Problem Set: Forces and Dynamics

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Part A: Fundamental Forces and Newton's Laws

Problem 1: Newton's Laws and Free Body Diagrams

Consider the fundamental forces and Newton's three laws of motion.

(a) State Newton's First Law. Show that if $\vec{F}_{\text{ext}} = 0$, then $\frac{d\vec{v}}{dt} = 0$.

(b) Derive Newton's Second Law in the form $\vec{F}_{\text{ext}} = m\vec{a}$ starting from the definition of momentum $\vec{p} = m\vec{v}$.

(c) State Newton's Third Law. For a system of two objects exerting forces on each other, show that the total momentum is conserved if no external forces act.

(d) A block of mass m rests on a horizontal surface with coefficient of static friction μ_s and kinetic friction μ_k. Draw a free body diagram and write the equations of motion:

- In the vertical direction
- In the horizontal direction when a force F is applied

(e) For $m = 5.0$ kg, $\mu_s = 0.4$, $\mu_k = 0.3$, and $F = 25$ N applied horizontally:

- Determine if the block moves
- Calculate the acceleration if it does move
- Find the friction force in both cases

Problem 2: Multiple Forces and Components

A force $\vec{F}$ can be decomposed into components: $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$.

(a) Show that for multiple forces, $\vec{F}_{\text{net}} = \sum_{i=1}^{n} \vec{F}_i$.

(b) Prove that the components satisfy:

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

(c) A particle experiences three forces: $\vec{F_1} = F_1(\cos\theta_1\hat{\imath} + \sin\theta_1\hat{\jmath})$, $\vec{F_2} = F_2(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath})$, and $\vec{F_3} = F_3(\cos\theta_3\hat{\imath} + \sin\theta_3\hat{\jmath})$. Write the net force and acceleration in component form.

(d) For $F_1 = 10$ N at $\theta_1 = 30$, $F_2 = 15$ N at $\theta_2 = 120$, and $F_3 = 8$ N at $\theta_3 = 240$ acting on a mass $m = 2.0$ kg:

- Calculate the net force components
- Find the magnitude and direction of acceleration

Part B: Contact Forces

Problem 3: Normal Force and Inclined Planes

Consider a block on an inclined plane at angle θ from the horizontal.

(a) Draw a free body diagram with gravity $m\vec{g}$, normal force $\vec{F}_N$, and friction $\vec{f}$.

(b) Define a coordinate system with x-axis along the incline and y-axis perpendicular to it. Find the equations of motion (x, y).

(c) For a block at rest on the incline, show that the static friction force is:

$$f_s = mg\sin\theta$$

and the maximum angle before slipping is:

$$\theta_{\text{max}} = \arctan(\mu_s)$$

(d) For a block sliding down with kinetic friction, derive the acceleration:

$$a = g(\sin\theta - \mu_k\cos\theta)$$

(e) A block of mass $m = 3.0$ kg initially rests on a $\theta = 25$ incline with $\mu_s = 0.5$ and $\mu_k = 0.4$:

- Does the block slide?
- If placed on a 35 incline, find the acceleration
- Calculate the normal force in both cases

Problem 4: Tension in Ropes and Pulleys

Tension is a force transmitted through a rope or cord.

(a) For a massless rope, show that tension must be uniform throughout. Explain why this fails for a massive rope.

(b) Consider an Atwood machine: two masses m_1 and m_2 connected by a rope over a massless pulley. Draw free body diagrams for both masses.

(c) Write the equations of motion for both masses.

(d) Solve for the acceleration and tension.

(e) For $m_1 = 2.0$ kg and $m_2 = 3.0$ kg:

- Calculate the acceleration
- Find the tension in the rope
- Determine the time to fall $h = 1.5$ m starting from rest

Part C: Friction

Problem 5: Static and Kinetic Friction

There are two types of friction: static (when surfaces are not sliding) and kinetic (when surfaces are sliding).

(a) The magnitude of friction satisfies $|f| \leq \mu|F_N|$. Explain the difference between static and kinetic friction coefficients: μ_s and μ_k.

(b) Show that static friction can vary: $0 \leq f_s \leq \mu_s F_N$, while kinetic friction is constant: $f_k = \mu_k F_N$.

