ECE 105: Classical Mechanics

Practice Final Exam

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Preface: This problem set is designed to prepare you for the final exam. Please, give yourself an honest attempt prior to searching for the solution; perhaps skip the question and come back to it. The difficulty of problems does not scale with question number. The challenge problems should NOT be attempted unless you are comfortable with the rest. Feel free to message me on the discord should you have questions.

I don't want to hear anything about partial derivatives. You can do it!

Topics Covered:

- Kinematics
- Forces and Newton's Laws
- Circular Motion
- Work, Energy, and Momentum
- Rotational Motion
- Oscillations
- Waves

Kinematics

Problem 1: Projectile Motion with Bouncing (10 marks)

A ball is thrown from the ground with initial speed v_0 at angle θ above the horizontal. The ball can either travel without bouncing, or it can bounce once off the ground. When the ball bounces, it leaves the ground at the same angle from the vertical as it arrived, and its speed after bouncing is α times its speed just before bouncing (where $0 < \alpha < 1$).

(a) **(3 marks)** For the case with no bounce, derive an expression for the total time the projectile is in the air and the horizontal distance travelled in terms of v_0, θ, and g.

(b) **(3 marks)** For the case with one bounce, the ball is thrown at some angle ϕ. Derive an expression for the total horizontal distance travelled (sum of both parabolic arcs) and the total time of flight in terms of v_0, ϕ, α, and g.

(c) **(3 marks)** Two players throw balls from the same position at the same initial speed, and both balls land at the same horizontal distance. One player throws at $\theta = 45$ with no bounce. The other player allows one bounce during the trajectory. If $\alpha = 0.8$, find the angle ϕ at which the second player must throw. Then calculate the ratio $t_{bounce}/t_{no\,bounce}$.

(d) **(1 mark)** Describe qualitatively how air resistance would affect the comparison in part (c).

Problem 2: River Crossing with Moving Target (10 marks)

A swimmer can swim at speed $v_s = 4$ km/hr in still water. The swimmer enters a river that flows with current speed $v_c = 2$ km/hr. The river is $W = 80$ m wide. At the same instant the swimmer starts, their friend begins walking along the opposite riverbank in the downstream direction with speed $v_f = 3$ km/hr.

(a) **(3 marks)** The swimmer wants to meet their friend at the opposite bank. Let θ be the angle the swimmer heads relative to the line directly across the river (positive θ means heading upstream). Derive an expression for the angle θ in terms of v_s, v_c, v_f, and W.

(b) **(2 marks)** Calculate the time for the swimmer to reach the other side.

(c) **(2 marks)** Calculate how far downstream from the starting point the swimmer meets the friend.

(d) **(1 mark)** If the swimmer wanted to cross in the minimum possible time, what direction should they head? Justify your answer briefly.

(e) **(2 marks)** Show that there exists a critical walking speed $v_{f,crit}$ above which the swimmer can never catch their friend at the opposite bank, and find an expression for $v_{f,crit}$ in terms of v_s and v_c.

Problem 3: Parametric Motion of The Cycloid (10 marks)

A wheel of radius a rolls without slipping along a horizontal surface. A point P is located on the rim of the wheel. The position of point P is described parametrically by:

$$x(\theta) = a(\theta - \sin\theta)$$

$$y(\theta) = a(1 - \cos\theta)$$

where θ is the angle through which the wheel has rotated. The wheel rotates with constant angular velocity $\omega = d\theta/dt$.

(a) **(2 marks)** Sketch the path of point P (the cycloid) for $0 \leq \theta \leq 4\pi$, indicating the positions where P touches the ground.

(b) **(3 marks)** Use the chain rule to find expressions for the velocity components $v_x = dx/dt$ and $v_y = dy/dt$ in terms of a, ω, and θ.

(c) **(3 marks)** Find expressions for the acceleration components $a_x = dv_x/dt$ and $a_y = dv_y/dt$ in terms of a, ω, and θ, and show that the magnitude of the acceleration is constant.

(d) **(2 marks)** At what values of θ is the speed of point P equal to zero? At what values is the speed maximum, and what is that maximum speed in terms of a and ω?

Problem 4: Two Projectiles Meeting (10 marks)

Ball A is dropped from rest from a height H above the ground. At the exact same instant, Ball B is thrown vertically upward from the ground with initial speed v_0.

(a) **(2 marks)** Write equations for the height of each ball as a function of time in terms of H, v_0, g, and t.

(b) **(2 marks)** Derive an expression for the time t_c at which the balls collide in terms of H and v_0.

(c) **(3 marks)** Derive an expression for the height y_c at which the collision occurs in terms of H, g, and v_0.

(d) **(3 marks)** Find the minimum initial speed $v_{0,min}$ required for Ball B to reach Ball A before Ball A hits the ground. Express your answer in terms of H and g.

Problem 5: Projectile Launched Up an Incline (10 marks)

A projectile is launched from the base of a frictionless inclined plane that makes angle α with the horizontal. The projectile is launched with speed v_0 at angle β above the inclined surface (so the total angle above horizontal is $\alpha + \beta$).

(a) **(3 marks)** Set up a coordinate system with the x-axis along the incline and the y-axis perpendicular to it. Derive expressions for the components of acceleration in this system in terms of g and α.

