CHAPTER 5

CONE, CYLINDER, CONICOID

Cone:

A cone is a surface generated by line (the lines are called generators) that passes through a fixed point and touches the given surface.

Note: The fixed point is called vertex of the cone.

Note: The lines are called generator of the cone.

General equation of a cone:

An equation,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

represents a cone with vertex at (u, v, w) if

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0.$$

Equation of cone having vertex at origin:

[2019 (Fall), 1999; 2001; 2003(Fall); 2006(Fall); 2008(Spring)–Short]

The equation,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represents a cone with vertex at origin.

Equation of cone having three mutually perpendicular generators:

[2000; 2002; 2004(Spring); 2007(Fall); 2008(Spring)–Short][2017 Fall]

An equation,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

have a set of three mutually perpendicular generators when $a + b + c = 0$.

Cylinder:

A cylinder is a locus of lines which remains parallel to a fixed line and intersects a given curve.

The lines are called generator of cylinder.

Equation of cylinder:

The equation of cylinder whose generators intersect the curve $x^2 + y^2 + 2fy + 2gx + c = 0$, $z = 0$ and are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is,

$$(nx - lz)^2 + (ny - mz)^2 + 2gn\,(nx - lz) + 2fx\,(ny - mz) + cn^2 = 0.$$

Note: If the generators are parallel to z–axis then $l = 0 = m$ and $n = 1$. So, the equation of cylinder is, **[2004 (Fall) – Short]**

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Equation of right circular cylinder:

The equation of right circular cylinder whose axis is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

and having radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - [a(x - x_1) + b(y - y_1) + c(z - z_1)]^2 = r^2$$

Note: If the axis is z–axis. Then $a = b = 0$ and $c = 1$. Also, $x_1 = y_1 = z_1 = 0$. So, the equation of right circular cylinder is,

$$x^2 + y^2 = r^2$$

Conicoid:

An equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represents a coincoid.

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CHAPTER 7

SERIES SOLUTIONS AND SPECIAL FUNCTIONS

Power series

The infinite series of the form,

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0)^1 + a_2(x - x_0)^2 + \dots\dots$$

where a_0, a_1, a_2 .... are constant, is called power series.

Particular case: If $x_0 = 0$ then,

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots$$

Some Formulae

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Exercise 7.1

Applying power series method, solve the following differential equations.

(1) $y' = 3y$

Solution: Given differential equation is,

$$y' = 3y \quad \dots\dots(i)$$

Let,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \quad \dots (ii)$$

be the solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

Putting the value of y and y' in (i),

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots = 3a_0 + 3a_1 x + 3a_2 x^2 + 3a_3 x^3 + 3a_4 x^3 + \dots$$

comparing coefficient of constant term, x, x^2

$$a_1 = 3a_0, \qquad 2a_2 = 3a_1 \Rightarrow a_2 = \frac{3}{2} a_1 = \frac{9}{2} a_0$$

$$3a_3 = 3a_2 \Rightarrow a_3 = a_2 = \frac{9}{2} a_0, \qquad 4a_4 = 3a_3 = \frac{3}{4} \times \frac{9}{2} a_0 = \frac{27}{8} a_0$$

and so on.

Putting the value of a_1, a_2, a_3 and a_4 in (ii),

$$y = a_0 + 3a_0x + \frac{9a_0}{2}x^2 + \frac{9a_0}{2}x^3 + \frac{27}{8}a_0x^4 + \ldots$$

$$= a_0\left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \ldots\right) = a_0e^{3x}$$

(2) $y' + 2y = 0$.

Solution: Given differential equation is,

$$y' + 2y = 0 \qquad \ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots(ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots\ldots$$

Putting the value of y an y' in (i) then,

$$a_1 + 2a_2x + 3a_3x^2 + \ldots + 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + \ldots = 0$$

$$\Rightarrow \quad (a_1 + 2a_0) + x(2a_2 + 2a_1) + x^2(3a_3 + 2a_2) + \ldots = 0$$

Equating each coefficient to zero,

$a_1 + 2a_0 = 0 \Rightarrow a_1 = -2a_0$; $\quad 2a_2 + 2a_1 = 0 \Rightarrow a_2 = -a_1 = 2a_0$;

$3a_3 + 2a_2 = 0 \Rightarrow a_3 = -\frac{2}{3}a_2 = -\frac{2}{3} \times 2a_0 = -\frac{4}{3}a_0$

and so on.

Substituting the value of $a_1, a_2, a_3, \ldots$ in (ii) then,

$$y = a_0 - 2a_0x + 2a_0x^2 - \frac{4}{3}a_0x^3 + \ldots = a_0\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\right)$$

$$= a_0e^{-2x}$$

(3) $y' - y = 0$.

