HW Problems for Assignment 1 - Lecture 2
Due 6:30 PM Tuesday, September 21, 2021
SOLUTIONS

1. Loss Distributions for a Hedged Put Option. As in the Black-Scholes model, assume the stock price has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $W = \{W_t\}_{t\leq T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate is $r > 0$. Let T be the maturity and K the strike of a put option, and set $P^{BS}(t,x)$ as the price of the call given $S_t = x$. I.e.

$$P^{BS}(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(K - S_T\right)^+ \middle| S_t = x\right]. \tag{0.1}$$

where $\mathbb{Q}$ is the risk neutral measure under which S has drift r. The Black-Scholes formula states (you DO NOT have to prove this)

$$P^{BS}(t,x) = x(N(d_1(T-t,x)) - 2) + Ke^{-r(T-t)}(1 - N(d_2(T-t,x))),$$

where N is the standard normal cdf and

$$\begin{aligned} d_1(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(\tau)\right), \\ d_2(\tau,x) &= d_1 - \sigma\sqrt{\tau}. \end{aligned}$$

Furthermore, with ϕ denoting the standard normal pdf we have

$$\begin{aligned} \delta(t,x) &= \partial_x P^{BS}(t,x) = N(d_1) - 1, \\ \gamma(t,x) &= \partial_{xx} P^{BS}(t,x) = \frac{\phi(d_1(T-t,x))}{x\sigma\sqrt{T-t}}, \\ \theta(t,x) &= \partial_t P^{BS}(t,x) = -\frac{\sigma}{2\sqrt{T-t}}x\phi(d_1(T-t,x)) \\ &\qquad + Kre^{-r(T-t)}(1 - N(d_2(T-t,x))). \end{aligned}$$

These are the "delta", "gamma" and "theta" respectively for the option.

At time t we are short M put options and long $M\delta(t,S_t)$ shares of S. Over the period $[t, t+\Delta]$ we hold the share position constant, writing $\lambda = \delta(t,S_t)$ to reinforce this fact. With this notation, the values of our portfolio at t and $t+\Delta$ are

$$\begin{aligned} V_t &= M\left(\lambda S_t - P^{BS}(t,S_t)\right), \\ V_{t+\Delta} &= M\left(\lambda S_{t+\Delta} - P^{BS}(t+\Delta, S_{t+\Delta})\right). \end{aligned}$$

(a) **(15 Points)** With $z_t = \ln(S_t)$, identify the full, linearized, and second order loss operators over $[t, t+\Delta]$ as a function of the log return $x = X_{t+\Delta}$. **Notes:**
  (i) Make sure to fully evaluate the linearized loss operator - there is a cool answer!.
  (ii) For the second order loss operator, only include the second derivative with respect to x: i.e. ignore the second order time derivative and second order time-space derivative.

(b) **(15 Points)** Write a simulation which identifies the loss distribution for the portfolio using the full, linearized and second order (with the adjustments in note (ii)) loss operators. As in class, produce a histogram approximation of the probability density functions. How well do the approximations work?

For parameters use $\mu = 0.16905$, $\sigma = 0.4907$, $r = 0.0011888$, $t = 0$, $T = .291667$, $\Delta = 10/252$ (ten day horizon), $S_0 = 152.51$, $K = 170$ and $M = 100$ options. Run $N = 100,000$ trials in your simulation.

Solution

(a) We have

$$V_t = M\left(\lambda e^{Z_t} - P^{BS}(t, e^{Z_t})\right) = f(t, Z_t).$$

From here, we immediately obtain the full loss operator

$$\begin{aligned} l_{[t]}(x) &= -M\lambda e^{z_t}\left(e^x - 1\right) \\ &\quad + M\left(P^{BS}(t+\Delta, e^{z_t+x}) - P^{BS}(t, e^{z_t})\right). \end{aligned}$$

As for the linearized loss operator, we first have

$$\begin{aligned} \partial_z f(t,z) &= M\left(\lambda e^z - \partial_z P^{BS}(t, e^z)\right) = M\left(\lambda e^z - \delta(t, e^z)e^z\right), \\ \partial_t f(t,z) &= -M\partial_t P^{BS}(t, e^z) = -M\theta(t, e^z). \end{aligned}$$

This gives the linearized loss operator

$$l_{[t]}^{lin}(x) = M\theta(t, e^{z_t})\Delta - Me^{z_t}\left(\lambda - \delta(t, e^{z_t})\right)x.$$