(c) Generally, $\mu_s > \mu_k$. Explain physically why this is true.

(d) Create a graph of friction force f versus applied force F_{app} for a block initially at rest. Mark the transition from static to kinetic friction.

(e) A block of mass m is pushed horizontally with force F on a surface with friction coefficients μ_s and μ_k. For what range of F does the block:

- Remain at rest?
- Move with acceleration?

(f) For typical material pairs, compare:

- Rubber on concrete: $\mu_s = 1.0$, $\mu_k = 0.8$
- Ice on ice: $\mu_s = 0.1$, $\mu_k = 0.03$
- Wood on wood: $\mu_s = 0.5$, $\mu_k = 0.3$

Which pair provides the most control for stopping?

Problem 6: Stacked Blocks

Consider a block of mass m_1 resting on top of a larger block of mass m_2. The lower block rests on a frictionless table. The coefficient of static friction between the two blocks is μ_s.

(a) A force F is applied horizontally to the **bottom** block (m_2). Draw free body diagrams for both blocks, explicitly showing the Newton's 3rd Law friction pair.

(b) Assuming they move together, show that the common acceleration is

$$a = \frac{F}{m_1 + m_2}$$

(c) The top block moves solely due to static friction. Show that the maximum acceleration possible before the top block slips is $a_{\text{max}} = \mu_s g$.

(d) Derive the maximum force F_{max} that can be applied to the bottom block such that the top block does not slip.

(e) If $F > F_{\text{max}}$, describe qualitatively what happens to the acceleration of both blocks.

Problem 7: Friction on Inclined Planes and Banking

Combine friction with inclined plane motion.

(a) A block of mass m rests on an incline of angle θ. Show that it remains stationary if:

$$\tan\theta \leq \mu_s$$

(b) For a block sliding down an incline, derive the acceleration:

$$a = g(\sin\theta - \mu_k \cos\theta)$$

(c) Find the maximum speed on a banked curve. A car moving at speed v on a banked curve of radius R and banking angle θ experiences:

- Normal force F_N
- Friction force f
- Centripetal acceleration $a_c = \frac{v^2}{R}$

(d) For maximum speed without slipping up the bank, derive the equations of motion (x, y). Then, solve the equations to derive v_{max}.

(e) For a curve with $R = 100$ m, $\theta = 15$, and $\mu_s = 0.6$:

- Calculate the maximum safe speed
- Compare with the speed for a banked curve with no friction: $v = \sqrt{Rg\tan\theta}$

Part D: Spring Forces

Problem 8: Hooke's Law and Spring Systems

A spring exerts a restoring force proportional to displacement.

(a) State Hooke's Law: $\vec{F} = -k\vec{x}$, where k is the spring constant and $\vec{x}$ is displacement from equilibrium.

(b) Show that the equation of motion for a mass-spring system is:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

(c) Verify that $x(t) = A\cos(\omega t + \phi)$ is a solution, where $\omega = \sqrt{k/m}$ is the angular frequency.

(d) Derive the period of oscillation:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

(e) A mass $m = 0.5$ kg is attached to a spring with $k = 200$ N/m. If stretched by $x_0 = 0.1$ m and released:

- Find the angular frequency and period
- Calculate the maximum speed (at equilibrium)
- Determine the maximum acceleration (at maximum displacement)

Problem 9: Energy in Spring Systems

Springs store potential energy when compressed or extended.

(a) Derive the elastic potential energy of a spring:

$$V = \frac{1}{2}kx^2$$

(b) For a mass-spring system, write the total mechanical energy.

(c) For this system, show that energy is conserved:

$$\frac{dE}{dt} = 0$$

(d) For maximum displacement A (amplitude), show that:

$$v_{\text{max}} = A\sqrt{\frac{k}{m}}$$

(e) A spring with $k = 400$ N/m is compressed by $x = 0.05$ m and used to launch a mass $m = 0.2$ kg:

- Calculate the stored potential energy
- Find the launch speed
- If launched vertically, what maximum height is reached?

Part E: Circular Motion and Centripetal Force

Problem 10: Uniform Circular Motion

An object moving in a circle at constant speed experiences centripetal acceleration.

