Xuyang Liu
xyangliu@bu.edu.

MF731 Homework 2 part 2.

1. Practise with ES

(a) $L \sim Exp(1/\theta) \Longrightarrow F(L) = \begin{cases} 1-e^{-\frac{1}{\theta}x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

set $1-e^{-\frac{1}{\theta}x} = \alpha \Longrightarrow x = -\theta \cdot \ln(1-\alpha)$

$\therefore VaR(\alpha) = -\theta \cdot \ln(1-\alpha)$

According to the definition,

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 -\theta \cdot \ln(1-\alpha)\, d\alpha$$
$$= \frac{\theta}{1-\alpha}\left[(1-\alpha) - (1-\alpha)\ln(1-\alpha)\right]$$
$$= \theta\left[1-\ln(1-\alpha)\right]$$

(b) $L \sim Pareto(1, \frac{1}{\xi}) \Longrightarrow F(T) = \begin{cases} 0, & T \leq 1 \\ 1-T^{-\frac{1}{\xi}}, & T > 1 \end{cases}$

Set $1-T^{-\frac{1}{\xi}} = \alpha \Longrightarrow VaR_\alpha(L) = \left(\frac{1}{1-\alpha}\right)^\xi$

$$\therefore ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \left(\frac{1}{1-\alpha}\right)^\xi d\alpha$$
$$= \frac{1}{1-\alpha} \cdot \frac{(1-\alpha)^{-\xi+1}}{1-\xi}$$
$$= \frac{(1-\alpha)^{-\xi}}{1-\xi}$$

2. Time aggregated risk measure for a constant weight portfolio of equities.

(a) For $V_t = (w^1, w^2, \cdots w^d) \cdot \begin{pmatrix} e^{z_t^1} \\ e^{z_t^2} \\ \vdots \\ e^{z_t^d} \end{pmatrix}$, we have return vector $X_{t+\Delta} = (X^1_{t+\Delta}, X^2_{t+\Delta}, \cdots X^d_{t+\Delta})$

Then $V_{t+\Delta} = (w^1, w^2, \cdots w^d) \cdot \begin{pmatrix} e^{z_t^1 + X^1_{t+\Delta}} \\ e^{z_t^2 + X^2_{t+\Delta}} \\ \vdots \\ e^{z_t^d + \cdots X^d_{t+\Delta}} \end{pmatrix} = w^T e^{X_{t+\Delta} + z_t} = V_t \cdot w^T e^{X_{t+\Delta}}$

Similarly, we have $V_{t+2\Delta} = V_{t+\Delta} \cdot w^T e^{X_{t+2\Delta}}$

$\therefore$ For $n = k$, $V_{t+k\Delta} = V_t \cdot \prod_{k=1}^{K} w^T e^{X_{t+k\Delta}}$

$\therefore L = -(V_t - V_{t+k\Delta}) = -V_t\left(\prod_{k=1}^{K} w^T e^{X_{t+k\Delta}} - 1\right)$

(b) For the detail, please see the code file.