Impenetrable wall + delta function potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & x < 0 < a \\ -\infty, & x = a \\ 0, & x > a \end{cases}$$

Bound state: $E < 0$, $k = \sqrt{-E}$
Region 1:

$$\psi_1(x) = Ae^{-kx} + Be^{kx}$$

Boundary condition: $\psi_1(0) = 0$, $A = -B$
Therefore,

$$\psi_1(x) = A(e^{-kx} - e^{kx})$$

Region 2:

$$\psi_2(x) = Ce^{-kx} + De^{kx}$$

Boundary condition: $\psi_2(\infty) = 0$
Therefore,

$$\psi_2(x) = Ce^{-kx}$$

Continuoty conditions:

$$\psi_1(a) = \psi_2(a)$$

$$\Delta \frac{d\psi}{dx} = -\frac{2m}{\hbar^2}\psi(a)$$

!!! Assume $\Delta \frac{d\psi}{dx} = -\psi(a)$ for simplicity
From the continuoty:

$$A(e^{-ka} - e^{ka}) = Ce^{-ka}$$

$$C = A(1 - e^{2ka})$$

Derivatives:

$$\left.\frac{d\psi_1}{dx}\right|_{x=a} = A(-ke^{-kx} - ke^{kx})\big|_{x=a} = -kA(e^{-ka} + e^{ka})$$

$$\left.\frac{d\psi_2}{dx}\right|_{x=a} = -kCe^{-ka} = -kA(1 - e^{2ka})e^{-ka} = -kA(e^{-ka} - e^{ka})$$

$$\Delta \frac{d\psi}{dx} = 2kAe^{ka} = -A(e^{-ka} - e^{ka})$$

Assume $a = 1$ for simplicity.

$$2ke^k = e^k - e^{-k}$$

Scattering states:

$$\psi_1(x) = Ae^{-ikx} + Be^{ikx}$$

$\psi_1(-1) = 0 = Ae^{ik} + Be^{-ik}$.

TODO: suppose both A and B are real.

$\frac{A}{B}e^{2ik} = -1$, therefore, $A = B$ or $A = -B$ to match magnitudes. TODO is that important? We should distinguish, I suppose.

Suppose $A = -B$

$2k = 2\pi n$, $k = \pi n, n \in \mathbb{Z}$

$$\psi_1(x) = A(e^{-i\pi nx} - e^{i\pi nx})$$
$$\psi_1'(x) = -i\pi nA(e^{-i\pi nx} + e^{i\pi nx})$$
$$\psi_1'(0) = -2i\pi nA$$

$$\psi_2(x) = Ce^{-ikx} + De^{ikx}$$

$\psi_1(0) = \psi_2(0) = 0$

$0 = C + D$, $D = -C$

$$\psi_2(x) = C(e^{-i\pi nx} - e^{i\pi nx})$$
$$\psi_2'(x) = -i\pi nC(e^{-i\pi nx} + e^{i\pi nx})$$
$$\psi_2'(0) = -2i\pi nC$$

$$\Delta\frac{d\psi}{dx} = -\psi(0) = 0$$

So, $-C + A = 0$, $A = C$

Ok, what does all that mean? For energies $\pi^2 n^2$, there is no delta barrier, there is just stationary point at zero?

1 Infinite well + step

$$V(x) = \begin{cases} \infty, & x < -1 \\ 0, & -1 < x < 0 \\ -V_0, & 0 < x < 1 \\ \infty, & x > 1 \end{cases}$$

