ECE 105: Classical Mechanics

Practice Final Exam Solutions Manual

Karl Keshavarzi – ECE 105, Classical Mechanics

karl.keshavarzi@uwaterloo.ca

Preface: This document contains solutions to all problems in the practice final exam. It does not contain the solutions to the challenge problems. Each solution includes derivations, explanations of key concepts, and verification where appropriate. Note that there could be multiple methods of solving the same problem; your solution does not have to match mine.

Kinematics

Solution 1: Projectile Motion with Bouncing

(a) No bounce case (3 marks):

For projectile motion with initial speed v_0 at angle θ, the initial velocity components are:

$$v_{0x} = v_0 \cos\theta, \quad v_{0y} = v_0 \sin\theta$$

The vertical position is $y = v_{0y}t - \frac{1}{2}gt^2$. At landing, $y = 0$:

$$0 = v_0 \sin\theta \cdot t - \frac{1}{2}gt^2 = t\left(v_0 \sin\theta - \frac{1}{2}gt\right)$$

The non-zero solution gives the total time of flight:

$$\boxed{T = \frac{2v_0 \sin\theta}{g}}$$

The horizontal distance (range) is:

$$R = v_{0x} \cdot T = v_0 \cos\theta \cdot \frac{2v_0 \sin\theta}{g} = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$

$$\boxed{R = \frac{v_0^2 \sin(2\theta)}{g}}$$

(b) One bounce case (3 marks):

For the first arc, launched at angle ϕ with speed v_0:

$$T_1 = \frac{2v_0 \sin\phi}{g}, \quad R_1 = \frac{v_0^2 \sin(2\phi)}{g}$$

At the bounce, the ball lands with speed v_0 (by symmetry of projectile motion). After bouncing:

- Speed becomes αv_0
- Angle remains ϕ (same angle from vertical means same angle above horizontal)

For the second arc:

$$T_2 = \frac{2(\alpha v_0) \sin\phi}{g} = \alpha T_1, \quad R_2 = \frac{(\alpha v_0)^2 \sin(2\phi)}{g} = \alpha^2 R_1$$

Total time of flight:

$$\boxed{T_{total} = T_1 + T_2 = \frac{2v_0 \sin\phi}{g}(1+\alpha)}$$

Total horizontal distance:

$$\boxed{R_{total} = R_1 + R_2 = \frac{v_0^2 \sin(2\phi)}{g}(1+\alpha^2)}$$

(c) Finding ϕ when $\alpha = 0.8$ (3 marks):

For the no-bounce case at $\theta = 45$:

$$R_{no\,bounce} = \frac{v_0^2 \sin(90)}{g} = \frac{v_0^2}{g}$$

Setting $R_{total} = R_{no\,bounce}$:

$$\frac{v_0^2 \sin(2\phi)}{g}(1+\alpha^2) = \frac{v_0^2}{g}$$

$$\sin(2\phi) = \frac{1}{1+\alpha^2} = \frac{1}{1+0.64} = \frac{1}{1.64} = 0.6098$$

$$2\phi = \arcsin(0.6098) = 37.6$$

$$\boxed{\phi = 18.8}$$

Time ratio:

$$t_{no\,bounce} = \frac{2v_0 \sin(45)}{g} = \frac{\sqrt{2}v_0}{g}$$

$$t_{bounce} = \frac{2v_0 \sin(18.8)}{g}(1+0.8) = \frac{2v_0(0.322)(1.8)}{g} = \frac{1.16v_0}{g}$$

$$\boxed{\frac{t_{bounce}}{t_{no\,bounce}} = \frac{1.16}{\sqrt{2}} = 0.82}$$

(d) Effect of air resistance (1 mark):

Air resistance would reduce the range of both trajectories. However, the bouncing ball spends less total time in the air and travels at lower average speeds (especially after the bounce), so it would experience less total drag. The no-bounce trajectory at 45° would be more affected by air resistance. Therefore, with air resistance, the bouncing player would need to throw at a **steeper angle** (larger ϕ) than calculated, and the time ratio would likely **increase**.

Solution 2: River Crossing with Moving Target

(a) Finding angle θ (3 marks):

Let the swimmer head at angle θ upstream from directly across. The velocity components of the swimmer relative to ground are:

- Across river (y-direction): $v_y = v_s \cos\theta$
- Along river (x-direction): $v_x = v_c - v_s \sin\theta$ (downstream positive)

Time to cross: $t = \frac{W}{v_s \cos\theta}$

During this time:

- Swimmer drifts downstream: $x_s = (v_c - v_s \sin\theta) \cdot t$
- Friend walks downstream: $x_f = v_f \cdot t$

For them to meet: $x_s = x_f$

$$(v_c - v_s \sin\theta) \cdot \frac{W}{v_s \cos\theta} = v_f \cdot \frac{W}{v_s \cos\theta}$$

$$v_c - v_s \sin\theta = v_f$$

$$\sin\theta = \frac{v_c - v_f}{v_s}$$

$$\boxed{\theta = \arcsin\left(\frac{v_c - v_f}{v_s}\right)}$$

(b) Time to cross (2 marks):

With $v_s = 4$ km/hr, $v_c = 2$ km/hr, $v_f = 3$ km/hr:

$$\sin\theta = \frac{2-3}{4} = -\frac{1}{4}$$

So $\theta = -14.48$ (negative means heading downstream, not upstream).

$$t = \frac{W}{v_s \cos\theta} = \frac{0.08 \text{ km}}{4 \text{ km/hr} \cdot \cos(-14.48)} = \frac{0.08}{4 \cdot 0.968} = 0.0207 \text{ hr}$$

$$\boxed{t = 74.4 \text{ s} \approx 1.24 \text{ min}}$$

(c) Distance downstream (2 marks):

$$x_f = v_f \cdot t = 3 \text{ km/hr} \times 0.0207 \text{ hr} = 0.0621 \text{ km}$$

$$\boxed{x = 62.1 \text{ m downstream}}$$

(d) Minimum crossing time (1 mark):

To minimize crossing time, the swimmer should head **directly across the river** ($\theta = 0$). This maximizes the component of velocity perpendicular to the banks ($v_y = v_s$), giving minimum time $t_{min} = W/v_s$. The downstream drift is irrelevant if we only want to minimize time to reach the opposite bank.

(e) Critical walking speed (2 marks):

From part (a): $\sin\theta = \frac{v_c - v_f}{v_s}$

For a solution to exist, we need $|\sin\theta| \leq 1$:

$$\left|\frac{v_c - v_f}{v_s}\right| \leq 1$$

$$|v_c - v_f| \leq v_s$$

If $v_f > v_c$, the swimmer must head downstream. The limit is:

$$v_f - v_c \leq v_s$$

$$v_f \leq v_s + v_c$$

The critical speed above which the swimmer cannot catch the friend is:

$$\boxed{v_{f,crit} = v_s + v_c}$$

For $v_f > v_{f,crit}$, the friend walks faster than the swimmer can possibly move downstream while still crossing the river.

Solution 3: Parametric Motion of The Cycloid

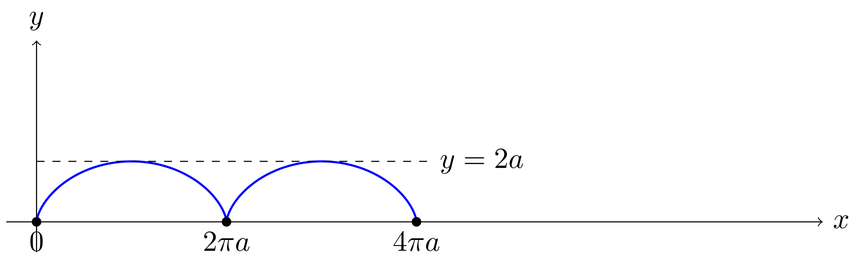
(a) Sketch of cycloid (2 marks):

The cycloid touches the ground when $y = 0$, which occurs when $\cos\theta = 1$, i.e., at $\theta = 0, 2\pi, 4\pi$.

At these points:

- $\theta = 0$: $(x, y) = (0, 0)$
- $\theta = 2\pi$: $(x, y) = (2\pi a, 0)$
- $\theta = 4\pi$: $(x, y) = (4\pi a, 0)$

The maximum height occurs when $\cos\theta = -1$, i.e., at $\theta = \pi, 3\pi$, giving $y_{max} = 2a$.



(b) Velocity components (3 marks):

Using the chain rule: $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$
$$\frac{dy}{d\theta} = a\sin\theta$$

Since $\omega = d\theta/dt$:

$$\boxed{v_x = \frac{dx}{dt} = a\omega(1 - \cos\theta)}$$
$$\boxed{v_y = \frac{dy}{dt} = a\omega\sin\theta}$$

(c) Acceleration components (3 marks):

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}[a\omega(1 - \cos\theta)] = a\omega \cdot \sin\theta \cdot \frac{d\theta}{dt} = a\omega^2 \sin\theta$$
$$\boxed{a_x = a\omega^2 \sin\theta}$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}[a\omega \sin\theta] = a\omega \cdot \cos\theta \cdot \frac{d\theta}{dt} = a\omega^2 \cos\theta$$

$$\boxed{a_y = a\omega^2 \cos\theta}$$

Magnitude of acceleration:

$$|a| = \sqrt{a_x^2 + a_y^2} = \sqrt{a^2\omega^4 \sin^2\theta + a^2\omega^4 \cos^2\theta} = a\omega^2\sqrt{\sin^2\theta + \cos^2\theta}$$

$$\boxed{|a| = a\omega^2 = \text{constant}}$$

This makes sense because point P undergoes circular motion about the instantaneous contact point.

(d) Speed analysis (2 marks):

Speed: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{a^2\omega^2(1-\cos\theta)^2 + a^2\omega^2 \sin^2\theta}$

$$v = a\omega\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = a\omega\sqrt{2(1-\cos\theta)} = a\omega\sqrt{4\sin^2(\theta/2)}$$

$$v = 2a\omega|\sin(\theta/2)|$$

Speed equals zero when $\sin(\theta/2) = 0$:

$$\boxed{\theta = 0, 2\pi, 4\pi, \ldots \text{ (when P touches the ground)}}$$

Speed is maximum when $|\sin(\theta/2)| = 1$, i.e., $\theta/2 = \pi/2, 3\pi/2, \ldots$:

$$\boxed{\theta = \pi, 3\pi, \ldots \text{ (at the top of each arc)}}$$

$$\boxed{v_{max} = 2a\omega}$$

Solution 4: Two Projectiles Meeting

(a) Height equations (2 marks):

Ball A (dropped from height H):

$$\boxed{y_A(t) = H - \frac{1}{2}gt^2}$$

Ball B (thrown upward from ground with speed v_0):

$$\boxed{y_B(t) = v_0 t - \frac{1}{2}gt^2}$$

(b) Collision time (2 marks):

At collision, $y_A = y_B$:

$$H - \frac{1}{2}gt^2 = v_0 t - \frac{1}{2}gt^2$$
$$H = v_0 t$$
$$\boxed{t_c = \frac{H}{v_0}}$$

Note: The $\frac{1}{2}gt^2$ terms cancel! This elegant result shows that the collision time is independent of g.

(c) Collision height (3 marks):

Substituting t_c into either equation (using Ball A):

$$y_c = H - \frac{1}{2}g\left(\frac{H}{v_0}\right)^2 = H - \frac{gH^2}{2v_0^2}$$
$$\boxed{y_c = H\left(1 - \frac{gH}{2v_0^2}\right)}$$

Or equivalently: $\boxed{y_c = H - \frac{gH^2}{2v_0^2}}$

(d) Minimum initial speed (3 marks):

For the collision to occur before Ball A hits the ground, we need $y_c > 0$:

$$H - \frac{gH^2}{2v_0^2} > 0$$
$$1 > \frac{gH}{2v_0^2}$$
$$v_0^2 > \frac{gH}{2}$$

$$\boxed{v_{0,min} = \sqrt{\frac{gH}{2}}}$$

Alternative interpretation: We also need the collision to occur before Ball A hits the ground, meaning $t_c < t_{ground}$ where $t_{ground} = \sqrt{2H/g}$ is when Ball A reaches the ground.

$$\frac{H}{v_0} < \sqrt{\frac{2H}{g}}$$

$$\frac{H^2}{v_0^2} < \frac{2H}{g}$$

$$v_0^2 > \frac{gH}{2}$$

This confirms: $\boxed{v_{0,min} = \sqrt{\frac{gH}{2}}}$

Solution 5: Projectile Launched Up an Incline

(a) Acceleration components in incline coordinates (3 marks):

With x-axis along the incline (positive up the slope) and y-axis perpendicular to it:

Gravity acts vertically downward with magnitude g. Decomposing into incline coordinates:

$$\boxed{a_x = -g\sin\alpha}$$

$$\boxed{a_y = -g\cos\alpha}$$

Both components are negative since gravity has components both down the slope and into the slope.