Solution: Given differential equation is,

$$y' - y = 0 \qquad \ldots\ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots\ldots\ldots(ii)$$

be the solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots$$

Putting the value of y and y' in (i) then

$$a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots - a_0 - a_1x - a_2x^2 - a_3x^3 \ldots\ldots\ldots\ldots = 0$$

$$\Rightarrow \quad (a_1 - a_0) + x(2a_2 - a_1) + x^2(3a_3 - a_2) + \ldots\ldots\ldots = 0.$$

Equating coefficient of like terms from both sides then,

$a_1 - a_0 = 0$, $\quad 2a_2 - a_1 = 0$, $\quad 3a_3 - a_2 = 0$ and so on.

$\Rightarrow a_1 = a_0$ $\quad \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$ $\quad \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{6}$

Substituting the value of $a_1, a_2, a_3, \ldots$ in (ii), we get

$$y = a_0 + a_0x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 \ldots\ldots = a_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots\right) = a_0e^x$$

(4) $y' = 2xy$. **[2017 Fall Q.No. 5 (a)]**

Solution: Given differential equation is,

$$y' = 2xy \qquad \ldots\ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \qquad \text{.......... (ii)}$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots\dots$$

Putting the value of y and y' is equation (i), we get

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots\dots = 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^3 + \dots\dots$$

Equating coefficient of like terms from both sides then,

$$a_1 = 0; \qquad 2a_2 = 2a_0 \Rightarrow a_2 = a_0;$$

$$3a_3 = 2a_1 \Rightarrow a_3 = \frac{2}{3}a_1 = 0; \qquad 4a_4 = 2a_2 \Rightarrow a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0$$

and so on.

Putting the value of $a_1, a_2, a_3, a_4, \dots$ in (ii), we get

$$y = a_0 + 0 + a_0x^2 + 0 + \frac{a_0}{2}x^4 + \dots = a_0\left(1 + x^2 + \frac{1}{2}x^4 + \dots\dots\right)$$

$$= a_0e^{x^2}$$

(5) $y' = -2xy$ **[1999, 2001 Q. No. 5(a) OR] [2004 Fall Q. No. 5(a)]**

Solution: Given differential equation is,

$$y' = -2xy \qquad \text{...........(i)}$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \qquad \text{.......... (ii)}$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots\dots\dots$$

Putting the value of y and y' in (i) then

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = -2a_0x - 2a_1x^2 - 2a_2x^3 - 2a_3^4 - \dots$$

Equating coefficient of like terms from both sides then,

$$a_1 = 0; \qquad 2a_2 = -2a_0 \Rightarrow a_2 = -a_0;$$

$$3a_3 = -2a_1 \Rightarrow a_3 = \frac{-2}{3}a_1 = 0; \qquad 4a_4 = -2a_2 \Rightarrow a_4 = \frac{-1}{2}a_2 = \frac{a_0}{2};$$

and so on.

Putting the value of $a_1, a_2, a_3, a_4, \dots$ in (ii) then,

$$y = a_0 + 0 - a_0x^2 + 0 - \frac{a_0}{2}x^4 + \dots = a_0\left(1 - x^2 - \frac{x^4}{2} - \dots\right) = a_0e^{-x^2}$$

(6) $xy' - 3y = k$ **[2012 Fall Q.No.5(b) OR]**

Solution: Given differential equation is,

$$xy' - 3y = k \qquad \text{.......(i)}$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \qquad \text{........ (ii)}$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

Putting the value of y_1y' in equation (i)

$$a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \dots - 3a_0 - 3a_1x - 3a_2x^2 - 3a_3x^3 - = k.$$

$$\Rightarrow a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \dots = k + 3a_0 + 3a_1x + 3a_2x^2 + 3a_3x^3 + \dots$$

Equating coefficient of like terms from both sides then,

$(3a_0 + k) = 0 \Rightarrow a_0 = \frac{-k}{3}$;

$a_1 = 3a_1 \Rightarrow a_1 = 0$;

$3a_2 = 2a_2 \Rightarrow a_2 = 0$;

and so on.

Putting the value of $a_0, a_1, a_2, \ldots$ in (ii) then

$$y = \frac{-k}{3}$$

[2011 Spring Q.No.5(a), 2000 Q.No.5(a) Or]

(7) $y'' + 9y = 0$.