Suppose $-V_0 < E < 0$

$$k = \sqrt{-E}$$

$$\psi_1(x) = Ae^{-kx} + Be^{kx}$$

$\psi_1(-1) = 0 = Ae^{k} + Be^{-k}$, therefore, $B = -Ae^{2k}$

$$\psi_1(x) = A(e^{-kx} - e^{2k}e^{kx})$$

$$\psi_1'(x) = -kA(e^{-kx} + e^{2k}e^{kx})$$

$$k' = \sqrt{E + V_0} = \sqrt{-k^2 + V_0}$$

$$\psi_2(x) = Ce^{-ik'x} + De^{ik'x}$$

$\psi_2(1) = 0 = Ce^{-ik'} + De^{ik'}$, therefore, $D = -Ce^{-2ik'}$

$$\psi_2(x) = C(e^{-ik'x} - e^{-2ik'}e^{ik'x})$$

$$\psi_1(0) = \psi_2(0)$$

$$A(1 - e^{2k}) = C(1 - e^{-2ik'})$$

$$\psi_1'(0) + \psi_2'(0)$$

$$-kA(1 + e^{2k}) = -ik'C(1 + e^{-2ik'})$$

Divide the second equation by the first:

$$-k\frac{1 + e^{2k}}{1 - e^{2k}} = -ik'\frac{1 + e^{-2ik'}}{1 - e^{-2ik'}}$$

$\frac{1+e^{-2ik'}}{1-e^{-2ik'}}$ is pure complex and equal to $\coth(ik') = -i\cot(k')$

$$k\coth(k) = k'\cot(k')$$

That actually has some solutions!

2 some weird basis

Particle in infinite well with $L = 1$.

$$(-\frac{d^2}{dx} - E)\psi(x) = \psi(x)$$

Let

$$f_1(x) = x(1 - x)$$

$$f_2(x) = x(\frac{1}{2} - x)(1 - x)$$

(without normalization). Orthonormal because f_1 is symmetric around 0.5 and f_2 is antisymmetric. Same applies to the derivatives.

$\langle f_1 \mid H - E \mid f_1 \rangle = 10 - E$

$\langle f_2 \mid H - E \mid f_2 \rangle = 42 - E$

Therefore, $E = 10$, $a_1 = 1$ and $E = 42$, $a_2 = 1$. These are clearly not solutions. Reason: incorrect assumption that $\psi(x) = a_1 f_1(x) + a_2 f_2(x)$.

TODO something to do with bound states?

TODO investigate scattering states.

3 Free particle + self-adjoint boundary conditions

Domain: $[0, a]$

Boundary conditions:

$$\psi(0) = 0$$

$$a\frac{\psi'(a)}{\psi(a)} = B$$

1. Negative energy case:

   $k = \sqrt{-E}$

   $$\psi(x) = A(e^{-kx} - e^{kx})$$

   $$a\frac{-ke^{-ka} - ke^{ka}}{e^{-ka} - e^{ka}} = B$$

   $$-ka\coth(-ka) = ka\coth(ka) = B$$

   For $B > 1$, there exists a single solution with $k > 0$.

2. Zero energy:

   $$\psi(x) = Ax$$

   $$a\frac{1}{a} = 1 = B$$

3. Positive energy:

   $k = \sqrt{E}$

   $$\psi(x) = A(e^{-ikx} - e^{ikx}) = -2IA\sin(kx)$$

$$a\frac{-ike^{-ika} - ike^{ika}}{e^{-ika} - e^{ika}} = B$$

$$-ika\coth(-ika) = -i^2 ka\cot(ka) = ka\cot(ka) = B$$

A lot of solutions for every value of B.

For $B < 1$: extra solution.

Suppose $B = -1.0$, $a = 1.0$.
No solutions for negative and zero energy.
$k\cot(k)$ tends to straight lines in the neighbourhood of zero.
$k_n \sim \frac{\pi}{2} + \pi n$
$u_n(1) \to 0$

$$R(E) = \frac{1}{a}\sum_{\lambda=1}^{\infty}\frac{u_\lambda^2(a)}{k_\lambda^2 - k^2}$$

