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Common Distributions in Financial Econometrics

- We saw the Normal and the Log-normal in a previous handout

- We now take a tour of the crucial univariate distributions that pop up all the time:

  Chi-square, Student-t, F, Bernoulli, Binomial

- These distributions can arise as those of a random process of interest or the estimator of a parameter such as mean, variance, ratio of variances, etc..

1. The Chi-Square Distribution

- A RV $x > 0$ has the chi-square $\chi^2(\nu)$ distribution with *ν degrees of freedom*, if its density is:

$$p(x \mid \nu) = \frac{x^{\frac{\nu}{2}-1} e^{-x/2}}{2^{\nu/2}\,\Gamma(\nu/2)}, \quad x > 0$$

- **Mean: E(x) = ν** **Variance: V(x) = 2ν** No proof

$E\left(\frac{x}{\nu}\right) = 1$

- The Gamma function in the integration constant is: $\Gamma(q) = \int_0^\infty u^{q-1} e^{-u} du$
  - Gamma function increases quickly and overflows for large ν
  - Gamma function is equal to the Factorial for integers: $\Gamma(n) = (n-1)!$
  - In computations, use the log of the gamma before taking the exponential: *lgamma* in R

- We already proved the density of the $\chi^2(1)$, the square of a unit normal. So: $\Gamma(1/2) = \sqrt{\pi}$

- Property 1: A $\chi^2(\nu)$ with ν degrees of freedom, can be written as the sum of squares of ν **independent** standardized Normal RVs. **[1]**

  $N_1^2 + N_2^2 + N_3^2 + N_4^2 + N_5^2$

- Property 2: A sum of **independent** chi-squares is distributed chi-square with degrees of freedom, the sum of the degrees of freedom:

  $$\chi^2(\nu_1) + \chi^2(\nu_2) + \ldots + \chi^2(\nu_m) \sim \chi^2(\nu_1+\nu_2 + .. + \nu_m) \qquad [2]$$

  [2] follows directly from [1] above

  No proof asked for [1], see Casella & Berger (proof by mgf)

- Shape of the chi-square:

  - As **all** distribution bounded on one side (here >0), the χ^2 is asymmetric with >0 skewness.

    The asymmetry is more pronounced if the bound matters more.

  - … When does the bound matter more ?

    Mode of the χ^2: dp/dx=0 **Mode = ν-2** **prove it.** What if $\nu \leq 2$?

    Skewness: $\sqrt{8/\upsilon}$ As $\boldsymbol{\upsilon} \to \infty$, the asymmetry disappears

    In R, use dchisq, rchisq to see the shape of the χ^2.

The sample variance of normal data is distributed as a χ^2

$$\sum_1^T \left(\frac{X_t-\mu}{\sigma}\right)^2 = \sum \frac{(X_t-\bar{X}+\bar{X}-\mu)^2}{\sigma^2} = \frac{\sum (X_t-\bar{X})^2}{\sigma^2} + \frac{\sum (\bar{X}-\mu)^2}{\sigma^2} + 2\,\frac{\cancel{[\sum (X_t-\bar{X})](\bar{X}-\mu)}}{\sigma^2}$$

$$\frac{(\bar{X}-\mu)^2}{\sigma^2/T}$$

- Consider $s^2 = \frac{1}{T-1}\sum_1^T (X_t-\bar{X})^2$, Let ν = *T-1*, Then $\frac{\nu s^2}{\sigma^2} = \sum_1^T \frac{(X_t-\bar{X})^2}{\sigma^2}$ [3]

$\chi^2(T-1)$ $\chi^2(1)$

[3] looks like a sum of T standardized squared Normals if X_t ~ iid N(μ,σ).

[3] is **not** a χ^2(T): The T deviations $X_t - \bar{X}$ are **not** independent, because they all contain $\bar{X}$

- Also: *s^2 and $\bar{X}$ are independent for the normal distribution (no proof, but know it)* [4]

- In fact, [3] ~ χ^2(ν=**T-1**) prove it using properties [2] and [4]

- Fundamental result for Normal data:

If X_i ~N(μ,σ), then $\frac{\nu s^2}{\sigma^2} \sim \chi^2(\nu)$, $with\ \nu = T-1$

$$s^2 \sim \frac{\sigma^2}{\nu}\chi^2(\nu) \qquad Var(\sigma^2) = \frac{\sigma^4}{\nu^2}\times 2\nu = \frac{2\sigma^4}{T-1}$$ **[5]**

- This **exact (small) sample** result requires the normality of the underlying data.

