1 Notations

1.1 Green's function

1.1.1 As the kernel of the resolvent

Consider the operator $L : X \to X$.

- Physics notation:
$$R(\lambda) = \frac{1}{\lambda I - L}$$

- Mathematics notation:
$$R(\lambda) = \frac{1}{L - \lambda I}$$

TODO: proof that it is an integral operator
$G(x, s; \lambda)$ is the kernel of $R(\lambda)$.

1.1.2 Green's function of the differential operator

Differential operator L, defined for distributions.
Green's function at point s is a solution (TODO what if there are many?) of

- Physics notation:
$$L_x G(x, s) = -\delta(x - s)$$

- Mathematics notation:
$$L_x G(x, s) = \delta(x - s)$$

That is, they differ up to sign. We use PHYSICS notation.
Eigenfunction expansion:

- Physics notation:
$$G(x, s) = -\sum_n \frac{\psi_n(x)\psi_n^*(s)}{\lambda_n}$$

- Mathematics notation:
$$G(x, s) = \sum_n \frac{\psi_n(x)\psi_n^*(s)}{\lambda_n}$$

2 Short intro into quantum physics

TODO

3 Schrodinger's equation

3.1 Time-dependent

$$i\hbar\partial_t\Psi(x,t) = H\Psi(x,t)$$

Particle in an electric (!!!) field:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(x,t)$$

TODO what about charge?

3.2 Time-independent

If $V(x,t)$ does not depend on time, we can separate the variables:
$\Psi(x,t) = \psi(x)T(t)$.

$$\begin{cases} i\hbar\partial_t T(t) = ET(t) \\ (-\frac{\hbar^2}{2m}\nabla^2 + V(x))\psi(x) = E\psi(x) \end{cases}$$

Now that we have time-independent states, we can represent any initial state ψ_0 (TODO: complete set of eigenstates) in its basis:

$$\psi_0 = \sum_\lambda A_\lambda \psi_\lambda$$

$$\psi(x,t) = \sum_\lambda A_\lambda \psi_\lambda(x) e^{-\frac{i}{\hbar}E_\lambda t}$$

4 Reflection coefficient and transmission coefficient

Reflection/transmission coefficients are quantities describing the behaviour of scattered wave (usually at infinity).

Probability current for spin-0 particle.

$$\boldsymbol{j}(x,t) = \frac{\hbar}{2mi}(\psi(x,t)^*\nabla\psi(x,t) - \psi(x,t)\nabla\psi^*(x,t))$$

TODO: write about stationary (scattering) states and omitting time.

$$T = \frac{|j_{trans}|}{|j_{inc}|}$$

$$R = \frac{|j_{refl}|}{|j_{inc}|}$$

These are position dependent and non-scalar values in general. Usually, geometry and symmetries of the scattering problem allows to define the asymptotic region and calculate its integral over scattering cross section, yielding a scalar value.

5 Free particle in magnetic field

$$H = \frac{1}{2m}(-i\hbar\nabla - qA)^2$$

Field $B = (0, 0, B)$, vector potential $A = (0, \frac{1}{2}Br, 0)$.
Polar coordinates:

$$(-i\hbar\nabla - qA)\psi = \begin{pmatrix} -i\hbar\frac{\partial\psi}{\partial r} \\ -i\hbar\frac{1}{r}\frac{\partial\psi}{\partial\theta} - q\frac{1}{2}Br\psi \\ -i\hbar\frac{\partial\psi}{\partial z} \end{pmatrix}$$

$$(-i\hbar\nabla - qA)\cdot F = -i\hbar\frac{1}{r}\frac{\partial(rF_r)}{\partial r} - i\hbar\frac{1}{r}\frac{\partial F_\theta}{\partial\theta} - q\frac{1}{2}BrF_\theta - i\hbar\frac{\partial F_z}{\partial z}$$

