Probability And Queing Theory

Probability:

① Experiment → Deterministic experiment
↳ Exp's that gives the predetermined outcomes. eg:- $H_2 + O_2 \rightarrow H_2O$.

→ probabilistic (or Random) experiment
eg:- Tossing a coin
where the possible outcomes: (i) either head (ii) or tail.

② Event (Outcomes): Result of trial

Sample space (S or Ω): set of possible outcomes
eg:- When we toss a coin twice.
⇒ So, the possible outcomes are: TT, HT, TH, HH

Trial: Individual performance of any experiment is called Trial. If we perform exp's once, trial = 1.

→ Simple Event :- Event having single outcomes.
Eg:- Rolling a die. possible outcomes $= \{1, 2, 3, 4, 5, 6\}$

→ Compound (Complex) Event
The event that have more than 1 result for an exp's is compound event.
Eg:- While rolling 2 die, the possible outcome $= \{1, 2, 3, 4, 5, 6\}$
Here, the possible outcomes to get odd outcome $= 3$

4) Equally likely Events

⇒ 2 or more Events are said to be equally likely events if they have same chances of occurence.

for eg:- In a single coin toss, the possible outcomes of getting head = 1 and getting tail is 1.

Hence, probability = $1/2$.

5) Mutually exclusive event

⇒ The event where if one event comes, no other outcomes is possible at the same time, it is said to be mutually exclusive event.

Eg:- getting head and tail at the same time is mutually exclusive event.

6) Exhaustive Cases or Events (n)

Total number of all possible outcomes of an experiment. for eg: events or exhaustive cases = 2 for a single coin toss.

7) Favourable cases or favourable events

⇒ The total no. of cases that favour the particular need of an event is called fav cases.

Eg:- Rolling a die

$S = \{1, 2, 3, 4, 5, 6\}$

Exhaustive cases, $n = 6$

Let E = getting odd = 3 (fav case).

(8) Independent event : The event that doesn't effect occurance of another (preceeding) event is called Independent event.
For eg:

(9) Dependent Event : vice versa
for eg:

Mathematical / classical defination of Probability

⇒ If there are 'n' number of exhaustive, mutually exclusive and equally likely events of a random experiment, out of 'n', 'm' number of cases are favourable to the event A (say), then the probability of happenning of an event A denoted P(A) and is defined as:

$$P(A) = \frac{\text{favourable cases } (m)}{\text{Exhaustive cases } (n)}$$

From Defination

The probability of non happenning of an event 'A' denoted by $P(\bar{A})$ or $P(A^c)$ is given by,

$$\therefore \boxed{P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)}$$

Remarks (1) $P(A) \geq 0$ and $P(\bar{A}) \geq 0$

$\therefore P(A) + P(\bar{A}) = 1$

i.e $p + q = 1$, where p = prob of happening

q = prob of not happening

(2) Prob of sample space $P(S) = 1$

(3) prob of impossible event $P(\phi) = 0$

(4) If event A and B are mutually exclusive event,

the probability of happening either A or B is given by,

$P(A \text{ or } B)$ or $P(A \cup B) = P(A) + P(B)$

similarly for three mutually exclusive events,

$P(A \text{ or } B \text{ or } C)$ / $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Qsn 1 If you flip a fair coin twice, find the probability of getting:

(a) 2 heads

(b) 1 head

(c) At least one head

(d) At most one head

(e) No head

Soln Here, Random experiment: flip a coin twice.

then, sample space is:

$S = \{HH, TH, HT, TT\}$

$\therefore$ Exhaustive cases $(n) = 4$

(1) 2 heads,

⇒ Let E_1 = Getting 2 heads

∴ favourable cases $(m) = 1$

∴ Probability $P(E_1) = \frac{m}{n} = 1/4$ //

(2) 1 head

→ Let E_2 = Getting 1 head

∴ Favourable cases $(m) = 2$

So, Probability $P(E_2) = \frac{m}{n} = \frac{2}{4} = 1/2$ //

(3) At least one head

⇒ Let E_3 = Getting at least one head

∴ favourable cases $(m) = 3$

∴ Probability $P(E_3) = \frac{m}{n} = \frac{3}{4}$

(4) At most one head.

↳ let E_4 = Getting at most one head

∴ favourable cases $(m) = 3$

∴ Probability $P(E_4) = \frac{m}{n} = \frac{3}{4}$

(5) No head

↳ let E_5 = Getting no head

∴ favourable cases $(m) = 1$

∴ Probability $P(E_5) = \frac{m}{n} = \frac{1}{4}$

Axiomatic Defination of Probability

⇒ Let s be the sample space of a random experiment and A be any event defined on sample space. Then

P(A) is the probability function satisfying the following axioms:

(a) $P(A) \geq 0$, Axiom of non-negativity

(b) $P(S) = 1$, Axiom of totality (certainty)

(c) Let $A_1, A_2, A_3 \ldots A_n$ be a finite or Infinite sequence of mutually exclusive events. Then, using the axiom of additivity,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$$

Similarly

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Example: Tossing a coin twice,

↳ sample space $S = \{HH, TT, HT, TH\}$

$\therefore$ Exhaustive case $n = 4$

Let E = Getting 1 head

$\therefore$ favourable cases $(m) = 2$

So, Probability $P(E) = \frac{m}{n} = \frac{2}{4} = 1/2$

$1! = 1$

$2! = 1 \times 2$

$3! = 1 \times 2 \times 3$

3×2

$4! = 4 \times$

Fundamental Rules of Counting

1) Addition Rule (mutually exclusive events only).

↳ When event A occurs in n_1 ways and event B occurs in n_2 ways and are mutually exclusive events; then either A or B (atleast one event) can occur in $(n_1 + n_2)$ ways.

Example: A bag contain 5 Red and 6 white balls. Then the total number of cases for either A or B is $5+6=11$

(b) Multiplication Rule

↳ When event A occurs in n_1 ways and event B occurs in n_2 ways, then both event A and B can occur simultaneously, occurs in $(n_1 \times n_2)$ ways

Basic Method of Counting Rules

(a) Permutation (Arrangement): Order matters

↳ to select r different objects from n diff objs,

$\therefore$ permutation, $^nP_r = \frac{n!}{(n-r)!}$

For repeated cases, permutation is given by,

$$P_{permu} = \frac{n!}{p!\,q!\,r!\,\ldots}$$

Eg: In how many ways the word "statistics" can be arranged?

↳ Soln Here word is: STASTICS.

Let n be the total no alphabets $n=8$

" 'p be the total no. of s $(p)=3$

$q=2$

$\therefore P = \frac{8!}{3! \times 2!}$

Combination : $n \rightarrow$ objects (order doesn't matters) $\rightarrow$ selection (drawn)
$\hookrightarrow r$ objects

$$^nC_r \text{ or } \binom{n}{r} \text{ or } C(n,r) = \frac{n!}{r!(n-r)!}$$

Eg: A box contains four white and six red balls. Two balls are drawn at random from the box. Find the probability that (order is not specified)
(i) Both are red balls
(ii) Both " white "
(iii) 1 red

Solution, Given,

No. of white balls = 4
No. of red balls = 6
Total number of balls = 10

→ Out of 10 balls, 2 can be selected in $^{10}C_2$ ways
{It is giving all of the possible outcomes while drawing randomly 2 balls out of 10}

∴ Exhaustive cases (n) $= {}^{10}C_2 = \frac{10!}{8! \times 2!}$
$= \frac{90}{2}$
$= 45$

(i) Sol: Let E = getting both red balls,
Out of 6 red balls, 2 red balls can be drawn in $^6C_2$ ways

∴ favourable cases (m) $= {}^6C_2 = \frac{6!}{2! \times 4!}$

∴ Prob that both are red $= 15/45 = \frac{1}{3} = 15$

(iii) Soln Let E_3 = 3 red and 3 white (same as 3 red)
out of 6 real balls, 3 white and out of 4 white balls,
3 white can be drawn in $^6C_1 \cdot {}^4C_1$ ways

$\therefore$ favourable cases $= \frac{6!}{5! \times 3!} \times \frac{4!}{3! \times 3!}$

$= 6 \times 4$

$= 24$

$\therefore$ probability that 3 red $= \frac{24}{45} = \frac{8}{15}$

(2) A lot of IC's of 10 good, 4 with minor defects and 2 with major defects. Two chips are selected randomly from the lot. What is the probability that atleast one chip is good?

Soln Given,

total no. of good chips = 10
" " " minor defected chips = 4
" " " major " " = 2

$\therefore$ Total no. of chips $= 10 + 4 + 2 = 16$

Out of 16 chips, 2 chips can be selected in $^{16}C_2$ ways

$\therefore$ Exhaustive cases $(n) = {}^{16}C_2 = \frac{16!}{2! \times 14!}$

$= \frac{16 \times 15}{2}$

$= 8 \times 15$

$= 120$

Let E = atleast one good chip
i.e. 1G or 2G where 1G means = 1G and 1 other

Favourable cases (m) = $^{10}C_1 \cdot {}^{6}C_1 + {}^{10}C_2$

$= 10 \times 6 + 45$

$= 60 + 45$

$= 105$

∴ Probability of getting atleast one good chip

$P(E) = \frac{m}{n} = \frac{105}{120}$

$= \frac{21}{24}$

Example : ② A lot of IC chips

① A box contains 75 good IC chips, 25 defective chips and 12 chips are selected randomly from the box. What is the probability that atleast one chips is defective.

↳ Sol: Given,

No. of good chips = 75

No. of defective chips = 25

∴ Total chips = 75+25 = 100

Out of 100 chips, 12 chips can be selected in $^{100}C_{12}$ ways.

=

=

∴ Exhaustive cases (n) = $^{100}C_{12}$

Let E = At least one chip is defective

1D & 11G, 2D & 10G ... 12D

1D and 11 good | 2D and 10 good |

and $\overline{E}$ = no defective (All good chips)

$\therefore$ favourable cases for $\overline{E}$ (m) = $^{75}C_{12}$

$\therefore P(\overline{E}) = \frac{m}{n}$

$= \frac{^{75}C_{12}}{^{100}C_{12}}$

$= 0.0248$

$\therefore$ The prob of at least one defective $P(E) = 1 - P(\overline{E})$

$= 1 - 0.024$

$= 0.9754$

(2) From a group of 15 chess players, 8 are selected to represent a group at a convention. What is the probability that the selected include three of the four best players in the group.

↳ Given, no. of best player = 4 / ordinary player = 11

total no. of chess player = 15

total no. of selected player = 8

Out of 15 players, 8 players can be selected in $^{15}C_8$ ways.

$\therefore$ Exhaustive case (n) = $^{15}C_8$.

Here, Let E = three of the four best player. (getting 3 best players) i.e 3 best and 5 ordinary

Favourable cases for E (m) = $^{4}C_3 \times {}^{11}C_5$

$\therefore$ Probability (P) = $\frac{m}{n} = \frac{^{4}C_3 \times {}^{11}C_5}{^{15}C_8}$

$= 0.2871$

(9) A box contains 15 IC chips with 5 defectives. If a random sample of 8 chips is drawn then find the probability of getting:

(a) all defectives
(b) 2 defectives
(c) one defectives
(d) No defective
(e) At least 1 defective
(f) At most one defective

Soln Given,

Total no. of chips = 15
Total no. of defective chips = 5
Total no. of undefective " = 15 − 5 = 10

out of 15 chips, 3 chips can be drawn in $^{15}C_3$ ways

∴ Exhaustive ~~chip~~ cases $(n) = {}^{15}C_3$

(a) all defectives, E = getting all defective chips

↳ out of 5 chips, all of the 3 defective chips can be drawn in $^{5}C_3$ ways

∴ favourable cases $(m) = {}^{5}C_3$

∴ probability of getting all defective chips

$$P(E) = \frac{m}{n} = \frac{^{5}C_3}{^{15}C_3} = \frac{2}{91} = 0.0219$$

(b) 2 defectives,

↳ Let E_1 = getting 2 defective (i.e also 1 undefective)

~~out of~~ ∴ favourable cases $(m) = {}^{5}C_2 \times {}^{10}C_1$

∴ Probability of getting 2 defectives

$$P(E_1) = \frac{m}{n} = \frac{^{5}C_2 \times {}^{10}C_1}{^{15}C_3} = \frac{20}{91} = 0.219$$

(c) one defective

↳ Let E_2 = getting one defective,

So, favourable cases $(m) = {}^5C_1 \times {}^{10}C_2$

∴ Probability $P(E_2) = \frac{m}{n} = \frac{{}^5C_1 \times {}^{10}C_2}{{}^{15}C_3} = \frac{45}{91} = 0.49$

(d) No defective,

↳ Let E_3 = Getting no defective (i.e getting undefective chips)

∴ Favourable cases $(m) = {}^{10}C_3$

∴ Probability $P(E_3) = \frac{m}{n} = \frac{{}^{10}C_3}{{}^{15}C_3} = 0.263$

(e) Al least 1 defective

↳ Let E_4 = getting atleast 1 defective chips

∴ favourable case $(m) = {}^5C_1 \times {}^{10}C_2 + {}^5C_2 \times {}^{10}C_1 + {}^5C_3$

So, probability $P(E_4) = \frac{m}{n} = 67/91 = 0.736$

(f) at most one defective,

↳ Let E_5 = getting at most one defective,

∴ favourable cases $(m) = {}^5C_1 \times {}^{10}C_2 + {}^{10}C_3$

∴ Probability $P(E_5) = \frac{m}{n} = \frac{69}{91} = 0.75$

Laws of probability

ⓐ Addition law
ⓑ Multiplication law

Addition law: (i) When A and B are two mutually exclusive events, the probability of getting either A or B (At least one event) is given by,

$$P(A \text{ or } B) = P(A \cup B).$$

Similarly, for three m.A.E (A, B and C),
Probability $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

(i) If A and B two (not mutually exclusive events), the probability of getting atleastone event, (either A or B or Both) is given by,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Similarly, for three not mutually exclusive events.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

(1) The probability that a new Airport will get an award for its design and awards for its efficient use of materials and both the award is respectively 0.16, 0.24 and 0.11. And the probability that new airport get atleast one award. Also what is the probability that It'll get only one of the 2 Awards.

Sol: Let, the event be defined as,

A = Award for its design
B = Award for its efficient use of materials.

Then, we have

$P(A) = 0.16$

$P(B) = 0.24$

$P(A \cap B) = 0.11$

$\therefore$ probability of getting at least one award is given, by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

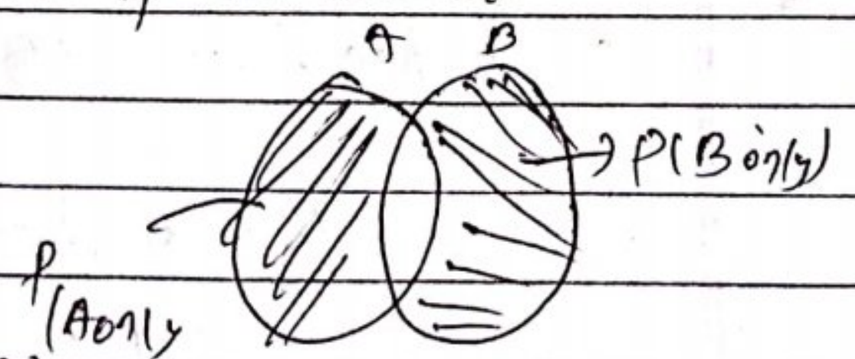
$= 0.16 + 0.24 - 0.11$

$= 0.40 - 0.11$

$= 0.29$ //

$\therefore$ Probability of getting only one of 2 award.

$P(A \text{ only or } B \text{ only})$



$= P(A \text{ only}) + P(B \text{ only})$

$= \cancel{P(A) - P(A \cap B) + P(B) - P(A \cap B)}$ $= P(A) - P(A \cap B)$

$= 0.16 + 0.24 - 2 \times 0.11$

$= \cancel{0.95}$

$= 0.40 - 0.22$

$= 0.18$ //

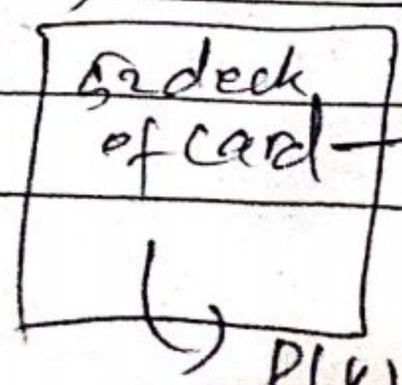
(b) Multiplications Law

↳ Let A and B are two independent events then, the probability of getting both events,

i.e. A and B is given by,

$P(A \cap B) = P(A) \times P(B)$

Example:



→ 1st draw of getting king (I) 4/52

now putting the obtained card in same deck,

∴ probability of getting king in 1st and 2nd draw both

$P(K_1 \cdot K_2) = P(K_1) \cdot P(K_2)$

$P(K_1 \cap K_2)$

Similarly, for three independent events, A, B and C

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

↳ Let A and B are two dependent events. Then the probability of getting both events (i.e. A and B) is given by

$P(A \cap B) = P(A) \cdot P(B/A)$ when AB ∃ 1st

$= P(B) \cdot P(A/B)$ " B " ∃ 1st.

$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P\left(C/_{A \cap B}\right)$

when the first occurence is A, (Let)

where $P(B/A)$ = conditional probability ← of event given that event A has already occured.

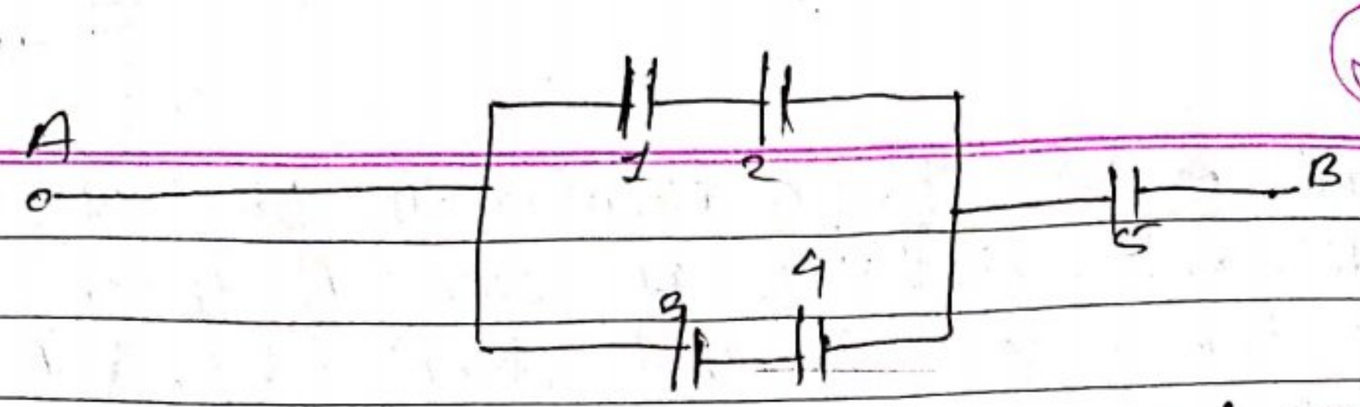
$P(A/B)$ = conditional probability of event A when the occurence of B B already occured.

Similarly for three dependent events: A, B and C)

↳ then, $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P\left(C/_{A \cap B}\right)$

Example:

VVI (1) For the circuit given below, the probability of closing each relay of the circuit is 0.7. Assume that the relay acts independently. What is the probability that a current will flow, between terminal A and B.



Soln Given that the probability of closing each relay of the ckt is,

0.7.

i.e. $P_1 = P_2 = P_3 = P_4 = P_5 = 0.7$

Let E_1 = closing of relay 1, and 2 and 5 (and 3, 4)

E_2 = " " " 3, 4 and 5

Then we have,

$P(E_1) = P(1, 2 \text{ and } 5)$

$= P_1 \times P_2 \times P_5$ (since relays acts independently)

$= 0.7 \times 0.7 \times 0.7$

$= 0.343$

Similarly,

$P(E_2) = P_3 \times P_4 \times P_5$

$= 0.7^3$ [relays acts independently]

$= 0.343$

$P(E_1 \cap E_2) = P(1, 2, 3, 4, 5)$

$= P_1 \times P_2 \times P_3 \times P_4 \times P_5$

$= 0.7^5$

$= 0.16807$

Thus, probability that a current will flow between terminal A and B is: $P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.343 \times 2 - 0.16807$

$= 0.51793.$

Conditional probability (developed on the basis of dependent event)

① The conditional probability of event A given that event B has already occured denoted by, $P(A/B)$ is defined as:

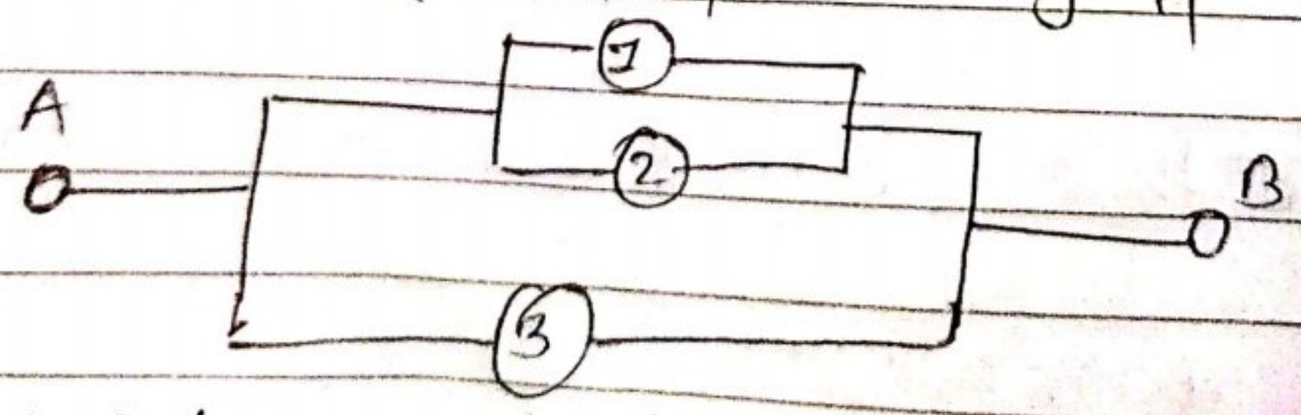
$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) \neq 0$$

$$\therefore P(A \cap B) = P(B) \cdot P(A/B)$$

② Similarly, the conditional probability of event B, given that A has already occured denoted by $P(B/A)$ is, defined as:

$$P(B/A) = \frac{P(B \cap A)}{P(A)}, \quad \text{provided that } P(A) \neq 0$$

Q2. Consider the following portion of electric circuit with three relays, current will flow from point A to point B, if there is atleast one closed path when relays are activated. The relays may malfunction and not close when activated. Suppose that the relays acts independently of one another and close properly when activated with probability of 0.9.



ⓐ What is the probability that current will flow when the relays are activated?

ⓑ Given that current flowed when relays were activated, what's the probability that relay ① function.

Soln Here, the probability, that the relay ~~works~~ closed properly = 0.9

(i) Let E_1 = closing of relay 1 and making path 1 closed.
E_2 = closing of " 2 " 2 "
E_3 = " " " 3 " " " 3 "

$P_1 = P_2 = P_3 = 0.9$ = prob of closing each relay independently

$\therefore P_{E_1} = P_{E_2} = P_{E_3} = 0.9$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

Since, the relays works independently,

$P(E_1 \cap E_2) = P(E_2 \cap E_3) = P(E_1 \cap E_3)$
$= 0.9 \times 0.9$
$= 0.81$

$P(E_1 \cap E_2 \cap E_3) = 0.9^3$
$= 0.81 \times 0.9$
$= 0.729$

So, the prob that current flow when relays are activated

$\therefore P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) + P(E_1 \cap E_2 \cap E_3) - P(E_1 \cup E_2) - P(E_2 \cup E_3) - P(E_3 \cup E_3)$
$= 0.9 + 0.9 + 0.9 + 0.729 - 3 \times 0.81$
$= 0.999$

(ii) Let E_1 = current flow = at least 1 closed
E_2 = relay 1 functions,

Then, we have probability of $f(E_1) = 0.999$ $P(E_1)$

So, the prob of relay 1 function, given that current flow when relay are activated

$P(E_2/E_1) = \frac{P(E_2 \cap E_2)}{P(E_1)} = \frac{P(E_2)}{P(E_1)}$ since $(E_2 \subset E_1)$ $= \frac{0.9}{0.999} = 0.9009$

Q3/ A manufacturer of airplane knows that the probability is 0.8 that an order will be ready for shipment on time and it is 0.7 that an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that, It was ready for shipment on time.

↳ Soln

Given,

① Probability than an order will be ready for shipment on time $P(S) = 0.8$

② Probability than an order will be ready for shipment and will be delivered on time $(P_{Sd}) = 0.7$

Asked,

Probability that such an order will be delivered on time, given that it was ready for shipment on time = ?

$P_{Sd} = 0.7$

$P(S) \times P_d = 0.7$

$P_d = 0.7/0.8$

$= 0.875$

$$P(d/S) = \frac{P(d \cap S)}{P(S)}$$

$$= \frac{P(d) \times P(S)}{P(S)}$$

$$= 0.875$$

Soln Given

(a) probability that an order will be ready for shipment on time = 0.8

(b) probability that an order will be ready for shipment and will be deliver on time = 0.7

Defining the event,

E_1 = order will be ready for shipment on time

E_2 = " " delivered on time.

To find,

probability that such an order will be delivered on time, given that It was ready for shipment on time = ?

$P(E_1) = 0.8$

$P(E_1 \cap E_2) = 0.7$

$$P(E_2/E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)}$$

$$= \frac{0.7}{0.8}$$

$$= 0.875$$

Bayer's Theorem (Inverse Probability)

Statement:

Let $E_1, E_2 --- E_n$ be 'n' mutually exclusive events, defined on sample space (S) with $P(E_i) \neq 0$ (i.e. have some chances of occurance). For any arbitrary event A (say) which is the subset of sample space (S) such that probability of $P(A) > 0$, Then the probability of getting E_i given that event A has already occured is given by,

$$P(E_i/A) = \frac{P(E_i) \times P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)}$$

proof: From defination of Conditional probability,

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i) P(A/E_i)}{P(A)} \quad \text{--- (i)}$$

Since A is a subset of sample space (S) and sample space (S) is given by,

$$S = E_1 \cup E_2 \cup E_3 \cup --- \cup E_n$$

then, we can write,

$$A = A \cap S$$

$$= A \cap [E_1 \cup E_2 \cup --- \cup E_n]$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup --- \cup (A \cap E_n)$$

using distributive law

Since $E_1, E_2 \ldots E_n$ are mutually exclusive events, so $(A \cap E_1), (A \cap E_2) \ldots (A \cap E_n)$ are also mutually exclusive event.

Now, taking probabilities on both sides, then, we have,

$$P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)]$$

$$= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$\therefore P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i) \quad \text{——— (2)}$$

From (i) and (ii),

$$\left[P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)} \right] \quad \text{——— (3)}$$

Here, $P(E_i)$ = prior probabilities

$P(A/E_i)$ = conditional probabilities

$P(E_i/A)$ = update form of prior probabilities,

Q1. In a bolt factory, A, B, C manufacture respectively 25%, 35% and 40% of the total outputs. Out of their outputs, 5%, 2% and 4% respectively are defective bolts. A bolt is drawn at random from the total outputs and found to be defective. What is the probability that it was manufactured by

(a) machine A
(b) " B
(c) " C

Sol: Defining the events,

E_1 = Bolts produce from machine A
E_2 = " " " B
E_3 = " " " C

The we have,

$P(E_1) = 0.25$
$P(E_2) = 0.35$
$P(E_3) = 0.40$

Also,

Let A = Defective Bolts

$P(A/E_1) = 0.05$ and $P(A/E_3) = 0.04$

$P(A/E_2) = 0.02$

(i) The prob that the defective Bolts B produced by machine A;

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

$$= \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \cdot 0.02 + 0.40 \times 0.04}$$

$$= 0.3521$$

(ii) $$P(E_2/A) = \frac{P(E_2)\, P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \times P(A/E_3)}$$

$$= \frac{0.35 \times 0.02}{0.25 \times 0.05 + 0.35 \times 0.02 + 0.40 \times 0.04}$$

$$= 0.1971$$

(iii) $P(E_3/A) = 0.4507$.

