PY2105 - Final Exam
Deadline: 28 November 2025

The final exam is a take-home exam, meaning you can use **all sources**, also beyond the examples and lecture notes on Canvas. The take home exam will follow the following scheme:

- Each student has to chose **one** of the four topics

- The projects will have several sub-exercises, where a larger number of solved sub-exercises result in a better mark. You should **not use any predefind solvers** for the differential equations, but program your own solution!

- The student will write a report about the project that contains the following:

  - Introduction/description of physical setting as well as a description of numerical algorithm
  - Presentation and discussion of the obtained results
  - Figures of the results (incl. label axis, fig. numbers, captions in a large enough font size etc.)
  - Figures and plots have to be made using a program (e.g. gnuplot or python)!

- In addition to the report, the student will also submit the corresponding written codes/programs via Canvas

- The report and the coding will be evaluated on a 50:50 basis

- The report should be written with the computer. No hand written report will be accepted!

- **The deadline for the final report is Friday, 28th of November at midnight! No late submissions will be accepted!**

Please submit all solutions electronically via Canvas. Any written text should be submitted as a **pdf**! If you write your report in word, convert it to a pdf before submission! Plots have to be made using a plotting program and be included in the report!

In the unlikely case of the Canvas upload not working, for whatever reason, please let me know immediately and send an email with your report and program to bbitsch@ucc.ie.

PY2105 - Final Exam: Problem 1
Orbital integration of Earth and Jupiter

According to Newton's law of gravitation the force acting on the Earth orbiting the sun is given as

$$F_{\mathrm{G}} = \frac{GM_{\odot}M_{\mathrm{E}}}{r^2} \ , \tag{1}$$

where G is the gravitational constant, $M_{\odot}$ is the stellar mass and M_{E} is the mass of the Earth. r is the distance between the planet and the sun. Let's make the assumption that the Sun's mass is sufficiently large, so that its motion can be neglected and that the sun is at the center of the coordinate system. We additionally make the assumption that the Sun and the Earth are in the same orbital plane.

The goal is to calculate the orbit of Earth as a function of time. Remember that we know from Newton's second law of motion:

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_{\mathrm{E}}} \ , \ \frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_{\mathrm{E}}}$$

a) Write the second order differential equations as two first-order differential equations that will allow an integration. Remember to look at the x and y components separately. Write down the numerical steps you will take.

b) Write a program that uses the Euler method to calculate the orbit of the Earth around the sun. The initial distance of the Earth to the sun is 1AU, while its initial velocity is given by the Keplerian velocity:

$$v_{\mathrm{K}} = \sqrt{\frac{GM_{\odot}}{r}}$$

Show how your results depend on the time step (use 3 different time steps) by making a plot of the orbital evolution of Earth.

c) Calculate the angular momentum $\mathbf{L}$ of Earth using:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \ ,$$

where $\mathbf{p} = m_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}$ is the linear momentum. Make a plot of the time evolution of the z component of $\mathbf{L}$. In addition plot also the total energy of the system as function of time and state how you calculate the total energy.

d) As discussed in the lecture, the Euler method is prone to large errors. Now use a Runge-Kutta 2nd and 4th order scheme and redo steps b) and c). Compare the orbital, angular momentum and energy evolution of the Euler and Runge-Kutta schemes as function of time.

e) Now expand the program to take Jupiter into account. You now have to deal with 2 planets orbiting around the sun. The gravitational force of the sun towards Jupiter can be calculated using eq.1, but using instead the mass and orbital distance of Jupiter (5.2 au). In addition you now have to take the force between Jupiter and the Earth into account:

$$F_{\mathrm{EJ}} = \frac{GM_{\mathrm{J}}M_{\mathrm{E}}}{r_{\mathrm{EJ}}^2} \ ,$$

where r_{EJ} corresponds to the distance between Earth and Jupiter. The corresponding differential equations will look very similar to a), but with an extra term for the interaction between Earth and Jupiter. Derive the differential equations and use the Runge-Kutta 4th order scheme to plot the evolution of both planets and how their energy and angular momentum changes with time. The initial velocities are again given by the Keplerian velocities. Start the planets on opposite sides of the sun.

HINT: Think carefully about the units you want to use, e.g. cm is not a good unit, instead use the astronomical unit au, which is the distance between the Earth and the sun. You can also make a sketch to understand how the forces work!

