Similar Question for Practice from Final Exam:

2000; 2004 Fall Q. No. 5(b)

Solve the following initial value problem, $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$

2007 Fall Q. No. 3(b)

Solve the following initial value problem, $y'' + 4y' + 5y = 0$, $y(0) = 2$, $y'(0) = -3$.

2002 Q. No. 5(b)

Solve the following initial value problem. $y'' + 2y' - 3y = 6e^{-2t}$, $y(0) = 2$, $y'(0) =$ 14.

2003 Fall Q. No. 5(b)

Solve $y'' + 4y' + 4y = \sin t$; $y(0) = 1$, $y'(0) = 3$.

2008 Spring Q. No. 5(b)

Solve the initial value problem: $y'' + y' - y = 14 + 2x - 2x^2$, $y(0) = 0$, $y'(0) = 0$.

2010 Spring Q. No. 5(b)

Solve the initial value problem: $y'' + y = 2\cos x$, where $y(0) = 3$ and $y'(0) = 4$.

2006 Fall Q. No. 5(b)

Solve the initial value problem $y'' - y' - 2y = 3e^{2x}$, $y(0) = 0$, $y'(0) = -2$

Short Questions

1999; 2001: Find the roots of the characteristic equation of the differential equation $y'' + p^2y = 0$.

Solution: Given differential equation is

$$y'' = p^2y = 0 \qquad \text{.........(1)}$$

The characteristic equation of (i) is

$$m^2 + p^2 = 0 \quad \Rightarrow \quad m^2 = -p^2 = (pi)^2$$

$$\Rightarrow \quad m = \pm pi$$

These are required root of characteristic equation of (i).

2000: Find the roots of the characteristic equation of the differential equation: $y'' - 2y' + 10y = 0$.

Solution: Given differential equation is

$$y'' - 2y' + 10y = 0 \qquad \text{...... (i)}$$

The characteristic equation of (i) is

$$m^2 - 2m + 10 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm \sqrt{(6i)^2}}{2} = 1 \pm 3i$$

These are required root of the characteristic equation of (i).

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Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 4x^2$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_1x^2 + c_2x + c_3 \quad \text{...... (v)}$$

Then,

$$y'_p = 2c_1x + c_2 \quad \text{and} \quad y''_p = 2c_1$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$2c_1 + 3(2c_1x + c_2) + 2(c_1x^2 + c_2x + c_3) = 4x^2$$

$$\Rightarrow \quad 2c_1 + 6c_1x + 3c_2 + 2c_1x^2 + 2c_2x + 2c_3 = 4x^2$$

$$\Rightarrow \quad 2c_1x^2 + x(6c_1 + 2c_2) + (2c_1 + 3c_2 + 2c_3) = 4x^2$$

Comparing coefficient on both side then,

$$2c_1 = 4, \qquad 6c_1 + 2c_2 = 0, \qquad 2c_1 + 3c_2 + 2c_3 = 0.$$

Solving we get, $c_1 = 2$, $c_2 = -6$, $c_3 = 7$.

Therefore, (v) becomes, $y_p = 2x^2 - 6x + 7$.

Now, general equation of (i) is,

$$y(x) = y_p + y_h(x) = c_1e^{-x} + c_2e^{-2x} + 2x^2 - 6x + 7.$$

2004 Fall; 2006 Fall Q. No. 4(b) OR

Solve by the method of variation of parameters: $y'' + y = \tan x$.

Solution: Given equation is,

$$y'' + y = \tan x. \quad \text{... (i)}$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0 \quad \text{... (iii)}$$

Its auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = (A \cos x + B \sin x) \quad \text{... (iv)}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x,$$

then, $$y'_1 = -\sin x \quad \text{and} \quad y_2' = \cos x$$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y'_2 - y_2y'_1 = \cos x . \cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1.$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \tan x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1 \int \left(\frac{y_2R}{W}\right) dx + y_2 \int \left(\frac{y_1R}{W}\right) dx$$

$$= -\cos x \int \frac{\sin x \tan x}{1} dx + \sin x \int \frac{\cos x . \tan x}{1}$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x \, dx$$

$$= -\cos x \int \frac{1 - \cos^2 x}{\cos x} dx + \sin x \, (-\cos x)$$

$$= -\cos x \int (\sec x - \cos x) \, dx - \sin x . \cos x$$

$$= -\cos x \, \{\log(\sec x + \tan x) - \sin x\} - \sin x . \cos x$$

$$= -\cos x \, \{\log(\sec x + \tan x)\} + \cos x \sin x - \sin x . \cos x$$

$$= -\cos x \, \{\log(\sec x + \tan x)\}.$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = A \cos x + B \sin x - \cos x \, \{\log(\sec x + \tan x)\}.$$

Similar Question for Practice from Final Exam:

2006 Spring Q. No. 4(b)

Solve: $y'' - 4y' + 4y = x^2 + e^{2x}$

2007 Fall Q. No. 4(b)

Solve the equation $y'' - 5y' + 6y = e^{2x}$

2010 Spring Q. No. 4(b)

Solve: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$.

2011 Fall Q. No. 4(b)

Solve $y'' + 2y' + 2y = 4e^{-x} \sec^3 x$.

Extra Questions (Long Questions)

1999; 2001 Q. No. 5(b)

Solve the following initial value, problem $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = -3$

Solution: Given that,

$$y'' + 5y' + 6y = 0 \quad \text{....... (i)}$$

$$y(0) = 2, \; y'(0) = -3 \quad \text{....... (ii)}$$

The auxiliary equation of of (i) is,

$$m^2 + 5m + 6 = 0 \Rightarrow (m + 2)(m + 3) = 0$$

$$\Rightarrow m = -2, -3$$

Here m has real and distinct values, so the solution of (i) is

$$y(x) = c_1e^{-2x} + c_2e^{-3x} \quad \text{... (iii)}$$

Using $y(0) = 2$ by (ii) to (vii) then

$$2 = c_1 + c_2 \quad \text{— (A)}$$

Differentiating (iii), we get,

$$y'(x) = -2c_1e^{-2x} - 3c_2e^{-3x}$$

Using $y'(0) = -3$ by (ii), then

$$-3 = -2c_1 - 3c_2 \quad \text{— (B)}$$

Solving (A) and (B) we get, $c_1 = 3$, $c_2 = -1$.

Now (vii) becomes

$$y(x) = 3e^{-2x} - e^{-3x}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 2y' + 2y = 0 \qquad \dots (iii)$$

The auxiliary equation of of (iii) is,

$$m^2 - 2m + 2 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm 2i.$$

Here m has complex values, so the solution of (iii) is

$$y_h(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) \qquad \dots (iv)$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 2e^x \cos x$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = e^x(c_3 \sin x + c_4 \cos x) \qquad \dots\dots (v)$$

Then,

$$y'_p = e^x(c_3 \cos x - c_4 \sin x) + e^x(c_3 \sin x + c_4 \cos x)$$
$$= e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)]$$
$$y''_p = e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)] + e^x[-(c_3 + c_4)\sin x + (c_3 - c_4)\cos x)]$$
$$= e^x[(c_3 + c_4 + c_3 - c_4)\cos x + (c_3 - c_4 - c_3 - c_4)\sin x)]$$
$$= e^x[2c_3 \cos x - 2c_4 \sin x)]$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$e^x[2c_3 \cos x - 2c_4 \sin x)] - 2e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)] = 2e^x \cos x$$
$$\Rightarrow \quad 2e^x[(c_3 - c_3 - c_4)\cos x + (-c_4 + c_3 - c_4)\sin x)] = 2e^x \cos x$$
$$\Rightarrow \quad [(-c_4)\cos x + (c_3 - 2c_4)\sin x)] = \cos x$$

Comparing coefficient on both side then,

$$-c_4 = 0 \quad \text{and} \quad c_3 - 2c_4 = 1$$

Solving we get

$$c_4 = 0 \text{ and } c_3 = \frac{1}{2}$$

So the equation (v) becomes,

$$y_p = \frac{1}{2}e^x \cos x$$

Now, general equation of (i) is,

$$y(x) = y_p + y_h(x) = c_1 + c_2 e^x - \frac{1}{2}e^x \cos x$$

<u>2000 Q. No. 4(b)</u>

Find the general solution of the differential equation, $y'' + 4y' + 3y = \sin x + 2\cos x$.

Solution: Given that,

$$y'' + 4y' + 3y = \sin x + 2\cos x \qquad \dots\dots (i)$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 4y' + 3y = 0 \qquad \dots (iii)$$

The auxiliary equation of of (iii) is,

$$m^2 + 4m + 3 = 0 \Rightarrow (m+3)(m+1) = 0 \Rightarrow m = -1, -3.$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-3x} \qquad \dots (iv)$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = \sin x + 2\cos x$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3 \cos x + c_4 \sin x \qquad \dots\dots (v)$$

Then,

$$y'_p = -c_3 \sin x + c_4 \cos x \quad \text{and} \quad y''_p = -c_3 \cos x - c_4 \sin x$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$-c_3 \cos x - c_4 \sin x - (-c_3 \sin x + c_4 \cos x) - 2(c_3 \cos x + c_4 \sin x) = \sin x + 2\cos x.$$
$$\Rightarrow \quad \cos x(-c_3 - c_4 - 2c_3) + \sin x(-c_4 + c_3 - 2c_4) = \sin x + 2\cos x.$$
$$\Rightarrow \quad \cos x(-3c_3 - c_4) + \sin x(-c_3 + 3c_4) = \sin x + 2\cos x.$$

Comparing the coefficient on both side, then,

$$-3c_3 - c_4 = 2, \qquad c_3 - 3c_4 = 1.$$

Solving we get, $c_3 = \frac{-1}{2}$ and $c_4 = \frac{-1}{2}$.

So the equation (v) becomes,

$$y_p = \frac{-1}{2}(\cos x + \sin x)$$

Now, general equation of (i) is,

$$y(x) = y_p + y_h(x) = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2}(\cos x + \sin x)$$

<u>2002 Q. No. 4(b)</u>

Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$.

Solution: Given that,

$$y'' + 3y' + 2y = 4x^2 \qquad \dots\dots (i)$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 3y' + 2y = 0 \qquad \dots (iii)$$

The auxiliary equation of of (iii) is,

$$m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -1, -2.$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} \qquad \dots (iv)$$

(16) $(D^2 + 4D + 4)\,y = \dfrac{2e^{-2x}}{x^2}$

Solution: Given equation is,

$$(D^2 + 4D + 4)\,y = \frac{2e^{-2x}}{x^2}$$

$$\Rightarrow \quad y'' + 4y' + 4y = \frac{2e^{-2x}}{x^2} \qquad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y' + 4y = 0 \qquad \dots (iii)$$

So, its auxiliary equation is

$$m^2 + 4m + 4 = 0 \Rightarrow (m + 2)^2 = 0 \Rightarrow m = -2, -2.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\,e^{-2x}$$

And, for particular solution, comparing (iv) with $y_h = c_1y_1 + c_2y_2$ then we get

$$y_1 = e^{-2x} \quad \text{and} \quad y_2 = xe^{-2x}$$

So, $\quad y_1' = -2e^{-2x} \quad$ and $\quad y_2' = (e^{-2x} - 2xe^{-2x})$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^{-2x}(e^{-2x} - 2xe^{-2x}) + xe^{-2x}.\,2e^{-2x}$$
$$= e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \frac{2e^{-2x}}{x^2}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{W}\right)dx + y_2\int\left(\frac{y_1R}{W}\right)dx$$
$$= -e^{-2x}\int\left(\frac{xe^{-2x}}{e^{-4x}}.\frac{2e^{-2x}}{x^2}\right)dx + xe^{-2x}\int\left(\frac{e^{-2x}}{e^{-4x}}.\frac{2e^{-2x}}{x^2}\right)dx$$
$$= -e^{-2x}\int\left(\frac{2}{x}\right)dx + xe^{-2x}\int\left(\frac{2}{x^2}\right)dx$$
$$= -2e^{-2x}\log x - 2e^{-2x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\,e^{-2x} - 2e^{-2x}\log x - 2e^{-2x})$$
$$= (c_1 + c_2x - 2\log x - 2)\,e^{-2x}$$

(17) $y'' + 4y = 4\tan 2x$

Solution: Given equation is,

$$y'' + 4y = 4\tan 2x. \qquad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y = 0 \qquad \dots (iii)$$

So, its auxiliary equation is,

$$m^2 + 4 = 0 \Rightarrow m^2 = 4i^2 \Rightarrow m = \pm 2i$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = A\cos 2x + B\sin 2x$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = \cos 2x \quad \text{and} \quad y_2 = \sin 2x$$

So, $\quad y_1' = -2\sin 2x \quad$ and $\quad y_2' = 2\cos 2x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = \cos 2x \times 2\cos 2x + \sin 2x.\,2\sin 2x$$
$$= 2(\cos^2 2x + \sin^2 2x)$$
$$= 2$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 4\tan 2x.$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{W}\right)dx + y_2\int\left(\frac{y_1R}{W}\right)dx$$
$$= -\cos 2x\int\left(\frac{\sin 2x\,4\tan 2x}{2}\right) + 4\sin 2x\int\left(\frac{\cos 2x.\,\tan 2x}{2}\right)dx$$
$$= -\cos 2x\int\left(\frac{2\sin^2 2x}{\cos 2x}\right)dx + 2\sin 2x\int \sin 2x\,dx$$
$$= -2\cos 2x\int\left(\frac{1 - \cos^2 2x}{\cos 2x}\right)dx + 2\sin 2x\int \sin 2x\,dx$$
$$= -2\cos 2x\int(\sec 2x - \cos 2x)\,dx + 2\sin 2x\int \sin 2x\,dx$$
$$= -2\cos 2x\left\{\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right\} + 2\sin 2x\,\frac{(-\cos 2x)}{2}$$
$$= -2\cos 2x\log(\sec 2x + \tan 2x) + \sin 2x.\,\cos 2x - \sin 2x\cos 2x$$
$$= \cos 2x\log(\sec 2x + \tan 2x)$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = A\cos 2x + B\sin 2x - \cos 2x\log(\sec 2x + \tan 2x)$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

Second Order Differential Equation

1999 Q. No. 4(b); 2001 Q. No. 4(b)

Find the general solution of the differential equation. $y'' - 2y' + 2y = 2e^x\cos x$.

Solution: Given that,

$$y'' - 2y' + 2y = 2e^x\cos x \qquad \dots\dots (i)$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$

$$= -e^{-x}\int\left(\frac{xe^{-x}e^{-x}\cos x}{e^{-2x}}\right) + xe^{-x}\int\left(\frac{e^{-x}e^{-x}\cos x}{e^{-2x}}\right)dx$$

$$= -e^{-x}\int x\cos x\,dx + xe^{-x}\int \cos x\,dx$$

$$= -e^{-x}(x\sin x + \cos x) + xe^{-x}\sin x$$

$$= -xe^{-x}\sin x - e^{-x}\cos x + xe^{-x}\sin x = -e^{-x}\cos x$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^{-x} - e^{-x}\cos x = (c_1 + c_2x - \cos x)e^{-x}$$

(14) $y'' - 2y' + y = \frac{e^x}{x^3}$ **[2015 Spring Q.No. 4(b)]**

Solution: Given equation is,

$$y'' - 2y' + y = \frac{e^x}{x^3} \quad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0 \quad \dots (iii)$$

So, its auxiliary equation is,

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)e^x \quad \dots (iv)$$

And, for particular solution, comparing (iv) with $y_h = c_1y_1 + c_2y_2$ then we get

$y_1 = e^x$ and $y_2 = xe^x$

So, $y_1' = e^x$ and $y_2 = e^x + xe^x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^x(e^x + xe^x) - xe^x.e^x = e^{2x} + xe^{-2x} + xe^{-2x} = e^{2x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \frac{e^x}{x^3}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$

$$= -e^x\int\left(\frac{xe^x}{e^{2x}}\frac{e^x}{x^3}\right)dx + xe^x\int\left(\frac{e^x}{e^{2x}}\frac{e^x}{x^3}\right)dx$$

$$= -e^x\int x^{-2}\,dx + xe^x\int x^{-3}\,dx$$

$$= -\frac{e^x}{-x} + \frac{xe^x}{-2x^2} = \frac{e^x}{2x}$$

Now (ii) becomes, $y(x) = y_h(x) + y_p = (c_1 + c_2x)e^x + \frac{e^x}{2x}$

(15) $(D^2 - 2D + 1)y = 3x^{3/2}e^x$

Solution: Given equation is,

$$(D^2 - 2D + 1)y = 3x^{3/2}e^x$$

$$\Rightarrow y'' - 2y' + y = 3x^{3/2}e^x \quad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0$$

So, its auxiliary equation is,

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)e^x$$

And, for particular solution, comparing (iv) with $y_h = c_1y_1 + c_2y_2$ then we get

$y_1 = e^x$ and $y_2 = xe^x$

So, $y_1' = e^x$ and $y_2' = e^x + xe^x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^x(e^x + xe^x) - xe^x.e^x = e^{2x} + xe^{-2x} + xe^{-2x} = e^{2x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 3x^{3/2}e^x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$

$$= -e^x\int\left(\frac{xe^x\,3x^{3/2}e^x}{e^{2x}}\right)dx + xe^x\int\left(\frac{e^x\,3x^{3/2}e^x}{e^{2x}}\right)dx$$

$$= -e^x\int 3x^{5/2}\,dx + xe^x\int 3x^{3/2}\,dx.$$

$$= -3e^x \times \frac{2}{7}x^{7/2} + 3xe^x.\frac{2}{5}x^{5/2} = -\frac{6}{7}e^x x^{7/2} + \frac{6}{5}e^x.x^{7/2}$$

$$= \frac{-30e^x x^{7/2} + 42e^x x^{7/2}}{35} = \frac{12e^x}{35}x^{7/2}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^x + \frac{12}{35}e^x x^{7/2}$$

$$= \left(c_1 + c_2x + \frac{12}{35}x^{7/2}\right)e^x$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^{-2x}\cos x$ and $y_2 = e^{-2x}\sin x$

So, $y_1' = -2e^{-2x}\cos x - \sin x\, e^{-2x}$ and $y_2' = -2e^{-2x}\sin x + e^{-2x}\cos x$

So, the Wronskian is,

$$\begin{aligned}W(y_1, y_2) &= y_1y_2' - y_2y_1'\\ &= e^{-2x}\cos x\,(e^{-2x}\cos x - 2e^{-2x}\sin x)\\ &\quad - e^{-2x}\sin x\,(-2e^{-2x}\cos x - \sin x e^{-2x})\\ &= e^{-4x}\end{aligned}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 10$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned}y_p &= -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx\\ &= -e^{-2x}\cos x\int\left(\frac{e^{-2x}\sin x\,10}{e^{-4x}}\right)dx + e^{-2x}\sin x\int\left(e^{-2x}\cos x.\frac{10}{e^{-4x}}\right)dx\\ &= -10e^{-2x}\cos x\int(e^{2x}\sin x)\,dx + 10e^{-2x}\sin x\int(e^{2x}\cos x)\,dx\\ &= -10e^{-2x}\cos x\,\frac{e^{2x}}{5}(2\sin x - \cos x)\\ &\qquad + 10e^{-2x}\sin x\,\frac{e^{2x}}{5}(2\cos x - \sin x)\\ &= -2\cos x\,(2\sin x - \cos x) + 2\sin x\,(2\cos x - \sin x)\\ &= -4\cos x\sin x + 2\cos^2 x + 4\sin x\cos x - 2\sin^2 x\\ &= 2(\cos^2 x + \sin^2 x) = 2\end{aligned}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = e^{-2x}(A\cos x + B\sin x) + 2$$

(12) $y'' - 4y + 4y = \dfrac{e^{2x}}{x}$

Solution: Given equation is,

$$y'' - 4y + 4y = \frac{e^{2x}}{x} \qquad \ldots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \ldots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 4y' + 4y = 0$$

So, its auxiliary equation is,

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\,e^{2x} \qquad \ldots (iv)$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$

So, $y_1' = 2e^{2x}$ and $y_2' = 2xe^{2x} + e^{2x}$

R =

Also,

So, the Wronskian is,

$$\begin{aligned}W(y_1, y_2) &= y_1y_2' - y_2y_1' = e^{2x}(2xe^{2x} + e^{2x}) - xe^{2x}(2e^{2x})\\ &= 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}\end{aligned}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \frac{e^{2x}}{x}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned}y_p &= -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx\\ &= -e^{2x}\int\left(\frac{xe^{2x}}{e^{4x}}.\frac{e^{2x}}{x}\right)dx + xe^{2x}\int\left(\frac{e^{2x}}{e^{4x}}.\frac{e^{2x}}{x}\right)dx\\ &= -xe^{2x} + x\log x\, e^{2x}\end{aligned}$$

Now (ii) becomes,

$$\begin{aligned}y(x) = y_h(x) + y_p &= (c_1 + c_2x)\,e^{2x} + x\log x e^{2x} - xe^{2x}\\ &= (c_1 + c_2x + x\log x - x)\,e^{2x}\end{aligned}$$

(13) $y'' + 2y' + y = e^{-x}\cos x$ **[2017 Fall Q.No. 5(b)]**

Solution: Given equation is,

$$y'' + 2y' + y = e^{-x}\cos x \qquad \ldots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \ldots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0 \qquad \ldots (iii)$$

So, its auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\,e^{-x} \qquad \ldots (iv)$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^{-x}$ and $y_2 = xe^{-x}$

So, $y_1' = -e^{-x}$, and $y_2' = e^{-x} - xe^{-x}$

So, the Wronskian is,

$$\begin{aligned}W(y_1, y_2) = y_1y_2' - y_2y_1' &= e^{-x}(e^{-x} - xe^{-x}) + xe^{-x}e^{-x}\\ &= e^{-2x} - xe^{2x} + xe^{-2x}\\ &= e^{-2x}\end{aligned}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = e^{-x}\cos x$$

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$
$$= -\cos x\int\left(\frac{\sin x.\sin x}{1}\right)dx + \sin x\int \sin x.\cos x\,dx$$
$$= -\cos x\int\left(\frac{1-\cos 2x}{2}\right)dx + \frac{\sin x}{2}\int \sin 2x\,dx$$
$$= -\cos x\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right) + \frac{\sin x}{2}\left(\frac{-\cos 2x}{2}\right)$$
$$= \frac{-x\cos x}{2} + \frac{\sin 2x.\cos x}{4} - \frac{\cos 2x.\sin x}{4}$$
$$= \frac{-x\cos x}{2} + \frac{1}{4}\sin(2x-4) = \frac{-x\cos x}{2} + \frac{\sin x}{4}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = A\cos x + B\sin x - \frac{x\cos x}{2} + \frac{\sin x}{4}.$$

(9) $y'' + 2y' + y = e^{-x}$

Solution: Given equation is,

$$y'' + 2y' + y = e^{-x} \qquad \dots (i)$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0 \qquad \dots (iii)$$

So, its auxiliary equation is,

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\,e^{-x}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^{-x}$ and $y_2 = xe^{-x}$

So, $y_1' = -e^{-x}$ and $y_2' = -xe^{-x} + e^{-x}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^{-x}(e^{-x} - xe^{-x}) - xe^{-x}(-e^{-x})$$
$$= e^{-2x} - xe^{-2x} + xe^{-2x} \quad = e^{-2x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = e^{-x}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$
$$= -e^{-x}\int\left(\frac{xe^{-x}e^{-x}}{e^{-2x}}\right)dx + xe^{-x}\int\left(\frac{e^{-x}e^{-x}}{e^{-2x}}\right)dx$$
$$= -e^{-x}\frac{x^2}{2} + xe^{-x}x \ = x^2e^{-x} - \frac{x^2}{2}e^{-x} = \frac{x^2e^{-x}}{2}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\,e^{-x} + \frac{x^2}{2}e^{-x}$$

(10) $y'' - y = e^x$

Solution: Given equation is,

$$y'' - y = e^x. \qquad \dots (i)$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - y = 0 \qquad \dots (iii)$$

So, its auxiliary equation is,

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

Here, m has real and distinct value, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{-x}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^x$ and $y_2 = e^{-x}$

So, $y_1' = e^x$ and $y_2' = -e^{-x}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^x \times e^{-x} - e^{-x}.e^x = -1 - 1 = -2$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = e^x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$
$$= -e^x\int\left(\frac{e^{-x}e^x}{-2}\right)dx + e^{-x}\int\left(\frac{e^xe^x}{-2}\right)dx$$
$$= \frac{xe^x}{2} - \frac{e^{-x}e^{2x}}{4} = \frac{xe^x}{2} - \frac{e^x}{4}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = c_1e^x + c_2e^{-x} + \frac{x}{2}e^x - \frac{e^x}{4}$$

(11) **$y'' + 4y' + 5y = 10$** **[2003 Fall Q. No. 4(b)]**

Solution: Given equation is,

$$y'' + 4y' + 5y = 10 \qquad \dots (i)$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y' + 5y = 0 \qquad \dots (iii)$$

So, its auxiliary equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = e^{-2x}(A\cos x + B\sin x) \qquad \dots (iv)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0 \qquad \text{... (iii)}$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

Here, m has real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\, e^x$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = e^x \quad \text{and} \quad y_2 = xe^x$$

So, $\quad y_1' = e^x \quad \text{and} \quad y_2' = xe^x$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 35e^x x^{3/2}$$

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= e^x (xe^x + e^x) - xe^x e^x \\ &= e^{2x} \end{aligned}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned} y_p &= -y_1 \int \left(\frac{y_2R}{w}\right) dx + y_2 \int \left(\frac{y_1R}{w}\right) dx \\ &= -e^x \int \left(\frac{xe^x\, 35e^x x^{3/2}}{e^{2x}}\right) dx + xe^x \int \left(\frac{e^x\, 35e^x x^{3/2}}{e^{2x}}\right) dx \\ &= -35e^x \int x^{5/2}\, dx + 35\, xe^x \int x^{3/2}\, dx \\ &= -35\, e^x \times \frac{2}{7} x^{7/2} + 35xe^x \times \frac{2}{5} x^{5/2} \\ &= -10x^{7/2} e^x + 14x^{5/2} e^x \\ &= 4e^x x^{7/2} \end{aligned}$$

Now (ii) becomes,

$$\begin{aligned} y(x) &= y_h(x) + y_p = (c_1 + c_2x)\, e^x + 4e^x x^{7/2} \\ &= e^x (c_1 + c_2x + 4x^{7/2}). \end{aligned}$$

(7) $y'' + y' = x$

Solution: Given equation is,

$$y'' + y' = x. \qquad \text{... (i)}$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y' = 0 \qquad \text{... (iii)}$$

Its auxiliary equation is,

$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1.$

Here, m has real and distinct value, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^{-x}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = 1, \quad$ and $\quad y_2 = e^{-x}$

So, $\quad y_1' = 0 \quad$ and $\quad y_2' = -e^{-x}$

So, the Wronskian is,

$W(y_1, y_2) = y_1y_2' - y_2y_1' = 1(-e^{-x}) - (e^{-x}\, 0) = -e^{-x}$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = x.$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned} y_p &= -y_1 \int \left(\frac{y_2R}{w}\right) dx + y_2 \int \left(\frac{y_1R}{w}\right) dx \\ &= -1 \int \left(\frac{e^{-x}x}{-e^{-x}}\right) + e^{-x} \int \left(\frac{x}{-e^{-x}}\right) dx \\ &= \int x\, dx + e^{-x} \int -x\, e^x .\, dx = \frac{x^2}{2} - e^{-x}(xe^x - e^x) \\ &= \frac{x^2}{2} - x + 1. \end{aligned}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{-x} + \frac{x^2}{2} - x + 1.$$

(8) $y'' + y = \sin x$

Solution: Given equation is,

$$y'' + y = \sin x. \qquad \text{... (i)}$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0 \qquad \text{... (iii)}$$

So, its auxiliary equation

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = A \cos x + B \sin x$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x$$

So, $\quad y_1' = -\sin x \quad$ and $\quad y_2' = \cos x$

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) = y_1y_2' - y_2y_1' &= \cos x \cos x + \sin x . \sin x \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \sin x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0$$

Its auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

Here, m has real and repeated values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\, e^{-x}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = xe^{-x}$$

So,

$$y_1' = -e^{-x} \quad \text{and} \quad y_2' = -xe^{-x} + e^{-x}$$

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= e^{-x}(-xe^{-x} + e^{-x}) - xe^{-x}(-e^{-x}) \\ &= -xe^{-2x} + e^{-2x} + xe^{-2x} = e^{-2x} \end{aligned}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 4e^{-x}\log x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned} y_p &= -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx \\ &= -e^{-x}\int\left(\frac{xe^{-x}\,4e^{-x}\log x}{e^{-2x}}\right)dx + xe^{-x}\int\left(\frac{e^{-x}\,4e^{-x}\log x}{e^{-2x}}\right)dx \\ &= -e^{-x}\int\left(\frac{4xe^{-2x}\log x}{e^{-2x}}\right)dx + xe^{-x}\int\left(\frac{4e^{-2x}\log x}{e^{-2x}}\right)dx \\ &= -4e^{-x}\left[\log x\int x\,dx - \int\left(\frac{d\log x}{dx}\int x\,dx\right)dx\right] + 4xe^{-x} \\ &\qquad \left\{\log x\int dx - \int\left(\frac{d\log x}{dx}\int dx\right)dx\right\} \\ &= -4e^{-x}\left(\frac{x^2}{2}\log x - \int\frac{1}{x}\,\frac{x^2}{2}dx\right) + 4xe^{-x}\left(x\log x - \int\frac{1}{x}.\,x\,dx\right) \\ &= -4e^{-x}\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + 4xe^{-x}(x\log x - x) \\ &= -2x^2e^{-x}\log x + x^2e^{-x} + 4x^2e^{-x}\log x - 4x^2e^{-x} = 2x^2e^{-x}\log x - 3x^2e^{-x} \end{aligned}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\,e^{-x} + x^2e^{-x}(2\log x - 3).$$

(5) $y'' + 2y' + 2y = 2e^{-x}\sec^3 x$ **[2016 Spring Q.No. 5 (b)]**

Solution: Given equation is,

$$y'' + 2y' + 2y = 2e^{-x}\sec^3 x \quad \dots (i)$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + 2y = 0$$

Its auxiliary equation is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{4i^2}}{2} = \frac{-2 \pm 2i}{2} = (-1 \pm i).$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = e^{-x}(A\cos x + B\sin x)$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = e^{-x}\cos x \quad \text{and} \quad y_2 = -e^{-x}\sin x$$

So,

$$y_1' = -e^{-x}\cos x - e^{-x}\sin x \quad \text{and} \quad y_2' = -e^{-x}\sin x + e^{-x}\cos x$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 2e^{-x}\sec^3 x$$

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= e^{-x}\cos x(-e^{-x}\sin x + e^{-x}\cos x) - e^{-x}\sin x(-e^{-x}\cos x - e^{-x}\sin x) \\ &= -e^{-2x}\cos x.\sin x + e^{-2x}\cos^2 x + e^{-2x}\sin x.\cos x + e^{-2x}\sin^2 x \\ &= e^{-2x} \end{aligned}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$\begin{aligned} y_p &= -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx \\ &= -e^{-x}\cos x\int\left(\frac{e^{-x}\sin x \times 2e^{-x}\sec^3 x}{e^{-2x}}\right)dx + \\ &\qquad e^{-x}\sin x\int\left(\frac{e^{-x}\cos x \times 2e^{-x}\sec^3 x}{e^{-2x}}\right)dx \\ &= -e^{-x}\cos x\int 2\tan x\sec^2 x\,dx + e^{-x}\sin x\int\sec^2 x\,dx \end{aligned}$$

Put, $\tan x = v$ then, $dv = \sec^2 x\,dx$. So.

$$\begin{aligned} &= -e^{-x}\cos x\int 2v\,dv + 2e^{-x}\sin x\int dv \\ &= -e^{-x}\cos x\,v^2 + 2e^{-x}\sin x\,v \\ &= -e^{-x}\cos x\tan^2 x + 2e^{-x}\sin x.\tan x \\ &= e^{-x}(-\cos x\tan^2 x + 2\sin x.\tan x) \\ &= e^{-x}\left(-\cos x.\frac{\sin^2 x}{\cos^2 x} + 2\sin x.\tan x\right) \\ &= e^{-x}(-\tan x.\sin x + 2\sin x.\tan x) = e^{-x}(\sin x\tan x). \end{aligned}$$

Now (ii) becomes,

$$\begin{aligned} y(x) = y_h(x) + y_p &= e^{-x}(A\cos x + B\sin x) + e^{-x}\sin x.\tan x \\ &= e^{-x}(A\cos x + B\sin x + \sin x.\tan x) \end{aligned}$$

(6) $y'' - 2y' + y = 35e^x x^{3/2}$

Solution: Given equation is,

$$y'' - 2y' + y = 35e^x x^{3/2} \quad \dots (i)$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

(2) $y'' - 2y' + y = \frac{12e^x}{x^3}$ **[2013 Fall Q.No. 4(b), 2012 Fall Q.No. 4(b)]**

Solution: Given equation is,

$$y'' - 2y' + y = \frac{12e^x}{x^3} \quad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0 \quad \dots (iii)$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

Here, m has repeated real values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\, e^x$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^x$ and $y_2 = xe^x$,

So, $y'_1 = e^x$ and $y'_2 = xe^x + e^x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^x(xe^x + e^x) - xe^x . e^x = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \frac{12e^x}{x^3}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$
$$= -e^x\int\left(\frac{xe^x}{e^{2x}} \times \frac{12e^x}{x^3}\right)dx + xe^x\int\left(\frac{e^x}{e^{2x}} . \frac{12e^x}{x^3}\right)dx$$
$$= -e^x\, 12\int x^{-2}\,dx + 12xe^x\int x^{-3}\,dx$$
$$= -12e^x\left(\frac{-1}{x}\right) + 12xe^x\left(\frac{-1}{2x^2}\right)$$
$$= \frac{12e^x}{x} - \frac{6e^x}{x} = \frac{6e^x}{x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^x + \frac{6e^x}{x}.$$

(3) $y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x}$

[2013 Spring Q.No. 4(b), 2009 Fall Q. No. 5(b). 2008 Spring Q. No. 4(b)]

Solution: Given equation is,

$$y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x} \quad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 4y' + 4y = 0 \quad \dots (iii)$$

Its auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$$

Here, m real and repeated value, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\, e^{2x}$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$

Then, $y'_1 = 2e^{2x}$ and $y'_2 = e^{2x} + 2xe^{2x}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^{2x}(e^{2x} + 2xe^{2x}) - xe^{2x} . 2e^{2x}$$
$$= e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = 6 + \frac{e^{2x}}{x}$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$
$$= -e^{2x}\int\left(\frac{xe^{2x}\left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}}\right) + xe^{2x}\int\left(\frac{e^{2x}\left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}}\right)dx$$
$$= -e^{2x}\int\left(\frac{6xe^{2x}}{e^{4x}} + \frac{xe^{4x}}{xe^{4x}}\right)dx + xe^{2x}\int\left(\frac{6e^{2x}}{e^{4x}} + \frac{e^{4x}}{xe^{4x}}\right)dx$$
$$= -e^{2x}\int(6xe^{-2x} + 1)\,dx + xe^{2x}\int(6e^{-2x} + x^{-1})\,dx$$
$$= -e^{2x}\left\{6\left(\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right) + x\right\} + xe^{2x}\left(\frac{6e^{-2x}}{-2} + \log x\right)$$
$$= 3x + \frac{3}{2} - xe^{2x} - 3x + xe^{2x}\log x$$
$$\Rightarrow \quad y_p = 1.5 - xe^{2x} + x\log xe^{2x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= (c_1 + c_2x)\, e^{2x} + x\log x\, e^{2x} - xe^{2x} + 1.5$$
$$= (c_1 + c_2x + x\log x - x)\, e^{2x} + 1.5$$

(4) $y'' + 2y' + y = 4e^{-x}\log x$

Solution: Given equation is,

$$y'' + 2y' + y = 4e^{-x}\log x \quad \dots (i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots (ii)$$

(ix) $y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x,\ y(0) = 0.2,\ y'(0) = 60.1$

Solution: Given that,

$$y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x \quad \dots(i)$$

$$y(0) = 0.2,\ y'(0) = 60.1 \quad \dots(ii)$$

Let the solution of (i) is

$$y = y_h + y_p \quad \dots(iii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 2y' + 5y = 0 \quad \dots(iv)$$

The auxiliary equation of of (iv) is,

$$m^2 + 2m + 5 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = (-1 \pm 2i)$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = e^{-x}(A\cos 2x + B\sin 2x) \quad \dots(v)$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$$

Clearly R has no repeated value to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_1e^{0.5x} + c_2\cos 4x + c_3\sin 4x \quad \dots(vi)$$

Then,

$$y_p' = 0.5c_1e^{0.5x} - 4c_2\sin 4x + 4c_3\cos 4$$

$$y_p'' = 0.25c_1e^{0.5x} - 16c_2\cos 4x - 16c_3\sin 4x$$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$0.25c_1e^{0.5x} - 16c_2\cos 4x - 16c_3\sin 4x + c_1e^{0.5x} - 8c_2\sin 4x + 8c_3\cos 4x + 5c_1e^{0.5x} + 5c_2\cos 4x + 5\,c_3\sin 4x = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$$

$$\Rightarrow 6.25c_1e^{0.5x} + (-11c_2 + 8c_3)\cos 4x - (11c_3 + 8c_2)\sin 4x = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$$

Comparing coefficient on both side then,

$$6.25c_1 = 1.25, \quad -11c_2 + 8c_3 = 40, \quad -11c_3 - 8c_2 = -55$$

Solving we get, $c_1 = \frac{1}{5}$, $c_2 = 0$ and $c_3 = 5$.

Then (iii) becomes,

$$y_p = \frac{1}{5}e^{0.5x} + 5\sin 4x$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p$$

$$= e^{-x}(A\cos 2x + B\sin 2x) + 0.2e^{0.5x} + 5\sin 4x) \quad \dots(vii)$$

Using $y(0) = 0.2$ by (ii) to (vii) then

$$0.2 = A + 2 \Rightarrow A = -1.8$$

And differentiating (iv) w. r. t. x., we get,

$$y'(x) = -2A\sin 2xe^{-x} - A\cos 2xe^{-x} - e^{-x}B\sin 2x + 2B\cos 2xe^{-x} + 20\cos 4x$$

Using $y'(0) = 60.1$ by (ii), then

$$60.1 = -A + 2B + 1 + 20$$

$$\Rightarrow 60.1 = 1.8 + 2B + 21$$

$$\Rightarrow 2B = 60.1 - 22.8$$

$$\Rightarrow B = \frac{37.3}{2} = 18.65$$

Now (vii) becomes

$$y(x) = e^{-x}(18.65\sin 2x - 1.8\cos 2x) + 0.2e^{0.5x} + 5\sin 4x$$

Exercise 6.12

Find a general solution of the following equating by method of variation of parameter.

