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MF731 Homework3

1. The "Exponential Premium Principal" Risk Measure.

(a) For $\rho_a(L) = \frac{1}{a}\log(E[e^{aL}])$, obviously it's a monotonic function.

Then for cash additivity, we have:

$$\rho_a(L-c) = \frac{1}{a}\log(E[e^{a(L-c)}]) = \frac{1}{a}\log(E[e^{aL}]) - c = \rho_a(L) - c$$

Therefore, for the convexity, for any $\lambda \in (0,1)$, we'll have

$$\rho_a(\lambda L_1 + (1-\lambda)L_2) = \frac{1}{a}\log(E[e^{a(\lambda L_1 + (1-\lambda)L_2)}])$$

Holder's inequality told us that: for $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}} \geq E[XY]$

Here we set $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$,

$$\rho_a(\lambda L_1 + (1-\lambda)L_2) \leq \frac{1}{a}\log(E[e^{aL_1}]^{\lambda} \cdot E[e^{aL_2}]^{1-\lambda})$$
$$= \lambda\rho_a(L_1) + (1-\lambda)\rho_a(L_2)$$

So ρ_a is a convex risk measure.

(b) For $L \sim N(0,1)$

$$\rho_a(\lambda L) = \frac{1}{a}\log(E[e^{a\lambda L}])$$
$$= \frac{1}{a}\log(e^{0+\frac{1}{2}(a\lambda)^2})$$
$$= \frac{1}{a}(\frac{1}{2}a^2\lambda^2) = \frac{1}{2}a\lambda^2$$

obviously, it's not homogenous.

2. For the detail, see the attached code file.

3.
(a) To prove same distribution, we need to show the characteristic function

For $a^T Z \Rightarrow E[e^{ira^T Z}] = E[E[e^{ir\sqrt{W}a^T \tilde{Z}}|W]] = E[e^{-\frac{1}{2}r^2 W|a|^2}]$

For $|a|Z^{(1)} \Rightarrow E[E[e^{ir|a|\sqrt{W}\tilde{Z}^{(1)}}|W]] = E[e^{-\frac{1}{2}r^2 W|a|^2}]$

Since $a^T Z$ and $|a|Z^{(1)}$ have same characteristic function, they have same distribution.

$\therefore \tilde{Z}$ is spherical

(b) (i) $L^{lin} = -\theta^T X = -\theta^T\mu - \theta^T A Z$, X is elliptically distributed $\longrightarrow$ Z is spherical.

$\therefore$ $a^T Z$ has same distribution with $|a|Z^{(1)}$, set $-A^T\theta = a$

$$\therefore |a| = \sqrt{|a|^2} = \sqrt{a^T a} = \sqrt{\theta^T A A \theta} = \sqrt{\theta^T \Sigma \theta}$$

$\therefore L^{lin}$ has the same distribution of $-\theta^T\mu + \sqrt{\theta^T\Sigma\theta}\, Z^{(1)}$

hence for any cash-additive, positively homogenous risk measure ρ.

$$\rho(L^{lin}) = -\theta^T\mu + \sqrt{\theta^T\Sigma\theta} \cdot \rho(Z^{(1)})$$

(ii) from (i), it's obvious that $R_M^{lin\,(i)}(\theta) = -\mu^{(i)} + \frac{\Sigma\theta^i}{\sqrt{\theta^T\Sigma\theta}} \cdot \rho(Z^{(1)})$

Thus, $R_{C\%}^{lin} = 100 \times \left(\frac{-\theta^i\mu^i + \frac{\theta^i\Sigma\theta^i}{\sqrt{\theta\Sigma\theta}}\rho(Z^1)}{-\theta^T\mu + \sqrt{\theta^T\Sigma\theta}\,\rho(Z^1)} \right)$

Therefore, we can make the same conclusion connecting $R_C'\%$ to the percentage contribution to variance, as we did when $X \sim N(\mu, \Sigma)$