(b) Range along incline (3 marks):

Initial velocity components in incline coordinates:

$$v_{0x} = v_0\cos\beta, \quad v_{0y} = v_0\sin\beta$$

Position equations:

$$x(t) = v_0\cos\beta\cdot t - \frac{1}{2}g\sin\alpha\cdot t^2$$
$$y(t) = v_0\sin\beta\cdot t - \frac{1}{2}g\cos\alpha\cdot t^2$$

The projectile lands when $y = 0$ (returns to the incline):

$$0 = t\left(v_0\sin\beta - \frac{1}{2}g\cos\alpha\cdot t\right)$$

$$t_{land} = \frac{2v_0\sin\beta}{g\cos\alpha}$$

Range along incline:

$$R = x(t_{land}) = v_0\cos\beta\cdot\frac{2v_0\sin\beta}{g\cos\alpha} - \frac{1}{2}g\sin\alpha\cdot\left(\frac{2v_0\sin\beta}{g\cos\alpha}\right)^2$$

$$R = \frac{2v_0^2\sin\beta\cos\beta}{g\cos\alpha} - \frac{2v_0^2\sin^2\beta\sin\alpha}{g\cos^2\alpha}$$

$$R = \frac{2v_0^2\sin\beta}{g\cos^2\alpha}\left(\cos\beta\cos\alpha - \sin\beta\sin\alpha\right)$$

$$\boxed{R = \frac{2v_0^2\sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha}}$$

(c) Angle for maximum range (2 marks):

To maximize R, we maximize $\sin\beta\cos(\alpha+\beta)$.

Using the product-to-sum identity:

$$\begin{aligned}\sin\beta\cos(\alpha+\beta) &= \frac{1}{2}[\sin(\beta+\alpha+\beta)+\sin(\beta-\alpha-\beta)] \\ &= \frac{1}{2}[\sin(2\beta+\alpha)-\sin\alpha]\end{aligned}$$

This is maximized when $\sin(2\beta+\alpha) = 1$, i.e., $2\beta+\alpha = 90$:

$$\boxed{\beta = \frac{90-\alpha}{2} = 45 - \frac{\alpha}{2}}$$

(d) Numerical calculation for $\alpha = 30$, $v_0 = 20$ m/s (2 marks):

Optimal angle: $\beta = 45 - 15 = 30$

Maximum range:

$$R_{max} = \frac{2(20)^2\sin 30\cos 60}{9.8\cos^2 30} = \frac{2(400)(0.5)(0.5)}{9.8(0.75)} = \frac{200}{7.35} = 27.2 \text{ m}$$

$$\boxed{R_{max} = 27.2 \text{ m}}$$

Time of flight:

$$t = \frac{2v_0\sin\beta}{g\cos\alpha} = \frac{2(20)\sin 30}{9.8\cos 30} = \frac{20}{8.49} = 2.36 \text{ s}$$

$$\boxed{t = 2.36 \text{ s}}$$

Solution 6: Object Falling Past a Window

(a) Velocity at top of window (3 marks):

Let v_{top} be the velocity at the top of the window. The object accelerates through the window:

$$H = v_{top}\tau + \frac{1}{2}g\tau^2$$

Solving for v_{top}:

$$\boxed{v_{top} = \frac{H}{\tau} - \frac{1}{2}g\tau}$$

(b) Height above window (3 marks):

The object was dropped from rest, so using $v^2 = 2gh$:

$$v_{top}^2 = 2gh$$

$$h = \frac{v_{top}^2}{2g} = \frac{1}{2g}\left(\frac{H}{\tau} - \frac{1}{2}g\tau\right)^2$$

$$\boxed{h = \frac{1}{2g}\left(\frac{H}{\tau} - \frac{g\tau}{2}\right)^2}$$

(c) Velocity at bottom of window (2 marks):

$$v_{bottom} = v_{top} + g\tau = \frac{H}{\tau} - \frac{1}{2}g\tau + g\tau$$

$$\boxed{v_{bottom} = \frac{H}{\tau} + \frac{1}{2}g\tau}$$

(d) Numerical values for $H = 2.0$ m, $\tau = 0.40$ s (2 marks):

Height above window:

$$v_{top} = \frac{2.0}{0.40} - \frac{1}{2}(9.8)(0.40) = 5.0 - 1.96 = 3.04 \text{ m/s}$$

$$h = \frac{(3.04)^2}{2(9.8)} = \frac{9.24}{19.6} = 0.472 \text{ m}$$

$$\boxed{h = 0.47 \text{ m}}$$

Velocity at bottom:

$$v_{bottom} = \frac{2.0}{0.40} + \frac{1}{2}(9.8)(0.40) = 5.0 + 1.96 = 6.96 \text{ m/s}$$

$$\boxed{v_{bottom} = 6.96 \text{ m/s}}$$

Forces and Newton's Laws

Solution 7: Two Blocks with Pulley and Friction

(a) Free body diagrams (3 marks):

Block M_1 on ramp:

- Weight $M_1 g$ (vertically down)
- Normal force N (perpendicular to ramp)
- Tension T (up the ramp)
- Kinetic friction f_k (down the ramp, opposing motion)

Block M_2 (hanging):

- Weight $M_2 g$ (down)
- Tension T (up)

(b) Kinetic friction force (1 mark):

The normal force on M_1: $N = M_1 g \cos\theta$

$$\boxed{f_k = \mu_k N = \mu_k M_1 g \cos\theta}$$

(c) System acceleration (2 marks):

For M_1 (up the ramp positive):

$$T - M_1 g \sin\theta - \mu_k M_1 g \cos\theta = M_1 a$$

For M_2 (downward positive):

$$M_2 g - T = M_2 a$$

Adding the equations:

$$M_2 g - M_1 g \sin\theta - \mu_k M_1 g \cos\theta = (M_1 + M_2) a$$

$$\boxed{a = \frac{g(M_2 - M_1 \sin\theta - \mu_k M_1 \cos\theta)}{M_1 + M_2}}$$

(d) Tension in rope (2 marks):

From the equation for M_2:

$$T = M_2 g - M_2 a = M_2(g - a)$$

$$T = M_2 g \left(1 - \frac{M_2 - M_1 \sin\theta - \mu_k M_1 \cos\theta}{M_1 + M_2}\right)$$

$$T = M_2 g \left(\frac{M_1 + M_2 - M_2 + M_1 \sin\theta + \mu_k M_1 \cos\theta}{M_1 + M_2}\right)$$

$$\boxed{T = \frac{M_1 M_2 g(1 + \sin\theta + \mu_k \cos\theta)}{M_1 + M_2}}$$

(e) Time to travel distance $L/2$ (2 marks):

Starting from rest with constant acceleration:

$$\frac{L}{2} = \frac{1}{2}at^2$$

$$t^2 = \frac{L}{a}$$

$$\boxed{t = \sqrt{\frac{L}{a}}}$$

Solution 8: Stacked Blocks with Friction

(a) Free body diagrams with force on lower block (2 marks):

Upper block m_1:

- Weight $m_1 g$ (down)
- Normal force N_1 from m_2 (up)
- Static friction f_s from m_2 (forward, in direction of motion)

Lower block m_2:

- Weight $m_2 g$ (down)
- Normal force N_{floor} from floor (up)
- Normal force N_1 from m_1 (down) - Newton's 3rd law pair
- Static friction f_s from m_1 (backward) - Newton's 3rd law pair
- Applied force F (forward)

(b) Common acceleration (2 marks):

If the blocks move together, treat them as one system:

$$F = (m_1 + m_2)a$$

$$\boxed{a = \frac{F}{m_1 + m_2}}$$

(c) Maximum force before slipping (3 marks):

The top block accelerates only due to friction:

$$f_s = m_1 a$$

Maximum static friction: $f_{s,max} = \mu_s N_1 = \mu_s m_1 g$

The top block slips when $f_s > f_{s,max}$:

$$m_1 a_{max} = \mu_s m_1 g$$

$$\boxed{a_{max} = \mu_s g}$$

Maximum applied force:

$$F_{max} = (m_1 + m_2)a_{max}$$

$$\boxed{F_{max} = \mu_s g(m_1 + m_2)}$$

(d) Force applied to upper block (3 marks):

Now friction on m_1 acts backward (opposing relative motion), and friction on m_2 acts forward.

For m_1: $F' - f_s = m_1 a$

For m_2: $f_s = m_2 a$

From m_2: $a = f_s/m_2$

The maximum friction is still $f_{s,max} = \mu_s m_1 g$

At maximum: $a_{max} = \frac{\mu_s m_1 g}{m_2}$

Maximum force:

$$F'_{max} = m_1 a_{max} + f_{s,max} = m_1 \cdot \frac{\mu_s m_1 g}{m_2} + \mu_s m_1 g = \mu_s m_1 g \left(\frac{m_1}{m_2} + 1\right)$$

$$\boxed{F'_{max} = \mu_s m_1 g \left(\frac{m_1 + m_2}{m_2}\right)}$$

Comparison:

$$\frac{F'_{max}}{F_{max}} = \frac{\mu_s m_1 g(m_1 + m_2)/m_2}{\mu_s g(m_1 + m_2)} = \frac{m_1}{m_2}$$

If $m_1 < m_2$, then $F'_{max} < F_{max}$: applying force to the lighter top block allows less total force before slipping.

Solution 9: Three Connected Blocks

(a) Free body diagrams (2 marks):

Block m_1 (rightmost):

- Applied force F (right)
- Tension T_{12} from string to m_2 (left)
- Weight $m_1 g$ (down), Normal N_1 (up)

Block m_2 (middle):

- Tension T_{12} from m_1 (right)
- Tension T_{23} from m_3 (left)
- Weight $m_2 g$ (down), Normal N_2 (up)

Block m_3 (leftmost):

- Tension T_{23} from m_2 (right)
- Weight $m_3 g$ (down), Normal N_3 (up)

(b) System acceleration (2 marks):

Treating all three blocks as one system (frictionless surface):

$$F = (m_1 + m_2 + m_3)a$$

$$\boxed{a = \frac{F}{m_1 + m_2 + m_3}}$$

(c) Tensions (3 marks):

For m_1: $F - T_{12} = m_1 a$

$$T_{12} = F - m_1 a = F - \frac{m_1 F}{m_1 + m_2 + m_3} = F\left(\frac{m_2 + m_3}{m_1 + m_2 + m_3}\right)$$

$$\boxed{T_{12} = \frac{F(m_2 + m_3)}{m_1 + m_2 + m_3}}$$

For m_3: $T_{23} = m_3 a$

$$\boxed{T_{23} = \frac{F m_3}{m_1 + m_2 + m_3}}$$

(d) Maximum force (3 marks):

The string between m_2 and m_3 breaks when $T_{23} = T_{max}$:

$$T_{max} = \frac{F_{max} \cdot m_3}{m_1 + m_2 + m_3}$$

$$\boxed{F_{max} = \frac{T_{max}(m_1 + m_2 + m_3)}{m_3}}$$

Solution 10: Double Pulley System on Incline

(a) Free body diagrams (2 marks):

Block m_1 on incline:

- Weight component down incline: $m_1 g \sin\theta$
- Weight component into incline: $m_1 g \cos\theta$
- Normal force N (perpendicular to incline)
- Tension T (up the incline)
- Kinetic friction $f_k = \mu_k m_1 g \cos\theta$ (down the incline)

Mass m_2 (hanging from movable pulley):

- Weight $m_2 g$ (down)
- Two rope segments each with tension T (up): total upward force $2T$

(b) Acceleration constraint (3 marks):

In a double pulley (block and tackle) system, when m_1 moves distance d up the incline, the rope on the m_1 side shortens by d. This rope passes around the movable pulley, so the pulley (and m_2) descends by $d/2$.

