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MF793 – Fall 2021
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RANDOM WALK, STATIONARITY, TIME SERIES MODELS

- What is a ***stationary*** random variable?
- Are financial data stationary ? What is a ***Random Walk***?
- Does the distribution of stock prices exist ? Does the distribution of stock returns exist?

  Start from the conditional distribution of P_t having just observed P_{t-1}. That must exist!
  Under what conditions does an unconditional distribution exist?

- Stationary AR(1) model, AR(p), **Moving Average** MA(q), mixed models
- Estimation

Read: Bruce Hansen 16.1-16.4
Greene 20.2-20.3

1 Some Definitions

Strict stationarity

A time series $\{r_t\}$ is strictly stationary if the joint distribution of $(r_{t+1}, \cdots, r_{t+k})$ is identical to that of $(r_{l+1}, \cdots, r_{l+k})$ for all t, l and k.

- Strict stationarity: Strong requirement that the distribution of any $(r_{t+1}, \cdots, r_{t+k})$ is invariant to any time shift
- Implies invariance **of all moments** under time shifts.

Weak stationarity

A time series $\{r_t\}$ is weakly stationary if both the mean of r_t and the covariance between r_t and r_{t-k} are time-invariant.

$$E[r_t] = \mu \text{ and } Cov(r_t, r_{t-k}) = \gamma_K \qquad \forall \textbf{ t and k}, \gamma\text{: time series notation}$$

We just introduced the ***autocovariance function*** γ_K. Note: $\gamma_0 \equiv Cov(r_t, r_t) = \sigma^2$

The ***autocorrelation function***: $\rho_K = \frac{Cov(r_t, r_{t-k})}{\sigma_t \sigma_{t-k}} = \frac{\gamma_k}{\gamma_0}$

Practical implications:

- For the past: Time plot of $\{r_t\}$ varies around a fixed level within a finite range.

- For the future: First 2 moments of future r_t are the same as in the past:

$$\mu = E[r_t] \qquad \sigma^2 = E[(r_t - \mu)^2] = \boldsymbol{\gamma_0}$$

- Weak stationarity requires that second moments exist.

- We often assume weak stationarity of financial time series **returns,** and growth rates of economic series.

- Covariance is not a function of t, only of the lag k Cov(rt+l,t+l-k)

2 The Random Walk

The **levels** of financial series are **non stationary** (Random Walk).

They are also **bounded** (in general >0)

Stock price
Stock index level
Exchange rate
Nominal interest rates
Macroeconomic series: CPI, GNP, GDP, IP

- A random variable (like a stock price or index) price P_t follows a Random Walk if

$P_{t+1} = \mu + P_t + \varepsilon_{t+1}$, $\quad\quad \varepsilon_t \sim (0, \sigma)$

$$E_t(\varepsilon_{t+1} \mid P_t, \varepsilon_{t-1}, \ldots) = 0$$

2A Existence of the Mean ? (not!)

- $E(P_{t+1} \mid P_t)$, the **conditional mean** always exists.
- Does $E(P_t)$ exist? NO!

$E_t(P_{t+1} \mid P_t) = \mu + P_t$ one-period ahead conditional mean

$$E_t(P_{t+2} \mid P_t) = E_t(\ E_{t+1}(P_{t+2} \mid P_{t+1})\ \mid P_t) = E_t(E_{t+1}(\ \mu + P_{t+1} + \varepsilon_{t+2} \mid P_{t+1}) \mid P_t)$$

$$= E_t(\quad \mu + P_{t+1} \quad \mid P_t) = \mu + E_t(\qquad\qquad)$$

$$= 2\mu + P_t$$

$$\mathbf{E(P_{t+k} \mid P_t) = k\,\mu + P_t} \qquad \textbf{[1]}$$

- Definition: **Unconditional Mean** (moment)

The unconditional mean is the limit of the
k-period ahead conditional mean as k goes to infinity

- $\lim_{k \to \infty} E(P_{t+k} \mid P_t)$? The unconditional mean of P_t does not exist
- What if $\mu=0$? $E(P_{t+k} \mid P_t) = \cancel{k\,\mu} + P_t = P_t$ … is a function of t. Violates weak stationarity

2B Existence of Variance ?

