Problem Set: Rotational Motion

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Part A: Rotational Kinematics

Problem 1: Angular Variables and Linear-Angular Relationships

A disk of radius r rotates about its central axis with angular position $\theta(t)$.

(a) Show that the linear velocity of a point at distance r from the axis is $v = r\omega$, where $\omega = d\theta/dt$.

(b) Derive the relationship between linear acceleration a and angular acceleration $\alpha = d\omega/dt$.

(c) For circular motion with constant angular acceleration α, derive the kinematic equations:

- $\omega = \omega_0 + \alpha t$
- $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
- $\omega^2 = \omega_0^2 + 2\alpha\theta$

(d) A wheel initially at rest accelerates with $\alpha = 3.0$ rad/s^2. After $t = 5.0$ s:

- Find the angular velocity
- Calculate the total angle rotated
- Determine the linear speed of a point $r = 0.25$ m from the axis

Problem 2: Connecting Rotational Motions

Two gears are in contact (or connected by a chain that does not slip). Gear A has radius R_A and Gear B has radius R_B.

(a) Since the tangential velocities at the contact point must match, show that:

$$R_A\omega_A = R_B\omega_B$$

(b) Gear A starts from rest and accelerates at $\alpha_A = 2.0$ rad/s^2. $R_A = 0.1$ m and $R_B = 0.3$ m.

- Find the angular acceleration of Gear B.
- Calculate the time required for Gear B to reach 100 rpm.
- Through how many radians does Gear A rotate during this time?

Part B: Moment of Inertia

Problem 3: Calculating Moment of Inertia

(a) Derive the moment of inertia of a uniform rod of mass M and length L about:

- Its center: $I_{center} = \frac{ML^2}{12}$
- One end: $I_{end} = \frac{ML^2}{3}$

(b) Use the parallel axis theorem $I = I_{cm} + Md^2$ to verify the relationship between I_{center} and I_{end}.

(c) A compound pendulum consists of a rod of mass m and length L with a point mass M at one end. Find the moment of inertia about the pivot at the other end.

(d) For $m = 0.5$ kg, $L = 1.0$ m, and $M = 2.0$ kg, calculate the moment of inertia.

Problem 4: Composite Objects

(a) State the general formula for moment of inertia: $I = \sum_i m_i r_i^2$ for discrete masses.

(b) A system consists of three point masses: $m_1 = 2$ kg at $(0, 0)$, $m_2 = 3$ kg at $(L, 0)$, and $m_3 = 1$ kg at $(L/2, L)$. Find I about the origin.

(c) Compare moments of inertia for various shapes with the same mass M and radius R:

- Solid disk: $I = \frac{1}{2}MR^2$
- Solid sphere: $I = \frac{2}{5}MR^2$
- Thin hoop: $I = MR^2$
- Hollow sphere: $I = \frac{2}{3}MR^2$

(d) Which shape would win a race rolling down an incline? Why?

Part C: Rotational Dynamics

Problem 5: Torque and Angular Acceleration

(a) Define torque as $\vec{\tau} = \vec{r} \times \vec{F}$ and show that $|\tau| = rF \sin\phi$.

(b) Derive the rotational equation of motion: $\tau = I\alpha$.

(c) A force F is applied tangentially to a disk of mass M and radius R. Find the angular acceleration.

(d) A pulley system has masses m_1 and m_2 connected by a rope over a pulley of mass M and radius R. Assuming m_2 hangs and descends, derive the acceleration:

$$a = \frac{g(m_2 - m_1 \sin\theta)}{m_1 + m_2 + I/R^2}$$

(e) For $m_1 = 2$ kg, $m_2 = 3$ kg, $M = 1$ kg (solid disk), $R = 0.2$ m, and $\theta = 30°$, find:

- The acceleration
- The tensions in the rope

Problem 6: Disk Rolling Down an Incline

A disk of mass m and radius R rolls without slipping down an incline of angle θ.

(a) Draw a free body diagram and identify all forces and torques.

(b) Write the equations of motion:

- Translation: $mg\sin\theta - f = ma$
- Rotation: $fR = I\alpha$

(c) Using the no-slip condition $a = R\alpha$, solve for:

$$a = \frac{g\sin\theta}{1 + I/(mR^2)}$$

(d) For a solid disk ($I = \frac{1}{2}mR^2$), find the friction force required.

(e) What is the minimum coefficient of static friction needed to prevent slipping?

Problem 7: Leaning Ladder

A uniform ladder of length L and mass m leans against a frictionless wall at an angle θ above the horizontal. The floor has static friction coefficient μ_s.

(a) Draw the free body diagram showing all forces (Normal force from wall, Normal force from floor, Friction, Gravity).

[Image of ladder free body diagram]

(b) Write the conditions for static equilibrium: $\sum F_x = 0$, $\sum F_y = 0$, $\sum \tau = 0$ (choose the base of the ladder as the pivot).

