Problem Set: Waves

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Part A: Wave Fundamentals and Mathematical Description

Problem 1: Wave Function Analysis

Consider a transverse wave on a string described by:

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where A is the amplitude, k is the wave number, ω is the angular frequency, and ϕ is the phase constant.

(a) Derive the wave velocity v in terms of ω and k.

(b) Show that the wave function satisfies the one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

(c) If a point on the string oscillates with maximum speed v_{max}, express v_{max} in terms of A and ω.

(d) For $A = 0.05$ m, wavelength $\lambda = 2.0$ m, and period $T = 0.5$ s, calculate:

- The wave number k
- The angular frequency ω
- The wave speed v
- The maximum transverse speed of a point on the string

Problem 2: Wave Properties from Graphs

A travelling wave at $t = 0$ has the profile shown by the function:

$$y(x,0) = A\sin(kx)$$

After time $t = t_1$, the wave has travelled a distance d to the right.

(a) Write the wave function at time t_1.

(b) If the wave completes n full cycles in time t_0, derive expressions for:

- The period T
- The frequency f
- The angular frequency ω

(c) Show that the relationship between wave speed, wavelength, and frequency is $v = f\lambda$.

(d) If $\lambda = 3.0$ m and the wave travels 15 m in 2.5 s, find the frequency.

Part B: Wave Velocity and Intensity

Problem 3: Waves on Strings

A transverse wave travels along a string with linear mass density μ under tension F_T.

(a) Using dimensional analysis, show that the wave velocity must have the form:

$$v = C\sqrt{\frac{F_T}{\mu}}$$

where C is a dimensionless constant.

(b) A string has mass m and length L. Express the wave velocity in terms of m, L, and F_T.

(c) For a guitar string with $m = 2.0$ g, $L = 65$ cm, and tension $F_T = 80$ N, calculate:

- The linear mass density
- The wave speed
- The time for a wave to travel the length of the string

Problem 4: Sound Intensity and Decibels

A point source emits sound power P uniformly in all directions.

(a) Show that the intensity I at distance r obeys the inverse square law:

$$I = \frac{P}{4\pi r^2}$$

(b) The sound level in decibels (dB) is defined as $\beta = 10\log_{10}(I/I_0)$, where $I_0 = 10^{-12}$ W/m^2 is the threshold of hearing. Show that doubling the distance from the source decreases the sound level by approximately 6 dB.

(c) A speaker emits 50 W of acoustic power. Calculate the intensity and sound level at:

- $r = 1.0$ m
- $r = 10.0$ m

Part C: Energy in Waves

Problem 5: Energy Transport

A sinusoidal wave $y(x,t) = A\sin(kx - \omega t)$ travels along a string with linear mass density μ.

(a) Show that the kinetic energy per unit length is:

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

(b) Derive the potential energy per unit length in terms of the string tension F_T and the slope $\partial y/\partial x$.

(c) Show that the average total energy per unit length is:

$$\left\langle\frac{dE}{dx}\right\rangle = \frac{1}{2}\mu\omega^2 A^2$$

(d) Find the power transmitted by the wave:

$$P = v\left\langle\frac{dE}{dx}\right\rangle$$

(e) For a wave with $A = 2.0$ mm, $f = 100$ Hz, on a string with $\mu = 5.0$ g/m under tension $F_T = 50$ N, calculate:

- The wave speed
- The average energy per unit length
- The power transmitted

Part D: Wave Interference and Superposition

Problem 6: Two-Wave Interference

Two sinusoidal waves with the same amplitude A, wavelength λ, and frequency f travel in the same direction:

$$y_1(x,t) = A\sin(kx - \omega t)$$
$$y_2(x,t) = A\sin(kx - \omega t + \phi)$$

(a) Use the principle of superposition to find the resultant wave $y_T(x,t)$.

(b) Show that the resultant can be written as:

$$y_T(x,t) = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

(c) For what values of ϕ does:

- Constructive interference occur (maximum amplitude)?
- Destructive interference occur (zero amplitude)?

(d) If $\phi = \pi/3$, $A = 0.10$ m, $\lambda = 2.0$ m, and $v = 10$ m/s, find:

- The amplitude of the resultant wave
- The frequency of oscillation
- Write the complete wave function

Problem 7: Standing Waves on a String

Two waves of equal amplitude travel in opposite directions:

$$y_1(x,t) = A\sin(kx - \omega t)$$

$$y_2(x,t) = A\sin(kx + \omega t)$$

(a) Show that the superposition creates a standing wave:

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

(b) Find the positions of nodes (where $y = 0$ always) and antinodes (maximum amplitude points).

(c) A string of length L is fixed at both ends. Show that standing waves can only exist for wavelengths:

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, ...$$

(d) For a guitar string with $L = 65$ cm and wave speed $v = 400$ m/s:

- Find the fundamental frequency ($n = 1$)
- Find the frequencies of the first three harmonics
- Calculate the wavelength of the third harmonic

Problem 8: Temporal Interference

Two sound waves with slightly different frequencies f_1 and f_2 arrive at the same point.

