

Problem Set: Kinematics

Karl Keshavarzi – ECE 105, Classical Mechanics

Part A: One-Dimensional Kinematics

Problem 1: Constant Acceleration Motion

A particle moves along a straight line with constant acceleration.

- (a) Starting from the definition of acceleration $a = \frac{dv}{dt}$, derive the kinematic equation:

$$v_f = v_i + at$$

- (b) Integrate the velocity to derive the position equation:

$$x = x_0 + v_i t + \frac{1}{2} a t^2$$

- (c) Derive the velocity-displacement relation by eliminating time:

$$v_f^2 = v_i^2 + 2a(x - x_0)$$

- (d) Show that the average velocity for constant acceleration is:

$$\bar{v} = \frac{v_i + v_f}{2}$$

- (e) A car traveling at $v_0 = 100$ m/s decelerates to rest over a distance $d = 10$ m:

- Calculate the acceleration
- Find the time taken to stop
- Express the deceleration in terms of g (take $g = 9.8$ m/s²)

Problem 2: Average Speed & Graphical Concepts (Classic Traps)

- (a) **The Harmonic Mean Trap:** A driver travels the first half of the *distance* of a trip at speed v_1 and the second half of the *distance* at speed v_2 . Show that the average speed for the total trip is:

$$\bar{v} = \frac{2v_1 v_2}{v_1 + v_2}$$

(Contrast this with traveling half the *time* at each speed, which yields the arithmetic mean).

(b) **Graphical Analysis:** A particle starts at $x = 0$ with velocity $v_0 > 0$. It is subject to a constant negative acceleration $-a$.

- Sketch the $v - t$ graph from $t = 0$ to $t = 2v_0/a$.
- Sketch the $x - t$ graph. Identify the time when the particle turns around.
- Explain the difference between *displacement* and *total distance traveled* using areas on your $v - t$ graph.

Part B: Free Fall and Vertical Motion

Problem 3: Projectile Launched Vertically

A projectile is thrown vertically upward with initial speed v_0 .

(a) Using $y = y_0 + v_0 t - \frac{1}{2}gt^2$, find the time to reach maximum height.

(b) Show that the maximum height is:

$$h_{\max} = \frac{v_0^2}{2g}$$

(c) Derive the total time of flight (up and down):

$$t_{\text{total}} = \frac{2v_0}{g}$$

(d) Show that the speed upon return to the launch height equals v_0 .

(e) A ball is thrown upward from height $h_0 = 1.5$ m with initial speed $v_0 = 19.2$ m/s:

- Find the maximum height above the ground
- Calculate the time to reach maximum height
- Determine the total time in the air
- Find the speed just before landing

Problem 4: Objects Falling from Rest

An object falls from rest under gravity.

(a) Derive the velocity after falling through height h :

$$v = \sqrt{2gh}$$

(b) Show that the distance fallen in time t is:

$$h = \frac{1}{2}gt^2$$

- (c) Find the velocity as a function of time: $v = gt$.
- (d) An object takes time t_1 to fall past a window of height H . Show that the height above the window from which it was dropped is:

$$h = \frac{gt_1^2}{8} \left(\frac{2H}{gt_1^2} - 1 \right)^2$$

- (e) An object falls past a window 2 m high in $t = 0.4$ s:

- Find the velocity at the top of the window
- Calculate the velocity at the bottom of the window
- Determine the height above the window from which it was dropped

Problem 5: Two-Body Vertical Collision

Two particles are moving under gravity along the same vertical line.

- (a) Ball A is dropped from rest from a height H . At the exact same instant, Ball B is thrown vertically upward from the ground with initial speed v_0 . Derive an expression for the time t_c when they collide.
- (b) Show that the height of collision is:

$$y_c = H \left(1 - \frac{gH}{2v_0^2} \right)$$

- (c) What is the minimum initial speed $v_{0,\min}$ required for Ball B to catch Ball A before Ball A hits the ground?

Part C: Two-Dimensional Kinematics

Problem 6: Projectile Motion Fundamentals

A projectile is launched with initial velocity v_0 at angle θ above horizontal.