- $Var(P_t)$? Only the conditional variance exists.

$$V(P_{t+1} \mid P_t) = V(\cancel{\mu} + P_t + \varepsilon_{t+1} \mid P_t) = \sigma^2$$ one-period ahead conditional variance

$$\begin{aligned} V(P_{t+2} \mid P_t) &= V(P_{t+1} + \varepsilon_{t+2} \mid P_t) \\ &= V(P_{t+1} \mid P_t) + \sigma^2 \\ &= V(P_t + \varepsilon_{t+1} \mid P_t) + \sigma^2 \\ &= 2\sigma^2 \end{aligned}$$

$$\mathbf{V(P_{t+k} \mid P_t) = k\,\sigma^2} \qquad \mathbf{[2]}$$

- **A Random walk does not have unconditional moments**

- We cannot estimate moments (means, variances) that do not exist !

  Sample average and sample variance of a dataset of financial prices do not correspond to any existing true moment, they are meaningless.

2C Remedy: Simply first-difference the series

- Most financial series are >0 by definition, model error must be consistent with this.

- Differenced model: $\Delta P_t = (P_t - P_{t-1}) = \mu + \varepsilon_t$,

  Need to model ε_t to eliminate possibilities of $P_t < 0$
  => cannot use unbounded shocks (e.g. normal):
  Distribution of ε_t must be a function of P_{t-1} to guarantee $P_t > 0$

- Bounded distributions are complicated: mean and variance interact

- Problem is more easily resolved in continuous time:

  If P_{t-1} is very small, the variance of ε_t is made proportional to P_{t-1} to eliminate chances of $P_t < 0$
  as in: $dP = \mu P\, dt + \sigma P\, dz$

  This does not work in discrete time, still a chance of $P_t < 0$ over a fixed time interval.

- **Remedy** in discrete time: Normalize by taking the difference of logarithms

$$\text{Log } P_t = \mu + \text{Log } P_{t-1} + \varepsilon_t, \quad E(\varepsilon_t) = 0$$

$$\ln P_t / P_{t-1} = r_t = \mu + \varepsilon_t$$

r_t: the continuous (aka log-) return (or growth rate)

$r_t = \mathrm{Log}(1+R_t)$, in contrast with the conventional return $R_t = (P_t - P_{t-1}) / P_t$

Conclusions:
Log-difference is the preferred transformation
Most realistic assumption: **r_t is normal, I.e. R_t is lognormal**

- For small returns (high frequency) $\mathrm{Log}(1+R_t) \approx R_t$ => $R_t \approx r_t$

When do we get small returns ….. when do we get big returns ?

3 Random Walk and Market Efficiency

The Random walk is the simplest representation of market efficiency. (It is not the only one)

- Price form: $\ln P_t = \mu + \ln P_{t-1} + \varepsilon_t$ $\quad E(\varepsilon_t \mid I_{t-1}) = 0$ [1]

  Returns form: $r_t = \mu + \varepsilon_t$ $\quad E(\varepsilon_t \mid I_{t-1}) = 0$

- $E(\varepsilon_t \mid I_{t-1}) = 0$ **I_{t-1}: all information available at time t-1**

- Three forms of efficiency:

  Weak form $I_{t-1} = R_{t-1}, R_{t-2}, R_{t-3}$, etc… **only past values of the series itself**
  Semi-strong form: I_{t-1} : all **publicly available** information
  Strong form: I_{t-1} : all public **and private** information

- Essentially: Returns are very hard to predict given past public information
  Warning: More complex finance models may allow for returns predictability in efficient markets

4 Crucial properties of a stationary series: the AR(1)

A stationary series does not need to be pure noise, it can have a predictable conditional mean.

The Stationary **A**uto**R**egressive model: AR(1)

$Y_{t+1} = \alpha + \phi Y_t + \varepsilon_{t+1}$ $\qquad E(\varepsilon_t \mid Y_{t-k}, k>0) = 0$

4.1 Unconditional Mean of Y_t

Sloppy way: $E(Y_{t+1}) = \alpha + \phi E(Y_t) \quad \Rightarrow \quad E(Y_t) = \alpha \ / (1-\phi)$ ... if it exists!