Q2. A manufacturing company produces steel pipes in 3 plants with daily production of 500, 1000, 2000 units. According to the past experience, it is known that the fraction of defective outputs produced by the three plants are respectively: 0.008, 0.01 and 0.005. if a pipe is selected at a random from a daily total production and found to be defective. from which plant, the defective pipe is expected to have been produced?

Ans) Here,

E_1 = production of steel pipes from plant 1

E_2 = " " " " 2

E_3 = " " " " 3

B = production of defective pipes,

$P(B/E_1) = 0.008$

$P(B/E_2) = 0.01$

$P(B/E_3) = 0.005$

$P(E_1) = 500/3500 = 0.14$

$P(E_2) = 1000/3500 = 0.28$

$P(E_3) = \frac{2000}{3500} = 0.57$

(i) The prob that the defective pipes is from plant (1),

$$P(E_1/B) = \frac{P(E_1) \cdot P(B/E_1)}{P(E_1) \cdot P(B/E_1) + \cdots + P(E_3) \cdot P(B/E_3)}$$

$$= \frac{0.14 \times 0.008}{0.14 \times 0.008 + 0.28 \times 0.01 + 0.57 \times 0.005}$$

$$= \frac{1.12 \times 10^{-3}}{6.77 \times 10^{-3}}$$

$$= 0.16$$

$$P(E_2/B) = \frac{P(E_2) \cdot P(B/E_2)}{6.77 \times 10^{-3}}$$

$$= \frac{0.28 \times 0.01}{6.77 \times 10^{-3}}$$

$$= 0.41$$

$$P(E_3/B) = \frac{0.005 \times 0.57}{6.77 \times 10^{-3}}$$

$$= 0.42$$

$\therefore$ From plant 3, defective

(3) A given lot of IC chips contains 2% defective chips. Each chip is tested before delivery. The tester itself is not totally reliable. Probability of tester say the chip is good, when it is actually good is 0.95 and the chip is defective when it is actually defective is 0.94. If a tested device is indicated to be defective, what is the prob that it is actually defective?

Defining the events,

E_1 = Defective chips

E_2 = Good chips

$P(E_1) = 0.02$

$P(E_2) = 1 - P(E_1) = 0.98$

A = Tester says chip is ~~good~~ defective

Then, $P(A/E_1) = 0.94$

$P(A/E_2) = 1 - 0.95 = 0.05$

Now, the probability that chip is actually defective when tester says its defective,

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{0.02 \times 0.94}{0.02 \times 0.94 + 0.98 \times 0.05}$$

$$= 0.277$$

(4) The contents of urns (I), (II) and (III) are as follows:

(a) 1 White, 2 black, and 3 red balls

(b) 2 white, 1 black " 1 " "

(c) 4 " , 5 " , 3 " "

One urn is choosen at random and two balls are drawn. They found to be white and red. What is the prob, that they come from urn (I)?

VVI

(5)

Soln Defining the events

E_1 = selecting Box I

E_2 = " " (II)

E_3 = " " (III)

Then, we have,

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

Also,

Let A = selecting one white and one red Balls

$$P(A/E_1) = \frac{\cancel{P(A) \cdot P(E_1/A)}}{\cancel{P(A)}}$$

$$= \frac{{}^1C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{3}{}$$

$$P(A/E_2) = \frac{{}^2C_1 \cdot {}^1C_1}{{}^4C_2}$$

$$P(A/E_3) = \frac{{}^4C_1 \cdot {}^3C_1}{{}^{12}C_2}$$

Now,

probability that they are from Box I,

i.e. $$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

=

VVI

(5) A binary communication channel carries data as of one of two types of signal denoted by 0 and 1. Due to noise, 0 transmitted is sometimes received as 1 and a transmitted 1 is sometimes received as zero(0) and probability is 0.91 that a transmitted 1 is received as 1. Further assume that probability of 0.45 of transmitted as a zero(0), if a signal is sent.

↳ For a given channel, assume a probability of 0.94, that a transmitted zero(0) is correctly received as 0.

↳ let T_0 = transmitted 0
T_1 = " 1
R_0 = Received 0
R_1 = " 1

Now, we have,

$P(T_0) = 0.45$

$P(T_1) = 1 - P(T_0)$
$= 0.55$

$P(R_0/T_0) = 0.94$

(Ex) $P(R_1/T_1) = 0.91$

Then we have,

$P(R_1/T_0)$ = prob of transmitting 1 when transmitted as 0

$= 1 - P(R_0/T_0)$
$= 0.06$

$P(R_0/T_1) = 1 - 0.91 = 0.09$

(i) Prob of R_1 $[P(R_1)] = P(T_1 \cap R_1) + P(T_0 \cap R_1)$

$= P(T_1) \cdot P(R_1/T_1) + P(T_0) \cdot P(R_1/T_0)$

$= 0.55 \times 0.91 + 0.45 \times 0.06$

$= 0.527$

(ii) a 1 was transmitted given that 1 was received

$\hookrightarrow P(T_1/R_1) = \dfrac{P(T_1) \cdot P(R_1/T_1)}{P(T_1) \cdot P(R_1/T_1) + P(T_0) \cdot P(R_1/T_0)}$

$= \dfrac{0.55 \times 0.91}{0.55 \times 0.91 + 0.45 \times 0.06}$

$= 0.948$

(iii) a 0 was ~~received~~ transmitted given that 0 was received

$\hookrightarrow P(T_0/R_0) = \dfrac{P(T_0) \cdot P(R_0/T_0)}{P(T_0) \cdot P(R_0/T_0) + P(T_1) \cdot P(R_0/T_1)}$

$= \dfrac{0.45 \times 0.06}{0.45 \times 0.06 + 0.55 \times 0.09}$

$= 0.35$

(iv) $P(\text{error}) = P(R_1 \cap T_0) + P(R_0 \cap T_1)$

$= P(T_0) \cdot P(R_1/T_0) + P(T_1) \cdot P(R_0/T_1)$

$\cancel{= P(R_1/T_0) + P(R_0/T_1)}$

$= 0.45 \times 0.06 + 0.55 \times 0.09$

$= 0.0765$

Q; Costremers are used to evaluate preliminary product designs in the past, 95% of highly succesfull products received good reviews 60% of moderately succesfull product received good reveiws and 10% of poor products " good reveiws. In addi tion 40% of products have been highly succes-full, 35% have been moderately succesful and 25% have been poor products,

(i) What's the probability that a product have a good reviews?
(ii) If a new design have a good reviews, what's the probability that it will be a highly succesful product?

Sol: Given,
95% of high succesfull prod got good review.
65% of moderately " " " " .
10% " poor " " " " .

Also,
40% of product have been high succesfull
35% of " " moderately "
25% " " " poorly succesfull.

Let E_1 = highly succesful
E_2 = moderately "
E_3 = poorly succesfull

$P(E_1) = 0.40$ | $P(E_3) = 0.25$
$P(E_2) = 0.35$

Let A = getting good review

$P(A/\varepsilon_1) = 0.95$

$P(A/\varepsilon_2) = 0.65$

$P(A/\varepsilon_3) = 0.20$

(i) Prob that a poor product have a good review

$\Rightarrow P(A \cap \varepsilon_1) + P(A \cap \varepsilon_2) + P(A \cap \varepsilon_3)$

$= P(\varepsilon_1) \cdot P(A/\varepsilon_1) + P(\varepsilon_2) P(A/\varepsilon_2) + P(\varepsilon_3) P(A/\varepsilon_3)$

$= 0.4 \times 0.95 + 0.35 \times 0.65 + 0.25 \times 0.20$

$= 0.632$

(ii) $P(\varepsilon_1/A) = \dfrac{P(\varepsilon_1) \cdot P(A/\varepsilon_1)}{P(\varepsilon_1) \cdot P(A/\varepsilon_1) + P(\varepsilon_2) \cdot P(A/\varepsilon_2) + P(\varepsilon_3) \cdot P(A/\varepsilon_3)}$

$= \dfrac{0.4 \times 0.95}{0.4 \times 0.95 + 0.35 \times 0.65 + 0.25 \times 0.1}$

Q6. Four technician regularly makes repairs when breakdown occurs on an automated production line. Janet who serves 20% of the breakdowns, make an incomplete repair 1 time in 20, Tom who serves 60% of breakdowns make an incomplete repair of 1 time in 10, Georgia who serves 15% of the breakdown make an incomplete repair of 1 time in 10 & peter who serves 5% of breakdown, make an incomplete repair of 1 time in 20.

For the next problem with the production line, diagnosed

as being due to an initial repair that was incomplete, what is the prob. that this initial repair was done by Jennet?

↳ Let

ε_1 = Jannet serving breakdown time

ε_2 = B-Tom " " "

ε_3 = Georgia " " "

ε_4 = Peter " " "

20f

Also, Let

A = Making an incomplete repair,

$P(\varepsilon_1) = 0.2$
$P(\varepsilon_2) = 0.6$
$P(\varepsilon_3) = 0.15$
$P(\varepsilon_4) = 0.05$

$P(A/\varepsilon_1) = \frac{{}^{20}C_1}{0.2}$

$P(A/\varepsilon_2) = \frac{{}^{10}C_1}{0.6}$

$P(A/\varepsilon_3) = \frac{{}^{10}C_1}{0.15}$

$P(A$

$P(A/\varepsilon_4) = \frac{{}^{20}C_1}{0.05}$

$$P\left(\frac{\varepsilon_1}{A}\right) = \frac{P(\varepsilon_1)\,P(A/\varepsilon_1)}{P(\varepsilon_1)\,P(A/\varepsilon_1) + P(\varepsilon_2)}$$

Random Variables (Short Notes)

↳ Random variable is a real valued function which is associated with each outcomes of a random experiment. In simple words, random variable is also known as the outcomes of a random experiment. It is denoted by capital letters X, Y, Z --- and the value taken by the random variable is denoted by x, y, z etc

eg: Tossing a coin twice

$S = \{HH, HT, TH, TT\}$

↳ No. of heads

$X =$ No. of heads

$x = 2, 1, 1, 0$

Some examples: (1) No. of student enrolled in cosmos college

2. Height of the students
3. No. of defective chips
4. Income
5. Weight etc

Experimental examples.

(a) Consider a random experiment of tossing a coin, assign 1 if head appears and 0 if tails appears.

Hence the number of heads is a random variable taking value 1 and 0 with probabilities 1/2 and 1/2 respectively.

i.e.

X = No. of heads	Prob
0	1/2
1	1/2

(b) Consider an exp, tossing a coin twice,
Sample space = {HH, HT, TH, TT}
Let X = No. of heads (Random variable)
Now, it may takes the values 0, 1, and 2

$X = x$	$P(X=x)$
0	$P(x=0) = 1/4$
1	$P(x=1) = 2/4$
2	$P(x=2) = 1/4$

random variable

Properties

(a) If X is a random variable, then aX is also a random variable, where a is a constant.

(b) If x and y are random variable, $x \pm y$
$x \pm y$, $ax \pm by$, xy and x/y $(x \cdot y^{-1})$ are also random variables.

Types of random variable

(a) Descrete random variable (b) Continuous Random variable

→ It is a random variable, which takes only integer values. In other words, it can be defined as ~~the~~ real valued function which is defined on descrete sample space.

Examples:
(1) No. of students enrolled every year in Cosmos College.
(2) No. of telephone calls per hour.
(3) No. of accidents per week in Kathmandu city.
(4) No. of defective items etc.

(i) **Probability function of Discrete R.V**

↳ Let X be a discrete r.v which takes the values $x_1, x_2, \ldots x_n$ with probabilities $P(X=x_i)$ or P_i or $P(x_i)$ where $i = 1, 2 \ldots n$

$X=x$	$P(X=x)$
0	1/2
1	1/2

Then, the probability function $P_i = P(X=x_i)$ is called Probability mass function (pmf), if satisfies the following conditions,

(i) $P(X=x_i) \geq 0, \quad \forall i = 1, 2, 3, \ldots n$

(ii) $\sum_{i=1} P(X=x_i) = 1$

Note
D.R.V, we use $\sum$

Example: Consider a random experiment of tossing a coin twice, find the Prob. distribution of no. of heads

sample space = {HH, HT, TH, TT}

Let X = No. of heads

Then X = discrete random variables which takes values 0, 1 and 2

Therefore,

$X=x$	$P(X=x)$
$(x=0)$ 0	1/4
$(x=1)$ 1	2/4
$(x=2)$ 2	1/4
	$\sum P(X=x) = 1$

(2) A fair coin is tossed 3 times, find the Prob distribution?

Sample space $(S) = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let X = No. of heads

Then, X is a descrete random variables which takes values 0, 1, 2, 3

Therefore the Probability distribution of X is:

$X=x$	$P(X=x)$
0	1/8
1	3/8
2	3/8
3	1/8
Total	$\Sigma P(X=x) = 1$

(3) A lot of 10 items contains 3 defectives from which a sample of 4 items is drawn without replacement. Let X be a random variable, representing the no. of defective items in the sample,

(i) find the probability distribution of X

(ii) " " " $(x<1)$

(iii) " " " $P(0<x<2)$

Soln Let X = No. of defective items.

Clearly, X may takes the values 0, 1, 2, 3

Now,

$$P(X=0) = \frac{{}^7C_4}{{}^{10}C_4}$$

$$P(X=1) = \frac{{}^7C_3 \cdot {}^3C_1}{{}^{10}C_4}$$

$$P(X=2) = \frac{{}^7C_2 \cdot {}^3C_2}{{}^{10}C_4}$$

$$P(X=3) = \frac{{}^7C_1 \cdot {}^3C_3}{{}^{10}C_4}$$

Therefore, the probability distribution of x is:

$X=x$	$P(X=x)$
0	1/6
1	1/2
2	3/10
3	1/15
$\Sigma P(X=x) = 1$	

(b) $P(x<1) = P(x=0) = 1/6$

(c) $P(0<x<2) = P(x=1) = 1/2$ //

Example : A random variable X has the following probability distribution,

$X=x$	0	1	2	3	4	5	6
$P(X=x)$	a	2a	3a	4a	3a	2a	a

Find the value of a

Sol:

$a+2a+3a+4a+3a+2a+a = 1$

$\therefore a = 1/16$

Since, the total probability of the random variable is 1, $\Sigma\{P(X=x)\} = 1$

(i) Find probability $P(x \le 2)$

(ii) $P(0 \le x \le 3)$ (iii) $P(2 \le x \le 4)$

Sol: We have, the probability distribution as:

$X=x$	0	1	2	3	4	5	6
$P(X=x)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

Soln $\therefore P(x \le 2) = 1/16 + 2/16 + 3/16$

$= \frac{6}{16}$

$\therefore P(0 \le x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= 1/16 + 2/16 + 3/16 + 4/16$

$= \frac{10}{16}$

$P(2 \le x < 4) = P(x=2) + P(x=3)$

$= 3/16 + \frac{4}{16}$

$= \frac{7}{16}$

So, all the required probabilities are find out.

Discrete Probability Distribution function (Cumulative Distribution function).

↳ Let X be a d.r.v with probability mass function, $P(X=x) \to p(x)$, $\forall x$ Then, the probability distribution function of X, denoted by $F(x)$, is defined as,

$$F(x) = P(X \le x) = \sum_{x}^{\infty} P(X=x)$$

(*) Properties of distribution function

↳ (i) $0 \le F(x) \le 1$, $\forall x$

(ii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

(iii) $F(+\infty) = \lim_{x \to +\infty} F(x) = 1$

(iv) $P(a < x \le b) = F(b) - F(a)$

(v) $F(x) > F(y)$ if $x > y$

Examples: A random variable X has the following prob. distribution,

$X=x$	0	1	2	3	4	5	6
$P(X=x)$	k	3k	2k	5k	7k	4k	8k

Find (i) k = ?
(ii) Prob that $X<2$
(iii) Prob $(1<x<5)$
(iv) find the distribution function of X.
(v) Find $F(2)$ and $F(4)$.

Sol:

We know, total probability = 1,

$$\sum_{x=0}^{6} P(X=x) = 1$$

$$\Rightarrow P(x=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6) = 1$$

$$\Rightarrow k+3k+2k+5k+7k+4k+8k = 1$$

$$\Rightarrow 30k = 1$$

$$\therefore k = 1/30$$

Therefore, the probability distribution becomes,

$X=x$	0	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{30}$	$3/30$	$2/30$	$5/30$	$7/30$	$4/30$	$8/30$

(ii) Prob$(X<2)$

$$\therefore P(X=0)+P(X=1)$$

$$= \frac{1}{30}+\frac{3}{30}$$

$$= \frac{4}{30}$$

$$= \frac{2}{25}$$

$P(1<x<5)$

$$= P(X=2)+P(X=3)+P(X=4)$$

$$= \frac{2}{30}+\frac{5}{30}+\frac{7}{30}$$

$$= \frac{14}{30}$$

$$= \frac{7}{15}$$

(iv) The distribution function of x is given by,

$X=x$	$P(X=x)$	$F(x)$
0	1/30	1/30
1	3/30	4/30
2	2/30	6/30
3	5/30	11/30
4	7/30	18/30
5	4/30	22/30
6	8/30	1

$\rightarrow F(0) = P(X=0) = 1/30$

$F(1) = P(X \leq 1) = P(X=0) + P(X=1)$

(v) $F(2) = 6/3$

$F(4) = 11/30$

Example: A random variable X has the following probability distribution,

$X=x$	-2	-1	0	1	2
$P(X=x)$	0.1	k	2k	0.3	k

(i) Find (k)

Soln As we know, the sum of all the individual probability is 1,

$$\therefore \sum_{x=-2}^{2} P(X=x) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2)$$

$$1 = 0.1 + k + 2k + 0.3 + k$$

$$1 = 0.4 + 4k$$

$$4k = 0.6$$

$$k = \frac{0.6}{4}$$

$$\therefore k = 0.15$$

(ii) $P(X \leq 0) = P(X=-2) + P(X=-1)$

$= 0.1 + k = 0.1 + 0.15 = 0.25$

$P(-1 \le X < 2)$

$= \cancel{P(X=-2)} + P(X=-1) + P(X=0) + P(X=1)$

$= 0.15 + (2 \times 0.15) + 0.3$

$= 0.15 + 0.3 + 0.3$

$= 0.75$

(iii) distribution function of X

$X=x$	$P(X=x)$	$F(x)$
-2	0.1	0.1
-1	0.15	0.25
0	0.3	0.55
1	0.3	0.85
2	0.15	1

(iv) $F(-1) = F(-2) + F(-1)$

$= 0.25$

(b) Continuous Random Variable

↳ A random variable is said to be a continuous random variable if it takes all possible values within a given range.

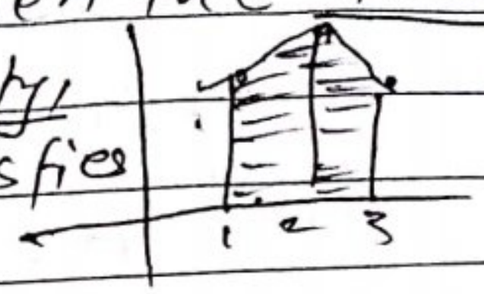
$(0, 1)$

ex: Heights of the individuals,
size of compact disc
Temperature recorded for a particular day.

If takes every possible value like int, decimals

Probability function of continuous r.v

↳ Let x be a continuous random variable with probability function $f(x)$, $\forall x$, then the probability function $f(x)$ is called as probability density function (pdf) if it satisfies the following conditions,



(i) $f(x) \geq 0$, $\forall x$

(ii) $\int_{x} f(x)\,dx = 1$ (Total prob = 1)

• Eg:- Given pdf (of continuous r.v)

$$f(x) = 4x^3; \quad 0 < x < 1$$
$$= 0; \text{ otherwise}$$

verify whether the given function is pdf or not?

Solution: The given function is:

$$f(x) = 4x^3; \quad 0 < x < 1$$

Then, obviously, $f(x) \geq 0$, $\forall x$

Now, for pdf,

Total pdf = 1,

$$\Rightarrow \int_{x} f(x)\,dx = 1$$

Therefore, $\int_{x} f(x)\,dx = \int_0^1 4x^3\,dx$

$$= 4\left[\frac{x^4}{4}\right]_0^1$$

$$\therefore \int_0^1 4x^3\,dx = 1$$

Hence, $f(x)$ satisfies the required condition,

Therefore $f(x)$ is pdf.

(2) verify whether the given prob, function is pdf or not. $f(x) = \frac{3}{4}x(2-x)$; $0 < x < 2$

Soln Here, the given function is,

$$f(x) = \frac{3}{4}x(2-x) \; ; \; 0 < x < 2$$

Then, obviously

$$f(x) \geq 0 \text{ for } 0 < x < 2$$

Now, for pdf

$$\int_{2} f(x)\,dx = 1 \text{ for } 0 \leq x < 2$$

i.e. $\int_0 \frac{3}{4}x(2-x)\,dx$

$$= \frac{3}{4}\int_0^2 (2x - x^2)\,dx$$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(4 - \frac{8}{3}\right) - (0-0)\right]$$

$$= \frac{\cancel{3}}{4} \times \frac{12-8}{\cancel{3}}$$

$$= \frac{4}{4}$$

$$= 1$$

Therefore $f(x)$ satisfies the required condition.

Hence $f(x)$ is pdf.

Distribution function of continous R.V

↳ Let X be a continuous r.v with pdf $f(x)$, $\forall x$ Then, the probability distribution function, of X is denoted by $F(x)$, and is defined as:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

It is also called cumulative distribution function.

(*) properties of $F(x)$:

(a) $0 \le F(x) \le 1$; $\forall x$

(b) $F(x)$ is a continuous fun, so $F'(x) = f(x)$, where $f(x)$ is prob density fun.

(c) $F(-\infty) = 0$

(d) $F(+\infty) = 1$

(e) $P(a \le x \le b) = P(a < x < b) = P(a \le x < b) = P(a < x \le b)$
$= F(b) - F(a)$

(f) $F(x) > F(y)$; if $x > y$

Q. Given, pdf is $f(x) = kx(1-x)$; $0 < x < 1$
$= 0$; otherwise

(i) Find k (2021 year)
(ii) $P(x < 0.4)$
(iii) $P(0.3 < x < 0.5)$

Soln
Here given pdf is,

$$f(x) = kx(1-x)$$
$$= k(x - x^2)$$

As we know that the probability of give pdf is

$$\int_x f(x)\,dx = 1 \;;$$

$$\therefore \int_0^1 k(x - x^2)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow k\left(\frac{1}{2}-\frac{1}{3}\right) = 1$$

$$\Rightarrow k\left(\frac{3-2}{6}\right) = 1$$

$$\Rightarrow k = 6$$

$$\therefore k = 6$$

Thus $k = 6$.

Therefore, the given pdf is:

$$f(x) = 6x(1-x) \;;\; 0<x<1$$
$$= 0 \quad \text{otherwise}$$

(ii) $P(x < 0.4)$

Sol^n $P(x<0.4) = \int_0^{0.4} f(x)\,dx$

$$= \int_0^{0.4} 6(x-x^2)\,dx$$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{0.4}$$

$$= 6\left[\frac{0.16}{2}-\frac{0.064}{3}\right]$$

$$= 6\times 0.058$$

$$= 0.352$$

$P(x > 0.5)$

$$= \int_{0.5}^{1} f(x)\,dx$$

$$= \int_{0.5}^{1} 6(x-x^2)\,dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{0.5}^{1}$$

$$= 6\left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{0.25}{2}-\frac{0.125}{3}\right)\right]$$

$= 6\left[\frac{1}{6} - \frac{(0.75 - 0.250)}{6}\right]$

$= 6 \times \frac{1 - 0.5}{6}$

$= 0.5$

(iii) $P(0.3 < x < 0.5)$

Soln Here, $P(0.3 < x < 0.5) = \int_{0.3}^{0.5} f(x)\,dx$

$= \int_{0.3}^{0.5} 6(x - x^2)\,dx$

$= \int_{0.3}^{0.5} 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)dx$

$= 0.284$

Bivariate Distribution (Joint probability Distribution)

↳ The probability Distribution of two random variables either both continuous or both discrete r.v is called Bivariate (Joint Probability Distribution).

Example: (a) Distribution of height and weight of a group

(A) Joint Prob Distribution of Descrete random variable (Descrete joint probability function).

↳ Let X and Y are two descrete random variables, with joint prob function (joint prob distribution) $P(X=x, Y=y)$ or $P(x, y)$, $\forall x$ and y. Then, the joint probability function $P(X=x, Y=y)$ or $P(x, y)$ is called joint Probability mass function, if it satisfies the following condition,

(i) $P(X=x, Y=y) \geq 0, \forall x, y$

(ii) $\sum_x \sum_y (P X=x, Y=y) = 1$

Marginal probability (Marginal Distribution) / Marginal probability Distribution function / MPMF

↳ Let X and Y are two descrete random variables with joint probability mass function $P(X=x, Y=y)$, $\forall x$ and y
Then,
the marginal distribution of X is denoted by $P(X=x)$ or $P(x)$ is given by,

$$P(X=x) = \sum_y P(X=x, Y=y)$$

Similarly, the margin distribution of y is denoted by, $P(Y=y)$ or $P(Y)$ and is given by,

$$P(Y=y) = \sum_x P(X=x, Y=y)$$

Conditional Distribution (CPD)

↳ Let X and y are two descrete r.v.s with joint Pmf $P(X=x, Y=y)$, $\forall x$ and y. Also, the marginal distribution of X and y are resp. $P(X=x)$ and $P(Y=y)$
Then, the conditional prob distribution of X, given (possible values of r.v)
$Y=y$ is denoted by,
$P(X=x/Y=y)$ or $P(X/y)$ and is ~~given~~ defined as: $P(X=x/Y=y) = \dfrac{P(X=x, Y=y)}{P(Y=y)}$
provided $P(Y=y) \neq 0$.

Similarly, the conditional prob distribution of Y given, $X=x$ is denoted by,

$$P(Y=y/X=x) \text{ or } P(Y=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

Condition for Independency

Two discrete r.v.s X and Y are said to be independant if,

$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$ otherwise it is not independant.

Joint distribution function of D.r.v's

↳ Let X and Y are two d.r.v's with joint pmf $P(X=x, Y=y)$, $\forall x$ and y, then, the joint distribution function of X and Y is denoted by, $F(x,y)$ and is defined as,

$$F(x,y) = P(X \le x, Y \le y) = \sum_{-\infty}^{x} \sum_{-\infty}^{y} P(X=x, Y=y)$$

Q1) Suppose X and Y are two discrete random variables with joint pmf

Y \ X	2	3
1	0.1	0.15
2	0.2	0.3
3	0.1	0.15

(i) Find the Marginal distribution of X and Y

(ii) Find the conditional prob. dist of X, given $Y=2$

(iii) Find the conditional prob mass fun of Y, given $X=3$.

(iv) Are X and Y independent?

(v) Find $F(2,2)$ and $F(3,2)$.

Ans (i) Marginal distribution of X, we know

$$P(X=x) = \sum_{y=1} P(X=x, Y=y)$$

for $X=2$,

$$P(X=2) = \sum_{y=1}^{3} P(X=2, Y=y)$$

$$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3)$$

$$= 0.1 + 0.2 + 0.1$$

$$= 0.4$$

for $X=3$

$$P(X=3) = \sum_{y=1}^{3} P(X=3, Y=y)$$

$$= P(X=3, Y=1) + P(X=3, Y=2) + P(X=3, Y=3)$$

$$= 0.15 + 0.3 + 0.15$$

$$= 0.6$$

$\therefore$ The marginal distribution of X is:

$X=x$	2	3
$P(X=x)$	0.4	0.6

(ii) Also, Marginal distribution of Y,

↳ we know that,

for $Y=1$,

$$P(Y=1) = \sum_{x=2}^{3} P(X=x, Y=1)$$

$$= P(X=2, Y=1) + P(X=3, Y=1)$$

$$= 0.1 + 0.15$$

$$= 0.25$$

for, $Y=2$,

$$P(Y=2) = \sum_{x=2}^{3} P(X=x, Y=2)$$

$$= P(X=2, Y=2) + P(X=2, Y=3)$$

$$= 0.2 + 0.3 = 0.5$$

for $(Y=3)$

$$P(Y=3) = \sum_{x=2}^{3} P(X=x, Y=3)$$

$$= P(X=2, Y=3) + P(X=3, Y=3)$$

$$= 0.1 + 0.15$$

$$= 0.25$$

Thus, the marginal distribution of Y:

$Y=y$	1	2	3
$P(Y=y)$	0.25	0.5	0.25

(ii) ⇒ Now, the conditional probability distribution of X when $Y=2$ is:

$$P(X=x/Y=2) = \frac{P(X=x, Y=2)}{P(Y=2)}$$

for $X=2$

$$P(X=2 / Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)}$$

$$= \frac{0.2}{0.5}$$

$$= 0.4$$

for $X=3$,

$$P(X=3 / Y=2) = \frac{P(X=3, Y=2)}{P(Y=2)}$$

$$= \frac{0.3}{0.5}$$

$$= 0.6$$

Thus, the condition probability distribution of X is,

$$P(X=2/Y=2) = 0.4$$

$$P(X=3/Y=2) = 0.6$$

(iii) ⇒ The conditional probability mass function of Y, when X=3, is:

(iv) ⇒ checking the independency,

$P(X=x, Y=y) = P(X=x) \cdot P(Y=x)$

when $x=2, y=1$,

$\cancel{P(X=}$

$\rightarrow P(Y=y/x=3) = \frac{P(X=3, Y=y)}{P(X=3)} = \frac{0.15}{0.6}$

Joint prob distribution of continuous R.V

↳ Let x and y are two continuous random variables with joint prob. function $f(x,y)$, $\forall x$ and $\forall y$. Then, the joint prob fun $f(x,y)$ is called joint prob ~~distribution~~ density function, if it is satisfies the following condition,

(i) $f(x,y) \geq 0$, $\forall x$ and y

(ii) $\int_x \int_y f(x,y)\,dx\,dy \Rightarrow$ i.e. $\int_y \left[\int_x f(x,y)\,dx\right] dy$

Marginal Distribution.

↳ Let x and y are two continuous random variables with joint pdf $f(x,y)$ $\forall x$ & y. Then the marginal distribution of X is denoted by $f(x)$ and is given by $f(x) = \int_y f(x,y)\,dy$.

& the marginal dist^n of Y is denoted by $f(y)$ and is given by,

$f(y) = \int_x f(x,y)\,dx$

Conditional density function

Let x and y are two continuous random variables, with joint pdf, $f(x,y)$, $\forall x$ and y. Also, the marginal distribution of x and y are ~~responsibility~~, respectively, $f(x)$ and $f(y)$:

Then, the conditional distributional of x given y is:

$$f(x/y) = \frac{f(x,y)}{f(y)} \; ; \text{ for } f(y) \neq 0$$

similarly, the conditional prob, y given x,

$$f(y/x) = \frac{f(x,y)}{f(x)} \quad \text{for } f(x) \neq 0$$

Condition for independency

Two continuous random variables x & y are said to be independent if

$$f(x,y) = f(x) \cdot f(y)$$

otherwise it is not independent.

Prob distribution fun (cummilative for 2 rand var)

Let x and y are two continuous random variables with joint pdf $f(x,y)$, $\forall x$ and $\forall y$, then the joint dist fun of x and y is:

Then, the jpd. f. of x and y is denoted by, $F(x,y)$ and is defined as:

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

Marginal distribution functions.

↳ Let X and Y be two continuous rand var with joint pdf $f(x,y)$, $\forall x$ and $\forall y$. and the joint prob distribution ~~is given~~ function of X and Y is given by $f(x,y)$, the marginal distribution fun of X is denoted by $F(x)$ and is defined as:

$$F(x) = P(X \le x, Y \le \infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

Similarly, the marginal distribution fun of y is denoted by,

$$F(y) = P(X \le \infty, Y < y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x,y)\,dx\,dy$$

<u>Example:</u> The joint prob density function of two rand var X and Y is given by,

$f(x,y) = 4xy$; $0 < x < 1$ and $0 < y < 1$ $= 0$; otherwise,

(i) Find the marginal density function X and Y.
(ii) Find the conditional density function of X, given Y and Y given X.
(iii) Check whether X and Y are independent.
(iv) Find $P(x \le 0.4, y \le 0.6)$ (= navaye pang kehi farak pardai)
(v) Find $P(X \le 0.4 / Y \le 0.6)$

Soln Given, joint pdf is:

$f(x,y) = 4xy$; $0 \le x < 1$ and $0 < y < 1$
$= 0$; otherwise

Marginal distribution of X,

↳ $f(x) = \int_y f(x,y)\,dy = \int_0^1 4xy\,dy = 4x\left[\frac{y^2}{2}\right]_0^1$

$\therefore f(x) = 2x \times 1 = 2x$,

Similarly, the marginal distribution of Y,

$$f(y) = \int_x f(x,y)\,dx$$

$$= \int_0^1 4xy\,dx$$

$$= 4y \left[\frac{x^2}{2}\right]_0^1$$

$$\therefore f(y) = 2y \quad ; \quad 0 < y < 1$$

(ii) Conditional density function of X, given Y and Y given X

↳ Soln $f(x/y)$ = cond dent fun ~~when~~ x given y:

$$= \frac{f(x,y)}{f(y)}$$

$$= \frac{4xy}{2y}$$

$$= 2x$$

Similarly, the cond prob fun y given x,

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{4xy}{2x}$$

$$\therefore \quad = 2y$$

(iii) checking whether X and Y are independent or not.

↳ $f(x,y) = f(x) \cdot f(y)$

$$4xy = 2x \cdot 2y$$

$0 < x < 1$ $\quad \therefore 4xy = 4xy$ Hence independent.

(iv) $P(x \le 0.4, y \le 0.6)$

Ans:

$$= \int_0^{0.4}\int_0^{0.6} f(x,y)\,dx\,dy$$

$$= \int_0^{0.4}\left[\int_0^{0.6} 4xy\,dy\right]dx$$

$$= \int_0^{0.4} 4x\left[\frac{y^2}{2}\right]_0^{0.6} dx$$

$$= \int_0^{0.4} 2x(0.6)^2\,dx$$

$$= 2\times 0.36\left[\frac{x^2}{2}\right]_0^{0.4}$$

$$= 0.36 \times 0.16$$

$$= 0.0576$$

(v) $P(x \le 0.4 / y \le 0.6)$

$$= \frac{P(x \le 0.4, y \le 0.6)}{P(y \le 0.6)} \quad \text{—(i)}$$

$P(y \le 0.6)$

$$= P(x \le \infty, y \le 0.6)$$

$$= \int_0^{1}\int_0^{0.6} f(x,y)\,dx\,dy$$

② The joint prob density fun of two random variable x and y, is given by

$$f(x,y) = e^{-(x+y)}, \quad x \geq 0, \; y \geq 0$$

Then find,

(i) Find the marginal density fun of X and Y,

(ii) Find the conditional density fun of X, given Y and Y given X

(iii) check whether X and Y are independent.

Soln Here given,

joint prob density fun of x, y

$$f(x,y) = e^{-(x+y)}, \quad x \geq 0, \; y \geq 0$$

i) The marginal density fun of X,

$$F(x) = P(X \leq x, Y \leq \infty)$$

$$= \int \int f(x,y)\, dx\, dy$$

(i), $f(x) = \int_y f(x,y)\, dx = \int_0^{\infty} e^{-(x+y)}\, dx$

$$= \frac{e^{-x+y}}{-x+y} \cdot \frac{x^2}{2}$$

Q. The joint prob density fun of two rand var X and Y is given by

$$f(x,y) = Axy \; ; \; 0<x<1$$
$$\text{and } 0<y<x$$

(i) Find A
(ii) Marginal distribution of x and y.
(iii) Condition distribution of x given y and Y given x.
(iv) Are x and Y independent?

↳ Since the given joint fun is,
Joint Pdf.

Therefore, $\iint_{xy} f(x,y)\,dx\,dy = 1$

$$\Rightarrow \int_x \left[\int_y f(x,y)\,dy\right] dx = 1$$

$$\Rightarrow \int_0^1 \int_0^x Axy\,dy\,dx = 1$$

$$\Rightarrow \int_0^1 Ax\left[\frac{y^2}{2}\right]_0^x dx = 1$$

$$\Rightarrow \int_0^1 \frac{Ax}{2} x^2\,dx = 1$$

$$\Rightarrow \frac{A}{2}\int_0^1 x^3\,dx = 1$$

$$\Rightarrow \frac{A}{2}\left[\frac{x^4}{4}\right]_0^1 = 1$$

$$\Rightarrow \frac{A}{8} \times 1 = 1$$

$$\therefore A = 8$$

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Therefore, the joint Pdf becomes,

$$f(x,y) = 8xy \; ; \; 0 < x < 1, \text{ and } 0 < y < x$$

(2) Marginal distribution of X,

$$f(x) = \int_y f(x,y)\,dy$$

$$= \int_0^x 8xy\,dy$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x$$

$$= 4x\,x^2$$

$$\therefore f(x) = 4x^3$$

Marginal distribution of Y,

$$f(y) = \int_x f(x,y)\,dx$$

$$= \int_y^1 f(x,y)\,dx$$

$$= \int_y^1 8xy\,dx$$

$$= 8y\left[\frac{x^2}{2}\right]_y^1$$

$$= 4y\left[1 - y^2\right]$$

$$\therefore f(y) = 4y - 4y^3$$

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Q. Suppose that the random variable x and y have joint pdf,

$$f(x,y) = kx(x-y) \; ; \; 0<x<2 \text{ and } -x<y<x$$
$$= 0 \; ; \text{ otherwise}$$

(a) Find the value of k,
(b) Find conditional probabilities.
(c) Find marginal density function of x and y.
(d) Find the conditional density fun of x given y & y given x.

Mathematical Expectation (Short notes) {Mean value of r. variable}

↳ Let X be a random variable, the expected value (or Mean) of random variable X, is denoted by,

E(X) called Expectation and is given by,

$$E(X) = \sum_x x \cdot P(X=x) \quad \text{if X is descrete r.v}$$

Similarly,

$$E(X) = \int_x x f(x)\,dx \; , \text{ if X is continuous.}$$

Q) Let X be a random variable, with the following prob distribution

$X=x$	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	2k	0.3	k

find E(X), find the mean of X,

Here, we know that the total probability is 1.

$$\therefore 0.1 + k + 0.2 + 0.3 + 2k + k = 1$$
$$\therefore 4k = 1 - 0.6 = 0.4/4 = 0.1$$
$$\therefore k = 0.1$$

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Q. Suppose that the random variable X and Y have joint pdf,

$$f(x,y) = kx(x-y) \; ; \; 0<x<2 \text{ and } -x<y<x$$
$$= 0 \; ; \text{ otherwise}$$

(a) Find the value of k,
(b) Find conditional probabilities.
(c) Find marginal density function of x and y.
(d) Find the conditional density fun of x given y & y given x.

Mathematical Expectation (Short notes) {Mean value of r. variable}

(expectation)

↳ Let X be a random variable, the expected value (or Mean) of random variable X, is denoted by,

$E(X)$ called Expectation and is given by,

$$E(X) = \sum_x x \cdot P(X=x) \quad \text{if X is descrete r.v}$$

Similarly,

$$E(X) = \int_x x f(x)\,dx \quad , \text{if X is continuous.}$$

Let X be a random variable, with the following prob distribution

$X=x$	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	2k	0.3	k

find E(X), find the mean of X,

Here, we know that the total probability is 1.

$$\therefore 0.1 + k + 0.2 + 0.3 + 2k + k = 1$$
$$\therefore 4k = 1 - 0.6 = 0.4/4 = 0.1$$
$$\therefore k = 0.1$$

Here, the given prob distribution is descrete in nature. so,

$$E(x) = \sum x \cdot P(X=x)$$
$$= -2\cdot(0.1) - 1\times 0.1 + 0\times 0.2 + 1\times 0.2 + 2\times 0.3 + 3\times 0.1$$

$$= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$$
$$= 0.8$$

→ Therefore, the prob distribution, becomes,

$x=x$	$P(X=x)$	$x\cdot P(X=x)$
-2	0.1	-0.2
-1	0.1	-0.1
0	0.2	0
1	0.2	0.2
2	0.3	0.6
3	0.1	0.3
		$\sum x P(X=x) = 0.8$

Thus, the mean (Expectation) of $X = 0.8$

(Q2) Let X be a continuous random variable with pdf

$f(x) = 2x, \quad 0 < x < 1$

find mean $[E(X)] = ?$

Soln Here the given random variable is continuous where $f(x) = 2x$, given $0 < x < 1$,

Thus, the expected value of X,

$$E(X) = \int_{x} x f(x)\,dx$$

$$= \int_0^1 x \cdot 2x\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} \times 1$$

$$\therefore E(x) = \frac{2}{3}$$

Hence, the mean (mathematical expection) of X is $\frac{2}{3}$.

Properties

(i) $E(a) = a$, where a is constant,

↳ $E(1) = 1$

(ii) $E(ax) = aE(X)$, where a is constant

↳ $E(2x) = 2E(x)$

(iii) $E(a+bX) = a + bE(X)$, where 'a' and 'b' are constant.

iv) $E(X \pm Y) = E(X) \pm E(Y)$

) Let X and Y are independent

$$E(XY) = E(X) \cdot E(Y)$$

(vi) Let $g(x)$ be a fun of rand variable, X, Then, the expectation of $g(X)$ is given by

$$E[g(x)] = \sum_{x} g(x) \cdot P(X = x)$$

for descrete

similarly,

$$E[g(x)] = \int_{x} g(x) f(x) dx$$

for continuous.

<u>Example,</u>

Let $g(x) = x^2$

Then,

$$E[g(x)] = E(X^2)$$

$$= \sum_{x} x^2 \cdot P(X = x)$$

similarly,

$$E[g(x)] = E(X^2)$$

$$= \int_{x} x^2 f(x) dx$$

Let X be a Random variable, with prob distribution,

$X = x$	-2	-1	0	1	2	3
$P(X=x)$	0.1	0.1	0.2	0.2	0.3	0.1

(i) find $E(X)$

Soln

$$E(x) = \sum x \cdot P(X = x)$$

$$= -2 \times 0.1 - 1 \times 0.1 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.1$$

$$= \cancel{-0.2} - 0.1 + \cancel{0.2} + 0.6 + 0.3$$

$$= 0.8$$

(ii) $E(X^2) =$

$= \cancel{E(X \cdot X)}$

$= \sum_x x^2 \cdot P(X=x)$

$= 4 \times 0.1 + 1 \times 0.1 + 1 \times 0.2 + 9 \times 0.3 + 9 \times 0.1$

$= 0.4 + 0.1 + 0.2 + 1.2 + 0.9$

$= 1.6 + 1.2$

$= 2.8$

(iii) $E(2+2x)$

Soln $= 2 + 2E(X)$

$= 2 + 2 \times 0.8$

$= 2 + 0.16$

$= 2.16$

(iv) $E(2X^2+3)$

$= 2E(X^2) + 3$

$= 3 + 2E(X^2)$

$= 3 + (2 \times 2.8)$

$= 3 + 5.6$

$= 8.6$ //

Q. For $f(x) = 2x$; $0 < x < 1$

Find $E(X^2)$ and $E(X+3)$

↳ second raw moment

Soln Here, $f(x) = 2x$ where $0 < x < 1$

To find, $E(X^2) = ?$

$E(X+3) = ?$

$\therefore E(X) = \int_{x} x f(x)\, dx$

$= \int_0^1 x(2x)\, dx$

$= 2\left[\frac{x^3}{3}\right]_0^1$

$= \frac{2}{3}$

$E(X^2) = \int_0^1 x^2 f(x)\, dx$

$= \int_0^1 x^2 \cdot 2x\, dx$

$= 2\int_0^1 x^3\, dx$

$= 2\left[\frac{x^4}{4}\right]_0^1 dx$

$= \left[\frac{x^4}{2}\right]_0^1$

$= \frac{1}{2}$ //

(2) Examples: Let x be a continuous random variable with with pdf $f(x) = xe^{-x}$ where $x \geq 0$,

(i) find $E(X)$

(ii) $E(X^2)$

Soln Here, given pdf of continuous random variable is;

$f(x) = xe^{-x}, \; x \geq 0$

Variance (measurement of variation)

i) Let x be a random variable, with expectation E(x), the variance of x is denoted by,
$V(X) = Var(X)$ and is given by,

$$V(X) = E[X - E(X)]^2$$

on simplification

$$V(X) = E(X^2) - [E(X)]^2$$

Properties

(i) $Var(a) = 0$, where a is constant

(ii) $Var(aX) = a^2 Var(X)$, a is const

(iii) $Var(aX+b) = a^2 Var(X)$, where $Var(b) = 0$, a & b are const.

(iv) $Var(x \pm y) = Var(x) + Var(y) \pm 2Cov(X, Y)$
where $Cov(x,y)$ = Covariance of X and y.
But if x and y are independent,

$$Var(X \pm Y) = Var(X) + Var(Y), \text{ since } Cov(X,Y) = 0$$

v) Covariance: It measures variation of 2 random variable simultaneously.
Let X and y are two rand variables with expectations E(X) and E(y) respectively, the covariance of X and y is, denoted by

$Cov(x,y) = \sigma_{xy}$ and is given by

$$Cov(x,y) = E[X - E(X)][y - E(y)]$$

on simplification, we get,

$$\boxed{Cov(x,y) = E(Xy) - E(X).E(y)}$$

for x, y descrete

Here, $E(xy) = \sum_x \sum_y xy\, P(X, Y)$ =

$$= \sum_x \sum_y xy\, P(X=x, Y=y)$$

for x, y, continuous,

$$E(X,Y) = \int_x \int_y xy\, f(x,y)\, dx\, dy$$

examples: Let X and Y are two random variables with probability distribution

y \ x	2	3
1	0.1	0.15
2	0.2	0.3
3	0.1	0.15

(i) find the marginal distribution of X and Y
(ii) Find the conditional distribution of X given $Y=1$
(iii) " " " " Y given $x=2$,
(iv) Are X and Y independent?
(v) $E(X)$, $E(Y)$, $V(X)$, $V(Y)$, $cov(x,y)$

Soln Here,
(i) for the marginal distribution of X

$$P(X=x), = \sum_y P(x=x, Y=y)$$

when, $x=2$,

$$P(X=2) = \sum_{y=1}^{3} P(X=2, Y=y)$$

$$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3)$$

$$= 0.1 + 0.2 + 0.1 = 0.4$$

when $x=3$

$$P(X=3) = \sum_{y=1}^{3} P(X=3, Y=y)$$

$$= 0.15 + 0.3 + 0.15$$

$$= 0.6$$

$\therefore$ Marginal distribution of X

$X=x$	2	3
$P(X=x$	0.4	0.6

for marginal distribution of Y,

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

when $y=1$

$$P(Y=1) = \sum_{x=2}^{3} P(X=x, Y=1)$$

$$= P(X=2, Y=1) + P(X=3, Y=1)$$

$$= 0.1 + 0.15$$

$$= 0.25$$

when $y=2$,

$$P(Y=2) = \sum_{x=2}^{3} P(X=x, Y=2)$$

$$= 0.2 + 0.3$$

$$= 0.5$$

when $Y=3$,

$$P(Y=3) = 0.1 + 0.15$$

$$= 0.25$$

$\therefore$ The marginal distribution of Y,

$Y=y$	1	2	3
$P(Y=y)$	0.25	0.5	0.25

(ii) For the conditional distribution of X given y = 1

$$P(X=x/Y=1) = \frac{P(X=x, Y=1)}{P(Y=1)} \quad \text{--- (i)}$$

when $x=2$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)}$$

$$= \frac{0.1}{0.25}$$

when $x=3$,

$$P(X=3/Y=1) = \frac{0.15}{0.25}$$

(iii) for the cond, prob of Y given X = 2,

$$P(Y=y/x=2) = \frac{P(Y=y, X=2)}{P(X=2)} \quad \text{--- (i)}$$

when $y=1$,

$$P(Y=1/x=2) = \frac{P(Y=1, X=2)}{P(X=2)}$$

$$= \frac{0.1}{0.4}$$

$$= \frac{1}{4}$$

when $y=2$,

$$P(Y=2/X=2) = \frac{0.2}{0.4} = \frac{1}{2}$$

when $y=3$

$$P(Y=3/X=2) = \frac{0.1}{0.4} = 1/4$$

(4) Are X and Y independent.

↳

(5) $E(X) = \sum_x x P(X=x)$

$= 2 \times 0.4 + 3 \times 0.6$

$= 0.8 + 1.8$

$= 2.6$

$E(Y) = \sum_y y P(Y=y)$

$= 1 \times 0.25 + 2 \times 0.5 + 3 \times 0.25$

$= 0.25 + 1 + 0.75$

$= 2$

$V(X) = E(X^2) - [E(X)]^2$

$= \sum x^2 P(X=x) - (2.6)^2$

$= 4 \times 0.4 + 9 \times 0.6 - (2.6)^2$

$= 1.6 + 5.4 - 6.76$

$= 0.24$

$V(Y) = E(Y^2) - [E(Y)]^2$

$= \sum y^2 P(Y=y) - (2)^2$

$= 0.25 + 2 + 9 \times 0.25 - 4$

$= 2.25 + 2.25 - 4$

$= 0.5$

$Cov(x,y) = E(xy) - E(X)\cdot E(y)$ —(i)

$E(xy) = \sum_x \sum_y xy P(X=x, Y=y)$

$= 2\times0.1 + 3\times0.15 + 4\times0.2 + 6\times0.3 + 6\times0.1 + 9\times0.15$

$= 5.2$

from (i)

$cov(x,y) = 5.2 - 2.6\times2$

$= 0$

$\therefore$ since $cov(x,y)=0$, so x and y are independent.

Q. Let X and y are the continuous random variable with Joint pdf,

$f(x,y) = Axy$; $0<x<1$ and $0<y<1$,

Find $E(X)$, $E(y)$, $V(x)$, $V(y)$ and $cov(x,y)$.

Soln Here

$f(x,y) = Axy$ — (A)

$E(X) = \int_x x\cdot f(x)\,dx$ —(i)

$f(x) = \int_y f(x,y)\,dy$

$= \int_0^1 Axy\,dy$

$= Ax\left[\frac{y^2}{2}\right]_0^1$

$= \frac{Ax}{2}\times1$

$= \frac{Ax}{2} = \frac{4x}{2} = 2x$

$f(y) = \int_x f(x,y)\,dx$

$= \int_0^1 Axy\,dx$

$= \frac{Ay}{2}\left[x^2\right]_0^1$

$= \frac{Ay}{2}$

$= \frac{4y}{2}$

$= 2y$

$\therefore E(x) = \int_x x f(x)\,dx$

$= \int_0^1 x \frac{Ax}{2}\,dx$

$= \frac{A}{2}\left[\frac{x^3}{3}\right]_0^1$

$= \frac{A}{6} \times 1$

$= A/6 = \frac{4}{6} = 2/3$

$\therefore E(y) = \int_0^1 y \frac{Ay}{2}\,dy$

$= \frac{A}{2}\left[\frac{y^3}{3}\right]_0^1$

$= \frac{A}{6} = \frac{4}{6} = \frac{2}{3}$

Here joint pdf $= 1$

$\int_x \int_y f(x,y)\,dy\,dx = 1$

$\int_0^1 \int_0^1 Axy\,dy\,dx = 1$

$\int_0^1 Ax\left[\frac{y^2}{2}\right]_0^1 dx = 1$

$\int_0^1 \frac{Ax}{2}\,dx = 1$

$\frac{A}{2}\left[\frac{x^2}{2}\right]_0^1 = 1$

$\frac{A}{4} = 1$

$A = 4$

$= \int_0^1 x\,2x\,dx$

$= 2\left[\frac{x^3}{3}\right]_0^1$

$= \frac{2}{3}$

$V(x) = E(x^2) - [E(x)]^2$

$= \int_0^1 x^2 f(x)\,dx - (2/3)^2$

$= \int_0^1 x^2\,2x\,dx - (2/3)^2$

$= 2\left[\frac{x^4}{4}\right]_0^1 - (2/3)^2$

$= \frac{1}{2} - (2/3)^2$

$= 1/18$

$V(y) = E(y^2) - [E(y)]^2$

$= \int_0^1 y^2 f(y)\,dy - (2/3)^2$

$= \int_0^1 y^2 \cdot 2y\,dy - (2/3)^2$

$= 2\left[\frac{y^4}{4}\right]_0^1 - \frac{4}{9}$

$= \frac{1}{18}$ //

$$cov(x,y) = E(xy) - E(x) \cdot E(y) \quad \text{--- (1)}$$

$$E(xy) = \int_x \int_y xy\, f(x,y)\, dx\, dy$$

$$= \int_x \int_y xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy\, Axy\, dx\, dy$$

$$= \int_0^1 \int_0^1 A x^2 y^2\, dx\, dy$$

$$= \int_0^1 A \left[\frac{x^3}{3}\right]_0^1 y^2\, dy$$

$$= \int_0^1 \frac{A}{3} y^2\, dy$$

$$= \frac{A}{3}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{A}{9} = \frac{4}{9}$$

from (i)

$$cov(x,y) = \frac{4}{9} - \frac{2}{3} \times \frac{2}{3}$$

$$= \frac{4}{9} - \frac{4}{9}$$

$$= 0$$

Moment of Random Variable

X

1> It is also mean of random variables, of different power.

1> Moments are the mean of different powers of the deviations of the given observations from any arbitrary value.

Let X be a random variable. Let r be any positive integer and 'c' be any arbitrary value (constant), then

The rth order of moments of random variable is, given by

$E[X-c]^r$ {Random variable's moments of r-th order}

where $r = 1, 2, 3 \ldots$

Raw moments (moments about the origin) i.e. $c = 0$.

Let X be a random variable, the r-th raw moment (moment about origin) of random variable X is denoted by μ_r' and is given by

$$\mu_r' = E(X)^r$$

where $r = 1, 2, 3, 4 \ldots \infty$

Central Moments

(Moments about the mean)

$c \Rightarrow$ mean

Let x be a random variable, the rth moment (moment about the mean) of random variable x is denoted by μ_r and is given by,

$$\mu_r = E[X - E(X)]^r$$

$$= E[X - \mu]^r$$

where $r = 1, 2, 3, 4 \ldots, \infty$

Raw moment continue

If x is a descrete random variable,

$$\mu_r' = E(X^r) \approx E(X)^r$$

$$= \sum_x x^r \cdot P(X = x)$$

Similarly, If x is continuous random variable,

$$\mu_r' = E(x^r) = \int_x x^r f(x)\,dx$$

(i) First order raw moment

$$\mu_1' = E(X) = \text{mean}$$

(ii) Secondorder raw moment

$$\mu_2' = E(X^2)$$

Note: (a) $\mu_0' = 1$ (b) $\mu_1' = E(X) = \mu$, the mean of X

central moment continue

↳ If x is descrete random var,

$$\mu_r = E(X-\mu)^r = \sum_X (x-\mu)^r P(X=x)$$

Similarly, if x is continuous random variable,

$$\mu_r = E(X-\mu)^r = \int_x (x-\mu)^r f(x)\,dx.$$

(*) for (r = 1) {First order central moment}

↳ $\mu_1 = E[X - E(X)]$

$= E(X) - E(X)$

$= \cancel{E(X)}\ 0$

For ($r=2$)

$\mu_2 = E[X - E(X)]^2 = \sigma^2$, variance of X,

<u>Note</u> : $\mu_0 = 1$

$\mu_1 = 0$

$\mu_2 = \sigma^2 = E[X - E(X)]^2$ (variance of x)

Relation between Raw moments and Central Moments

(a) $\mu_0 = \mu_0' = 1$
(b) $\mu_1 = \mu_1' - \mu_1' = 0$
(c) $\mu_2 = \mu_2' - (\mu_1')^2$
} VVI (need for our syllabus)

(d) $\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

(e) $\mu_4 = \mu_4' - 4\mu_3'\cdot\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

(Q.) Let x be a random variable with pdf.

$$f(x) = xe^{-x}; \; x \geq 0$$

(i) Find first four order of raw and central moment.
(ii) Find mean and variance of X

Solution Given pdf is

$$f(X) = xe^{-x} \; ; x \geq 0$$

Then, the rth raw moments given by

$$\mu_r' = E(X)^r = \int_x x^r f(X)\,dx$$

for $r=1$,

$$\mu_1' = \int_0^\infty x \; xe^{-x}\,dx = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left[x^2 \int e^{-x} dx - \int \left\{ \frac{d}{dx} x^2 \int e^{-x} dx \right\} dx \right]_0^\infty$$

$$= \left[x^2 \frac{e^{-x}}{-1} - \int 2x \frac{e^{-x}}{-1} dx \right]_0^\infty$$

$$= \left[x^2 e^{-x} + 2 \left\{ \frac{x e^{-x}}{-1} - \frac{e^{-x}}{-1} \right\} \right]_0^\infty$$

$$= \left[x^2 e^{-x} - 2x e^{-x} + 2 e^{-x} \right]_0^\infty$$

$$\therefore \int_0^\infty x^2 e^{-x} dx$$

$$= \int_0^\infty x^{3-1} e^{-x} dx$$

$$= \Gamma 3$$

$$= 2!$$

$$= 2$$

For $r=2$,

$$\mu_2' = E(X^2)$$

$$= \int_x x^2 f(x) dx$$

$$= \int_0^\infty x^2 x e^{-x} dx$$

$$= \int_0^\infty x^3 e^{-x} dx$$

$$= \Gamma 4$$

$$= 3!$$

$$= 6$$

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

$$= (n-1)!$$

Here the c.o. of x in e^{-x}, must be 1.

For $r=3$,

$$\mu_3' = \int_x x^3 f(x) dx$$

$$= \int_0^\infty x^3 x e^{-x} dx$$

$$= \int_0^\infty x^4 e^{-x} dx$$

$$= \Gamma 5$$

$$= 4!$$

$$= 24$$

For $r=4$,

$$\Gamma 6$$

$$= 5!$$

$$= 120$$

μ_1'

(ii) Now for central moments

↳ Here the $\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= 6 - (2)^2$$
$$= 6 - 4$$
$$\therefore \mu_2 = 2$$

Also, $\mu_3 = \mu_3' - 3\mu_1' \cdot \mu_2' + 2(\mu_1')^3$

$$= 24 - 3 \cdot 2 \cdot 6 + 2 \cdot (2)^3$$
$$= 24 + 16 - 36$$
$$= 4$$

So, the required central moment is also calculated.

(iii) Now, for the mean and variance of X,

↳ mean $(\mu_1') = E(X) = \mu = 2$

variance $(\mu_2) = \sigma^2 = E(X - E(X))^2$ — (i)

$$E(X) = \int_0^\infty x \cdot f(x)\,dx$$
$$= \int_0^\infty x \cdot x e^{-x}\,dx$$
$$= \int_0^\infty x^2 e^{-x}\,dx$$
$$= \Gamma 3 = 2! = 2$$

For (ii)

$$= E(X)^2 - E(X)^2$$
$$= 6 - 2^2$$
$$= 6 - 4$$
$$= 2$$

② Let X be a rand var with pdf,

$$f(x) = \theta e^{-\theta x} \; ; x \geq 0$$

(i) Find first four order of raw and central moments.
(ii) Find, mean and variance of X.

Soln Given, pdf is,

$$f(x) = \theta e^{-\theta} \; ; x \geq 0$$

The, r^{th} raw moment is given by,

$$\mu_r' = E(x^r)$$

short notes

Moment Generating Function

The MGF of a r.v is the function which is used to find the moments of a random variable.

Let X be a random variable, Then the moment generating function (MGF) of r.v. X is denoted by, and is given by,

$$\boxed{M_x(t) = E(e^{tx})} \; t \in R \quad \text{——— (i)},$$

If 'X is descrete r.v,

MGF, $M_x(t) = E(e^{tx})$

$$\therefore M_x(t) = \sum_x e^{tx} P(X=x)$$

Here $P(X=x)$ or $P(x)$ = pmf of X.

Similarly, if X is continuous r.v, MGF of X is,

$$M_x(t) = E(e^{tx})$$

$$= \int_x e^{tx} f(x)\,dx$$

Here, $f(x)$ = pdf of c.r.v X

Now, the rth Raw moments (i.e, raw moments is, moment about the origin) of r.v. x is given by,

$$\mu_r' = \frac{d^r M_x(t)}{dt^r}\Big|_{t=0}$$

Characteristic Function (VVI)

↳ It also helps us to find the moment of R.V, but the main difference between MGF and CF is that CF exists for each and every value (distribution) of R.V, ~~but~~ where as the MGF doesn't exists for all the distribution

Let X be a random variable, the characteristic function of x, is denoted by $\phi_x(t)$ and is given by,

$$\phi_x(t) = E\left(e^{itx}\right) \text{ for } t \in R \text{ (Real num)}.$$

If x is a d.r.v, then,

$$\phi_x(t) = E\left(e^{itx}\right) = \sum_x e^{itx} P(X=x)$$

Here, $P(X=x)$ or $P(X)$ = pmf of x,

Similarly,

If X is c.r.v, Then

$$\phi_x(t) = E\left(e^{itx}\right) = \int_x e^{itx} f(x)\,dx$$

Here, $f(x)$ = pdf of x

Now, the rth raw moment (~~moment about the origin~~) of r.v. x, is given by,

$$\mu_r' = \frac{1}{i^r} \frac{d^r[\phi_x(t)]}{dt^r}\Big|_{t=0}$$

Q. If X is a c.r.v with pdf,

$$f(x) = \theta e^{-\theta x}, \quad x \geq 0$$
$$= 0 \quad ; \text{otherwise}$$

(i) find the MGF of X.
(ii) find mean and Variance.

Sol. Here, the given pdf of c.r.v, is,

$$f(x) = \theta e^{-\theta x} \quad ; x \geq 0$$
$$= 0 \quad ; \text{otherwise},$$

So, the moment generating function of x is,

$$M_x(t) = \int_0^\infty e^{tx} \cdot f(x)\, dx \qquad \frac{1}{e^\infty}$$

$$= \int_0^\infty e^{tx}\, \theta e^{-\theta x}\, dx$$

$$= \int_0^\infty \theta e^{-(\theta - t)x}\, dx$$ [common have to be taken since if we don't take, ∞ will be an]

$$= \theta \left[\frac{e^{-(\theta - t)x}}{-(\theta - t)} \right]_0^\infty$$

$$= \frac{\theta}{-(\theta - t)} \left[e^{-\infty} - e^0 \right]$$

$$= \frac{\theta}{-(\theta - t)} \left[\frac{1}{e^\infty} - 1 \right]$$

$$= \frac{\theta}{-(\theta - t)} \times -1 = \frac{\theta}{\theta - t}$$

$$\therefore M_x(t) = \frac{\theta}{\theta - t}$$

Now, ~~mean~~ variance,

$$V(x) = E(X^2) - [E(X)]^2$$

$$= \int_0^\infty x^2 \cdot f(x)\,dx - \left[\int_0^\infty x f(x)\,dx\right]^2$$

$$= \int_0^\infty x^2 \theta e^{-\theta x}\,dx - \left[\int_0^\infty x\,\theta e^{-\theta x}\,dx\right]^2$$

$$= \theta \int_0^\infty x^2 e^{-\theta x}\,dx - \left[\theta \int_0^\infty x\,e^{-\theta x}\,dx\right]^2$$

~~∴ Mean~~

$$\therefore \text{Mean}[E(X)] = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x\,\theta e^{-\theta x}\,dx$$

$$= \theta \int_0^\infty x e^{-\theta \cdot x}\,dx$$

Now, the rth order raw moment is given by,

$$\mu_r' = \frac{d^r}{dt^r}\left[M_x(t)\right]\Big|_{t=0}$$

$$\frac{\theta - t}{\theta}$$

$$= 1 - t/\theta$$

for $r = 1$

$$\mu_1' = \frac{d}{dt}\left[\theta/\theta - t\right]$$

$$= \frac{d}{dt}\left[1 - \theta/t\right]$$

$$= 0 - \frac{d(\theta t^{-1})}{dt}$$

$$= 0 - \theta \frac{d}{dt}\left[t^{-1}\right]$$

$$= 0 - \theta(-1)\, t^{-2}$$

$$= \frac{\theta}{t^2}$$

$$\mu_1' = \frac{d}{dt}\left[\frac{\theta}{\theta - t}\right]$$

$$= \theta \frac{d}{d(\theta - t)}\left[\theta - t\right]^{-1} \times \frac{d(\theta - t)}{dt}$$

$$= \theta \times \frac{-1}{(\theta - t)^2} \times -1$$

$$= \frac{\theta}{(\theta - t)^2}$$

$$= \frac{\theta}{(\theta - 0)^2}$$

$$= \frac{\theta}{\theta^2} = 1/\theta$$

$$\mu_1' = \frac{\theta \cdot (-1) - (\theta - t)\,0}{(\theta - t)^2}$$

$$= \frac{-\theta}{(\theta - t)^2}$$

$\therefore$ Mean $= 1/\theta$

$$\mu_2' = \text{variance} = \frac{d^2}{dt^2}\left[\theta/\theta - t\right]\Big|_{t=0} = \mu_2' - (\mu_1')^2$$

$$= \frac{d}{dt}\left[\frac{\theta}{(\theta - t)^2}\right]\Big|_{t=0}$$

$$= \frac{\theta(-2)}{(\theta - t)^3} \times (-1) = \frac{2\theta}{(\theta - t)^3} = \frac{2\theta}{\theta^3} = \frac{2}{\theta^2}$$

$$= \frac{2}{\theta^2} - (1/\theta)^2$$

$$= \frac{2}{\theta^2} - 1/\theta^2 = 1/\theta^2$$

c.r.v

(2) If X is a ~~characteristics function~~, with pdf,

$$f(x) = \theta e^{-\theta x} \; ; \; x \geq 0$$
$$= 0 \; ; \text{ otherwise,}$$

(i) find c.f of X

Soln Here,

the given pdf of c.r.v is,

$$f(x) = \theta e^{-\theta x}$$

$\therefore$ Characteristic function, $\phi_x(t) = E(e^{itx})$

$$= \int_0^\infty e^{itx} f(x)\,dx$$

$$= \int_0^\infty e^{itx}\, \theta e^{-\theta x}\,dx$$

$$= \theta \int_0^\infty e^{itx} e^{-\theta x}\,dx$$

$$= \theta \int_0^\infty e^{-(\theta - it)x}\,dx$$

$$= \theta \left[\frac{e^{-(\theta - it)x}}{-(\theta - it)} \right]_0^\infty$$

$$= \frac{\theta}{-(\theta - it)} (0 - 1)$$

$$= \frac{\theta}{\theta - it}$$

$\therefore$ characteristic fun of X, $\phi_x(t) = \dfrac{\theta}{\theta - it}$

$\therefore$ Mean $(= \mu_1') = \dfrac{1}{i} \dfrac{d}{dt} \left[\theta / \theta - it \right]$

$$= \theta \times \frac{\theta - 1}{(\theta - it)^2} \times -i$$

$$= \frac{i\theta}{(\theta - it)^2}$$

exam

(5) Find the c.f of uniform distribution,

$$f(x) = \frac{1}{b-a} \; ; \; a<x<b$$

Also, find mean and variance;

(6) Let X be a random variable with prob distribution,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \; x=0,1,2,\ldots$$

(i) find MGF of X
(ii) find mean and variance.

Soln Here we have,

$$f(x) = \frac{1}{b-a} \; ; \; a<x<b$$

To find, c.f ⇒
mean ⇒
variance ⇒

Now,

$$c.f = \phi_x(t) = \int_a^b e^{itx} f(x)\,dx$$

$$= \int_a^b e^{itx} \frac{1}{(b-a)}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{itx}}{it}\right]_a^b$$

$$= \frac{1}{it(b-a)}\left[e^{itb} - e^{ita}\right]$$

$$\therefore c.f = \phi_x(t) = \frac{1}{(b-a)it}\left[e^{itb} - e^{ita}\right]$$

Also, mean $(\mu_1^{\prime}) = E[e^{itx}] =$

(2) variance $= \mu_2' - (\mu_1')^2$ — (i)

Here,

$$\mu_2' = \frac{1}{2i} \frac{d^2}{dt^2}\left[\frac{\theta}{\theta - it}\right]$$

$$= \frac{\theta}{2i} \frac{d}{dt}\left[\frac{i\theta}{(\theta - it)^2}\right]$$

$$= \frac{\theta^2}{2} \frac{d}{dt}\left[\frac{1}{(\theta - it)^2}\right]$$

$$\cancel{= \frac{\theta^2}{2} \times 2(\theta - it)}$$

$$= \frac{\theta^2}{2} \times \frac{-2}{(\theta - it)^3} \times -i$$

$$= \left.\frac{i\theta^2}{(\theta - it)^3}\right|_{t=0} = \frac{i\theta^2}{\theta^3} = \frac{i}{\theta} \;//$$

$\therefore$ variance $= \mu_2' - (\mu_1')^2$

$$= \frac{i\theta^2}{(\theta - it)^3} - \left[\frac{i\theta}{(\theta - it)^2}\right]^2$$

$$= \frac{i\theta^2}{(\theta - it)^3} + \frac{\theta^2}{(\theta - it)^4}$$

$$= \frac{i\theta^2}{\theta^3} + \frac{\theta^2}{\theta^4}$$

$$= \frac{i}{\theta} + \frac{1}{\theta^2} \;//$$

→ continuous

(3) Find the c.f of the uniform distribution $f(x) = \frac{1}{b-a}$; $a < x < b$. Also find mean and variance.

↳ Here, the given function is:

$$f(x) = \frac{1}{b-a} \quad ; \quad a < x < b$$

$\therefore$ c.f of the given v.d $\phi_x(t) = \int_x e^{itx} f(x)\,dx$

$$= \int_a^b e^{itx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \int_a^b e^{itx}\,dx$$

$$= \frac{1}{b-a} \left[\frac{e^{itx}}{it}\right]_a^b$$

$$= \frac{1}{(b-a)\,it} \left(e^{itb} - e^{ita}\right)$$

$$\phi_x(t) = \frac{e^{itb} - e^{ita}}{(b-a)\,it}$$

Now, mean $= \mu_1' = \frac{1}{i} \frac{d}{dt}\left[\phi_x(t)\right]$

$$= \frac{1}{i} \frac{d}{dt}\left[\frac{e^{itb} - e^{ita}}{(b-a)\,it}\right]$$

$$= \frac{1}{i(b-a)} \frac{d}{dt}\left[\frac{e^{ibt} - e^{iat}}{it}\right]$$

(4) Let x be a random variable, with prob distribution,

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

(i) Find MGF of x

(ii) Find mean & variance.

Soln So, Here the MGF of x,

$$MGF = M_x(t) = E(e^{tx})$$

$$\therefore M_x(t) = \sum_x e^{tx} P(X=x)$$

$$= \sum_x \frac{e^{tx} e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_0^\infty \frac{e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_0^\infty \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\left[\therefore M_x(t) = e^{\lambda(e^t - 1)}\right]$$

expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$\sum_0^\infty \frac{(e^t \lambda)^x}{x!}$$

Probability Distribution

1. Discrete probability distribution.
2. Continue " " .

→ Binomial Distribution
→ Poisson Distribution

Bernoulli Distribution (by James Bernoulli)

↳ Single performance of a random experiment is Bernoulli Trial

↳ only two possible outcomes i.e. success and failure
where success is required outcomes
failure is outcomes that is not required.

→ success is denoted by 1 i.e. P
→ Failure is denoted by 0 i.e. q

$\therefore p+q = 1$

R. variable → $\begin{cases} x=1 \rightarrow p \\ x=0 \rightarrow q \end{cases}$

These r.v follows Bernoulli trial, then,

If $X \sim B(1, p)$ ← prob of success,

$$\left[P(X=x) = p q^{1-x} \right], \text{ where } x=0, 1, 2$$

Note:- Binomial distribution is extension of Bernoulli distribution

VVI

VVI Short notes

Binomial Distribution

Conditions for Binomial distribution

(i) The no. of trials (sample size), n is fixed & finite.

(ii) The outcomes of the trial are binary (only 2 possible outcomes)

(iii) The probability of success is denoted by p and is constant for each trial, such that $p+q=1$

$\Rightarrow q = 1-p$ (Prob of failure)

(iv) Every trials are independent.

(Discrete Distribution)

Probability function of Binomial Distribution

↳ Let X be a random variable, that representant the number of success which follows Binomial distribution with parameters 'n' and 'p', then its probability mass function is given by,

$$P(X=x) = {}^nC_x \, p^x q^{n-x}, \text{ where } x = 0, 1, \ldots n$$

$$p(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where n = no. of trials (fixed)

X = no. of required success

p = probability of success (constant)

such that

$q = 1-p$

Note: If a random variable X follows binomial distribution with parameters n and p, we can write

$$\left[X \sim B(n, p)\right]$$

X follows Binomial distribution with n and p.

Properties :

① It is descrete probability distribution.
② R.v of Binomial distribution is No. of success
③ Mean of Binomial distribution is np
i.e. $E(X) = np$
④ Variance of Binomial distribution is npq.
⑤ Standard deviation is $\sqrt{var} = \sqrt{npq}$

Examples: Determine the binomial distribution for which the mean 4 and variance is 3.

↳ $n = ?$
$p = ?$

Let X be a random variable which follows Binomial distribution with parameter n and p

[Let $X \sim B(n, p)$]

$E(X) = np = 4$
$p = 4/n$ —— (i)

$V(X) = npq = 3$
$\therefore n \cdot \frac{4}{n} \cdot q = 3$

$\therefore q = 3/4$

$p + q = 1$
$\therefore p = 1 - q = 1 - 3/4$
$= 1/4$

$np = 4$
So, $n = 4/p = 4/(1/4) = 16$ //

Example: For a Binomial distribution with $n=4$ and $p=0.45$

(i) $P(X=0)$

(ii) $P(X=2)$

(iii) $P(X\leq 2)$

(iv) $P(X>2)$

Soln Let $X \sim B(n,p)$, Then its pmf is:

$$P(X=x)=\binom{n}{x}p^x q^{n-x}$$

Here, $n=4$, $p=0.45$

$\therefore q=0.55$

$$\therefore P(X=x)=\binom{4}{x}(0.45)^x(0.55)^{4-x}$$

when $x=0$,

$$P(X=0)=\binom{4}{0}(0.45)^0(0.55)^4$$

$$P(X=2)=\binom{4}{2}(0.45)^2(0.55)^2$$

$$P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)$$

$$= \quad + \binom{4}{1}(0.45)^1(0.55)^3 +$$

VVI Examquestion

2021 fall/Sp. Rule:- 7×30

(1) According to a front page article in the wall street journal, 30% of all students in American universities miss classes due to drinking. If 10 students are randomly choosen, what is the probability that (i) exactly 3 students miss the classes due to drinking;

(ii) At most 3 students miss the class due to drinking

(iii) None of them miss the classes due to drinking.

Soln Given,

No. of student $(n) = 10$

probability of missing class $(p) = 30\% = 0.30$

$\therefore q = 1 - p = 0.70$

Let X = no. of students who miss the class,

$\therefore X \sim B(10, p)$

Then,

$$P(X=x) = {}^{n}C_{x}\, p^{x}\, q^{n-x} \quad \text{where } n = 0, 1, 2 \ldots n$$

(i) Exactly 3 student miss the classes

$$\therefore P(X=3) = {}^{10}C_{3}\, p^{3}\, q^{10-3}$$

$$= \frac{10!}{3! \times 7!} (0.3)^{3} (0.7)^{7}$$

$$\therefore P(X=3) = 0.26$$

(ii) At most 3 student miss the class due to drinking:

↳ Soln $(X \leq 3)$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) \quad \text{--(ii)}$$

$$= 0.648$$

$P(X=0) = {}^{10}C_{0}\, p^{0}\, q^{10-0}$ $= 0.028$	$P(X=1) = 0.121$ $P(X=2) = 0.233$	$P(X=3) = 0.266$

(iii) None of them,

$P(X=0) = 0.028$

2020 fall

② The normal rate of covid virus infection in certain city is known to be 35%. In an experiment with 10 persons, calculate the probability that

ⓐ Non of them are infected.

ⓑ At least 2 are infected.

ⓒ At most 2 are infected

ⓓ Exactly 5 are infected. (or 5 are infected)

Soln Here,

Total no. of person $(n) = 10$

probability of covid infection $(P) = 35\% = 0.35$

So, $q = 1 - P = 1 - 0.35$

$= 0.65$

ⓐ Non of them are infected $(P_{X=0})$.

↳

↳ Here we let X = no. of people who are covid infected.

So, we have

$X \sim B(n, p)$

Then,

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$

So, ⓐ None of them are infected $P(X=0)$

↳ $P(X=0) = {}^{10}C_0 \, (0.35)^0 (0.65)^{10-0}$

$= 0.013$

ⓑ At least 2 are infected :

↳ $P(X \geq 2)$

Here,

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [0.013 + 0.072]$

$P(X \geq 2) = 0.914$

(c) At most 2 are infected,

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= 0.013 + 0.072 + 0.175$

$= \cancel{0.175}$

$P(X \leq 2) = 0.260 = 0.26$

(d) Exactly 5 are infected

$P(X=5) = {}^{10}C_5 (0.35)^5 (0.65)^{10-5}$

$P(X=5) = 0.153$

2019 Fall

[Q7b (2b)] → almost same

(*) An information source emits a five digit message into a channel in a binary code. Each digit is choosen independently and ~~each~~ 1 is 1 with probability 0.3. Find the probability that the message contains

(a) exactly 3 1's

(b) four 1's

(c) Atleast two 1's

(d) ~~At least~~ none of the 1's

Sol: Here we are provided,

Total no. of digits message (n) = 5

Probability of getting 1 (P) = 0.3

$\therefore$ probability of not getting 1 $(q) = 1 - p = 1 - 0.3$
$= 0.7$

Let X = no. of 1's that we want,
$\therefore X \sim B(n, p)$

So, we have
$P(X = x) = {}^nC_x \, p^x q^{n-x}$ —— (i)

(i) Exactly 3 1's
$P(X = 3) = {}^5C_3 \, (0.3)^3 (0.7)^{5-3}$
$\therefore P(X = 3) = \cancel{0.266}$ 0.132 //

(ii) Four 1's
↳ So, $P(X$

$P(X = 4) = {}^5C_4 \, (0.3)^4 (0.7)^{5-4}$
$= 0.02835$

(iii) At least 2 1's
↳ $P(X \geq 2)$
$= \cancel{P(X=2) + P(X=3) + P(X=}$
$= 1 - P(X < 2)$
$= 1 - [P(X=0) + P(X=1) + \cancel{P(X=2)}]$
$= 1 - [0.16 + 0.36]$
$= 1 - 0.52$
$P(X \geq 2) = 0.479$ //

2015
3(a)

80m

(iv) none of them are 1's

$\Rightarrow P(X=0) = {}^5C_0 (0.3)^0 (0.7)^5$

$= 1 \times 1 \times 0.1168$

$= 0.1680$

So $P(X=0) = 0.1680$

2015 SP
3(a) It is known that 5% of the screws manufactured by an automatic machine are defective. If a sample of 20 screws is selected at random, find the prob that the sample contains,

(a) Exactly 2 defective screws
(b) At least 2 defective screws
(c) At most 2 defective screws
(d) No defective screws

Soln Here,
total no. of screws $(n) = 20$
probability of manufactured screws are defective $(p) = 5\% = 0.05$
probability of " are non defective $(q) = 1 - 0.05 = 0.95$

Here Let X = no. of defective screws,
$\therefore X \sim B(n, p)$
we have,
$P(X=x) = {}^nC_x \, p^x q^{n-x}$

$\therefore$ (i) Exactly 2 defective
$P(X=2) = {}^{20}C_2 (0.05)^2 (0.95)^{20-2}$

$= 0.188$

(ii) $P(X \geq 2) = 1 - P(X < 2)$
$= 1 - [P(X=0) + P(X=1)]$ $= 1 - (0.35 + 0.37)$
$= 0.27$

(iii) $P(x \le 2) = P(X=0) + P(X=1) + P(X=2)$
$= 0.35 + 0.37 + 0.188$
$= 0.908$

(iv) $P(X=0) = 0.35$

Poisson Distribution

When $n \ge 30$
& p is very very small,

→ It is used in Rare case Events,
e.g. (i) No of accidents per week
(ii) No. of suicide per year. etc

Events are defined on interval of time.

Poison, is the limiting case of Binomial Distribution under the following conditions,

1. If the no. of trials (sample size) is indefinitely large i.e $n \to \infty$
2. If the probability of success (p) is very small, i.e. $p \to 0$
3. $\boxed{np = \lambda,}$ is finite

Probability function (pmf)

↳ Let 'X' be a random variable that represents the number of success which follows "poisson distribution" with parameter λ, the probability of getting success is given by,

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

Here, x = required no. of success
λ = Parameter of the distribution (mean or variance)
e = exponential constant = 2.7183

If X follows poisson distribution with parameter λ then we can write as,

$$X \sim P(\lambda)$$

Properties

(a) Mean of poison distribution λ
i.e. $E(X) = \lambda$

(b) Variance of poisson distribution is:
$V(X) = \lambda$

Note: (a) Mean and variance are equal in poissons distribution
(b) Standard deviation of poisson distribution is $\sqrt{\lambda}$

Example

(a) Between the hour of 2pm and 4pm, the average no. of phone calls per minute is 9. Switch board of a company is 2.5. Find the probability that during any given moment

(1) No calls at all
(2) at least 2 calls
(3) at most 2 calls
(4) calls during any interval of 2 moments.