- (a) Write the position components as functions of time:

$$x = v_0 \cos \theta \cdot t$$

$$y = v_0 \sin \theta \cdot t - \frac{1}{2}gt^2$$

- (b) Derive the time of flight for launch and landing at the same height:

$$t_{\text{flight}} = \frac{2v_0 \sin \theta}{g}$$

(c) Show that the maximum range occurs at $\theta = 45$ and equals:

$$R_{\max} = \frac{v_0^2}{g}$$

(d) Derive the maximum height:

$$h_{\max} = \frac{v_0^2 \sin^2 \theta}{2g}$$

(e) A projectile is launched at $v_0 = 20$ m/s at angle $\theta = 45$:

- Calculate the time of flight
- Find the maximum height
- Determine the range
- Find the velocity (magnitude and direction) at $t = 1.0$ s

Problem 7: Projectile Motion from Elevated Position

A projectile is launched from height h above the ground.

(a) Write the trajectory equation:

$$y = h + x \tan \theta - \frac{gx^2}{2v_0^2 \cos^2 \theta}$$

(b) Find the time of flight by solving:

$$0 = h + v_0 \sin \theta \cdot t - \frac{1}{2}gt^2$$

(c) Derive the range R in terms of h , v_0 , θ , and g .

(d) Show that for $\theta = 0$ (horizontal launch), the time of flight is:

$$t = \sqrt{\frac{2h}{g}}$$

(e) An object is thrown horizontally at $v_0 = 15.63$ m/s from a height $h = 2.0$ m:

- Find the time to hit the ground
- Calculate the horizontal distance traveled
- Determine the velocity components at impact
- Find the angle of impact below horizontal

Problem 8: Motorcycle Jump Problem

A motorcyclist approaches a ramp to jump across a gap.

- (a) A motorcycle traveling horizontally at speed v_m drives off a cliff of height $h = 20$ m. The horizontal distance to the landing point is $D = 120$ m. Derive the required speed:

$$v_m = D \sqrt{\frac{g}{2h}}$$

- (b) If the ramp is at angle θ , show that the condition to land at horizontal distance D is:

$$D = \frac{v_0^2 \sin 2\theta}{g} + v_0 \cos \theta \sqrt{\frac{2h}{g}}$$

(for small height differences)

- (c) Discuss two approaches: (1) assume $v > v_{\max}$ and find how far the motorcycle goes, or (2) assume it makes the jump and find the required speed.
- (d) For $h = 20$ m, $D = 120$ m, and $v_m = 20$ km/hr = 5.56 m/s:

- Determine if the motorcycle makes the jump
- If not, find the actual distance traveled
- Calculate the required speed to just make the jump

Part D: Relative Motion

Problem 9: Reference Frames and Relative Velocity

Motion appears different in different reference frames.

- (a) Define relative velocity. For three objects A, B, and ground G, show that:

$$\vec{v}_{AB} = \vec{v}_{AG} - \vec{v}_{BG}$$

- (b) Prove the vector addition property:

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

- (c) A person walks on a moving train. If the train moves at velocity \vec{v}_T and the person walks at velocity \vec{v}_P relative to the train, find the person's velocity relative to the ground.
- (d) For a boat crossing a river with current, derive the condition for the boat to land directly opposite its starting point.
- (e) A train travels east at $v_T = 72$ km/hr. A package falls from the train:

- From the ground frame, describe the package's trajectory
- From the train frame, describe the package's trajectory
- If the package falls from height $h = 60$ m, find where it lands relative to the drop point

Problem 10: Rain and Umbrella

- Rain is falling vertically at speed v_r relative to the ground. A person runs horizontally with speed v_p . Derive the angle θ (with the vertical) at which the person must hold their umbrella to get the best protection.
- A wind starts blowing, causing the rain to fall at speed v_r at an angle α with the vertical. If the person runs *against* the wind at speed v_p , derive the new condition for the umbrella angle.

Problem 11: River Crossing Problems

A boat crosses a river with width w and current speed v_c .

