Consider an autonomous continuous time chaotic system-

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

An input signal $u \in \mathbb{R}$ is used to stabilize the system / synchronize the master slave system. The control is injected with a vector field $g(x)$.

$$\dot{x} = f(x) + g(x) \cdot u$$

Suppose there is a stabilizing state feedback $\gamma(x)$ such that the closed loop

$$\dot{x} = f(x) + g(x) \cdot \gamma(x) \qquad \cdots (1)$$

is asymmetric stable.

Converse of Lyapunov stability theorem:

there exists a C^{∞} Lyapunov function $V(x)$ and class K function $\alpha_i \in K$ $i \in (1, 2, 3)$ such that

(i) $\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$ - bounded

(ii) $V(x) \le -\alpha_3(\|x\|)$ - converge with negative rate

Choose a nonlinear sliding surface of the form:

$$\sigma(x) = \frac{dV(x)}{dt} g(x) = 0$$

the following relations hold for the system (1)

(1) $$\dot{V}(x) = \frac{dV(x)}{dt} \cdot f(x) + \frac{dV(x)}{dt} \cdot g(x) \cdot \gamma(x)$$

(2) $$\dot{V}(x) = \frac{dV(x)}{dt} \cdot f(x)$$

(3) $$\dot{V}(x) \le -\alpha_3(\|x\|)$$

choose the feedback control to ensure the attractivity of the surface and guarantee the sliding behaviour.

→ the necessary condition $\sigma \cdot \dot{\sigma} < 0$ must be satisfied.

1) choose $\dot{\sigma} = -w_1 \sigma - w_2 \operatorname{sgn}(\sigma)$, $w_1 > 0$ & $w_2 > 0$

$$\rightarrow \sigma \dot{\sigma} = \sigma(-w_1 \sigma - w_2 \operatorname{sgn}(\sigma))$$

$$= -w_1 \sigma^2 - w_2 (\sigma) < 0$$

Knowing that on the sliding surface

$$\dot{\sigma} = \frac{d\sigma}{dx} \cdot f(x) + \frac{d\sigma}{dx} \cdot g(x) \cdot \boxed{u} = -w_1 \sigma - w_2 \operatorname{sgn}(\sigma)$$

$$u = \frac{-w_1 \sigma - w_2 \operatorname{sgn}(\sigma) - \frac{d\sigma}{dx} \cdot f(x)}{\frac{d\sigma}{dx} \cdot g(x)}$$

the state will be attracted to the surface $\sigma = 0$, and slide along $\sigma(x) = 0$ to the equilibrium point.

<u>Robustness</u>

Consider $\dot{x} = f(x) + \Delta f(x) + g(x) \cdot u$

→ prove the attractivity and sliding condition $\sigma \cdot \dot{\sigma} < 0$ still applies.

Recall

$$\dot{\sigma} = \frac{d\sigma}{dx} \cdot \tilde{f}(x) + \frac{d\sigma}{dx} \cdot g(x) \cdot u$$

$$\rightarrow \dot{\sigma} = \frac{d\sigma}{dx} \cdot f(x) + \frac{d\sigma}{dx} \cdot \Delta f(x) + \boxed{\frac{d\sigma}{dx} \cdot g(x) \cdot u}$$

Consider

$$u = \frac{-w_1 \cdot \sigma - w_2 \operatorname{sgn}(\sigma) - \frac{d\sigma}{dx} \cdot f(x)}{\frac{d\sigma}{dx} \cdot g(x)}$$

$$\rightarrow \dot{\sigma} = \frac{d\sigma}{dx}\cdot f(x) + \frac{d\sigma}{dx}\cdot \Delta f(x) + \left(-w_1\cdot\sigma - w_2\,\mathrm{sgn}(\sigma) - \frac{d\sigma}{dx}\cdot f(x)\right)$$

$$= -w_1\sigma - w_2\,\mathrm{sgn}(\sigma) + \frac{d\sigma}{dx}\cdot \Delta f(x)$$

Let $\quad w_2 > \delta_2 > \left\| \frac{d\sigma}{dx}\cdot \Delta f(x) \right\|$

$$\begin{aligned}
\sigma\dot{\sigma} &= \sigma\left(-w_1\sigma - w_2\,\mathrm{sgn}(\sigma) + \frac{d\sigma}{dx}\cdot \Delta f(x)\right) \\
&= -w_1\sigma^2 - w_2(\sigma) + \sigma\cdot\frac{d\sigma}{dx}\cdot \Delta f(x) \\
&\leq -w_1\sigma^2 - w_2|\sigma| + \sigma\cdot\delta_2 \\
&\leq -w_1\sigma^2 - \underbrace{(w_2 - \delta_2)}_{w_2-\delta_2>0}|\sigma|
\end{aligned}$$

$\therefore \quad \sigma\dot{\sigma} \leq 0$

Consider a slave system

$$\begin{cases} \dot{x} = f(x) + g(x)\cdot u \\ y = h(x) \end{cases}$$

to achieve synchronization, $y_m = y$, or $y_m - y = 0$

consider a nonlinear transformation $z = \phi(x)$ such that

$$\begin{cases} z_1 = h(x) \\ z_2 = L_f^1 h(x) \\ z_3 = L_f^2 h(x) \\ \vdots \\ z_n = L_f^{n-1} h(x) \end{cases}$$

Using z as the new variable,

$$\begin{cases} \dot z_i = z_{i+1} & i = 1, 2, \dots, n-1 \\ \dot z_n = F(z) + G(z)u \end{cases}$$

$z_1 = h(x)$

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$\dot z_1 = \dot h(x) = \mathcal{L}_f^1 h(x)$

$z_i = \mathcal{L}_f^{i-1} h(x)$

$\dot z_i = z_{i+1} = \mathcal{L}_f^{i-1} \cdot \mathcal{L}_f h(x)$

$= \mathcal{L}_f^i h(x)$

with $y = z_1$ and

$$F(z) = \mathcal{L}_f^n h(\underbrace{\phi^{-1}(z)}_{x})$$

$$G(z) = \mathcal{L}_g \cdot \mathcal{L}_f^n h(\phi^{-1}(z))$$

Define tracking error $e = Y_m - z$

$$Y_m = \begin{pmatrix} y_m \\ \dot y_m \\ \vdots \\ y_m^{(n-1)} \end{pmatrix}_{n \times 1} \quad z = \begin{pmatrix} y = z_1 \\ z_2 = \dot z_1 \\ z_3 = \dot z_2 \\ \vdots \\ z_{n-1} = \dot z_{n-2} \end{pmatrix}_{n \times 1}$$

the error dynamics are -

$$\dot e_i = e_{i+1}$$

$$\dot e_n = y_m^{(n)} + F(\overbrace{Y_m - e}^{z}) + G(Y_m - e)u$$

Smooth controller : $\gamma(e) = \frac{1}{G(z)} \cdot (-F(z) + y_m^{(n)} + K^T e)$

where $K^T = (k_1 \ k_2 \ \dots \ k_n)$ with closed loop $\dot e = Ae$

$$\underbrace{\begin{pmatrix} \dot e_1 \\ \dot e_2 \\ \vdots \\ \dot e_n \end{pmatrix}}_{\dot e} = \underbrace{\begin{pmatrix} 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}}_{e}$$

Characteristics eqt:

$$p(s) = s^n + k_n s^{n-1} + k_{n-1} s^{n-2} + \dots + k_2 s + k_1$$

should have negative real parts. Indeed we have

$$\lim_{t\to\infty} e(t) = 0 \iff \lim_{t\to\infty} z(t) = Y_m$$

Taking the inverse transformation: $z = \phi(x) \to x = \phi^{-1}(z)$

$$\lim_{t\to\infty} z(t) = Y_m \to \phi^{-1}(Y_m) = X_m$$

$$\phi^{-1}(z) = x$$

$\Rightarrow \lim_{t\to\infty} x(t) = X_m$ synchronization achieved.

Recall $\dot{e} = Ae \to$ linear system $\to$ use Lyapunov Function

$V = e^T P e$ $P > 0$ positive definite

$\to \dot{V} = e^T Q e$ $Q > 0$

Construct the sliding surface:

$$\sigma_e = -\frac{dV(e)}{de} \cdot G(\overbrace{Y_m - e}^{z}) = 0$$

Recall

$$u(x) = \frac{-\frac{d\sigma}{dx} f(x) - w_1 \sigma(x) - w_2 \operatorname{sgn}(\sigma)}{\frac{d\sigma}{dx} g(x)}$$

$$\to u(e) = \frac{-\frac{d\sigma_e}{de} F(e) - w_1 \sigma_e - w_2 \operatorname{sgn}(\sigma_e)}{\frac{d\sigma}{de} G(e)}$$

with $F(e) = \begin{pmatrix} e_2 \\ e_3 \\ \vdots \\ e_n \\ y_m^{(n)} - F(Y_m - e) \end{pmatrix}$ $G(e) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -G(Y_m - e) \end{pmatrix}$

u(e) will force the error dynamics slides on the surface $\sigma_e = 0$ and converge to the equilibrium point with $e = 0$.

