HW Problems for Assignment 2 - Part 1

1. VaR for a Portfolio of Microsoft, Apple and Google Stocks. The file "StockData.xlsx" contains prices for Microsoft, Apple, and Google over the period 9/1/19 – 8/31/21. Column 1 has the date; Column 2 Microsfot closing prices; Column 3 Apple closing prices; and Column 4 Google closing prices. Additionally, as of 8/31/21, the market capitalizations of Microsoft, Apple and Google were 2.269, 2.510 and 1.940 trillion dollars respectively. Our goal is to compute the Value at Risk for a market capitalization weighted portfolio of these three stocks. The portfolio size is $\$1,000,000$ and the risk factor changes are the log returns X.

(a) **(5) Points** Estimate the mean vector and covariance matrix for the daily log returns using EWMA. Start with some initial value μ_0, Σ_0 (e.g. the sample mean and variance, you can also set both to 0, it will not matter much) and then update via the EWMA formulas

$$\begin{aligned}\mu_{t+\Delta} &= \lambda\mu_t + (1-\lambda)X_t \\ \Sigma_{t+\Delta} &= \theta\Sigma_t + (1-\theta)(X_t - \mu_t)(X_t - \mu_t)^{\mathrm{T}}.\end{aligned}$$

This will give you an estimate for μ, Σ as of 8/31/21, which is then used over the next period 8/31/21 – 9/1/21. Use $\lambda = \theta = 0.97$.

(b) **(15 Points)** Estimate the VaR, for a 95% confidence, of the market cap weighted portfolio in the following ways.

(i) Using the empirical distribution of the log returns and the loss operators $l_{[t]}$, $l_{[t]}^{lin}$, $l_{[t]}^{quad}$.

(ii) Assuming the log returns are normally distributed with the estimated mean vector and covariance matrix, and using loss operators $l_{[t]}$, $l_{[t]}^{lin}$, $l_{[t]}^{quad}$. For the full and quadratic loss operators you will have to run a simulation sampling the normal random variables to obtain the VaR. For the linear loss operator you can obtain an exact VaR estimate using the methodology discussed in class. For the simulation, sample $N = 100,000$ normals.

How do the VaR estimates compare? Is any one (or more than one) estimate different from the rest? If so, please explain why you think this is the case. **Note:** instead of EWMA for estimating the mean and covariance matrix, try using the standard estimators. Do you see a change? What do you think is happening.

2. (20 Points) VaR and Time Aggregation. In this exercise you will replicate the results in lecture, where we estimated the $K = 10$ day VaR for a hedged call option in the Black-Scholes model. The only difference is that here we have sold a put, rather than a call, option.

We have sold a put option with strike κ and maturity T, and are delta hedging in the underlying stock. The asset has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where μ and $\sigma > 0$ are constant, and W is a Brownian motion under the physical measure $\mathbb{P}$. The risk free rate is the constant $r > 0$.

At the current time t our position in the stock is $h_t = \partial_s P^{BS}(t, S_t)$, where $P^{BS}(t, s)$ is the Black-Scholes call price given $S_t = s$. We have no initial position in the money market. As such, the portfolio value is

$$V_t = h_t S_t + 0 - P^{BS}(t, S_t).$$

As h_t is constant over $[t, t+\Delta]$, at the end of the day the portfolio value is

$$V_{t+\Delta} = h_t S_{t+\Delta} + 0 \times e^{rt} - P^{BS}(t+\Delta, S_{t+\Delta}).$$

The one period loss is $L_{t+\Delta} = V_t - V_{t+\Delta}$, and we can think of is as a function of the log return $X_{t+\Delta} = \log(S_{t+\Delta}) - \log(S_t)$.

We then follow the re-balancing rule as described in class. Namely, over $[T_{k-1}, T_k]$, $T_k = t + k\Delta, k = 2, ..., K$ we

- We hold $h_{T_k} = \partial_s P^{BS}(T_{k-1}, S_{T_{k-1}})$ shares of the stock.
- If $Y_{T_{k-1}}$ was the amount in the money market at the end of the previous period, the new amount at the beginning of the current period is $Y^{new}_{T_{k-1}} = Y_{T_{k-1}} - (h_{T_k} - h_{T_{k-1}})S_{T_{k-1}}$ which has end of period value $Y_{T_k} = Y^{new}_{T_{k-1}} e^{r\Delta}$.
- At T_k the portfolio value is

$$V_{T_k} = h_{T_k} S_{T_k} + Y_{T_k} - P^{BS}(T_k, S_{T_k}),$$

  and the loss is $L_k = V_{T_{k-1}} - V_{T_k}$.

The above methodology produces losses $\{L_k\}_{k=1}^K$ which are then aggregated into the total loss $L_{T_K} = \sum_{k=1}^K L_k$. We can sample this loss by sampling the log returns $\{X_{t+k\Delta}\}_{k=1}^K$.

Using the above methodology, for a lot of 100 puts (i.e. multiply your losses by 100):

(1) Estimate the one day VaR over $[t, t+\Delta]$.
(2) Estimate the K day VaR over $[t, t+K\Delta]$.
(3) Compare the K day VaR with $\sqrt{K}$ times the one day VaR, to see how the square root of time rule holds up.

For parameters, use $\alpha = .95$, $t = 0$, $K = 10$, $\Delta = 1/252$, $T = .292$, $S_t = 152.51$, $\kappa = 170$, $r = 0.119\%$, $\sigma = 49.07\%$ and $\mu = 16.91\%$. For your simulations, use $40,000$ runs.

For error checking purposes, I obtained $\mathrm{VaR}^{10}_{.95} \approx 88$ and a $\sqrt{10}\mathrm{VaR}_{.95} \approx 97$.

3. (20 Points) Backtesting VaR. In this exercise, you will replicate the results given in lecture on the backtesting of VaR. Specifically, we will backtest two different methods for estimating VaR, for a portfolio which keeps a constant \$1 in the S&P 500 index throughout time.

The historical prices are given in the file

"SP_Prices.csv".

The first column contains the date, the second the closing price for that date. Data is from 8/31/07 - 8/31/17 sorted oldest to newest.

Using the above data set, produce a series of one-day VaR estimates (using the full loss operator) and exceedances using

(a) The empirical distribution method.

(b) The normal log returns method with EWMA updating.

Follow the lecture notes to implement the above. For both methods above use a four year rolling window, so the first VaR estimate will take place after the 1010^{th} oldest log return.

Notes:

(1) You will need an initial estimate for $\hat{\mu}, \hat{\sigma}$ to obtain the first VaR estimate. To obtain this, it is fine to take the sample mean and sample variance for the first 1010 data points.

Obtain the total number of exceedances for each method and compare them with what was obtained in class. Additionally, produce the $1-\beta$ confidence interval and see if the number of exceedances falls within this range. Use parameter values $\alpha = .95$, $\beta = .05$, and $\lambda = \theta = .97$ EWMA values, to fit the models.