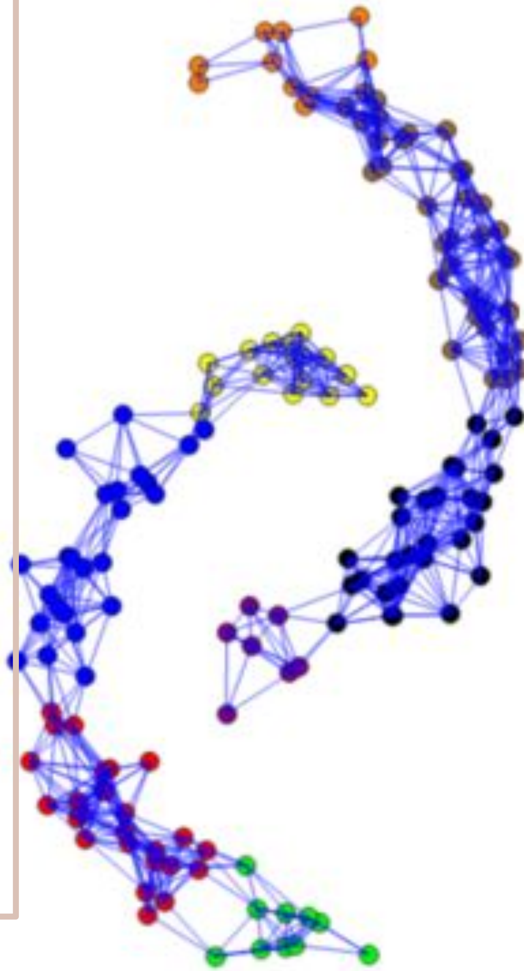
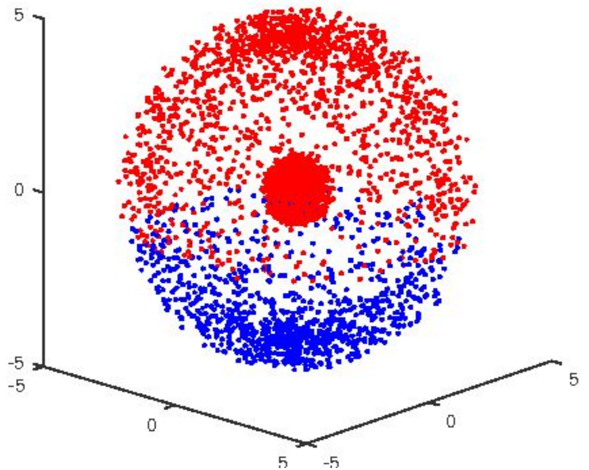
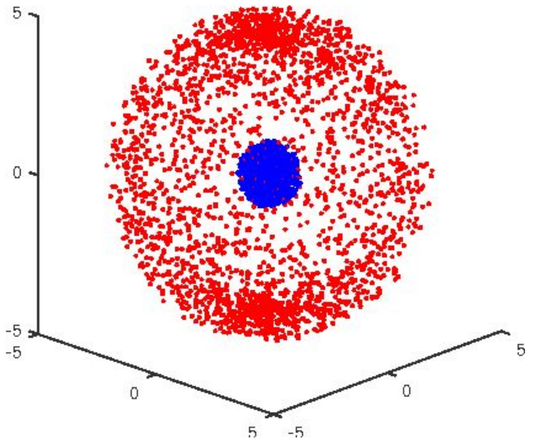
Multiclass spectral clustering

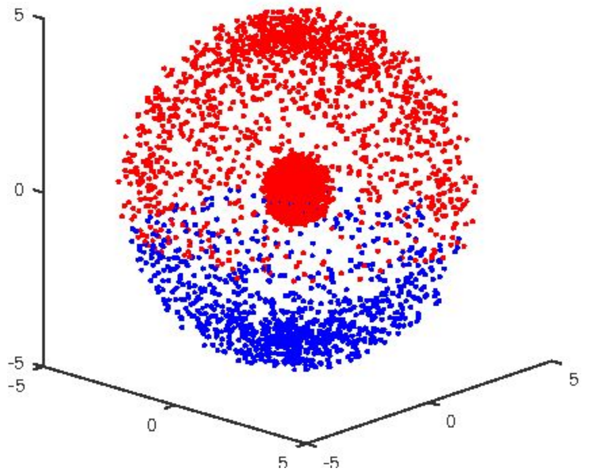
Karlo Grozdanić
Mislav Jelašić
Luka Karlić