(b) **(3 marks)** Derive an expression for the range R along the incline in terms of v_0, g, α, and β.

(c) **(2 marks)** Find the angle β that maximizes the range along the incline in terms of α.

(d) **(2 marks)** For $\alpha = 30$ and $v_0 = 20$ m/s, calculate the maximum range along the incline and the time of flight.

Problem 6: Object Falling Past a Window (10 marks)

An object is dropped from rest from an unknown height above a window. The window has height H, and the object takes time τ to fall past the window (from top edge to bottom edge).

(a) **(3 marks)** Let h be the height above the top of the window from which the object was dropped. Derive an expression for the velocity v_{top} at the top of the window in terms of H, τ, and g.

(b) **(3 marks)** Derive an expression for the height h above the window from which the object was dropped in terms of H, τ, and g.

(c) **(2 marks)** Find an expression for the velocity at the bottom of the window in terms of H, τ, and g.

(d) **(2 marks)** An object falls past a 2.0 m tall window in 0.40 s. Calculate:

- The height above the window from which it was dropped
- The velocity at the bottom of the window

Forces and Newton's Laws

Problem 7: Two Blocks with Pulley and Friction (10 marks)

A block of mass M_1 sits at the bottom of a ramp of angle θ and length L. The block is connected by a rope that passes over a frictionless, massless pulley to a hanging block of mass M_2. The coefficient of kinetic friction between M_1 and the ramp is μ_k. Assume M_2 is heavy enough that the system accelerates with M_2 going down.

(a) **(3 marks)** Draw free body diagrams for both blocks, clearly labeling all forces including friction, tension, normal force, and weight.

(b) **(1 mark)** Write an expression for the kinetic friction force on M_1 in terms of M_1, g, θ, and μ_k.

(c) **(2 marks)** Apply Newton's second law to each block and derive an expression for the acceleration of the system in terms of M_1, M_2, g, θ, and μ_k.

(d) **(2 marks)** Derive an expression for the tension T in the rope in terms of M_1, M_2, g, θ, and μ_k.

(e) **(2 marks)** If the block M_1 starts at rest at the bottom of the ramp, find an expression for the time it takes for M_1 to travel halfway up the ramp (a distance $L/2$) in terms of L and a.

Problem 8: Stacked Blocks with Friction (10 marks)

A block of mass m_1 rests on top of a larger block of mass m_2. The lower block rests on a frictionless horizontal surface. The coefficient of static friction between the two blocks is μ_s.

(a) **(2 marks)** A horizontal force F is applied to the **lower** block (m_2). Draw free body diagrams for both blocks, explicitly showing the Newton's third law friction pair.

(b) **(2 marks)** Assuming the blocks move together, derive an expression for the common acceleration in terms of F, m_1, and m_2.

(c) **(3 marks)** The top block accelerates only due to friction. Derive the maximum acceleration the system can have before the top block slips in terms of μ_s and g, and hence derive the maximum force F_{max} that can be applied in terms of μ_s, g, m_1, and m_2.

(d) **(3 marks)** Now suppose the force F is applied to the **upper** block (m_1) instead. Derive the maximum force F'_{max} before slipping occurs in terms of μ_s, g, m_1, and m_2, and compare to the previous case.

Problem 9: Three Connected Blocks (10 marks)

Three blocks with masses m_1, m_2, and m_3 are connected by light strings and pulled along a frictionless horizontal surface by a force F applied to m_1. The blocks are arranged as: m_3 — m_2 — m_1 — $\rightarrow F$

(a) **(2 marks)** Draw free body diagrams for all three blocks.

(b) **(2 marks)** Derive an expression for the acceleration of the system in terms of F, m_1, m_2, and m_3.

(c) **(3 marks)** Find expressions for the tension T_{12} in the string between m_1 and m_2, and the tension T_{23} in the string between m_2 and m_3, in terms of F, m_1, m_2, and m_3.

(d) **(3 marks)** If the string between m_2 and m_3 can withstand a maximum tension T_{max}, find the maximum force F that can be applied in terms of T_{max}, m_1, m_2, and m_3.

Problem 10: Double Pulley System on Incline (10 marks)

A block of mass m_1 rests on an incline of angle θ with coefficient of kinetic friction μ_k. The block is connected by a rope to a double pulley system: the rope goes over a pulley at the top of the incline, down to a movable pulley, and back up to a fixed anchor point. A mass m_2 hangs from the movable pulley. Assume $m_2 \gg m_1$ so that m_1 accelerates up the incline. All pulleys are massless and frictionless.

(a) **(2 marks)** Draw free body diagrams for the block m_1 and the hanging mass m_2.

(b) **(3 marks)** Due to the double pulley arrangement, if m_1 moves a distance d up the incline, m_2 moves a distance $d/2$ downward. Derive the constraint relating the accelerations a_1 and a_2.

(c) **(3 marks)** Write Newton's second law equations for both masses. Using the constraint from part (b), derive an expression for the acceleration a_1 of the block up the incline in terms of m_1, m_2, g, θ, and μ_k.

(d) **(2 marks)** Derive an expression for the tension T in the rope in terms of m_1, m_2, g, θ, and μ_k.

Problem 11: Rope Pulling a Sled (10 marks)

A uniform rope of total mass m and length L lies in a straight line on a frictionless horizontal surface. One end of the rope is attached to a sled of mass M. A constant horizontal force F is applied to the free end of the rope, pulling the entire system.