- How wrong is [5] if the data are not normal? In R, simulate non-normal data and look ! See the R lecture note and the χ^2 widget.

- Practice: Use the change of variable rule to find the pdf of s^2, the pdf of s, the distribution of the **precision h = 1/s²**.

  They all belong to the *Gamma family of distributions*, crucial in Bayesian analysis.

- From [5], we can now show that **$\text{Var}(s^2) = \frac{2\sigma^4}{T-1}$** $\qquad \frac{vs^2}{\sigma^2} \sim \chi^2$

  Prove it using the variance of the χ^2

- **Confidence interval for the variance** using the χ^2 distribution. We have

$$P\left(\chi^2_{0.025} < \frac{\nu s^2}{\sigma^2} < \chi^2_{0.975}\right) = 0.95$$

$$P\left(vs^2/\chi^2_{0.975} < \sigma^2 < vs^2/\chi^2_{0.025}\right) = 0.95$$

- Contrast the exact sample distribution in [5] with the following **large sample** (asymptotic) normal approximation. It does not require the data to be normal but requires a large sample.

$$\mathbf{s^2 \overset{.}{\sim} N(\sigma^2, \ 2\sigma^4/T)}$$

$$s^2 \mp 1.96\sqrt{\frac{2\sigma^2}{T}}$$

  How large a sample? In, R simulate s^2 and check its normality. See R lecture note.

2. The Student-t Distribution

Salient facts:

- Used to model fatter tails (higher probabilities of extreme events) than the normal for stocks and other financial series returns.
- Arises in the estimation of the mean of normally distributed random variables
- Standardized version has 1 crucial parameter: **Degrees of freedom ν**
- As $\nu \to \infty$, the Student-t converges to the normal
- The lower ν, the "fatter" the tails.

Definition:

If Y~N(0,1), Z~$\chi^2(\nu)$, Y and Z are independent, then

$t = \frac{Y}{\sqrt{Z/\nu}}$ **is said to have a Student-t distribution with ν degrees of freedom.**

Density:

$$p(t|\nu) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\nu}\,\Gamma(1/2)\,\Gamma(\nu/2)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \qquad -\infty < t < \infty,$$

- There is a non-standardized version of the Student-t with non-zero mean and a scale

  Exercise: Use the change of variable rule to find the density of $\mathbf{x = \theta + \sigma t}$

Crucial Properties

- Symmetric, all odd moments are zero

- Variance: $\mathbf{V(t_\nu) = \nu / (\nu-2)}$ it is infinite if $\nu \leq 2$.

  The variance of the standardized t **is larger than 1**, **variance inflation factor**

- Kurtosis: $K(t_\nu) = 3 + 6 / (\nu-4)$, it is infinite if $\nu \leq 4$.

- **Moments of order ν and above do not exist**.

$E\left(t^{\nu+k}\right) = \infty$

$k \geq 0$

- Student-t(**ν=1) is the Cauchy density, its mean does not exist** $p(x) = \frac{1}{\pi(1+x^2)}$

Fat-tailness and financial returns

Can the t generate the fat tails we see in financial returns ?

Kurtosis implied by the Student-t is either 9 (ν=5) or infinite!

We often find kurtosis higher than 9: Do stock returns have infinite kurtosis?

- **The Student-t is the sampling density of the sample mean for normal data**

Recall: $t = \frac{Y}{\sqrt{\chi^2(\nu)/_\nu}}$ Estimate the mean and standard deviation of T=ν+1 normal data,

we have $Y = \frac{(\bar{R}-\mu)}{\sigma/\sqrt{N}} \sim N(0,1)$ and $\frac{\nu s^2}{\sigma^2} \sim \chi^2(\nu)$

- Can easily shows that $\frac{(\bar{R}-\mu)}{s/\sqrt{N}}$ **~ t(ν)** **Prove it**

  Needed piece of the proof omitted for now: $\bar{R}$ and s are independent

- Is t(ν) very different from its normal approximation? Check quantiles in R, for T=20, 40, 60, 100

- The t converges to the Normal as ν → ∞, $\lim_{\nu\to\infty}(1+t^2/\nu)^{-(\nu+1)/2}$? **Prove it**

$e^{-\frac{\nu+1}{2}\cdot\log(1+\frac{t^2}{\nu})} \approx e^{-\frac{\nu+1}{2}\cdot\frac{t^2}{\nu}} = e^{-\frac{t^2}{2}}$