PY2105 - Final Exam: Problem 2
Orbital evolution of a binary system

Most stars in the galaxy are actually in binary configurations, where two objects of similar mass orbit each other. Their gravitational interactions can be described by Newton's law of gravitation. In this case we want to have N bodies that we want to integrate. The acceleration on an individual body i is given by

$$a_{\mathrm{i}}(t) = \sum_{j \neq i}^{N} GM_{\mathrm{j}} \frac{\mathbf{r}_{\mathrm{ij}}}{|r_{\mathrm{ij}}|^3} \ , \tag{2}$$

where $r_{\mathrm{ij}} = r_{\mathrm{j}} - r_{\mathrm{i}}$ is the distance between bodies i and j, while M_{j} is the mass of body j. Let's first make the assumption that the two objects orbit in the same plane. The goal is to simulate the motion of the two stars. Remember that we know from Newton's second law of motion:

$$\frac{d^2 x_i}{dt^2} = \frac{F_{G_i,x}}{M_{\mathrm{i}}} \ , \ \frac{d^2 y_i}{dt^2} = \frac{F_{G_i,y}}{M_{\mathrm{i}}}$$

a) Write the second order differential equations as two first-order differential equations that will allow an integration for both bodies. Remember to look at the x and y components separately. Write down the numerical steps that you will take.

b) Write a program that uses the Euler method to calculate the orbit of both stars around each other. Show how your results depend on the time step (use 3 different time steps) by making a plot of the orbital evolution of the objects. Use here a binary star system with 0.5 Solar masses for each object and use an initial orbital separation of 5 AU. It is recommended to start the stars at (-2.5,0) and (2.5,0) respectively. The initial velocities of the objects can be calculated by equating the gravitational force with the centripetal force. Make the assumptions that the velocities are directed towards (0,-1) and (0,1) respectively (to have circular orbits around the center of mass at the origin).

c) Calculate the angular momentum $\mathbf{L}$ of the objects using:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \ ,$$

where $\mathbf{p} = m\mathbf{v}$ is the linear momentum. Make a plot of the time evolution of the z component of $\mathbf{L}$. In addition plot also the total energy of the system as function of time and state how you calculate the total energy.

d) As discussed in the lecture, the Euler method is prone to large errors. Now use a Runge-Kutta 2nd and 4th order scheme and redo steps b) and c). Compare the orbital, angular momentum and energy evolution of the Euler and Runge-Kutta schemes as function of time.

e) Binary star systems come in different flavors, where the stars do not need to have the same mass. Using your best resolution (for the time step) and the Euler and Runge-Kutta 4th order algorithm, calculate the orbits of binary stars systems with masses of 0.9 and 0.1 Solar masses respectively. Plot the angular momentum and total energy evolution of the system. Then change it to masses of 0.99 and 0.01 solar masses. Discuss the differences of the evolution of these systems compared b). What do you notice about the orbits?

HINT: Think carefully about the units you want to use, e.g. cm is not a good unit, instead use the astronomical unit au, which is the distance between the Earth and the sun. You can also make a sketch to understand how the forces work!

PY2105 - Final Exam: Problem 3
Temperature Diffusion

Most offices in the Kane building have single glass windows, which allow shielding from water drops, but also allows heat to (basically) freely escape from the room. As winter is approaching it is good to know where it is hottest in a room. Make the assumption that the outside temperature is 5°C and the whole room cooled down to that temperature over night. After Professor B. entered the room, he turn up the heating at the other side of the room to 25°C. Make the unrealistic assumption that the heater reaches 25°C instantly. Use a one dimensional approach, where you divide the room with length of 3 meters into 10 segments. The heat conduction equation is given as

$$\alpha \frac{\partial^2 T}{\partial^2 x} = \frac{\partial T}{\partial t} \ ,$$

where T is the temperature, x the radial coordinate and t the time. Let's assume $\alpha = 0.001$.

a) Derive the analytical solutions to the problem in equilibrium and make a plot of the temperature distribution as function of distance.

b) Write a program that uses the explicit method as discussed in the lecture to derive the evolution of the temperature. Use $\Delta t = 5$s and integrate for 120min. Plot the temperature as function of distance after 30s, 5min, 10min, 60min and 120min as well as the analytical solution.

c) The office of Prof. B. is not only a straight line, but has a 2nd dimension. Assume (for simplicity) that Prof. B.'s office is square in shape (3m on each side). You now have to solve the 2D heat equation:

$$\frac{\partial T}{\partial t} - \alpha \left(\frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} \right) = 0 \ .$$

Derive the algorithm of the explicit method that you have to use.

d) Write a program that uses the explicit method to solve the 2D heat equation. Make the assumption that the windows cover the full side, while the heater is only 1.2m in length. Use a 10×10 grid and plot the final temperature distribution after 2 hours. Did you reach the equilibrium temperature at this stage?

e) Redo step b), but use the implicit method. Compare your result to the explicit method at early times (e.g. 30 seconds) and after 120 min. What difference do you observe? Hint: you can do part e) without solving c) and d)!

