2}\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}\,(\sqrt{1-x^2}\,.\sqrt{1-y^2} - xy)}{\sqrt{1-x^2}\,(\sqrt{1-x^2}\sqrt{1-y^2} - xy)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \quad \text{........ (ii)}$$

Again differentiating w. r. t. (ii) x,

$$\frac{d^2y}{dx^2} = \frac{-\left[\sqrt{1-x^2} \times \frac{-2y}{2\sqrt{1-y^2}} \times \frac{dy}{dx} - \sqrt{1-y^2} \times \frac{-2x}{2\sqrt{1-x^2}}\right]}{(1-x^2)}$$

$$= \frac{-\left[\frac{-y\sqrt{1-x^2}}{\sqrt{1-y^2}}\frac{dy}{dx} + x\sqrt{1-y^2}/\sqrt{1-x^2}\right]}{(1-x^2)}$$

$$= \frac{\left[\frac{y\sqrt{1-x^2}}{\sqrt{1-y^2}} \times -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} - \frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}\right]}{(1-x^2)} \quad \text{Using eq}^{n}.\text{ (ii)}$$

$$= \frac{-[y\sqrt{1-x^2} + x\sqrt{1-y^2}]}{(1-x^2)^{3/2}}$$

$$= -\frac{a}{(1-x^2)^{3/2}} \quad [\text{using eq}^{n}.\text{ (i)}]$$

(33) If $u = x(x + y) + y(x + y)$ proved that: $\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x . \partial y} + \frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = x(x + y) + y(x + y)$

$\Rightarrow u = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = 2x + 2y, \qquad \frac{\partial u}{\partial y} = 2x + 2y$$

Again,

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial x\, \partial y} = 2 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = 2.$$

Now,

$$\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x . \partial y} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 \times 2 + 2 = 4 - 4 = 0.$$

(34) If $u = \tan^{-1}\frac{y}{x}$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = \tan^{-1}\frac{y}{x}$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times -\frac{y}{x^2} = -\frac{y}{x^2} \times \frac{x^2}{x^2 + y^2} = -\frac{y}{x^2 + y^2}$$

And, $$\frac{\partial u}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x} = \frac{1}{x} \times \frac{x^2}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

Again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{-y}{(x^2 + y^2)^2} \times -1 \times 2x = \frac{2xy}{(x^2 + y^2)^2} \qquad \text{.......(i)}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{x}{(x^2 + y^2)^2} \times 2y = -\frac{2xy}{(x^2 + y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0.$$

(35) If $y = f(x + ct) + q(x - ct)$ show that $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution: Let, $y = f(x + ct) + q(x - ct)$

Differentiating partially then,

$$\frac{\partial y}{\partial t} = f'(x + ct) \times c + q'(x - ct) \times -c = cf'(x + ct) - cq'(x - ct)$$

And, $$\frac{\partial y}{\partial x} = f'(x + ct).1 + q'(x - ct).1 = f'(x + ct) + q'(x - ct)$$

Also,

$$= c^2\{f''(x+ct) + q''(x-ct)\} \quad \text{....... (i)}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 f''(x+ct) + c^2 q''(x-ct)$$

$$\text{And, } \frac{\partial^2 y}{\partial x^2} = f''(x+ct) + q''(x-ct) \quad \text{..... (ii)}$$

Now, from (i) and (ii),

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

(36) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$ show that $x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution: Let, $\quad u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}} + \frac{y}{x^2+y^2}$$

Multiplying by x,

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{...... (i)}$$

Now,

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{x^2}{y^2}} \times \left(-\frac{x}{y^2}\right)$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} - \frac{x}{(x^2+y^2)}$$

Multiplying by 'y'

$$y\frac{\partial u}{\partial y} = \frac{-xy}{y\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$= -\frac{-x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{..... (ii)}$$

Adding (i) and (ii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

(38) If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$ show that $\frac{\partial z}{\partial v} = \frac{\cos y - e^v x \sin y}{x^2}$

Solution: Let, $\quad z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = \left(-\frac{\cos y}{x^2}\right), \quad \frac{\partial z}{\partial y} = \left(-\frac{\sin y}{x}\right), \quad \frac{\partial x}{\partial v} = -1, \quad \frac{\partial y}{\partial v} = e^v$$

