

Orbital Integration of Earth and Jupiter

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1 Introduction

This report examines various methods of solving differential equations numerically by computer programme to simulate the Earth and Jupiter orbiting the Sun. Specifically, the Euler method, Runge-Kutta 2nd order (RK2) and Runge-Kutta 4th order (RK4) methods are used and compared for their accuracy and stability. The orbit of earth around the Sun was first simulated with each of the aforementioned methods. Jupiter was then added to the system and its gravitational affect on the orbit of the earth examined.

2 Orbit of Earth Around the Sun

2.1 Mathematical Analysis

From Newton's universal law of gravitation, the gravitational force F_E acting on the Earth as it orbits the Sun is given by

$$F_E = -\frac{GM_E M_\odot}{r^2} \hat{\mathbf{r}} \quad (1)$$

where G is the universal gravitational constant, M_\odot the mass of the sun, M_E the mass of the earth, r the distance between the Earth and the Sun and $\hat{\mathbf{r}}$ the unit vector pointing from the Earth to the Sun. The acceleration a_E that the earth experiences at any time during its orbit around the Sun is then given from Newton's second law of motion by

$$a_E = \frac{F_E}{M_E} = -\frac{GM_\odot}{r^2} \hat{\mathbf{r}} = -\frac{GM_\odot \vec{r}}{r^3} \quad (2)$$

where \vec{r} is the vector pointing from the Earth to the Sun.

It is now assumed that the mass of the Sun is sufficiently large relative to that of the Earth such that its motion can be ignored. Setting the Sun's position to the centre of the Cartesian coordinate system, the x and y components of the Earth's position as it orbits the Sun can now be written as

$$\frac{d^2x}{dt^2} = -\frac{GM_\odot x}{r^3} \quad \frac{d^2y}{dt^2} = -\frac{GM_\odot y}{r^3} \quad (3, 4)$$

Now in order to solve these differential equations numerically, they need to be split into two separate first order differential equations. By setting the velocities in the x and y directions to

$$\frac{dx}{dt} = v_x \quad \frac{dy}{dt} = v_y \quad (5, 6)$$

equations 3 and 4 can be rewritten in terms of the acceleration a_x and a_y as

$$a_x = \frac{dv_x}{dt} = -\frac{GM_\odot x}{r^3} \quad a_y = \frac{dv_y}{dt} = -\frac{GM_\odot y}{r^3} \quad (7, 8)$$

This set of four differential equations are now all of first order and can thus be solved using the numerical integration methods.

2.2 Euler Method

2.2.1 Implementation

The Euler method is a simple technique which the next value y_{i+1} of first order differential equations over step h of the form

$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0 \quad (9)$$

using the following equation

$$y_{i+1} = y_i + f(x_i, y_i)h \quad (10)$$

In order to solve the above differential equations with C++ in a nice manner using the Euler method, a state vector was defined as an array which stores the Earth's x, y position in the first two indexes and the Earth's velocity components v_x, v_y in the last two indexes. A function 'Earth Orbit' was then defined which takes in the current state vector and returns the derivative of the state vector, namely the velocity v_x, v_y and acceleration a_x, a_y . (This function corresponds to the $f(x_i, y_i)$ function of equation 10). The acceleration is calculated from equation 7 and 8. The new velocities can simply be passed on from the old state vector due to the nature of equations 5 and 6. Below is the code of this function.

```
\begin{minted}[language=cpp]{}
typedef std::array<double, 4> State;

State EarthOrbit(State s)
{
    //calculate the magnitude of r to find acceleration later
    double r = std::sqrt(s[0] * s[0] + s[1] * s[1]);

    //calcualte the acceleration for the x and y components
    double a_x = (-G * M_Sun * s[0]) / (r * r * r);
    double a_y = (-G * M_Sun * s[1]) / (r * r * r);

    //return the derivative state using the velocities of the input state vector
    return {s[2], s[3], a_x, a_y};
}
\end{minted}
```

The function 'EulerStep' was then defined which takes in a function (the 'EarthOrbit' function in this case) and simply implements equation 10 as show here

```
\begin{minted}[language=cpp]{}
State EulerStep(State (*f)(State s), State& s, double dt)
{
    return s + f(s) * dt;
}
\end{minted}
```

This function with a specified step size dt was then run in a loop over a certain time frame to numerically solve the required differential equations using the Euler method.