Solution: Given differential equation is,

$$y'' + 9y = 0 \quad \text{...........(i)}$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \quad \text{...... (ii)}$$

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

and, $y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots$

Putting the value of y and y" in (i) then,

$$2a_2 + 6a_3x + 12x_4x^2 + \ldots + 9a_0 + 9a_1 + 9a_2x^2 + 9a_3x^3 + \ldots = 0$$

$$\Rightarrow \quad (2a_2 + 9a_0) + x(6a_3 + 9a_1) + x^2(12a_4 + 9a_2) + \ldots = 0$$

Equating coefficient of like terms from both sides then,

$2a_2 + 9a_0 = 0$ $\quad$ $6a_3 + 9a_1 = 0$ $\quad$ $12a_4 + 9a_2 = 0$ and so on.

$\Rightarrow a_2 = \frac{-9}{2}a_0$ $\quad$ $\Rightarrow a_3 = \frac{-9}{6}a_1$ $\quad$ $\Rightarrow a_4 = \frac{-9}{12}a_2 = \frac{27}{8}a_0$

Putting the value of a_2, a_3, a_4, in (ii),

$$y = a_0 + a_1x - \frac{9}{2}a_0x^2 - \frac{3}{2}a_1x^3 + \frac{27}{8}a_0x^4 + \ldots\ldots.$$

$$= a_0\left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \ldots\ldots.\right) + a_1\left(x - \frac{3}{2}x^3 + \ldots\ldots.\right)$$

$$= a_0\cos 3x + a_1\sin 3x..$$

(8) $y'' + y = 0$.

[2014 Spring Q.No. 5 (a),2006 Spring, 2008 Fall, 2011 Fall Q. No. 5(a)]

Solution: Given differential equation is,

$$y'' + y = 0 \quad \text{........(i)}$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \quad \text{....... (ii)}$$

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots.$$

and $y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots.$

Putting the value of y and y" in eq^n. (i)

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots. + a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots. = 0$$

$$\Rightarrow \quad (2a_2 + a_0) + x(6a_3 + a_1) + x^2(12a_4 + a_2) + \ldots\ldots. = 0$$

Equating coefficient of like terms from both sides then,

$2a_2 + a_0 = 0$ $\quad$ $6a_3 + a_1 = 0$ $\quad$ $12a_4 + a_2 = 0$ and so on.

$\Rightarrow a_2 = \frac{-a_0}{2}$ $\quad$ $\Rightarrow a_3 = \frac{-a_1}{6}$ $\quad$ $\Rightarrow a_4 = \frac{-a_2}{12} = -\frac{a_0}{2} \times \frac{-1}{12} = \frac{a_0}{24}$

Putting the value of a_2, a_3, a_4, ... in (ii) then,

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{24}x^4 + \ldots\ldots.$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + a_1\left(x - \frac{x^3}{6} + \ldots\ldots\right).$$

(9) $y' = 3x^2y$. [2004 Spring Q. No. 5(a) OR]

Solution: Given differential equation is,

$$y' = 3x^2y \qquad \ldots\ldots\ldots\ldots (i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots \qquad \ldots\ldots(ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\ldots\ldots$$

Putting the value of y and y' in eq^n. (i)

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots.. = 3a_0x^2 + 3a_1x^3 + 3a_2x^4 + 3a_3x^5 + 3a_4x^6 + \ldots..$$

Equating coefficient of like terms from both sides then,

$a_1 = 0,\quad 2a_2 = 0 \quad 3a_3 = 3a_0 \quad 4a_4 = 3a_1 \quad 5a_5 = 3a_2$ and so on.

$$\Rightarrow a_2 = 0. \quad \Rightarrow a_3 = a_0 \quad \Rightarrow a_4 = \frac{3}{4}a_1 \Rightarrow a_5 = \frac{3}{5}a_2 = 0.$$

Putting the value of $a_1, a_2, a_3, \ldots\ldots.$ in (ii) then,

$$y = a_0 + a_0x^3 + \ldots\ldots..$$
$$= a_0(1 + x^3 + \ldots\ldots.)$$
$$= a_0e^{x^3}$$

(10) $y'' + 4y = 0$.

