Problem Set: Work, Energy, and Momentum

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Part A: Work and Kinetic Energy

Problem 1: Work-Energy Theorem Fundamentals

A particle of mass m moves along a straight line under the influence of a force F.

(a) Starting from Newton's second law, derive the work-energy theorem:

$$W_{\text{net}} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta KE$$

(b) Show that for a constant force F acting over a displacement d, the work done is $W = Fd\cos\theta$ where θ is the angle between $\vec{F}$ and $\vec{d}$.

(c) A variable force $F(x) = F_0\left(1 - \frac{x}{L}\right)$ acts on the particle from $x = 0$ to $x = L$. Calculate the total work done.

(d) Verify your result by calculating the work using integration:

$$W = \int_{x_i}^{x_f} F(x)\,dx$$

(e) For $m = 2.0$ kg, $F_0 = 50$ N, $L = 4.0$ m, and $v_i = 3.0$ m/s:

- Calculate the work done by the force
- Find the final velocity v_f
- Determine the average force over the displacement

Problem 2: Power and Variable Forces

Power is defined as the rate at which work is done: $P = \frac{dW}{dt}$.

(a) Show that $P = \vec{F} \cdot \vec{v}$ for instantaneous power.

(b) A particle moves with velocity $v(t) = v_0 + at$ under a constant force F. Derive the power as a function of time.

(c) Show that the average power over time interval Δt is:

$$\overline{P} = \frac{W}{\Delta t} = \frac{\Delta KE}{\Delta t}$$

(d) A car of mass m accelerates from rest with constant power P_0. Show that:

- The velocity increases as $v(t) = \left(\frac{2P_0 t}{m}\right)^{1/2}$
- The displacement varies as $x(t) = \frac{2}{3}\left(\frac{2P_0}{m}\right)^{1/2} t^{3/2}$

(e) For $m = 1500$ kg and $P_0 = 75$ kW:

- Find the velocity after $t = 10$ s
- Calculate the displacement after 10 s
- Determine the acceleration at $t = 5$ s

Part B: Conservative Forces and Potential Energy

Problem 3: Potential Energy and Force

A force is conservative if work done is path-independent, allowing us to define potential energy V

(a) For a conservative force in one dimension, the relationship between force and potential energy is:

$$F(x) = -\frac{dV}{dx}$$

Use this to derive the potential energy stored in a spring obeying Hooke's Law ($F = -kx$), assuming $V(0) = 0$.

(b) Conversely, show that the work done by a conservative force can be written as the negative change in potential energy:

$$W = \int_{x_i}^{x_f} F(x)dx = -\Delta U$$

(c) The Lennard-Jones potential describes the interaction between two atoms:

$$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

Find the expression for the force $F(r)$ between the atoms.

(d) Find the equilibrium distance r_{eq} where the net force is zero.

(e) A particle moves in a potential $V(x) = \alpha x^4$. Find the force acting on the particle at $x = 2$ m if $\alpha = 3$ J/m^4.

Problem 4: Mechanical Energy Conservation

For conservative systems, total mechanical energy $E = KE + PE$ is conserved.

(a) Starting from the work-energy theorem, derive the principle of conservation of mechanical energy for conservative forces ($E_i = E_f$).

(b) A particle of mass m moves in a potential $U(x) = \frac{1}{2}kx^2 - \frac{1}{3}bx^3$. Write the total energy equation.

(c) Show that for a particle moving under gravity near Earth's surface ($U = mgh$):

$$v = \sqrt{v_0^2 + 2g(h_0 - h)}$$

(d) For motion in a vertical circle of radius R, find the minimum velocity at the top required to maintain contact (assuming only gravity and normal force act).

(e) A particle slides down a frictionless incline of height $h = 5.0$ m and angle $\theta = 30°$:

- Find the final velocity at the bottom
- Calculate the time taken to reach the bottom
- Compare with free fall from the same height

Part C: Non-Conservative Forces and Energy Dissipation

Problem 5: Friction and Energy Loss

When friction is present, mechanical energy is not conserved.