$\nu \to \infty$

$\lim_{x\to 0}(\log(1+x)) \approx x$

$\lim_{\nu\to\infty}\frac{\nu+1}{\nu} = 1$

3. The F distribution

Definition: If $Y_1 \sim \chi^2(\nu_1)$ and $Y_2 \sim \chi^2(\nu_2)$ are independent, then the ratio

$$X = \frac{Y_1/\nu_1}{Y_2/\nu_2}$$

$\frac{\chi^2_{v_1}}{Y_1}$ $\sim \frac{s_1^2}{\sigma_1^2}$

has the Fischer (F) distribution with degrees of freedom (ν_1, ν_2).

- $E(X) = \nu_2 / (\nu_2 - 2)$ No proof

$\lim_{v_2 \to \infty} F_{v_1 v_2} = \frac{\chi^2_{v_1}}{v_1}$

- What happens when the degrees of freedom increase, numerator, denominator, ratio ?

Where does it arise in Financial Econometrics?

- To compare the variances of two samples, test whether variance changes through subsamples

  Greg Chow's test.

  The **variance ratio** is easy to compute and has an F distribution under the null hypothesis that the two true variances are equal, and the data are normal:

$$\mathrm{VR} = \frac{s_1^2}{s_2^2} = \frac{\nu_1 s_1^2 / \sigma_1^2 \nu_1}{\nu_2 s_2^2 / \sigma_2^2 \nu_2} \sim F(\nu_1, \nu_2)$$

When is the second equality true ?

Careful: The two chi-squares must be independent!

- Jim Poterba and Larry Summers use variance ratios to assess the predictability of stock returns.

If stocks are not predictable, the aggregation of variance must be perfect.

Use M-period returns R_M: estimate σ_M: s_M with $1+\nu_M$ observations.

Use one-period returns R: estimate σ_1: s_1 with $1+\nu_1 = M(1+\nu_M)$ obs.

Null hypothesis H_0: $\sigma_M^2 = M\sigma_1^2$, or $VR_{PS} = {s_M^2}\big/{M s_1^2} = 1$

The distribution of this VR is in fact not trivial in small sample, it is not an F!

Generally we find VR a bit higher than 1 for horizons below one year (momentum)

VR a bit lower than 1 for horizons above one year (reversals)

4. The Binomial distribution

- **Bernoulli** trial: $X_i \sim Bern(p)$: $p(X_i=1) = p$

  $E(X_i) = p * 1 + (1-p) * 0 = \mathbf{p}$

  $E(X_i^2) = p * 1 + 0 = p$

  $\mathbf{V(X_i)} = p - p^2 = \mathbf{p(1-p)}$

- **Binomial:** Sum of n independent Bernoulli trials

  $$p(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in [0, n] \text{ integer}$$

  E(X) = np

  V(X) = n p (1-p)

- **Application in Finance:**

  1. Think of an i.i.d. sample of n managers with probability p of beating the market

  2. Approximation of the normal distribution: Binomial converges to a Normal distribution as the number of steps increases.

     Binomial trees in derivatives pricing: Increase the number of steps for a given option maturity

     Then the Binomial price converges to the Log-normal based Black-Scholes price

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{np(1-p)}} \qquad\qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = 3 + \frac{1-6p(1-p)}{np(1-p)}.$$

As n gets large, Skewness goes to zero, Kurtosis to 3, no matter what p is.

In option pricing, one can choose p to calibrate a desired variance of the limiting normal

- **Application**: Estimation of p via the sample proportion random variable: X/n.

  N=250 Fund managers, X=140 beat the market

  Can we reject the null H_0 that the probability of a manager beating the market is 0.5?

  $$\hat{\boldsymbol{p}} = \frac{\boldsymbol{X}}{\boldsymbol{n}} = \;\; 0.56$$

  $$\boldsymbol{V}(\hat{\boldsymbol{p}}) = \frac{np(1-p)}{n^2} = \;\; \frac{\boldsymbol{p}(\boldsymbol{1}-\boldsymbol{p})}{\boldsymbol{n}}$$

- Normal approximation for the Binomial with large n:

$$\hat{\boldsymbol{p}} \dot{\sim} \boldsymbol{N}\left(\boldsymbol{p}\,,\ \frac{\boldsymbol{p}(\boldsymbol{1}-\boldsymbol{p})}{\boldsymbol{N}}\right)$$