Chapter 2 Atmospheric Thermodynamics

Atmospheric Thermodynamics is to study the state variables (P, T, ρ, Γ) with the (vertical) flow.

1. **Virtual Temperature** T_v is to include for the thermodynamic effect of moisture in air. Given $\rho = \rho_d' + \rho_v'$ and $p = p_d + e$. To write the equation $p = \rho R_d T_v$,

$$T_v = \frac{T}{1 - \left(\frac{e}{p}\right)(1 - \varepsilon)} \tag{1}$$

Note: $\rho < \rho_d$ and $T_v > T$.

2. **Geopotential** Φ, which is to eliminate ρ in calculation, and take account of the effect of hydrostatic pressure and temperature in moving up/down of an air parcel, is the *work* taken against gravitation to perform the process.

Given $dp = \rho g dz$ and $p = \rho R_d T_v$, $d\Phi = gdz = -vdp$,

Geopotential height Z is

$$Z = \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g(z)dz\,, \qquad \Delta Z = \frac{R_d}{g_0}\int_{p_2}^{p_1} T_v \frac{dp}{p} \tag{2}$$

At higher position, it has a lower gravity g(z) and hence requires less energy to move an air parcel upward.

3. **Scaled Height** is the height to have pressure decreased by a factor of *e* (=2.718) given an *isothermal & dry atmosphere*. It is to simplify the geopotential calculation with constant T_v. Recall (2),

$$Z_2 - Z_1 = H \ln\left(\frac{p_1}{p_2}\right), \text{given } H = \frac{R_d T_v}{g_0} = 29.3 T_v$$

$$p_2 = p_1 \exp\left(-\frac{Z_2 - Z_1}{H}\right) \tag{3}$$

Note: When $Z_2 - Z_1 = \{0, H, 2H, \ldots\}$, $p_2/p_1 = \{1, e^{-1}, e^{-2}, \ldots\}$ P and ρ of heavier gases fall off more rapidly with height (with diff. R).

4. Given the Geopotential Equation (2), it is better to have constant T_v, i.e.

$$\bar{T}_v = \int_{\ln p_2}^{\ln p_1} T_v\, d(\ln p) \Big/ \int_{\ln p_2}^{\ln p_1} d\,(\ln p)$$

Hypsometric Equation:

$$Z_2 - Z_1 = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{p_2}{p_1}\right) = \bar{H} \ln\left(\frac{p_2}{p_1}\right) \tag{4}$$

The difference of geopotential height ($Z_2 - Z_1$) between two levels is the **thickness**. Tropics has a higher $\bar{T}_v$ and hence larger thickness. It requires less energy to reach the same height. A **pressure correction** is required, given $p_0 = p_g \exp(g_o Z_g / R_d T_v)$ and $p_0 - p_g = p_g (g_o Z_g / R_d T_v)$. It means pressure falls about 1mbar for every 8m of vertical ascent.

5. Given the **Lapse Rate** Γ, such that $T = T_0 - \Gamma z$ and hydrostatic equation $dp = -\rho g dz$ and ideal gas law $p = \rho RT$,

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)} dz \to z = \frac{T_0}{\Gamma}\left(1 - \left(\frac{p}{p_0}\right)^{\frac{R\Gamma}{g}}\right) \tag{5}$$

It includes the effect of lapse rate (temp gradient) and pressure difference in height measurement with pressure.

6. **First Law of Thermodynamics**: $dq = du + dw = c_v dT + pdv$
   It gives $c_p = c_v + R$, at the expense of constant pressure, i.e. $vdP = 0$ and
$$dq = d(h + \Phi) = c_p dT + gdz \tag{6}$$
   Recall $d\Phi = gdz = -vdp$.
   Consider an **air parcel** in the atmosphere, it is
   - **Thermal insulated** from its environment so its state changes **adiabatically** as it sinks or rises.
   - Always at **same pressure** at same level, such that it is in hydrostatic equilibrium
   - Move slowly enough, such that kinetic energy is negligible in its total energy.

7. The dry air **adiabatic lapse rate** is given by
$$dq = c_p dT + gdz = 0 \rightarrow -\left(\frac{dT}{dz}\right)_{dry} = \frac{g}{c_p} = \Gamma_d = 9.8 \text{ K km}^{-1} \tag{7}$$

8. **Potential temperature** θ is the air parcel with pressure p and temperature T moves **adiabatically** to the sea level with pressure p_0. Given the first law: $dq = c_p dT - vdp = 0$ and ideal gas law: $pv = RT$, **Poisson Equation**:
$$\frac{c_p}{R}\frac{dT}{T} - \frac{dp}{p} = 0 \rightarrow \theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \tag{8}$$
   It is to provide a common basis for comparison of energy of air parcel in different pressure condition. Note: air parcel subject to adiabatic transformation has a *constant* potential temperature.

9. To remove the vapor partial pressure *e* in the virtual temperature equation (1), a **mixing ratio** w is defined as $w = m_w/m_d$.
   The vapor pressure *e* is given by
$$e = \frac{\frac{m_w}{M_w}}{\frac{m_w}{M_w} + \frac{m_d}{M_d}} = \left(\frac{w}{\varepsilon + w}\right)p$$
   Substitute into Equation (1), the virtual temperature is only affected by the mixing ratio, i.e. the water content of the air parcel.
$$T_v = T\frac{\varepsilon + w}{\varepsilon(1 + w)} \tag{9}$$

10. Consider a box of dry air at T(K) and the floor is covered with pure water. Water evaporates, and the vapor pressure increases. Eventually it reaches an equilibrium state such that rate of condensation is equal to rate of evaporation, and a **saturated** vapor pressure e_s is exerted to the plane surface of water. Define the **saturated mixing ratio** as $w_s = m_{ws}/m_d$ and with the ideal gas law,
$$w_s = 0.622\frac{e_s}{p} \tag{10}$$
    Relative Humidity, the ratio of actual mixing ratio w to the saturation mixing w_s, is to see the percentage of water the air holds as compared to its maximum, i.e. RH = w/w_s.

11. Consider the latent heat effect $dq = -Ldw_s$ at a **saturated adiabatic process**.
    Given the first law: $dq = -Ldw_s = c_p dT + gdz$
$$\frac{dT}{dz} = -\frac{L}{c_p}\frac{dw_s}{dz} - \frac{g}{c_p} \rightarrow \Gamma_s = -\frac{dT}{dz} = \frac{\Gamma_d}{1 + (L/c_p)(dw_s/dT)} \tag{11}$$
    It is noted that:
    - $\Gamma_s = \Gamma_s(p, T)$ is not a constant.

