

Problem Set: Work, Energy, and Momentum

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Part A: Work and Kinetic Energy

Problem 1: Work-Energy Theorem Fundamentals

A particle of mass m moves along a straight line under the influence of a force F .

- (a) Starting from Newton's second law, derive the work-energy theorem:

$$W_{\text{net}} = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

- (b) Show that for a constant force F acting over a displacement d , the work done is $W = Fd \cos \theta$ where θ is the angle between \vec{F} and \vec{d} .
- (c) A variable force $F(x) = F_0(1 - \frac{x}{L})$ acts on the particle from $x = 0$ to $x = L$. Calculate the total work done.
- (d) Verify your result by calculating the work using integration:

$$W = \int_{x_i}^{x_f} F(x) dx$$

- (e) For $m = 2.0$ kg, $F_0 = 50$ N, $L = 4.0$ m, and $v_i = 3.0$ m/s:

- Calculate the work done by the force
- Find the final velocity v_f
- Determine the average force over the displacement

Problem 2: Power and Variable Forces

Power is defined as the rate at which work is done: $P = \frac{dW}{dt}$.

- (a) Show that $P = \vec{F} \cdot \vec{v}$ for instantaneous power.
- (b) A particle moves with velocity $v(t) = v_0 + at$ under a constant force F . Derive the power as a function of time.

(c) Show that the average power over time interval Δt is:

$$\bar{P} = \frac{W}{\Delta t} = \frac{\Delta KE}{\Delta t}$$

(d) A car of mass m accelerates from rest with constant power P_0 . Show that:

- The velocity increases as $v(t) = \left(\frac{2P_0t}{m}\right)^{1/2}$
- The displacement varies as $x(t) = \frac{2}{3} \left(\frac{2P_0}{m}\right)^{1/2} t^{3/2}$

(e) For $m = 1500$ kg and $P_0 = 75$ kW:

- Find the velocity after $t = 10$ s
- Calculate the displacement after 10 s
- Determine the acceleration at $t = 5$ s

Part B: Conservative Forces and Potential Energy

Problem 3: Potential Energy and Force

A force is conservative if the work done is path-independent, allowing us to define potential energy U .

(a) For a conservative force in one dimension, the relationship between force and potential energy is:

$$F(x) = -\frac{dU}{dx}$$

Use this to derive the potential energy stored in a spring obeying Hooke's Law ($F = -kx$), assuming $U(0) = 0$.

(b) Conversely, show that the work done by a conservative force can be written as the negative change in potential energy:

$$W = \int_{x_i}^{x_f} F(x)dx = -\Delta U$$

(c) The Lennard-Jones potential describes the interaction between two atoms:

$$U(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

Find the expression for the force $F(r)$ between the atoms.

(d) Find the equilibrium distance r_{eq} where the net force is zero.

(e) A particle moves in a potential $U(x) = \alpha x^4$. Find the force acting on the particle at $x = 2$ m if $\alpha = 3$ J/m⁴.

Problem 4: Mechanical Energy Conservation

For conservative systems, total mechanical energy $E = KE + PE$ is conserved.

- Starting from the work-energy theorem, derive the principle of conservation of mechanical energy for conservative forces ($E_i = E_f$).
- A particle of mass m moves in a potential $U(x) = \frac{1}{2}kx^2 - \frac{1}{3}bx^3$. Write the total energy equation.
- Show that for a particle moving under gravity near Earth's surface ($U = mgh$):

$$v = \sqrt{v_0^2 + 2g(h_0 - h)}$$

- For motion in a vertical circle of radius R , find the minimum velocity at the top required to maintain contact (assuming only gravity and normal force act).
- A particle slides down a frictionless incline of height $h = 5.0$ m and angle $\theta = 30^\circ$:
 - Find the final velocity at the bottom
 - Calculate the time taken to reach the bottom
 - Compare with free fall from the same height

Part C: Non-Conservative Forces and Energy Dissipation

Problem 5: Friction and Energy Loss

When friction is present, mechanical energy is not conserved.

- For a particle moving with friction force $f = \mu_k N$, show that:

$$\Delta E_{\text{mech}} = -f \cdot d = -\mu_k N d$$

where d is the distance traveled.

- A block of mass m slides down an incline of angle θ and length L with kinetic friction coefficient μ_k . Derive the final velocity.
- Show that the fraction of initial potential energy dissipated is:

$$\frac{E_{\text{dissipated}}}{E_{\text{initial}}} = \frac{\mu_k \cos \theta}{\sin \theta}$$

- For what angle θ is exactly half the energy dissipated?
- A hockey puck of mass $m = 0.17$ kg slides on ice with $\mu_k = 0.05$ at initial velocity $v_0 = 15$ m/s:
 - Calculate the stopping distance
 - Find the time to stop

- Determine the power dissipated as a function of time

Problem 6: Springs and Damping

A mass-spring system with damping has energy dissipation.

