Pi Calculator

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Motivation

This project was an idea that Valentin and Karthik came up with over winter break during 2020. The objective of this project is to develop an application that allows the user to enter in the number of decimal places of π desired and the application outputs an approximation correct to the desired amount of decimal places. It's important to note that we did not use the built in value of π from programming languages such as Python as that would defeat the purpose. This project is interesting as π (the ratio of a circle's circumference to its diameter) shows up in interesting applications. For example, the Euler identity is as follows

$$e^{i \cdot \pi} = -1$$

Also, another well known identity is below:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Approximating π to a given amount of decimal places is a challenge and there is a rich history. According to the Exploratorium, the ancient Babylonians and Egyptians were the first to attempt to approximate π before Archimedes. If you would like to learn more about the history, the link is in the citations. Our application provides a brief exploration of this topic through looking at two well known methods to approximate π and analyzing the performance.

Methods

0.1 Taylor Series Basic

We explore the possibility of approximating π through use of a Taylor series. As a refresher, Taylor series are a way to approximate a function using a summation of polynomials and higher order derivatives. The general formula for a Taylor series centered at $x = a$ is as shown:

$$f(x) \approx f(a) + \frac{f'(a) \cdot (x-a)}{1!} + \frac{f''(a) \cdot (x-a)^2}{2!} + ...$$

In regards to π, it's nice to observe that the Taylor series of $\arctan{()}\, x)$ can help.

Observe that

$$\frac{d}{dx}(\arctan{(x)}) = \frac{1}{1+x^2} \implies \arctan{(x)} = \int (\frac{1}{1+x^2})dx$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

The above expression represents an infinite geometric series with $a_0 = 1$ and $r = -x^2$.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ... = \sum_{i=0}^{\infty}(-1)^i \cdot (x^{2i})$$

Therefore,

$$\int \frac{1}{1+x^2}dx = \int [\sum_{i=0}^{\infty}(-1)^i \cdot (x^{2i})]dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{i=0}^{\infty}(-1)^i \cdot \frac{x^{2i+1}}{2i+1}$$

We have established that

$$\boxed{\arctan{(x)} = \sum_{i=0}^{\infty}(-1)^i \cdot \frac{x^{2i+1}}{2i+1}}$$

The above series is helpful because we know that $\arctan{(1)} = \frac{\pi}{4}$ by the unit circle and the range of outputs in the graph of the $\arctan{(}function$

Therefore,

$$\boxed{\frac{\pi}{4} = \arctan(1) = \sum_{i=0}^{\infty} (-1)^i \cdot \frac{1}{2i+1}}$$

$\pi = 4\arctan(()\,1)$ so we have established a Taylor series approximation to π. Earlier, we mentioned that the goal was to be able to calculate an approximation to π to n decimal places of accuracy. This Taylor series is what we call an alternating series since the sign of the terms alternate as shown above. If we consider the magnitude of each term (ie: $\frac{1}{2i+1}$), we know the following:

$$\frac{1}{2i+1} > \frac{1}{2\cdot(i+1)+1}$$

$$\frac{1}{2i+1} \geq 0 \;\; \forall i \geq 0 \in Z^+$$

From the above observations, it follows that $\lim_{i\to\infty} \frac{1}{2i+1} = 0$ and by the Alternating Series Test, the series for $\arctan(()\,1)$ converges. Now, we are able to use the Alternating Series Estimation Theorem to find the desired error in our approximation of π.

Since the magnitude of the terms are decreasing and the sign of the term alternates, the error is at most the first neglected term in our estimation, which is the $(n+1)th$ term. For our approximation of π,

$$|E| = |\pi - \sum_{i=0}^{\infty} (-1)^n \cdot \frac{1}{2i+1}| < \frac{1}{2n+3}$$

Since we want the first n decimal places of π correct, our error should be at most $10^{-(n+1)}$ since we are able to have the $(n+1)th$ decimal place value vary. Let k represent the number of terms needed to achieve an error of at most $10^{-(n+1)}$

$$\frac{1}{2k+3} < 10^{-(n+1)} \implies 10^{n+1} < 2k+3 \implies k > \frac{10^{n+1}-3}{2}$$

An equivalent way of looking at this estimation is to simply find the first term that is less than $10^{-(n+1)}$ and you will get a comparable formula for the number of terms that need to be calculated.