- $dw_s/dT > 0$, then $\Gamma_s < \Gamma_d$
- $\Gamma_s = 6 \sim 7$ K km^{-1} in middle troposphere.

12. **Equivalent potential temperature** is to include the effect of moisture / saturation into the potential temperature (i.e. adiabatic process) consideration.
Given the first law:

$$dq = c_p dT - v dp \rightarrow \frac{dq}{T} = c_p \frac{dT}{T} - R\frac{dp}{p}$$

and the definition of potential temperature:

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \rightarrow c_p \frac{d\theta}{\theta} = c_p \frac{dT}{T} - R\frac{dp}{p}$$

Substitute latent heat equation $dq = -L dw_s$ and assumes L, c_p, T are relatively constant,

$$d\left(-\frac{Lw_s}{c_p T}\right) = \frac{d\theta}{\theta} \rightarrow \ \theta_e = \theta \exp\left(\frac{Lw_s}{c_p T}\right) \tag{12}$$

Given an air parcel heated up by the ground, and the buoyancy force drives it up (i.e. a constant θ process) to the **lifting condensation layer**, the air parcel is saturated. The air parcel is further driven up and water is removed to release latent heat and cool down the air parcel. The dry air moves down ($p \uparrow T \uparrow$) with higher density and lower temperature.

13. Consider an *unsaturated* air parcel with a small upward displacement dz in a **stably stratified** atmosphere.
The air parcel experiences a downward force

$$(\rho' - \rho)g \text{ per unit volume, or} (\rho' - \rho)g/\rho \text{ per unit mass}$$

Given the ideal gas law $p = \rho RT$, and $(T - T') = (\Gamma_d - \Gamma)dz$

$$F = \frac{g(\Gamma_d - \Gamma)dz}{T} \tag{13}$$

If $\Gamma_d > \Gamma$, the force restores its position and it is **positively stable**.
Consider also its **potential temperature** θ. Given $-L\, dw_s = c_p dT + gdz$ and

$$-\frac{L}{c_p T} dw_s = \frac{d\theta}{\theta}$$

The static stability in terms of potential temperature is given by

$$\frac{1}{\theta}\frac{d\theta}{dz} = \frac{1}{T}(\Gamma_d - \Gamma) \tag{14}$$

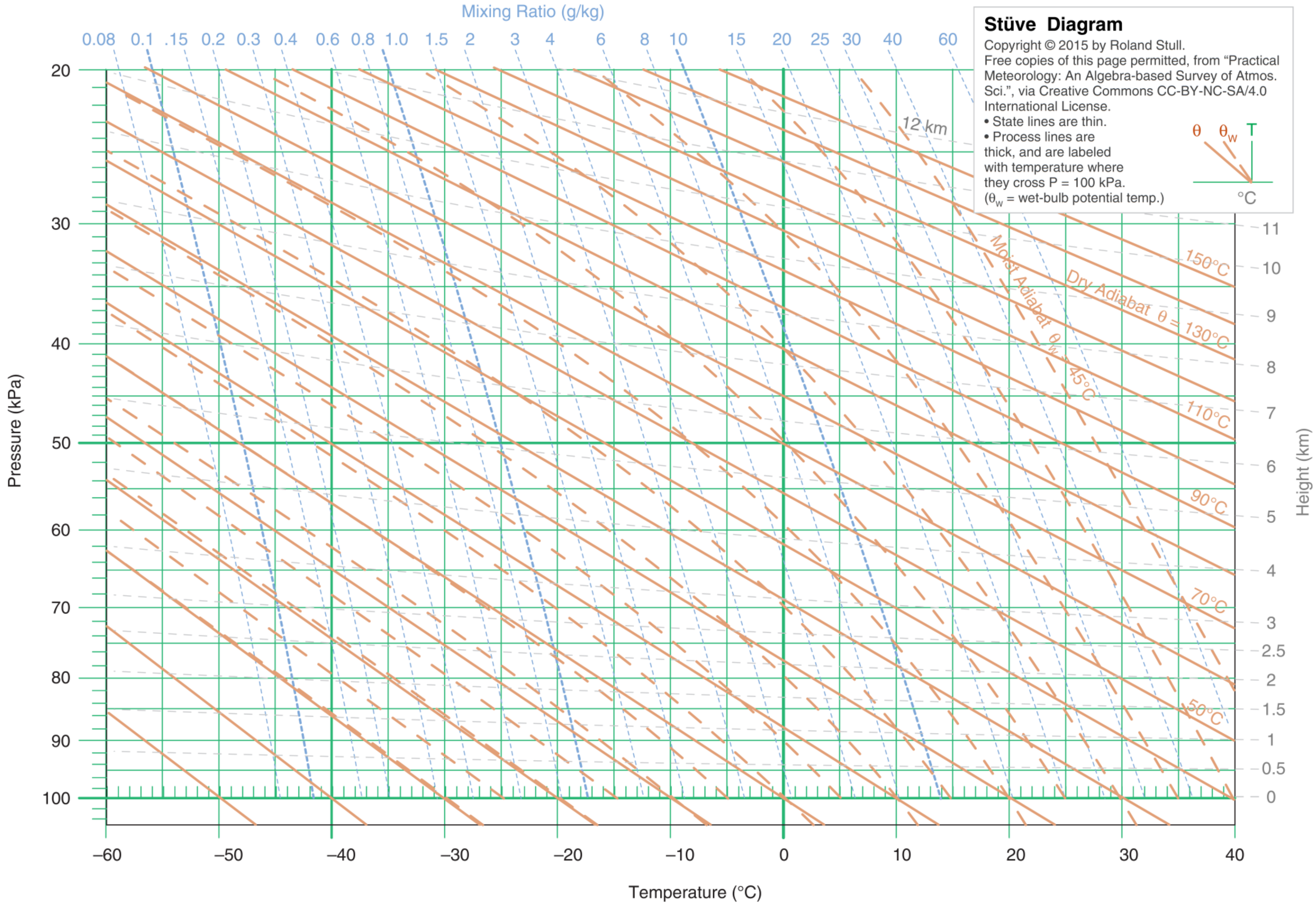
It indicates its stability with $\Gamma_d > \Gamma$, and hence positive $d\theta/dz$.