- For a spring force $F = -kx$ and damping force $F_d = -bv$, write the equation of motion (Newton's 2nd Law).
- Show that the rate of energy dissipation is $\frac{dE}{dt} = -bv^2$.
- A mass m is released from rest at position x_0 on a spring with constant k . Find the maximum velocity if there is no damping.
- If a damping force $F_d = -bv$ is added, derive the condition under which the mass just reaches equilibrium without oscillating (critical damping condition from differential equations).
- For $m = 0.5$ kg, $k = 200$ N/m, $x_0 = 0.1$ m, and $b = 4.0$ N·s/m:
 - Calculate the undamped maximum velocity
 - Find the energy dissipated in the first half cycle (approximation)
 - Determine if the system is underdamped, critically damped, or overdamped

Part D: Linear Momentum and Impulse

Problem 7: Impulse-Momentum Theorem

The impulse-momentum theorem relates force and momentum change.

- Starting from Newton's second law, derive:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt = \Delta \vec{p} = m \vec{v}_f - m \vec{v}_i$$

- Show that for a constant force, $J = F\Delta t$.
- A time-varying force $F(t) = F_0 \sin(\omega t)$ acts on a particle of mass m from $t = 0$ to $t = T$. Calculate the impulse and change in velocity.
- For a collision where force varies as $F(t) = F_{\max} \left(1 - \frac{t}{\tau}\right)$ during contact time τ , find the total impulse by integrating.
- A baseball of mass $m = 0.145$ kg is struck by a bat with average force $F_{\text{avg}} = 8000$ N over contact time $\Delta t = 1.2$ ms:
 - Calculate the impulse
 - If the ball was initially moving at 40 m/s toward the bat, find its rebound velocity

- Determine the average acceleration during contact

Problem 8: Conservation of Momentum

In the absence of external forces, total momentum is conserved.

- Prove that if $\sum \vec{F}_{\text{ext}} = 0$, then $\frac{d\vec{p}_{\text{total}}}{dt} = 0$.
- For a two-particle system, show that the center of mass moves with constant velocity if no external forces act.
- A system explodes into two fragments. Show that the fragments move in opposite directions with momenta of equal magnitude.
- Derive the relationship between the kinetic energies of the two fragments in terms of their masses.
- A stationary object of mass $M = 5.0$ kg explodes into two pieces: $m_1 = 2.0$ kg moving at $v_1 = 12$ m/s:
 - Find the velocity of the second piece
 - Calculate the kinetic energy of each fragment
 - Determine the total kinetic energy added by the explosion

Part E: Collisions

Problem 9: Elastic Collisions

In elastic collisions, both momentum and kinetic energy are conserved.

- For a one-dimensional elastic collision between masses m_1 and m_2 with initial velocities v_1 and v_2 , derive the final velocities:

$$v'_1 = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2}, \quad v'_2 = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}$$
- Show that for equal masses ($m_1 = m_2$), the velocities are exchanged.
- For $m_1 \gg m_2$ with m_2 initially at rest, show that $v'_1 \approx v_1$ and $v'_2 \approx 2v_1$.
- Derive the expression for the relative velocity of approach and separation: $v_{\text{separation}} = -v_{\text{approach}}$.
- A neutron ($m_n = 1.0$ u) collides elastically with a stationary carbon nucleus ($m_C = 12.0$ u) at $v_n = 2.0 \times 10^7$ m/s:
 - Find the final velocities of both particles
 - Calculate the fraction of kinetic energy transferred

- Determine the scattering angle for the neutron (maximum possible)

Problem 10: Perfectly Inelastic Collisions

In perfectly inelastic collisions, objects stick together. Kinetic energy is not conserved.

- (a) For a perfectly inelastic collision between m_1 (velocity v_1) and m_2 (velocity v_2), derive the final common velocity:

$$v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

- (b) Calculate the kinetic energy lost in such a collision:

$$\Delta KE = KE_i - KE_f = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2$$

- (c) A car of mass $m_1 = 1200$ kg traveling at $v_1 = 25$ m/s collides with a stationary truck of mass $m_2 = 2400$ kg. They lock bumpers and slide together.

- Find the final velocity of the wreckage.
- Calculate the total kinetic energy lost.
- What fraction of the initial energy was dissipated?