- If the boat's speed relative to water is v_b at angle θ upstream from perpendicular, show that the velocity components relative to shore are:

$$v_x = v_c - v_b \sin \theta$$

$$v_y = v_b \cos \theta$$

- Find the angle θ needed to travel straight across:

$$\sin \theta = \frac{v_c}{v_b}$$

(only possible if $v_b > v_c$)

- For the minimum time crossing, show that $\theta = 0$ (perpendicular to current).
- Derive the drift distance for perpendicular crossing:

$$d = v_c \cdot \frac{w}{v_b}$$

- A boat with speed $v_b = 5$ m/s crosses a river of width $w = 500$ m with current $v_c = 2$ m/s:
 - If heading perpendicular to current, find the drift distance and crossing time
 - Find the angle to head upstream to cross straight across
 - Calculate the crossing time for straight-across trajectory
 - Determine the fastest possible crossing time

Problem 12: Wind and Airplane Navigation

An airplane's velocity relative to ground depends on wind velocity.

- State the velocity addition formula:

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$$

where P = plane, A = air, G = ground.

- (b) An airplane flies in still air at speed v_0 . A wind of speed v_w blows at angle ϕ to the airplane's heading. Derive the ground speed.
- (c) For an airplane heading north at airspeed v_0 with eastward wind v_w , show that the actual heading is:

$$\theta = \arctan\left(\frac{v_w}{v_0}\right)$$

east of north.

- (d) To travel directly north despite the wind, show the plane must head at angle:

$$\theta = \arcsin\left(\frac{v_w}{v_0}\right)$$

west of north.

- (e) An airplane has airspeed $v_0 = 300$ km/hr. A wind blows at $v_w = 50$ km/hr from the southwest (toward NE):
- If heading north, find the actual ground velocity (magnitude and direction)
 - Determine the heading needed to travel due north
 - Calculate the ground speed when traveling due north
 - Find the heading for maximum ground speed

Part E: Circular Motion and Acceleration

Problem 13: Uniform Circular Motion

A particle moves in a circle of radius R with constant speed v .

- (a) Show that the centripetal acceleration is:

$$a_c = \frac{v^2}{R}$$

directed toward the center.

- (b) Express the centripetal acceleration in terms of angular velocity ω :

$$a_c = \omega^2 R$$

where $v = \omega R$.

- (c) Derive the period of one revolution:

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\omega}$$

- (d) For motion with tangential acceleration a_t in addition to centripetal acceleration, show that the total acceleration is:

$$|\vec{a}_{\text{tot}}| = \sqrt{a_t^2 + a_c^2}$$

- (e) A particle moves on a circular path of radius $R = 500$ m with speed $v = 6.0$ m/s and tangential acceleration $a_t = 0.3$ m/s 2 :

- Calculate the centripetal acceleration
- Find the magnitude of total acceleration
- Determine the angle of total acceleration from the radial direction
- Calculate the period of revolution

Problem 14: Radius of Curvature

The concept of centripetal acceleration $a_c = v^2/R$ applies to any curved path, where R is the instantaneous radius of curvature and a_c is the component of acceleration perpendicular to velocity.

- (a) A projectile is launched with speed v_0 at angle θ . At the peak of its trajectory, the speed is $v_x = v_0 \cos \theta$ and the total acceleration is g (downward). Show that the radius of curvature at the top is:

$$R_{\text{top}} = \frac{v_0^2 \cos^2 \theta}{g}$$

- (b) At the launch point ($t = 0$), the velocity is v_0 and the component of gravity perpendicular to the velocity is $g \cos \theta$. Show that the radius of curvature at launch is:

$$R_{\text{launch}} = \frac{v_0^2}{g \cos \theta}$$

Problem 15: Non-Uniform Circular Motion

Consider circular motion with changing speed.

- (a) Define tangential acceleration:

$$a_t = \frac{dv}{dt} = R\alpha$$

where $\alpha = \frac{d\omega}{dt}$ is angular acceleration.

- (b) For constant angular acceleration α , derive the angular kinematic equations:

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \theta_0 + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

- (c) Show that the total acceleration makes angle ϕ with the radius where:

$$\tan \phi = \frac{a_t}{a_c}$$

- (d) A wheel starts from rest and rotates with constant angular acceleration α for time t . Find the ratio of tangential to centripetal acceleration as a function of time.

- (e) A car enters a circular track of radius $R = 50$ m at speed $v_i = 10$ m/s and accelerates tangentially at $a_t = 2.0$ m/s² for $t = 5.0$ s:

- Calculate the final speed
- Find the centripetal acceleration at $t = 0$ and $t = 5$ s
- Determine the total acceleration magnitude at $t = 5$ s
- Calculate the angle swept out during acceleration

Part F: Complex Kinematic Problems

Problem 16: Chase Problems

Two objects move relative to each other.

