Problem Set: Oscillation

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Part A: Spring Systems

Problem 1: Horizontal Spring-Mass System

A mass m slides on a frictionless horizontal surface, attached to a spring k.

(a) Starting from Newton's second law ($F = -kx$), derive the differential equation: $\ddot{x} + \frac{k}{m}x = 0$.

(b) Show that $x(t) = A\cos(\omega t + \phi)$ is a solution.

(c) For a mass $m = 0.5$ kg and $k = 200$ N/m, displaced 0.1 m from equilibrium and released from rest:

- Find the angular frequency ω.
- Write the specific equation of motion $x(t)$.
- Find the maximum velocity and acceleration.

Problem 2: Vertical Spring-Mass System

A spring of constant k hangs vertically. A mass m is attached to it and allowed to come to equilibrium.

(a) Draw free body diagrams for the mass at (1) the un-stretched spring position and (2) the equilibrium position y_{eq}.

(b) Show that the equilibrium stretch is $\Delta L = mg/k$.

(c) If the mass is displaced by y from this *new* equilibrium position, show that the net restoring force is $F_{net} = -ky$ (i.e., gravity cancels out of the restoring force term).

(d) Conclude that the frequency of oscillation ω is identical to the horizontal case.

Problem 3: Springs in Series and Parallel

Derive the effective spring constant k_{eq} for the following configurations:

(a) **Parallel:** Two springs k_1 and k_2 are attached side-by-side to the same mass m. (Displacements are equal, forces add).

$$k_{eq} = k_1 + k_2$$

(b) **Series:** Two springs k_1 and k_2 are connected end-to-end, with the mass m at the end of the chain. (Forces are equal, displacements add).

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

(c) A mass $m = 2$ kg is oscillating on two springs in series, each with $k = 100$ N/m. Calculate the period.

Part B: Energy and Momentum Applications

Problem 4: Energy Analysis

(a) For a harmonic oscillator with amplitude A, show that the total energy is $E = \frac{1}{2}kA^2$.

(b) Show that the velocity at any position x is $v = \pm\omega\sqrt{A^2 - x^2}$.

(c) At what position x is the kinetic energy exactly equal to the potential energy?

(d) If the amplitude is doubled, how does the total energy change? How does the maximum velocity change?

Problem 5: Collision-Induced Oscillation

A wooden block of mass M is attached to a spring k and rests on a frictionless table. A bullet of mass m moving at velocity v_0 strikes the block and embeds itself (perfectly inelastic collision).

(a) Use conservation of momentum to find the velocity V_f of the system immediately after the collision (at the equilibrium position).

(b) This velocity V_f becomes the maximum velocity (v_{max}) of the subsequent oscillation. Use energy conservation to find the amplitude A of the oscillation.

(c) Show that:

$$A = \frac{mv_0}{\sqrt{k(M+m)}}$$

(d) For $M = 1.0$ kg, $m = 0.01$ kg, $v_0 = 400$ m/s, and $k = 400$ N/m, calculate the amplitude and period.

Problem 6: Block on an Oscillating Plate

A block rests on top of a horizontal plate which oscillates vertically with simple harmonic motion: $y(t) = A\cos(\omega t)$.

(a) Draw the free body diagram of the block. Note that the normal force N varies with time.

(b) Write Newton's Second Law for the block: $N - mg = ma = m(-\omega^2 y)$.

(c) The block loses contact with the plate when $N \to 0$. Show that this happens when the downward acceleration of the plate exceeds g.

(d) Derive the condition for the maximum amplitude A_{max} allowed for the block to stay on the plate at a given frequency f:

$$A_{max} = \frac{g}{(2\pi f)^2}$$

Part C: Angular Oscillators

Problem 7: The Simple Pendulum

A mass m is suspended by a string of length L.

(a) Apply the small angle approximation ($\sin\theta \approx \theta$) to derive the SHM equation $\ddot{\theta} + \frac{g}{L}\theta = 0$.

(b) Show that the period is $T = 2\pi\sqrt{L/g}$.

(c) A grandfather clock has a period of $T = 2.0$ s. Calculate the required length L.

(d) If the clock is moved to the Moon ($g \approx g_{earth}/6$), what is the new period?

Problem 8: The Physical Pendulum

A rigid body of mass M pivots about a point distance d from its center of mass.

(a) Write the rotational equation of motion $\tau = I\alpha$.

(b) Show that for small angles, the period is $T = 2\pi\sqrt{\frac{I}{Mgd}}$.

(c) A uniform rod of length L pivots about one end ($I = \frac{1}{3}mL^2$). Find its period.

Problem 9: The Torsional Pendulum

A disk of rotational inertia I hangs from a wire. When twisted by angle θ, the wire exerts a restoring torque $\tau = -\kappa\theta$.

(a) Derive the differential equation of motion.

(b) Show that the period is $T = 2\pi\sqrt{I/\kappa}$.

(c) This system is often used in watches (balance wheels). If I doubles, how must κ change to keep the same period?

Part D: Advanced and Optional Topics

Problem 10: The U-Tube Oscillator (Fluids)

A liquid of density ρ and total length L oscillates in a U-tube of area A.

(a) Show that the restoring force is the weight of the unbalanced liquid column height $2x$: $F = -2\rho Agx$.

(b) Using $F = m_{total}a$, derive the frequency $\omega = \sqrt{2g/L}$.

Problem 11: Tunnel Through the Earth (Gauss's Law)

A tunnel is drilled through the center of the Earth (assume uniform density).

(a) Use Gauss's Law to show gravity inside the Earth is $F(r) = -\frac{GMm}{R^3}r$.

(b) Since $F \propto -r$, this is SHM. Find the period of oscillation.

(c) Evaluate the period in minutes ($R = 6371$ km).

Problem 12: Damped Oscillations

A damped oscillator obeys $m\ddot{x} + b\dot{x} + kx = 0$.

(a) For underdamping (b is small), the solution is $x(t) = Ae^{-bt/2m}\cos(\omega' t)$. Sketch this motion.

(b) Define the time constant $\tau = m/b$. How is this related to the energy decay?