HW Problems for Assignment 1 - Lecture 2
Due 6:30 PM Tuesday, September 21, 2021

1. Loss Distributions for a Hedged Put Option. As in the Black-Scholes model, assume the stock price has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $W = \{W_t\}_{t \leq T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate is $r > 0$. Let T be the maturity and K the strike of a put option, and set $P^{BS}(t,x)$ as the price of the call given $S_t = x$. I.e.

$$P^{BS}(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(K - S_T\right)^+ \middle| S_t = x\right]. \tag{0.1}$$

where $\mathbb{Q}$ is the risk neutral measure under which S has drift r. The Black-Scholes formula states (you DO NOT have to prove this)

$$P^{BS}(t,x) = x(N(d_1(T-t,x)) - 1) + Ke^{-r(T-t)}(1 - N(d_2(T-t,x))),$$

where N is the standard normal cdf and

$$\begin{aligned}
d_1(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(\tau)\right),\\
d_2(\tau,x) &= d_1 - \sigma\sqrt{\tau}.
\end{aligned}$$

Furthermore, with ϕ denoting the standard normal pdf we have

$$\begin{aligned}
\delta(t,x) &= \partial_x P^{BS}(t,x) = N(d_1) - 1,\\
\gamma(t,x) &= \partial_{xx} P^{BS}(t,x) = \frac{\phi(d_1(T-t,x))}{x\sigma\sqrt{T-t}},\\
\theta(t,x) &= \partial_t P^{BS}(t,x) = -\frac{\sigma}{2\sqrt{T-t}}x\phi(d_1(T-t,x))\\
&\qquad + Kre^{-r(T-t)}(1 - N(d_2(T-t,x))).
\end{aligned}$$

These are the "delta", "gamma" and "theta" respectively for the option.

At time t we are short M put options and long $M\delta(t,S_t)$ shares of S. Over the period $[t, t+\Delta]$ we hold the share position constant, writing $\lambda = \delta(t,S_t)$ to reinforce this fact. With this notation, the values of our portfolio at t and $t+\Delta$ are

$$\begin{aligned}
V_t &= M\left(\lambda S_t - P^{BS}(t,S_t)\right),\\
V_{t+\Delta} &= M\left(\lambda S_{t+\Delta} - P^{BS}(t+\Delta, S_{t+\Delta})\right).
\end{aligned}$$

(a) **(15 Points)** With $z_t = \ln(S_t)$, identify the full, linearized, and second order loss operators over $[t, t+\Delta]$ as a function of the log return $x = X_{t+\Delta}$. **Notes:**

(i) Make sure to fully evaluate the linearized loss operator - there is a cool answer!.
(ii) For the second order loss operator, only include the second derivative with respect to x: i.e. ignore the second order time derivative and second order time-space derivative.

(b) **(15 Points)** Write a simulation which identifies the loss distribution for the portfolio using the full, linearized and second order (with the adjustments in note (ii)) loss operators. As in class, produce a histogram approximation of the probability density functions. How well do the approximations work?

For parameters use $\mu = 0.16905$, $\sigma = 0.4907$, $r = 0.0011888$, $t = 0$, $T = .291667$, $\Delta = 10/252$ (ten day horizon), $S_0 = 152.51$, $K = 170$ and $M = 100$ options. Run $N = 100,000$ trials in your simulation.

2. Practice with VaR. Explicitly compute $\mathrm{VaR}_\alpha(L)$ assuming L has the following distributions/probability distribution functions (pdfs).

(a) **(7 Points)** L is a "double-sided" exponential with threshold l_0 in that L has pdf

$$f(l) = \frac{ab}{ae^{-bl_0} + be^{al_0}} \left(e^{al} 1_{l \leq l_0} + e^{-bl} 1_{l > l_0} \right); \qquad l \in \mathbb{R},$$

where $a, b > 0$. Here, you may assume $\alpha \geq b/(b + ae^{-(a+b)l_0})$.

(b) **(6 Points)** L is a binomial random variable with n number of trials and p probability of success on any given trial. Give an explicit answer when $n = 6$, $p = 1/2$ and $\alpha = .9$.

(c) **(7 Points)** $L = Y/Z$ where Y and Z are independent exponential random variables with means $1/\lambda$ (for Y) and $1/\theta$ (for Z).