	Dry air	Moist air
First Law	$dq = c_p dT + gdz$	$dq = c_p dT + gdz + Ldw_s$
Adiabatic Process (dq = 0)	$dq = c_p dT + gdz = 0 \rightarrow -\frac{dT}{dz} = \frac{g}{c_p}$	$c_p dT + gdz = -Ldw_s$
Potential Temperature	$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$	$\theta_e = \theta \exp\left(\frac{Lw_s}{c_p T}\right)$
Vertical Motion	$\frac{d\theta}{dz} = 0$	$\frac{d\theta_e}{dz} = 0$
Lapse Rate	$-\frac{dT}{dz} = \frac{g}{c_p}$	$\frac{dT}{dz} = -\frac{L}{c_p}\frac{dw_s}{dz} - \frac{g}{c_p}$

Stüve Diagram

- State lines are thin.
- Process lines are thick, and are labeled with temperature where they cross P = 100 kPa.

(θ_w = wet-bulb potential temp.)



Chapter 2 Atmospheric Thermodynamics

Formula

Ideal Gas Law: $p = \rho RT$ or $pv = RT$

Hydrostatic Equation: $dp = \rho g dz$ (0)

Virtual Temperature:
$$T_v = \frac{T}{1 - \left(\frac{e}{p}\right)(1 - \varepsilon)} \quad (1)$$

Geopotential Height:
$$Z = \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g(z)dz\,, \qquad \Delta Z = \frac{R_d}{g_0}\int_{p_2}^{p_1} T_v \frac{dp}{p} \quad (2)$$

Scaled Height:
$$p_2 = p_1 \exp\left(-\frac{Z_2 - Z_1}{H}\right) \quad (3)$$

Hypsometric Equation:
$$Z_2 - Z_1 = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{p_2}{p_1}\right) = \bar{H} \ln\left(\frac{p_2}{p_1}\right)$$

given: (4)
$$\bar{T}_v = \int_{\ln p_2}^{\ln p_1} T_v\, d(\ln p) \Big/ \int_{\ln p_2}^{\ln p_1} d\,(\ln p)$$

Lapsed Rate:
$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)} dz \rightarrow z = \frac{T_0}{\Gamma}\left(1 - \left(\frac{p}{p_0}\right)^{\frac{R\Gamma}{g}}\right) \quad (5)$$

First Law:
$$dq = d(h + \Phi) = c_p dT + g dz \quad (6)$$

Dry Adiabatic Lapse Rate:
$$dq = c_p dT + g dz = 0 \rightarrow -\left(\frac{dT}{dz}\right)_{dry} = \frac{g}{c_p} = \Gamma_d = 9.8\ \mathrm{K\,km^{-1}} \quad (7)$$

Effective Temperature & **Poisson Equation**:
$$\frac{c_p}{R}\frac{dT}{T} - \frac{dp}{p} = 0 \rightarrow \theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \quad (8)$$

Virtual Temperature with mixing ratio:
$$T_v = T\frac{\varepsilon + w}{\varepsilon(1 + w)} \quad (9)$$

Saturated Mixing Ratio:
$$w_s = 0.622\frac{e_s}{p} \quad (10)$$

Saturated Adiabatic Process:
$$\frac{dT}{dz} = -\frac{L}{c_p}\frac{dw_s}{dz} - \frac{g}{c_p} \rightarrow \Gamma_s = -\frac{dT}{dz} = \frac{\Gamma_d}{1 + (L/c_p)(dw_s/dT)} \quad (11)$$

Effective Potential Temperature:
$$d\left(-\frac{Lw_s}{c_p T}\right) = \frac{d\theta}{\theta} \rightarrow \theta_e = \theta \exp\left(\frac{Lw_s}{c_p T}\right) \quad (12)$$

Net force in stably stratified environment:
$$F = \frac{g(\Gamma_d - \Gamma)dz}{T} \quad (13)$$

Stability in Potential Energy form:
$$\frac{1}{\theta}\frac{d\theta}{dz} = \frac{1}{T}(\Gamma_d - \Gamma) \quad (14)$$