(a) **(2 marks)** Find the acceleration of the entire system (rope + sled) in terms of F, m, and M.

(b) **(3 marks)** Let x be the distance measured from the sled along the rope. Consider the portion of the rope between the sled and position x. By applying Newton's second law to this portion, derive an expression for the tension $T(x)$ in the rope as a function of position in terms of F, m, M, L, and x.

(c) **(3 marks)** Verify that your expression gives the correct tension at $x = 0$ (at the sled) and at $x = L$ (at the free end where the force is applied).

(d) **(2 marks)** At what position along the rope is the tension equal to half of the applied force F? Express your answer in terms of m, M, and L.

Problem 12: Spring-Coupled Blocks with Friction (10 marks)

A block of mass m_1 rests on a horizontal surface with coefficient of kinetic friction μ_k. The block is connected to a wall by a spring with spring constant k (initially at its natural length). On the other side, the block is connected by a light rope over a massless, frictionless pulley to a hanging block of mass m_2, where $m_2 \gg m_1$. The system is released from rest.

(a) **(2 marks)** Draw free body diagrams for both blocks, clearly labeling all forces including friction, tension, spring force, and weights.

(b) **(5 marks)** Let x be the displacement of m_1 from its initial position (equal to the stretch of the spring). Write Newton's second law for both blocks and derive, but do not solve, the equation of motion in terms of m_1, m_2, k, μ_k, g, and x.

(c) **(3 marks)** Determine the expression for μ_k in terms of m_1, m_2, and g for which the system remains at rest.

Circular Motion

Problem 13: Circular Motion on a Banked Track (10 marks)

A car of mass m travels at constant speed v around a circular track of radius R. The track is banked at angle θ to the horizontal. The coefficient of static friction between the tires and the track is μ_s.

(a) **(2 marks)** Draw a free body diagram for the car, clearly showing all forces and the coordinate system used.

(b) **(2 marks)** Using Newton's second law, derive an expression for the *ideal* speed (no friction needed) in terms of g, R, and θ.

(c) **(3 marks)** For speeds below v_{ideal}, friction prevents the car from sliding down the bank. Derive an expression for the **minimum** speed v_{min} the car can travel without sliding down in terms of g, R, θ, and μ_s.

(d) **(2 marks)** Derive an expression for the **maximum** speed v_{max} before the car slides up the bank in terms of g, R, θ, and μ_s.

(e) **(1 mark)** Under what condition on θ and μ_s can the car remain stationary on the banked track?

Problem 14: The Conical Pendulum (10 marks)

A mass m is suspended from a string of length L. The mass revolves in a horizontal circle at constant speed, with the string making a constant angle θ with the vertical.

(a) **(2 marks)** Draw a free body diagram showing the tension T and weight mg. Note that the acceleration is horizontal, directed toward the center of the circle.

(b) **(3 marks)** Apply Newton's second law in the vertical and horizontal (radial) directions. Derive the equations relating T, θ, m, g, v, and the radius $r = L\sin\theta$.

(c) **(2 marks)** Derive an expression for the speed v in terms of g, L, and θ.

(d) **(3 marks)** Derive an expression for the period of revolution in terms of g, L, and θ. What happens to the period as $\theta \to 0$? As $\theta \to 90$?

Problem 15: Block Sliding on a Hemisphere (10 marks)

A block of mass m is released from rest at the top of a frictionless hemisphere of radius R. The block slides down the curved surface.

(a) **(4 marks)** Define an appropriate coordinate system with the angle ϕ measured from the vertical. Using conservation of energy and Newton's second law in the radial direction, derive the angle ϕ at which the block loses contact with the sphere. Hence, find the height above the ground at which the block leaves the sphere in terms of R.

(b) **(2 marks)** Find an expression for the speed of the block at the moment it leaves the hemisphere in terms of g and R.

(c) **(4 marks)** Now suppose the block does not start from rest, but is given an initial horizontal speed v_0 at the top of the hemisphere. Derive the value of v_0 for which the block *immediately* loses contact with the sphere in terms of g and R.

Problem 16: Ball in Vertical Circle with Energy Loss (10 marks)

A small ball of mass m is attached to a string of length R and swung in a vertical circle. Due to air resistance, the ball loses speed as it goes around.

(a) **(2 marks)** At the bottom of the circle, the speed is v_b and the tension is T_b. Derive an expression for T_b in terms of m, g, v_b, and R.

(b) **(2 marks)** At the top of the circle, the speed is v_t and the tension is T_t. Derive an expression for T_t in terms of m, g, v_t, and R.

(c) **(3 marks)** The string can withstand a maximum tension T_{max}. If the ball barely makes it over the top (minimum speed), find the minimum speed at the bottom in terms of g and R, and compare T_b to T_t in this case.

(d) **(3 marks)** If the ball loses energy ΔE due to air resistance during one complete cycle, and starts at the bottom with speed v_0, find the speed at the bottom after one complete revolution in terms of m, v_0, and ΔE. For what value of ΔE (in terms of m, g, and R) will the ball fail to complete the loop?

Work, Energy, and Momentum

Problem 17: Conservative Forces and Potential Energy (10 marks)

A particle of mass m moves along the x-axis under the influence of a force:

$$F(x) = 2x - 4x^3$$

where x is in meters and F is in Newtons.