[2016 Fall Q.No. 5 (a), 2013 Fall Q.No. 5 (a), 2009 Spring Q. No. 5(a)]

Solution: Given differential equation is,

$$y'' + 4y = 0 \qquad \ldots\ldots\ldots..(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots. (ii)$$

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots..$$

and $y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots\ldots$

Putting the value of y and y'' in (i) then,

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots. + 4a_0 + 3a_1x + 4a_2x^2 + 4a_3x^3 + \ldots\ldots.. = 0$$
$$\Rightarrow (2a_2 + 4a_0) + x(6a_3 + 4a_1) + x^2(12a_4 + 4a_2) + \ldots\ldots = 0$$

Equating coefficient of like terms from both sides then,

$2a_2 + 4a_0 = 0 \qquad 6a_3 + 4a_1 = 0 \qquad 12a_4 + 4a_2 = 0$ and so on.

$$\Rightarrow a_2 = -2a_0 \qquad \Rightarrow a_3 = \frac{-2}{3}a_1 \qquad \Rightarrow a_4 = \frac{-1}{3}a_2 = \frac{2}{3}a_2$$

Putting the value of $a_2, a_3, a_4, \ldots..$ in (ii) then,

$$y = a_0 + a_1x - 2a_0a^2 - \frac{2}{3}a_1x^3 + \frac{2}{3}a_0x^4 + \ldots\ldots$$
$$= a_0\left(1 - 2x^2 + \frac{2}{3}x^4\right) + a_1\left(x - \frac{2}{3}x^3 + \ldots\ldots\ldots\right)$$
$$= a_0\cos 2x + \frac{1}{2}a_1 \sin 2x.$$

(11) $(1 + x)\,y' = y$.

[2018 Spring Q.No. 5(a), 2017 Spring Q.No. 5 (a), 2003 Fall Q. No. 5(a)]

Solution: Given differential equation is,

$$(1 + x)\,y' = y \qquad \ldots\ldots.(i)$$

Let, $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ .... (ii)

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots$$

Putting the value of y and y' in (i) then,

$$(1 + x)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

$$\Rightarrow a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots + a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \ldots$$

$$= + a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

$$\Rightarrow a_1 + x(2a_2 + a_1) + x^2(3a_3 + 2a_2) + x^2(4a_4 + 3a_3) + \ldots$$

$$= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

Equating coefficient of like terms from both sides then,

$a_1 = a_0$ $\quad 2a_2 + a_1 = a_1$ $\quad 3a_3 + 2a_2 = a_2$ $\quad 4a_4 + 3a_3 = a_3$ and so on

$\Rightarrow a_2 = 0$ $\quad \Rightarrow a_3 = \frac{-a_2}{3} = 0$ $\quad \Rightarrow 4a_4 = -2a_3$

$\Rightarrow a_4 = \frac{-1}{2} a_3 = 0$

Putting the value of $a_1, a_2, a_3, \ldots\ldots$ in (2) then,

$$y = a_0 + a_0x + 0 + 0 + 0 + \ldots$$

$$= a_0(1 + x).$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

Similar Question for Practice from Final Exam:

2002 Q. No. 5(a)

Find a power series solution of the differential equation $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)y = 0$

2002 Q. No. 5(a) OR; 2006 Fall; 2008 Spring; 2010 Spring Q. No. 5(a), 2013 Spring Q.No. 5 (a), 2015 Fall Q.No. 5 (a), 2015 Spring Q.No. 5 (a), 2016 Spring Q.No. 5 (a), 2019 Fall Q.No. 5 (a),

Solve by power series method: y" = 4y.

2002 Q. No. 5(b)

Solve the initial value problem $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$; y(0) = 10, y'(0) = 0 by power series solution.

2000(OR); 2007 Fall Q. No. 5(a)

Solve y" = 9y by using power series method.

2009 Spring Q. No. 5(a), 2014 Fall Q.No. 5 (a), 2009 Fall Q.No. 5(a)

Solve: y" = 8y by power series method.

2018 Fall Q.No. 5(a)

Solve the differential equation : $(1 + x^2)y'' + xy' - y = 0$, by using power series methods.

Legendre's Equation:

The second order differential equation of the form

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

is known as Legendre's equation.

Note: The solution of above equation is Legendre's function.

Legendre's Polynomial:

The polynomial,

$$P_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(2n-2m)!^m}{2^{2n}\, m!\,(n+m)!\,(n-m)!} x^{n-2m}$$

is called the Legendre's polynomial of order n.

Solution of Legendre's Equation: [2007 Fall Q. No. 5(a) OR]

We have Legendre's equation as

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \ldots\ldots (1)$$

Let,
$$y = \sum_{m=0}^{\infty} a_m x^m \quad \ldots\ldots (2)$$

be the solution of (1).