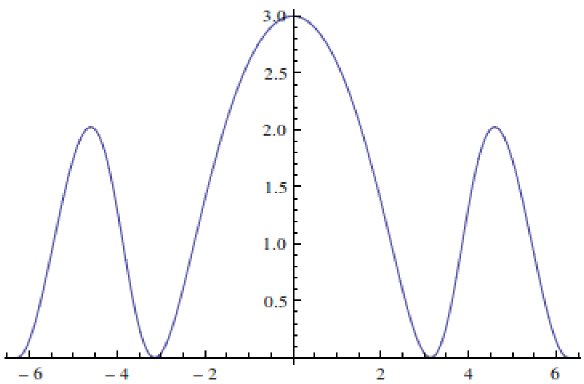
3.1 Bounds on the sum

Suppose $a = 1$.
Let k_n be the nth positive root of $x\cot(x) = B$.
Let b_n be its asymptotic approximation: $b_n = \frac{pi}{2} + \pi n$
If $B < 0$, $b_n < k_n$
If $B > 0$, $b_n > k_n$
Value of nth eigenfunction square on the boundary is

$$\psi_n^2(a) = \frac{1}{4(\frac{a}{2} - \frac{\sin(2k_n a)}{4k_n})}(2\sin(k_n a))^2 = \frac{\sin^2 k_n}{\frac{1}{2} - \frac{\sin(2k_n)}{4k_n}}$$

As function of k:

```
g[k_] := Sin[k]^2 / (0.5 - (Sin[2 * k]) / (4 * k))
Plot[g[k], {k, -2 * Pi, 2 * Pi}]
```



That is, for $B < 0$, $\psi^2(b_n) > \psi^2(k_n)$ if k_n is slightly greater than $\frac{\pi}{2}$.
So, starting from $i = n$ for which $k_i^2 > k^2$:

$$R_i = \sum_{i=n}^{\infty}\frac{\psi_i^2(1.0)}{k_i^2 - k^2} < \sum\frac{\phi_i^2(1.0)}{b_i^2 - k^2} = \sum\frac{2.0}{b_i^2 - k^2}$$

$$\sum \frac{1}{(\pi/2+\pi i)^2-k^2} = \frac{1}{\pi^2}\sum \frac{1}{i^2+i+1/4-\frac{k^2}{\pi^2}}$$

$$D = \sqrt{1-(1-4\frac{k^2}{\pi^2})} = 2\frac{k}{\pi}$$

$$i_{1,2} = -\frac{1}{2} \pm \frac{k}{\pi}$$

$$\frac{1}{i^2+i+1/4-\frac{k^2}{\pi^2}} = \frac{\pi}{2k}\left(\frac{1}{i+0.5-\frac{k}{\pi}} - \frac{1}{i+0.5+\frac{k}{\pi}}\right)$$

$$R_n \leq -\frac{1}{\pi k}(-PG(0.5+n+\frac{k}{\pi}) + PG(0.5+n-\frac{k}{\pi}))$$

Where PG is the polygamma function.
Seems to be working..

3.2 Bounds on the sum: more general case

$$ak\cot(ak) = B$$

$$k_n \sim \frac{1}{a}(\frac{\pi}{2}+\pi n)$$

$$\psi_n^2(a) = \frac{4\sin^2(k_n a)}{2a - \frac{\sin(2k_n a)}{k_n}}$$

Asymptotically, $\psi_n^2(a) \leq 2/a$.

$$\left(\frac{1}{a}(\frac{\pi}{2}+\pi i)\right)^2 - k^2 \leq k_i^2 - k^2$$

$$\frac{1}{a^2}\left((\frac{\pi}{2}+\pi i)^2 - a^2k^2\right)$$

$$R \leq \frac{1}{a}\sum \frac{2/a}{\frac{1}{a^2}\left((\frac{\pi}{2}+\pi i)^2-a^2k^2\right)} = \sum \frac{2}{(\frac{\pi}{2}+\pi i)^2-a^2k^2} = \frac{1}{\pi^2}\frac{\pi}{ak}\sum \frac{1}{i+0.5-\frac{ak}{\pi}} - \frac{1}{i+0.5+\frac{ak}{\pi}}$$

$$R_{n_0} \leq \frac{1}{\pi^2}\frac{\pi}{ak}(PG(n_0+0.5+\frac{ak}{\pi}) - PG(n_0+0.5-\frac{ak}{\pi}))$$