(a) **(2 marks)** Show that this force is conservative by finding a potential energy function $V(x)$ such that $F(x) = -dV/dx$. Choose $V(0) = 0$.

(b) **(2 marks)** Sketch the potential energy function $V(x)$ for $-1.5 \leq x \leq 1.5$. Identify all equilibrium points and classify each as stable or unstable.

(c) **(2 marks)** Describe qualitatively the motion of a particle with total mechanical energy $E > 0$ that starts near $x = 0$.

(d) **(2 marks)** If the total mechanical energy is $E = 0$, find the kinetic energy of the particle at the positions $x = +1$ and $x = -1$.

(e) **(2 marks)** Qualitatively describe the motion for a particle that starts near $x = -1$ with total energy in the range $-\frac{1}{4} < E < 0$.

Problem 18: Loop-the-Loop (10 marks)

A small block of mass m is released from rest at height h above the ground on a frictionless track. The track includes a vertical circular loop of radius R.

(a) **(2 marks)** At the top of the loop, draw a free body diagram and write Newton's second law in the radial direction.

(b) **(2 marks)** The block barely maintains contact at the top when the normal force $N = 0$. Derive an expression for the minimum speed at the top of the loop in terms of g and R.

(c) **(3 marks)** Using conservation of energy, derive an expression for the minimum release height h_{min} for the block to complete the loop (measured from the bottom of the loop) in terms of R.

(d) **(3 marks)** If the block is released from height $h = 3R$, find the speed at the bottom of the loop in terms of g and R, and the normal force on the block at the top and bottom in terms of m and g.

Problem 19: Two-Dimensional Elastic Collision (10 marks)

A particle of mass m_1 moves with velocity v_0 in the $+x$ direction and collides elastically with a stationary particle of mass m_2. After the collision, m_1 moves at angle θ_1 above the x-axis, and m_2 moves at angle θ_2 below the x-axis.

(a) **(2 marks)** Write the conservation of momentum equations for the x and y components in terms of m_1, m_2, v_0, v_1, v_2, θ_1, and θ_2.

(b) **(1 mark)** Write the conservation of kinetic energy equation in terms of m_1, m_2, v_0, v_1, and v_2.

(c) **(3 marks)** For the special case $m_1 = m_2$, show that the two particles move at right angles to each other after the collision.

(d) **(4 marks)** For $m_1 = m_2$ and $\theta_1 = 30$, find θ_2 and the final speeds v_1 and v_2 in terms of v_0. Verify that energy is conserved.

Problem 20: The Ballistic Pendulum (10 marks)

A bullet of mass m is fired horizontally with speed v_0 into a wooden block of mass M that is suspended from a string of length L. The bullet embeds in the block, and the combined system swings upward, reaching a maximum angle θ_{max} from the vertical.

(a) **(2 marks)** Explain why momentum is conserved during the collision but kinetic energy is not.

(b) **(2 marks)** Using conservation of momentum, derive an expression for the velocity V of the block-bullet system immediately after the collision in terms of m, M, and v_0.

(c) **(2 marks)** Using conservation of energy for the swinging phase, derive an expression for the initial bullet speed v_0 in terms of m, M, g, L, and θ_{max}.

(d) **(2 marks)** Derive an expression for the fraction of kinetic energy lost in the collision in terms of m and M.

(e) **(2 marks)** A 10 g bullet embeds in a 2.0 kg block suspended from a 1.5 m string. The system swings to $\theta_{max} = 20$. Calculate the initial speed of the bullet and the energy lost.

Problem 21: Collision and Spring Compression (10 marks)

A block of mass m_1 slides on a frictionless surface with speed v_0 and collides with a stationary block of mass m_2 that is attached to a spring of constant k (the spring is initially at its natural length). The collision is perfectly elastic.

(a) **(3 marks)** Find expressions for the velocities v_1 and v_2 of both blocks immediately after the elastic collision (before the spring compresses) in terms of m_1, m_2, and v_0.

(b) **(3 marks)** After the collision, block m_2 compresses the spring. Derive an expression for the maximum compression x_{max} of the spring in terms of m_1, m_2, v_0, and k.

(c) **(2 marks)** What is the velocity of block m_2 when the spring returns to its natural length? Express your answer in terms of m_1, m_2, and v_0.

(d) **(2 marks)** For the case $m_1 = m_2 = m$, simplify your expressions and verify that total mechanical energy is conserved throughout.

Problem 22: Work by Variable Force (10 marks)

A particle of mass m starts from rest at $x = 0$. A position-dependent force acts on it:

$$F(x) = F_0 \left(1 - \frac{x}{L}\right)$$

where F_0 and L are constants.

(a) **(3 marks)** Calculate the work done by this force as the particle moves from $x = 0$ to $x = L$ in terms of F_0 and L.

(b) **(2 marks)** Using the work-energy theorem, find the speed of the particle at $x = L$ in terms of F_0, L, and m.

(c) **(3 marks)** Derive an expression for the speed as a function of position $v(x)$ in terms of F_0, L, m, and x.

(d) **(2 marks)** At what position x is the speed maximum (in terms of L)? What is the maximum speed in terms of F_0, L, and m?