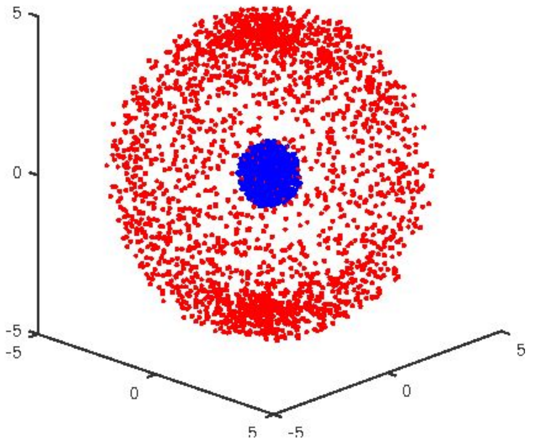






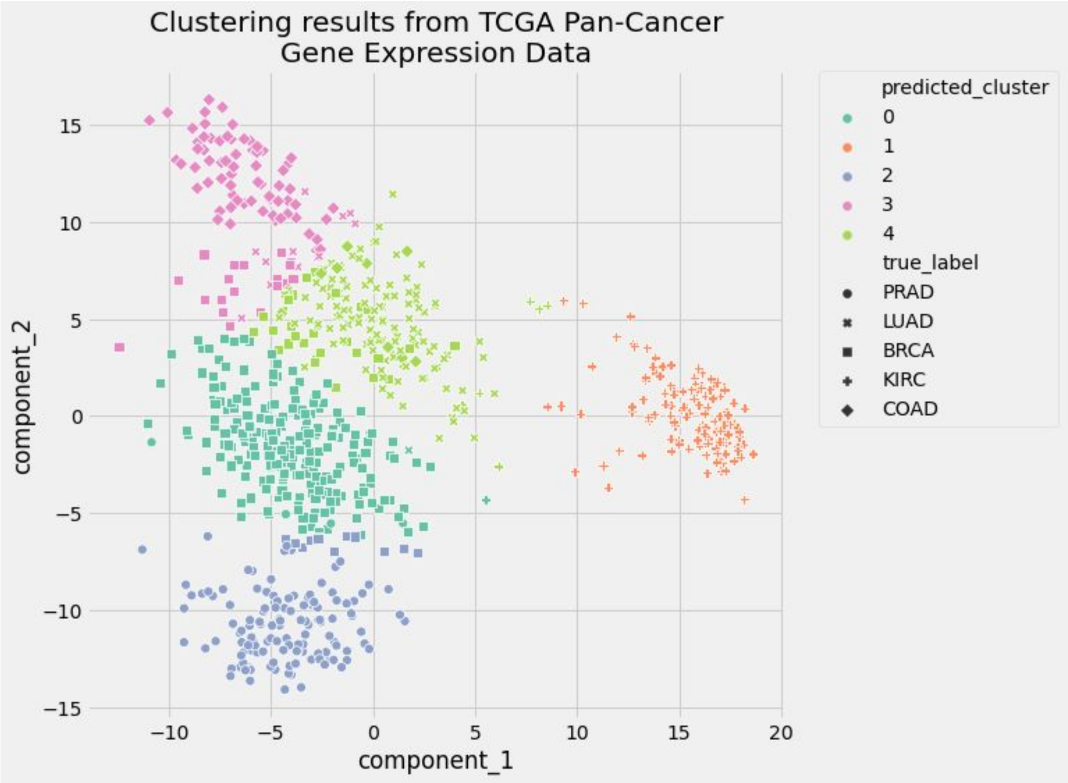
SADRŽAJ





SADRŽAJ

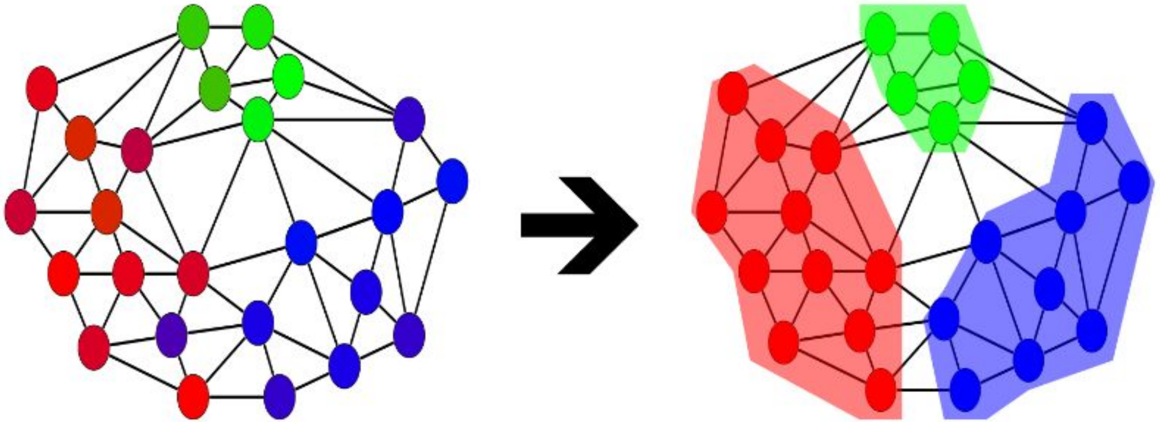
KLASIČNI PROBLEM KLUSTERIRANJA

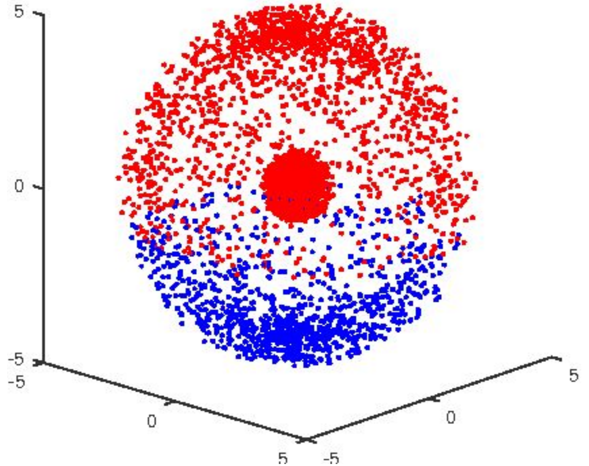


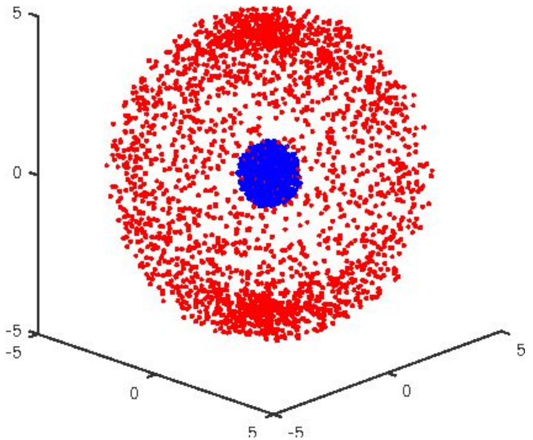
Težinski graf

Definiramo težinski graf kao uređenu trojku $\mathbb{G} = (\mathbb{V}, \mathbb{E}, W)$, gdje je $\mathbb{V} = \{v_1, v_2, ..., v_n\}$ konačan skup vrhova, $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ skup bridova koji ih povezuju i $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ matrica težina koja opisuje vjerojatnost da dva vrha pripadaju istoj grupi (matrica susjedstva).

Klasteriranje N točaka u K grupa svodi se na traženje K-particije grafa u oznaci $\Gamma_{\mathbb{V}}^{K} = \{\mathbb{V}_1, ..., \mathbb{V}_K\}$, odnosno familije nepraznih skupova $\mathbb{V}_1, \mathbb{V}_2, ..., \mathbb{V}_K$ takvih da je $\mathbb{V} = \cup_{i=1}^{K} \mathbb{V}_i$ i $\mathbb{V}_i \cap \mathbb{V}_j = \emptyset$, $\forall i \neq j$.







SADRŽAJ

Mjere povezanosti

$$links(\mathbb{A},\mathbb{B}) = \sum_{i\in\mathbb{A},j\in\mathbb{B}} W(i,j).$$

STUPANJ POVEZANOSTI SKUPOVA

$$degree(\mathbb{A}) = links(\mathbb{A},\mathbb{V}) = \sum_{i\in\mathbb{A},j\in\mathbb{V}} W(i,j).$$

STUPANJ VRHA

$$linkratio(\mathbb{A},\mathbb{B}) = \frac{links(\mathbb{A},\mathbb{B})}{degree(\mathbb{A})}.$$

RELATIVNA POVEZANOST SKUPOVA

$$knassoc(\Gamma_{\mathbb{V}}^{K}) = \frac{1}{K}\sum_{l=1}^{K} linkratio(\mathbb{V}_l,\mathbb{V}_l)$$

NORMIRANA ASOCIJACIJA

$$kncuts(\Gamma_{\mathbb{V}}^{K}) = \frac{1}{K}\sum_{l=1}^{K} linkratio(\mathbb{V}_l,\mathbb{V}\setminus\mathbb{V}_l)$$

NORMIRANI REZ

Funkcije cilja - minimizacija / maksimizacija

$knassoc(\Gamma_{\mathbb{V}}^{K})$ i $kncuts(\Gamma_{\mathbb{V}}^{K})$ realni brojevi između 0 i 1, čiji zbroj iznosi točno 1 zbog jednakosti $degree(\mathbb{V}_l) = links(\mathbb{A}, \mathbb{V}) + links(\mathbb{A}, \mathbb{V} \setminus \mathbb{V}_l)$.

$$\epsilon(\Gamma_{\mathbb{V}}^{K}) = knassoc(\Gamma_{\mathbb{V}}^{K}) \longrightarrow max$$

$$\varphi(\Gamma_{\mathbb{V}}^{K}) = kncuts(\Gamma_{\mathbb{V}}^{K}) \longrightarrow min$$

REPREZENTACIJA NORMALIZIRANIH REZOVA

Označimo sa $\mathrm{X} \in \mathbb{R}^{N \times K}$ binarnu matricu particije pridruženu particiji $\Gamma_{\mathbb{V}}^{K}$. Preciznije, $X = [X_1, ..., X_K]$, gdje je X_l binarni indikator od V_l:

$$X(i,l) = \langle i \in \mathbb{V}_l \rangle = \begin{cases} 1, & i \in \mathbb{V}_l \\ 0, & inače \end{cases}, \; i \in \mathbb{V}, \; l \in [K].$$

MATRICA PARTICIJE

$$D = Diag(W 1_N).$$

DIJAGONALNA MATRICA STUPNJEVA

$$links(\mathbb{V}_l, \mathbb{V}_l) = X_l^T W X_l \qquad degree(\mathbb{V}_l) = X_l^T D X_l$$

