$$= \frac{2}{(1+at)^2} \times \left[\frac{1}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2} \times (1+at)^3 - 1$$

$$= 2(1+at) - 1$$

$$= 2at + 1$$

$$Var[X(t)] = E[X^2(t)] - \left(E[X(t)]\right)^2 = 2at + 1 - (1)^2 = 2at$$

Which is not a constant.

$\therefore \{X(t)\}$ is not a stationary.

Problems on Markov Chain

Problem 3.6:

An engineer analyzing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals follow a highly distorted signal, with no recognizable signal between, whereas 20 out of 23 recognizable signals follow recognizable signals, with no highly distorted signal between. Given that only highly signals are not recognizable, find the fraction of signals that are highly distorted.

(N/D 2010),(N/D 2014)

Solution:

Let Highly distorted $\rightarrow$ D and

Recognizable $\rightarrow$ R.

$\therefore$ The state space is $\{D, R\}$.

The tpm of the Markov chain is

$$P = \begin{array}{c c} & \begin{array}{cc} D & R \end{array} \\ \begin{array}{c} D \\ R \end{array} & \begin{bmatrix} \frac{1}{15} & - \\ - & \frac{20}{23} \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cc} D & R \end{array} \\ \begin{array}{c} D \\ R \end{array} & \begin{bmatrix} \frac{1}{15} & \frac{14}{15} \\ \frac{3}{23} & \frac{20}{23} \end{bmatrix} \end{array}$$

$$\Rightarrow a = \frac{14}{15} \text{ and } b = \frac{3}{23} \qquad \Rightarrow a+b = \frac{14}{15}+\frac{3}{23} = \frac{367}{345}$$

Steady state distribution $= \left[\dfrac{b}{a+b}, \dfrac{a}{a+b}\right]$

$$= \left[\frac{3/23}{367/345}, \frac{14/15}{367/345}\right] = \left[\frac{45}{367}, \frac{322}{367}\right]$$

$\therefore$ The fraction of signals that are highly distorted is $\dfrac{45}{367}$.

Problem 3.7:

An observer at a lake notices that when fish are caught, only 1 out of 9 trout is caught after another trout, with no other fish between, whereas 10 out of 11 non-trout are caught following non-trout, with no trout between. Assuming that all fish are equally likely to be caught, what fraction of fish in the lake is trout? (N/D 2012)

Solution:

Let trout fish $\rightarrow$ T and

non-trout $\rightarrow$ N.

$\therefore$ The state space is $\{T, N\}$.

The tpm of the Markov chain is

$$P = \begin{array}{c} \\ T \\ N \end{array} \begin{array}{c} \begin{array}{cc} T & N \end{array} \\ \begin{bmatrix} \frac{1}{9} & - \\ - & \frac{10}{11} \end{bmatrix} \end{array} = \begin{array}{c} \\ T \\ N \end{array} \begin{array}{c} \begin{array}{cc} T & N \end{array} \\ \begin{bmatrix} \frac{1}{9} & \frac{8}{9} \\ \frac{1}{11} & \frac{10}{11} \end{bmatrix} \end{array}$$

$\Rightarrow a = \frac{8}{9}$ and $b = \frac{1}{11}$ $\qquad \Rightarrow a + b = \frac{8}{9} + \frac{1}{11} = \frac{97}{99}$

Steady state distribution $= \left[\frac{b}{a+b}, \frac{a}{a+b}\right]$

$$= \left[\frac{1/11}{97/99}, \frac{8/9}{97/99}\right] = \left[\frac{9}{97}, \frac{88}{97}\right]$$

$\therefore$ The fraction of trout fish is $\frac{9}{97}$.

