Distribution Analysis – Voltage Drop, Load Model, Power Flow

[Course Time = 9 hours]

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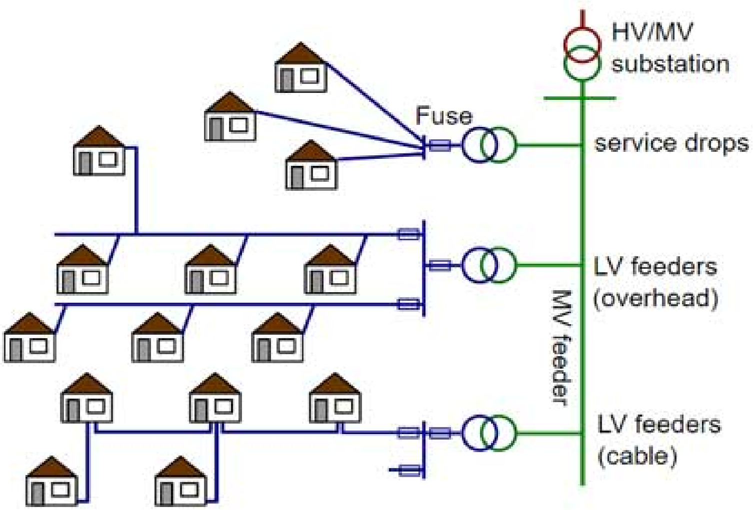
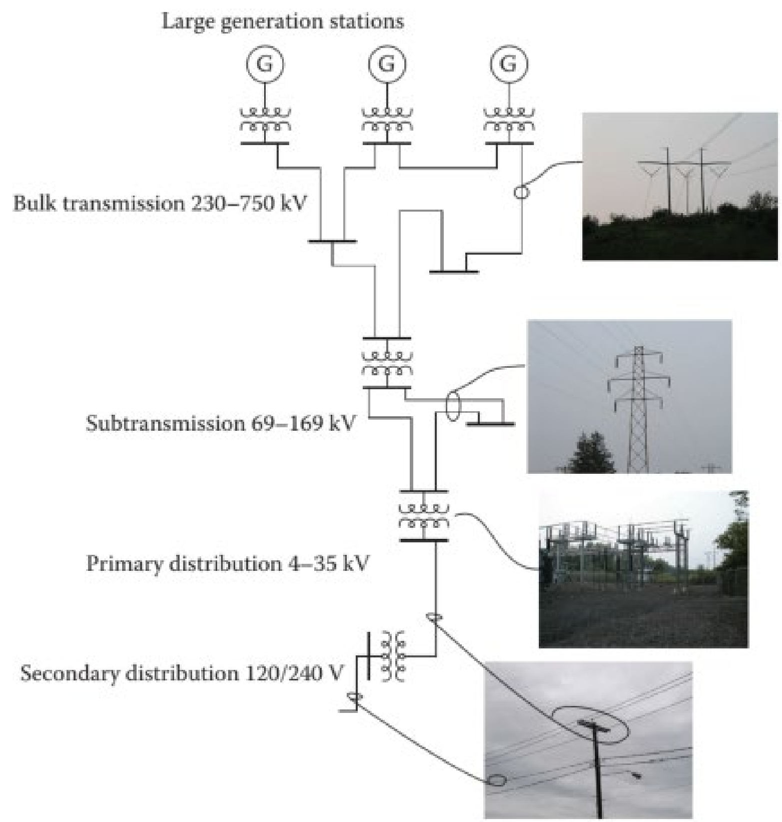
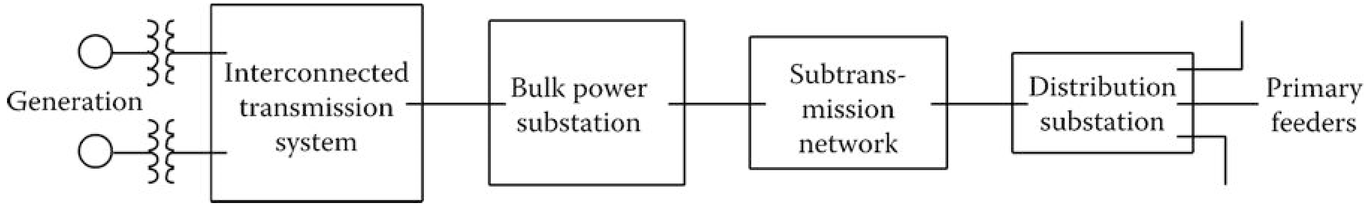
Power Distribution

- Electric power distribution is portion of power delivery infrastructure that takes electricity from highly meshed, high-voltage transmission circuits and delivers it to customers.
- Primary distribution lines are "medium-voltage" circuits, normally thought of as 600 V to 35 kV.
- From distribution transformer, secondary distribution circuits connect to end user.





Power Infrastructure and Role of Power Distribution



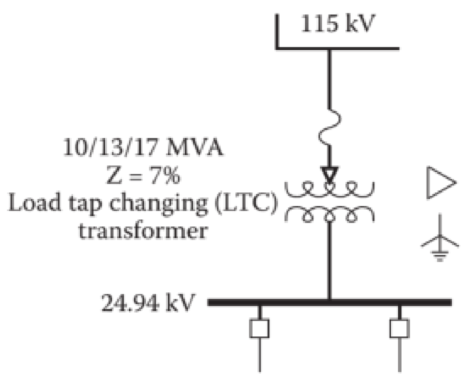
- Generation: 1kV-30 kV
- Ultra High Voltage Transmission: 500kV-765kV
- High Voltage Transmission: 230kV-345kV
- Sub-transmission system: 69kV-169kV
- Distribution system: 120V-35kV

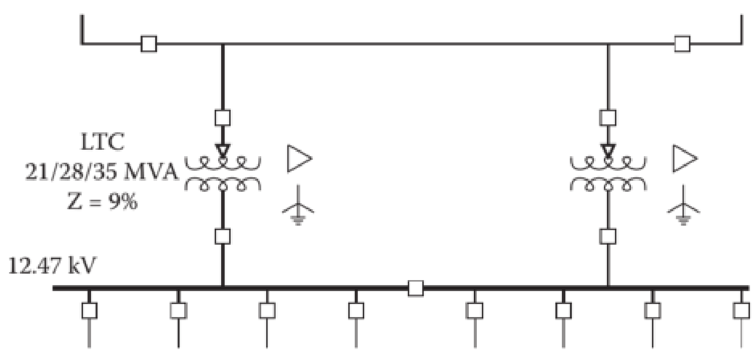
What are main differences between transmission and distribution systems?

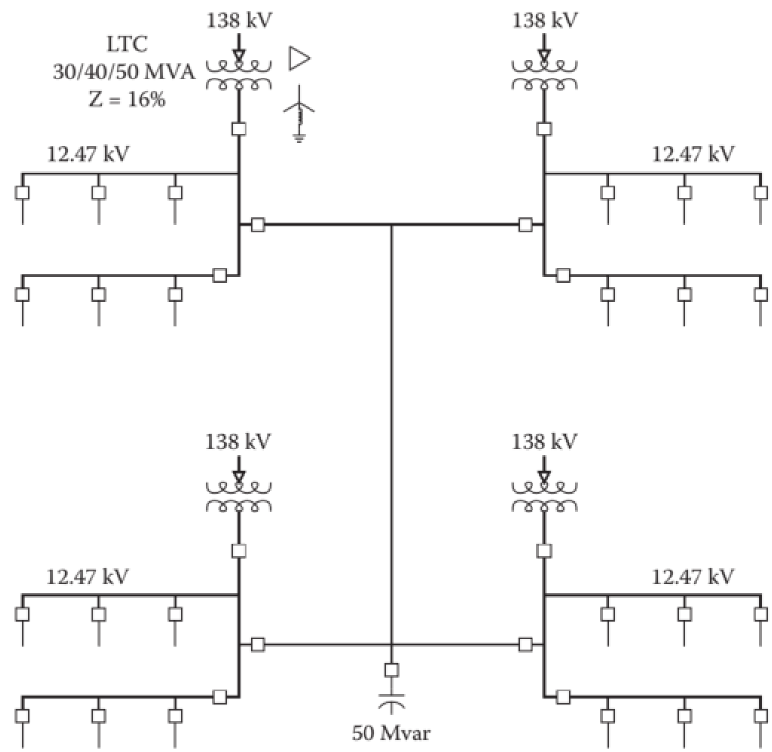
- Meshed vs Radial
- Balanced vs Unbalanced
- Voltage levels
- R/X ratios

Distribution Substation

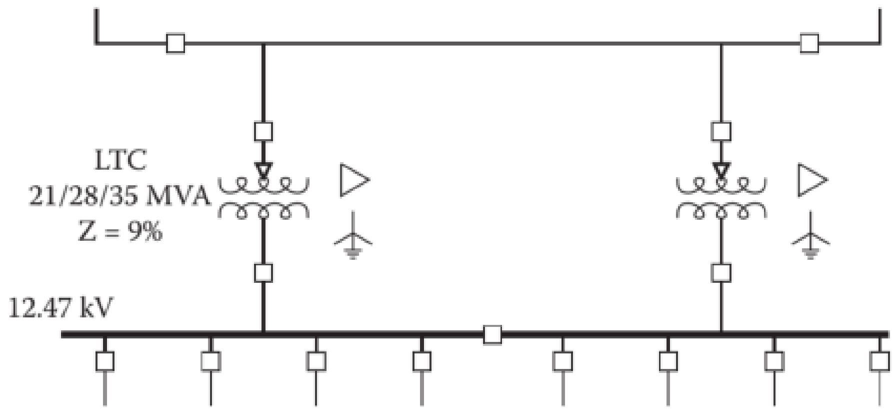
- Distribution substations come in many sizes and configurations.
- A small rural sub-station may have a nominal rating of 5MVA while an urban station may be over 200MVA. figures show examples of small, medium, and large substations.
- As much as possible, many utilities have standardized substation layouts, transformer sizes, relaying systems, and automation and SCADA (supervisory control and data acquisition) facilities.







Distribution Substation



- Substations with two transformers are very common. To ensure system security, it is often supplied with interconnectors connecting two primary substation (11kV or 22kV).
- Considering fault level at distribution board (11kV or 22kV), it is possible to parallel 2 to 3 transformers. Parallel transformer allow no loss of load in case of fault, even without any auto-switching systems.
- Transformers are possibly earthed with neutral earthing resistor (NER) to reduce voltage dip during fault and fault current in SLG fault.
- Normally, utilities size transformers so that if either transformer fails, remaining unit can carry entire substation’s load, i.e. firm capacity. Utility practices vary on how much safety margin is built into this calculation, and load growth can eat into redundancy.

Distribution Substation

Voltage transformation

A distribution substation is to step down voltage to distribution voltage level. Standard substation designs will call for two or more three-phase transformers.

Voltage regulation

To maintain user's voltages within an acceptable range, voltage at substation needs to vary as load varies.

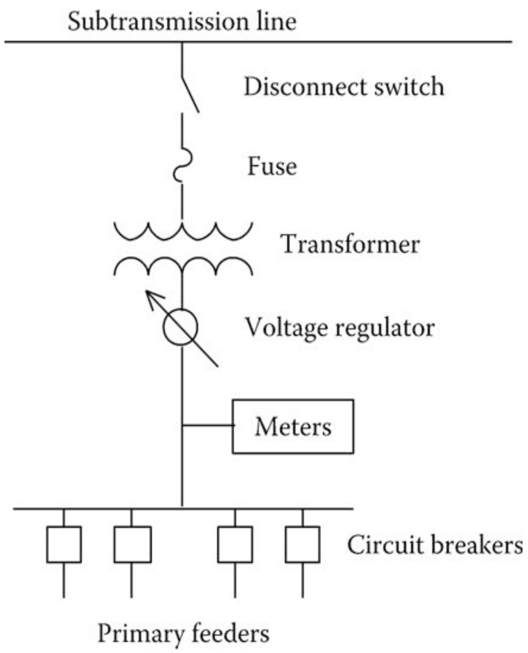
Transformers are often provided with “on load tap changer” (OLTC), which can select taps at HV neutral side to alter turn ratio and hence respective voltage output. Transformers are provided with Automatic Voltage Regulator (AVR) for voltage control through OLTC.