(c) Show that the minimum angle θ_{min} to prevent slipping is given by:

$$\tan\theta_{\text{min}} = \frac{1}{2\mu_s}$$

(d) For a ladder with $m = 10$ kg and $\mu_s = 0.4$:

- Calculate the minimum angle.
- If $\theta = 60°$, find the magnitude of the friction force.

Part D: Angular Momentum

Problem 8: Angular Momentum Conservation

(a) Define angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and show that for rotation, $L = I\omega$.

(b) State the principle of angular momentum conservation when $\vec{\tau}_{ext} = 0$.

(c) A figure skater with arms extended ($I_1 = 5.0$ kg·m²) pulls arms in ($I_2 = 2.0$ kg·m²). If initial $\omega_1 = 2$ rad/s:

- Find the final angular velocity
- Calculate the change in kinetic energy
- Explain where the extra energy comes from

(d) A disk rotating at ω_0 has another identical disk dropped onto it. Find the final ω.

Problem 9: Collision Problems

A rod of mass m and length ℓ is initially at rest, pivoted at one end. A ball of mass M moving with velocity v_0 strikes the rod at distance d from the pivot.

(a) Write the angular momentum conservation equation regarding the pivot point.

(b) For a perfectly inelastic collision, derive the final angular velocity:

$$\omega_f = \frac{Mv_0 d}{I_{rod} + Md^2}$$

(c) Calculate ω_f for $m = 1$ kg, $\ell = 2$ m, $M = 0.5$ kg, $d = 1.5$ m, $v_0 = 4$ m/s.

(d) Find the energy lost in the collision.

Part E: Rotational Energy

Problem 10: Energy Methods and Rolling

(a) Show that rotational kinetic energy is $KE_{rot} = \frac{1}{2}I\omega^2$.

(b) For an object with translation and rotation, derive an expression for the total kinetic energy.

(c) A cylinder rolls down a hill of height h. Using energy conservation, find the final velocity:

$$v = \sqrt{\frac{2gh}{1 + I/(mR^2)}}$$

(d) Compare final velocities for sliding vs. rolling for the same height.

(e) A 5 kg cylinder rolls from rest down a 2 m high incline. Calculate:

- Final linear velocity
- Final angular velocity
- Fraction of energy in rotation

Problem 11: Rolling Sphere with Energy Approach

A solid sphere of radius R rolls without slipping down an incline of angle θ.

(a) Show that the condition for rolling without slipping is $v_{cm} = R\omega$.

(b) Express the total kinetic energy as $KE_{total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

(c) For a solid sphere ($I = \frac{2}{5}mR^2$), show that $KE_{total} = \frac{7}{10}mv^2$.

(d) Using conservation of energy ($mgh = \Delta KE$), derive the acceleration down the ramp.

Problem 12: The Yo-Yo

A yo-yo of mass m, inner radius r, and outer radius R with $I = \frac{1}{2}m(r^2 + R^2)$ descends under gravity.

(a) Write the equations of motion for the yo-yo ($\sum F = ma$ and $\sum \tau = I\alpha$).

(b) Find the linear acceleration of the center of mass.

(c) Calculate the tension in the string.

(d) For $m = 0.1$ kg, $r = 0.01$ m, $R = 0.05$ m:

- Find the acceleration
- Calculate string tension
- Determine time to descend 1 m

Part G: Advanced and Optional Topics

Problem 13: Precession and Gyroscopic Motion

(a) For a spinning top with angular momentum $\vec{L}$, explain precession.

(b) Show that precession angular velocity is:

$$\Omega = \frac{\tau}{L} = \frac{mgr}{I\omega}$$

(c) A bicycle wheel ($I = 0.5$ kg·m^2) spins at 10 rad/s. When held horizontally by one end of its axle (0.3 m from center), find the precession rate.

Problem 14: Variable Moment of Inertia

(a) Consider a rotating platform where mass can move radially. If I changes, what is conserved?

(b) A person on a rotating platform moves from radius $r_1 = 2$ m to $r_2 = 0.5$ m. If initial $\omega_1 = 1$ rad/s and person mass is 70 kg, platform $I_{platform} = 100$ kg·m^2:

- Find final angular velocity
- Calculate work done by the person

Problem 15: The Physical Pendulum

A rigid body of mass M is pivoted at a point distance d from its center of mass.

(a) Write the equation of motion for small oscillations: $\tau = -Mgd\theta = I_{pivot}\alpha$.

(b) Derive the period of oscillation:

$$T = 2\pi\sqrt{\frac{I_{pivot}}{Mgd}}$$

(c) Show that for a simple pendulum ($I_{pivot} = Md^2$), this reduces to $T = 2\pi\sqrt{d/g}$.

Problem 16: The Bowling Ball

A ball of radius R and mass M is thrown with initial speed v_0 and zero angular velocity ($\omega_0 = 0$) onto a surface with kinetic friction coefficient μ_k.

(a) Write the equations of motion for translation ($a = -\mu_k g$) and rotation ($\alpha = \mu_k mgR/I$).

(b) Find the time t when pure rolling begins (when $v(t) = R\omega(t)$).

(c) Show that the final speed is $v_f = \frac{5}{7}v_0$ (for a solid sphere).