- (a) A police car initially at position $x_P = 0$ traveling at $v_P = 90$ km/hr (25 m/s) pursues a truck initially at $x_T = 50$ m ahead, traveling at $v_T = 70$ km/hr. Write the position functions and find the catch-up time.
- (b) If the police car accelerates from rest at a_P , write the position equation:

$$x_P(t) = \frac{1}{2}a_P t^2$$

and solve for when it catches the truck at constant speed.

- (c) For the police car decelerating before impact, incorporate deceleration starting at time t_1 when the speeds match.
- (d) Generalize to find the minimum initial distance d_{\min} for the police car to catch up without collision.
- (e) A police car decelerates from $v_0 = 252$ km/hr to match a truck's speed $v_T = 70$ km/hr with deceleration $a = -25$ m/s². Initial separation is $d_0 = 50$ m:

- Find the time when speeds match
- Calculate positions when speeds match
- Determine if collision occurs
- Find the total time and distance for the police car to catch the truck

Problem 17: Optimal Angle Problems

Finding angles that maximize or minimize quantities.

- (a) For projectile motion, derive the angle θ that maximizes range on level ground:

$$\frac{dR}{d\theta} = 0 \Rightarrow \theta = 45^\circ$$

- (b) For a projectile launched from height h , show that the optimal angle is:

$$\theta_{\text{opt}} < 45^\circ$$

- (c) A projectile must clear a wall of height h_w at horizontal distance d_w . Derive the condition on launch angle θ .

- (d) For landing at distance D down a slope of angle α , show that the optimal launch angle is:

$$\theta_{\text{opt}} = 45^\circ + \frac{\alpha}{2}$$

- (e) A ski jumper leaves a ramp at $v_0 = 20$ m/s. The landing slope is at angle $\alpha = 40^\circ$ below horizontal:

- Find the optimal launch angle for maximum distance
- Calculate the maximum distance along the slope
- Determine the time of flight
- Find the velocity at landing

Problem 18: Monkey and Hunter Problem

A classic problem demonstrating projectile motion.

- (a) A monkey hangs from a branch at height h and horizontal distance x_T from a hunter. The hunter aims directly at the monkey. Show that the bullet will hit the monkey if the monkey drops the instant the gun fires.

- (b) Prove this by showing both the bullet and monkey fall the same vertical distance:

$$\Delta y_{\text{bullet}} = \Delta y_{\text{monkey}} = \frac{1}{2}gt^2$$

- (c) Write the condition for the bullet to reach the monkey's drop position:

$$x_T = v_0 \cos \theta \cdot t$$

$$h - \frac{1}{2}gt^2 = v_0 \sin \theta \cdot t - \frac{1}{2}gt^2$$

- (d) Show that $\tan \theta = h/x_T$ is required (aiming directly at the monkey).

- (e) For $h = 20$ m, x_T = (horizontal distance), and $v_0 = 50$ m/s:

- Find the required aiming angle
- Calculate the time to intercept
- Determine the height at which they meet
- Verify the result using the trajectory equation

Problem 19: Motion on Inclined Planes

Kinematics on inclined surfaces.

- (a) For a block sliding down a frictionless incline at angle θ , show that the acceleration along the incline is:
$$a = g \sin \theta$$
- (b) A ball rolling down an incline has acceleration $a = \frac{5}{7}g \sin \theta$ (due to rotational inertia). Derive the time to travel distance L down the incline.
- (c) For motion up and down an incline with different accelerations (due to friction), show that the total time differs from frictionless motion.
- (d) Two particles start from rest at the top of differently angled frictionless ramps of the same height h . Show that they reach the bottom with the same speed but different times.
- (e) A cart starts at rest at the top of a $\theta = 30^\circ$ incline of length $L = 50$ m:
 - Calculate the acceleration down the incline
 - Find the time to reach the bottom
 - Determine the final speed
 - Compare with free fall time from height $h = L \sin \theta$

Part G: Advanced and Optional Topics

Problem 20: Motion with Changing Acceleration

Consider motion where acceleration changes with time or position.