4 Trash

$$\psi(x) = e^{-ikx} + Ue^{ikx}$$

$$\frac{\psi'(a)}{\psi(a)} = X$$

$$U = e^{-2ika}\frac{ik+X}{ik-X}$$

5 Delta potential

Suppose we have delta potential at $x = d$: $-a\delta(x-d)$

$$\psi_1(x) = e^{-ikx} - e^{ikx}$$

, we omit the A coefficient and calculate the normalization later.

$$\psi_2(x) = Be^{-ikx} + Ce^{ikx}$$

$$\begin{cases} \psi_1(d) = \psi_2(d) \\ \psi_2'(d) - \psi_1'(d) = -a\psi_1(d) \end{cases}$$

After solving, we get:

$$B = \frac{iae^{2idk} - ia + 2k}{2k}$$

$$C = -\frac{e^{-2idk}\left(iae^{2idk} - ia + 2ke^{2idk}\right)}{2k}$$

Suppose $d = 1.0$ and $a = 1.0$

$$\psi_2(x) = \frac{\left(ie^{2ik} - i + 2k\right)e^{-ikx}}{2k} - \frac{\left(ie^{2ik} + 2ke^{2ik} - i\right)e^{-2ik+ikx}}{2k}$$

6 Random definitions

Resonance: pole of resolvent/Green's function continued meromorphically to the lower half-plance.

Real part: energy of the resonance, imaginary part: rate of decay.

6.1 Resolvent of the free particle

$$G(x,s;E) = \int_0^\infty \frac{\psi_\lambda(x)\psi_\lambda^*(s)}{E-\lambda}d\lambda$$

$\psi_\lambda(x) =$

7 Question

Green's function is kernel of the resolvent:
$R_\lambda(f) = \int G(x,s)f(s)ds$
What is incoming and outgoing Green's function/resolvent?

8 Cylinder delta scattering

$$\Delta\Psi(r,\theta,z) = E\Psi(r,\theta,z)$$

Symmetric w.r.t. to θ, separate variables:

$$\Psi_m(E;r,\theta,z) = \frac{e^{im\theta}}{\sqrt{2\pi}}\psi_m(E;r,z)$$

Each m defines a 2D scattering problem.

$$\left(-\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}\right) + V(r,z)\right)\psi_m(E;r,z) = E\psi_m(E;r,z)$$

Again, separate variables:

$$\psi_m(E;r,z) = \phi(r)\varphi(z)$$

$\phi(r)$: radial equation

$$-\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + V_\perp(r)\right)\phi(r) = E_\perp\phi(r)$$

Solutions:

$$\phi_n^m(r) = \frac{\sqrt{2}}{RJ_{|m|+1}(x_{mn})}J_m(x_{mn}\frac{r}{R}), n = 1,2\dots$$

$$E_{\perp n}^m = \left(\frac{x_{mn}}{R}\right)^2, n = 1,2\dots$$

$\varphi(z)$: one-dimensional spatial equation

$$\left(-\frac{d^2}{dz^2} + V_s\right)\varphi(z) = (E - E_\perp)\varphi(z)$$

For fixed n,m two linearly independent solutions.

