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MF 793 – Fall 2021

This is a Individual Problem Set 4

Due Sunday November 8th on GradeScope

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- This **individual** problem set will reward you for studying LN9 and LN10.
- It must be handwritten to get a **check**.
- You need to answer **all** the questions
- Answer questions in the space provided

Problem 1: The theoretical β

LN 9 computes the optimal β when the linear model y = β x + ϵ, approximates an unknown CEF E(y|x). Generalize this result when the linear model approximation is y = α + β x + ε. I.e., solve for α and β that minimize the MSE, $E(\varepsilon^2)$ as a function of the true moments of x and y. **10 pts**

$y = \alpha + \beta x + \varepsilon$, using the property, we know $E(\varepsilon) = 0$

$\Rightarrow$ The approximation tells us that $\bar{y} = \alpha + \beta \bar{x} \Rightarrow \alpha = \bar{y} - \beta \bar{x}$

$\therefore MSE = E(\varepsilon^2) = E[(y - \alpha - \beta x)^2] = E[((y-\bar{y}) - \beta(x-\bar{x}))^2]$

$= E[(y-\bar{y})^2] + \beta^2 E[(x-\bar{x})^2] - 2\beta \cdot E[(y-\bar{y})(x-\bar{x})]$

$= Var(y) + \beta^2 Var(x) - 2\beta\, Cov(x, y)$

To minimize MSE, let $\frac{\partial MSE}{\partial \beta} = 0 \Rightarrow 2\beta Var(x) - 2Cov(x,y) = 0$

$$\Rightarrow \begin{cases} \beta = \frac{Cov(x,y)}{Var(x)} \\ \alpha = \bar{y} - \frac{Cov(x,y)}{Var(x)} \bar{x} \end{cases}$$

Problem 2: The link between standard deviation and option prices

Don't even think of zooming into an interview and not knowing the Black-Scholes-Merton model. A Call C (no dividends) trades on a stock of value S. Maturity is τ years, exercise price X. The risk free rate at the time of trading is r_F (continuously compounded annual).

a) Write, no proof, the Black-Scholes formula for these known components of the call price.
8 pts

d_1 =

$$\frac{\ln\frac{S}{X} + (r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

d_2 =

$$\frac{\ln\frac{S}{X} + (r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Δ =

$$\Phi(d_1) = \Phi\left(\frac{\ln\frac{S}{X} + (r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

BS(X, S, τ, r_F, σ) =

$$S \cdot N(d_1) - Xe^{-rT}N(d_2)$$

b) GOOGL trades at $2,960. The 1-month call with exercise price of $2,960 trades at $106. Assume a 0% risk-free rate over that horizon. What is GOOGL's ***implied standard deviation*** (annualized) for this call. You must use R to answer this problem **8pts**

ISD = 0.3110576

What would be the ISD if the call was trading 10% higher, at $116.60 for the same stock price?

ISD = 0.3421874

What % change in ISD (over the one at $106) does this represent:

% increase in ISD = 0.1000773 ---> 10.008%

How does this % increase compare too the % increase in Call price?

Answer: the % increase in ISD is about the same as the % increase in Call price, but there're still some small differences. 10.008% is a litter higher than 10%, which means the change of call price leads to less change in ISD. In other word, the % increase in ISD might lead to larger % increase in call price in real world.

c) Type in the R code you used to find the Implied Standard Deviation? **4pts**

```
t <- 1/12
S <- 2960
X <- 2960
callreal <- 116.60  # we can change the parameter here
rf <- 0
upper <- 1
lower <- 0
sigma <- 0.5

d1 <- (log(S/X)+(rf+0.5*(sigma**2))*t)/sigma/(t**0.5)
d2 <- d1 - sigma*(t**0.5)
call <- S*pnorm(d1) - X * exp(-rf*t)*pnorm(d2)
call
while (abs(call-callreal)>=0.0001){
 if ((call-callreal)<0){
   lower<- sigma
 }
 if ((call-callreal)>0){
   upper <- sigma
 }
 if ((call-callreal)==0){
   break
 }
 sigma <- (lower+upper)/2
 d1 <- (log(S/X)+(rf+0.5*(sigma**2))*t)/sigma/(t**0.5)
 d2 <- d1 - sigma*(t**0.5)
 call <- S*pnorm(d1) - X * exp(-rf*t)*pnorm(d2)
}
call1 <- 0.3110576
call2 <- 0.3421874
(call2-call1)/call1  #% increase in ISD
```