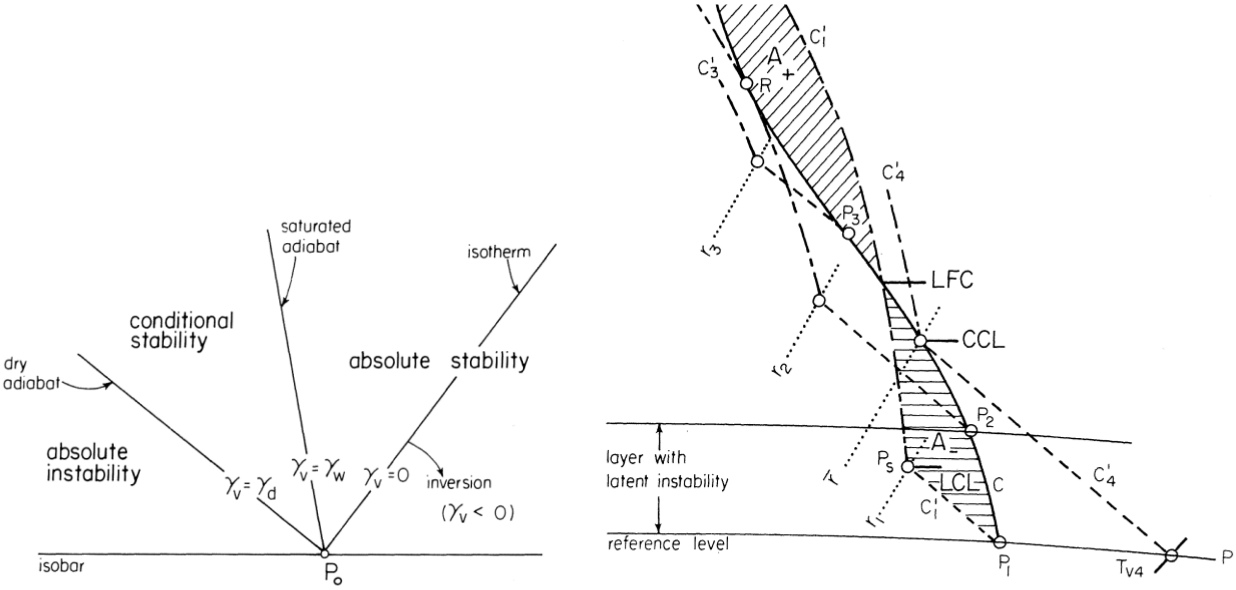


Figure 1(a) Stability with Lapsed Rate, (b) Latent Instability

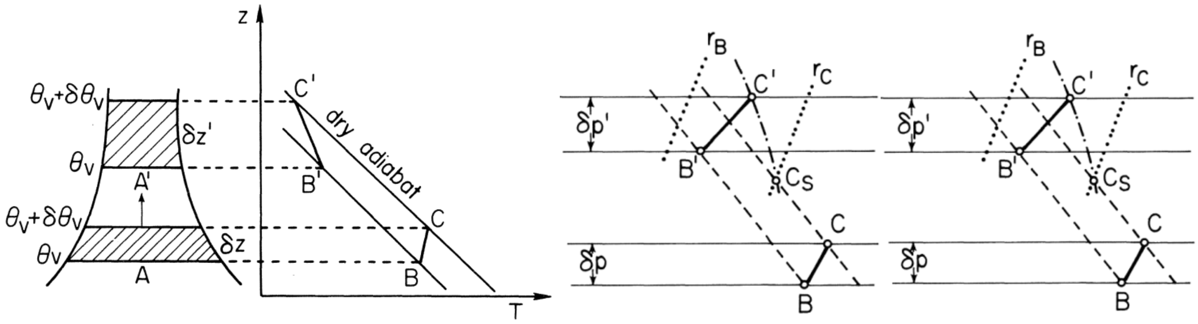


Figure 2(a) Potential instability, non-saturated layer
(b) Potentially unstable layer becoming partly saturated. (c) Potentially stable layer becoming partly saturated.

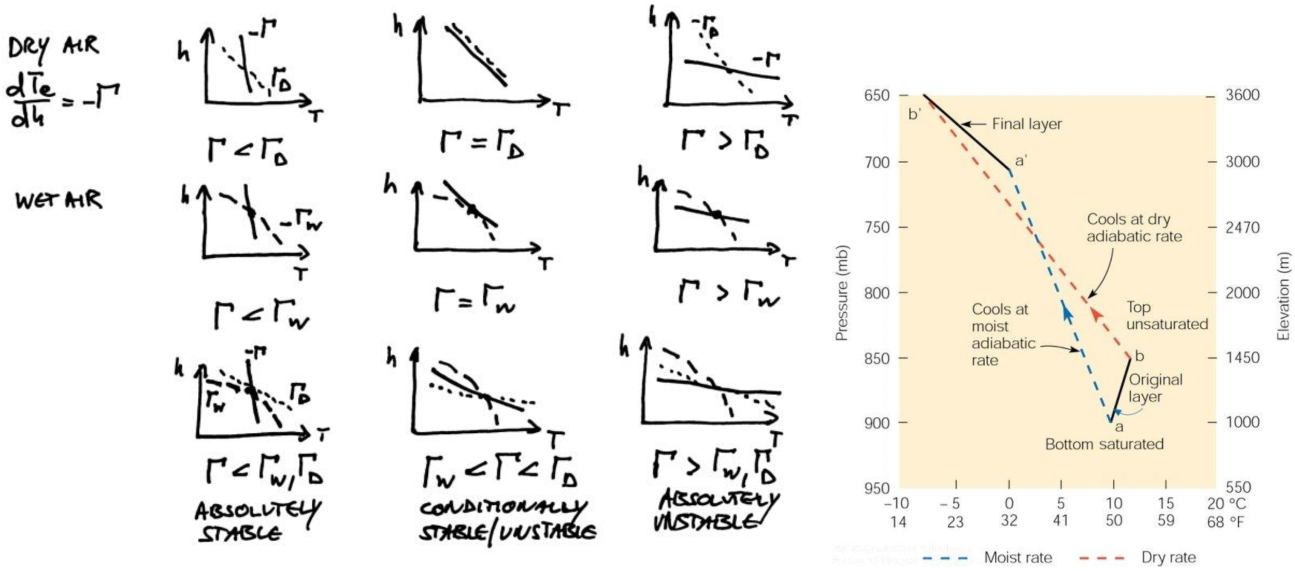


Figure 3(a) z-T diagram for wet air in different stability condition (b) Convective Instability

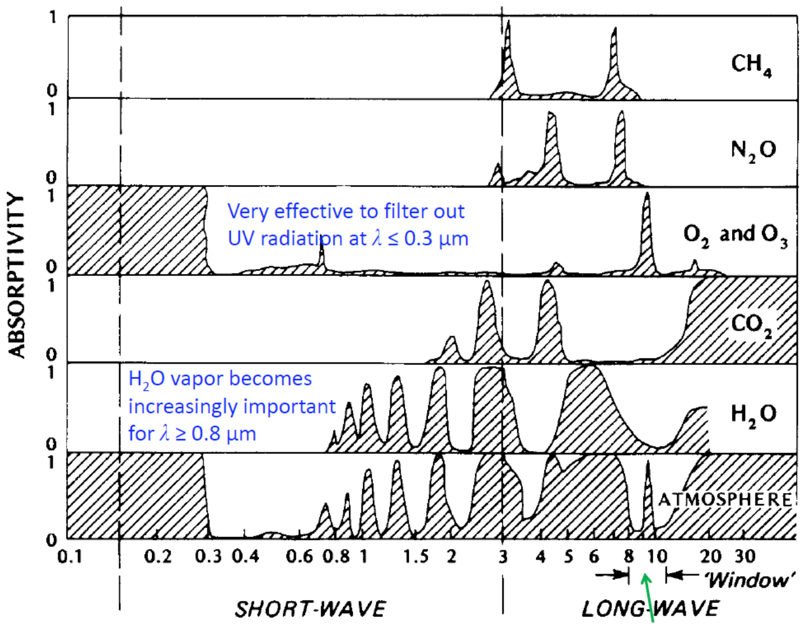
Chapter 3 – Energy and Mass Balance

The only energy to the Earth is **radiation** from sun. It is **reflected** by air or cloud ($I_0/4$ = 342W/m^2) water, as large as 48% reflected and 23% **absorbed**) in atmosphere and **transmitted** to ground with 70% (25%) of the source at clear sky (cloudy) condition. At daytime, **short wave radiation** is transmitted to the Earth; At night, **long wave radiation** is released from Earth. Different constituent components have its **absorptivity** in different wavelength.