(1) $y'' + y = \sec x$ **[2006 Spring Q. No. 4(b) OR]**

Solution: Given equation is,

$$y'' + y = \sec x. \quad \dots(i)$$

This is second order non–homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots(ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0 \quad \dots(iii)$$

Its auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

Here, m has complex value, so the solution of (iii) is

$$y_h(x) = (A\cos x + B\sin x) \quad \dots(iv)$$

And, for particular solution, comparing (iv) with $y_h = Ay_1 + By_2$ then we get

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x,$$

then, $$y_1' = -\sin x \quad \text{and} \quad y_2' = \cos x$$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y'$$

$$= \cos x.\cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1.$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get

$$R = \sec x$$

By method of variation of parameter solution, the particular solution of the given equation (i) is,

$$y_p = -y_1\int\left(\frac{y_2R}{w}\right)dx + y_2\int\left(\frac{y_1R}{w}\right)dx$$

$$= -\cos x\int\left(\frac{\sin x \sec x}{1}\right)dx + \sin x\int\left(\frac{\cos x.\sec x}{1}\right)dx$$

$$= -\cos x\int \tan x\,dx + \sin x\int dx$$

$$= -\cos x\log(\sec x) + x\sin x$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = A\cos x + B\sin x - \cos x\log(\sec x) + x\sin x.$$

$$= \cos x\,(A - \log(\sec x)) + \sin x\,(B + x).$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' = 0 \qquad \dots \text{(iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1 + c_2e^{-x} \qquad \dots \text{(v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 2 + 2x + x^2$$

Clearly R has repeated value 2 (polynomial function) for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = (c_3x^2 + c_4x + c_5)\,x = c_3x^3 + c_4x^2 + c_5x \qquad \dots\dots \text{(vi)}$$

Then,

$$y_p' = 3c_3x^2 + 2c_4x + c_5$$

and $\quad y''_p = 6c_3x + 2c_4$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$6c_3 + 2c_4 + 3c_3x^2 + 3c_4x + c_5 = 2 + 2x + x^2$$

$$\Rightarrow 3c_3x^2 + x(6c_3 + 2c_4) + (2c_4 + c_5) = 2 + 2x + x^2$$

Comparing coefficient on both side then,

$$3c_3 = 1,\quad 6c_3 + 2c_4 = 2,\quad 2c_4 + c_5 = 2.$$

Solving we get, $c_3 = \frac{1}{3}$, $c_5 = 2$ and $c_4 = 0$

Then (vi) becomes,

$$y_p = \frac{1}{3}x^3 + 2x$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{-x} + \frac{x^3}{3} + 2x \qquad \dots\dots \text{(vii)}$$

Using $y(0) = 8$ by (ii) to (vii) then

$$8 = c_1 + c_2 \qquad \dots\dots \text{(A)}$$

And differentiating (vii) w. r. t. x. we get,

$$y'(x) = -c_2e^{-2x} + x^2 + 2$$

Using $y'(0) = -1$ by (ii), then

$$-1 = c_2 + 2 \Rightarrow c_2 = 3.$$

Then (A) gives, $\qquad c_1 = 5.$

Now (vii) becomes

$$y(x) = 5 + 3e^{-x} + \frac{x^3}{3} + 2x.$$

(viii) $y'' + 2y' + y = e^{-x}$, $y(0) = -1$, $y'(0) = 1$

[2017 Spring Q.No. 5(b), 2015 Fall Q.No. 5(b), 2012 Fall Q.No. 5(a), 2008 Fall Q. No. 5(b)]

Solution: Given that,

$$y'' + 2y' + y = e^{-x} \qquad \dots\dots \text{(i)}$$

$$y(0) = -1,\ y'(0) = 1 \qquad \dots\dots \text{(ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \dots \text{(iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 2y' + y = 0 \qquad \dots \text{(iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

Here m has real and repeated values, so the solution of (iv) is

$$y_h(x) = (c_1 + c_2x)\,e^{-x} \qquad \dots \text{(v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{-x}$$

Clearly R has repeated value e^{2x} for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3x^2e^{-x} \qquad \dots\dots \text{(vi)}$$

Then,

$$y'_p = 2c_3xe^{-x} - c_3x^2e^{-x}$$

$$y''_p = 2c_3(-e^{-x}x + e^{-x}) - c_3(2xe^{-x} - x^2e^{-x})$$

$$= -2c_3xe^{-x} + 2c_3e^{-x} - 2c_3xe^{-x} + c_3x^2e^{-x}$$

$$= -2c_3xe^{-x} + 2c_3e^{-x} + c_3x^2e^{-x}$$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$-4c_3xe^{-x} + 2c_3e^{-x} + c_3x^2e^{-x} + 4c_3xe^{-x} - 2c_3x^2e^{-x} + c_3x^2e^{-x} = e^{-x}$$

$$\Rightarrow 2c_3e^{-x} = e^{-x}$$

$$\Rightarrow c_3 = \frac{1}{2}$$

Then (iii) becomes,

$$y_p = \frac{1}{2}x^2e^{-x}$$

Now, general equation of (i) is,

$$y_p = y_h(x) + y_p = (c_1 + c_2x)\,e^{-x} + \frac{1}{2}x^2e^{-x} \qquad \dots\dots \text{(vii)}$$

Using $y(0) = -1$ by (ii) to (vii) then

$$-1 = c_1$$

And differentiating (iv) w. r. t. x, we get,

$$y'(x) = -c_1e^{-x} + c_2e^{-x} - c_2xe^{-x} + xe^{-x} - \frac{1}{2}x^2e^{-x}$$

Using $y'(0) = -2$ by (ii), then

$$1 = -c_1 + c_2 \Rightarrow c_2 = 0.$$

Now (vii) becomes

$$y(x) = -e^{-x} + \frac{1}{2}x^2\,e^{-x}$$

$$= \left(\frac{x^2}{2} - 1\right)e^{-x}$$

Clearly R has repeated value e^{-2x} for once to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_3xe^{-2x} + c_4x \qquad \text{.....(vi)}$$

Then,

$$y'_p = -2c_3xe^{-2x} + c_3e^{-2x} + c_4$$
$$y''_p = 4c_3xe^{-2x} - 2c_3e^{-2x} - 2c_3e^{-2x}$$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$4c_3xe^{-2x} - 4c_3e^{-2x} - 4c_3xe^{-2x} - 4c_4x = e^{-2x} - 2x$$
$$\Rightarrow -4c_3e^{-2x} - 4c_4x = e^{-2x} - 2x$$

Comparing coefficient on both side then,

$$-4c_3 = 1, \qquad -4c_4 = -2$$

Solving we get, $c_3 = \frac{1}{2}, c_4 = \frac{1}{2}$

Then (iii) becomes, $y_p = \frac{-1}{4}xe^{-2x} + \frac{x}{2}$

Now, general equation of (i) is,

$$y_p = y_h(x) + y_p = c_1e^{-2x} + c_2e^{2x} - \frac{x}{4}e^{-2x} + \frac{x}{2} \qquad \text{.....(vii)}$$

Using y(0) = 0 by (ii) to (vii) then

$$0 = c_1 + c_2 \qquad \text{— (A)}$$

And differentiating (vii) w. r. t. x. we get,

$$y'(x) = -2c_1e^{-2x} + 2c_2e^{2x} - \frac{1}{4}(-2xe^{-2x} + e^{-2x}) + \frac{1}{2}$$
$$\Rightarrow y'(x) = -2c_1e^{-2x} + 2c_2e^{2x} + \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + \frac{1}{2}$$

Using y'(0) = 0 by (ii) then

$$0 = -2c_1 + 2c_2 - \frac{1}{4} + \frac{1}{2}$$
$$\Rightarrow -2c_1 + 2c_2 + \frac{1}{4} = 0 \qquad \text{— (B)}$$

Solving (A) and (B) we get,

$$c_1 = \frac{1}{16}, c_2 = -\frac{1}{16}$$

Now (vii) becomes

$$y(x) = \frac{1}{16}e^{-2x} + \frac{1}{16}e^{2x} - \frac{xe^{-2x}}{4} + \frac{x}{2}$$
$$= -\frac{1}{8}\left(\frac{e^{2x} - e^{-2x}}{2}\right) - \frac{x}{4}e^{-2x} + \frac{x}{2}$$
$$= -\frac{1}{8}\sinh 2x + \frac{x}{2} - \frac{x}{4}e^{-2x}$$

(vi) $y'' + 1.2y' + 0.36y = 4e^{-0.6x}$, $y(0) = 0$, $y'(0) = 1$.

Solution: Given that,

$$y'' + 1.2y' + 0.36y = 4e^{-0.6x} \qquad \text{.......(i)}$$
$$y(0) = 0, y'(0) = 1 \qquad \text{.......(ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 1.2y' + 0.36y = 0 \qquad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 + 1.2m + 0.36 = 0$$
$$\Rightarrow (m + 0.6)^2 = 0$$
$$\Rightarrow m = -0.6, -0.6$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = (c_1 + c_2x)e^{-0.6x} \qquad \text{... (v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 4e^{-0.6x}$$

Clearly R has repeated value $e^{-0.6x}$ for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_2x^2e^{-0.6x} \qquad \text{........ (vi)}$$

Then,

$$y'_p = -0.6c_2x^2e^{-0.6x} + 2c_2xe^{-0.x}$$
$$y''_p = -0.6c_2(2xe^{-0.6x} - x^20.6e^{-0.6x}) + 2c_2(-0.6xe^{-0.6x} + e^{-0.6x})$$
$$= -1.2c_2xe^{-0.6} + 0.36c_2x^2e^{-0.6x} - 1.2c_2xe^{-0.6x} + 2c_2e^{-0.6x}$$
$$= 0.36c_2x^2e^{-0.6x} - 2.4c_2xe^{-0.6x} + 2c_2e^{-0.6x}$$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$0.36c_2x^2e^{-0.x} - 2.4xc_2xe^{-0.6x} + 2c_2e^{-0.6x} - 0.72c_2x^2e^{-0.6x} + 2.4c_2xe^{-0.6x} + 0.36c_2x^2e^{-0.6x} = 4e^{-0.6x}$$
$$\Rightarrow 2c_2e^{-0.6x} = 4e^{-0.6x}$$
$$\Rightarrow c_2 = 2.$$

Then (vi) becomes, $y_p = 2x^2e^{-0.6x}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^{-0.6x} + 2x^2e^{-0.6x} \qquad \text{.......... (iv)}$$

Using y(0) = 0 by (ii) to (vii) then

$$0 = c_1$$

And differentiating (iv) w. r. t. x. we get,

$$y'(x) = 0.6c_1e^{-0.6x} + c_2e^{-0.6x} - 0.6c_2xe^{-0.6x} + 4xe^{-0.6x} - 1.2x^2e^{-0.6x}$$

Using y'(0) = 1 by (ii), then

$$1 = 0.6c_1 + c_2$$
$$\Rightarrow c_2 = 1 \qquad \text{[Being } c_1 = 0\text{]}$$

Now (vii) becomes

$$y(x) = xe^{-0.6x} + 2x^2e^{-0.6x}$$
$$= (x + 2x^2)e^{-0.6x}$$

(vii) $y'' + y' = 2 + 2x + x^2$, $y(0) = 8$, $y'(0) = -1$

Solution: Given that,

$$y'' + y' = 2 + 2x + x^2 \qquad \text{....... (i)}$$
$$y(0) = 8, y'(0) = -1 \qquad \text{....... (ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (iii)}$$

$\Rightarrow \quad \sin 2x(-6c_3 - 2c_4) + \cos 2x(-6c_4 + 2c_3) = -6\sin 2x - 18\cos 2x$

Comparing coefficient on both side then,

$$-6c_3 - 2c_4 = -6, \qquad -6c_4 + 2c_3 = -18.$$

Solving we get, $c_3 = 0$ and $c_4 = 3$.

Then (vi) becomes, $y_p = 3\cos 2x$

Now, general equation of (i) is,

$$y = y_h(x) + y_p = c_1e^{-2x} + c_2e^{x} + 3\cos 2x \qquad \text{......(vii)}$$

Using y(0) = 2 by (ii) to (vii) then

$$2 = c_1 + c_2 + 3 \Rightarrow c_1 + c_2 = -1 \qquad \text{......(A)}$$

And differentiating (vii) w. r. t. x., we get,

$$y'(x) = -2c_1e^{-2x} + c_2e^{x} - 6\sin 2x$$

Using y'(0) = 2 by (ii), then

$$2 = -2c_1 + c_2 \qquad \text{......(B)}$$

Solving (A) and (B) we get, $c_2 = 0$ and $c_1 = -1$.

Now (vii) becomes

$$y(x) = -e^{-2x} + 3\cos 2x$$

$$\Rightarrow y(x) = 3\cos 2x - e^{-2x}$$

(iv) $y'' + 1.5y' - y = 12x^2 + 6x^3 - x^4, y(0) = 4, y'(0) = -8$

Solution: Given that,

$$y'' + 1.5y' - y = 12x^2 + 6x^3 - x^4 \qquad \text{.......(i)}$$

$$y(0) = 4, y'(0) = -8 \qquad \text{.......(ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 1.5y' - y = 0 \qquad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 + \frac{3m}{2} - 1 = 0 \Rightarrow 2m^2 + 3m - 2 = 0$$

$$\Rightarrow 2m^2 + 4m - m - 2 = 0$$

$$\Rightarrow 2m(m+2) - 1(m+2) = 0$$

$$\Rightarrow (m+2)(2m-1) = 0$$

$$\Rightarrow m = -2, \frac{1}{2}$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1e^{-2x} + c_2e^{0.5x} \qquad \text{... (v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 72x^2 + 6x^3 - x^4$$

Clearly R has repeated value e^{2x} for once to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_3x^4 + c_4x^3 + c_5x^2 + c_6x + c_7 \qquad \text{....... (vi)}$$

Then,

$$y'_p = 4c_3x^3 + 3c_4x^2 + 2c_5x + c_6$$

$$y''_p = 12c_3x^2 + 6c_4x + 2c_5$$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$12c_3x^2 + 6c_4x + 2c_5 + 6c_3x^3 + \frac{9}{2}c_4x^2 + 3c_5x + \frac{3}{2}c_6 - c_3x^4 - c_4x^3 - c_5x^2 - c_6x - c_7 = 72x^2 + 6x^3 - x^4$$

$$\Rightarrow -c_3x^4 + x^3(6c_3 - c_4) + x^2\left(12c_3 + \frac{9}{2}c_4 - c_5\right) + x$$

$$(6c_4 + 3c_5 - c_6) + \left(2c_5 + \frac{3}{2}c_6 - c_7\right) = 12x^2 + 6x^3 - c^4$$

Comparing coefficient on both side then,

$$-c_3 = -1, \qquad 6c_3 - c_4 = 6, \qquad 12c_3 + \frac{9}{2}c_4 - c_5 = 12,$$

$$6c_4 + 3c_5 - c_6 = 0, \qquad 2c_3 + \frac{3}{2}c_6 - c_7 = 0$$

Solving we get,

$$c_3 = 1, \; c_4 = 0, \; c_5 = 0, \; c_6 = 0, \; c_7 = 0.$$

Then (iii) becomes, $y_p = x^4$

Now, general equation of (i) is,

$$y = y_h(x) + y_p = c_1e^{-2x} + c_2e^{-0.5x} + x^4 \qquad \text{....... (vii)}$$

Using y(0) = 4 by (ii) to (vii) then

$$4 = c_1 + c_2 \qquad \text{— (A)}$$

And differentiating (iv) w. r. t. x. we get,

$$y'(x) = -2c_1e^{-2x} + 0.5\,c_2e^{0.5x} + 4x^3$$

Using y'(0) = –8 by (ii), then

$$-8 = -2c_1 + \frac{1}{2}c_2 \qquad \text{— (B)}$$

Solving (A) and (B) we get, $c_2 = 0$ and $c_1 = 4$.

Now (vii) becomes,

$$y(x) = 4e^{-2x} + x^4$$

(v) $y'' - 4y = e^{-2x} - 2x, y(0) = 0, y'(0) = 0$

Solution: Given that,

$$y'' - 4y = e^{-2x} - 2x \qquad \text{....... (i)}$$

$$y(0) = 0, y'(0) = 0 \qquad \text{....... (ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 4y = 0 \qquad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 - 4 = 0 \Rightarrow m = \pm 2.$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1e^{-2x} + c_2e^{2x} \qquad \text{... (v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{-2x} - 2x$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 3e^{2x}$$

Clearly R has repeated value e^{2x} for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3xe^{2x} \quad \text{...... (vi)}$$

Then, $y'_p = 2c_3xe^{2x} + c_3e^{2x}$

$y''_p = 4c_3xe^{2x} + 2c_3e^{2x} + 2c_3e^{2x} = 4c_3xe^{2x} + 4c_3e^{2x}$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$4c_3xe^{2x} + 4c_3e^{2x} - 2c_3xe^{2x} - c_3e^{2x} - 2c_3xe^{2x} = 3e^{2x}$$

$\Rightarrow 3c_3e^{2x} = 3e^{2x}$

$\Rightarrow c_3 = 1$

Then (vi) becomes, $\quad y_p = xe^{2x}$

Now, general equation of (i) is,

$$y = y_h(x) + y_p = c_1e^{2x} + c_2e^{-x} + xe^{2x} \quad \text{...... (vii)}$$

Using $y(0) = 0$ by (ii) to (vii) then

$$0 = c_1 + c_2 \quad \text{— (A)}$$

Differentiating (vii), we get,

$$y'(x) = 2c_1e^{2x} - c_2e^{-x} + 2xe^{2x} + e^{2x}$$

Using $y'(0) = -2$ by (ii), then

$-2 = 2c_1 - c_2 + 1$

$\Rightarrow 2c_1 - c_2 + 3 = 0 \quad \text{— (B)}$

Solving (A) and (B) we get, $c_1 = -1$, $c_2 = 1$.

Now (vii) becomes

$$y(x) = -e^{2x} + e^{-x} + xe^{2x}$$

(ii) $y'' + y' - 2y = 14 + 2x - 2x^2, y(0) = 0, y'(0) = 0$ [2008 Spring Q. No. 5(b)]

Solution: Given that,

$$y'' + y' - 2y = 14 + 2x - 2x^2 \quad \text{....... (i)}$$

$$y(0) = 0, y'(0) = 0 \quad \text{....... (ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' - 2y = 0 \quad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$m^2 + m - 2 = 0 \Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2, 1.$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1e^{-2x} + c_2e^{x} \quad \text{... (v)}$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 14 + 2x - 2x^2$$

Clearly R has non-repeated term to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3x^2 + c_4x + c_5 \quad \text{...... (vi)}$$

Then,

$y'_p = 2c_3x + c_4$ and $y''_p = 2c_3$.

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$2c_3 + 2c_3x + c_4 - 2c_3x^2 - 2c_4x - 2c_5 = 14 + 2x - 2x^2$$

$$\Rightarrow -2c_3x^2 + x(2c_3 - 2c_4) + (2c_3 + c_4 - 2c_5) = 14 + 2x - 2x^2$$

Comparing the coefficients we get

$-2c_3 = -2$, $\quad 2c_3 - 2c_4 = 2$, $\quad 2c_3 + c_4 - 2c_5 = 14.$

Solving we get, $\quad c_3 = 1, c_4 = 0, \quad c_5 = -6.$

Then (vi) becomes, $\quad y_p = x^2 - 6.$

Now, general equation of (i) is,

$$y = y_h(x) + y_p = c_1e^{-2x} + c_2e^{x} + x^2 - 6 \quad \text{...... (vii)}$$

Using $y(0) = 0$ by (ii) to (vii) then

$$0 = c_1 + c_2 - 6 \Rightarrow c_1 + c_2 = 6 \quad \text{...... (A)}$$

And differentiating (vii) w. r. t. x. we get,

$$y = -2c_1e^{-2x} + c_2e^{x} + 2x$$

Using $y'(0) = 0$ by (ii), then

$$0 = -2c_1 + c_2 \quad \text{— (B)}$$

Solving (A) and (B) we get, $c_1 = 2$ and $c_2 = 4$.

Now (vii) becomes,

$$y(x) = 2e^{-3x} + 4e^{x} + x^2 - 6.$$

(iii) $y'' + y' - 2y = -6\sin2x - 18\cos2x, y(0) = 2, y'(0) = 2.$ [2006 Spring Q. No. 5(b)]

Solution: Given that,

$$y'' + y' - 2y = -6\sin2x - 18\cos2x \quad \text{.......(i)}$$

$$y(0) = 2, y'(0) = 2 \quad \text{........ (ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' - 2y = 0 \quad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$m^2 + m - 2 = 0 \Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2, 1.$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1e^{-2x} + c_2e^{x} \quad \text{... (v)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = -6\sin2x - 18\cos2x$$

Clearly R has repeated value e^{2x} for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3\sin2x + c_4\cos2x \quad \text{...... (vi)}$$

Then, $\quad y'_p = 2c_3\cos2x - 2c_4\sin2x$

$y''_p = -4c_3\sin2x - 4c_4\cos2x$

Since (vi) is particular solution of (i), so, (vi) satisfies (i). That is,

$$-4c_3\sin2x - 4c_4\cos2x + 2c_3\cos2x - 2c_4\sin2x - 2c_3\sin2x - c_4\cos2x = -6\sin2x - 18\cos2x$$

(xxv) $y'' - 3y' = e^{3x} - 12x$

Solution: Given that,

$$y'' - 3y' = e^{3x} - 12x \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 3y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 3m = 0 \Rightarrow m(m-3) = 0 \Rightarrow m = 0, 3$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^{3x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{3x} - 12x$$

Clearly R has repeated value e^{3x} for once to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_3xe^{3x} - (c_4x + c_5)\,x$$
$$= c_3xe^{3x} - c_4x^2 - c_5x \quad \text{...... (v)}$$

Then,

$$y'_p = 3c_3xe^{3x} + c_3e^{3x} - 2c_4x - c_5$$
$$y''_p = 9c_3xe^{3x} + 3c_3e^{3x} + c_3e^{3x} - 2c_4$$
$$= 9c_3xe^{3x} + 4c_3e^{3x} - 2c_4$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$9c_3xe^{3x} + 4c_3e^{3x} - 2c_4 - 3(3c_3xe^{3x} + c_3e^{3x} - 2c_4 - 5) = e^{3x} - 12x$$

$$\Rightarrow \quad c_3e^{3x} + 6c_4x + (-2c_4 + 3c_5) = e^{3x} - 12x$$

Comparing coefficient on both side then,

$$c_3 = 1, \quad 6c_4 = -12 \quad 4 + 3c_5 = 0$$

Solving we get, $c_3 = 1, c_4 = -2, c_5 = \dfrac{3}{-4}$

Then, (v) becomes, $y_p = xe^{3x} + 2x^2 + \dfrac{4}{3}x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{3x} + ce^{3x} + 8x^2 + \frac{4}{3}x$$

(xxvi) $y'' - y' = x^2$

Solution: Given that,

$$y'' - y' = x^2 \quad \text{....... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1.$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^x \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = x^2$$

Clearly R has repeated value x^2 (which is polynomial) for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = (c_3x^3 + c_4x^2 + c_5x + c_6)\,x$$
$$= c_3x^4 + c_4x^3 + c_5x^2 + c_6x \quad \text{....... (v)}$$

Then,

$$y'_p = 4c_3x^3 + 3c_4x^2 + 2c_5x + c_6$$
$$y''_p = 12c_3x^2 + 6c_4x + 2c_5$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$12c_3x^2 + 6c_4x + 2c_5 - 4c_3x^3 - 3c_4x^2 - 2c_5x - c_6 = x^3$$

$$\Rightarrow -4c_3x^3 + x^2(12c_3 - 3c_4) + x(6c_4 - 2c_5) + (2c_5 - c_6) = x^3$$

Comparing coefficient on both side then,

$$-4c_3 = 1, \quad 12c_3 - 3c_4 = 0, \quad 6c_4 - 2c_5 = 0, \quad 2c_5 - c_6 = 0.$$

Solving we get,

$$c_3 = \frac{-1}{4}, \; c_4 = -1, c_5 = -3, c_6 = -6.$$

Then, (v) becomes,

$$y_p = \frac{-1}{4}x^4 - x^3 - 3x^2 - 6x$$

Now, general equation of (i) is,

$$y_p = y_n(x) + y_p = c_1 + c_2e^x + \frac{-1}{4}x^4 - x^3 - 3x^2 - 6x$$

3. Solve the following initial value problems.

(i) $y'' - y' - 2y = 3e^{2x}$, $y(0) = 0$, $y'(0) = -2$.

[2015 Spring, Q. No. 5(b), 2014 Spring Q. No. 5(b), 2006 Fall Q. No. 5(b)]

Solution: Given that,

$$y'' - y' - 2y = 3e^{2x} \quad \text{....... (i)}$$
$$y(0) = 0, y'(0) = -2 \quad \text{....... (ii)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (iii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y' - 2y = 0 \quad \text{... (iv)}$$

The auxiliary equation of of (iv) is,

$$m^2 - m - 2 = 0$$
$$\Rightarrow (m-2)(m+1) = 0$$
$$\Rightarrow m = 2, -1.$$

Here m has real and distinct values, so the solution of (iv) is

$$y_h(x) = c_1e^{2x} + c_2e^{-x} \quad \text{... (v)}$$

$$\Rightarrow m = 5, -1$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^{5x} + c_2e^{-x} \quad \text{... (iv)}$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^x + 4$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3e^x + c_4 \quad \text{...... (v)}$$

Then, $y'_p = c_3e^x$ and $y''_p = c_3e^x$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3e^x - 4c_3e^x - 5c^3e^x - 5c_4 = e^x + 4$$

$$\Rightarrow \quad -8c_3e^x - 5c_4 = e^x + 4$$

Comparing coefficient on both side then,

$$-8c_3 = 1 \Rightarrow c_3 = \frac{-1}{8} \quad \text{and} \quad -5c_4 = 4 \Rightarrow c_4 = \frac{-4}{5}$$

Then, (v) becomes, $\quad y_p = -\frac{1}{8}e^x - \frac{4}{5}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^{5x} + c_2e^{-x} - \frac{e^x}{8} - \frac{4}{5}$$

(xxiii) $y'' - y' - 6y = e^{-x} - 7\cos x$

Solution: Given that,

$$y'' - y' - 6y = e^{-x} - 7\cos x \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y' - 6y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - m - 6 = 0$$

$$\Rightarrow (m-3)(m+2) = 0$$

$$\Rightarrow m = 3, -2$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^{3x} + c_2e^{-2x} \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{-x} - 7\cos x$$

Clearly R has repeated value e^x for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3e^{-x} - (c_4\cos x + c_5\sin x) \quad \text{...... (v)}$$

Then,

$$y'_p = -c_3e^{-x} - (c_4 - \sin x + c_5\cos x)$$

$$y''_p = c_3e^{-x} - (-c_4\cos x - c_5\sin x) = c_3e^{-x} + c_4\cos x + c_5\sin x$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3e^{-x} + c_4\cos x + c_5\sin x - (-c_3e^{-x} + c_4\sin x - c_5\cos x)$$
$$-6(c_3e^{-x} - c_4\cos x - c_5\sin x) = e^{-x} - 7\cos x$$

$$\Rightarrow \quad -4c_3e^{-x} + \cos x\,(7c_4 + c_5) + \sin x\,(7c_5 - c_4) = e^{-x} - 7\cos x$$

Comparing coefficient on both side then,

$$-4c_3 = 1, \qquad 7c_4 + c_5 = -7, 7c_5 - c_4 = 0$$

Solving we get, $\quad c_3 = -\frac{1}{4}, c_4 = \frac{-49}{50}$ and $c_5 = -7$

Then, (ii) becomes, $\quad y_p = \frac{-1}{4} + \frac{7}{50}\cos x + \frac{49}{50}\sin x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^{3x} + c_2e^{-2x} - \frac{1}{4}e^{-x} + \frac{7}{50}\cos x + \frac{49}{50}\sin x$$

(xxiv) $y'' + 5y' = 15x^2$

Solution: Given that,

$$y'' + 5y' = 15x^2 \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 5y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 5m = 0 \Rightarrow m(m+5) = 0 \Rightarrow m = 0, -5$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^{-5x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 15x^2$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = (c_3x^2 + c_4x + c_5)x = c_3x^3 + c_4x^2 + c_5x \quad \text{... (ii)}$$

Then,

$$y'_p = 3c_3x^2 + 2c_4x + c_5$$

$$y''_p = 6c_3x + 2c_4$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$6c_3x + 2c_4 + 15c_3x^2 + 10c_4x + 5c_5 = 15x^2$$

$$\Rightarrow \quad 15c_3x^2 + x(6c_3 + 10c_4) + (2c_4 + 5c_5) = 15x^2$$

Comparing coefficient on both side then,

$$15c_3 = 15,\ 6c_3 + 10c_4 = 0, \qquad 2c_4 + 5c_5 = 0$$

Solving we get, $c_3 = 1$, $c_4 = \frac{-3}{5}$ and $c_5 = \frac{6}{25}$

Then, (ii) becomes, $y_p = x^3 - \frac{3}{5}x^2 + \frac{6}{25}x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{-5x} + x^3 - \frac{3}{5}x^2 + \frac{6}{25}x$$

$$y_h(x) = c_1e^x + c_2e^{-x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^x$$

Clearly R has repeated value e^x for once to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_3xe^x \quad \text{...... (v)}$$

Then,

$$y'_p = c_3xe^x + c_3e^x$$

$$y''_p = c_3xe^x + c_3e^x + c_3e^x = c_3e^x + 2c_3e^x$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3xe^x + 2c_3e^x - c_3xe^x = e^x$$

$$\Rightarrow 2c_3 = 1$$

$$\Rightarrow c_3 = \frac{1}{2}$$

Then, (ii) becomes, $y_p = \frac{x}{2} e^x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^x + c_2e^{-x} + \frac{x}{2} e^x$$

(xx) $y'' + 4y' + 5y = 10$

Solution: Given that,

$$y'' + 4y' + 5y = 10 \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 4y' + 5y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 4m + 5 = 0$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{4^2 - 4.1.5}}{2/1} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 + 2i}{2} = (-2 \pm i)$$

Here m has complex values, so the solution of (iii) is

$$y_h(x) = e^{-2x}(A \cos x + B \sin x)$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{-x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 10$$

Clearly R has repeated value e^x for once to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_1 \quad \text{...... (v)}$$

Then, $y'_p = 0$ and $y''_p = 0$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$0 + 0 + 5c_1 = 10. \Rightarrow c_1 = 2$$

Then, (ii) becomes, $y_p = 2.$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = e^{-2x}(A \cos x + B \sin x) + 2$$

(xxi) $y'' - y' = e^x + e^{-x}$

Solution: Given that,

$$y'' - y' = e^x + e^{-x} \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^x \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^x + e^{-x}$$

Clearly R has repeated value e^x for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3xe^x + c_4e^{-x} \quad \text{...... (v)}$$

Then, $y'_p = c_3(xe^x + e^x) + (-1)\, c_4e^{-x}$

$$y''_p = c_3xe^x + c_3e^x + c_3e^x + c_4e^{-x}$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3xe^x + 2c_3e^x + c_4e^{-x} - c_3xe^x - c_3e^x + c_4e^{-x} = e^x + e^{-x}$$

$$\Rightarrow c_3e^x + 2c_4e^{-x} = e^x + e^{-x}$$

Comparing coefficient on both side then,

$$c_3 = 1 \text{ and } 2c_4 = 1 \Rightarrow c_4 = \frac{1}{2}$$

Then, (v) becomes, $y_p = xe^x + \frac{1}{2}e^{-x}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^x + xe^x + \frac{1}{2}e^{-x}$$

(xxii) $y'' - 4y' - 5y = e^x + 4$

Solution: Given that,

$$y'' - 4y' - 5y = e^x + 4 \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 4y' - 5y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 4m - 5 = 0$$

$$\Rightarrow (m-5)(m+1) = 0$$

Comparing coefficient on both side then,

$2c_3 = 1$ and $2c_3 + c_4 = 0$

Solving we get, $c_3 = \frac{1}{2}$ and $c_4 = -1$.

Then, (ii) becomes, $y_P = \frac{1}{2}x^2 - x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{-x} + \frac{1}{2}x^2 - x$$

(xvii) $y'' + y = \sin x$

Solution: Given that,

$$y'' + y = \sin x \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

Here m has complex values, so the solution of (iii) is

$$y_h(x) = A\cos x + B\sin x \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = \sin x$$

Clearly R has repeated value sinx for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = x(c_1\sin x + c_2\cos x) \quad \text{...... (v)}$$

Then,

$$y'_p = (c_1\sin x + c_2\cos x) + x(c_1\cos x - c_2\sin x)$$
$$= c_1\sin x + c_2\cos x + c_1x\cos x - c_2x\sin x$$
$$y''_p = c_1\cos x - c_2\sin x + c_1\{x(-\sin x) + \cos x\} - c_2\{x\cos x + \sin x\}$$
$$= c_1\cos x - c_2\sin x - c_1x\sin x + c_1\cos x - c_2x\cos x - c_2\sin x$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$2c_1\cos x - 2c_2\sin x - c_1s\sin x - c_2x\cos x + c_1x\sin x + c_2x\cos x = \sin x$$
$$\Rightarrow \quad 2c_1\cos x - 2c_2\sin x = \sin x$$

Comparing coefficient of sinx and cosx from both side then,

$2c_1 = 0$ and $-2c_2 = 1$

Solving we get, $c_1 = 0,\ c_2 = \frac{-1}{2}$

Then, (v) becomes, $y_P = -\frac{1}{2}x\cos x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = A\cos x + B\sin x - \frac{1}{2}x\cos x$$

(xviii) $y'' + 2y' + y = e^{-x}$

Solution: Given that,

$$y'' + 2y' + y = e^{-x} \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 2y' + y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$$
$$\Rightarrow m = -1, -1$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\,e^{-x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{-x}$$

Clearly R has repeated value e^{-x} for twice to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3x^2e^{-x} \quad \text{....... (v)}$$

Then,

$$y'_p = c_3(2xe^{-x} - x^2e^{-x})$$
$$y''_p = c_3[2(-xe^{-x} + e^{-x}) - (2xe^{-x} + x^2e^{-x})]$$
$$= c_3(2e^{-x} - 4xe^{-x} + x^2e^{-x})$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3(2e^{-x} - 4xe^{-x} + x^2e^{-x} + 4xe^{-x} - 2x^2e^{-x} + x^2e^{-x}) = e^{-x}$$
$$\Rightarrow \quad c_3e^{-x}(2 - 4x + x^2 + 4x - 2x^2 + x^2) = e^{-x}$$
$$\Rightarrow \quad 2c_3e^{-x} = e^{-x}$$

Comparing coefficient on both side then, $c_3 = \frac{1}{2}$

Then, (v) becomes, $y_P = \frac{1}{2}x^2e^{-x}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^{-x} + \frac{1}{2}x^2e^{-x}$$

(xix) $y'' - y = e^x$

Solution: Given that,

$$y'' - y = e^x \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$$

Here m has real and repeated values, so the solution of (iii) is

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 10y' + 25y = 0 \qquad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 10m + 25 = 0$$

$\Rightarrow \quad (m+5)^2 = 0$

$\Rightarrow \quad m = -5, -5$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)\, e^{-5x}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = e^{-5x}$$

Clearly R has repeated term e^{-5x} for twice to the independent solution of y_h therefore choose the particular solution of (i) is

$$y_p = c_3x^2e^{-5x} \qquad \text{......... (iv)}$$

Then,

$$y'_p = c_3(-5x^2e^{-5x} + 2xe^{-5x})$$
$$y''_p = -5c_3(-5x^2e^{-5x} + 2xe^{-5x}) + 2c_3(-5xe^{-5x} + e^{-5x})$$
$$= 25c_3x^2e^{-5x} - 10c_3xe^{-5x} - 10c_3xe^{-5x} + 2c_3e^{-5x}$$
$$= 25c_3x^2e^{-5x} - 20c_3xe^{-5x} + 2c_3e^{-5x}$$

Since (iv) is particular solution of (i), so, (iv) satisfies (i). That is,

$$25c_3x^2e^{-5x} - 20c_3xe^{-5x} + 2c_3e^{-5x} - 50c_3x^2e^{-5x} + 20c_3xe^{-5x} + 25c_3x^2e^{-5x} = e^{-5x}$$

$\Rightarrow \quad 2c_3e^{-5x} = e^{-5x}$

This gives, $c_3 = \frac{1}{2}$

Then (iv) becomes, $\quad y_p = \frac{1}{2}x^2e^{-5x}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\, e^{-5x} + \frac{1}{2}x^2e^{-5x}$$

(xv) $y'' + 8y' + 16y = 64 \cosh 4x$

Solution: Given that,

$$y'' + 8y' + 16y = 64 \cosh 4x \qquad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 8y' + 16y = 0 \qquad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 8m + 16 = 0 \Rightarrow (m+4)^2 = 0$$

$\Rightarrow \quad m = -4, -4$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)e^{-4x} \qquad \text{... (iv)}$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 64 \cosh 4x = 32(e^{4x} + 3^{-4x})$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = Ae^{4x} + Bx^2e^{-4x} \qquad \text{...... (v)}$$

Then,

$$y'_p = 4Ae^{4x} - 4Bx^2e^{-4x} + 2Bxe^{-4x}$$
$$y''_p = 16Ae^{4x} + 16Bx^2e^{-4x} - 8Bxe^{-4x} + 2Be^{-4x} - 8Bxe^{-4x}$$
$$= 16Ae^{4x} + 16Bx^2e^{-4x} - 16Bxe^{-4x} + 2B^{-4x}$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$(16Ae^{4x} + 16Bx^2e^{-4x} - 16Bxe^{-4x} + 2Be^{-4x}) + 8(4Ae^{4x} - 4Bx^2e^{-4x} + 2Bxe^{-4x}) + 16(Ae^{4x} + Bx^2e^{-4x}) = 32e^{4x} + 32e^{-4x}$$

$$\Rightarrow (16A + 32A + 16A)\, e^{4x} + xe^{-4x}(-16B + 16B) + x^2e^{-4x}(16B - 32B + 16B) + (2B)\, e^{-4x} = 32e^{4x} + 32e^{-4x}$$

$$\Rightarrow \quad 64Ae^{4x} + 2Be^{-4x} = 32e^{4x} + 32e^{-4x}$$

Comparing coefficient on both side then,

$64A = 32 \Rightarrow A = \frac{1}{2}$ and $2B = 32 \Rightarrow B = 16.$

Then, (v) becomes, $y_p = \frac{1}{2}e^{4x} + 16x^2e^{-4x}$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\, e^{-4x} + \frac{1}{2}e^{4x} + 16x^2e^{-4x}$$

(xvi) $y'' + y' = x$ **[2009 Spring Q. No. 4(b)]**

Solution: Given that,

$$y'' + y' = x \qquad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' = 0 \qquad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^{-x} \qquad \text{... (iv)}$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = x$$

Clearly R has repeated value x (being polynomial) for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = (c_3x + c_4)\, x = c_3x^2 + c_4x \qquad \text{...... (v)}$$

Then,

$$y'_p = 2c_3x + c_4 \quad \text{and} \quad y''_p = 2c_3$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$2c_3 + 2c_3x + c_4 = x$$

$$\Rightarrow \quad 2c_3x + (2c_3 + c_4) = x$$

Then, $y'_p = 2c_1x + c_2$ and $y''_p = 2c_1$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$2c_1 + 2(2c_1x + c_2) + 10(c_1x^2 + c_2x + c_3) = 25x^2 + 3$$

$$\Rightarrow 2c_1 + 4c_1x + 2c_2 + 10c_1x^2 + 10c_2x + 10c_3 = 25x^2 + 3$$

$$\Rightarrow 10c_1x^2 + x(4c_1 + 10c_2) + (2c_1 + 2c_2 + 10c_3) = 25x^2 + 3$$

Comparing coefficient on both side then,

$$10c_1 = 25, \quad 4c_1 + 10c_2 = 0, \quad 2c_1 + 2c_2 + 10c_3 = 3.$$

Solving we get, $c_1 = \frac{5}{2}$, $c_2 = -1$, $c_3 = 0$.