Problem 23: Block Launched onto a Spring (10 marks)

A block of mass m is launched with initial speed v_0 along a frictionless horizontal surface toward a spring of spring constant k attached to a wall. After compressing the spring, the block rebounds and travels back along the surface, then up a frictionless ramp inclined at angle θ.

(a) **(2 marks)** Find the maximum compression x_{max} of the spring in terms of m, v_0, and k.

(b) **(3 marks)** Derive an expression for the speed of the block as a function of the spring compression x while the block is in contact with the spring in terms of m, v_0, k, and x.

(c) **(3 marks)** After rebounding from the spring, the block travels up the ramp. Derive an expression for the maximum height h the block reaches along the ramp in terms of v_0 and g.

(d) **(2 marks)** Now suppose the ramp has coefficient of kinetic friction μ_k. If the block travels a distance d along the ramp before momentarily stopping, derive an expression for the initial speed v_0 in terms of d, θ, μ_k, and g.

Problem 24: Explosion and Center of Mass (10 marks)

A projectile of mass M is launched with speed v_0 at angle θ above the horizontal. At the highest point of its trajectory, an internal explosion splits the projectile into two pieces of masses m and $M - m$. Immediately after the explosion, the piece of mass m is stationary (zero velocity).

(a) **(2 marks)** What is the velocity of the projectile at the highest point, just before the explosion, in terms of v_0 and θ?

(b) **(3 marks)** Using conservation of momentum, find the velocity of the piece of mass $(M - m)$ immediately after the explosion in terms of M, m, v_0, and θ.

(c) **(2 marks)** Show that the center of mass of the system continues on the original parabolic trajectory.

(d) **(3 marks)** Derive expressions for where each piece lands (horizontal distance from launch point) in terms of M, m, v_0, θ, and g.

Rotational Motion

Problem 25: Rolling Motion on an Incline (10 marks)

A solid sphere of mass m and radius R is released from rest at the top of a ramp of height h and angle θ. The sphere rolls without slipping. For a solid sphere, $I = \frac{2}{5}mR^2$.

(a) **(2 marks)** Draw a free body diagram of the sphere, clearly showing all forces including friction.

(b) **(3 marks)** Write Newton's second law for translation and rotation. Derive an expression for the acceleration of the center of mass in terms of g and θ.

(c) **(2 marks)** Find an expression for the friction force in terms of m, g, and θ, and derive the minimum coefficient of static friction needed to prevent slipping in terms of θ.

(d) **(3 marks)** Using energy conservation, derive an expression for the speed at the bottom in terms of g and h. What fraction of the total kinetic energy is rotational?

Problem 26: Angular Momentum and Collisions (10 marks)

A uniform rod of mass M and length L is pivoted at one end and hangs vertically at rest. A ball of mass m moving horizontally with speed v_0 strikes the rod at a distance d from the pivot and sticks to it. For the rod, $I_{rod} = \frac{1}{3}ML^2$.

(a) **(2 marks)** Explain why angular momentum about the pivot is conserved during the collision.

(b) **(3 marks)** Derive an expression for the angular velocity ω immediately after the collision in terms of m, M, L, d, and v_0.

(c) **(2 marks)** Derive an expression for the kinetic energy lost in the collision in terms of m, M, L, d, and v_0.

(d) **(3 marks)** After the collision, the rod-ball system swings upward. Using energy conservation, derive an expression for the maximum angle θ_{max} the rod swings from vertical in terms of m, M, L, d, v_0, and g.

Problem 27: Atwood Machine with Massive Pulley (10 marks)

An Atwood machine consists of two masses m_1 and m_2 ($m_2 > m_1$) connected by a massless rope over a pulley of mass M and radius R. The pulley is a solid disk, so $I = \frac{1}{2}MR^2$.

(a) **(2 marks)** Draw free body diagrams for both masses and for the pulley.

(b) **(3 marks)** Write Newton's second law for each mass and the rotational equation for the pulley. Derive an expression for the acceleration of the system in terms of m_1, m_2, M, and g.

(c) **(2 marks)** Derive expressions for the two tensions T_1 and T_2 in terms of m_1, m_2, M, and g.

(d) **(3 marks)** Using energy methods, verify your expression for acceleration by applying conservation of energy as m_2 descends a height h.

Problem 28: The Yo-Yo (10 marks)

A yo-yo of mass m has an inner radius r (where the string is wound) and an outer radius R. The moment of inertia about its center is $I = \frac{1}{2}m(r^2 + R^2)$. The yo-yo is released from rest and unwinds as it falls.

(a) **(2 marks)** Draw a free body diagram of the yo-yo showing tension and weight.

(b) **(3 marks)** Write Newton's second law for translation and rotation. Using the constraint that the string unwinds at rate $v = r\omega$, derive an expression for the acceleration of the center of mass in terms of g, r, and R.

(c) **(2 marks)** Derive an expression for the tension in the string in terms of m, g, r, and R.

(d) **(3 marks)** For $m = 0.1$ kg, $r = 0.01$ m, and $R = 0.05$ m:

- Calculate the acceleration
- Find the tension
- Determine the time to descend 1 m from rest

Problem 29: Two Disks and Angular Momentum (10 marks)

A disk of mass M and radius R rotates freely about a vertical axis through its center with angular velocity ω_0. A second disk of mass m and radius r is dropped onto it from a small height, landing concentrically (same axis).