In fact, it exists only if $|\phi| < 1$, how do we know ? By doing it the ..

Formal way:

$$E_t(Y_{t+2} \mid Y_t) = E_t(\alpha + \phi Y_{t+1} + \varepsilon_{t+2} \mid Y_t)$$

$$= \alpha + \phi E_t(Y_{t+1} + \varepsilon_{t+2} \mid Y_t) = \alpha + \phi E_t(Y_{t+1} \mid Y_t) \qquad [1]$$

$$= \alpha + \phi E_t(\alpha + \phi Y_t + \varepsilon_{t+1} \mid Y_t) = \alpha(1+\phi) + \phi^2 Y_t$$

Note that in [1] we see the recursion: $\mathbf{E_t(Y_{t+k} \mid Y_t) = \alpha + \phi E_t(Y_{t+k-1} \mid Y_t)}$

$$E_t(Y_{t+3} \mid Y_t) = \alpha + \phi\ E_t(Y_{t+2} \mid Y_t) = \alpha + \phi\ [\alpha(1+\phi) + \phi^2 Y_t]$$

$$= \alpha(1 + \phi + \phi^2) + \phi^3 Y_t$$

$$E_t(Y_{t+k} \mid Y_t) = \alpha(1+\phi + \dots \phi^{k-1}) + \phi^k Y_t \qquad [1]$$

$$= \alpha \frac{1-\phi^k}{1-\phi} + \phi^k Y_t$$

- Unconditional Mean if it exists:

Is the limit of [1] as $k \to \infty$
Limit of series: $1+\phi + \dots \phi^{k-1}$?

$1+\phi + \dots \phi^{k-1}$ what if $|\phi|=1$? $|\phi|>1$?

$$\lim_{k \to \infty} E_t(Y_{t+k} \mid Y_t) = \frac{\alpha}{1-\phi} \qquad \text{if } |\phi|<1$$

4.2 Unconditional Variance of Y_t

<u>Sloppy</u> $V(Y_t) = \phi^2 \ V(Y_{t-1}) + \sigma^2_\varepsilon \Rightarrow$ **$V(Y_t) = \sigma^2_\varepsilon / (1-\phi^2)$** ... if it exists!

Again, need $|\phi| < 1$, but we don't see if from here

<u>Serious</u>: Compute k-period ahead conditional variance and take the limit

$V_t(Y_{t+1} \mid Y_t) = V_t(\alpha + \phi Y_t + \varepsilon_{t+1} \mid Y_t) = \sigma^2_\varepsilon$

$V_t(Y_{t+2} \mid Y_t) = V_t(\alpha + \phi Y_{t+1} + \varepsilon_{t+2} \mid Y_t) = \phi^2 V_t(Y_{t+1} \mid Y_t) + \sigma^2_\varepsilon$

$= \sigma^2_\varepsilon (1 + \phi^2)$

$V_t(Y_{t+k} \mid Y_t) = \sigma^2_\varepsilon (1 + \ldots + (\phi^2)^{k-1})$

$$\lim_{k \to \infty} V_t(Y_{t+k} \mid Y_t) = \sigma^2_\varepsilon / (1-\phi^2)$$

4.3 Autocovariance $Cov(Y_t , Y_{t-1})$

$$Cov(Y_t , Y_{t-1}) = Cov(\phi\, Y_{t-1} + \varepsilon_t , Y_{t-1}) = \phi\, Cov (Y_{t-1} , Y_{t-1}) + Cov(\varepsilon_t , Y_{t-1})$$

$$= \phi\ \sigma^2{}_Y$$

$$Cov(Y_t , Y_{t-2}) = Cov(\phi\, Y_{t-1} + \varepsilon_t , Y_{t-2}) = \phi\ Cov(Y_{t-1} + \varepsilon_t , Y_{t-2}) =$$

$$= \phi^2\ \sigma^2{}_Y$$

$$Cov(Y_t , Y_{t-k}) = \phi\, Cov(Y_{t-1} , Y_{t-k}) \quad = \phi^2\, Cov(Y_{t-2} , Y_{t-k})$$

$$= \phi^k\ \sigma^2{}_Y$$

4.4 Autocorrelation of the stationary AR(1)

$$\rho_k = Cov(Y_t, Y_{t-k}) \;/\; \sigma_{Yt}\,\sigma_{Yt-k} = \phi^k\;\sigma^2_{Yt} / \sigma_{Yt}\,\sigma_{Yt-k} = \phi^k$$

