Problem Set: Kinematics

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Part A: One-Dimensional Kinematics

Problem 1: Constant Acceleration Motion

A particle moves along a straight line with constant acceleration.

(a) Starting from the definition of acceleration $a = \frac{dv}{dt}$, derive the kinematic equation:

$$v_f = v_i + at$$

(b) Using integration, derive the position equation:

$$x = x_0 + v_i t + \frac{1}{2}at^2$$

(c) Derive the velocity-displacement relation:

$$v_f^2 = v_i^2 + 2a(x - x_0)$$

(d) Show that the average velocity for constant acceleration is:

$$\bar{v} = \frac{v_i + v_f}{2}$$

(e) A car traveling at $v_0 = 100$ m/s decelerates to rest over a distance $d = 10$ m:

- Calculate the acceleration
- Find the time taken to stop
- Express the deceleration in terms of g (take $g = 9.8$ m/s^2)

Problem 2: Average Speed & Graphical Concepts

(a) **The Harmonic Mean Trap:** A driver travels the first half of the *distance* of a trip at speed v_1 and the second half of the *distance* at speed v_2. Show that the average speed for the total trip is:

$$\bar{v} = \frac{2v_1 v_2}{v_1 + v_2}$$

(Contrast this with traveling half the *time* at each speed, which yields the arithmetic mean).

(b) **Graphical Analysis:** A particle starts at $x = 0$ with velocity $v_0 > 0$. It is subject to a constant negative acceleration $-a$.

- Sketch the $v - t$ graph from $t = 0$ to $t = 2v_0/a$.
- Sketch the $x - t$ graph. Identify the time when the particle turns around.
- Explain the difference between *displacement* and *total distance traveled* using areas on your $v - t$ graph.

Part B: Free Fall and Vertical Motion

Problem 3: Projectile Launched Vertically

A projectile is thrown vertically upward with initial speed v_0.

(a) Using $y = y_0 + v_0 t - \frac{1}{2}gt^2$, find the time to reach maximum height.

(b) Show that the maximum height is:

$$h_{\text{max}} = \frac{v_0^2}{2g}$$

(c) Derive the total time of flight (up and down):

$$t_{\text{total}} = \frac{2v_0}{g}$$

(d) Show that the speed upon return to the launch height equals v_0.

(e) A ball is thrown upward from height $h_0 = 1.5$ m with initial speed $v_0 = 19.2$ m/s:

- Find the maximum height above the ground
- Calculate the time to reach maximum height
- Determine the total time in the air
- Find the speed just before landing

Problem 4: Objects Falling from Rest

An object falls from rest under gravity.

(a) Derive the velocity after falling through height h:

$$v = \sqrt{2gh}$$

(b) Show that the distance fallen in time t is:

$$h = \frac{1}{2}gt^2$$

(c) Find the velocity as a function of time.

(d) An object takes time t_1 to fall past a window of height H. Show that the height above the window from which it was dropped is:

$$h = \frac{gt_1^2}{8}\left(\frac{2H}{gt_1^2} - 1\right)^2$$

(e) An object falls past a window 2 m high in $t = 0.4$ s:

- Find the velocity at the top of the window
- Calculate the velocity at the bottom of the window
- Determine the height above the window from which it was dropped

Problem 5: Two-Body Vertical Collision

Two particles are moving under gravity along the same vertical line.

(a) Ball A is dropped from rest from a height H. At the exact same instant, Ball B is thrown vertically upward from the ground with initial speed v_0. Derive an expression for the time t_c when they collide.

(b) Show that the height of collision is:

$$y_c = H\left(1 - \frac{gH}{2v_0^2}\right)$$

(c) What is the minimum initial speed $v_{0,\text{min}}$ required for Ball B to catch Ball A before Ball A hits the ground?

Part C: Two-Dimensional Kinematics

Problem 6: Projectile Motion Fundamentals

A projectile is launched with initial velocity v_0 at angle θ above horizontal.