Therefore, (ii) becomes, $y_p = \frac{5}{2}x^2 - x$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = e^{-x}(A\cos 3x + B\sin 3x) + \frac{5}{2}x^2 - x$$

(xii) $y'' + y' - 6y = -6x^3 + 3x^2 + 6x$

Solution: Given that,

$$y'' + y' - 6y = -6x^3 + 3x^2 + 6x \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' - 6y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + m - 6 = 0$$

$$\Rightarrow (m + 3)(m - 2) = 0$$

$$\Rightarrow m = -3, 2$$

Here m has real and distinct values, so the solution of (iii) is

So, its solution is, $y_h(x) = c_1e^{-3x} + c_2e^{2x}$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = -6x^3 + 3x^2 + 6x$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3x^3 + c_4x^2 + c_5x + c_6 \quad \text{...... (iv)}$$

Then, $y'_p = 3c_3x^2 + 2c_4x + c_5$ and $y''_p = 6c_3x + 2c_4$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$(6c_3x + 2c_4) + (3c_3x^2 + 2c_4x + c_5) - 6(c_3x^3 - c_4x^2 - c_5x - c_6) = 6x^3 + 3x^2 + 6x$$

$$\Rightarrow -6c_3x^3 + x^2(3c_3 - 6c_4) + x(6c_3 + 2c_4 - 6c_5) + (2c_4 + c_5 - 6c_6) = 6x^3 + 3x^2 + 6x$$

Comparing coefficient on both side then,

$$-6c_3 = -6, \quad 3x_3 - 6c_4 = 3, \quad 6c_3 + 2c_4 - 6c_5 = 6, \quad 2c_4 + c_5 - 6c_6 = 0$$

Solving we get,

$$c_3 = 1, c_4 = 0, c_5 = 0, c_6 = 0.$$

So that (iv) becomes

$$y_p = x^3$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^{-3x} + c_23^{2x} + x^3$$

(xiii) $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$

Solution: Given that,

$$y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 2y' - 35y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 2m - 35 = 0$$

$$\Rightarrow (m + 7)(m - 5) = 0$$

$$\Rightarrow m = -7, 5$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^{-7x} + c_2e^{5x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 12e^{5x} + 37\sin 5x$$

Clearly R has repeated term e^{5x} for once to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3xe^{5x} + c_4\sin 5x + c_5\cos 5x \quad \text{....... (v)}$$

Then,

$$y'_p = c_3(5xe^{5x} + e^{5x}) + 5c_4\cos 5x - 25c_5\sin 5x$$

$$y''_p = c_3(25xe^{5x} + 5e^{5x} + 5e^{5x}) - 25c_4\sin 5x - 25c_5\cos 5x$$

$$\Rightarrow y''_p = 25c_3xe^{5x} + 10c_3e^{5x} - 25c_4\sin 5x - 25c_5\cos 5x$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$[25c_3xe^{5x} + 10c_3e^{5x} - 25c_4\sin 5x - 25c_5\cos 5x] + 2[c_3(5xe^{5x} + e^{5x}) + 5c_4\cos 5x - 5c_5\sin 5x] - 35[c_3xe^{5x} + c_4\sin 5x + c_5\cos 5x] = 12e^{5x} + 37\sin 5x$$

$$\Rightarrow xe^x(25c_3 + 10c_3 - 35c_3) + e^{5x}(10c_3 + 2c_3) + (-25c_4 - 10c_5 - 35c_4)\sin 5x + (-25c_5 + 10c_4 - 35c_5) = 12e^{5x} + 37\sin 5x$$

$$\Rightarrow xe^x(0) + e^{5x}(12c_3) + (-25c_4 - 60c_5)\sin 5x + (-60c_5 + 15c_4)\cos 5x = 12e^{5x} + 37\sin 5x$$

Comparing coefficient on both side then,

$$12c_3 = 12, \quad -60c_4 - 10c_5 = 37, \quad -60c_5 + 10c_4 = 0$$

Solving we get, $c_3 = 1$, $c_4 = -0.6$ and $c_5 = -0.1$.

So (ii) becomes

$$y_p = xe^{5x} - 0.6\sin 5x - 0.1\cos 5x$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^{-7x} + c^2e^{5x} - 0.6\sin 5x - 0.10\cos 5x$$

(xiv) $y'' + 10y' + 25y = e^{-5x}$ **[2019 Fall Q.No. 4(b)]**

Solution: Given that,

$$y'' + 10y' + 25y = e^{-5x} \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

Solving we get, $c_4 = 2, c_5 = -3, c_6 = 15, c_7 = -8$

So, equation (v) becomes,

$$y_p = 2x^3 - 3x^2 + 15x - 8$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^x + c_2e^{-x} + c_3e^{-2x} + 2x^3 - 3x^2 + 15 - 8$$

(ix) $y'' + 4y = \sin 3x$

Solution: Given that, $y'' + 4y = \sin 3x$ ...... (i)

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 4y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 = -4 \Rightarrow m = \pm 2i$$

Here m has complex values, so the solution of (iii) is

$$y_h(x) = A\cos 2x + B\sin 2x \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = \sin 3x$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_1 \sin 3x + c_2 \cos 3x \quad \text{...... (v)}$$

Then, $y'_p = 3c_1\cos 3x - 3c_2\sin 3x$

$y''_p = -9c_1\sin 3x - 9c_2\cos 3x$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$-9c_1\sin 3x - 9c_2\cos 3x + 4(c_1\sin 3x + c_2\cos 3x) = \sin 3x$$

$$\Rightarrow -5c_1\sin 3x - 5c_2\cos 3x = \sin 3x$$

Comparing coefficient on both side then,

$$-5c_1 = 1 \quad \text{and} \quad -5c_2 = 0$$

Solving we get, $c_1 = -\frac{1}{5}$, $c_2 = 0$.

Therefore, (v) becomes, $y_p = \frac{-1}{5}\sin 3x$.

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = A\cos 2x + B\sin 2x - \frac{1}{5}\sin 3x$$

(x) $y'' + 3y' = 28\cosh 4x$

Solution: Given that,

$$y'' + 3y' = 28\cosh 4x \quad \text{....... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 4y' + 3y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 3m = 0 \Rightarrow m(m+3) = 0 \Rightarrow m = 0, -3$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1 + c_2e^{-3x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 28\cosh 4x = 14(e^{4x} + e^{-4x})$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3 e^{4x} + c_4 e^{-4x} \quad \text{...... (v)}$$

Then,

$$y'_p = 4c_3 e^{4x} - 4c_4 e^{-4x} \quad \text{and} \quad y''_p = 16c_3 e^{4x} + 16c_4 e^{-4x}$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$16c_3 e^{4x} + 16c_4 e^{-4x} + 3(4c_3 e^{4x} - 4c_4 e^{-4x}) = 14(e^{4x} + e^{-4x})$$

$$\Rightarrow 28c_3 e^{4x} + 4c_4 e^{-4x} = 14(e^{4x} + e^{-4x})$$

Comparing coefficient on both side then,

$$28c_3 = 14, \quad 4c_4 = 14$$

Solving we get, $c_4 = \frac{1}{2}$, $c_3 = \frac{7}{2}$

So that (v) becomes,

$$y_p = \frac{1}{2}e^{4x} + \frac{7}{2}e^{-4x}$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1 + c_2e^{-3x} + \frac{1}{2}e^{4x} + \frac{7}{2}e^{-4x}$$

(xi) $y'' + 2y' + 10y = 25x^2 + 3$

Solution: Given that,

$$y'' + 2y' + 10y = 25x^2 + 3 \quad \text{..... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 2y' + 10y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 2m + 10 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = (-1 \pm 3i).$$

Here m has complex values, so the solution of (iii) is

$$y_h(x) = e^{-x}(A\cos 3x + B\sin 3x) \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 25x^2 + 3$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_1x^2 + c_2x + c_3 \quad \text{....... (v)}$$

$$R = e^x \sin x$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = e^x(c_3 \sin x + c_4 \cos x) \quad \text{...... (v)}$$

Then,

$$y'_p = e^x(c_3 \cos x - c_4 \sin x) + e^x(c_3 \sin x + c_4 \cos x)$$
$$= e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)]$$
$$y''_p = e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)] + e^x[-(c_3 + c_4)\sin x + (c_3 - c_4)\cos x)]$$
$$= e^x[(c_3 + c_4 + c_3 - c_4)\cos x + (c_3 - c_4 - c_3 - c_4)\sin x)]$$
$$= e^x[2c_3 \cos x - 2c_4 \sin x)]$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$e^x[2c_3 \cos x - 2c_4 \sin x)] - 2e^x[(c_3 + c_4)\cos x + (c_3 - c_4)\sin x)] = e^x \sin x$$
$$\Rightarrow e^x[(2c_3 - 2c_3 - 2c_4)\cos x + (-2c_4 + 2c_3 - 2c_4)\sin x)] = e^x \sin x$$
$$\Rightarrow [(-2c_4)\cos x + (2c_3 - 4c_4)\sin x)] = \sin x$$

Comparing coefficient on both side then,

$$-2c_4 = 0 \quad \text{and} \quad 2c_3 - 4c_4 = 1$$

Solving we get

$$c_4 = 0 \text{ and } c_3 = \frac{1}{2}$$

So the equation (v) becomes,

$$y_p = \frac{1}{2}e^x \sin x$$

Now, general equation of (i) is,

$$y(x) = y_p + y_h(x) = c_1 + c_2e^x - \frac{1}{2}e^x \sin x$$

(vii) $y'' + y' = x^2 + 2x + 4$ **[2004 Spring Q. No. 4(b)]**

Solution: Given that,

$$y'' + y' = x^2 + 2x + 4 \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, 1$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^{0x} + c_2e^{-x} = c_1 + c_2e^{-x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = x^2 + 2x + 4$$

Clearly R has constant term (which is part of polynomial) to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = x(c_3x^2 + c_4x + c_5)$$
$$= c_3x^3 + c_4x^2 + c_5x \quad \text{...... (v)}$$

Then, $y'_p = 3c_3x^2 + 2c_4x + c_5$ and $y''_p = 6c_3x + 2c_4$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$6c_3x + 2c_4 + 3c_3x^2 + 2c_4x + c_5 = x^2 + 2x + 4$$
$$\Rightarrow 3c_3x^2 + x(6c_3 + 2c_4) + (2c_4 + c_5) = x^2 + 2c + 4.$$

Comparing coefficient on both side then,

$$3c_3 = 1, \qquad 6c_3 + 2c_4 = 2, 2c_4 + c_5 = 4$$

Solving we get, $c_3 = \frac{1}{3}, c_4 = 0$ and $c_5 = 5$.

So, the equations (v) becomes, $y_p = \frac{1}{3}x^3 + 4x$

Now, general equation of (i) is,

$$y(x) = y_p + y_h(x) = c_1 + c_2e^{-x} + \frac{1}{3}x^3 + 4x$$

(viii) $y''' + 2y'' - y' - 2y = 1 - 4x^3$

Solution: Given that, $y''' + 2y'' - y' - 2y = 1 - 4x^3$ ...... (i)

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y''' + 2y'' - y' - 2y = 0 \quad \text{... (iii)}$$

The auxiliary equation of (iii) is,

$$m^3 + 2m^2 - m - 2 = 0$$
$$\Rightarrow m^2(m+2) - 1(m+2) = 0$$
$$\Rightarrow (m^3 - 1)(m+2) = 0$$
$$\Rightarrow (m+1)(m-1)(m+2) = 0$$
$$\Rightarrow m = 1, -1, -2$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{-x} + c_3e^{-2x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 1 - 4x^3$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_4x^3 + c_5x^2 + c_6x + c_7 \quad \text{...... (v)}$$

Then,

$$y'_p = 3c_4x^2 + 2c^5x + c_6$$
$$y''_p = 6c_4x + 2c_5$$
$$\text{and } y'''_p = 6c_4.$$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$6c_4 + 2(6c_4 + 2c_5) - (3c_4x^2 + 2c_5x + c_6) - 2(c_4x^3 + c_5x^2 + c_6x + c_7) = 1 - 4x^3$$
$$\Rightarrow -2c_4x^3 + x^2(-3c_4 - 2c_5) + (12c_4 - 2c_5 - 2c_6) + (6c_4 + 4c_5 - c_6 - 2c_7) = -4x^3 + 1$$

Comparing coefficient on both side then,

$$-2c_4 = -4, \qquad -3c_4 - 2c_5 = 0,$$
$$12c_4 - 2c_5 - 2c_6 = 0 \qquad 6c_4 + 4c_5 - c_6 - 2c_7 = 1.$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3 \cos x + c_4 \sin x \quad \text{...... (v)}$$

Then, $y'_p = -c_3 \sin x + c_4 \cos x$

$y''_p = -c_3 \cos x - c_4 \sin x$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$-c_3\cos x - c_4\sin x - (-c_3\sin x + c_4\cos x) - 2(c_3\cos x + c_4\sin x) = 10\cos x$$

$$\Rightarrow \cos x(-c_3 - c_4 - 2c_3) + \sin x(-c_4 + c_3 - 2c_4) = 10\cos x$$

$$\Rightarrow \cos x(-3c_3 - c_4) + \sin x(-c_3 + 3c_4) = 10\cos x$$

Comparing the coefficient on both side, then,

$$-3c_3 - c_4 = 10, \quad c_3 - 3c_4 = 0.$$

Solving we get, $c_3 = -3$ and $c_4 = -1$.

Thus, equation (v) becomes

$$y_p = -3\cos x - \sin x$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p = c_1e^{2x} + c_2e^{-x} - 3\cos x - \sin x$$

(iv) $y'' - 3y' + 2y = 4x + e^x$

Solution: Given that,

$$y'' - 3y' + 2y = 4x + e^x \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 4y' + 3y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = 2, 1$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{3x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 4x + e^x$$

Clearly R has e^x is only repeated for once and second one is not repeated term to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3x + c_4xe^x = x(c_3 + c_4e^x) \quad \text{...... (v)}$$

Then,

$$y'_p = c_3 + c_4e^x + c_4xe^x$$

and $y''_p = c_4e^x + c_4xe^x + c_4e^x = 2c_4e^x + c_4xe^x$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$2c_4e^x + c_4xe^x - 3(c_3 + c_4e^x + c_4xe^x) + 2(c_3x + c_4xe^x) = 4x + e^x$$

$$\Rightarrow (-3c_3) + x(2c_3) + e^x(2c_4 - 3c_4) + xe^x(c_4 - 3c_4 + 2c_4) = 4x + e^x$$

$$\Rightarrow (-3c_3) + x(2c_3) + e^x(2c_4 - 3c_4) = 4x + e^x$$

(v) $y'' + 4y' + 4y = 18\cosh x$

Solution: Given that,

$$y'' + 4y' + 4y = 18\cosh x \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 4y' + 4y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$$

Here m has real and repeated values, so the solution of (iii) is

$$y_h(x) = (c_1 + c_2x)e^{-2x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 18\cosh x = 9(e^x + e^{-x})$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3e^x + c_4e^{-x} \quad \text{...... (v)}$$

Then, $y'_p = c_3e^x - c_4e^{-x}$ and $y''_p = c_3e^x + c_4e^{-x}$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$c_3e^x + c_4e^{-x} + 4(c_3e^x - c_4e^{-x}) + 4(c_3e^x + c_4e^{-x}) = 9(e^x + e^{-x})$$

$$\Rightarrow 9c_3e^x + c_4e^{-x} = 9(e^x + e^{-x})$$

Comparing coefficient on both side then,

$9c_3 = 9$ and $c_4 = 9$

Solving we get, $c_1 = 1$ and $c_4 = 9$.

Then, the equation (iii) becomes

$$y_p = e^x + 9e^{-x}$$

Now, general equation of (i) is,

$$y(x) = y_n(x) + y_p = (c_1 + c_2x)e^{-2x} + e^x + 9e^{-x}$$

(vi) $y'' - 2y' = e^x \sin x$

Solution: Given that,

$$y'' - 2y' = e^x \sin x \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 2y' = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2.$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^0 + c_2e^{2x}$$

$$\Rightarrow y_h(x) = c_1 + c_2e^{2x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' + 4y = 0$$

The auxiliary equation of of (iii) is, ... (iii)

The auxiliary equation of homogeneous part of (i) is,

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = A \cos 2x + B \sin 2x \qquad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 8x^2$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_1x^2 + c_2x + c_3 \qquad \text{...(v)}$$

Then, $y'_p = 2c_1x + c_2$ and $y''_p = 2c_1$

Since (v) is particular solution of (i), so; (v) satisfies (i). That is,

$$2c_1 + 4(c_1x^2 + c_2x + c^3) = 8x^2$$

$$\Rightarrow 2c_1 + 4c_1x^2 + 4c_2x + 4c_1x^2 = 8x^2$$

$$\Rightarrow (2c_1 + 4c_3) + 4c_2x + 4c_1x^2 = 8x^2$$

Comparing the coefficient on the both side

$$2c_1 + 3c_3 = 0, \quad 4c_2 = 0 \quad \text{and} \quad 4c_1 = 8.$$

Solving we get, $c_1 = 2$, $c_2 = 0$ and $c_3 = 1$.

Therefore (v) becomes

$$y_p = 2x^2 - 1$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p(x)$$

$$= A \cos 2x + B \sin 2x + 2x_2 - 1$$

(iii) $y'' - y' - 2y = 10 \cos x$ **[2004 Spring; 2006 Fall Q. No. 4(b)]**

Solution: Given that,

$$y'' - y' - 2y = 10 \cos x \qquad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - y' - 2y = 0 \qquad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - m - 2 = 0$$

$$\Rightarrow (m - 2)(m + 1) = 0$$

$$\Rightarrow m = 2, -1$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{3x} \qquad \text{... (iv)}$$

And, comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 10 \cos x$$

Exercise 6.11

1. **Find the particular integral of the following**
For the particular integral we need solution of homogeneous equation corresponds to given equation which is find out as in 6.10. Adding the solution of homogeneous equation and the particular solution, we find the general solution of the given non-homogeneous equation of second degree and first order, as Q. 2 Exercise 6.11. So, here we skip Q.1.

(i) $y'' - y = 3e^{2x}$ (ii) $y'' + y' - 2y = 14 + 2x - 2x^2$

(iii) $y'' + 9y = 17e^{-5x}$ (iv) $y'' - 6y' + 9y = 2e^{3x}$

(v) $y'' + 3y' + 4y = -6.8\sin x$

(2) **Find the general solution of the following**

(i) $y'' - 4y' + 3y = 10e^{-2x}$ **[2009 Fall Q. No. 4(b)]**

Solution: Given that,

$$y'' - 4y' + 3y = 10e^{-2x} \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

The homogeneous equation corresponds to (i) is

$$y'' - 4y' + 3y = 0 \quad \text{... (iii)}$$

The auxiliary equation of of (iii) is,

$$m^2 - 4m + 3 = 0 \Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 1, 3$$

Here m has real and distinct values, so the solution of (iii) is

$$y_h(x) = c_1e^x + c_2e^{3x} \quad \text{... (iv)}$$

Comparing (i) with $y'' + Py' + Qy = R$ then we get,

$$R = 10e^{-2x}$$

Clearly R is not repeated to the independent solution of y_h, therefore choose the particular solution of (i) is

$$y_p = c_3e^{-2x} \quad \text{...... (v)}$$

Then, $y'_p = -2c_3e^{-2x}$ and $y''_p = 4c_3e^{-2x}$

Since (v) is particular solution of (i), so, (v) satisfies (i). That is,

$$4c_3e^{-2x} + 8c_3e^{-2x} + 3c_3e^{-2x} = 10e^{-2x}$$

$$\Rightarrow 15c_3e^{-2x} = 10e^{-2x}$$

$$\Rightarrow c_3 = \frac{10}{15} = \frac{2}{3}$$

Thus, equation (v) becomes, $y_p = \frac{2}{3}e^{-2x}$

Now, the solution of (i) is,

$$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{3x} + \frac{2}{3}e^{-2x}$$

$' + 4y = 8x^2$

: Given that,

$$y'' + 4y = 8x^2 \quad \text{...... (i)}$$

Let the solution of (i) is

$$y = y_h + y_p \quad \text{... (ii)}$$

8. If $\frac{d^4x}{dt^4} = m^4x$

Solution: Given that,

$$x^{iv} - m^4x = 0. \qquad \text{......(i)}$$

The auxiliary equation of (i) is

$$u^4 - m^4 = 0 \Rightarrow (u^2)^2 - (m^2)^2 = 0$$
$$\Rightarrow (u^2 + m^2)(u^2 - m^2) = 0$$

Since $u^2 + m^2 = 0$ and $u^2 - m^2 = 0$

$\Rightarrow u^2 = (im)^2$, $\quad \Rightarrow u^2 = m^2$

$\Rightarrow u = \pm mi$, $\quad \Rightarrow u = \pm m$

Here m has distinct values, so the general solution of given equation (i) is,

$$x = c_1e^{mt} + c_2e^{-mt} + c_3 \cos mt + c_4 \sin mt$$

9. **$y''' - 7y' - 6y = 0$**

Solution: Given that,

$$y''' - 7y' - 6y = 0 \qquad \text{.......(i)}$$

The auxiliary equation of (i) is

$$m^3 - 7m - 6 = 0$$
$$\Rightarrow m^3 + m^2 - m^2 - m - 6m - 6 = 0$$
$$\Rightarrow m^2(m + 1) - m(m + 1) - 6(m + 1) = 0$$
$$\Rightarrow (m + 1)(m^2 - m - 6) = 0$$
$$\Rightarrow (m + 1))(m^2 - 3m + 2m - 6) = 0$$
$$\Rightarrow (m + 1)\{m(m-3) + 2(m - 3)\} = 0$$
$$\Rightarrow (m + 1)(m - 3)(m + 1) = 0$$
$$\Rightarrow m = -1, 3, -2$$

Here m has real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{-x} + c_2e^{3x} + c_3e^{-2x}$$

10. **$y''' - 4y'' + 4y' = 0$**

Solution: Given that,

$$y''' - 4y'' + 4y' = 0 \qquad \text{.......(i)}$$

The auxiliary equation of (i) is

$$m^3 - 4m^2 + 4m = 0 \Rightarrow m(m^2 - 4m + 4) = 0$$
$$\Rightarrow m(m - 2)^2 = 0$$
$$\Rightarrow m = 0, 2, 2$$

Here m has two real repeated and one distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1 + (c_2 + c_3x)e^{2x}$$

$$\Rightarrow \quad (m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

for $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{4}i}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

for $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4}i}{2} = \frac{2 + \pm 2i}{2} = 1 \pm i.$$

Here m complex values, so the general solution of given equation (i) is,

$$y(x) = e^{-x}(A\cos x + D\sin x) + e^{x}(C\cos x + D\sin x).$$

5. $(D^3 + 6d^2 + 11D + 6) = y = 0$

Solution: Given that,

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0 \qquad \text{...... (i)}$$

The auxiliary equation of (i) is

$$m^3 + 6m^2 + 11m + 6 = 0$$
$$\Rightarrow \quad m^3 + m^2 + 5m^2 + 5m + 6m + 6 = 0$$
$$\Rightarrow \quad m^2(m+1) + 5m(m+1) + 6(m+1) = 0$$
$$\Rightarrow \quad (m+1)(m^2 + 5m + 6) = 0$$
$$\Rightarrow \quad (m+1)(m^2 + 2m + 3m + 6) = 0$$
$$\Rightarrow \quad (m+1)\{m(m+2) + 3(m, +2)\} = 0$$
$$\Rightarrow \quad (m+1)(m+2)(m+3) = 0$$
$$\Rightarrow \quad m = -1, -2, -3$$

Here m real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$$

6. $y''' + 3y'' + 3y' - y = 0$

Solution: Given that,

$$y''' + 3y'' + 3y' - y = 0 \qquad \text{...... (i)}$$

The auxiliary equation of (i) is

$$m^3 - 3m^2 + 3m - 1 = 0$$
$$\Rightarrow \quad (m-1)^3 = 0$$
$$\Rightarrow \quad m = 1, 1, 1$$

Here m has real and repeated values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x + c_3x^2)e^x$$

7. $y^{iv} + 8y'' + 16y = 0$

Solution: Given that,

$$y^{iv} + 8y'' + 16y = 0 \qquad \text{....... (i)}$$

The auxiliary equation of (i) is

$$m^4 + 8m^2 + 16 = 0$$
$$\Rightarrow \quad (m^2)^2 + 2.m^2.4 + (4)^2 = 0$$
$$\Rightarrow \quad (m^2 + 4)^2 = 0$$

for $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m^2 = (\pm 2i)^2 \Rightarrow m = 0 \pm 2i$

Here m has repeated complex values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$$

Exercise 6.10

Find the general solution of the following differential equation

1. **$y''' - y'' - y' + 2y = 0.$**

Solution: Given that,

$$y''' - y'' - y' + 2y = 0 \quad \text{.......(i)}$$

The auxiliary equation of (i) is

$$m^3 - 2m^2 - m + 2 = 0$$
$$\Rightarrow m^2(m-2) - 1(m-2) = 0$$
$$\Rightarrow (m-2)(m^2-1) = 0$$
$$\Rightarrow (m-2)(m-1)(m+1) = 0$$
$$\Rightarrow m = 2, 1, -1$$

Here m has real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{2x} + c_2e^{x} + c_3e^{-x}$$

2. **$y''' - y' = 0$**

Solution: Given that,

$$y''' - y' = 0 \quad \text{.......(i)}$$

The auxiliary equation of (i) is

$$m^3 - m = 0 \quad \Rightarrow \quad m(m^2-1) = 0$$
$$\Rightarrow \quad m = 0, 1, -1$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1 + c_2e^{x} + c_3e^{-x}$$

3. **$y^{iv} - 5y'' + 4y = 0$**

Solution: Given that,

$$y^{iv} - 5y'' + 4y = 0 \quad \text{.......(i)}$$

The auxiliary equation of (i) is

$$m^4 - 5m^2 + = 0 \Rightarrow m^4 - 4m^2 - m^2 + 4 = 0$$
$$\Rightarrow m^2(m^2-4) - 1(m^2-4) = 0$$
$$\Rightarrow (m^2-4)(m^2-1) = 0$$
$$\Rightarrow (m-2)(m+2)(m-1)(m+1) = 0$$
$$\Rightarrow m = 2, -2, 1, -1$$

Here m has real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3e^{x} + c_4e^{-4}$$

4. **$(d^4 + 4)y = 0$**

Solution: Given that,

$$\frac{d^4y}{dx^4} + 4y = 0 \quad \text{......(i)}$$

The auxiliary equation of (i) is

$$m^4 + 4 = 0 \quad \Rightarrow \quad (m^2)^2 + (2)^2 = 0$$
$$\Rightarrow (m^2+2)^2 - 4m^2 = 0$$
$$\Rightarrow (m^2+2)^2 - (2m)^2 = 0$$

Here m has two real and repeated values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x)e^{\frac{-x}{3}} \qquad \text{... (iii)}$$

Using the given value y(0) = 4 to (iii) we get

$$4 = c_1e^0 \Rightarrow c_1 = 4$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = \frac{-1}{3}c_1e^{\frac{-x}{3}} + c_2\left(-\frac{1}{3}xe^{\frac{-x}{3}} + e^{\frac{-x}{3}}\right)$$

Using the given value $y'(0) = \frac{-13}{3}$ to (iii) we get

$$-\frac{13}{3} = -\frac{1}{3}c_1 + c_2 \Rightarrow -\frac{13}{3} = -\frac{4}{3} + c_2$$

$$\Rightarrow c_2 = \frac{4}{2} - \frac{13}{3} = \frac{-9}{3} = -3$$

Now, the equations (iii) becomes,

$$y(x) = (4 - 3x)e^{\frac{-x}{3}}.$$

This is the solution of the given equation (i) satisfying (ii).

(xiii) y" – y' –2y = 0, y(0) = –4, y'(0) = –17

[2016 Fall Q.No. 5(b), 2013 Fall Q.No. 5(b)]

Solution: Given equation is,

$$y'' - y' - 2y = 0 \qquad \text{...... (i)}$$

$$y(0) = -4,\ y'(0) = -17 \qquad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$m^2 - m - 2 = 0$$

$$\Rightarrow (m-2)(m+1) = 0$$

$$\Rightarrow m = 2, -1$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{2x} + c_2e^{-x} \qquad \text{... (iii)}$$

Using the given value y(0) = –4 to (iii) we get

$$-4 = c_1 + c_2 \qquad \text{... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = 2c_1e^{2x} - c_2e^{-x}$$

Using the given value y'(0) = –17 to (iii) we get

$$-17 = 2c_1 - c_2 \qquad \text{... (B)}$$

Solving the equations (A) and (B) we get,

$$c_1 = -7,\ c_2 = 3.$$

Now, the equations (iii) becomes,

$$y(x) = 3e^{-x} - 7e^{2x}$$

This is the solution of the given equation (i) satisfying (ii).

Solving the equations (A) and (B) we get,

$$c_1 = 0, \quad c_2 = 2$$

Now, the equations (iii) becomes,

$$y(x) = 2e^{-1.3x}$$

This is the solution of the given equation (i) satisfying (ii).

(xi) **$4y'' - 4y' - 3y = 0$, $y(-2) = e$, $y'(-2) = -\frac{e}{2}$**

Solution: Given equation is,

$$4y'' - 4y' - 3y = 0 \qquad \text{...... (i)}$$

$$y(-2) = e, \; y'(-2) = -\frac{e}{2} \qquad \text{....... (ii)}$$

The auxiliary equation of (i) is,

$$4m^2 - 4m - 3 = 0$$

$$\Rightarrow (2m - 3)(2m + 1) = 0$$

$$\Rightarrow m = -\frac{3}{2}, \frac{-1}{2}$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1 e^{\frac{3}{2}x} + c_2 e^{\frac{-1}{2}x} \qquad \text{...... (iii)}$$

Using the given value $y(-2) = e$ to (iii) we get

$$e = c_1 e^{-3} + c_2 e. \qquad \text{... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = \frac{3}{2} c_1 e^{\frac{3}{2}x} - \frac{1}{2} c_2 e^{\frac{-1}{2}x}$$

Using the given value $y'(-2) = \frac{-e}{2}$ to (iii) we get

$$\frac{-e}{2} = \frac{3}{2} c_1 e^{-3} - \frac{1}{2} c_2 e$$

$$\Rightarrow -e = 3c_1 e^{-3} - c_2 e \qquad \text{... (B)}$$

Solving the equations (A) and (B) we get,

$$c_1 = 0, c_2 = 1$$

Now, the equations (iii) becomes,

$$y(x) = e^{-0.5x}$$

This is the solution of the given equation (i) satisfying (ii).

(xii) **$9y'' + 6y' + y = 0$, $y(0) = 4$, $y'(0) = \frac{-13}{3}$**

Solution: Given equation is,

$$9y'' + 6y' + y = 0 \qquad \text{...... (i)}$$

$$y(0) = 4, \; y'(0) = \frac{-13}{3} \qquad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$9m^2 + 6m + 1 = 0 \Rightarrow (3m + 1)^2 = 0 \Rightarrow m = \frac{-1}{3}, \frac{-1}{3}$$

(ix) $8y'' - 2y' - y = 0, y(0) = -0.2, y'(0) = -0.325$

Solution: Given equation is,

$$8y'' - 2y' - y = 0 \quad \text{...... (i)}$$

$$y(0) = -0.2,\ y'(0) = -0.325 \quad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$8m^2 - 2m - 1 = 0$$

$$\Rightarrow (2m-1)(4m+1) = 0$$

$$\Rightarrow m = \frac{1}{2}, -\frac{1}{4}$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{\frac{x}{2}} + c_2e^{\frac{-x}{4}} \quad \text{...... (iii)}$$

Using the given value $y(0) = -0.2$ to (iii) we get

$$-0.2 + c_1 + c_2 \quad \text{... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = \frac{1}{2}c_1e^{\frac{x}{2}} - \frac{1}{4}c_2e^{\frac{x}{2}}$$

Using the given value $y'(0) = -0.325$ to (iii) we get

$$-0.325 = \frac{1}{2}c_1 - \frac{1}{4}c_2$$

$$\Rightarrow 2c_1 - c_2 = -1.3 \quad \text{... (B)}$$

Solving the equations (A) and (B) we get,

$$c_1 = -0.5 \text{ and } c_2 = 0.3$$

Now, the equations (iii) becomes,

$$y(x) = 0.3e^{\frac{-x}{4}} - 0.5e^{\frac{x}{2}}$$

This is the solution of the given equation (i) satisfying (ii).

(x) $\mathbf{y'' + 2.2y' + 1.17y = 0, y(0) = 2, y'(0) = -2.60}$

Solution: Given equation is,

$$y'' + 2.2y' + 1.17y = 0 \quad \text{...... (i)}$$

$$y(0) = 2,\ y'(0) = -2.60 \quad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$m^2 + 2.2m + 1.17 = 0$$

$$\Rightarrow m = \frac{-2.2 \pm \sqrt{2.2^2 - 4.1.1.17}}{2.1} = \frac{-2.2 \pm \sqrt{0.16}}{2} = \frac{-2.2 \pm 0.4}{2}$$

$$\Rightarrow m = -0.90, -1.30$$

Here m has complex values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{-0.9x} + c_2e^{-1.3x} \quad \text{...... (iii)}$$

Using the given value $y(0) = 2$ to (iii) we get

$$2 = c_1 + c_2 \quad \text{... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = -0.9c_1e^{-0.9x} - 1.3c_2e^{-1.3x}$$

Using the given value $y'(0) = -2.60$ to (iii) we get

$$-2.60 = 0.9c_1 - 1.3c_2 \quad \text{... (B)}$$

(vii) $y'' - 4y' + 3y = 0, y(0) = -1, y'(0) = -5$

Solution: Given equation is, $y'' - 4y' + 3y = 0$ **[2019 Fall Q.No. 5 (b)]**

$$y(0) = -1, y'(0) = -5 \qquad \text{...... (i)}$$

The auxiliary equation of (i) is, ... (ii)

$$m^2 - 4m + 3 = 5$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, 1.$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{3x} + c_2e^x$$

Using the given value $y(0) = -1$ to (iii) we get ...... (iii)

$$-1 = c_1 + c_2$$

Differential equation (iii) w. r. t. x, then, ... (A)

$$y'(x) = 3c_1e^{3x} + c_2e^x$$

Using the given value $y'(0) = -5$ to (iii) we get

$$-5 = 3c_1 + c_2 \qquad \text{... (B)}$$

Solving the equations (A) and (B) we get,

$$c_1 = -2 \text{ and } c_2 = 1$$

Now, the equations (iii) becomes,

$$y(x) = e^x - 2e^{3x}$$

This is the solution of the given equation (i) satisfying (ii).

(viii) $y'' + 4y' + 4y = 0$ $y(0) = 1, y'(0) = 1$

Solution: Given equation is, $y'' + 4y' + 4y = 0$ ...... (i)

$$y(0) = 1, y'(0) = 1 \qquad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow (m+2)^2 = 0$$

$$\Rightarrow m = -2, -2.$$

Here m has two real and repeated values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x)e^{-2x} \qquad \text{...... (iii)}$$

Using the given value $y(0) = 1$ to (iii) we get

$$1 = c_1$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = -2c_1e^{-2x} + c_2(-2xe^{-2x} + e^{-2x})$$

Using the given value $y'(0) = 1$ to (iii) we get

$$1 = -2c_1 + c_2 \qquad \text{....... (A)}$$

$$\Rightarrow c_2 = 3 \qquad \text{[Using (A)]}$$

Now, the equations (iii) becomes,

$$y(x) = (1 + 3x)e^{-2x}$$

This is the solution of the given equation (i) satisfying (ii).

$$2 = e^0 (A - \sin x + B \cos x) + 2(\cos x + B \sin x)e^{2x}$$

$$\Rightarrow \quad 2 = B + 2A$$

$$\Rightarrow \quad 2 = B + 2 \quad [\because A = 1]$$

$$\Rightarrow \quad B = 0$$

Therefore, the equation (iii) becomes,

$$y(x) = e^{2x}\cos x.$$

This is the solution of the given equation (i) satisfying (ii).

(v) $y'' - 4y' + 4y = 0, y(0) = 3, y'(0) = 1$ **[2009 Spring Q. No. 5(b)]**

Solution: Given equation is,

$$y'' - 4y' + 4y = 0 \qquad \text{...... (i)}$$

with

$$y(0) = 3, \; y'(0) = 1 \qquad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$m^2 - 4m + 4 = 0 \;\Rightarrow\; (m-2)^2 = 0 \;\Rightarrow\; m = 2, 2.$$

Here m has two real and repeated values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x)e^{2x} \qquad \text{...... (iii)}$$

Using the given value $y(0) = 3$ to (iii) we get

$$3 = (c_1 + c_2 0)e^0 \;\Rightarrow\; c_1 = 3.$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = 2c_1e^{2x} + c_2(x.\, 2e^{2x} + e^{2x})$$

$$\Rightarrow \quad y'(x) = 2c_1e^{2x} + 2c_2xe^{2x} + c_2e^{2x}$$

Using the given value $y'(0) = 1$ to (iii) we get

$$1 = 2 \times 3e^0 + 0 + c_2e^0$$

$$\Rightarrow \quad c_2 = -5$$

Now, the equations (iii) becomes,

$$y(x) = (3 - 5x)\, e^{2x}$$

This is the solution of the given equation (i) satisfying (ii).

(vi) $y'' - y = 0, y(0) = 6, y'(0) = -4$

Solution: Given equation is,

$$y'' - y = 0 \qquad \text{...... (i)}$$

$$y(0) = 6, \; y'(0) = -4 \qquad \text{...... (ii)}$$

The auxiliary equation of (i) is,

$$m^2 - 1 = 0 \;\Rightarrow\; m = \pm 1$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^x + c_2e^{-x} \qquad \text{...... (iii)}$$

Using the given value $y(0) = 6$ to (iii) we get

$$6 = c_1 + c_2 \qquad \text{... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = c_1e_x - c_2e^{-x}$$

Using the given value $y'(0) = -4$ to (iii) we get

$$-4 = c_1 - c_2 \qquad \text{... (B)}$$

Solving the equations (A) and (B) we get,

$$c_1 = 1 \text{ and } c_2 = 5$$

Now, the equations (iii) becomes,

$$y(x) = e^x + 5e^{-x}$$

This is the solution of the given equation (i) satisfying (ii).

Using the given value y' (0) = 14 to (iii) we get

$$14 = c_2(-3e^0 + e^0) - 3c_1$$

$$\Rightarrow \quad 14 = -2c_2 + 12$$

$$\Rightarrow \quad c_2 = -1. \qquad \text{[Using (A)]}$$

Therefore, the equation (iii) becomes,

$$y(x) = (-4 + 2x)^{-3x}$$

$$\Rightarrow \quad y(x) = (2x - 4)e^{-3x}$$

This is the solution of the given equation (i) satisfying (ii).

(iii) $y'' + y' - 2y = 0, y(0) = 3, y'(0) = 0$

Solution: Given equation is,

$$y'' + y' - 2y = 0 \qquad \text{...... (i)}$$

with $\quad y(0) = 3, y'(0) = 0 \qquad$ ...... (ii)

The auxiliary equation of (i) is,

$$m^2 + m - 2 = 0$$

$$\Rightarrow \quad (m + 2)(m - 1) = 0$$

$$\Rightarrow \quad m = -2, 1$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{-2x} + c_2e^x \qquad \text{...... (iii)}$$

Using the given value y(0) = 1 to (iii) we get

$$c_1 + c_2 = 3 \qquad \text{...... (A)}$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = -2c_1e^{-2x} + c_2e^x$$

Using the given value y' (0) = 0 to (iii) we get

$$c_2 - 2c_1 = 0 \qquad \text{...... (B)}$$

Solving (A) and (B) we get,

$$c_1 = 1 \quad \text{and} \quad c_2 = 2.$$

Therefore, the equation (iii) becomes,

$$y(x) = e^{-2x} + 2e^x$$

This is the solution of the given equation (i) satisfying (ii).