This can be in form of a three-phase gang-operated regulator or individual phase regulators that operate independently.

Protection

Substation are protected against occurrence of short circuits, a.k.a. faults. The simplest designs from HV to LV are fuse or OCEF protection. It is often time coordinated, a.k.a. discriminated or graded with IDMT characteristics curves.

Individual feeder circuit breakers or reclosers are used to provide interruption of short circuits that occur outside substation.



Distribution Substation

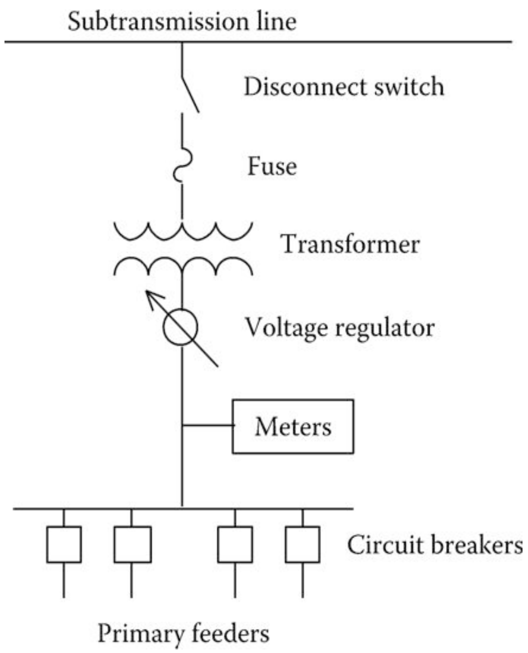
Metering and Instrumentation

To provide observability to system control engineers (SCEs), system are equipped with transducer to measure and transfer the analog signal to SCADA systems. Additional state estimation can be performed to validate the switch status.

Every substation has some form of metering. This may be an ammeter displaying present value of substation current as well as minimum and maximum currents that have occurred over a specific time period.

Digital recording meters are becoming very common. These meters record minimum, average, and maximum values of current, voltage, power, power factor, etc., over a specified time range. Typical time ranges are 15 min, 30 min, and 1 h.

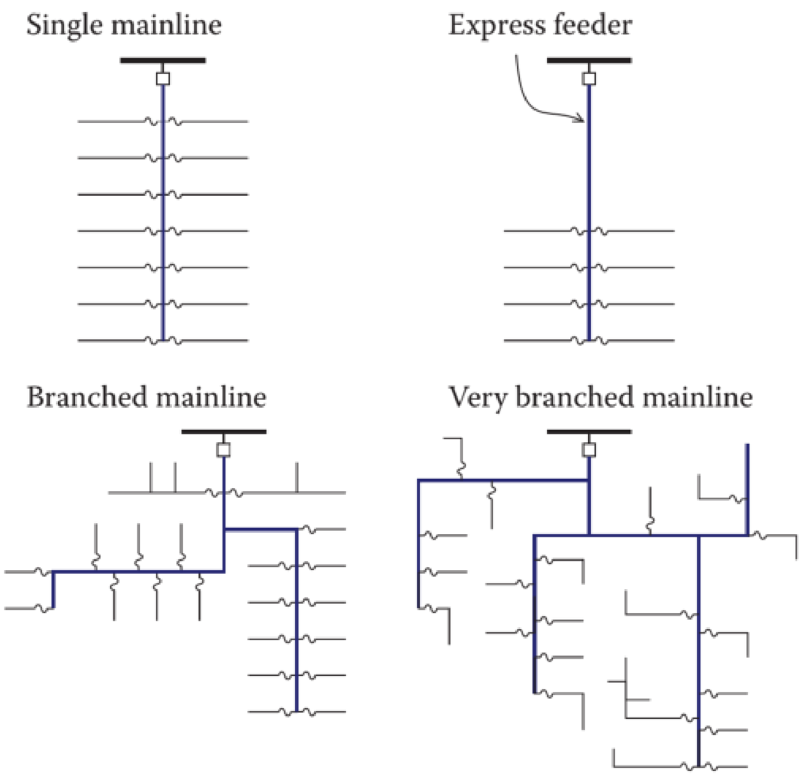
It is essential to have metering as it provides the bills.

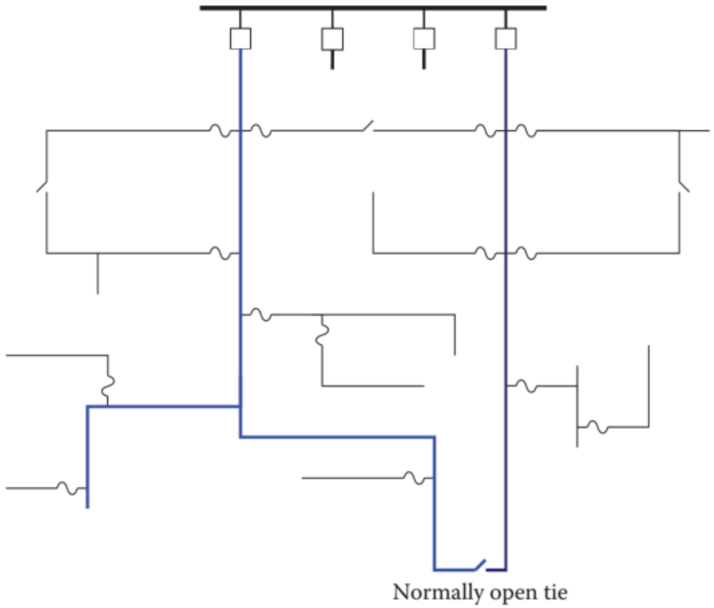


Primary Distribution Configurations

- The most common distribution substations (e.g. 11kV/380V, Dyn11, 1.5MVA, 6%) are four-wire, grounded systems: three-phase conductors plus a grounded neutral.
- Single-phase loads are served by transformers connected between one phase and neutral. neutral acts as a returning conductor and as an equipment safety ground (it is grounded periodically and at all equipment).
- A single-phase line has one phase conductor and neutral (220V), and a two-phase line has two phases and neutral (380V in maximum). Some distribution primaries are three phase systems (with no neutral). On these, single-phase loads are connected phase to phase.
- Most distribution circuits are radial (both primary and secondary). Radial circuits have many advantages over networked circuits including
  - Easier fault current protection
  - Lower fault currents over most of circuit
  - Easier voltage control
  - Easier prediction and control of power flows
  - Lower cost

Primary Distribution Configurations



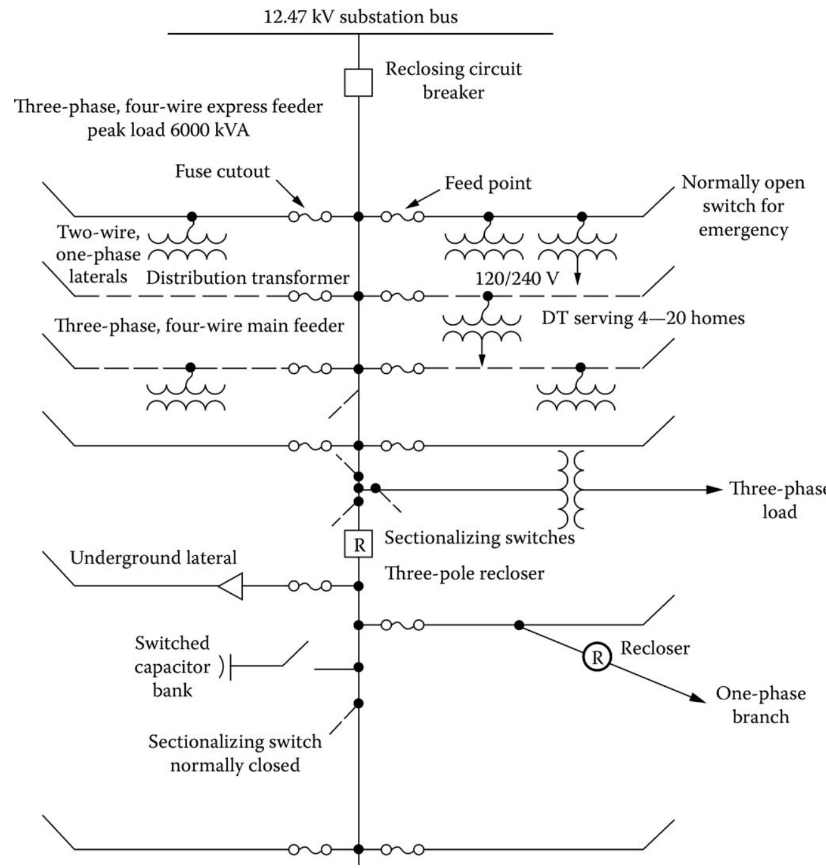


- The arrangements depend on street layouts, shape of area covered by circuit, obstacles (such as lakes), and where big loads (load centers) are.
- For improved reliability, radial circuits are often provided with normally open (N.O.) tie points to other circuits. Normally, these switches are manually operated, but some utilities use automated reclosers to isolate faults and restore power to unfaulted lines.

Primary Distribution Configurations

Voltage Levels

- Unless otherwise specified, voltages are given as line-to-line voltages; this follows normal industry practice.
- The major voltage classes are 11kV, 22kV and 33kV. A voltage class is to describe voltage applied to a set of distribution equipment; it is not actual system voltage. For example, a 15-kV insulator is suitable for application on any 15-kV class voltage, including 11kV system.
- The dividing line between distribution and sub-transmission is often gray. Some lines act as both sub-transmission and distribution circuits. A 34.5-kV circuit may feed a few 12.5-kV distribution substations, but it may also serve some load directly. Some utilities would refer to this as sub-transmission.