Part F: Systems of Particles and Center of Mass

Problem 11: Center of Mass Dynamics

The center of mass of a system obeys simple equations of motion.

- (a) For a system of n particles, define the center of mass position \vec{r}_{cm} . Derive the equation of motion: $M\vec{a}_{\text{cm}} = \sum \vec{F}_{\text{ext}}$.
- (b) Show that internal forces do not affect the motion of the center of mass.
- (c) For a two-particle system with masses m_1 and m_2 separated by distance d , find the center of mass location relative to m_1 .
- (d) A system consists of two particles with momenta \vec{p}_1 and \vec{p}_2 . Show that $KE_{\text{total}} = KE_{\text{cm}} + KE_{\text{rel}}$.
- (e) Two skaters, $m_1 = 50$ kg and $m_2 = 70$ kg, push off from rest. If m_1 moves at $v_1 = 2.8$ m/s:
- Find the velocity of m_2
 - Calculate the center of mass velocity
 - Determine the total kinetic energy

Part G: Classic Applications

Problem 12: Ballistic Pendulum

A ballistic pendulum measures projectile velocity through collision and energy conversion.

- (a) A projectile of mass m embeds in a pendulum bob of mass M , which then swings to height h . Derive the initial projectile velocity:

$$v_0 = \frac{m+M}{m} \sqrt{2gh}$$

- (b) Show that the fraction of initial kinetic energy lost in the collision is $\frac{\Delta KE}{KE_0} = \frac{M}{m+M}$.

- (c) For what ratio M/m is exactly 90% of the energy lost?

- (d) If the pendulum swings through angle θ_{\max} , express v_0 in terms of θ_{\max} and pendulum length L .

- (e) A bullet of mass $m = 10$ g embeds in a pendulum of mass $M = 2.0$ kg and length $L = 1.5$ m. The pendulum swings to $\theta_{\max} = 25^\circ$:

- Find the bullet's initial velocity
- Calculate the energy lost in the collision
- Determine the average force during embedding if contact time is $\Delta t = 2.0$ ms

Problem 13: Atwood Machine (Energy Method)

Two masses connected by a string over a pulley form an Atwood machine.

- (a) For masses m_1 and m_2 ($m_1 > m_2$) connected by a massless string over a massless pulley, derive the acceleration using Newton's laws:

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2}$$

- (b) Find the tension in the string.

- (c) Show that the acceleration can be written as $a = g \left(\frac{\Delta m}{M} \right)$ where $\Delta m = m_1 - m_2$ and $M = m_1 + m_2$.

- (d) **Using energy methods**, find the velocity after mass m_1 descends a distance h .

- (e) For $m_1 = 5.0$ kg, $m_2 = 3.0$ kg, and $h = 2.0$ m:

- Calculate the acceleration
- Find the tension in the string
- Determine the velocity after descending distance h

Problem 14: Loop-the-Loop

A particle slides down a frictionless track and enters a circular loop.

- (a) For a particle of mass m at the top of a vertical loop of radius R , derive the minimum velocity needed to maintain contact:

$$v_{\text{top}} = \sqrt{gR}$$

- (b) Using energy conservation, find the minimum release height h for the particle to complete the loop:

$$h = \frac{5R}{2}$$

- (c) Show that at the bottom of the loop, the normal force is $N = mg + \frac{mv_{\text{bottom}}^2}{R}$.

- (d) If the particle is released from height $h = 3R$, find the normal force at the bottom and top of the loop.

- (e) For $R = 1.0$ m and $m = 0.5$ kg:

- Calculate the minimum release height
- Find the velocity at the top of the loop for this height
- Determine the normal force at the top

Problem 15: Two-Dimensional Collisions

Momentum is conserved in all directions for 2D collisions.

- (a) For a 2D elastic collision, write the conservation equations for momentum (x and y components) and kinetic energy.

- (b) A particle of mass m_1 moving with velocity v_1 strikes a stationary particle of mass m_2 . After collision, m_1 moves at angle θ_1 and m_2 at angle θ_2 . For an elastic collision with $m_1 = m_2$, show that $\theta_1 + \theta_2 = 90^\circ$.

- (c) Derive the scattering angle relationship for unequal masses in terms of impact parameter.

- (d) A billiard ball moving at $v_1 = 5.0$ m/s strikes an identical stationary ball. After collision, the first ball moves at 30° to its original direction:

- Find the angle at which the second ball moves
- Calculate the final velocities of both balls
- Verify that kinetic energy is conserved

Part H: Advanced and Optional Topics

Problem 16: Formal Conservative Forces (Vector Calculus)

- Show that for a conservative force, $\oint \vec{F} \cdot d\vec{r} = 0$ around any closed path.
- Prove that a conservative force can be written as $\vec{F} = -\nabla U$
- For a central force $\vec{F} = f(r)\hat{r}$, show that $\nabla \times \vec{F} = 0$
- Given $U(x, y) = \frac{1}{2}k(x^2 + y^2)$, find the force components using partial derivatives.