8.1 Next attempt

Fix channels m and n:
Left region:

$$\psi_n^1(r,z) = \exp(Ik_n z)\phi_n(r) + \sum_{t=1}^{\infty} R_{nt}\exp(-ik_t z)\phi_t(r)$$

Right region:

$$\psi_n^2(r,z) = \sum_{t=1}^{\infty} T_{nt}\exp(ik_t z)\phi_t(r)$$

Boundary conditions at $z = 0$: for any r:

$$\psi_n^1(r,0) = \psi_n^2(r,0)$$

$$\partial_z\psi^2(r,0) - \partial_z\psi_n^1(r,0) = u\psi_n^1(r,0)$$

Substituting expressions:

$$\phi_n(r) + \sum_{t=1}^{\infty} R_{nt}\phi_t(r) = \sum_{t=1}^{\infty} T_{nt}\phi_t(r)$$

$$\sum_{t=1}^{\infty} T_{nt}ik_t\phi_t(r) - (ik_n\phi_n(r) - \sum_{t=1}^{\infty} R_{nt}ik_t\phi_t(r)) = u\sum_{t=1}^{\infty} T_{nt}\phi_t(r)$$

For each q, multiply by $\phi_q(r)$ and integrate from 0 to R.
For $q = n$:

$$1 + R_{nn} = T_{nn}$$

$$k_nT_{nn} - (k_n - k_nR_{nn}) = \frac{u}{i}T_{nn}$$

$$\begin{cases} R = \frac{2k_n}{2k_n+iu} \\ T = \frac{-iu}{2k_n+iu} \end{cases}$$

For $q \neq n$:

$$0 + R_{nq} = T_{nq}$$

$$k_qT_{nq} - (0 - k_qR_{nq}) = \frac{u}{i}T_{nq}$$

$$\begin{cases} R = 0 \\ T = 0 \end{cases}$$

9 Cylindric delta well, surrounded by host material

Fix channels m and n:
Left region:

$$\psi_n^1(r,z) = \exp(ik_n z)\phi_n(r) + \sum_{t=1}^{\infty} R_t \exp(-ik_t z)\phi_t(r)$$

Right region:

$$\psi_n^2(r,z) = \sum_{t=1}^{\infty} T_t \exp(ik_t z)\phi_t(r)$$

Boundary conditions at $z = 0$:
$\forall 0 \le r \le RR : \psi_n^1(r,0) = \psi_n^2(r,0)$
$\forall 0 \le r \le R : \partial_z \psi^2(r,0) = \partial_z \psi_n^1(r,0) + u\psi_n^1(r,0)$
$\forall R \le r \le RR : \partial_z \psi^2(r,0) = \partial_z \psi_n^1(r,0)$
Substituting expressions:
$\forall 0 \le r \le RR : \phi_n(r) + \sum\limits_{t=1}^{\infty} R_t\phi_t(r) = \sum\limits_{t=1}^{\infty} T_t\phi_t(r)$
$\forall 0 \le r \le R : \sum\limits_{t=1}^{\infty} T_t k_t \phi_t(r) = k_n\phi_n(r) - \sum\limits_{t=1}^{\infty} R_t k_t\phi_t(r) + \frac{u}{i}\sum\limits_{t=1}^{\infty} T_t\phi_t(r)$
$\forall R \le r \le RR : \sum\limits_{t=1}^{\infty} T_t k_t \phi_t(r) = k_n\phi_n(r) - \sum\limits_{t=1}^{\infty} R_t k_t\phi_t(r)$
For each q, multiply first equation by $r\phi_q(r)$ and integrate from 0 to RR:

- for $q = n$: $1 + R_n = T_n$
- for $q \neq n$:$R_q = T_q$

For each q, multiply second and third equation by $r\phi_q(r)$, integrate second from 0 to R, third from R to RR, and add them up:
$\int\limits_0^{RR} \sum\limits_{t=1}^{\infty} T_t k_t r\phi_t(r)\phi_q(r) = \int\limits_0^{RR} k_n r\phi_n(r)\phi_q(r) - \int\limits_0^{RR} \sum\limits_{t=1}^{\infty} R_t k_t r\phi_t(r)\phi_q(r) + \int\limits_0^{R} \frac{u}{i}\sum\limits_{t=1}^{\infty} T_t r\phi_t(r)\phi_q(r)$