Therefore: $\boxed{a_2 = \frac{a_1}{2}}$ or equivalently $\boxed{a_1 = 2a_2}$

(c) Acceleration derivation (3 marks):

For m_1 (up incline positive):

$$T - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = m_1 a_1$$

For m_2 (downward positive):

$$m_2 g - 2T = m_2 a_2 = m_2 \cdot \frac{a_1}{2}$$

From the second equation:

$$T = \frac{m_2 g - \frac{m_2 a_1}{2}}{2} = \frac{m_2 g}{2} - \frac{m_2 a_1}{4}$$

Substituting into the first equation:

$$\frac{m_2 g}{2} - \frac{m_2 a_1}{4} - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = m_1 a_1$$

$$\frac{m_2 g}{2} - m_1 g(\sin\theta + \mu_k \cos\theta) = a_1 \left(m_1 + \frac{m_2}{4}\right)$$

$$\boxed{a_1 = \frac{g\left(\frac{m_2}{2} - m_1(\sin\theta + \mu_k\cos\theta)\right)}{m_1 + \frac{m_2}{4}} = \frac{g(2m_2 - 4m_1(\sin\theta + \mu_k\cos\theta))}{4m_1 + m_2}}$$

(d) Tension (2 marks):

$$T = \frac{m_2 g}{2} - \frac{m_2 a_1}{4} = \frac{m_2}{4}(2g - a_1)$$

Substituting a_1:

$$\boxed{T = \frac{m_1 m_2 g(2 + \sin\theta + \mu_k\cos\theta)}{4m_1 + m_2}}$$

Solution 11: Rope Pulling a Sled

(a) System acceleration (2 marks):

The total mass being accelerated is $(m + M)$. Applying Newton's second law to the entire system:

$$F = (m + M)a$$

$$\boxed{a = \frac{F}{m + M}}$$

(b) Tension as function of position (3 marks):

Consider the portion of the rope from the sled ($x = 0$) to position x along the rope, plus the sled itself.

Mass of rope segment from 0 to x: $\frac{m}{L} \cdot x$

Total mass of sled plus rope segment: $M + \frac{mx}{L}$

The tension $T(x)$ at position x must accelerate this mass:

$$T(x) = \left(M + \frac{mx}{L}\right) a = \left(M + \frac{mx}{L}\right) \frac{F}{m + M}$$

$$\boxed{T(x) = \frac{F(M + mx/L)}{m + M} = \frac{F(ML + mx)}{L(m + M)}}$$

(c) Verification at boundaries (3 marks):

At $x = 0$ (at the sled):

$$T(0) = \frac{F \cdot M}{m + M}$$

This is correct: the tension at the sled must accelerate only the sled mass M at acceleration $a = F/(m + M)$, giving $T = Ma = FM/(m + M)$. ✓

At $x = L$ (at the free end):

$$T(L) = \frac{F(M + m)}{m + M} = F$$

This is correct: the tension at the free end equals the applied force F. ✓

(d) Position where tension equals $F/2$ (2 marks):

$$\frac{F}{2} = \frac{F(ML + mx)}{L(m + M)}$$

$$\frac{L(m + M)}{2} = ML + mx$$

$$\frac{Lm + LM}{2} = ML + mx$$
$$\frac{Lm - LM}{2} = mx$$
$$x = \frac{L(m - M)}{2m}$$

$$\boxed{x = \frac{L(m - M)}{2m}}$$

Note: If $M > m$, then $x < 0$, meaning the tension never drops to $F/2$ along the rope. If $M = m$, then $x = 0$ (at the sled). If $M < m$, then $0 < x < L/2$.

Solution 12: Spring-Coupled Blocks with Friction

(a) Free body diagrams (2 marks):

Block m_1 on surface:

- Weight $m_1 g$ (down)
- Normal force $N = m_1 g$ (up)
- Spring force kx (toward wall, opposing displacement)
- Tension T (away from wall, toward pulley)
- Kinetic friction $f_k = \mu_k m_1 g$ (toward wall, opposing motion)

Block m_2 (hanging):

- Weight $m_2 g$ (down)
- Tension T (up)

(b) Equation of motion (5 marks):

For m_1 (positive direction away from wall):

$$T - kx - \mu_k m_1 g = m_1 a = m_1 \ddot{x}$$

For m_2 (positive direction downward):

$$m_2 g - T = m_2 a = m_2 \ddot{x}$$

Since the rope is inextensible, both blocks have the same acceleration magnitude.

Adding the two equations:

$$m_2 g - kx - \mu_k m_1 g = (m_1 + m_2)\ddot{x}$$

Rearranging:

$$\boxed{(m_1 + m_2)\ddot{x} + kx = m_2 g - \mu_k m_1 g = g(m_2 - \mu_k m_1)}$$

This is a harmonic oscillator equation with shifted equilibrium. It can be rewritten as:

$$\ddot{x} + \frac{k}{m_1 + m_2} x = \frac{g(m_2 - \mu_k m_1)}{m_1 + m_2}$$

(c) Condition for system to remain at rest (3 marks):

For the system to remain at rest with the spring at natural length ($x = 0$):

For m_2: $T = m_2 g$

For m_1: The tension must be balanced by static friction (no spring force initially):

$$T \leq \mu_s m_1 g$$

$$m_2 g \leq \mu_s m_1 g$$

For the system to be on the verge of moving (static friction at maximum with kinetic friction coefficient):

$$\boxed{\mu_k = \frac{m_2}{m_1}}$$

More precisely: The system remains at rest if $\mu_s \geq m_2/m_1$. The critical kinetic friction coefficient (if the system were moving and came to rest) would be $\mu_k = m_2/m_1$.

Circular Motion

Solution 13: Circular Motion on a Banked Track

(a) Free body diagram (2 marks):

Forces on car:

- Weight mg (vertically down)
- Normal force N (perpendicular to banked surface)
- Friction force f (along the banked surface, direction depends on speed)

Choose coordinates: x pointing toward center of circle (horizontal), y pointing vertically up.

(b) Ideal speed (no friction) (2 marks):

With no friction, only N and mg act:

Vertical: $N\cos\theta = mg$

Horizontal (centripetal): $N\sin\theta = \frac{mv^2}{R}$

Dividing:

$$\tan\theta = \frac{v^2}{gR}$$

$$\boxed{v_{ideal} = \sqrt{gR\tan\theta}}$$

(c) Minimum speed (friction prevents sliding down) (3 marks):

At speeds below v_{ideal}, friction points up the slope to prevent the car from sliding down.

Vertical: $N\cos\theta - mg + f\sin\theta = 0$, so $N\cos\theta + f\sin\theta = mg$

Horizontal: $N\sin\theta - f\cos\theta = \frac{mv^2}{R}$

At minimum speed, $f = \mu_s N$:

$N\cos\theta + \mu_s N\sin\theta = mg \Rightarrow N = \frac{mg}{\cos\theta + \mu_s\sin\theta}$

$N\sin\theta - \mu_s N\cos\theta = \frac{mv_{min}^2}{R}$

$N(\sin\theta - \mu_s\cos\theta) = \frac{mv_{min}^2}{R}$

$$v_{min}^2 = \frac{gR(\sin\theta - \mu_s\cos\theta)}{\cos\theta + \mu_s\sin\theta} = gR\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

$$\boxed{v_{min} = \sqrt{gR\left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right)}}$$

(d) Maximum speed (friction prevents sliding up) (2 marks):

At high speeds, friction points down the slope:

$N\cos\theta - \mu_s N\sin\theta = mg$

$N\sin\theta + \mu_s N\cos\theta = \frac{mv_{max}^2}{R}$

$$\boxed{v_{max} = \sqrt{gR\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)}}$$

(e) Condition for car to remain stationary (1 mark):

For $v = 0$, the car must not slide down. This requires friction to balance the downhill component of gravity:

$\mu_s N\cos\theta \geq N\sin\theta$ (friction up slope balances gravity component)

$$\boxed{\mu_s \geq \tan\theta}$$

Solution 14: The Conical Pendulum

(a) Free body diagram (2 marks):

Forces on mass m:

- Tension T along the string, at angle θ from vertical
- Weight mg vertically downward

The acceleration is horizontal, directed toward the center of the circular path (centripetal acceleration).

(b) Newton's second law equations (3 marks):

Vertical direction (equilibrium):

$$\boxed{T\cos\theta = mg}$$

Horizontal/radial direction (centripetal):

$$\boxed{T\sin\theta = \frac{mv^2}{r} = \frac{mv^2}{L\sin\theta}}$$

From the first equation: $T = \frac{mg}{\cos\theta}$

Substituting into the second:

$$\frac{mg\sin\theta}{\cos\theta} = \frac{mv^2}{L\sin\theta}$$

$$g\tan\theta = \frac{v^2}{L\sin\theta}$$

(c) Speed expression (2 marks):

$$v^2 = gL\sin\theta\tan\theta = gL\frac{\sin^2\theta}{\cos\theta}$$

$$\boxed{v = \sqrt{gL\sin\theta\tan\theta} = \sin\theta\sqrt{\frac{gL}{\cos\theta}}}$$

(d) Period of revolution (3 marks):

The circumference of the circle is $2\pi r = 2\pi L\sin\theta$.

Period:

$$T = \frac{2\pi r}{v} = \frac{2\pi L\sin\theta}{\sin\theta\sqrt{gL/\cos\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L\cos\theta}{g}}}$$

Limiting cases:

As $\theta \to 0$: $\cos\theta \to 1$, so $T \to 2\pi\sqrt{L/g}$

This is the period of a simple pendulum of length L — makes sense since for small angles, the conical pendulum approaches a simple pendulum.

As $\theta \to 90$: $\cos\theta \to 0$, so $T \to 0$

The period approaches zero, meaning the mass must move infinitely fast to maintain a horizontal string. This is physically impossible (would require infinite centripetal force).

Solution 15: Block Sliding on a Hemisphere

(a) Angle where block loses contact (4 marks):

Let ϕ be the angle from the vertical. At angle ϕ:

Height descended: $h = R - R\cos\phi = R(1 - \cos\phi)$

Conservation of energy (starting from rest at top):

$$\frac{1}{2}mv^2 = mgR(1 - \cos\phi)$$

$$v^2 = 2gR(1 - \cos\phi)$$

Radial direction (toward center):

$$mg\cos\phi - N = \frac{mv^2}{R}$$

The block loses contact when $N = 0$:

$$mg\cos\phi = \frac{mv^2}{R} = \frac{m \cdot 2gR(1 - \cos\phi)}{R}$$

$$\cos\phi = 2(1 - \cos\phi) = 2 - 2\cos\phi$$

$$3\cos\phi = 2$$

$$\boxed{\phi = \arccos\left(\frac{2}{3}\right) \approx 48.2}$$

Height above ground where block leaves:

$$h_{leave} = R\cos\phi = \frac{2R}{3}$$

$$\boxed{h = \frac{2R}{3}}$$

(b) Speed when leaving hemisphere (2 marks):

$$v^2 = 2gR(1 - \cos\phi) = 2gR\left(1 - \frac{2}{3}\right) = \frac{2gR}{3}$$

$$\boxed{v = \sqrt{\frac{2gR}{3}}}$$

(c) Initial speed for immediate loss of contact (4 marks):

At the top ($\phi = 0$), if the block has horizontal speed v_0:

The centripetal acceleration is v_0^2/R (directed downward at the top).