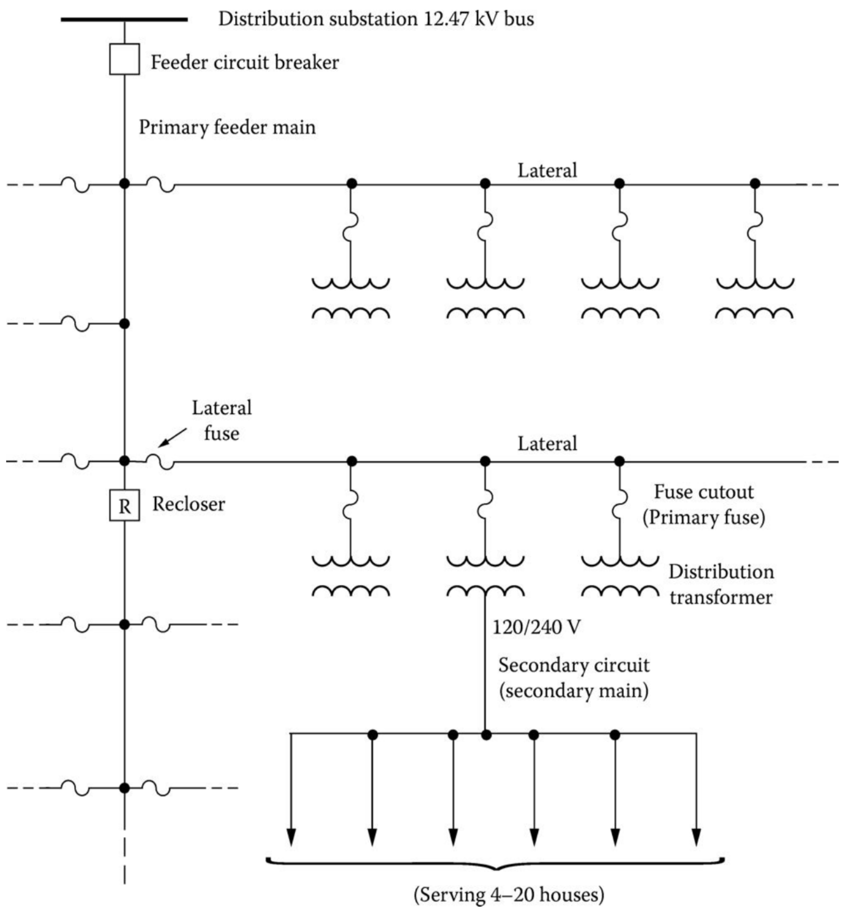
Primary Distribution Configurations

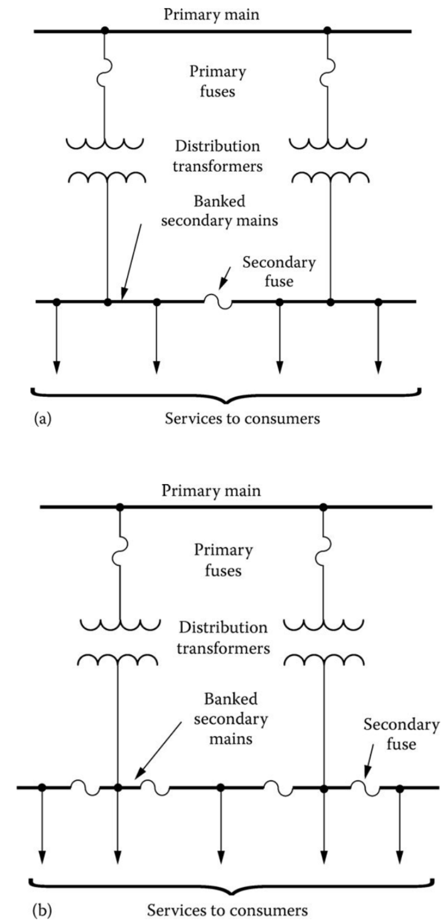
Primary Voltage Level

Advantage	**Disadvantage**
Voltage drop - A higher-voltage circuit has less voltage drop for a given power flow.	Reliability - An important disadvantage of higher voltages: longer circuits mean more customer interruptions.
Capacity - A higher-voltage system can carry more power for a given ampacity	Crew safety and acceptance - Crews do not like working on higher-voltage distribution systems.
Losses - For a given level of power flow, a higher-voltage system has fewer line losses.	Equipment cost - From transformers to cable to insulators, higher-voltage equipment costs more.
Reach - With less voltage drop and more capacity, higher-voltage circuits can cover a much wider area.	
Fewer substations - Because of longer reach, higher-voltage distribution systems need fewer substations.	

- A 34.5 kV mainline is typically 30-mile long.
- A 12.5 kV mainline is typically 8-mile long.
- Overall, 11kV to 22kV class voltages provide a good balance between cost, reliability, safety, and reach. Although a 22kV circuit does not naturally provide long reach, with voltage regulators and feeder capacitors, it can be stretched to reach 20 mi or more.

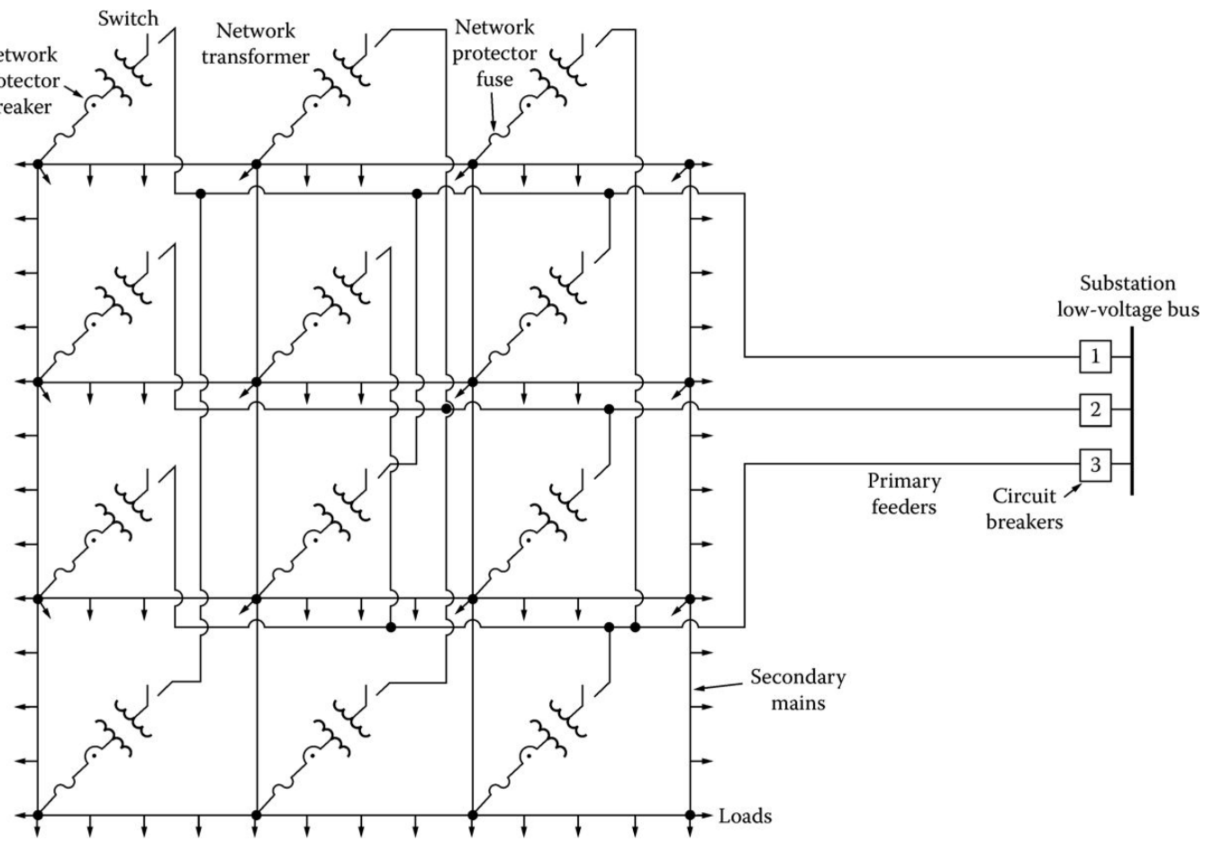
Secondary Distribution Configurations





Secondary banking: Parallel connection of secondary sides of two or more distribution transformers, which are supplied from same primary feeder, is sometimes practiced in residential and light-commercial areas where services are relatively close to each other, and therefore, required spacing between transformers is little.

Secondary Distribution Configurations

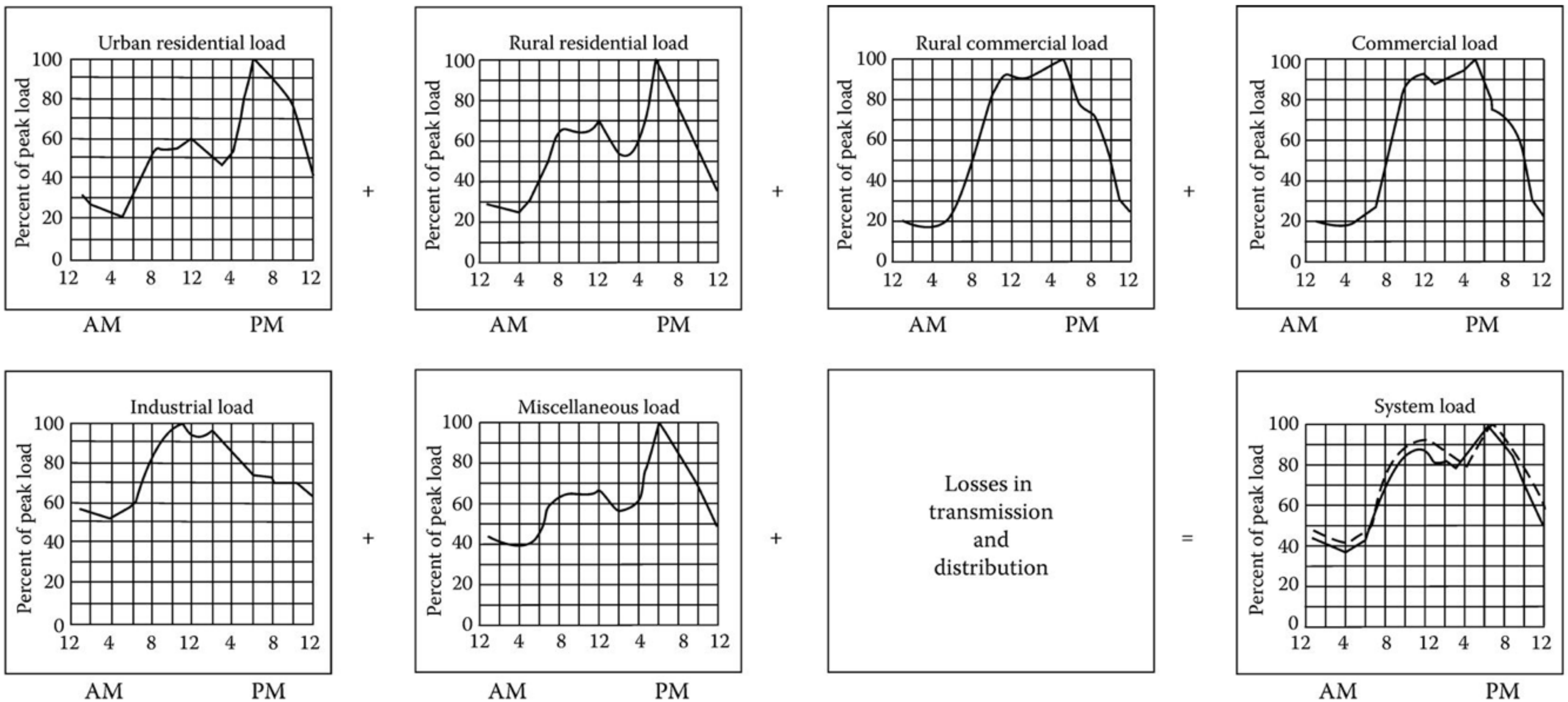


- Secondary systems are radial designed except for some specific service areas such as downtown areas or business districts, military installations and hospitals.
- Reliability and service-continuity are far more important than economic considerations. Therefore, secondary systems are designed in grid-or mesh-type network configurations in those areas.