(a) For circular motion with radius R and constant speed v, derive the centripetal acceleration:

$$\vec{a}_c = -\frac{v^2}{R}\hat{r}$$

pointing toward the center.

(b) Show that the magnitude of centripetal force required is:

$$F_c = m\frac{v^2}{R}$$

(c) Express the centripetal acceleration in terms of angular velocity ω:

$$a_c = \omega^2 R$$

where $v = \omega R$.

(d) Derive the period T of one complete revolution:

$$T = \frac{2\pi}{\omega}$$

(e) A car of mass $m = 1200$ kg travels around a circular track of radius $R = 50$ m at $v = 20$ m/s:

- Calculate the required centripetal force
- Find the coefficient of friction needed to provide this force
- Determine the period of one lap

Problem 11: The Conical Pendulum

A mass m is suspended by a string of length L. The mass revolves in a horizontal circle with constant speed v, and the string makes a constant angle θ with the vertical.

(a) Draw a free body diagram for the mass. Note that the acceleration is horizontal, toward the center of the circle.

(b) Apply Newton's Second Law in the vertical and horizontal directions to show:

$$T\cos\theta = mg$$

$$T\sin\theta = \frac{mv^2}{R}$$

where $R = L\sin\theta$ is the radius of the circle.

(c) Divide the equations to derive an expression for the speed v independent of mass:

$$v = \sqrt{gL \sin\theta \tan\theta}$$

(d) Show that the period of revolution, T, obeys the expression:

$$T = 2\pi \sqrt{\frac{L\cos\theta}{g}}$$

Problem 12: Vertical Circular Motion

Consider an object moving in a vertical circle, such as a mass on a string or a roller coaster loop.

(a) Write the force equation at the top of the circle:

$$T + mg = m\frac{v_{\text{top}}^2}{R}$$

(b) Do the same, but this time, at the bottom of the circle:

$$T - mg = m\frac{v_{\text{bottom}}^2}{R}$$

(c) For a mass on a string, find the minimum speed at the top to maintain tension ($T \geq 0$):

$$v_{\text{min}} = \sqrt{gR}$$

(d) Using energy conservation, find an equation wich relates speeds at top and bottom.

(e) For a roller coaster loop of radius $R = 10$ m:

- Find the minimum speed at the top
- Calculate the speed at the bottom if starting from rest at height $h = 25$ m
- Determine the normal force on a $m = 70$ kg passenger at top and bottom

Part F: Systems of Particles

Problem 13: Center of Mass

For a system of particles, the center of mass moves as if all mass were concentrated there.

(a) Define the center of mass position:

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$

(b) Show that the velocity of the center of mass is:

$$\vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{M}$$

(c) Prove that the total momentum equals:

$$\vec{p}_{\text{total}} = M\vec{v}_{\text{cm}}$$

(d) Derive the equation of motion for the center of mass:

$$\vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}$$

where $\vec{F}_{\text{ext}}$ is the net external force (internal forces cancel by Newton's third law).

(e) Three particles have masses $m_1 = 2.0$ kg at position $(0, 0)$, $m_2 = 3.0$ kg at $(L, 0)$, and $m_3 = 1.0$ kg at $(L/2, L)$ where $L = 2.0$ m:

- Find the center of mass position
- If m_1 moves with velocity $(v, 0)$ and others are stationary, find $\vec{v}_{\text{cm}}$ for $v = 3.0$ m/s

Part G: Advanced and Optional Topics

Problem 14: Universal Gravitation

Newton's law of universal gravitation states that every point mass attracts every other point mass.

(a) State Newton's law of gravitation:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

where $G = 6.674 \times 10^{-11}$ N·m²/kg² is the gravitational constant.

(b) Show that for an object on Earth's surface, the gravitational force simplifies to $F = mg$ where:

$$g = \frac{GM_E}{R_E^2}$$

with $M_E = 5.972 \times 10^{24}$ kg and $R_E = 6.371 \times 10^6$ m.

(c) Calculate the value of g at Earth's surface and verify $g \approx 9.8$ m/s².