(iv) $y'' - 4y' + 5y = 0, y(0) = 1, y'(0) = 2$ **[2011 Fall Q. No. 5(b)]**

Solution: Given equation is,

$$y'' - 4y' + 5y = 0 \qquad \text{...... (i)}$$

with $\quad y(0) = 1, y'(0) = 2 \qquad$ ...... (ii)

The auxiliary equation of (i) is,

$$m^2 - 4m + 5 = 0$$

$$\Rightarrow \quad m = \frac{+4 \pm \sqrt{16 - 4.1.5}}{2.1}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4i}}{2} = \frac{4 \pm 2i}{2} = (2 \pm i)$$

Here m has complex values, so the general solution of given equation (i) is,

$$y(x) = e^{2x}(A\cos x + B\sin x) \qquad \text{...... (iii)}$$

Using the given value y(0) = 1 to (iii) we get

$$1 = e^0(A\cos 0 + B\sin 0) \Rightarrow A = 1.$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = e^{2x}(-A\sin x + B\cos x) + (A\cos x + B\sin x)e^{2x}$$

Using the given value y' (0) = 2 to (iii) we get

(xx) $y'' - 2\sqrt{2}\,y' + 2y = 0$

Solution: Given equation is,

$$y'' - 2\sqrt{2}\,y' + 2y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 2\sqrt{2}m + 2 = 0 \Rightarrow (m - \sqrt{2})^2 = 0 \Rightarrow m = \sqrt{2}, \sqrt{2}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{\sqrt{2}x}$$

2. Solve the following initial value problems

(i) $y'' - 16y = 0, y(0) = 1, y'(0) = 20$

Solution: Given equation is,

$$y' - 16y = 0 \qquad \ldots\ldots (i)$$

with $$y(0) = 1, y'(0) = 20 \qquad \ldots\ldots (ii)$$

The auxiliary equation of (i) is,

$$m^2 - 16 = 0 \Rightarrow m = \pm 4$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = c_1e^{4x} + c_2e^{-4x} \qquad \ldots\ldots (iii)$$

Using the given value $y(0) = 1$ to (iii) we get

$$1 = c_1e^0 + c_2e^0 \quad \Rightarrow c_1 + c_2 = 1 \qquad \ldots\ldots (A)$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = 4c_1e^{4x} - 4c_2e^{-4x}$$

Using the given value $y'(0) = 20$ to (iii) we get

$$20 = 4c_1 - 4c_2 \quad \Rightarrow c_1 - c_2 = 5 \qquad \ldots\ldots (B)$$

Solving the equations (A) and (B) we get,

$$c_1 = 3 \text{ and } c_2 = -2.$$

Therefore, the equation (iii) becomes,

$$y(x) = 3e^{4x} - 2e^{-4x}$$

This is the solution of the given equation (i) satisfying (ii).

(ii) $y'' + 6y' + 9y = 0, y(0) = -4, y'(0) = 14$

Solution: Given equation is,

$$y'' + 6y' + 9y = 0 \qquad \ldots\ldots (i)$$

with $$y(0) = -4, y'(0) = 14 \qquad \ldots\ldots (ii)$$

The auxiliary equation of (i) is,

$$m^2 + 6m + 9 = 0 \Rightarrow (m + 3)^2 = 0$$
$$\Rightarrow m = -3, -3.$$

Here m has two real and distinct values, so the general solution of given equation (i) is,

$$y(x) = (c_1 + c_2x)e^{-3x} \qquad \ldots\ldots (iii)$$

Using the given value $y(0) = 1$ to (iii) we get

$$-4 = (c_1 + 0)e^0 \qquad \Rightarrow c_1 = -4. \qquad \ldots\ldots (A)$$

Differential equation (iii) w. r. t. x, then,

$$y'(x) = c_2(-3xe^{-3x} + e^{-3x}) - 3c_1e^{-3x}$$

(xvi) $16y'' - p^2y = 0$

Solution: Given equation is,

$$16y'' - p^2y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$16m^2 - p^2 = 0 \Rightarrow 16m^2 = p^2$$

$$\Rightarrow m = \pm\frac{p}{4}$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{\frac{p}{4x}} + c_2e^{\frac{-p}{4x}}$$

(xvii) $25y'' + 40y' + 16y = 0$

Solution: Given equation is,

$$25y'' + 40y' + 16y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$16m^2 - 8m + 5 = 0$$

$$\Rightarrow m = \frac{8 \pm \sqrt{8^2 - 4.16.5}}{2.16} = \frac{8 \pm \sqrt{64 - 320}}{32} = \frac{8 \pm \sqrt{-256}}{32}$$

$$= \frac{8 \pm \sqrt{256i^2}}{32} = \frac{8 \pm 16i}{32} = \frac{1 \pm 2i}{4}$$

Here m has complex values, so the general solution of given equation (1) is,

$$y(x) = e^{\frac{1}{4x}}\left(A\cos\frac{1}{2^x} + B\sin\frac{x}{2}\right).$$

(xviii) $17y'' - 8y' + 5y = 0$

Solution: Given equation is

$$17y'' - 8y' + 5y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$16m^2 - 8m + 5 = 0$$

$$\Rightarrow m = \frac{8 + \sqrt{64 - 320}}{32} = \frac{8 + \sqrt{-256}}{32} = \frac{8 + i16}{32} = \frac{1}{4} \pm i\frac{1}{2}$$

Here m has complex values, so the general solution of given equation (1) is,

$$y = e^{x/4}\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$$

(xix) $y'' - 9p^2y = 0$

Solution: Given equation is,

$$y'' - 9p^2y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 9p^2 = 0 \Rightarrow m^2 - 9p^2 = 0 \Rightarrow m = \pm 3p$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = (c_1e^{3px} + c_2e^{-3px}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{\frac{5}{3x}}$$

(xii) $y'' + 2ky' + k^2y = 0$ **where k is a constant.**

Solution: Given equation is,

$$y'' + 2ky' + k^2y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 2km + k^2 = 0 \;\Rightarrow\; (m + k)^2 = 0$$
$$\Rightarrow\; m = -k, -k.$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{-kx}$$

(xiii) $y'' - 3y' - 4y = 0$

Solution: Given equation is,

$$y'' - 3y' - 4y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 3m - 4 = 0 \;\Rightarrow\; (m - 4)(m + 1) = 0$$
$$\Rightarrow\; m = 4, -1.$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{4x} + c_2e^{-x}$$

(xiv) $y'' - 4y' + y = 0$

Solution: Given equation is,

$$y'' - 4y' + y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow\; m = \frac{4 \pm \sqrt{16^2 - 4.1.1}}{2.1} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = (2 \pm \sqrt{3})$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$$

(xv) $y'' + 6y' + 9y = 0$

Solution: Given equation is,

$$y'' + 6y' + 9y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 6m + 9 = 0 \;\Rightarrow\; (m + 3)^2 = 0$$
$$\Rightarrow\; m = -3, -3$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{-3x}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{2x}$$

(viii) $4y'' + 4y' - 3y = 0$

Solution: Given equation is,

$$4y'' + 4y' - 3y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$4m^2 + 4m - 3 = 0 \Rightarrow (2m+3)(2m-1) = 0$$

$$\Rightarrow m = \frac{-3}{2}, \frac{1}{2}$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{\frac{-3}{2x}} + c_2e^{\frac{1}{2x}}$$

(ix) $2y'' - 9y' = 0$

Solution: Given equation is,

$$2y'' - 9y' = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$2m^2 - 9m = 0 \Rightarrow m(2m-9) = 0$$

$$\Rightarrow m = 0, \frac{9}{2}$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^0 + c_2e^{\frac{9}{2x}}$$

$$\Rightarrow y(x) = c_1 + c_2e^{\frac{9}{2x}}$$

(x) $y'' + 9y' + 20y = 0$

Solution: Given equation is,

$$y'' + 9y' + 20y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 9m + 20 = 0 \Rightarrow (m+4)(m+5) = 0$$

$$\Rightarrow m = -5, -4.$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{-5x} + c_2e^{-4x}$$

(xi) $9y'' - 30y' + 25y = 0$

Solution: Given equation is,

$$9y'' - 30y' + 25y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$9m^2 - 30m + 25 = 0 \Rightarrow (3m-5)^2 = 0$$

$$\Rightarrow m = \frac{5}{3}, \frac{5}{3}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)\, e^{4x}$$

(iii) $y'' + y' + 0.25y = 0$

Solution: Given equation is,

$$y'' + y' + 0.25y = 0 \qquad \text{... (1)}$$

The auxiliary equation of (1) is,

$$m^2 + m + \frac{1}{4} = 0 \quad \Rightarrow \quad 4m^2 + 4m + 1 = 0$$

$$\Rightarrow \quad (2m + 1)^2 = 0$$

$$\Rightarrow \quad m = -\frac{1}{2}, -\frac{1}{2}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)\, e^{\frac{1}{2x}}$$

(iv) $8y' - 2y' - y = 0$

Solution: Given equation is,

$$8y' - 2y' - y = 0 \qquad \text{... (1)}$$

The auxiliary equation of (1) is,

$$8m^2 - 2m - 1 = 0 \quad \Rightarrow \quad (2m - 1)(4m + 1) = 0$$

$$\Rightarrow \quad m = \frac{1}{2}, -\frac{1}{4}$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1 e^{\frac{1}{2x}} + c_2 e^{\frac{1}{4x}}$$

(v) $2y'' + 10y' + 25y = 0$

Solution: Given equation is,

$$2y'' + 10y' + 25y = 0 \qquad \text{... (1)}$$

The auxiliary equation of (1) is,

$$2m^2 + 10m + 25 = 0$$

$$\Rightarrow \quad m = \frac{-10 \pm \sqrt{10^2 - 4.2.25}}{2.2} = \frac{-10 \pm \sqrt{100}}{4} = \frac{-10 \pm \sqrt{100i^2}}{4}$$

$$= \frac{-10 \pm 10i}{4} = \frac{-5 \pm 5i}{2}$$

Here m has one real and two imaginary values, so the general solution of given equation (1) is,

$$y(x) = e^{\frac{-5}{2x}} \left(A \cos \frac{5}{2}x + B \sin \frac{5}{2}x\right).$$

(vii) $y'' - 4y' + 4y = 0$

Solution: Given equation is,

$$y'' - 4y' + 4y = 0 \qquad \text{... (1)}$$

The auxiliary equation of (1) is,

$$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m - 2)^2 = 0$$

$$\Rightarrow \quad m = 2, 2.$$

Now, the equation (iii) becomes,

$$y(x) = 0$$

This is the solution of (i) satisfies (ii).

(iii) $y'' - 4y' + 3y = 0,\ y(0) = -1,\ y'(0) = -5$

Solution: Given equation is, $y'' - 4y' + 3y = 0$ ...... (i)

$$y(0) = -1,\ y'(0) = -5 \quad \text{...... (ii)}$$

The auxiliary equation of (1) is,

$$m^2 - 4m + 3 = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, 1$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{3x} + c_2e^{x} \quad \text{... (A)}$$

Since, by (ii), we have $y(0) = -1$ then (iii) gives,

$$-1 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = -1 \quad \text{... (B)}$$

Now, differentiating (iii) we get,

$$y'(x) = 3c_1e^{3x} + c_2e^{x}$$

Since, by (ii), we have $y'(0) = -5$ then

$$-5 = 3c_1e^{3x} + c_2e^{x} \Rightarrow 3c_1 + c_2 = -5 \quad \text{... (B)}$$

Solving (A) and (B) we get,

$$c_1 = -2 \text{ and } c_2 = 1$$

Now, the equation (i) becomes

$$y(x) = -2e^{3x} + e^{x}$$

$$\Rightarrow y(x) = e^{x} - 2e^{3x}$$

This is the solution of (i) satisfies (ii).

Exercise 6.9

1. **Solve the following differential equations.**

(i) $y'' - 25y = 0$

Solution: Given equation is,

$$y'' - 25y = 0 \quad \text{...(1)}$$

The auxiliary equation of (1) is,

$$m^2 - 25 = 0 \Rightarrow m^2 = (\pm 5)^2$$

$$\Rightarrow m = \pm 5$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{5x} + c_2e^{-5x}$$

(ii) $y'' - 8y' + 16 = 0$

Solution: Given equation is,

$$y'' - 8y' + 16 = 0 \quad \text{... (1)}$$

The auxiliary equation of (1) is,

$$m^2 - 8m + 16 = 0 \Rightarrow (m-4)^2 = 0$$

$$\Rightarrow m = 4, 4.$$

The auxiliary equation of (1) is,

$$2m^2 + 5m - 12 = 0$$

$$\Rightarrow (m+4)(2m-3) = 0$$

$$\Rightarrow m = -4,\ 3/2$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{-4x} + c_2e^{\frac{3}{2x}}$$

3. Solve the following initial value problems.

(i) $y'' - y = 0, y(0) = 6, y'(0) = 4$

Solution: Given equation is, $y'' - y = 0$ ...... (i)

$y(0) = 6, y'(0) = 4$ ...... (ii)

The auxiliary equation of (1) is,

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{x} + c^2e^{-x} \quad \text{...... (iii)}$$

Since, by (ii) we have y(0) = 6 then (iii) gives,

$$6 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = 6 \quad \text{.... (A)}$$

Now, differentiating (iii) we get,

$$y'(x) = c_1e^{x} - c_2e^{-x}$$

Since, by (ii) we have y'(0) = 4 then

$$-4 = c_1e^0 = c_2e^0 \Rightarrow c_1 - c_2 = -4 \quad \text{.... (B)}$$

Solving (A) and (B) we get,

$$c_1 = 1,\ c_2 = 5$$

Now, the equations (iii) becomes

$$y(x) = e^{x} + 5e^{-x}$$

This is the solution of (i) satisfies (ii).

(ii) $y'' - 3y' + 2y = 0, y(0) = 0, y'(0) = 0$

Solution: Given equation is, $y'' - 3y' + 2y = 0$ ...... (i)

$y(0) = 0, y'(0) = 0$ ...... (ii)

The auxiliary equation of (1) is,

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = 2, 1$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{2x} + c_2e^{x} \quad \text{...... (iii)}$$

Since, by (ii), we have y(0) = 0 then (iii) gives,

$$0 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = 0 \quad \text{... (A)}$$

Now, differentiating (iii) we get,

$$y'(x) = 2c_1e^{2x} + c_2e^{x}$$

Since, by (ii), we have y'(0) = 0 then

$$0 = 2c_1e^0 + c_2e^0 \Rightarrow 2c_1 + c_2 = 0 \quad \text{... (B)}$$

Solving (A) and (B) we get,

$$c_1 = 0 \text{ and } c_2 = 0$$

2. Find a general solution of the following

(i) $y'' - a^2y = 0$

Solution: Given equation is,

$$y'' - a^2y = 0 \quad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 - a^2 = 0 \Rightarrow (m)^2 = (\pm a)^2$$
$$\Rightarrow m = a, -a$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{ax} + c_2e^{-ax}$$

(ii) $y'' - 4y' + 3y = 0$

Solution: Given equation is,

$$y'' - 4y' + 3y = 0 \quad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 4m + 3 = 0$$
$$\Rightarrow (m-3)(m-1) = 0$$
$$\Rightarrow m = 3, 1$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{3x} + c_2e^{x}$$

(iii) $y'' + y' = 0$

Solution: Given equation is,

$$y'' + y' = 0 \quad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0$$
$$\Rightarrow m = 0, -1$$

Here m has two real and distinct values, so the general solution of given equation (1) is,

$$y(x) = c_1e^{0} + c_2e^{-x}$$
$$\Rightarrow y(x) = c_1 + c_2e^{-x}$$

(iv) $16y'' + 24y' + 9y = 0$ **[2002 Q. No. 4(b)]**

Solution: Given equation is,

$$16y'' + 24y' + 9y = 0 \quad \dots (1)$$

The auxiliary equation of (1) is,

$$16m^2 + 24m + 9 = 0$$
$$\Rightarrow (4m + 3)^2 = 0$$
$$\Rightarrow m = -\frac{3}{4}, -\frac{3}{4}$$

Here m has two real and repeated values, so the general solution of given equation (1) is,

$$y(x) = (c_1 + c_2x)e^{\frac{-3}{4x}}$$

(v) $2y'' + 5y' - 12 = 0$

Solution: Given equation is,

$$2y'' + 5y' - 12 = 0 \quad \dots (1)$$

$$(m-2-i)(m-2+i)=0$$
$$\Rightarrow (m-2)^2-(\pm i)^2=0$$
$$\Rightarrow (m-2)^2+1=0$$
$$\Rightarrow m^2-4m+5=0$$

So, its different equation is,

$$y''-4y'+5y=0.$$

(iii) e^{-2x}, 1

Solution: Given that the independent solutions of a differential equation are,

$$e^{-2x}, 1 \text{ i.e. } e^{-2x}, e^{0x}$$

This means the equation has two real and non–repeated roots

$$m=-2, 0$$

Therefore the auxiliary equation of the equation is,

$$(m+2)(m-0)=0$$
$$\Rightarrow m^2+2m=0$$

So, its different equation is,

$$y''+2y'=0.$$

Exercise 6.8

1. Are the following functions linearly dependent or independent on the gien interval?

(i) cosx, sinx (any interval)

Solution: Let, $y_1 = \cos x$ and $y_2 = \sin x$

Now,

$$\frac{y_1(x)}{y_2(x)}=\frac{\cos x}{\sin x}=\cot x,\ \text{which is not a constant.}$$

So, cosx and sinx are linearly independent.

(ii) x^2, x^3 $(o<x<1)$

Solution: Let, $y_1=x^2$ and $y_2=x^3$

Now,

$$\frac{y_1(x)}{y_2(x)}=\frac{x^2}{x^3}=\frac{1}{x}\quad \text{,which is not a constant.}$$

So, x^2-4 and $-3x^2+12$ are linearly dependent.

(iv) 1, e^{4x} $(x<0)$

Solution: Let, $y_1=1$ and $y_2=e^{4x}$

Now,

$$\frac{y_1(x)}{y_2(x)}=\frac{1}{e^{4x}}=e^{-4x}\quad \text{,which is not a constant.}$$

So, 1 and e^{4x} are linearly dependent.

(v) logx, $\log x^2$ $(x>0)$

Solution: Let, $y_1=\log x$ and $y_2=\log x^2=2\log x$

Now,

$$\frac{y_1(x)}{y_2(x)}=\frac{\log x}{2\log x}=\frac{1}{2}\quad \text{,which is a constant.}$$

So, logx and $\log x^2$ are linearly dependent.

$$\Rightarrow \left(m+\frac{1}{2}\right)^2 = i^2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \left(m+\frac{1}{2}\right)^2 = \left(\pm i\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow m+\frac{1}{2} = \pm i\frac{\sqrt{3}}{2}$$

So, $m_1 = i\frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(i\sqrt{3}-1)$ and $m_2 = i\frac{\sqrt{3}}{2} - \frac{1}{2} = -\frac{1}{2}(i\sqrt{3}+1)$

Thus, m has two imaginary roots. So the independent solutions of (1) are,

$$\left.\begin{aligned} y_1 &= e^{m1x} = e^{\frac{1}{2}}(i\sqrt{3}+1)x \\ y_2 &= e^{m2x} = e^{\frac{1}{2}}(i\sqrt{3}+1) \end{aligned}\right\}$$

(9) $y'' - 2y + 4y = 0$.

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0. \quad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 - 2m + 4 = 0$$

$$\Rightarrow (m-1)^2 = -3$$

$$\Rightarrow (m-1)^2 = (\pm i\sqrt{3})^2$$

$$\Rightarrow m - 1 = \pm i\sqrt{3}$$

Thus, m has two imaginary roots. So the independent solutions of (1) are,

$$\left.\begin{aligned} y_1 &= e(1+i\sqrt{3})x \\ y_2 &= e^{m}2^{x} = e(1-i\sqrt{3}x) \end{aligned}\right\}$$

B. Find a differential equation of the form y" + ay' + by = 0 for which the following functions are the solutions.

(i) e^{2x}, e^{-2x}

Solution: Given that the independent solutions of a differential equation are,

$$e^{2x}, e^{-2x}.$$

This means the equation has two real and non–repeated roots

$$m = -2, 2$$

Therefore the auxiliary equation of the equation is,

$$(m+2)(m-2) = 0$$

$$\Rightarrow m^2 - 4 = 0$$

So, its different equation is,

$$y'' - 4y = 0.$$

(ii) $e^{(2+i)x}, e^{(2-i)x}$

Solution: Given that the independent solutions of a differential equation are,

$$e^{(2+i)x}, e^{(2-i)x}.$$

This means the equation has imaginary roots

$$m = (2+i), (2-i)$$

Therefore the auxiliary equation of the equation is, Here, m has non–repeated real values. So the independent solutions of (1) are,

$$y_1 = e^x \quad \text{and} \quad y_2 = e^{-2x}$$

(5) $y'' + w^2y = 0$.

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + w^2y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 + w^2 = 0 \Rightarrow m^2 = -w^2 \Rightarrow m = -iw.$$

Thus, m has two imaginary roots. So the independent solutions of (1) are,

$$y_1 = e^{-iwx} \quad \text{and} \quad y_2 = e^{iwx}$$

(6) $y'' - y' - 2y = 0$.

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 - m - 2 = 0$$
$$\Rightarrow m^2 - 2m + m - 2 = 0$$
$$\Rightarrow m(m-2) + 1(m-2) = 0$$
$$\Rightarrow (m-2)(m+1) = 0$$
$$\Rightarrow m = 2, -1.$$

Here, m has non–repeated real values. So the independent solutions of (1) are,

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = e^{-x}$$

(7) $y'' + 2y' - 3y = 0$.

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0 \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 2m - 3 = 0$$
$$\Rightarrow m^2 + 3m - m - 3 = 0$$
$$\Rightarrow m(m+3) - 1(m+3) = 0$$
$$\Rightarrow (m+3)(m-1) = 0$$
$$\Rightarrow m = -3, 1$$

Here, m has non–repeated real values. So the independent solutions of (1) are,

$$y_1 = e^{-3x} \quad \text{and} \quad y_2 = e^{m_2x} = e^x$$

(8) $y'' + y' + y = 0$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0. \qquad \dots (1)$$

The auxiliary equation of (1) is,

$$m^2 + m + 1 = 0$$
$$\Rightarrow m^2 + 2m\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \frac{3}{4} = 0$$
$$\Rightarrow \left(m + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

Exercise 6.7

A. Find solutions (Independent Solution(s)) of the following differential equation.

(1) $y'' + 5y' + 6y = 0$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 5m + 6 = 0$$
$$\Rightarrow \quad m^2 + 2m + 3m + 6 = 0$$
$$\Rightarrow \quad (m+2)(m+3) = 0$$
$$\Rightarrow \quad m = -2, -3.$$

Here, m has non–repeated real values. So the independent solutions of (1) are,

$$y_1 = e^{mx} = e^{-2x} \quad \text{and} \quad y_2 = e^{mx} = e^{-3x}$$

(2) $y'' + 6y' + 9y = 0$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + 6m + 9 = 0$$
$$\Rightarrow \quad m^2 + 3m + 3m + 9 = 0$$
$$\Rightarrow \quad m(m+3) + 3(m+3) = 0$$
$$\Rightarrow \quad (m+3)(m+3) = 0$$
$$\Rightarrow \quad m = -3, -3.$$

Here, m has repeated real values. So the independent solutions of (1) are,

$$y_1 = e^{mx} = e^{-3x} \quad \text{and} \quad y_2 = e^{mx} = xe^{-3x}$$

(3) $y'' + y' = 0.$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1$$

Here, m has repeated real values. So the independent solutions of (1) are,

$$y_1 = e^{o} = 1 \quad \text{and} \quad y_2 = e^{-x}$$

(4) $y'' + y' - 2y = 0.$

Solution: Given differential equation is,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \qquad \ldots (1)$$

The auxiliary equation of (1) is,

$$m^2 + m - 2 = 0$$
$$\Rightarrow \quad m^2 + 2m - m - 2 = 0$$
$$\Rightarrow \quad m(m+2) - 1(m+2) = 0$$
$$\Rightarrow \quad (m+2)(m-1) = 0$$
$$\Rightarrow \quad m = 1, -2.$$

$$= e^{-2z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

Then (i) reduces to,

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + P\frac{dy}{dz} + Qy = R.$$

$$\Rightarrow \frac{d^2y}{dz^2} + (P-1)\frac{dy}{dz} + Qy = R. \qquad \ldots \text{(ii)}$$

Which is linar differential equation of second order in y and z.

B. Reducible to linear Diff. eq^n – Legendre's Form

An equation of the form

$$(a+bx)^2\frac{d^2y}{dx^2} + P(a+bx)\frac{dy}{dx} + Qy = R \qquad \ldots \text{(iii)}$$

Where, P, Q are constants and R is function of x or constant, is called legender's linear differential equation.

Process

Put $a + bx = e^z$ i.e. $z = \log(a + bx)$

Then

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \left(\frac{b}{a+bx}\right)\frac{dy}{dz} \Rightarrow (a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = b\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \left(\frac{b^2}{a+bx}\right)\left(-e^{-z}\frac{dy}{dz} + e^{-z}\frac{d^2y}{dz^2}\right)$$

$$= \frac{e^{-z}b^2}{a+bx}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$= = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow (a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

Then (iii) reduces to,

$$b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + Pb\frac{dy}{dz} + Qy = R.$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{P-b}{b}\right)\frac{dy}{dz} + \frac{Q}{b^2} = \frac{R}{b^2} \qquad \ldots \text{(iv)}$$

Which is linear differential equation of second order in y and z.

Particular Integral (P.I)

The integral factor of non-homogeneous linear differential equation which is free from dependent variable as well as arbitral constant value and that satisfies the equation is called the particular integral (P.I) of the equation. Normally, it is denoted by y_P or P.I.

Choose a P.I. for the equation

By the method of undetermined coefficients, we choose the P.I. for the given non-homogeneous equation as:

Term in R	Choose of y_P or P.I.
$C_1 e^{ax}$	$A e^{ax}$
$C_1 x^n$	$A_1 x^n + A_2 x^{n-1} + \ldots + A_n$
$C_1 \cos ax$ or $C_1 \sin ax$	$A \cos ax + B \sin ax$
$C_1 \cos ax + C_2 \sin ax$	$A \cos ax + B \sin ax$

Note:

1. If two or more terms are in multiplied form in R then choose y_P for each and arrange in their respective multiple form.
2. If the term is R is repeated with the solution of homogeneous part of the equation, the multiply by x to the respective y_P for the time of repetitions.

Method of variation of parameter of Wronskian's Method to determining the particular integral of a non homogeneous linear differential equation

Consider a linear equation.

$$y'' + Py' + Qy = R \quad \ldots (i)$$

Let the solution of homogeneous from of (i) i.e. $y'' + Py' + Qy = 0$ is,

$$y_n = C_1 y_1 + C_2 y_2 \quad \ldots (ii)$$

Let, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ which is called Wronskian of y_1 and y_2.

By the method of variation of parameter, the particular integral of (i) is given

$$y_P = \text{P.I.} = -y_1 \left(\frac{y_2 R}{W}\right) dx + y_2 \int \left(\frac{y_1 R}{W}\right) dx$$

Then the solution of (1) is,

$$y = P_n + \text{P.I. (or } y_P)$$

A. Reducible to linear Diff. eq^n – Cauchy's Form

Consider an equation of the form

$$x^2 y'' + Pxy' + Qy = R \quad \ldots (i)$$

Where P, Q are constants and R is function of x or constant is called Cauchy's linear differential equation of second order.

Process

Put $x = e^z$ i.e. $z = \log(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = e^{-z} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dz}\left(\dfrac{dz}{dx}\right)\dfrac{dz}{dx} = \dfrac{1}{x}\dfrac{d}{dz}\left(e^{-z}\dfrac{dy}{dz}\right) = e^{-z}\left[e^{-z}(-1)\dfrac{dy}{dz} + e^{-z}\dfrac{d^2y}{dz^2}\right]$

Linear differential equation of second order

A differential equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R$$

Where P, Q, R are function of x or constant is called a linear differential equation of second order.

Note: If R = 0 then the above equation is known as homogenous are if R ≠ 0 then the equation is called non-homogeneous.

Example: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ fi homogeneous

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$ fi non-homogeneous

Auxiliary equation of homogenous equation of second order

Consider a second order differential equation.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \ldots (i)$$

This implies $m^2 + Pm + Q = 0$ . . . (ii)

Here (ii) is known as auxiliary equation of (i)

Solution of homogeneous linear diff. equation of Second order

The auxiliary equation of (i) is (ii) is above article. Clearly (ii) is quadratic in m. So, m has 2 roots.

Condition I: If m has both real and repeated roots say m = a, a then the solution of (i) will be

$$y = (C_1 + x\,C_2)\,e^{ax}$$

Where C_1 and C_2 are constants.

Condition II: If m has real and district roots say m = a, b then general solution of (i) will be

$$y = C_1\,e^{ax} + C_2\,e^{bx}$$

Condition III: If ma has complex roots like m = ± ib then the solution of (i) will be

$$y = C_1 \cos bx + C_2 \sin bx$$

And if m has complex roots like m = a ± ib then the solution of (i) will be

$$y = e^{ax}\,(C_1 \cos bx + C_2 \sin bx)$$

Linearly dependent or independent solution of a second order diff. equation

Consider a second order linear diff. equation.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \ldots (i)$$

Let the general solution of (i) is,

$$y = C_1y_1 + C_2y_2 \qquad \ldots (ii)$$

Hence y_1 and y_2 are also solutions of (i). The then solutions y_1 and y_2 are linearly dependent if the ratio of y_1 and y_2 is free from x i.e. constant. For otherwise the solutions are linearly independent.

For example, *See exercise 6.8.*

This is a linear differential equation of first order whose I.F. is,

$$\text{I.F.} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Now, multiplying (i) by I.F. and then integrating, we get,

$$ye^{\tan x} = \int e^{\tan x} \sec^2 x \tan x\, dx + c$$

Put $\tan x = u$ then $\sec^2 x\, dx = du$. So,

$$y\, e^{\tan x} = \int e^u u\, du + c = ue^u - e^u + c = \tan x\, e^{\tan x} - e^{\tan x} + c$$

$$\Rightarrow \quad y = \tan x - 1 + c\, e^{-\tan x}$$

2008 Fall: Solve $(x + 1)y' = x(y^2 + 1)$.

Solution: Here,

$$(x + 1)\, y' = x\,(y^2 + 1)$$

$$\Rightarrow \quad \frac{dy}{1 + y^2} = \frac{x}{x + 1}\, dx = \left(1 - \frac{1}{x + 1}\, dx\right)$$

Integrating,

$$\tan^{-1}(y) = x - \log(x + 1) + c$$

2009 Spring: Solve: $\frac{dy}{dx} = (y - x)^2$.

Solution: Here,

$$\frac{dy}{dx} = (y - x)^2$$

Put $y - x = u$, then, $\frac{dy}{dx} - 1 = \frac{du}{dx}$. Then,

$$\frac{du}{dx} + 1 = u^2$$

$$\Rightarrow \quad \frac{du}{u^2 - 1} = dx$$

Integrating we get,

$$\frac{1}{2}\log\left(\frac{u - 1}{u + 1}\right) = x + c$$

$$\Rightarrow \quad \log\left(\frac{y - x - 1}{y - x + 1}\right) = e^{2(x + c)}$$

$$\Rightarrow \quad \log\left(\frac{y - x - 1}{y - x + 1}\right) = Ae^{2x}$$

Other Short Questions

2004 Fall: Show that the equation $2(y \sin 2x + \cos 2x)\,dx = \cos 2x\,dy$ is exact.

Solution: Given that,

$$2(y \sin 2x + \cos 2x)\,dx = \cos 2x\,dy \qquad \ldots\ldots \text{(i)}$$

Comparing above equation with $Mdx + Ndy = 0$ then,

$$M = 2(y \sin 2x + \cos 2x)$$

and $\quad -\cos 2x$

So, $\quad \dfrac{\partial M}{\partial y} = 2 \sin 2x$

and $\quad \dfrac{\partial N}{\partial x} = 2 \sin 2x$

Thus, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. So, (i) is exact.

2004 Fall: Solve: $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$.

Solution: Given that,

$$x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$$

$$\Rightarrow \frac{y}{\sqrt{1+y^2}}dy + \frac{x}{\sqrt{1+x^2}}dx = 0 \qquad \ldots\ldots \text{(i)}$$

Put, $1 + y^2 = u^2$ and $1 + x^2 = v^2$ then $2y\dfrac{dy}{dx} = 2u\dfrac{du}{dx} \Rightarrow y\dfrac{dy}{dx} = u\dfrac{du}{dx}$.

So, (i) become,

$$\frac{u}{u}du + \frac{v}{v}dv = 0$$

$$\Rightarrow du + dv = 0$$

Integrating we get,

$$u + v = C$$

$$\Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} = C$$

2007 Fall: Solve: $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Solution: Here,

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + y \sec^2 x = \sec^2 x \tan x \qquad \ldots\ldots \text{(i)}$$

Short Questions

2002: Solve: $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0.$

Solution: Here, $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$

Integrating we get,

$$\tan^{-1}(x) + \tan^{-1}(y) = C_1$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = C_1$$

$$\Rightarrow \frac{x+y}{1-xy} = \tan C_1$$

$$\Rightarrow x + y = C(1 - xy) \qquad \text{for } C = \tan(C_1)$$

$$\Rightarrow y(1 + Cx) = C - x$$

$$\Rightarrow y = \frac{C-x}{1+Cx}$$

2002: What is meant by integrating factor. Write down the condition for the differential equation $Mdx + Ndy = 0$ to be exact.

Solution: See the definition.

See the condition for exactness.

2003 Fall: Find integrating factor of $\frac{dy}{dx} + \frac{y}{x} = x$.

Solution: Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = x$$

$$\Rightarrow y' + \frac{y}{x} = x \qquad \text{....... (i)}$$

Comparing (i) with $y' + Py = Q$ then

$$P = \frac{1}{x} \quad \text{and} \quad Q = x$$

Now, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int (1/x)\,dx} = e^{\log(x)} = x.$$

<u>Similar Question for Practice from Final Exam:</u>

2000: Show that: $\frac{1}{x^2+y^2}$ is an integrating factor of $x\,dy - y\,dx = \frac{1}{x^2}$

2006 Fall: Find integrating factor of $\frac{dy}{dx} + \cot x\, y = \cos x$

2006 Spring: Find integrating factor of $\frac{dy}{dx} + y\tan x = \sec x$.

2009 Fall: Find integrating factor of $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

2009 Fall Q. No. 4(a)

Define order and degree of ordinary differential equation. Solve the initial value problem: $y' + \frac{y}{x} = x^2$; $y(1) = 0$.

Solution: For the first part see definition.

For second Part: Given equation is

$$y' + \frac{y}{x} = x^2 \qquad \text{........ (i)}$$

$$\text{with} \quad y(1) = 0 \qquad \text{......... (ii)}$$

Clearly, the equation (i) is a linear differential equation of first order.

Comparing (i) with $y' + Py = Q$ then we get

$$P = \frac{1}{x} \quad \text{and} \quad Q = x^2$$

So, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int (1/x)\,dx} = e^{\log x} = x$$

Now, multiplying (i) by I.F. and then taking integration w. r. t. x, then,

$$yx = \int x^3\,dx + C = \frac{x^4}{4} + C \qquad \text{.......(iii)}$$

And by (ii), we have $y(1) = 0$. Then (iii) gives,

$$0 = \frac{1}{4} + C \Rightarrow C = -\frac{1}{4}$$

Then (iii) becomes,

$$yx = \frac{x^4}{4} - \frac{1}{4}$$

$$\Rightarrow 4xy = x^4 - 1.$$

This is required solution of (i).

2014 Spring Q.No. 4(a), 2017 Spring Q.No. 4(a)

Define Bernoulli equation. And Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = e^x (1 + x) \sec y$.

2017 Fall Q. No. 4(a)

Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = e^x (1 + x) \sec y$.

2018 Fall Q.No. 4(a), 2016 Spring Q.No. 4(a), 2015 Spring Q.No. 4(a), 2013 Spring Q.No. 4(a)

Solve: $\frac{dy}{dx} + \frac{1}{x} \sin 2y = x^3 \cos^2 y$

2018 Spring Q.No. 4(a)

Define order and degree of ordinary differential with example. Solve: $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}$ where $y(0) = 4$.

2019 Fall Q. No. 4(a)

Define order and degree of ordinary differential with equation. Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = e^x (1 + x) \sec y$.

Now, multiplying (i) by I.F. and then taking integration w. r. t. x then,

$$y \sec x = \int \sec^2 x \, dx + C = \tan x + C$$

$$\Rightarrow \quad y = \sin x + C \cos x$$

This is required solution of (i).

<u>2006 Spring Q. No. 4(a)</u>

Define the first order linear differential equations with suitable example and solve: $x\dfrac{dy}{dx} + y = y^2 \log x$.

Solution: See the definition.

For problem, see Q. 6, Exercise 6.5.

<u>2008 Fall Q. No. 4(a)</u>

Define order and degree of the differential equation with suitable example. Check exactness condition of the differential equation: $(2\cos y + 4x^2)\,dx = x \sin y\,dy$, if it is not exact find integrating factor (IF) and then solve it by using IF.

Solution: See the definition.

For problem, see Q. A(vi), Exercise 6.3.

<u>2008 Fall; 2010 Spring Q. No. 4(a)</u>

Solve: $\dfrac{dy}{dx} + \dfrac{\sin 2y}{x} = x^3 \cos^2 y$.

Solution: Give differential equation is,

$$y' + \frac{\sin 2y}{x} = x^3 \cos^2 y$$

$$\Rightarrow \quad \sec^2 y\, y' + \frac{2\sin y \cos y}{\cos^2 y . x} = x^3$$

$$\Rightarrow \quad \sec^2 y\, y' + 2 \tan y \frac{1}{x} = x^3 \qquad \text{...... (i)}$$

Put $\tan y = u$ then $\sec^2 y\, y' = u'$, then (i) becomes

$$u' + \frac{2u}{x} = x^3 \qquad \text{...... (ii)}$$

This is a linear differential equation of first order is a whose integrating factor is,

$$\text{I.F.} = e^{\int (2/x)\,dx} = e^{2\log x} = x^2$$

Now, multiplying on both sides of (ii) by I.F. and then integrating we get,

$$ux^2 = \int x^5\,dx + \frac{c}{6} = \frac{x^6}{6} + \frac{c}{6}$$

$$\Rightarrow \quad 6x^2 \tan y = x^6 + c.$$

This is the solution of given equation.

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$\text{i.e.}\quad u\frac{1}{x} = -\int \frac{1}{x^2}\,dx + c = -\frac{x^{-3}}{-3} + c = \frac{1}{3x^3} + c$$

$$\Rightarrow\quad \frac{1}{yx} = \frac{1}{3x^3} + c$$

$$\Rightarrow\quad 3x^2 = y(1 + 3Cx^2)$$

2002 Set I & II; 2004 Spring; 2011 Fall Q. No. 4(a), 2013 Fall Q.No. 4(a), 2014 Fall Q.No. 4(a)

Solve $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$.

Solution: Given that,

$$\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$$

$$\Rightarrow\quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2} \qquad \text{...... (i)}$$

Put $\frac{1}{\log y} = u$ then, $\frac{-1}{y(\log y)^2}\frac{dy}{dx} = \frac{du}{dx}$, then (i) reduces to,

$$\frac{-du}{dx} + \frac{1}{x}u = \frac{1}{x^2}$$

$$\Rightarrow\quad \frac{du}{dx} - \frac{u}{x} = \frac{-1}{x^2} \qquad \text{...... (ii)}$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log(x^{-1})} = \frac{1}{x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$\text{i.e.}\quad ux^{-1} = \int \frac{-1}{x^3}\,dx + c = \frac{x^{-2}}{2} + c$$

$$\Rightarrow\quad 2ux^{-1} = x^{-2} + c$$

$$\Rightarrow\quad \frac{2}{\log y} = \frac{1}{x^2} + c\,.$$

This is the solution of given equation.

2004 Fall; 2006 Fall Q. No. 4(a)

Solve: $y' + y\tan x = \sec x$

Solution: Given equation is

$$y' + y\tan x = \sec x \qquad \text{.........(i)}$$

This is first order linear differential equation of first order.