Radial equation at top:

$$mg - N = \frac{mv_0^2}{R}$$

For immediate loss of contact, $N = 0$:

$$mg = \frac{mv_0^2}{R}$$

$$\boxed{v_0 = \sqrt{gR}}$$

Solution 16: Ball in Vertical Circle with Energy Loss

(a) Tension at bottom (2 marks):

At the bottom, centripetal acceleration is directed upward (toward center). Newton's second law in radial direction:

$$T_b - mg = \frac{mv_b^2}{R}$$

$$\boxed{T_b = mg + \frac{mv_b^2}{R} = m\left(g + \frac{v_b^2}{R}\right)}$$

(b) Tension at top (2 marks):

At the top, both tension and weight point toward the center (downward):

$$T_t + mg = \frac{mv_t^2}{R}$$

$$\boxed{T_t = \frac{mv_t^2}{R} - mg = m\left(\frac{v_t^2}{R} - g\right)}$$

(c) Minimum speed conditions (3 marks):

At minimum speed at top, $T_t = 0$:

$$0 = \frac{mv_t^2}{R} - mg$$

$$v_{t,min} = \sqrt{gR}$$

For a frictionless case, energy conservation from bottom to top:

$$\frac{1}{2}mv_b^2 = \frac{1}{2}mv_t^2 + mg(2R)$$

With $v_t = \sqrt{gR}$:

$$\frac{1}{2}v_b^2 = \frac{1}{2}gR + 2gR = \frac{5gR}{2}$$

$$\boxed{v_{b,min} = \sqrt{5gR}}$$

Comparing tensions at minimum speed:

$$T_t = 0$$

$$T_b = m\left(g + \frac{5gR}{R}\right) = 6mg$$

$$\boxed{T_b = 6mg, \quad T_t = 0 \quad \Rightarrow \quad T_b - T_t = 6mg}$$

(d) Speed after one revolution with energy loss (3 marks):

Energy at start: $E_0 = \frac{1}{2}mv_0^2$

Energy after one revolution: $E_1 = E_0 - \Delta E = \frac{1}{2}mv_0^2 - \Delta E$

If $E_1 = \frac{1}{2}mv_1^2$:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2 - \Delta E$$

$$\boxed{v_1 = \sqrt{v_0^2 - \frac{2\Delta E}{m}}}$$

The ball fails to complete the loop if it can't reach the top with minimum speed. The minimum energy needed at the bottom to complete the loop is:

$$E_{min} = \frac{1}{2}m(5gR) = \frac{5mgR}{2}$$

The ball fails when the energy after reaching the bottom again is less than E_{min}. More directly, if energy loss during one half-cycle (bottom to top) prevents reaching minimum speed at top:

Energy at bottom: $\frac{1}{2}mv_0^2$

Energy needed at top: $\frac{1}{2}m(gR) + mg(2R) = \frac{5mgR}{2}$

If $\Delta E/2$ is lost going up (assuming uniform loss):

$$\frac{1}{2}mv_0^2 - \frac{\Delta E}{2} < \frac{5mgR}{2}$$

Critical energy loss (over full cycle) for failure:

$$\boxed{\Delta E_{max} = m\left(v_0^2 - 5gR\right)}$$

Work, Energy, and Momentum

Solution 17: Conservative Forces and Potential Energy

(a) Potential energy function (2 marks):

For a conservative force: $F(x) = -\frac{dV}{dx}$

$$V(x) = -\int F(x)\,dx = -\int (2x - 4x^3)\,dx = -x^2 + x^4 + C$$

With $V(0) = 0$: $C = 0$

$$\boxed{V(x) = x^4 - x^2}$$

(b) Sketch and equilibrium analysis (2 marks):

Equilibrium points occur where $F(x) = 0$:

$$2x - 4x^3 = 2x(1 - 2x^2) = 0$$

$$x = 0, \quad x = \pm\frac{1}{\sqrt{2}} = \pm 0.707$$

Classification (using second derivative of V):

$$\frac{d^2V}{dx^2} = 4x^2 \cdot 3 - 2 = 12x^2 - 2$$

At $x = 0$: $\frac{d^2V}{dx^2} = -2 < 0 \Rightarrow$ **unstable** (local maximum of V)

At $x = \pm\frac{1}{\sqrt{2}}$: $\frac{d^2V}{dx^2} = 12(0.5) - 2 = 4 > 0 \Rightarrow$ **stable** (local minima of V)

$V(0) = 0$ (local max), $V(\pm 1/\sqrt{2}) = (1/4) - (1/2) = -1/4$ (local min)

(c) Motion with $E > 0$ starting near $x = 0$ (2 marks):

With $E > 0$, the particle has energy above the local maximum at $x = 0$. Since $V(x) \to +\infty$ as $x \to \pm\infty$, there exist turning points where $E = V(x)$.

For $E > 0$, solving $x^4 - x^2 = E$ gives turning points at large $|x|$.

The particle will oscillate between these outer turning points, passing through all three equilibrium points. The motion is **bounded oscillation** which spans all potential energy wells.

(d) Kinetic energy at $x = \pm 1$ when $E = 0$ (2 marks):

$$K = E - V = 0 - V(\pm 1) = 0 - (1 - 1) = 0$$

$V(1) = 1^4 - 1^2 = 1 - 1 = 0$

$$\boxed{K = E - V(1) = 0 - 0 = 0}$$

At $x = \pm 1$, these are turning points when $E = 0$.

(e) Motion near $x = -1$ with $-1/4 < E < 0$ (2 marks):

With E between the minimum ($-1/4$) and the local maximum (0), the particle is trapped in one of the potential wells.

Starting near $x = -1$ (which is near $x = -1/\sqrt{2} \approx -0.707$), the particle will undergo **bounded oscillation** in the left potential well, oscillating between two turning points on either side of $x = -1/\sqrt{2}$. It cannot escape to the right well because it doesn't have enough energy to cross the barrier at $x = 0$.

Solution 18: Loop-the-Loop

(a) Free body diagram and equation at top (2 marks):

At the top of the loop, forces on the block:

- Normal force N (pointing downward, toward center)
- Weight mg (pointing downward)

Both forces point toward the center. Newton's second law (radial):

$$\boxed{N + mg = \frac{mv^2}{R}}$$

(b) Minimum speed at top (2 marks):

The block barely maintains contact when $N = 0$:

$$mg = \frac{mv_{min}^2}{R}$$

$$\boxed{v_{min} = \sqrt{gR}}$$

(c) Minimum release height (3 marks):

Using energy conservation from release height h to top of loop (height $2R$):

$$mgh = \frac{1}{2}mv_{top}^2 + mg(2R)$$

With $v_{top} = v_{min} = \sqrt{gR}$:

$$gh_{min} = \frac{1}{2}gR + 2gR = \frac{5gR}{2}$$

$$\boxed{h_{min} = \frac{5R}{2}}$$

(d) Speeds and normal forces for $h = 3R$ (3 marks):

Speed at bottom (using energy from $h = 3R$ to bottom):

$$mg(3R) = \frac{1}{2}mv_b^2$$

$$v_b = \sqrt{6gR}$$

$$\boxed{v_{bottom} = \sqrt{6gR}}$$

Speed at top:

$$mg(3R) = \frac{1}{2}mv_t^2 + mg(2R)$$

$$gR = \frac{1}{2}v_t^2$$

$$v_t = \sqrt{2gR}$$

Normal force at top:

$$N_t + mg = \frac{mv_t^2}{R} = \frac{m(2gR)}{R} = 2mg$$

$$\boxed{N_{top} = mg}$$

Normal force at bottom (normal force points up, weight points down):

$$N_b - mg = \frac{mv_b^2}{R} = \frac{m(6gR)}{R} = 6mg$$

$$\boxed{N_{bottom} = 7mg}$$

Solution 19: Two-Dimensional Elastic Collision

(a) Conservation of momentum equations (2 marks):

x-component:

$$\boxed{m_1 v_0 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2}$$

y-component:

$$\boxed{0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2}$$

(b) Conservation of kinetic energy (1 mark):

$$\boxed{\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2}$$

(c) Proof that particles move at right angles when $m_1 = m_2$ (3 marks):

For $m_1 = m_2 = m$:

Momentum: $\vec{p}_0 = \vec{p}_1 + \vec{p}_2$, so $m\vec{v}_0 = m\vec{v}_1 + m\vec{v}_2$

$$\vec{v}_0 = \vec{v}_1 + \vec{v}_2$$

Energy: $v_0^2 = v_1^2 + v_2^2$

Squaring the momentum equation:

$$|\vec{v}_0|^2 = |\vec{v}_1 + \vec{v}_2|^2 = v_1^2 + v_2^2 + 2\vec{v}_1 \cdot \vec{v}_2$$

$$v_0^2 = v_1^2 + v_2^2 + 2v_1 v_2 \cos(\theta_1 + \theta_2)$$

Comparing with energy equation:

$$v_1^2 + v_2^2 = v_1^2 + v_2^2 + 2v_1 v_2 \cos(\theta_1 + \theta_2)$$

$$0 = 2v_1 v_2 \cos(\theta_1 + \theta_2)$$

Since $v_1, v_2 \neq 0$ (in general):

$$\cos(\theta_1 + \theta_2) = 0$$

$$\boxed{\theta_1 + \theta_2 = 90}$$

(d) Specific case: $m_1 = m_2$, $\theta_1 = 30$ (4 marks):

From part (c): $\theta_2 = 90 - 30 = 60$

$$\boxed{\theta_2 = 60}$$

From y-momentum: $v_1 \sin 30 = v_2 \sin 60$

$$v_1 \cdot \frac{1}{2} = v_2 \cdot \frac{\sqrt{3}}{2}$$

$$v_1 = \sqrt{3} v_2$$

From x-momentum: $v_0 = v_1 \cos 30 + v_2 \cos 60$

$$v_0 = \sqrt{3} v_2 \cdot \frac{\sqrt{3}}{2} + v_2 \cdot \frac{1}{2} = \frac{3v_2}{2} + \frac{v_2}{2} = 2v_2$$

$$v_2 = \frac{v_0}{2}, \quad v_1 = \frac{\sqrt{3} v_0}{2}$$

$$\boxed{v_1 = \frac{\sqrt{3}}{2} v_0 \approx 0.866 v_0, \quad v_2 = \frac{v_0}{2}}$$

Verification of energy conservation:

$$v_1^2 + v_2^2 = \frac{3v_0^2}{4} + \frac{v_0^2}{4} = v_0^2 \quad \checkmark$$

Solution 20: The Ballistic Pendulum

(a) Why momentum is conserved but KE is not (2 marks):

During the collision:

- **Momentum is conserved** because there are no external horizontal forces during the brief collision. The string tension is vertical, and gravity is negligible during the very short collision time.
- **Kinetic energy is NOT conserved** because the collision is inelastic, the bullet embeds in the block. Energy is dissipated as heat, sound, and deformation of the materials.