Problem 3: Understanding (vector) derivative of inner products and quadratic forms

You need to be comfortable with quadratic forms, inner and outer products, in econometrics and portfolio optimization. This includes their derivatives.
For example, you need to understand the "normal equation", LN10, p.8.

a) In the Sum of Squares, top of p.8, verify that Z=X'Y has length k. Set k=2, consider a column vector $Z=(z_1, z_2)'$, the vector $\beta' = (\beta_1, \beta_2)$, and the scalar quantity $s=\beta'Z$. Write each component of the 2x1 vector $\partial s/\partial\beta \equiv (\partial s/\partial\beta_1, \partial s/\partial\beta_2)$. **6pts**

$$\frac{\partial\left([\beta_1, \beta_2]\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)}{\partial\beta_1} = \frac{\partial(\beta_1 z_1 + \beta_2 z_2)}{\partial\beta_1} = z_1$$

$\partial s/\partial\beta_1 =$

$$\frac{\partial(\beta_1 z_1 + \beta_2 z_2)}{\partial\beta_2} = z_2$$

$\partial s/\partial\beta_2 =$

$$\frac{\partial s}{\partial\beta} = \frac{\partial s}{\partial\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Z$$

Then compare $\partial s/\partial\beta$ with Z:

b) Let's study the second part of the SSE on p.8. Take a column vector $\beta=(\beta_1, \beta_2)'$, and a matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

The scalar quantity $Q = \beta'H\beta$ is a quadratic form in β. Compute it (write it out) as a function of the h's and the βs. **4pts**

$Q = \beta'H\beta =$

$$(\beta_1, \beta_2)\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = (\beta_1 h_{11} + \beta_2 h_{21}, \beta_1 h_{12} + \beta_2 h_{22})\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$
$$= \beta_1^2 h_{11} + \beta_1\beta_2 h_{21} + \beta_1\beta_2 h_{12} + \beta_2^2 h_{22}$$

c) Use your result in b) to write each component of the 2x1 vector $\left({\partial Q}/{\partial \beta_1}, {\partial Q}/{\partial \beta_2}\right) \equiv {\partial Q}/{\partial \beta}$.
4pts

$${\partial Q}/{\partial \beta_1} = \frac{\partial(\beta_1^2 h_{11} + \beta_1\beta_2 h_{21} + \beta_1\beta_2 h_{12} + \beta_2^2 h_{22})}{\partial \beta_1} = 2\beta_1 h_{11} + \beta_2 h_{21} + \beta_2 h_{12}$$

$${\partial Q}/{\partial \beta_2} = \frac{\partial(\beta_1^2 h_{11} + \beta_1\beta_2 h_{21} + \beta_1\beta_2 h_{12} + \beta_2^2 h_{22})}{\partial \beta_2} = 2\beta_2 h_{22} + \beta_1 h_{21} + \beta_1 h_{12}$$

d) Write out each component of the 2x1 vector: $2H\beta$ in function of the h's and β's. **4 pts**

$$2H\beta = 2\cdot\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 2\begin{pmatrix} \beta_1 h_{11} + \beta_2 h_{12} \\ \beta_1 h_{21} + \beta_2 h_{22} \end{pmatrix}$$

e) Compare your results in c) and d):
When is the derivative of $\beta' H\beta$ with respect to β equal to $2H\beta$? **2pts**

$$\frac{\partial \beta' H \beta}{\partial \beta} = \frac{\partial Q}{\partial \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}} = \begin{pmatrix} 2\beta_1 h_{11} + \beta_2 h_{21} + \beta_2 h_{12} \\ 2\beta_2 h_{22} + \beta_1 h_{21} + \beta_1 h_{12} \end{pmatrix}$$