Comparing (i) with $y' + Py = Q$ then we get,

$$P = \tan x \quad \text{and} \quad Q = \sec x$$

So, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

First Order Differential Equation

1999 Q. No. 4(a); 2001 Q. No. 4(a)

Show that the differentiation equation: $\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0$ is exact and solve it.

Solution: Given equation is

$$\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0 \qquad \text{.......(i)}$$

Comparing (i) with $Mdx + Ndy = 0$ then we get,

$$M = \sinh x \cos y \quad \text{and} \quad N = -\cosh x \sin y$$

Then,

$$\frac{\partial M}{\partial y} = -\sinh x \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sinh x \sin y$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, the equation (i) is exact.

Therefore solution of (i) is

$$\int M dx + \int (\text{terms of N free from x})\, dy = C$$

$$\Rightarrow \int \sinh x \cos y\, dx + \int 0\, dy = C$$

[$\because$ N has no term which is not included x]

$$\Rightarrow \cos y \int \sinh x\, dx = C$$

$$\Rightarrow \cos y \cosh x = C$$

This is required solution of (i)

2000 Q. No. 4(a); 2007 Fall Q. No. 4(a)

Solve the differential equation $y' + \frac{y}{x} = \frac{y^2}{x}$.

Solution: Given equation is

$$y' + \frac{y}{x} = \frac{y^2}{x}$$

$$\Rightarrow \frac{1}{y^2} y' + \frac{1}{xy} = \frac{1}{x} \qquad \text{............(i)}$$

Put $u = \frac{1}{y} = u$ then $-\frac{1}{y^2} y' = u'$. Then, the equation (i) reduces to,

$$-u' + \frac{u}{x} = \frac{1}{x}$$

$$\Rightarrow u' - \frac{u}{x} = -\frac{1}{x} \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = -\frac{1}{x} \quad \text{and} \quad Q = -\frac{1}{x}$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\, dx} = e^{\int (-1/x)\, dx} = e^{-\log x} = e^{\log (x^{-1})} = \frac{1}{x}$$

(4) $y' - 2xy = -2x$

Solution: Given that,

$$\frac{dy}{dx} - 2xy = -2x \qquad \text{...... (i)}$$

Comparing the equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = -2x \quad \text{and} \quad Q = -2x.$$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - 2xy = 0 \Rightarrow \frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{y} = 2x\,dx.$$

Integration we get,

$$\log y = x^2 + \log c$$

$$\Rightarrow y = ce^{x^2}$$

Set, $v(x) = e^{x^2}$ then, we get, $y = cv$

Now, using formula of method of variation of parameter,

$$y = v\left(\int \frac{Q}{v} dx + c\right) = e^{x^2}\left\{\int \frac{-2x}{e^{x^2}} dx + c\right\} = e^{x^2}\{-\int(2xe^{-x^2} + c)\}$$

$$= e^{x^2}(e^{-x^2} + c) = 1 + ce^{x^2}$$

(5) $y' + 3y = e^{-2x}$

Solution: Given that,

$$\frac{dy}{dx} + 2y = e^{-2x} \qquad \text{...... (i)}$$

Comparing the equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = 2 \quad \text{and} \quad Q = e^{-2x}$$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} + 2y = 0$$

$$\Rightarrow \frac{dy}{y} = -2dx$$

Integrating we get,

$$\log y = -2x + \log c$$

$$\Rightarrow y = ce^{-2x}$$

Set, $v(x) = e^{-2x}$ then, $y = cv$.

Now, using formula of method of variation of parameter,

$$y = v\left(\int \frac{Q}{v} dx + c\right)$$

$$= e^{-2x}\left\{\int \frac{e^{-2x}}{e^{-2x}} dx + c\right\}$$

$$= e^{-2x}(x + c).$$

(2) $y' - y = x$

Solution: Given that,

$$\frac{dy}{dx} - y = x, \qquad \text{......(i)}$$

Comparing the equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = -1 \text{ and } Q = x$$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - y = 0 \Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = dx$$

Integrating we get,

$$\log y = x + \log c \Rightarrow y = ce^x$$

Set, $v(x) = e^x$ then $y = cv$.

Now, using formula of method of variation of parameter,

$$y = v\left(\int \frac{Q}{v} dx + c\right) = e^x\left(\int \frac{x}{e^x} dx + c\right) = e^x\left(\int xe^{-x} dx + c\right)$$

$$= e^x(-xe^{-x} - e^x + c) = -x - 1 + ce^x$$

$$= ce^x - x - 1.$$

(3) $xy' - 2y = x^4$

Solution: Given that,

$$x\frac{dy}{dx} - 2y = x^4.$$

$$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = x^3 \qquad \text{......(i)}$$

Comparing the equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = -\frac{2}{x} \text{ and } Q = x^3$$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - \frac{2y}{x} = 0 \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

Integrating we get,

$$\log y = 2\log x + \log c$$

$$\Rightarrow \log y = \log x^2 + \log c$$

$$\Rightarrow \log y = \log cx^2$$

$$\Rightarrow y = cx^2$$

Set, $v = x^2$ then $y = cv$.

Now, using formula of method of variation of parameter,

$$y = v\left(\int \frac{Q}{v} dx + c\right) = x^2\left(\int \frac{x^3}{x^2} dx + c\right) = x^2\left(\frac{x^2}{2} + c\right)$$

$$= \frac{x^4}{2} + cx^2$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $u\, e^{x^2/2} = \int x^3 e^{x^2/2}\, dx + c$

Put, $t = \frac{x^2}{2}$ then $dt = x dx$. Then,

$$ue^{x^2/2} = \int 2t\, e^t\, dt + c = 2[te^t - e^t] + c$$

$$= 2e^t (t-1) + c = 2e^{x^2/2}\left(\frac{x^2}{2} - 1\right) + c$$

$$\Rightarrow \quad u = 2\left(\frac{x^2}{2} - 1\right) + ce^{-x^2/2}$$

$$\Rightarrow \quad \frac{-1}{y} = x^2 - 2 + ce^{-x^2/2}$$

Exercise 6.6

Solve by the method of variation of parameters.

(1) $\quad y' - \frac{2}{x} y = x^2 \cos 3x$

Solution: Given that,

$$y' - \frac{2}{x} y = x^2 \cos 3x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2}{x} y = x^2 \cos 3x \qquad \text{...... (i)}$$

Comparing the equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = -\frac{2}{x}, \quad \text{and} \quad Q = x^2 \cos 3x$$

Now, corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - \frac{2}{x} y = 0 \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

Integrating we get,

$$\log y = 2 \log x + \log c$$

$$\Rightarrow \log y = \log (cx^2)$$

$$\Rightarrow y = cx^2$$

Set, $v(x) = x^2$ then, $y = cv$.

Now, using formula of method of variation of parameter,

$$y = v\left(\int \frac{Q}{v} dx + c\right)$$

$$\Rightarrow \quad y = x^2 \int \left(\frac{x^2 \cos 3x}{x^2}\right) dx + cx^2$$

$$\Rightarrow \quad y = x^2 \left(\frac{\sin 3x}{3} + c\right).$$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \cos x \qquad \text{...... (i)}$$

Put $u = \frac{1}{y^2} = y^{-2}$ then $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}\frac{du}{dx} = \frac{1}{y^3}\frac{dy}{dx}$.

Then, the equation (i) reduces to,

$$-\frac{1}{2}\frac{du}{dx} + y\tan x = \cos x$$

$$\Rightarrow \quad \frac{du}{dx} - 2u\tan x = -2\cos x \qquad \text{....... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = -2\tan x \qquad \text{and} \qquad Q = -2\cos x$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-2\tan x)\,dx} = e^{-2\log\sec x} = e^{\log(\sec x)^{-2}} = (\sec x)^{-2} = \frac{1}{\sec^2 x}.$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $$u\frac{1}{\sec^2 x} = \int\left(-2\cos x\,\frac{1}{\sec^2 x}\right)dx + c$$

$$\Rightarrow \quad u\cos^2 x = -2\int\cos^3 x\,dx + c$$

$$\Rightarrow \quad u\cos^2 x = -2\int\left(\frac{\cos 3x + 3\cos x}{4}\right)dx + c$$

$$\Rightarrow \quad u\cos^2 x = -\frac{1}{2}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c$$

$$\Rightarrow \quad \frac{1}{y^2}\cos^2 x = -\frac{1}{2}\frac{(\sin 3x + 9\sin x)}{3} + c$$

$$\Rightarrow \quad \cos^2 x = -y^2\left(\frac{\sin 3x + 9\sin x}{6} + c\right).$$

17. $(x^3y^2 + xy)\,dx = dy$

Solution: Given that, $(x^3y^2 + xy)\,dx = dy$

$$\Rightarrow \quad \frac{dy}{dx} - xy = x^3y^2$$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3$$

Put $u = \frac{-1}{y}$ then $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces to,

$$\frac{du}{dx} + ux = x^3 \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = x \qquad \text{and} \qquad Q = x^3$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int x\,dx} = e^{x^2/2}$$

Put, $u = -\frac{1}{y} = -y^{-1}$. Then, $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces,

$$\frac{du}{dx} + u = 2xe^{-x} \qquad \text{......(ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = 1 \quad \text{and} \quad Q = 2xe^{-x}$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int dx} = e^x.$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $ue^x = \int 2xe^{-x} e^x\,dx + c = 2\int x\,dx + c$

$\Rightarrow \frac{-1}{y} e^x = x^2 + c$

$\Rightarrow e^x = -y(x^2 + c).$

15. $\tan y\, y' + \tan x = \cos y \,.\, \cos^2 x$

Solution: Given that,

$$\tan y\, y' + \tan x = \cos y \cos^2 x$$

$\Rightarrow \frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x$

$\Rightarrow \sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \qquad \text{......(i)}$

Put $u = \sec y$ then $\frac{du}{dx} = \sec y \tan y \frac{dy}{dx}$. Then, the equation (i) reduces to,

$$\frac{du}{dx} + u \tan x = \cos^2 x \qquad \text{......(ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = \tan x \quad \text{and} \quad Q = \cos^2 x$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $u \sec x = \int \cos^2 x \sec x\,dx + c$

$\Rightarrow u \sec x = \int \cos x\,dx + c$

$\Rightarrow \sec y \sec x = \sin x + c$

$\Rightarrow \sec y = (\sin x + c)\cos x.$

16. $y' + y \tan x = y^3 \cos x$

Solution: Given that,

$$y' + y \tan x = y^3 \cos x$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (x^2)\,dx} = e^{x^3/3}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

$$\text{i.e.}\quad ue^{x^3/3} = \int dx + c$$

$$\Rightarrow\quad ue^{x^3/3} = x + c$$

$$\Rightarrow\quad -\frac{1}{y}e^{x^3/3} = x + c$$

$$\Rightarrow\quad y(x + c) = e^{x^3/3}$$

. $e^y(y' + 1) = e^x$

lution: Given that,

$$e^y(y' + 1) = e^x$$

$$\Rightarrow\quad e^y\frac{dy}{dx} + e^y = e^x \qquad \text{...... (i)}$$

Put $u = e^y$ then $\frac{du}{dx} = e^y\frac{dy}{dx}$. Then, the equation (i) reduces to,

$$\frac{du}{dx} + u = e^x \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = 1 \text{ and } \quad Q = e^x$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int dx} = e^x.$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

$$\text{i.e.}\quad ue^x = \int e^{2x}\,dx + \frac{c}{2}$$

$$\Rightarrow\quad e^y e^x = \frac{e^{2x}}{2} + \frac{c}{2}$$

$$\Rightarrow\quad 2e^{x+y} = e^{2x} + c.$$

$y(2xy + e^x)\,dx = e^x\,dy$

ion: Given that,

$$y(2xy + e^x)\,dx = e^x\,dy$$

$$\Rightarrow\quad \frac{2xy^2 + ye^x}{e^x} = \frac{dy}{dx}$$

$$\Rightarrow\quad \frac{dy}{dx} = 2xy^2e^{-x} + y$$

$$\Rightarrow\quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x} \qquad \text{...... (i)}$$

$$\Rightarrow \frac{du}{dx} + (-1)u = (2x-1) \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = -1 \quad \text{and} \quad Q = 2x - 1$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (-1)\,dx} = e^{-x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $ue^{-x} = \int (2x-1)e^{-x}\,dx + c$

$\Rightarrow ue^{-x} = -(2x-1)\,e^{-x} - 2e^{-x} + c$

$\Rightarrow u = -2x + 1 - 2 + ce^{x}$

$\Rightarrow \frac{1}{y^3} = -(2x+1) + ce^{x}$

$\Rightarrow y^{-3} = ce^{x} - 2x - 1.$

11. $y' = \frac{1}{6e^y - 2x}$

Solution: Given that,

$$y' = \frac{1}{6e^y - 2x} \Rightarrow \frac{dy}{dx} = \frac{1}{6e^y - 2x}$$

$$\Rightarrow \frac{dx}{dy} = 6e^y - 2x$$

$$\Rightarrow \frac{dx}{dy} + 2x = 6e^y \qquad \text{.......... (i)}$$

This is linear differential equation in x whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int 2\,dy} = e^{2y}$$

Now, multiplying (i) by I.F. and then taking integration on both sides then,

$$xe^{2y} = \int e^{2y}\,6e^y\,dy + c = 2\int e^{3y}\,(3dy) + c = 2e^{3y} + c$$

$$\Rightarrow x = 2e^y + ce^{-2y}.$$

12. $y' - x^2y = y^2e^{-x^3/3}$

Solution: Given that,

$$y' - x^2y = y^2e^{-x^3/3}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{x^2}{y} = e^{-x^3/3}$$

Put $u = -\frac{1}{y} = -y^{-1}$. Then, $\frac{du}{dx} = y^{-2}\frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$.

Then, the equation (i) reduces,

$$\frac{du}{dx} + ux^2 = e^{-x^3/3} \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = x^2 \quad \text{and} \quad Q = e^{-x^3/3}$$

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (2/x)\,dx} = e^{2\log x} = x^2$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $ux^2 = -\int 20x^4\,dx + c$

$\Rightarrow y^{-4}x^2 = -4x^5 + c$

$\Rightarrow x^2 = 4x^5y^4 + cy^4$

9. $y' + 2y = y^2$

Solution: Given that,

$$y' + 2y = y^2$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + 2\frac{1}{y} = 1 \qquad \text{...... (i)}$$

Put $u = \frac{1}{y}$ then $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces to,

$$-\frac{du}{dx} + 2u = 1$$

$$\Rightarrow \frac{du}{dx} + (-2)u = -1.$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = -2 \quad \text{and} \quad Q = -1$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (-2)\,dx} = e^{-2x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $ue^{-2x} = -\int e^{-2x}\,dx + c = -\frac{e^{-2x}}{-2} + c$

$\Rightarrow 2u = 1 + 2ce^{2x}$

$\Rightarrow \frac{2}{y} = 1 + 2ce^{2x}$

$\Rightarrow 2 = y(1 + 2ce^{2x}).$

10. $y' + \frac{y}{3} = \left(\frac{1-2x}{3}\right)y^4$

Solution: Given that,

$$y' + \frac{y}{3} = \left(\frac{1-2x}{3}\right)y^4$$

$$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3y^3} = \left(\frac{1-2x}{3}\right) \qquad \text{...... (i)}$$

Put $u = \frac{1}{y^3} = y^{-3}$ then $\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \Rightarrow -\frac{1}{3}\frac{du}{dx} = \frac{1}{y^4}\frac{dy}{dx}$.

Then, the equation (i) reduces to,

$$-\frac{1}{3}\frac{du}{dx} + \frac{1}{3}u = \left(\frac{1-2x}{3}\right)$$

$$\Rightarrow \cot x \operatorname{cosec} x \frac{dy}{dx} + \frac{1}{x} \operatorname{cosec} y = \frac{1}{x^2} \quad \text{......(i)}$$

Put $u = \operatorname{cosec} y$. Then, $\frac{du}{dx} = -\operatorname{cosec} y \cot y \frac{dy}{dx}$. Then, the equation (i) reduces,

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{-1}{x} u\right) = \left(\frac{-1}{x^2}\right) \quad \text{......(ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = -\frac{1}{x}, Q = -\frac{1}{x^2}$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (-1/x)\,dx} = e^{-\log x} = \frac{1}{x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $u\frac{1}{x} = -\int \left(\frac{1}{x^3}\right) dx$

$$\Rightarrow \frac{u}{x} = -\frac{x^{-2}}{-2} + c$$

$$\Rightarrow \frac{\operatorname{cosec} y}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow \operatorname{cosec} y = \frac{1 + 2cx^2}{2x^2}$$

$$\Rightarrow 2x = 1\,(1 + 2cx^2) \sin y.$$

8. $\mathbf{2xy' = 10x^3y^5 + y}$

Solution: Given that,

$$2xy' = 10x^3y^5 + y$$

$$\Rightarrow \frac{1}{y^5}\frac{dy}{dx} - \frac{1}{2xy^4} = 5x^2 \quad \text{......(i)}$$

Put $u = y^{-4}$. Then, $\frac{du}{dx} = -4y^{-5}\frac{dy}{dx}$. Then, the equation (i) reduces,

$$-\frac{1}{4}\frac{du}{dx} - \frac{u}{2x} = 5x^2$$

$$\Rightarrow \frac{du}{dx} + \frac{4u}{2x} = -20x^2$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{2}{x}\right) u = (-20x^2). \quad \text{......(ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = \frac{2}{x} \quad \text{and} \quad Q = -20x^2$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\Rightarrow\quad -\frac{x}{y} = c\sqrt{x} - 1$$

$$\Rightarrow\quad \frac{x}{y} = 1 - c\sqrt{x}.$$

6. $xy' + y = y^2 \log x$

Solution: Given that,

$$x\frac{dy}{dx} + y = y^2 \log x$$

$$\Rightarrow\quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x \qquad \text{...... (i)}$$

Put $u = \frac{1}{y} = y^{-1}$. Then, $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces,

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x}\log x$$

$$\Rightarrow\quad \frac{du}{dx} + \left(-\frac{1}{x}\right)u = \left(-\frac{1}{x}\log x\right). \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = \frac{-1}{x} \quad \text{and} \quad Q = -\frac{1}{x}\log x$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-1/x)\,dx} = e^{-\log x} = \frac{1}{x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $u\frac{1}{x} = -\int\left(\frac{1}{x}\log x\frac{1}{x}\right)dx$

$$\Rightarrow\quad \frac{u}{x} = -\left[\log x \int x^{-2}dx - \int\left(\frac{d(\log x)}{dx}\int x^{-2}\,dx\right)dx\right]$$

$$= -\left[-\frac{\log x}{x} - \int\left(\frac{1}{x}\times\left(\frac{-1}{x}\right)\right)dx\right] = -\left[-\frac{\log x}{x} + \int x^{-2}\,dx\right]$$

$$= -\left[-\frac{\log x}{x} + \frac{1}{x}\right] + c = \frac{\log x}{x} + \frac{1}{x} + c.$$

$$\Rightarrow\quad \frac{1}{xy} = \frac{\log x + 1}{x} + c$$

$$\Rightarrow\quad cxy + y(\log x + 1) = 1.$$

7. $y' + \frac{1}{x}\text{Tany} = \frac{1}{x^2}\text{Tany Siny}$

Solution: Given that,

$$y' + \frac{1}{x}\text{Tany} = \frac{1}{x^2}\text{Tany Siny}$$

$$\Rightarrow\quad \frac{1}{\text{Tany Siny}}\frac{dy}{dx} + \frac{\text{Tany}}{x\,\text{Tany Siny}} = \frac{1}{x^2}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log(x^{-1})} = \frac{1}{x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

$$\text{i.e.} \quad u\frac{1}{x} = -\int \frac{1}{x^3}\,dx + c$$

$$\Rightarrow \quad \frac{u}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow \quad \frac{2ux^2}{x} = 1 + 2cx^2$$

$$\Rightarrow \quad \frac{1}{e^y} 2x = 1 + 2c\,x^2$$

$$\Rightarrow \quad 2x = (1 + 2cx^2)\,e^y$$

5. $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Solution: Given that,

$$2\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$$

$$\Rightarrow \quad \frac{2}{y^2}\frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2} \qquad \text{......(i)}$$

Put $u = -\frac{1}{y}$ then $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces,

$$2\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{u}{2x} = \frac{1}{2x^2} \qquad \text{......(ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$$P = \frac{1}{2x} \quad \text{and} \quad Q = \frac{1}{2x^2}$$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(1/2x)\,dx} = e^{\log x^{1/2}} = \sqrt{x}$$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

$$\text{i.e.} \quad u\sqrt{x} = \int \sqrt{x}\,\frac{1}{2x^2}\,dx + c = \frac{1}{2}\int x^{-3/2}\,dx + c$$

$$= \frac{1}{2}\left(\frac{x^{-1/2}}{-1/2}\right) + c$$

$$\therefore\ \text{I.F.} = e^{\int p\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Now, multiplying (ii) by I.F. and then taking integration on both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $\quad u\,e^{x^2} = \int 2x\,e^{x^2}\,dx + c$

$\Rightarrow \quad u\,e^{x^2} = e^{x^2} + c$

$\Rightarrow \quad y^2 e^{x^2} = e^{x^2} + c$

$\Rightarrow y^2 = 1 + ce^{-x^2}$

$x\dfrac{dy}{dx} + y\log y = xye^x$ **[2012 Fall Q.No. 4(a), 2009 Spring Q. No. 4(a)]**

olution: Given that,

$$x\frac{dy}{dx} + y\log y = xye^x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x \qquad \text{...... (i)}$$

Put $u = \log(y)$ then $\dfrac{du}{dx} = \dfrac{1}{y}\dfrac{dy}{dx}$. So, the equation (i) reduces to,

$$\frac{du}{dx} + \frac{1}{x}u = e^x \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\dfrac{du}{dx} + Pu = Q$ then we get,

$P = \dfrac{1}{x}$ and $Q = e^x$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$\text{I.F.} = e^{\int p\,dx} = e^{\int (1/x)\,dx} = e^{\log x} = x.$

Now, multiplying (ii) by I.F. and then taking integration on both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $\quad ux = \int e^x\,x\,dx + c$

$\Rightarrow \quad ux = xe^x - e^x + c$

$\Rightarrow \quad (\log y)\,x = xe^x - e^x + c.$

$\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$

lution: Given that,

$$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$$

$$\Rightarrow \quad \frac{1}{e^y}\frac{dy}{dx} + \frac{1}{xe^y} = \frac{1}{x^2} \qquad \text{...... (i)}$$

Put $u = \dfrac{1}{e^y} = e^{-y}$ then $\dfrac{du}{dx} = -e^{-y}\dfrac{dy}{dx} \Rightarrow -\dfrac{du}{dx} = \dfrac{1}{e^y}\dfrac{dy}{dx}$. Then, the equation (i) reduces to,

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{du}{dx} + \left(-\frac{1}{x}\right)u = \left(-\frac{1}{x}\right) \qquad \text{...... (ii)}$$

Exercise 6.5

Solve the following differential equation:

1. $\frac{dy}{dx} - y\tan x = -y^2 \sec x.$ **[2003 Fall Q. No. 4(b)]**

Solution: Given that,

$$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \qquad \text{...... (i)}$$

Put $u = \frac{1}{y}$ then $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$. Then, the equation (i) reduces to,

$$\frac{du}{dx} + u\tan x = \sec x \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$P = \tan x$ and $Q = \sec x$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

Now, multiplying (ii) by I.F. and then taking integration onboth sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e.
$$u\sec x = \int \sec^2 x\,dx + c$$
$$\Rightarrow u\sec x = \tan x + c$$
$$\Rightarrow \frac{1}{y}\sec x = \tan x + c$$
$$\Rightarrow \sec x = y(\tan x + c).$$

2. $\frac{dy}{dx} + xy = xy^{-1}.$

Solution: Given that,

$$\frac{dy}{dx} + xy = xy^{-1}$$

$$\Rightarrow y\frac{dy}{dx} + xy^2 = x \qquad \text{...... (i)}$$

Put $u = y^2$ then $\frac{du}{dx} = 2y\frac{dy}{dx}$. Then, the equation (i) reduces to,

$$\frac{1}{2}\frac{du}{dx} + xu = x$$

$$\Rightarrow \frac{du}{dx} + 2x\,u = 2x \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{du}{dx} + Pu = Q$ then we get,

$P = 2x$ and $Q = 2x$

Clearly (ii) is a linear differential equation of first order in u, whose integrating factor (I.F.) is

(x) $\frac{dy}{dx} + y \cot x = 5e^{\cos x}, y\left(\frac{\pi}{2}\right) = -4$

Solution: Given that, $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$ ...... (1)

And, $y\left(\frac{\pi}{2}\right) = -4$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 5e^{\cos x}$.

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y \sin x = \int 5e^{\cos x} \sin x\,dx + c$ .... (3)

Put $u = \cos x$ then $\frac{du}{dx} = -\sin x \Rightarrow -du = \sin x\,dx$. Then (3) becomes,

$$y \sin x = -5\int e^{u}\,du + c = -5e^{u} + c = -5e^{\cos x} + c$$

$\Rightarrow y \sin x + 5e^{\cos x} = c$ ...... (4)

Using (2) to (4), then (4) gives,

$$-4 \sin\frac{\pi}{2} + 5e^{\cos\left(\frac{\pi}{2}\right)} = c$$

$\Rightarrow -4 + 5 = c$

$\Rightarrow c = 1$

Now (4) becomes,

$$y \sin x + 5e^{\cos x} = 1.$$

(xi) $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}, y(0) = 4$

Solution: Given that, $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}$ ...... (1)

And, $y(0) = 4$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = -\tan x$ and $Q = 3e^{-\sin x}$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-\tan x)\,dx} = e^{-\log \sec x} = e^{\log \cos x} = \cos x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y \cos x = \int 3e^{-\sin x} \cos x\,dx + c$ ... (3)

Put, $u = -\sin x$, then $\frac{du}{dx} = -\cos x \Rightarrow -du = \cos x\,dx$. Then (3) becomes,

$$y \cos x = -3\int e^{u}\,du + c = -3e^{u} + c$$
$$= -3e^{-\sin x} + c \quad \text{...... (4)}$$

Using (2), then (4) gives,

$4 \cos 0 = -3e^{-\sin 0} + c$

$\Rightarrow 4 + 3 = c \Rightarrow c = 7$

Now (4) becomes,

$$y \cos x = 7 - 3e^{-\sin x}$$

(x) $\frac{dy}{dx} + y \cot x = 5e^{\cos x}, y\left(\frac{\pi}{2}\right) = -4$

Solution: Given that, $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$ ...... (1)

And, $y\left(\frac{\pi}{2}\right) = -4$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 5e^{\cos x}$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y \sin x = \int 5e^{\cos x} \sin x\,dx + c$ .... (3)

Put $u = \cos x$ then $\frac{du}{dx} = -\sin x \Rightarrow -du = \sin x\,dx$. Then (3) becomes,

$$y \sin x = -5 \int e^u\,du + c = -5e^u + c = -5e^{\cos x} + c$$

$$\Rightarrow y \sin x + 5e^{\cos x} = c \quad \text{...... (4)}$$

Using (2) to (4), then (4) gives,

$$-4 \sin\frac{\pi}{2} + 5e^{\cos\left(\frac{\pi}{2}\right)} = c$$

$$\Rightarrow -4 + 5 = c$$

$$\Rightarrow c = 1$$

Now (4) becomes,

$$y \sin x + 5e^{\cos x} = 1.$$

(xi) $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}, y(0) = 4$

Solution: Given that, $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}$ ...... (1)

And, $y(0) = 4$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = -\tan x$ and $Q = 3e^{-\sin x}$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (-\tan x)\,dx} = e^{-\log \sec x} = e^{\log \cos x} = \cos x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y \cos x = \int 3e^{-\sin x} \cos x\,dx + c$ ... (3)

Put, $u = -\sin x$, then $\frac{du}{dx} = -\cos x \Rightarrow -du = \cos x\,dx$. Then (3) becomes,

$$y \cos x = -3 \int e^u\,du + c = -3e^u + c$$
$$= -3e^{-\sin x} + c \quad \text{...... (4)}$$

Using (2), then (4) gives,

$$4 \cos 0 = -3e^{-\sin 0} + c$$

$$\Rightarrow 4 + 3 = c \Rightarrow c = 7$$

Now (4) becomes,

$$y \cdot \cos x = 7 - 3e^{-\sin x}$$

$$\Rightarrow\quad y = \frac{1}{10}[3\sin x - \cos x] + ce^{-3x} \qquad \text{...... (3)}$$

Using (2) to (3), then (3) gives,

$$0.3 = \frac{1}{10}[3 - 0] + c \;\Rightarrow\; c = 0.$$

Now (3) becomes,

$$y = \frac{1}{10}[3\sin x - \cos x].$$

(viii) $\frac{dy}{dx} = (1 + y^2),\ y(0) = 0$

Solution: Given that,

$$\frac{dy}{dx} = 1 + y^2$$

$$\Rightarrow \frac{dy}{1+y^2} = dx$$

Integrating we get,

$$\tan^{-1}(y) = x + c \qquad \text{...... (1)}$$

$$\Rightarrow\quad y = \tan(x + c)$$

And, using the given value $y(0) = 0$ to (1) then (i) gives,

$$0 = \tan(c) \Rightarrow 0 = c.$$

Now, (1) becomes,

$$y = \tan x.$$

(ix) $\frac{dy}{dx} + y\cot x = 4x\cos x,\ y\left(\frac{\pi}{2}\right) = 0$

Solution: Given that, $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$ ...... (1)

And, $y\left(\frac{\pi}{2}\right) = 0$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \cot x \qquad \text{and} \qquad Q = 4x\operatorname{cosec} x$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\sin x = \int 4x\operatorname{cosec} x \sin x\,dx + c$

$$\Rightarrow\quad y\sin x = \int 4x\,dx + c$$

$$\Rightarrow\quad y\sin x = 2x^2 + c \qquad \text{...... (3)}$$

Using (2) to (3), then (3) gives,

$$0 = 2\left(\frac{\pi}{2}\right)^2 + c \;\Rightarrow\; c = \frac{-\pi^2}{2}$$

Now (3) becomes,

$$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\frac{dy}{dx} = 2(y-1)\tanh 2x$$

$$\Rightarrow \frac{dy}{dx} - 2y\tanh 2x = -2\tanh 2x \qquad \text{...... (1)}$$

And $$y(0) = 4 \qquad \text{...... (2)}$$

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = -2\tanh 2x \quad \text{and} \quad Q = -2\tanh 2x$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-2\tanh 2x)\,dx} = e^{-\log(\cosh 2x)} = \frac{1}{(\cos h\, 2x)}$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $$y\frac{1}{(\cosh 2x)} = -2\int \tanh 2x \frac{1}{\cosh 2x}\, dx + c$$

$$\Rightarrow \frac{y}{(\cosh 2x)} = \int\left(\frac{-2\sinh 2x}{(\cosh 2x)^2}\right) dx + c$$

Put, $v = \cosh 2x$ then $\frac{dv}{dx} = -2\sinh 2x \Rightarrow dv = 2\sinh 2x\, dx.$

Then,

$$\frac{y}{(\cosh 2x)} = -\int\frac{dv}{v^2} + c = \frac{v^{-1}}{-1} + c = \frac{1}{v} + c = \frac{1}{\cosh 2x} + c$$

$$\Rightarrow y = 1 + c\cosh 2x \qquad \text{... (3)}$$

Using (2) to (3), then (3) gives,

$$4 = 1 + c(\cosh 0) \Rightarrow c = 4 - 1 = 3$$

Now (3) becomes,

$$y = 1 + 3\cosh 2x.$$

(vii) $\frac{dy}{dx} + 3y = \sin x,\ y\left(\frac{\pi}{2}\right) = 0.3$

Solution: Given that,

$$\frac{dy}{dx} + 3y = \sin x \qquad \text{...... (1)}$$

And, $$y\left(\frac{\pi}{2}\right) = 0.3 \qquad \text{...... (2)}$$

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = 3 \quad \text{and} \quad Q = \sin x$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int 3\,dx} = e^{3x}$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $$ye^{3x} = \int \sin x\, e^{3x}\, dx + c = \frac{e^{3x}}{9+1}[3\sin x - \cos x] + c$$

$$= \frac{e^{3x}}{10}[3\sin x - \cos x] + c$$

$$y' + \frac{y}{x} = x^2 \qquad \text{...... (1)}$$

And, $\quad y(1) = 1 \qquad$ .... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get;

$$P = \frac{1}{x} \qquad \text{and} \qquad Q = x^2$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $\quad yx = \int x^3\, dx + c$

$$\Rightarrow \quad xy = \frac{x^4}{4} + c \qquad \text{...... (3)}$$

Using (2) to (3), then (iii) gives,

$$1 = \frac{1}{4} + c \Rightarrow 1 = \frac{1}{4} + c \quad \Rightarrow c = \frac{3}{4}$$

Now (3) becomes,

$$xy = \frac{x^4}{4} + \frac{3}{4}$$

$$\Rightarrow \quad 4xy = x^4 + 3.$$

(v) $\frac{dy}{dx} + 4y = 20,\ y(0) = 2$

Solution: Given that,

$$\frac{dy}{dx} + 4y = 20 \qquad \text{...... (1)}$$

And, $\quad y(0) = 2 \qquad$ ...... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = 4 \qquad \text{and} \qquad Q = 20$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int 4\,dx} = e^{4x}$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $\quad ye^{4x} = \int 20\, e^{4x}\, dx + c$

$$\Rightarrow \quad ye^{4x} = 5e^{4x} + c$$

Using (2) to (3), then (3) gives, ... (3)

$$2 \times e^{4\times 0} = 5e^{4\times 0} + c \Rightarrow 2 = 5 + c \quad \Rightarrow c = -3$$

Now (3) becomes,

$$ye^{4x} = 5e^{4x} - 3$$

$$\Rightarrow \quad y = 5 - 3e^{-4x}$$

(vi) $\frac{dy}{dx} = 2(y - 1)\tanh 2x,\ y(0) = 4$

Solution: Given that,

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$\begin{aligned}
\text{i.e.}\quad y\, e^x &= \int (x^2 + 2x + 1)\, e^x\, dx = \int (x+1)^2 e^x\, dx \\
&= (x+1)^2 e^x - 2(x+1)\, e^x + 2e^x + c \\
&= [(x+1)^2 - 2(x+1) + 2]e^x + c = (x^2+1)\, e^x + c \\
\Rightarrow\quad y &= x^2 + 1 + ce^{-x} \qquad \text{...... (3)}
\end{aligned}$$

Using (2), then (3) gives,

$$0 = 0 + 1 + c$$

$$\Rightarrow \quad c = -1$$

Now (3) becomes,

$$y = x^2 + 1 - e^{-x}$$

(iii) $xy' - 3y - x^4(e^x + \cos x) - 2x^2,\ y(\pi) = \pi^3 e^\pi + 2\pi^2$

Solution: Given that,

$$xy' - 3y - x^4(e^x + \cos x) - 2x^2$$

$$\Rightarrow \quad x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3}{x}y = x^3(e^x + \cos x) - 2x \qquad \text{...... (1)}$$

And, $\quad y(\pi) = \pi^3 e^\pi + 2\pi^2 \qquad$ .... (2)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{3}{x} \quad \text{and} \quad Q = x^3(e^x + \cos x) - 2x$$

Then the integrating factor of (1) is,

$$\text{I.F.} = e^{\int p\, dx} = e^{-3\int \frac{1}{x} dx} = e^{-3\log x} = e^{\log(x^{-3})} = \frac{1}{x^3}$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y\frac{1}{x^3} = \int \{x^3(e^x + \cos x) - 2x\}\frac{1}{x^3}\, dx + c$$

$$\Rightarrow \quad \frac{y}{x^3} = \int \left(e^x + \cos x - \frac{2}{x^2}\right) dx + c$$

$$\Rightarrow \quad \frac{y}{x^3} = e^x + \sin x + 2x^{-1} + c$$

$$\Rightarrow \quad y = x^3(e^x + \sin x + 2x^{-1} + c) \qquad \text{...... (3)}$$

Using (2) to (3), then (3) gives,

$$\pi^3 e^\pi + 2\pi^2 = \pi^3(e^\pi + \sin \pi + 2\pi^{-1} + c)$$

$$\Rightarrow \quad \pi^3 e^\pi + 2\pi^2 = e^\pi \pi^3 + 2\pi^2 + \pi^3 c$$

$$\Rightarrow \quad c = 0$$

Now (3) becomes,

$$y = x^3\left(e^x + \sin x + \frac{2}{x}\right).$$

(iv) $y' + \frac{y}{x} = x^2,\ y(1) = 1$

Solution: Given that,

2. Solve the following initial value problems.

(i) $x^2y + 2xy - x + 1 = 0, y(1) = 0$

Solution: Given that, $x^2y + 2xy - x + 1 = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y - \frac{1}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = \left(\frac{1}{x} - \frac{1}{x^2}\right) \quad \ldots\ldots (1)$$

and $\quad y(1) = 0 \quad \ldots\ldots (2)$

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \left(\frac{2}{x}\right) \quad \text{and} \quad Q = \left(\frac{1}{x} - \frac{1}{x^2}\right)$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log(x)} = x^2$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$\text{i.e. } yx^2 = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\,dx + c$$

$$\Rightarrow yx^2 = \int(x-1)\,dx + c$$

$$\Rightarrow x^2y = \left(\frac{x^2}{2} - x\right) + c$$

$$\Rightarrow y = \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2} \quad \ldots\ldots (3)$$

Using (2), then (3) gives,

$$0 = \frac{1}{2} - \frac{1}{1} + \frac{c}{1^2}$$

$$\Rightarrow 1 - \frac{1}{2} = c \Rightarrow c = \frac{1}{2}$$

Now (3) becomes,

$$y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$$

(ii) $y' + y = (x+1)^2, y(0) = 0$

Solution: Given that,

$$y' + y = (x+1)^2 \Rightarrow \frac{dy}{dx} + y = (x+1)^2$$

$$\Rightarrow \frac{dy}{dx} + y = x^2 + 2x + 1 \quad \ldots\ldots (1)$$

And, $y(0) = 0 \quad \ldots\ldots (2)$

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = 1 \quad \text{and} \quad Q = x^2 + 2x + 1$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int dx} = e^x$$

$$\frac{dy}{dx}=\frac{y}{2y\log y+y-x}$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{2y\log y+y-x}{y}=2\log y+1-\frac{x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{x}{y}=2\log y+1 \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $x'+Px=Q$ then we get,

$$P=\frac{1}{y} \qquad \text{and} \qquad Q=2\log y+1$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.}=e^{\int p\,dy}=e^{\int (1/y)\,dy}=e^{\log y}=y$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{x\times \text{I.F.}=\int Q\times \text{I.F.}\,dy+c}$$

$$\text{i.e. } xy=\int(2y\log y+1)\,y\,dy+c=\int(2y\log y+y)\,dy+c$$

$$=\log y\int 2y\,dy-\int\left(\frac{d\log y}{dy}\int 2y\,dy\right)dy+\int y\,dy+c$$

$$=\log y\,y^2-\int\frac{1}{y}y^2\,dy+\frac{y^2}{2}+c=y^2\log y-\frac{y^2}{2}+\frac{y^2}{2}+c$$

$$=y^2\log y+c$$

$$\Rightarrow \quad x=y\log y+\frac{c}{y}$$

(xxii) $\mathbf{x^2\frac{dy}{dx}=3x^2-2xy+1}$

Solution: Given that, $x^2\frac{dy}{dx}=3x^2-2xy+1$

$$\Rightarrow \quad \frac{dy}{dx}=3-\frac{2y}{x}+\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(\frac{2}{x}\right)y=\left(3+\frac{1}{x^2}\right) \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $y'+Py=Q$ then we get,

$$P=\left(\frac{2}{x}\right) \qquad \text{and} \qquad Q=\left(3+\frac{1}{x^2}\right)$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.}=e^{\int p\,dx}=e^{\int\frac{2}{x}dx}=e^{2\log(x)}=x^2$$