(d) For an object at height h above Earth's surface, show that:

$$g(h) = g_0 \left(\frac{R_E}{R_E + h}\right)^2$$

where g_0 is the surface gravity.

(e) What is the gravitational field (acceleration due to gravity) experienced by:

- An object at $h = 1000$ m above Earth's surface?
- A satellite at $h = 400$ km (ISS altitude)?
- The Moon at distance $r = 3.844 \times 10^8$ m from Earth?

Problem 15: Gravitational Versus Inertial Mass

The concept of gravitational mass versus inertial mass is fundamental to general relativity.

(a) Define inertial mass m_I from Newton's second law: $\vec{F} = m_I \vec{a}$.

(b) Define gravitational mass m_G from Newton's law of gravitation: $F = G\frac{m_G m'_G}{r^2}$.

(c) Experimentally, it is found that $m_I = m_G$ to extremely high precision. This is the *principle of equivalence*. State what this principle implies about acceleration in a gravitational field.

(d) Show that in a uniform gravitational field $\vec{g}$, all objects fall with the same acceleration:

$$\vec{a} = \frac{m_G}{m_I}\vec{g} = \vec{g}$$

regardless of mass (assuming $m_I = m_G$).

(e) How does this explain why a hammer and feather fall at the same rate in vacuum?

Problem 16: Drag Force and Terminal Velocity

In real fluids, objects experience a drag force that opposes motion. Assume air resistance is proportional to velocity: $\vec{F}_d = -b\vec{v}$.

(a) Write Newton's Second Law for an object of mass m falling vertically from rest. Show that the differential equation of motion is:

$$mg - bv = m\frac{dv}{dt}$$

(b) Determine the *terminal velocity* v_t (the speed when acceleration reaches zero).

(c) Integrate the differential equation to find velocity as a function of time:

$$v(t) = v_t\left(1 - e^{-\frac{bt}{m}}\right)$$

(d) Sketch the $v - t$ graph and identify the asymptotic behavior.

Problem 17: Block on Accelerating Wedge

A block of mass m rests on a frictionless wedge of mass M and angle θ. The wedge can slide on a frictionless horizontal surface.

(a) Draw free body diagrams for both the block and wedge. Identify all forces.

(b) Let a_W be the acceleration of the wedge to the right, and $a_{B,\mathrm{rel}}$ be the acceleration of the block relative to the wedge (down the slope). Write the equations of motion:

For the block in the horizontal direction:

$$F_N \sin\theta = ma_x$$

where $a_x = a_W + a_{B,\mathrm{rel}} \cos\theta$.

For the block in the vertical direction:

$$F_N \cos\theta - mg = ma_y$$

where $a_y = -a_{B,\mathrm{rel}} \sin\theta$.

For the wedge:

$$-F_N \sin\theta = Ma_W$$

(c) Solve the system to find:

$$a_W = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

$$a_{B,\mathrm{rel}} = \frac{g(M+m)\sin\theta}{M + m\sin^2\theta}$$

(d) For $m = 2.0$ kg, $M = 5.0$ kg, and $\theta = 30$:

- Calculate the wedge acceleration
- Find the block's acceleration relative to the wedge
- Determine the normal force between block and wedge

Problem 18: Pulley with Mass

Reconsider the Atwood machine, but now the pulley has mass M and radius R, with moment of inertia $I = \frac{1}{2}MR^2$ (solid disk).

(a) For the pulley, write the torque equation:

$$\tau = I\alpha = (T_2 - T_1)R$$

where T_1 and T_2 are the tensions on each side.

(b) The no-slip condition gives $a = R\alpha$ where a is the linear acceleration of the masses.

(c) Write the equations of motion for masses m_1 and m_2:

$$T_1 - m_1 g = m_1 a$$
$$m_2 g - T_2 = m_2 a$$

(d) Solve the system to find:

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}$$
$$T_1 = m_1(g + a), \quad T_2 = m_2(g - a)$$

(e) For $m_1 = 2.0$ kg, $m_2 = 3.0$ kg, $M = 1.0$ kg, and $R = 0.2$ m:

- Calculate the acceleration
- Find the tensions T_1 and T_2
- Compare with the massless pulley case from Problem 4