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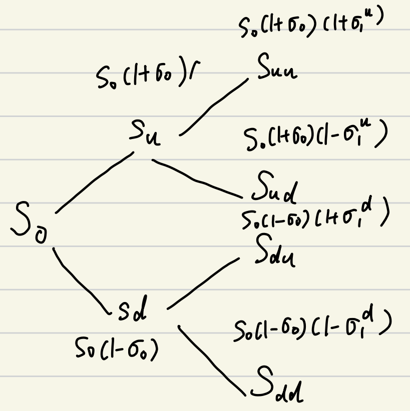
MF731 Homework 5

1. Calibrating to At the Money Options.

(a)



As the tree I put on the Left:

- At time 1: $C^m(1,S_0) = \tilde{\mathbb{E}}[(S_1-S_0)^+]$

$$\Rightarrow \frac{\lambda_1 S_0}{4} = \frac{1}{2} S_0 \cdot \sigma_0 \Rightarrow \sigma_0 = \frac{\lambda_1}{2}$$

- At time 2: $C^m(2,S_0) = \tilde{\mathbb{E}}[(S_2-S_0)^+]$

$$\Rightarrow \frac{\lambda_2 S_0}{4} = \frac{1}{4}\Big[\big(S_0(1+\sigma_0)(1+\sigma_1^u)-S_0\big)^+ + \big(S_0(1+\sigma_0)(1-\sigma_1^u)-S_0\big)^+ + \big(S_0(1-\sigma_0)(1+\sigma_1^d)-S_0\big)^+ + \underbrace{\big(S_0(1-\sigma_0)(1-\sigma_1^d)-S_0\big)^+}_{0}\Big]$$

$$\Rightarrow \frac{\lambda_2 S_0}{4} = \frac{1}{4}S_0\Big[\big(\sigma_1^u+\tfrac{\lambda_1}{2}+\tfrac{\lambda_1}{2}\sigma_1^u\big)^+ + \big(-\sigma_1^u+\tfrac{\lambda_1}{2}-\tfrac{\lambda_1}{2}\sigma_1^u\big)^+ + \big(\sigma_1^d-\tfrac{\lambda_1}{2}-\tfrac{\lambda_1}{2}\sigma_1^d\big)^+\Big]$$

$$\Rightarrow \lambda_2 = \tfrac{1}{2}\big((2+\lambda_1)\sigma_1^u+\lambda_1\big) + \tfrac{1}{2}\big((-\lambda_1-2)\sigma_1^u+\lambda_1\big)1_{\sigma_1^u\le\frac{\lambda_1}{\lambda_1+2}} + \tfrac{1}{2}\big((2-\lambda_1)\sigma_1^d-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_1^d<1}$$

$$\Rightarrow \lambda_2-\lambda_1 = \tfrac{1}{2}\big((2+\lambda_1)\sigma_u^1-\lambda_1\big) - \tfrac{1}{2}\big((2+\lambda_1)\sigma_u-\lambda_1\big)1_{\sigma_u^1\le\frac{\lambda_1}{\lambda_1+2}} + \tfrac{1}{2}\big((2-\lambda_1)\sigma_d-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1} \quad (1)$$

$$\Rightarrow \lambda_2-\lambda_1 = \tfrac{1}{2}\big((2+\lambda_1)\sigma_1^u-\lambda_1\big)1_{\frac{\lambda_1}{2+\lambda_1}<\sigma_u^1<1} + \tfrac{1}{2}\big((2-\lambda_1)\sigma_d^1-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1} \quad (2)$$

(b) $\because \lambda_2>\lambda_1$, $\therefore 2\lambda_2-\lambda_1>\lambda_1$, Assume $\sigma_u^1 > \frac{2\lambda_2-\lambda_1}{\lambda_1+2}$, then $\sigma_1^u > \frac{\lambda_1}{2+\lambda_1}$

$$\Rightarrow 2(\lambda_2-\lambda_1) = (2+\lambda_1)\sigma_1^u-\lambda_1 + \big((2-\lambda_1)\sigma_d^1-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1}$$

① $$\sigma_1^u = \frac{2\lambda_2-\lambda_1-\big((2-\lambda_1)\sigma_1^d-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1}}{2+\lambda_1} \le \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$$

that means $\sigma_1^u \le \frac{2\lambda_1-\lambda_1}{\lambda_1+2}$, which not follow the assumption that $\sigma_u^1 > \frac{2\lambda_1-\lambda_1}{\lambda_1+2}$

Therefore, no solution for $(0,1)$ if $\sigma_1^u > \frac{2\lambda_2-\lambda_1}{\lambda_2+2}$

② for $0<\sigma_1^u\le\frac{2\lambda_2-\lambda_1}{2+\lambda_1}$

(i) when $\sigma_1^u < \frac{\lambda_1}{2+\lambda_1} \Rightarrow \lambda_2-\lambda_1 = \big((2-\lambda_1)\sigma_1^d-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1}$

If we want a solution, we set $\sigma_d^1 > \frac{\lambda_1}{2-\lambda_1} \Rightarrow \sigma_1^d = \frac{2\lambda_2-\lambda_1}{2-\lambda_1}$

(ii) when $\frac{\lambda_1}{2+\lambda_1}<\sigma_1^u<\frac{2\lambda_2-\lambda_1}{2+\lambda_1} \Rightarrow$ Similarly, we must set $\sigma_1^d>\frac{\lambda_1}{2-\lambda_1}$,

Then we have a solution $\sigma_1^d = \frac{2\lambda_2-(2+\lambda_1)\sigma_1^u}{2-\lambda_1}$

(iii) when $\sigma_1^u = \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$, $\Rightarrow \frac{1}{2}\big((2-\lambda_1)\sigma_d^1-\lambda_1\big)1_{\frac{\lambda_1}{2-\lambda_1}<\sigma_d^1<1} = 0$

which means $\sigma_1^d \in \left(0, \frac{\lambda_1}{2-\lambda_1}\right)$

2. For the detail, please see the code file.