Rossler system –

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) + u \end{cases} \rightarrow \dot{x} = f(x) + g(x)u$$

Fixed Point / Equilibrium Point – put all 3 eqts into a

$$\begin{cases} 0 = -x_2 - x_3 \\ 0 = x_1 + a x_2 \\ 0 = b + x_3(x_1 - c) + u \end{cases} \rightarrow \begin{aligned} x_1 &= \frac{c \pm \sqrt{c^2 - 4ab}}{2} \\ x_2 &= -\left(\frac{c \pm \sqrt{c^2 - 4ab}}{2a}\right) \\ x_3 &= \frac{c \pm \sqrt{c^2 - 4ab}}{2a} \end{aligned}$$

$\#\ x_{eq}^{1} = \frac{c + \sqrt{c^2 - 4ab}}{2} \begin{pmatrix} 1 \\ -\frac{1}{a} \\ \frac{1}{a} \end{pmatrix}$

$\downarrow$

$$\det \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix}$$

Put $a = b = 0.2$, $c = 5.7$:

$\boxed{x_1, x_2 = 0.0980 \pm j\,0.9551}$; $x_3 = -6.9905$

unstable pole

$$V_1 = \begin{pmatrix} 0.072 \\ -0.0721 - j0.7033 \\ 0.0028 - j0.0004 \end{pmatrix}$$ unstable manifold.

$$V_2 = \begin{pmatrix} 0.072 \\ -0.0721 - j\,0.7033 \\ 0.0028 - j\,0.0004 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0.1389 \\ -0.0193 \\ 0.0901 \end{pmatrix}$$

Choose V_{31} and V_{32} with reference to the stable V_3 and normal to σ_1 and σ_2 –

$$\sigma_1(x) = V_{31}^T (x - x_{eq}^{1})$$
$$\sigma_2(x) = V_{32}^T (x - x_{eq}^{1})$$

Let $V_i(j)$ = jth component of V_i.

choose $\quad V_{31} = \left(-\frac{1}{V_3(1)}, -\frac{1}{V_3(2)}, 0\right)^T$ and

$$V_{32} = \left(\frac{1}{V_3(2)}, \frac{1}{V_3(1)}, -\frac{V_3^2(1)+V_3^2(2)}{V_3(1)\,V_3(2)\,V_3(3)}\right)$$

Consider the second order sliding mode with $\ddot{\sigma}_1 \rightarrow$ to reach sliding mode $\sigma_1 = 0$ and $\dot{\sigma}_1 = 0$

$$\ddot{\sigma}_1 = V_{31}(1)\cdot \underbrace{(-\dot{x}_2 - \dot{x}_3)}_{=\ddot{x}_1} + V_{32}(2)\cdot \underbrace{(\dot{x}_1 + a\dot{x}_2)}_{\ddot{x}_2} + 0\,(\ldots)$$

$$= \boxed{V_{31}(1)}\cdot\left[-(x_1+ax_2) - (b - x_3(x_1-c)\boxed{-u})\right)$$

$$+ V_{32}(1)\cdot\left(-(x_2+x_3) + a(x_1+ax_2)\right)$$

$$= V_{31}(1)\left(-(x_1+ax_2) - (b - x_3(x_1-c)\right) + V_{32}(1)\cdot\left(-(x_2+x_3) + a(x_1+ax_2)\right)$$

$$+ \boxed{V_{31}(1)\cdot u}$$

$$= A(x) + \boxed{B(x)\cdot u}$$

Apply

$$u = \frac{-A(x) - w_{11}\,\sigma_1(x) - w_{12}\,\mathrm{sgn}(\sigma_1(x)) - w_{21}\,\dot{\sigma}_1(x) - w_{22}\,\mathrm{sgn}(\dot{\sigma}_1(x))}{B(x)}$$

to make σ_1 attractive, and

$$u = \frac{-\frac{d\sigma}{dx}f(x) - w_1\,\sigma_2(x) - w_2\,\mathrm{sgn}(x)}{\frac{d\sigma}{dx}g(x)}$$

to make σ_2 attractive.

new sliding mode control with nonlinear transformation.-

Select $h(x) = x_2$. Define state transformation:

$$\phi(x) = z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + a x_2 \\ a x_1 + (a^2 - 1) x_2 - x_3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ F(z) + G(z) u \end{pmatrix},$$ where $G(z) = -1$

Recall
$$\dot z_1 = z_2$$
$$\dot z_2 = z_3$$
$$\dot z_3 = F(z) + G(z) u$$

Controller: $\gamma(e) = \frac{1}{G(z)} \left(-F(z) + y_m^{(n)} + k^T e\right)$

→ leading to close loop system $\dot e = Ae$

$$\begin{pmatrix} \dot e_1 \\ \dot e_2 \\ \dot e_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_3 & -k_2 & -k_1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \rightarrow \text{linear}$$

$$p(s) = s^3 + k_3 s^2 + k_2 s + k_1 = 0$$

→ construct a sliding surface:

$$\sigma_e = \frac{d\sigma(e)}{de} \cdot G(Y_m - e) \text{ and}$$

and the corresponding input:

$$\boxed{u = \frac{-\frac{d\sigma(e)}{de} F(z) - w_1 \sigma - w_2 \,\mathrm{sgn}(\sigma)}{\frac{d\sigma(e)}{de} G(z)}}$$

σ and $\frac{d\sigma}{de}$ independent of system parameter.

and $F(z) = \begin{pmatrix} e_2 \\ e_3 \\ y_m^{(n)} - F(Y_m - e) \end{pmatrix}$ $\quad G(z) = \begin{pmatrix} 0 \\ 0 \\ -G(Y_m - e) \end{pmatrix}$

Hence, select $K = \begin{pmatrix} 8 \\ 12 \\ 6 \end{pmatrix}$ and choose $V = e^T P e$ with $P = \frac{1}{512}\begin{pmatrix} 4880 & 3160 & 160 \\ & 5150 & 360 \\ & & 275 \end{pmatrix} > 0$

and $\dot{V} = -5\|e\|^2 < 0$

the sliding surface becomes $\sigma_e = \frac{-1}{256}\left(160e_1 + 360e_2 + 270e_3\right)$

\#

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) + u \end{cases}$$

$\dot{x} = F(x) + G(x) \cdot u$

$\rightarrow G(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\rightarrow \sigma_e = -\frac{dV(e)}{de} G(y) = 0$

$V = e^T P e$

$\frac{dV}{de} = e^T (P + P^T) \rightarrow \because P$ is symmetric

$\rightarrow \frac{dV}{de} = 2e^T P$

$" -\frac{dV(e)}{de} G(y)$

$\rightarrow \sigma_e = -2e^T P \cdot G(y)$

$$= -2 \cdot (e_1\ e_2\ e_3) \frac{1}{512}\begin{pmatrix} 4880 & 3160 & \boxed{160} \\ 3160 & 5360 & \boxed{370} \\ 160 & 370 & \boxed{275} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{256}(e_1\ e_2\ e_3)\begin{pmatrix} 160 \\ 370 \\ 275 \end{pmatrix}$$

$$= -\frac{160e_1 + 370e_2 + 275e_3}{256}$$

note: there was an important assumption -

fixing at high level leads to unwanted chattering

$\left\| \frac{d\sigma_e}{de} \cdot F(e) \right\|_1 < \boxed{W_2}$ bounded for $\quad - \quad u(e) = \frac{-W_1\sigma_e - W_2\,\mathrm{sgn}(\sigma_e)}{\frac{d\sigma_e}{de} G(e)}$

→ Adapted sliding gain - $\omega_2(t) > \left\| \frac{d\sigma_e}{dt} F(e) \right\|$

Consider the system $\dot{\sigma}_e = \frac{d\sigma_e}{dt} \cdot F(e) + \frac{d\sigma_e}{dt} G(e) \cdot u$

with adaptive linear controller -

$$u_{lin}(\sigma_e) = \hat{K}(t)^T \sigma_e(t)$$

where $\hat{K}(t)$ is adjusted adaptively such that

$$u^*_{lin}(t) = \underbrace{\hat{K}(t)^{*T}}_{\text{optimal gain vector}} \sigma_e(t) = u_{io}(t)$$

\# $u_{io}(t)$ = input-output linearizing controller

$$= \frac{1}{G(z)}\left(-F(z) + y_m^{(n)} + K^T e\right)$$

\# Recall the system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \xrightarrow{z = \phi(x)} \begin{cases} \dot{z}_i = z_{i+1} \quad i = 1, 2, \dots, n-1 \\ \boxed{\dot{z}_n = F(z) + G(z)u} \\ y = z_1 \end{cases}$$

nonlinear continuous-time dynamics → linearized state-space

→ $\dot{z}_n = F(z) + G(z)u$ where $u_{lin} = \hat{K}^T \sigma_e$

$$= F(z) + G(z)u_{lin} - G(z)u_{io} \boxed{+ G(z)u_{io}}$$

$$= \cancel{F(z)} + G(z)u_{lin} \boxed{-\cancel{F(z)} + y_m^{(n)} + K^T e} - G(z)u_{io}$$

$$= y_m^{(n)} + K^T e + G(z)(u_{lin} - u_{io})$$

define $B_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ G(z) \end{pmatrix} \rightarrow$

$$\dot{e} = A_c e + B_c (u_{io} - u_{lim})$$

$$\dot{e} = A_c e + B_c (u^*_{lim} - u_{lim}) \quad \cdots (9)$$

$$\dot{e} = A_c e + B_c (\tilde{k}^* - \tilde{k})^T e$$

Consider the state space system

$$\begin{cases} \dot{z}_i = z_{i+1} \\ \dot{z}_n = F(z) + G(z) u \\ y = z_1 \end{cases}$$

Assume z is measurable and y_m is a smooth trajectory to be tracked.