Here differentiating with respect to x, we get,

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Substituting these values in equation (1) we get,

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

where $k = n(n+1)$

By writing the first expression as two separate series, we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow 2a_1 + 3.2a_3x + 4.3a_4x^2 + \ldots\ldots + (s+2)(s+1)a_s + 2x^s + \ldots\ldots - 2.1a_2x^2 - \ldots\ldots + \ldots\ldots - 2.1a_1x - 2.2a_2x^2 - \ldots\ldots - s(s-1)a_sx^s - \ldots\ldots + ka_0 + ka_1x + ka_2x^2 + \ldots\ldots - 2sa_sx^s - \ldots\ldots = 0.$$

Comparing the coefficients of x^0, x, x^s, we get

$$2a_2 + ka_0 = 0 \Rightarrow 2a_2 + n(n+1)a_0 = 0 \quad \ldots\ldots (3)$$

$$6a_3 + [-2+k]a_1 = 0 \Rightarrow 6a_3 + [-2+n(n+1)]a_1 = 0 \quad \ldots\ldots (4)$$

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + k]a_s = 0$$

$$\Rightarrow (s+2)(s+1)a_{s+2} + [-s^2 - s + n(n+1)]a_s = 0 \quad \ldots\ldots (5)$$

Thus, $a_{s+2} = -\dfrac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s$ for $s = 0, 1, 2, 3, \ldots$

From equation (3), (4) and (5) we get,

$$a_2 = -\frac{n(n+1)}{2!}a_0; \qquad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4.3}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0;$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

Substituting the coefficients in equation (2), we get

$$y = a_0 + a_1x + \frac{(-n)(n+1)}{2!}a_0x^2 + \frac{(-)(n-1)(n+2)}{3!}a_1x^3 + \frac{(n-2)n(n+1)(n+3)}{4!}a_0x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1x^5 + \ldots\ldots$$

$$\Rightarrow \quad y = a_0\left(1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right) \qquad \ldots\ldots (6)$$

$$\Rightarrow \quad y = a_0y_1 + a_1y_1$$

where

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\ldots$$

And, $$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots\ldots\ldots\ldots$$

These y_1 and y_2 be power series, which are convergent for $|x| < 1$. Thus $y = a_0y_1 + a_1y_2$ is the Legendre solution of the given Legendre's equation (1).

Definition of Bessel's Function of First Kind:

The Bessel's function of first kind of order n is denoted by $J_n(x)$ and is defined as,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!(n+m)!}$$

Bessel's Equation:

A differential equation of second order of the form

$$x^2y'' + xy' + (x^2 - g^2)\, y = 0 \qquad \ldots\ldots (1)$$

where g is real and non-negative number; is said to be Bessel equation.

Bessel Function of first kind of order n:

The function of the form,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!(n+m)!}$$

is called Bessel function of first kind of order n.

Solution of Bessel Equation:

Consider a Bessel's equation,

$$x^2y'' + xy' + (x^2 - g^2)\, y = 0 \qquad \ldots\ldots (1)$$

where g is real and non-negative number.

Let $$y = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad \ldots\ldots (2)$$

with $(a_0 \neq 0)$, be a solution of (1). Then,

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)\, a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - g^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Equating the coefficient of x^{s+r} to zero, we get

$$(s+r)(s+r-1)a_s + (s+r)\, a_s + a_{s-2} - g^2 a_s = 0 \qquad \ldots\ldots (3)$$

For s=0, we get,

$$r(r-1)a_0+ra_0-g^2a_0=0 \Rightarrow (r^2-r+r-g^2)=0$$
$$\Rightarrow (r^2-g^2)=0$$
$$\Rightarrow (r-g)(r+g)=0 \Rightarrow r=g,-g$$

Let the roots of r is, $r_1 = g$ and $r_2 = -g$

For $r = g$ we have $(s+r)(s+r-1)a_s+(s+r)a_s+a_{s-2}-g^2a_s=0$

$$\Rightarrow (s^2+2sr+r^2-s-r+s+r-g^2)a_s+a_{s-2}=0$$
$$\Rightarrow (s^2+2sr+r^2-g^2)a_s+a_{s-2}=0$$
$$\Rightarrow [(s+r)^2-g^2]a_s+a_{s-2}=0$$
$$\Rightarrow (s+r-g)(s+r+g)a_s+a_{s-2}=0$$

If $r = g$ then $s(s+2g)a_s + a_{s-2} = 0$ ...... (4)

Since, $a_1 = 0$ and $g \geq 0$, it gives $a_3 = 0$, $a_5 = 0$, ... successively.

