Advantage of L-stability compared to A-stability

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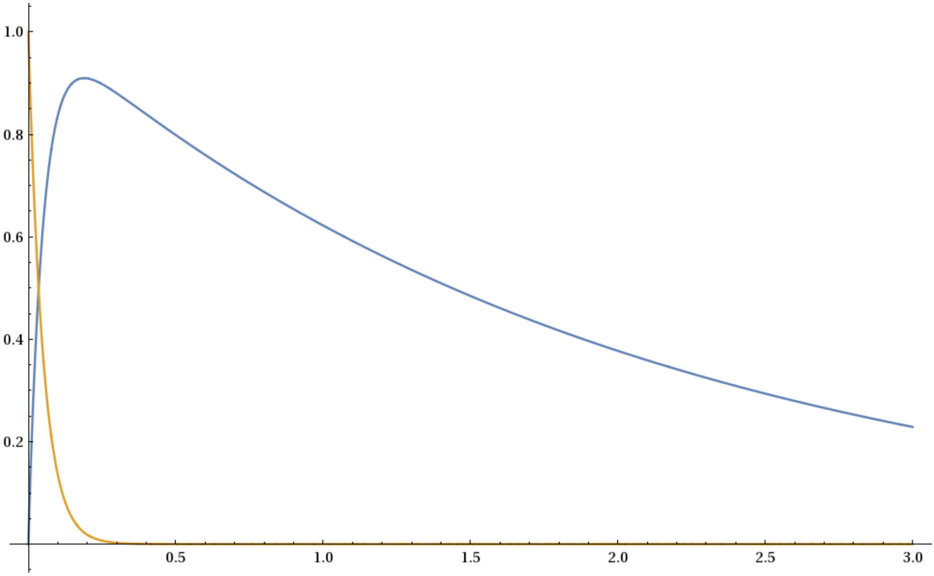
Consider the following model stiff problem

$$\dot{x} = -0.5x + 20y \dot{y} = -20y x(0) = 0, \quad y(0) = 1$$

The plot of $x(t), y(t)$



has two regions

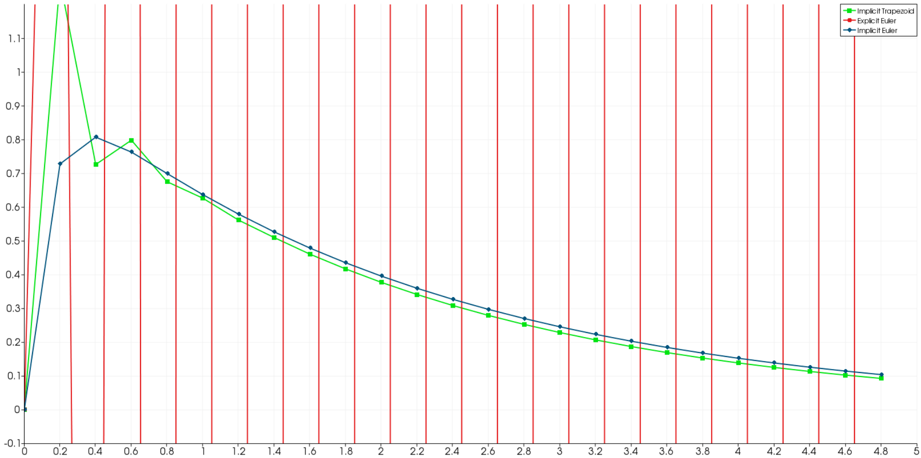
$0 \leq t \leq 0.5$. Here $y(t)$ vanishes quickly transforming into $x(t)$. I would refer to this region (maybe incorrectly) as a boundary layer.

$t > 0.5$. Here $y(t)$ is almost zero and does not affect the slow decay of $x(t)$.

Assume that we have a well-tested program with an explicit integration method that can integrate this system with an *automated step control* feature. We expect the step to be small in boundary layer (since $x(t), y(t)$ change quickly there) and to be big in the normal region, since nothing quick happen there.

But actually the program will choose the step to be small everywhere. Even if one manually increases the step in the normal region it results in an instability (look at the red plot, it's explicit Euler method). Using A-stable (implicit midpoint) or L-stable (implicit Euler) method show almost no difference.

But what if we don't want to reduce the time step in the boundary layer? Then A-stable but not L-stable methods show oscillations. Even while the oscillations are bounded and do not grow like in explicit methods, they are undesirable and can yield physically incorrect solutions (for example, violate $x(t) > 0$). But the L-stable methods correctly represent the solution qualitatively when the time step is large. This actually is due to $r(\lambda\,\Delta\,t) \approx 0$ for big timesteps $\Delta\,t \gg \lambda^{-1}$ (just like the behavior of $e^{\lambda\,\Delta\,t} \approx 0$).



The other important application of L-stable methods is for time integration of partially discretized PDEs, for example the heat equation

$$u_t = u_{xx}$$

For the true solution the $\sin kx$ initial solution should decay as fast as $e^{-k^2 t}$, but for A-stable method it would decay almost equally as $q^{t\,/\,\Delta\,t}$ where $q = r(-\infty)$. The speed of decay almost does not depend on k provided it is large.

For the L-stable method the stability function $r(z)$ has zero limit at $z = -\infty$, thus numerical solution decays faster for bigger values of k.