MODELIRANJE PROBLEMA

$$(PNCX)\begin{cases} maksimiziraj\ \epsilon(X) = \frac{1}{K}\sum_{l=1}^{K}\frac{X_l^T W X_l}{X_l^T D X_l} \\ uz\ X \in \{0,1\}^{N\times K},\ X1_K = 1_N \end{cases}$$

$$Z = f(X) = X(X^T D X)^{-\frac{1}{2}}.$$

FUNKCIJA NORMALIZACIJE

$$X = f^{-1}(Z) = Diag(diag^{-\frac{1}{2}}(ZZ^T))Z.$$

$$(PNCZ)\begin{cases} maksimiziraj\ \epsilon(Z) = \frac{1}{K}tr(Z^T W Z) \\ uz\ Z^T D Z = I_K \end{cases}$$

REZULTATI SPEKTRALNE ANALIZE

$$Wx = \lambda Dx \quad \longleftrightarrow \quad (D^{-\frac{1}{2}} W D^{-\frac{1}{2}})(D^{\frac{1}{2}} x) = \lambda (D^{\frac{1}{2}} x)$$

Propozicija 1 *(Ortonormalna invarijantnost) Neka je $R \in \mathbb{R}^{K \times K}$ ortogonalna matrica. Ako je Z rješenje PNCZ programa, tada je rješenje i cijeli skup $\{ZR : R^T R = I_K\}$. Dodatno, $\epsilon(ZR) = \epsilon(Z)$.*

Propozicija 3 *(Ograničenost odozgo) Za bilo koji prirodni broj K vrijedi:*

$$\max \epsilon(\Gamma_{\mathbb{V}}^{K}) \leq \max_{Z^T D Z = I_K} \epsilon(Z) = \frac{1}{K} \sum_{l=1}^{K} s_l$$

$$\max_{Z^T D Z = I_{K+1}} \epsilon(Z) \leq \max_{Z^T D Z = I_K} \epsilon(Z)$$

$$P = D^{-1} W$$ **NORMALIZIRANA MATRICA TEŽINA**

REZULTATI SPEKTRALNE ANALIZE

Propozicija 2 *(Optimalno rješenje) Neka su* $V = [V_1, ..., V_N]$ *i* $S = Diag(s_1, ..., s_N)$, $s_1 \geq ... \geq s_N$, *matrice svojstvenih vektora i svojstvenih vrijednosti od* P, *odnosno rješenja svojstvene zadaće* $PV = PS$. *Par* (V, S) *može se dobiti preko ortonormalne matrice* $\bar{V} = [\bar{V}_1, ..., \bar{V}_N]$ *koja rješava simetrični problem svojstvenih vrijednosti matrice* $D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$:

$$V = D^{-\frac{1}{2}} \bar{V}$$

$$D^{-\frac{1}{2}} W D^{-\frac{1}{2}} \bar{V} = \bar{V} S, \ \bar{V}^T \bar{V} = I_N$$

Štoviše, matrice V *i* S *su realne te svakih* K *različitih svojstvenih vektora čine kandidata za lokalno optimalno rješenje s funkcijom cilja*

$$\epsilon([V_{\pi_1}, ..., V_{\pi_K}]) = \frac{1}{K} \sum_{l=1}^{K} s_{\pi_l},$$

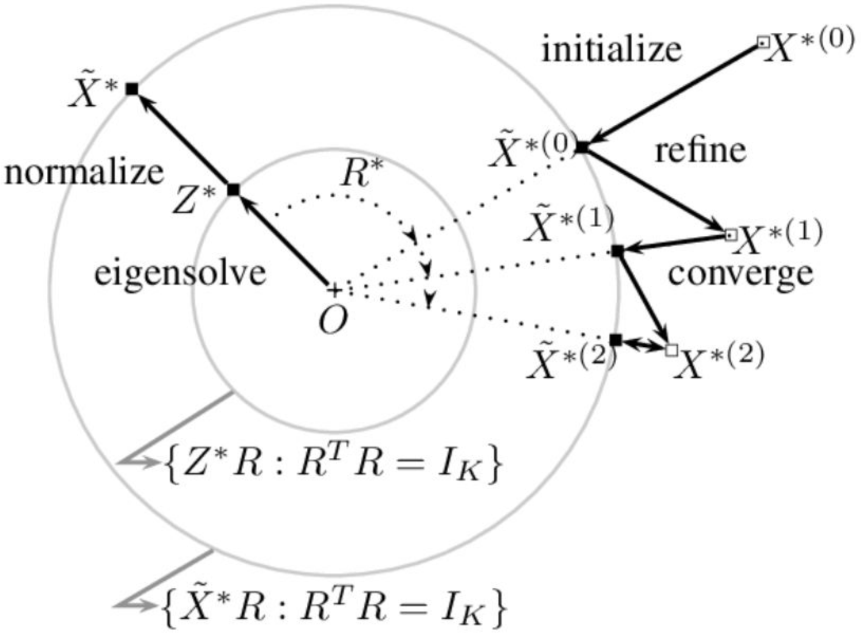
gdje je π *vektor indeksa od* K *različitih elemenata skupa prirodnih brojeva* $\{1, ..., N\}$. *Globalni optimum se stoga postiže za* $\pi = [1, ..., K]$ *što daje rješenje problema:*

$$Z^* = [V_1, ..., V_K]$$

$$\Lambda^* = Diag([s_1, ..., s_K])$$

$$\epsilon(Z^*) = \frac{1}{K} tr(\Lambda^*) = \max_{Z^T D Z = I_K} \epsilon(Z)$$

NALAŽENJE OPTIMALNOG RJEŠENJA



$$\{\tilde{X}^*R : \tilde{X}^* = f^{-1}(Z^*),\ R^TR = I_K\}$$

$$\{Z^*R : R^TR = I_K, PZ^* = Z^*\Lambda\}$$

NALAŽENJE OPTIMALNOG RJEŠENJA

$$(POD) \begin{cases} minimiziraj\ \phi(X, R) = \|X - \tilde{X}^* R\|_F^2 \\ uz\ X \in \{0,1\}^{N\times K},\ X1_K = 1_N,\ R^T R = I_K \end{cases}$$

$$\|X - \tilde{X}^* R\|_F^2 = \|X\|_F^2 + \|\tilde{X}^*\|_F^2 - 2tr(XR^T \tilde{X}^{*T})$$

$$(PODX) \begin{cases} minimiziraj\ \phi(X) = \|X - \tilde{X}^* R^*\|_F^2 \\ uz\ X \in \{0,1\}^{N\times K}, X1_K = 1_N, \end{cases}$$

$$(PODR) \begin{cases} minimiziraj\ \phi(R) = \|X^* - \tilde{X}^* R\|_F^2 \\ uz\ R^T R = I_K, \end{cases}$$

NALAŽENJE OPTIMALNOG RJEŠENJA

Teorem 4 *Neka je* $\tilde{X} = \tilde{X}^* R^*$. *Optimalno rješenje programa PODX je dano s:*

$$X(i,l) = \langle l = \arg\max_{k \in [K]} \tilde{X}(i,k) \rangle, \; i \in \mathbb{V}$$

Teorem 5 *Rješenje programa PODR dano je SVD dekompozicijom matrice* $X^* \tilde{X}^*$:

$$R^* = \tilde{U} U^T,$$

$$X^* \tilde{X}^* = U \Omega \tilde{U}^T, \; \Omega = Diag(\omega_1, ..., \omega_K),$$

gdje je $U^T U = I_K$, $\tilde{U}^T \tilde{U} = I_K$ *i* $\omega_1 \geq ... \geq \omega_K$.

