CHAPTER 3

STRAIGHT LINE

Definition: The straight line is determined by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2.$$

Equation of a line in symmetrical form

(i) through the point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(ii) through the point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

(iii) through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Note: From (i) and (ii), we see $a = x_2 - x_1$, $b = y_2 - y_1$, $c = z_2 - z_1$

and, $l = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$, $n = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$.

General point of the line:

Let

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

be the given line then,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$$

gives

$$x = ar + x_1, \quad y = br + y_1, \quad z = cr + z_1$$

be general point of the line.

NOTE that for different values of r, we get different points on the line.

Transformation of equation of a line from general form to symmetrical form:

Let $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ be the equation of line. Then its symmetrical form is,

(i) If we assume $z = 0$,

$$\frac{x - \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right)}{b_1c_2 - b_1c_1} = \frac{y - \left(\dfrac{a_1d_1 - a_1d_2}{a_1b_2 - a_2b_1}\right)}{a_2c_1 - a_1c_2} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

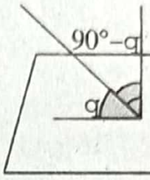
(ii) If we assume $y = 0$,

$$\frac{x-\left(\frac{c_1d_2-c_2d_1}{a_1c_2-a_2c_1}\right)}{b_1c_2-b_2c_1}=\frac{y-0}{a_2c_1-a_1c_2}=\frac{z-\left(\frac{a_1d_2-a_2d_1}{a_1c_2-a_2c_1}\right)}{a_1b_2-a_2b_1}$$

(iii) If we assume $x = 0$,

$$\frac{x-0}{b_1c_2-b_2c_1}=\frac{y-\left(\frac{c_2d_1-c_1d_2}{b_1c_2-b_2c_1}\right)}{a_2c_1-a_1c_2}=\frac{z-\left(\frac{c_1d_2-c_2d_1}{a_1c_2-a_2c_1}\right)}{a_1b_2-a_2b_1}.$$

Angle between a plane and a line:



If q be the angle between a plane $ax + by + cz + d = 0$ and a line, $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ then the angle between the normal and line will be $\left(\frac{p}{2}-q\right)$, so

$$\cos\left(\frac{p}{2}-q\right)=\sin q=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

where l, m, n are direction ratios of the line.

Note:

(i) If the plane is parallel to the line then, $al + bm + cn = 0$.

(ii) If the plane is penpendicular to the line then $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.

Note: The condition for origin to lie is the acute angle is $a_1a_2 + b_1b_2 + c_1c_2 < 0$.

The condition for origin to lie in the obtuse angle is $a_1a_2 + b_1b_2 + c_1c_2 > 0$.

Note: If q be the angle between any one of given plane and a bisector plane and if $q < \frac{p}{4}$ then the bisector bisects the acute angle and another plane bisects obtuse angle. Consequently, of $q > \frac{p}{4}$ then the bisector bisects the obtuse angle and the another plane bisects acute angle.

Condition to represents a pair of planes:

An equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes if the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

is satisfied.

Angle between two planes:

The angle between two planes represents by an equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ be,

$$\tan q=\frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

Note: If $q=\frac{p}{2}$ then $a + b + c = 0$.

Angle between a line and plane:

Let,

$$ax + by + cz + d = 0 \qquad \text{.... (i)}$$

be a plane and

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{.... (ii)}$$

be a line

Case I

$$\cos(90 - q) = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

$$\Rightarrow \sin q = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Case II

If $q = 0$, line and plane are parallel,

$$al + bm + cn = 0.$$

Plane containing a line:

Plane containing a line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where, $al + bm + cn = 0$

Condition for a line to lie on the plane:

A line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c}$ lie on a plane $a_2x + b_2y + c_2z + d = 0$ if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \text{and} \quad a_2x_1 + b_2y_1 + c_2z_1 + d = 0$$

Equation of plane containing a line:

The equation of plane containing the line $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ is,

$$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$$

where k is some constant value.

Coplanar lines:

Two (or more) lines are said to be coplanar if they lie in the same plane.

Condition for coplanar lines:

The lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar if the condition,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

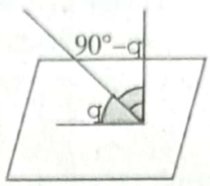
is satisfied.

Note: If two lines intersect each other then they will be coplanar.

Note: If the lines are not in symmetrical form, then first we transform them into in symmetrical form whenever needed.

Equation of plane containing the two lines:

The equation of plane containing the lines, $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is



$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Or,

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Shortest Distance:

If two lines are neither parallel nor intersect to each other then such lines are called skew lines.

The shortest distance between two skew lines is the perpendicular distance between them.

Length of shortest distance between the lines:

The shortest distance between the lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ be

$$d = a_1 \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

where, $a_1 = \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_3 - c_3a_1)^2 + (a_1b_2 - a_2b_1)^2}$

Equation of plane containing the line:

Equation of plane containing the first line which having shortest distance is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a & b & c \end{vmatrix} = 0 \qquad \ldots (1)$$

where a, b, c are direction ratios of the line of shortest distance.

AND

Equation of plane containing the second line and shortest distance is,

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_2 & b_2 & c_2 \\ a & b & c \end{vmatrix} = 0 \qquad \ldots (2)$$

where a, b, c are direction ratios of the line of a shortest distance.

Exercise 3.1

1. Find the value of k, such that the lines $\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2}$ and $\frac{x-2}{2k} = \frac{y-1}{3} = \frac{2-1}{4}$ are perpendicular.

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2} \quad \text{and} \quad \frac{x-2}{2k} = \frac{y-1}{3} = \frac{2-1}{4} \qquad \ldots\ldots (i)$$

Comparing the lines with $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ then we get,

$a_1 = 2, \quad b_1 = 4k, \quad c_1 = 2 \quad$ and $\quad a_2 = 2k, \quad b_2 = 3, \quad c_2 = 4.$

Given that the lines are perpendicular to each other. So, using condition of perpendicularity

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$\Rightarrow (2)(2k) + (4k)(3) + (2)(4) = 0$$
$$\Rightarrow 4k + 12k + 8 = 0$$
$$\Rightarrow 16k = -8$$
$$\Rightarrow k = \frac{-1}{2}.$$

Thus for $k = \frac{-1}{2}$, the given lines (i) will be perpendicular to each other.

2. **Find the distance of the point (1, -3, 5) from the plane 3x – 2y + 6z = 15 along a line with direction cosines proportional to (2, 1, –2).**

[2016 Fall Q. 1 (a)]

Solution: The equation of line passing through point (1, -3, 5) and having direction ratio (2, 1, -2) is

$$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-5}{-2}=r \quad \text{(suppose)} \qquad \ldots\ldots \text{(i)}$$

So, $\quad x = 2r + 1, y = r - 3, z = -2r + 5.$

Let (i) meets the given plane $3x - 2y + 6z = 15$ at M where M is at $(2r + 1, r - 3, -2r + 5)$ for some fixed r. So,

$$3(2r + 1) - 2(r - 3) + 6(2r + 5) = 15$$
$$\Rightarrow 6r + 3 - 2r + 6 - 12r + 30 = 15$$
$$\Rightarrow -8r = -24$$
$$\Rightarrow r = 3$$

Therefore, the point is,

$$x = 2(3) + 1 = 7, y = 3 - 3 = 0, z = -2(3) + 5 = -1.$$

i.e. (7, 0, –1).

Thus, point of intersection is (7, 0, –1).

Now, distance between (1, –3, 5) and (7, 0, –1) is

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(7-1)^2+(0+3)^2+(-1-5)^2}$$
$$= \sqrt{6^2+3^2+(-6)^2} = \sqrt{36+9+36} = \sqrt{81} = 9.$$

Thus, the required distance is 9 units.

3. **Find the points in which the line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}$ out the surface $11x^2 - 5y^2 + z^2 = 0$.**

Solution: Given line is,

$$\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}=r \quad \text{(suppose)} \qquad \ldots\ldots \text{(i)}$$

So, the general point of the line (i) is, $(-r - 1, 5r + 12, 2r + 7)$.

If this point lies on the surface $11x^2 - 5y^2 + z^2 = 0$. Then,

$$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$$
$$\Rightarrow 11(r^2 + 2r + 1) - 5(25r^2 + 120r + 144) + 4r^2 + 28r + 49 = 0$$
$$\Rightarrow 11r^2 + 22r + 11 - 125r^2 - 600r - 720 + 4r^2 + 28r + 49 = 0$$

$\Rightarrow \quad -110r^2 - 550r - 660 = 0$

$\Rightarrow \quad r^2 + 5r + 6 = 0$

$\Rightarrow \quad (r + 2)(r + 3) = 0$

$\Rightarrow \quad r = -2, -3.$

Then the point becomes, (1, 2, 3) at $r = -2$ and (2, –3, 1) at $r = -3$.

Thus, the required points are (1, 2, 3) and (2, –3, 1) which lie on the line as well as in the given surface.

4. Find the point where the line joining (1, –3, 4), (9, 2, -1) cuts the plane $x - y + 2z = 3$.

Solution: The equation of line joining the given points (1, –3, 4) and (9, 2, –1) is

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

$$\frac{x-1}{9-1} = \frac{y+3}{3+3} = \frac{z-4}{-1-4}$$

$$\Rightarrow \quad \frac{x-1}{8} = \frac{y+3}{6} = \frac{z-4}{-5} = r \text{ (suppose)} \qquad \dots \text{(i)}$$

So, the general point of the line (i) is,

$x = 8r + 1,\ y = 6r - 3,\ z = -5r + 4.$

Since the line (i) cuts the plane $x - y + 2z = 3$, so the point $(8r + 1, 6r - 3, -5r + 4)$ lies on the plane $x - y + 2z = 3$ for some fixed r. Therefore,

$8r + 1 - (6r - 3) + 2(-5r + 4) = 3$

$\Rightarrow \quad 8r + 1 - 6r - 3 - 10r + 8 = 3$

$\Rightarrow \quad -8r = -9$

$\Rightarrow \quad r = \frac{9}{8}$

Then,

$$x = 8\left(\frac{9}{8}\right) + 1 = 10, \qquad y = 6\left(\frac{9}{8}\right) - 3 = \frac{27}{4} - 3 = \frac{15}{4},$$

$$z = -5\left(\frac{9}{8}\right) + 4 = -\frac{45+32}{8} = \frac{-13}{8}$$

Thus, at the point $(x, y, z) = \left(10, \frac{15}{4}, \frac{-13}{8}\right)$ the line cuts the plane.

5. Find the distance to the point (–1, –5, –10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

Solution: Given line is,

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r \text{ (suppose)} \qquad \dots\dots \text{(i)}$$

Given that the line (i) intersects the plane $x - y + z = 5$. So, the general point of the line (i) is, $(3r + 2, 4r - 1, 12r + 2)$ lies on the plane $x - y - z = 5$ for some fixed r. Then,

$3r + 2 - (4r - 1) + (12r + 2) = 5$

$\Rightarrow \quad 3r + 2 - 4r + 1 + 12r + 2 = 5$

$\Rightarrow \quad 11r = 0$

$\Rightarrow \quad r = 0.$

Therefore, the point is

$(0 + 2, 0 - 1, 0 + 2)$ i.e. $(2, -1, 2)$.

That is the line (i) meets the plane at $P(2, -1, 2)$.

Now, the distance between the point $P(2, -1, 2)$ and $M(-1, -5, -10)$ is

$$PM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9 + 16 + 144}$$
$$= \sqrt{169} = 13.$$

Thus, the distance of the point $(-1, -5, -10)$ from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is 13 units.

6. Find the two points on the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2}$ either side of $(2, -3, -5)$ and at a distance 3 from it. **[2008 Fall Q. No. 1(a)]**

Solution: Given line is,

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2} = r \text{ (suppose)} \qquad \ldots \text{(i)}$$

So, the general point of the line (i) is,

$$(x = r + 2, y = 2r - 3, z = 2r - 5)$$

Given that the distance between the line (i) and the point $(2, -3, -5)$ be 3. So, the distance between the point $(r + 2, 2r - 3, 2r - 5)$ and $(2, -3, -5)$ is 3.