Let $\tilde{k}(t)$ be the solution of $\dot{\tilde{k}} = \gamma e^T P B_c e$

with $\tilde{k}(0)$ is a Hurwitz vector and $\gamma > 0$.

Let $P > 0$ be the positive definite solution of $A_c^T P + P A_c = -Q$ $\quad Q > 0$

then the control law $u_{lim}(t) = \tilde{k}(t)^T e(t)$ leads to asymptotic tracking.

$$V(e, \tilde{k}) = e^T P e + \frac{1}{\gamma} (\tilde{k}^* - \tilde{k})^T (\tilde{k}^* - \tilde{k})$$

$\alpha = y^T x$

$\frac{\partial \alpha}{\partial z} = x^T \frac{\partial y}{\partial z} + y^T \frac{\partial x}{\partial z}$

$$\dot{V}(e, \tilde{k}) = e^T (A_c^T P + P A_c) e + 2 e^T P B_c (\tilde{k}^* - \tilde{k})^T e - \frac{2}{\gamma} (\tilde{k}^* - \tilde{k})^T \dot{\tilde{k}}$$

$$= -e^T Q e - 2 (\tilde{k}^* - \tilde{k})^T \left(\frac{\dot{\tilde{k}}}{\gamma} - e^T P B_c e \right)$$

$\rightarrow$ choose $\frac{\dot{\tilde{k}}}{\gamma} - e^T P B_c e = 0 \rightarrow \dot{\tilde{k}} = \gamma e^T P B_c e \rightarrow \dot{V} = -e^T Q e$

→ $\dot{V}(e, \tilde{K}) = -e^T Q e$

$\leq -\lambda_{min}(Q)\|e\|^2$

min eigenvalue at Q. → system(5) is Lyapunov stable

→ integrating : $V(t) \leq V(0) - \lambda_{min}(Q)\int_0^t e^T e \, dt$

Barbalat's Lemma -

if $f(t)$ has finite limit as $t \to \infty$

and $\dot{f}(t)$ is uniformly continuous ($\ddot{f}(t)$ is bounded.

then $\dot{f}(t) \to 0$ as $t \to \infty$

$$\underbrace{V(t)}_{} \leq \underbrace{V(0)}_{\text{finite}} - \lambda_{min}(Q) \underbrace{\int_0^t e^T e \, dt}_{}$$

if $\int_0^t e^T e \, dt$ has finite limit and $e^T e$

is uniformly continuous

→ $\lim_{t \to 0} e(t) = 0$ and $\lim_{t \to 0} y(t) = y_m$ → tracked.

$u_{lin} = \tilde{K}(t)^T e(t)$ can be substituted by a PID controller

$$u_{pid} = K_p e_1(t) + K_i \int_{t_0}^{t} e_1(\tau) d\tau + K_d \frac{de_1(t)}{dt} = \tilde{K}(t)^T \pi(e_1)$$

$(K_p \;\; K_i \;\; K_d)$ $\begin{pmatrix} e_1(t) \\ \int e_1(\tau) d\tau \\ \dot{e}_1(\tau) \end{pmatrix}$

for PID controller - $\tilde{K}(t) \in \mathbb{R}^3$ rather than $\mathbb{R}^n$

if $n = 2$ → PD controller

Another adaptive controller –

$$u_{lin}(\sigma_e) = \frac{-\frac{d\sigma_e}{dt}F(e) - k\sigma_e}{\frac{d\sigma_e}{dt}G(e)}$$

$$u^*_{ad}(\sigma_e) = \underbrace{\hat{k}^*(t)}\,\sigma_e(t) = u_{lin}(\sigma_e)$$

$\hat{k}^*(t)$ is a constant limit of $\hat{k}(t)$

The adaptive linear controller with $u_{ad}(\sigma_e) = \hat{k}(t)\sigma_e(t)$ where

$$\hat{k}(t) = -\int_0^t \gamma \frac{d\sigma_e}{de} G(e)\,\sigma_e^2\,dt, \quad \gamma > 0$$

yields to $\sigma_e = 0$.

<u>Proof</u>

$$\dot{\sigma}_e = \frac{d\sigma}{de}F(e) + \frac{d\sigma}{de}G(e)\cdot u_{ad}$$

$$= \frac{d\sigma}{de}F(e) + \frac{d\sigma}{de}G(e)\cdot u_{lin}\, \boxed{-\frac{d\sigma_e}{de}G(e)(u_{lin}-u_{ad})}$$

$$= \frac{d\sigma_e}{de}F(e) + \frac{d\sigma}{de}G(e)\cdot \frac{-\frac{d\sigma_e}{de}F(e) - k\sigma_e}{\frac{d\sigma}{de}G(e)}$$

$$-\frac{d\sigma_e}{de}G(e)(u_{lin}-u_{ad})$$

$$= -k\sigma_e - \frac{d\sigma_e}{de}G(e)(u_{lin}-u_{ad})$$

$$\dot{\sigma}_e = -k\sigma_e - \frac{d\sigma_e}{de}G(e)\cdot(\hat{k}(t)^* - \hat{k}(t)\cdot\sigma_e$$

Consider a Lyapunov function

$$V(\sigma_e, \tilde{K}) = \frac{1}{2}\sigma_e^2 + \frac{1}{2\gamma}(\tilde{K}^* - \tilde{K})^2$$

$$\dot{V}(\sigma_e, \tilde{K}) = \boxed{\dot{\sigma}_e}\sigma_e - \frac{1}{\gamma}(\underbrace{\tilde{K}^*}_{\text{constant}} - \tilde{K})\dot{\tilde{K}}$$

$$= \left(-K\sigma_e - \frac{d\sigma_e}{de}G(e)(\tilde{K}^* - \tilde{K})\sigma_e\right)\sigma_e$$

$$- \frac{1}{\gamma}(\tilde{K}^* - \tilde{K})\dot{\tilde{K}}$$

$$= -K\sigma_e^2 - (\tilde{K}^* - \tilde{K})\left(\frac{d\sigma_e}{de}G(e)\sigma_e^2 - \frac{\dot{\tilde{K}}}{\gamma}\right)$$

$$\dot{\tilde{K}} = -\frac{d}{dt}\int_0^t \gamma \frac{d\sigma_e}{dt}G(e)\sigma_e^2 dt$$

$$-(\tilde{K}^* - \tilde{K})\left(\frac{d\sigma_e}{de}G(e)\sigma_e^2 - \frac{\dot{\tilde{K}}}{\gamma}\right)$$

$$= -(\tilde{K}^* - \tilde{K})\left(\frac{d\sigma_e}{de}G(e)\sigma_e^2 - \frac{1}{\gamma}\cdot\gamma\frac{d\sigma_e}{dt}G(e)\sigma_e^2\right)$$

$$= 0$$

$$\rightarrow \dot{V}(\sigma_e, \tilde{K}) = -K\sigma_e^2 < 0$$

$\rightarrow \dot{\sigma}_e = -K\sigma_e - \frac{d\sigma_e}{de}G(e)(\tilde{K}^* - \tilde{K})\cdot\sigma_e$ is stable

\# $\dot{V} = -K\sigma_e^2 \rightarrow V\Big|_0^t = -\int_0^t K\sigma_e^2 dt \rightarrow \int_0^t \sigma_e^2 dt = \frac{V(0) - V(t)}{K}$

Consider $u_{ad}(\sigma_e) = \hat{k}(t)\,\sigma_e(t)$ and $u_{cin}(\sigma_e) = \dfrac{-\frac{d\sigma_e}{de}F(e) - k\sigma_e}{\frac{d\sigma_e}{de}G(e)}$

when $\hat{k}(t)$ reaches its limit, $u_{ad}(\sigma_e) = \hat{k}(t)\,\sigma_e(t) = u_{cin}(\sigma_e)$

$$\rightarrow \quad \hat{k}(t)\,\sigma_e(t) = \frac{-\boxed{\frac{d\sigma_e}{de}F(e)} - k\sigma_e}{\frac{d\sigma_e}{de}G(e)}$$

$$\rightarrow \quad \left(\hat{k}\cdot\frac{d\sigma_e}{de}G(e) + k\right)\sigma_e = -\frac{d\sigma_e}{de}F(e)$$

select sliding condition $\omega_2(t) = \left|\frac{d\sigma_e}{de}G(e)\,\hat{k}(t) + k\right|\,|\sigma_e| + \eta \qquad \eta > 0$

with

$$u(e) = \frac{-\omega_1 e - \omega_2(t)\,\mathrm{sgn}(\sigma_e)}{\frac{d\sigma_e}{dt}\cdot G(e)}$$

#

Controllers

- PID
- Linear Quadratic Regulator (LQR)
- Linear Quadratic Gaussian (LQG)

extension to linear system
→ stability only achieved at small region from equilibrium point.

- Backstepping
- Sliding mode control (SMC) – kind of variable structure control (VSC)

SMC – General Form: $s(x,t) = \left(\frac{d}{dt} + \lambda x\right)^{n-1} e$

$x \in \mathbb{R}^n$ variable state

$e = x - x_d$ tracking error

λx → dynamics of tracking surface

– Attractiveness: make a positive scalar function for the state variable and choose the control law to decrease this function

$\dot{V}(x) < 0$ with $V(x) > 0$

select $V(x) = \frac{1}{2} s^2$ → check $\dot{V} = s\dot{s} < 0$

– control $u(t) = u_{eq}(t) + u_{sw}(t)$

compensate the uncertainty of model

equivalent control
to determine the behavior of system with ideal sliding regime

$u_{eq}(t)$ - found by invariance condition

$s=0$ and $\dot{s}=0$

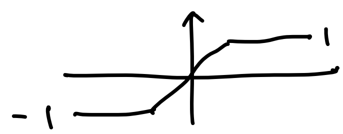
$u_{sw}(t)$ - calculated by checking condition of attractiveness.

$$u_{sw}(t) = -k_1 \,\mathrm{sgn}(s) - k_2 s$$

$$k_1 > 0,\ k_2 > 0$$

Avoid chattering with continuous (smooth control signal)

$$u_{sw} = -k_1 \,\mathrm{sgn}(s) - k_2 s$$



Employ sigmoid function $\mathrm{sigm}(s) = \dfrac{s}{\|s\| + \alpha}$

↑ small constant to smooth the continuity.