1. Each material has its properties in radiation:

$$\Psi_\lambda + \alpha_\lambda + \zeta_\lambda = 1 \tag{1}$$

given Ψ_λ = transmittivity, α_λ = reflectivity and ζ_λ = absorptivity.



O_3 is able to absorb (filter out) short-wave radiation with $\lambda \leq 0.3$ um.
H_2O (cloud) absorbs most **short-wave radiation** in atmosphere with $\lambda \in (0.8\text{ um}, 3\text{um})$ at daytime; It absorbs (blocks any escape) **long-wave radiation** in atmosphere at night.
Greenhouse gases such as CH_4, N_2O, *CO_2 and *O_3 mainly block the long-wave radiation at night. It provides a small **atmospheric window**.

2. Radiation from the sun is reflected from the cloud $K_{Ac}^{\uparrow}$ and air in atmosphere $K_{Aa}^{\uparrow}$, absorbed by the cloud and air, i.e. K_{Ac}^{*} and K_{Aa}^{*}, and reflected and absorbed by earth, i.e. $K_{E}^{\uparrow}$ and K_{E}^{*}. The balance equation is given by

$$\overline{K}_{EX} = K_{Ac}^{\uparrow} + K_{Aa}^{\uparrow} + K_{Ac}^{*} + K_{Aa}^{*} + K_{E}^{\uparrow} + K_{E}^{*} \tag{2}$$

The day time surface budget is

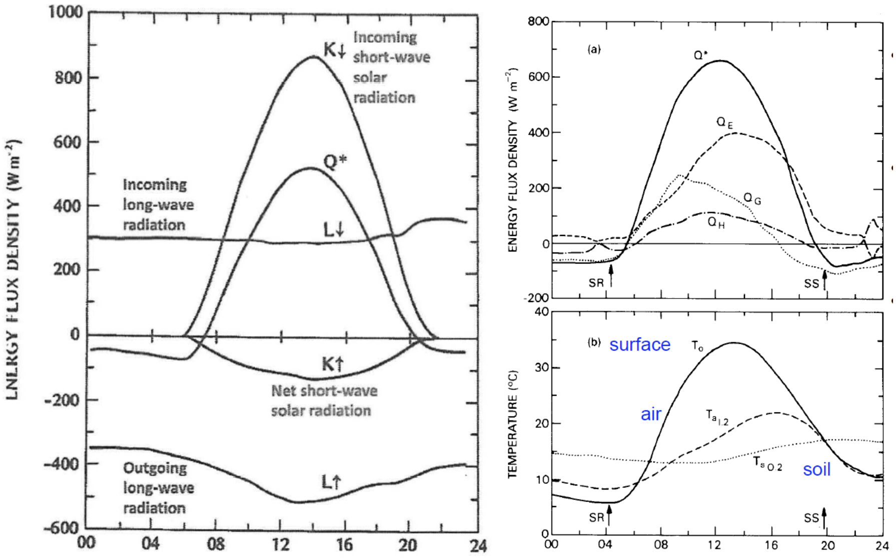
$$Q^{*} = (K\downarrow - K\uparrow) + (L\downarrow - L\uparrow) = Q_H + Q_E + Q_G \tag{3}$$

where (which is absorbed and released at all time)

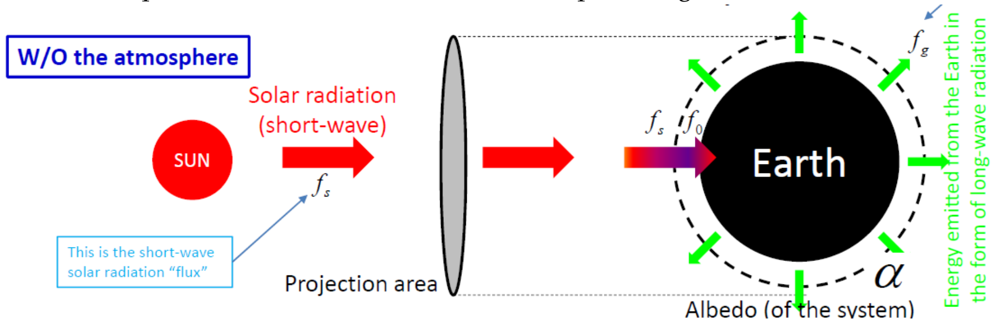
$$L\uparrow = \varepsilon\sigma T_o^4 + (1-\varepsilon)L\downarrow$$

Note:

- Short-wave radiation has the relationship $K\uparrow = \alpha K\downarrow$.
- From the Q graph, most heat is driven by Q_E, the latent heat. The net heat is absorbed from ground to soil at daytime and released at nighttime as an energy storage.



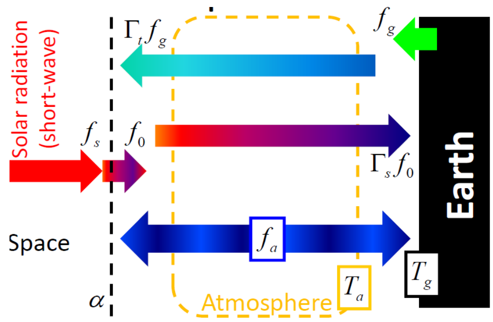
3. The equilibrium temperature with and without the atmosphere is given as follows.



Assume Earth is a black body and a is the Earth radius.
Short wave energy received from sun to Earth: $F_0 = f_0\, \pi a^2 = (1-\alpha) f_s \pi a^2$
Long wave energy released from Earth to sun: $F_g = f_g 4\pi a^2 = \sigma T_g^4 (4\pi a^2)$

$$F_0 = F_g \rightarrow T_g^4 = \frac{(1-\alpha) f_s}{4\sigma} \tag{4}$$

<u>With the atmosphere</u>



The energy balance over and below the atmosphere is given by:

$$f_0 = f_a + \Gamma_t f_g; \;\; f_g = f_a + \Gamma_s f_0$$

It gives

$$f_g = \frac{1+\Gamma_s}{1+\Gamma_t} f_0; \;\; f_g = \sigma T^4; \;\; f_0 = \frac{(1-\alpha) f_s}{4}$$

Hence,

$$T_g^4 = \frac{1-\alpha}{4\sigma} \frac{1+\Gamma_s}{1+\Gamma_t} f_s \tag{5}$$

Chapter 4 Atmospheric Fluid Dynamics

A wide variety of flows occurs in the atmosphere, given **vertical flow** is mainly driven by the **buoyancy** (temperature) and hydrostatic pressure, and horizontal flow is driven by **Coriolis force**.