(b) Velocity after collision (2 marks):

Conservation of momentum (horizontal):

$$mv_0 = (m + M)V$$

$$\boxed{V = \frac{mv_0}{m + M}}$$

(c) Initial bullet speed in terms of swing angle (2 marks):

After the collision, the block-bullet system swings up. Using energy conservation:

$$\frac{1}{2}(m + M)V^2 = (m + M)gh$$

where $h = L(1 - \cos\theta_{max})$ is the height risen.

$$\frac{1}{2}V^2 = gL(1 - \cos\theta_{max})$$

$$V = \sqrt{2gL(1 - \cos\theta_{max})}$$

Substituting $V = mv_0/(m + M)$:

$$\frac{mv_0}{m + M} = \sqrt{2gL(1 - \cos\theta_{max})}$$

$$\boxed{v_0 = \frac{m + M}{m}\sqrt{2gL(1 - \cos\theta_{max})}}$$

(d) Fraction of KE lost (2 marks):

Initial KE: $K_i = \frac{1}{2}mv_0^2$

Final KE: $K_f = \frac{1}{2}(m + M)V^2 = \frac{1}{2}(m + M)\left(\frac{mv_0}{m+M}\right)^2 = \frac{m^2v_0^2}{2(m+M)}$

Fraction lost:
$$\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{m^2 v_0^2/[2(m+M)]}{\frac{1}{2}mv_0^2} = 1 - \frac{m}{m+M}$$
$$\boxed{\text{Fraction lost} = \frac{M}{m+M}}$$

(e) Numerical calculation (2 marks):

Given: $m = 0.010$ kg, $M = 2.0$ kg, $L = 1.5$ m, $\theta_{max} = 20$

$$h = L(1 - \cos 20) = 1.5(1 - 0.940) = 1.5(0.060) = 0.090 \text{ m}$$

$$V = \sqrt{2(9.8)(0.090)} = \sqrt{1.764} = 1.33 \text{ m/s}$$

$$v_0 = \frac{m+M}{m}V = \frac{2.01}{0.01}(1.33) = 201 \times 1.33 = 267 \text{ m/s}$$
$$\boxed{v_0 \approx 267 \text{ m/s}}$$

Energy lost:
$$\Delta K = K_i - K_f = \frac{1}{2}mv_0^2 - \frac{1}{2}(m+M)V^2$$
$$= \frac{1}{2}(0.01)(267)^2 - \frac{1}{2}(2.01)(1.33)^2$$
$$= 356.4 - 1.78 = 354.6 \text{ J}$$
$$\boxed{\Delta K \approx 355 \text{ J}}$$

Or using the fraction: $\Delta K = \frac{M}{m+M} \cdot \frac{1}{2}mv_0^2 = \frac{2.0}{2.01} \cdot 356.4 \approx 355$ J

Solution 21: Collision and Spring Compression

(a) Velocities after elastic collision (3 marks):

For a 1D elastic collision with m_2 initially at rest, the standard results are:

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} v_0$$
$$v_2 = \frac{2m_1}{m_1 + m_2} v_0$$

$$\boxed{v_1 = \frac{m_1 - m_2}{m_1 + m_2} v_0, \quad v_2 = \frac{2m_1}{m_1 + m_2} v_0}$$

(b) Maximum spring compression (3 marks):

After the collision, m_2 moves with velocity v_2 and compresses the spring. At maximum compression, all of m_2's kinetic energy is stored in the spring:

$$\frac{1}{2} m_2 v_2^2 = \frac{1}{2} k x_{max}^2$$

$$x_{max} = v_2 \sqrt{\frac{m_2}{k}} = \frac{2m_1}{m_1 + m_2} v_0 \sqrt{\frac{m_2}{k}}$$

$$\boxed{x_{max} = \frac{2m_1 v_0}{m_1 + m_2} \sqrt{\frac{m_2}{k}}}$$

(c) Velocity when spring returns to natural length (2 marks):

The spring-mass system is conservative. When the spring returns to its natural length, all the potential energy converts back to kinetic energy. Since there's no friction, m_2 returns to its original speed v_2 but now moving in the opposite direction (away from the wall):

$$\boxed{v_2' = -\frac{2m_1}{m_1 + m_2} v_0}$$

(The negative sign indicates motion away from the wall, opposite to the initial direction.)

(d) Special case $m_1 = m_2 = m$ (2 marks):

When $m_1 = m_2$:

$$v_1 = \frac{m - m}{2m} v_0 = 0$$

$$v_2 = \frac{2m}{2m} v_0 = v_0$$

So m_1 stops completely and m_2 moves with speed v_0.

$$x_{max} = v_0 \sqrt{\frac{m}{k}}$$

Energy verification:

- Initial KE: $\frac{1}{2} m v_0^2$
- After collision: $\frac{1}{2}(0)^2 + \frac{1}{2} m v_0^2 = \frac{1}{2} m v_0^2$ ✓
- At max compression: $\frac{1}{2} k x_{max}^2 = \frac{1}{2} k \cdot \frac{m v_0^2}{k} = \frac{1}{2} m v_0^2$ ✓

$$\boxed{\text{Energy is conserved throughout: } E = \frac{1}{2} m v_0^2}$$

Solution 22: Work by Variable Force

(a) Work done from $x = 0$ to $x = L$ (3 marks):

$$W = \int_0^L F(x)\,dx = \int_0^L F_0 \left(1 - \frac{x}{L}\right) dx$$

$$W = F_0 \left[x - \frac{x^2}{2L}\right]_0^L = F_0 \left(L - \frac{L^2}{2L}\right) = F_0 \left(L - \frac{L}{2}\right)$$

$$\boxed{W = \frac{F_0 L}{2}}$$

(b) Speed at $x = L$ (2 marks):

Using work-energy theorem, starting from rest:

$$W = \Delta KE = \frac{1}{2}mv^2 - 0$$

$$\frac{F_0 L}{2} = \frac{1}{2}mv^2$$

$$\boxed{v = \sqrt{\frac{F_0 L}{m}}}$$

(c) Speed as function of position (3 marks):

Work done from 0 to x:

$$W(x) = \int_0^x F_0 \left(1 - \frac{x'}{L}\right) dx' = F_0 \left[x' - \frac{x'^2}{2L}\right]_0^x = F_0 \left(x - \frac{x^2}{2L}\right)$$

From work-energy theorem:

$$\frac{1}{2}mv^2 = F_0 x \left(1 - \frac{x}{2L}\right)$$

$$\boxed{v(x) = \sqrt{\frac{2F_0 x}{m}\left(1 - \frac{x}{2L}\right)} = \sqrt{\frac{F_0}{mL}\left(2Lx - x^2\right)}}$$

(d) Position and value of maximum speed (2 marks):

Speed is maximum when $\frac{d(v^2)}{dx} = 0$:

Taking the derivative of $v^2 = \frac{F_0}{mL}(2Lx - x^2)$:

$$\frac{d(v^2)}{dx} = \frac{F_0}{mL}(2L - 2x) = 0 \Rightarrow x = L$$

So maximum speed is at $x = L$:

$$\boxed{v_{max} = \sqrt{\frac{F_0 L}{m}} \text{ at } x = L}$$

Solution 23: Block Launched onto a Spring

(a) Maximum compression (2 marks):

Using energy conservation (frictionless surface):

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_{max}^2$$

$$\boxed{x_{max} = v_0\sqrt{\frac{m}{k}}}$$

(b) Speed as function of compression (3 marks):

At compression x, energy conservation gives:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$v^2 = v_0^2 - \frac{k}{m}x^2$$

$$\boxed{v(x) = \sqrt{v_0^2 - \frac{k}{m}x^2}}$$

(c) Maximum height on frictionless ramp (3 marks):

After rebounding from the spring, the block has the same speed v_0 (energy conserved). On the frictionless ramp, all kinetic energy converts to potential energy:

$$\frac{1}{2}mv_0^2 = mgh$$

$$\boxed{h = \frac{v_0^2}{2g}}$$

(d) Initial speed with friction on ramp (2 marks):

With friction, the work-energy theorem gives:

$$\frac{1}{2}mv_0^2 = mgh + W_{friction}$$

The height reached: $h = d\sin\theta$

Work done by friction: $W_{friction} = \mu_k N \cdot d = \mu_k mg\cos\theta \cdot d$

$$\frac{1}{2}mv_0^2 = mgd\sin\theta + \mu_k mgd\cos\theta$$

$$\frac{1}{2}v_0^2 = gd(\sin\theta + \mu_k\cos\theta)$$

$$\boxed{v_0 = \sqrt{2gd(\sin\theta + \mu_k\cos\theta)}}$$

Solution 24: Explosion and Center of Mass

(a) Velocity at highest point (2 marks):

At the highest point of projectile motion, the vertical component of velocity is zero. Only the horizontal component remains:

$$\boxed{v_{top} = v_0 \cos\theta \text{ (horizontal)}}$$

(b) Velocity of $(M - m)$ piece after explosion (3 marks):

Conservation of momentum (explosion is internal):

Before: $\vec{p}_{before} = M v_0 \cos\theta \, \hat{i}$

After: The piece of mass m is stationary, so:

$$M v_0 \cos\theta = m(0) + (M - m)v'$$

$$\boxed{v' = \frac{M v_0 \cos\theta}{M - m}}$$

This velocity is horizontal, in the original direction of motion.

(c) Center of mass continues on original trajectory (2 marks):

The explosion involves only internal forces. By Newton's third law, internal forces come in equal and opposite pairs and cannot change the momentum of the system. Therefore:

$$\vec{p}_{system} = M\vec{v}_{cm} = \text{constant}$$

Since no external horizontal forces act (gravity is vertical), the horizontal velocity of the center of mass remains $v_0 \cos\theta$. The center of mass continues to follow the original parabolic trajectory as if no explosion occurred.

$$\boxed{\text{The CM follows the original trajectory because internal forces don't affect total momentum.}}$$

(d) Landing positions (3 marks):

Time from highest point to ground: At highest point, height is $H = \frac{v_0^2 \sin^2\theta}{2g}$

Time to fall: $H = \frac{1}{2} g t_f^2 \Rightarrow t_f = \frac{v_0 \sin\theta}{g}$

Horizontal distance from launch to highest point: $x_{peak} = v_0 \cos\theta \cdot \frac{v_0 \sin\theta}{g} = \frac{v_0^2 \sin\theta \cos\theta}{g}$

Piece of mass m (stationary after explosion):

Falls straight down from the highest point.

$$\boxed{x_m = \frac{v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{2g}}$$

Piece of mass $(M - m)$:

Travels horizontally at speed $v' = \frac{Mv_0 \cos\theta}{M-m}$ for time t_f:

$$x_{M-m} = x_{peak} + v' \cdot t_f = \frac{v_0^2 \sin\theta\cos\theta}{g} + \frac{Mv_0\cos\theta}{M-m} \cdot \frac{v_0 \sin\theta}{g}$$

$$x_{M-m} = \frac{v_0^2 \sin\theta\cos\theta}{g}\left(1 + \frac{M}{M-m}\right) = \frac{v_0^2 \sin\theta\cos\theta}{g} \cdot \frac{2M-m}{M-m}$$

$$\boxed{x_{M-m} = \frac{v_0^2 \sin(2\theta)}{2g} \cdot \frac{2M-m}{M-m}}$$

Rotational Motion

Solution 25: Rolling Motion on an Incline

(a) Free body diagram (2 marks):

Forces on the sphere:

- Weight mg acting at center, vertically downward
- Normal force N perpendicular to incline, at contact point
- Static friction f_s up the incline, at contact point (prevents slipping)

(b) Newton's second law and acceleration (3 marks):

Translation (along incline, positive down):

$$mg\sin\theta - f_s = ma$$

Rotation (about center of mass):

$$f_s \cdot R = I\alpha = \frac{2}{5}mR^2 \cdot \alpha$$

Rolling constraint: $a = R\alpha$, so $\alpha = a/R$

From rotation equation:

$$f_s = \frac{2}{5}mR \cdot \frac{a}{R} = \frac{2}{5}ma$$

Substituting into translation:

$$mg\sin\theta - \frac{2}{5}ma = ma$$

$$mg\sin\theta = \frac{7}{5}ma$$

$$\boxed{a = \frac{5g\sin\theta}{7}}$$

(c) Friction force and minimum μ_s (2 marks):

$$f_s = \frac{2}{5}ma = \frac{2}{5}m \cdot \frac{5g\sin\theta}{7} = \frac{2mg\sin\theta}{7}$$

$$\boxed{f_s = \frac{2mg\sin\theta}{7}}$$

For rolling without slipping: $f_s \leq \mu_s N = \mu_s mg\cos\theta$

$$\frac{2mg\sin\theta}{7} \leq \mu_s mg\cos\theta$$

$$\boxed{\mu_{s,min} = \frac{2\tan\theta}{7}}$$

(d) Speed at bottom and rotational KE fraction (3 marks):

Using energy conservation:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{2}{5}mR^2\cdot\frac{v^2}{R^2}$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2$$

$$\boxed{v = \sqrt{\frac{10gh}{7}}}$$

Rotational KE: $KE_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{5}mv^2$

Total KE: $KE_{total} = \frac{7}{10}mv^2$

Fraction:

$$\boxed{\frac{KE_{rot}}{KE_{total}} = \frac{\frac{1}{5}mv^2}{\frac{7}{10}mv^2} = \frac{2}{7}}$$

Solution 26: Angular Momentum and Collisions

(a) Why angular momentum is conserved (2 marks):

During the brief collision, the only external forces are gravity (acting at the center of mass of the rod and at the ball) and the reaction force at the pivot.