SMC
- classical SMC = linear SMC
- Terminal SMC (TSMC) → create a nonlinear "terminal attractor term" in the sliding manifold.

$u = u_{eq} + u_{sw}$ → corrective term to uncertainties from time varying sliding surface $s=0$

↓

equivalent control law to maintain $\dot{s}=0$ if the system dynamics a well known.

Consider a second order nonlinear system —

control input

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u + \delta(x) \end{cases}$$

$\neq 0$

uncertainty and disturbance s.t. $|\delta(x)| \le \delta_{max}$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = system state

→ Select sliding surface as

$$s(x,t) = \left(\frac{d}{dt} + \lambda\right)^{n-1} e$$

Second order system — $s(x,t) = \lambda e + \dot{e} \rightarrow \dot{s} = \lambda \dot{e} + \ddot{e}$

$$= \lambda \dot{e} + \dot{x}_2 - \ddot{x}_{1d}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u \end{cases} \rightarrow u_{eq} = \frac{1}{g(x)}(\ddot{x}_{1d} - f(x) + \lambda \dot{e})$$

$\ddot{x}_1 = f(x) + g(x)u = \cancel{f(x)} + g(x) \cdot \frac{1}{g(x)}(\ddot{x}_{1d} - \cancel{f(x)} + \lambda \dot{e})$

$= \ddot{x}_{1d} + \lambda \dot{e} \rightarrow 0 = (\ddot{x}_{1d} - \ddot{x}_1) + \lambda \dot{e} = \ddot{e} + \lambda \dot{e}$

$\rightarrow \dot{s} = 0$

why select such ueq control

Select a Lyapunov function to determine the corrective term $u_{sw}(t)$ —

by exponential reaching law

$$V(s,t) = \frac{1}{2}s^2 > 0 \rightarrow \dot{V}(s,t) = s\dot{s} < 0, \quad s(t) \neq 0$$

one possible u_{sw} is

$$u_{sw} = -k\,\mathrm{sgn}(s)$$

$$\rightarrow u = \underbrace{\frac{1}{g(x)} \cdot (\ddot{x}_{1d} - f(x) - \lambda \dot{e})}_{u_{eq}} + (-k\,\mathrm{sgn}(s))$$

TSMC – guarantee finite time convergence
(instead of infinite time convergence for SMC)

Consider again the second-order system
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x) + g(x)u + \delta(x) \end{cases}$$

Select a TSM sliding manifold: $s = x_2 + \beta x_1^{q/p}$

where $\beta > 0$ is a design constant, and p, q are positive odd integers to satisfy $q < p$ and $1 < p/q < 2$.

β and η = design constant.

control law:
$$u = -\frac{1}{g(x)}\left(f(x) + \beta \frac{q}{p} x_1^{q/p-1} x_2 + (\delta_{max} + \eta)\,\mathrm{sgn}(s)\right)$$

Second order sliding mode control via measurement noise

- Robot Manipulator System –
  (Lagrangian Formulation)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

Gravitational torque

+ve definite inertia matrix

Coriolis and Centrifugal Forces

control torque or input

- State Equation –
$$\begin{cases} \frac{dx}{dt} = f(x) + g(x)\cdot u \\ y = q \end{cases} \qquad x = \begin{bmatrix} q^T \\ \dot{q}^T \end{bmatrix} \in \mathbb{R}^{2n}$$

$u = \tau \in \mathbb{R}^n$

→ tracking control to achieve input torque τ such that q follows reference q_d

Tracking error: $e = q - q_d$

Sliding condition: $S(e) = \dot{e} + 2\lambda e + \lambda^2 \int e \, dt = 0$

Lyapunov Stability Requirement:

$$S^T \cdot \dot{S} < 0$$

Define control $u = \underbrace{u_{eq}}_{\text{ensure convergence to sliding plane.}} + \underbrace{u_{sw}}_{\text{deviation on sliding surface}}$

$$\dot{S} = F(x) + G(x) u = 0$$

$$u_{eq} = -(G(x))^{-1} F(x)$$

the discontinuities term u_{sw} is defined as

$$u_{sw} = -(G(x))^{-1} K \operatorname{sgn}(S)$$

$$\rightarrow \quad \dot{S} = F(x) + G(x)\left(-(G(x))^{-1}(F(x) + K \operatorname{sgn}(S)\right)$$

$$= -K \operatorname{sgn}(S)$$

$$S\dot{S} = S \cdot \left(-K \operatorname{sgn}(S)\right) = -K|S| < 0$$

Second order SMC (SOSMC) - $\dot{s} = \sigma \rightarrow \Delta u = (G(x))^{-1} \sigma$

Dynamic Control Behavior on Sliding Plane -

$$\begin{cases} \dot{S} = \sigma \\ \dot{\sigma} = -a_0 S - a_1 \sigma + \underset{\substack{\uparrow \\ \text{new control variable}}}{v} \end{cases} \rightarrow \quad \ddot{S} = -a_0 S - a_1 \dot{S} + v$$

$\rightarrow$ second order

characteristic equation: $\left(\frac{d}{dt}+\mu\right)^{2n} e = 0 \rightarrow (s+\mu)^{2n} = 0$

$\mu > 0$

$$\begin{cases} \dot{s} = \sigma \\ \dot{\sigma} = -a_0 s - a_1 \sigma + v \end{cases}$$

put $z = \begin{pmatrix} s \\ \sigma \end{pmatrix}$

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

null matrix

$$\dot{z} = \begin{pmatrix} 0 & I \\ -\mu^2 I & -2\mu I \end{pmatrix} z + \begin{pmatrix} 0 \\ I \end{pmatrix} v$$

Discontinuity term $v = -K \,\mathrm{sgn}(\tau^T P S)$ ← ??

diagonal positive definite matrix

$$P\Phi + \Phi^T P = -Q \quad P > 0 \quad Q > 0$$

Stability Analysis: $V = \frac{1}{2}(\mu^2 S^T S + \sigma^T \sigma)$

$$\dot{V} = -2\mu \sigma^T \sigma - \sum_i k_i |\rho_i| \le 0$$

$$\rho = \tau^T P S$$

Disturbance: 1) input disturbance

$$\ddot{q} = M^{-1}(q)\left(\tau - C(q,\dot{q}) - G(q)\right) + \boxed{b_1(t)}$$

where $\|b_1(t)\| \le d_1$

2) measurement disturbance

$$x_m = x(t)\left(1 + \boxed{b_2(t)}\right) = x(t) + \Delta x(t)$$

multiplicative bounded noise

The controller uses measure state $x_m = x + \Delta x$ instead.
So the sliding mode becomes -

$$\begin{cases} u = u_{eqm} + \Delta u_m \\ u_{eqm} = G(x_m)^{-1} F(x_m) \\ \Delta u_{eqm} = [G(x_m)]^{-1} \sigma \end{cases}$$

$$\rightarrow \dot{s} = F(x) + G(x)\Big(\underbrace{-G(x_m)^{-1} F(x_m)}_{u_{eqm}} + \underbrace{G(x_m)^{-1}\sigma}_{\Delta u_{eqm}}\Big) + h_1$$

$$= F(x) - G(x) G(x_m)^{-1} F(x_m) + G(x) G(x_m)\sigma$$

$$= \frac{\partial f}{\partial x}\Delta x - \sum \frac{\partial G}{\partial x_i} G(x_m)^{-1} F(x_m)\Delta x_i + O(\Delta x)^2 \quad \text{''} M_1 \Delta x + O(\Delta x)^2$$

$$+ \sigma + \sum \frac{\partial G}{\partial x_i} G(x_m)^{-1}\sigma \cdot \Delta x_i + O(\Delta x)^2 \quad \text{''} M_2 \Delta x + O(\Delta x)^2$$

$$= \sigma + M\Delta x + O(\Delta x)^2$$

Nonlinear Affine System -

$$\dot{x}(t)_{n\times 1} = f(x)_{n\times m} + g(x)_{n\times m}\, u(t)_{m\times 1}$$

Define a generalized error vector $S(x)$, the sliding surface is described by -

$$S = \{x \in \mathbb{R}^n \mid s(x) = (s_1(x) \quad s_2(x) \quad \cdots \quad s_m(x))^T = 0\}$$

Controller -

$$u = u_{eq} + \Delta u$$

1. Sliding expression: $\dot{s}(x) = \frac{\partial s}{\partial x}\left(f(x) + g(x)u\right) = 0$

$$\rightarrow \quad u_{eq} = -\left(\frac{\partial s}{\partial x} g(x)\right)^{-1} \cdot \frac{\partial s}{\partial x} f(x)$$

2. Resulted dynamics: $\dot{x} = f(x) + g(x)u$

$$\rightarrow \quad \dot{x} = f(x) + g(x)\left(-\frac{\partial s}{\partial x} g(x)\right)^{-1} \cdot \frac{\partial s}{\partial x} f(x)$$

$$= \left[I - g(x)\left(\frac{\partial s}{\partial x} g(x)\right)^{-1}\right] \cdot f(x)$$

3. discontinuity term: $\Delta u = -\left(\frac{\partial s}{\partial x} g(x)\right)^{-1} W \,\mathrm{sgn}(s)$

4. High order SMC: $s = \dot{s} = \ddot{s} = \cdots = s^{(n-1)} = 0$

Standard sliding surface in state space $\mathbb{R}^n$ -

$$s(x) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \underbrace{\tilde{x}(t)}_{q(t) - q_d(t)}$$

Cooperative control -

1. define tracking error: $\varepsilon_{li}(t) = q_i(t) - q_l(t)$

$q_l(t) \in \mathbb{R}^n$ denotes the position of leader robot.