ALGORITAM (NC)

Dana je matrica težina W i broj elemenata particije K:

1. Izračunaj matricu stupnjeva $D = Diag(W1_N)$
2. Nađi optimalno rješenje Z^* svojstvenog problema:

$$D^{-\frac{1}{2}} W D^{-\frac{1}{2}} \bar{V}_{[K]} = \bar{V}_{[K]} Diag(s_{[K]}), \bar{V}_{[K]}^T \bar{V}_{[K]} = I_{[K]}$$

$$Z^* = D^{-\frac{1}{2}} \bar{V}_{[K]} \ (opcionalno)$$

3. Normaliziraj Z^*: $\tilde{X}^* = Diag(diag^{-\frac{1}{2}}(Z^* Z^{*T}))Z^*$
4. Inicijaliziraj X^* preko R^* na sljedeći način:

$$R_1^*[\tilde{X}^*(i,1), ..., \tilde{X}^*(i,K)]^T, \ i \in [N] \ slučajan$$

$$c = 0_{\mathbb{N} \times 1}$$

$$Za \ k = 2, ..., K \ radi:$$

$$c = c + abs(\tilde{X}^* R_{k-1}^*)$$

$$R_k^* = [\tilde{X}^*(i,1), ..., \tilde{X}^*(i,K)]^T, \ i = arg \ min \ c$$

ALGORITAM (NC)

5. Postavi parametar kovergencije $\bar{\phi}^* = 0$.

6. Nađi optimalno diskretno rješenje X^*:

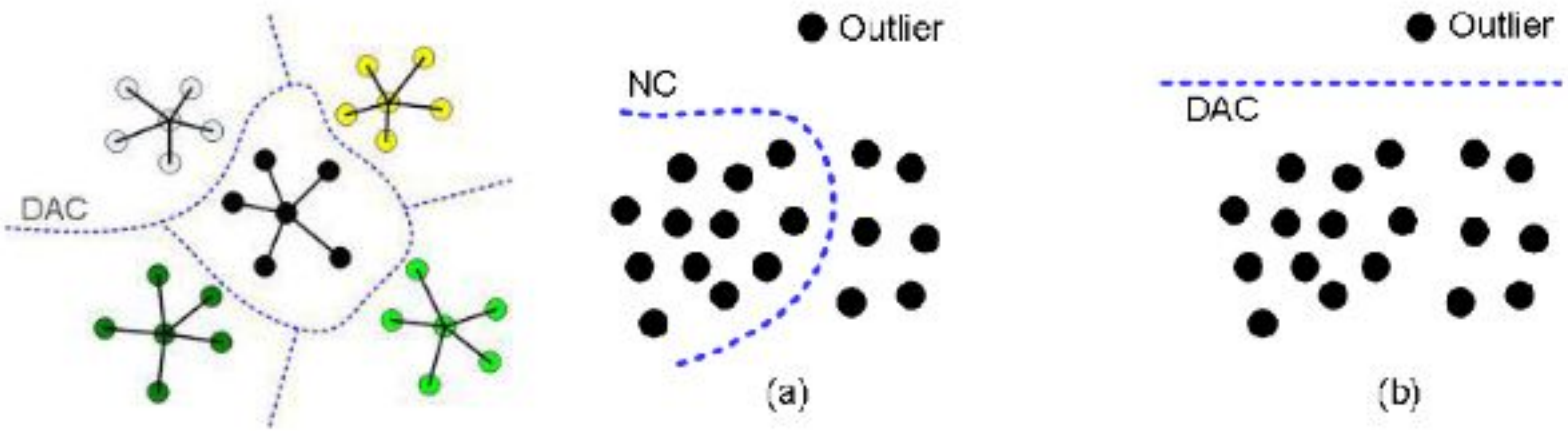
$$\tilde{X} = \tilde{X}^* R^*$$
$$X^*(i,l) = \langle l = arg\, max_{k\in[K]} \tilde{X}(i,K)\rangle,\ i \in \mathbb{V},\ l \in [K].$$

7. Nađi optimalnu ortogonalnu matricu R^* pomoću SVD dekompozicije:

$$X^{*T}\tilde{X}^* = U\Omega\tilde{U}^T,\ \Omega = Diag(\omega)$$
$$\bar{\phi} = tr(\Omega)$$
$$Ako\ je\ |\bar{\phi} - \bar{\phi}^*| < mašinska\ preciznost,\ stani\ i\ ispiši\ X^*$$
$$\bar{\phi}^* = \bar{\phi}$$
$$R^* = \tilde{U}U^T$$

8. Idi na korak 6.

Metoda bazinara na diskriminantnoj analizi - DACA



Metoda bazinara na diskriminantnoj analizi - DACA

$$w_{ij}^* = \begin{cases} 1, & ako\, j \in N_k(i)\ ili\ i \in N_k(j) \\ 0, & inače, \end{cases}$$

Dodajemo informaciju o blizini točaka

$$\widehat{W} = (\widehat{w}_{ij})_{N \times N} = (w_{ij} \cdot w_{ij}^*)_{N \times N}$$

$$(DAC) \begin{cases} maksimiziraj\ g(X) = \frac{1}{K}\sum_{n=1}^{K} \frac{X_n^T \widehat{W} X_n}{X_n^T Q X_n} = \frac{1}{K}\sum_{n=1}^{K} \frac{[X_n(X_n^T X_n)^{-\frac{1}{2}}]^T \widehat{W} [X_n(X_n^T X_n)^{-\frac{1}{2}}]}{[X_n(X_n^T X_n)^{-\frac{1}{2}}]^T Q [X_n(X_n^T X_n)^{-\frac{1}{2}}]} \\ uz\ X \in \{0,1\}^{N \times K}, X1_K = 1_N, \end{cases}$$

$$P = X(X^T X)^{-\frac{1}{2}} \quad \Longrightarrow \quad (DAC) \begin{cases} maksimiziraj\ \ h(P) = \frac{1}{K} tr\{(P^T Q P)^{-1}(P^T \widehat{W} P)\} \\ uz\ \ P^T P = I_K \end{cases}$$

Metoda bazinara na diskriminantnoj analizi - DACA

$$\Uparrow \quad X_n^T \widehat{W} X_n = \sum_{i \in \mathbb{V}_n} \sum_{j \in B_i} w_{ij}; \quad X_n^T Q X_n = \sum_{i \in \mathbb{V}_n} \sum_{j \in \bar{B}_i} w_{ij}, \quad \Downarrow$$

kompaktnost unutar iste klase

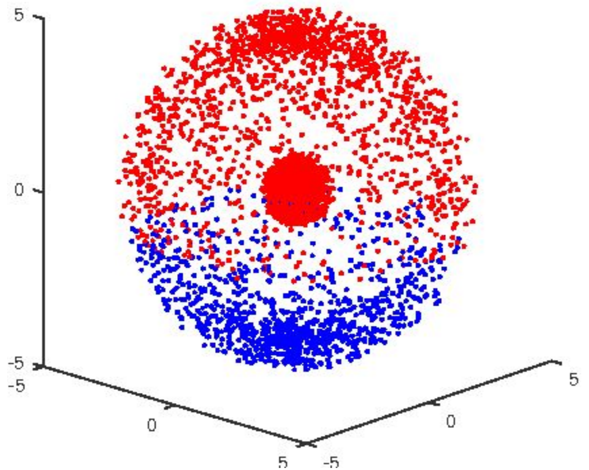
separabilnost između različitih klasa

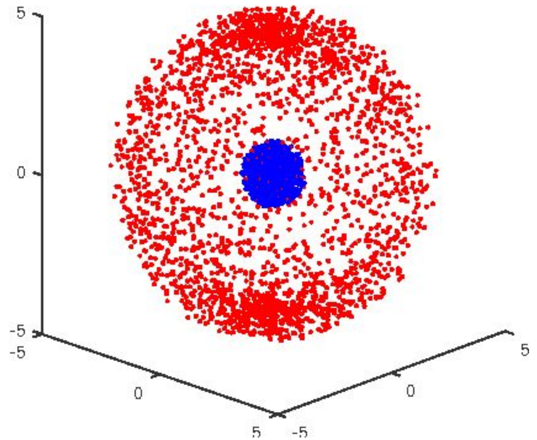
$$B_i = \{u | u \in \mathbb{V}_n,\ i \in N_k(u) \ \ ili \ \ u \in N_k(i)\},\ \bar{B}_i = \{u | u \notin \mathbb{V}_n,\ i \in N_k(u) \ \ ili \ \ u \in N_k(i)\},$$