HINT: You can solve the matrix for the implicit method via Gaussian elimination, or you can use a more efficient method, e.g. Thomas' algorithm. Even wikipedia gives a good intro to that. Remember to set your boundary conditions correctly!

PY2105 - Final Exam: Problem 4
Tunguska Event

Eyewitnesses observed a bright flash followed by a huge explosion in a remote area of the Siberian Taiga on June 30, 1908. More than a thousand square kilometers of forest were flattened. Today it is known as the Tunguska event (named after the nearby river). What could have caused such an enormous release of energy? The most likely explanation is an airburst of a medium sized asteroid several kilometers above the surface of the Earth. To estimate the energy released by the impact of an asteroid, let us consider the simple case of radial infall toward the center of the Earth, i.e. in vertical direction. The equation of motion for free fall in the gravitational potential of Earth is given by

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM_{\mathrm{E}}}{r^2} \quad \text{or} \quad \frac{\partial^2 h}{\partial t^2} = -\frac{GM_{\mathrm{E}}}{(R_{\mathrm{E}} + h)^2} \; ,$$

where $h = r - R_{\mathrm{E}}$ is the vertical height above the surface. R_{E} and M_{E} correspond to the Earth's radius and mass, while G is the gravitational constant. Assuming a total energy of $E = 0$ (zero velocity at infinity), energy conservation implies

$$v \equiv -\frac{dh}{dt} = \sqrt{\frac{2GM_{\mathrm{E}}}{R_{\mathrm{E}} + h}} \; ,$$

where v is the infall velocity. The impact velocity at the surface of Earth is given by $v_0 = \sqrt{2GM_{\mathrm{E}}/R_{\mathrm{E}}}$. Make the assumption that the asteroid causing the Tunguska event was spherical with 50m in diameter and had a mass of 10^5 tons, corresponding to a kinetic Energy of 2.3×10^{16} J. To compute the motion of an asteroid in Earth's atmosphere, the following equation of motion has to be solved:

$$\frac{\partial^2 h}{\partial t^2} = -\frac{GM_{\mathrm{E}}}{(R_{\mathrm{E}} + h)^2} + \frac{1}{2m}\rho_{\mathrm{air}}(h)C_{\mathrm{D}}A\left(\frac{dh}{dt}\right)^2 \; ,$$

where $A = \pi R^2$ is the cross section. The 2nd term on the right hand side corresponds to the air resistance. The drag coefficient for a spherical body is $C_{\mathrm{D}} \approx 0.5$. The dependence of the density of Earth's atmosphere on altitude can be approximated by the barometric height formula:

$$\rho_{\mathrm{air}}(h) = 1.3\mathrm{kg/m}^3 \times \exp\left(-\frac{h}{8.4\mathrm{km}}\right)$$

Use an initial height of $h_0 = 300$km.

a) Write the equation in such a way that you can numerically integrate it.

b) Write a program that uses the Euler method to estimate the fall of the asteroid onto Earth (use 3 different time step sizes). Make readable plots of the height of the asteroid and its velocity as function of time as well as a plot of the velocity of the asteroid as function of its height.

c) The energy loss of the asteroid before it hits the ground is given by the dissipation rate:

$$E_{\mathrm{diss}}(t) = -\frac{C_{\mathrm{D}}A}{2}\int_0^t \rho_{\mathrm{air}}(h)\left(\frac{dh}{dt}\right)^3 dt' \; .$$

Use the trapezoidal rule to evaluate this integral in your code and compare to the asteroids kinetic Energy and discuss the implications.

d) As discussed in the lecture, the Euler method is prone to large errors. Now use a Runge-Kutta 2nd and 4th order scheme to redo setps b) and c) and compare your results.

e) The air resistance increases for smaller objects. Redo step d), but use a meteoroid with R=25 cm (assume the same density as the asteroid). Plot the altitute and velocity of the meteoroid along the asteroid data using the Runge-Kutta 4th order scheme and interpret the differences compared to the asteroid. Plot the impact time and speed. Estimate how much energy the meteoroid loses up to impact and discuss what this implies.