Mathematical Induction

Mathematical induction is a proof technique which is used to prove a statement, a formula, or a theorem is true for all natural numbers.

Weak Induction

Prove the base case, assume inductive hypothesis $n = k$, then prove $k + 1$ holds (inductive step).

If $p(k)$ is true, then $p(k + 1)$ is true.

Strong Induction

Similar to weak induction, but all previous values of k are also assumed.

If $p(i)$ is true $\forall i : i \leq k$, then $p(k + 1)$ is true.

Well Ordering Property

Every non-empty set of non-negative integers has a least element.

If $a \in A$, where $a = \{a : a = \text{ positive values of } A\}$. The least element of A is 1, since it is the least positive element.

It is important to know the least element in mathematical induction since the base case always starts with the least element.

Recursive Definitions and Structural Induction

Recursion is a method of defining a function, sequence, or object in terms of itself.

Recursively Defined Functions

Arithmetic Sequence is an order of numbers that has a pattern regarding the difference of the numbers that are beside each other.

$$\{5, 9, 13, 17, 21, \ldots\}$$

Finding the n-th term of an arithmetic sequence is given by the formula:

$$a_n = a_1 + (n - 1)d$$

Where d is the difference between the terms.

Geometric Sequence is an order of numbers that has a pattern regarding the ration of the numbers that are beside each other.

$$\{2, 4, 8, 16, 32, \ldots\}$$

Finding the n-th term of a geometric sequence is given by the formula:

$$a_n = a_1 r^{n-1}$$

Where r is the common ratio between the terms.

Recursively Defined Sets and Structures

Assume S is a set. To recursively define S, we use two steps to define S. The *basis step* defines the initial collection of elements. The *recursive step* defines a rule in forming new elements from known values in the set.

Consider $S \subseteq \mathbb{Z}$; $3 \in S$ and if $x \in S$ and $y \in S$, then $x + y \in S$.

In this example all values in S are multiples of 3.

$$\begin{aligned}
3 &\in S \\
3 + 3 \in S &\implies 6 \in S \\
3 + 6 \in S &\implies 9 \in S \\
3 + 9 \in S &\implies 12 \in S \\
6 + 9 \in S &\implies 15 \in S
\end{aligned}$$

Recursive Algorithms

Algorithms are instructions or set of rules to be followed to achieve something in the field of programming. An algorithm is called recursive if a problem is solved by reducing the same problem, but its input size is getting smaller everytime the algorithm runs.

$$\begin{aligned}
f(0) &= 1, n = 0, 1, 2, 3, \ldots \\
f(n+1) &= f(n)^2 + f(n) + 1
\end{aligned}$$

Binomial Coefficients and Identities

The number of r-combinations from a set of n elements is often denoted by $\binom{n}{r}$. This relation is also called a binomial coefficient as it occurs as coefficients of a binomial expansion.

The Binomial Theorem

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.

Expand $(x+y)^3$.

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

Theorem 1: Binomial Theorem. Let x and y be variables, and n be a non-negative integer. Then:

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^y \\ &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n\end{aligned}$$

Corollary 1. Let n be a non-negative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Corollary 2. Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Pascal's Identity and Triangle

Theorem 2: Pascal's Identity. Let n and k be positive integers where $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Other Identities Involving Binomial Coefficients

Theorem 3: Vandermonde's Identity. Let m, n, and r be non-negative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Corollary 4 follows from Vandermonde's Identity.

Corollary 4. If n is a non-negative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Theorem 4. Let n and r be non-negative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$