Typical distribution circuit parameters

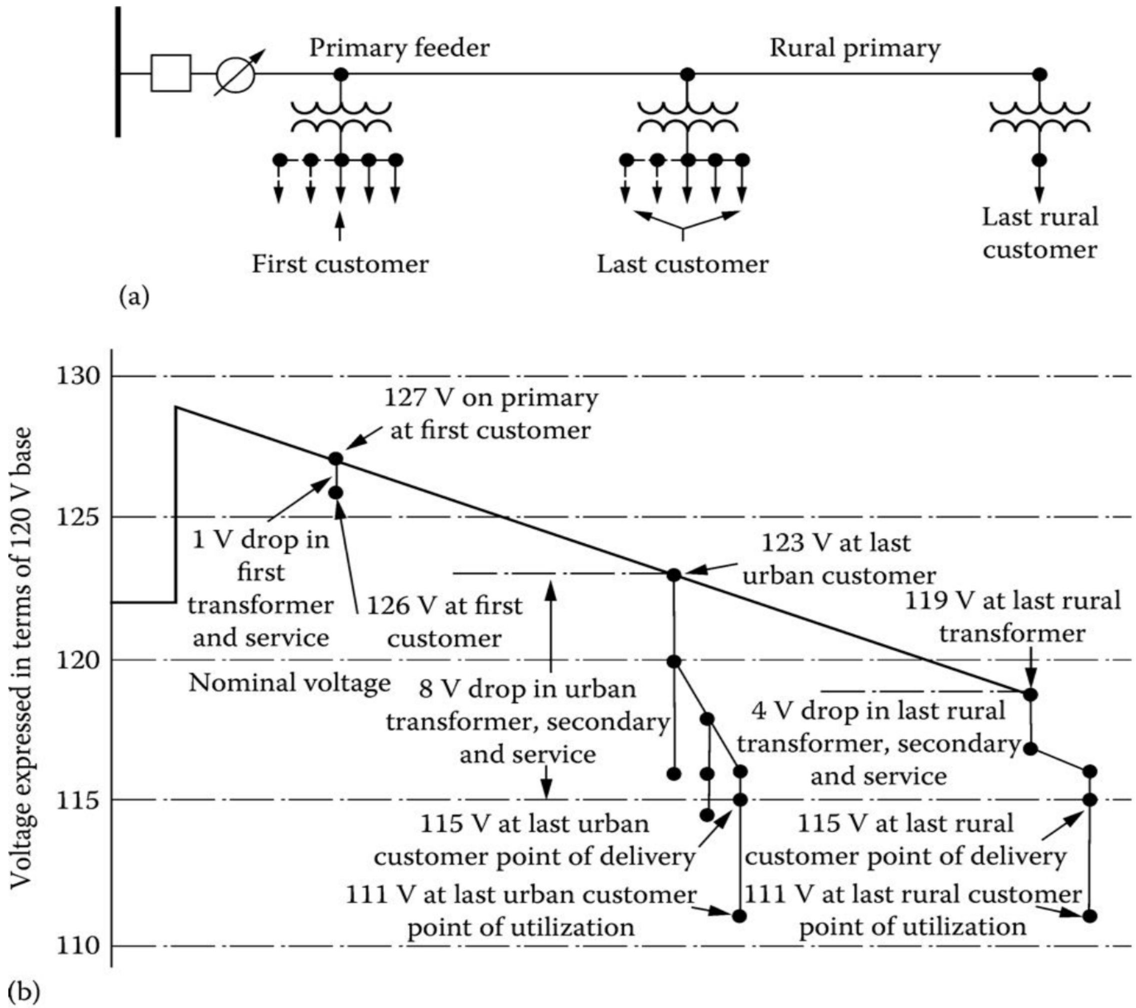
	Most Common Value	Other Common Values
Substation Characteristics		
Voltage	12.47 kV	4.16, 4.8, 13.2, 13.8, 24.94, and 34.5 kV
Number of station transformers	2	1 to 6
Substation transformer size	21 MVA	5 to 60 MVA
Number of feeders per bus	4	1 to 8
Feeder Characteristics		
Peak current	400 A	100 to 600 A
Peak load	7 MVA	1 to 15 MVA
Power factor	0.98 lagging	0.8 lagging to 0.95 leading
Number of customers	400	50 to 5000
Length of feeder mains	4 mi	2 to 15 mi
Length including laterals	8 mi	4 to 25 mi
Area covered	25 mi^2	0.5 to 500 mi^2
Mains wire size	500 kcmil	4/0 to 795 kcmil
Lateral tap wire size	1/0	#4 to 2/0
Lateral tap peak current	25 A	5 to 50 A
Lateral tap length	0.5 mi	0.2 to 5 mi
Distribution transformer size (1 ph)	25 kVA	10 to 150 kVA

Load Profiles



- Loads are often divided in terms of its function: residential, industrial and commercial (RIC). Due to its function, the load profile could have a particular shapes, e.g. high load at night and low load at daytime for residential.
- Loads are often divided in terms of its electrical characteristics: constant impedance, constant current and constant power (ZIP load). The characteristics provides insights in distribution system modelling, e.g. under fault, voltage recovery and protection studies.

Voltage Drop at Peak Load



- The main goal for voltage studies is to ensure the voltage profile does not violate supply rule.

Approximate Method of Analysis

- A distribution feeder provides service to loads over un-transposed line segments. This combination leads to the three-phase line currents and the line voltages being unbalanced.
- To analyze these conditions precisely, it is necessary to model all three phases of the feeder as accurate. Due to the lack and inaccuracy of information in distribution, some approximate methods of modeling and analysis can be employed.
- All approximate methods of modeling and analysis will assume perfectly balanced three-phase systems. It is assumed balanced loads and balanced line segment without coupling.

Voltage Drop

A line-to-neutral equivalent circuit of a three-phase line segment serving a balanced three-phase load is shown.

KVL: $$V_S = V_L + (R + jX)I = V_L + RI + jXI \tag{1}$$

The dashed lines in Fig.2 represent the **real** and **imaginary parts** of the voltage drop. The voltage drop down the line is defined as the difference between the magnitudes of the source and the load voltages:

$$\Delta V = |V_S| - |V_L| \tag{2}$$

The angle between the source voltage and the load voltage (δ) is very small. Because of that, the voltage drop between the source and load is approximately equal to the **real part** of the impedance drop.

The definition of voltage drop: $$\Delta V \cong Re(Z * I) \tag{3}$$

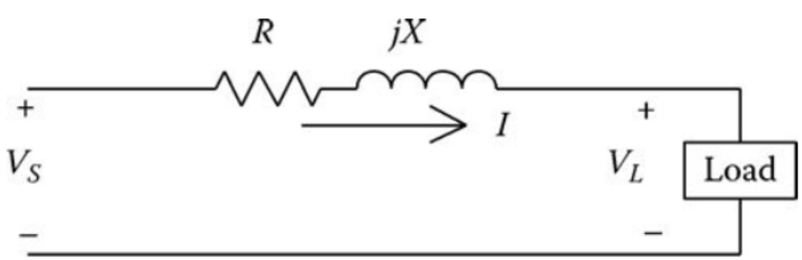


Fig.1 Customer demand curve

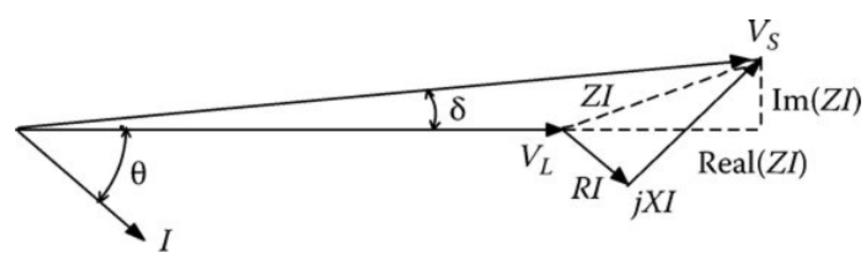


Fig.2 Customer demand curve

Example 1

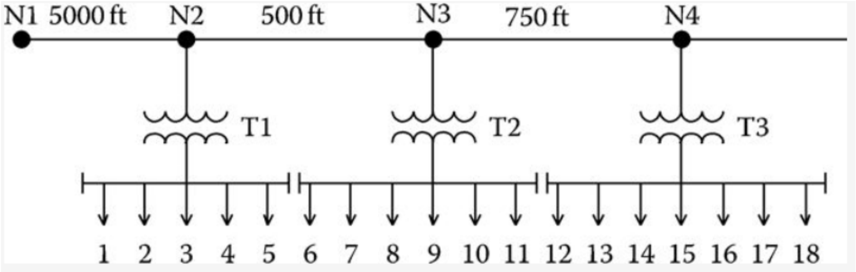
The impedance of the first line segment is $Z_{12} = 0.2841 + j0.5682\ \Omega$. The current flowing through the line segment is $I_{12} = 43.0093\angle -25.8419$ A. The voltage at node N1 is $V_1 = 2400\angle 0.0$ V.

Determine

(a) Exact voltage at node N2 and hence the actual voltage drop across node N1 and N2.

(b) The estimated voltage drop across node N1 and N2.

(c) The estimation error.



Solution

The exact voltage at node N2 is computed to be

$$V_2 = 2400\angle 0.0 - (0.2841 + j0.5682)43.01\angle -25.8419 = 2378.41\angle -0.4015\text{V}$$

The voltage drop between the nodes is then

$$\Delta V = 2400.0000 - 2378.4098 = 21.5902\ V$$

From Equation (3), $\Delta V = Re[(0.2841 + j0.5682)43.0093\angle -25.8419] = 21.6486\ V$

Estimation Error: $\varepsilon = \frac{21.5902 - 21.6486}{21.5902} \times 100\% = -0.27\%$

Line Impedance and K Factor

For approximate modeling of a line segment, it is assumed that **line segment is transposed**. With this assumption only the positive sequence impedance of the line segment needs to be determined. A typical three-phase line configuration is shown in Fig.4.

$$z_{+} = r + j0.12134 * \ln\left(\frac{D_{eq}}{GMR}\right) \Omega/\text{mile} \tag{4}$$

$$D_{eq} = \sqrt[3]{D_{ab} * D_{bc} * D_{ca}}\ (\text{ft}) \tag{5.a}$$

GMR is the conductor geometric mean radius (from tables) in feet.

Another voltage drop approximation is by "K" factor. There will be two types of K factors, one for voltage drop and the other for voltage rise calculations.

$$K_{drop} = \frac{\text{Percent\_Voltage\_Drop}}{\text{kVA\_mile}} \tag{5.b}$$

The K_{drop} factor is determined by computing the percent voltage drop down a line that is 1 mile long and serving a balanced three-phase load of 1 kVA. The percent voltage drop is referenced to the nominal line-to-neutral voltage of the line. **To calculate this factor, power factor of the load must be assumed.**

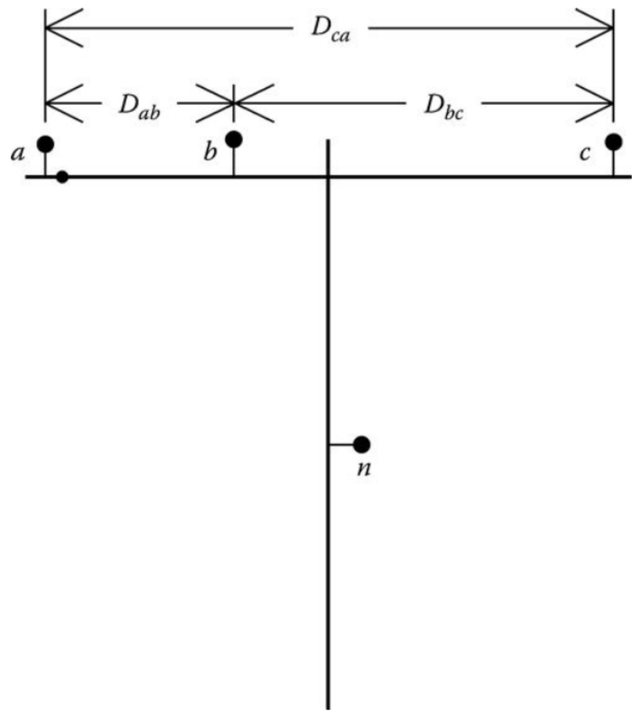


Fig.4 Three-phase line configuration

Example 2 – Voltage Drop with K Factor

A three-phase line segment has the configuration shown in Fig.4. The spacings between conductors are $\mathrm{D_{ab}} = 2.5 \text{ ft}$, $\mathrm{D_{bc}} = 4.5 \text{ ft}$, $\mathrm{D_{ca}} = 7.0 \text{ ft}$, The conductors of the line are 336,400 26/7 ACSR.

a) Determine the positive sequence impedance of the line in Ω/mile.

b) Compute the K_{drop} factor assuming a load power factor of 0.9 lagging and a nominal voltage of 12.47 kV (line to line).