Algoritam DACA

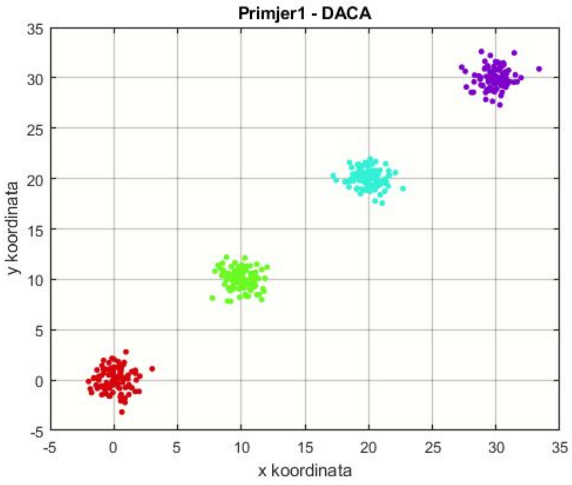
Dan je skup podataka $Z = \{z_1, z_2, ..., z_n\}$ i broj elemenata particije K:

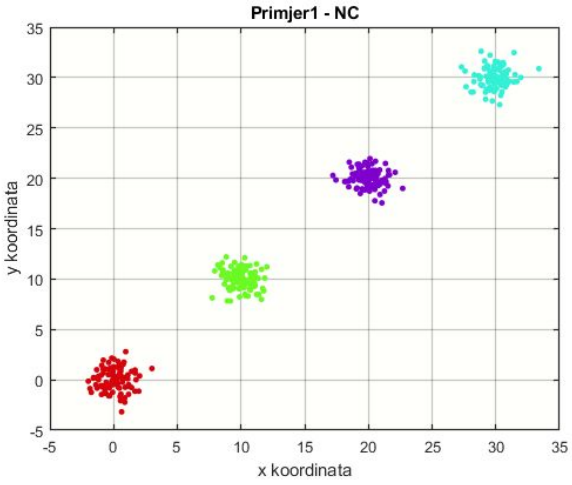
1. Kreiraj graf $G = (V, E, W, W^*)$, gdje je $V = \{1, ..., N\}$ skup vrhova, $E \subseteq V \times V$ skup bridova, $W^* = (w^*_{ij})_{N\times N}$ definirana ranije, $W = (w_{ij})_{N\times N}$, npr., $w_{ij} = \exp(-\text{dist}(z_i, z_j)/2\sigma^2)$, gdje je σ faktor skaliranja, a dist$(\cdot)$ prestavlja neku funkciju udaljenosti.
2. Izračunaj $\widehat{W} = (\widehat{w}_{ij})_{N\times N} = (w_{ij} \cdot w^*_{ij})_{N\times N}$ i $Q = \widehat{D} - \widehat{W}$, $\widehat{D}$ dijagonalna dana s $\widehat{d}_{ii} = \sum_j \widehat{w}_{ij}$ za $1 \le i, j \le N$. Ako je Q singularna matrica, zamjenjujemo ju s $Q + \epsilon I_N$.
3. Odredi $\tilde{P}$ kao K najvećih normaliziranih svojstvenih vektora $Q^{-1}\widehat{W}$.
4. Nađi kandidata za rješenje particije grafa: $\tilde{X} = Diag(diag^{-\frac{1}{2}}(\tilde{P}\tilde{P}^T))\tilde{P}$
5. Izvedi iterativnu proceduru poboljšanja rješenja na $\tilde{X}$ opisanu u algoritmu (NC).