So to evaluate the coefficient of even numbers s=2m. Put s = 2m in equation (4) we get,

$$(2m+2g)2ma_{2m}+a_{2m-2}=0$$
$$\Rightarrow a_{2m}=\frac{1}{2^2m(g+m)}a_{2m-2}; \qquad \text{for } m=1,2,3,\ldots\ldots$$

Thus we get,

$$a_2=\frac{-a_0}{2^2(g+1)} \quad \text{and} \quad a_4=\frac{(-a_2)}{2^2\,2(g+2)}$$

Therefore, $a_4=\dfrac{a_0}{2^4\,2!(g+1)(g+2)}$

So in general,

$$a_{2m}=\frac{(-1)^m a_0}{2^{2m}m!(g+1)(g+2)\ldots..(g+m)}, \qquad \text{for } m=1,2,\ldots\ldots$$

Put g = n, then,

$$a_{2m}=\frac{(-1)^m a_0}{2^{2m}m!(n+1)(n+2)\ldots..(n+m)}$$

Here a_0 is still arbitrary. Let us choose $a_0 = \dfrac{1}{2^n n!}$, because $n!(n+1)\ldots\ldots(n+m)=(n+m)!$.

Then, $a_{2m}=\dfrac{(-1)^n}{2^{2m+n}m!(n+m)!}$ for m = 1, 2, 3, ...

Substituting these values of coefficients in equation (2) we get,

$$y=x^n\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}$$

Let y is denoted by $J_n(x)$. That is,

$$J_n(x)=x^n\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}$$

This is the solution of Bessel's equation (1).

Some Remarks on Bessel's Function of First Kind:

1. Show that $J_{-n}(x) = (-1)^n J_n(x)$.

Solution: We have,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!(n+m)!}$$

Put n = -n we get,

$$J_{-n}(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n}\, m!(-n+m)!}$$

$$= \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n}\, m!(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!s!} \quad \text{when } s = m-n$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n}(n+s)!s!} = (-1)^n J_n(x)$$

Thus, $J_{-n}(x) = (-1)^n J_n(x)$.

2. Show that $\frac{d}{dx}[x^g J_g(x)] = x^g J_{y-1}(x)$

[2004(Spring)–Short; 2004 Spring Q. No. 5(b)]

Solution: We have,

$$x^g J_g(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+g} m! G(g+m+1)}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}[x^g J_g(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+g) x^{2m+2g-1}}{2^{2m+g} m! G(g+m+1)}$$

$$= x^g x^{g-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+g+1} m! G(g+m)} = x^g J_{y-1}(x)$$

$$\Rightarrow \frac{d}{dx}[x^g J_g(x)] = x^g J_{y-1}(x).$$

3. Show that $\frac{d}{dx}[x^{-g} J_g(x)] = -x^{-g} J_{y+1}(x)$

Solution: We have,

$$x^{-g} J_g(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+g} m! G(g+m+1)}$$

Differentiating with respect to x, we get,

$$\frac{d}{dx}[x^{-g} J_g(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m+g} m! G(g+m+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+g-1}(m-1)! G(g+m+1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2(m-1)+1}}{2^{2(m-1)+g+1}(m-1)!G(g+m-1+2)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+g+1} s!G(g+s+2)} \text{ by putting } s = m-1$$

$$= -x^{-g} \sum_{m=0}^{\infty} \frac{(-1)^s x^{2s+1+g}}{2^{2s+g+1} s!G(g+s+1+1)} = -x^{-g} J_{y+1}(x).$$

Thus, $\frac{d}{dx}[x^{-g}J_g(x)] = -x^{-g}J_{y+1}(x)$.

4. **Show that $gx^{g-1}J_g(x) + x^gJ¢_g(x) = x^gJ_{g-1}(x)$**

Solution: We have,

$$\frac{d}{dx}[x^gJ_g(x)] = x^gJ_{g-1}(x) \qquad \text{[By 2]}$$

$$\Rightarrow \quad x^gJ'_g(x) + gx^{g-1}J_{g-1}(x) = x^gJ_{g-1}(x)$$

5. **Show that $J_{g-1}(x) + J_{g+1}(x) = \frac{2g}{x}J'_g(x)$**

Solution: We have,

$$\frac{d}{dx}[x^gJ_g(x)] = x^gJ_{g-1}(x) \qquad \text{...... (1)}$$

and $$\frac{d}{dx}[x^{-g}J_g(x)] = -x^{-g}J_{g+1}(x) \qquad \text{...... (2)}$$

From (1), $$gx^{g-1}J_g(x) + x^gJ'_g(x) = x^gJ_{g-1}(x)$$

$$\Rightarrow \frac{g}{x}J_g(x) + J'_g(x) = J_{g-1}(x) \qquad \text{...... (3)}$$

From equation (2),

$$-gx^{-g-1}J_g(x) + x^{-g}J'_g(x) = -x^{-g}J_{g+1}(x)$$

$$\Rightarrow \frac{-g}{x}J_g(x) + J'_g(x) = -J_{g+1}(x) \qquad \text{...... (4)}$$