(a) Write the position components as functions of time:

$$x = v_0 \cos\theta \cdot t$$

$$y = v_0 \sin\theta \cdot t - \frac{1}{2}gt^2$$

(b) Derive the time of flight for launch and landing at the same height:

$$t_{\text{flight}} = \frac{2v_0 \sin\theta}{g}$$

(c) Show that the maximum range occurs at $\theta = 45$ and equals:

$$R_{\text{max}} = \frac{v_0^2}{g}$$

(d) Derive the maximum height:

$$h_{\text{max}} = \frac{v_0^2 \sin^2 \theta}{2g}$$

(e) A projectile is launched at $v_0 = 20$ m/s at angle $\theta = 45$:

- Calculate the time of flight
- Find the maximum height
- Determine the range
- Find the velocity (magnitude and direction) at $t = 1.0$ s

Problem 7: Projectile Motion from Elevated Position

A projectile is launched from height h above the ground.

(a) Derive the trajectory equation:

$$y = h + x \tan \theta - \frac{gx^2}{2v_0^2 \cos^2 \theta}$$

(b) Find the time of flight by solving:

$$0 = h + v_0 \sin \theta \cdot t - \frac{1}{2} g t^2$$

(c) Derive the range R in terms of h, v_0, θ, and g.

(d) Show that for $\theta = 0$ (horizontal launch), the time of flight is:

$$t = \sqrt{\frac{2h}{g}}$$

(e) An object is thrown horizontally at $v_0 = 15.63$ m/s from a height $h = 2.0$ m:

- Find the time to hit the ground
- Calculate the horizontal distance traveled
- Determine the velocity components at impact
- Find the angle of impact below horizontal

Problem 8: Motorcycle Jump Problem

A motorcyclist approaches a ramp to jump across a gap.

(a) A motorcycle traveling horizontally at speed v_m drives off a cliff of height $h = 20$ m. The horizontal distance to the landing point is $D = 120$ m. Derive the required speed:

$$v_m = D\sqrt{\frac{g}{2h}}$$

(b) If the ramp is at angle θ, show that the condition to land at horizontal distance D is:

$$D = \frac{v_0^2 \sin 2\theta}{g} + v_0 \cos\theta\sqrt{\frac{2h}{g}}$$

(for small height differences)

(c) Discuss two approaches: (1) assume $v > v_{\max}$ and find how far the motorcycle goes, or (2) assume it makes the jump and find the required speed.

(d) For $h = 20$ m, $D = 120$ m, and $v_m = 20$ km/hr $= 5.56$ m/s:

- Determine if the motorcycle makes the jump
- If not, find the actual distance traveled
- Calculate the required speed to just make the jump

Part D: Relative Motion

Problem 9: Reference Frames and Relative Velocity

Motion appears different in different reference frames.

(a) Define relative velocity. For three objects A, B, and ground G, show that:

$$\vec{v}_{AB} = \vec{v}_{AG} - \vec{v}_{BG}$$

(b) Show the vector addition property:

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

(c) A person walks on a moving train. If the train moves at velocity $\vec{v}_T$ and the person walks at velocity $\vec{v}_P$ relative to the train, find the person's velocity relative to the ground.

(d) For a boat crossing a river with current, derive the condition for the boat to land directly opposite its starting point.

(e) A train travels east at $v_T = 72$ km/hr. A package falls from the train:

- From the ground frame, describe the package's trajectory
- From the train frame, describe the package's trajectory
- If the package falls from height $h = 60$ m, find where it lands relative to the drop point

Problem 10: Rain and Umbrella

(a) Rain is falling vertically at speed v_r relative to the ground. A person runs horizontally with speed v_p. Derive the angle θ (with the vertical) at which the person must hold their umbrella to get the best protection.

(b) A wind starts blowing, causing the rain to fall at speed v_r at an angle α with the vertical. If the person runs *against* the wind at speed v_p, derive the new condition for the umbrella angle.

Problem 11: River Crossing Problems

A boat crosses a river with width w and current speed v_c.

(a) If the boat's speed relative to water is v_b at angle θ upstream from perpendicular, show that the velocity components relative to shore are:

$$v_x = v_c - v_b \sin\theta$$

$$v_y = v_b \cos\theta$$

(b) Find the angle θ needed to travel straight across:

$$\theta = \arcsin\left(\frac{v_c}{v_b}\right)$$

(only possible if $v_b > v_c$)

(c) For the minimum time crossing, show that $\theta = 0$ (perpendicular to current).

(d) Derive the drift distance for perpendicular crossing:

$$d = v_c \cdot \frac{w}{v_b}$$

(e) A boat with speed $v_b = 5$ m/s crosses a river of width $w = 500$ m with current $v_c = 2$ m/s:

- If heading perpendicular to current, find the drift distance and crossing time
- Find the angle to head upstream to cross straight across
- Calculate the crossing time for straight-across trajectory
- Determine the fastest possible crossing time

Problem 12: Wind and Airplane Navigation

An airplane's velocity relative to ground depends on wind velocity.