Therefore,

$$3 = \sqrt{(2 - r - 2)^2 + (-3 - 2r + 3)^2 + (-5 - 2r + 5)^2}$$
$$\Rightarrow \quad 3 = \sqrt{r^2 + 4r^2 + 4r^2}$$
$$\Rightarrow \quad 3 = \sqrt{9r^2}$$
$$\Rightarrow \quad 9r^2 = 9$$
$$\Rightarrow \quad r^2 = 1$$
$$\Rightarrow \quad r = \pm 1.$$

Then at $r = 1$, the general point is, $(x, y, z) = (3, -1, -3)$.

Then at $r = -1$, the general point is, $(x, y, z) = (1, -5, -7)$.

Thus the required points are $(3, -1, -3)$ and $(1, -5, -7)$.

7. Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Solution: The equation of the line through $(1, -2, 3)$ and parallel to line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (suppose)} \qquad \ldots\ldots \text{(i)}$$

Let (i) meets the given plane at M where $M(x = 2r + 1, y = 3r - 2, z = -6r + 3)$ for some fixed r.

The line (i) is parallel to given line and meets the given plane $x - y + z = 5$.

Therefore, the point $(2r + 1, 3r - 2, -6r + 3)$ lies on the plane $x - y + z = 5$ that is,

$$2r + 1 - (3r - 2) + (-6r + 3) = 5$$
$$\Rightarrow \quad 2r + 1 - 3r + 2 - 6r + 3 = 5$$
$$\Rightarrow \quad -7r = -1$$

$$\Rightarrow \quad r = \frac{1}{7}$$

Then, $x = 2r + 1 = 2\left(\frac{1}{7}\right) + 1 = \frac{9}{7}$,

$$y = 3r - 2 = 3\left(\frac{1}{7}\right) - 2 = \frac{-11}{7}$$

and $z = -6r + 3 = -6\left(\frac{1}{7}\right) + 3 = \frac{15}{7}$

Therefore, the point where the given line meets the plane at $M(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7})$.

Now, the distance between point P(1, –2, 3) and $M(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7})$ is

$$d = \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(\frac{-11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2}$$

$$= \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1.$$

8. Find the equation to the line passing through (–1, –2, –3) and the perpendicular to each of the lines $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6}$.

Solution: The equations of the line passing through (–1, 2, –3) and having direction ratio (a, b, c) are,

$$\frac{x+1}{a} = \frac{x+2}{b} = \frac{z+3}{c} \qquad \text{...... (i)}$$

Given that the line (i) perpendicular to line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6}$. So,

$$3a + 4b + 5c = 0$$
$$4a + 5b + 6c = 0$$

Solving by cross multiplication we get,

$$\frac{a}{-1} = \frac{b}{2} = \frac{c}{-1} = 1.$$

Thus, $a = -1, b = 21, c = -1$.

Then, equation (i) becomes

$$\frac{x+1}{-1} = \frac{x+2}{21} = \frac{z+3}{-1}$$

$$\Rightarrow \quad \frac{x+1}{1} = \frac{x+2}{-2} = \frac{z+3}{1}$$

This is the equation of required line.

9. Show that the equation of the line which is perpendicular from the point (1, 6, 3) to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$ and the foot of the perpendicular is (1, 3, 5) and the length of perpendicular is $\sqrt{13}$.

Solution: Given line is,

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r \text{ (let)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, (r, 2r + 1, 3r + 2)

Let M be the foot of the perpendicular from the given point P(1, 6, 3) on the given line (i). Then, M(r, 2r + 1, 3r +2) for some fixed r. Therefore the direction ratios of the line joining P and M is, r − 1, 2r + 1 − 6, 3r + 2 − 3

i.e. r − 1, 2r − 5, 3r − 1.

Since the line PM is perpendicular to line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ so,

$$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

$$(r-1)(1) + (2r-5)(2) + (3r-1)(3) = 0$$

$$\Rightarrow \quad r - 1 + 4r - 10 + 9r - 3 = 0$$

$$\Rightarrow \quad 14r = 14$$

$$\Rightarrow \quad r = 1$$

Then, $x = 1$, $y = 3$, $z = 5$. Therefore M = (1, 3, 5).

And, the direction ratio of the perpendicular line is

$$(r-1, 2r-5, 3r-1) = (1-1, 2-5, 3-1) = (0, -3, 2).$$

Then the equation of the line through (1, 6, 3) and having direction ratios 0, −3, 2 be,

$$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}.$$

Now, distance of perpendicular line,

$$PM = \sqrt{(1-1)^2 + (6-3)^2 + (3-5)^2} = \sqrt{13}$$

Thus, distance of the perpendicular line is $\sqrt{13}$.

[2017 Fall Q.No. 1(a) OR]

Find the equation of the line through (1, 6, 3) which is perpendicular to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Solution: See first part of above question.

10. Find the equation to the line through (−1, 3, 2) and perpendicular to the plane x + 2y + 2z = 3, the length of perpendicular and the co–ordination of its foot. **[2007 Fall; 2009 Spring Q. No. 1(a)]**

Solution: The equation of the line PM (perpendicular) passing through (−1, 3, 2) and perpendicular to the plane is x + 2y + 2z = 3. Therefore, the direction ratios perpendicular from P are 1, 2, 2.

$$\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2} = r \text{ (suppose)} \qquad \ldots\ldots \text{(i)}$$

Let (i) meets the plane at M then M will be at (r − 1, 2r +3, 2r + 2) for some fixed r. So,

$$r - 1 + 2(2r + 3) + 2(2r + 2) = 3$$

$$\Rightarrow \quad r - 1 + 4r + 6 + 4r + 4 = 3$$

$$\Rightarrow \quad 9r = -6$$

$$\Rightarrow \quad r = \frac{-2}{3}$$

Then, $x = r - 1 = \frac{-2}{3} - 1 = \frac{-5}{3}$, $y = 2r + 3 = 2\left(\frac{-2}{3}\right) + 3 = \frac{5}{3}$,

$$z = 2r + 2 = 2\left(\frac{-2}{3}\right) + 2 = \frac{2}{3}.$$

Thus, the coordinate of the perpendicular foot is M $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$

Now, the length of distance between point $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$ and (–1, 3, 2) is

$$\sqrt{\left(-1+\frac{5}{3}\right)^2+\left(3-\frac{5}{3}\right)^2+\left(2-\frac{2}{3}\right)^2} = \sqrt{\frac{4}{9}+\frac{16}{9}+\frac{16}{9}} = \sqrt{\frac{36}{9}} = 2 \text{ unit.}$$

Thus, the equation of the line is $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}$, the length of perpendicular is 2 units and the coordinate of the foot is $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$.

11. Find the image of the point P (1, 3, 4) in the plane 2x – y + z + 3 = 0.

[2015 Fall Q. 1 (b), 2014 Spring Q. 1(a)]

Solution: The equation of line passing through (1, 3, 4) and perpendicular to the plane $2x - y + z + 3 = 0$ is,

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r \text{ (suppose)} \qquad \text{...... (i)}$$

Then the general point of (i) is, $(x, y, z) = (2r+1, 3-r, r+4)$.

Since, if the point $Q(2r+1, 3-r, r+4)$ is the image of P then the middle point R lies on the plane $2x - y + z + 3 = 0$.

Here, the coordinate of R is,

$$R\left(\frac{2r+1+1}{2}, \frac{3-r+3}{2}, \frac{r+4+4}{2}\right) = R\left((r+1), 3-\frac{r}{2}, 4+\frac{r}{2}\right)$$

Since the point lies on the plane $2x - y + z + 3 = 0$, so,

$$2(r+1) - 3 + \frac{r}{2} + 4 + \frac{r}{2} + 3 = 0.$$

$$\Rightarrow \quad 2r + 2 + 4 + r = 0.$$

$$\Rightarrow \quad r = -2.$$

Then the coordinate of Q is,

$$(2r+1, 3-r, r+4) = (-4+1, 3+2, -2+4)$$
$$= (-3, 5, 2)$$

Thus, the image of the point P (1, 3, 4) in the plane $2x - y + z + 3 = 0$ be (–3, 5, 2).

Exercise 3.2

1. Change the equation x + y + z + 1 = 0, 4x + y – 2z + 2 = 0 in symmetrical form. **[2017 Fall Q.No. 1(a)]**

Solution: Given equation of line is,

$$x + y + z + 1 = 0 = 4x + y - 2z + 2. \qquad \text{...... (i)}$$

Then we have to change (i) in symmetrical form.

Put z = 0 then,

$$x + y + 1 = 0 \quad \text{and} \quad 4x + y + 2 = 0 \qquad \text{...... (ii)}$$

Solving these equations we get,

$$x = \frac{-1}{3}, \quad \text{and} \quad y = \frac{1}{3} - 1 = \frac{-2}{3}$$

Thus, the point on the line is $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$.

Let the directions ratio are a, b, c. Then, (ii) gives,

$$a + b + c = 0$$
$$4a + b - 2c = 0$$

Solving by cross multiplication,

$$\frac{a}{-2-1}=\frac{b}{4+2}=\frac{c}{1-4}$$

$$\Rightarrow \frac{a}{-3}=\frac{b}{6}=\frac{c}{-3}=1 \text{ (assume)}$$

Then, $a=-21,\ b=61,\ c=-31.$

So, the equation of line passing through $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ and having the direction ratio a = 31, b = 61, c = -31 be,

$$\frac{x+(1/3)}{-31}=\frac{y+(2/3)}{61}=\frac{z-0}{-31}$$

$$\Rightarrow \frac{3x+1}{-9}=\frac{3y+2}{18}=\frac{z}{-3}$$

$$\Rightarrow \frac{3x+1}{3}=\frac{3y+2}{-6}=\frac{z}{1}$$

This is the equation of line (i) in symmetrical form.

2. **Find the equation of the plane through (–1, 1, –1) and perpendicular to the line x – 2y + z = 4, 4x + 3y – z + 4 = 0.**

Solution: Let the equation of plane through (-1,1,-1) is

$$a(x+1)+b(y-1)+c(z+1)=0 \qquad \ldots\ldots \text{(i)}$$

Since equation (i) is perpendicular to the line x - 2y + z = 4, 4x + 3y - z + 4 = 0. Therefore,

$$a-2b+c=0$$
$$4a+3b-c=0$$

So, by cross multiplication

$$\frac{a}{2-3}=\frac{b}{4+1}=\frac{c}{3+8}$$

$$\Rightarrow \frac{a}{-1}=\frac{b}{5}=\frac{c}{11}=1 \text{ (let)}$$

This gives, $a=-1$, $b=51$, $c=111$. Putting the value of a, b, c in equation (i) then,

$$-1(x+1)+51(y-1)+111(z+1)=0$$
$$\Rightarrow -x-1+5y-5+11z+11=0$$
$$\Rightarrow x-5y-11z=5.$$

This is the equation of required plane.

3. **Find the equation of the line passing through (2, 3, 4) parallel to the line x – 2y + z = 4, 4x + 3y – z + 4 = 0.**

Solution: Let the equation of the line passing through (2, 3, 4) and the direction ratio (a, b, c) be

$$\frac{x-2}{a}=\frac{y-3}{b}=\frac{z-4}{c} \qquad \ldots\ldots\ldots \text{(i)}$$

Since (i) is parallel to the given line x – 2y + z = 4. So, (i) is perpendicular to the normal of given plane, so

$$a-2b+c=0$$
$$4a+3b-c=0$$

Solving by cross multiplication

$$\frac{a}{2-3}=\frac{b}{4+1}=\frac{c}{3+8}=1 \text{ (let)}$$

$\Rightarrow$ $a = -1$, $b = 51$ and $c = 111$.