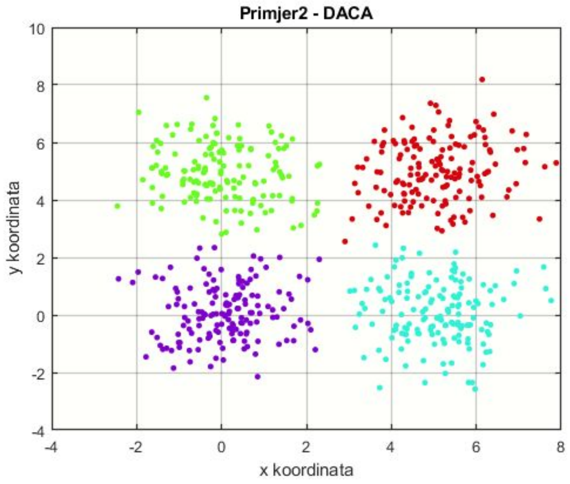


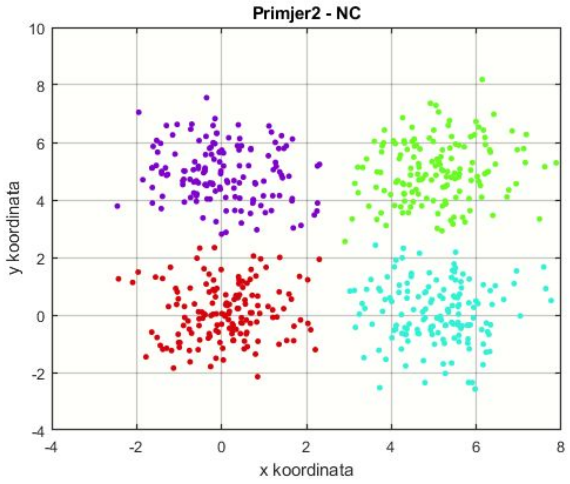
SADRŽAJ

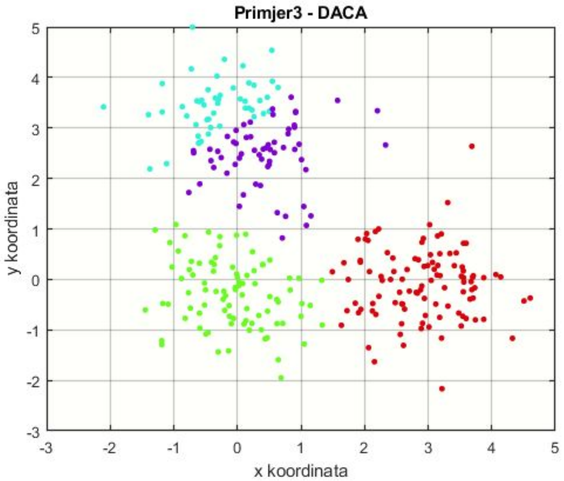


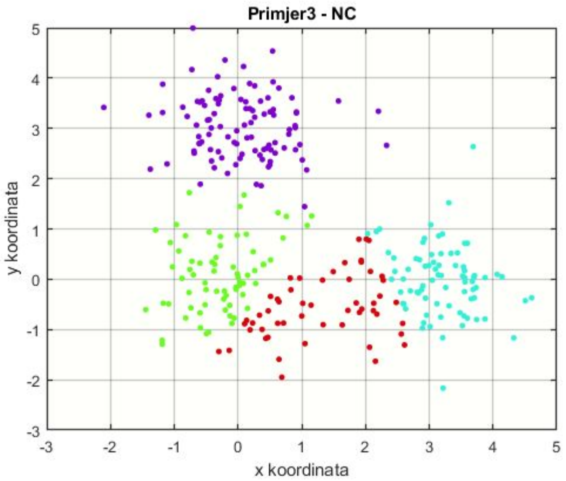


Na lakšim primjerima, oba algoritma jednako rade

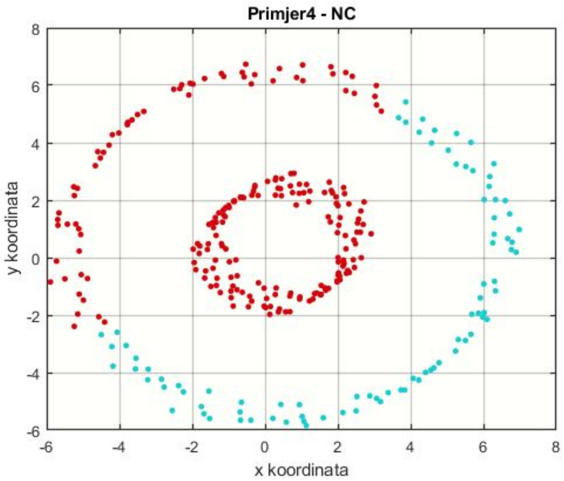


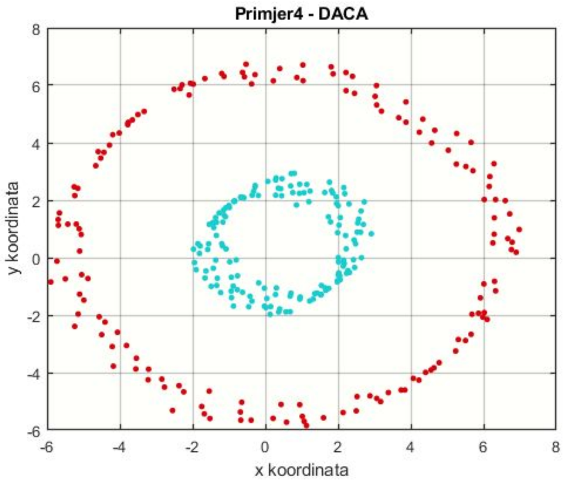


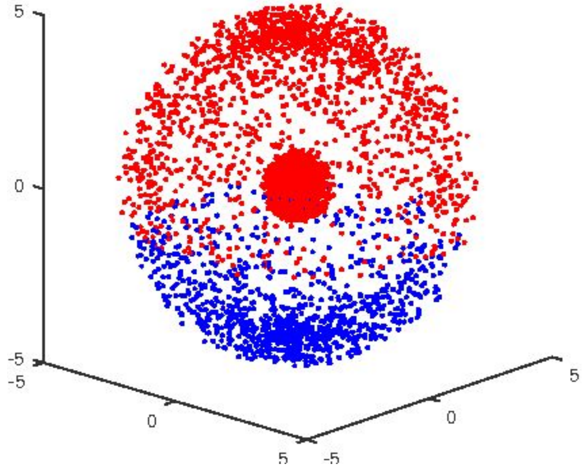


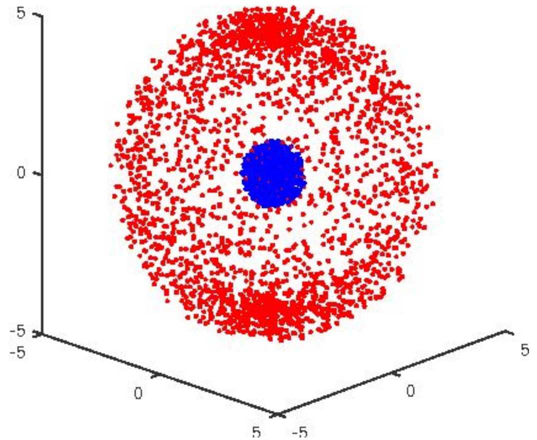


Na težim primjerima DACA ispada uspješnija









SADRŽAJ

KLASIČNI PROBLEM SEMANTIČKE SEGMENTACIJE

- poseban oblik klasteriranja
- bitno područje interesa u računalnom vidu

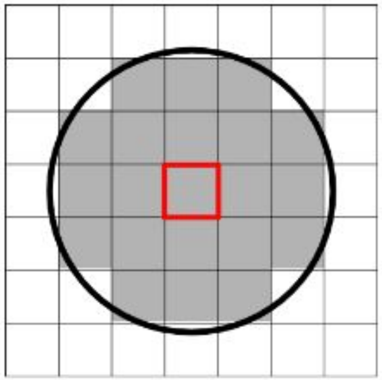
CILJ: grupirati piksele koji čine smislenu cjelinu

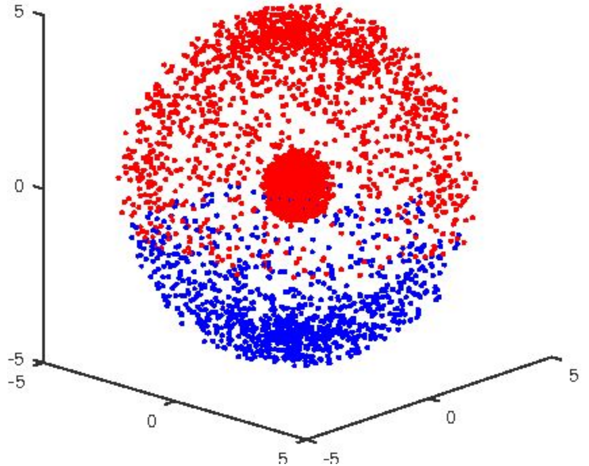
UDALJENOST MEĐU PIKSELIMA

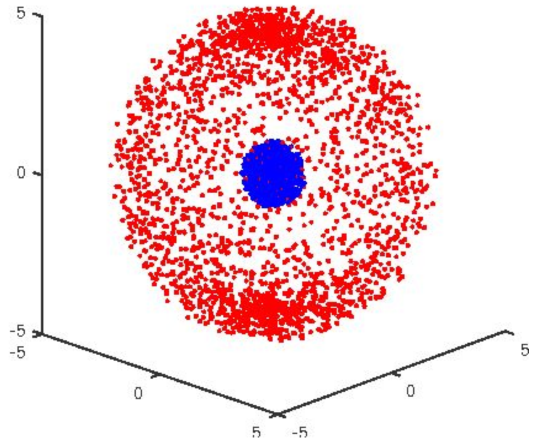
Što je **sigma_f** veći, to će biti veći utjecaj udaljenosti među pikselima na konačni rezultat

$$w_{ij} = e^{-\frac{\|f_i - f_j\|_2^2}{\sigma_f}} \cdot e^{-\frac{\|x_i - x_j\|_2^2}{\sigma_x}} \cdot \gamma_r(i,j), \quad \gamma_r(i,j) = \begin{cases} 1, & \text{za } \|x_i - x_j\|_2 < r \\ 0, & \text{inače} \end{cases}$$

Što je **sigma_f** veći, to će biti veći utjecaj razlike u bojama na konačni rezultat



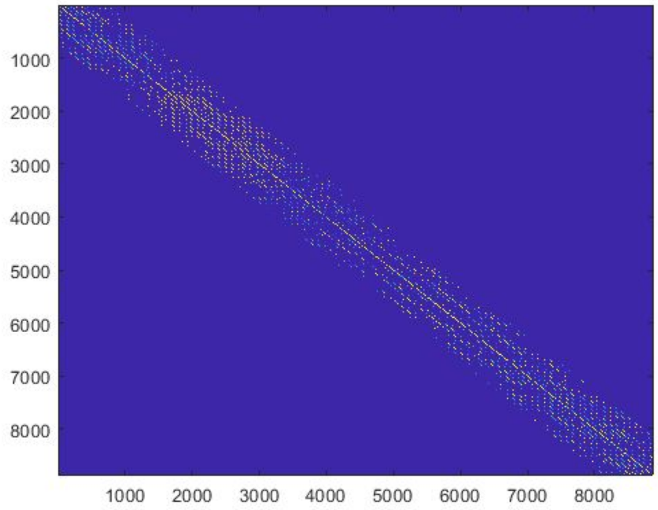




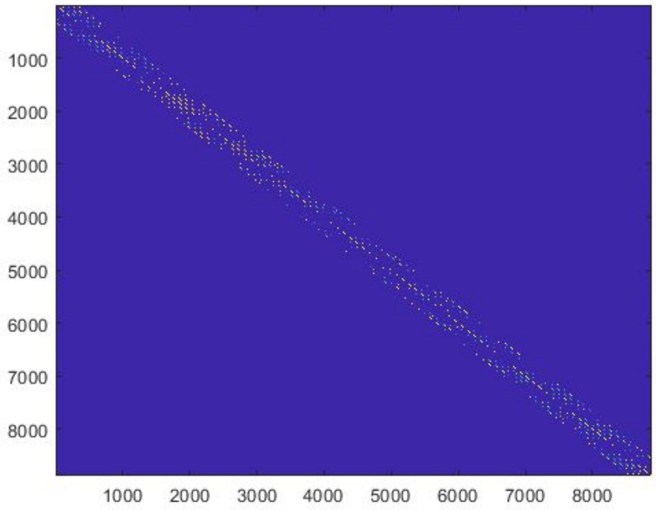
SADRŽAJ

Heuristika za dobivanje dobrih rezultata

1. Odaberi početne parametre za generiranje matrice W tako da sigma_f bude iz iz [0.5, 4], sigma_x iz [5, 20], radius otprilike 10-20% visine ili širine slike.
2. Nacrtaj matricu W te korigiraj parametre.