Solution

From Conductor Data: $r = 0.306 \ \Omega/\text{mile}$, $GMR = 0.0244 \text{ ft}$

Compute the equivalent spacing:

$$D_{eq} = \sqrt[3]{2.5 * 4.5 * 7.0} = 4.2863 \text{ (ft)}$$

$$z_{positive} = 0.306 + j0.12134 \ln\left(\frac{4.2863}{0.0244}\right) = 0.306 + j0.6272 \ \Omega/\text{mile}$$

The current taken by 1 kVA at 0.9 lagging power factor is given by

$$I = \frac{1 \text{ kVA}}{\sqrt{3}(kV_{LL})} \angle -\cos^{-1}(PF) = \frac{1}{\sqrt{3}(12.5)} \angle -\cos^{-1}(0.9)$$

$$= 0.046299\angle - 25.84 \text{ A}$$

Example 2 – Voltage Drop with K Factor

The voltage drop is computed to be

$$\Delta V = Re[ZI] = Re[(0.306 + j0.6272)(0.046299\angle - 25.84)] = 0.025408\ \mathrm{V}$$

The nominal line-to-neutral voltage is

$$V_{LN} = \frac{12470}{\sqrt{3}} = 7199.6\ V$$

The K_{drop} factor is then

$$\mathrm{K_{drop}} = \frac{0.025408}{7199.6} 100\% = 0.00035291\%$$

The assumed power factor of 0.9 lagging is a good approximation of the power factor for a feeder serving a predominately residential load.

The $\mathrm{K_{drop}}$ factor can be used to quickly compute the approximate voltage drop down a line section. For example, assume a load of 7500 kVA is to be served at a point 1.5 miles from the substation. Using the $\mathrm{K_{drop}}$ factor computed, the percent voltage drop down the line segment is computed to be

$$V_{\mathrm{drop}} = \mathrm{K_{drop}} * kVA_{mile} = 0.00035291 * 7500 * 1.5 = 3.9702\%$$

Suppose now that the utility has a maximum allowable voltage drop of 3.0%. Then the load that can be served 1.5 miles from the substation is given by

$$kVA_{total} = \frac{3.0\%}{0.000035291 * 1.5} = 5694.76\ kVA$$

Example 3

A three-segment feeder is shown in Fig.5.

If K_{drop} factor for the line segments is 0.00003591, determine percent voltage drop from N0 to N3.

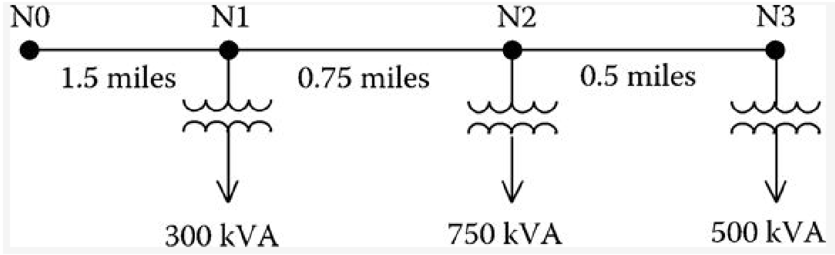


Fig.5 Three segment feeder

Solution

The total kVA flowing in segment N0 to N1 is $kVA_{01} = 300 + 750 + 500 = 1550 \text{ kVA}$

The percent voltage drop from N0 to N1 is $V_{drop01} = 0.00003591 * 1550 * 1.5 = 0.8205\%$

The total kVA flowing in segment N1 to N2 is $kVA_{12} = 750 + 500 = 1250 \text{ kVA}$

The percent voltage drop from N1 to N2 is $V_{drop12} = 0.00003591 * 1250 * 0.75 = 0.3308\%$

The kVA flowing in segment N2 to N3 is $kVA_{23} = 500\ kVA$

The percent voltage drop in segment N2 to N3 is $V_{drop23} = 0.00003591 * 500 * 0.5 = 0.0882\%$

The total percent voltage drop from N0 to N3 is $V_{drop\_total} = 0.8205 + 0.3308 + 0.0882 = 1.2396\%$

K Factors

- The K_{rise} factor is similar to the K_{drop} factor except now the "load" is a shunt capacitor. When a leading current flows through an inductive reactance, there will be a voltage rise, rather than a voltage drop, across the reactance. This is illustrated by the phasor diagram in Fig.6.

$$V_{rise} = \left|Re(ZI_{cap})\right| = X * \left|I_{cap}\right| \tag{6}$$

- In equation (6) it is necessary to take the magnitude of the real part of ZI so that the voltage rise is a positive number. The K_{rise} factor is defined the same as for the K_{drop} factor:

$$K_{rise} = \frac{\text{Percent\_Voltage\_Rise}}{\text{kvar\_mile}} \tag{7}$$

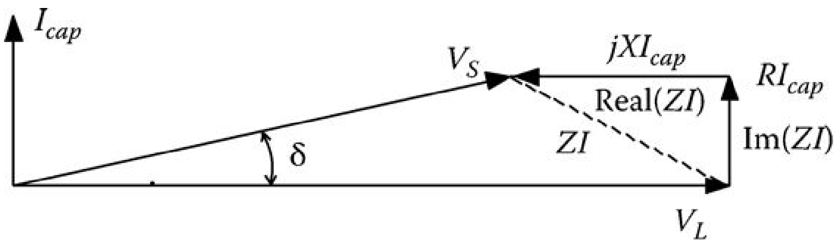


Fig.6 Voltage rise phasor diagram

Example 4 – Voltage Rise

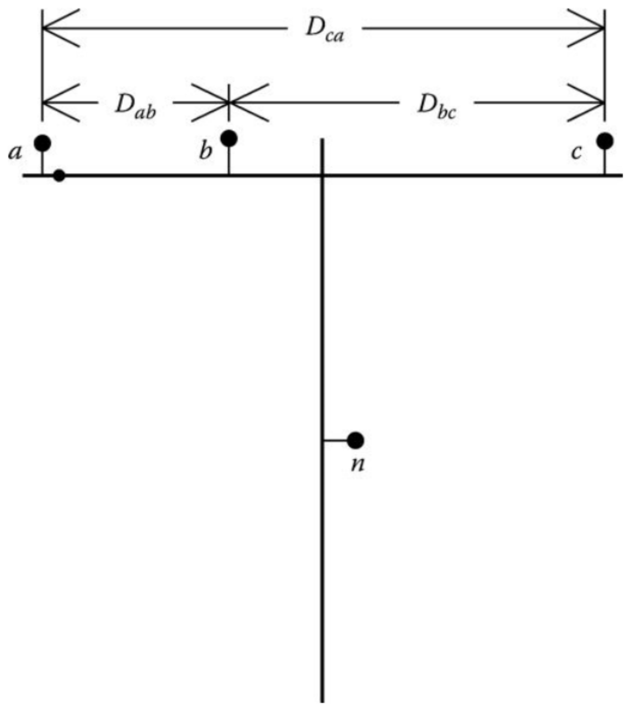
Calculate the K_{rise} factor for the line of Fig.4.

Solution

The impedance of 1 mile of line was computed to be:

$$Z = 0.306 + j0.6272\ \Omega$$

The current taken by a 1 kVAR three-phase capacitor bank is given by



$$I = \frac{1\text{ kVAR}}{\sqrt{3}(kV_{LL})}\angle 90^o = \frac{1}{\sqrt{3}(12.47)}\angle 90^o = 0.046299\angle 90^o\text{ A}$$

The voltage rise per kVAR-mile is computed to be

$$V_{rise} = \left|Re\left(ZI_{cap}\right)\right|$$
$$= |\mathrm{Re}((0.306 + j0.6272)0.046299\angle 90^o)|$$
$$= 0.029037\text{ V}$$

The nominal line-to-neutral voltage is

$$V_{LN} = \frac{12470}{\sqrt{3}} = 7199.6\ V$$

The K_{rise} factor is then

$$K_{rise} = \frac{\text{Percent\_Voltage\_Rise}}{\text{kvar\_mile}} = \frac{0.029037}{7199.6}(100\%)$$

= 0.00041331% /kVAR-mile

Example 5

Determine the rating of a three-phase capacitor bank to limit the voltage drop in Example 3 to 2.5%.

Solution

The percent voltage drop in Example 3 was computed to be 3.9702%. To limit the total voltage drop to 2.5%, the required voltage rise due to a shunt capacitor bank is

$$V_{rise} = 3.9702 - 2.5 = 1.4702\ \%$$

The required rating of the shunt capacitor is

$$kvar = \frac{V_{rise}}{K_{rise} * mile} = \frac{1.4702}{0.00040331 * 1.5} = 2430.18 \text{ kvar}$$

Uniformly Distributed Loads (UDL)

It can be assumed that loads are uniformly distributed along a line.

When the loads are uniformly distributed, it is not necessary to model each load to determine the total voltage drop from the source end to the last load.

Fig.7 shows a generalized line with **n** uniformly distributed loads.

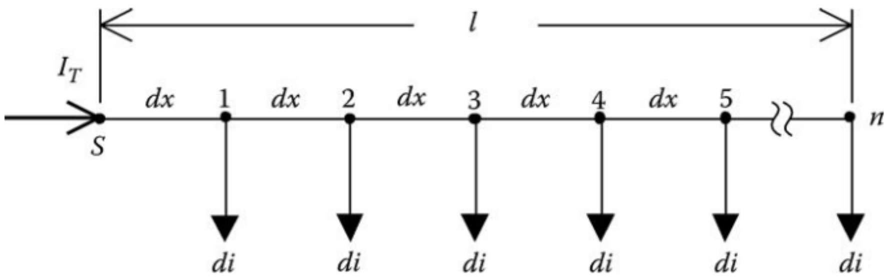


Fig.7 Generalized Line with n UDL

Fig.7 shows n uniformly spaced loads dx miles apart. The loads are all equal and will be treated as constant current loads with a value of di. The total current into the feeder is I_T. It is desired to determine the total voltage drop from the source node (**S**) to the last node n.

Let -
l be the length of the feeder
z = r + jx be the impedance of the line in Ω/mile
dx be the length of each line section
di be the load currents at each node
n be the number of nodes and number of line sections
I_T be the total current into the feeder

Uniformly Distributed Loads (UDL)

The load currents are given by $$di = \frac{I_T}{n} \quad (8)$$

The voltage drop in the first line segment is given by $$V_{drop1} = Re\{z * dx * (n * di)\} \quad (9)$$

The voltage drop in the second line segment is given by $$V_{drop2} = Re\{z * dx * [(n-1) * di]\} \quad (10)$$

The total voltage drop from the source node to the last node is then given by

$$V_{drop\_total} = V_{drop\_1} + V_{drop\_2} + \cdots + V_{drop\_n} \quad (11)$$

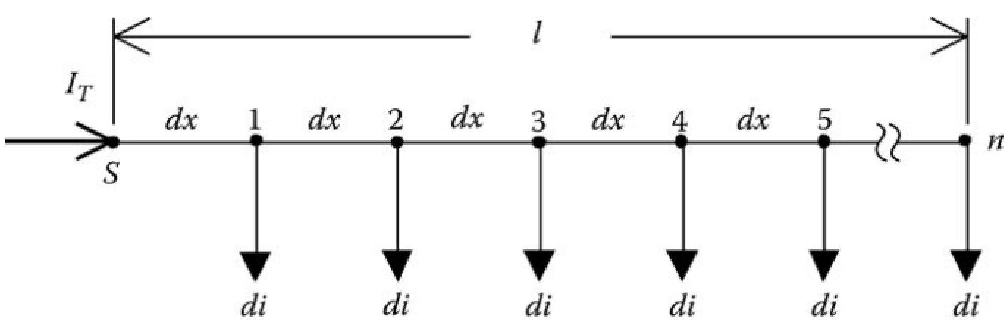
$$V_{drop\_total} = Re\{z * dx * di * [n + (n-1) + (n - 2 + \cdots + (1))]\} \quad (12)$$

Given that

$$n + (n-1) + (n-2) + \cdots + 1 = \frac{n * (n+1)}{2}$$

Using the expansion, Equation (11) becomes

$$V_{drop\_total} = Re\left\{z * dx * di * \left[\frac{n * (n+1)}{2}\right]\right\} \quad (13)$$



Uniformly Distributed Loads (UDL)

The incremental distance and incremental distance is

$$dx = \frac{l}{n} \qquad (14) \qquad\qquad di = \frac{I_T}{n} \qquad (15)$$

Substituting Equations (14) and (15) into Equation (13) results in

$$V_{drop\_total} = \text{Re}\left\{z * \frac{l}{n} * \frac{I_T}{n} * \left[\frac{n * (n+1)}{2}\right]\right\} = \text{Re}\left\{z * l * I_T * \frac{1}{2}(\frac{n+1}{n})\right\}$$

$$V_{drop\_total} = \text{Re}\left\{\frac{1}{2} * Z * I_T\ (1 + \frac{1}{n})\right\} \qquad (16)$$

where Z = z · l.