Putting the value of a, b, c in equation (i)

$$\frac{x-2}{-1}=\frac{y-3}{51}=\frac{z-4}{111}$$

$$\Rightarrow \frac{x-2}{-1}=\frac{y-3}{5}=\frac{z-4}{11}$$

This is the equation of required line.

4. Find the angle between the lines in which the plane $x - y + z = 5$ is cut by the planes $2x + y - z = 3$ and $2x + y + 3z - 1 = 0$.

Solution: Given planes are,

$$2x + y - z = 3 \quad \text{........ (i)}$$

$$2x + 2y + 3z - 1 = 0 \quad \text{......... (ii)}$$

Given that the plane $x - y + z = 5$ cuts the planes (i) and (ii). Let direction ratio of line (i) is $a_1, b_1\ c_1$ and direction ratio of line (ii) is a_2, b_2, c_2. So,

$a_1 - b_1 + c_1 = 0$ $\qquad$ $a_2 - b_2 + c_2 = 0$

$2a_1 + b_1 - c_1 = 0$ $\qquad$ $2a_2 + 2b_2 + 3c_2 = 0$

Solving by cross multiplication we get,

$$\frac{a_1}{1-1}=\frac{b_1}{2+1}=\frac{c_1}{1+2} \qquad \frac{a_2}{-3-2}=\frac{b_2}{2-3}=\frac{c_2}{2+2}$$

$$\Rightarrow \frac{a_1}{0}=\frac{b_1}{3}=\frac{c_1}{3} \qquad \Rightarrow \frac{a_2}{-5}=\frac{b_2}{-1}=\frac{c_2}{4}$$

Thus, $(a_1, b_1, c_1) = (0, 3, 3)$ $\qquad$ Thus, $(a_2, b_2, c_2) = (-5, -1, 4)$

Let q be the angle between the lines then,

$$\cos q = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{0\times -5 + 3\times -1 + 3\times 4}{\sqrt{0^2 + 3^2 + 3}\sqrt{(-5)^2 + (-1)^2 + 4^2}}$$

$$= \frac{-3 + 12}{\sqrt{9+9}\sqrt{25 + 1 + 16}} = \frac{9}{\sqrt{18\times 42}} = \frac{3}{\sqrt{84}} = \frac{3}{2\sqrt{21}}$$

$$q = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right).$$

Thus, the angle between the given lines is $\cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$.

5. Prove that the lines $x = ay + b$, $z = cy + d$ and $x = a^1y + b^1$, $z = a^1y + d^1$ are perpendicular it $aa^1 + cc^1 + 1 = 0$.

Solution: Given line is,

$$x = ay + b, \quad z = cy + d$$

$$\Rightarrow y = \frac{x-b}{a}, y = \frac{z-d}{c}$$

This implies, $$\frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c} \quad \text{... (i)}$$

And, another given line is,

$$x = a^1y + b^1, z = c^1y + d^1$$

$$\Rightarrow y = \frac{x-b^1}{a^1}, y = \frac{z-d^1}{c}$$

This implies, $\frac{x-b^{1}}{a^{1}}=\frac{y-0}{1}=\frac{z-d^{1}}{c^{1}}$ ... (ii)

The line (i) and (ii) are perpendicular only if,

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$\Rightarrow \quad aa^{1}+1.1+cc^{1}=0$$
$$\Rightarrow \quad aa^{1}+cc^{1}+1=0.$$

6. **Prove that the lines x = -2y + 7, z = 3y + 10 and x = 5y – 1, z = 3y – 6 are perpendicular to each other.**

Solution: Given line is,

$$x=-2y+7, \quad z=3y+10$$
$$\Rightarrow \quad -2y=x-7,\ 3y=z-10$$
$$\Rightarrow \quad y=\frac{x-7}{-2},\ y=\frac{z-10}{3}$$

This implies, $\frac{x-7}{-2}=\frac{y-0}{1}=\frac{z-10}{3}$ ...... (i)

And, another given line is,

$$x=5y-1, \quad z=3y-6$$
$$\Rightarrow \quad 5y=x+1,\ 3y=z+6$$
$$\Rightarrow \quad y=\frac{x+1}{5},\ y=\frac{z+6}{3}$$

This implies, $\frac{x+1}{5}=\frac{y-0}{1}=\frac{z+6}{3}$ ...... (ii)

The line (i) and (ii) are perpendicular only if,

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$\Rightarrow \quad (-2)(5)+(1)(1)+(3)(3)=0$$
$$\Rightarrow \quad -10+1+9=0$$
$$\Rightarrow \quad 0=0.$$

This proves that the lines are perpendicular to each other.

7. **Find the co–ordinate of the foot of the line which is the perpendicular from the origin on the straight line given by x + 2y + 3z + 4 = 0, x + y + z + 1 = 0.**

[2015 Spring Q.No. 1(a)]

Solution: Given line is the intersection of two planes,

$$x+2y+3z+4=0 \quad \text{and} \quad x+y+z+1=0$$

Put $z=0$, then,

$$x+2y+4=0 \quad \text{and} \quad x+y+1=0$$

Solving these two planes we get,

$$y=-3,\ x=2.$$

Therefore, the given line passes through the point (2, –3, 0).

Also, let direction ratio is a, b, c then,

$$a+2b+3c+4=0 \quad \text{and} \quad a+b+c+1=0$$

Solving by cross multiplication

$$\frac{a}{2-3}=\frac{b}{3-1}=\frac{c}{2-1}=1 \ \text{(let)}$$
$$\Rightarrow \quad a=-1,\ b=21,\ c=1$$

Now, the equation of line passing through (2, -3, 0) and having direction ratio (-1,21,-1) is,

$$\frac{x-2}{-1} = \frac{y+3}{21} = \frac{z}{-1}$$

$$\Rightarrow \quad \frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = r \text{ (let)} \qquad \ldots \text{(i)}$$

Let M be the foot of perpendicular from O(0, 0, 0) on (i) then

$$M = (r + 2, -2r - 3, r)$$

for some fixed r.

Then the direction ratios of OM are

$$r + 2, -2r - 3, r$$

Since the line joining origin and (i) are perpendicular to line (i). So,

$$(r + 2)(1) + (-2r - 3)(-2) + (r)(1) = 0$$

$$\Rightarrow \quad r + 2 + 4r + 6 + r = 0$$

$$\Rightarrow \quad r = -\frac{6}{8} = -\frac{4}{3}$$

Then,

$$x = r + 2 = -\frac{4}{3} + 2 = \frac{2}{3},$$

$$y = -2r - 3 = -2(-\frac{4}{3}) - 3 = \frac{8}{3} - 3 = -\frac{1}{3}$$

and $\quad z = -\frac{4}{3}$

Thus, $(x, y, z) = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

Hence, the coordinate of the foot of the perpendicular from the origin on the given line is, $\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

Exercise 3.3

1. **Find the value of k such that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x - 3y + z = 3$.**

Solution: Given line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x - 3y + z = 3$.

So, line is perpendicular to the normal of the plane (i.e. to the direction ratios of the plane). Therefore,

$$\boxed{al + bm + cn = 0}$$

i.e. $(2)(2) + (5)(-3) + (k)(1) = 0$

$$\Rightarrow \quad 4 - 15 + k = 0$$

$$\Rightarrow k = 11.$$

Thus, for k = 11 the given line and the plane are in the form of parallel.

2. **Find the equation of the plane parallel to the line $x - 2 = \frac{y-1}{3} = \frac{z-3}{2}$ containing (0, 0, 0) and (–3, 1, 2).**

Solution: Let us consider the equation plane is $ax + by + cz + d = 0$ $\qquad \ldots$ (i)

As the plane passes through (0, 0, 0), then d = 0. Then, (i) becomes,

$$ax + by + cz = 0 \qquad \ldots. \text{(ii)}$$

Also, the plane (i) passes through (–3, 1, 2) then (ii) gives,

$$-3a + b + 2c = 0 \quad \text{.... (iii)}$$

And, given that the plane is parallel to the line $\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-3}{2}$. Then,

$$\boxed{al + bm + cn = 0}$$

$$a + 3b + 2c = 0 \quad \text{..... (iv)}$$

Solving the equation (iii) and (iv), by cross multiplication

$$\frac{a}{2-6} = \frac{b}{2+6} = \frac{c}{-9-1} = 1$$

$$\Rightarrow \quad a = -41, b = 81, c = 101$$

Now, equation (ii) becomes

$$-41x + 81y - 101 = 0$$

$$\Rightarrow \quad 2x - 4y + 5z = 0.$$

This is equation of required plane.

3. **Find the equation of a plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$.**

Solution: Given line is,

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \quad \text{...... (i)}$$

Let the equation of plane containing the line is (i) is,

$$a(x-1) + b(y+1) + c(z-3) = 0 \quad \text{...... (ii)}$$

Let the normal of the required plane be a, b, c. Then,

$$2a + (-1)b + 4c = 0$$

$$\Rightarrow \quad 2a - b + 4c = 0 \quad \text{...... (iii)}$$

And, given that the plane (ii) is perpendicular to the plane $x + 2y + z = 12$ then,

$$(a)(1) + (b)(2) + (1)(c) = 0$$

$$\Rightarrow \quad a + 2b + c = 0 \quad \text{...... (iv)}$$

Solving equation (iii) and (iv), by cross multiplication,

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1} = 1 \text{ (let)}$$

$$\Rightarrow \quad a = -91, b = 21, c = 51$$

Now equation (ii) becomes,

$$-91(x-1) + 21(y+1) + 51(z-3) = 0$$

$$\Rightarrow \quad -9(x-1) + 2(y+1) + 5(z-3) = 0$$

$$\Rightarrow \quad -9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$\Rightarrow \quad 9x - 2y - 5z + 4 = 0.$$

Thus, the equation of required plane is $9x - 2y - 5z + 4 = 0$.

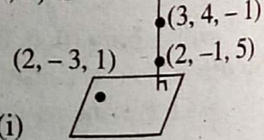
> **If plane $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular. Then, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.**

4. **Find the equation of the plane through (2, –3, 1) normal to the line joining (3, 4, –1) and (2, –1, 5).**

Solution: The equation of line joining (3, 4, –1) and (2, –1, 5) is

$$\frac{x-3}{3-2} = \frac{y-4}{4-(-1)} = \frac{z-(-1)}{-1-5}$$

$$\Rightarrow \quad \frac{x-3}{1} = \frac{y-4}{5} = \frac{z+1}{-6} \quad \text{... (i)}$$



That is, the direction ratios of the line (i) are 1, 5, –6.

Now the equation of the plane through (2, –3, 1) and is normal to (i) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

i.e. $1(x - 2) + 5(y + 3) - 6(z - 1) = 0$

$\Rightarrow$ $x - 2 + 5y + 15 - 6z + 6 = 0$

$\Rightarrow$ $x + 5y - 6z + 19 = 0.$

This is the equation of required plane.

5. Find the equation of the line through (2, –1, –1) is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.

Solution: The equation of line through (2, –1, –1) is

$$\frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c} \quad \text{...... (i)}$$

where, a, b and c are the direction ratio of the line.

Since the equation (i) is parallel to the plane $4x + y + z + 2 = 0$. Then,

$$4a + b + c = 0 \quad \text{...... (ii)}$$

Again, given a line,

$$2x + y = 0 = x - z + 5$$

This gives,

$$x = -\frac{y}{2} \quad \text{and} \quad x = z - 5$$

Therefore,

$$\frac{x}{1} = \frac{-y}{2} = \frac{z-5}{1} \quad \text{...... (iii)}$$

Given that the line (i) is perpendicular to the line (iii). So,

$$\boxed{a_1a_2+b_1b_2+c_1c_2 = 0}$$

$\Rightarrow$ $(a)(1) + (-2)(b) + (c)(1) = 0$

$\Rightarrow$ $a - 2b + c = 0 \quad \text{....... (iv)}$

From equation (ii) and (iv)

$$\frac{a}{1+2} = \frac{b}{1-4} = \frac{c}{-8-1} = 1$$

$\Rightarrow$ $a = 31, b = -31, \text{and } c = -91$

Putting the value of a, b and c in equation (i) then

$$\frac{x-2}{31} = \frac{y+1}{-31} = \frac{z+1}{-91}$$

$$\Rightarrow \frac{x-2}{1} = \frac{y+1}{-1} = \frac{z+1}{-3}.$$

This is the equation of required line.