Now,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \left(-\frac{\cos y}{x^2}\right)\frac{\partial x}{\partial v} + \left(-\frac{\sin y}{x}\right)\frac{\partial y}{\partial v} = \left(-\frac{\cos y}{x^2}\right)(-1) + \left(-\frac{\sin y}{x}\right)\times e^v$$

$$= \frac{\cos y}{x^2} - \frac{e^v \sin y}{x}$$

$$= \frac{\cos y - e^v x \sin y}{x^2}$$

Thus, $\frac{\partial z}{\partial v} = \frac{\cos y - e^v x \sin y}{x^2}$.

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

2002 Q. No. 2(b), 2015 Spring 2(a), 2017 Spring 2(a), 2019 Fall 2(a)

State Euler's theorem for partial derivative of homogeneous function of two variables. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Solution: First Part: See the statement of Euler's theorem for partial derivative of homogeneous function of two variables, P.

Second Part:

Let, $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let, $f = \sin u$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x}, \quad \text{and} \quad \frac{\partial u}{\partial x} = \cos u\frac{\partial u}{\partial y}$$

And, $f = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}\,(\sqrt{t}+\sqrt{y})} = t^{1/2}.f$$

This shows that f is homogeneous function of degree $(n) = \frac{1}{2}$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}.f \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow \cos u\left(x\frac{\partial u}{\partial x} + y\frac{1}{2}\right) = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{1}{2} = \frac{1}{2}\tan u$$

Thus, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

2003 Fall Q. No. 2(b)

State and prove Euler's theorem for a homogeneous function of two variable of degree n and hence if $v = \frac{xy}{x+y}$, show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: Let $v = \frac{xy}{x+y}$

Set x by tx and y by ty then

$$v = \frac{t^2 xy}{t(x+y)} = t\left(\frac{xy}{x+y}\right)$$

This shows that v is a homogeneous function of degree (n) = 1.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial v} + y\frac{\partial v}{\partial y} = nv \Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$$

2004 Spring Q. No. 2(b)

What is homogeneous function? State and prove Euler's theorem on homogeneous function of two variables.

Solution: First Part: See definition of homogeneous function.

Second Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

2006 Fall Q. No. 2(b), 2012 Fall Q.No. 2(a)

State and prove Eulers theorem for homogeneous function of two variables with degree n. If $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: Let $f = \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$. For $f = \sin u$.

Differentiating f w. r. to x and y then

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

Also, for $f = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, set x by tx and y by ty.

So, $$f = \frac{t^{1/2}(\sqrt{x}-\sqrt{y})}{t^{1/2}(\sqrt{x}+\sqrt{y})} = t^0\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right).$$

This shows that f is homogeneous of degree (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \Rightarrow \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0.\ f = 0$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

2006 Spring Q. No. 2(b)

Define partial derivative of a function at a point. If $u = \log\sqrt{x^2+y^2+z^2}$, show that: $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$.

Solution: First Part: See definition of partial derivative of a function.

Second Part: See Exercise 10.1 Q. No. 6.

2008 Spring Q. No. 2(b)

Define homogeneous function of two variables. Verify Euler's theorem for $u = x^n\sin(y/x)$.

Solution: First Part: See definition of homogeneous function, P.

Second Part: Let $u = x^n \sin\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = (tx)^n \sin\left(\frac{ty}{tx}\right) = t^n x^n \sin\left(\frac{y}{x}\right) = t^n.u$$

This shows that u is homogeneous function of degree (n) = n.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = nx^{n-1}\sin\left(\frac{y}{x}\right) + x^n\cos\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right) \quad \text{And} \quad \frac{\partial u}{\partial y} = x^n\times\cos\left(\frac{y}{x}\right)\times\frac{1}{x}$$

$$= nx^{n-1}\sin\left(\frac{y}{x}\right) - x^{n-2}y\cos\left(\frac{y}{x}\right) \qquad = x^{n-1}\cos\left(\frac{y}{x}\right)$$

So,

$$x\frac{\partial u}{\partial x} = nx^n\sin\left(\frac{y}{x}\right) - x^{n-1}y\cos\left(\frac{y}{x}\right) \quad \text{And} \quad y\frac{\partial u}{\partial y} = x^{n-1}y\cos\left(\frac{y}{x}\right)$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\sin\left(\frac{y}{x}\right) - x^{n-1}y\cos\left(\frac{y}{x}\right) + x^{n-1}y\cos\left(\frac{y}{x}\right)$$

$$= nx^n\sin\left(\frac{y}{x}\right)$$

$$= n.u$$

Hence, the Euler's theorem is verified.

