# Choice under Uncertainty: Aversion to Risk and the Utility Function

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# Goal of this Note

This condensed note introduces decision making under uncertainty

- Some decisions only make sense taking into account risk preferences
- We introduce the utility function.
- .. the Certainty Equivalent and the risk premium
- .. show how this makes sense ... St Petersburg Paradox

# Saint Petersburg Game - Bernoulli goes to the Casino

- Game:
  Toss coin, count number of tosses needed before Heads comes up.
  Payoff:

| Tails | | Probability | Payoff | Profit |
|---|---|---|---|---|
| 0 | H | $1/2$ | $\$ \ 2^0 = 1$ | $1 - f$ |
| 1 | TH | $1/2^2$ | $\$ \ 2^1 = 2$ | $2 - f$ |
| 2 | TTH | $1/2^3$ | $\$ \ 2^2 = 4$ | $4 - f$ |
| . | . | . | . | . |
| N | ... | $1/2^{N+1}$ | $\$ \ 2^N$ | $2^N - f$ |

- What fee are you willing to pay for the right to play the game?

- Observed: Players never want to pay more than a few dollars ... but:

$$E(\widetilde{Payoff}) = \sum_{i=0}^{\infty} p(i) \times W_i = 1/2 \times 1 + 1/4 \times 2 +$$

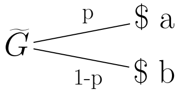
# Expected Value is an incomplete description of the game

- ???
- Expected value of wealth does not account for risk
- ... investors are risk averse.

# Utility of Rational Risk Averse Investors - Von Neumann & Morgenstern

- **If** Investors 1) can assess risky bets (gambles), and 2) are rational:
  1. Can compare gambles: $G_1 \succ G_2$ or $G_2 \succ G_1$
  2. Consistent comparisons: If $G_1 \succ G_2$ and $G_2 \succ G_3$, then $G_1 \succ G_3$
  3. and a few more such axioms of behavior ...

- **Then:**
  Von Neumann and Morgenstern show that gambles can be ranked using expected value rule and a utility function: the concept of expected utility




$$
\begin{aligned}
E[\widetilde{G}] &= p\,a + (1-p)\,b \\
E[U(\widetilde{G})] &= pU(a) + (1-p)U(b)
\end{aligned}
$$

# Utility MUST be concave: Classic Logarithmic Utility

- Risk Aversion: we prefer $E(\widetilde{W})$ to the risky wealth itself

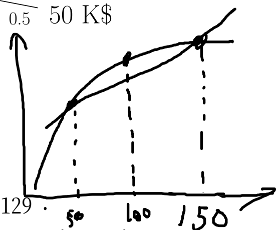
  $\rightarrow$ Utility is concave: $U(E(\widetilde{W})) > E(U(\widetilde{W}))$

- The logarithm function is concave (.. has other desirable properties)

- Consider the risky wealth

$$\widetilde{W} \begin{cases} \xrightarrow{0.5} 150 \text{ K\$} \\ \xrightarrow{0.5} 50 \text{ K\$} \end{cases}$$

- Then:




$$E(\widetilde{W}) = 100K\$$$

$$U(E(\widetilde{W})) = \log 100K = 11.5129$$

$$EU(\widetilde{W}) = 0.5\log(50K) + 0.5\log(150K) = 11.3691$$

# Certainty Equivalent and Risk Premium

- Certainty Equivalent:

Guaranteed amount CE$ providing the same utility as the risky wealth
Investor is indifferent between CE$ for sure and the risky wealth

$$U(CE\$) = EU(\widetilde{W})$$

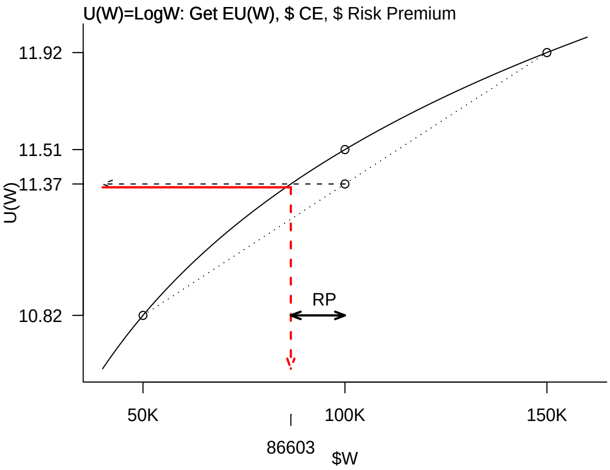
$CE = U^{-1}(EU(\widetilde{W})) = e^{11.3691} = \$\ 86603$

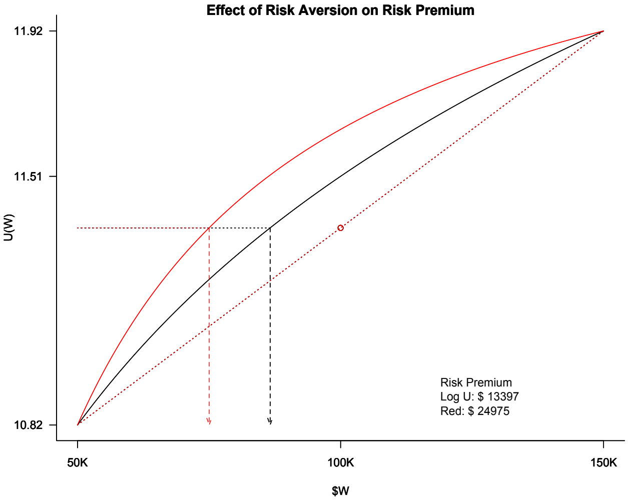
- Risk Premium:

$$\$RP = \$E(\widetilde{W}) - \$CE$$

$\text{RP} = \$\ 13397$

# Log Utility vs Wealth






In finance, we will show how to to quantify risk aversion

Back to Saint-Petersburg:

- Use U(W) = Log (W)

- Then EU(W) is:

$$
\begin{aligned}
EU(W) &= \frac{1}{2}U(\$1) + \frac{1}{4}U(\$2) + \ldots + \frac{1}{2^{N+1}}U(2^N) + \ldots \\
&= \log(2)\underbrace{\left[\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \ldots + \frac{N}{2^{N+1}} + \ldots\right]}_{1} \\
&= \log(2) = U(CE) = 0.693
\end{aligned}
$$

- Certainty Equivalent: