HW Problems for Assignment 3
Due 6:30 PM Tuesday, October 19, 2021
SOLUTIONS

1. (10 Points) The "Exponential Premium Principal" Risk Measure. In this exercise we introduce another risk measure, which shares certain properties with Value at Risk and Expected Shortfall, but which is not a coherent risk measure. Namely, fix a parameter $\alpha > 0$ and, for a random variable L define the "exponential premium principal" risk measure with parameter α by the formula

$$\varrho_\alpha(L) := \frac{1}{\alpha}\log\left(E\left[e^{\alpha L}\right]\right).$$

(a) Show that ϱ_α is a convex risk measure. To show convexity, use Holder's inequality.
(b) By considering $L \sim N(0,1)$ show that ϱ_α is not positive homogenous, and hence not coherent.

Solution:

(a) It is clear that ϱ_α is monotonic. As for cash additivity, we have

$$\begin{aligned}\varrho_\alpha(L-c) &= \frac{1}{\alpha}\log\left(E\left[e^{\alpha(L-c)}\right]\right) = \frac{1}{\alpha}\log\left(e^{-\alpha c}E\left[e^{\alpha L}\right]\right)\\ &= -c + \frac{1}{\alpha}\log\left(E\left[e^{\alpha L}\right]\right) = \varrho_\alpha(L) - c.\end{aligned}$$

Lastly, for convexity, let $\lambda \in (0,1)$. We have

$$\begin{aligned}\varrho_\alpha\left(\lambda L_1 + (1-\lambda)L_2\right) &= \frac{1}{\alpha}\log\left(E\left[e^{\alpha(\lambda L_1 + (1-\lambda)L_2)}\right]\right)\\ &= \frac{1}{\alpha}\log\left(E\left[e^{\alpha\lambda L_1}\times e^{\alpha(1-\lambda)L_2}\right]\right)\end{aligned}$$

We now use Holder's inequality which states that if $p, q > 1$ are such that $1/p + 1/q = 1$ and X, Y are non-negative random variables then $E[XY] \le E[X^p]^{1/p}\, E[Y^q]^{1/q}$. Here, we take $p = 1/\lambda, q = 1/(1-\lambda)$, $X = e^{\alpha\lambda L_1}$ and $Y = e^{\alpha(1-\lambda)L_2}$. This gives

$$\begin{aligned}&\varrho_\alpha\left(\lambda L_1 + (1-\lambda)L_2\right)\\ &\quad\le \frac{1}{\alpha}\log\left(E\left[e^{\alpha L_1}\right]^{\lambda}\times E\left[e^{\alpha L_2}\right]^{1-\lambda}\right),\\ &\quad= \lambda\frac{1}{\alpha}\log\left(E\left[e^{\alpha L_1}\right]\right) + (1-\lambda)\frac{1}{\alpha}\log\left(E\left[e^{\alpha L_2}\right]\right),\\ &\quad= \lambda\varrho_\alpha(L_1) + (1-\lambda)\varrho_\alpha(L_2).\end{aligned}$$

Convexity follows.

(b) When $L \sim N(0,1)$ we have, for $\lambda > 0$ that

$$\varrho_\alpha(\lambda L) = \frac{1}{\alpha}\log\left(E\left[e^{\alpha\lambda L}\right]\right) = \frac{1}{2}\alpha\lambda^2.$$

As this is not linear in λ positive homogeneity fails.

2. (25 Points) Component risk measures for an equity portfolio. In this exercise, we will compute component risk measures for an equally weighted portfolio of five stocks: Walmart, Target, Costco, Citigroup and JP Morgan. Closing price data is in the file "Five_Stock_Prices.csv". In the file, the first column is the date (in Excel numeric format) while columns 2-6 give the stock price data. Data is sorted oldest to newest.

The hypothetical portfolio is fixed at 15 million dollars, and allocates 20% in each stock. As such the respective dollar positions are kept constant throughout time at $3 million in each stock.

We use linearized losses and normally distributed log returns, with EWMA updating. As before, we use the oldest M returns to obtain an estimate for μ, Σ, and then update there-after according to the EWMA procedure.

In this setting, compute on a daily basis throughout time the percent component ϱ for ϱ equal to

(1) Value at Risk.
(2) A spectral risk measure with exponential weighting function $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1} e^{\gamma u}, 0 \leq u \leq 1$.

Additionally, compute the percent contribution to the loss variance across time.

Begin your calculations after the first M periods so, with N log returns, you will have $N + 1 - M$ component risk measure estimates.

Note that for each day, the above quantities are vectors with five components (one for each stock). Output a time evolution plot for each of the above values, over the range $t - (N - M)\Delta, \dots, t$. Each plot will have five graphs.