Problem 3.8:

A man either drives a car (or) catches a train to go to office each day. He never goes two days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. Find (1) the probability that he takes a train on the third day and (2) the probability that he drives to work in the long run. (N/D 2011),(N/D 2015)

Solution:

The tpm of the Markov chain is

$$P = \begin{array}{c} \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cc} C & T \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \end{array}$$

The initial state distribution is

$$P^{(1)} = \begin{array}{c} \begin{matrix} C & T \end{matrix} \\ \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix} \end{array} \qquad P^{(1)} - \textit{On the first day travel}$$

(1)

$$P^{(2)} = P^{(1)}P = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{12} & \frac{1}{12} \end{bmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{bmatrix} \frac{11}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{13}{24} & \frac{11}{24} \end{bmatrix}$$

$\therefore$ P (The man takes a train on the third day) $= \dfrac{11}{24}$.

(2)

The tpm of the Markov chain is

$$P = \begin{array}{c c} & \begin{matrix} C & T \end{matrix} \\ \begin{matrix} C \\ T \end{matrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow a = \dfrac{1}{2}$ and $b = 1$ $\qquad\qquad \Rightarrow a + b = \dfrac{1}{2} + 1 = \dfrac{3}{2}$

$$\text{Steady state distribution} = \begin{bmatrix} \frac{b}{a+b}, & \frac{a}{a+b} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3/2}, & \frac{1/2}{3/2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}, & \frac{1}{3} \end{bmatrix}$$

$\therefore$ P (The man drives (car) to work in the long run) $= \dfrac{2}{3}$.

Problem 3.9:

A salesman territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city-A, then the next day he sells in city-B. However if he sells in either city-B or city-C, the next day he is twice as likely to sell in city-A as in the other city. In the long run how often does he sell in each of the cities? (M/J 2012),(N/D 2013)

Solution:

The tpm of the Markov chain is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

Let $\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$ be the steady state distribution of the chain.

By the property, we have

$$\pi P = \pi \quad \ldots(1)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad \ldots(2)$$

$$(1) \Rightarrow (\pi_1 \quad \pi_2 \quad \pi_3)\begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 = \pi_1 \quad \ldots(3)$$

$$\pi_1 + \frac{1}{3}\pi_3 = \pi_2 \quad \ldots(4)$$

$$\frac{1}{3}\pi_2 = \pi_3 \quad \ldots(5)$$

Substituting (5) in (3), we have

$$(3) \Rightarrow \frac{2}{3}\pi_2 + \frac{2}{3}\left(\frac{1}{3}\pi_2\right) = \pi_1$$

$$\Rightarrow \pi_1 = \frac{8}{9}\pi_2 \qquad \ldots(6)$$

Substituting π_1 and π_3 in (2), we have

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\frac{8}{9}\pi_2 + \pi_2 + \frac{1}{3}\pi_2 = 1$$

$$\pi_2\left(\frac{8}{9} + 1 + \frac{1}{3}\right) = 1 \Rightarrow \pi_2 = \frac{9}{10}$$

$$(6) \Rightarrow \pi_1 = \frac{8}{9}\left(\frac{9}{20}\right) = \frac{8}{20}$$

$$(5) \Rightarrow \pi_3 = \frac{1}{3}\left(\frac{9}{20}\right) = \frac{3}{20}$$

$\therefore$ The steady-state distribution of the chain is $\pi = \left(\frac{8}{20} \quad \frac{9}{20} \quad \frac{3}{20}\right)$.

The percentage value is $\pi = \left(\frac{8}{20}\times 100 \quad \frac{9}{20}\times 100 \quad \frac{3}{20}\times 100\right)$

$$= (40\% \quad 45\% \quad 15\%)$$

Hence in the long run, he sells 40% of the item in city A, 45% of the item in city B and 15% of the item in city C.

Problem 3.10:

Find the limiting-state probabilities associated with the following transition probability matrix $\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$. (A/M 2011)

Solution:

Let $P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$.

Consider $\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$ be the steady state distribution of the chain.