- for $q = n$: $T_n k_n = k_n - R_n k_n + \frac{u}{i}\sum\limits_{t=1}^{\infty} T_t \int\limits_0^{R} r\phi_t(r)\phi_n(r)$
- for $q \neq n$: $T_q k_q = -R_q k_q + \frac{u}{i}\sum\limits_{t=1}^{\infty} T_t \int\limits_0^{R} r\phi_t(r)\phi_q(r)$

Substitute $T_n = R_n + 1, T_q = R_q$:

- for $q = n$: $(R_n + 1)k_n = k_n - R_n k_n + \frac{u}{i}(\sum\limits_{t=1}^{\infty} R_t \int\limits_0^{R} r\phi_t(r)\phi_n(r) + \int\limits_0^{R} r\phi_n(r)\phi_n(r))$
- for $q \neq n$: $R_q k_q = -R_q k_q + \frac{u}{i}(\sum\limits_{t=1}^{\infty} R_t \int\limits_0^{R} r\phi_t(r)\phi_q(r) + \int\limits_0^{R} r\phi_n(r)\phi_q(r))$

Simplify,

- for $q = n$: $2R_n k_n - \frac{u}{i} \sum\limits_{t=1}^{\infty} R_t \int\limits_0^R r\phi_t(r)\phi_n(r) = \frac{u}{i} \int\limits_0^R r\phi_n(r)\phi_n(r)$
- for $q \neq n$: $2R_q k_q - \frac{u}{i} \sum\limits_{t=1}^{\infty} R_t \int\limits_0^R r\phi_t(r)\phi_q(r) = \frac{u}{i} \int\limits_0^R r\phi_n(r)\phi_q(r)$

DO NOT FORGET TO SET $u = \frac{2\mu}{\hbar^2} u$ IN REAL WORLD CALCULATIONS.

10 Zero width slit, 2D geometry

Domain of the adjoint operator:

$$u(r) = \begin{cases} \beta_1 G_1^E(r, r_{12}, k_0) + u_1(r), & r \in \Omega_1 \\ \beta_{12} G^I(r, r_{12}, k_0) + \beta_{23} G^I(r, r_{23}, k_0) + u_2(r), & r \in \Omega_2 \\ \beta_3 G_3^E(r, r_{23}, k_0) + u_3(r), & r \in \Omega_3 \end{cases}$$

k_0 is some regular (TODO: what's that) value of the spectral parameter.
$u_i \in W_2^2(\Omega_i)$ (TODO what's W_2^2?)
Boundary conditions: conservation of flux:

$$\begin{cases} \beta_1 = -\beta_{12} \\ \beta_3 = -\beta_{23} \\ u_1(r_{12}) = u_2(r_{12}) \\ u_3(r_{23}) = u_2(r_{23}) \end{cases}$$

Search solution of scattering problem in form:

$$u(r, k) = \begin{cases} \alpha_1 G_1^E(r, r_{12}, k) + \tilde{u}(x, k), & x \in \Omega_1 \\ \alpha_{12} G^I(r, r_{12}, k) + \alpha_{23} G^I(r, r_{23}, k), & x \in \Omega_1 \\ \alpha_3 G_3^E(r, r_{23}, k), & x \in \Omega_1 \end{cases}$$

10.1 Particle in a 2D box: von Neumann boundary conditions

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) = E\psi(x, y)$$

$$\begin{cases} \psi(x, y) = X(x)Y(y) \\ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} X(x) = E_x X(x) \\ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} Y(x) = E_y Y(y) \\ E = E_x + E_y \end{cases}$$

$$\begin{cases} X(x) = A_x \sin(k_x x) + B_x \cos(k_x x) \\ Y(y) = A_y \sin(k_y y) + B_y \cos(k_y y) \\ k_x = \sqrt{\frac{2mE_x}{\hbar^2}} \\ k_y = \sqrt{\frac{2mE_y}{\hbar^2}} \end{cases}$$