(a) For a particle moving with friction force $f = \mu_k N$, show that:

$$\Delta E_{\text{mech}} = -f \cdot d = -\mu_k N d$$

where d is the distance traveled.

(b) A block of mass m slides down an incline of angle θ and length L with kinetic friction coefficient μ_k. Derive the final velocity.

(c) Show that the fraction of initial potential energy dissipated is:

$$\frac{E_{\text{dissipated}}}{E_{\text{initial}}} = \frac{\mu_k}{\tan\theta}$$

(d) For what angle θ is exactly half the energy dissipated?

(e) A hockey puck of mass $m = 0.17$ kg slides on ice with $\mu_k = 0.05$ at initial velocity $v_0 = 15$ m/s:

- Calculate the stopping distance
- Find the time to stop
- Determine the power dissipated as a function of time

Problem 6: Springs and Damping

A mass-spring system with damping has energy dissipation.

(a) For a spring force $F = -kx$ and damping force $F_d = -bv$, write the equation of motion (Newton's 2nd Law).

(b) Show that the rate of energy dissipation is $\frac{dE}{dt} = -bv^2$.

(c) A mass m is released from rest at position x_0 on a spring with constant k. Find the maximum velocity if there is no damping.

(d) If a damping force $F_d = -bv$ is added, derive the condition under which the mass just reaches equilibrium without oscillating (critical damping condition from differential equations).

(e) For $m = 0.5$ kg, $k = 200$ N/m, $x_0 = 0.1$ m, and $b = 4.0$ N·s/m:

- Calculate the undamped maximum velocity
- Find the energy dissipated in the first half cycle (approximation)
- Determine if the system is underdamped, critically damped, or overdamped

Part D: Linear Momentum and Impulse

Problem 7: Impulse-Momentum Theorem

The impulse-momentum theorem relates force and momentum change.

(a) Starting from Newton's second law, derive:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$$

(b) Show that for a constant force, $J = F\Delta t$.

(c) A time-varying force $F(t) = F_0 \sin(\omega t)$ acts on a particle of mass m from $t = 0$ to $t = T$. Calculate the impulse and change in velocity.

(d) For a collision where force varies as $F(t) = F_{\max}\left(1 - \frac{t}{\tau}\right)$ during contact time τ, find the total impulse by integrating.

(e) A baseball of mass $m = 0.145$ kg is struck by a bat with average force $F_{\text{avg}} = 8000$ N over contact time $\Delta t = 1.2$ ms:

- Calculate the impulse
- If the ball was initially moving at 40 m/s toward the bat, find its rebound velocity
- Determine the average acceleration during contact

Problem 8: Conservation of Momentum

In the absence of external forces, total momentum is conserved.

(a) Prove that if $\sum \vec{F}_{\text{ext}} = 0$, then $\frac{d\vec{p}_{\text{total}}}{dt} = 0$.

(b) For a two-particle system, show that the center of mass moves with constant velocity if no external forces act.

(c) A system explodes into two fragments. Show that the fragments move in opposite directions with momenta of equal magnitude.

(d) Derive the relationship between the kinetic energies of the two fragments in terms of their masses.

(e) A stationary object of mass $M = 5.0$ kg explodes into two pieces: $m_1 = 2.0$ kg moving at $v_1 = 12$ m/s:

- Find the velocity of the second piece
- Calculate the kinetic energy of each fragment
- Determine the total kinetic energy added by the explosion

Part E: Collisions

Problem 9: Elastic Collisions

In elastic collisions, both momentum and kinetic energy are conserved.

(a) For a one-dimensional elastic collision between masses m_1 and m_2 with initial velocities v_1 and v_2, derive the final velocities:

$$v_1' = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2}, \quad v_2' = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}$$

(b) Show that for equal masses ($m_1 = m_2$), the velocities are exchanged.

(c) For $m_1 \gg m_2$ with m_2 initially at rest, show that $v_1' \approx v_1$ and $v_2' \approx 2v_1$.