(a) Let $y_1 = A\cos(2\pi f_1 t)$ and $y_2 = A\cos(2\pi f_2 t)$. Use trigonometric identities to show:

$$y_{total} = 2A\cos(2\pi f_{mod} t)\cos(2\pi f_{avg} t)$$

where $f_{mod} = (f_1 - f_2)/2$ and $f_{avg} = (f_1 + f_2)/2$.

(b) Explain why the "beat frequency" (frequency of amplitude maxima) is $f_{beat} = |f_1 - f_2|$, not f_{mod}.

(c) Two tuning forks are struck. One has a frequency of 440 Hz. A beat frequency of 3 Hz is heard. What are the possible frequencies of the second fork?

Part E: Wave Reflection and Transmission

Problem 9: Boundary Conditions

A wave pulse travels along a string and encounters a boundary where the string properties change.

(a) State the two boundary conditions that must be satisfied at the junction between two strings with different linear mass densities μ_1 and μ_2.

(b) For an incident wave $y_i = A_i \sin(k_1 x - \omega t)$ in medium 1, write expressions for:

- The reflected wave y_r
- The transmitted wave y_t

(c) Derive the amplitude reflection coefficient:

$$R = \frac{A_r}{A_i} = \frac{v_1 - v_2}{v_1 + v_2}$$

(d) Show that the power is conserved: $P_i = P_r + P_t$

(e) A wave travels from a light string ($\mu_1 = 2.0$ g/m) to a heavy string ($\mu_2 = 8.0$ g/m) under the same tension. Calculate:

- The ratio of wave speeds v_1/v_2
- The reflection coefficient
- The percentage of incident power transmitted

Problem 10: Standing Waves in Air

Sound waves can form standing waves in pipes.

(a) **Open-Open Pipe:** A pipe of length L is open at both ends.

- Explain why there must be displacement antinodes at both ends.
- Show that this implies the length must be an integer number of half-wavelengths: $L = n\frac{\lambda}{2}$.
- Derive the frequency formula: $f_n = n\frac{v}{2L}$ for $n = 1, 2, 3...$

(b) **Open-Closed Pipe:** A pipe is open at one end and closed at the other.

- Explain why there is an antinode at the open end and a node at the closed end.
- Show that this implies the length must be an odd integer number of quarter-wavelengths: $L = m\frac{\lambda}{4}$ (where m is odd).
- Derive the frequency formula: $f_n = n\frac{v}{4L}$ for $n = 1, 3, 5...$

(c) An organ pipe is 3.0 m long. Calculate the fundamental frequency if:

- Both ends are open ($v_{sound} = 343$ m/s).
- One end is closed.

Part F: Advanced and Optional Topics

Problem 11: Wave Packets and Group Velocity

A wave packet is formed by superposing waves with slightly different frequencies:

$$y(x,t) = A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t)$$

(a) Show that this can be written as:

$$y(x,t) = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\sin\left(\bar{k}x - \bar{\omega}t\right)$$

where $\Delta k = k_2 - k_1$, $\Delta\omega = \omega_2 - \omega_1$, $\bar{k} = (k_1 + k_2)/2$, $\bar{\omega} = (\omega_1 + \omega_2)/2$.

(b) Identify the envelope (modulation) and carrier wave components.

(c) Define and derive the group velocity $v_g = \frac{d\omega}{dk}$.

(d) For waves in deep water where $\omega = \sqrt{gk}$ (where g is gravitational acceleration):

- Find the phase velocity $v_p = \omega/k$
- Find the group velocity v_g
- Show that $v_g = v_p/2$

(e) For $g = 9.8$ m/s^2 and $\lambda = 100$ m, calculate both velocities.

Problem 12: Doppler Effect

A source emitting waves of frequency f_s moves with velocity v_s toward a stationary observer. The wave speed in the medium is v.

(a) Derive the observed frequency:

$$f_o = f_s\left(\frac{v}{v - v_s}\right)$$

(b) What happens when v_s approaches v?

(c) For a moving observer with velocity v_o toward a stationary source, show:

$$f_o = f_s\left(\frac{v + v_o}{v}\right)$$

(d) Derive the general Doppler formula for both source and observer in motion.

(e) An ambulance siren emits sound at $f_s = 1000$ Hz. The ambulance approaches at 30 m/s. Given $v_{sound} = 343$ m/s:

- Calculate the frequency heard as it approaches
- Calculate the frequency heard as it recedes

- Find the frequency shift

Problem 13: Sound Speed in Gases

Sound waves in a gas are longitudinal waves with speed:

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

where γ is the heat capacity ratio, P is pressure, and ρ is density.

(a) Show that this can also be written as $v = \sqrt{\frac{\gamma RT}{M}}$ for an ideal gas.

(b) Calculate the speed of sound in air at 20°C given $\gamma = 1.40$ and $M = 28.97$ g/mol.

Problem 14: The Shock Wave (Sonic Boom)

Consider a source moving at velocity $v_s > v_{sound}$.

(a) Explain qualitatively the formation of a shock wave (Mach cone).

(b) Show that the half-angle of the Mach cone is given by $\sin\theta = \frac{v_{sound}}{v_s} = \frac{1}{\text{Mach Number}}$.

(c) A jet flies at Mach 2.0. Calculate the cone angle.