By the property, we have

$$\pi P = \pi \qquad \ldots(1)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad \ldots(2)$$

$$(1) \Rightarrow \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$0.4\pi_1 + 0.3\pi_2 + 0.3\pi_3 = \pi_1 \qquad \ldots(3)$$

$$0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_2 \qquad \ldots(4)$$

$$0.1\pi_1 + 0.4\pi_2 + 0.5\pi_3 = \pi_3 \qquad \ldots(5)$$

The above three equations (3), (4) and (5) can be rewrite as

$$-0.6\pi_1 + 0.3\pi_2 + 0.3\pi_3 = 0 \qquad \ldots(6)$$

$$0.5\pi_1 - 0.7\pi_2 + 0.2\pi_3 = 0 \qquad \ldots(7)$$

$$0.1\pi_1 + 0.4\pi_2 - 0.5\pi_3 = 0 \qquad \ldots(8)$$

Now solving (6), (7) and (8) with the help of (2).

$$(6) \Rightarrow \quad -0.6\pi_1 + 0.3\pi_2 + 0.3\pi_3 = 0$$

$$(2) \times 0.6 \Rightarrow 0.6\pi_1 + 0.6\pi_2 + 0.6\pi_3 = 0.6$$

$$0.9\pi_2 + 0.9\pi_3 = 0.6$$

$$\Rightarrow 0.3\pi_2 + 0.3\pi_3 = 0.2 \qquad \ldots(9)$$

$$(7) \Rightarrow \quad 0.5\pi_1 - 0.7\pi_2 + 0.2\pi_3 = 0$$

$$(2) \times -0.5 \Rightarrow -0.5\pi_1 - 0.5\pi_2 - 0.5\pi_3 = -0.5$$

$$-1.2\pi_2 - 0.3\pi_3 = -0.5$$

$$\Rightarrow 1.2\pi_2 + 0.3\pi_3 = 0.5 \qquad \ldots(10)$$

Solving (9) and (10), we get

$\pi_2 = 0.3$ and $\pi_3 = 0.3$

Substituting π_2 and π_3 values in (2), we get

$\pi_1 + 0.3 + 0.3 = 1 \Rightarrow \pi_1 = 0.4$

Hence $\pi = \begin{pmatrix} 0.4 & 0.3 & 0.3 \end{pmatrix}$.

Problem 3.11:

Consider a Markov chain with transition probability matrix $P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$. Find the limiting probabilities of the system. (M/J 2014)

Solution:

(Similar to previous problem)

Problem 3.12:

Consider a Markov chain with 3 states and transition probability matrix $P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix}$. Find the stationary probabilities of the chain. (N/D 2017)

Solution:

(Similar to Problem 3.10)

Problem 3.13:

The transition probability matrix of a Markov chain $\{X(t)\}$, $n = 1,2,3,\ldots$, having three states 1, 2 and 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $p^{(0)} = (0.7 \quad 0.2 \quad 0.1)$. Find (1) $p[X_2 = 3]$

(2) $p[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$.

(M/J 2012),(N/D 2013),(M/J 2014)

Solution:

Given the transition probability matrix with three states 1,2 and 3

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \quad \text{and} \quad p^{(0)} = (0.7 \quad 0.2 \quad 0.1).$$

$$\Rightarrow P^2 = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(1)

$$p[X_2 = 3] = \sum_{i=1}^{3} p[X_2 = 3 / X_0 = i] \times p[X_0 = i]$$

$$= p_{13}^{(2)} \times p[X_0 = 1] + p_{23}^{(2)} \times p[X_0 = 2] + p_{33}^{(2)} \times p[X_0 = 3]$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$

$$= 0.182 + 0.068 + 0.029 = 0.279$$

(2)

$$p[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$$

$$= p[X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2] \times p[X_2 = 3, X_1 = 3, X_0 = 2]$$

$$= p_{32}^{(1)} \times p[X_2 = 3, X_1 = 3, X_0 = 2]$$

$$= p_{32}^{(1)} \times p[X_2 = 3 / X_1 = 3, X_0 = 2] \times p[X_1 = 3, X_0 = 2]$$

$$= p_{32}^{(1)} \times p_{33}^{(1)} \times p[X_1 = 3, X_0 = 2]$$

$$= p_{32}^{(1)} \times p_{33}^{(1)} \times p[X_1 = 3 / X_0 = 2] \times p[X_0 = 2]$$

$$= p_{32}^{(1)} \times p_{33}^{(1)} \times p_{23}^{(1)} \times p[X_0 = 2]$$

$$= 0.4 \times 0.3 \times 0.2 \times 0.2 = 0.0048$$

Problem 3.14:

The following is the transition probability matrix of a Markov chain with state space $\{0, 1, 2, 3, 4\}$. Specify the classes, and determine which classes are transient and which are recurrent.