(a) **(2 marks)** What quantity is conserved during the collision? Why is kinetic energy not conserved?

(b) **(3 marks)** If the moment of inertia of the first disk is $I_1 = \frac{1}{2}MR^2$ and the second disk is $I_2 = \frac{1}{2}mr^2$, derive an expression for the final angular velocity ω_f of the combined system in terms of M, R, m, r, and ω_0.

(c) **(3 marks)** Derive an expression for the kinetic energy lost in the collision in terms of M, R, m, r, and ω_0. Express it as a fraction of the initial kinetic energy.

(d) **(2 marks)** For $M = 2.0$ kg, $R = 0.3$ m, $m = 0.5$ kg, $r = 0.1$ m, and $\omega_0 = 10$ rad/s, calculate the final angular velocity and the energy lost.

Problem 30: The Bowling Ball Problem (10 marks)

A bowling ball of mass m and radius R is thrown with initial speed v_0 and *no* initial rotation ($\omega_0 = 0$). It slides on a surface with kinetic friction coefficient μ_k. For a solid sphere, $I = \frac{2}{5}mR^2$.

(a) **(2 marks)** Initially the ball slides. Write the equations for the linear acceleration (deceleration) and angular acceleration due to friction in terms of μ_k, g, and R.

(b) **(3 marks)** Derive expressions for the linear velocity $v(t)$ and angular velocity $\omega(t)$ as functions of time in terms of v_0, μ_k, g, R, and t.

(c) **(3 marks)** Find the time t^* when pure rolling begins (when $v = R\omega$) in terms of v_0, μ_k, and g.

(d) **(2 marks)** Derive an expression for the final speed of the ball once pure rolling is achieved in terms of v_0.

Oscillations

Problem 31: Simple Harmonic Motion (10 marks)

A mass m is attached to a horizontal spring with spring constant k on a frictionless surface. The mass oscillates with amplitude A.

(a) **(2 marks)** Derive the differential equation of motion and find an expression for the angular frequency ω in terms of k and m.

(b) **(2 marks)** Derive an expression for the total mechanical energy in terms of k and A, and show that it is constant.

(c) **(2 marks)** Derive an expression for velocity as a function of position $v(x)$ in terms of ω, A, and x.

(d) **(2 marks)** At what position(s) x (in terms of A) is the kinetic energy exactly equal to the potential energy?

(e) **(2 marks)** A bullet of mass m_b moving at speed v_b embeds itself in a wooden block of mass M that is at rest on the spring at its equilibrium position. Using conservation of momentum and energy, derive an expression for the amplitude of the resulting oscillation in terms of m_b, M, v_b, and k.

Problem 32: The Physical Pendulum (10 marks)

A uniform rod of mass M and length L is pivoted about a horizontal axis through one end.

(a) **(3 marks)** Draw a free body diagram of the rod when displaced by a small angle θ. Write the equation of rotational motion about the pivot.

(b) **(3 marks)** Using the small angle approximation, derive an expression for the period of oscillation in terms of L and g.

(c) **(2 marks)** Compare this to the period of a simple pendulum of length L. Which has the longer period and by what factor?

(d) **(2 marks)** Derive an expression for the distance d from the center of mass at which the rod should be pivoted to minimize the period of oscillation in terms of L.

Problem 33: Block on Vibrating Platform (10 marks)

A block rests on a horizontal platform that oscillates vertically with simple harmonic motion: $y(t) = A\cos(\omega t)$.

(a) **(2 marks)** Draw the free body diagram of the block. Write Newton's second law for the block.

(b) **(3 marks)** The block stays on the platform as long as the normal force $N > 0$. Show that the block loses contact when the downward acceleration of the platform exceeds g.

(c) **(3 marks)** Derive an expression for the maximum amplitude A_{max} for which the block stays on the platform at a given frequency f in terms of g and f.

(d) **(2 marks)** For $f = 5$ Hz, calculate the maximum amplitude. If $A = 1.5$ cm, at what point in the cycle does the block lose contact?

Problem 34: Combined Spring Systems (10 marks)

(a) **(3 marks)** Two springs with constants k_1 and k_2 are connected in **parallel** (side by side) to a mass m. Derive an expression for the effective spring constant in terms of k_1 and k_2, and the period of oscillation in terms of k_1, k_2, and m.

(b) **(3 marks)** The same two springs are now connected in **series** (end to end). Derive an expression for the effective spring constant in terms of k_1 and k_2, and the period of oscillation in terms of k_1, k_2, and m.

(c) **(2 marks)** For $k_1 = 100$ N/m, $k_2 = 200$ N/m, and $m = 0.5$ kg, calculate the period for both configurations.

(d) **(2 marks)** A mass hangs from two springs in series. If the mass is pulled down and released, show that the energy oscillates between the springs and find what fraction of the total potential energy is stored in each spring when the system is at maximum displacement in terms of k_1 and k_2.

Problem 35: Spring Oscillating Sphere (10 marks)

A solid sphere of mass m and radius R is attached to a horizontal spring with spring constant k. The spring is attached to the center of the sphere, and the sphere rolls without slipping on a horizontal surface. The sphere is displaced from equilibrium and released.