6. Find the equation of the straight line lying in the plane $x - 2y + 4z - 51 = 0$ and intersecting the straight line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-6}{7}$ at right angle.

Solution: Given line is,

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-6}{7} = r(\text{let}) \quad \text{...... (i)}$$

Then $(3r + 2, 4r + 1, 7r + 6)$ be the general point of (i) for fixed r.

Let (i) intersect the plane

$$x - 2y + 4z - 51 = 0 \quad \text{...... (ii)}$$

Therefore,
$$3r + 2 - 8r + 2 + 28r + 24 - 51 = 0$$
$$\Rightarrow 23r = 23$$
$$\Rightarrow r = 1$$

Therefore, the common point of (i) and (ii) is (5, 3, 13).

Given that the required line is lying in the plane (ii). So, the normal of (ii) i.e. 1, –2, 4 is parallel to the required line and also (ii) in perpendicular to (i).

Therefore, $(1, -2, 4)\cdot(3, 4, 7)$ is parallel to (ii).

Here,
$$(1, -2, 4)\cdot(3, 4, 7) = \begin{vmatrix} 1 & -2 & 4 \\ 3 & 4 & 7 \end{vmatrix} = (-30, 5, 10) = 5(-6, 1, 2).$$

Now, the equation of the line which is passing through (5, 3, 13) and is parallel to the line having direction ratios (–6, 1, 2) is
$$\frac{x-5}{-6} = \frac{y-3}{1} = \frac{z-13}{2}.$$

7. Find the equation of line through (a, b, d) parallel, to the plane $lx + my + nz = P$, $l_1x + m_1y + n_1z = P$.

Solution: The equation of line passing through (a, b, d) and having the direction ratio (a, b, c) is
$$\frac{x-a}{a} = \frac{y-b}{b} = \frac{z-d}{c} \qquad \text{......... (i)}$$

Since equation (i) is parallel to the plane $lx + my + nz = p$. Then,
$$al + bm + cn = 0 \qquad \text{........ (ii)}$$

Similarly, equation (i) is parallel to $l_1x + m_1y + n_1z = p$. Then,
$$al_1 + bm_1 + cn_1 = 0 \qquad \text{........ (iii)}$$

Solving equation (ii) and (iii) we get,
$$\frac{a}{mn_1 - m_1n} = \frac{b}{l_1n - ln_1} = \frac{c}{lm_1 - l_1m} = 1 \text{ (Say)}$$
$$\Rightarrow a = 1\,(mn_1 - m_1n),$$
$$b = 1\,(l_1n - ln_1),$$
$$c = 1\,(lm_1 - l_1m)$$

Now, equation (i) becomes
$$\frac{x-a}{1\,(mn_1 - m_1n)} = \frac{y-b}{1\,(l_1n - ln_1)} = \frac{z-d}{1\,(lm_1 - l_1m)}$$
$$\Rightarrow \frac{x-a}{(mn_1 - m_1n)} = \frac{y-b}{(l_1n - ln_1)} = \frac{z-d}{(lm_1 - l_1m)}.$$

This is the equation of required line.

8. Find the equations to the planes through the line $2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6$, parallel to the coordinate axes.

Solution: The equation of plane passes through the lines, $2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6$ is
$$2x + 3y - 5z - 4 + 1\,(3x - 4y + 3z - 6) = 0$$
$$\Rightarrow 2x + 3y - 5z - 4 + 3x1 - 4y1 + 3z1 - 61 = 0 \qquad \text{....... (i)}$$
$$\Rightarrow x\,(2 + 31) + y\,(3 - 41) + z\,(-5 + 51) - 4 - 61 = 0$$

(a) If the equation (i) is parallel to the x–axis having direction ratios 1, 0, 0 then,
$$al + bm + cn = 0$$

$$\Rightarrow \quad (2+3l)(1)+(3-4l)(0)+(-5+5l)(0)=0$$
$$\Rightarrow \quad 2+3l=0$$
$$\Rightarrow \quad l=\frac{-2}{3}.$$

Then, the equation (i) becomes

$$2x+3y-5z-4+\frac{-2}{3}(3x-4y+5z-6)=0$$
$$\Rightarrow \quad 6x+9y-15z-12-6x+8y-10z+12=0$$
$$\Rightarrow \quad 17y-25z=0$$

(b) If the equation (i) is parallel to the y–axis having direction ratios 0, 1, 0 then,

$$(2+3l)(0)+(3-4l)(1)+(-5+5l)(0)=0$$
$$\Rightarrow \quad 3-4l=0$$
$$\Rightarrow \quad l=\frac{3}{4}$$

Then, equation (i) becomes

$$2x+3y-5z-4+\frac{3}{4}(3x-4y+5z-6)=0$$
$$\Rightarrow \quad 8x+12y-20z-16+9x-12y+15z-18=0$$
$$\Rightarrow \quad 17x-5z-34=0$$

(c) If the equation (i) is parallel to the z–axis having direction ratios 0, 0, 1 then

$$(2+3l)(0)+(3-4l)(0)+(-5+5l)(1)=0$$
$$\Rightarrow \quad -5+5l=0$$
$$\Rightarrow \quad l=1$$

Then, equation (i) becomes

$$2x+3y-5z-4+3x-4y+5z-6=0$$
$$\Rightarrow \quad 5x-y-10=0.$$

Thus the equations of required planes are $17y-25z=0$, $17x-5z-34=0$, $5x-y-10=0$.

9. Find the equation of plane through the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ parallel to y–axis. **[2004 Spring Q. No. 1(a)]**

Solution: Given that the equation of a line is,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \qquad \text{...... (i)}$$

Now, the equation of a plane containing (i) and having direction ratios a, b, c is

$$a(x-1)+b(y-2)+c(z-3)=0 \qquad \text{..... (ii)}$$

Since the direction ratios of the line is normal to itself. So, by the condition of perpendicularity,

$$2a+3b+4c=0 \qquad \text{...... (iii)}$$

Given that the plane (ii) is parallel to y–axis. So, the condition of parallelism gives,

$$a(0)+b(1)+c(0)=0$$
$$\Rightarrow \quad b=0 \qquad \text{...... (iv)}$$

Solving equation (iii) and (iv) we get,

$$\frac{a}{0-4}=\frac{b}{0-0}=\frac{c}{2-0}$$

This gives, $(a, b, c) = (-4, 0, 2)$.

Now, equation (ii) becomes

$$-4(x-1) + 0(y-2) + 2(z-3) = 0$$
$$\Rightarrow -4x + 4 + 2z - 6 = 0$$
$$\Rightarrow -4x + 2z - 2 = 0$$
$$\Rightarrow 2x - z + 1 = 0.$$

Thus, the equation of required plane is $2x - z + 1 = 0$.

10. Find the equation of the plane through the points (2, 2, 1), (1, –2, 3) and parallel to the line joining the points (2, 1, –3), (–1, 5, –8). **[2016 Fall Q. 1 (a)]**

Solution: The plane through the points (2, 2, 1) is,

$$a(x-2) + b(y-2) + c(z-1) = 0 \qquad \text{...... (i)}$$

As (i) passes through the points (1, –2, 3) then,

$$a(1-2) + b(-2-2) + c(3-1) = 0$$
$$\Rightarrow -a - 4b + 2c = 0$$
$$\Rightarrow a + 4b - 2c = 0 \qquad \text{...... (ii)}$$

The equation of line joining points (2, 1, –3) and (–1, 5, –8) is

$$\frac{x-2}{2-(-1)} = \frac{y-1}{1-5} = \frac{z-(-3)}{-3-(-8)}$$
$$\Rightarrow \frac{x-2}{3} = \frac{y-1}{-4} = \frac{z+3}{5} \qquad \text{...... (iii)}$$

Given that the plane (i) to is parallel to the line (iii). So, by the condition of parallelism,

$$3a - 4b + 5c = 0 \qquad \text{...... (iv)}$$

Solving equation (ii) and (iv) we get

$$\frac{a}{20-8} = \frac{b}{-6-5} = \frac{c}{-4-12} = 1$$
$$\Rightarrow a = 121, b = -111, c = -161$$

Now equation (iii) becomes

$$121(x-2) - 111(y-2) - 161(z-1) = 0$$
$$\Rightarrow 12x - 24 - 11y + 22 - 16z + 16 = 0$$
$$\Rightarrow 12x - 11y - 16z + 14 = 0.$$

Thus the equation of required plane is $12x - 11y - 16z + 14 = 0$.

Exercise 3.4

1. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Also, obtain the equation of plane containing them. **[2003 Fall Q. No. 1(a)]**

Solution: Given line is,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \qquad \text{...... (i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 3)$ and having the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \qquad \text{...... (ii)}$$

Then the line (i) passes through the point $(x_2, y_2, z_2) = (2, 3, 4)$ and having the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1(15-16) - 1(10-12) + 1(8-9)$$

$$= -1 + 2 - 1 = 0$$

This shows that the lines are coplanar.

And, the equation of plane containing the lines (i) and (ii) is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(15-16) - (y-2)(10-12) + (z-3)(8-9) = 0$$

$$\Rightarrow -x+1+2y-4-z+3=0$$

$$\Rightarrow x-2y+z=0.$$

Thus, the equation of the plane containing the lines be $x - 2y + z = 0$.

2012 Fall Q. No. 1(a):

Examine whether the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar? If so, find the equation of plane containing them.

Solution: Similar to Q1.

2. **Show that the lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7} = y-4 = \frac{z-5}{3}$ are coplanar. Find their common point and equation of plane in which they lie.**

Solution: Given line is,

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5} \qquad \text{...... (i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (5, 7, -3)$ and having the direction ratio $(l_1, m_1, n_1) = (4, 4, -5)$.

Also, given line is,

$$\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3} \qquad \text{...... (ii)}$$

Then the line (ii) passes through the point $(x_2, y_2, z_2) = (8, 4, 5)$ and having the direction ratio $(l_2, m_2, n_2) = (7, 1, 3)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 3(12+5) - (-3)(12+35) + 8(4-28)$$

$$= 51 + 141 - 192$$

$$= 0$$

This shows that the lines are coplanar.

For common point, let,

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = r \text{ (say)} \qquad \text{...... (iii)}$$

So, the general point of (iii) is,

$$(x_1, y_1, z_1) = (4r+5, 4r+7, -5r-3) \qquad \text{... (*)}$$

And,

$$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3} = r^1 \text{ (say)} \qquad \text{...... (iv)}$$

So, the general point of (iv) is,

$$(x_2, y_2, z_2) = (7r^1 + 8, r^1 + 4, 3r^1 + 5) \qquad \text{... (**)}$$

Since the lines (iii) and (iv) meets at a point. So, the general points (*) and (**) will be identical for some fixed value of r and r^1. That is,

$$x_1 = x_2, \qquad y_1 = y_2, \qquad z_1 = z_2$$

Then,

$$4r + 5 = 7r^1 + 8 \qquad \text{..... (a)}$$

$$4r + 7 = r^1 + 4 \qquad \text{..... (b)}$$

$$-5r - 3 = 3r^1 + 5 \qquad \text{..... (c)}$$

Solving (a) and (b) we get,

$$r^1 = -1$$

So that, $(7r^1 + 8, r^1 + 4, 3r^1 + 5) = (-7+8, -1+4, -3+5) = (1, 3, 2)$.

Thus (1, 3, 2) be the common point of (i) and (ii).