0.2 Improved Taylor Series

Our approximation to π using the above Taylor Series does not perform well computationally as the number of decimal places increases. It is evident that a larger amount of terms need to be summed up to get π approximated to more decimal places. More details on that will be mentioned in the Performance section. One way to improve on our computational performance while getting comparable precision could be to use a slightly different Taylor series. We want another way of expressing $arctan 1$ that gives a faster converging series. The following lemma will help us

Lemma 1. $4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \frac{\pi}{4}$

We take advantage of the following identity

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$$

Let $a = \arctan(A)$ and $b = \arctan(B)$

$$a-b = \arctan\left(\frac{A-B}{1+AB}\right) \implies \arctan(A) - \arctan(B) = \arctan\left(\frac{A-B}{1+AB}\right)$$

Likewise,

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right)$$

Observe that

$$4 \arctan\left(\frac{1}{5}\right) = 2\cdot(\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{5}\right)) = 2\cdot(\arctan\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right)) = 2\cdot(\arctan\left(\frac{5}{12}\right))$$

$$2\cdot(\arctan\left(\frac{5}{12}\right)) = \arctan\left(\frac{5}{12}\right) + \arctan\left(\frac{5}{12}\right) = \arctan\left(\frac{\frac{10}{12}}{\frac{119}{144}}\right) = \arctan\left(\frac{120}{119}\right)$$

$$4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = \arctan\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}}\right)$$

$$\arctan\left(\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}\right)=\arctan\left(\frac{\frac{28561}{28441}}{\frac{28561}{28441}}\right)=\arctan(1)=\frac{\pi}{4}$$

The above identity is known as Machin's formula. We can still leverage the Taylor series representation to approximate π.

$$\pi=4(4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right))=16\sum_{i=0}^{\infty}(-1)^i\cdot\frac{(\frac{1}{5})^{2i+1}}{2i+1}-4\sum_{i=0}^{\infty}(-1)^i\cdot\frac{(\frac{1}{239})^{2i+1}}{2i+1}$$

Condensing the series, we get that

$$\boxed{\pi=\sum_{i=0}^{\infty}(-1)^i\cdot\frac{16*(\frac{1}{5})^{2i+1}-4*(\frac{1}{239})^{2i+1}}{2i+1}}$$

Observe that the above series satisfies the criteria for being a convergent alternating series. We are able to apply the Alternating Series Estimation Theorem. As mentioned above, the desired error is $10^{-(n+1)}$ where n is the number of decimal places desired. The series above is hard to manipulate in order to solve for the number of terms, but when we graph the magnitude of the term in the alternating series, the graph can be approximated with the function $(\frac{1}{25})^i$ (comparably). Therefore,

$$(\frac{1}{25})^k<10^{-(n+1)}\implies k<\log_{\frac{1}{25}}\left(10^{-(n+1)}\right)\implies k<-(n+1)\cdot\frac{1}{-2\log_{10}(5)}$$

Performance

Our application compares the performance of the Machin Algorithm against the conventional $\arctan(x)$ Taylor Series through the help of a Matplotlib visualization. If we graph the number of terms as a function of the number of decimal places desired, we see that the conventional $\arctan(x)$ Taylor series produces an exponential curve whereas the Machin Algorithm produces a relatively linear curve. In other words, the Machin Algorithm is able to acheive comparable precision with less terms needed to be summed up. The link to our app is https://valistelea.github.io/PiCalculator/PiCalcProject.html?.

Next Steps

Overall, this project was very satisfying in terms of fostering a greater appreciation towards the power of calculus and Taylor series. Future iterations of this project might explore other algorithms used to approximate π and compare performance of those algorithms to the ones already in our application.

Works Cited

References

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