# MF850: Advanced Computational Methods

## Problem Set 4

If you run into problems, try to verify each component of your code on simple cases to make sure they work independently.

**Due date:** Thursday November 24, 2022, at noon Boston time.

**Instructions:** You submit on Blackboard. You may solve this assigment in groups of two. A submission is constituted by answers to the problems along with the code used. A file called `hw4.jl` should contain your code, or your entry point if you separate your code into multiple files. This file should run without errors from a fresh Julia instance/REPL. The code must be formatted by loading the package `JuliaFormatter` and running `format` on the submission files. In other words, submissions in notebook format are not accepted (but you may of course develop in them before creating the submission).
*Hint:* Running `format(".")` runs the formatter on every `.jl` file recursively in the current directory.

Please contact the instructor or a TA if you have questions regarding these instructions or if you find the problem formulation unclear.

**Problem 4.1** When pricing an American option, the dynamic programming principle states that (with time-discretized decision-making)

$$V_{t-\Delta t} = \max\{P(X_{t-\Delta t}), e^{-r\Delta t}\mathbb{E}_{t-\Delta t}[V_t]\},$$

where $P(X_t)$ denotes the payoff of executing the option at time $t$ and $\mathbb{E}_t$ is the risk-neutral expectation conditioned at time $t$.

One method of solving this is to evaluate $\mathbb{E}_{t-\Delta t}[V_t]$ by Monte–Carlo. However, a naive implementation is likely to suffer from performance issues. The following method uses Monte–Carlo in an efficient way.

Given simulated asset prices $X_t$ *Least Squares Monte–Carlo* is the method of iterating the following steps, starting with the known value $V_T = P(X_T)^+$.

1. **Construct polynomials of asset prices**

   Construct polynomial variables as $N$-vectors $x_d = X_{t-\Delta t}^d$ for $d = 0, 1, \ldots, p$ and $p \geq 5$;

2. **Regress continuation value on current asset prices**

   Using the value $V_t$, run the least squares regression $V_t = \sum_d \beta_d x_d + e$ and compute the estimate $\hat{V}_t = \sum_d \beta_d x_d$, i.e., regress $V_t$ on the $x_d$ variables and compute the prediction;

   *Hint:* Use the backslash operator to run the linear regression.

3. **Determine whether to execute**

   Compute the estimated option value at time $t - \Delta t$ as either

   $$V_{t-\Delta t} = \max\{P(X_{t-\Delta t}), e^{-r\Delta t}\hat{V}_t\} \tag{TvR}$$

   or

   $$V_{t-\Delta t} = \begin{cases} P(X_t) & \text{if } P(X_{t-\Delta t}) \geq e^{-r\Delta t}\hat{V}_t \\ e^{-r\Delta t}V_t & \text{otherwise.} \end{cases} \tag{LS}$$

   *Note:* In the case of (LS), $V_0$ is not a constant. The final option value approximation is given by $\max\{V_0, P(X_0)\}$.

For simplicity, consider a Black–Scholes market of one stock $S$, with $r = 0.01$, $\sigma = 0.1$ and a put option with maturity $T = 1$ and strike $K = 95$, i.e., $P_t = (K - S_t)^+$.

(a) With $\Delta t = T$ and $t = T$, simulate asset prices $X_t$ conditioned on $X_{t-\Delta t} = 100$ a total of $N = 100'000$ times. Evaluate a one-step American option by completing the steps above.

Compute the two solutions using each of (TvR) and (LS).

*Note:* Because of the constant starting point, all $x_d$ are constant and only $x_0$ is needed. However, $\hat{V}_t$ is still unique and this case does not need special treatment when using Julia's backslash operator.

(b) Simulate $N = 100'000$ paths starting at $X_0$ of $n$ steps until $T$. Let $\Delta t = T/n$.

Repeat the procedure above for multiple time steps $t = T, T - \Delta t, \ldots, 0$. Use the value $V_{t-\Delta t}$ found at the end of each time for the next iteration.

Run the two computations with (TvR) and (LS).

*Note:* The variables $x_d$ are no longer constant, except for $t = 0$.

(c) Using the (LS) variation, modify the regression step by only including samples for which $P(X_{t-\Delta t}) > 0$.

Why do you think only including these samples makes sense?

The method using (LS) provides better performance and results, as it avoids estimation errors at each time step when the option is not executed, cf. [Tv01; LS01].

# References

[LS01] Francis A Longstaff and Eduardo S Schwartz. "Valuing American options by simulation: a simple least-squares approach". In: *The review of financial studies* 14.1 (2001), pp. 113–147.

[Tv01] John N Tsitsiklis and Benjamin van Roy. "Regression methods for pricing complex American-style options". In: *IEEE Transactions on Neural Networks* 12.4 (2001), pp. 694–703.