And, the equation of plane containing the lines (i) and (ii) is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (x-5)(12+5) - (y-7)(12+35) + (z+3)(4-28) = 0$$

$$\Rightarrow (x-5).17 - (y-7).47 + (z+3).(-24) = 0$$

$$\Rightarrow 17x - 85 - 47y + 329 - 24z - 72 = 0$$

$$\Rightarrow 17x - 47y - 24z + 172 = 0.$$

Thus, the equation of the plane containing the lines is

$$17x - 47y - 24z + 172 = 0.$$

3. Show that the lines x – 1 = 2y – 4 = 3z and 3x – 5 = 4y – 9 = 3z meet in a point and the equation of the plane is which they lie is 3x – 8y + 3z + 13 = 0.

Solution: Given lines is,

$$x - 1 = 2y - 4 = 3z$$

$$\Rightarrow \frac{x-1}{6} = \frac{2y-6}{6} = \frac{3z}{6}$$

$$\Rightarrow \frac{x-1}{6} = \frac{y-3}{3} = \frac{z-0}{2} \quad \text{......(i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 0)$ and having the direction ratio $(l_1, m_1, n_1) = (6, 3, 2)$

And given line is

$$3x - 5 = 4y - 9 = 3z$$

$$\Rightarrow \frac{3x-5}{12} = \frac{4y-9}{12} = \frac{3z}{12}$$

$$\Rightarrow \frac{x-(5/3)}{4} = \frac{y-(9/4)}{3} = \frac{z-0}{4}$$

Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(\frac{5}{3}, \frac{9}{4}, 0\right)$ and having the direction ratio $(l_2, m_2, n_2) = (4, 3, 4)$.

If the line (i) and line (ii) meet at a point then they should be coplanar.

That is,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Here,

$$\begin{vmatrix} \frac{5}{3} - 1 & \frac{9}{4} - 2 & 0 - 0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{4} & 0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix}$$

$$= \frac{2}{3}(12 - 6) - \frac{1}{4}(24 - 8) + 0$$

$$= \frac{2}{3}(6) - \frac{1}{4}(16)$$

$$= 4 - 4 = 0.$$

This show the lines are coplanar. Since the lines are not parallel, so they intersect.

And the equation of plane containing the lines is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x - 1 & y - 2 & z - 0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 0.$$

$$\Rightarrow (x - 1)(12 - 6) - (y - 2)(24 - 8) + z(18 - 12) = 0$$

$$\Rightarrow 6(x - 1) - 16(y - 2) + 6z = 0$$

$$\Rightarrow 6x - 6 - 16y - 32 + 6z = 0$$

$$\Rightarrow 6x - 16y + 6z - 38 = 0$$

$$\Rightarrow 3x - 8y + 3z + 13 = 0.$$

This shows that the plane 3x – 8y + 3z + 13 = 0 contains the lines (i) and (ii).

4. **Show that the lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{1-z}{2}$ and $3x-2y+z+5=0=2x+3y+4z-4$ are coplanar. Find their point of intersection and the plane in which they lie.**

Solution: Given lines are

$$\frac{x+4}{3}=\frac{y+6}{5}=\frac{1-z}{2} \quad \text{...... (i)}$$

And, $$3x-2y+z+5=0=2x+3y+4z-4 \quad \text{...... (ii)}$$

First we set (ii) in symmetrical form. For this set z = 0, then

$$3x-2y+z+5=0 \quad \text{..... (*)}$$

$$2x+3y+4z-4=0 \quad \text{..... (**)}$$

Solving (*) and (**) we get,

$$x=-\frac{7}{13} \text{ and } y=\frac{22}{13}$$

Therefore the line (ii) passes through the point $\left(\frac{-7}{13},\frac{22}{13},0\right)$.

Let direction ratio of lines is (a, b, c) of (ii) then

$$3a-2b+c=0 \quad \text{....... (a)}$$

$$2a+3b+4c=0 \quad \text{....... (b)}$$

Solving equation (a) and (b), by cross multiplication we get,

$$\frac{a}{-8-3}=\frac{b}{2-12}=\frac{c}{9+4}$$

$$\Rightarrow \quad (a,b,c)=(-11,-10,13)$$

Now, equation of the line passes through the point $\left(\frac{-7}{13},\frac{22}{13},0\right)$ and have direction ratio (-11, -10, 13) is

$$\frac{x+(7/13)}{-11}=\frac{y-(22/13)}{-10}=\frac{z-0}{13} \quad \text{...... (iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (-4,-6,1)$ and having the direction ratio $(l_1, m_1, n_1) = (3, 5, -2)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(\frac{-7}{13},\frac{22}{13},0\right)$ and having the direction ratio $(l_2, m_2, n_2) = (-11,-10,13)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

Now,

$$\begin{vmatrix} \frac{-7}{13}+14 & \frac{22}{13}+6 & 0-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix}=\begin{vmatrix} \frac{45}{13} & \frac{100}{13} & -1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix}$$

$$=\frac{45}{13}(65-20)-\frac{100}{13}(39-22)-1(-30+55)$$

$$=\frac{45}{13}(45)-\frac{100}{13}(17)-25=0$$

Hence the given lines are coplanar.

For point of intersection, let,

$$\frac{x+4}{3}=\frac{y+6}{5}=\frac{1-z}{2}=r \text{ (let)} \qquad \text{...... (iv)}$$

Then the general point of (iv) is, $(x = 3r-4,\ y = 5r-6,\ z = 1-2r)$.

And,

$$\frac{x+(7/13)}{-11}=\frac{y-(22/13)}{-10}=\frac{z-0}{13}=r^1 \qquad \text{...... (v)}$$

Then the general point of (v) is, $(x_2 = -11r^1 - \frac{7}{13},\ y_2 = -10r^1 + \frac{22}{13},\ z_2 = 13r^1)$

For point of intersection, at least one point of the line (iv) and (v) is same.

So, $x_1 = x_2,\quad y_1 = y_2\quad$ and $\quad z_1 = z_2$

That is, $3r-4=-11r^1-\frac{7}{13}$ ...... (a)

$$5r-6=-10r^1+\frac{22}{13} \qquad \text{...... (b)}$$

$$-2r+1=13r^1 \qquad \text{........ (c)}$$

Solving the equation (a) and (b) we get,

$$r^1=-\frac{3}{13}$$

Then (c) gives, $-2r = 13\left(\frac{-3}{13}\right)-1 \Rightarrow\ r=2$

Therefore the point of intersection is $(3r-4, 5r-6, -2r+1) = (2, 4, -3)$.

And the equation of plane containing the lines be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

$$\Rightarrow \begin{vmatrix} x+4 & y+6 & z-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix}=0$$

$\Rightarrow (x+4)(65-20)-(y+6)(39-22)+(z-1)(-35+55)=0$

$\Rightarrow 45(x+4)-17(y+6)+25(z-1)=0$

$\Rightarrow 45x+180-17y-102+25z-25=0$

$\Rightarrow 45x-17y+25z+53=0.$

This is the equation of required plane that contains the given line lines (i) and (ii).

5. Prove that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z-11$ are coplanar and find the point it intersection and the equation of the plane containing them. [2017 Spring Q.No. 1(a)]

Solution: Given lines are

$$\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3} \qquad \text{...... (i)}$$

And, $x+2y+3z-8=0=2x+3y+4z-11$ ...... (ii)

First we set (ii) in symmetrical form. For this set $z = 0$, then

$$x+2y-8=0 \qquad \text{.... (*)}$$

$$2x+3y-11=0 \qquad \text{..... (**)}$$

Solving (*) and (**) we get,

$x = -2$ and $y = 5$.

Therefore the line (ii) passes through the point (-2, 5, 0).

Let direction ratio of lines is (a, b, c) of (ii) then

$$a + 2b + 3c = 0 \qquad \text{...... (a)}$$
$$2a + 3b + 4c = 0 \qquad \text{....... (b)}$$

Solving equation (a) and (b), by cross multiplication we get,

$$\frac{a}{8-9} = \frac{b}{6-4} = \frac{c}{3-4} = 1$$

$$\Rightarrow \quad a = -1, b = 21, c = -1$$

Now, equation of the line passes through the point (-2, 5, 0)and have direction ratio (-1, 21, -1) is

$$\frac{x+2}{-1} = \frac{y-5}{21} = \frac{z-0}{-1}$$

$$\Rightarrow \quad \frac{x+2}{-1} = \frac{y-5}{2} = \frac{z-0}{-1} \qquad \text{...... (iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (-1, -1, -1)$ and having the direction ratio $(l_1, m_1, n_1) = (1, 2, 3)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = (-2, 5, 0)$ and having the direction ratio $(l_2, m_2, n_2) = (-1, 2, -1)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} -2-(-1) & 5-(-1) & 0-(-1) \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 6 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= -1(-2-6) - 6(-1+3) + 1(2+2)$$
$$= 8 - 12 + 4 = 0.$$

Hence the given lines are coplanar.

For point of intersection, let,

$$\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} = r \text{ (let)} \qquad \text{...... (iv)}$$

Then the general point of (iv) is,

$$(r-1, 2r-1, 3r-1).$$

And,

$$\frac{x+2}{-1} = \frac{y-5}{2} = \frac{z-0}{-1} = r^1 \qquad \text{...... (v)}$$

Then the general point of (v) is, $(-r^1 - 2, 2r^1 + 5, -r^1)$.

For point of intersection, at least one point of the line (iv) and (v) is same.

So, $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

That is,

$$r - 1 = -r^1 - 2 \qquad \text{...... (a)}$$
$$2r - 1 = 2r^1 + 5 \qquad \text{...... (b)}$$
$$3r - 1 = -r^1 \qquad \text{........ (c)}$$

Solving the equation (a) and (b) we get,

$$r = 1.$$

Therefore the point of intersection is $(r - 1, 2r - 1, 3r - 1) = (0, 1, 2)$.

And the equation of plane containing the lines be,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x+1 & y+1 & z+1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)(-8) - (y+1)(2) + (z+1)(4) = 0$$

$$\Rightarrow -8x - 8 - 2y - 2 + 4z + 4 = 0$$

$$\Rightarrow -8x - 2y + 4z - 6 = 0$$

$$\Rightarrow 4x + y - 2z + 3 = 0$$

This is the equation of required plane that contains the given line lines (i) and (ii).

6. Find the equation of the plane containing $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5}$.

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3} \qquad \ldots\ldots \text{(i)}$$

And $$\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5} \qquad \ldots\ldots \text{(ii)}$$

Then the general point of (i) is, $(x_1, y_1, z_1) = (1, 1, 3)$ and its direction ratio is, $(l_1, m_1, n_1) = (2, 2, 3)$.

And, the general point of (ii) is, $(x_2, y_2, z_2) = (2, 1, 4)$ and its direction ratio is, $(l_2, m_2, n_2) = (3, 4, 5)$.

Now, equation of plane containing (i) and (ii) is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-3 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(10-12) - (y-1)(10-9) + (z-3)(8-6) = 0$$

$$\Rightarrow -2(x-1) - 1(y-1) + 2(z-3) = 0$$

$$\Rightarrow -2x + 2 - y + 1 + 2z - 6 = 0$$

$$\Rightarrow -2x - y + 2z - 3 = 0$$

$$\Rightarrow 2x + y - 2z + 3 = 0.$$

This is the equation of required plane that contains the given line lines (i) and (ii).

7. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar. Also find their point of contact. **[2009 Fall Q. No. 1(a) OR]**

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \qquad \ldots\ldots \text{(i)}$$

And, $$4x - 3y + 1 = 0 = 5x - 3z + 2 \qquad \ldots\ldots \text{(ii)}$$

The equation can be written as,

$4x - 3y + 1 = 0$ and $5x - 3z + 2 = 0$

$\Rightarrow 4x = 3y - 1$ $\qquad$ $\Rightarrow 5x = 3z - 2$

$\Rightarrow x = \frac{3y-1}{4}$ $\qquad$ $\Rightarrow x = \frac{3z-2}{5}$

Then the line (ii) can be written in the symmetrical form as,

$$\frac{x-0}{1}=\frac{3y-1}{4}=\frac{3z-2}{5}$$

$$\Rightarrow \frac{x}{1}=\frac{y-(1/3)}{(4/3)}=\frac{z-(2/3)}{(5/3)} \qquad \text{...... (iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 3)$ and having the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(0, \frac{1}{3}, \frac{2}{3}\right)$ and having the direction ratio $(l_2, m_2, n_2) = \left(1, \frac{4}{3}, \frac{5}{3}\right)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} 0-1 & (1/3)-2 & (2/3)-3 \\ 2 & 3 & 4 \\ 1 & (4/3) & (5/3) \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -5/3 & -7/3 \\ 2 & 3 & 4 \\ 1 & 4/3 & 5/3 \end{vmatrix}$$

$$= -1\left(\frac{15}{3}-\frac{16}{3}\right) + \frac{5}{3}\left(\frac{10}{3}-4\right) - \frac{7}{3}\left(\frac{8}{3}-3\right) = \frac{1}{3} - \frac{10}{9} + \frac{7}{9}$$

$$= \frac{3-10+7}{9} = 0.$$

Hence, the given lines are coplanar.

For point of intersection, let,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r \text{ (let)} \qquad \text{...... (iv)}$$

Then the general point of (iv) is, $(2r + 1, 3r + 2, 4r + 3)$.

And,

$$\frac{x}{1}=\frac{y-1/3}{4/3}=\frac{z-2/3}{5/3}=r^1 \qquad \text{...... (v)}$$

Then the general point of (v) is, $\left(r^1, \frac{4r^1}{3}+\frac{1}{3}, \frac{5r^1}{3}+\frac{2}{3}\right)$.

For point of intersection, at least one point of the line (iv) and (v) is same.

So,

That is, $x_1 = x_2,\quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2$

$$2r + 1 = r_1 \qquad \text{...... (a)}$$

$$3r + 2 = \frac{4r}{3} + \frac{1}{3} \qquad \text{...... (b)}$$

$$4r + 3 = \frac{5r^1}{3} + \frac{2}{3} \qquad \text{...... (c)}$$

Solving the equation (a) and (b) we get,

$$r^1 = -1 \text{ and } r = -1.$$

Therefore the point of contact is,

$$(2r + 1, 3r + 2, 4r + 3) = (-1, -1, -1).$$

Exercise 3.5

1. Find the magnitude and equation of the line of shortest distance between the line of shortest distance the lines $\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **[2013 Spring Q.No. 1(a) OR]**

Solution: Given line is,

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7} \quad \text{...... (i)}$$

The point of line (i) is, $(x_1, y_1, z_1) = (8, -9, 10)$ and the direction cosines of (i) are $(l_1, m_1, n_1) = (3, -16, 7)$.

Another given line is,

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \quad \text{...... (ii)}$$

The point of line (ii) is, $(x_2, y_2, z_2) = (15, 29, 5)$ and the direction cosines of (ii) are $(l_2, m_2, n_2) = (3, 8, -5)$.

Let l, m, n be the direction cosines of shortest distance between (i) and (ii).

Since the line of shortest distance is perpendicular to both lines (i) and (ii), so

$$3l - 16m + 7n = 0$$
$$3l + 8m - 5n = 0$$

Solving these equations by method of cross multiplication,

$$\frac{l}{80-56}=\frac{m}{21+15}=\frac{n}{24+48} \Rightarrow \frac{l}{24}=\frac{m}{36}=\frac{n}{72}$$
$$\Rightarrow \frac{l}{2}=\frac{m}{3}=\frac{n}{6} = k \text{ (let)}$$
$$\Rightarrow l = 2k, m = 3k, n = 6k$$

Since we have,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (2k)^2 + (3k)^2 + (6k)^2 = 1$$
$$\Rightarrow 49k^2 = 1$$
$$\Rightarrow k = \frac{1}{7} \text{ (taking the +ve sign only)}$$

Then, $l = \frac{2}{7}, m = \frac{3}{7}, n = \frac{6}{7}$.

Now, the length of shortest distance (SD) is,

$$SD = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (15-8)\frac{2}{7} + (29+9)\frac{3}{7} + (5-10)\frac{6}{7}$$
$$= 7\left(\frac{2}{7}\right) + 38\left(\frac{3}{7}\right) - 5\left(\frac{6}{7}\right) = \frac{14+114+-30}{7} = \frac{98}{7} = 14.$$

And, the equation of shortest distance be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 2/7 & 3/7 & 6/7 \end{vmatrix} = 0 = \begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2/7 & 3/7 & 6/7 \end{vmatrix}$$

$$\Rightarrow \quad (x-8)\left(\frac{-96}{7}-3\right)-(y+9)\left(\frac{18}{7}-2\right)+(z-10)\left(\frac{9}{7}-\frac{32}{7}\right)=0=(x-15)$$

$$\left(\frac{48}{7}+\frac{15}{7}\right)-(y-29)\left(\frac{18}{7}+\frac{10}{7}\right)+(z-5)\left(\frac{9}{7}-\frac{16}{7}\right)$$

$$\Rightarrow \quad (x-8)(-96-21)-(y+9)(18-14)+(z-9)(9+32)=0=(x-15)\frac{63}{7}-$$

$$(y-29)\frac{28}{7}+(z-5)\left(\frac{-7}{7}\right)$$

$\Rightarrow$ $-117(x-8)-4(y+9)+41(z-10)=0=(x-15)9-(y-29)4+$ $(z-5)(-1)$.

$\Rightarrow$ $117x+4y-41z-490=0=9x-4y-z-135+116+5.$

$\Rightarrow$ $117x+4y-41z-490=0=9x-4y-z-14.$

Thus, the magnitude of the shortest distance is 14 unit and the equation of the line of shortest distance is, $117x+4y-41z-490=0=9x-4y-z-14.$

2. Show that the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ is $3\sqrt{30}$. Find out the equation of the line of shortest distance.

2013 Spring Q.No. 1(a)

Find the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. Also, find the equation of the line of shortest distance.

Solution: Given line is,

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \quad \ldots\ldots\ldots \text{(i)}$$

The point of line (i) is, $(x_1, y_1, z_1) = (3, 8, 3)$ and the direction ratio of (i) are $(l_1, m_1, n_1) = (3, -1, 1)$.

Another given line is,

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \quad \ldots\ldots\ldots \text{(ii)}$$

The point of line (ii) is, $(x_2, y_2, z_2) = (-3, -7, 6)$ and the direction ratio of (ii) are $(l_2, m_2, n_2) = (-3, 2, 4)$.

Let l, m, n be the direction ratio of shortest distance between (i) and (ii).

Since the line of shortest distance is perpendicular to both lines (i) and (ii), so

$$3l-m+n=0$$
$$-3l+2m+4n=0$$

Solving by cross multiplication, we get,

$$\frac{l}{-4-2}=\frac{m}{-3-12}=\frac{n}{6-3}$$

$$\Rightarrow \quad \frac{l}{-6}=\frac{m}{-15}=\frac{n}{3}$$

$$\Rightarrow \quad \frac{l}{-2}=\frac{m}{-5}=\frac{n}{1}=k$$

This gives, $l=-2k$, $m=-5k$, $n=k$.

Since we have,

$$l^2 + m^2 + n^2 = 1 \Rightarrow 4k^2 + 25k^2 + k^2 = 1$$
$$\Rightarrow 30k^2 = 1$$
$$\Rightarrow k = \frac{1}{\sqrt{30}} \quad \text{(taking the +ve positive sign)}$$

Then, $l = \frac{-2}{\sqrt{30}}$, $m = \frac{-5}{\sqrt{30}}$, $n = \frac{1}{\sqrt{30}}$.

Now, the length of shortest distance (SD) is,

$$\begin{aligned} SD &= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n \\ &= (-3-3)\left(\frac{-2}{\sqrt{30}}\right) + (-7-8)\left(\frac{-5}{\sqrt{30}}\right) + (6-3)\left(\frac{1}{\sqrt{30}}\right) \\ &= \frac{12}{\sqrt{30}} + \frac{75}{\sqrt{30}} + \frac{3}{\sqrt{30}} = \frac{12+75+3}{\sqrt{30}} = \frac{90}{\sqrt{30}} = 3\sqrt{30} \end{aligned}$$

And, the equation of shortest distance be,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ -2 & -5 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ -2 & -5 & 1 \end{vmatrix}$$

$$\Rightarrow (x-3)(-1+5) - (y-8)(3+2) + (z-3)(-15-2) = 0 = (x+3)(2+20) - (y+7)(-3+8) + (z-6)(15+4)$$
$$\Rightarrow (x-3)\,4 - (y-8)\,5 + (z-3)(-17) = 0 = (x+3)\,(22) - (y+7)\,5 + (z-6)\,19.$$
$$\Rightarrow 4x - 5y - 17z - 12 + 40 + 51 = 0 = 22x - 5y + 19z + 66 - 35 - 114.$$
$$\Rightarrow 4x - 5y - 17z + 79 = 0 = 22x - 3y + 19z - 83.$$

Thus, the magnitude of the shortest distance is $3\sqrt{30}$ unit and the equation of the line of shortest distance is, $4x - 5y - 17z + 79 = 0 = 22x - 3y + 19z - 83$.

3. Find the shortest distance between the lines $x = y + 4 = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y}{2} = z$.

Solution: Given lines are,

$$\frac{x-0}{1} = \frac{y+4}{1} = \frac{z-0}{3} \quad \text{...... (i)}$$

And
$$\frac{x-1}{3} = \frac{y-0}{2} = \frac{z-0}{1} \quad \text{...... (ii)}$$

Comparing the lines with $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ then we get

(a) the line (i) passes through the point $(x_1, y_1, z_1) = (0, -4, 0)$ and its direction ratio are $(l, m, n) = (1, 1, 3)$.

(b) the line (ii) passes through the point $(x_1, y_1, z_1) = (1, 0, 0)$ and its direction ratio be $(l, m, n) = (3, 2, 1)$.

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$l + m + 3n = 0$$
$$3l + 2m + n = 0$$

Solving by cross – multiplication

$$\frac{l}{1-6} = \frac{m}{9-1} = \frac{n}{2-3}$$

$$\Rightarrow \quad \frac{l}{-5} = \frac{m}{8} = \frac{n}{-1} = k \text{ (let)}$$

$$\Rightarrow \quad l = 5k, m = 8k, n = -k$$

We have, $l^2 + m^2 + n^2 = 1$. So,

$$(-5k)^2 + (8k)^2 + (-k)^2 = 1$$
$$\Rightarrow \quad 25k^2 + 64k^2 + k^2 = 1$$
$$\Rightarrow \quad 90k^2 = 1$$
$$\Rightarrow \quad k = \pm\frac{1}{\sqrt{90}}$$

Taking the positive sign,

$$l = \frac{-5}{\sqrt{90}}, m = \frac{8}{\sqrt{90}}, n = \frac{-1}{\sqrt{90}}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (1-0)\left(\frac{-5}{\sqrt{90}}\right) + (0+4)\left(\frac{8}{\sqrt{90}}\right) + (0-0)\left(\frac{-1}{\sqrt{90}}\right)$$
$$= \frac{-5}{\sqrt{90}} + \frac{32}{\sqrt{90}} + 0 = \frac{27}{\sqrt{90}} = \frac{27}{3\sqrt{10}} = \frac{9}{\sqrt{10}}.$$

4. **Find the shortest distance between the lines $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5}$ find also the equation of shortest distances.**

[2016 Spring Q. No. 1(a) OR] [2008 Spring Q. No. 1(a)]

2015 Spring Q. No. 1(a) OR

Find the magnitude and equation of the shortest distance between the lines $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5}$.

Solution: Given that,

$$\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7} \qquad \text{..... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (5, 7, 3)$ and it has the direction ratio $(l_1, m_1, n_1) = (3, -16, 7)$.