Now, multiplying (1) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y\times \text{I.F.}=\int Q\times \text{I.F.}\,dx+c}$$

$$\text{i.e. } y\,x^2=\int\left(3+\frac{1}{x^2}\right)x^2\,dx+c$$

$$\Rightarrow \quad yx^2=\int(3x^2+1)\,dx+c$$

$$\Rightarrow \quad yx^2=x^3+x+c$$

$$\Rightarrow \quad y=x+\frac{1}{x}+\frac{c}{x^2}$$

(xix) $x\dfrac{dy}{dx} + y = e^x - xy$

Solution: Given that, $x\dfrac{dy}{dx} + y = e^x - xy \Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{e^x}{x} y$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x} + 1\right) = \frac{e^x}{x} \quad \ldots\ldots(1)$$

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{x} + 1 \qquad \text{and} \qquad Q = \frac{e^x}{x}$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int\left(\frac{1}{x}+1\right)dx} = e^{(\log x + x)} = e^{\log x} e^x = xe^x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,xe^x = \int \dfrac{e^x}{x}\,xe^x\,dx + c$

$$\Rightarrow yxe^x = \frac{e^{2x}}{2} + c$$

$$\Rightarrow xy = \frac{e^x}{2} + ce^{-x}$$

(xx) $ye^y\,dy = (y^2 + 2xe^y)\,dy$

Solution: Given that

$$ye^y\,dx = (y^3 + 2x\,e^y)\,dy \quad \ldots\ldots..(1)$$

$$\Rightarrow \frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y} = y^2e^{-y} + x\left(\frac{2}{y}\right)$$

$$\Rightarrow \frac{dx}{dy} + x\left(\frac{-2}{y}\right) = y^2e^{-y} \quad \ldots\ldots..(2)$$

Comparing (1) with the equation $x' + Px = Q$ then we get,

$$P = \left(\frac{-2}{y}\right) \qquad \text{and} \qquad Q = y^2e^{-y}$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dy} = e^{\int(-2/y)\,dy} = e^{-2\log y} = \frac{1}{y^2}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{x \times \text{I.F.} = \int Q \times \text{I.F.}\,dy + c}$$

i.e. $x\dfrac{1}{y^2} = \int y^2.\,e^{-y}.\,\dfrac{1}{y^2}\,dy + c$

$$\Rightarrow \frac{x}{y^2} = \int e^{-y}\,dy + c = \frac{e^{-y}}{-1} + c = (c - e^{-y})$$

$$\Rightarrow x = y^2(c - e^{-y})$$

(xxi) $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}$

Solution: Given that,

(xiv) **Repeated to (xi)**

(xv) **Repeated to (xii)**

(xvi) $(x^2+1)\dfrac{dy}{dx}+2xy=x^2$

Solution: Given that, $(x^2+1)\dfrac{dy}{dx}+2xy=x^2$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{(x^2+1)}=\frac{x^2}{(x^2+1)} \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $y'+Py=Q$ then we get,

$$P=\frac{2x}{(x^2+1)} \quad \text{and} \quad Q=\frac{x^2}{x^2+1}$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.}=e^{\int p\,dx}=e^{\int \frac{2x}{x^2+1}dx}=e^{\log(x^2+1)}=(x^2+1)$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y\times \text{I.F.}=\int Q\times \text{I.F.}\,dx+c}$$

i.e. $y(x^2+1)=\displaystyle\int \frac{x^2}{x^2+1}\times(x^2+1)\,dx+c$

$\Rightarrow y(x^2+1)=\displaystyle\int x^2\,dx+c$

$\Rightarrow y(x^2+1)=\dfrac{x^3}{3}+c$

(xvii) $(1+x^3)\dfrac{dy}{dx}+6x^2y=1+x^2$

Solution: Given that, $(1+x^3)\dfrac{dy}{dx}+6x^2y=1+x^2$

$$\Rightarrow \frac{dy}{dx}+\frac{6x^2y}{1+x^3}=\frac{1+x^2}{1+x^3} \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $y'+Py=Q$ then we get,

$$P=\frac{6x^2}{1+x^3} \quad \text{and} \quad Q=\frac{1+x^2}{1+x^3}$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.}=e^{\int p\,dx}=e^{\int \frac{6x^2}{x^3+1}dx}=e^{2\log(1+x^3)}=e^{\log(1+x^3)^2}=(1+x^3)^2$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y\times \text{I.F.}=\int Q\times \text{I.F.}\,dx+c}$$

i.e. $y(1+3^3)^2=\displaystyle\int \frac{1+x^2}{(1+x^3)}\times(1+x^3)^2\,dx+c$

$\Rightarrow y(1+x^3)^2=\displaystyle\int (1+x^3+x^2+x^5)\,dx+c$

$\Rightarrow y(1+x^3)^2=\left(x+\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^6}{6}\right)+c.$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \tanh x\,dx} = e^{\log\cosh x} = \cosh x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \cosh x = \int \frac{-e^x}{\cosh x} \times \cosh x\, dx + c$$

$$\Rightarrow \quad y \cosh x = -e^x + c$$

(xii) $(x - 2y)\,dy + y\,dx = 0.$

Solution: Given that, $(x - 2y)\,dy + y\,dx = 0.$

$$\Rightarrow \frac{dx}{dy} = -\frac{x - 2y}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 2 \qquad \text{...... (1)}$$

Comparing (1) with the equation $x' + Px = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 2.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{x \times \text{I.F.} = \int Q \times \text{I.F.}\, dy + c}$$

i.e. $xy = \int 2y\, dy + c$

$$\Rightarrow xy = y^2 + c.$$

(xiii) $x\,dy + y\,dx = y\,dy$

Solution: Given that, $x\,dy + y\,dx = y\,dy$

$$\Rightarrow \frac{x\,dy}{y\,dy} + \frac{y\,dx}{y\,dy} = 1$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 1 \qquad \text{...... (1)}$$

Comparing (1) with the equation $x' + Px = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 1.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{x \times \text{I.F.} = \int Q \times \text{I.F.}.\, dy + c}$$

i.e. $xy = \int y\, dy + c$

$$\Rightarrow xy = \frac{y^2}{2} + \frac{c}{2}$$

$$\Rightarrow 2xy = y^2 + c$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \tanh x\,dx} = e^{\log \cosh x} = \cosh x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

$$y \cosh x = \int \frac{-e^x}{\cosh x} \times \cosh x\,dx + c$$

$$\Rightarrow \quad y \cosh x = -e^x + c$$

(xii) $(x - 2y)\,dy + y\,dx = 0.$

Solution: Given that, $(x - 2y)\,dy + y\,dx = 0.$

$$\Rightarrow \frac{dx}{dy} = -\frac{x - 2y}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 2 \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $x' + Px = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 2.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{x \times \text{I.F.} = \int Q \times \text{I.F.}\,dy + c}$$

i.e. $xy = \int 2y\,dy + c$

$$\Rightarrow \quad xy = y^2 + c.$$

(xiii) $x\,dy + y\,dx = y\,dy$

Solution: Given that, $x\,dy + y\,dx = y\,dy$

$$\Rightarrow \frac{x\,dy}{y\,dy} + \frac{y\,dx}{y\,dy} = 1$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 1 \qquad \ldots\ldots (1)$$

Comparing (1) with the equation $x' + Px = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 1.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{x \times \text{I.F.} = \int Q \times \text{I.F.}\,dy + c}$$

i.e. $xy = \int y\,dy + c$

$$\Rightarrow \quad xy = \frac{y^2}{2} + \frac{c}{2}$$

$$\text{I.F.} = e^{\int p\,dx} = e^{-\int \frac{x}{1+x}dx} = e^{-\int \frac{1+x-1}{1+x}dx} = e^{-\int \left(1-\frac{1}{1+x}\right)dx}$$
$$= e^{-(x-\log(1+x))} = e^{\log(1+x)}\, e^{-x}$$
$$= (1+x)\, e^{-x}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $y.(1+x)\, e^{-x} = \int \frac{1-x}{1+x} \times (1+x)\, e^{-x}\, dx + c$

$\Rightarrow\ y\,(1+x)\, e^{-x} = \int (e^{-x} - xe^{-x})\, dx + c$

$\Rightarrow\ y\,(1+x)e^{-x} = -e^{-x} - (-xe^{-x}\ \ e^{-x}) + c$

$\Rightarrow\ y\,(1+x)e^{-x} = -e^{-x} + xe^{-x} + e^{-x} + c$

$\Rightarrow\ y(1+x) = x + ce^{x}$

(x) $(1-x^2)\frac{dy}{dx} - xy = 1$

Solution: Given that, $(1-x^2)\frac{dy}{dx} - xy = 1.$

$\Rightarrow\ \frac{dy}{dx} - \frac{x}{1-x^2}\,y = \frac{1}{1-x^2}$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{x}{1-x^2} \quad \text{and} \quad Q = \frac{1}{1-x^2}.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{-\int \frac{x}{1+x^2}dx} = e^{-\frac{1}{2}\log(1+x2)} = e^{\frac{1}{2}\log(1-x^2)}$$
$$= e^{\log(1-x^2)^{1/2}} = \sqrt{(1-x^2)}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

i.e. $y\,(1-x^2)^{1/2} = \int \frac{1}{(1-x^2)} \times (1-x^2)^{1/2}\, dx + c$

$\Rightarrow\ y\,(1-x^2)^{1/2} = \int (1-x^2)^{-1/2}\, dx + c$

$\Rightarrow\ y\,(1-x^2)^{1/2} = \int \frac{1}{\sqrt{(1-x^2)}}\, dx + c$

$\Rightarrow\ y\sqrt{(1-x^2)} = \sin^{-1} x + c.$ $\left[\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}\right]$

(xi) $\cosh x\, dy + (y \sinh x + e^x)\, dx = 0.$

Solution: Given that, $\cosh x\, dy + (y \sinh x + e^x)\, dx = 0$

$\Rightarrow\ \frac{dy}{dx} + \frac{(y \sinh x + e^x)}{\cosh x} = 0$

$\Rightarrow\ \frac{dy}{dx} + y \tanh x = \frac{-e^x}{\cosh x}$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \tanh x \quad \text{and} \quad Q = \frac{-e^x}{\cosh x}$$

(vii) $xy' - 2y = x^3 e^x$

Solution: Given that, $xy' - 2y = x^3 e^x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \frac{-2}{x} \quad \text{and} \quad Q = x^2 e^x.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(-2/x)\,dx} = e^{-2\log x} = \frac{1}{x^2}.$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\dfrac{1}{x^2} = \int x^2 e^x \dfrac{1}{x^2}\,dx + c$

$\Rightarrow \dfrac{y}{x^2} = e^x + c$

$\Rightarrow y = x^2 e^x + cx^2$

(viii) $x^2y' + 2xy + \sinh 3x$

Solution: Given that, $x^2y' + 2xy + \sinh 3x$

$\Rightarrow y' + \dfrac{2}{x}y = \dfrac{1}{x^2}\sinh 3x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = \frac{2}{x} \quad \text{and} \quad Q = \frac{1}{x^2}\sinh 3x.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int(2/x)\,dx} = e^{2\log x} = x^2.$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,x^2 = \int \dfrac{1}{x^2}\sinh 3x \times x^2\,dx + c$

$\Rightarrow yx^2 = \dfrac{\cosh 3x}{3} + c$

$\Rightarrow 3yx^2 = \cosh 3x + c$

(ix) $(1+x)\dfrac{dy}{dx} - xy = 1 - x$

Solution: Given that, $(1+x)\dfrac{dy}{dx} - xy = 1 - x$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{x}{(1+x)}y = \dfrac{(1-x)}{(1+x)}$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{x}{1+x} \quad \text{and} \quad Q = \frac{(1-x)}{(1+x)}.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

(iv) $y' + y\cot x = 2\cos x$ [2008 Spring Q. No. 4(a)]

Solution: Given that, $y' + y\cot x = 2\cos x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 2\cos x$.

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\sin x = \int 2\cos x \sin x\,dx + c$

$\Rightarrow y\sin x = \int \sin 2x\,dx + c$

$\Rightarrow y\sin x = -\dfrac{\cos 2x}{2} + c$

$\Rightarrow 2y\sin x + \cos 2x + c.$

(v) $y' + ky = e^{-kx}$

Solution: Given that, $y' + ky = e^{-kx}$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = k$ and $Q = e^{-kx}$.

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int k\,dx} = e^{kx}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,e^{kx} = \int e^{-kx} e^{kx}\,dx + c$

$\Rightarrow y\,e^{kx} = \int dx + c$

$\Rightarrow ye^{kx} = x + c$

$\Rightarrow y = (x + c)\,e^{-kx}$

(vi) $y' + 2y\tan x = \sin x$

Solution: Given that, $y' + 2y\tan x = \sin x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$P = 2\tan x$ and $Q = \sin x$.

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int 2\tan x\,dx} = e^{2\log \sec x} = \sec^2 x.$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\sec^2 x = \int \sin x \sec^2 x\,dx + c$

$\Rightarrow y\sec^2 x = \int \tan x \sec x\,dx + c$

$\Rightarrow y\sec^2 x = \sec x + c$

$\Rightarrow y\sec^2 x - \sec x = c$

Exercise 6.4

1. Solve the following differential equation:

(i) $y' + 2y = 4x$

Solution: Given that, $y' + 2y = 4x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = 2 \quad \text{and} \quad Q = 4x.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int 2\,dx} = e^{2x}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,e^{2x} = \int 4x\,e^{2x}\,dx + c$

$\Rightarrow \quad y\,e^{2x} = 4x\dfrac{e^{2x}}{2} - 4\dfrac{e^{2x}}{4} + c$

$\Rightarrow \quad y\,e^{2x} = 2xe^{2x} - e^{2x} + c$

$\Rightarrow \quad y = 2x - 1 + ce^{-2x}$

(ii) $y' - y = 3$

Solution: Given that, $y' - y = 3$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = -1 \quad \text{and} \quad Q = 3.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int (-1)\,dx} = e^{-x}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,e^{-x} = \int 3\,e^{-x}\,dx + c$

$\Rightarrow \quad ye^{-x} = -3e^{-x} + c$

$\Rightarrow \quad y = -3 + ce^{x}.$

(iii) $y' + 2y = 6e^x$

Solution: Given that, $y' + 2y = 6e^x$ ...... (1)

Comparing (1) with the equation $y' + Py = Q$ then we get,

$$P = 2 \quad \text{and} \quad Q = 6e^x.$$

The equation (1) is linear differential equation of first order whose integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int 2\,dx} = e^{2x}$$

Now, multiplying (1) by I.F. and then taking integration on both sides, so that (1) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\,dx + c}$$

i.e. $y\,e^{2x} = \int 6e^x\,e^{2x}\,dx + c$

$\Rightarrow \quad y\,e^{2x} = 6\int e^{3x}\,dx + c$

$\Rightarrow \quad y\,e^{2x} = 6\dfrac{e^{3x}}{3} + c$

$\Rightarrow \quad y = 2e^{x} + ce^{-2x}$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Here,

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(y^4+2y)}(y^3 - 4 - 4y^3 - 2) = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y}$$

Then the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int(-3/y)\,dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$$

Multiplying equation (1) by I.F., then

$$\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$$

$$\Rightarrow \quad (y + 2y^{-2})\,dx + (x + 2y - 4xy^{-3})\,dy = 0$$

This is exact. So, its solution is,

$$\int M dx + \int (\text{terms of N which is free from x})\,dy = c$$

$$\Rightarrow \quad \int (y + 2y^{-2})\,dx + \int 2y\,dy = c$$

$$\Rightarrow \quad \left(y + \frac{2}{y^2}\right)x + y^2 = c.$$

(vii) $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Solution: Given that,

$$(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0 \quad \text{.........(1)}$$

Comparing it with Mdx + Ndy = 0 then we get,

$$M = 3xy - 2ay^2 \quad \text{and} \quad N = x^2 - 2axy$$

So, $$\frac{\partial M}{\partial y} = 3x - 4ay \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x - 2ay$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Here,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(x-2ay)}(3x - 4ay - 2x + 2ay)$$

$$= \frac{x - 2ay}{x(x-2ay)} = \frac{1}{x}$$

Then the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int(1/x)\,dx} = e^{\log x} = x$$

Multiplying equation (1) by I.F., then

$$x\,(3axy - 2ay^2)\,dx + x(x^2 - 2axy)\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \quad \int x\,(3ay - 2ay^2)\,dx + \int 0\,dy = c$$

$$\Rightarrow \quad 3ay\int x^2\,dx - 2ay^2\int x\,dx = c$$

$$\Rightarrow \quad ayx^3 - ay^2x^2 = c$$

$$\Rightarrow \quad ax^2(xy - y^2) = c.$$

Integrating we get,

$$\frac{v^{-3}}{-3} + \frac{1}{4}\log(v^4) + \log(x) = c$$

$$\Rightarrow -\frac{x^3}{3y^3} + \frac{1}{4}\log\left(\frac{y^4}{x^4}\right) + \log(x) = c.$$

$$\Rightarrow \frac{-x^3}{3y^3} + \frac{1}{4}\log(y^4) - \frac{1}{4}\log(x^4) + \log(x) = C$$

$$\Rightarrow \frac{-x^3}{3y^3} + \frac{4}{4}\log(y) - \frac{4}{4}\log(x) + \log(x) = C$$

$$\Rightarrow \frac{-x^3}{3y^3} + \log(y) = C$$

(v) $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$

Solution: Given that,

$$(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0 \quad ....(1)$$

Comparing it with Mdx + Ndy = 0 then we get,

$$M = x^2 + y^2 + 1 \quad \text{and} \quad N = -2xy$$

So, $$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Here,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}(2y + 2y) = \frac{4y}{-2xy} = -\frac{2}{x}$$

Then the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int -2/x\,dx} = e^{-2\log x} = x^{-2} = \frac{1}{x^2}$$

Multiplying equation (1) by I.F., then

$$\left(\frac{x^2 + y^2 + 1}{x^2}\right)dx - \left(\frac{2xy}{x^2}\right)dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int\left(\frac{x^2 + y^2 + 1}{x^2}\right)dx\,dx + \int 0\,dy = c$$

$$\Rightarrow \int (1 + y^2 x^{-2} + x^{-2})\,dx = c$$

$$\Rightarrow x + y^2\left(\frac{x^{-1}}{-1}\right) + \frac{x^{-1}}{-1} = c$$

$$\Rightarrow x - \frac{y^2}{x} - \frac{1}{x} = c$$

(vi) $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)dy = 0$

Solution: Given that,

$$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)dy = 0 \quad ....(1)$$

Comparing it with Mdx + Ndy = 0 then we get,

$$M = y^4 + 2y \quad \text{and} \quad N = xy^3 + 2y^4 - 4x$$

So, $$\frac{\partial M}{\partial y} = 4y^3 + 2 \quad \text{and} \quad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = x + c$$

$$\Rightarrow y = x\tan(x + c)$$

This is the solution of given equation.

(ii) $y(2xy + e^x)\,dx = e^x\,dy$

Solution: Given that,

$$y(2xy + e^x)\,dx = e^x\,dy$$

$$\Rightarrow 2xy^2\,dx + ye^x\,dx = e^x\,dy$$

$$\Rightarrow 2x\,dx = \frac{e^x\,dy - ye^x\,dx}{y^2}$$

$$\Rightarrow d(x^2) = -d\left(\frac{e^x}{y}\right)$$

Integrating we get,

$$x^2 = -\frac{e^x}{y} + c$$

$$\Rightarrow x^2 + \frac{e^x}{y} = c.$$

(iii) $x\,dy - y\,dx = xy^2\,dx$

Solution: Given that,

$$xdy - ydx = xy^2\,dx$$

$$\Rightarrow \frac{xdy - ydx}{y^2} = x\,dx$$

$$\Rightarrow -\left(\frac{y\,dx - x\,dy}{y^2}\right) = d\left(\frac{x^2}{2}\right)$$

$$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$$

Integrating we get,

$$-\frac{x}{y} = \frac{x^2}{2} - c$$

$$\Rightarrow \frac{x^2}{2} + \frac{x}{y} = c.$$

(iv) $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **[2015 Fall Q.No. 4(a)]**

Solution: Given that,

$$x^2y\,dx - (x^3 + y^3)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$$

This is a homogeneous equation. So, put $y = vx$. Then $\frac{dy}{dx} = v + x\,.\frac{dv}{dx}$. Therefore,

$$v + x\frac{dv}{dx} = \frac{x^3v}{x^3 + v^3x^3} = \frac{v}{1 + v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1 + v^3} - v = \frac{v - v - v^4}{1 + v^3} = -\frac{v^4}{1 + v^3}$$

$$\Rightarrow \left(\frac{1 + v^3}{v^4}\right)dv + \frac{dx}{x} = 0$$

$$\Rightarrow \left(\frac{1}{v^4} + \frac{1}{4}\cdot\frac{4v^3}{v^4}\right)dv + \frac{dx}{x} = 0$$

$$\Rightarrow \int [(x+1)e^x - e^y\} \, dx + \int 0 \, dy = c$$
$$\Rightarrow \int (xe^x + e^x - e^y) \, dx = c$$
$$\Rightarrow xe^x - e^x + e^x - e^y x = c$$
$$\Rightarrow xe^x - xe^y = c \quad \text{......(3)}$$

Since $y(1) = 0$ then (3) gives

$$1.e^1 = 1.e^0 = c \quad \Rightarrow (e-1) = c$$

Now (3) becomes,

$$xe^x - xe^y = e - 1.$$

(viii) 2 sin 2x sinh y dx – cos 2x cosh y dy = 0, y (0) = 1

Solution: Given that,

$$2 \sin 2x \sinh y \, dx - \cos 2x \cosh y \, dy = 0 \quad \text{......(1)}$$
$$y(0) = 1 \quad \text{......(2)}$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = 2 \sin 2x \sinh y \quad \text{and} \quad N = -\cos 2x \cosh y$$

So, $\frac{\partial M}{\partial y} = 2 \sin 2x \cosh y,$ and $\frac{\partial N}{\partial x} = 2 \sin 2x, \cosh y$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (1) is exact. Then,

$$\int M \, dx + \int (\text{terms of N not containing x}) \, dy = c$$
$$\Rightarrow \int 2 \sin 2x \sinh y \, dx + \int 0 \, dy = c$$
$$\Rightarrow 2 \sinh y \int \sin 2x \, dx = c$$
$$\Rightarrow 2 \sinh y \left(-\frac{\cos 2x}{2}\right) = c$$
$$\Rightarrow -\sinh y \cos 2x = c \quad \text{......(3)}$$

Since, $y(0) = 1$ then (iii) gives,

$$-\sinh 1 \cos 0 = c \quad \Rightarrow c = -\sinh 1$$

Now, equation (3) becomes,

$$-\sinh y \cos 2x = -\sinh 1$$
$$\Rightarrow \sinh y \cos 2x = \sinh 1$$

E. Solve the following differential equations:

(i) x dy – y dx = (x² + y²) dx

Solution: Given that

$$x \, dy - y \, dx = (x^2 + y^2) \, dx \quad \text{.........(1)}$$

Put, $x = r\cos\theta$, $y = r\sin\theta$ then $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. So,

$$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{x\frac{dy}{dx} - y}{x^2}\right) = \frac{1}{x^2 + y^2}\left(x\frac{dy}{dx} - y\right) = \frac{x \, dy - y \, dx}{r^2 \, dx}$$

$$\Rightarrow r^2 \, d\theta = x \, dy - y \, dx$$

Then (1) becomes,

$$r^2 d\theta = r^2 \, dx$$
$$\Rightarrow d\theta = dx$$

Integrating we get,

$$\theta = x + c$$

(vi) $2xy\,dy = (x^2+y^2)\,dx,\ y(1)=2$

Solution: Given that,

$$(x^2+y^2)\,dx - 2xy\,dy = 0 \qquad \text{...... (1)}$$
$$y(1) = 2 \qquad \text{...... (2)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$$M = (x^2+y^2) \quad \text{and} \quad N = -2xy$$

So, $$\frac{\partial M}{\partial y} = (0+2y) = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x).$$

Therefore, $\text{I.F.} = e^{\int f(x)\,dx} = e^{\int (-2/x)\,dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying equation (1) by I.F., then

$$\frac{(x^2+y^2)}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0.$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M\,dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int\left(1+\frac{y^2}{x^2}\right)dx + \int 0\,dy = c$$

$$\Rightarrow \left(x - \frac{y^2}{x}\right) = c \qquad \text{...... (3)}$$

Since y(1) = 2 then (3) gives, $c = \left(1 - \frac{2^2}{1}\right) = -3$

Now, (3) becomes,

$$x - \frac{y^2}{x} = -3 \Rightarrow x^2 - y^2 = -3x$$
$$\Rightarrow y^2 = x^2 + 3x$$
$$\Rightarrow y = \sqrt{x^2+3x}.$$

(vii) $[(x+1)e^x - e^y]\,dx = xe^y dy,\ y(1) = 0$

Solution: Given that,

$$[(x+1)e^x - e^y]\,dx - xe^y\,dy = 0 \qquad \text{...... (1)}$$
$$y(1) = 0 \qquad \text{...... (2)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$$M = (x+1)e^x - e^y \quad \text{and} \quad N = -xe^y$$

So, $$\frac{\partial M}{\partial y} = 0 - e^y = -e^y \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^y$$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then,

$$\int M\,dx + \int (\text{terms of N not containing})\,dy = c$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int \tan x\,dx} = e^{-\log \sec x} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

Multiplying equation (1) by I.F.,

$$2 \cos x\,dx + \sec x \cos x \cos y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \quad 2 \int \cos x\,dx + \int \cos y\,dy = c$$

$$\Rightarrow \quad 2 \sin x + \sin y = c \qquad \text{...... (3)}$$

Since, $y(0) = 0$, then (3) gives,

$$2 \sin 0 + \sin 0 = c \Rightarrow c = 0.$$

Now equation (3) becomes

$$2 \sin x + \sin y = 0.$$

(v) $2 \sin y\,dx + \cos y\,dy = 0,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$2 \sin y\,dx + \cos y\,dy = 0 \qquad \text{....... (1)}$$

$$y(0) = \frac{\pi}{2} \qquad \text{...... (2)}$$

Comparing (i) with $M dx + N dy = 0$ then,

$$M = 2 \sin y, \quad \text{and} \quad N = \cos y$$

So, $$\frac{\partial M}{\partial y} = 2 \cos y, \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 \cos y - 0}{\cos y} = \frac{2 \cos y}{\cos y} = 2 = f(x)$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{2\int dx} = e^{2x}$$

Multiplying equation (1) by I.F., then

$$2 \sin y\, e^{2x}\,dx + e^{2x} \cos y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \quad \int 2e^{2x} \sin y\,dx + \int 0\,dy = c$$

$$\Rightarrow \quad 2 \sin y \int e^{2x} = c$$

$$\Rightarrow \quad 2\frac{e^{2x}}{2} \sin y = c$$

$$\Rightarrow \quad e^{2x} \sin y = c \qquad \text{...... (3)}$$

Since, $y(0) = \frac{\pi}{2}$, then (3) gives,

$$e^0 \sin \frac{\pi}{2} = c \Rightarrow c = 1.$$

Then (3) becomes,

$$e^{2x} \sin y = 1$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int \tan x\,dx} = e^{-\log \sec x} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

Multiplying equation (1) by I.F.,

$$2\cos x\,dx + \sec x \cos x \cos y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow\ 2\int \cos x\,dx + \int \cos y\,dy = c$$

$$\Rightarrow\ 2\sin x + \sin y = c \qquad \ldots\ldots (3)$$

Since, y(0) = 0, then (3) gives,

$$2\sin 0 + \sin 0 = c \Rightarrow c = 0.$$

Now equation (3) becomes

$$2\sin x + \sin y = 0.$$

(v) $2\sin y\,dx + \cos y\,dy = 0,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$2\sin y\,dx + \cos y\,dy = 0 \qquad \ldots\ldots (1)$$

$$y(0) = \frac{\pi}{2} \qquad \ldots\ldots (2)$$

Comparing (i) with Mdx + Ndy = 0 then,

$$M = 2\sin y, \quad \text{and} \quad N = \cos y$$

So, $$\frac{\partial M}{\partial y} = 2\cos y, \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2\cos y - 0}{\cos y} = \frac{2\cos y}{\cos y} = 2 = f(x)$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{2\int dx} = e^{2x}$$

Multiplying equation (1) by I.F., then

$$2\sin y\, e^{2x}\,dx + e^{2x}\cos y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow\ \int 2e^{2x}\sin y\,dx + \int 0\,dy = c$$

$$\Rightarrow\ 2\sin y \int e^{2x} = c$$

$$\Rightarrow\ 2\frac{e^{2x}}{2}\sin y = c$$

$$\Rightarrow\ e^{2x}\sin y = c \qquad \ldots\ldots (3)$$

Since, $y(0) = \frac{\pi}{2}$, then (3) gives,

$$e^0 \sin\frac{\pi}{2} = c \Rightarrow c = 1.$$

Then (3) becomes,

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int \tan x\, dx} = e^{-\log \sec x} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

Multiplying equation (1) by I.F.,

$$2 \cos x\, dx + \sec x \cos x \cos y\, dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\, dy = c$$

$$\Rightarrow \quad 2 \int \cos x\, dx + \int \cos y\, dy = c$$

$$\Rightarrow \quad 2 \sin x + \sin y = c \qquad \ldots\ldots (3)$$

Since, $y(0) = 0$, then (3) gives,

$$2 \sin 0 + \sin 0 = c \Rightarrow c = 0.$$

Now equation (3) becomes

$$2 \sin x + \sin y = 0.$$

(v) $2 \sin y\, dx + \cos y\, dy = 0,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$2 \sin y\, dx + \cos y\, dy = 0 \qquad \ldots\ldots (1)$$

$$y(0) = \frac{\pi}{2} \qquad \ldots\ldots (2)$$

Comparing (i) with $Mdx + Ndy = 0$ then,

$$M = 2 \sin y, \quad \text{and} \quad N = \cos y$$

So,

$$\frac{\partial M}{\partial y} = 2 \cos y, \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 \cos y - 0}{\cos y} = \frac{2 \cos y}{\cos y} = 2 = f(x)$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{2\int dx} = e^{2x}$$

Multiplying equation (1) by I.F., then

$$2 \sin y\, e^{2x}\, dx + e^{2x} \cos y\, dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\, dy = c$$

$$\Rightarrow \quad \int 2e^{2x} \sin y\, dx + \int 0\, dy = c$$

$$\Rightarrow \quad 2 \sin y \int e^{2x} = c$$

$$\Rightarrow \quad 2 \frac{e^{2x}}{2} \sin y = c$$

$$\Rightarrow \quad e^{2x} \sin y = c \qquad \ldots\ldots (3)$$

Since, $y(0) = \frac{\pi}{2}$, then (3) gives,

$$e^0 \sin \frac{\pi}{2} = c \Rightarrow c = 1.$$

Then (3) becomes,

$$e^{2x} \sin y = 1$$

$\Rightarrow \quad \frac{3x^2}{x^3}dx + \frac{4y^3}{y^4}dy = 0$

$\Rightarrow \quad \frac{3}{x}dx + \frac{4}{y}dy = 0$

Integrating we get,

$3\log x + 4\log y = \log c$

$\Rightarrow \quad \log(x^3y^4) = \log c$

$\Rightarrow \quad x^3y^4 = c$ .... (i)

When y(i) = 2, (i) given

$16 = c$

Therefore, (i) becomes $x^3y^4 = 16$.

(iii) $y' = \frac{1-x}{1+y}$ y(1) = 0.

Solution: Given that,

$$y' = \frac{1-x}{1+y} \qquad \text{...... (1)}$$

$$y(1) = 0 \qquad \text{...... (2)}$$

Then (1) becomes,

$$\frac{dy}{dx} = \frac{(1-x)}{(1+y)}$$

$$\Rightarrow \quad (1+y)\,dy = (1-x)dx$$

Integrating,

$$\int(1+y)\,dy = \int(1-x)\,dx$$

$$\Rightarrow \quad y + \frac{y^2}{2} = x - \frac{x^2}{2} + c$$

$$\Rightarrow \quad \frac{2y+y^2}{2} = \frac{2x-x^2}{2} + c$$

$$\Rightarrow \quad x^2 - 2x + y^2 + 2y = c \qquad \text{... (3)}$$

Since, y(1) = 0 then (3) gives,

$$1 - 2 = c \quad \Rightarrow c = -1$$

Therefore (3) becomes,

$$x^2 - 2x + y^2 + 2y = -1$$

$$\Rightarrow \quad (x-1)^2 + (y+1)^2 = 1.$$

(iv) $2dx + \sec x \cos y\, dy = 0$, y(0) = 0.

Solution: Given that,

$$2dx + \sec x \cos y\, dy = 0 \qquad \text{....... (1)}$$

$$y(0) = 0 \qquad \text{....... (2)}$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$M = 2$ and $N = \sec x \cos y$

So, $\frac{\partial M}{\partial y} = 0$, and $\frac{\partial N}{\partial x} = \cos y . \sec x . \tan x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 - \cos y . \sec x . \tan x}{\sec x . \cos y} = \frac{-\cos y . \sec x . \tan x}{\sec x . \cos y}$$

$$= -\tan x$$

$$y(0) = \frac{2}{3} \quad \text{...... (2)}$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = (y-1) \quad \text{and} \quad N = (x-3)$$

So, $$\frac{\partial M}{\partial y} = 1, \quad \text{and} \quad \frac{\partial N}{\partial x} = 1$$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (1) is exact. Then, the solution of (1) is,

$$\int M\,dx + \int (\text{terms of N not containing dy}) = c$$

$$\Rightarrow \int (y-1)\,dx - 3\int dy = c$$

$$\Rightarrow (y-1)x - 3y = c$$

$$\Rightarrow x(y-1) - 3y = c \quad \text{...... (3)}$$

Since, $y(0) = \frac{2}{3}$, then (3) gives,

$$0(y-1) - 3 \times \frac{2}{3} = c$$

$$\Rightarrow c = -2$$

Therefore (3) becomes,

$$x(y-1) - 3y = -2 \Rightarrow xy - x - 3y + 2 = 0$$

$$\Rightarrow xy - 3y - x + 3 - 1 = 0$$

$$\Rightarrow xy - 3y - x + 3 = 1$$

$$\Rightarrow y(x-3) - 1(x-3) = 1$$

$$\Rightarrow (x-3)(y-1) = 1.$$

(ii) $3x^2y^4dx + 4x^3y^3dy = 0$, $y(1) = 2$

Solution: Given that,

$$3x^2y^4dx + 4x^3y^3dy = 0 \quad \text{...... (1)}$$

$$y(1) = 2 \quad \text{...... (2)}$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = 3x^2y^4 \quad \text{and} \quad N = 4x^3y^3$$

So, $$\frac{\partial M}{\partial y} = 12x^2y^3, \quad \text{and} \quad \frac{\partial N}{\partial x} = 12x^2y^3$$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (1) is exact. Then, its solution is,

$$\int M\,dx + \int (\text{terms of N not containing dy}) = c$$

$$\Rightarrow \int 3x^2y^4\,dx + \int 0\,dy = c$$

$$\Rightarrow 3y^4 \times \frac{x^2}{3}\,c$$

$$\Rightarrow c = x^3y^4 \quad \text{...... (3)}$$

Since, y(1) = 2 then (3) gives,

$$c = 1 \times 2^4 = 16$$

Therefore (3) becomes,

$$x^3y^4 = 16.$$

Alternatively:

$$3x^2y^4dx + 4x^3y^3dy = 0$$

(iv) $xy' + y + 4 = 0$

Solution: Given that,

$$xy' + y + 4 = 0$$

$$\Rightarrow x\frac{dy}{dx} + (y + 4) = 0$$

$$\Rightarrow xdy + (y + 4)\,dx = 0. \qquad \ldots (1)$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = (y + 4) \quad \text{and} \quad N = x$$

So, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (1) is exact. Then, its solution is,

$$\int M\,dx + \int (\text{terms of N not containing dy}) = c$$

$$\Rightarrow \int (y + 4)\,dx + \int 0 = c$$

$$\Rightarrow (y + 4)\,x = c$$

$$\Rightarrow xy + 4x = c$$

$$\Rightarrow xy = c - 4x$$

$$\Rightarrow y = \frac{c}{x} - 4.$$

C. Solve the following

(i) $(x + y)\,dx + (y - x)\,dy = 0$

Solution: Given that,

$$(x + y)\,dx + (y - x)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x}{y + x} = \frac{-(1 - y/x)}{1 + y/x} \qquad \ldots\ldots (1)$$

So, (1) is homogeneous equation. Put $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$.

