Introduction to Moment Estimation

Eric Jacquier

Boston University Questrom School of Business

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Outline

- Is the usual sample mean a biased estimator of the **true unknown mean**?
- ... What is its **precision**?
- Sample variance: biased or unbiased estimation ?
- Estimating skewness and kurtosis: sample skewness and sample kurtosis
- How precisely can we estimate (stock and portfolio) mean returns?
- t-test and confidence intervals
- A different question: predicting return's **shortfall risk**
- ... and the **predictive density** of a portfolio return

Estimating the mean: Is the sample mean biased?

We do not know the true moments, we estimate them from data by computing sample quantities.

- Sample of T returns: $R_1, \ldots, R_t, \ldots, R_T$
- The average, aka the **sample mean** is

$$\overline{R} \equiv \widehat{\mu} = \frac{1}{T} \sum_{1}^{T} R_t \tag{1}$$

  It is a portfolio of returns! Why equal weights?

- Is the sample mean an unbiased estimate of the mean?

$$\begin{aligned} E(\overline{R}) &= E(\frac{1}{T} \sum_{1}^{T} R_t) = \frac{1}{T} E(R_1 + \ldots + R_T) = \frac{1}{T}(\mu_1 + \ldots + \mu_T) \\ &= \frac{1}{T} T \mu = \mu \qquad \text{unbiased} \end{aligned} \tag{2}$$

- What assumption did we use?
- Using the sample mean again and again, we find the true mean on average.

Precision of the sample mean?

- Variance of the sample mean:

$$
\begin{aligned}
V(\overline{R}) &= V\left(\frac{1}{T}\sum_{1}^{T} R_t\right) = \frac{1}{T^2} V(R_1 + \ldots + R_T) \\
&= \frac{1}{T^2}(\sigma_1^2 + \ldots + \sigma_T^2 + 0 + \ldots + 0) \\
&= \frac{1}{T^2} T\sigma^2 = \frac{\sigma^2}{T} \qquad (3)
\end{aligned}
$$

- Why the zeros ?
  Returns are NOT correlated with one another. In time series talk, they have zero autocorrelation.

- The sample is identically and independently distributed **(i.i.d)**.

- Key for precision: Number of observations, variance of the underlying data

- Is the formula in (3) feasible?
  No! We need to estimate σ^2 as well:

Estimating the variance

Sample Variance

$$\widehat{\sigma^2} = s^2 = \frac{1}{T-1}\sum_t (R_t - \overline{R})^2 \tag{4}$$

- Why $\frac{1}{T-1}$, why not $\frac{1}{T}$?
  - Can show that $\frac{1}{T-1}$ is unbiased. **Prove it !**
  - But we can show that $\frac{1}{T}$ is most precise in large samples (a property of maximum likelihood estimator)
- How to estimate Standard Deviation: $\widehat{\sigma} = \sqrt{s^2}$?
  Question:
  If (4) estimates σ^2 with no bias, does the square-root of (4) estimate σ with no bias?
- How precisely can we estimate variances and standard deviations?
  We can show that (no proof):

$$var(\widehat{\sigma}) \approx \frac{\sigma^2}{2T} \tag{5}$$

$$var(\widehat{\sigma^2}) \approx \frac{2\sigma^4}{T} \tag{6}$$

- Sample Skewness and Sample Kurtosis:

$$\widehat{Sk} = \frac{1}{T}\sum_t \left(\frac{R_t - \overline{R}}{s}\right)^3$$
$$\widehat{K} = \frac{1}{T}\sum_t \left(\frac{R_t - \overline{R}}{s}\right)^4$$

- Unbiased for very large samples.
- How precise are the estimators? (no proof)

$$Var(\widehat{Sk}) \approx \frac{6}{T}$$
$$Var(\widehat{K}) \approx \frac{24}{T}$$

Example: How precisely can I estimate a mean return

- 60 annual returns, $\overline{R} = 0.1, s = 0.2$

$$V(\overline{R}) = \frac{\sigma^2}{T} = \frac{0.2^2}{60}$$

- Can we reject the hypothesis that the mean return is 6 % ?
- Distribution of $\overline{R}$: $\bar{R} \sim (\mu, \frac{\sigma^2}{T})$

  Normally distributed if the stock return is normally distributed.