We can simplify this by recalling that $\lambda = \delta(t, S_t) = \delta(t, e^{z_t})$. Therefore,

$$l_{[t]}^{lin}(x) = l_{[t]}^{lin} = M\theta(t, e^z)\Delta,$$

which corresponds to the fact we are delta hedged and is not even a function of x! As for the second order loss operator, as per the note, we only keep the second order spatial derivative, which is

$$\partial_z^2 f(t,z) = Me^z\left(\lambda - \delta(t, e^z)\right) - M\gamma(t, e^z)e^{2z}.$$

As $\lambda = \delta(t, e^{z_t})$ we have

$$\begin{aligned} l_{[t]}^{quad}(x) &= l_{[t]}^{lin}(x) + \frac{1}{2}M\gamma(t, e^{z_t})e^{2z_t}x^2, \\ &= M\theta(t, e^z)\Delta + \frac{1}{2}M\gamma(t, e^{z_t})e^{2z_t}x^2. \end{aligned}$$

(b) See the Matlab file "Put_Option_Dist.m". We see that the second order approximation distribution almost exactly coincides with the full loss distribution.

2. Practice with VaR. Explicitly compute $\mathrm{VaR}_\alpha(L)$ assuming L has the following distributions/probability distribution functions (pdfs).

(a) **(7 Points)** L is a "double-sided" exponential with threshold l_0 in that L has pdf

$$f(l) = \frac{ab}{ae^{-bl_0} + be^{al_0}} \left(e^{al} 1_{l \le l_0} + e^{-bl} 1_{l > l_0} \right); \qquad l \in \mathbb{R},$$

where $a, b > 0$. Here, you may assume $\alpha \ge b/(b + ae^{-(a+b)l_0})$.

(b) **(6 Points)** L is a binomial random variable with n number of trials and p probability of success on any given trial. Give an explicit answer when $n = 6$, $p = 1/2$ and $\alpha = .9$.

(c) **(7 Points)** $L = Y/Z$ where Y and Z are independent exponential random variables with means $1/\lambda$ (for Y) and $1/\theta$ (for Z).

Solution:

(a) Note that $F(l_0) = b/(b + a^{-(a+b)l_0})$ so $\alpha \ge F(l_0)$ and hence $\mathrm{VaR}_\alpha(L) \ge l_0$. Thus, for $l \ge l_0$

$$F(l) = \frac{b + ae^{-(a+b)l_0}\left(1 - e^{-b(l-l_0)}\right)}{b + ae^{-(a+b)l_0}}.$$

Solving $F(l) = \alpha$ yields, for $\alpha \ge F(l_0)$,

$$\mathrm{VaR}_\alpha(L) = -\frac{1}{b} \log\left(\frac{1}{a} e^{al_0} \left(b + ae^{-(a+b)l_0} \right) (1 - \alpha) \right).$$

(b) Recall the characterization from class of $\mathrm{VaR}_\alpha(L) = x_0$ if and only if $F(x_0) \ge \alpha$ and $F(x) < \alpha$ for $x < x_0$, where F is the cdf of L. Therefore, we have

$$\mathrm{VaR}_\alpha(L) = \min\left\{ k \mid \sum_{j=0}^{k} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \ge \alpha \right\}.$$

For the specification, we can explicitly compute the cdf which takes the values

$$\sum_{j=0}^{k} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$$

at $k = 0, 1, ..., n$. Here, we obtain

$$F(0) = \frac{1}{64}, F(1) = \frac{7}{64}, F(2) = \frac{22}{64}, F(3) = \frac{42}{64};$$
$$F(4) = \frac{57}{64}, F(5) = \frac{63}{64}, F(6) = 1.$$

Since $63/64 \approx 0.984$ and $57/64 \approx 0.891$ we see that $\mathrm{VaR}_\alpha(L) = 5$.

(c) For a fixed $Z = z$, L has cdf $1 - e^{-\lambda\tau z}$ for $\tau \geq 0$. Using this and conditioning we see that

$$\begin{aligned}\mathbb{P}\left[L \leq \tau\right] = \mathbb{E}\left[1_{L\leq\tau}\right] = \mathbb{E}\left[\mathbb{E}\left[1_{L\leq\tau}\middle|Z\right]\right] = \mathbb{E}\left[1 - e^{-\lambda\tau Z}\right];\\ = 1 - \int_0^\infty \theta e^{-\theta z - \lambda\tau z} dz = 1 - \frac{\theta}{\theta + \lambda\tau}.\end{aligned}$$

We thus see that $\mathrm{VaR}_\alpha\left(L\right) = \frac{\alpha\theta}{\lambda(1-\alpha)}$.