Therefore (1) becomes

$$u + x\frac{du}{dx} = \frac{-1 + u}{1 + u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{-1 + u}{1 + u} - u = \frac{-1 - u^2}{1 + u}$$

$$\Rightarrow \frac{1 + u}{1 + u^2}\,du = -\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1 + u^2} + \frac{u}{1 + u^2}\right) du = -\frac{dx}{x}$$

Integrating both side

$$\tan^{-1}(u) + \frac{1}{2}\log(1 + u^2) = -\log(x) + A$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\log(x^2 + y^2) = A$$

D. Solve the following initial value problems.

(I) $(y - 1)dx + (x - 3)\,dy = 0,\ y(0) = \frac{2}{3}$

Solution: Given that,

$$(y - 1)dx + (x - 3)\,dy = 0 \qquad \ldots\ldots (1)$$

So, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1 + y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1 + y - 1}{y} = \frac{y}{y} = 1 = f(y)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int dx} = e^x$$

Multiplying (1) by I.F. then

$$ye^y\,dx + xe^y(1 + y)\,dy = 0.$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int ye^y\,dx + \int 0\,dy = c$$

$$\Rightarrow ye^y \int dx + 0 = c$$

$$\Rightarrow xye^y = c.$$

(iii) $\frac{3y\cos 3x\,dx - \sin 3x\,dy}{y^2} = 0$

Solution: Given that,

$$\frac{3y\cos 3x\,dx - \sin 3x\,dy}{y^2} = 0$$

$$\Rightarrow 3y\cos 3x\,dx - \sin 3x\,dy = 0 \qquad \ldots\ldots (1)$$

Comparing (1) with Mdx + Ndy = 0 then,

$M = 3y\cos 3x$ and $N = -\sin 3x$

So, $\frac{\partial M}{\partial y} = 3\cos 3x$ and $\frac{\partial N}{\partial x} = -3\cos 3x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3\cos 3x + 3\cos 3x}{-\sin 3x} = -\frac{6\cos 3x}{\sin 3x} = -6\cot 3x = f(y)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx}\, e^{\int dx} = e^{-6\int \cot 3x\,dx}$$

$$= e^{-\frac{6}{3}\log(\sin 3x)} = e^{\log(\sin 3x)^{-2}} = (\sin 3x)^{-2}$$

Multiplying (1) by I.F. then

$$3y\cos 3x \frac{1}{\sin^2 3x}\,dx - \frac{\sin 3x}{\sin^2 3x}\,dy = 0.$$

$$\Rightarrow 3y\cot 3x \,.\, \operatorname{cosec} 3x\,dx - \operatorname{cosec} 3x\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int 3y\cot 3x.\operatorname{cosec} 3x\,dx + \int 0\,dy = c$$

$$\Rightarrow 3y\frac{\operatorname{cosec} 3x}{3} + 0 = c$$

$$\Rightarrow \frac{y}{\sin 3x} = c$$

$$\Rightarrow y = c\sin 3x.$$

(viii) $x^{-1}\cosh y\,dx + \sinh y\,dy = 0$ ...(i)

Solution: Given that,

$$x^{-1}\cosh y\,dx + \sinh y\,dy = 0 \qquad ...(1)$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = x^{-1}\cosh y \quad \text{and} \quad N = \sinh y$$

So, $\dfrac{\partial M}{\partial y} = \dfrac{1}{x}\sinh y$ and $\dfrac{\partial N}{\partial x} = 0$

Thus, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\frac{1}{x}\sin hy}{\sin hy} = \frac{1}{x}$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx}\; e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Multiplying (1) by I.F. then

$$\frac{1}{x} \times x\cosh y\,dx + x\sinh y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int \cosh y\,dx + \int 0\,dy = c$$

$$\Rightarrow \cosh y \int dx = c$$

$$\Rightarrow x\cosh y = c.$$

B. Solve the following differential equation:

(i) $(1 + x^2)\,dy + 2xy\,dx = 0$

Solution: Given that,

$$(1 + x^2)\,dy + 2xy\,dx = 0 \qquad ......(1)$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = (1 + x^2) \quad \text{and} \quad N = 2xy$$

So, $\dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$

Thus, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. So, (i) is exact. So its solution is

$$\int M dx + \int (\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int 2xy\,dx + \int dy = c$$

$$\Rightarrow 2y \times \frac{x^2}{2} + y = c.$$

$$\Rightarrow y(1 + x^2) = c.$$

(ii) $y\,dx + x(1 + y)\,dy = 0$

Solution: Given that,

$$y\,dx + x(1 + y)\,dy = 0 \qquad ......(1)$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = y \quad \text{and} \quad N = x(1 + y)$$

(vi) $(2\cos y + 4x^2)\,dx = x\sin y\,dy$

Solution: Given that,

$$(2\cos y + 4x^2)dx - x\sin y\,dy = 0 \qquad \text{...... (1)}$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = 2\cos y + 4x^2 \quad \text{and} \quad N = -x\sin y$$

So, $$\frac{\partial M}{\partial y} = -2\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin y$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2\sin y + \sin y}{-x\sin y} = \frac{-\sin y}{-x\sin y} = \frac{1}{x} = f(x)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Multiplying (1) by I.F. then

$$(2x\cos y + 4x^3)\,dx - x^2\sin y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int(2x\cos y + 4x^3)\,dx + \int 0\,dy = c$$

$$\Rightarrow 2\cos y\int x\,dx + 4\int x^3\,dx = c$$

$$\Rightarrow 2\cos y\,\frac{x^2}{2} + 4\frac{x^4}{4} = c$$

$$\Rightarrow c = x^2\cos y + x^4$$

(vii) $2x\tan y\,dx + \sec^2 y\,dy = 0$

Solution: Given that,

$$2x\tan y\,dx + \sec^2 y\,dy = 0 \qquad \text{...... (i)}$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = 2x\tan y \quad \text{and} \quad N = \sec^2 y$$

So, $$\frac{\partial M}{\partial y} = 2x\sec^2 y, \quad \text{and} \quad \frac{\partial N}{\partial x} = 0.$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x\sec^2 y - 0}{\sec^2 y} = 2x.$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying (1) by I.F. then

$$2xe^{x^2} - \tan y\,dx + e^{x^2}\sec^2 y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int 2xe^{x^2}\tan y\,dx + \int 0\,dy = c$$

$$\Rightarrow \tan y\int 2x\,e^{x^2}\,dx = c$$

$$\Rightarrow e^{x^2}\tan y = c \qquad \left[\text{put } u = x^2,\ e^u\frac{dy}{dx} = 2x \Rightarrow du = 2x\,dx\right]$$

Multiplying (1) by I.F. then

$$y\frac{\cos x}{\sqrt{\sin x}}dx + 2\frac{\sin x}{\sqrt{\sin x}}dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\, dy = c$$

$$\Rightarrow \int y\frac{\cos x}{\sqrt{\sin x}}dx + \int 0\, dy = c$$

$$\Rightarrow y\int \frac{\cos x}{\sqrt{\sin x}}dx = c$$

Put v = sin x dv = cos dx

$$y\int\frac{dv}{\sqrt{v}} = c \Rightarrow y\frac{v^{1/2}}{\frac{1}{2}} = c$$

$$\Rightarrow 2y\, v^{1/2} = c$$

$$\Rightarrow 2y\sqrt{\sin x} = c$$

$$\Rightarrow y^2 \sin x = c.$$

(v) 2 cosh x cosy dx = sinh x sin y dy

Solution: Given that,

$$2\cosh x \cos y\, dx - \sinh x \sin y\, dy = 0 \qquad \text{...... (1)}$$

Comparing (1) with Mdx + Ndy = 0 then,

$$M = 2\cosh x \cos y \quad \text{and} \quad N = -\sinh x \sin y$$

So, $\frac{\partial M}{\partial y} = -2\cosh x \sin y$ and $\frac{\partial N}{\partial x} = -\cosh x \sin y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2\cosh x \sin y + \cosh x \sin y}{-\sinh x \sin y}$$

$$= \frac{-\cosh x \sin y}{-\sinh x \sin y}$$

$$= \coth x$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int \coth x\, dx} = e^{\log \sinh x} = \sinh x$$

Multiplying (1) by I.F. then

$$2\cosh x \sinh x \cos y\, dx - \sinh^2 x \sin y\, dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\, dy = c$$

$$\Rightarrow \int 2\cosh x \sinh x \cos y + \int 0\, dy = c$$

$$\Rightarrow 2\cos y\int \cosh x \sinh x\, dx = c$$

Put, v = sinh x. So, dv = cosh x dx. So that,

$$2\cos y\int v\, dv = c$$

$$\Rightarrow 2\cos y\frac{v^2}{2} = c$$

$$\Rightarrow \sinh^2 x \cos y = c.$$

$$\Rightarrow \frac{y}{x^2}dx + \left(-\frac{1}{x}\right)dy.$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c_1$$

$$\Rightarrow \int\frac{y}{x^2}du + \int 0\,dy = c_1$$

$$\Rightarrow y\frac{-1}{x} = c_1$$

$$\Rightarrow c_1 = -\frac{y}{x}$$

$$\Rightarrow y = cx \quad \text{for } c = -c_1.$$

(iii) $\mathbf{2dx - e^{y-x}\,dy = 0.}$

Solution: Given that,

$$2dx - e^{y-x}\,dy = 0$$

$$\Rightarrow 2dx - e^y . e^{-x}\,dy = 0 \quad \text{...... (1)}$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = 2 \quad \text{and} \quad N = -e^y e^{-x}$$

So, $\frac{\partial M}{\partial y} = 0$, and $\frac{\partial N}{\partial x} = e^y e^{-x}$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{0 - e^y e^{-x}}{-e^y e^{-x}} = \frac{-e^y e^{-x}}{-e^y e^{-x}} = 1$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int dx} = e^x$$

Multiplying (1) by I.F. then

$$2e^x\,dx - e^y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \int 2e^x\,dx + \int(-e^y)\,dy = c$$

$$\Rightarrow 2e^x - e^y = c.$$

(iv) $\mathbf{y\cos x\,dx + 2\sin x\,dy = 0.}$

Solution: Given that,

$$y\cos x\,dx + 2\sin x\,dy = 0 \quad \text{...(1)}$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = y\cos x \quad \text{and} \quad N = 2\sin x$$

So, $\frac{\partial M}{\partial y} = \cos x$, and $\frac{\partial N}{\partial x} = 2\cos x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\cos x - 2\cos x}{2\sin x} = -\frac{1}{2}\cot x = f(x)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int\frac{1}{2}\cot x} = e^{-\frac{1}{2}\log\sin x} = (\sin x)^{-1/2} = \frac{1}{\sqrt{\sin x}}$$

$$\Rightarrow \quad \frac{y}{x^2}dx + \left(-\frac{1}{x}\right)dy.$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c_1$$

$$\Rightarrow \quad \int\frac{y}{x^2}du + \int 0\,dy = c_1$$

$$\Rightarrow \quad y\frac{-1}{x} = c_1$$

$$\Rightarrow \quad c_1 = -\frac{y}{x}$$

$$\Rightarrow \quad y = cx \qquad \text{for } c = -c_1.$$

(iii) $2dx - e^{y-x}\,dy = 0$.

Solution: Given that,

$$2dx - e^{y-x}\,dy = 0$$

$$\Rightarrow \quad 2dx - e^y \cdot e^{-x}\,dy = 0 \qquad \ldots\ldots (1)$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = 2 \quad \text{and} \quad N = -e^y e^{-x}$$

So, $\frac{\partial M}{\partial y} = 0$, and $\frac{\partial N}{\partial x} = e^y e^{-x}$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{0 - e^y e^{-x}}{-e^y e^{-x}} = \frac{-e^y e^{-x}}{-e^y e^{-x}} = 1$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int dx} = e^x$$

Multiplying (1) by I.F. then

$$2e^x\,dx - e^y\,dy = 0$$

Which is exact differential equation, so its solution is

$$\int Mdx + \int(\text{terms of N not containing})\,dy = c$$

$$\Rightarrow \quad \int 2e^x\,dx + \int(-e^y)\,dy = c$$

$$\Rightarrow \quad 2e^x - e^y = c.$$

(iv) $y\cos x\,dx + 2\sin x\,dy = 0$.

Solution: Given that,

$$y\cos x\,dx + 2\sin x\,dy = 0 \qquad \ldots(1)$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = y\cos x \quad \text{and} \quad N = 2\sin x$$

So, $\frac{\partial M}{\partial y} = \cos x$, and $\frac{\partial N}{\partial x} = 2\cos x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (1) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\cos x - 2\cos x}{2\sin x} = -\frac{1}{2}\cot x = f(x)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int\frac{1}{2}\cot x} = e^{-\frac{1}{2}\log\sin x} = (\sin x)^{-1/2} = \frac{1}{\sqrt{\sin x}}$$

Exercise 6.3

A. Find I.F and solve

(i) $3ydx + 2xdy = 0$

Solution: Given that,

$$3ydx + 2xdy = 0 \qquad \ldots\ldots (1)$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = 3y \quad \text{and} \quad N = 2x$$

So, $$\frac{\partial M}{\partial y} = 3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2.$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3-2}{2x} = \frac{1}{2x}$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2}\log x} = x^{1/2}$$

Multiplying (1) by I.F. then

$$3y\,x^{1/2}\,dx + 2x\;x^{1/2}\,dy = 0.$$

Which is exact differential equation, so its solution is

$$\int M\,dx + \int (\text{terms of N not containing x})\,dy = 2c^2$$

$$\Rightarrow \int 3x^{1/2}y\,dx + \int 0\,dy = 2c^2$$

$$\Rightarrow 3y\,\frac{2}{3}x^{3/2} = 2c^2$$

$$\Rightarrow 2c^2 = 2x^{3/2}y$$

$$\Rightarrow c = x^3y^2.$$

(ii) $xdy - ydx = 0$

Solution: Given that,

$$xdy - ydx = 0$$

$$\Rightarrow -y\,dx + xdy = 0$$

$$\Rightarrow ydx + (-x)\,dy = 0 \qquad \ldots\ldots (1)$$

Comparing (1) with $Mdx + Ndy = 0$ then,

$$M = y \quad \text{and} \quad N = -x$$

So, $$\frac{\partial M}{\partial y} = 1, \quad \text{and} \quad \frac{\partial N}{\partial x} = -1$$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1+1}{-x} = -\frac{2}{x} = f(x)$$

Here, the integrating factor (I.F.) of (1) is,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^{-2} = \frac{1}{x^2}$$

Multiplying (1) by I.F. then

$$\frac{y}{x^2}\,dx + \left(-\frac{x}{x^2}\right)dy = 0$$

(vi) $x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$

Solution: Given equation is

$$x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sqrt{1+\frac{y^2}{x^2}} \qquad \text{.....(1)}$$

This is homogeneous equation.

Put $y = xu$ then $\frac{dy}{dx} = x\frac{du}{dx} + u$. Then (1) becomes

$$x\frac{du}{dx} + u = u - \sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = -\frac{dx}{x}$$

Integrating we get

$$\log(u + \sqrt{u^2+1}) = -\log(x) + \log c$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2}) + c\right]$$

$$\Rightarrow u + \sqrt{u^2+1} = \frac{c}{x}$$

$$\Rightarrow \frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = \frac{c}{x}$$

$$\Rightarrow y + \sqrt{x^2+y^2} = c$$

(vii) $\mathbf{x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx}$

Solution: Given equation is

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \sqrt{1+\frac{y^2}{x^2}} \qquad \text{........(1)} \quad [\because \text{dividing by } x\,dx]$$

This is homogeneous equation.

Put $\frac{y}{x} = u$ then $y = xu$. So, $\frac{dy}{dx} = u + x\frac{dy}{dx}$. Then (1) becomes,

$$u + x\frac{du}{dx} - u = \sqrt{1+u^2}$$

$$\Rightarrow x\frac{du}{dx} = \sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = \frac{dx}{x}$$

Integrating we get,

$$\log(u + \sqrt{1+u^2}) = \log(x) + \log(c)$$

$$\Rightarrow u + \sqrt{1+u^2} = cx$$

$$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$$

(vi) $x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$

Solution: Given equation is

$$x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sqrt{1+\frac{y^2}{x^2}} \quad \text{.....(1)}$$

This is homogeneous equation.

Put $y = xu$ then $\frac{dy}{dx} = x\frac{du}{dx} + u$. Then (1) becomes

$$x\frac{du}{dx} + u = u - \sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = -\frac{dx}{x}$$

Integrating we get

$$\log(u + \sqrt{u^2+1}) = -\log(x) + \log c$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2}) + c\right]$$

$$\Rightarrow u + \sqrt{u^2+1} = \frac{c}{x}$$

$$\Rightarrow \frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = \frac{c}{x}$$

$$\Rightarrow y + \sqrt{x^2+y^2} = c$$

(vii) $\mathbf{x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx}$

Solution: Given equation is

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \sqrt{1+\frac{y^2}{x^2}} \quad \text{........(1)} \quad [\because \text{dividing by } x\,dx]$$

This is homogeneous equation.

Put $\frac{y}{x} = u$ then $y = xu$. So, $\frac{dy}{dx} = u + x\frac{dy}{dx}$. Then (1) becomes,

$$u + x\frac{du}{dx} - u = \sqrt{1-u^2}$$

$$\Rightarrow x\frac{du}{dx} = \sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = \frac{dx}{x}$$

Integrating we get,

$$\log(u + \sqrt{1+u^2}) = \log(x) + \log(c)$$

$$\Rightarrow u + \sqrt{1+u^2} = cx$$

$$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$$

$$x \sin\left(\frac{y}{x}\right) dy = \left(y \sin\frac{y}{x} - x\right) dx$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) \cdot \frac{dy}{dx} = \frac{y}{x} \sin\left(\frac{y}{x}\right) - 1 \qquad \text{.......(1)}$$

This is homogeneous equation.

Put $y = ux$ then $\frac{dy}{dx} = u + x.\frac{du}{dx}$. Then (1) becomes,

$$\sin u. \left(u + x\frac{du}{dx}\right) = u \sin u - 1$$

$$\Rightarrow \quad u \sin u + x \sin u \frac{du}{dx} = u \sin u - 1$$

$$\Rightarrow \quad \sin u\, du = -\frac{dx}{x}$$

Integrating we get,

$$-\cos u = -\log(x) + c$$

$$\Rightarrow \quad \cos\left(\frac{y}{x}\right) = \log(x) - c$$

$$\Rightarrow \quad \log(x) = \cos\left(\frac{y}{x}\right) + c$$

(v) $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Solution: Given equation is

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

$$\Rightarrow \quad (1 + e^{x/y})\frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0 \qquad \text{.......(1)}$$

This is homogeneous equation.

Put $x = yu$. Then $\frac{dx}{dy} = u + y\frac{du}{dy}$. So (1) becomes,

$$(1 + e^u)\left(u + y\frac{du}{dy}\right) + e^u(1 - u) = 0$$

$$\Rightarrow \quad u + ue^u + (1 + e^u)y\frac{du}{dy} + e^u - ue^u = 0$$

$$\Rightarrow \quad \left(\frac{1 + e^u}{u + e^u}\right) du = -\frac{dy}{y} \qquad \text{......(2)}$$

Set $u + e^u = t$ then $(1 + e^u)\, du = dt$. So, (2) becomes,

$$\frac{dt}{t} = -\frac{dy}{y}$$

Integrating we get,

$$\log(t) = -\log(y) + \log(c)$$

$$\Rightarrow \quad t = \frac{c}{y}$$

$$\Rightarrow \quad u + e^u = \frac{c}{y}$$

$$\Rightarrow \quad \frac{x}{y} + e^{x/y} = \frac{c}{y}$$

$$\Rightarrow \quad x + y\, e^{x/y} = c.$$

$$\Rightarrow (1-v)\left(v + x\frac{dv}{dx}\right) = v(1+v)$$

$$\Rightarrow (1-v)\,x\frac{dv}{dx} = v(1+v) - v(1-v) = v(1+v-1+v) = 2v^2$$

$$\Rightarrow \frac{1-v}{v^2}dv = \frac{2dx}{x}$$

$$\Rightarrow \left(v^{-2} - \frac{1}{v}\right)dv = 2\frac{dx}{x}$$

Integrating we get,

$$\frac{v^{-1}}{-1} - \log(v) = 2\log(x) + \log c$$

$$\Rightarrow -\frac{1}{v} - \log(v) = 2\log(x) + \log c$$

$$\Rightarrow -\frac{x}{y} - \log\left(\frac{y}{x}\right) = 2\log(x) + \log(c)$$

$$\Rightarrow -\frac{x}{y} - \log(y) + \log(x) = 2\log(x) + \log(c)$$

$$\Rightarrow -\frac{x}{y} - \log(y) = \log(x) + \log(c)$$

$$\Rightarrow -\frac{x}{y} = \log(cxy)$$

$$\Rightarrow cxy = e^{-x/y}$$

This is the solution of given equation.

(iii) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Solution: Given equation is

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \quad \text{.........} (1)$$

This is homogeneous equation.

So, put $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$. Then (1) becomes,

$$u + x\frac{du}{dx} = u + \tan u$$

$$\Rightarrow x\frac{du}{dx} = \tan u$$

$$\Rightarrow \left(\frac{\cos u}{\sin u}\right)du = \frac{dx}{x}$$

Integrating we get,

$$\log(\sin u) = \log(x) + \log(c)$$

$$\Rightarrow \sin u = cx$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = cx$$

(iv) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$

Solution: Given equation is

Since we have $y(2) = 4$. So, (2) gives,

$$\sqrt{16+16} = c.4 \Rightarrow 4\sqrt{2} = c.4 \Rightarrow c = \sqrt{2}$$

Therefore (2) becomes,

$$\sqrt{y^2 + 4x^2} = \sqrt{2}.\, x^2$$

$$\Rightarrow y^2 + 4x^2 = 2x^4 \qquad [\because \text{squaring on both sides}]$$

$$\Rightarrow y = \sqrt{2x^4 - 4x^2}$$

This is the solution of given equation.

C. Find the solution of the following homogeneous differential equations.

(i) $\frac{dy}{dx} + \frac{y}{x} = \left(\frac{y}{x}\right)^2$

Solution: Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = \left(\frac{y}{x}\right)^2 \qquad \text{......(1)}$$

This is homogeneous equation of first order.

Put, $y = xu$. Then $\frac{dy}{dx} = u + x.\frac{du}{dx}$. Then (1) becomes,

$$u + x.\frac{du}{dx} + u = u^2$$

$$\Rightarrow x.\frac{du}{dx} = u^2 - 2u$$

$$\Rightarrow \frac{du}{u^2 - 2u} = \frac{dx}{x}$$

$$\Rightarrow \frac{du}{u(u-2)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{u-2} - \frac{1}{u}\right) du = \frac{dx}{x}$$

Integrating we get,

$$\frac{1}{2}[\log(u-2) - \log(u)] = \log(x) + \log(c_1)$$

$$\Rightarrow \log\left(\frac{u-2}{u}\right) = 2\log(c_1x)$$

$$\Rightarrow \frac{u-2}{u} = cx^2 \qquad \text{for } c_1^2 = c$$

$$\Rightarrow \frac{y-2x}{y} = cx^2$$

$$\Rightarrow y - 2x = cx^2y$$

This is the solution of given equation.

(ii) $\mathbf{x(x-y)\,dy = y(x+y)dx}$

Solution: Given equation is,

$$x(x-y)\,dy = y(x+y)\,dx$$

$$\Rightarrow x(x-y)\frac{dy}{dx} = y(x+y) \qquad \text{..........(i)}$$

This is a homogeneous equation. So, put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So that,

$$x(x - vx)\left(v + x\frac{dv}{dx}\right) = vx(x + vx)$$

$$\Rightarrow \quad x^2u' = 3x^4\cos^2 u$$
$$\Rightarrow \quad u' = 3x^2\cos^2 u$$
$$\Rightarrow \quad \sec^2 u\, du = 3x^2 dx$$

Taking integration on both sides,

$$\int \sec^2 u\, du = \int 3x^2 dx$$
$$\Rightarrow \quad \tan u = x^3 + C$$
$$\Rightarrow \quad \tan\left(\frac{y}{x}\right) = x^3 + C \qquad \text{.... (2)}$$

Since we have, $y(1) = 0$. So, (2) gives,

$$\tan 0 = 1 + C \Rightarrow C = -1.$$

Therefore (2) becomes,

$$\tan\left(\frac{y}{x}\right) = x^3 - 1$$
$$\Rightarrow \quad \frac{y}{x} = \tan^{-1}(x^3 - 1)$$
$$\Rightarrow \quad y = x\tan^{-1}(x^3 - 1).$$

(vi) $xyy' = 2y^2 + 4x^2, \; y(2) = 4$

Solution: Given equation is

$$xyy' = 2y^2 + 4x^2, \quad y(2) = 4$$

Here,

$$xy\,y' = 2y^2 + 4x^2$$
$$\Rightarrow \quad \left(\frac{y}{x}\right)y' = 2\left(\frac{y}{x}\right)^2 + 4 \qquad \text{....... (1)} \quad [\because \text{dividing by } x^2]$$

This is a homogeneous equation. So, put $y = ux$ then,

$$y' = u + x\,u'.$$

Then (1) becomes,

$$u(u + xu') = 2u^2 + 4$$
$$\Rightarrow \quad u^2 + xu\,u' = 2u^2 + 4$$
$$\Rightarrow \quad xu\,u' = u^2 + 4$$
$$\Rightarrow \quad \left(\frac{u}{u^2 + 4}\right) du = \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{2}\left(\frac{2u}{u^2 + 4}\right) du = \frac{dx}{x}$$

Integrating we get,

$$\frac{1}{2}\log(u^2 + 4) = \log(x) + \log(c)$$
$$\Rightarrow \quad \sqrt{u^2 + 4} = cx$$
$$\Rightarrow \quad \sqrt{\left(\frac{y}{x}\right)^2 + 4} = cx$$
$$\Rightarrow \quad \sqrt{y^2 + 4x^2} = cx^2 \qquad \text{...... (2)}$$

$$\Rightarrow \quad u^2 + 5u = -x + c$$
$$\Rightarrow \quad (x-2y)^2 + 5(x-2y) = -x + c$$
$$\Rightarrow \quad (x-2y)^2 + 6x - 10y = c \qquad \text{...... (2)}$$

Since we have, $y(2) = 2.5$. So, (2) gives

$$(-3)^2 + 12 - 25 = c \Rightarrow c = -4.$$

Therefore, (2) becomes,

$$(x-2y)^2 + 6x - 10y + 4 = 0.$$

This is the solution of given equation.

(iv) $y' - x\tan(y-x) = 1,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$y' - x\tan(y-x) = 1 \qquad \text{.......... (1)}$$

with $\qquad y(0) = \frac{\pi}{2} \qquad$ ....(2)

Put $y - x = u$ then $y' - 1 = u'$ then (1) becomes

$$u' + 1 - x\tan u = 1$$
$$\Rightarrow \quad u' - x\tan u = 0$$
$$\Rightarrow \quad \frac{du}{\tan u} = x\,dx$$

Taking integration on both sides,

$$\int \frac{du}{\tan u} = \int x\,dx$$
$$\Rightarrow \quad \log(\sin u) = \frac{x^2}{2} + \frac{C}{2}$$
$$\Rightarrow \quad 2\log(\sin(y-x)) = x^2 + C \qquad \text{......... (3)}$$

Since $y(0) = \frac{\pi}{2}$ then (3) gives us,

$$2\log\left(\sin\frac{\pi}{2}\right) = 0 + C$$
$$\Rightarrow \quad C = 0 \qquad [\because \log(1) = 0]$$

Therefore (3) becomes,

$$\log(\sin(y-x)) = \frac{x^2}{2}$$
$$\Rightarrow \quad \sin(y-x) = e^{x^2/2}$$

This is the solution of given equation.

(v) $xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right),\ y(1) = 0.$

Solution: Given equation is

$$xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right) \qquad \text{with} \qquad y(1) = 0$$

Here,

$$xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right) \qquad \text{..... (1)}$$

Put, $y = xu$ then $y' = u + xu'$. Then (1) becomes,

$$x(u + xu') = xu + 3x^4\cos^2 u$$

(ii) $y' = \frac{y-x}{y-x-1}$, $y(-5) = 5$

Solution: Given equation is

$$y' = \frac{y-x}{y-x-1} \qquad \text{with } y(-5) = 5$$

Here,

$$y' = \frac{y-x}{y-x-1} = \frac{(y-x)}{(y-x)-1} \qquad \text{...... (i)}$$

Put, $y - x = u$ then $y' - 1 = u'$. So, (1) becomes,

$$1 + u' = \frac{u}{u-1}$$

$$\Rightarrow \quad u' = \frac{u}{u-1} - 1 = \frac{1}{u-1}$$

$$\Rightarrow \quad (u-1)\,du = dx$$

Taking integration on both sides,

$$\int (u-1)\,du = \int dx$$

$$\Rightarrow \quad \frac{u^2}{2} - u = x + A$$

$$\Rightarrow \quad u^2 - 2u = 2x + 2A$$

$$\Rightarrow \quad (y-x)^2 - 2(y-x) = 2x + C \quad \text{for } 2A = C$$

$$\Rightarrow \quad (y-x)^2 - 2y = C \qquad \text{...... (2)}$$

Since we have $y(-5) = 5$. So, (2) gives

$$(5+5)^2 - 10 = C$$

$$\Rightarrow \quad C = 90.$$

Therefore, (2) becomes,

$$(y-x)^2 - 2y = 90.$$

This is the solution of given equation.

(iii) $(2x - 4y + 5)y' + (x - 2y + 3) = 0$, $y(2) = 2.5$

Solution: Given equation is

$$(2x - 4y + 5)y' + (x - 2y + 3) = 0 \qquad \text{with } y(2) = 25$$

Here,

$$(2x - 4y + 5)y' + (x - 2y + 3) = 0$$

$$\Rightarrow \quad y' = \frac{x - 2y + 3}{2(x-2y) + 5} \qquad \text{..... (1)}$$

Put $x - 2y = u$ then $1 - 2y' = u' \Rightarrow y' = \frac{1-u'}{2}$. Then (1) becomes,

$$\frac{1-u'}{2} = \frac{u+3}{2u+5}$$

$$\Rightarrow \quad 1 - u' = \frac{2u+6}{2u+5}$$

$$\Rightarrow \quad u' = 1 - \frac{2u+6}{2u+5} = \frac{-1}{2u+5}$$

$$\Rightarrow \quad (2u+5)\,du = -\,dx$$

Taking integration on both sides,

$$\int (2u+5)\,du = -\int dx$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{1}{x^2+y^2}\left(\frac{x\,dy - y\,dx}{dx}\right)$$

$$\Rightarrow r^2 d\theta = x\,dy - y\,dx$$

Then (1) becomes,

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{1-r^2}{r^2}}$$

$$\Rightarrow \frac{dr}{d\theta} = \sqrt{1-r^2}$$

$$\Rightarrow \frac{dr}{\sqrt{1-r^2}} = d\theta$$

Taking integration on both sides,

$$\int \frac{dr}{\sqrt{1-r^2}} = \int d\theta$$

$$\Rightarrow \sin^{-1}(r) = \theta + C$$

$$\Rightarrow r = \sin(\theta + C)$$

$$\Rightarrow r^2 = \sin^2(\theta + C)$$

$$\Rightarrow x^2 + y^2 = \sin^2\left(\tan^{-1}\left(\frac{y}{x}\right) + C\right).$$

This is the general solution of given equation (1).

B. Solve the following initial value problem.

(i) $2x^2yy' = \tan(x^2y^2) - 2xy^2,\ y(1) = \sqrt{\frac{\pi}{2}}$

Solution: Given equation is

$$2x^2yy' = \tan(x^2y^2) - 2xy^2 \quad \text{with } y(1) = \sqrt{\frac{\pi}{2}}$$

Here, $\quad 2x^2yy' = \tan(x^2y^2) - 2xy^2$

$$\Rightarrow 2x^2yy' + 2xy^2 = \tan(x^2y^2) \qquad \text{......... (1)}$$

Put, $x^2y^2 = u$ then $2x^2y\,y' + 2xy^2 = u'$. Then (1) becomes,

$$u' = \tan u \quad \Rightarrow \frac{du}{\tan u} = dx$$

$$\Rightarrow \left(\frac{\cos u}{\sin u}\right) du = dx$$

Taking integration on both sides,

$$\int \left(\frac{\cos u}{\sin u}\right) du = \int dx$$

$$\Rightarrow \log(\sin u) = x + C$$

$$\Rightarrow \sin u = e^{(x+C)}$$

$$\Rightarrow \sin(x^2y^2) = e^{x+C} \qquad \text{......(2)}$$

Since, we have $y(1) = \sqrt{\frac{\pi}{2}}$. So, (1) gives

$$\sin\left(\frac{\pi}{2}\right) = e^{1+C}$$

$$\Rightarrow 1 = e^{1+C} \Rightarrow 1 + C = \log(1) = 0 \Rightarrow C = -1.$$

Thus, (2) becomes, $\quad \sin(x^2y^2) = e^{x-1}$.

This is the solution of given equation.

Put $x + y = u$ then $1 + \frac{dy}{dx} = \frac{du}{dx}$. Then (1) becomes,

$$(u^2 + 1)\left(\frac{du}{dx} - 1\right) = u$$

$$\Rightarrow \quad (u^2 + 1)\frac{du}{dx} = (u + u^2 + 1)$$

$$\Rightarrow \quad \left(\frac{u^2 + 1}{u^2 + u + 1}\right) du = dx$$

$$\Rightarrow \quad \left(1 - \frac{u}{u^2 + u + 1}\right) du = dx$$

Taking integration on both sides,

$$\int du - \int \frac{u\,du}{u^2 + u + 1} = \int dx \qquad \text{.......(2)}$$

Here,

$$\int \frac{u\,du}{u^2 + u + 1} = \int \left(\frac{2u + 1 - 1}{u^2 + u + 1}\right) du + \int \frac{du}{\left(u + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u + \frac{4}{2}}{\frac{\sqrt{3}}{2}}\right) + A = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + A$$

Then (2) becomes,

$$u - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) = x + C$$

$$\Rightarrow \quad x + y - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 2y + 1}{\sqrt{3}}\right) = x + C$$

$$\Rightarrow \quad y - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 2y + 1}{\sqrt{3}}\right) = C.$$

This is the general solution of given equation (1).

(xvi) $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{1 - (x^2 + y^2)}{x^2 + y^2}}$

Solution: Here,

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - (x^2 + y^2)}{x^2 + y^2}} \qquad \text{........ (1)}$$

Put $x = r\cos\theta$, $y = r\sin\theta$. Then $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Then,

$$2r\frac{dr}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \quad r\,dr = x\,dx + y\,dy$$

and

$$\frac{d\theta}{dx} = \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2}\right)\left(\frac{x\frac{dy}{dx} - y}{x^2}\right)$$

$$\Rightarrow \frac{du}{(e^u - 2)} = dx$$

$$\Rightarrow \frac{du}{(e^u - 2)} = dx$$

$$\Rightarrow \frac{1}{2}\left(\frac{2e^{-u}\,du}{(1 - 2e^{-u})}\right) = dx$$

Taking integration we get

$$\frac{1}{2}\int\left(\frac{2e^{-u}\,du}{(1 - 2e^{-u})}\right) = \int dx$$

$$\Rightarrow \frac{1}{2}\log(1 - 2e^{-u}) = x + C$$

$$\Rightarrow 1 - 2e^{-u} = Ae^{2x} \qquad \text{for } A = e^{2C}$$

$$\Rightarrow 1 - 2e^{-(x+y)} = Ae^{2x}$$

This is the general solution of given equation (1).

(xiv) $\frac{dy}{dx} + 1 = e^{x-y}$

Solution: Given equation is

$$\frac{dy}{dx} + 1 = e^{x-y} \qquad \text{........ (1)}$$

Put $x - y = u$ then $1 - \frac{dy}{dx} = \frac{du}{dx}$. Then (1) becomes,

$$1 - \frac{du}{dx} + 1 = e^u$$

$$\Rightarrow \frac{du}{dx} = 2 + e^u$$

$$\Rightarrow \frac{du}{(2 + e^u)} = dx$$

Taking integration we get

$$\int\frac{du}{(2 + e^u)} = \int dx$$

$$\Rightarrow \int\frac{e^{-u}\,du}{(2e^{-u} + 1)} = \int dx$$

$$\Rightarrow \frac{1}{2}\log(2e^{-u} + 1) = x + C$$

$$\Rightarrow 2e^{-u} + 1 = Ae^{2x} \qquad \text{for } A = e^{2C}$$

$$\Rightarrow e^{y-x} = Ae^{2x}$$

$$\Rightarrow e^y = Ae^x$$

This is the general solution of given equation (1).

(xv) $(x^2 + 2xy + y^2 + 1)\frac{dy}{dx} = x + y$

Solution: Given equation is

$$(x^2 + 2xy + y^2 + 1)\frac{dy}{dx} = x + y$$

$$\Rightarrow [(x + y)^2 + 1]\,.\,\frac{dy}{dx} = (x + y) \qquad \text{..... (1)}$$

Put $y - x = u$ then $y' - 1 = u'$. Then (1) becomes

$$1 + u' = \frac{u+1}{u+5}$$

$$\Rightarrow \quad u' = \frac{u+1}{u+5} - 1 = \frac{-4}{u+5}$$

$$\Rightarrow \quad (u+5)\,du = -4dx$$

Taking integration we get

$$\int (u+5)\,du = -4\int dx$$

$$\Rightarrow \quad \frac{u^2}{2} + 5u = -4x + A$$

$$\Rightarrow \quad u^2 + 10u = -x + 2A$$

$$\Rightarrow \quad (y-x)^2 + 10(y-x) = -x + C \qquad \text{for } 2A = C$$

$$\Rightarrow \quad (y-x)^2 + 10y - 9x = C$$

This is the general solution of given equation (1).

(xii) $y' = \dfrac{1-2y-4x}{1+y+2x}$

Solution: Given equation is

$$y' = \frac{1-2y-4x}{1+y+2x} = \frac{1-2(y+2x)}{1+(y+2x)} \qquad \text{...... (1)}$$

Put, $y + 2x = u$ then $y' + 2 = u'$. Then (1) becomes,

$$u' - 2 = \frac{1-2u}{1+u}$$

$$\Rightarrow \quad u' = \frac{1-2u}{1+u} + 2 = \frac{1-2u+2+2u}{1+u} = \frac{3}{1+u}$$

$$\Rightarrow \quad (1+u)\,du = 3dx$$

Taking integration we get

$$\int (1+u)\,du = 3\int dx$$

$$\Rightarrow \quad u + \frac{u^2}{2} = 3x + C$$

$$\Rightarrow \quad 2u + u^2 = 6x + C$$

$$\Rightarrow \quad 2y + 4x + (y+2x)^2 = 6x + c$$

$$\Rightarrow \quad (y+2x)^2 + 2y - 2x = C$$

This is the general solution of given equation (1).

(xiii) $\dfrac{dy}{dx} + 1 = e^{x+y}$

Solution: Given equation is

$$\frac{dy}{dx} + 1 = e^{x+y} \qquad \text{........ (1)}$$

Put $x + y = u$ then $1 + \dfrac{dy}{dx} = \dfrac{du}{dx}$. Then (1) becomes,

$$1 + \frac{du}{dx} + 1 = e^u$$

$$\Rightarrow \quad \frac{du}{dx} = e^u - 2$$

(ix) $y' = (y-x)^2$

Solution: Given equation is

$$y' = (y-x)^2 \qquad \text{.......(1)}$$

Put, $y - x = u$ then $y' - 1 = u'$. Then (1) becomes,

$$1 + u' = u^2$$

$$\Rightarrow \quad \frac{du}{u^2 - 1} = dx$$

Taking integration we get

$$\int \frac{du}{u^2-1} = \int dx$$

$$\Rightarrow \quad \frac{1}{2}\log\left(\frac{u-1}{u+1}\right) = x + \log(A)$$

$$\Rightarrow \quad \log\left(\frac{u-1}{u+1}\right) = 2x + \log(A^2)$$

$$\Rightarrow \quad \frac{u-1}{u+1} = A^2 e^{2x}$$

$$\Rightarrow \quad u - 1 = (u+1)\,Ce^{2x} \qquad \text{for } C = A^2$$

$$\Rightarrow \quad u(1 - ce^{2x}) = 1 + Ce^{2x}$$

$$\Rightarrow \quad u = \frac{1 + Ce^{2x}}{1 - Ce^{2x}}$$

$$\Rightarrow \quad y = x + \left(\frac{1 + Ce^{2x}}{1 - Ce^{2x}}\right)$$

This is the general solution of given equation (1).

(x) $xy' = e^{-xy} - y$

Solution: Given equation is

$$xy' = e^{-xy} - y \qquad \text{...... (1)}$$

Put $xy = u$ then $xy' + y = u'$ $\Rightarrow$ $xy' = u' - \frac{u}{x}$. Then (1) becomes,

$$u' - \frac{u}{x} = e^{-u} - \frac{u}{x}$$

$$\Rightarrow \quad u' = e^{-u}$$

$$\Rightarrow \quad e^{u}\,du = dx$$

Taking integration we get

$$\int e^{u}\,du = \int dx$$

$$\Rightarrow \quad e^{u} + x + C$$

$$\Rightarrow \quad e^{xy} = x + C$$

$$\Rightarrow \quad xy = \log(x + C)$$

$$\Rightarrow \quad y = \frac{1}{x}\log(x + C)$$

This is the general solution of given equation (1).