Equation (16) gives the general equation for computing the total voltage drop from the source to the last node n for a line of length l. In the limiting case where n goes to infinity, the final equation becomes

$$V_{drop\_total} = \text{Re}\left\{\frac{1}{2} * Z * I_T\ \right\} \qquad (17)$$

Interpretation

- total line distributed load can be lumped at the midpoint of the lateral
- lump one-half of the total line load at the end of the line (node **n**)

Uniformly Distributed Loads (UDL)

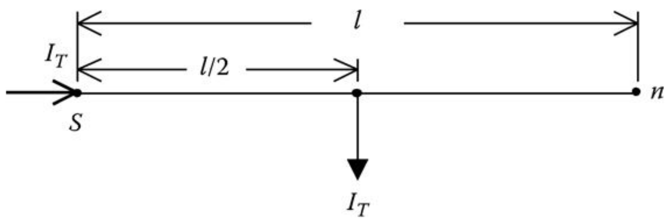


Fig.8 Load lumped at the midpoint

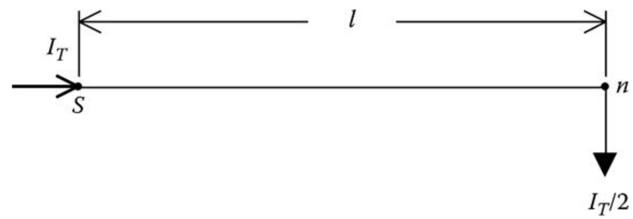


Fig.9 One-half load lumped at the end

Of equal importance in the analysis of a distribution feeder is the power loss. If the model of Fig.8 is used to compute the total three-phase power loss down the line, the result is

$$P_{loss} = 3 * |I_T|^2 * \frac{R}{2} = \frac{3}{2} * |I_T|^2 * \mathrm{R} \tag{18}$$

When the model of Fig.9 is used to compute the total three-phase power loss, the result is

$$P_{loss} = 3 * \left|\frac{I_T}{2}\right|^2 * \mathrm{R} = \frac{3}{4} * |I_T|^2 * \mathrm{R} \tag{19}$$

None of the above result is correct.

To derive the correct model for power loss, reference is made to Fig.7 and the definitions for the parameters in that figure. The total three-phase power loss down the line will be the sum of the power losses in each short segment of the line. For example, the three-phase power loss in the first segment and the second segment are

$$P_{loss1} = 3 * (r * dx) * |(n * di)|^2 \tag{20}$$

$$P_{loss2} = 3 * (r * dx) * |(n - 1) * di|^2 \tag{21}$$

Uniformly Distributed Loads (UDL)

The total power loss over the length of the line is then given by

$$P_{loss\_total} = 3 * (r * dx) * |di|^2 * [n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1^2] \tag{22}$$

Given the sum formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n * (n+1) * (2n+1)}{6} \tag{23}$$

Substituting Equations (14), (15), and (23) into Equation (22) gives

$$dx = \frac{l}{n} \quad (14) \qquad\qquad di = \frac{I_T}{n} \quad (15)$$

$$P_{loss\_total} = 3 * \left(r * \frac{l}{n}\right) \left|\frac{I_T}{n}\right|^2 \left[\frac{n * (n+1) * (2n+1)}{6}\right] \tag{24}$$

Simplifying the equation,

$$P_{loss\_total} = 3|I_T|^2 R \left[\frac{(n+1)(2n+1)}{6n^2}\right] = 3|I_T|^2 R \left[\frac{2n^2 + 3n + 1}{6n^2}\right]$$

$$P_{loss\_total} = 3|I_T|^2 R \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right] \tag{25}$$

where R = r · l is the total resistance per phase of the line segment.

Uniformly Distributed Loads (UDL)

Equation (25) gives the total three-phase power loss for a discrete number of nodes and line segments. For a truly uniformly distributed load, the number of nodes goes to infinity $n \to \infty$.

$$P_{loss\_total} = 3|I_T|^2 R \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right] \tag{25}$$

$$P_{loss\_total} = 3\left[\frac{1}{3}|I_T|^2 R\right] \tag{26}$$

A circuit model for Equation (26) is given in Figure 3.9.

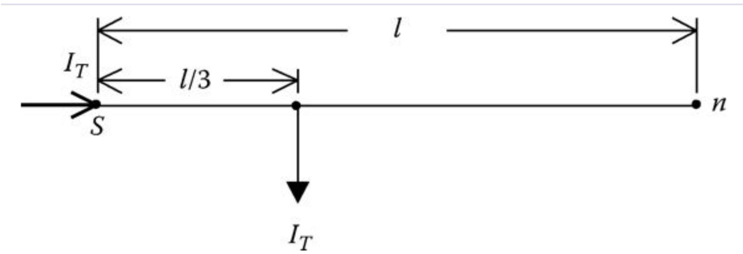


Fig.10 Power loss model

It is obvious that the same model cannot be used for both voltage drop and power loss calculations.

Uniformly Distributed Load – Exact Lumped Load Model

- In the previous sections lumped load models were developed. The first models of Voltage Drop section can be used for the computation of the total voltage drop down the line. Power Loss section developed a model that will give the correct power loss of the line. What is needed is one model that will work for both voltage drop and power loss calculations.
- Fig.11 shows the general configuration of the "exact" model that will give correct results for voltage drop and power loss.

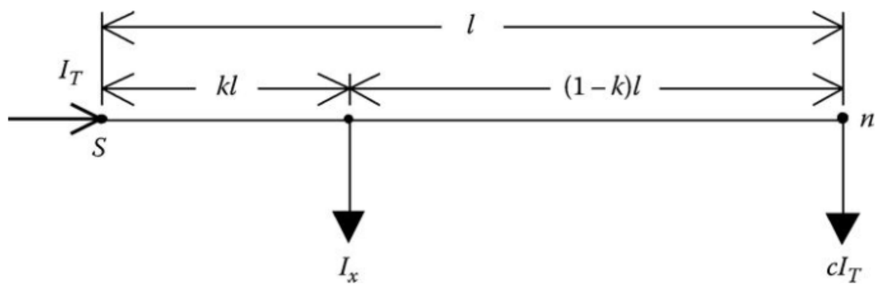


Fig.11 General exact lumped load model

- In Fig.11 the total voltage drop down the line is given by

$$V_{drop\_total} = Re\,[kI_TZ + (1-k)cI_TZ] \tag{27}$$

where

Z is the total line impedance in ohms
k is factor of total line length where first part of the load current is modeled
c is factor of total current to be placed at the end of the line such that $I_T = I_x + c \cdot I_T$

Uniformly Distributed Load – Exact Lumped Load Model

In Voltage Drop section, it was shown that the total voltage drop down the line is given by

$$V_{drop\_total} = Re\left[\frac{1}{2} * Z * I_T\right] \tag{28}$$

Set Equation (17) equal to Equation (27):

$$Re\left[\frac{1}{2} I_T Z\right] = Re\left[kI_T Z + (1-k)cI_T Z\right] \tag{29}$$

The terms inside the brackets on both sides of the equal side need to be set equal, that is

$$\frac{1}{2} I_T Z = kI_T Z + (1-k)cI_T Z \tag{30}$$

Divide both sides of the equation by ZI_T:

$$\frac{1}{2} = [k + (1-k)c] \rightarrow k = \frac{0.5 - c}{1 - c} \tag{32}$$

The same procedure can be followed for the power loss model. The total three-phase power loss in Fig.11 is given by

$$P_{loss\_total} = 3\left[k|I_T|^2 R + (1-k)(c|I_T|)^2 R\right] \tag{33}$$

$$3\left[\frac{1}{3}|I_T|^2 R\right] = 3\left[k|I_T|^2 R + (1-k)(c|I_T|)^2 R\right] \rightarrow \frac{1}{3} = \left[k + c^2 - kc^2\right] = \left[k(1-c^2) + c^2\right] \tag{34}$$

Solving,

$$c = \frac{1}{3}, k = \frac{1}{4} \tag{38}$$

Uniformly Distributed Load – Exact Lumped Load Model

The interpretation of Equations (38) is that one-third of the load should be placed at the end of the line and two-thirds of the load placed one-fourth of the way from the source end. Figure 3.12 gives the final exact lumped load model.

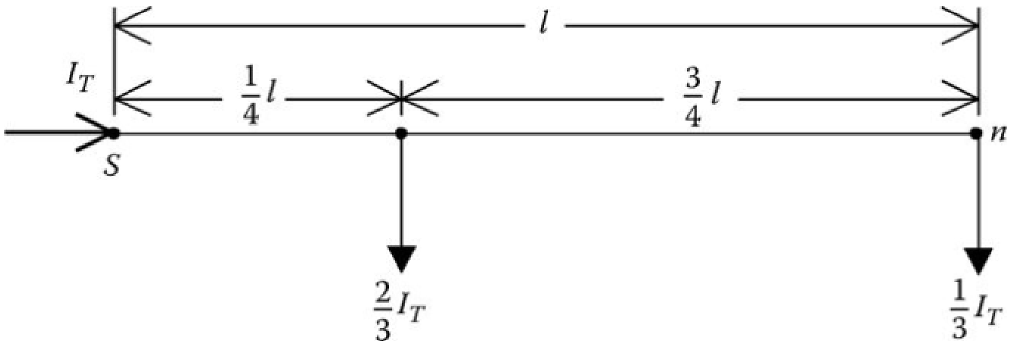


Fig.12 Exact lumped load model

Lumping Loads in Geometric Configurations

- Feeder areas can be represented by geometric configurations such as rectangles, triangles, and trapezoids. By assuming a constant load density in the configurations, approximate calculations can be made for computing the voltage drop and total power losses. The approximate calculations can aid in the determination of the maximum load that can be served in a specified area at a given voltage level and conductor size. For all the geographical areas to be evaluated, the following definitions will apply:

  where

  - D represents the load density in kVA/mile^2.
  - PF represents the assumed lagging power factor.
  - z represents the line impedance in Ω/mile.
  - l represents the length of the area.
  - w represents the width of the area.
  - kV^{LL} represents the nominal line-to-line voltage in kV.
  - It will also be assumed that the loads are modeled as constant current loads.

- A rectangular area of length **l** and width **w** is to be served by a primary main feeder. The feeder area is assumed to have a constant load density with three-phase laterals uniformly tapped off of the primary main. Fig.13 shows a model for the rectangular area.
- Fig.13 represents a rectangular area of constant load density being served by a three-phase main running from node **n** to node **m**.