$\Rightarrow$ Obviously, when h is symmetric, They would be equal.

f) What kind of matrix verifies this condition? **2pts**

Answer: covariance matrix

Problem4: OLS estimator, The X'X matrix

You need to understand the structure of (X'X). The OLS estimator is $\hat{\boldsymbol{\beta}}_{OLS}$= $(X'X)^{-1}X'Y$ and its variance **V($\hat{\beta}_{OLS}$)** = $\sigma^2(X'X)^{-1}$. For a simple regression with intercept and one X variable, we have $\boldsymbol{\beta} = (\alpha, \beta)$.

a) Write each component of the 2x2 matrix (X'X), with only the sample size T and sample statistics: $\bar{X}, \bar{Y}, \widehat{\sigma_{XY}}, \widehat{\sigma_X}, \widehat{\sigma_Y}$. **4pts**

$$y = \alpha + \beta X \Rightarrow \underset{Y}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}} = \underset{X}{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{pmatrix}} \underset{\beta}{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

a) $$X'X = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_2 & X_3 & \cdots & X_T \end{pmatrix} \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \\ \vdots & \vdots \\ 1 & X_T \end{pmatrix} = \begin{pmatrix} T, & \sum X_i \\ \sum X_i, & \sum X_i^2 \end{pmatrix} = \begin{pmatrix} T, & T\bar{X} \\ T\bar{X}, & T\hat{\sigma}_X^2 + T\bar{X}^2 \end{pmatrix}$$

for $\Rightarrow \{D(x) = E(x^2) - E^2(x)\}$

b) Invert (X'X). Look at Var($\hat{\beta}$), what leads to a more precisely estimated **slope** coefficient ? **4pts**

Det (X'X) =

$$DET(X'X) = T^2(\hat{\sigma}_X^2 + \bar{X}^2 - \bar{X}^2) = T^2\hat{\sigma}_X^2$$

σ^2 (X'X)$^{-1}$ =

$$\sigma^2(X'X)^{-1} = \frac{\sigma^2 T}{T^2\hat{\sigma}_X^2} \begin{pmatrix} \hat{\sigma}_X^2 + \bar{X}^2, & -\bar{X} \\ -\bar{X}, & 1 \end{pmatrix} = \frac{\sigma^2}{T\hat{\sigma}_X^2} \begin{pmatrix} \hat{\sigma}_X^2 + \bar{X}^2, & -\bar{X} \\ -\bar{X}, & 1 \end{pmatrix}$$

c) What makes the slope coefficient estimator more precise? **2pts**

Answer: from the V($\hat{\beta}_{OLS}$) = $\sigma^2(X'X)^{-1}$, it's easy to find that larger size T, larger sample variance of X and smaller variance σ^2 makes slope coefficient estimator more precise.

Problem 5: The X'X matrix and the sample covariance of the X variables.
a) Write X'X in the case of an intercept and 2 variables, X_1 and X_2, X'X is 3x3. Compute its determinant <u>only as function of the sample statistics of X_1 and X_2 and the sample size T</u>. **4pts**