(a) State the velocity addition (relative motion) formula:

$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$$

where P = plane, A = air, G = ground.

(b) An airplane flies in still air at speed v_0. A wind of speed v_w blows at angle ϕ to the airplane's heading. Derive the ground speed.

(c) For an airplane heading north at airspeed v_0 with eastward wind v_w, show that the actual heading is:

$$\theta = \arctan\left(\frac{v_w}{v_0}\right)$$

east of north.

(d) To travel directly north despite the wind, show the plane must head at angle:

$$\theta = \arcsin\left(\frac{v_w}{v_0}\right)$$

west of north.

(e) An airplane has airspeed $v_0 = 300$ km/hr. A wind blows at $v_w = 50$ km/hr from the southwest (toward NE):

- If heading north, find the actual ground velocity (magnitude and direction)
- Determine the heading needed to travel due north
- Calculate the ground speed when traveling due north
- Find the heading for maximum ground speed

Part E: Circular Motion and Acceleration

Problem 13: Uniform Circular Motion

A particle moves in a circle of radius R with constant speed v.

(a) Show that the centripetal acceleration is:

$$a_c = \frac{v^2}{R}$$

directed toward the center.

(b) Express the centripetal acceleration in terms of angular velocity ω:

$$a_c = \omega^2 R$$

where $v = \omega R$.

(c) Derive the period of one revolution:

$$T = \frac{2\pi}{\omega}$$

(d) For motion with tangential acceleration a_t in addition to centripetal acceleration, show that the total acceleration is:

$$|\vec{a}_{\text{tot}}| = \sqrt{a_t^2 + a_c^2}$$

(e) A particle moves on a circular path of radius $R = 500$ m with speed $v = 6.0$ m/s and tangential acceleration $a_t = 0.3$ m/s²:

- Calculate the centripetal acceleration
- Find the magnitude of total acceleration
- Determine the angle of total acceleration from the radial direction
- Calculate the period of revolution

Problem 14: Radius of Curvature

The concept of centripetal acceleration $a_c = v^2/R$ applies to any curved path, where R is the instantaneous radius of curvature and a_c is the component of acceleration perpendicular to velocity.

(a) A projectile is launched with speed v_0 at angle θ. At the peak of its trajectory, the speed is $v_x = v_0 \cos\theta$ and the total acceleration is g (downward). Show that the radius of curvature at the top is:

$$R_{\text{top}} = \frac{v_0^2 \cos^2\theta}{g}$$

(b) At the launch point ($t = 0$), the velocity is v_0 and the component of gravity perpendicular to the velocity is $g\cos\theta$. Show that the radius of curvature at launch is:

$$R_{\text{launch}} = \frac{v_0^2}{g\cos\theta}$$

Part F: Complex Kinematic Problems

Problem 15: Chase Problems

Two objects move relative to each other.

(a) A police car initially at position $x_P = 0$ traveling at $v_P = 90$ km/hr (25 m/s) pursues a truck initially at $x_T = 50$ m ahead, traveling at $v_T = 70$ km/hr. Write the position functions and find the catch-up time.

(b) If the police car accelerates from rest at a_P, write the position equation:

$$x_P(t) = \frac{1}{2} a_P t^2$$

and solve for when it catches the truck at constant speed.

(c) For the police car decelerating before impact, incorporate deceleration starting at time t_1 when the speeds match.

(d) Generalize to find the minimum initial distance $d_{\min}$ for the police car to catch up without collision.

(e) A police car decelerates from $v_0 = 252$ km/hr to match a truck's speed $v_T = 70$ km/hr with deceleration $a = -25$ m/s². Initial separation is $d_0 = 50$ m:

- Find the time when speeds match
- Calculate positions when speeds match
- Determine if collision occurs
- Find the total time and distance for the police car to catch the truck

Problem 16: Optimal Angle Problems

Finding angles that maximize or minimize quantities.

(a) For projectile motion, derive the angle θ that maximizes range on level ground:

$$\frac{dR}{d\theta} = 0 \Rightarrow \theta = 45$$

(b) For a projectile launched from height h, show that the optimal angle is:

$$\theta_{\text{opt}} < 45$$

(c) A projectile must clear a wall of height h_w at horizontal distance d_w. Derive the condition on launch angle θ.