  In large sample: approximately normally distributed even if the stock return is not normally distributed.

$$\frac{\bar{R} - \mu}{\sigma/\sqrt{T}} \sim N(0,1)$$

$$\frac{\overline{R} - \mu}{(s/\sqrt{T})} \sim t_{T-1}$$

a Student-t with $\nu =$ T-1 degrees of freedom. (Proof later in Distributions Lecture Note)

Precision of the sample mean: Student t-test approach

Student t-test (sometimes approximated by a normal Z test)

- Under the null hypothesis H_0, $\mu = 0.06$,

$$t(\nu = 60 - 1) \approx Z = \frac{0.10 - 0.06}{(0.2/\sqrt{60})} = 1.55$$

- Z measures how many standard deviations our estimate (0.10) is from the null hypothesis (0.06).

  Standard deviations ... of what?

  Look up a table, the 5% cutoff of the normal is +/- 1.96.

  With 60 years of data, a 10% estimate can not statistically reject the hypothesis that the true mean return is 6%

Welcome to doing statistics in finance!

Does a confidence interval cover the null hypothesis?

- If Z is normal (what if it is Student-t?): $Prob[-1.96 < Z < 1.96] = 0.95$

$$\begin{aligned}
Prob[-1.96 < \frac{\overline{R}-\mu}{s/\sqrt{T}} < 1.96] &= 0.95 \\
Prob[-1.96\frac{s}{\sqrt{T}} < \overline{R}-\mu < 1.96\frac{s}{\sqrt{T}}] &= 0.95 \\
Prob[\overline{R}-1.96\frac{s}{\sqrt{T}} < \mu < \overline{R}+1.96\frac{s}{\sqrt{T}}] &= 0.95
\end{aligned}$$

- 95% confidence interval: [0.10 - 0.051 , 0.10 + 0.051]. It contains 14%.
- We do not reject the hypothesis that 14% is the true mean index return.

A different question: Shortfall Risk

Given our estimates, what is the probability the index will return less than 5% next year?

- Ignore for now that the estimates are only noisy estimates of the true parameter. Take the numbers as given.
- The question is about the distribution of the index return itself, not about the distribution of the sample mean: $Prob(\widetilde{R} < 0.05) = ?$
- As per our estimates $\widetilde{R} \sim N(0.1, 0.2)$. The question is:

$$Prob\left(\frac{\widetilde{R}-\mu}{\sigma} < \frac{0.05-0.1}{0.2}\right)$$

$Prob(Z < -0.25) = 0.40$. Look up in a table, or ask R.

How about a relative shortfall, e.g. 2 random portfolios, $P(\tilde{R}_1 - \tilde{R}_2 < 0.05)$?

Predicting future returns accounting for mean estimation error

We did not incorporate the uncertainty in parameter estimates.
The **predictive density** of the return combines the uncertainty in the future return itself and that of the parameter estimators.

- Predictive densities are crucial to portfolio management. They are much more natural in Bayesian statistics, but we can use some easy intuition.

- Given our information up to T, the future return R_{T+1} is $\mathrm{N}(\hat{\mu}, \sigma)$.

$$R_{T+1} = \hat{\mu} + \sigma\epsilon_{T+1}, \quad with\ \epsilon_t \sim i.i.d. N(0,1) \ \ and\ \hat{\mu} \sim N(\mu, \frac{\sigma}{\sqrt{T}})$$

- The predictive density of R_{T+1} has moments:

$$\begin{aligned} E[R_{T+1}] &= E(\hat{\mu}) + 0 = \mu \\ V[R_{T+1}] &= V(\hat{\mu}) + V(\sigma\epsilon) = \sigma^2(\frac{1}{T} + 1) \end{aligned}$$

Why no cross-term in the variance? $+ 2cov(\ , \ ) = 0$

- The uncertainty in $\hat{\sigma}$ is a second order effect, it will wait until the Student-t distribution.