Sol: Given
average no. of phone calls per minute
$(\lambda) = 2.5$

Let X = no. of getting calls.
Now, from poissons distribution,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(X=x) = \frac{e^{-2.5}(2.5)^{x}}{x!}$$

where $x = 0, 1, 2, 3, \ldots$

(i) No calls at all

$$\hookrightarrow P(X=0) = \frac{e^{-2.5}(2.5)^{0}}{0!}$$

$$\therefore P(X=0) = 0.082$$

(ii) at least 2 calls,

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - \left[0.082) + \frac{e^{-2.5}(2.5)^{1}}{1}\right]$$
$$= 1 - [0.082 + 0.205]$$
$$= 1 - 0.28$$
$$= 0.7127$$

(iii) at most 2 calls

$$\hookrightarrow P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$
$$= 0.082 + 0.250 + \frac{e^{-2.5}(2.5)^{2}}{2!}$$
$$= 0.082 + 0.205 + 0.2565$$
$$\therefore P(X \leq 2) = 0.54$$

(iv) 4 calls per 2 minute

↳ Average call per minute $(x) = 2.5$

" " 2 " $(x) = 2.5 \times 2$
$= 5$ per 2 minute

Now $P(X=4) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

$= \frac{e^{-5}5^{4}}{4!}$

$= 0.1754$

(b) A life insurance sellsman sells the average of 3 life insurance policies per week. use poisson's law to calculate the probabilities that in a given week he will sell.

(i) some policies

(ii) 2 or more policies but less than 5 policies

(iii) Assuming that there are 5 working days per week. What is the propabilities that in a given day, he'll sell 1 policie?

Soln Average insurance policies per week $(\lambda) = 3$

Let X = insurance policies

Now, using poisson law,

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(X=x) = \frac{e^{-3}(3)^{x}}{x!}$$

where $x = 0, 1, 2, 3 \ldots$

(i) same policies,

$P(X \geq 1) = 1 - P(X=0)$
$= 1 - e^{-3}\, 3^{0}/0! = 0.9502$

(ii) Two or more policies but less than 5

Soln $P(2 \le x < 5) = P(X=2) + P(X=3) + P(X=4)$

$$= \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!}$$

$$= 0.616$$

(iii) Average insurance policies, per 5 days = 3

Average insurance policies in day $(\lambda) = 3/5$

$$P(X=1) = \frac{e^{-0.6}(3/5)^1}{1!}$$

$$= 0.329$$

(e) 7% of jobs

Normal Distribution (Almost sure question)

Date
Page

- Developed
- It is a type of continuous probability distribution widely used for all type of data.

Probability function

Let X be a random variable which follows normal distributions with parameters (μ) mean and variance (σ^2), then

The pdf is given by

$f(x) \propto f(x; \mu, \sigma^2)$ {prob fun of X with parameter: mean(μ) and varian(σ^2)}

$$\boxed{\therefore f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

Here σ and μ are parameters of normal distribution.

Note: If X follows normal distributions with parameter σ^2 and μ, the we can write

$\boxed{X \sim N(\mu, \sigma^2)}$ ⇒ reads: X follows Normal distribution with μ and σ^2

follows

→ Here X range varies from $-\infty$ to ∞

Standard Normal Distribution

{Needed since the μ and σ varies in Normal Dis}

Let $X \sim N(\mu, \sigma^2)$ then,

Define a new variable $\boxed{Z = \frac{X-\mu}{\sigma}}$

This Random variable is called Standard Normal variate (variable)

& its distribution is Standard Normal Distribution.

$\therefore$ Pdf of Z, $\boxed{f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2}}$; $-\infty < Z < \infty$

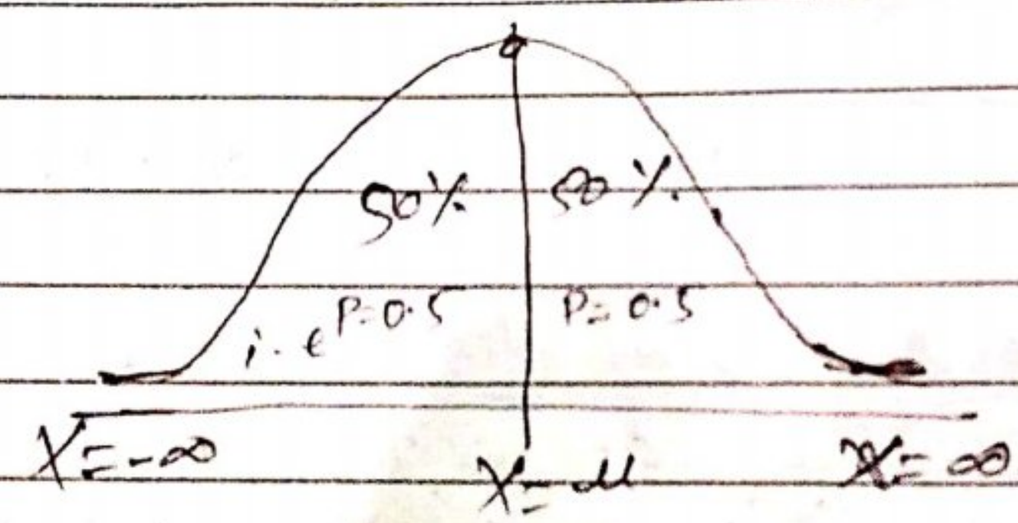
Note:- S.N. Distribution is the normal distribution with mean $(\mu) = 0$ and variance $(\sigma^2) = 1$

$\boxed{\{Z \sim N(0, 1)\}}$

Characteristics of N. Distribution (Also applicable to S.N.D)

⇒ (a) It is continuous prob. distribution

(b) ~~It~~ is curve of N. Distribution is bell shaped and symmetrical at $x = \mu$



(c) Mean, median, mode are same

(d) The total area (probability), under the curve is always 1

(e) The normal curve is asymptotic to the x-axis {i.e. the tail of the curve tends to x-axis but never touches}

(f) Area property:

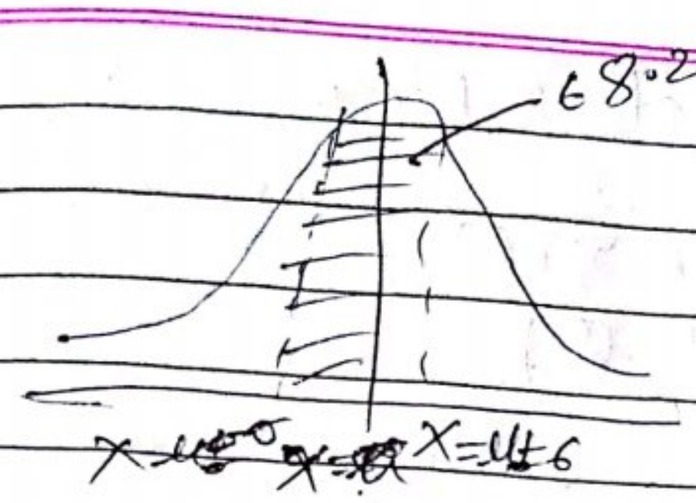
(i) If $x = \mu \pm \sigma$, $z = \frac{\mu \pm \sigma - \mu}{\sigma}$

$\therefore z = \pm 1$

$\therefore P(\mu - \sigma \le x \le \mu + \sigma) = P(-1 \le z \le 1)$

$= 68.26\%$ of total

68.26% of total



i.e. 68.26% of total area is occupied whenever 1σ distribution is done

(ii) If $x=\mu\pm 2\sigma$, $z=\pm 2$

$$P(\mu-2\sigma \le x \le \mu+2\sigma) = P(-2\le z\le 2) = 0.9544$$

95.44% of total area

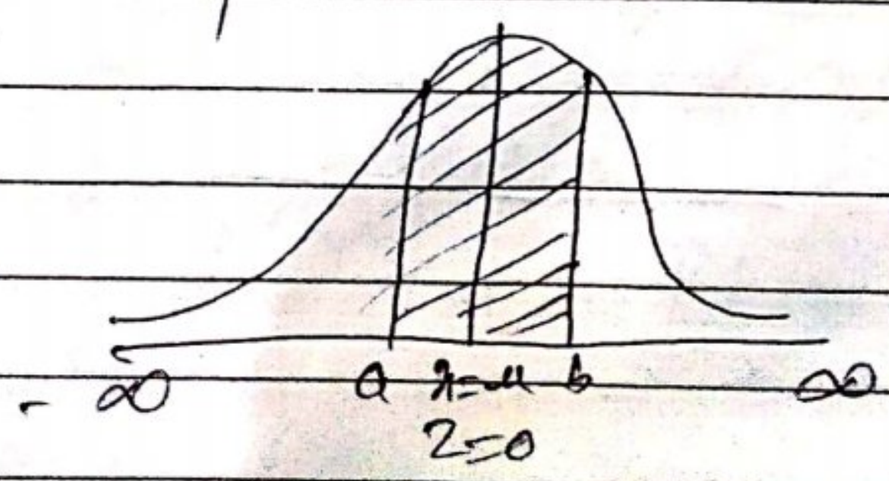
(iii) If $x=\mu\pm 3\sigma$, $z=\pm 3$

$$P(\mu-3\sigma \le x \le \mu+3\sigma) = P(-3\le z\le 3)$$
$$= 0.997$$

Computation of Area under the normal curve

① $P(a\le x\le b)$

if $a<\mu$ and $b>\mu$



∴ expressing in terms of z,

$$P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right) = P(-z_1 \le z \le z_2)$$
$$= P(-z_1 \le z < 0) + P(0 \le z \le z_2)$$
$$= P(0 \le z \le z_1) + P(0 \le z \le z_2)$$

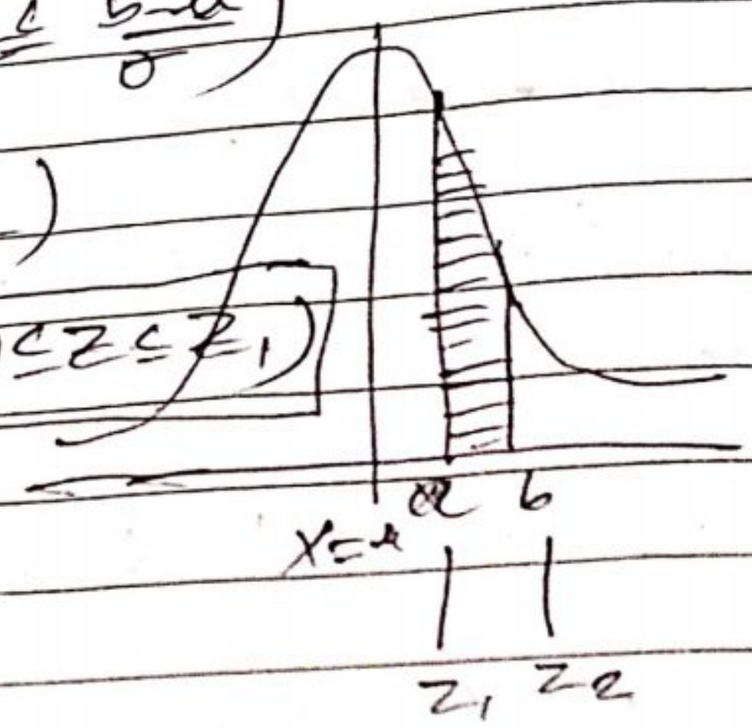
which is find from Normal table,

(ii) $P(a \le x \le b)$, where $(a, b > \mu)$,

$P(a \le x \le b) = P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right)$

$= P(z_1 \le z \le z_2)$

$\boxed{= P(0 \le z \le z_2) - P(0 \le z \le z_1)}$



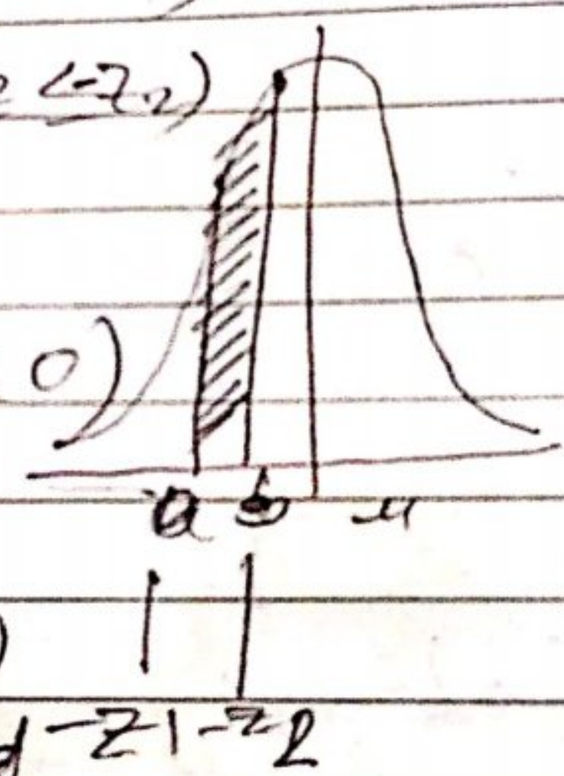
(iii) If $a, b < \mu$, $P(a \le x \le b)$

↳ $P(a \le x \le b) = P(-z_1 \le z \le -z_2)$

$= P(-z_1 \le z \le 0)$
$- P(-z_2 \le z \le 0)$

$= P(0 \le z \le z_1)$
$- P(0 \le z \le z_2)$

{Due to the symmetrical distribution}



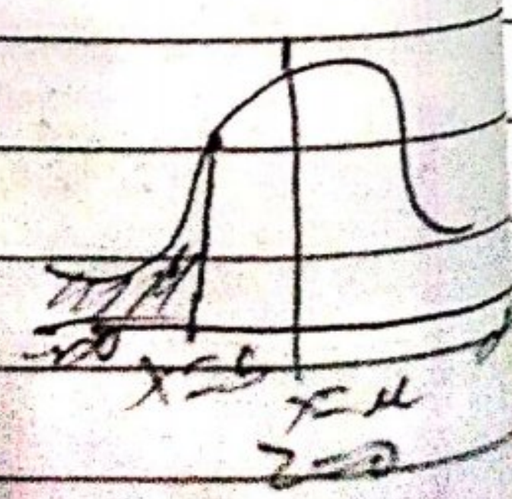
(2) $P(X \le b)$

(i) (if) $b < \mu$

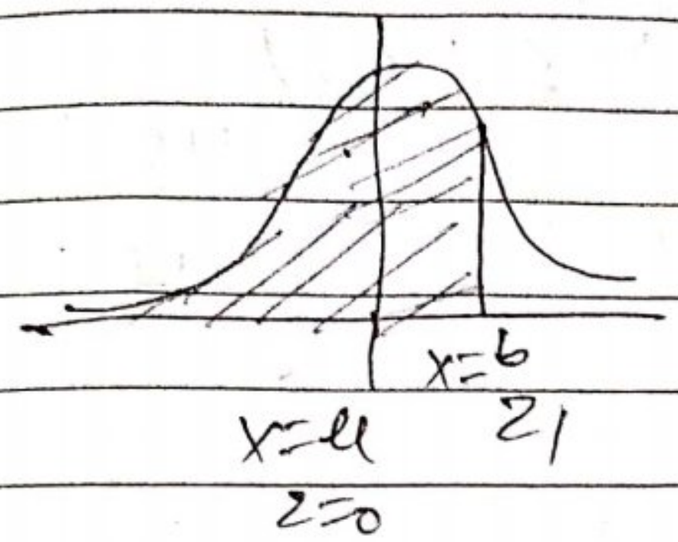
$P(X \le b) = P(z < -z_1)$

$= 0.5 - P(-z_1 < z < 0)$

$= 0.5 - P(0 < z < z_1)$

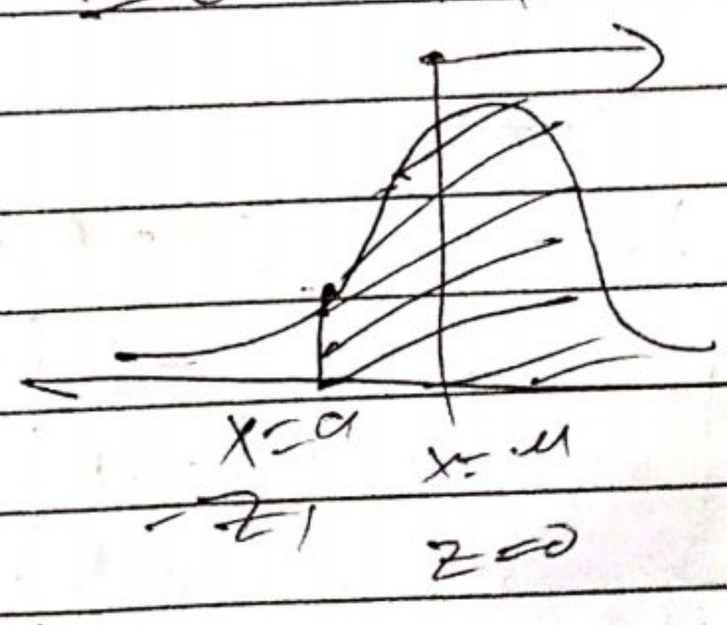


(ii) if $b > \mu$



$p(x < b) = p(z < z_1)$
$= 0.5 + p(0 < z < z_1)$

(5) if $(x > a)$
(i) if $(a < \mu)$
$\hookrightarrow p(x > a) = p(z \geq -z_1)$
$= 0.5 + p(-z_1 < z < 0)$
$= 0.5 + p(0 < z < z_1)$



(b) if $a > \mu$
$p(x \geq a) = p(z \geq z_1)$
$= 0.5 - p(0 < z < z_1)$

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Page

Example: μ σ^2

(1) Let $X \sim N(15, 16)$, find $P(X \leq 12)$, $P(12 \leq x \leq 17)$, $P(15 \leq X \leq 17)$, $P(X \geq 17)$, $P(X \geq 12)$, $P(X \leq 17)$, $P(17 \leq X \leq 18)$, $P(12 \leq X \leq 104)$

$P(X)$ or $P(X <)$ } Same

Sol^n: Let ~~X~~ ~~fall~~

$X \sim N(\mu, \sigma^2)$, we have

$\mu = 15$

$\sigma^2 = 16$

$\therefore \sigma = \sqrt{16} = 4$

So, define a new variable,

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$= \frac{X-15}{4}$$

(i) $P(X \leq 12)$

When $X = 12$,

$$Z = \frac{12-15}{4}$$

$$= \frac{-3}{4}$$

$$= -0.75$$

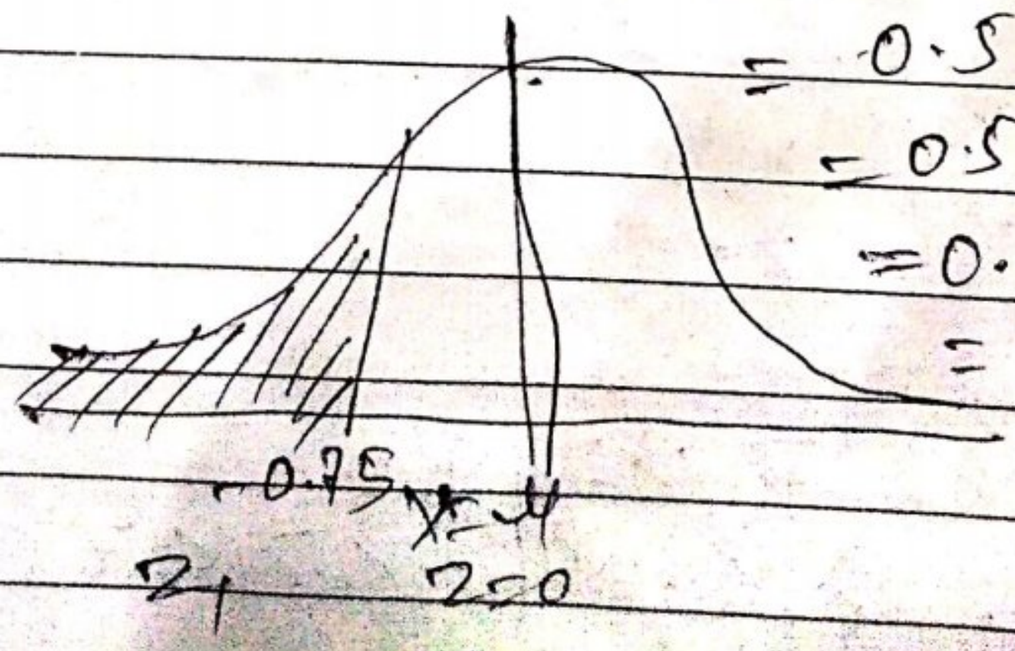
$\therefore P(X \leq 12) = P(Z \leq -0.75)$

$= 0.5 - P(-0.75 \leq Z \leq 0)$

$= 0.5 - P(0 \leq Z \leq 0.75)$ [Due to symmetry]

$= 0.5 - 0.2734$ (from S.N.T)

$= 0.2266$

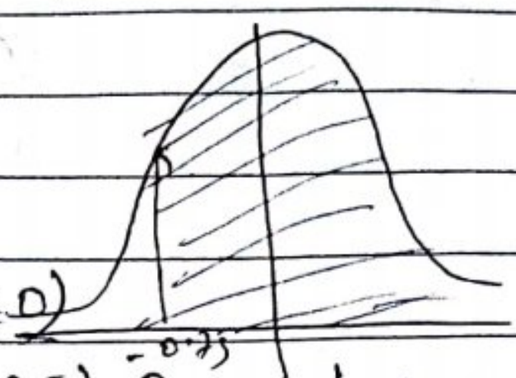


Standard N.T

(ii) $P(X \geq 12)$

↳ when $x = 12$, $z = \frac{12-15}{4} = -0.75$

$P(X \geq 12) = P(z \geq -0.75)$



$= \cancel{P(0.5}$

$= 0.5 + P(-0.75 \leq z \leq 0)$

$= 0.5 + P(0 \leq z \leq 0.75)$ Due to symmetry

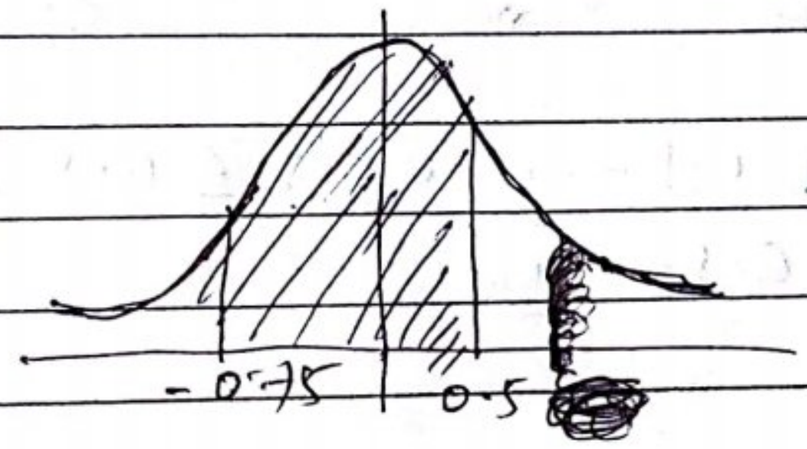
$= 0.5 + 0.2734$ (from table)

$= 0.734$

(iii) $P(12 \leq x \leq 17)$

↳ when $x = 12$, $z = -0.75$

when $x = 17$, $z = \frac{17-15}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$



$P(-0.75 \leq x \leq 0.5)$

$= P(-0.75 \leq x \leq 0) + P(0 \leq z \leq 0.5)$ (Due to symmetry

$= P(0 \leq x \leq 0.75) + P(0 \leq z \leq 0.5)$

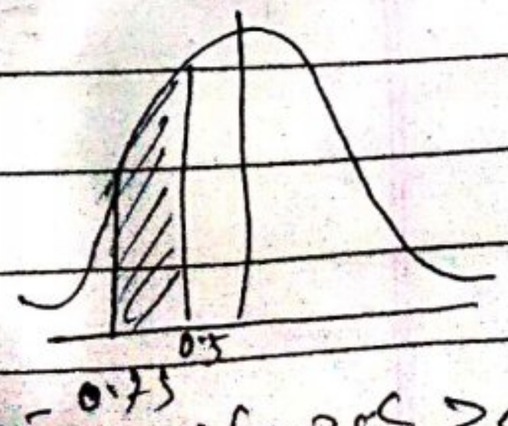
$= 0.2734 + 0.1915$ (~~Due to~~ from table)

$= 0.4649$

(iv) $P(12 \leq x \leq 14)$

when $x = 12$, $z = -0.75$

$x = 14$, $z = \frac{14-15}{2} = -0.25$



$P(-0.75 \leq z \leq -0.25) = \cancel{P(-0.75 \leq z \leq 0) - P(0.25 \leq z \leq 0)}$

$= P(0 \leq z \leq 0.75) - P(0 \leq z \leq 0.25)$ $= 0.2734 - 0.0987$

$= 0.1747$

$P(X \geq 17)$

Soln Here $x = 17$

$$\therefore z = \frac{x-\mu}{\sigma}$$

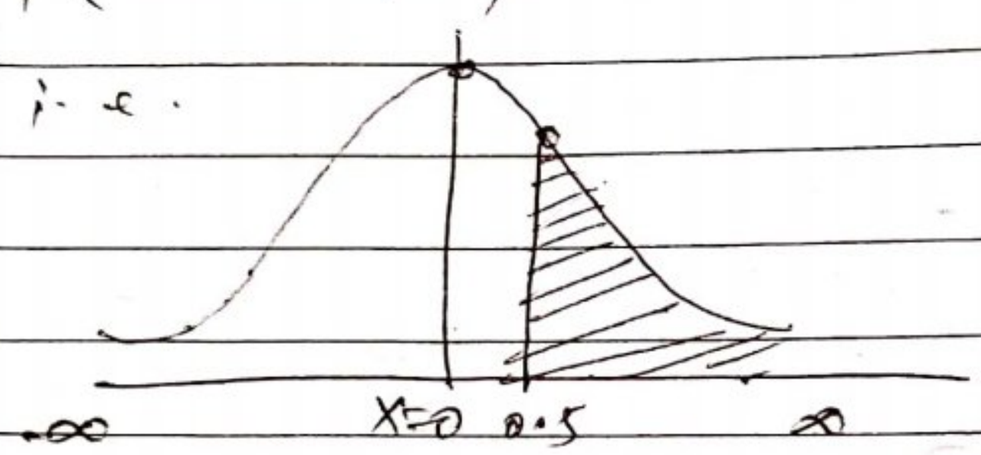
$$= \frac{17-15}{4}$$

$$= \frac{2}{4}$$

$$= 1/2 = 0.5$$

$\therefore P(Z \geq 0.5)$

i.e.



$$\therefore P(Z \geq 0.5) = 0.5 - P(0 \leq Z \leq 0.5)$$

$$= 0.5 - 0.1915$$

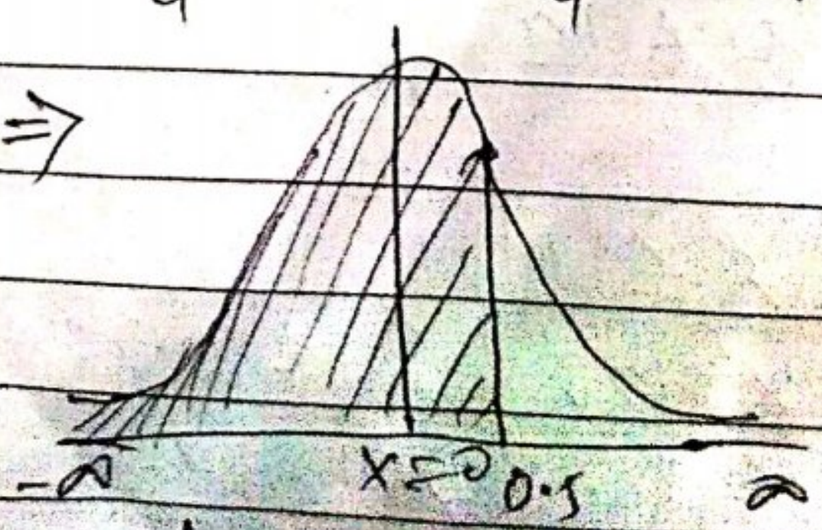
$$= 0.3085$$

$P(X \leq 17)$

Soln Here we have $x = 17$, so

$$z = \frac{x-\mu}{\sigma} = \frac{17-15}{4} = \frac{2}{4} = 0.5$$

$\therefore P(Z \leq 0.5) \Rightarrow$



$$\therefore P(Z \leq 0.5) = 0.5 + P(0 \leq Z \leq 0.5)$$

$$= 0.5 + 0.1915$$

$P(15 \le X \le 17)$

= Here $X = 15$

So, $Z = \frac{X-\mu}{\sigma}$

$= \frac{15-15}{4}$

$= \frac{0}{4}$

$= 0$

Also, $X = 17$

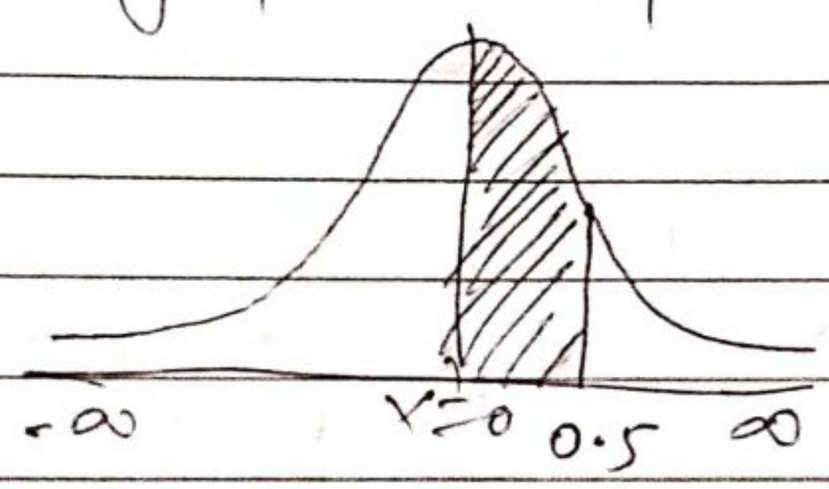
$\therefore Z = \frac{17-15}{4}$

$= \frac{2}{4}$

$= 0.5$

$\therefore P(15 \le X \le 17) = P(0 \le Z \le 0.5)$

So, the graphical representation is,



$\therefore P(0 \le Z \le 0.5) = 0.1915$ (from table)

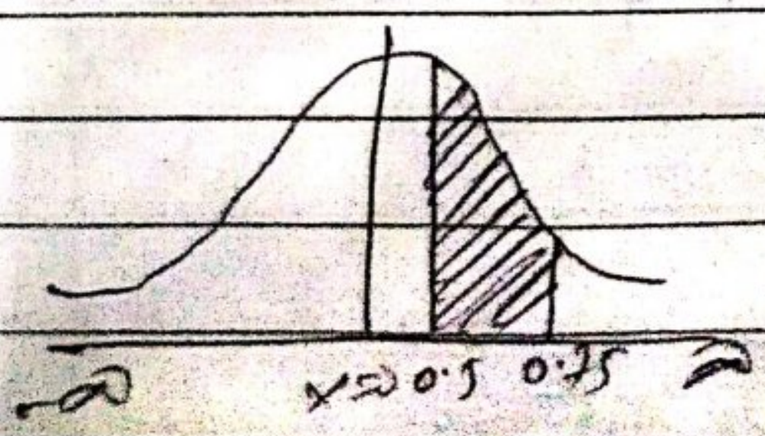
$P(17 \le X \le 18)$

$\hookrightarrow \therefore Z = \frac{17-15}{4} = \frac{2}{4} = 0.5$

$Z = \frac{18-15}{4} = \frac{3}{4} = 0.75$

$\therefore P(17 \le X \le 18) = P(0.5 \le Z \le 0.75)$

i.e.