(d) For landing at distance D down a slope of angle α, show that the optimal launch angle is:

$$\theta_{\text{opt}} = 45 + \frac{\alpha}{2}$$

(e) A ski jumper leaves a ramp at $v_0 = 20$ m/s. The landing slope is at angle $\alpha = 40$ below horizontal:

- Find the optimal launch angle for maximum distance
- Calculate the maximum distance along the slope
- Determine the time of flight
- Find the velocity at landing

Problem 17: Monkey and Hunter Problem

A classic problem demonstrating projectile motion.

(a) A monkey hangs from a branch at height h and horizontal distance x_T from a hunter. The hunter aims directly at the monkey. Show that the bullet will hit the monkey if the monkey drops the instant the gun fires.

(b) Prove this by showing both the bullet and monkey fall the same vertical distance:

$$\Delta y_{\text{bullet}} = \Delta y_{\text{monkey}} = \frac{1}{2}gt^2$$

(c) Write the condition for the bullet to reach the monkey's drop position:

$$x_T = v_0 \cos\theta \cdot t$$

$$h - \frac{1}{2}gt^2 = v_0 \sin\theta \cdot t - \frac{1}{2}gt^2$$

(d) Show that $\tan\theta = h/x_T$ is required (aiming directly at the monkey).

(e) For $h = 20$ m, $x_T =$ (horizontal distance), and $v_0 = 50$ m/s:

- Find the required aiming angle
- Calculate the time to intercept
- Determine the height at which they meet
- Verify the result using the trajectory equation

Problem 18: Motion on Inclined Planes

Kinematics on inclined surfaces.

(a) For a block sliding down a frictionless incline at angle θ, show that the acceleration along the incline is:

$$a = g\sin\theta$$

(b) A ball rolling down an incline has acceleration $a = \frac{5}{7}g\sin\theta$ (due to rotational inertia). Derive the time to travel distance L down the incline.

(c) For motion up and down an incline with different accelerations (due to friction), show that the total time differs from frictionless motion.

(d) Two particles start from rest at the top of differently angled frictionless ramps of the same height h. Show that they reach the bottom with the same speed but different times.

(e) A cart starts at rest at the top of a $\theta = 30$ incline of length $L = 50$ m:

- Calculate the acceleration down the incline
- Find the time to reach the bottom
- Determine the final speed
- Compare with free fall time from height $h = L\sin\theta$

Part G: Advanced and Optional Topics

Problem 19: Non-Uniform Circular Motion

Consider circular motion with changing speed.

(a) Define tangential acceleration:

$$a_t = \frac{dv}{dt} = R\alpha$$

where $\alpha = \frac{d\omega}{dt}$ is angular acceleration.

(b) For constant angular acceleration α, derive the angular kinematic equations:

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \theta_0 + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

(c) Show that the total acceleration makes angle ϕ with the radius where:

$$\tan\phi = \frac{a_t}{a_c}$$

(d) A wheel starts from rest and rotates with constant angular acceleration α for time t. Find the ratio of tangential to centripetal acceleration as a function of time.

(e) A car enters a circular track of radius $R = 50$ m at speed $v_i = 10$ m/s and accelerates tangentially at $a_t = 2.0$ m/s² for $t = 5.0$ s:

- Calculate the final speed
- Find the centripetal acceleration at $t = 0$ and $t = 5$ s
- Determine the total acceleration magnitude at $t = 5$ s
- Calculate the angle swept out during acceleration

Problem 20: Motion with Changing Acceleration

Consider motion where acceleration changes with time or position.

(a) For acceleration as a function of time $a(t) = a_0 + bt$, derive expressions for $v(t)$ and $x(t)$ given initial conditions v_0 and x_0.

(b) A particle has position-dependent acceleration $a(x) = -kx$. Show that this leads to simple harmonic motion.

(c) For a particle with $a = -\alpha v$ (velocity-dependent acceleration), derive the velocity as a function of time:

$$v(t) = v_0 e^{-\alpha t}$$

(d) Find the total distance traveled as $t \to \infty$ for the case in part (c).

(e) A rocket has thrust that decreases linearly: $a(t) = a_0(1 - t/T)$ for $0 \leq t \leq T$. For $a_0 = 15$ m/s² and $T = 10$ s:

- Find the maximum velocity
- Calculate the distance traveled during powered flight
- Determine the average acceleration