2. define cross coupling error:

$$\varepsilon_{2i} = \sum_{j \neq i}^{p} k_{ij} \left[(q_i - q_M) - (q_j - q_M) \right] = \sum_{j \neq 1}^{q} k_{ij} (q_i - q_j)$$

3. define global error: $e_i = \varepsilon_{1i} + \int_{t_0}^{t} \varepsilon_{2i}(\tau)\, d\tau$

Define standard sliding surface

$$S(x) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \Rightarrow S_i = \dot{e}_i + \lambda_i e_i \quad i = 1, 2, \ldots$$

<u>SOSMC</u> $\quad \dot{S}_i = \sigma_i \Rightarrow u = u_{eq} + \left(\frac{\partial s}{\partial x} g(x)\right)^{-1} \sigma$

Consider the representation of the system -

$$\begin{cases} \dot{S}_i = \sigma_i \\ \dot{\sigma}_i = -a_0 S_i - a_1 \sigma_i + v_i \end{cases} \rightarrow \text{SISO system}$$

$$\rightarrow \begin{pmatrix} \dot{S} \\ \sigma \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}}_{\phi = \text{Hurwitz matrix}} \begin{pmatrix} S \\ \sigma \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_i$$

diagonal positive matrix

$v_i = -v_{0i}\, \mathrm{sgn}(w_i)$

linear combination of S_i

$\rightarrow P\phi + \phi^T P = -Q \quad P > 0$ and $Q > 0$

$w_i = L \cdot S_i$
$= \Gamma^T \cdot P \cdot S_i$

$$s_i = \dot{e}_i + \lambda_i e_i \; ; \quad i = 1, 2, \ldots$$

$$\dot{s}_i = \sigma_i$$

master $\qquad\qquad e_i = \varepsilon_{1i} + \int \varepsilon_{2i}\, d\tau$

$$\rightarrow \quad \dot{s}_i = \ddot{e}_i + \lambda_i \dot{e}_i = \underbrace{\ddot{q}_i - \ddot{q}_d}_{\ddot{e}_i} + \lambda_i \underbrace{(\dot{\varepsilon}_{1i} + \varepsilon_{2i})}_{\dot{e}_i}$$

$$= \ddot{q}_i - \ddot{q}_d + \lambda_i \left(\dot{q}_i - \dot{q}_d + \sum_{j \neq i} k_{ij} (q_i - q_j) \right)$$

$$= \sigma_i$$

Recall the dynamic equation $\quad M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i$

$$\ddot{q}_i = M_i(q_i)^{-1} \left(\tau_i - C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) \right)$$

$$\rightarrow \quad \sigma_i = \overbrace{M_i(q_i)^{-1} \left(\tau_i - C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) \right)}^{\ddot{q}_i} - \ddot{q}_d$$

$$+ \lambda_i \left(\dot{q}_i - \dot{q}_d + \sum_{j \neq i} k_{ij} (q_i - q_j) \right)$$

$$\tau_i = C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) + M_i(q_i) \left(\sigma_i + \ddot{q}_d - \lambda_i (\dot{q}_i - \dot{q}_d) + \sum_{j \neq i} k_{ij} (q_i - q_j) \right)$$

Consider the Lyapunov Function:

$$V = \sum_i V_i = \sum_i s_i^T P s_i \quad \leftarrow \text{switching function}$$

$$\dot{V} = \sum_i \dot{s}_i^T P s_i + \sum_i s_i^T P \dot{s}_i \qquad \dot{s}_i = \Phi s_i + \Gamma v_i$$

$$= \sum_i (\Phi s_i + \Gamma v_i) P s_i + s_i^T P (\Phi s_i + \Gamma v_i)$$

$$= \sum_i s_i^T (\Phi^T P + P\Phi) s_i + 2 \sum_i v_i \Gamma^T P s_i$$

$\Phi^T P + P\Phi = -Q$, $P > 0$, $Q > 0$

$v_i \underbrace{\Gamma^T P}_{} s_i = v_i \underbrace{L s_i}_{\omega_i}$ $\quad v_i = -v_{0i}\,\mathrm{sgn}(\omega_i)$

$\rightarrow v_i \Gamma^T P s_i = v_i \omega_i = -v_{0i}\,\omega_i\,\mathrm{sgn}(\omega_i)$

$= -v_{0i}|\omega_i|$

$\rightarrow \dot{V} = -\sum_i s_i^T Q s_i - 2\sum_i v_{0i}|\omega_i|$

< 0

v_{0i} = diagonal positive matrix

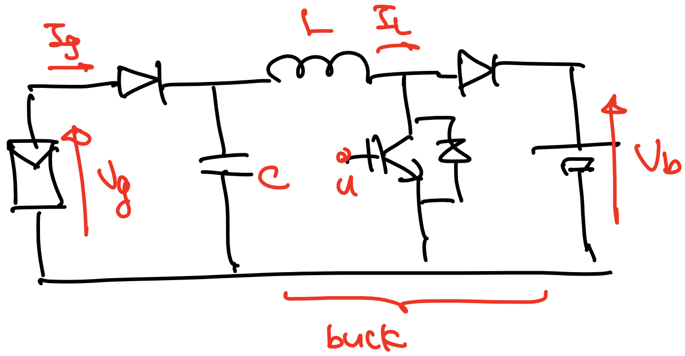
→ the dynamic system would remain stable, with Lyapunov stability criterion.

Define $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ and $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - G(q)) \end{pmatrix}$$

$$q_d = \begin{pmatrix} q_{d1} \\ q_{d2} \\ q_{d3} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{6}\sin(1.5\pi t) \\ \frac{\pi}{4}\sin \pi t \\ 0.2\,\mathrm{sgn}(0.8 - t) \end{pmatrix}$$

PV Battery system with Fuzzy sliding mode



$$C\frac{dV_g}{dt} = I_g - I_L$$

$$L\frac{dI_L}{dt} = V_g - (1-u')V_b$$

$$u' \begin{cases} 0 = \text{OFF} \\ 1 = \text{ON} \end{cases}$$

Put $x_1 = V_g$, $x_2 = I_L$, $y = V_g$ $\rightarrow$

$$\begin{cases} \dot{x}_1 = \frac{1}{C} I_g - \frac{1}{C} x_2 \\ \dot{x}_2 = \frac{1}{L} x_1 - \underbrace{\frac{V_b}{L}}_{\text{constant}} \cdot \underbrace{\boxed{u'}}_{\text{input}} \\ y = x_1 \end{cases}$$

Sliding mode control recap.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \qquad \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = \alpha(x) + \gamma(x)u \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$\alpha(x)$, $\gamma(x)$ nonlinear,
$\gamma(x) \neq 0$
$\gamma(x) > \gamma_0 > 0$

→ control law $u = u(x)$ such that $x_i \rightarrow 0$

→ Sliding surface $s = k_1 x_1 + k_2 x_2 \dots + k_{n-1} x^{n-1} + x_n$
with coefficient $k = (k_1\ k_2 \dots k_{n-1})$
chosen from the polynomial $p(t) = k_1 + k_2 t + \dots + k_{n-1} t^{n-2} + t^{n-1}$
is a Hurwitz one.

On the surface, the dynamics is governed by

$$s = 0, \quad x_n = -\sum_{i=1}^{n-1} k_i x_i$$

the result is asymptotic stable and $x_i \to 0$

It can also be written as differential eqt.

$$x_n = -\sum_{i=1}^{n-1} k_i x_i \to x_1^{(n-1)} = -(k_1 x_1 + k_2 \dot{x}_1 + \dots + k_{n-1} x_1^{(n-2)})$$

and it follows $x_1 \to 0$, $x_2 = \dot{x}_1 \to 0 \dots$

→ dynamics of the state $x^T = (x_1 \ x_2 \ x_3 \dots x_n)^T$ converge to the origin

1) Choose a proper sliding surface
2) impose $S = 0$ by a control law $u = u(x)$

Choose $V = \frac{1}{2} s^2$ as the candidate Lyapunov Function.

$$\dot{V} = s\dot{s} = s(k_1 \dot{x}_1 + k_2 \dot{x}_2 + \dots + k_{n-1} \dot{x}_{n-1} + \dot{x}_n)$$

$$= s(k_1 x_2 + k_2 x_3 + \dots + k_{n-1} x_n + \alpha(x) + \gamma(x) u)$$

Choosing $u = -\frac{1}{\gamma(x)} (k_1 x_2 + k_2 x_3 + \dots + k_{n-1} x_n + \alpha(x) + v)$

$$\dot{V} = s\dot{s} = +sv$$

* goal is to have a controller u such that $\dot{V} < 0$

Define $v = -\beta(x) \operatorname{sgn}(s)$

$$\dot{V} = -s \underset{\beta(x)}{\operatorname{sgn}(s)} = -|s| \beta(x)$$

$$\to \dot{V} = -|s| \beta(x) \leq -|s| \beta_0 \underline{\underline{< 0}}$$

Hence, the trajectory reaches the surface $s = 0$ at finite time

Problem

1. Control law forces the system to reach the surface with $w\,sgn(s)$. Once the system reaches the surface, it remains on the surface for a while, and leave it due to inertia, hysteresis and delay.