Subtracting (4) from (3) we get,

$$\frac{2g}{x}J_g(x) = J_{g-1}(x) + J_{g+1}(x).$$

6. **Show that $J_{g-1}(x) - J_{g+1}(x) = 2J'_g(x)$** **[2003 Fall Q. No. 5(a) OR]**

Solution: We have,

$$\frac{d}{dx}[x^gJ_g(x)] = x^gJ_{g-1}(x)$$

$$\Rightarrow \frac{g}{x}J_g(x) + J'_g(x) = J_{g-1}(x) \qquad \text{...... (1)}$$

And $$\frac{d}{dx}[x^{-g}J_g(x)] = -x^{-g}J_{g+1}(x)$$

Also, $$\frac{-g}{x}J_g(x) + J'_g(x) = -J_{g+1}(x) \qquad \text{...... (2)}$$

Adding (1) and (2) we get,

$$2J'_g(x) = J_{g-1}(x) - J_{g+1}(x).$$

7. Show that $\int x^g J_{g-1}(x)\,dx = x^g J_g(x) + c$

Solution: We have,

$$\frac{d}{dx}[x^g J_g(x)] = x^g J_{g-1}(x)$$

Integrating with respects to x, we get,

$$\int x^g J_{g-1}(x)\,dx = x^g J_g(x) + c.$$

8. Show that $\int x^{-g} J_{g+1}(x)\,dx = -x^{-g} J_g(x) + c$

Solution: We have,

$$\frac{d}{dx}[x^{-g} J_g(x)] = -x^{-g} J_{g+1}(x)$$

Integrating with respect to x, we get

$$x^{-g} J_g(x) + c = -\int x^{-g} J_{g+1}(x)\,dx$$

$$\Rightarrow \int x^{-g} J_{g+1}(x)\,dx = -x^{-g} J_g(x) + c$$

9. Show that $\int J_{g+1}(x)dx = \int J_{g-1}(x)dx - 2J_g(x)$

Solution: We have,

$$\int J_{g-1}(x) - J_{g+1}(x) = 2J'_g(x)$$

Integrating both side with respects to x

$$\int J_{g+1}(x)\,dx - \int J_{g-1}(x)dx = 2J'_g(x)$$

$$\Rightarrow \int J_{g+1}(x)dx = \int J_{g-1}(x)dx - 2J_g(x).$$

10. Show that $xJ_r'(x) = rJ_r(x) - xJ_{r+1}(x)$

Solution: Since we have,

$$\frac{d}{dx}(x^{-r} J_r(x)) = -x^{-r} J_{r+1}$$

$$\Rightarrow x^{-r} J_r'(x) - r x^{-r-1} J_r(x) = -x^{-r} J_{r+1}$$

$$\Rightarrow x^{-r}[J_r'(x) - rx^{-1} J_r(x)] = -x^{-r} J_{r+1}$$

$$\Rightarrow J_r'(x) - r x^{-1}(x) = -J_{r+1}$$

$$\Rightarrow xJ_r'(x) = rJ_r(x) - xJ_{r+1}(x).$$

Exercise 7.2

(1) Show that $J_0'(x) = -J_1(x)$.

Proof: We have,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n}\, m!\,(n+m)!}$$

For, n = 1,

$$J_1(x)\ \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1}\, m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} \quad \text{.........(i)}$$

For n = 0, $\quad J_0(x)\ \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!\, m!} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{64+36} \quad \text{.........(ii)}$

Differentiating w. r. t. x, then

$$J_0'(x) = 0 - \frac{2x}{4} + \frac{4x^3}{64} - \frac{6x^5}{64 + 36} + \ldots\ldots$$

$$\Rightarrow \quad J_0'(x) = -\left(\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} \ldots\ldots\right)$$

$$\Rightarrow \quad J_0'(x) = -J_1(x) \qquad \text{(using (i)}$$

Alternative method:

Since we have,

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Set $n = 0$ then,

$$xJ_0'(x) = 0 - x\,J_1(x)$$

$$\Rightarrow \quad J_0'(x) = -J_1(x)$$

2. **Show that, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$**

Solution: Since we have,

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

Set $n = 2$ then,

$$J_1(x) - J_3(x) = 2J_2'(x)$$

$$\Rightarrow \quad J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

3. **Repeated question to 1**

4. **Show that $J_1'(x) = J_0(x) - x^{-1}J_1(x)$**

Solution: Since we have,

$$nJ_n(x) + xJ_n'(x) = xJ_{n-1}(x)$$

Set $n = 1$, then

$$J_1(x) + xJ_1'(x) = xJ_0(x)$$

$$\Rightarrow \quad x^{-1}J_1(x) + J_1'(x) = J_0(x)$$

$$\Rightarrow \quad J_1'(x) = J_0(x) - x^{-1}J_1(x)$$

5. **Evaluate**

(i) $\int J_3(x)\,dx$ **(ii)** $\int x^3 J_2(x)\,dx$ **(iii)** $\int J_5(x)\,dx$

Solution:

(i) Since we have,

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + C \qquad \ldots\ldots\text{(i)}$$

and $$\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x) \qquad \ldots\ldots\text{(ii)}$$

Set $n = 0$ in (i) then,

$$\int J_1(x)\,dx = -J_0(x) + C \qquad \ldots\ldots\text{(iii)}$$

And set $n = 2$ in (ii) then,

$$\int J_3(x)\,dx = \int J_1(x)\,dx - 2J_2(x)$$

$$= J_0(x) + C - 2J_2(x) \qquad [\because \text{ using (iii)}]$$

$$= -2J_2(x) - J_0(x) + C$$

(ii) Since we have,

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + C$$

Set n = 3 then,

$$\int x^3 J_2(x)\,dx = x^3 J_3(x) + C$$

[2002 – Sh...

(iii) Set, n = 4 in (ii) then

$$\int J_5(x)\,dx = \int J_3(x)\,dx - 2J_4(x)$$

$$= -2J_2(x) - J_0(x) + C - 2J_4(x) \quad [\because \text{ using Q. 1}]$$

$$= -2J_4(x) - 2J_2(x) - J_0(x) + C$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

1999 Q. No. 5(a)

Write down the Legendre's equation and its general solution. Also, define the Legendre's polynomial of order 2.

Solution: See the Legendre's equation.

Second Part: See the solution of Legendre's equation.

Third Part: Since we have the Legendre's polynomial of order n is,

$$P_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(2n-2m)!^m}{2^{2n} m!\,(n+m)!\,(n-m)!} x^{n-2m}$$

Set n = 2, then,

$$P_2(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(4-2m)!^m}{2^4 m!\,(2+m)!\,(2-m)!} x^{2-2m}$$

2000 Q. No. 5(a)

Write down the Leendre's and Bessel equation and then also write down the general solution of the Legendre' equation and Bessel function of first kind $J_p(x)$.

Solution: See the Legendre's equation and Bessel's equation.

Second Part: See the solution of Legendre's equation.

Third Part: See the solution of Bessel's equation.

2001 Q. No. 5(a)

Write down the Legendre's equation and its general solution. Also define the Legendre's polynomial of order n and then find Legendre's polynomial of order 2.

Solution: See Solution of 1999.

2002 Q. No. 5(a), 2012 Fall Q. No. 5(a), 2013 Fall Q. No. 5(a), 2013 Spring Q. No. 5(a), 2014 Spring Q. No. 5(a) OR, 2015 Spring Q. No. 5(a) OR

Define Bessel function of the first kind. Show that: $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$.

Solution: See the definition of Bessel's function.

See the result 2.

2004 Spring; 2009 Spring: 2010 Spring (OR) Q. No. 5(a)

What is Legendre's equation? Find its solution.

Solution: See definition of Legendre's equation.

See the solution of Legendre's equation.

2004 Fall Q. No. 5(a) OR

Define Bessele's function of first kind of order p. Prove that:

(i) $x J'_n(x) = -n J_n(x) + x J_{n-1}(x)$ **(ii)** $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Solution: See definition of Bessel's function. And for problem, see result 2.

2006 Fall Q. No. 5(a) OR, 2018 Fall Q. No. 5(a) OR

Define Bessel function of the first kind. Also show that $\frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x)$.

Solution: See definition of Bessel's function. And for problem, see result 3.

2006 Spring Q. No. 5(a) OR

Define Bessel equation and its solution. Show that: $J_{-n}(x) = (-1)^n J_n(x)$

Solution: See definition and for problem, see result 1.

2008 Spring Q. No. 5(a) OR

Define Bessel function of the first kind. Also show that $J_0'(x) = -J_1(x)$.

Solution: See definition and for problem, see Q. No. 1, Exercise 7.2.

2008 Fall; 2011 Fall Q. No. 5(a) OR, 2019 Fall Q. No. 5(a) OR

Write the Bessel's function of first kind of order n. Prove that:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$$

Solution: See 2000 with replacing **v** by n.

2009 Fall Q. No. 5(b)

What is Legendre's equation? Find its Legendre's polynomials.

Solution: See definition and for second part, see result.

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