$P(0.5 \le Z \le 0.7) = P(0 \le Z \le 0.75) - P(0 \le Z \le 0.5)$

$= 0.2734 - 0.1915$

$= 0.0819$

2015SP

3(b) The heights of boys in a particular age follows a normal distributions with mean (μ) = 150.3 cm and standard deviation (σ) = 5 cm. Find the probability that a boy picked at random from this age group has height,

(i) less than 150 cm (iv) Between 150 and 158 cm

(ii) less than 159 cm

(iii) more than 158 cm

Soln Let X represent the height of the boys then

Given

$$X \sim N(\mu, \sigma^2)$$

where μ = mean = 150.3 cm

σ^2 = Variance = $\sigma \cdot \sigma$ = stand deviation^2 = $5^2 = 25$

Also, σ = Standard deviation = 5 cm

$$\therefore X \sim N(150.3, 25)$$

So, Let us define a new variable,

$$Z = \frac{X - \mu}{\sigma}$$

$$\therefore Z = \frac{X - 150.3}{5}$$

(i) $P(X \leq 150\text{cm})$

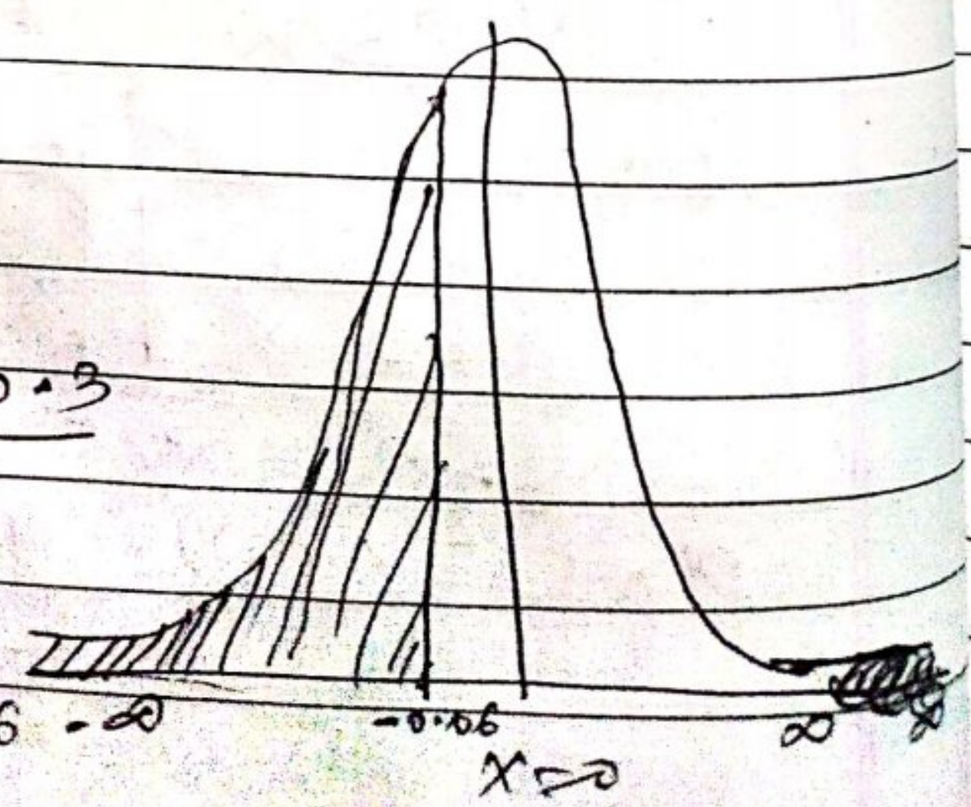
Soln We have $X = 150$ cm

$$\therefore Z = \frac{X - 150.3}{5}$$

$$= \frac{150 - 150.3}{5}$$

$$= \frac{-0.3}{5}$$

$$= -0.06$$



$$\therefore P(X \leq 150) = P(Z < -0.06)$$

$= 0.5 - P(-0.06 \le Z \le 0)$

$= 0.5 - P(0 \le Z \le 0.06)$ [from symmetric property]

$= 0.5 - 0.0239$ [From S.N. Table]

$= 0.4761$ //

(ii) $P(X \le 153\text{ cm})$

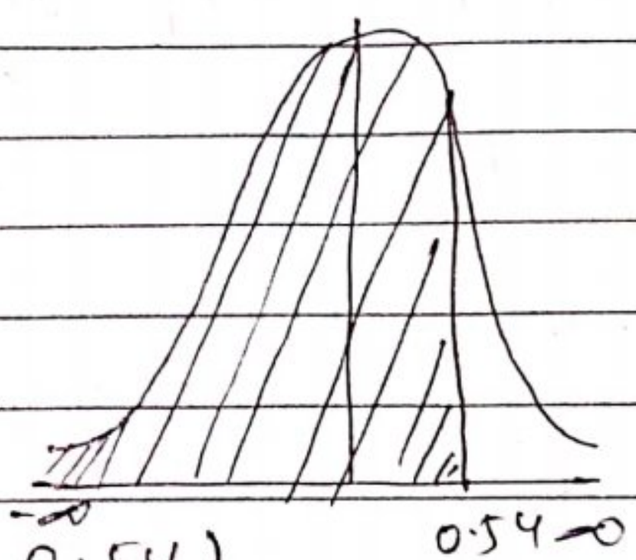
Soln we have

$X = 153\text{ cm}$

$\therefore Z = \frac{153 - 150.3}{5}$

$= \frac{2.7}{5}$

$= 0.54$



$\therefore P(X < 153) = P(Z < 0.54)$

$= 0.5 + P(0 \le Z \le 0.54)$

$= 0.5 + 0.2054$

$= 0.7054$ //

(iii) more that 158cm

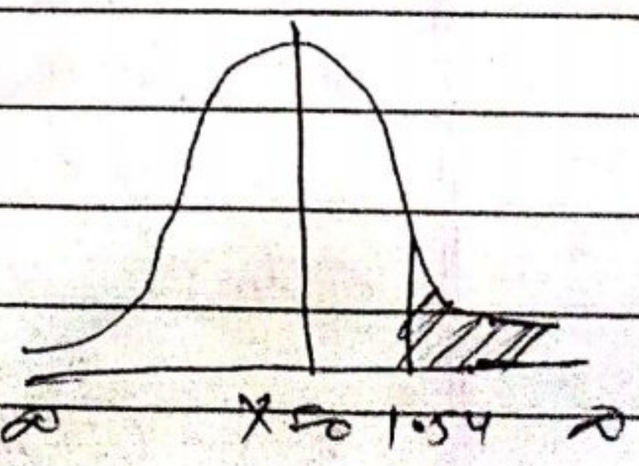
i.e $P(X \ge 158\text{ cm})$

b) we have $X = 158\text{cm}$

$\therefore Z = \frac{158 - 150.3}{5}$

$= \frac{7.7}{5} = \frac{77}{50} =$

$= 1.54$



$\therefore P(X > 158) = P(Z > 1.54)$

$= 0.5 - P(0 \le Z \le 1.54)$

$= 0.5 - 0.4382$

$= 0.0618$ //

(iv) $P(150 \le X \le 158)$

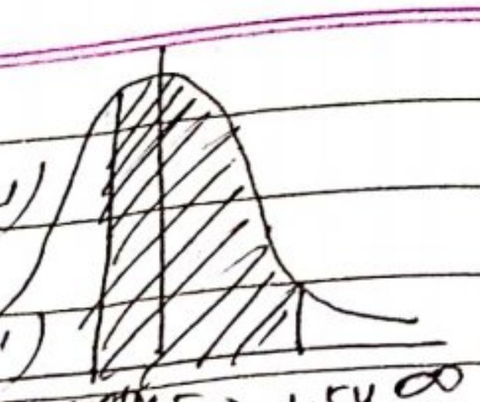
$Z = -0.06$ (when $X = 150$)

$Z = 1.54$ (when $X = 158$)

$P(-0.06 \le Z \le 1.54)$

$= P(-0.06 \le Z \le 0) + P(0 \le Z \le 1.54)$

$= P(0 \le Z \le 0.06) + P(0 \le Z \le 1.54)$



$= 0.0239 + 0.4382$

$= 0.4621$

Thus, all of the required probability are found out.

From here, we gonna find the value of X. when μ, σ, Prob

2021 fall 8a Agile is an organization whose member passes IQ that are in the top 2% of the population. It is known that IQ's are normally distributed with mean of 100 and standard deviation of 16. Find minimum IQ's needed to be an Agile member.

Soln $X =$ IQ's score

$\mu =$ mean $= 100$

$\sigma =$ standard Deviation $= 16$

Let x be a minimum IQ for an Agile member

i.e. $P(X > x) = 0.02$ —(i)

Define $Z = \frac{X - \mu}{\sigma} = \frac{X - 100}{16}$

when $X = x$, $Z = \frac{x - 100}{16} = Z_1$ (say) —— (ii)

Now,

$P(X > x) = 0.02$

$P(Z > Z_1) = 0.02$

or $0.5 - P(0 < Z < Z_1) = 0.02$

or $P(0 < Z < Z_1) = 0.48$

0.4803

0.4798

Age, income, height, score, test ⇒ follows N. Distribution generally.

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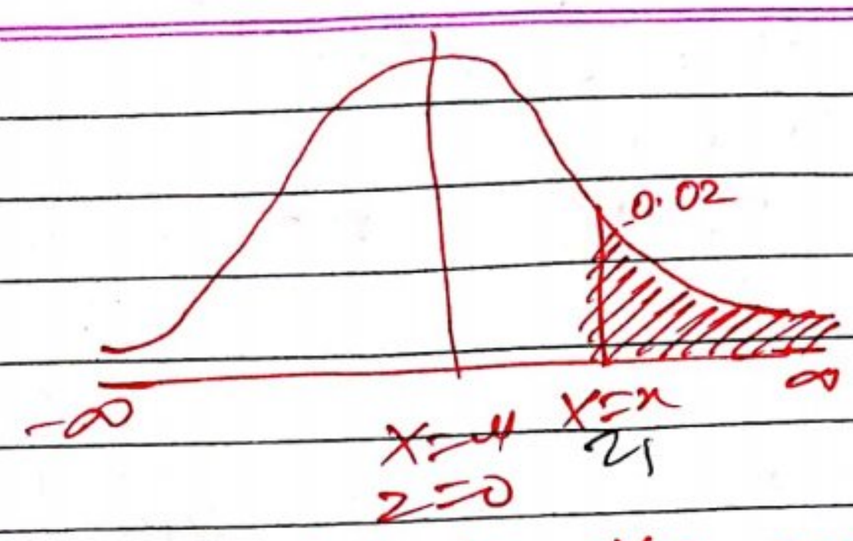
$\therefore z_1 = 2.05$

since, $z = \frac{X-100}{16}$

$\therefore X = 16z + 100$

$= 16 \times 2.05 + 100$

Thus, $X = 132.8$



2018 SP In a test administrated to 1000 students, the average score was 60 & standard deviation 10, Find

(a) The number of students exceeding a score of 50

(b) The number of students lying between 44 and 74

(c) The value of score exceeded by the top 100 students

(d) The max^m value of score obtained by 200 poor student.

Sol: Here Let $X =$ test score

we have, $\mu = 60$

$\sigma = 10$

we let x be the minimum score,

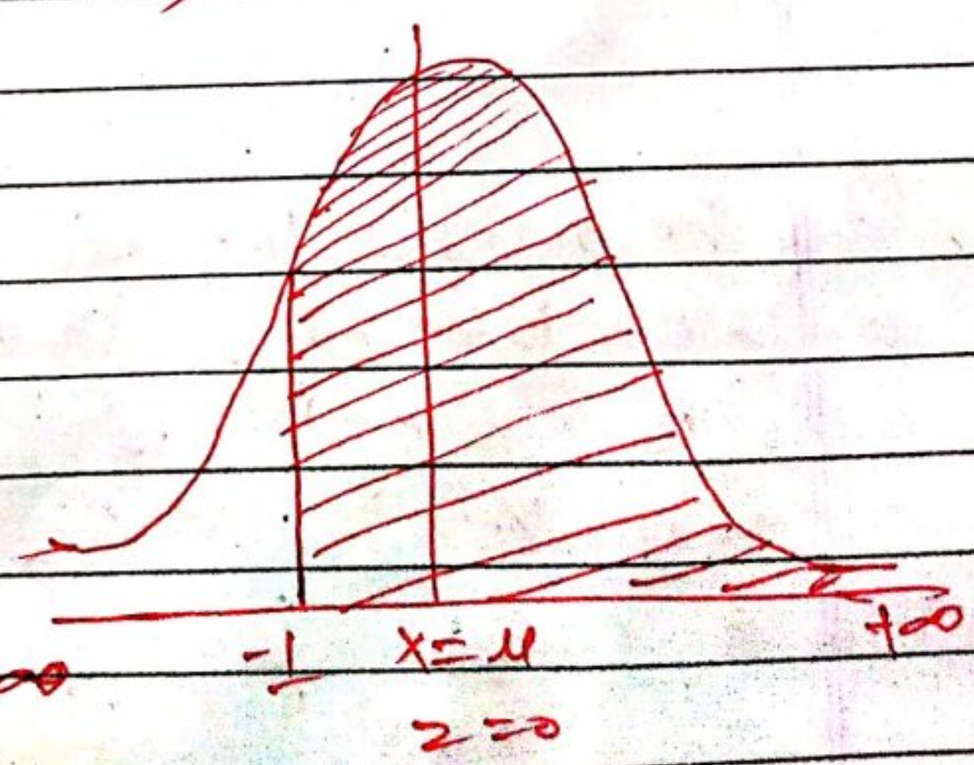
Define $z = \frac{X-\mu}{\sigma} = \frac{X-60}{10}$ — (i)

(a) $P(X > 50)$

we have $x = 50$

$z = \frac{X-60}{10} = \frac{50-60}{10}$

$= -1$



$\therefore P(X > 50) = P(z > -1)$

So, $P(z > -1) = 0.5 + P(-1 < z < 0)$

$= 0.5 + P(0 < z < 1)$ {from symmetry}

$= 0.5 + 0.3413$

$= 0.8413$

Formula { Expected frequency = $N * P(X=x)$
where N = total population
$P(X=x)$ = Prob }

(b) $P(44 < X < 74)$

Soln We have

$X = 44 \therefore z = \frac{44-60}{10}$

$= \frac{-16}{10}$

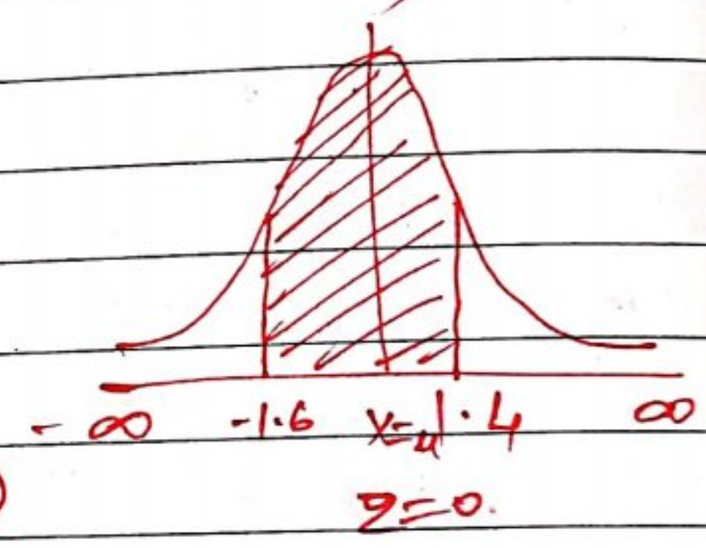
$= -1.6$

$X = 74, \therefore z = \frac{74-60}{10}$

$= \frac{14}{10}$

$= 1.4$

So, $P(44 < X < 74) = P(-1.6 < z < 1.4)$



$\therefore P(-1.6 < z < 1.4)$

$\cancel{= \{0.5 - P(-1.6 < z < 0)\} + \{0.5 -}$

$= P(-1.6 < z < 0) + P(0 < z < 1.4)$

$= P(0 < z < 1.6) + P(0 < z < 1.4)$

$= 0.4452 + 0.4192$

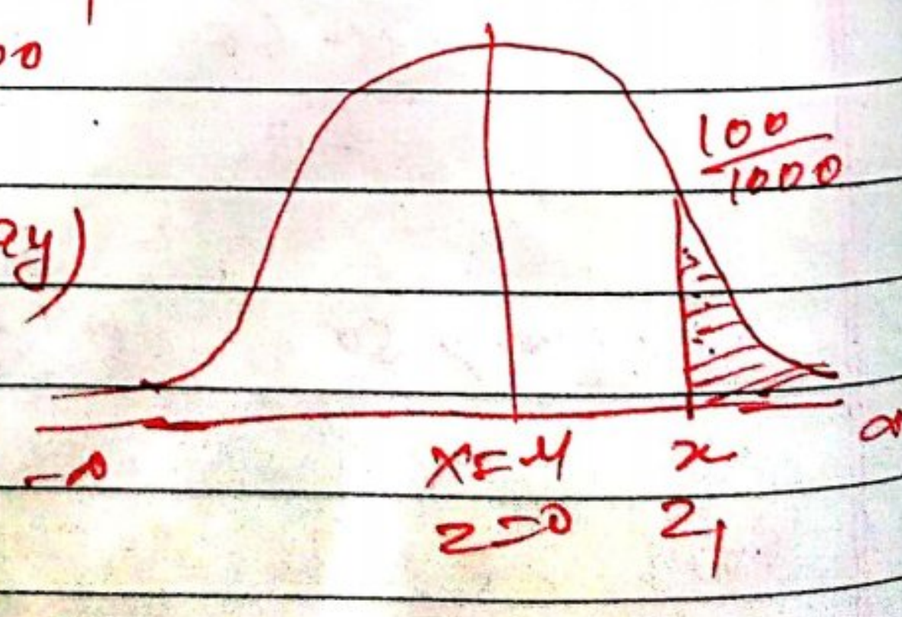
$= 0.8644$

(c) The no. of value exceeded by top 100 students.

Here total no. of student = 1000

$\therefore P(X > x) = 0.1$

When $(X=x)$, $z = \frac{x-60}{10} = z_1$ (say)



$\therefore P(X > x) = 0.1$

So, $P(z > z_1) = 0.1$

or $0.5 - P(0 < z < z_1) = 0.1$

or $P(0 < z < z_1) = 0.4$

$\therefore z_1 = 1.28$

$\therefore z = \frac{x-60}{10}$

or $10 \times 1.28 + 60 = x$

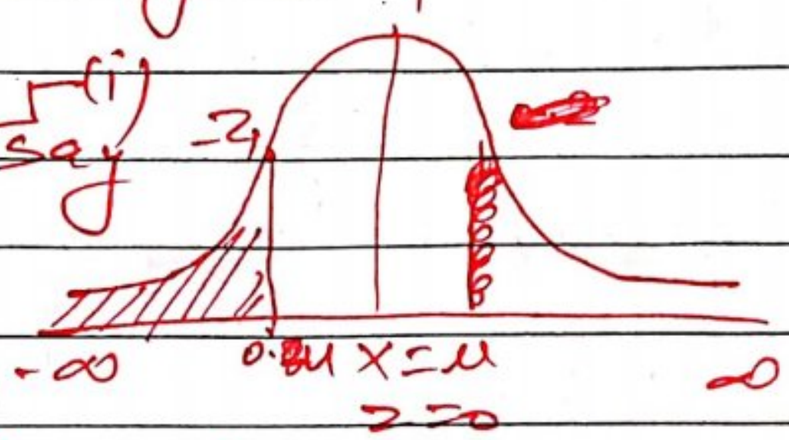
or $x = 12.8 + 60$

$\therefore x = 72.8$

(d) The maxm value of score obtained by 200 poor student,

b) $\therefore P(X > x) = \frac{200}{1000} = 0.2$

when $X = x$, $z = \frac{x-60}{10} = z_1$ say (i)



$\therefore P(X > x) = 0.2$

or $P(Z > z_1) = 0.2$

or $0.5 - P(z < Z < 0) = 0.2$

or $0.5 - P(0 < Z < -z_1) = 0.2$

or $0.3 = P(0 < Z < -z_1)$

$\therefore -z_1 = 0.84$

So, $z_1 = -0.84$

So, from (i),

$z = \frac{x-60}{10} = 10 \times -0.84 + 60$

or $x = -8.4 + 60$

$\therefore x = 51.6$

Q. In a normal distribution, 31% of items are under 45 and 8% are over 64. Find the mean and standard deviation of the distributions.

Q. In a normal distribution 7% of items are under 35 and 89% are under 63, what are the mean and standard deviation of distribution?

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7st.

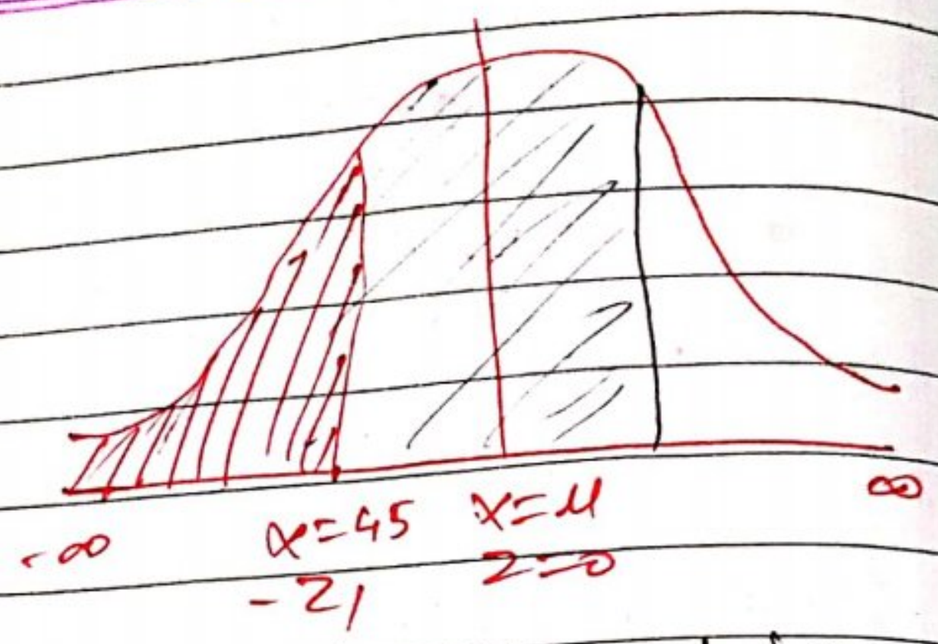
↳ $X \sim N(\mu, \sigma^2)$

Then we have

$P(X<45) = 0.31$ —— (i)

$P(X>64) = 0.08$ —— (ii)

find, $\mu, \sigma = ?$



define,

$$z = \frac{x-\mu}{\sigma}$$

when $x = 45$,

$$z = \frac{45-\mu}{\sigma} = -z_1 \text{ (say)} \quad \text{—— (iii)}$$

{ if the batai garya (iii) eye more than 50, is area but since $P < 31\%$ so, the shaded red portion is an

$P(X<45) = 0.31$

so, $P(z < -z_1) = 0.31$

or $0.5 - P(-z_1 < z < 0) = 0.31$

or $0.5 - P(0 < z < z_1) = 0.31$

or $P(0 < z < z_1) = 0.50 - 0.31$

or $P(0 < z < z_1) = 0.19$

$\therefore z_1 = 0.5$

Here, from (iii)

~~0.5 = 45~~

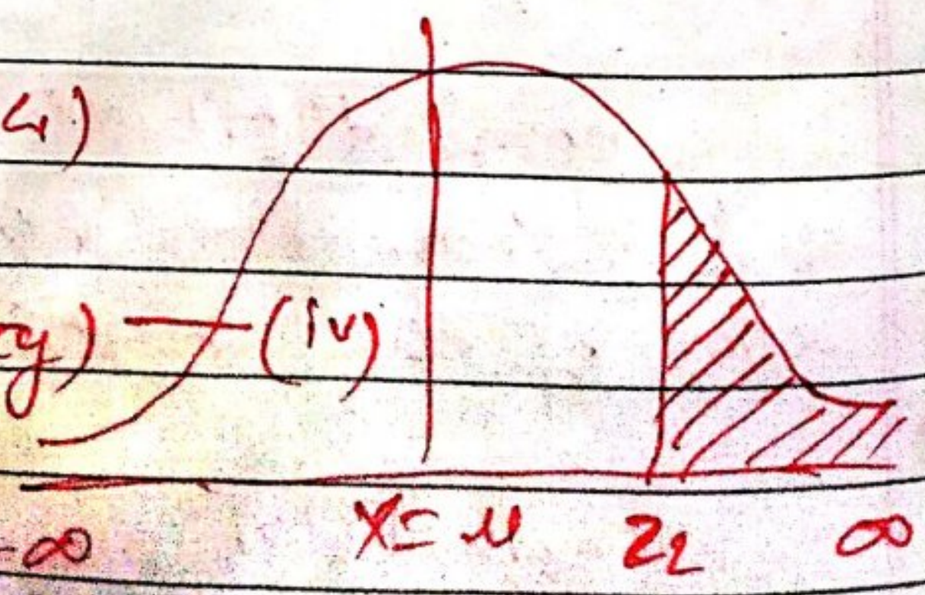
$$\frac{45-\mu}{\sigma} = -0.5$$

or $45 - \mu = -0.5\sigma$ —— (a)

for eqn (ii) (~~over~~ 8% are over 64)

when $x = 64$,

$$z = \frac{64-\mu}{\sigma} = z_2 \text{ (say)} \quad \text{—— (iv)}$$



$P(X>64) = 0.08$

or, $P(Z > z_2) = 0.08$

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or $0.5 - P(0 < Z < Z_2) = 0.08$

or $P(0 < Z < Z_2) = 0.42$

$\therefore Z_2 = 1.41$

So, from (iv),

$$Z = \frac{64 - \mu}{\sigma}$$

$1.41\sigma = 64 - \mu$

or $\mu = 64 - 1.41\sigma$ —— (5)

From (2),

$45 - \mu = -0.56\sigma$

or $45 - 64 + 1.41\sigma = -0.56\sigma$

or $1.97\sigma = 19$

$\therefore \sigma = 9.64$

From (5),

$\mu = 64 - 1.41 \times 9.64$

$= 64 - 13.59$

$\therefore \mu = 50.41$

Gamma Distribution

↳ Consumption of electricity in Big city

↳ " " Milk or water in Big city

↳ Electronic device life time.