2. Chattering problem - $s<0 \rightarrow sgn(s)=-1 \rightarrow$ inverse action to push back to the sliding plane.

→ Boundary layer method → not to impose $s=0$, use $|s| \le \varepsilon$

   1. when $|s| > \varepsilon$, the control law forces the system to reach the surface with $sgn|s| > 1$

   2. when $|s| < \varepsilon$, the control law can be freely modified.

   Standard solution: $w = \rho(x)\,sgn(s)$

   $\rightarrow w = \rho(x)\,sat(\frac{s}{\varepsilon})$

Consider a tracking problem:
$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad x \in \mathbb{R}^n$$

transform to state space with change of variable

smooth vectorial nonlinear function

→ transformation with feedback linearization

Introducing Lie derivative.

$$L_{\tau(x)}\, w(x) = \frac{\partial w(x)}{\partial x} \cdot \tau(x)$$

where $\tau(x)$ and $w(x)$ are vector field.

Define the relative degree ρ for the system as:

$$L_g h(x) = L_g L_f h(x) = L_g L_f^2 h(x) = \cdots = L_g L_f^{\rho-2} h(x) = 0$$

$$L_g L_f^{\rho-1} h(x) \geq a > 0 \qquad \forall x \in D$$

The control objective is to make output y tracks the reference signal $r(t)$, where $r(t)$, $\dot{r}(t)$, $\ddot{r}(t)$, ..., $r^{(\rho)}(t)$ are known and bounded for all $t \geq 0$ and $r^{(\rho)}(t)$ is piecewise continuous function of t.

→ Apply input-output linearization, the system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \longrightarrow \Sigma' \begin{cases} \dot{\eta} = f_0(\eta, \xi) \leftarrow \text{zero dynamics} \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \qquad \vdots \\ \dot{\xi}_{\rho-1} = \xi_\rho \\ \dot{\xi}_\rho = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\, u \\ y = \xi_1 \end{cases}$$

by the change of variable

$$\begin{bmatrix} \eta \\ \hline \xi \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \hline \psi(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ \hline h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix} = T(x)$$

where $\frac{\partial \phi_i}{\partial x} g(x) = 0$ for $1 \leq i \leq n-\rho \quad \forall x \in D$

Define the reference for the system $\quad R = \begin{pmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{pmatrix}$

Apply the change of variable $e = \xi - R$

$$\Sigma_e \begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \vdots = \vdots \\ \dot{e}_{\rho-1} = e_\rho \\ \dot{e}_\rho = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u - r^{(\rho)}(t) \end{cases}$$

It is a linear system with controllable canonical form

→ use the linear controller $e_\rho = -(k_1 e_1 + \dots + k_{\rho-1} e_{\rho-1})$

such that $s^{\rho-1} + k_{\rho-1} s^{\rho-2} + \dots + k_2 s + k_1$

is a Hurwitz

→ select the sliding surface $s = \sum_{i=1}^{\rho-1} k_i e_i + e_\rho = 0$

Recall $\dot{e}_p = L_f^p h(x) + L_g L_f^{p-1} h(x) u - r^{(p)}(t)$

and $\dot{s} = k_1 \dot{e}_1 + k_2 \dot{e}_2 + \dots + k_{p-1} \dot{e}_{p-1} + \dot{e}_p$

$$= k_1 \dot{e}_1 + k_2 \dot{e}_2 + \dots + k_{p-1} \dot{e}_{p-1} + L_f^p h(x) + L_g L_f^{p-1} h(x) u - r^{(p)}(t)$$

Select $u = \frac{-1}{L_g L_f^{p-1} h(x)} \left(k_1 \dot{e}_1 + k_2 \dot{e}_2 + \dots + k_{p-1} \dot{e}_{p-1} + L_f^p h(x) - r^{(p)}(t) \right) + v$

Hence

$\dot{s} = v$ and select $v = -\beta(x)\,\mathrm{sgn}(s)$

$$V = \frac{1}{2} s^2 \rightarrow \dot{V} = s\dot{s} = sv = -s \cdot \beta(x)\,\mathrm{sgn}(s) = -|s|\,\beta(x) \leq -\beta_0 |s| < 0$$

Sliding mode system for PV system Design

Recall PV System -

$$\Sigma : \begin{cases} \dot{x}_1 = -\frac{1}{C} x_2 + \frac{\overline{I_g}}{C} \\ \dot{x}_2 = \frac{1}{L} x_1 - \frac{\overline{V_b}}{L} u \\ y = x_1 \end{cases}$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{C} x_2 + \frac{\overline{I_g}}{C} \\ \frac{1}{L} x_1 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} 0 \\ -\frac{\overline{V_b}}{L} \end{pmatrix}}_{g(x)} u$$

$$y = \underbrace{x_1}_{h(x)}$$

① Perform state transformation:

Lie derivative

\# $L_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$

$$h(x) = y = x_1$$

$$L_f h(x) = \dot{x}_1 = \frac{\overline{I_g}}{C} - \frac{1}{C} x_2$$

$$L_g h(x) = 0$$

$$L_f^2 h(x) = \underbrace{\left(\frac{1}{C}\frac{\partial \bar{I}_q}{\partial x_1} \quad -\frac{1}{C}\right)}_{\frac{\partial}{\partial x} L_f h(x)} \underbrace{\begin{pmatrix} \frac{\bar{I}_q}{C} - \frac{x_2}{C} \\ \frac{x_1}{L} \end{pmatrix}}_{f(x)}$$

$$\frac{\partial}{\partial x} L_f h(x) = \left(\frac{\partial}{\partial x_1} \cdot \left(\frac{\bar{I}_q}{C} - \frac{1}{C}x_2\right) \quad \frac{\partial}{\partial x_2}\left(\frac{\bar{I}_q}{C} - \frac{1}{C}x_2\right)\right)$$

$$= \left(\frac{1}{C}\frac{\partial \bar{I}_q}{\partial x_1} \quad -\frac{1}{C}\right)$$

$$= \frac{1}{C}\frac{\partial \bar{I}_q}{\partial x_1}\left(\frac{\bar{I}_q}{C} - \frac{x_2}{C}\right) - \frac{x_1}{LC}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\bar{I}_q}{C}\right)^2 = 2\frac{\bar{I}_q}{C}\frac{\partial \bar{I}_q / C}{\partial x_1}$$

$$= \frac{1}{C}\frac{\partial \bar{I}_q}{\partial x_1} \cdot \frac{\bar{I}_q}{C} - \frac{x_1}{LC}$$

$$= \frac{1}{2}\left(\frac{\dot{\bar{I}}_q}{C}\right) - \frac{x_1}{LC}$$

$$L_g L_f h(x) = L_g\left(\frac{\bar{I}_q}{C} - \frac{x_2}{C}\right)$$

$$\# \; L_f h(x) = \frac{\partial h(x)}{\partial x} \cdot f(x)$$

$$g(x) = \begin{pmatrix} 0 \\ -\bar{V}_b / L \end{pmatrix}$$

$$= \left(\frac{1}{C}\frac{\partial \bar{I}_q}{\partial x_1} \quad -\frac{1}{C}\right)\begin{pmatrix} 0 \\ -\frac{\bar{V}_b}{L} \end{pmatrix}$$

$$= \frac{\bar{V}_b}{LC} \neq 0$$

The relative degree is $\rho = 2$ → the system is completely linearizable with the transformation

$$T = \begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{\bar{I}_q}{C} - \frac{x_2}{C} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Define $r(t)$ as the reference signal for

$$y = x_1 = \xi_1$$

and introduce the following error:

$$\begin{cases} e_1 = \xi_1 - r \\ e_2 = \xi_2 - \dot{r} \end{cases}$$

$$\# \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

$$\& \quad L_f h(x) = \frac{\partial h(x)}{\partial x} \cdot f(x)$$

the error dynamics would be

$$\begin{cases} \dot{e}_1 = \dot{\xi}_1 - \dot{r} = \xi_2 - \dot{r} \\ \dot{e}_2 = \dot{\xi}_2 - \ddot{r} \end{cases}$$

$$= L_f^2 h(x) + L_g L_f h(x) u - \boxed{\ddot{r}}$$

$$= \frac{1}{2}\left(\frac{\dot{I_g}}{c}\right) - \frac{x_1}{LC} + \frac{\bar{V_b}}{LC} u - \ddot{r}$$

$$\Sigma \begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = L_f^2 h(x) + L_g L_f h(x) u \\ y = \xi_1 \end{cases}$$

$$\& \quad e = \xi - R$$

$$\Sigma_e^r = \begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{e}_1 = e_2 \\ \dot{e}_2 = L_f^2 h(x) + L_g L_f h(x) u \boxed{- r^{(2)}(t)} \end{cases}$$