(a) **(3 marks)** Let x be the displacement of the center of the sphere from equilibrium. Using the rolling without slipping condition, relate the angular velocity ω of the sphere to the velocity v of its center. Write expressions for the total kinetic energy (translational + rotational) and potential energy of the system in terms of m, v, k, and x.

(b) **(3 marks)** Using energy conservation or Newton's second law combined with the rotational equation, derive the equation of motion for $x(t)$.

(c) **(2 marks)** Derive an expression for the angular frequency ω_{osc} and period T of small oscillations in terms of k and m. Compare this to the period of a sliding block (no rotation) of the same mass on the same spring.

(d) **(2 marks)** For $m = 2.0$ kg, $R = 0.1$ m, and $k = 140$ N/m, calculate the period of oscillation. If the sphere is released from rest at $x_0 = 0.05$ m, find the maximum speed of the center of the sphere.

Waves

Problem 36: Wave Function Analysis (10 marks)

A transverse wave on a string is described by:

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

(a) **(2 marks)** Derive an expression for the wave velocity v in terms of ω and k.

(b) **(3 marks)** Show that this wave function satisfies the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

(c) **(2 marks)** Find an expression for the maximum transverse speed v_{max} of a point on the string in terms of A and ω.

(d) **(3 marks)** For $A = 0.05$ m, $\lambda = 2.0$ m, and $T = 0.5$ s, calculate k, ω, the wave speed, and the maximum transverse speed.

Problem 37: Standing Waves on a String (10 marks)

A string of length L is fixed at both ends. The string has linear mass density μ and is under tension T.

(a) **(2 marks)** Using dimensional analysis, derive an expression for the wave speed on the string in terms of T and μ.

(b) **(3 marks)** For standing waves with nodes at both ends, derive the allowed wavelengths λ_n in terms of L and n, and the corresponding harmonic frequencies f_n in terms of n, L, T, and μ.

(c) **(2 marks)** Now pretend it is magically opened at one end, and fixed at another. For this situation, derive the same as part (b) in terms of the appropriate variables.

(d) **(3 marks)** A guitar string has $L = 0.65$ m, $\mu = 3.0 \times 10^{-3}$ kg/m, and fundamental frequency 330 Hz. Calculate the tension, the wavelength of the fundamental, and the third harmonic frequency.

Problem 38: Wave Superposition and Interference (10 marks)

Two sinusoidal waves with the same amplitude A, wavelength λ, and frequency f travel in the same direction but with a phase difference ϕ:

$$y_1 = A\sin(kx - \omega t)$$

$$y_2 = A\sin(kx - \omega t + \phi)$$

(a) **(3 marks)** Using the identity $\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$, derive an expression for the total wave $y_{total} = y_1 + y_2$ in terms of A, k, ω, t, x, and ϕ.

(b) **(2 marks)** For what values of ϕ does constructive interference occur? Destructive interference?

(c) **(2 marks)** If $\phi = \pi/3$, what is the amplitude of the resultant wave in terms of A?

(d) **(3 marks)** Two waves have slightly different frequencies f_1 and f_2. At a fixed point, $y_1 = A\cos(2\pi f_1 t)$ and $y_2 = A\cos(2\pi f_2 t)$. Derive an expression for the superposition in terms of A, f_1, f_2, and t, and explain why the "beat frequency" is $|f_1 - f_2|$.

Problem 39: Standing Waves in Pipes (10 marks)

Sound waves can form standing waves in pipes. The speed of sound is $v = 343$ m/s.

(a) **(3 marks) Open-Open Pipe:** A pipe of length L is open at both ends. Explain why there are antinodes at both ends. Derive an expression for the allowed frequencies f_n in terms of n, v, and L.

(b) **(3 marks) Open-Closed Pipe:** A pipe is open at one end and closed at the other. Explain why there is an antinode at the open end and a node at the closed end. Derive an expression for the allowed frequencies f_n in terms of n, v, and L, and explain why only odd harmonics exist.

(c) **(2 marks)** An organ pipe is 2.0 m long. Calculate the fundamental frequency if both ends are open, and if one end is closed.

(d) **(2 marks)** If you want to create a pipe whose fundamental is 440 Hz (A4) with one end closed, how long should it be?

Problem 40: Energy in Waves (10 marks)

A sinusoidal wave $y(x,t) = A\sin(kx - \omega t)$ travels along a string with linear mass density μ under tension T.

(a) **(2 marks)** Derive an expression for the kinetic energy per unit length in terms of μ and $\partial y/\partial t$.

(b) **(3 marks)** The potential energy per unit length is related to the stretching of the string. Derive an expression for the average total energy per unit length in terms of μ, ω, and A.

(c) **(2 marks)** The power transmitted by the wave is $P = v \cdot \langle dE/dx \rangle$. Derive an expression for the power in terms of μ, T, ω, and A.

(d) **(3 marks)** A wave with $A = 2.0$ mm, $f = 100$ Hz travels on a string with $\mu = 5.0$ g/m under tension $T = 50$ N. Calculate the wave speed, the average energy per unit length, and the power transmitted.

Challenge Problems

These problems combine multiple concepts and are designed to be more difficult than typical exam problems.

Challenge Problem 1: The Double Atwood Machine (15 marks)

A pulley of negligible mass is suspended from the ceiling by a string. Over this pulley hangs a mass m_1 on one side and *another pulley* (also massless) on the other side. Over the second pulley hang masses m_2 and m_3.