1. Mass Balance:
   With a small change in x, i.e. $x \to x + dx$:

$$(\rho(x)u(x) - \rho(x+dx)u(x+dx)dydz = -\frac{\partial}{\partial x}(\rho u)dx \ \cdot dydz$$

   Including the inflow and outflow of y and z direction, the net inflow is $-\nabla \cdot (\rho\vec{u})\Delta V$
   It must be equal to the rate of increase of mass in the box, i.e. $\partial/\partial t\ (\rho \Delta V)$
   The **continuity equation** is

$$-\nabla \cdot (\rho\vec{u})\Delta V = \frac{\partial}{\partial t}(\rho \Delta V) \to \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0$$

   Given the material derivative, or advective derivative:

$$\frac{\partial}{\partial t} + \vec{u} \cdot \nabla = \frac{D}{Dt}$$

   It can also be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0 \to \frac{\partial \rho}{\partial t} + \rho\nabla \cdot \vec{u} + \vec{u}\nabla\rho = 0 \to \frac{D\rho}{Dt} + \vec{u}\nabla\rho = 0 \qquad (1)$$

2. Given the acceleration of a particle with path $\vec{r}(t)$:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\vec{u}$$

   which is a *non-linear* term.
   (a) The **pressure force** in x-direction is given by:

$$F_p = pdydz - \left(p + \frac{\partial p}{dx}dx\right)dydz = \frac{\partial p}{dx}dV$$

   Hence, the 3D vector pressure force is given by $F_p = -dV(\nabla p)$

   (b) The **gravity force** is given by $F_g = -\dot{m}g\vec{k} = -\rho dVg\vec{k}$
   (c) The **viscous force** is given by the kinetic theory,

$$\tau = \mu\frac{\partial u}{dz} \to F_\tau = [\tau(z+dz) - \tau(z)]dxdy = \frac{\partial \tau}{dz}dxdydz = \mu\frac{\partial^2 u}{dz^2}dV$$

   The 3D vector shear is given by $F_\tau = dV\mu\nabla^2\vec{u}$

   The momentum equation, i.e. **Navier-Stokes Equation**, is given by:

$$(\rho dV)\vec{a} = -dV(\nabla p) - \rho dVg\vec{k} + dV\mu\nabla^2\vec{u}$$

   Rearrange the terms,

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{u} - g\vec{k} \qquad (2)$$

3. Consider a **rotating reference frame** R, $\vec{A}$ changes from $\vec{A}(t)$ to $\vec{A}(t+dt) = \vec{A}(t) + d\vec{A}_R$. At the inertial frame, $d\vec{A}_I = d\vec{A}_R + \vec{\Omega} \times \vec{A}dt$. Hence,

$$\left(\frac{d\vec{A}}{dt}\right)_I = \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times \vec{A}$$

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_I = \left(\frac{d^2\vec{A}}{dt^2}\right)_R + 2\vec{\Omega} \times \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{A})$$

Substitute $\vec{A}(t)$ as $\vec{r}(t)$,

$$\vec{a}_I = \vec{a}_R + 2\vec{\Omega} \times u_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad (3)$$

The acceleration in inertial frame I is the acceleration in rotating frame R, Coriolis acceleration $2\vec{\Omega} \times u_R$ and Centrifugal acceleration $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$.

4. Include the Coriolis and Centrifugal acceleration, the **Navier-Stokes Equation** is given by

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + 2\vec{\Omega} \times u + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \frac{\mu}{\rho}\nabla^2\vec{u} - g\vec{k}$$

Note that the centrifugal acceleration is too small.
Introduce small incremental distance in the spherical coordinate:

- Eastward: $dx = r\cos\phi \ d\lambda$
- Northward: $dy = rd\phi$
- Vertical distance: $r = a + z \to dr = dz$

It gives the Navier-Stokes Equation in rotating frame: (F_x, F_y, F_z are frictional force in eastward, northward and upward direction)

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x$$
$$\frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{1}{\rho}\frac{\partial p}{\partial y} = F_y$$
$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\phi + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = F_y$$

Assume:

- $r \approx a$
- $|u| < 100\text{ms}^{-1}$, $\Omega a \approx 465\text{ms}^{-1}$ (Except near the pole $\cos\phi \to 0$) $\to \frac{|u|}{a\cos\phi} \ll 2\Omega$
- Vertical velocity is much less than horizontal except near the equator ($\sin\phi \to 0$) $\to$ $|v\sin\phi| \ll |w\cos\phi\,|$
- Introduce the Coriolis parameter $f = 2\Omega\sin\phi$
- $\frac{u^2+v^2}{a} \approx 0$, $2\Omega u\cos\phi \approx 0$ as compared to g
- F_z is negligible

The **Navier-Stokes Equation (rotating frame)** is simplified to:

$$\frac{Du}{Dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x$$
$$\frac{Dv}{Dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = F_y \quad (4)$$
$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

Note:

F-Plane Approximation: Take $f = 2\Omega\cos\phi$ as a constant, i.e. $f_0 = 2\Omega\cos\phi_0$. The geostrophic equation acts as a system rotating about z-axis.

***β*-Plane Approximation**: For large system where ϕ cannot be put as a constant, linearization applies: $f = f_0 + \beta y = f_0 + \frac{2\Omega\cos\phi_0}{a}y$. It allows variation (linearly) with northward distance y.

5. **Geostropic and Hydrostropic Approximation**
Given the scales for approximation:

Scale	Symbol	Typical magnitude
Horizontal scale	L	1,000 km = 10^6 m
Vertical scale	H	10 km = 10^4 m
Horizontal velocity	U	10 m sec^{-1}
Vertical velocity	W	10^{-2} m sec^{-1}
Timescale	T	1 day ~ 10^5 sec
Surface density	ρ	1 kg m^{-3}
Earth's radius	a	6.4×10^6 m
2 × rotation rate	2Ω	10^{-4} sec^{-1}
Acceleration of gravity	g	10 m sec^{-2}

Consider the vertical momentum equation:

$$\frac{Dw}{Dt} \approx 0 \rightarrow \frac{\partial p}{\partial z} = -\rho g \tag{5a}$$