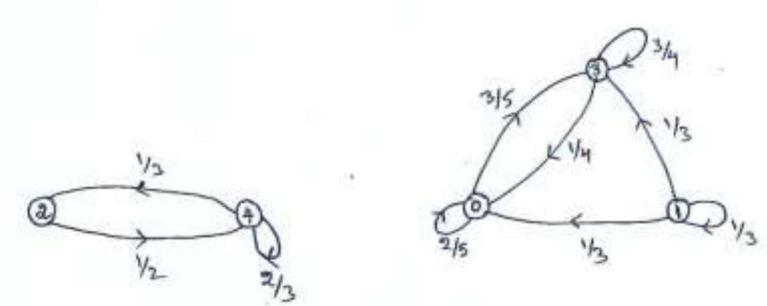
Give reasons $P = \begin{bmatrix} \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$. (N/D 2010)

Solution:

Given tmp for the state {0, 1, 2, 3, 4} is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \end{array}$$

Transition State Diagram of the Markov Chain:



From the transition state diagram, states 2 and 4 communicate each other because we can reach the states 2 to 4 and 4 to 2. Similarly 0 and 3 also communicate each other.

$\therefore$ $\{2, 4\}$ and $\{0, 3\}$ are form classes and thus $\{2, 4\}$ and $\{0, 3\}$ are recurrent states.

From 1 we can reach 0 and 3, but we cannot reach 1 from them and the system goes from 1 to 1 with probability $\frac{1}{3}$.

$\therefore$ The state 1 is transient (non recurrent).

Problem 3.15:

Find the nature of the states of the Markov chain with the tpm

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$ (M/J 2013),(A/M 2018)

Solution:

(Similar to previous problem)

Problem 3.16:

The following is the transition probability matrix of a Markov chain with state space $\{1, 2, 3, 4, 5\}$. Specify the classes, and determine which classes are transient and which are recurrent

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/5 & 0 & 3/5 \end{pmatrix}.$$ (N/D 2012)

Solution:

(Similar to previous problem)

Problem 3.17:

Let the Markov Chain consisting of the states 0, 1, 2, 3 have the transition probability matrix $P=\begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Determine which states are transient and which are recurrent by defining transient and recurrent states. (A/M 2010)

Solution:

(Similar to previous problem)

Problem 3.18:

A gambler has Rs.2. He bets Rs.1 at a time and wins Rs.1 with probability $1/2$. He stops playing if he loses Rs.2 or wins Rs.4. (1) What is the tpm of the related Markov chain? (2) What is the probability that he has lost his money at the end of 5 plays?
(M/J 2013)

Solution:

Let $\{X_n\}$ represent the amount with the player at the end of the nth round of the play.

The state space is $\{0, 1, 2, 3, 4, 5, 6\}$.

The player stops playing if he losses Rs. 2 or wins Rs.4.

We observe that if the player wins Rs.4, then he will have Rs. 6.

(1) TPM of the Markov chain:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(2) Money at at the end of 5 plays:

Since the player has got Rs. 2 initially, the initial probability distribution of $\{X_n\}$ is

$$P^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P^{(1)} = P^{(0)} P$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$P^{(2)} = P^{(1)}P$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}$$

Similarly,

$$P^{(3)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$

$$P^{(4)} = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix}$$

$$P^{(5)} = \begin{pmatrix} \frac{3}{8} & \frac{5}{32} & 0 & \frac{9}{32} & 0 & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$

$\Rightarrow P(\text{Player has lost money at the end of 5th plays})$

$$= P(X_5 = 0) = \frac{3}{8}$$