2.2.2 Results

The Euler integration method was run for the Earth around the Sun for a simulated time of 5 years for three different step sizes of a day, a half day and a quarter of a day as shown in figure 1. The initial x position was set to 1 Astronomical Unit (AU) which is defined as the mean distance between the Sun and the Earth while the initial y position was set to 0. The initial velocity of the Earth was set to the Keplerian velocity in the positive y direction given by

$$v_{y0} = \sqrt{\frac{GM_E}{r}} \quad (11)$$

The initial velocity in the x direction v_{x0} was set to zero.

Euler Integration of Earth's Orbit using Different Step Sizes

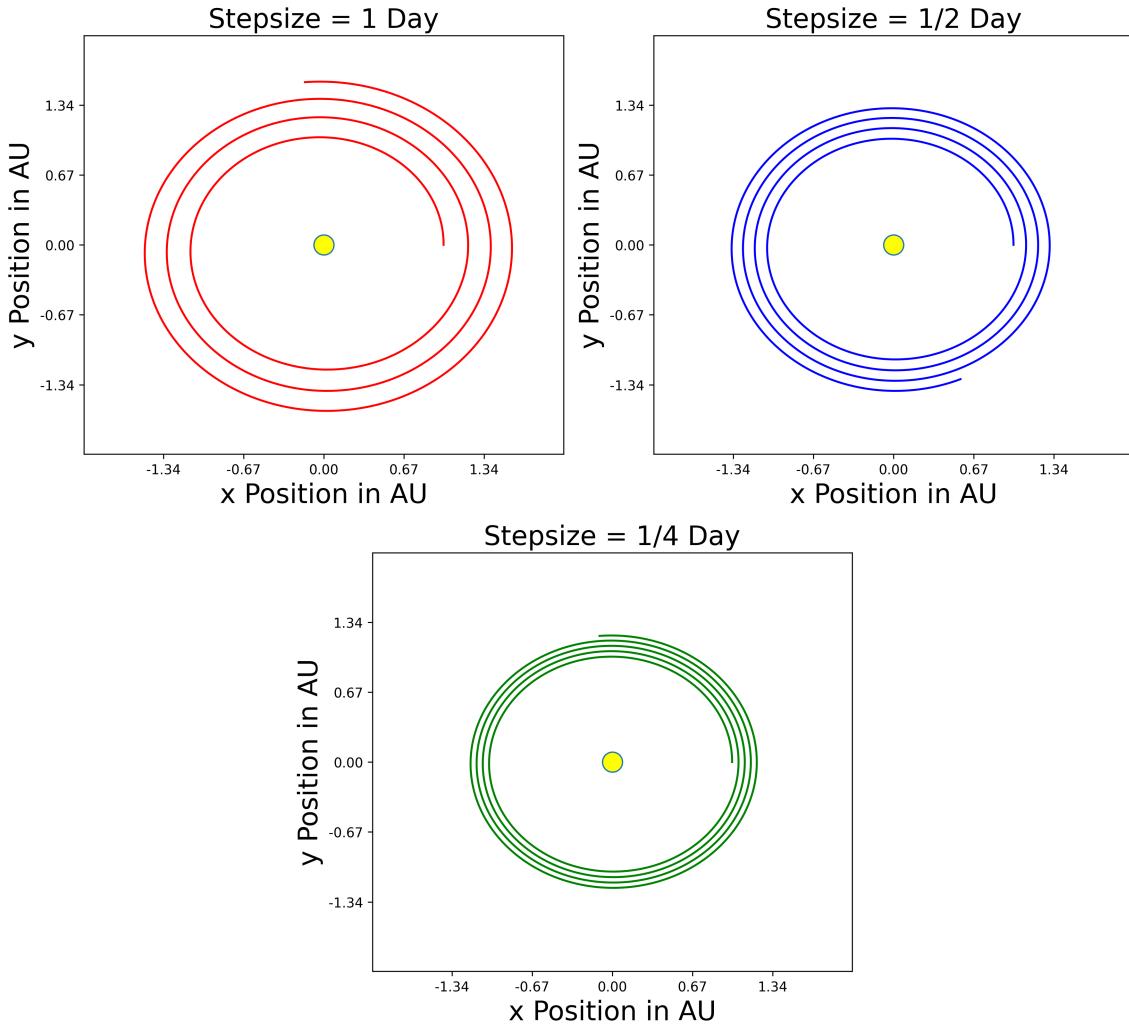


Figure 1: The Euler integration method performed for 3 different step sizes. The Earth's initial position was set to 1 AU to the right of Sun which is represented by the yellow circle in each plot. The initial velocity was set to the Keplerian velocity