- Link with the R^2 of the autoregression

The simple regression $\quad y = \alpha + x\beta + \varepsilon \qquad E(\varepsilon \mid x) = Cov(\varepsilon, x) = 0$

.. is a variance decomposition $\quad \sigma^2_y = \beta^2\,\sigma^2_x + \sigma^2_\varepsilon$

$$\beta = \frac{Cov(xy)}{Var(x)} = \rho_{XY}\,\frac{\sigma_Y}{\sigma_X} \qquad\qquad R^2 = \frac{\beta^2\sigma_X^2}{\sigma_Y^2} = \rho_{XY}^2$$

- **Autoregression** (common in Finance for prediction): $R_t = \alpha + \beta\,R_{t-1} + \varepsilon_t$,

Returns are stationary, but may still be predictable: $0 < \beta < 1$

Conditionally $\qquad E(R_t \mid R_{t-1}) = \alpha + \beta\,R_{t-1} \qquad V(R_t \mid R_{t-1}) = V(\varepsilon)$

Unconditionally: $\qquad E(R_t) = E(R_{t-1}) = E(R) \qquad V(R_t) = V(R_{t-1}) = V(R)$

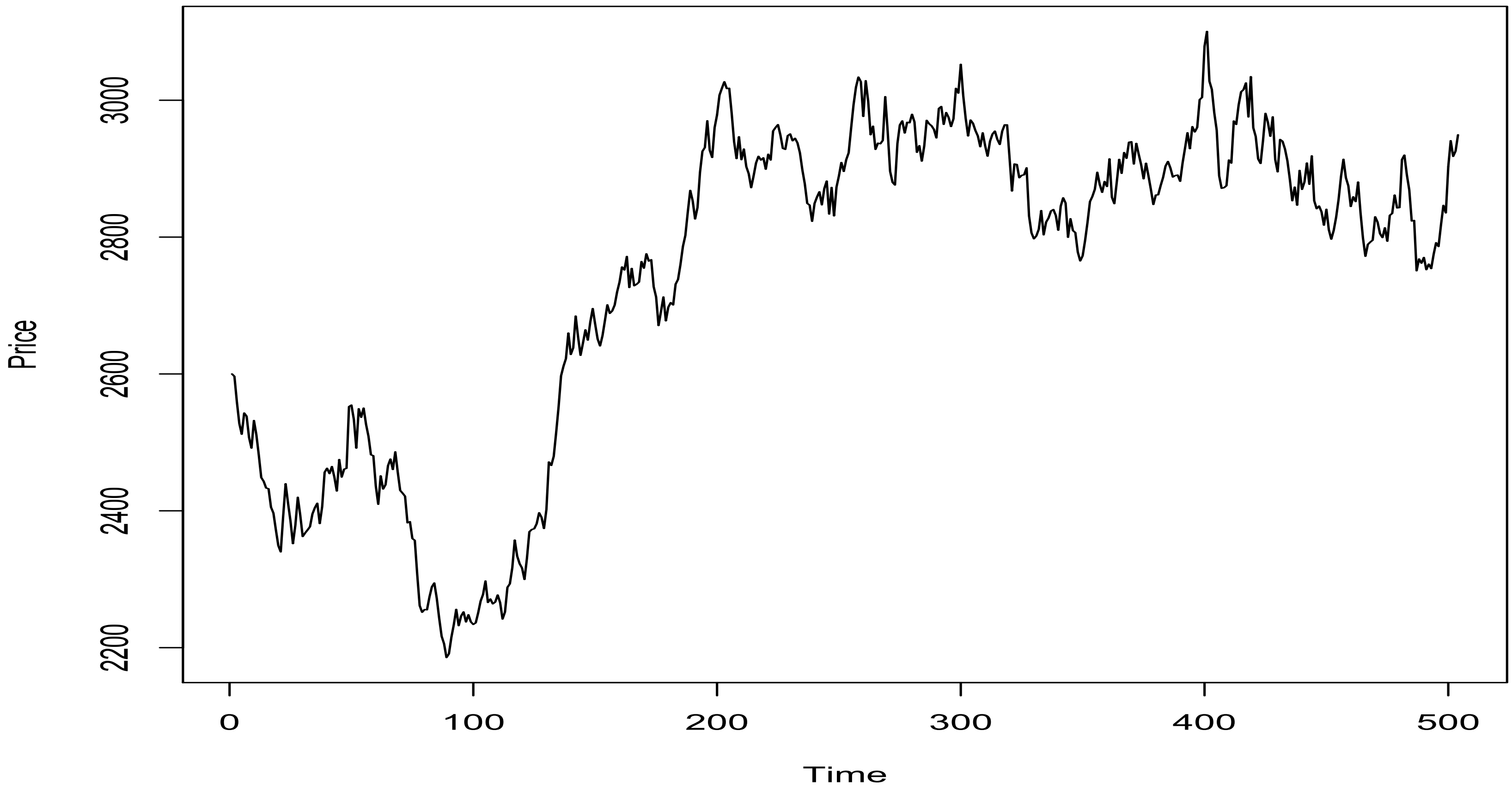
$$\beta = \rho_{R_t,R_{t-1}}\,\frac{\sigma_{Rt}}{\sigma_{Rt-1}} \equiv \rho_1$$