(α and β)

Let X be a random variable. Then the random variable X is said to have a gamma distribution with two parameters α and β, if its prob density fun

$\alpha \rightarrow$ location parameter $\Rightarrow$ conc^n of data
$\beta \rightarrow$ scale " $\Rightarrow$ define shape

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-x/\beta} \; ; x \geq 0 \text{ and } \alpha, \beta > 0$$

And we can write as:

$$X \sim G(\alpha, \beta)$$

But sometimes the gamma distribution is defined based on single parameter α (i.e. $\beta = 1$) and its probability density function is given by,

$$f(x) = \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x} \; ; x \geq 0 \text{ and } \alpha > 0$$

(ii) Mean and Variance of Gamma Distribution

▷ Mean, $E(X) = \alpha\beta$

& Variance, $V(X) = \alpha \cdot \beta^2$

Eg: ① Suppose that the life time of a certain kind of computer is a random variable having gamma distribution with $\alpha = 2$ and $\beta = 6$. Find the mean life time of computers. Also find the prob that such a computer will last more than 10 years.

Soln

Let X = life time of computer (in years)

$X \sim G(\alpha, \beta)$, then

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-x/\beta} \; ; x \geq 0$$

Also, we have, $\alpha = 2$, $\beta = 6$

Now, $$f(x) = \frac{1}{6^2 \Gamma 2} x^{2-1} e^{-x/6}$$

$$= \frac{1}{36} x e^{-x/6} \; ; x \geq 0$$

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$\therefore$ Probability that the computer will last more than 10 years.

i.e. $P(X>10) = 1 - P(X \le 10)$

$$= 1 - \int_0^{10} f(x)\,dx$$

$$= 1 - \int_0^{10} \frac{1}{36} x e^{-x/6}\,dx$$

$$= 1 - \frac{1}{36} \int_0^{10} x e^{-x/6}\,dx$$

VI

(2) The daily consumption of milk in kathmandu city in excess 200000 ltr, is approximately distributed as gamma distribution with $\alpha = 2$ and $\beta = 10^5$. The city has a daily stock of milk of 300000 ltr. What is the prob that the stock is insufficient on a particular day

Soln Let X = consumptions of milk

Also, Let y = consumption of milk excess 200000

$$y = X - 200000 \quad \text{--- (i)}$$

Which follows Gamma Distribution with $\alpha = 2$ and $\beta = 10^5$.

Then its pdf is:

$$f(y) = \frac{1}{\beta^{\alpha}\Gamma\alpha} y^{\alpha-1} e^{-y/\beta} \; ; y \ge 0$$

$$= \frac{1}{(10^5)^2 \Gamma 2} y^{2-1} e^{-y/10^5} \; ; y \ge 0$$

$$= \frac{1}{(10)^{10}} y e^{-y/10^5}$$

Now, The probability that the dailly stock of milk is insufficient {i.e when consumption is > stock (300000)}

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i.e- $P(X > 300000) = P(Y + 200000 > 300000)$

$= P(Y > 100000)$

$= \int_{100000}^{\infty} f(y)\,dy$

$= 1 - \int_{0}^{100000} \frac{1}{(10)^{10}} y e^{-y/10^5} dy$

$= 1 - \frac{1}{(10)^{10}} \int_{0}^{100000} \underset{(u)}{y} \underset{(v)}{e^{-y/10^5}} dy$

Soln

$= 1 - \frac{1}{(10)^{10}} \left\{ \left[y \int e^{-y/10^5} dy \right]_0^{100000} - \left[\left(\frac{dy}{dy} \int e^{-\frac{y}{10^5}} dy \right) dy \right]_0^{100000} \right\}$

$= 1 - \frac{1}{(10)^{10}} \left\{ \left[y \frac{e^{-y/10^5}}{\frac{-1}{10^5}} \right]_0^{100000} - \left[\frac{e^{-\frac{y}{10^5}}}{\left(\frac{-1}{10^5}\right)^2} \right]_0^{100000} \right\}$

$= 1 - \frac{1}{10^{10}} \left\{ -10^5 \left(y e^{-y/10^5} \right)_0^{100000} - (10^5)^2 \left[e^{-y/10^5} \right]_0^{100000} \right\}$

$= 1 - \frac{1}{10^{10}} \left\{ -10^5 \left(10^5 e^{-1} - 0 \right) - (10^5)^2 \left(e^{-1} - 1 \right) \right\}$

$= 1 - \frac{1}{10^{10}} \left\{ -\frac{10^{10}}{e} - \frac{10^{10}}{e} + 10^{10} \right\}$

$= 1 - \frac{1}{10^{10}} \times$

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Q. In a certain city, the daily consumptions of a electric power (in millions of kwhrs) can be treated as a random variable having a gamma distribution with $\alpha = 3$ and $\beta = 2$. If the power plant of the city has 12 million kwhrs. What is the probability that this power supply will be inadequate in any given day.

Soln Let X = Daily consumption of electric power (in millions of kwhrs)

Here,

$$X \sim G(\alpha, \beta)$$

where $\alpha = 3$

$\beta = 2$, then

we know, the probability ~~distri~~ density function is:

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-x/\beta} \; ; x \geq 0$$

$$\text{or } f(x) = \frac{1}{2^3 \Gamma 3} x^{3-1} e^{-x/2}$$

$$= \frac{1}{8 \times 2} x^2 e^{-x/2}$$

$$\therefore f(x) = \frac{1}{16} x^2 e^{-x/2} \; ; x \geq 0$$

Here, the electric power capacity = 12 million kwhrs,

Now,

Probability that this power supply will be inadequate in any given day $P(X > 12)$

$$= 1 - P(X < 12)$$

$$= 1 - \int_0^{12} f(x)\,dx$$

$$= 1 - \int_0^{12} \frac{1}{16} x^2 e^{-x/2}\,dx$$

$$= 1 - \frac{1}{16} \int_0^{12} \underset{(u)}{x^2} \underset{(v)}{e^{-x/2}}\,dx$$

$$= 1 - \frac{1}{16} \left\{ \left[x^2 \int e^{-x/2}\,dx \right]_0^{12} - \left[\int \left(\frac{dx^2}{dx} \int e^{-x/2}\,dx \right) dx \right]_0^{12} \right\}$$

$$= 1 - \frac{1}{16} \left\{ \left[x^2 \frac{e^{-x/2}}{-1/2} \right]_0^{12} - \left[\int 2x \frac{e^{-x/2}}{-1/2}\,dx \right]_0^{12} \right\}$$

$$= 1 - \frac{1}{16} \left\{ \left[-2x^2 e^{-x/2} \right]_0^{12} - \left[-4 \left\{ \frac{e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{(-1/2)^2} \right\} \right]_0^{12} \right\}$$

$$= 1 - \frac{1}{16} \left\{ -288 e^{-6} + 4 \left\{ (-24 e^{-6} - 4 e^{-6}) - \left(0 - \frac{1}{1/4} \right) \right\} \right\}$$

$$= 1 - \frac{1}{16} \left\{ -0.71 - 0.27 + 4 \right\}$$

$$= 1 - 0.18875$$

$$= 0.81125$$

Beta Distribution

Beta function

(i) $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

(ii) $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

A random variable X is said to have Beta distribution with parameters α and β. Then its pdf is

$$f(x) = \frac{1}{B(\alpha, \beta)}$$

Chebyshev's Inequality (Imp)

↳ Sometimes probability distribution is unknown for a random variable. but $E(x)=\mu$ and $V(X)=\sigma^2$ are known. In this case, chebyshev's Inequality is used to compute the prob for upper and lower bound.

→ chebyshev's inequality describes how the variance measures the degree concentration of prob near mean

→ $E(X)=\mu$.

→ In other word, chebyshev inequality provide upper bound (or lower bound) to the prob of getting a value that deviated from the mean by more than or less than K times standard deviation.

$$P(|X - E(x)| \geq K\sigma) = ?$$

$$P(|X - E(x)| \leq K\sigma) = ?$$

R. variable mean bata kati le deviate vako xa vanne chu ktimes σ le dinxa.

<u>Statement</u> : Let X be a random variable with mean (μ) and finite variance (σ^2) $\{\sigma^2 > 0\}$, then for any real number $k>0$

$$P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Alternatively, we can write, for any real number $k>0$

$$P(|X - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

$Y = \mu + k\sigma$

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proof : Consider a continuous r.v (X) with p.d.f $f(x)$,
By defination of variance, we have

$$V(X) = \sigma^2 = E[X - E(X)]^2$$

$$= E[X-\mu]^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{(i)}$$

$|x| > 2$

$x > 2$
$x < -2$

we have,

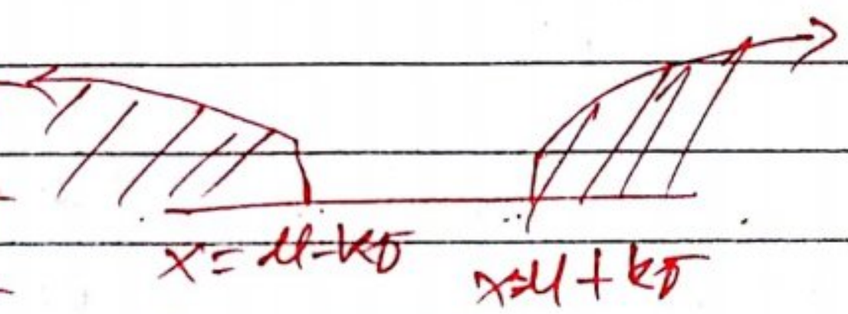
$$|X-\mu| \geq k\sigma$$

$\therefore X-\mu > k\sigma$

i.e $X > \mu + k\sigma$

And, $X-\mu < -k\sigma$

i.e. $X < \mu - k\sigma$



from (i),

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{—— (\#)}$$

Because the integral $\int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx > 0$

variance,

Now, for 1st integral, Range is

$$-\infty \leq X \leq \mu - k\sigma$$

$$\Rightarrow X \leq \mu - k\sigma$$

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or $\mu - x \geq k\sigma$ —— (ii)

$(\mu - x)^2 \geq k^2\sigma^2$ —— ⓐ

Also, $\mu + k\sigma \leq x \leq \infty$ —— for 2nd integral

or $x \geq \mu + k\sigma$

or $x - \mu \geq k\sigma$ —— (iii)

or $(x-\mu)^2 \geq k^2\sigma^2$ —— ⓑ

from ⓐ and ⓑ as,

$(x-\mu)^2 = (\mu - x)^2$

$\therefore$ final outcome $= (x-\mu)^2 \geq k^2\sigma^2$

Now, from eq^n #

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$$

$$\text{or } \cancel{\sigma^2} \geq k^2\cancel{\sigma^2}\left[\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx\right]$$

$$\text{or } \frac{1}{k^2} \geq P(x \leq \mu - k\sigma) + P(x \geq \mu + k\sigma) \longrightarrow$$

$|x| > 2$

$x > 2$

$x < -2$

$$\text{or } \frac{1}{k^2} \geq P(|x-\mu| \geq k\sigma) \longrightarrow$$

$$\left\{\therefore P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}\right\}$$

Equivalently,

$$\left[P(|x-\mu| < k\sigma) > 1 - \frac{1}{k^2}\right]$$

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Corollary : Let X be a random variable with mean(μ) and finit variance (σ^2) Then, for any real number $k>0$ and if $k\sigma = c$

$$\therefore P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \text{ for } c>0$$

Equivalently,

$$P(|X-\mu| < c) \geq 1 - \frac{\sigma^2}{c^2}$$

Q. A random variable X has a mean of 10 and variance 4 and unknown prob distribution. Find the value of c such that :

$$P(|X-\mu| \geq c) \geq 0.04$$

Soln Given that

mean(μ) = 10

variance (σ^2) = 4

$\therefore \sigma = 2$

Also,

$$P(|X-10| \geq c) \leq 0.04 \quad \text{—(i)}$$

using chebyshev's inequality,

$$P(|X-10| \geq c) \leq \frac{\sigma^2}{c^2}$$

$$\text{or } P(|X-10| \geq c) \leq \frac{4}{c^2} \quad \text{—(ii)}$$

compairing eqn (i) and (ii), we get

$$\frac{4}{c^2} = 0.04$$

$$\text{or } c^2 = \frac{4}{0.04}$$

$$c = \sqrt{100} = 10$$

$$\therefore c = 10$$

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Q. A random variable X with unknown prob distribution has mean 8 and variance 9. Find (i) $P(-4 < X < 20)$ {less than} → lower bound
(ii) $P(|X-8| \geq 6)$ → upper bound

Sol: Given that

mean $(\mu) = 8$

variance $(\sigma^2) = 9$

$\therefore \sigma = 3$

(i) using chebyshev's inequality,

$$P(|X-\mu| < c) \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P(|X-8| < c) \geq 1 - \frac{9}{c^2}$$

$$\Rightarrow P(-c < X-8 < c) \geq 1 - \frac{9}{c^2}$$

$$\Rightarrow P(8-c < X < c+8) \geq 1 - \frac{9}{c^2} \quad \text{—(i)}$$

Now, for, $P(-4 < X < 20)$

we have,

$-4 = 8 - c$ and $c + 8 = 20$

$\therefore c = 12$

$$\therefore P(-4 < X < 20) = 1 - \frac{9}{12^2}$$

$$= 1 - \frac{9}{144}$$

$$= \frac{144-9}{144}$$

$$= \frac{135}{144}$$

$$= 0.9375$$

$\frac{1350}{144}$

(ii) $P(|X-8| \geq 6)$

Sol: we have $P(|X-8\mu| \geq c) \leq \frac{\sigma^2}{c^2}$

er bound

$\therefore P(|X-8| \geq 6) = \frac{9}{6^2}$

$= \frac{9}{36}$

$= \frac{3}{12}$

$\left[\therefore P(|X-8| \geq 6) = 0.25\right]$

8. Let X is a random variable with mean = 11, and variance = 9. using chebesiv inequality, find
(i) Lower bound for $P(6 < X < 16)$
(ii) The value of c such that $P(|x-11| \geq c) \leq 0.09$.

iid ⇒ $X_1, X_2, X_3 \ldots X_n$ are independent
and μ ~~and variance~~ of all r.v are equal
and σ (variance) of r.v is equal

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Weak Law of large numbers

large pop^n → μ → mean of pop^n

sample → $\bar{X}$ → sample mean

for $n \to \infty$,

or, for large n $P[\bar{X}_n = \mu]$ —— (*)

i.e, for $n \to \infty$, $\bar{X}_n \xrightarrow{P} \mu$, i.e $\bar{X}_n - \mu \xrightarrow{P} 0$

Then, Weak law of large number states that,

$$P(|\bar{X}_n - \mu| < \epsilon) \longrightarrow 1 \left\{ \begin{array}{l} \text{as } \epsilon \text{ almost } 0 \\ \epsilon \approx 0 \end{array} \right.$$

as $n \to \infty$

So, we can write

$$P(|\bar{X}_n - \mu| \geq \epsilon) = 1 - P(|\bar{X}_n - \mu| < \epsilon)$$

$$= 1 - 1$$

$$\therefore P(|\bar{X}_n - \mu| \geq \epsilon) = 0 \quad \text{—— (+)}$$

Statement: Let $\{X_n\}$ be a sequence of independently identically distributed random variables with mean $E(X_i) = \mu$ and $V(X_i) = \sigma^2$
Then for very small positive ϵ, we have,

$$P(|\bar{X}_n - \mu| \geq \epsilon) \longrightarrow 0 \text{ as } n \to \infty$$

Chebyshev's B asked alot

(Proof B not asked most times)

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$$

equivalently,

$$P(|\bar{X}_n - \mu| < \epsilon) \to 1 \text{ as } n \to \infty$$

WLLN can also be stated as Chebyshev's Inequality,

Let $\{X_n\}$ be a sequence of iid random variables, with $E(X_i) = \mu$ and $V(X_i) = \sigma^2 < \infty$ (finite)

$$\text{Let } \bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$\therefore E(\bar{X}_n) = E\left(\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}\right)$$

{as sample can be as many as we want say (n), so the mean of sample B calculated}

notneeded

Note: Static → The numeric value calculated from sample, for eg: sample mean ($\bar{X}$)
sample variance (S^2)
sample std deviation (S)

Also, static is also considered a r.v.

$$= \frac{1}{n}\left[E(X_1) + E(X_2) + \cdots + E(X_n)\right]$$

$$= \frac{1}{n} E[X_1 + X_2 + \cdots + X_n] = \frac{1}{n} \times n\mu = \mu$$

for indi $\Rightarrow V(X+Y) = V(X) + V(Y)$

for dep $\Rightarrow V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$

Also, $V(\bar{X}_n) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)$

$$= \frac{1}{n^2} V(X_1 + X_2 + X_3 + \cdots + X_n)$$

$$= \frac{1}{n^2}\left[V(X_1) + V(X_2) + V(X_3) + \cdots + V(X_n)\right]$$

$$= \frac{1}{n^2} \times n\sigma^2$$

$$\boxed{\therefore V(\bar{X}_n) = \frac{\sigma^2}{n}}$$

Now, WLLN states as:

$$P(|\bar{X}_n - E(\bar{X}_n)| \geq \varepsilon) \leq \frac{V(\bar{X}_n)}{\varepsilon^2}$$

$$\boxed{P(|\bar{X}_n - E(\bar{X}_n)| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}}$$

equivalently,

$$P(|\bar{X}_n - E(\bar{X}_n)| < \varepsilon) \geq 1 - \frac{V(\bar{X}_n)}{\varepsilon^2}$$

$$\boxed{P(|\bar{X}_n - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}}$$

2021 fall Estimate the no. of times that a fair coin should be tossed to ensure that the probability that between 48% and 52% of the result of head is 0.99

Soln Let X = no. of heads
and let p and q are the probability of getting heads respectively.

Then, for single toss, of fair coin,

$p = 1/2$

$q = \frac{1}{2} = 1 - p = 1 - 1/2$
$= 1/2$

→ process depends on time
→ result or outcome depends on time.

Markov Process (~~stochastic~~ stochastic process) [Random process]

↳ In probability theory, A markov process is a random process with memoryless property (i.e. Markov property) That is, the future state of random process depends only on current state and not on the states in the past.

mathematically,
A random process $\{X(t), t \in T\}$ is called Markov process if,

$$P[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, (X_{t_{n-1}}) = x_{n-1}, \ldots, X(t_0) = x_0]$$

$$P[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n]$$

where $t_0 < t_1 < t_2, \ldots t_n < t_{n+1}$ and $x_0, x_1, x_2 \ldots x_n, x_{n+1}, \ldots$ are the states of process.
↓
Result

time and result could be continuous or descrete,

(#) that Random process is called markove, if future outcomes depends upon present condition (not previous outcomes, as these results are neglected.

(#) Some example of Markov process
(a) Board game with dice as (Ludo) since the next step that we change is based on present position.

(b) Weather prediction model
(c) Brand loyalty

Markov chain

↳ It is derived from markov process.
↳ A descrete state markov process is called Markov chain.
↳ Mathematically, it is defined as,
↳ Here both time and result must be descrete,

$$P[X_{n+1} = y / X_n = x, X_{n-1} = x_{n-1}, \ldots X_0 = x_0]$$

↑ step ↑ state

$$= P[X_{n+1} = y / X_n = x]$$

Mathematically, → time representation (i.e. discrete time)

denoted by, $X_n, n \geq 0$

Let $\{X_n, n \geq 0\}$ be a markov chain. If $X_n = i$, we say, the random process is in state i at time n. (step n).

Then, the conditional probability

$$P[X_{n+1} = j / X_n = i]$$ is called one

↑ step ↑ state

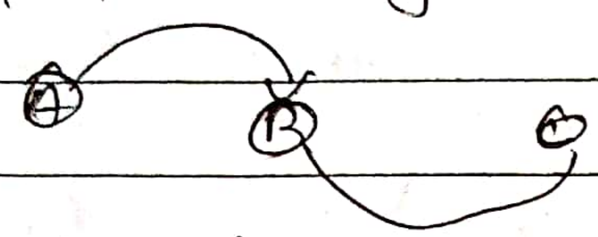
step transition probability from state i to state j at time $(n+1)$ or $(n+1)$ step

n step transition probability is the prob of upcoming nth day is find out by using present prob.

one step

$P_{ij}(n, n+1)$

Remarks : (i) If one step transition prob is independent of n i.e. $P_{ij}(n, n+1) = P_{ij}(m, m+1)$ is called homogenous markov chain. It have stationary transition probability.



$P(A \text{ to } B) = P(B \text{ to } C)$

$P_{ij}(2,3) = P_{ij}(4,5)$ (no changes is seen)

And the one step transition prob of homo markov chain is denoted by P_{ij}.

(ii) The matrix of one step transition prob i.e. $P = [P_{ij}]$ is called transition prob of matrix (tpm) $\rightarrow$ $\begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix}$

(iii) $P_{ij} = P_{ij}^{(1)}$ is called one step transition probability.

(iv) For homogenous markov chain, the conditional probability $P[X_n = j / X_0 = i]$

assuming that step starts from 0.

That is, the prob that process in in state j at step n given that it was in state i

at step 0, is called n step transition probability.
· And it is denoted by $(P_{ij})^n$

i.e. $(P_{ij})^n = P[X_n = j / X_0 = i]$

<u>Defn</u> Let $\{X_n, n \geq 0\}$ be a homogenous markov chain with descrete state $S = \{0, 1, 2, \ldots\}$, Then the one step transition probability from state i to state j is defined as,

$$P_{ij} = P[X_{n+1} = j / X_n = i]; \; i \geq 0, j \geq 0, \forall n$$

Then the transition probability matrix (tpm) of the homo markov chain $\{X_n, n \geq 0\}$ is given by,

$$P = P_{ij} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where $P_{ij} \geq 0$ and total prob of particular row $= 1$ $P_{R_0} = 1$, $P_{R_1} = 1$

and $\sum_{j=0}^{\infty} P_{ij} = 1$ for all $i = 0, 1, 2, \ldots \infty$

assumed start = 1

For finite state space, $S = \{1, 2, \ldots m\}$, then the tpm is given by,

$$P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \\ P_{m1} & P_{m2} & \cdots & P_{m} \end{bmatrix} \text{ square}$$

where, $P_{ij} \geq 0$ and $\sum_{j=1}^{m} P_{ij} = 1$ for all $i = 1, 2, \ldots m$

<u>Example</u> : Simple weather model,
Let rainy day = state 0
and sunny day = state 1

$$\begin{array}{cc} & \text{state of } X_{n+1} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \text{state of } X_n \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix} \end{array}$$

Here, $P_{ij} > 0$ and $\sum_{j=0}^{1} P_{ij} = 1$ for all $i = 0, 1$)

$\Rightarrow$ It is a random variable w.r.t time.

Let $\{X_n, n \geq 0\}$ be a homogenous markov chain.
Let $P_i(n) = P\{X_n = i\}$ be the probability that the process is in state i at step n.
state (outcomes)
step (time)
And Let $P(n) = [P_0(n) \; P_1(n) \; P_2(n) \; \ldots]$
is prob distribution of n step.

(H) Queuing Theory *3 qsn.

where, $\sum_{k=0}^{\infty} P_k(n) = 1$

Then,
$P_i(0) = P\{X_0 = i\}$; $i \geq 0$ are called initial state probabilities.
and
$P(0) = [P_0(0), P_1(0) \; P_2(0) - - - - - -]$ is called initial state probability distribution.

where, $\sum_{k=0}^{\infty} P_k(0) = 1$

~~Also P~~ Note: The prob distribution of markov chain is completely determined ~~#~~ by the tpm and initial prob distribution.

Example :- Consider 2 state Markov chain. Suppose that step 0 represent rainy day and state 1 represents sunnyday with the given tpm,

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix} \end{array}$$

rainy, sunnyday

Soln Suppose that initially the whether is known to be rainy i.e. $X_0 = 0$, the initial prob distribution is given by,

$P(0) = [P_0(0) \quad P_1(0)]$

$= [1 \quad 0]$ {initial prob distribution}

Every state ma tpm change navayema, it is said to be Homogenous markov chain

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Thus, the whether on day 1 can be predict as,

$$P(1) = P(0) \cdot P \quad \left\{ \begin{matrix} \text{initial} \\ \text{prob} \\ \text{dist} \end{matrix} \times \text{tpm} \right)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

Thus, there 50% chance that rainy day will be followed by ~~sunny~~ rainy day.

Again, the whether on day 2 can be predicted as

$$P(2) = P(1) \cdot P$$

$$= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0.7 \end{bmatrix}$$

Similarly, the whether on day 3 can be predicted as:

$$P(3) = P(2) \cdot P$$

$$=$$

In general, the whether on day 'n', can be predicted as:

$$\left\{ P(n) = P(n-1) \cdot P \right\}$$

Related Qns

202 There are 3 brands of beauty products A, B, C in a shop. It has been found that, a person when purchase of a particular brand will continue to buy, same brand or switch over to another brand during his next purchase of a beauty product in a shop. The TPM associated with 3 brands is given below:

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

If the initial distribution of purchase of the brands a, b and c is $[0.4 \quad 0.4 \quad 0.2]$

(a) Determine the distribution of the brands after 2 purchase.