Taking torques about the pivot: the pivot force has zero moment arm, and gravity's impulse is negligible during the short collision time. Therefore, **angular momentum about the pivot is conserved** during the collision.

(b) Angular velocity after collision (3 marks):

Before collision:

$$L_{before} = mv_0 \cdot d$$

(The ball has linear momentum mv_0 at distance d from pivot)

After collision (ball sticks to rod):

$$L_{after} = I_{total}\omega$$

where $I_{total} = I_{rod} + I_{ball} = \frac{1}{3}ML^2 + md^2$

Conservation: $mv_0 d = \left(\frac{1}{3}ML^2 + md^2\right)\omega$

$$\boxed{\omega = \frac{mv_0 d}{\frac{1}{3}ML^2 + md^2}}$$

(c) Kinetic energy lost (2 marks):

Initial KE: $KE_i = \frac{1}{2}mv_0^2$

Final KE: $KE_f = \frac{1}{2}I_{total}\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2 + md^2\right)\omega^2$

Substituting ω:

$$KE_f = \frac{1}{2}\left(\frac{1}{3}ML^2 + md^2\right) \cdot \frac{m^2 v_0^2 d^2}{\left(\frac{1}{3}ML^2 + md^2\right)^2} = \frac{m^2 v_0^2 d^2}{2\left(\frac{1}{3}ML^2 + md^2\right)}$$

$$\boxed{\Delta KE = \frac{1}{2}mv_0^2 - \frac{m^2 v_0^2 d^2}{2\left(\frac{1}{3}ML^2 + md^2\right)} = \frac{1}{2}mv_0^2\left(1 - \frac{md^2}{\frac{1}{3}ML^2 + md^2}\right)}$$

(d) Maximum swing angle (3 marks):

After collision, the system swings up. Using energy conservation:

$$\frac{1}{2}I_{total}\omega^2 = \Delta PE$$

The center of mass of the rod rises by $\frac{L}{2}(1 - \cos\theta_{max})$, and the ball rises by $d(1 - \cos\theta_{max})$:

$$\frac{1}{2}I_{total}\omega^2 = Mg\frac{L}{2}(1 - \cos\theta_{max}) + mgd(1 - \cos\theta_{max})$$

$$\frac{1}{2}I_{total}\omega^2 = \left(\frac{MgL}{2} + mgd\right)(1 - \cos\theta_{max})$$

$$\boxed{1 - \cos\theta_{max} = \frac{I_{total}\omega^2}{MgL + 2mgd} = \frac{m^2v_0^2d^2}{(MgL + 2mgd)\left(\frac{1}{3}ML^2 + md^2\right)}}$$

$$\boxed{\theta_{max} = \arccos\left(1 - \frac{m^2v_0^2d^2}{(MgL + 2mgd)\left(\frac{1}{3}ML^2 + md^2\right)}\right)}$$

Solution 27: Atwood Machine with Massive Pulley

(a) Free body diagrams (2 marks):

Mass m_1:

- Weight $m_1 g$ (down)
- Tension T_1 (up)

Mass m_2:

- Weight $m_2 g$ (down)
- Tension T_2 (up)

Pulley:

- Tension T_1 creating torque (one side)
- Tension T_2 creating torque (other side)

Note: $T_1 \neq T_2$ because the pulley has mass and rotates.

(b) Acceleration derivation (3 marks):

For m_1 (taking up as positive, $m_2 > m_1$ so m_1 accelerates up):

$$T_1 - m_1 g = m_1 a$$

For m_2 (taking down as positive):

$$m_2 g - T_2 = m_2 a$$

For pulley (net torque = $I\alpha$):

$$T_2 R - T_1 R = I\alpha = \frac{1}{2} M R^2 \cdot \frac{a}{R} = \frac{1}{2} M R a$$

$$T_2 - T_1 = \frac{1}{2} M a$$

Adding all three equations:

$$m_2 g - m_1 g = m_1 a + m_2 a + \frac{1}{2} M a$$

$$(m_2 - m_1) g = \left(m_1 + m_2 + \frac{M}{2} \right) a$$

$$\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{M}{2}}}$$

(c) Tensions (2 marks):

From m_1: $T_1 = m_1(g + a) = m_1 g \left(1 + \frac{m_2 - m_1}{m_1 + m_2 + M/2}\right)$

$$\boxed{T_1 = \frac{m_1 g(2m_2 + M/2)}{m_1 + m_2 + M/2}}$$

From m_2: $T_2 = m_2(g - a) = m_2 g \left(1 - \frac{m_2 - m_1}{m_1 + m_2 + M/2}\right)$

$$\boxed{T_2 = \frac{m_2 g(2m_1 + M/2)}{m_1 + m_2 + M/2}}$$

(d) Energy verification (3 marks):

As m_2 descends height h: m_1 rises height h, pulley rotates through $\theta = h/R$.

Energy conservation:

$$m_2 gh = m_1 gh + \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \frac{1}{2}I\omega^2$$

$$(m_2 - m_1)gh = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2} \cdot \frac{1}{2}MR^2 \cdot \frac{v^2}{R^2}$$

$$(m_2 - m_1)gh = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)v^2$$

Using $v^2 = 2ah$:

$$(m_2 - m_1)g = \left(m_1 + m_2 + \frac{M}{2}\right)a$$

This confirms: $\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + M/2}}$ ✓

Solution 28: The Yo-Yo

(a) Free body diagram (2 marks):

Forces on yo-yo:

- Weight mg acting at center (downward)
- Tension T in string at inner radius r (upward)

The string applies a torque about the center of mass.

(b) Acceleration derivation (3 marks):

Translation (downward positive):

$$mg - T = ma$$

Rotation (about center):

$$Tr = I\alpha = \frac{1}{2}m(r^2 + R^2)\alpha$$

Constraint (string unwinds): $a = r\alpha$, so $\alpha = a/r$

From rotation:

$$T = \frac{I\alpha}{r} = \frac{\frac{1}{2}m(r^2 + R^2) \cdot \frac{a}{r}}{r} = \frac{m(r^2 + R^2)a}{2r^2}$$

Substituting into translation:

$$mg - \frac{m(r^2 + R^2)a}{2r^2} = ma$$

$$g = a\left(1 + \frac{r^2 + R^2}{2r^2}\right) = a\left(\frac{2r^2 + r^2 + R^2}{2r^2}\right) = a\left(\frac{3r^2 + R^2}{2r^2}\right)$$

$$\boxed{a = \frac{2gr^2}{3r^2 + R^2}}$$

(c) Tension (2 marks):

$$T = m(g - a) = mg\left(1 - \frac{2r^2}{3r^2 + R^2}\right) = mg\left(\frac{r^2 + R^2}{3r^2 + R^2}\right)$$

$$\boxed{T = \frac{mg(r^2 + R^2)}{3r^2 + R^2}}$$

(d) Numerical calculations (3 marks):

Given: $m = 0.1$ kg, $r = 0.01$ m, $R = 0.05$ m

$$a = \frac{2(9.8)(0.01)^2}{3(0.01)^2 + (0.05)^2} = \frac{2(9.8)(0.0001)}{0.0003 + 0.0025} = \frac{0.00196}{0.0028} = 0.70 \text{ m/s}^2$$

$$\boxed{a = 0.70 \text{ m/s}^2}$$

$$T = \frac{(0.1)(9.8)(0.0001 + 0.0025)}{0.0028} = \frac{(0.98)(0.0026)}{0.0028} = 0.91 \text{ N}$$

$$\boxed{T = 0.91 \text{ N}}$$

Time to descend 1 m from rest:

$$1 = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2}{a}} = \sqrt{\frac{2}{0.70}} = 1.69 \text{ s}$$

$$\boxed{t = 1.69 \text{ s}}$$

Solution 29: Two Disks and Angular Momentum

(a) Conserved quantity (2 marks):

Angular momentum is conserved because there are no external torques about the rotation axis during the collision.

Kinetic energy is NOT conserved because when the second disk lands on the first, they initially have different angular velocities and friction between them does work until they rotate together. Energy is dissipated as heat due to the rubbing/slipping between surfaces.

(b) Final angular velocity (3 marks):

Conservation of angular momentum:

$$L_{before} = L_{after}$$

$$I_1\omega_0 + I_2(0) = (I_1 + I_2)\omega_f$$

$$\frac{1}{2}MR^2 \cdot \omega_0 = \left(\frac{1}{2}MR^2 + \frac{1}{2}mr^2\right)\omega_f$$

$$\boxed{\omega_f = \frac{MR^2}{MR^2 + mr^2}\omega_0}$$

(c) Energy lost (3 marks):

Initial KE: $KE_i = \frac{1}{2}I_1\omega_0^2 = \frac{1}{4}MR^2\omega_0^2$

Final KE: $KE_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2}\left(\frac{1}{2}MR^2 + \frac{1}{2}mr^2\right)\left(\frac{MR^2\omega_0}{MR^2+mr^2}\right)^2$

$$KE_f = \frac{1}{4}(MR^2 + mr^2) \cdot \frac{M^2R^4\omega_0^2}{(MR^2 + mr^2)^2} = \frac{M^2R^4\omega_0^2}{4(MR^2 + mr^2)}$$

Energy lost:

$$\Delta KE = KE_i - KE_f = \frac{1}{4}MR^2\omega_0^2 - \frac{M^2R^4\omega_0^2}{4(MR^2 + mr^2)}$$

$$= \frac{MR^2\omega_0^2}{4}\left(1 - \frac{MR^2}{MR^2 + mr^2}\right) = \frac{MR^2\omega_0^2}{4} \cdot \frac{mr^2}{MR^2 + mr^2}$$

$$\boxed{\Delta KE = \frac{MR^2mr^2\omega_0^2}{4(MR^2 + mr^2)}}$$

Fraction lost:

$$\boxed{\frac{\Delta KE}{KE_i} = \frac{mr^2}{MR^2 + mr^2}}$$

(d) Numerical values (2 marks):

Given: $M = 2.0$ kg, $R = 0.3$ m, $m = 0.5$ kg, $r = 0.1$ m, $\omega_0 = 10$ rad/s

$MR^2 = 2.0(0.09) = 0.18$ kg·m^2, $mr^2 = 0.5(0.01) = 0.005$ kg·m^2

$$\omega_f = \frac{0.18}{0.18 + 0.005}(10) = \frac{0.18}{0.185}(10) = 9.73 \text{ rad/s}$$

$$\boxed{\omega_f = 9.73 \text{ rad/s}}$$

$$\Delta KE = \frac{(0.18)(0.005)(100)}{4(0.185)} = \frac{0.09}{0.74} = 0.122 \text{ J}$$

$$\boxed{\Delta KE = 0.12 \text{ J}}$$

Solution 30: The Bowling Ball Problem

(a) Linear and angular acceleration (2 marks):

The friction force acts backward (opposing the sliding motion):

$$f = \mu_k mg$$

Linear acceleration (friction decelerates translation):

$$ma = -\mu_k mg$$

$$\boxed{a = -\mu_k g}$$

Angular acceleration (friction creates torque that speeds up rotation):

$$I\alpha = fR = \mu_k mgR$$

$$\frac{2}{5}mR^2\alpha = \mu_k mgR$$

$$\boxed{\alpha = \frac{5\mu_k g}{2R}}$$

(b) Velocity and angular velocity as functions of time (3 marks):

Linear velocity (starting from v_0):

$$v(t) = v_0 + at = v_0 - \mu_k gt$$

$$\boxed{v(t) = v_0 - \mu_k gt}$$

Angular velocity (starting from $\omega_0 = 0$):

$$\omega(t) = \omega_0 + \alpha t = 0 + \frac{5\mu_k g}{2R}t$$

$$\boxed{\omega(t) = \frac{5\mu_k g}{2R}t}$$

(c) Time when pure rolling begins (3 marks):