(xi) $y' = \dfrac{y - x + 1}{y - x + 5}$

Solution: Given equation is

$$y' = \frac{y - x + 1}{y - x + 5} = \frac{(y-x)+1}{(y-x)+5} \qquad \text{.......(1)}$$

Put $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$. Then,

$$u + x\frac{du}{dx} = \sqrt{x^2 + x^2u^2} + u \quad \Rightarrow \quad x\frac{du}{dx} = x\sqrt{1+u^2}$$

$$\Rightarrow \quad \frac{du}{\sqrt{1+u^2}} = dx$$

Integrating we get,

$$\log(u + \sqrt{1+u^2}) = x + \log c \quad \Rightarrow \quad u + \sqrt{1+u^2} = ce^x$$

$$\Rightarrow \quad \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = ce^x$$

$$\Rightarrow \quad y + \sqrt{x^2 + y^2} = cxe^x$$

(viii) $\quad \cos(x + y)\, dy = dx$

Solution: Given equation is

$$\cos(x + y)\, dy = dx \qquad \text{.......(1)}$$

Put $x + y = u$ then $1 + y' = u'$. Then (1) becomes,

$$\cos u\,(u' - 1) = 1$$

$$\Rightarrow \quad \cos u\, u' = 1 + \cos u$$

$$\Rightarrow \quad (\cos u)\frac{du}{dx} = 1 + \cos u$$

$$\Rightarrow \quad \left(\frac{\cos u}{1 + \cos u}\right)\frac{du}{dx} = 1$$

$$\Rightarrow \quad \left(\frac{\cos^2(u/2) - \sin^2(u/2)}{2\cos^2(u/2)}\right) du = dx$$

$$\Rightarrow \quad \left(1 - \tan^2\left(\frac{u}{2}\right)\right) du = 2\, dx$$

$$\Rightarrow \quad \left(2 - \sec^2\frac{u}{2}\right) du = 2\, dx$$

Taking integration we get

$$\int\left(2 - \sec^2\frac{u}{2}\right) du = 2\int dx$$

$$\Rightarrow \quad 2u - \frac{\tan(u/2)}{(1/2)} = 2x + 2C$$

$$\Rightarrow \quad 2u - 2\tan\left(\frac{u}{2}\right) = 2x + 2C$$

$$\Rightarrow \quad u - \tan\frac{u}{2} = x + C$$

$$\Rightarrow \quad x + y - \tan\left(\frac{x+y}{2}\right) = x + C$$

$$\Rightarrow \quad y - \tan\left(\frac{x+y}{2}\right) = C$$

$$\Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = y - C.$$

This is the general solution of given equation (1).

$$\Rightarrow \log(x^2+y^2) - \log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = 2A$$

$$\Rightarrow \log(x^2+y^2) - \log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = 2A$$

$$\Rightarrow \log\left(\frac{x^2+y^2}{x^2}\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = C \quad \text{for } C = 2A.$$

This is the general solution of given equation (1).

(vi) $y' = \sin(x+y) + \cos(x+y)$

Solution: Given equation is

$$y' = \sin(x+y) + \cos(x+y) \qquad \text{...... (1)}$$

Put $x + y = u$ then $1 + y' = u'$. Then (1) becomes,

$$u' - 1 = \sin u + \cos u$$

$$\Rightarrow \frac{du}{1 + \sin x + \cos x} = dx$$

Taking integration we get

$$\int \frac{du}{1 + \sin x + \cos x} = \int dx \qquad \text{...... (2)}$$

Set, $\tan\left(\frac{u}{2}\right) = t$ then $\sec^2\left(\frac{u}{2}\right)\frac{du}{2} = dt \Rightarrow du = \frac{2\,dt}{1+t^2}$. Also,

$$\sin u = \frac{2t}{1+t^2} \quad \text{and} \quad \cos u = \frac{1-t^2}{1+t^2}.$$

Then,

$$\int \frac{du}{1 + \sin x + \cos x} = \int dx$$

$$\Rightarrow \int \frac{2\,dt/(1+t^2)}{1 + \{2t//(1+t^2)\} + \{(1-t^2)/(1+t^2)\}} = \int dx$$

$$\Rightarrow \int \frac{2\,dt}{(1+t^2) + 2t + (1-t^2)} = \int dx$$

$$\Rightarrow \frac{2}{2}\int \frac{dt}{1+t} = \int dx$$

$$\Rightarrow \log(1+t) + C = x$$

$$\Rightarrow \log\left(1 + \tan\frac{u}{2}\right) + C = x$$

$$\Rightarrow \log\left(1 + \tan\left(\frac{x+y}{2}\right)\right) = x - C$$

This is the general solution of given equation (1).

(vii) $xy' - y = x\sqrt{x^2+y^2}$

Solution: Given equation is

$$xy' - y = x\sqrt{x^2+y^2}$$

$$\Rightarrow x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{x^2+y^2} + \frac{y}{x}$$

(iv) $x^2y' = y^2 + xy + x^2$

Solution: Given equation is

$$x^2y' = y^2 + xy + x^2 \qquad .....(1)$$

$$\Rightarrow \quad y' = \frac{y^2}{x^2} + \frac{y}{x} + 1 \qquad ....(2)$$

This is a homogeneous differential equation of first order. Put $y = xu$. Then, $y' = u + xu'$. Therefore (2) becomes,

$$u + xu' = u^2 + u + 1$$

$$\Rightarrow \quad xu' = u^2 + 1$$

$$\Rightarrow \quad \frac{du}{u^2+1} = \frac{dx}{x}$$

Taking integration we get

$$\int \frac{du}{u^2+1} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}(u) = \log(x) + C$$

$$\Rightarrow \quad u = \tan(\log(x) + C)$$

$$\Rightarrow \quad \frac{y}{x} = \tan(\log x + C)$$

$$\Rightarrow \quad y = x\tan(\log(x) + Cx.$$

This is the solution of (1).

(v) $y' = \frac{y-x}{y+x} = \frac{y/x-1}{y/x+1}$

Solution: Given equation is

$$y' = \frac{y-x}{y+x} = \frac{(y/x)-1}{(y/x)+1} \qquad .....(1)$$

This is a homogeneous differential equation of first order. Put $y = xu$. Then, $y' = xu' + u$. So, (1) becomes,

$$xu' + u = \frac{u-1}{u+1}$$

$$\Rightarrow \quad xu' = \frac{u-1}{u+1} - u = \frac{u-1-u^2-u}{u+1} = -\left(\frac{u^2+1}{u+1}\right)$$

$$\Rightarrow \quad \left(\frac{u+1}{u^2+1}\right) du = -\frac{dx}{x}$$

$$\Rightarrow \quad \left[\frac{1}{2}\left(\frac{2u}{u^2+1}\right) + \frac{1}{u^2+1}\right] du = -\frac{dx}{x}$$

Taking integration we get

$$\frac{1}{2}\int\left(\frac{2u}{u^2+1}\right) du + \int \frac{du}{u^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log(u^2+1) + \tan^{-1}(u) = -\log(x) + A$$

$$\Rightarrow \quad \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\log(x) + A$$

$$\Rightarrow \quad \log(x^2+y^2) - 2\log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = -2\log(x) + 2A$$

$$\Rightarrow \quad \log(x^2+y^2) - 2\log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = -\log(x^2) + 2A$$

(ii) $(x+y)^2 y' = a^2$

Solution: Given equation is

$$(x+y)^2 y' = a^2 \quad .....(1)$$

Put $x + y = u$ then $1 + y' = u'$ where $u' = \frac{du}{dx}$. Then (1) becomes,

$$u^2(u'-1) = a^2 \Rightarrow u^2 u' = a^2 + u^2$$

$$\Rightarrow \left(\frac{u^2}{a^2+u^2}\right) u' = 1$$

$$\Rightarrow \left(1 - \frac{a^2}{a^2+u^2}\right) du = dx$$

Taking integration on both sides, then

$$\int\left(1 - \frac{a^2}{a^2+u^2}\right) du = \int dx$$

$$\Rightarrow u - a^2\left(\frac{1}{a}\right)\tan^{-1}\left(\frac{u}{a}\right) = x + C$$

$$\Rightarrow u - a\tan^{-1}\left(\frac{u}{a}\right) = x + C$$

$$\Rightarrow x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + C$$

$$\Rightarrow y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{a}\right) = \frac{y-C}{a}$$

$$\Rightarrow x + y = a\tan\left(\frac{y-C}{a}\right).$$

This is the solution of (1).

(iii) $xy' = x + y$

Solution: Given equation is

$$xy' = x + y \quad .....(1)$$

$$\Rightarrow y' = 1 + \frac{y}{x} \quad .....(2)$$

This is a homogeneous differential equation of first order. Put $y = xu$. Then, $y' = u + xu'$. Therefore (2) becomes,

$$u + xu' = 1 + u$$

$$\Rightarrow x\frac{du}{dx} = 1$$

$$\Rightarrow$$

Taking integration on both sides, then

$$\int du = \int \frac{dx}{x}$$

$$\Rightarrow u = \log(x) + C$$

$$\Rightarrow \frac{y}{x} = \log(x) + C$$

$$\Rightarrow y = x\log(x) + Cx$$

This is the general solution of (1).

Here,

$$y' = -\frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

Taking integration on both sides, then

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \log(y) = -\log(x) + \log(C)$$

$$\Rightarrow \log(xy) = \log(C)$$

$$\Rightarrow xy = C \quad \text{.....(i)}$$

Since we have $y(1) = 1$. So (i) gives,

$$(1)(1) = C \Rightarrow C = 1$$

Then (i) becomes,

$$xy = 1 \Rightarrow y = \frac{1}{x}.$$

This is the solution of (i).

Exercise 6.2

A. Find the general solution of the following equations:

(i) $(x + y + 1)y' = 1$.

Solution: Given equation is

$$(x + y + 1)y' = 1 \quad \text{... (1)}$$

Put $x + y + 1 = u$ then $1 + y' = u'$ where $u' = \frac{du}{dx}$. Then (1) becomes,

$$u(u' - 1) = 1$$

$$\Rightarrow uu' - u = 1$$

$$\Rightarrow \frac{uu'}{1+u} = 1$$

$$\Rightarrow \left(1 - \frac{1}{1+u}\right) u' = 1$$

$$\Rightarrow \left(1 - \frac{1}{1+u}\right) du = dx$$

Taking integration on both sides, then

$$\int \left(1 - \frac{1}{1+u}\right) du = \int dx$$

$$\Rightarrow u - \log(1 + u) = x + A$$

$$\Rightarrow (x + y + 1) - \log(1 + x + y + 1) = x + A$$

$$\Rightarrow y - \log(x + y + 1) = x + A - x - 1 = A + 1 = C \text{ (let)}$$

$$\Rightarrow y - \log(x + y + 2) = C$$

This is the general solution of (1).

Taking integration on both sides, then

$$\int y^{-2}\, dy = \int 2e^{-x} (1 + x)\, dx$$

$$\Rightarrow \frac{y^{-1}}{-1} = 2 \int e^{-x} (x + 1)\, dx$$

$$\Rightarrow -\frac{1}{y} = 2 \left[(x + 1)\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{(-1)^2}\right)\right] + C$$

[∵ applying integrating by parts]

$$= -2\,e^{-x}(x + 1 + 1) + C$$

$$= -2e^{x}(x + 2) + C \qquad \ldots (i)$$

Since we have $y(0) = \frac{1}{6}$ then (i) gives,

$$-6 = -2e^{0}(0 + 2) + C$$

$$\Rightarrow -6 = -4 + C$$

$$\Rightarrow C = -2$$

Therefore (i) becomes,

$$-\frac{1}{y} = -2e^{-x}(x + 2) - 2$$

$$\Rightarrow y = \frac{1}{2\,e^{-x}(x + 2) + 2}$$

This is the solution of (i).

3. $y' \cosh^2 x - \sin^2 y = 0,\ y(0) = \frac{\pi}{2}$

Solution: Given equation is $y' \cosh^2 x - \sin^2 y = 0$ with $y(0) = \frac{\pi}{2}$

Here,

$$y' \cosh^2 x - \sin^2 y = 0$$

$$\Rightarrow \frac{dy}{\sin^2 y} - \frac{dx}{\cosh^2 x} = 0$$

$$\Rightarrow \operatorname{cosec}^2 y\, dy - \operatorname{sech}^2 x\, dx = 0$$

Taking integration on both sides, then

$$\int \operatorname{cosec}^2 y\, dy - \int \operatorname{sech}^2 x\, dx = C$$

$$\Rightarrow -\cot y - \tanh x = C$$

$$\Rightarrow \cot y + \tanh x = C \qquad \ldots (i)$$

Since we have, $y(0) = \frac{\pi}{2}$. Then (i) gives,

$$\cot\left(\frac{\pi}{2}\right) + \tanh(0) = C \Rightarrow 0 + 0 = C \Rightarrow C = 0$$

Then (i) becomes,

$$\cot y + \tanh x = 0$$

$y' = -\frac{y}{x},\ y(1) = 1$

lution: Given equation is

$$y' = -\frac{y}{x} \quad \text{with } y(1) = 1$$

$$xy' = 3y \quad \ldots (2)$$

Here,

$$y = cx^3$$

So, $\quad y' = 3cx^2$

Then, $\quad xy' = 3cx.x^2 = 3cx^3 = 3y$

$$\Rightarrow \quad xy' = 3y$$

This shows that (1) is the solution of (2).

6. $x^2 + 4y^2 = c \qquad\qquad 4yy' + x = 0$

Solution: Given equation is

$$x^2 + 4y^2 = c \quad \ldots (1)$$

Then we have to show (1) is the solution of

$$4yy' + x = 0 \quad \ldots (2)$$

Here,

$$x^2 + 4y^2 = c$$

So, $\quad 2x + 8y.y' = 0$

$$\Rightarrow \quad x + 4yy' = 0$$

$$\Rightarrow \quad 4yy' + x = 0$$

This shows that (1) is the solution of (2).

D. Solve the following initial value problems.

(1) $xy' + y = 0, y(2) = -2$

Solution: Given equation is

$$xy' + y = 0 \quad \text{with } y(2) = -2$$

Here,

$$xy' + y = 0 \quad \Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

Taking integration on both sides, then

$$\int\frac{dy}{y} + \int\frac{dx}{x} = \log (C)$$

$$\Rightarrow \quad \log(y) + \log(x) = \log(c)$$

$$\Rightarrow \quad xy = c \quad \ldots (i)$$

Since we have $y(2) = -2$. So,

$$2(-2) = C \quad \Rightarrow \quad C = -4$$

Then (i) becomes,

$$xy = -4 \quad \Rightarrow \quad xy + 4 = 0$$

This is the required solution.

2. $e^x y' = 2(x + 1) y^2, y(0) = \frac{1}{6}$

Solution: Given equation is

$$e^x y' = 2(x + 1) y^2 \quad \text{with} \quad y(0) = \frac{1}{6}$$

Here,

$$e^x y' = 2(x + 1) y^2$$

$$\Rightarrow \quad \frac{dy}{y^2} = \frac{2(x + 1)}{e^x} dx$$

$$\Rightarrow \quad y^{-2} dy = 2e^{-x} (1 + x) dx$$

So, $y' = -ce^{-x} + 2x - 2$

Then,

$$y' + y = -ce^{-x} + 2x - 2 + ce^{-x} + x^2 - 2x = x^2 - 2$$
$$\Rightarrow \quad y' + y = x^2 - 2. \text{ Thus, (2) is satisfied.}$$

This shows (1) is solution of (2).

2. $\mathbf{y = e^x + ax^2 + bx + c}$ $\qquad\qquad$ $\mathbf{y''' = e^x}$

Solution: Given equation is

$$y = e^x + ax^2 + bx + c \qquad \ldots (1)$$

Then we have to show (1) is solution of

$$y''' = e^x \qquad \ldots (2)$$

Here, by (1),

$$y = e^x + ax^2 + bx + c$$

So, $y' = e^x + 2ax + b$

And, $y'' = e^x + 2a$

Also, $y''' = e^x$

This shows that (1) is the solution of (2).

3. $\mathbf{x^2 + y^2 = 1}$ $\qquad\qquad$ $\mathbf{x + y\,y' = 0.}$

Solution: Given equation is

$$x^2 + y^2 = 1 \qquad \ldots (1)$$

Then we shall show that (1) satisfies the equation

$$x + y\,y' = 0 \qquad \ldots (2)$$

Here

$$x^2 + y^2 = 1$$

Differentiating w. r. t. x, then

$$2x + 2y\,y' = 0$$
$$\Rightarrow \quad x + yy' = 0$$

This shows that (1) satisfies (2). So, (1) is the solution of (2).

4. $\mathbf{y = ce^{-2x} + 14}$ $\qquad\qquad$ $\mathbf{y' + 2y = 2.8}$

Solution: Given equation is

$$y = ce^{-2x} + 14 \qquad \ldots (1)$$

Then we have to show (1) is the solution of

$$y' + 2y = 2.8 \qquad \ldots (2)$$

Here,

$$y = ce^{-2x} + 1.4$$

So, $y' = -2ce^{-2x}$

Then,

$$y' + 2y = -2c\,e^{-2x} + 2ce^{-2x} + 2.8$$
$$\Rightarrow \quad y' + 2y = 2.8$$

This shows that (1) satisfies (2). So, (1) is the solution of (2).

5. $\mathbf{y = cx^3}$ $\qquad\qquad$ $\mathbf{xy' = 3y}$

Solution: Given equation is

$$y = cx^3 \qquad \ldots (1)$$

Then we have to show (1) is the solution of

Taking integration on both sides, then

$$L\int \frac{dI}{I} = R\int dt \qquad \ldots (1)$$

$$\Rightarrow \quad L\log(I) + Rt = C$$

Since we have $I(0) = I_0$. Then (1) gives

$$L\log(I_0) + R(0) = C$$

$$\Rightarrow \quad C = L\log(I_0)$$

Then, (1) becomes,

$$L\log(I) + Rt = L\log(I_0)$$

$$\Rightarrow \quad \log(I) + \left(\frac{R}{L}\right)t = \log I_0$$

$$\Rightarrow \quad \log(I) - \log(I_0) = -\left(\frac{R}{L}\right)t$$

$$\Rightarrow \quad \log\left(\frac{I}{I_0}\right) = \left(\frac{-R}{L}\right)t$$

$$\Rightarrow \quad I = I_0\, e^{-(Rt/L)}$$

(vi) $dr\sin\theta = 2r\cos\theta\, d\theta,\ r\left(\frac{\pi}{2}\right) = 2$

tion: Given equation is

$$dr\sin\theta = 2r\cos\theta\, d\theta \quad \text{with } r\left(\frac{\pi}{2}\right) = 2.$$

Here,

$$dr.\sin\theta = 2r\cos\theta\, d\theta$$

$$\Rightarrow \quad \frac{dr}{r} = 2\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

Taking integration on both sides, then

$$\int \frac{dr}{r} = 2\int\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$\Rightarrow \quad \log(r) = 2\log(\sin\theta) + \log(C) = \log(\sin^2\theta) + \log(C)$$

$$\Rightarrow \quad r = C\sin^2\theta \qquad \ldots (1)$$

Since we have, $r\left(\frac{\pi}{2}\right) = 2$. So, (1) gives

$$2 = C\sin^2\left(\frac{\pi}{2}\right) \quad \Rightarrow \quad c = 2$$

Then, (1) becomes,

$$r = 2\sin^2\theta.$$

C. Show that the given function is a solution of given differential equation. (Here a, b, c are arbitrary constants).

(1) $y = c\,e^{-x} + x^2 - 2x \qquad y' + y = x^2 - 2.$

Solution: Given equation is

$$y = c\,e^{-x} + x^2 - 2x \qquad \ldots (1)$$

Then we have to show (1) is solution of

$$y' + y = x^2 - 2 \qquad \ldots (2)$$

Here, by (1),

$$y = ce^{-x} + x^2 - 2x$$

Here,

$$2xy' = 3y$$

$$\Rightarrow \quad 2\frac{dy}{y} = \frac{3}{x}dx$$

Taking integration on both sides, then

$$2\int\frac{dy}{y} = 3\int\frac{dx}{x}$$

$$\Rightarrow \quad 2\log(y) = 3\log(x) + \log C$$

$$\Rightarrow \quad y^2 = Cx^3 \qquad \dots (1)$$

Since $y(1) = 4$. Then (1) gives,

$$(4)^2 = C(1)^3 \quad \Rightarrow \quad C = 16.$$

Therefore, (1) becomes,

$$y^2 = 16x^3$$

$$\Rightarrow \quad y = 4x^{3/2}$$

(iv) $xy\,y' = y + 2,\ y(2) = 0$.

Solution: Given equation is

$$xy\,y' = y + 2 \text{ with } y(2) = 0$$

Here,

$$xy\,y' = y + 2 \quad \Rightarrow \quad \left(\frac{y}{y+2}\right)dy = \frac{dx}{x}$$

$$\Rightarrow \quad \left(1 - \frac{2}{y+2}\right)dy = \frac{dx}{x}$$

Taking integration on both sides, then

$$\int\left(1 - \frac{2}{y+2}\right)dy = \int\frac{dx}{x}$$

$$\Rightarrow \quad y - 2\log(y+2) = \log(x) + \log C$$

$$= \log(Cx) \qquad \dots (1)$$

Since we have, $y(2) = 0$. Then (1) gives

$$0 - 2\log(2) = \log(2c)$$

$$\Rightarrow \quad \log(4)^{-1} = \log(2c)$$

$$\Rightarrow \quad \frac{1}{4} = 2c$$

$$\Rightarrow \quad c = \frac{1}{8}$$

Then (1) becomes,

$$y - 2\log(y+2) = \log\left(\frac{x}{8}\right).$$

(v) $Li + RI = 0,\ I(0) = I_0$ whre $i = \frac{dI}{dt}$, (L & R are constants)

Solution: Given equation is

$$Li + RI = 0 \quad \text{with } I(0) = I_0 \text{ whre } i = \frac{dI}{dt}, \quad \text{(L \& R are constants)}$$

Here, $\quad Li + RI = 0 \quad \Rightarrow \quad L\frac{dI}{dt} + RI = 0$

$$\Rightarrow \quad \frac{L\,dI}{I} + R\,dt = 0$$

B. Solve the following initial value problems.

(i) $\log(y') = 3x + 4y, y(0) = 0$

Solution: Given equation is

$$\log(y') = 3x + 4y \quad \text{with } y(0) = 0$$

Here,

$$\log(y') = 3x + 4y$$
$$\Rightarrow \quad y' = e^{3x+4y} = e^{3x} \cdot e^{4y}$$
$$\Rightarrow \quad e^{-4y}\,dy = e^{3x}\,dx$$

Taking integration on both sides, then

$$\int e^{-4y}\,dy = \int e^{3x}\,dx$$
$$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + A$$
$$\Rightarrow \quad 3e^{-4y} = -4e^{3x} - 12A$$
$$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = C \qquad \text{for } -12A = C \qquad \ldots (1)$$

Since, $y(0) = 0$. So, (1) gives,

$$4e^{0} + 3e^{-0} = C \Rightarrow 3 + 4 = C \Rightarrow C = 7.$$

Then (1) becomes

$$4e^{3x} + 3e^{-4y} = 7.$$

(ii) $y' = y\tan 2x, y(0) = 2$

Solution: Given equation is

$$y' = y\tan 2x \quad \text{with } y(0) = 2$$

Here,

$$y' = y\tan 2x$$
$$\Rightarrow \quad \frac{dy}{y} = \tan 2x\,dx$$

Taking integration on both sides, then

$$\int \frac{dy}{y} = \int \tan 2x\,dx$$
$$\Rightarrow \quad \log(y) = \frac{\log(\sec 2x)}{2} + \log A$$
$$\Rightarrow \quad 2\log(y) = \log(\sec 2x) + 2\log A$$
$$\Rightarrow \quad \log(y^2) = \log(\sec 2x) + \log C \qquad \text{for } C = A^2$$
$$\Rightarrow \quad y^2 = C\sec(2x) \qquad \ldots (1)$$

Since, $y(0) = 2$ then (1) gives,

$$(2)^2 = C\sec 0 \quad \Rightarrow \quad C = 4$$

Then (1) becomes,

$$y^2 = 4\sec 2x$$
$$\Rightarrow \quad y^2 = \frac{4}{\cos 2x}$$
$$\Rightarrow \quad y = \frac{2}{\sqrt{\cos 2x}}$$

(iii) $2xy' = 3y, y(1) = 4$

Solution: Given equation is

$$2xy' = 3y \quad \text{with } y(1) = 4$$

$$-\int\frac{dt}{t} = \int\frac{dy}{y}$$

$\Rightarrow\ -\log(t) = \log(y) - \log(C)$

$\Rightarrow\ \log(y) + \log(t) = \log(C)$

$\Rightarrow\ \log(ty) = \log(C)$

$\Rightarrow\ ty = C$

$\Rightarrow\ (1+e^{-x})y = C$

(x) $(1-x^2)(1-y)\,dx = (1+y)\,xy\,dy$

Solution: Given equation is

$$(1-x^2)(1-y)\,dx = (1+y)\,xy\,dy$$

$$\Rightarrow \left(\frac{1-x^2}{x}\right)dx = \left(\frac{1+y}{1-y}\right)y\,dy$$

$$\Rightarrow \left(\frac{1-x^2}{x}\right)dx = \left(\frac{y+y^2}{1-y}\right)dy$$

$$\Rightarrow \left(\frac{1}{x}-x\right)dx = \left(-y-2+\frac{2}{1-y}\right)dy$$

Taking integration on both sides then,

$$\int\left(\frac{1}{x}-x\right)dx = \int\left(-y-2+\frac{2}{1-y}\right)dy$$

$$\Rightarrow \log(x) - \frac{x^2}{2} = -\frac{y^2}{2} - 2y - 2\log(1-y) + C$$

$$\Rightarrow \log(x) - \frac{x^2}{2} + \frac{y^2}{2} + 2y + 2\log(1-y) = C$$

(xi) $(a^2+y^2)\,xdx + y(x^2-a^2)dy = 0$

Solution: Given equation is

$$(a^2+y^2)\,xdx + y(x^2-a^2)dy = 0$$

$$\Rightarrow \left(\frac{x}{x^2-a^2}\right)dx + \left(\frac{y}{y^2+a^2}\right)dy = 0$$

$$\Rightarrow \left(\frac{2x}{x^2-a^2}\right)dx + \left(\frac{2y}{y^2+a^2}\right)dy = 2(0) = 0$$

Taking integration on both sides,

$$\int\left(\frac{2x}{x^2-a^2}\right)dx + \int\left(\frac{2y}{y^2+a^2}\right)dy = 0$$

Put $x^2 - a^2 = u$ and $y^2 + a^2 = v$ then $2x\,dx = du$ and $2y\,dy = dv$. So,

$$\int\frac{du}{u} + \int\frac{dv}{v} = 0$$

$\Rightarrow\ \log(u) + \log(v) = A$

$\Rightarrow\ \log(uv) = A$

$\Rightarrow\ uv = e^A = C$ (let)

$\Rightarrow\ (x^2-a^2)(y^2+a^2) = C$

$$\int\left(\frac{2t}{2\sqrt{t^2}}\right)dt = \int\frac{dx}{x}$$

$$\Rightarrow \int dt = \int\frac{dx}{x}$$

$$\Rightarrow t = \log(x) + C$$

$$\Rightarrow \sqrt{y-1} = \log(x) + C$$

$$\Rightarrow y - 1 = (\log(x) + C)^2$$

$$\Rightarrow y = 1 + (\log(x) + C)^2$$

(vii) $e^{x-y}\,dx + e^{y-x}\,dy = 0$. **[2010 Spring– Short]**

Solution: Given equation is

$$e^{x-y}\,dx + e^{y-x}\,dy = 0$$

$$\Rightarrow \left(\frac{e^x}{e^y}\right)dx + \left(\frac{e^y}{e^x}\right)dy = 0$$

$$\Rightarrow e^{2x}\,dx + e^{2y}\,dy = 0$$

Taking integration on both sides then,

$$\int e^{2x}\,dx + \int e^{2y}\,dy = 0$$

$$\Rightarrow \frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{C}{2}$$

$$\Rightarrow e^{2x} + e^{2y} = C$$

(viii) $x\cos y\,dy = (x\,e^x\log(x) + e^x)\,dx$

Solution: Given equation is

$$x\cos y\,dy = (x\,e^x\log(x) + e^x)\,dx$$

$$\Rightarrow \cos y\,dy = \left(\frac{x\,e^x\log(x) + e^x}{x}\right)dx = \left[e^x\log(x) + \frac{e^x}{x}\right]dx$$

Taking integration on both sides then,

$$\int\cos y\,dy = \int e^x\left[\log(x) + \frac{1}{x}\right]dx$$

$$\Rightarrow \sin y = e^x\log(x) + C \qquad \left[\because \int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C\right]$$

(ix) $y\,dx = (1 + e^x)\,dy$

Solution: Given equation is

$$y\,dx = (1 + e^x)\,dy$$

$$\Rightarrow \frac{dx}{1 + e^x} = \frac{dy}{y}$$

$$\Rightarrow \left(\frac{e^{-x}}{(e^{-x} + 1)}\right)dx = \frac{dy}{y}.$$

Taking integration on both sides then,

$$\int\left(\frac{e^{-x}}{(e^{-x} + 1)}\right)dx = \int\frac{dy}{y}$$

Put $(e^{-x} + 1) = t$, then $(-e^{-x})\,dx = dt$. So,

$$\Rightarrow \left(\frac{1}{x}+1\right) dx + \left(\frac{1}{y}+1\right) dy = 0$$

Taking integration on both sides then,

$$\int\left(\frac{1}{x}+1\right) dx + \int\left(\frac{1}{y}+1\right) dy = 0$$

$$\Rightarrow x + \log(x) + y + \log(y) = C$$

$$\Rightarrow x + y + \log(xy) = C.$$

(iv) tany dx + tanx dy = 0

Solution: Given equation is

$$\tan y\, dx + \tan x\, dy = 0$$

$$\Rightarrow \frac{dx}{\tan x} + \frac{dy}{\tan y} = 0 \quad [\because \text{dividing both sides by } \tan x \tan y]$$

$$\Rightarrow \left(\frac{\cos x}{\sin x}\right) dx + \left(\frac{\cos y}{\sin y}\right) dy = 0$$

Taking integration on both sides then,

$$\int\left(\frac{\cos x}{\sin x}\right) dx + \int\left(\frac{\cos y}{\sin y}\right) dy = 0$$

$$\Rightarrow \log(\sin x) + \log(\sin y) = A$$

$$\Rightarrow \log(\sin x \cdot \sin y) = A$$

$$\Rightarrow \sin x \cdot \sin y = e^A = C \quad \text{(let)}$$

(v) y' = y tanh x

Solution: Given equation is

$$y' = y \tanh x \Rightarrow \frac{dy}{dx} = y \tanh x$$

$$\Rightarrow \frac{dy}{y} = \tanh x\, dx = \left(\frac{\sinh x}{\cosh x}\right) dx$$

Taking integration on both sides then,

$$\int \frac{dy}{y} = \int\left(\frac{\sinh x}{\cosh x}\right) dx$$

$$\Rightarrow \log(y) = \log(\cosh x) + \log(C)$$

$$\Rightarrow y = C \cosh x.$$

(vi) $y' = 2x^{-1}\sqrt{y-1}$

Solution: Given equation is

$$y' = 2x^{-1}\sqrt{y-1} \Rightarrow \frac{dy}{dx} = \frac{2\sqrt{y-1}}{x}$$

$$\Rightarrow \frac{dy}{2\sqrt{y-1}} = \frac{dx}{x}$$

Taking integration on both sides then

$$\int \frac{dy}{2\sqrt{y-1}} = \int \frac{dx}{x}$$

Put $(y-1) = t^2$ then $dy = 2t\, dt$. Therefore,

Exercise 6.1

A. Find the general solution of the following:

(i) $y' - 2y + a = 0$

Solution: Given equation is

$$y' - 2y + a = 0 \Rightarrow \frac{dy}{dx} = 2y - a$$

$$\Rightarrow \frac{2\,dy}{2y - a} = 2\,dx$$

Taking integration on both sides

$$\int \frac{2\,dy}{2y - a} = 2\int dx.$$

$$\Rightarrow \log(2y - a) = 2x + \log(A)$$

$$\Rightarrow \log\left(\frac{2y - a}{A}\right) = 2x$$

$$\Rightarrow \frac{2y - a}{A} = e^{2x}$$

$$\Rightarrow y = Ce^{2x} + \frac{a}{2} \quad \text{for } C = \frac{A}{2}.$$

(ii) $(x \log x)\,y'$ for $x > 0$.

Solution: Given equation is

$$(x \log x)\,y' = y \Rightarrow y' = \frac{y}{x \log x} \quad \text{for } x > 0.$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x \log x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x \log x}$$

Taking integration on both sides then,

$$\int \frac{dy}{y} = \int \frac{dx}{x \log x}$$

Put $\log(x) = t$ then $\left(\frac{1}{x}\right)dx = dt$. So,

$$\int \frac{dy}{y} = \int \frac{dt}{t}$$

$$\Rightarrow \log(y) = \log(t) + \log(C)$$

$$\Rightarrow \log(y) = \log(Ct)$$

$$\Rightarrow y = Ct$$

$$\Rightarrow y = C\log(x)$$

(iii) $(1 + x)y\,dx + (1 + y)x\,dy = 0$

Solution: Given equation is

$$(1 + x)y\,dx + (1 + y)x\,dy = 0$$

$$\Rightarrow \left(\frac{1 + x}{x}\right)dx + \left(\frac{1 + y}{y}\right)dy = 0 \quad [\because \text{dividing both sides by } xy]$$

Integrating factor:

If a differential equation is not exact, some time the equation may be exact if the equation is multiplied by a function. Such function is called an integrating factor (I. F.).

Linear differential equation:

A differential equation of the form, $y' + Py = Q$ where, P and Q are free from y, is called a linear differential equation of first order in y.

Note: If $Q \doteq 0$ in above equation, then the equation is called a homogeneous linear differential equation.

A differential equation of the form $y'' + Py' + Qy = R$ where, P, Q and R are free from y, is called homogenous linear differential equation of second order in y.

❑ **Solution process (Linear differential equation)**

Consider a diff. equation

$$\frac{dy}{dx} + Py = Q \text{ .... (i)}$$

Where P and Q are free from x.

Choose an integrating factor (I.F.) for (i) is,

$$\text{I.F.} = e^{\int P dx}$$

Multiplying (i) by I.F. then we get

$$e^{\int P dx}\frac{dy}{dx} + Py\, e^{\int P dx} = Qe^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Taking integration we get,

$$ye^{\int P dx} = \int Qe^{\int P dx}\, dx + C$$

i.e. $\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + C}$

Here (ii) is the solution of (i).

Bernoulli's equations:

A differential equation of the form $y' + Py = Qy^n$, where, P and Q are free from y, is called a Bernoulli's equation of first order in y.

❑ **Solution process (Reducible to Linear differential equation)**

Consider an equation

$$\frac{dy}{dx} + Py = Qy^n \text{ .... (i)}$$

$$\Rightarrow \left(\frac{1}{y^n}\right)\frac{dy}{dx} + \left(\frac{1}{y^{n-1}}\right)P = Q \text{ .... (ii)}$$

Put $\frac{1}{y^{n-1}} = u$ then $(1-n)\frac{1}{y^n}\frac{dy}{dx} = \frac{du}{dx}$

Then (ii) reduces to, $\frac{du}{dx} + P(1-n)u = Q(1-n)$ .... (iii)

Which is linear diff. equation of first order. This will solve as process mentioned above.

$x = X + h,\ y = Y + k$ for some fixed value of h, k.

Such that $\left.\begin{aligned} ah + bk + c = 0 \\ \text{and } Ah + Bk + C = 0 \end{aligned}\right\}$ ..... (ii)

And $\frac{dy}{dx} = \frac{d(Y+k)}{d(X+h)} = \frac{dY}{dX}$

Then (i) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{A(X+h) + B(Y+k) + C} = \frac{aX + bY}{AX + BY} \text{ ...... (iii) [}\because\text{ Using (ii)}$$

Which is homogenous linear diff. equation of first order.

Put $Y = VX$ then $\frac{dY}{dX} = V + X\frac{dV}{dX}$

Then (iii) reduces to $V + X\frac{dV}{dX} = \frac{aX + bVX}{AX + BVX} = \frac{a + bV}{A + BV}$

$\Rightarrow\quad X\frac{dV}{dX} = \frac{a + bV}{A + BV} - V = \frac{a + (b - A)V - BV^2}{A + BV}$

$\Rightarrow\quad \left(\frac{A + BV}{a + (b - A)V - BV^2}\right) dV = \frac{dX}{X}$

Taking integration, $\int \left(\frac{A + BV}{a + (b - A)V - BV^2}\right) dV = \int \frac{dX}{X}$

This gives the solution of (iii).

After substituting the value of V and then X, Y, h, k, we get the solution of (i).

Exact differential equation

A differential equation of the form, $M\,dx + N\,dy = 0$, is called exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial X}$, is satisfied.

- **Solution process (Exact differential equation)**

Consider an equation $Mdx + Ndy = 0$ .... (i)

Let (ii) is exact. Then the solution of (i) is

$$\int_{y\text{-constant}} Mdx + \int(\text{term of N free from x})\, dy = C$$

- **Solution process (Reducible of Exact differential equation)**

Consider an equation $Mdx + Ndy = 0$ .... (i)

Let (i) is not exact. That is $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Then if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is free from x then the integrating factor of (i) is

$$\text{I.F.} = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy}$$

And if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is free from y then the integrating factor of (i) is

$$\text{I.F.} = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$$

Now, multiplying (i) by I.F. then

$$(M \times \text{I.F.})\, dx + (N \times \text{I.F.})dy = 0$$

which is exact diff. equation.

❏ Solution process (Homogenous equation of first order)

Given equation is

$$y' = g\left(\frac{y}{x}\right) \text{ .... (i)}$$

Put $\frac{y}{x} = u$ i.e. $y = ux$. Then $y' = u + x\frac{du}{dx}$. Then (i) becomes

$$u + x\frac{du}{dx} = g(u)$$

$$\Rightarrow \quad x\frac{du}{dx} = g(u) - u$$

$$\Rightarrow \quad \frac{du}{g(u) - u} = \frac{dx}{x}$$

Taking integration

$$\int \frac{du}{g(u) - u} = \int \frac{dx}{x}$$

This givens the solution of (i).

❏ Solution process (Equation Reducible to Separable)

Consider an equation, $\frac{dy}{dx} = \frac{ax + by + c}{AX + By + C}$ .... (i)

Where $\frac{a}{A} = \frac{b}{B}$

Put $Ax + By = \lambda\,(ax + by)$ for same scalar value λ.

Here (i) can be written as,

$$\frac{dy}{dx} = \frac{ax + by + c}{\lambda(ax + by) + C} \text{ .... (ii)}$$

Put $ax + by = t$ then $a + b\frac{dy}{dx} = \frac{dt}{dx}$.

Then (ii) becomes,

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{\lambda.t + C}$$

$$\Rightarrow \quad \frac{dt}{dx} - a = \frac{b(t + c)}{\lambda t + C}$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{b(t + c)}{\lambda t + C} + a = \frac{b(t + c) + a(\lambda t + c)}{\lambda t + C}$$

$$\Rightarrow \quad \left(\frac{\lambda t + C}{b(t + c) + a(\lambda t + c)}\right) dt = dx \text{ .... (iii)}$$

Which is separable from. After integrating (iii) we get the solution of (i).

❏ Solution process (Equation Reducible to Homogenous form)

Consider an equation

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C} \text{ .... (i)}$$

with $\frac{a}{A} \neq \frac{b}{B}$. Then put

CHAPTER 6

ORDINARY DIFFERENTIAL EQUATIONS (ODE)

Ordinary differential equations:

An ordinary equation is an equation that involves a single independent variable and its derivatives.

Order of different equation:

The order of a differential equation is the highest number of derivative involved in the differential equation.

Degree of a differential equation:

The degree of a differential equation after it has been made free from radicals is the power of highest order derivative that involved in the differential equation.

Solution of a differential equation:

A solution of a differential equation is a relation among the variables which is free from the derivative satisfying the given differential equation.

General solution:

A general solution of a differential equation is a solution that has arbitrary constants equal in number with the order of the equation.

Particular solution:

Any solution of a differential equation obtained from general solution by giving particular values of the constants, is called the particular solution.

Initial value problem

Any equation with initial condition is an initial value problem.

Example: An equation, $(x + 1)\, y' = 2y,\ y(o) = 1$, is an initial value problem of first order.

An equation

$$y'' + y' - 2y = 0,\ y(0) = 3, y'(0) = 0$$

is an initial value problem of second order.

Homogeneous equation:

The equation of the form $y' = g\left(\frac{y}{x}\right)$ is known as homogeneous equation of first order.

A differential equation of the form $y'' + Py' + Q = 0$, is known as homogeneous equation of second order where P and Q are constants or function of independent variables of the equation.

Note: If the differential equation is not homogeneous then it is called non-homogeneous.