Lumping Loads in Geometric Configurations

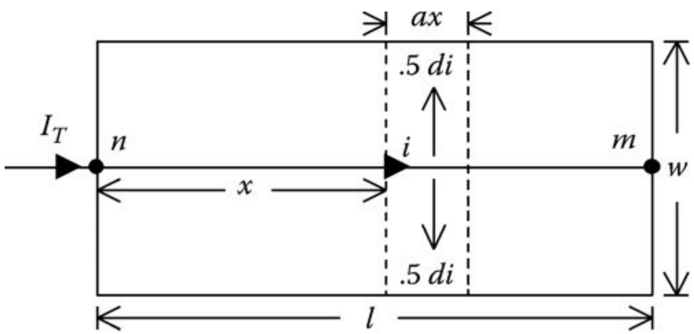


Fig.13 Constant load density rectangular area

- The total current entering the area is given by

$$I_T = \frac{DLw}{\sqrt{3}kV_{LL}} \angle -\cos^{-1}(PF) \tag{39}$$

- An incremental segment is located x miles from node n. The incremental current serving the load in the incremental segment is given by

$$di = \frac{I_T}{l} \text{ A/mile} \tag{40}$$

- The current in the incremental segment is given by

$$i = I_T - x * di = I_T - x * \frac{I_T}{l} = I_T\left(1 - \frac{x}{l}\right) \tag{41}$$

- The voltage drop in the incremental segment is

$$dV = Re(z * i * dx) = Re(z * I_T * (1 - \frac{x}{l}) * dx) \tag{42}$$

Lumping Loads in Geometric Configurations

The total voltage drop down the primary main feeder is

$$V_{drop} = \int_0^l dv = Re[z * I_T * \int_0^l (1 - \frac{x}{l}) * dx] \quad (41)$$

Evaluating the integral and simplifying,

$$V_{drop} = Re\left[z * I_T * \frac{1}{2} * l\right] = Re\left[\frac{1}{2} * Z * I_T\right] \quad (43)$$

where Z = z · l.

$$V_{drop\_total} = Re\left\{\frac{1}{2} * Z * I_T\right\} \quad (17)$$

- Equation (43) gives the same result as that of Equation (17) which was derived for loads uniformly distributed along a feeder. The only difference here is the way the total current (I_T) is determined.
- The bottom line is that the total load of a rectangular area can be modeled at the centroid of the rectangle as shown in Fig.14. The voltage drop computed to the load point will represent the total voltage drop from node **n** to node **m**.

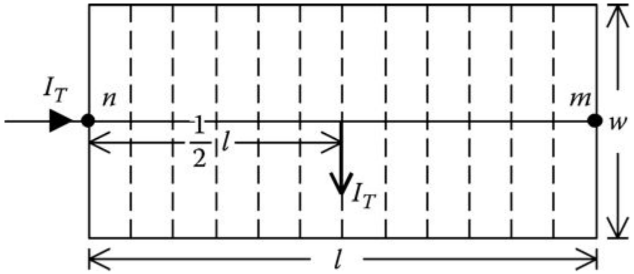


Fig.14 Rectangle voltage drop model

Lumping Loads in Geometric Configurations

- A similar derivation can be done to determine the total three-phase power loss down the feeder main. The power loss in the incremental length is

$$\mathrm{dp} = 3|i|^2 r dx = 3|I_T|^2\left(1-\frac{x}{l}\right)^2 r dx = 3r|I_T|^2\left(1-2\frac{x}{l}+\left(\frac{x}{l}\right)^2\right)dx$$

- The total three-phase power loss down the primary main is

$$P_{\mathrm{loss}} = \int_0^l dp = 3|I_T|^2 r\int_0^l 1-2\frac{x}{l}+\left(\frac{x}{l}\right)^2 dx = 3\left[\frac{1}{3}R|I_T|^2\right] \tag{44}$$

where R = r · l.

- Equation (44) gives the same result as that of Equation (26). The only difference, again, is the way the total current I_T is determined. The model for computing the total three-phase power loss of the primary main feeder is shown in Fig.15. Once again, it must be understood that the power loss computed using the model of Fig.15 represents the total power loss from node **n** to node **m**.

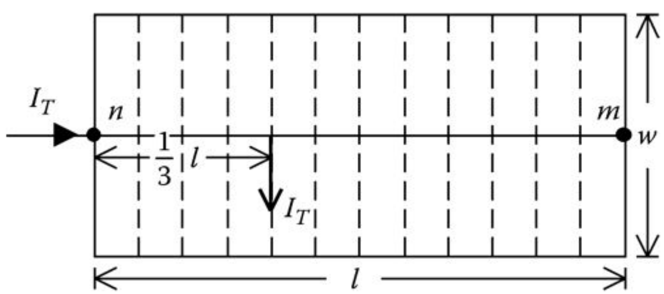


Fig.15 Rectangle power loss model

Example 5 – Voltage Drop by Geometric Estimation

It is proposed to serve a rectangular area of length 10,000 ft and width of 6,000 ft. The load density of the area is 2500 kVA/mile2 with a power factor of 0.9 lagging. The primary main feeder uses 336,400 26/7 ACSR on a pole configured as shown in Example 2, Fig.4.

a) What minimum standard nominal voltage level can be used to serve this area without exceeding a voltage drop of 3% down the primary main? The choices of nominal voltages are 4.16 and 12.47 kV.

b) Compute also the total three-phase power loss.

Solution

From Example 2, the impedance of the line was computed to be $z = 0.306 + j0.6272\ \Omega/\text{mile}$.

The length and width of the area in miles are

$$l = \frac{10{,}000}{5{,}280} = 1.8939\ \text{miles}, w = \frac{6{,}000}{5{,}280} = 1.1364\ \text{miles}$$

The total area of the rectangular area is $\mathrm{A} = lw = 2.1522\ \text{mile}^2$

The total load of the area is $\mathrm{kV}A = DA = 2500 * 2.1522 = 5380.6\ kVA$

The total impedance of the line segment is

$$\mathrm{Z}= zl = (0.306 + j0.6272)1.8939 = 0.5795 + j1.1879\Omega$$

For a nominal voltage of 4.16 kV, the total area current is

$$I_T = \frac{kVA}{\sqrt{3} * kV_{LL}} = \frac{5380.6}{\sqrt{3} * 4.16}\angle - \cos^{-1}(0.9) = 746.7\angle - 25.84\ A$$

Example 5 – Voltage Drop by Geometric Estimation

The total voltage drop down the primary main is

$$V_{drop} = Re\left[\frac{1}{2} * Z * I_T\right] = Re\left[\frac{1}{2} * (0.5795 + j1.1879) * (746.7\angle - 25.84)\right] = 338.1V$$

The nominal line-to-neutral voltage is $V_{LN} = \dfrac{4160}{\sqrt{3}} = 2401.8\ V$

The percent voltage drop is $V_{\%} = \dfrac{V_{drop}}{V_{LN}} * 100\% = \dfrac{388.1}{2401.8} * 100\% = 16.16\%$

The nominal voltage of 4.16 kV will not meet the criteria of a voltage drop less than 3.0%. For a nominal voltage of 12.47 kV, the total area current is

$$I_T = \frac{kVA}{\sqrt{3} * kV_{LL}} = \frac{5380.6}{\sqrt{3} * 12.47}\angle - \cos^{-1}(0.9) = 249.1\angle - 25.84\ A$$

The total voltage drop down the primary main is

$$V_{drop} = Re\left[\frac{1}{2} * Z * I_T\right] = Re\left[\frac{1}{2} * (0.5795 + j1.1879) * (249.1\angle - 25.84)\right] = 129.5V$$

The nominal line-to-neutral voltage is $V_{LN} = \dfrac{12{,}470}{\sqrt{3}} = 7{,}199.6\ V$

The percent voltage drop is $V_{\%} = \dfrac{V_{drop}}{V_{LN}} * 100\% = \dfrac{129.5}{7{,}199.6} * 100\% = 1.80\%$

Example 5 – Voltage Drop by Geometric Estimation

- The nominal voltage of 12.47 kV is more than adequate to serve this load. It would be possible at this point to determine how much larger the area could be and still satisfy the 3.0% voltage drop constraint.
- For the 12.47 kV selection, the total three-phase power loss down the primary main is

$$P_{\text{loss}} = 3 * \left[\frac{\left(\frac{1}{3}\right) * R * |I_T|^2}{1000} \right] = \left[\frac{\left(\frac{1}{3}\right) * 0.5795 * (249.1)^2}{1000} \right] = 35.965 \text{ kW}$$

Lumping Loads in Geometric Configurations

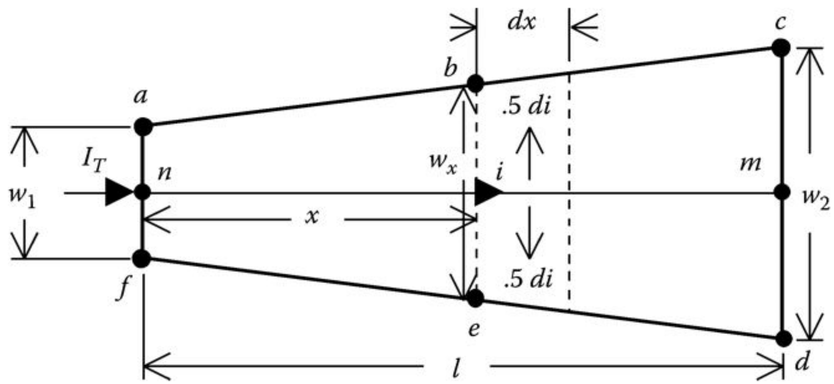


Fig.17 General trapezoid

Voltage Drop for Trapezoid Area

$$V_{drop} = Re\left[Z * I_T * \left(\frac{w_1 + 2w_2}{3 * (w_1 + w_2)}\right)\right] \tag{61}$$

- Equation (61) is very general and is used in the following to determine the models for the rectangular and triangular areas.

Rectangle

For a rectangular area the two widths w_1 and w_2 will be equal. Let

$$w_1 = w_2 = w \tag{62}$$

Substitute Equation (62) into Equation (61):

$$V_{drop} = Re\left[Z * I_T * \left(\frac{w + 2w}{3 * (w + w)}\right)\right] = Re\left[Z * I_T * \frac{3w}{6w}\right] = Re\left[\frac{1}{2} * Z * I_T\right]$$

Lumping Loads in Geometric Configurations

$$V_{drop} = Re\left[Z * I_T * \left(\frac{w_1 + 2w_2}{3 * (w_1 + w_2)}\right)\right] \tag{61}$$

Triangle

For a triangular area the width w_1 will be zero. Let

$$w_1 = 0 \tag{64}$$

Substitute Equation (64) into Equation (61):

$$V_{drop} = Re\left[Z * I_T * \left(\frac{0 + 2w_2}{3 * (0 + w_2)}\right)\right] = Re\left[\frac{2}{3} * Z * I_T\right] \tag{65}$$

Equation (65) is the same as was derived for the triangular area.

Power Loss for Trapezoid

$$P_{loss} = 3 * \left\{R * |I_T|^2 * \left[\frac{8w_2^2 + 9 * w_1 * w_2 + 3w_1^2}{15 * (w_1 + w_2)^2}\right]\right\} \tag{69}$$

where R = r · l.

- The rectangular and triangular areas are special cases of Equation (69).

Load Models

- The loads on a distribution system are typically specified by the complex power consumed. This demand can be specified as kW and kVAr.
- The voltage specified will always be the voltage at the LV bus of the distribution substation. This creates a problem since the current requirement of the loads can not be determined without knowing the voltage. For this reason, modified ladder iterative technique must be employed.
- Loads on a distribution feeder can be modeled as wye connected or delta connected. The loads can be with any degree of unbalanced.
- The ZIP models are
    - Constant impedance (Z)
    - Constant current (I)
    - Constant real and reactive power (constant P)
    - Any combination of the above
- The load models developed are to be used in the iterative process of a power-flow program where the load voltages are initially assumed.
- One of the results of the power-flow analysis is to replace the assumed voltages with the actual operating load voltages.
- All models are initially defined by a complex power per phase and an assumed line-to-neutral voltage (wye load) or an assumed line-to-line voltage (delta load). The units of the complex power can be in volt-amperes and volts or per-unit volt-amperes and per-unit voltages.

