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MF731 HW1 part 2

1.(a) $L_{t+\Delta} = -(V_{t+\Delta} - V_t) = -[M(\lambda S_t - P^{BS}(t, S_t)) - M(\lambda S_{t+\Delta} - P^{BS}(t+\Delta, S_{t+\Delta}))]$

① Full: $L_{t+\Delta} = -M[\lambda e^{Z_t}(e^{X_{t+\Delta}} - 1) - (P^{BS}(t+\Delta, e^{Z_t + X_{t+\Delta}}) - P^{BS}(t, e^{Z_t}))]$

② For linear:

<i> $e^{Z_t + X_{t+\Delta}} = e^{Z_t} + e^{Z_t} \cdot X_{t+\Delta}$

<ii> $P^{BS}(t+\Delta, e^{Z_t + X_{t+\Delta}}) = P^{BS}(t, e^{Z_t}) + \partial_{Z_t} P^{BS}(t, e^{Z_t}) \cdot X_{t+\Delta}$

$= P^{BS}(t, e^{Z_t}) + e^{Z_t} \cdot \delta(t, e^{Z_t}) \cdot X_{t+\Delta} + \theta(t, e^{Z_t}) \cdot \Delta$

$\Rightarrow L^{Lin}_{t+\Delta} = -M[\lambda e^{Z_t} X_{t+\Delta} - e^{Z_t} \cdot \lambda X_{t+\Delta} - \theta(t, S_t) \cdot \Delta)$

$= M \cdot \theta(t, e^{Z_t}) \cdot \Delta$

③ For Second:

(i) $e^{Z_t + X_{t+\Delta}} = e^{Z_t} + e^{Z_t} \cdot X_{t+\Delta} + \frac{1}{2} e^{Z_t} \cdot X_{t+\Delta}^2$

(ii) $P^{BS}(t+\Delta, e^{Z_t + X_{t+\Delta}}) = P^{BS}(t, e^{Z_t}) + \lambda e^{Z_t} \cdot X_{t+\Delta} + \theta(t, e^{Z_t}) \cdot \Delta$

$+ \frac{1}{2} \underbrace{\partial_z^2 P^{BS}(t, e^{Z_t})}_{(iii)} \cdot X_{t+\Delta}^2$

(iii) $\partial_{Z_t} \delta(t, e^{Z_t}) \cdot e^{Z_t} = \lambda e^{Z_t} + e^{2Z_t} \cdot \gamma(t, e^{Z_t})$

$\Rightarrow$ For Second:

$L^{Qua}_{t+\Delta} = -M[\frac{1}{2} \lambda e^{Z_t} \cdot X_{t+\Delta}^2 - \theta(t, e^{Z_t}) \cdot \Delta - \frac{1}{2}(\lambda e^{Z_t} + e^{2Z_t} \cdot \gamma(t, e^{Z_t})) \cdot X_{t+\Delta}^2]$

$= M[\theta(t, e^{Z_t}) \cdot \Delta + \frac{1}{2} e^{2Z_t} \gamma(t, e^{Z_t}) \cdot X_{t+\Delta}^2]$

2.(a) $f(l) = \begin{cases} \frac{ab}{ae^{-bl_0} + be^{al_0}} e^{al} & (l \le l_0) \\ \frac{ab}{ae^{-bl_0} + be^{al_0}} e^{-bl} & (l > l_0) \end{cases} \xrightarrow{CDF} F(l) = \begin{cases} \frac{be^{al}}{ae^{-bl_0} + be^{al_0}} & (l \le l_0) & [>0] \\ \frac{-a e^{-bl}}{ae^{-bl_0} + be^{al_0}} & (l > l_0) & [<0] \end{cases}$

$F(l) = \alpha \rightarrow F^{-1}(\alpha) = \begin{cases} \ln[\frac{(ae^{-bl_0} + be^{al_0})\alpha}{b}] \frac{1}{a} & (l \le l_0) \\ -\ln[\frac{(a^{-bl_0} + be^{al_0})\alpha}{-a}] \frac{1}{b} & (l > l_0) \end{cases}$

$\because \alpha \geq \frac{be^{al_0}}{be^{al_0} + ae^{-bl_0}} \Rightarrow VaR = F^{-1}(\alpha) = -\ln(\frac{1}{a} e^{al_0}(b + ae^{-(a+b)l_0})(1-\alpha)) \frac{1}{b}$

(b) For $L \sim (6, \frac{1}{2})$, $\alpha = 0.9$

Then the CDF of $L = F(L) = \sum_{i=0}^{\lfloor x \rfloor} \binom{6}{i} (\frac{1}{2})^6 = \sum_{i=0}^{\lfloor x \rfloor} C_6^i (\frac{1}{2})^6$

$⓪ = C_6^0 \cdot (\frac{1}{2})^6 = 0.015625 < 0.9$

$① = C_6^0 \cdot (\frac{1}{2})^6 + C_6^1 \cdot (\frac{1}{2})^6 = 0.109375 < 0.9$

$② = ① + C_6^2 \cdot (\frac{1}{2})^6 = 0.34375 < 0.9$

$③ = ② + C_6^3 \cdot (\frac{1}{2})^6 = 0.65625 < 0.9$

$④ = ③ + C_6^4 \cdot (\frac{1}{2})^6 = 0.890625 < 0.9$

$⑤ = ④ + C_6^5 \cdot (\frac{1}{2})^6 = 0.984375 > 0.9$

$\therefore VaR(F)_\alpha = \inf [F(L) \geq 0.9] = 5$

(c) PDF of Y and Z: $Y = \lambda e^{-\lambda x}, x \geq 0$

$Z = \theta e^{-\theta y}, y \geq 0$

For $L = \frac{Y}{Z}$:

$$F(L) = P(L \leq l)$$
$$= P(\frac{Y}{Z} \leq l) = \int_{y=0}^{\infty} \int_{x=0}^{ly} \lambda \theta e^{-\lambda x} e^{-\theta y} \, dx \, dy$$
$$= 1 - \frac{\theta}{\theta + \lambda l}$$
$$VaR(F)_\alpha = F^{-1}(\alpha) = \frac{\alpha \theta}{\lambda - \alpha \lambda}.$$

PS: For 1.(b) see the attached code file [MF731 HW1.2.ipynb]