(a) **(3 marks)** Draw free body diagrams for all three masses and both pulleys.

(b) **(5 marks)** Let a_1 be the acceleration of m_1, and a_2, a_3 be the accelerations of m_2 and m_3 respectively. Note that a_2 and a_3 are related to a_1 through the constraint that the second pulley moves with the string on the right side of the first pulley. Derive the constraint equation relating the accelerations.

(c) **(5 marks)** Find expressions for the accelerations a_1, a_2, and a_3 in terms of m_1, m_2, m_3, and g.

(d) **(2 marks)** For what relationship between the masses (in terms of m_1, m_2, and m_3) is m_1 in equilibrium?

Challenge Problem 2: Sliding on a Moving Hemisphere (15 marks)

A hemisphere of mass M and radius R rests on a frictionless floor. A small block of mass m is placed at the top of the hemisphere and given a slight push. Both the hemisphere and the block can move.

(a) **(3 marks)** As the block slides down, the hemisphere slides in the opposite direction. Using conservation of momentum, relate the velocity of the hemisphere V to the velocity of the block relative to the hemisphere in terms of m and M.

(b) **(5 marks)** Using conservation of energy, derive an expression for the speed of the block relative to the ground when it has descended to angle θ from the vertical in terms of m, M, g, R, and θ.

(c) **(5 marks)** Derive an expression for the angle θ at which the block leaves the hemisphere in terms of m and M. Show that this angle corresponds to a smaller height than the fixed hemisphere case.

(d) **(2 marks)** In the limit $M \to \infty$, verify that your result reduces to the fixed hemisphere case.

Challenge Problem 3: The Falling Spool (15 marks)

A spool of mass m consists of a cylinder of radius r with two large flanges of radius R. A string is wound around the inner cylinder, and the free end is attached to the ceiling. The spool is released from rest. The moment of inertia of the spool about its axis is I.

(a) **(3 marks)** Draw a free body diagram and write the equations for translation and rotation.

(b) **(4 marks)** The string unwinds such that $a = r\alpha$ (where a is the downward acceleration of the center of mass). Derive an expression for the acceleration in terms of m, g, r, and I.

(c) **(3 marks)** Derive an expression for the tension in the string in terms of m, g, r, and I.

(d) **(3 marks)** If the spool is a uniform solid cylinder (ignoring the flanges for the moment of inertia calculation), so $I = \frac{1}{2}mr^2$, what is the acceleration in terms of g?

(e) **(2 marks)** Compare this to a yo-yo and explain why a spool with $R > r$ has a different behavior than if $R = r$.

Challenge Problem 4: Oscillating Cylinder in Curved Track (15 marks)

A solid cylinder of mass m and radius r rolls without slipping inside a cylindrical track of radius R (where $R > r$). The cylinder is displaced slightly from the bottom and released.

(a) **(3 marks)** Let θ be the angle that the line from the center of the track to the center of the cylinder makes with the vertical. Derive an expression for the height h of the center of the cylinder above the lowest point in terms of R, r, and θ.

(b) **(4 marks)** Using energy conservation with rolling (both translational and rotational kinetic energy), derive an expression for the angular frequency of small oscillations in terms of g, R, and r.

(c) **(4 marks)** What would the angular frequency be if the cylinder were sliding (no rolling, frictionless)? Express your answer in terms of g, R, and r.

(d) **(4 marks)** Compare the rolling and sliding cases. Which has the longer period and by what factor? Explain physically why this makes sense.

Challenge Problem 5: Three-Body Collision (15 marks)

Three identical pucks of mass m are on a frictionless air table. Puck 1 moves with velocity v_0 in the $+x$ direction and simultaneously collides with Pucks 2 and 3, which are initially at rest and touching each other. The line connecting the centers of Pucks 2 and 3 makes an angle of 90 (they are arranged symmetrically about the x-axis). All collisions are elastic.

(a) **(3 marks)** By symmetry, what can you say about the motion of Pucks 2 and 3 after the collision?

(b) **(4 marks)** Write the conservation of momentum (in x and y) and conservation of energy equations in terms of m, v_0, and the final velocities.

(c) **(5 marks)** Solve for the final velocities of all three pucks in terms of v_0.

(d) **(3 marks)** At what angle do Pucks 2 and 3 move relative to the x-axis?

Challenge Problem 6: The Brachistochrone (15 marks)

A bead slides frictionlessly on a wire from point A at height h to point B at the origin. What shape of wire minimizes the travel time?

(a) **(3 marks)** Using conservation of energy, derive an expression for the speed of the bead at height y below A in terms of g and y.

(b) **(4 marks)** Consider a straight wire (inclined plane) from A to B. Find the time of travel in terms of h, d (the horizontal distance), and g.

(c) **(4 marks)** The actual solution is a cycloid (the curve traced by a point on a rolling wheel). The parametric equations are:

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

Derive an expression for the arc length element $ds = \sqrt{dx^2 + dy^2}$ in terms of a, θ, and $d\theta$.

(d) **(4 marks)** Using $v = \sqrt{2gy}$ and the arc length element, set up the time integral and show that it simplifies to a remarkably simple form where the time is proportional to the angle parameter.

End of Problem Set

Good luck on your final exam!