The slope coefficient of an autoregression is the first order autocorrelation

1

5 Detecting random walk and non-stationarity

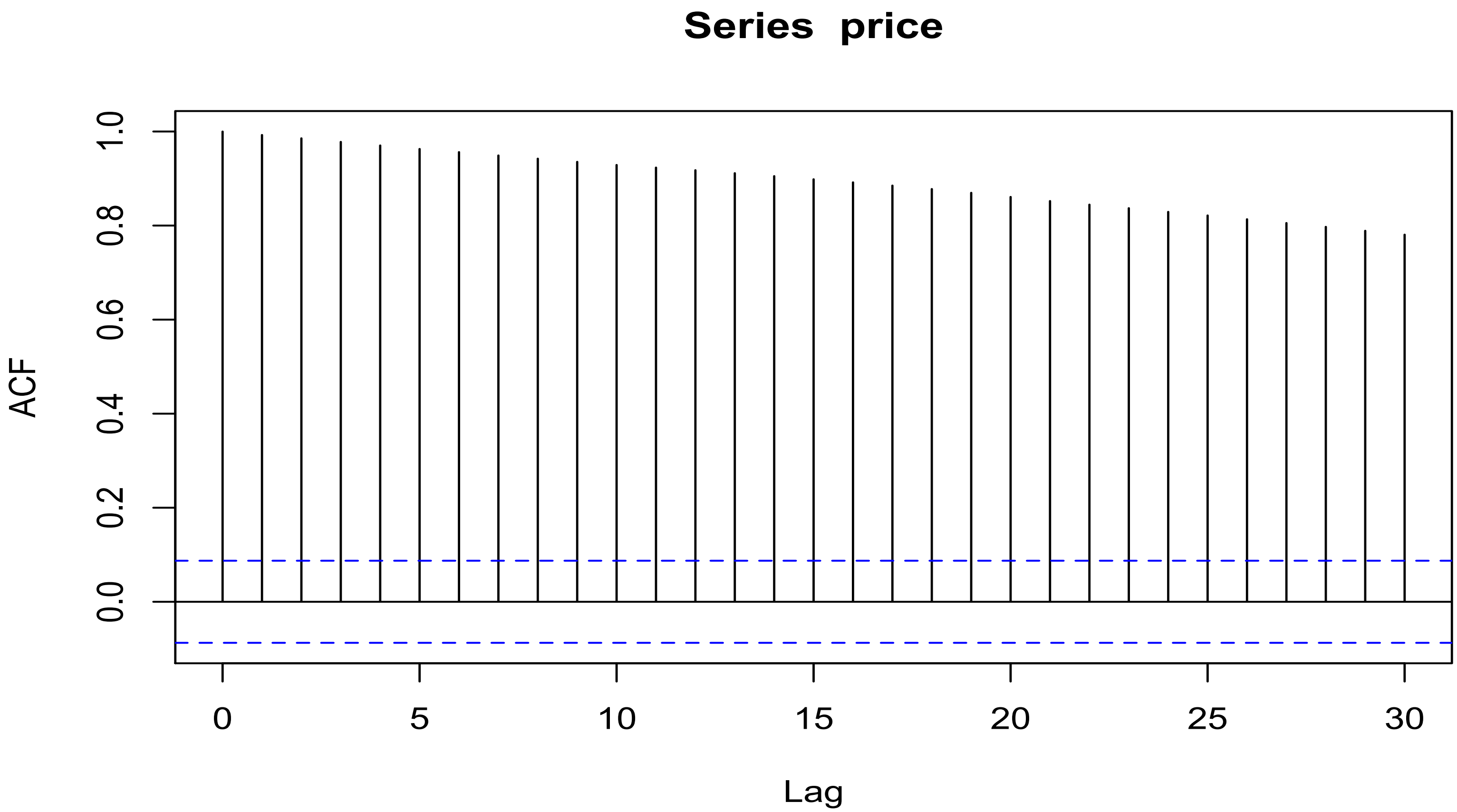
The S&P500 (! ?) over 2 years.



- Can't use estimates of the mean and variance since they don't exist !

- Intuition is in $\rho_k = \phi^k$: The **A**uto**C**orrelation **F**unction (ACF) must die out if $|\phi| < 1$

- Plot the autocorrelation function vs. k (autocorrelogram)



- This S&P was a fake, just a pure log-random walk simulated as:

```
mu    <- 0.10 /252
sig   <- 0.15/sqrt(252)
lpt   <- rep(0,504)          # Do two years, create log price vector
epst  <- rnorm(504,0, 1)
lpt[1]<- log(2600)           # Starting value
for (i in 2:504) { lpt[i] <- lpt[i-1] + mu + sig*epst[i]}
price <- exp(lpt)
ts.plot(price, ylab="Price")
acf(price,lag.max=30)
```

- Contrast with the ACF of a stationary, <u>even strongly</u> autocorrelated, series:

```
yy    <- rep(0,504)
yy[1] <- 0
phi   <- 0.95             $ high ϕ but stationary
epst  <- rnorm(504,0, 1)
for (i in 2:504)   { yy[i] <- phi*yy [i-1] + epst[i] * sig   }
acf(yy,lag.max=30)
```

- Same for more complex but stationary models ARMA:

  Autocorrelations are more complex for the first few lags,

  But they still die out very quickly after.

6 Generalization

6.1 The AR(p) model

AR(2): $R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + \varepsilon_t$ $E(\varepsilon_t \mid R_{t-k}, \forall k>0) = 0$

Noise (shock) unrelated to past information

AR(p): $R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + \ldots + \phi_p R_{t-p} + \varepsilon_t$

- Stationarity conditions less simple

AR(2), we can show that we need ϕ_1 and ϕ_2 to be in the triangle:

$\phi_1 > -1$ $\phi_2 < 1 + \phi_1$ $\phi_2 < 1 - \phi_1$

- Autocorrelation function decays fast after lag p for a stationary AR(p)

The ACF of an AR(P) looks a bit like that of an AR(1), but not as simple as: $\rho_k = \phi^k$

All ARs, even AR(1)s **have non-zero autocorrelation up to any lag**. **[1]**

ACF decays quickly but is never exactly zero.

How can we distinguish the ACF of an AR(1) from an AR(2), from and AR(3) ?