For horizontal momentum equation:

$$\frac{\partial u}{\partial t} \approx 10^{-4}, \qquad u\frac{\partial u}{\partial x} \approx 10^{-4}, \qquad w\frac{\partial u}{\partial z} \approx 10^{-4}, \qquad fv = 2\Omega \sin\phi\, v \approx 10^{-3}$$

$$\begin{cases} \dfrac{Du}{Dt} - fv + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = F_x \\ \dfrac{Dv}{Dt} + fu + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = F_y \end{cases} \rightarrow \begin{cases} fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \end{cases} \tag{5b}$$

Note: The Coriolis force in y-direction is balanced by pressure gradient in x-direction. Hence, rotating flow is along isobar (not from high pressure to low pressure)

6. Substitute the ideal gas law: $p = \rho RT$

$$\frac{\partial}{\partial z}: \; fv = \frac{RT}{p}\frac{\partial p}{\partial x} = RT\frac{\partial \ln p}{\partial x}$$
$$\frac{\partial}{\partial x}: \; -\frac{g}{RT} = \frac{\partial \ln p}{\partial z}$$

Based on the cross differentiation, the **thermal wind shear equation** is derived.

$$f\frac{\partial v}{\partial z} = \frac{g}{T}\frac{\partial T}{\partial x}, \qquad f\frac{\partial u}{\partial z} = -\frac{g}{T}\frac{\partial T}{\partial y} \tag{6}$$

If the wind turns anticlockwise (clockwise) with height, the wind on average blows across the isotherms from the cold to the warm; cold advection (warm to cold; warm advection) side

7. **Gradient Wind Balance**:
Consider a vortex centred at a point P, near the latitude ϕ and use the polar coordinates r, φ in horizontal plane.
The velocity vector is given by $\vec{u} = V(r)\vec{\imath}_\varphi$
Coriolis force: $-\rho f\vec{k} \times \vec{u} = \rho f V(r)\vec{\imath}_r$
Centrifugal force: $\rho\, V^2/r\, \vec{\imath}_r$
Sum of Coriolis force and Centrifugal force balances the radial pressure gradient:

$$\frac{V^2}{r} + fV = \frac{1}{\rho}\frac{dp}{dr} = G \tag{7}$$

Note: Gradient-Wind Balance does NOT cancel the centrifugal force as geostropic balance does.
Solving the quadratic equation,

$$V(r) = -\frac{rf}{2} \pm \left(\frac{r^2 f^2}{r} + Gr\right)^{\frac{1}{2}}$$

$$G = \frac{1}{\rho}\frac{dp}{dr} \geq -r\frac{f^2}{4}$$

Note: It means that the pressure drop in radial direction cannot be too rapid if gradient wind balance is to occur.

8. Meteorologists employs pressure p instead of height z as vertical coordinate as synoptic-scale height cannot be measured and Navier-Stoke Equation in p is simpler.
Consider a differential column of air: $dp = -\rho g dz$ (one-to-one mapping)
Hydrostatic Equation (in pressure form):

$$g\left(\frac{\partial z}{\partial p}\right)_{x,y,t} = -\left(\frac{1}{\rho}\right) = -\frac{RT}{p} \rightarrow \left(\frac{\partial gz}{\partial p}\right)_{x,y,t} = \left(\frac{\partial \Phi}{\partial p}\right)_{x,y,t} = -\frac{RT}{p} \tag{8a}$$

For Geostropic Equation (in pressure form):

$$fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{\partial z}{\partial p}\left(-\frac{gp}{RT}\right) = -g\left(-\frac{\partial z}{\partial x}\right) \rightarrow fv = \left(\frac{\partial \Phi}{\partial x}\right)_{y,p,t}$$
$$fu = -\left(\frac{\partial \Phi}{\partial y}\right)_{x,p,t} \tag{8b}$$

For the geometric vertical velocity w, it is introduced a pressure velocity ω.

$$\omega = \frac{Dp}{Dt} = -\rho g w \tag{8c}$$

The continuity equation is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \rightarrow \left(\frac{\partial u}{\partial x}\right)_{y,p,t} + \left(\frac{\partial v}{\partial y}\right)_{x,p,t} + \left(\frac{\partial w}{\partial p}\right)_{x,y,t} \tag{8c}$$

9. The 1st Law of Thermodynamics is given by: $dq = du + dw \rightarrow TdS = dU + pdV$
With ideal gas law:

$$TdS = c_p dT - \frac{1}{\rho}dp$$

Divide the equation with dt and let $dt \rightarrow 0$:
Consider a moving (Lagrangian) mass, i.e. time derivative with D/Dt, the **thermodynamic equation** becomes:

$$Q = T\frac{DS}{Dt} = c_p\frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} \tag{9a}$$

With adiabatic motion, i.e. Q = 0:

$$\frac{DT}{Dt} = \frac{1}{\rho c_p}\frac{Dp}{Dt}$$

The temperature of a moving blob of air:

- Increases if it descends (move to higher pressure: $P \uparrow T \uparrow$)

If the blob moves adiabatically, it also means constant potential temperature θ.
Recall definition of potential temperature θ:

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$$

Substitute back to thermodynamic equation, it, thermodynamic equation in potential temperature form:

$$\frac{D\theta}{Dt} = \frac{Q}{c_p}\left(\frac{p}{p_0}\right)^{-\frac{R}{c_p}} \tag{9b}$$

Chapter 7 Stratification & Shallow Water Equation

Stratification is the fluid parcels with different densities. With gravity, heavier fluid goes below those of lower density. Gradient of properties in vertical affects the velocity field.

1. **Static Stability**:

   Given an air parcel $\rho(z)$ moves h upwards with background density $\rho(z+h)$, the downward force is given by:

$$\rho(z)V\frac{d^2h}{dt^2} = g[\rho(z+h) - \rho(z)]V$$

$$\frac{d^2h}{dt^2} - \left(-\frac{g}{\rho_0}\frac{d\rho}{dz}h\right) = \frac{d^2h}{dt^2} + N^2h = 0 \tag{1}$$

   where N^2 is the Brunt-Vaisala (BV) frequency.

   Troposphere has a more **unstable** ($N^2<0$) environment at near-ground region with T increases with height. The **turbulent** flow is mainly driven by the **ground temperature**, which is heated by short-wave radiation. **Stratosphere,** on the other hand, is more **stable**, and in **weak turbulent flow**.