(d) Derive the expression for the relative velocity of approach and separation: $v_{\text{separation}} = -v_{\text{approach}}$.

(e) A neutron ($m_n = 1.0$ u) collides elastically with a stationary carbon nucleus ($m_C = 12.0$ u) at $v_n = 2.0 \times 10^7$ m/s:

- Find the final velocities of both particles
- Calculate the fraction of kinetic energy transferred
- Determine the scattering angle for the neutron (maximum possible)

Problem 10: Perfectly Inelastic Collisions

In perfectly inelastic collisions, objects stick together. Kinetic energy is not conserved.

(a) For a perfectly inelastic collision between m_1 (velocity v_1) and m_2 (velocity v_2), derive the final common velocity:

$$v_f = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

(b) Calculate the kinetic energy lost in such a collision:

$$\Delta KE = KE_i - KE_f = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_1 - v_2)^2$$

(c) A car of mass $m_1 = 1200$ kg traveling at $v_1 = 25$ m/s collides with a stationary truck of mass $m_2 = 2400$ kg. They lock bumpers and slide together.

- Find the final velocity of the wreckage.
- Calculate the total kinetic energy lost.
- What fraction of the initial energy was dissipated?

Part F: Systems of Particles and Center of Mass

Problem 11: Center of Mass Dynamics

The center of mass of a system obeys simple equations of motion.

(a) For a system of n particles, define the center of mass position $\vec{r}_{\text{cm}}$. Derive the equation of motion: $M\vec{a}_{\text{cm}} = \sum \vec{F}_{\text{ext}}$.

(b) Show that internal forces do not affect the motion of the center of mass.

(c) For a two-particle system with masses m_1 and m_2 separated by distance d, find the center of mass location relative to m_1.

(d) A system consists of two particles with momenta $\vec{p}_1$ and $\vec{p}_2$. Show that $KE_{\text{total}} = KE_{\text{cm}} + KE_{\text{rel}}$.

(e) Two skaters, $m_1 = 50$ kg and $m_2 = 70$ kg, push off from rest. If m_1 moves at $v_1 = 2.8$ m/s:

- Find the velocity of m_2
- Calculate the center of mass velocity
- Determine the total kinetic energy

Part G: Classic Applications

Problem 12: Ballistic Pendulum

A ballistic pendulum measures projectile velocity through collision and energy conversion.

(a) A projectile of mass m embeds in a pendulum bob of mass M, which then swings to height h. Derive the initial projectile velocity:

$$v_0 = \frac{m + M}{m}\sqrt{2gh}$$

(b) Show that the fraction of initial kinetic energy lost in the collision is $\frac{\Delta KE}{KE_0} = \frac{M}{m+M}$.

(c) For what ratio M/m is exactly 90% of the energy lost?

(d) If the pendulum swings through angle θ_{max}, express v_0 in terms of θ_{max} and pendulum length L.

(e) A bullet of mass $m = 10$ g embeds in a pendulum of mass $M = 2.0$ kg and length $L = 1.5$ m. The pendulum swings to $\theta_{\text{max}} = 25°$:

- Find the bullet's initial velocity
- Calculate the energy lost in the collision
- Determine the average force during embedding if contact time is $\Delta t = 2.0$ ms

Problem 13: Atwood Machine (Energy Method)

Two masses connected by a string over a pulley form an Atwood machine.

(a) For masses m_1 and m_2 ($m_1 > m_2$) connected by a massless string over a massless pulley, derive the acceleration using Newton's laws:

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2}$$

(b) Find the tension in the string.

(c) Show that the acceleration can be written as $a = g\left(\frac{\Delta m}{M}\right)$ where $\Delta m = m_1 - m_2$ and $M = m_1 + m_2$.

(d) **Using energy methods**, find the velocity after mass m_1 descends a distance h.

(e) For $m_1 = 5.0$ kg, $m_2 = 3.0$ kg, and $h = 2.0$ m:

- Calculate the acceleration
- Find the tension in the string
- Determine the velocity after descending distance h

Problem 14: Loop-the-Loop

A particle slides down a frictionless track and enters a circular loop.