Also, given line is,

$$\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5} \qquad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (9, 13, 15)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 3, -5)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$3l - 16m + 7n = 0$$
$$3l + 8m - 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{80-56} = \frac{m}{21+15} = \frac{n}{24+48} \Rightarrow \frac{l}{24} = \frac{m}{36} = \frac{n}{72} = k$$
$$\Rightarrow l = 24k,\ m = 36k,\ n = 72k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (24k)^2 + (36k)^2 + (72k)^2 = 1$$
$$\Rightarrow 576\,k^2 + 1296\,k^2 + 5184\,k^2 = 1$$
$$\Rightarrow 7056\,k^2 = 1$$
$$\Rightarrow k = \frac{1}{84} \quad \text{,taking +ve sign only.}$$

So that, $l = 24\times\frac{1}{84} = \frac{2}{7}$, $m = 36\times\frac{1}{84} = \frac{3}{7}$, $n = 72\times\frac{1}{84} = \frac{6}{7}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (9-5)\times\frac{2}{7} + (13-7)\times\frac{3}{7} + (15-3)\times\frac{6}{7} = \frac{8+18+72}{7} = \frac{98}{7} = 14.$$

Thus the shortest distance between the lines is 14 units.

Next for equation of shortest distance,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2/7 & 3/7 & 6/7 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2/7 & 3/7 & 6/7 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow (x-5)(-96-21) - (y-7)(18-14) + (z-3)(9+32) = 0 = (x-9)(48+15) - (y-13)(18+10) + (z-15)(9-16)$$
$$\Rightarrow (x-5)(-117) - (y-7)4 + (z-3)41 = 0 = (x-9)9 - (y-13)4 + (z-15)(-1)$$
$$\Rightarrow -117(x-5) - 4(y-7) + 41(z-3) = 0 = (x-9)9 - (y-13)4 - (z-15).$$
$$\Rightarrow -117x - 4y + 41z + 490 = 0 = 9x - 4y - z - 14.$$

5. Find the magnitude and equation of the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$. [2009 Fall Q. No. 1(a)]

Solution: Given that,

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \quad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (0, 0, 0)$ and it has the direction ratio $(l_1, m_1, n_1) = (2, 1, -2)$.

Also, given line is,

$$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z-(-2)}{2} \quad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (2, 1, -2)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, -5, 2)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2l - 3m + n = 0$$
$$3l - 5m + 2n = 0.$$

Solving by cross multiplication we get,

$$\frac{l}{-6+5} = \frac{m}{3-4} = \frac{n}{-10+9}$$

$$\Rightarrow \quad \frac{l}{-1} = \frac{m}{-1} = \frac{n}{-1} = k$$

$$\Rightarrow \quad l = -k, m = -k, n = -k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \quad \Rightarrow \quad k^2 + k^2 + k^2 = 1 \quad \Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

Now, taking negative sign (because the direction ratios have negative sign)

$$l = -\frac{1}{\sqrt{3}}, m = -\frac{1}{\sqrt{3}}, n = -\frac{1}{\sqrt{3}}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$$

$$= (2-0)\left(\frac{-1}{\sqrt{3}}\right) + (1-0)\left(\frac{-1}{\sqrt{3}}\right) + (-2-0)\left(\frac{-1}{\sqrt{3}}\right)$$

$$= \frac{-2}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ (neglecting –ve sign)}$$

Thus the shortest distance between the lines is $\frac{1}{\sqrt{3}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & -3 & 1 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{vmatrix}$$

$$\Rightarrow \left(\frac{-1}{\sqrt{3}}\right)\begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 = \left(\frac{-1}{\sqrt{3}}\right)\begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow$ $x(-3-1) - y(2-1) + z(2+3) = 0 = (x-2)(-5-2) - (y-1)(3-2) + (z+2)(3+5).$

$\Rightarrow$ $-4x - y + 5z = 0 = -7x - y + 8z + 31$

$\Rightarrow$ $4x + y - 5z = 0 = 7x + y - 8z - 31.$

This is the equation of the required line.

6. Find the length and equation of the shortest distance between the lines $x - y + z = 0 = 2x - 3y + 4z$ and $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$

Solution: Given that,

$$x - y + z = 0 = 2x - 3y + 4z$$

When, $z = 0$ then $\quad x - y = 0 \quad \ldots\ldots (i)$

and $\quad 2x - 3y = 0 \quad \ldots\ldots (ii)$

Solving equation (i) and (ii) then we get,

$$y = 0 \text{ and } x = 0$$

Thus the point is, $\quad (x_1, y_1, z_1) = (0, 0, 0)$.

Let direction ratio of the line is, (a, b, c). Then,

$$a - b + c = 0 \quad \ldots\ldots (iii)$$

$$2a - 3b + 4c = 0 \quad \ldots\ldots (iv)$$

Solving by cross multiplication

$$\frac{a}{-4+3} = \frac{b}{2-4} = \frac{c}{-3+2}$$

$$\Rightarrow \frac{a}{-1} = \frac{b}{-2} = \frac{c}{-1}$$

$$\Rightarrow (a, b, c) = (-1, -2, -1)$$

Now the equation of line passing through (0, 0, 0) and having direction ratios $-1, -2, -1$ be,

$$\frac{x-0}{-1} = \frac{y-0}{-2} = \frac{z-0}{-1} \quad \ldots\ldots (v)$$

Another given line is,

$$x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$$

When, $z = 0$ then we get,

$$x + y - 3 = 0 \quad \ldots\ldots (vi)$$

$$2x + 3y - 4 = 0 \quad \ldots\ldots (vii)$$

Solving (vi) and (vii) we get,

$$x = 5 \text{ and } y = -2.$$

Thus the point be, $(x_2, y_2, z_2) = (5, -2, 0)$.

Let direction ratio of the line is, (l, m, n). Then,

$$l + m + 2n = 0 \quad \ldots\ldots (*)$$

$$2l + 3m + 3n = 0 \quad \ldots\ldots (**)$$

Solving above equation by cross multiplication we get,

$$\Rightarrow \frac{l}{3-6} = \frac{m}{4-3} = \frac{n}{3-2}$$

$$\Rightarrow \frac{l}{-3} = \frac{m}{1} = \frac{n}{1}$$

$$\Rightarrow (l, m, n) = (-3, 1, 1).$$

Thus the equation of line passing through (5, −2, 0) and having direction ratios −3, 1, 1 be,

$$\frac{x-5}{-3} = \frac{y+2}{1} = \frac{z-0}{1} \quad \ldots\ldots (viii)$$

From line (v) and (viii), $(x_1, y_1, z_1) = (0, 0, 0)$ and $(l_1, m_1, n_1) = (-1, -2, -1)$.

Also, $(x_2, y_2, z_2) = (5, -2, 0)$ and $(l_2, m_2, n_2) = (-3, 1, 1)$.

Let l, m, n, be the direction ratio of the line that measure the shortest distance between the given lines. So,

$$-l - 2m - n = 0 \quad \ldots\ldots (ix)$$

$$-3l + m + n = 0 \quad \ldots\ldots (x)$$

From equation (ix) and (x) by cross multiplication

$$\frac{l}{-2+1} = \frac{m}{3+1} = \frac{n}{-1-6}$$

$$\Rightarrow \quad \frac{l}{-1} = \frac{m}{4} = \frac{n}{-7} = k$$

$$\Rightarrow \quad l = -k,\ m = 4k \text{ and } n = -7k.$$

Since we have,

$$l^2 + m^2 + n^2 = 1 \quad \Rightarrow \quad (-k)^2 + 4k)^2 + (-7k)^2 = 1$$

$$\Rightarrow \quad k^2 + 16k^2 + 49k^2 = 1$$

$$\Rightarrow \quad 66k^2 = 1$$

$$\Rightarrow \quad k = \pm\frac{1}{\sqrt{66}} \text{ (taking the +ve sign).}$$

Thus, $l = \frac{-1}{\sqrt{66}}$, $m = \frac{4}{\sqrt{66}}$, $n = \frac{-7}{\sqrt{66}}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$$

$$= (5-0)\left(\frac{-1}{\sqrt{66}}\right) + (-2-0)\left(\frac{4}{\sqrt{66}}\right) + 0\left(\frac{-7}{\sqrt{66}}\right)$$

$$= -\frac{5}{\sqrt{66}} - \frac{8}{\sqrt{66}} = -\frac{13}{\sqrt{66}}.$$

$$= \frac{13}{\sqrt{66}} \quad \text{(neglecting –ve sign)}$$

Thus the shortest distance between the lines is $\frac{13}{\sqrt{66}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \quad \frac{1}{\sqrt{66}}\begin{vmatrix} x & y & z \\ -1 & -2 & -1 \\ -1 & 4 & -7 \end{vmatrix} = 0 = \frac{1}{\sqrt{66}}\begin{vmatrix} x-5 & y+2 & z-0 \\ -3 & 1 & 1 \\ -1 & 4 & -7 \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} x & y & z \\ -1 & -2 & -1 \\ -1 & 4 & -7 \end{vmatrix} = 0 = \begin{vmatrix} x-5 & y+2 & z-0 \\ -3 & 1 & 1 \\ -1 & 4 & -7 \end{vmatrix}$$

$$\Rightarrow \quad x\,(14+4) - y\,(7-1) + z\,(-4-2) = 0 = (x-5)(-7-4) - (y+2)(21+1) + z\,(-12+1)$$

$$\Rightarrow \quad 18x - 6y - 6z = 0 = -11.(x-5) - 22.(y+2) - 11z$$

$$\Rightarrow \quad 18x - 6y - 6z = 0 = -11x + 55 - 22y - 44 - 11z$$

$$\Rightarrow \quad 18x - 6y - 6z = 0 = -11x - 22y - 11z + 11$$

$$\Rightarrow \quad 3x - y - z = 0 = x + 2y + z - 1$$

This is the equation of the required line.

7. Show that the shortest distance between the y-axis and $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ is $-\dfrac{bd' - b'd}{\sqrt{(ba' - b'a)^2 + (bc' - b'c)^2}}$.

Solution: we have the y-axis is

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{0} \qquad \text{...... (i)}$$

Another given line is,

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \qquad \text{... (ii)}$$

The plane containing (ii) is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0 \qquad \text{... (iii)}$$

Let the line (i) is parallel to (iii) then the normal to (iii) is perpendicular to (i). Therefore,

$$0(a + ka') + 1(b + kb') + 0(c + c') = 0$$

$$\Rightarrow \quad k = \frac{-b}{b'}$$

Therefore (iii) becomes

$$ax + by + cz + d - \frac{b}{b'}(a'x + b'y + c'z + d') = 0$$

$$\Rightarrow \quad (ab' - a'b)x + (cb' - c'b)z + (db' - d'b) = 0 \qquad \text{... (iv)}$$

Thus (iv) contains (ii).

The origin contained in y-axis. Therefore the shortest distance from y-axis to (ii) is the perpendicular form origin to (ii). So,

$$SD = -\frac{bd' - b'd}{\sqrt{(ba' - b'a)^2 + (bc' - b'c)^2}}$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

<u>1999; 2001 Q. No. 1(a)</u>

Find the shortest distance between the lines, $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$.

Solution: Given that,

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4} \qquad \text{...... (i)}$$

Clearly the line (i) passes through the point (x, y, z) = (3, 4, 5) and it has the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5} \qquad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (4, 5, 7)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2l + 3m + 4n = 0$$
$$3l + 4m + 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

$$\Rightarrow \quad \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=k$$

$$\Rightarrow \quad l=-k,\ m=2k,\ n=-k$$

We know that,

$$l^2+m^2+n^2=1 \Rightarrow (-k)^2+(2k)^2+(-k)^2=1$$

$$\Rightarrow 6k^2=1$$

$$\Rightarrow k=\frac{1}{\sqrt{6}} \quad \text{,taking +ve sign only.}$$

So that, $l=\frac{1}{\sqrt{6}}$, $\quad m=-2\frac{1}{\sqrt{6}}$, $\quad n=\frac{1}{\sqrt{6}}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n$$

$$=(4-3)\left(\frac{1}{\sqrt{6}}\right)+(5-4)\left(\frac{-2}{\sqrt{6}}\right)+(7-5)\left(\frac{1}{\sqrt{6}}\right)$$

$$=\frac{1-2+2}{\sqrt{6}}=\frac{1}{\sqrt{6}}.$$

Thus the shortest distance between the lines is $\frac{1}{\sqrt{6}}$ units.