Pure rolling occurs when $v = R\omega$:

$$v_0 - \mu_k gt^* = R \cdot \frac{5\mu_k g}{2R}t^*$$

$$v_0 - \mu_k gt^* = \frac{5\mu_k g}{2}t^*$$

$$v_0 = \mu_k gt^* + \frac{5\mu_k g}{2}t^* = \frac{7\mu_k g}{2}t^*$$

$$\boxed{t^* = \frac{2v_0}{7\mu_k g}}$$

(d) Final speed (2 marks):

$$v_f = v_0 - \mu_k g t^* = v_0 - \mu_k g \cdot \frac{2v_0}{7\mu_k g} = v_0 - \frac{2v_0}{7} = \frac{5v_0}{7}$$

$$\boxed{v_f = \frac{5}{7}v_0}$$

Verification: $R\omega_f = R \cdot \frac{5\mu_k g}{2R} \cdot \frac{2v_0}{7\mu_k g} = \frac{5v_0}{7} = v_f$ ✓

Oscillations

Solution 31: Simple Harmonic Motion

(a) Differential equation and angular frequency (2 marks):

Newton's second law: $F = ma = -kx$

$$m\ddot{x} = -kx$$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

This is SHM with angular frequency:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

(b) Total mechanical energy (2 marks):

At maximum displacement $x = A$, velocity is zero:

$$E = KE + PE = 0 + \frac{1}{2}kA^2$$

$$\boxed{E = \frac{1}{2}kA^2}$$

At any position: $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

Since $v = \omega\sqrt{A^2 - x^2}$ and $\omega^2 = k/m$:

$$E = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}kx^2 = \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Energy is constant. ✓

(c) Velocity as function of position (2 marks):

From energy conservation:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v^2 = \frac{k}{m}(A^2 - x^2) = \omega^2(A^2 - x^2)$$

$$\boxed{v(x) = \omega\sqrt{A^2 - x^2}}$$

(d) Position where KE = PE (2 marks):

$$KE = PE$$

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

Since $KE + PE = \frac{1}{2}kA^2$:

$$2 \cdot \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$x^2 = \frac{A^2}{2}$$

$$\boxed{x = \pm\frac{A}{\sqrt{2}} = \pm\frac{A\sqrt{2}}{2}}$$

(e) Bullet embedding in block (2 marks):

Conservation of momentum during collision:

$$m_b v_b = (m_b + M)V$$

$$V = \frac{m_b v_b}{m_b + M}$$

This velocity V becomes the maximum velocity of oscillation. At equilibrium, all energy is kinetic:

$$\frac{1}{2}(m_b + M)V^2 = \frac{1}{2}kA^2$$

$$A = V\sqrt{\frac{m_b + M}{k}} = \frac{m_b v_b}{m_b + M}\sqrt{\frac{m_b + M}{k}}$$

$$\boxed{A = \frac{m_b v_b}{\sqrt{k(m_b + M)}}}$$

Solution 32: The Physical Pendulum

(a) Free body diagram and equation of motion (3 marks):

Forces on rod:

- Weight Mg at center of mass (distance $L/2$ from pivot)
- Reaction force at pivot (creates no torque about pivot)

Torque about pivot (taking counterclockwise positive, θ positive for displacement to the right):

$$\tau = -Mg\frac{L}{2}\sin\theta$$

Rotational equation:

$$I_{pivot}\ddot{\theta} = -Mg\frac{L}{2}\sin\theta$$

where $I_{pivot} = \frac{1}{3}ML^2$ (moment of inertia about end).

$$\boxed{\frac{1}{3}ML^2\ddot{\theta} = -\frac{MgL}{2}\sin\theta}$$

(b) Period of oscillation (3 marks):

For small angles, $\sin\theta \approx \theta$:

$$\frac{1}{3}ML^2\ddot{\theta} = -\frac{MgL}{2}\theta$$
$$\ddot{\theta} = -\frac{3g}{2L}\theta$$

This is SHM with $\omega^2 = \frac{3g}{2L}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$\boxed{T = 2\pi\sqrt{\frac{2L}{3g}}}$$

(c) Comparison to simple pendulum (2 marks):

Simple pendulum of length L: $T_{simple} = 2\pi\sqrt{\frac{L}{g}}$

Ratio:

$$\frac{T_{rod}}{T_{simple}} = \frac{2\pi\sqrt{2L/3g}}{2\pi\sqrt{L/g}} = \sqrt{\frac{2}{3}} \approx 0.816$$

$$\boxed{\text{The rod has the shorter period by a factor of } \sqrt{2/3} \approx 0.816}$$

(d) Pivot point to minimize period (2 marks):

For a rod pivoted at distance d from the center of mass:

$$I = I_{cm} + Md^2 = \frac{1}{12}ML^2 + Md^2$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + d^2}{gd}}$$

Minimize $T^2 \propto \frac{L^2/12+d^2}{d}$:

$$\frac{d}{d(d)}\left(\frac{L^2}{12d} + d\right) = -\frac{L^2}{12d^2} + 1 = 0$$

$$d^2 = \frac{L^2}{12}$$

$$d = \frac{L}{2\sqrt{3}} = \frac{L\sqrt{3}}{6}$$

$$\boxed{d = \frac{L}{2\sqrt{3}} = \frac{L\sqrt{3}}{6} \approx 0.289L \text{ from center of mass}}$$

Solution 33: Block on Vibrating Platform

(a) Free body diagram and Newton's second law (2 marks):

Forces on block:

- Weight mg (downward)
- Normal force N (upward)

The block accelerates with the platform. Platform acceleration:

$$a_{platform} = \frac{d^2y}{dt^2} = -A\omega^2\cos(\omega t)$$

Newton's second law (taking up as positive):

$$\boxed{N - mg = ma = -mA\omega^2\cos(\omega t)}$$

$$\boxed{N = m(g - A\omega^2\cos(\omega t))}$$

(b) Condition for losing contact (3 marks):

The block stays on the platform as long as $N > 0$:

$$m(g - A\omega^2\cos(\omega t)) > 0$$

$$g > A\omega^2\cos(\omega t)$$

The most critical moment is when $\cos(\omega t) = 1$ (platform at highest point, accelerating downward fastest):

$$g > A\omega^2$$

The block loses contact when the downward acceleration of the platform exceeds g:

$$\boxed{|a_{max}| = A\omega^2 > g}$$

(c) Maximum amplitude at given frequency (3 marks):

For the block to stay on: $A\omega^2 \leq g$

With $\omega = 2\pi f$:

$$A(2\pi f)^2 \leq g$$

$$\boxed{A_{max} = \frac{g}{4\pi^2 f^2}}$$

(d) Numerical calculation for $f = 5$ Hz (2 marks):

$$A_{max} = \frac{9.8}{4\pi^2(25)} = \frac{9.8}{986.96} = 0.00993 \text{ m}$$

$$\boxed{A_{max} = 0.99 \text{ cm}}$$

If $A = 1.5$ cm $= 0.015$ m (which exceeds A_{max}), the block loses contact.

It loses contact when $N = 0$:

$$g = A\omega^2 \cos(\omega t)$$

$$\cos(\omega t) = \frac{g}{A\omega^2} = \frac{9.8}{0.015 \times 4\pi^2 \times 25} = \frac{9.8}{14.8} = 0.662$$

$$\omega t = \arccos(0.662) = 48.5$$

$$\boxed{\text{The block loses contact when } \cos(\omega t) = 0.66, \text{ i.e., at } \omega t \approx 48.5 \text{ from the top}}$$

Solution 34: Combined Spring Systems

(a) Parallel springs (3 marks):

When springs are in parallel, both springs extend by the same amount x, and their forces add:

$$F_{total} = k_1 x + k_2 x = (k_1 + k_2)x$$

$$\boxed{k_{eff,parallel} = k_1 + k_2}$$

Period:

$$\boxed{T_{parallel} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}}$$

(b) Series springs (3 marks):

When springs are in series, the force through each spring is the same (F), but extensions add:

$$x_{total} = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$$

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\boxed{k_{eff,series} = \frac{k_1 k_2}{k_1 + k_2}}$$

Period:

$$\boxed{T_{series} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}}$$

(c) Numerical values (2 marks):

Given: $k_1 = 100$ N/m, $k_2 = 200$ N/m, $m = 0.5$ kg

Parallel:

$$k_{eff} = 100 + 200 = 300 \text{ N/m}$$

$$T_{parallel} = 2\pi\sqrt{\frac{0.5}{300}} = 2\pi\sqrt{0.00167} = 0.257 \text{ s}$$

$$\boxed{T_{parallel} = 0.26 \text{ s}}$$

Series:

$$k_{eff} = \frac{100 \times 200}{300} = 66.7 \text{ N/m}$$

$$T_{series} = 2\pi\sqrt{\frac{0.5}{66.7}} = 2\pi\sqrt{0.0075} = 0.544 \text{ s}$$

$$\boxed{T_{series} = 0.54 \text{ s}}$$

(d) Energy distribution in series springs (2 marks):

At maximum displacement, all energy is potential energy stored in the springs.

Each spring has the same force F. The potential energy in each spring is:

$$PE_1 = \frac{1}{2}k_1x_1^2 = \frac{1}{2}k_1\left(\frac{F}{k_1}\right)^2 = \frac{F^2}{2k_1}$$

$$PE_2 = \frac{1}{2}k_2x_2^2 = \frac{F^2}{2k_2}$$

Fractions:

$$\frac{PE_1}{PE_{total}} = \frac{F^2/2k_1}{F^2/2k_1 + F^2/2k_2} = \frac{1/k_1}{1/k_1 + 1/k_2} = \frac{k_2}{k_1 + k_2}$$

$$\boxed{\text{Fraction in spring 1: } \frac{k_2}{k_1 + k_2}, \quad \text{Fraction in spring 2: } \frac{k_1}{k_1 + k_2}}$$

The softer spring stores more energy!

Solution 35: Spring Oscillating Sphere

(a) Energy expressions (3 marks):

Rolling without slipping: $v = R\omega_{sphere}$, so $\omega_{sphere} = v/R$

Translational KE: $KE_{trans} = \frac{1}{2}mv^2$

Rotational KE: $KE_{rot} = \frac{1}{2}I\omega_{sphere}^2 = \frac{1}{2} \cdot \frac{2}{5}mR^2 \cdot \frac{v^2}{R^2} = \frac{1}{5}mv^2$

$$\boxed{KE_{total} = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2}$$

$$\boxed{PE = \frac{1}{2}kx^2}$$

(b) Equation of motion (3 marks):

Total energy (conserved):

$$E = \frac{7}{10}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Taking time derivative:

$$\frac{7}{10}m(2v\dot{v}) + \frac{1}{2}k(2x\dot{x}) = 0$$

$$\frac{7}{5}mv\dot{v} + kx\dot{x} = 0$$

Since $v = \dot{x}$:

$$\frac{7}{5}m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\frac{7}{5}m\ddot{x} + kx = 0$$

$$\boxed{\ddot{x} + \frac{5k}{7m}x = 0}$$

(c) Angular frequency and period (2 marks):

$$\omega_{osc} = \sqrt{\frac{5k}{7m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{7m}{5k}}}$$

For a sliding block (no rotation): $T_{slide} = 2\pi\sqrt{\frac{m}{k}}$

Comparison:

$$\frac{T_{roll}}{T_{slide}} = \sqrt{\frac{7m/5k}{m/k}} = \sqrt{\frac{7}{5}} \approx 1.18$$

$$\boxed{\text{Rolling period is longer by factor } \sqrt{7/5} \approx 1.18}$$

(d) Numerical values (2 marks):

Given: $m = 2.0$ kg, $R = 0.1$ m, $k = 140$ N/m, $x_0 = 0.05$ m

$$T = 2\pi\sqrt{\frac{7(2.0)}{5(140)}} = 2\pi\sqrt{\frac{14}{700}} = 2\pi\sqrt{0.02} = 0.89 \text{ s}$$

$$\boxed{T = 0.89 \text{ s}}$$

Maximum speed occurs at $x = 0$. Energy conservation:

$$\frac{1}{2}kx_0^2 = \frac{7}{10}mv_{max}^2$$

$$v_{max} = \sqrt{\frac{5kx_0^2}{7m}} = \sqrt{\frac{5(140)(0.05)^2}{7(2.0)}} = \sqrt{\frac{1.75}{14}} = 0.354 \text{ m/s}$$

$$\boxed{v_{max} = 0.35 \text{ m/s}}$$

Waves

Solution 36: Wave Function Analysis

(a) Wave velocity (2 marks):