(a) For a particle of mass m at the top of a vertical loop of radius R, derive the minimum velocity needed to maintain contact:

$$v_{\text{top}} = \sqrt{gR}$$

(b) Using energy conservation, find the minimum release height h for the particle to complete the loop:

$$h = \frac{5R}{2}$$

(c) Show that at the bottom of the loop, the normal force is $N = mg + \frac{mv_{\text{bottom}}^2}{R}$.

(d) If the particle is released from height $h = 3R$, find the normal force at the bottom and top of the loop.

(e) For $R = 1.0$ m and $m = 0.5$ kg:

- Calculate the minimum release height
- Find the velocity at the top of the loop for this height
- Determine the normal force at the top

Problem 15: Two-Dimensional Collisions

Momentum is conserved in all directions for 2D collisions.

(a) For a 2D elastic collision, write the conservation equations for momentum (x and y components) and kinetic energy.

(b) A particle of mass m_1 moving with velocity v_1 strikes a stationary particle of mass m_2. After collision, m_1 moves at angle θ_1 and m_2 at angle θ_2. For an elastic collision with $m_1 = m_2$, show that $\theta_1 + \theta_2 = 90°$.

(c) Derive the scattering angle relationship for unequal masses in terms of impact parameter.

(d) A billiard ball moving at $v_1 = 5.0$ m/s strikes an identical stationary ball. After collision, the first ball moves at 30° to its original direction:

- Find the angle at which the second ball moves
- Calculate the final velocities of both balls
- Verify that kinetic energy is conserved

Part H: Advanced and Optional Topics

Problem 16: Formal Conservative Forces (Vector Calculus)

(a) Show that for a conservative force, $\oint \vec{F} \cdot d\vec{r} = 0$ around any closed path.

(b) Prove that a conservative force can be written as $\vec{F} = -\nabla U$

(c) For a central force $\vec{F} = f(r)\hat{r}$, show that $\nabla \times \vec{F} = 0$

(d) Given $V(x, y) = \frac{1}{2}k(x^2 + y^2)$, find the force components using partial derivatives.

Problem 17: Coefficient of Restitution

(a) Define the coefficient of restitution $e = \frac{|v_2' - v_1'|}{|v_1 - v_2|}$. Show that $e = 1$ for elastic and $e = 0$ for perfectly inelastic collisions.

(b) A ball of mass m drops from height h onto a floor with coefficient of restitution e. Show that it rebounds to height $h' = e^2 h$.

(c) A car ($m_1 = 1200$ kg, $v_1 = 25$ m/s) collides with a stationary car ($m_2 = 1000$ kg). For $e = 0.40$:

- Find the final velocities of both cars.
- Calculate the kinetic energy lost.

Problem 18: Variable Mass Systems (The Rocket Equation)

(a) For a rocket ejecting mass at rate $\frac{dm}{dt}$ with exhaust velocity v_e, derive the thrust equation:

$$F_{\text{thrust}} = v_e \frac{dm}{dt}$$

(b) Show that the rocket equation is $\Delta v = v_e \ln\left(\frac{m_0}{m_f}\right)$.

(c) A rocket has initial mass $m_0 = 2000$ kg (1500 kg fuel), exhaust velocity $v_e = 2500$ m/s. Calculate the final velocity.

Problem 19: Potential Energy Diagrams and Stability

(a) Define equilibrium points where $F(x) = -dU/dx = 0$.

(b) Explain the condition for stable ($\frac{d^2U}{dx^2} > 0$) vs. unstable ($\frac{d^2U}{dx^2} < 0$) equilibrium.

Problem 20: The Falling Chain

A uniform chain of length L and mass M slides off a table.

(a) Show that the driving force is $F(x) = \frac{M}{L} x g$ where x is the hanging length.

(b) Use conservation of energy to find the velocity of the chain as it leaves the table.