↳ Given,

$$TPM = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} = P$$

Initial distribution of purchase of brands a, b, c is:

$$P_0 = [0.4 \quad 0.4 \quad 0.2]$$

So, the distribution of the brands after 2 purchase,

$$P(2) = P(1) \cdot P$$

$$= P(0) \cdot P \cdot P$$

$$= [0.4 \quad 0.4 \quad 0.2] \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$= [0.4 \quad 0.4 \quad 0.2] \begin{bmatrix} 0.4 & 0.24 & 0.36 \\ 0.36 & 0.28 & 0.36 \\ 0.36 & 0.24 & 0.4 \end{bmatrix}$$

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(Q2) A man either drives a car or catches a train or a bus to go to office everyday; suppose that the transition probability matrix for this markov chain is given by,

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.5 & 0.2 & 0.3 \\ 1 & 0.4 & 0.5 & 0.1 \\ 2 & 0.2 & 0.6 & 0.2 \end{array}$$

where, the states 0, 1, and 2 denotes a car, train and bus respectively. Suppose that on the first day of a week, the man tossed a fair coin and took a train to work if a head appears, or a bus to work if a tail appears:

Find the probability that:

(a) He'll take a train on the 3rd day of week.

(b) He'll take a bus to work on the 4th day of week

Soln

Given:

$$TPM\ (P) = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

Initial distribution = The distribution of day 1 as:

$$P(1) = [0 \quad 0.5 \quad 0.5]$$

So, for prob of taking a train on the 3rd day of week:

$$\cancel{P(3) = P(2) \cdot P(1) \cdot P}$$

$$\cancel{= P(1) \cdot P \cdot P(1) \cdot P}$$

$$\cancel{= P(1)^2 \cdot P}$$

$$= [0 \quad 0.5 \quad 0.5]^2 \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

$P(2) = P(1) \cdot P$
$P(3) = P(2) \cdot P$

$=$

$P(3) = P(2)\, P(1) \cdot P \cdot P$

$P(2) = P_1 \cdot P$

$P(3) = P(2) \cdot P$

$= P_1 \cdot P \cdot P$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}^2$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} \begin{pmatrix} 0.39 & 0.38 & 0.23 \\ 0.42 & 0.39 & 0.19 \\ 0.38 & 0.46 & 0.16 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 & 0.42 & 0.175 \end{pmatrix}$$

Chapman-Kolmogorov Theorem

$\Rightarrow$ Let $\{X_n, n \geq 0\}$ be a homogenous Markov chain with TPM $P = [P_{ij}]$ and n-step tpm,

$$P^{(n)} = [P_{ij}^{(n)}]$$

where,
$$P_{ij}^{(n)} = P\{X_n = j / X_0 = i\}$$

and,
$$P_{ij} = P_{ij}^{(1)}$$

Then, the following properties hold:

(i) $P^{(n+m)} = P^{(n)} \cdot P^{(m)}$

(ii) $P^{(n)} = P^n$ i.e. n state transition probability matrix = multiplication of 'n' Transition prob matrix

Irreducible Markov chain

A homogenous markov chain is known as, Irreducible if its transition prob matrix is irreducible, i.e. the n-step tpm, P^n has all positive entries i.e. $P_{ij}^{(n)} > 0$

Steady-state Distribution of Markov

⇒ Let $\{X_n, n \geq 0\}$ be a homogenous markov chain, with tpm P. If there exists a probability vector (or prob distribution) π, such that,

$$\pi P = \pi$$

Then, π is called steady state distribution of Markov chain,

where $\pi = [\pi_1, \pi_2, \pi_3, ---]$

and $\sum_{i=1}^{\infty} \pi_i = 1$

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Qn. The one step transition probability matrix is:

$$\begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$$

, prove that it is irreducible & Also find the steady state probabilities (or steady state distribution).

Soln Let the one-step tpm is:

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$$

Now, the 2 step tpm is given by

$$P^2 = P^2 = P \cdot P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.28 & 0.22 \\ 0.2 & 0.66 & 0.14 \\ 0.6 & 0.12 & 0.28 \end{bmatrix}^2$$

Since $P_{ij} > 0$, so it is irreducible

<u>For steady state distribution</u>

Let $\pi = [\pi_1, \pi_2, \pi_3]$ be a steady state distribs of markov chain, which is obtained by solving following eqns.

$$\pi P = \pi, \text{ and } \pi_1 + \pi_2 + \pi_3 = 1$$

Now, for equivalently for $\pi p = \pi$, we have,

$$[\pi_1, \pi_2, \pi_3] \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix} = [\pi_1, \pi_2, \pi_3]$$

Then we have,

$$\begin{bmatrix} 0.6\pi_1 + 0.1\pi_2 + 0.6\pi_3 \\ 0.2\pi_1 + 0.8\pi_2 + 0\pi_3 \\ 0.2\pi_1 + 0.1\pi_2 + 0.4\pi_3 \end{bmatrix} =$$

$$0.6\pi_1 + 0.1\pi_2 + 0.6\pi_3 = \pi_1$$
$$0.2\pi_1 + 0.8\pi_2 + 0\pi_3 = \pi_2$$
$$0.2\pi_1 + 0.1\pi_2 + 0.4\pi_3 = \pi_2$$

~~solving~~

Also, $\pi_1 + \pi_2 + \pi_3 = 1$,

$\pi_1 = 7/5$ | $\pi_3 = \frac{14}{5}$

$\pi_2 = 8/5$

$\pi_1 = 0.4$

$\pi_2 = 0.4$

$\pi_3 = 0.2$

Queuing Theory

Queuing Theory is a mathematical study of Queue.

Queue : Queue is a waiting line for service; Queue are commonly observed at traffic light, Ticket Counter, doctor's clinic.

Queueing Theory is considered as a branch of operation Research as the queing models are often used in business for making decisions about the resources needed for providing services to the customer.

Queueing Theory has its apps : in intelligent Transportation system
(b) Telecommunication (c) Different network, (d) Traffic flow management system.

(e) Queueing Theory is used to derive mathematical Model of customer needing service & we use that model to predict the ~~Queue length~~ no. of customers in Queue (queue length) and in the system (line length) and waiting times of a

customers in the queue and in the system.

⇒ A simple queueing model is shown in figure as below.

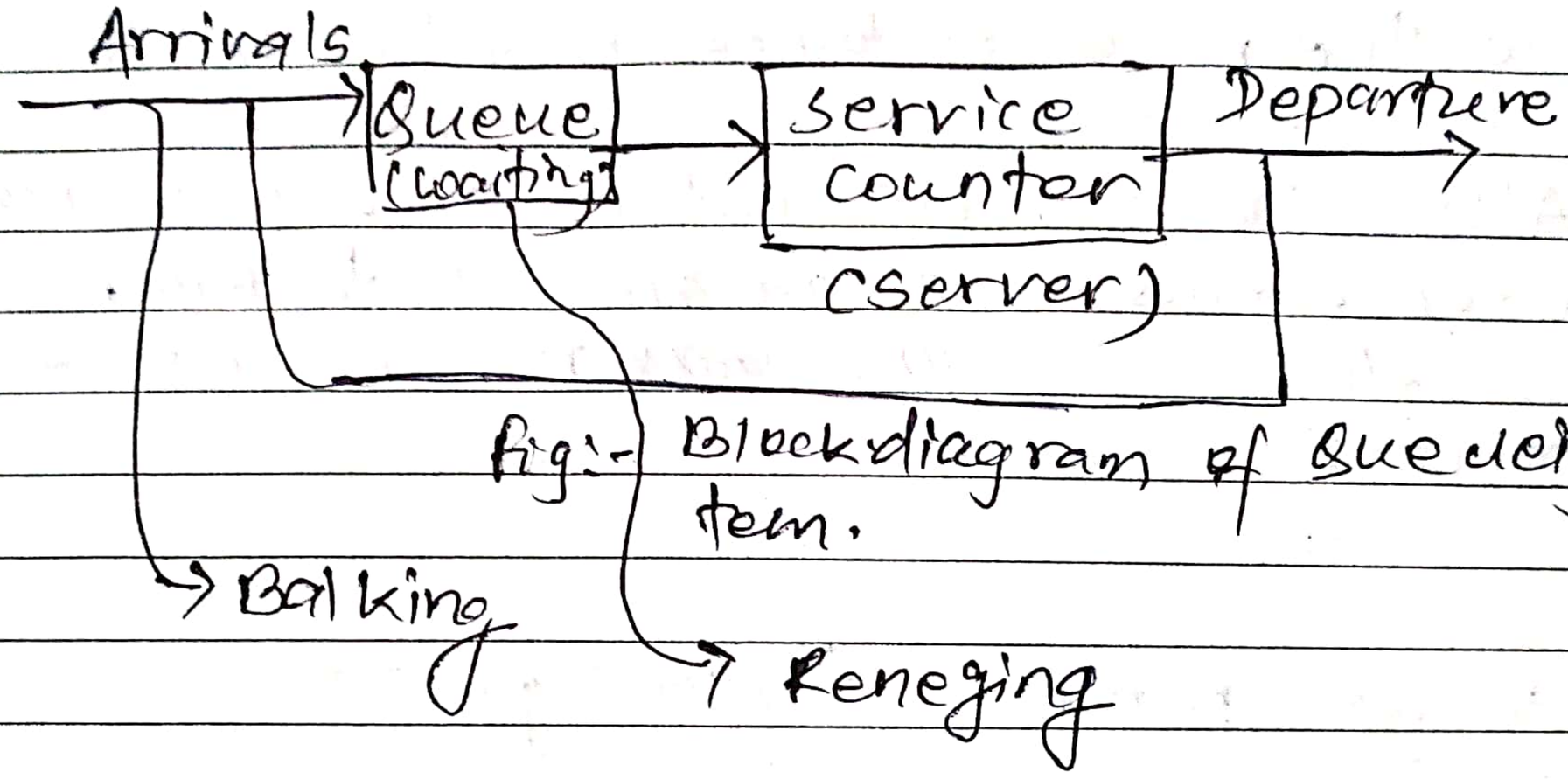

fig:- Block diagram of queueing system.

Basic Components (or characteristics) of queueing system

↳ There are following basic components of the queueing system.

⇒ (i) Arrival or input patterns
(ii) Service Mechanism or pattern
(iii) Queue Discipline
(iv) System capacity

ⓐ **Arrival patterns:** (i) Arrival rate (follows poisson's distribution)
(ii) Inter arrival

↳ The no. of customers per unit time is known as Arrival rate. i.e. Arrival rate = λ * Customer/unit time.

Interarrival time (follows exponential Distribution)
↳ The time between 2 consecutive arrivals is called interarrival time.

$$\text{Interarrival time} = \frac{1}{\lambda} \text{ time.}$$

Here, the no. of customers may be finite or infinite sources.

Also, the arrival pattern should indicate the no. of queues that are allowed to form, the max queue length and the max no. of customers requiring services.

Service pattern ~~(follows poisson~~

(i) Service Rate (follows poisson Distro)
↳ No. of service / unit time.

S.R = (μ) customer / unit time
↳ no. of

(ii) Service time (follows expo distro)
↳ Time required to complete the service for a customer is called service time.

$$\text{service time} = \frac{1}{\mu} \text{ time.}$$

The service pattern should indicate the no. of service counters (servers / channel).

and may have one or many servers in series or in parallel form.

Note: No. of servers is denoted by s.

Queueing Discipline

The rule by which the customers are selected for the service is called queue discipline.

Queue Discipline are:

(i) FIFO or FCFS → first serve
First come

This is the most commonly used queue discipline

(ii) LIFO or LCFS

This is mostly seen in inventory system.

(iii) SIRO ⇒ selection in Random order.

(iv) PIR or SIP ⇒ selection in priority.

System capacity

Max^m no. of customers in the queueing system can be either finite or infinite.

If any no. of customers are allowed to join the system, the system capacity is infinite. then the system capacity is infinite. sometime only limited customers are allowed in the system so that when the limiting value is reached no further customers are allowed enter in the system.

For Short Notes

Kendall's Notation of Queueing System

↳ Kendell's Notation provides a very convenient description of Queueing System.

This is notation of a Queueing system in the ~~following~~ form:

$(a/b/c):(d/e)$

where

a = Inter-Arrival time distribution.
b = Service time distribution.
c = No. of servers
d = system capacity. (may be finite or infinite)
e = Queue discipline

In Kendell's Notation a and b usually take one of the following distributions,

M = Exponential Distribution (or Markovian)
G = General Distributions
D = Deterministic Distribution.
E_K = Erlang distribution (Gamma Distribution)

The four important Queueing Systems described by Kendall's Notation are as follows:

(i) $(M/M/1):(\infty/FIFO)$
(ii) $(M/M/1):(N/FIFO)$ — finite system capacity.
(iii) $(M/M/S):(\infty/FIFO)$ — multiple servers

(iv) $(M/M/S):(N/FIFO)$

finite system capacity (or k sometimes)

Terminologies

(a) n = No. of customers (or units) in the system

(b) K or N = System capacity

(c) λ = Mean Arrival Rate $\Rightarrow 1/\lambda$ = mean interarrival time

(d) μ = Mean Service Rate $\Rightarrow \frac{1}{\mu}$ = mean service time

(e) $\rho = \frac{\lambda}{\mu}$, Traffic Intensity or Utilization factor probability that the server is busy, (show the busyness of server)

(f) P_0 = Probability that there is no customers or (units) in the system
(prob that the server is idle)

$\therefore$ $P_n \Rightarrow$ Prob that there is 'n' no. of customers in the queueing system

(g) S or C = No. of servers

L_s = Expected (or Average) no. of customers in the system (or predicted no. of " " " system)

(h) L_q = Expected No. of customers in queue.

(i) W_s = Expected waiting time of a customer in a system. W_q = " " " " queue.

(k) P_w = Prob that an arriving customer has to wait for service

$\rightarrow P(n \geq s)$

Study of Models

Model (I): $(M/M/1):(\infty/FIFO)$

service time distri (expo)

system capa (∞)

single server

Inter Arrival time distro (exponential distro)

Here, This model consists single server with ∞ system capacity.

→ Assumption

(i) Mean Arrival Rate

i.e. $\lambda_n = \lambda$

(ii) Mean Service Rate,

$\mu_n = \mu$

→ denotes busy for single server.

Traffic Intensity (Alway find firstly in numerical)

$\rho = \frac{\lambda}{\mu}$

(i) $P_0 = 1 - \rho = \bar{\rho}$ (Reverse of traffic intensity)

(ii) $P_n = (1-\rho)\rho^n$

i.e., $P_n = P_0 \rho^n$

$P_0 = \rho$ for single server model.

(iii) $L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{1-\lambda} = \frac{\lambda}{\mu-\lambda}$

$\frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}} = \frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

(iv) $L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

(v) $W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$

(vi) $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$

(vii) Prob that the no. of customers 'n' in the system exceeds a given number 'k'

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i.e $P(n>5)$

↳ $P(n>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$

or $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k}$

(viii) $P(W_s > t)$: given time vanda waiting time badi rayema,

$P(W_s > t) = e^{(\lambda-\mu)t}$

(ix) $P(W_q > t)$: $\rho\, e^{(\lambda-\mu)t}$

(x) $P_w = \rho$

Date
Page

single server

for model (i)

Q1. A supermarket has 1 cashier. During peak hours customers arrive at rate of 20/hour. The average no. of customers that can be processed by the casier is 24/hour.

Find:

a) The probability that the casier is idle (free)

b) Average no. of customer in supermarket

c) Average no. of customer in queue.

d) Average time a ~~cashier~~ customer spends in the supermarket.

e) The average time customer spends in queue for the service

Soln

Given

no. of server = 1

system capacity = ∞

So, the queueing model is:

$(M/M/1):(\infty/FIFO)$, Then we have

Average arrival rate = 20/hour. = λ

Average service rate = 24/hour. = μ

Traffic intensity $(\rho) = \lambda/\mu$

$= \frac{20}{24}$

$= 5/6$

$= 0.8$

(i) $P_0 = 1-\rho = 1-0.8$

$= 0.2$

(ii) Average.

$$L_s = \frac{\rho}{1-\rho}$$

$$= \frac{5/6}{1-5/6}$$

$$= 5 = (\text{customer/hours})$$

(iii) $$L_q = \frac{\rho^2}{1-\rho}$$

$$= \frac{25/36}{1-5/6}$$

$$= \frac{25}{6} \text{ customer/hr}$$

Always ($L_s > L_q$)

(iv) $$W_s = \frac{L_s}{\lambda}$$

$$= \frac{5}{20}$$

$$= 1/4 \text{ hours}$$

(v) $$W_q = \frac{L_q}{\lambda} = \frac{25/6}{20} \text{ hours}$$

Q2. The arrival of customer in the only single teller counter of a bank is poisson distributed at the rate of 25 customers/hour. The tellers takes on an average of 2 minutes to cash the cheque. The service time has been shown to be exponentially distributed,

find @ % of time, the teller is busy.

(b) And the prob that arrival has to wait,
(c) Probability of at least 10 customers in bank
(d) L_s, L_q, W_s, W_q

Soln Given,

no. of server = no. of teller Counter = 1

Average rate of arrival of customer = 25/hr. = λ

Average service ~~rate~~ time = 2 mins $= \frac{2}{60} = 1/\mu$

$= \frac{1}{30}$ hr

$\therefore$ Average service rate = 30/hr = μ

$\therefore$ So the queueing Model is:

$(M/M/1) : (\infty/FIFO)$

$\therefore$

(Q3) A Zerox machine is maintained in an office and operated by a secretary who does other jobs also. The service rate is poisson distributed with a mean service rate of 10 jobs / hr. Generally the requirements for use are random over the entire 8 hour day but arrive at a rate of 5 jobs / hour. Several people have noted that a waiting line developes occassionally and have questioned the office policy of maintaining only one Zerox machine. If the time of a secretary is valued at Rs 100 / hour. Find :

a) Utilization of the Zerox machine } 5 Same
b) Prob that an arrival has to wait. } Pw
c) The mean no of jobs in the system
d) The mean time spend by an ~~customer~~ arrival in the system (Ws)
e) Mean no. of jobs in the queue.
f) Average waiting of a job in queue.
g) The average cost / day due to waiting and operating the machine.

Soln

For model II

$(M/M/S) : (\infty/FIFO)$

- M → Interarrival time distribution
- M → service time distribution
- S → multiple no. of server
- ∞ → system capacity
- FIFO → queue discipline

Assumptions

(i) Mean arrival rate

i.e. $\lambda_n = \lambda$

(ii) Mean service rate

$\mu_n = n\mu$ if $n < S$

or $S\mu$ if $n \geq S$

For steady state condition:

We have, $\lambda < S\mu$

Thus, $\boxed{\text{Traffic intensity } (\rho) = \frac{\lambda}{S\mu}}$

(i) $$P_0 = \left[\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \cdot \frac{S\mu}{S\mu - \lambda} \right]^{-1}$$

OR

$$P_0 = \left[1 + \frac{S\rho}{1!} + \frac{(S\rho)^2}{2!} + \cdots + \frac{S\rho}{S!(1-\rho)} \right]^{-1}$$

where, $\rho = \frac{\lambda}{S\mu}$

For $S = 2$

$$P_0 = \left[1 + \frac{S\rho}{1!} + \frac{S\rho}{2!(1-\rho)} \right]^{-1}$$

for $s=3$

$$P_0 = \left[1 + \frac{s\rho}{1!} + \frac{(s\rho)^2}{2!} + \frac{s\rho}{3!(1-\rho)}\right]^{-1}$$

(ii) $P_n = \frac{s^n \rho^n}{n!} P_0$ if $n<s$

$= \frac{s^s \rho^n}{s!} P_0$ if $n \geq s$

(iii) $P(n \geq s)$ or P_W = Probability that all servers are busy.

(i.e Probability that an arriving will have to wait)

$$\left\{ = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda} P_0 \right\} \Longrightarrow (\#)$$

where

s = no. of servers

μ = Mean service rate

λ = Mean arrival rate

(iv) $L_q = \left[\frac{1}{(s-1)!} \cdot \left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\lambda\mu}{(s\mu-\lambda)^2}\right] P_0$

or,

$$L_q = \frac{1}{s \cdot s!} \frac{(\lambda/\mu)^{s+1}}{(1-\lambda/s\mu)^2} \cdot P_0$$

(v) $L_s = L_q + \frac{\lambda}{\mu}$ OR

$(L_s = L_q + s\rho)$

Note: Always focus on servers and system capacity.

6] $W_q = \frac{L_q}{\lambda}$

7] $W_s = \frac{L_s}{\lambda}$

(Qn) A travel centre has 3 service counters to receive people who visit to book air tickets. The customers arrives in a poission's distribution with the average arrival 100 persons in 10 hour services. It has been estimated that the service time follows exponential distribution. The average service time is 15 minutes.

Find,

(i) Expected number of customers in the system (L_s) = ?

(ii) Expected number of customers in the queue (L_q) = ?

(iii) Average time a customer spends in a system (W_s) = ?

(iv) Expected waiting time for a customer in queue (W_q) = ?

(v) Probability that a customer must wait before he gets service (P_w) = ?

↳ Here

The queueing model is:

$(M/M/S) : (\infty/FIFO)$

We have,

$S = 3$

mean arrival rate = 100 persons per 10 hour

$\therefore \lambda =$ 10 person per hour.

Mean service time $\left(\frac{1}{\mu}\right) = 15$ minutes

$= \frac{15}{60}$ per hour

$\therefore \mu = 4$ pr/hr

$\therefore$ Traffic intensity $(\rho) = \frac{\lambda}{s\mu} = \frac{10}{4 \times 3} = \frac{10}{12} = 5/6$

$$P_0 = \left[1 + s\rho + \frac{(s\rho)^2}{2!} + \frac{(s\rho)^s}{3!\,(1-\rho)}\right]^{-1}$$

$$\text{or } P_0 = \left[1 + 5/2 + \frac{(5/2)^2}{2!} + \frac{(5/2)^3}{3!\,(1-5/6)}\right]^{-1}$$

$\therefore P_0 = 4/89$

$\therefore L_s = L_q + \frac{\lambda}{\mu}$

$$L_q = \frac{1}{s \cdot s!} \frac{(\lambda/\mu)^{s+1}}{\left(1 - \frac{\lambda}{s\mu}\right)^2} P_0$$

$$= \frac{1}{3 \cdot 3!} \frac{(10/4)^{3+1}}{\left(1 - \frac{10}{12}\right)^2} 0.044$$

$$= \frac{1}{18} \times \frac{\frac{625}{16}}{1 - 1/36} \times 0.044$$

$\therefore L_q = 3.4375$ (i.e persons per hour)

$$W_s = \frac{L_s}{\lambda} = \frac{3.9375}{10}$$

$= 0.39375$ hrs,

$L_s = L_q + \frac{\lambda}{\mu}$

$= 3.4375 + \frac{10}{4}$ $W_q = \frac{L_q}{\lambda}$

$= 5.9375$ (persons per hours)

$P(n \geq s)$ or $P(W)$

i.e.

$P(n \geq 3) = \frac{1}{s!}\left(\lambda/\mu\right)^s \left(\frac{s\mu}{s\mu - \lambda}\right) \cdot P_0$

$= \frac{1}{3!}\left(\frac{10}{4}\right)^3 \left(\frac{12}{12-10}\right) \times 0.044$

$= \frac{1}{6} \times \frac{125}{8} \times 6 \times 0.044$

$= 0.6875$

A petrol pump has 4 pumps. The service time follows exponential distribution with a mean of 6 minutes and car arrives for service in a possion process at the rate of 30 cars per hour.

(i) What is the probability that an arrival will have to wait (P_W)

(ii) Find the average waiting time in the queue (W_q), average time spent in the system (W_s).

(iii) For what percentage of time would the pumps be idle on an average.

→ The queueing model is:

$(M/M/S) : (\infty/FIFO)$

where

$S = \mu$

Given,

Mean, service time $(1/\mu) = 6$ cars per minus,

$\therefore \left(\frac{1}{\mu}\right) = \frac{6}{60}$ cars per hr

$\mu = 10$ cars per hour.

$\therefore$ Mean service rate $(\mu) = 10$ cars / hour.

Mean arrival rate $(\lambda) = 30$ cars per hours.

Here,

traffic intensity $(\rho) = \lambda / S\mu = \frac{30}{4 \times 10} = 3/4$

$$P_0 = \left[1 + \frac{(S\rho)}{1!} + \frac{(S\rho)^2}{2!} + \frac{(S\rho)^3}{3!} + \frac{(S\rho)^4}{4!(1-\rho)}\right]^{-1}$$

$= 2/53 = 0.0377$

$P(n \geq 4)$

Stochastic process : Random process

Markov process

It is the process where the outcomes depends upon time.
Markov process is the random process where the outcomes depends upon time i.e. It is a ~~memory less~~ process having memory less property (i.e. markov property) i.e. future state of random process depends upon current state not in past state.

Markov process

- It is the process where the outcomes depends upon time.
- This process is also known as, Random, Stochastic process
- It is a process with memory less prop (markov property) where the outcome of future state depends upon the outcome of present state not of past state.

Mathematically,

A random process, $P\{X(t), t \in T\}$ is called Markov process, if

~~$P\{X_{(n+1)} t_{n+1}$~~

$$P\{X(t_{n+1})\} = X_{n+1}\left[X(t_n) = x_n,\ X(t_{n-1}) = x_{n-1},\ X(t_{n-2}) = x_{n-2} \cdots\right]$$

$$= X_{n+1}\left[X(t_n) = x_n\right]$$

where $t_0 < t_1 < t_2 \cdots < t_{n+1}$ and $x_0, x_1, x_2 \cdots x_n, x_{n+1}$ are the states (Result) of process.

25/8/17

Defining the event,

E_1 = selecting Box 1

E_2 = selecting Box 2

E_3 = " " 3

Then,

$P(E_1) = 1/3$, $P(E_2) = 1/3$, $P(E_3) = 1/3$

Let A = selecting 1 W and 1 R,

Then, $P(A/E_1) = \frac{{}^1C_1 \, {}^2C_1}{{}^6C_2}$

$= \frac{1 \times 2}{15}$

$= 2/15$

$P(A/E_2) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2}$

$= \frac{2 \times 3}{15}$

$= 6/15$

$= \frac{2}{5}$

$P(A/E_3) = {}^3C_1$

→ Discrete Random variable
→ continuous Random variable

Random variable ⇒ X, Y, Z | value taken by R.V ⇒ x_1, y_1, z

prob fcn. of D.R.V

↳ Let X be a D.R.V, which takes the values, $x_1, x_2, \ldots x_n$, with probabilities $P(X=x_i) = P(x_i) = P(X)$ where $i = 1, \ldots n$

$P(X=x_i)$	$P(X=x)$
0	1/2
1	1/2

So, The probability function $P(X=x_i)$ is said to be probability mass function (pmf) if it satisfies the following condition,

$$\sum_{i=1}^{n} P(X=x_i) = 1$$

$$P(X=x_i) \geq 0 \quad \forall i = 1, 2 \ldots n$$

Starting — Normal Distribution

Gamma, Beta distri ⇒ rare
N. Distrib ⇒ asked always

Snotes ⇒ 2-out of 2 (5 marks)
numerical ⇒ 3 (1ab, 2ab, 3ab) ⇒ (7+8) × 3