- Distinguishing AR(1) from an AR(p): the **Partial AutoCorrelation Function** **PACF**

AR(1) $\rho_k = \phi^k \neq 0$ Autocorrelation decays fast but is not zero at low lags

AR(2): $R_t = \phi_0 + \phi_1 R_{t-1} + \phi_2 R_{t-2} + \varepsilon_t$ $E(\varepsilon_t \mid R_{t-k}, k>0) = 0$

PACF: PACF$_1$: Regress R_t on R_{t-1} => $p_1 = \widehat{\phi_1}$

PACF$_2$: Regress $R_t - p_1 R_{t-1}$ on R_{t-2} => $p_2 = \widehat{\phi_2}$

PACF$_3$: Regress $R_t - p_1 R_{t-1} - p_2 R_{t-2}$ on R_{t-3} => $p_3 = \hat{0}$

PACF estimates drop to zero past the AR lag. We plot the PACFs

- In R: acf(mydata, type="partial") #

6.2 The moving average model (MA)

MA(1): $R_t = \mu + a_t + \theta_1 a_{t-1}$, $a_t \sim$ i.i.d $N(0,\sigma_a)$

MA(q): $R_t = \mu + a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}$

- ACF of the MA(q):

$Cov(R_t, R_{t-1}) = Cov(a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}, \; a_{t-1} + \theta_1 a_{t-2} + \ldots + \theta_q a_{t-q} + \theta_q a_{t-1-q})$

$Cov(R_t, R_{t-q}) = Cov(a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}, \; a_{t-q} + \theta_1 a_{t-q-1} + \ldots + \theta_q a_{t-q-q})$

$= \theta_q \sigma^2_a$

$\mathbf{Cov(R_t, R_{t-q-1})} = Cov(a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}, \; a_{t-q-1} + \theta_1 a_{t-q-1-1} + \ldots)$ **=**

The ACF of the MA(q) drops to exactly zero at lag q+1

- Moments of the MA(q)

$E(R_t) = \mu$ $Var(R_t) = \sigma^2_a (1 + \theta_1^2 + \ldots + \theta_q^2)$

The MA is always covariance stationary

6.3 ARMA(p,q)

$R_t = \mu + \phi_1 R_{t-1} + \ldots + \phi_p R_{t-p} + a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}$

Parcimony:

Long lag in AR can be approximated by short lag MA, and vice-versa (no proof)

ARMA model with **small p,q** can be an effective modeling strategy

6.4 ARIMA(p,d,q)

I : *integrated*

d: Number of times needed to difference to obtain stationarity, typically d=1

Refers to the fact that we may want to difference the original series (price, index level, etc..) before applying an ARMA to it.

If you "pre-differenced" your series, for example, computed stock or index returns from price, you don't need this.

Example: GDP is non stationary, log-difference it and study GDP growth rate
CPI is non stationary, log-difference it and study inflation

7 Estimation

7.1 Estimating ρ_k

- Estimator of ρ_1: $E(\widehat{\rho_1}) \dot{\sim} \frac{-1}{T}$, $V(\hat{\rho}_1) \dot{\sim} \frac{1}{T}$ $\hat{\rho} \dot{\sim} N(\frac{-1}{T}, \frac{1}{T})$ **under H_0: ρ=0** No proof
  Good for k≪T and T large

  Slightly better approximation: $\hat{\rho}_i \dot{\sim} N(\frac{-(T-i)}{T(T-1)}, \frac{T-i}{T(T-2)})$ under H_0: ρ_i=0

- Testing a joint null H_0: $\rho_1 = \rho_2 = \ldots = \rho_k = 0$

  Box-Pierce: $\sum_{i=1}^{k} \frac{(\hat{\rho}_i - 0)^2}{1/T} \dot{\sim} \chi^2(k)$

  The $\hat{\rho}_i$s are not dependent from one another, ∀ i≠j

  Box-Ljung: Similar but uses (T-i)/T(T-2) for variance

7.2 Estimating parameters of AR and MA.

- Estimating parameters of AR, MA, ARMA models require transformation (state space form), maximum or quasi-maximum likelihood methods

- Pure AR models can be estimated by OLS approximately in large samples.

- MA models require specific likelihood type and time series estimation techniques

- In R: packages tseries, command arima, straightforward, a bit like a regression output.