For a traveling wave $y = A\sin(kx - \omega t + \phi)$:

A point of constant phase moves such that $kx - \omega t = \text{constant}$

Taking the derivative: $k\frac{dx}{dt} - \omega = 0$

$$\boxed{v = \frac{dx}{dt} = \frac{\omega}{k}}$$

(b) Verification of wave equation (3 marks):

Given: $y = A\sin(kx - \omega t + \phi)$

$$\frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t + \phi)$$
$$\frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(kx - \omega t + \phi) = -\omega^2 y$$

$$\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t + \phi)$$
$$\frac{\partial^2 y}{\partial x^2} = -Ak^2\sin(kx - \omega t + \phi) = -k^2 y$$

Check wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$
$$-\omega^2 y = \left(\frac{\omega}{k}\right)^2(-k^2 y) = -\omega^2 y \quad \checkmark$$

$$\boxed{\text{The wave equation is satisfied.}}$$

(c) Maximum transverse speed (2 marks):

Transverse velocity: $v_y = \frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t + \phi)$

Maximum occurs when $|\cos| = 1$:

$$\boxed{v_{max} = A\omega}$$

(d) Numerical calculations (3 marks):

Given: $A = 0.05$ m, $\lambda = 2.0$ m, $T = 0.5$ s

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2.0} = \pi \text{ rad/m}$$
$$\boxed{k = 3.14 \text{ rad/m}}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s}$$
$$\boxed{\omega = 12.57 \text{ rad/s}}$$

$$v = \frac{\omega}{k} = \frac{4\pi}{\pi} = 4 \text{ m/s}$$
$$\boxed{v = 4.0 \text{ m/s}}$$

$$v_{max} = A\omega = 0.05 \times 4\pi = 0.628 \text{ m/s}$$
$$\boxed{v_{max} = 0.63 \text{ m/s}}$$

Solution 37: Standing Waves on a String

(a) Wave speed from dimensional analysis (2 marks):

Wave speed v depends on tension T [N = kg·m/s^2] and linear density μ [kg/m].

$$[v] = \text{m/s}$$

Try $v = T^a \mu^b$:

$$\text{m/s} = (\text{kg} \cdot \text{m/s}^2)^a \cdot (\text{kg/m})^b$$

For kg: $0 = a + b \Rightarrow b = -a$

For m: $1 = a - b = a + a = 2a \Rightarrow a = 1/2$

For s: $-1 = -2a = -1$ ✓

$$\boxed{v = \sqrt{\frac{T}{\mu}}}$$

(b) Fixed at both ends (3 marks):

Boundary conditions: nodes at $x = 0$ and $x = L$

The wavelength must satisfy: $L = n\frac{\lambda_n}{2}$ where $n = 1, 2, 3, ...$

$$\boxed{\lambda_n = \frac{2L}{n}}$$

Frequencies:

$$f_n = \frac{v}{\lambda_n} = \frac{v \cdot n}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

$$\boxed{f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \quad n = 1, 2, 3, ...}$$

(c) Fixed at one end, open at other (2 marks):

Boundary conditions: node at closed end, antinode at open end

$$L = \frac{\lambda_n}{4}, \frac{3\lambda_n}{4}, \frac{5\lambda_n}{4}, ... = \frac{(2n-1)\lambda_n}{4}$$

$$\boxed{\lambda_n = \frac{4L}{2n-1}}$$

$$\boxed{f_n = \frac{(2n-1)v}{4L} = \frac{2n-1}{4L}\sqrt{\frac{T}{\mu}}, \quad n = 1, 2, 3, ...}$$

Only odd harmonics exist.

(d) Guitar string calculation (3 marks):

Given: $L = 0.65$ m, $\mu = 3.0 \times 10^{-3}$ kg/m, $f_1 = 330$ Hz

Tension:

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

$$T = 4L^2 f_1^2 \mu = 4(0.65)^2(330)^2(3.0 \times 10^{-3})$$

$$T = 4(0.4225)(108900)(0.003) = 550.5 \text{ N}$$

$$\boxed{T \approx 551 \text{ N}}$$

Fundamental wavelength:

$$\lambda_1 = 2L = 2(0.65) = 1.30 \text{ m}$$

$$\boxed{\lambda_1 = 1.30 \text{ m}}$$

Third harmonic:

$$f_3 = 3f_1 = 3(330) = 990 \text{ Hz}$$

$$\boxed{f_3 = 990 \text{ Hz}}$$

Solution 38: Wave Superposition and Interference

(a) Superposition of two waves (3 marks):

$$y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx - \omega t + \phi)$$

Using the identity $\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$:

Let $\alpha = kx - \omega t + \phi$ and $\beta = kx - \omega t$

$$\alpha - \beta = \phi, \quad \alpha + \beta = 2(kx - \omega t) + \phi$$

$$y_{total} = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

$$\boxed{y_{total} = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)}$$

(b) Constructive and destructive interference (2 marks):

Amplitude of resultant: $A_{result} = 2A\cos(\phi/2)$

Constructive interference: Maximum amplitude when $\cos(\phi/2) = \pm 1$

$$\frac{\phi}{2} = n\pi \Rightarrow \boxed{\phi = 2n\pi = 0, 2\pi, 4\pi, ... \text{ (even multiples of } \pi)}$$

Destructive interference: Zero amplitude when $\cos(\phi/2) = 0$

$$\frac{\phi}{2} = \frac{\pi}{2} + n\pi \Rightarrow \boxed{\phi = \pi + 2n\pi = \pi, 3\pi, 5\pi, ... \text{ (odd multiples of } \pi)}$$

(c) Resultant amplitude for $\phi = \pi/3$ (2 marks):

$$A_{result} = 2A\cos\left(\frac{\pi/3}{2}\right) = 2A\cos\left(\frac{\pi}{6}\right) = 2A \cdot \frac{\sqrt{3}}{2}$$

$$\boxed{A_{result} = \sqrt{3}A \approx 1.73A}$$

(d) Beats (3 marks):

$$y_1 + y_2 = A\cos(2\pi f_1 t) + A\cos(2\pi f_2 t)$$

Using $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$:

$$y_{total} = 2A\cos\left(2\pi\frac{f_1+f_2}{2}t\right)\cos\left(2\pi\frac{f_1-f_2}{2}t\right)$$

$$\boxed{y_{total} = 2A\cos\left(\pi(f_1-f_2)t\right)\cos\left(\pi(f_1+f_2)t\right)}$$

The result is a wave at the average frequency $\frac{f_1+f_2}{2}$ with an amplitude that varies at frequency $\frac{|f_1-f_2|}{2}$.

The **beat frequency** is $|f_1 - f_2|$ because the intensity (proportional to amplitude squared) varies at twice the amplitude modulation frequency:

$$I \propto \cos^2\left(\pi(f_1-f_2)t\right)$$

which has period $\frac{1}{|f_1-f_2|}$, giving beat frequency $|f_1 - f_2|$.

$$\boxed{f_{beat} = |f_1 - f_2|}$$

Solution 39: Standing Waves in Pipes

(a) Open-Open Pipe (3 marks):

At open ends, air is free to move, creating pressure nodes and displacement antinodes. Both ends are antinodes.

The distance between adjacent antinodes is $\lambda/2$, so:

$$L = n \cdot \frac{\lambda_n}{2}, \quad n = 1, 2, 3, ...$$

$$\lambda_n = \frac{2L}{n}$$

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}$$

$$\boxed{f_n = \frac{nv}{2L}, \quad n = 1, 2, 3, ... \text{ (all harmonics)}}$$

(b) Open-Closed Pipe (3 marks):

Open end: antinode (air moves freely) Closed end: node (air cannot move)

Distance from node to adjacent antinode is $\lambda/4$:

$$L = \frac{\lambda_1}{4}, \frac{3\lambda_3}{4}, \frac{5\lambda_5}{4}, ... = \frac{(2n-1)\lambda_n}{4}$$

$$\lambda_n = \frac{4L}{2n-1}$$

$$f_n = \frac{v}{\lambda_n} = \frac{(2n-1)v}{4L}$$

$$\boxed{f_n = \frac{(2n-1)v}{4L}, \quad n = 1, 2, 3, ...}$$

Only odd harmonics exist because the asymmetric boundary conditions (node at one end, antinode at the other) require wavelengths that fit an odd number of quarter-wavelengths.

(c) 2.0 m organ pipe (2 marks):

Open-open: $f_1 = \frac{v}{2L} = \frac{343}{2(2.0)} = 85.75$ Hz

$$\boxed{f_{1,open} = 85.8 \text{ Hz}}$$

Open-closed: $f_1 = \frac{v}{4L} = \frac{343}{4(2.0)} = 42.875$ Hz

$$\boxed{f_{1,closed} = 42.9 \text{ Hz}}$$

(d) Pipe length for 440 Hz with one end closed (2 marks):

$$f_1 = \frac{v}{4L}$$
$$L = \frac{v}{4f_1} = \frac{343}{4(440)} = 0.195 \text{ m}$$

$$\boxed{L = 0.195 \text{ m} = 19.5 \text{ cm}}$$

Solution 40: Energy in Waves

(a) Kinetic energy per unit length (2 marks):

Consider a small element of string with length dx and mass $dm = \mu\, dx$.

The transverse velocity is $v_y = \frac{\partial y}{\partial t}$.

Kinetic energy of element: $dKE = \frac{1}{2}dm \cdot v_y^2 = \frac{1}{2}\mu\, dx \left(\frac{\partial y}{\partial t}\right)^2$

$$\boxed{\frac{dKE}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2}$$

(b) Average total energy per unit length (3 marks):

For $y = A\sin(kx - \omega t)$:

$$\frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t)$$

Kinetic energy density: $\frac{dKE}{dx} = \frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)$

The potential energy comes from stretching. It can be shown that:

$$\frac{dPE}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}TA^2k^2\cos^2(kx - \omega t)$$

Since $v = \omega/k = \sqrt{T/\mu}$, we have $T = \mu\omega^2/k^2$:

$$\frac{dPE}{dx} = \frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)$$

Total energy density: $\frac{dE}{dx} = \mu A^2\omega^2\cos^2(kx - \omega t)$

Time average: $\langle\cos^2\rangle = \frac{1}{2}$

$$\boxed{\left\langle\frac{dE}{dx}\right\rangle = \frac{1}{2}\mu A^2\omega^2}$$

(c) Power transmitted (2 marks):

$$P = v \cdot \left\langle\frac{dE}{dx}\right\rangle = \sqrt{\frac{T}{\mu}} \cdot \frac{1}{2}\mu A^2\omega^2 = \frac{1}{2}\mu A^2\omega^2\sqrt{\frac{T}{\mu}}$$

$$\boxed{P = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2}$$

(d) Numerical calculations (3 marks):

Given: $A = 2.0$ mm $= 0.002$ m, $f = 100$ Hz, $\mu = 5.0$ g/m $= 0.005$ kg/m, $T = 50$ N

Wave speed:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{50}{0.005}} = \sqrt{10000} = 100 \text{ m/s}$$

$$\boxed{v = 100 \text{ m/s}}$$

Angular frequency: $\omega = 2\pi f = 2\pi(100) = 628.3$ rad/s

Average energy per unit length:

$$\left\langle \frac{dE}{dx} \right\rangle = \frac{1}{2}(0.005)(0.002)^2(628.3)^2 = \frac{1}{2}(0.005)(4 \times 10^{-6})(394784)$$

$$= 3.95 \times 10^{-3} \text{ J/m}$$

$$\boxed{\left\langle \frac{dE}{dx} \right\rangle = 3.95 \text{ mJ/m}}$$

Power:

$$P = v \cdot \left\langle \frac{dE}{dx} \right\rangle = 100 \times 3.95 \times 10^{-3} = 0.395 \text{ W}$$

$$\boxed{P = 0.40 \text{ W}}$$

End of Solutions

Good luck on your final exam!