In addition to the above portfolio value and weights, use $M = 50$ trail days, a VaR confidence of $\alpha = .99$, a spectral risk aversion of $\gamma = 25$, and EWMA parameters of $\lambda = .94$, $\theta = .96$.

As we discussed in class, the component risk measure plots should be very close to the variance contribution plot. Is this the case?

Solution: See the Matlab file "Risk_Measure_Components.m".

3. Spherical and elliptical random variables. This exercise shows that many of the conclusions on component risk measures, and risk-measure based optimal investment, extend beyond the Gaussian setting.

We say a random vector $Z \in \mathbb{R}^d$ is *spherical* if for all $a \in \mathbb{R}^d$ the random variables $a^{\mathrm{T}}Z$ and $|a|Z^{(1)}$ have the same distribution, where $Z^{(1)}$ is the first component of Z. In other words $\mathbb{P}\left[a^{\mathrm{T}}Z \le \tau\right] = \mathbb{P}\left[|a|Z^{(1)} \le \tau\right]$ for all $\tau \in \mathbb{R}$. For more information on spherical random variables, see Chapter 6.3 of the class textbook.

We say a random vector $X \in \mathbb{R}^d$ is *elliptical* if $X = \mu + AZ$ where $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and Z is spherical. Lastly, denote by $\Sigma = AA^{\mathrm{T}}$.

(a) (**5 Points**) Assume $Z = \sqrt{W} \times \widetilde{Z}$ where $\widetilde{Z} \sim N(0, 1_d)$, and $W \ge 0$ is a random scalar independent of $\widetilde{Z}$. Show that Z is spherical. In particular, taking $W \equiv 1$ shows that $\widetilde{Z}$ is spherical.

(b) Assume a one period model with d stocks, whose log returns X are elliptically distributed. Write θ as the vector of dollar positions.

  (i) (**5 Points**) Show that the linearized losses L^{lin} have the same distribution as
$$-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times Z^{(1)},$$
  and hence for any cash-additive, positively homogenous risk measure ϱ
$$\varrho(L^{lin}) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times \varrho(Z^{(1)}).$$

  (ii) (**5 Points**) Argue why part (i) means we can make the same conclusions connecting $\mathcal{R}^{'}_{\mathrm{C},\%}$ to the percentage contribution to variance, as we did when $X \sim N(\mu, \Sigma)$.

Solution:

(a) The characteristic function of $a^{\mathrm{T}}Z$ is
$$E\left[e^{i\gamma a^{\mathrm{T}}Z}\right] = E\left[E\left[e^{i\gamma\sqrt{W}a^{\mathrm{T}}\widetilde{Z}} \mid W\right]\right] = E\left[e^{-\frac{1}{2}\gamma^2 W|a|^2}\right],$$
where we have used the independence of W and $\widetilde{Z}$. Similarly, the characteristic function of $|a|Z^{(1)}$ is (since $\widetilde{Z}^{(1)} \sim N(0,1)$)
$$E\left[e^{i\gamma|a|Z^{(1)}}\right] = E\left[E\left[e^{i\gamma|a|\sqrt{W}\widetilde{Z}^{(1)}} \mid W\right]\right] = E\left[e^{-\frac{1}{2}\gamma^2 W|a|^2}\right].$$
Since the characteristic functions are the same, the distributions coincide.

(b) (i) We know $L^{lin} = -\theta^{\mathrm{T}}X = -\theta^{\mathrm{T}}\mu - \theta^{\mathrm{T}}AZ$, where Z is spherical. Now, write $a = -A^{\mathrm{T}}\theta$. Since Z is spherical, the distribution of $a^{\mathrm{T}}Z$ coincides with that of $|a|Z^{(1)}$. But
$$|a| = \sqrt{|a|^2} = \sqrt{a^{\mathrm{T}}a} = \sqrt{\theta^{\mathrm{T}}AA^{\mathrm{T}}\theta} = \sqrt{\theta^{\mathrm{T}}\Sigma\theta}.$$
Thus, L^{lin} has the same distribution as $-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}Z^{(1)}$. The result about $\varrho(L^{lin})$ is immediate from cash-additivity and positive homogeneity.

(ii) From part (i) we conclude

$$\mathcal{R}_{\mathrm{M}}^{lin,(i)}(\theta) = -\mu^{(i)} + \frac{(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}} \times \varrho(Z^{(1)}),$$

so

$$\mathcal{R}_{\mathrm{C},\%}^{lin,(i)}(\theta) = 100 \times \left(\frac{-\theta^{(i)}\mu^{(i)} + \frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}}\varrho(Z^{(1)})}{-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}\varrho(Z^{(1)})} \right).$$

Thus, if $\mu = 0$ then the percent component risk measure does not depend upon ϱ, and in fact just measures the contribution to portfolio variance.