Define the sliding surface $s = k_1 e_1 + e_2$

Set the control law:

$$u = \frac{1}{L_g L_f h(x)} \left(k_1 e_2 + L_f^2 h(x) - \ddot{r} \right) + v$$

$$= \frac{1}{L_g L_f^{(\rho-1)} h(x)} \left[\left(k_1 \dot{e}_1 + k_2 \dot{e}_2 + \dots + k_{\rho-1} \dot{e}_\rho \right) + L_f^{(\rho)} h(x) - r^{(\rho)}(t) \right] + v$$

$$= -\frac{LC}{\bar{V_b}} \Big(k_1 e_2 + \underbrace{\frac{1}{2}\left(\frac{\dot{I_g}}{c}\right) - \frac{x_1}{LC}}_{L_f^2 h(x)} - \ddot{r} \Big) + v$$

Lyapunov Stability:

$$\dot{V} = S\dot{S} = s(k_1\dot{e}_1 + \dot{e}_2)$$

$$= S\left(k_1 e_2 + \frac{1}{2}\left(\frac{\dot{I}_g}{C}\right) - \frac{x_1}{LC} + \frac{\bar{V}_b}{LC}u - \ddot{r}\right)$$

Define $u = \underbrace{-\frac{LC}{\bar{V}_b}\left(k_1 e_2 + \frac{1}{2}\left(\frac{\dot{I}_g}{C}\right) - \frac{x_1}{LC} - \ddot{r}\right)}_{\text{nominal part}} + \underbrace{v}_{\text{equivalent part.}}$

$$\dot{V} = S\dot{S} = S\left(\frac{\bar{V}_b}{LC}v\right) = -\frac{\bar{V}_b}{LC}\beta(x)|s| < 0$$

Select $v = -\beta\, \mathrm{sgn}(s) \rightarrow \beta(x) > 0$ and $\forall x \in D$

In conclusion:

$$u = -\frac{LC}{\bar{V}_b}\left(k_1 e_2 + \frac{\dot{I}_g}{2C} - \frac{x_1}{LC} - \ddot{r}\right) + v$$

$$v = -\beta(x)\,\mathrm{sgn}(s) \qquad \beta(x) = K \cdot \frac{LC}{\bar{V}_b}$$

to improve the chattering problem: $\mathrm{sgn}(s) \rightarrow \mathrm{sat}\left(\frac{s}{\varepsilon}\right)$

$$u = -\frac{LC}{\bar{V}_b}\left(k_1 e_2 + \frac{\dot{I}_g}{2C} - \frac{x_1}{LC} - \ddot{r}\right) + v$$

$$v = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right) \qquad \beta(x) = K \cdot \frac{LC}{\bar{V}_b}$$

control performance highly dependent on system parameter and coefficient.

→ control law optimized for parameter set P_1 is not applicable to P_2.

Global Performance Improvement:

1) design optimal sliding mode control law for different operating point.

2) use of fuzzy inference to define proper control input by combine different law

System Parameter → Optimal Sliding law

P_i → L_i

most important parameter: solar irradiance → λ

ABC algorithm: Artificial Bee Colonies

possible soln of optimization problem = position of food source

nectar amount of food source = fitness of associated solution

SB | EB | OB

scout bee (random search), employee bees (greedy search), onlooker bee (probability search)

ABC code: Initiate Population

while (requirement not met)

{ place EB on their food source for greedy search

place OB on the food source depending on nectar amount.

Randomly send out SB to discover new food source

Memorize the best source so far

}

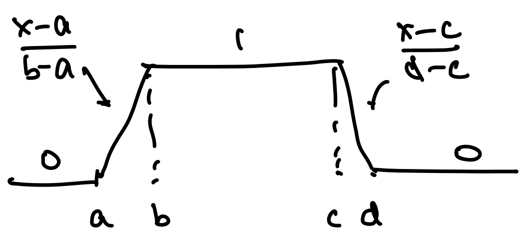
Fuzzy Inference

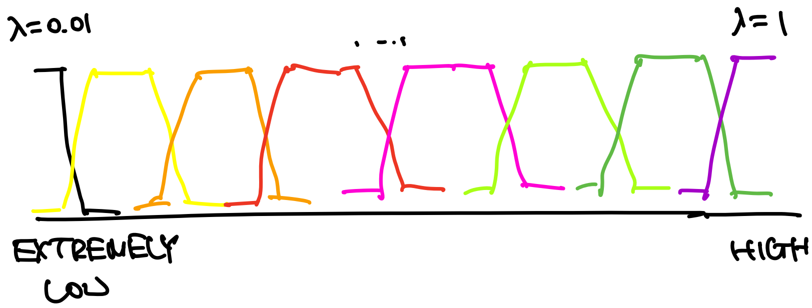
$\lambda \rightarrow$ control parameter

zero-order Takagi Sugeno Fuzzy Inference

$\lambda \rightarrow 8$ fuzzy sets

$\mu_A(x) =$





Recall
$$u = -\frac{LC}{\sqrt{V_b}}\left(k_1 e_2 + \frac{\dot{I}_g}{2C} - \frac{x_1}{LC} - \ddot{r}\right) - \overbrace{K \cdot \frac{LC}{\sqrt{V_b}}}^{\beta(x)} \operatorname{sat}\left(\frac{s}{\varepsilon}\right)$$

$\rightarrow$ parameter $\{k_1, \underbrace{\beta, \varepsilon}\}$ for different λ

change the slope of sliding plane

leading to nonlinear slope inside boundary layer

1. PI control

Commonly used → acceptable steady state & (e_{ss})
(~90% of PV system) transient response (ξ → overshoot)
stability.

slow transient, oscillation → due to PWM delay.

unsuitable input (duty cycle) → controller cannot find the right output to avoid oscillation
→ characterized with strong solar irradiance

oscillation → battery damage.

2. Backstepping

e.g. 3D Vaidyanathan jerk chaotic system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = a\xi_1 - b\xi_2 - c\xi_3 - \xi_1^2 - \xi_2^2 + v \end{cases}$$

$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ = state

$a, b, c > 0$

← backstepping control to be designed.

"chaotic and dissipative system with a strange attractor"

$(a, b, c) = (7.3, 4, 0.9)$ with $x(0) = (0.3, 0.2, 0.3)$

Lyapunov exponent : $\lambda_1 = 0.181$; $\lambda_2 = 0$; $\lambda_3 = -1.181$

$$\begin{cases} \dot{\xi}_1 = \boxed{\xi_2} \quad \cdots (a) \\ \dot{\xi}_2 = \xi_3 \quad \cdots (b) \\ \dot{\xi}_3 = a\xi_1 - b\xi_2 - c\xi_3 - \xi_1^2 - \xi_2^2 + u \quad \cdots (c) \end{cases}$$

virtual controller → $\dot{\xi}_1 = w_1$ ← first order system with w_1 as control

\# $V_1(\xi_1) = \frac{1}{2}\xi_1^2$ $\quad \dot{V}_1 = \xi_1\dot{\xi}_1 = \xi_1 w_1$ → choose $w_1 = -\xi_1$

→ $\dot{V}_1 = -\xi_1^2 < 0$ → globally asymptotic stable

→ Required ξ_2 forced to track virtual controller w_1 $\quad$ \# $\xi_2(0) \neq w_1(0)$

→ integrator backstepping control to regulate the output -

$$z_1 = \xi_2 - w_1 = \xi_2 + \xi_1$$

$$\xi_2 = z_1 - \xi_1$$

→ with the system (a): $\dot{\xi}_1 = \xi_2$

$$\dot{\xi}_1 = \xi_2 = z_1 - \xi_1$$

note that $\quad \dot{z}_1 = \dot{\xi}_1 + \dot{\xi}_2 = \underbrace{\xi_3}_{\dot{\xi}_2} + \underbrace{z_1 - \xi_1}_{\dot{\xi}_1}$

In the coordinate system - (ξ_1, z_1, ξ_2)

$$\dot{\xi}_1 = z_1 - \xi_1$$

$$\dot{z}_1 = z_1 - \xi_1 + \xi_3$$

$$\dot{\xi}_3 = a\xi_1 - \xi_1^2 - b\overbrace{(z_1 - \xi_1)}^{\xi_2} - \overbrace{(z_1 - \xi_1)^2}^{\xi_2^2} - c\xi_3 + u$$

In (b): $\dot{\xi}_2 = \xi_3$

the 2D nonlinear system is $\boxed{\begin{aligned}\dot{\xi}_1 &= z_1 - \xi_1 \\ \dot{z}_1 &= z_1 - \xi_1 + \omega_2\end{aligned}}$ system

$$V_2(\xi_1, z_1) = V_1(z_1) + \frac{1}{2}z_1^2 = \frac{1}{2}(\xi_1^2 + z_1^2)$$

$$\dot{V}_2 = \xi_1\dot{\xi}_1 + z_1\dot{z}_1 = \xi_1(z_1 - \xi_1) + z_1(z_1 - \xi_1 + \omega_2)$$

$$\rightarrow \dot{V}_2 = -\xi_1^2 + z_1(z_1 + \omega_2)$$

Put $\boxed{\omega_2 = -2z_1}$ $\rightarrow$ $\dot{V}_2 = -\xi_1^2 - z_1^2 < 0$ $\rightarrow$ globally asymptotic stable

controller

Again, we need ξ_3 to track controller $\omega_2(t)$

$$z_2 = \xi_3 - \omega_2 = \xi_3 + 2z_1 = \xi_3 + 2(\xi_2 + \xi_1)$$
$$= 2\xi_1 + 2\xi_2 + \xi_3$$