2. **Flow Condition** and **Hydraulic Jump**

   Given the density perturbation within Δz as:

$$\Delta\rho = \frac{d\rho}{dz}\Delta z = \frac{\rho_0 N^2}{g}$$

   The hydrostatic pressure is calculated:

$$\Delta P = \Delta\rho\, gH = \rho_0 N^2 H\Delta z$$

   Navier Stoke Equation can be reduced to (dp/dz induces change in velocity)

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \sim \frac{1}{\rho_0}\frac{\partial p}{\partial x} \to \frac{U^2}{L} = \frac{\Delta P}{\rho_0 L}$$

$$U^2 = N^2 H\Delta z$$

   Ratio of vertical convergence to horizontal divergence:

$$\frac{\frac{W}{H}}{\frac{U}{L}} = \frac{\Delta z}{H} = \frac{U^2}{N^2H^2} = Fr^2$$

   - **Hydraulic Jump**, which is mainly dependent on **initial fluid velocity**, is a dissipation of energy from **supercritical** (Fr > 1) flow to **subcritical flow** (Fr < 1)

   Given the **governing equation**:

   Bernoulli Equation just below the interface:

$$\frac{p_1}{\rho_1} + \frac{U^2}{2} + g(h + h_1) = \text{constant}$$

   Continuity:

$$U_1 h_1 = Q = \text{constant}$$

   Pressure just above the interface:

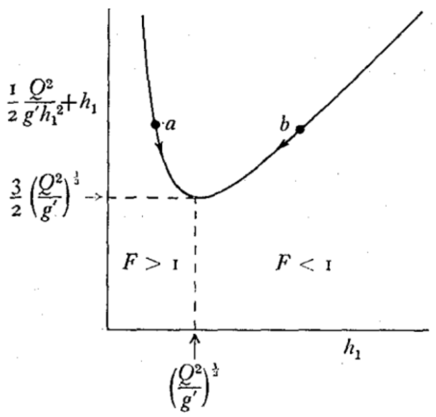
$$\frac{p_1}{\rho_2} + g(h + h_1) = \text{constant}$$

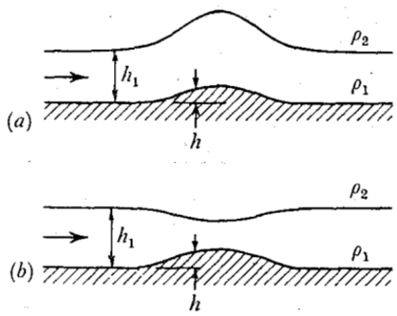
   Eliminate p_1 and U:

$$\frac{1}{2}\frac{Q^2}{g' h_1^2(x)} + h_1(x) = C - h(x)$$

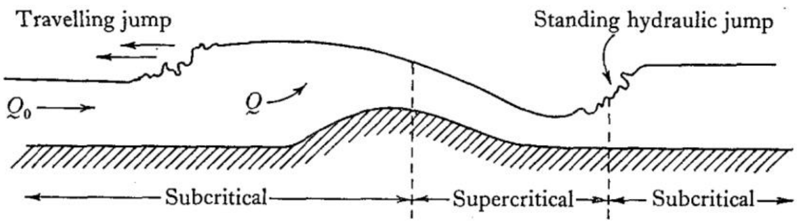
   Froude number Fr:

$$Fr^2 = \frac{Q^2}{g' h_1^3} = \frac{U^2}{g' h_1}$$





- Fr = 1: Flow velocity ($Uh = Q, 1 = U/\sqrt{gh_1}$) is equal to the infinitesimal long wave on the layer
- Fr > 1: (Supercritical) It has enough KE to go through any obstacles without any sacrificial to PE.
- Fr < 1: (Subcritical) Short wave can remain at rest relative to the obstacles
  When stationary wave appears downstream, longer waves can propagate upstream damped by friction.
  If h is too high such that h > h_C, the fluid has no more PE (blocking).



Apply the continuity equation and momentum to state before (h_1) and after (h_2):

$$h_c^3 = \frac{1}{2}h_1h_2(h_1 + h_2) \tag{2a}$$

Final depth:

$$h_2 = \frac{1}{2}h_1\left(\sqrt{1 + 8Fr_1^2} - 1\right) \tag{2b}$$

The energy dissipated is:

$$\frac{(h_2 - h_1)^3 g'Q}{4h_1h_2} \tag{2c}$$

- It causes **sudden change in pressure** with a **change of wind speed**
- Condensation and precipitation would intensify hydraulic jump effect
- Moist supercritical flow passes over a mountain when thickening and lifting can give rise to heavy orographic rainfall.

3. **Single Layer Model (Shallow-Water Reduced Gravity Model)**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3a}$$
$$g' = g\,(\rho_2 - \rho_1)/\rho_0$$

Internal Waves assume inviscid flow ($\mu = 0$) and decomposed density into $\rho_0 + \bar{\rho}(z) + \rho'(x, y, z, t)$. Wave solution is $e^{j(k_x x + k_y y + k_z z - wt)}$ and wave frequency is $w^2 = N^2(k_x^2 + k_y^2)/(k_x^2 + k_y^2 + k_z^2)$

Lee Wave is the generation of **internal waves** by wind blowing over the **irregular terrain**. It obeys a superposition of wind and background internal wave flow, i.e. $z = b(x + Ut) = H\cos(k_x x - wt)$. The dispersion relation is $k_x^2 = N^2/U^2 - k_z^2$.

Radiating Wave (N> k_xU) has a sufficient **strong stratification** or long topographic wavelength. Energy **radiating upward support the wave** and allow drag force $\rho_0\overline{u'w'}|_{z=0}$. $k_x = +\sqrt{N^2/U^2 - k_z^2}$ as only the wave with upward group velocity is physically relevant with energy source at the bottom.

Trapped Wave (N> k_xU) is a **weak stratification** with imaginary vales of k_z. Hence, all terms have the damping coefficient e^{-az}. There is no upward energy loss, and hence the drag force is equal to zero.

If the <u>KE loss > PE gain</u> in stratified fluid, i.e. $\frac{1}{8}(\rho_2 - \rho_1)gH^2 < \frac{1}{8}\rho_0(U_1 - U_2)^2 H$, **Kelvin-Helmholtz instability** occurs with wave number k unstable, i.e. $(\rho_2^2 - \rho_1^2)g < \rho_1\rho_2 k(U_1 - U_2)^2$. The discontinuity of density causes unstable environment and wave breaking and mixing occurs to make the environment stable.

Monin-Obukhov Length Scale (L) is to represent the depth with neutral stability (Ri = 0, $\varsigma = z/L$ = 0). When buoyancy is -ve with +ve Ri and ς, stable stratification occurs.