X'X =

$$y=\alpha+\beta_1 X_1+\beta_2 X_2 \Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & X_{11} & X_{2T} \\ 1 & X_{12} & X_{22} \\ 1 & X_{13} & X_{23} \\ \vdots & \vdots & \vdots \\ 1 & X_{1T} & X_{2T} \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix}}_{\beta}$$

a)

$$X'X = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_{12} & \cdots & & X_{1T} \\ X_{21} & X_{22} & \cdots & & X_{2T} \end{pmatrix} \begin{pmatrix} 1 & X_{11} & X_{21} \\ 1 & X_{12} & X_{22} \\ \vdots & \vdots & \vdots \\ 1 & X_{1T} & X_{2T} \end{pmatrix} = \begin{pmatrix} T, & \sum_1^T X_{1i}, & \sum_1^T X_{2i} \\ \sum_1^T X_{1i}, & \sum_1^T X_{1i}^2, & \sum_1^T X_{1i}X_{2i} \\ \sum_1^T X_{2i}, & \sum_1^T X_{1i}X_{2i}, & \sum_1^T X_{2i}^2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} T, & T\cdot E(X_1), & T\cdot E(X_2) \\ T\cdot E(X_1), & T(\widehat{Var}(X_1)+E(X_1)^2), & T(\widehat{Cov}(X_1,X_2)+E(X_1)E(X_2)) \\ T\cdot E(X_2), & T(\widehat{cov}(X_2,X_1)+E(X_1)E(X_2)), & T(\widehat{Var}(X_2)+E(X_2)^2) \end{pmatrix}$$

Det =

$$\xrightarrow{-1\cdot E(X_1)+2} \begin{vmatrix} T & TE(X_1) & TE(X_2) \\ 0 & T\widehat{Var}(X_1) & T\widehat{Cov}(X_1X_2) \\ T\cdot E(X_2) & T(\widehat{cov}(X_1,X_2)+E(X_1)E(X_2)), & T(\widehat{Var}(X_2)+E(X_2)^2) \end{vmatrix} \xrightarrow{L1\cdot E(X_2)\cdot -1+L3} \begin{vmatrix} T & T\cdot E(X_1) & TE(X_2) \\ 0 & T\cdot\widehat{Var}(X_1) & T\widehat{Cov}(X_1,X_2) \\ 0 & T\cdot\widehat{Cov}(X_2,X_1) & T\widehat{Var}(X_2) \end{vmatrix}$$

$$= T\cdot\left(T^2\,\widehat{Var}(X_1)\,\widehat{Var}(X_2) - T^2(\widehat{Cov}(X_1,X_2))^2\right)$$

$$= T^3\cdot\left(\widehat{Var}(X_1)\cdot\widehat{Var}(X_2) - (\widehat{Cov}(X_1,X_2))^2\right)$$

b) Write $X_d'X_d$ the 2x2 matrix where X_1 and X_2 are written in deviation form $X_1 - \mathbf{1}\,\overline{X_1}$, $X_2 - \mathbf{1}\overline{X_2}$, where **1** is a Nx1 vector of ones, so there is no intercept anymore. Compute its determinant. Compare the two determinants. **4pts**

$X_d'X_d$ **=**

$$: \begin{pmatrix} X_{11}-\bar{X}_1, & X_{12}-\bar{X}_1, & \cdots, & X_{1T}-\bar{X}_1 \\ X_{21}-\bar{X}_2, & X_{22}-\bar{X}_2, & \cdots, & X_{2T}-\bar{X}_2 \end{pmatrix} \begin{pmatrix} X_{11}-\bar{X}_1, & X_{21}-\bar{X}_2 \\ X_{12}-\bar{X}_1, & X_{22}-\bar{X}_2 \\ \vdots & \vdots \\ X_{1T}-\bar{X}_1, & X_{2T}-\bar{X}_2 \end{pmatrix} = \begin{pmatrix} T\cdot \widehat{Var}(X_1), & T\cdot \widehat{Cov}(X_1,X_2) \\ T\cdot \widehat{Cov}(X_1,X_2), & T\cdot \widehat{Var}(X_2) \end{pmatrix}$$

Det =

$$T^2\left(\widehat{Var}(X_1)\,\widehat{Var}(X_2) - \left(\widehat{Cov}(X_1,X_2)\right)^2 \right)$$

c) The two determinants in a) and b) are ….

Answer : Obviously, $\text{Det}(X_d'X_d)$ ✖ $T = \text{Det}(X'X)$