Boundary conditions:

$$\begin{cases} X'(0) = 0, A_x = 0 \\ X'(L_x) = 0, k_x L_x = \pi n_x \\ Y'(0) = 0, A_y = 0 \\ Y'(L_y) = 0, k_y L_y = \pi n_y \end{cases}$$

$$\begin{cases} \psi(x, y) = \frac{2}{\sqrt{L_x L_y}} \cos(k_x x) \cos(k_y y) \\ k_x = \frac{\pi n_x}{L_x} \\ k_y = \frac{\pi n_y}{L_y} \\ E = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) \end{cases}$$

Let $n = n_x$, $m = n_y$.

10.1.1 Green's function

$$G(x, y, x_s, y_s; E) = \sum_{n,m=1}^{\infty} \frac{\psi_{nm}(x, y)\psi^*_{nm}(x_s, y_s)}{E_{nm} - E}$$

TODO investigate convergence on the boundary

$$G(x, y, x_0, y_0; E) = G(x, y, \frac{L_x}{2}, 0; E) = \sum_{n,m=1}^{\infty} \frac{4}{L_x L_y} \frac{\cos(k^x_n x) \cos(k^y_m y) \cos(\frac{\pi}{2} m)}{E_{nm} - E}$$

$$G(x_0, y_0, x_0, y_0; E) = \frac{4}{L_x L_y} \sum_{n,m=1}^{\infty} \frac{\cos(\frac{\pi}{2} n) \cos(\frac{\pi}{2} n)}{E_{nm} - E} = \frac{4}{L_x L_y} \sum_{n,m=1}^{\infty} \frac{\cos^2(\frac{\pi}{2} n)}{E_{nm} - E}$$

$$G(x_0, y_0, x_0, y_0; E) = \frac{4}{L_x L_y} \sum_{n=1}^{\infty} \cos^2(\frac{\pi}{2} n) \sum_{m=1}^{\infty} \frac{1}{E_{nm} - E} = \frac{4}{L_x L_y} \sum_{n'=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{E_{(2n')m} - E}$$

10.2 1D free particle

We have two fold-eigenvalue degeneracy: for each E, there is $\psi^E_+(x) = N_E e^{ikx}$ and $\psi^E_-(x) = N_E e^{-ikx}$, where $k = \frac{\sqrt{2mE}}{\hbar}$. ψ_+ and ψ_- are orthonormal: $\int \psi_+(x)\psi^*_-(x)dx = \int |N_E|^2 e^{2i\sqrt{E}x} dx = \int |N_E|^2 \frac{1}{2} e^{i\sqrt{E}x} = \frac{1}{2}|N_E|^2 2\pi\delta(k)$, which is zero for non-zero E.

Eigenstates normalization: $\langle\psi_E|\,|\psi_{E'}\rangle = \int\limits_{x=-\infty}^{\infty} \psi_E(x)\psi^*_{E'}(x) = N_E N^*_{E'} \int\limits_{x=-\infty}^{\infty} e^{i(k-k')x} dx = N_E N^*_{E'} 2\pi\delta(k - k')$

Next, use $\delta(g(x)) = \frac{\delta(x-x_0)}{g'(x_0)}$ (the function has only one root).

$\langle\psi_E|\,|\psi_{E'}\rangle = N_E N^*_{E'} 2\pi\delta(\frac{\sqrt{2mE}}{\hbar} - \frac{\sqrt{2mE'}}{\hbar}) = N_E N^*_{E'} 2\pi\sqrt{\frac{2E}{m}}\hbar\delta(E - E')$

It has to be equal to $\delta(E-E')$. For $E \neq E'$, the normalization could be arbitrary, for $E = E'$ the coefficient should be equal to 1:

$|N_E|^2 2\pi\sqrt{\frac{2E}{m}}\hbar = 1$, therefore, $|N_E| = \left(\frac{m}{2E}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}}$

