Resonant states for quantum ring with two infinite leads

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Abstract. A scattering model of one dimensional quantum graph Γ consisting of incoming and outgoing channels with a ring scatterer is studied. Completeness of the resonant states of Γ in L_2 on the ring is proved.

1. Introduction

Quantum graph is a widely used model of nanosystem [1-4]. If the graph Γ consists of finite number of edges and all edges have finite lengths then the Hamiltonian has purely discrete spectrum. The system of eigenfunctions is complete in $L_2(\Gamma)$. If the graph contains semi-infinite edges, one has non-empty continuous spectrum and resonances generated by the eigenvalues of the initial Hamiltonian of finite graph. For many applications, it is important to know a space in which the resonant states form a complete system. In the present paper we determine this space for a graph with two infinite leads and one loop using Sz.-Nagy model [5].

2. Scattering model

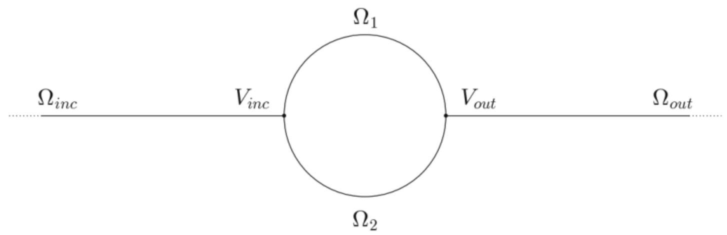

Figure 1. Quantum graph Γ consisting of four edges $\Omega_{inc}, \Omega_{out}, \Omega_1, \Omega_2$. $\Omega_1$and $\Omega_2$represent the 1D ring connected to lead Ω_{inc} via the vertex V_{inc} and to lead Ω_{out} via V_{out}.

The wavefunctions on the resonator ring arcs Ω_1 and Ω_2 take the most general form:

$$\psi_1(x) = Q_1 e^{ikx} + W_1 e^{-ikx}, \qquad \psi_2(x) = Q_2 e^{ikx} + W_2 e^{-ikx}$$

We are interested in the scattering of a wave coming from the left lead ($x < -d$) to the right one ($x > d$):

$$\psi_{inc}(x) = e^{ikx} + Re^{-ikx}, \qquad \psi_{out}(x) = T e^{ikx},$$

where R is the reflection coefficient, T is the transmission coefficient. Then, we impose boundary conditions at vertices V_{inc}, V_{out} (so-called δ-coupling [1]):

$$\psi_{inc}(d) = \psi_1(-d) = \psi_2(-d), \qquad \psi_{out}(d) = \psi_1(d) = \psi_2(d)$$
$$-\psi'_{inc}(-d) + \psi'_1(-d) + \psi'_2(-d) = a\psi_{inc}(0), \qquad \psi'_{out}(d) - \psi'_1(d) - \psi'_2(d) = a\psi_{out}(d),$$

where a is a real valued coupling constant. With the use of boundary conditions, we obtain system of linear equations which leads to expressions for the reflection and transmission coefficients:

$$\begin{cases} R = -\dfrac{4ak\cos^2(kd) + (a^2 - 3k^2)\cos(kd)\sin(kd) - 2ak}{(4i\,k^2 - 4ak)\sin^2(kd) + (a^2 - 2\,i\,a\,k - 5\,k^2)\cos(kd)\sin(kd) + 2ak - 2ik^2} \\ T = -\dfrac{2ik^2}{(4i\,k^2 - 4ak)\sin^2(kd) + (a^2 - 2\,i\,a\,k - 5\,k^2)\cos(kd)\sin(kd) + 2ak - 2ik^2} \end{cases}$$

Due to the symmetry of the scattering problem w.r.t. the origin, scattering matrix has the form $S = \begin{pmatrix} R & T \\ T & R \end{pmatrix}$, and,

$$\det S = \frac{((a^2 - 5k^2)\cos(kd)\sin(kd) - 4ak\sin^2(kd) + 2ak) + i(2ak\cos(kd)\sin(kd) - 4k^2\sin^2(kd) + 2k^2)}{((a^2 - 5k^2)\cos(kd)\sin(kd) - 4ak\sin^2(kd) + 2ak) - i(2ak\cos(kd)\sin(kd) - 4k^2\sin^2(kd) + 2k^2)}$$

We establish the completeness of the resonant states in the space of square integrable functions on the ring. To show this, we have to prove that S is a Blaschke-Potapov product [6], that is,

$$\lim_{r\to 1-0} \int\limits_{L_r} \log|\det S(z)|\frac{dz}{(z-1)^2} = 0,$$

where L_r is the image of the curve $|\zeta| = r < 1$ under the map $z = i\frac{1+\zeta}{1-\zeta}$. Substituting $\det S$, which we calculated above, and estimating the integral, one obtains the desired result.

3. Acknowledgments

This work was partially financially supported by the Government of the Russian Federation (grant 074-U01), by Ministry of Science and Education of the Russian Federation (GOSZADANIE 2014/190, Projects No 14.Z50.31.0031 and No. 1.754.2014/K), by grant MK-5001.2015.1 of the President of the Russian Federation and DFG Grant NE~1439/3-1

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