Note that in (ξ_1, z_1, z_2) coordinate

the non-linear system in (ξ_1, z_1, ξ_2)

$$\begin{cases}\dot{\xi}_1 = z_1 - \xi_1 \\ \dot{z}_1 = z_1 - \xi_1 \boxed{+\xi_3} \\ \dot{\xi}_3 = a\xi_1 - \xi_1^2 - b(\underbrace{z - \xi_1}_{\xi_2}) \\ \quad - (\underbrace{z - \xi_1}_{\xi_1})^2 - c\xi_3 + v\end{cases}$$

Put ξ_3 as virtual controller

$\omega_2 = -2z_1$

$z_2 = \xi_3 - \omega_2 = \xi_2 + 2z_1$

$\xi_3 = z_2 - 2\xi_1 - 2\xi_3$

$$\begin{aligned}\dot{\xi}_1 &= z_1 - \xi_1 \\ \dot{z}_1 &= -\xi_1 \boxed{-z_1 + z_2} \\ \dot{z}_2 &= (a + b - 2)\xi_1 \\ &\quad + (2c - b - 2)z_1 \\ &\quad + (2 - c)z_2 \\ &\quad - \xi_1^2 - (z_1 - \xi_1)^2 + v\end{aligned}$$

$$V(\xi_1, z_1, z_2) = V_2(\xi_1, z_1) + \frac{1}{2}z_2^2 = \frac{1}{2}(\xi_1^2 + z_1^2 + z_2^2)$$

$$\dot{V} = \xi_1\dot{\xi}_1 + z_1\dot{z}_1 + z_2\dot{z}_2$$

$$= \xi_1(z_1 - \xi_1)$$
$$+ z_1(-\xi_1 - z_1 + z_2)$$
$$+ z_2[\underbrace{(a+b-2)\xi_1 + (2c-b-1)z_1 + (2-c)z_2) - \xi_1^2 - (z_1-\xi_2)^2 + v}_{\triangleq S}]$$

$$= -\xi_1^2 - z_1^2 + z_2 S$$

consider the integrator backstepping control as –

$$v = -(a+b-2)\xi_1 + (2c-b-1)z_1 - (2-c+k)z_2 + \xi_1^2 - (z_1-\xi_1)^2$$
$(k>0)$

$$\rightarrow \quad \dot{V} = -\xi_1^2 - z_1^2 - kz_2^2$$

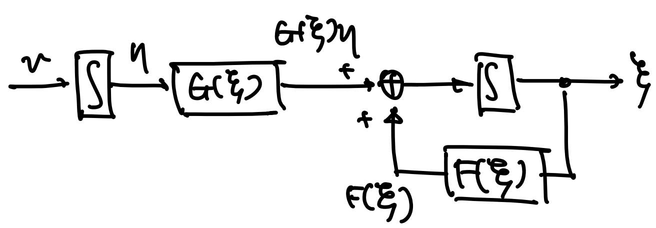
Integrating Backstepping Control: $\dot{\xi} = F(\xi) + G(\xi)\boxed{\eta}$ virtual controller

$\dot{\eta} = v$

$x = (\xi, \eta) \in \mathbb{R}^{n+1}$ state vector

v = control input

→ design goal – find backstepping control law v such that $x(t) \to 0$ as $t \to \infty$



we assume there exists a smooth feedback $\eta = \varphi(\xi)$ with $\varphi(0) = 0$ such that $\xi = 0$ is asymptotic stable of the first system $\dot{\xi} = F(\xi) + G(\xi)\varphi(\xi)$

We also assume there exists a Lyapunov function $V_1(\xi)$ such that

$$\frac{\partial V_1}{\partial \xi}\left[F(\xi)+G(\xi)\varphi(\xi)\right] \le -W(\xi)$$

where $W(\xi)$ is positive definite in $\mathbb{R}^n$

$$\begin{cases} \dot{\xi} = F(\xi) + G(\xi)\boxed{\eta} \\ \dot{\eta} = v \end{cases}$$

virtual control

$$\rightarrow \begin{cases} \dot{\xi} = F(\xi) + G(\xi)\varphi(\xi) + G(\xi)(\eta - \varphi(\xi)) \\ \dot{\eta} = v \end{cases}$$

Introduce the change of variable $y = \eta - \varphi(\xi)$

state

error

control

Design Goal → find v such that $y(t) \to 0$ as $t \to \infty$

$$\rightarrow \begin{cases} \dot{\xi} = \left[F(\xi) + G(\xi)\varphi(\xi)\right] + G(\xi)\,y \\ \dot{y} = v - \dot{\varphi}(\xi) \end{cases}$$

F, G, φ → known

$$\rightarrow \quad \dot{\varphi}(\xi) = \frac{\partial \varphi}{\partial \xi}\left[F(\xi) + G(\xi)\eta\right]$$

and set $u = v - \dot{\varphi}(\xi)$

$$\rightarrow \begin{cases} \dot{\xi} = [F(\xi) + G(\xi)\varphi(\xi)] + G(\xi)y \\ \dot{y} = u \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\xi} = F(\xi) + G(\xi)\eta \\ \dot{\eta} = v \end{cases}$$

<u>Advantage</u>

If $u=0$, the system is asymptotically stable at $\xi = 0$

control Lyapunov Function $V_1(\xi)$ is utilized for stabilization of overall control function

$$V(\xi, \eta) = \boxed{V_1(\xi)} + \frac{1}{2}y^2 = V_1(\xi) + \frac{1}{2}(\eta - \varphi(\xi))^2$$

positive definite function

$\rightarrow V_1(\xi) \geq 0 \;\; \forall \xi \in \mathbb{R}^n$ and $V_1(\xi) = 0 \Leftrightarrow \xi = 0$

show $V(\xi, \eta) = 0 \Leftrightarrow (\xi, \eta) = (0, 0)$

$\rightarrow \xi = 0$ and $\eta = 0$

$\Leftrightarrow V_1(0) = 0$ and $\varphi(\xi = 0) = 0$

$\rightarrow$ 1. V is a positive definite function on $\mathbb{R}^{n+1}$. ✓

2. Proof $\dot{V}$ is a negative definite function. i.e.

$$\dot{V} = \frac{\partial V_1}{\partial \xi_1}[F(\xi) + G(\xi)\varphi(\xi)] + \frac{\partial V_1}{\partial z} G(\xi) y + yv$$

$$\leq -W(\xi) + \frac{\partial V_1}{\partial z} G(\xi) y + yu$$

→ choose the backstepping control u as

$$\boxed{u = -\frac{\partial V_1}{\partial \xi} G(\xi) y - ky}$$ control u

→ $\dot{V} \le -W(\xi) - ky^2$

Recall $u = v - \dot{\varphi}(\xi)$

$$v = u + \boxed{\dot{\varphi}(\xi)} = \overbrace{\boxed{\frac{\partial \varphi}{\partial \xi}[F(\xi) + G(\xi)\eta]}}^{\dot{\varphi}(\xi)} \overbrace{\boxed{-\frac{\partial V_1}{\partial \xi} G(\xi)}}^{u}$$

$$\boxed{-k(\xi - \varphi(\xi))} \qquad (k > 0)$$

stabilize the equilibrium $(\xi, \eta) = (0,0)$ of the system

$$\begin{cases} \dot{\xi} = F(\xi) + G(\xi)\eta \\ \dot{\eta} = v \end{cases}$$ ← integral backstepping.

with the total Lyapunov function

$$V(\xi, \eta) = V_1(\xi) + \frac{1}{2}(\eta - \varphi(\xi))^2$$

Backstepping Control in General System -

$$* \quad \begin{cases} \dot{\xi} = F(\xi) + G(\xi)\eta \\ \dot{\eta} = \alpha(\xi, \eta) + \beta(\xi, \eta) u \end{cases}$$

take the control input u as

$$u = \frac{1}{\beta(\xi, \eta)} (v - \alpha(\xi, \eta))$$

$$\rightarrow \begin{cases} \dot{\xi} = F(\xi) + G(\xi)\eta \\ \dot{\eta} = v \end{cases}$$

Recursive Application of Backstepping control in
Strict Feedback Form -

virtual control

$$\begin{cases} \dot{\xi} = F_0(\xi) + G_0(\xi)\boxed{\eta_1} \\ \dot{\eta}_1 = F_1(\xi, \eta_1) + G_1(\xi, \eta_1)\boxed{\eta_2} \\ \dot{\eta}_2 = F_2(\xi, \eta_1, \eta_2) + G_2(\xi, \eta_1, \eta_2)\boxed{\eta_3} \\ \vdots = \vdots \\ \dot{\eta}_{k-1} = F_{k-1}(\xi, \eta_1, \dots, \eta_{k-1}) + G_{k-1}(\xi, \eta_1, \dots, \eta_{k-1})\boxed{\eta_k} \\ \dot{\eta}_k = F_k(\xi, \eta_1, \dots, \eta_k) + G_k(\xi, \eta_1, \dots, \eta_k)\boxed{u} \end{cases}$$

where $\xi, \eta = \{\eta_1, \eta_2, \dots, \eta_k\} \in \mathbb{R}$ are states and $u \in \mathbb{R}$ is the input

$\underline{F}$ and $\underline{G}$ are known <u>smooth</u> functions

$\dot{\xi}_i$ only depends on state $\xi, \eta_1, \dots, \eta_i$.

Feedback linearization approach → cancellation of <u>useful</u> nonlinear terms

Backstepping more flexible control → does not require final input-output dynamics is a linear system.