10.2.1 Green's function

$$G(x,s;E) = \frac{1}{2\pi\hbar}\left(\frac{m}{2}\right)^{1/2}\left(\int_0^\infty \frac{1}{\sqrt{\lambda}}\frac{e^{i\frac{\sqrt{2m\lambda}}{\hbar}(x-s)}}{\lambda - E}d\lambda + \int_0^\infty \frac{1}{\sqrt{\lambda}}\frac{e^{-i\frac{\sqrt{2m\lambda}}{\hbar}(x-s)}}{\lambda - E}d\lambda\right)$$

Change the variable of integration:

$$\begin{cases} k = \frac{\sqrt{2m\lambda}}{\hbar} \\ \lambda = \frac{\hbar^2 k^2}{2m} \\ d\lambda = 2\frac{\hbar^2}{2m}kdk \end{cases}$$

$$\begin{aligned} G(x,s;E) &= C(\int + \int) \\ &= C\int \frac{1}{\frac{\hbar k}{\sqrt{2m}}}\frac{e^{ikx}}{\frac{\hbar^2k^2}{2m} - E}2\frac{\hbar^2}{2m}kdk + C\int \\ &= C2\frac{\sqrt{2m}}{\hbar}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^2 - \frac{2mE}{\hbar^2}}dk \\ &= \frac{1}{2\pi\hbar}\sqrt{\frac{m}{2}}2\frac{\sqrt{2m}}{\hbar}\int \\ &= \frac{2m}{\hbar^2}\frac{i}{2k_0}e^{ik_0|x-s|} \end{aligned}$$

, where $k_0 = \frac{\sqrt{2mE}}{\hbar}$. That was for positive energy. For negative energy:

$k_0 = \frac{\sqrt{2m|E|}}{\hbar}$.

$$\int_{-\infty}^{\infty}\frac{e^{ik|x-s|}}{k^2 - E}dk = 2\pi i \operatorname*{Res}_{k=ik_0} f(k) = \frac{e^{iik_0|x-s|}}{2ik_0} = 2\pi\frac{e^{-k_0|x-s|}}{2k_0}$$

$$G(x,s;E) = \frac{2m}{\hbar^2}\frac{1}{2k_0}e^{-k_0|x-s|}$$

That is, we have to choose branch of the square root with positive imaginary part.

10.3 Pipe

Height of the pipe H.

$$\begin{cases} X(x) = N_E e^{ik_x x} \\ Y(y) = \frac{2}{\sqrt{H}} \cos(k_y y) \\ k_x = \frac{\sqrt{2\mu E}}{\hbar} \\ k_y = \frac{\pi m}{H} \end{cases}$$

TODO don't forget the energy normalization and wavefunction normalization

10.3.1 Green's function

Suppose $M(E)$ is the count of open transversal modes for the given energy E.

$$\begin{aligned} G(x, y, x_s, y_s; E) &= \int_0^\infty \frac{\sum_{m=0}^{M(\lambda)} Y_m(y) Y_m^*(y_s) X'(x) X'^*(x_s)}{\lambda - E} d\lambda \\ &= \sum_{m=0}^{\infty} \int_{E_m^Y}^{\infty} \frac{Y_m(y) Y_m^*(y_s) X'(x) X'^*(x_s)}{\lambda - E} d\lambda \\ &= \sum_{m=0}^{\infty} \int_0^{\infty} \frac{Y_m(y) Y_m^*(y_s) X(x) X^*(x_s)}{\lambda + E_m^y - E} d\lambda \\ &= \sum_{m=0}^{\infty} Y_m(y) Y_m^*(y_s) \int_0^{\infty} \frac{X(x) X^*(x_s)}{\lambda - (E - E_m^y)} d\lambda \\ &= \sum_{m=0}^{\infty} Y_m(y) Y_m^*(y_s) G^x(x, x_s; E - E_m^y) \end{aligned}$$

TODO make clear that negative wavevectors were taken into account.