Problem 17: Coefficient of Restitution

- Define the coefficient of restitution $e = \frac{|v'_2 - v'_1|}{|v_1 - v_2|}$. Show that $e = 1$ for elastic and $e = 0$ for perfectly inelastic collisions.
- A ball of mass m drops from height h onto a floor with coefficient of restitution e . Show that it rebounds to height $h' = e^2 h$.
- A car ($m_1 = 1200$ kg, $v_1 = 25$ m/s) collides with a stationary car ($m_2 = 1000$ kg). For $e = 0.40$:
 - Find the final velocities of both cars.
 - Calculate the kinetic energy lost.

Problem 18: Variable Mass Systems (The Rocket Equation)

- For a rocket ejecting mass at rate $\frac{dm}{dt}$ with exhaust velocity v_e , derive the thrust equation:
$$F_{\text{thrust}} = v_e \frac{dm}{dt}$$
- Show that the rocket equation is $\Delta v = v_e \ln \left(\frac{m_0}{m_f} \right)$.
- A rocket has initial mass $m_0 = 2000$ kg (1500 kg fuel), exhaust velocity $v_e = 2500$ m/s. Calculate the final velocity.

Problem 19: Potential Energy Diagrams and Stability

- Define equilibrium points where $F(x) = -dU/dx = 0$.
- Explain the condition for stable ($\frac{d^2U}{dx^2} > 0$) vs. unstable ($\frac{d^2U}{dx^2} < 0$) equilibrium.

Problem 20: The Falling Chain

A uniform chain of length L and mass M slides off a table.

- Show that the driving force is $F(x) = \frac{M}{L}xg$ where x is the hanging length.
- Use conservation of energy to find the velocity of the chain as it leaves the table.

Numerical Answers

Problem 1(e): $W = 100 \text{ J}$, $v_f = 7.35 \text{ m/s}$, $F_{\text{avg}} = 25 \text{ N}$

Problem 2(e): $v = 31.6 \text{ m/s}$, $x = 211 \text{ m}$, $a = 2.5 \text{ m/s}^2$

Problem 4(e): $v = 9.90 \text{ m/s}$, $t = 2.02 \text{ s}$, $t_{\text{freefall}} = 1.01 \text{ s}$

Problem 5(e): $d = 229 \text{ m}$, $t = 30.6 \text{ s}$, $P(t) = -mg\mu_k v(t)$

Problem 6(e): $v_{\text{max}} = 2.0 \text{ m/s}$, $\Delta E = 0.68 \text{ J}$, underdamped

Problem 7(e): $J = 9.6 \text{ N}\cdot\text{s}$, $v_f = 26 \text{ m/s}$, $a = 5.5 \times 10^4 \text{ m/s}^2$

Problem 8(e): $v_2 = -8.0 \text{ m/s}$, $KE_1 = 144 \text{ J}$, $KE_2 = 96 \text{ J}$, $KE_{\text{added}} = 240 \text{ J}$

Problem 9(e): $v'_n = 1.54 \times 10^7 \text{ m/s}$, $v'_C = 3.08 \times 10^6 \text{ m/s}$, fraction = 28.4%

Problem 10(c): $v_f = 8.33 \text{ m/s}$, $\Delta KE = 2.5 \times 10^5 \text{ J}$ (66.7% lost)

Problem 11(e): $v_2 = -2.0 \text{ m/s}$, $v_{\text{cm}} = 0.35 \text{ m/s}$, $KE_{\text{total}} = 336 \text{ J}$

Problem 12(e): $v_0 = 393 \text{ m/s}$, $E_{\text{lost}} = 768 \text{ J}$, $F_{\text{avg}} = 1.97 \times 10^3 \text{ N}$

Problem 13(e): $a = 2.45 \text{ m/s}^2$, $T = 36.8 \text{ N}$, $v = 3.13 \text{ m/s}$

Problem 14(e): $h = 2.5 \text{ m}$, $v_{\text{top}} = 3.13 \text{ m/s}$, $N_{\text{top}} = 0 \text{ N}$

Problem 15(d): $\theta_2 = 60^\circ$, $v'_1 = 2.5 \text{ m/s}$, $v'_2 = 4.33 \text{ m/s}$