$$\begin{aligned}
2mH\psi &= -\hbar^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) \\
&- \hbar^2\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + i\hbar\frac{1}{r}q\frac{1}{2}Br\frac{\partial\psi}{\partial\theta} + q\frac{1}{2}Bri\hbar\frac{1}{r}\frac{\partial\psi}{\partial\theta} + q^2\frac{1}{4}B^2r^2\psi \\
&- \hbar^2\frac{\partial^2\psi}{\partial z^2} \\
&= \hat{p}_r^2 - \frac{i\hbar}{r}\hat{p}_r \\
&+ (\hat{p}_\theta - \frac{1}{2}qBr)^2 \\
&+ \hat{p}_z^2
\end{aligned}$$

Commutes with p_z and seems to commute with p_λ.
Eigenvectors of p_z:

$$\psi(r, \theta, z) = \xi(r, \theta)e^{\frac{i}{\hbar}p_z z}$$

Eigenvectors of p_λ:

$$\psi(r, \theta, z) = \xi(r, z)e^{\frac{i}{\hbar}p_\lambda\theta}$$

Periodicity requires $\frac{1}{\hbar}p_\lambda 2\pi = 2\pi k, k \in \mathbb{Z}$, therefore, $p_\lambda = \hbar k, k \in \mathbb{Z}$.

$$\psi(r, \theta, z) = R(r)e^{ik\theta}e^{\frac{i}{\hbar}p_z z}$$

Substitute in the equation:

$$\begin{aligned}
2mH\psi = -\hbar^2&\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right) = e^{\cdots}(-\hbar^2R''(r) - \frac{\hbar^2}{r}R'(r)) \\
&+ (\frac{1}{r}\hbar k - \frac{1}{2}qBr)^2\psi \\
&+ p_z^2\psi \\
&= 2mE\psi
\end{aligned}$$

Divide everything by $e^{\cdots}$:

$$\left(-\hbar^2\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}\right)+\frac{1}{r^2}\hbar^2k^2-qB\hbar k+\frac{1}{4}q^2B^2r^2+p_z^2-2mE\right)R(r)=0$$

Divide by $-\hbar^2$:

$$\left(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}-\frac{1}{r^2}k^2-\frac{1}{4}\frac{1}{\hbar^2}q^2B^2r^2+\frac{1}{\hbar}qBk-\frac{p_z^2}{\hbar^2}+\frac{2mE}{\hbar^2}\right)R(r)=0$$

Use substitution: $R(r)=\frac{U(r)}{\sqrt{r}}$

$$\left(\frac{1}{\sqrt{r}}\frac{\partial^2 U}{\partial r^2}+\frac{1}{4r^{5/2}}U(r)\right)+\left(-\frac{1}{r^2}k^2-\frac{1}{4}\frac{1}{\hbar^2}q^2B^2r^2+\frac{1}{\hbar}qBk-\frac{p_z^2}{\hbar^2}+\frac{2mE}{\hbar^2}\right)\frac{U(r)}{\sqrt{r}}=0$$

Multiply all by $\sqrt{r}$:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{4r^2}-\frac{1}{r^2}k^2-\frac{1}{4}\frac{1}{\hbar^2}q^2B^2r^2+\frac{1}{\hbar}qBk-\frac{p_z^2}{\hbar^2}+\frac{2mE}{\hbar^2}\right)U(r)=0$$

5.1 Notes

5.1.1 Cylindrical coordinates

$$\boldsymbol{r}=(r,\theta,z)$$

$$\hat{p_r}=-i\hbar\frac{\partial}{\partial r}$$

$$\hat{p_\theta}=-i\hbar\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\hat{p_z}=-i\hbar\frac{\partial}{\partial z}$$

$$\nabla f=\begin{pmatrix}\frac{\partial f}{\partial r}\\ \frac{1}{r}\frac{\partial f}{\partial\theta}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$

$$\nabla\cdot F=\frac{1}{r}\frac{\partial(rF_r)}{\partial r}+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}$$