- (a) For acceleration as a function of time $a(t) = a_0 + bt$, derive expressions for $v(t)$ and $x(t)$ given initial conditions v_0 and x_0 .
- (b) A particle has position-dependent acceleration $a(x) = -kx$. Show that this leads to simple harmonic motion.
- (c) For a particle with $a = -\alpha v$ (velocity-dependent acceleration), derive the velocity as a function of time:
$$v(t) = v_0 e^{-\alpha t}$$

- (d) Find the total distance traveled as $t \rightarrow \infty$ for the case in part (c).
- (e) A rocket has thrust that decreases linearly: $a(t) = a_0(1 - t/T)$ for $0 \leq t \leq T$. For $a_0 = 15$ m/s² and $T = 10$ s:
 - Find the maximum velocity
 - Calculate the distance traveled during powered flight
 - Determine the average acceleration

Numerical Answers

Problem 1(e): $a = -500 \text{ m/s}^2 \approx -51g$; $t = 0.2 \text{ s}$

Problem 3(e): $h_{\max} = 20.3 \text{ m}$; $t_{\text{up}} = 1.96 \text{ s}$; $t_{\text{total}} = 4.18 \text{ s}$; $v_f = 20.5 \text{ m/s}$

Problem 4(e): $v_{\text{top}} = 3.04 \text{ m/s}$; $v_{\text{bottom}} = 5.96 \text{ m/s}$; $h = 0.47 \text{ m}$

Problem 6(e): $t = 2.89 \text{ s}$; $h = 10.2 \text{ m}$; $R = 40.8 \text{ m}$; $v = 17.0 \text{ m/s}$ at $\theta = 32.6$ below horizontal

Problem 7(e): $t = 0.64 \text{ s}$; $d = 10.0 \text{ m}$; $v_x = 15.63 \text{ m/s}$, $v_y = -6.27 \text{ m/s}$; $\theta = -21.8$

Problem 8(d): Does not make jump; $d_{\text{actual}} = 11.2 \text{ m}$; $v_{\text{required}} = 59.4 \text{ m/s} = 214 \text{ km/h}$

Problem 9(e): Parabola from ground frame; straight down from train frame; $x = 145 \text{ m}$ from drop point

Problem 11(e): $d = 200 \text{ m}$, $t = 100 \text{ s}$ (perpendicular); $\theta = 23.6$ (straight across); $t = 109 \text{ s}$ (straight); $t_{\min} = 100 \text{ s}$

Problem 12(e): $|\vec{v}_{PG}| = 337 \text{ km/hr}$ at $\theta = 6.01 \text{ E of N}$; heading: $\theta = 9.59 \text{ W of N}$; $v_G = 296 \text{ km/hr}$; straight east for max

Problem 13(e): $a_c = 0.072 \text{ m/s}^2$; $|\vec{a}| = 0.309 \text{ m/s}^2$; $\phi = 13.5$; $T = 523 \text{ s}$

Problem 15(e): $v_f = 20 \text{ m/s}$; $a_c(0) = 2.0 \text{ m/s}^2$, $a_c(5) = 8.0 \text{ m/s}^2$; $|\vec{a}(5)| = 8.25 \text{ m/s}^2$; $\theta = 0.75 \text{ rad}$

Problem 16(e): $t = 22.6 \text{ s}$; $x_P = -31.25 \text{ m}$ (behind), $x_T = 50 \text{ m}$; No collision; $t_{\text{total}} = 22.6 \text{ s}$

Problem 17(e): $\theta_{\text{opt}} = 65$; $d_{\max} = 97.96 \text{ m}$; $t = 4.38 \text{ s}$; $v_{\text{land}} = 24.5 \text{ m/s}$

Problem 18(e): $\theta = \arctan(h/x_T)$; calculated based on given x_T

Problem 19(e): $a = 4.9 \text{ m/s}^2$; $t = 4.52 \text{ s}$; $v = 22.1 \text{ m/s}$; $t_{\text{freefall}} = 2.26 \text{ s}$

Problem 20(e): $v_{\max} = 75 \text{ m/s}$; $d = 500 \text{ m}$; $\bar{a} = 7.5 \text{ m/s}^2$