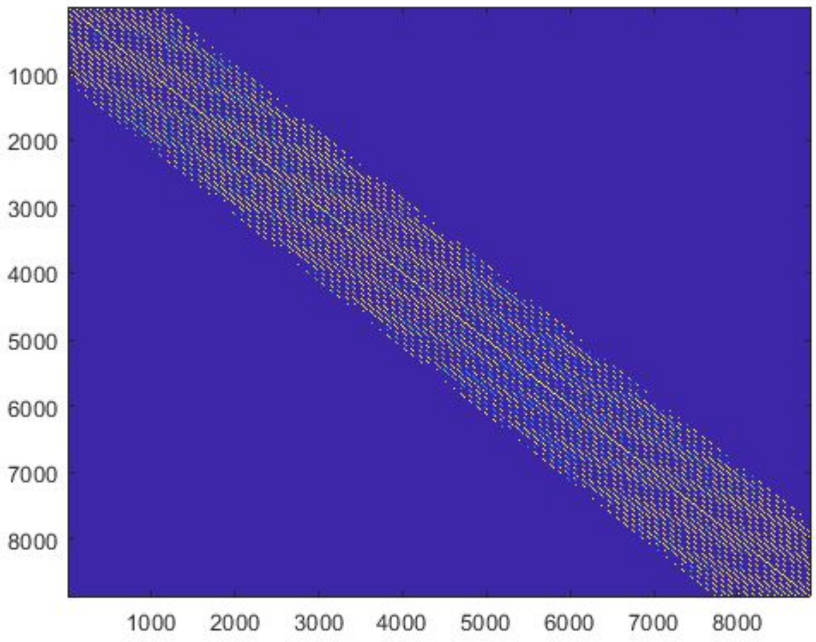


Dobra matrica W



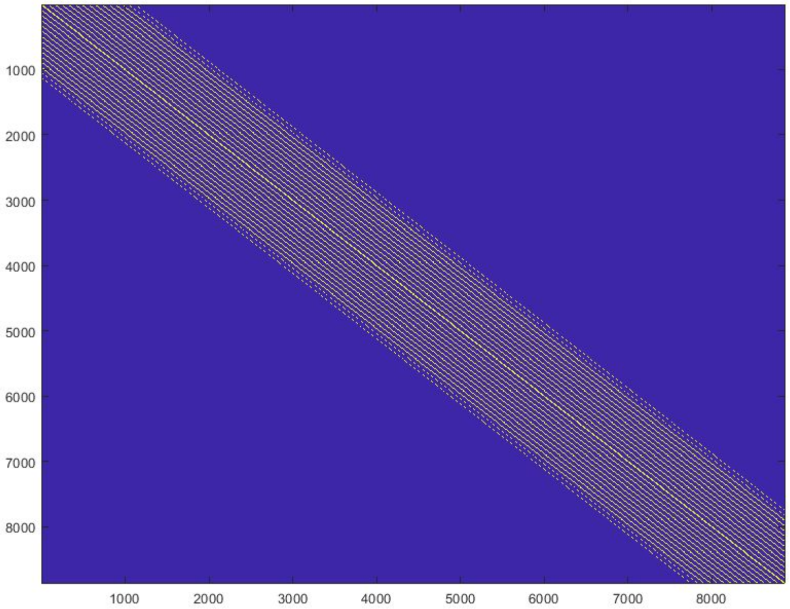
Radijus premali

Heuristika za dobivanje dobrih rezultata

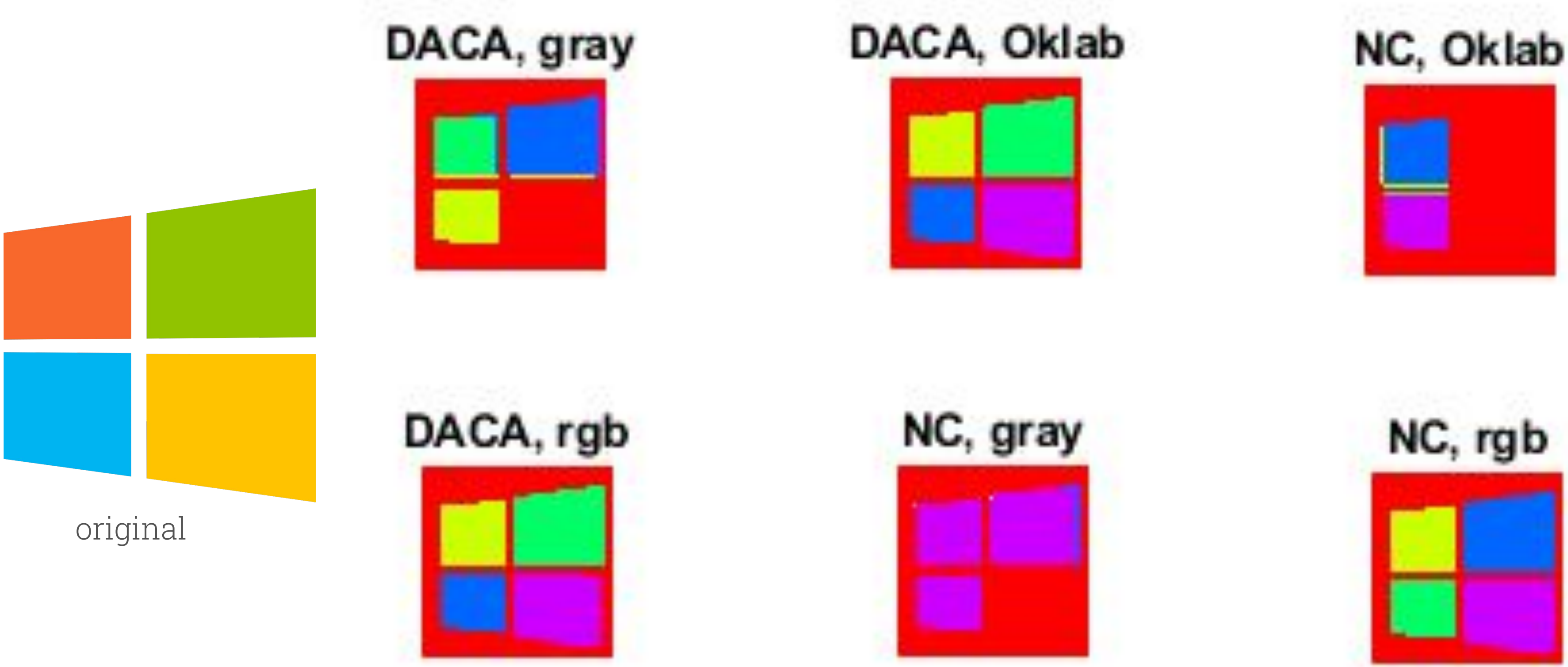


Radijus prevelik

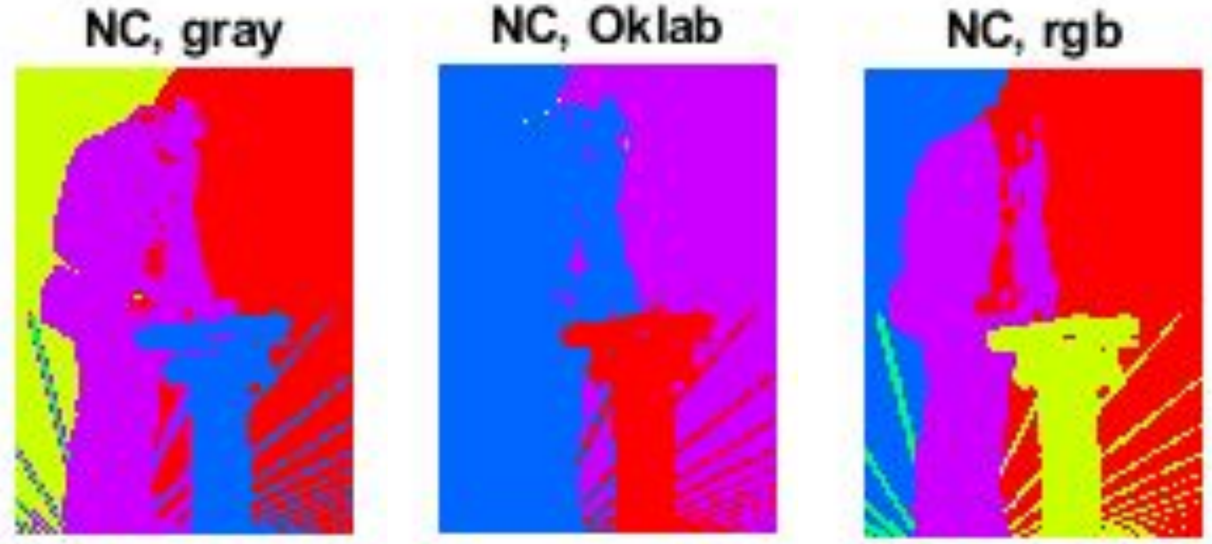
Sigma prevelik



Windows logo

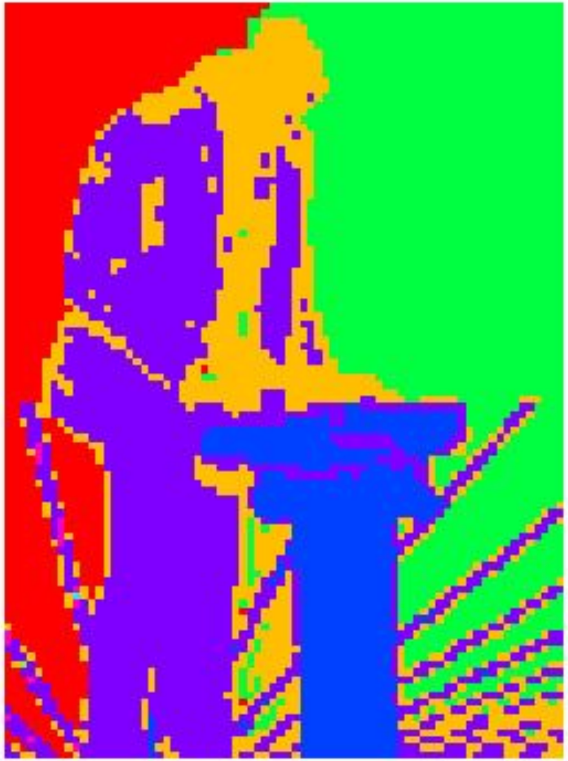


Kanye West - Runaway



Kanye West - Runaway

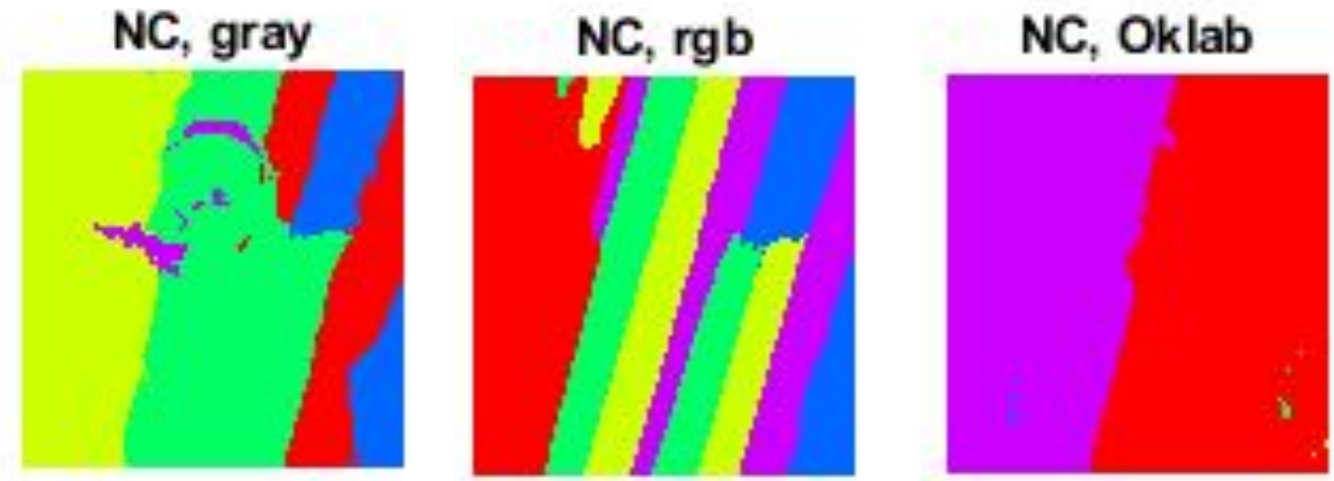
Kanye West - Runaway + računanje fizičke udaljenosti piksela



grayscale

Oklab

Kanye West - The Life of Pablo



Kanye West - The Life of Pablo

HVALA NA PAŽNJI