2009 Fall Q. No. 2(a)

State and prove Euler's theorem for a homogeneous function of two variable of degree n and hence if $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$, $x \neq y$. Show that: $x\frac{\partial u}{\partial u} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 3.

2016 Fall Q. No. 2(a)

State and prove Euler's theorem for a homogeneous function of two variable of degree n. If $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$, $x \neq y$. Then find the value of $x\frac{\partial u}{\partial u} + y\frac{\partial u}{\partial y}$.

2008 Spring Q. No. 2(a); 2011 Fall Q. No. 2(b), 2017 Fall Q.No. 2(a), 2018 Fall Q.NO. 2(b)

State and prove Euler's theorem for homogeneous function of two variables.

If $U = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that: $x\frac{\partial U}{\partial 2x} + y\frac{\partial U}{\partial y} = 1$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 4.

2011 Spring Q. No. 2(a)

If $u = x^3 + y^3 + z^3 - 3xyz$. Prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

2013 Fall Q.No. 2(a), 2015 Fall Q.No 2(a)

State and prove Euler's theorem on homogeneous function of two variable of degree n. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 2.

2014 Fall Q.No. 2(a)

State and prove Euler's theorem on homogeneous function of two variable.

Using it show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$, where $\sin u = \frac{x^2+y^2}{x+y}$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 5.

2014 Spring Q.No. 2(a)

State and prove Euler's theorem on homogeneous function of two variable and hence if $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$ and show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 3.

SHORT QUESTIONS

1999, 2001: If $u = \tan^{-1}\frac{y}{x}$ then $\partial u = \ldots$

Solution: Let $u = \tan^{-1}\left(\frac{y}{x}\right)$. Then,

$$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x^2}\right) = \frac{1}{x^2+y^2}$$

Now,

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy = \frac{1}{x^2+y^2}(-y\,dx + dy).$$

2004 Spring; 2008 Spring: Find $\frac{dy}{dx}$ if $x^3 + y^3 - 3axy = 0$ (using partial derivative).

2006 Fall: If $u = x^2 + y^2 + z^2$, show that: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$.

Solution: Here, $u = x^2 + y^2 + z^2$

Replace x by tx, y by ty and z by tz. Then,

$$u = t^2(x^2 + y^2 + z^2)$$

This shows that u is a homogeneous function of order (n) = 2.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \Rightarrow x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$$

2006 Spring: If $v = x^3 + y^3 + z^3$ than verify that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 3v$.

Solution: Here, $v = x^3 + y^3 + z^3$

Replace x by tx, y by ty and z by tz. Then,

$$u = t^3(x^3 + y^3 + z^3)$$

This shows that u is a homogeneous function of order (n) = 3.

Then by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nu \Rightarrow x\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 3v.$$

2009 Spring: If $V = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then find the value of $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$.

Solution: Let, $v = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

Replace x by tx, y by ty and z by tz. Then

$$v = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx} = t^0\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$$

This shows that v is homogeneous of order (n) = 0. Then by Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{y} + z\frac{\partial v}{\partial z} = nv \Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0.$$

2009 Fall: If $f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{x}{x}$, then show that: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0$.

Solution: See 2009 Spring with replacing V by f.

2010 Spring, 2011 Fall: Verify the Eulers' theorem for $f(x, y) = x/y$.

Solution: Let $f = \frac{x}{y}$

Set x by tx and y by ty. Then,

$$f = \frac{tx}{ty} = t^{0}\left(\frac{x}{y}\right)$$

This shows that f is homogeneous of order (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 0 \qquad \text{........ (i)}$$

And we have, $f = \frac{x}{y}$

So, $\frac{\partial f}{\partial x} = \frac{y}{y^2}$ And $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$.

Now,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{xy}{y^2} - \frac{xy}{y^2} = 0$$

This verifies (i).

Thus, f verifies the Euler's theorem.

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