2002 Q. No. 1(a)

Find the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=2$.

Solution: The given line is

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=r\ (\text{say}) \qquad \text{............(1)}$$

So, the general point of (1) is

$$(r+3,\ 2r+4,\ 2r+5) \qquad \text{.........(2)}$$

Given that the line (1) meets the plane $x+y+z=2$. Then the point (2) is the common point of (1) and (2). Therefore,

$$r+3+2r+4+2r+5=2$$

$$\Rightarrow 5r+12=2$$

$$\Rightarrow r=-2$$

So the point of intersection of the line and the plane is

$$(-2+3,\ -4+4,\ -4+5) \Rightarrow (1, 0, 1)$$

Now, length of (3, 4, 5) from (1, 0, 1) is

$$d=\sqrt{(3-1)^2+(4-0)^2+(5-1)^2}=\sqrt{4+16+16}=\sqrt{36}=6.$$

2002 Q. No. 1(a)

Find the equation of the plane containing the line $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and parallel to the line $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$.

Solution: Given line is

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \text{........(1)}$$

Clearly, the line (1) passes through he point (x_1, y_1, z_1).
Then the plane containing the line (1) passes through the point (x_1, y_1, z_1).
Now, the equation of plane through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{........(2)}$$

So, $$al_1 + bm_1 + cn_1 = 0 \quad \text{.........(3)}$$

By question the plane (2) is parallel to the line

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \quad \text{..........(4)}$$

So, $$al_2 + bm_2 + cn_2 = 0 \quad \text{..........(5)}$$

Solving (3) and (5) we get,

$$\frac{a}{m_1n_2 - m_2n_1} = \frac{b}{n_1l_2 - n_2l_1} = \frac{c}{l_1m_2 - l_2m_1}$$

Thus (2) becomes,

$$(m_1n_2 - m_2n_1)(x - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_1m_2 - l_2m_1)(z - z_1) = 0$$

This is the equation of required plane that contains (1) and is parallel to (2).

2002 Q. No. 1(a) OR

Find shortest distance between the lines $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-6}{5}$. Also find the equation of the shortest distance.

Solution: Given that, $$\frac{x-1}{2} = \frac{y-4}{3} = \frac{z-5}{4} \quad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (1, 4, 5)$ and it has the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.
Also, given line is,

$$\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-6}{5} \quad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (4, 5, 6)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$
Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).
Then,

$$2l + 3m + 4n = 0$$
$$3l + 4m + 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9} \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = k$$
$$\Rightarrow l = -k, m = 2k, n = -k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (-k)^2 + (2k)^2 + (-k)^2 = 1$$
$$\Rightarrow 6k^2 = 1$$
$$\Rightarrow k = \frac{1}{\sqrt{6}} \text{ ,taking +ve sign only.}$$

So that, $l = \frac{1}{\sqrt{6}}$, $m = -2\frac{1}{\sqrt{6}}$, $n = \frac{1}{\sqrt{6}}$

Now, the shortest distance between (i) and (ii) is

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (4-1)\left(\frac{1}{\sqrt{6}}\right) + (5-4)\left(\frac{-2}{\sqrt{6}}\right) + (6-5)\left(\frac{1}{\sqrt{6}}\right)$$

$$= \frac{3+1+1}{\sqrt{6}} = \frac{5}{\sqrt{6}}.$$

Thus the shortest distance between the lines is $\frac{5}{\sqrt{6}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix} = 0 = \begin{vmatrix} x-4 & y-5 & z-6 \\ 3 & 4 & 5 \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x-4 & y-5 & z-6 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix}$$

$\Rightarrow$ $(x-1)(3+8) - (y-4)(2-4) + (z-5)(-4-3) = 0 = (x-4)(4+10) - (y-5)(3-5) + (z-6)(-6-4).$

$\Rightarrow$ $(x-1)11 - (y-4)(-2) + (z-5)(-7) = 0 = (x-4)14 - (y-5)(-2) + (z-6)(-10)$

$\Rightarrow$ $11x + 2y - 7z - 11 - 8 + 35 = 0 = 14x + 2y - 10z - 56 - 10 + 60.$

$\Rightarrow$ $11x + 2y - 7z + 16 = 0 = 14x + 2y - 10z - 6.$

This is he equation of required line.

Similar Question for Practice from Final Exam:

2003 Fall Q. No. 1(a) OR

Find the magnitude and the equation of S.D. between $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$

2007 Fall Q. No. 1(a) OR

Find the shortest distance between the lines $\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$. Also find the equation of shortest distance.

2009 Spring Q. No. 1(a) OR

Find the shortest distance between the lines $x = y + 4 = \frac{z}{3}$ and $\frac{x-1}{3} = \frac{y}{2} = z$. Find also the equation of shortest distance.

2010 Spring Q. No. 1(a) OR

Find the shortest distance between the lines, $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{3} = \frac{15-z}{5}$. Also find the equation of shortest distance.

2018 Spring Q. No. 1(a)

Find the length of perpendicular from the (3, –1, 11) to the $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Also, find the equation of perpendicular.

2018 Fall Q.No. 1(a) OR; 2018 Spring Q.No. 1(a) OR

Find the shortest distance between the z-axis and ax + by + cz + d = 0 = a'x + b'y + c'z + d'.

Solution: we have the y-axis is

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1} \quad \ldots\ldots (i)$$

Another given line is,

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \quad \ldots (ii)$$

The plane containing (ii) is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0 \quad \ldots (iii)$$

Let the line (i) is parallel to (iii) then the normal to (iii) is perpendicular to (i). Therefore,

$$0(a + ka') + 0(b + kb') + 1(c + c') = 0$$

$$\Rightarrow \quad k = \frac{-c}{c'}$$

Therefore (iii) becomes

$$ax + by + cz + d - \frac{c}{c'}(a'x + b'y + c'z + d') = 0$$

$$\Rightarrow \quad (ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \quad \ldots (iv)$$

Thus (iv) contains (ii).

The origin contained in y-axis. Therefore the shortest distance from y-axis to (ii) is the perpendicular form origin to (ii). So,

$$SD = -\frac{c'd - d'c}{\sqrt{(ca' - c'a)^2 + (bc' - b'c)^2}}$$

OTHER QUESTIONS

2019 Fall Q.No. 1(a)

Define shortest distance between two skew lines in space. Find the length and the equation of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is $3\sqrt{30}$

Solution: See Ex 3.5 Q.No. 2.

2018 Fall Q.No. 1(a)

Find the image of the point P (1, 2, 3) in the plane 2x – y + z + 3 = 0.

Solution: The equation of line passing through (1, 2, 3) and perpendicular to the plane $2x - y + z + 3 = 0$ is,

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{1} = r \text{ (suppose)} \quad \ldots\ldots (i)$$

Then the general point of (i) is, $(x, y, z) = (2r + 1, 2 - r, r + 3)$.

Since, if the point $Q(2r + 1, 2 - r, r + 3)$ is the image of P then the middle point R lies on the plane $2x - y + z + 3 = 0$.

Here, the coordinate of R is,

$$R\left(\frac{2r+1+1}{2}, \frac{2-r+2}{2}, \frac{r+3+3}{2}\right) = R\left(r+1, 1-\frac{r}{2}, 3+\frac{r}{2}\right)$$

Since the point lies on the plane $2x - y + z + 3 = 0$, so,

$$2(r+1) - 1 + \frac{r}{2} + 3 + \frac{r}{2} + 3 = 0.$$

$$\Rightarrow \quad 3r + 7 = 0.$$

$$\Rightarrow \quad r = \frac{-7}{3}.$$

Then the coordinate of Q is,

$$(2r+1, 2-r, r+3) = (-11/3, 13/3, 2/3)$$

Thus, the image of the point P (1, 2, 3) in the plane $2x - y + z + 3 = 0$ be $(-11/3, 13/3, 2/3)$.

2011 Spring Q. No. 1(a) OR

Find the equation of shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+1}{4}$.

Solution: Similar to above.

2011 Fall Q. No. 1(a) OR; 2014 Spring Q. No. 1(a) OR

Define shortest distance between two skew lines in space. Find the length and equation of shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

Solution: See definition of S.D. and for second part process as above.

2013 Fall Q. No. 1(a)

Find the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$. Also, find the equations of shortest distance.

Solution: See 2011 Fall.

2004 Fall Q. No. 1(a)

Find the equation of the plane through the points (1, 0, –1) and (3, 2, 2) and parallel to the line: $x - 1 = \frac{1-y}{2} = \frac{z-2}{3}$.

Solution: Given line is

$$\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3} \qquad \text{.........(1)}$$

Since the equation of plane through (1, 0, –1) is

$$a(x-1) + b(y-0) + c(z+1) = 0 \qquad \text{........(2)}$$

Given that the plane also passes through (3, 2, 2). Then (2) gives,

$$a(2-1) + 2b + c(2+1) = 0$$

2006 Fall Q. No. 1(a) OR

Prove that the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ and $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ intersect. Find also their point of intersection and plane through them.

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = r \text{ (let)} \qquad \text{........(1)}$$

and

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = r \text{ (let)} \qquad \text{........(2)}$$

Then the general point of (1) is $(2r + 1, -3r - 1, 8r - 10)$ and of (2) is,

$$(r' + 4, -4r' - 3, 7r' - 1).$$

If the lines are intersect then they should have at least a common point. So,

$$2r + 1 = r' + 4, \qquad -3r - 1 = -4r' - 3 \qquad 8r - 10 = 7r' - 1$$

$$\Rightarrow r' = 2r - 3, \qquad \Rightarrow 3r = 4r' + 2 \qquad \Rightarrow 8r = 7r' + 9$$

Solving first two equations, we get

$$r = 2 \quad \text{and} \quad r' = 1.$$

Then the third equations becomes

$$8(2) = 7(1) + 9$$

$$\Rightarrow 16 = 7 + 9$$

$$\Rightarrow 16 = 16$$

This proves that (i) and (ii) intersects each other.

And, the general point of (i) with r = 2, we get, (5, –7, 6).

This is the point of contact of (1) and (2),

Also, the equation of plane containing the line (1) and (2) is

$$\begin{vmatrix} x-1 & y+1 & z+10 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(-21+32) - (y+1)(14-8) + (z+10)(-8+3) = 0$$

$$\Rightarrow 11(x-1) - 6(y+1) - 5(z+10) = 0$$

$$\Rightarrow 11x - 6y - 5z - 67 = 0$$

This is the equation of required plane.

2015 Fall Q. No. 1(a)

Find the equation of plane through (a, b, g) and the line $x = py + q = rz + s$.

Solution: *Similar to Ex. 3.3 Q. 7*

SHORT QUESTIONS

2003 Fall: Write down the equations of the line through (2, 1, 3) and (4, 2, 4).

Solution: Since we have the equation of line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

So, the equation of line through (2, 1, 3) and (4, 2, 4) be

$$\frac{x-2}{4-2}=\frac{y-1}{2-1}=\frac{z-3}{4-3}$$

$$\Rightarrow \frac{x-2}{2}=y-1=z-3$$

2010 Spring: Find the equation of the line through (1, 2, 3) and normal to the plane $2x + 3y - z = 4$.

Solution: Since we have the equation of line through (x_1, y_1, z_1) and is normal to the plane $ax + by + cz + d = 0$ is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

So, the equation of line through (1, 2, 3) and normal to the plane $2x + 3y - z = 4$ is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{-1}.$$

2018 Spring: Find the equation of the line through (1, 5, 3) and normal to the plane $2x + 3y + 7z = 0$.

Solution: Since we have the equation of line through (1, 5, 3) and is normal to the plane $2x + 3y + 7z = 0$ is

$$\frac{x-1}{2}=\frac{y-5}{3}=\frac{z-3}{7}.$$

□□□