Wye Connected Loads

The notation for the specified complex powers and voltages are as follows:

- Phase a: $|S_a|\angle\theta_a = P_a + jQ_a$ and $|V_{an}|\angle\delta_a$
- Phase b: $|S_b|\angle\theta_b = P_b + jQ_b$ and $|V_{bn}|\angle\delta_b$
- Phase c: $|S_c|\angle\theta_c = P_c + jQ_c$ and $|V_{cn}|\angle\delta_c$

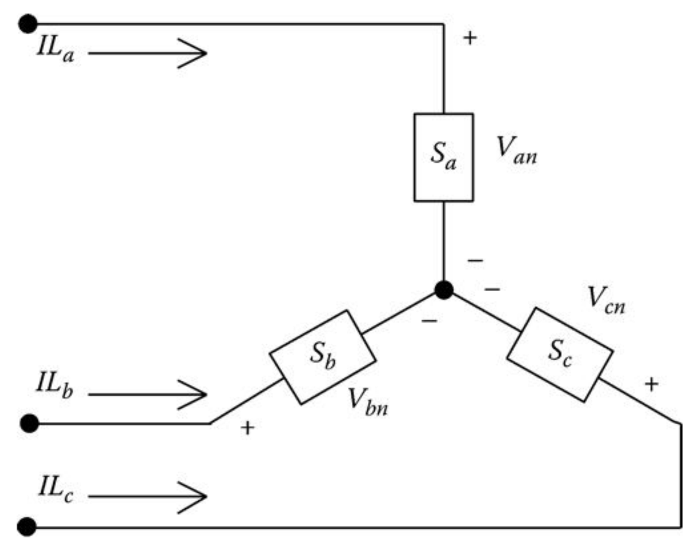
The line currents for constant real and reactive power loads (PQ loads) are given by

$$IL_a = \left(\frac{S_a}{V_{an}}\right)^* = \frac{|S_a|}{|V_{an}|}\angle\delta_a - \theta_a = |IL_a|\angle\delta_a$$

$$IL_b = \left(\frac{S_b}{V_{bn}}\right)^* = \frac{|S_b|}{|V_{bn}|}\angle\delta_b - \theta_b = |IL_b|\angle\delta_b$$

$$IL_c = \left(\frac{S_c}{V_{cn}}\right)^* = \frac{|S_c|}{|V_{cn}|}\angle\delta_c - \theta_c = |IL_c|\angle\delta_c$$

In this model, the line-to-neutral voltages will change during each iteration until convergence is achieved.



Constant Impedance Load (Z)

- The "constant load impedance" is first determined from the specified complex power and assumed (e.g., nominal voltage) line-to-neutral voltages:

$$Z_a = \frac{|V_{an}|^2}{S_a^*} = \frac{|V_{an}|^2}{|S_a|}\angle\theta_a = |Z_a|\angle\theta_a$$

$$Z_b = \frac{|V_{bn}|^2}{S_b^*} = \frac{|V_{bn}|^2}{|S_b|}\angle\theta_b = |Z_b|\angle\theta_b$$

$$Z_c = \frac{|V_{cn}|^2}{S_c^*} = \frac{|V_{cn}|^2}{|S_c|}\angle\theta_c = |Z_c|\angle\theta_c$$

- The load currents as a function of the "constant load impedances" are given by

$$IL_a = \frac{V_{an}}{Z_a} = \frac{|V_{an}|}{|Z_a|}\angle\delta_a - \theta_a = |IL_a|\angle\alpha_a$$

$$IL_b = \frac{V_{bn}}{Z_b} = \frac{|V_{bn}|}{|Z_b|}\angle\delta_b - \theta_b = |IL_b|\angle\alpha_b$$

$$IL_c = \frac{V_{cn}}{Z_c} = \frac{|V_{cn}|}{|Z_c|}\angle\delta_c - \theta_c = |IL_c|\angle\alpha_c$$

- In this model, the line-to-neutral voltages will change during each iteration, but the impedance computed in Equation (5) will remain constant.

Constant Current Load (I)

$$IL_a = \left(\frac{S_a}{V_{an}}\right)^* = \frac{|S_a|}{|V_{an}|}\angle\delta_a - \theta_a = |IL_a|\angle\delta_a$$

$$IL_b = \left(\frac{S_b}{V_{bn}}\right)^* = \frac{|S_b|}{|V_{bn}|}\angle\delta_b - \theta_b = |IL_b|\angle\delta_b$$

$$IL_c = \left(\frac{S_c}{V_{cn}}\right)^* = \frac{|S_c|}{|V_{cn}|}\angle\delta_c - \theta_c = |IL_c|\angle\delta_c$$

- In this model, the magnitudes of the currents are computed according to Equations (4) and then held constant while the angle of the voltage (δ) changes resulting in a changing angle on the current so that the power factor of the load remains constant:

$$IL_a = |IL_a|\angle\delta_a - \theta_a$$

$$IL_b = |IL_b|\angle\delta_b - \theta_b$$

$$IL_c = |IL_c|\angle\delta_c - \theta_c$$

  where

  - δ_{abc} represents the voltage angles (across an impedance)
  - θ_{abc} represents the power factor angles

- Combination loads can be modeled by assigning a percentage of the total load to each of the three aforementioned load models (ZIP). The total line current entering the load is the sum of the three components.

Example 1 – ZIP Load with Iterative Procedures

Complex powers of a wye-connected load are

$$[S_{abc}] = \begin{bmatrix} 2236.1\angle 26.6 \\ 2506.0\angle 28.6 \\ 2101.4\angle 25.3 \end{bmatrix} \text{ kVA}$$

The load is specified to be 50% constant complex power, 20% constant impedance, and 30% constant current. The nominal line-to-line voltage of the feeder is 12.47 kV.

Assume the nominal voltage and compute the component of load current attributed to each component of the load and the total load current. The assumed line-to-neutral voltages at the start of the iterative routine are

$$[VLN_{abc}] = \begin{bmatrix} 7200\angle 0 \\ 7200\angle -120 \\ 7200\angle 120 \end{bmatrix} V$$

The component of currents due to the constant complex power is

$$Ipq_i = \left(\frac{S_i \cdot 1000}{VLN_i}\right)^* \cdot 0.5 = \begin{bmatrix} 155.3\angle -26.6 \\ 174.0\angle -148.6 \\ 146.0\angle 94.7 \end{bmatrix} A$$

The constant impedances for that part of the load are computed as

$$Z_i = \frac{VLN_i^2}{S_i^* \cdot 1000} \cdot 0.5 = \begin{bmatrix} 20.7 + j10.4 \\ 18.2 + j9.9 \\ 22.3 + j10.6 \end{bmatrix} \Omega$$

Example 1 – ZIP Load with Iterative Procedures

For the first iteration, the currents due to the constant impedance portion of the load are

$$Iz_i = \left(\frac{VLN_i}{Z_i}\right) \cdot 0.2 = \begin{bmatrix} 62.1\angle -26.6 \\ 69.6\angle -148.6 \\ 58.4\angle 94.7 \end{bmatrix} A$$

The magnitudes of the constant current portion of the load are

$$IM_i = \left|\left(\frac{S_i \cdot 1000}{VLN_i}\right)^*\right| \cdot 0.3 = \begin{bmatrix} 93.2 \\ 104.4 \\ 87.6 \end{bmatrix} A$$

The contribution of the load currents due to the constant current portion of the load is

$$II_i = IM_i \angle \delta_i - \theta_i = \begin{bmatrix} 93.2\angle -26.6 \\ 104.4\angle -148.6 \\ 87.6\angle 94.7 \end{bmatrix} A$$

The total load current is the sum of the three components:

$$[I_{abc}] = [I_{pq}] + [I_z] + [I_I] = \begin{bmatrix} 310.6\angle -26.6 \\ 348.1\angle -148.6 \\ 292.0\angle 94.7 \end{bmatrix} A$$

Determine the currents at the start of the second iteration. Load voltage after first iteration are:

$$[VLN] = \begin{bmatrix} 6850.0\angle -1.9 \\ 6972.7\angle -122.1 \\ 6886.1\angle 117.5 \end{bmatrix} V$$

Example 1 – ZIP Load with Iterative Procedures

The steps are repeated with the exceptions that the impedances of the constant impedance portion of the load will not be changed and the magnitude of the currents for the constant current load will not change.

The constant complex power portion of the load currents is

$$Ipq_i = \left(\frac{S_i \cdot 1000}{VLN_i}\right)^* \cdot 0.5 = \begin{bmatrix} 163.2\angle -28.5 \\ 179.7\angle -150.7 \\ 152.7\angle 92.1 \end{bmatrix} A$$

The currents due to the constant impedance portion of the load are

$$Iz_i = \left(\frac{VLN_i}{Z_i}\right) \cdot 0.2 = \begin{bmatrix} 59.1\angle -28.5 \\ 67.4\angle -150.7 \\ 55.9\angle 92.1 \end{bmatrix} A$$

The currents due to the constant current portion of the load are

$$II_i = IM_i\angle\delta_i - \theta_i = \begin{bmatrix} 93.2\angle -28.5 \\ 104.4\angle -150.7 \\ 87.6\angle 92.1 \end{bmatrix} A$$

The total load currents at the start of the second iteration will be

$$[I_{abc}] = [I_{pq}] + [I_z] + [I_I] = \begin{bmatrix} 315.5\angle -28.5 \\ 351.5\angle -150.7 \\ 296.2\angle 92.1 \end{bmatrix} A$$

The currents for the constant power loads have increased because the voltages are reduced from the original assumption. The currents for the constant impedance load have decreased because the impedance stayed constant but the voltages are reduced.

Delta Connected Load

The model for a delta-connected load is shown.

The notations for the specified complex powers and voltages in Fig.2 are as follows:

Phase ab: $|S_{ab}|\angle\theta_{ab} = P_{ab} + jQ_{ab}$ and $|V_{ab}|\angle\delta_{ab}$

Phase bc: $|S_{bc}|\angle\theta_{bc} = P_{bc} + jQ_{bc}$ and $|V_{bc}|\angle\delta_{bc}$

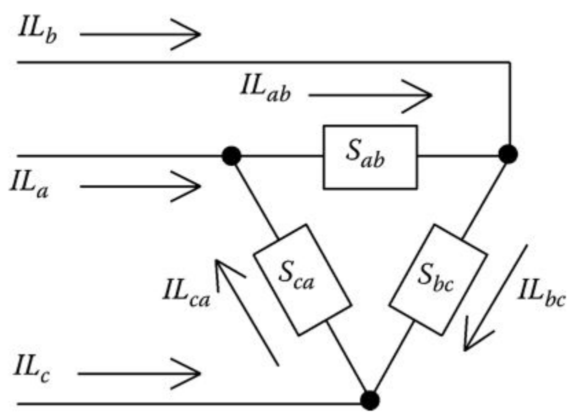
Phase ca: $|S_{ca}|\angle\theta_{ca} = P_{ca} + jQ_{ca}$ and $|V_{ca}|\angle\delta_{ca}$

Current in Delta Load can be represented as:

$$IL_{ab} = \left(\frac{S_{ab}}{V_{ab}}\right)^* = \frac{|S_{ab}|}{|V_{ab}|}\angle\delta_{ab} - \theta_{ab} = |IL_{ab}|\angle\alpha_{ab}$$

$$IL_{bc} = \left(\frac{S_{bc}}{V_{bc}}\right)^* = \frac{|S_{bc}|}{|V_{bc}|}\angle\delta_{bc} - \theta_{bc} = |IL_{bc}|\angle\alpha_{bc}$$

$$IL_{ca} = \left(\frac{S_{ca}}{V_{ca}}\right)^* = \frac{|S_{ca}|}{|V_{ca}|}\angle\delta_{ca} - \theta_{ca} = |IL_{ca}|\angle\alpha_{ca}$$



In this model, the line-to-line voltages will change during each iteration resulting in new current magnitudes and angles at the start of each iteration.

Constant Power Load (P):

The currents in the delta-connected loads are

$$IL_{ab} = \left(\frac{S_{ab}}{V_{ab}}\right)^* = \frac{|S_{ab}|}{|V_{ab}|}\angle\delta_{ab} - \theta_{ab} = |IL_{ab}|\angle\alpha_{ab}$$

$$IL_{bc} = \left(\frac{S_{bc}}{V_{bc}}\right)^* = \frac{|S_{bc}|}{|V_{bc}|}\angle\delta_{bc} - \theta_{bc} = |IL_{bc}|\angle\alpha_{bc}$$

$$IL_{ca} = \left(\frac{S_{ca}}{V_{ca}}\right)^* = \frac{|S_{ca}|}{|V_{ca}|}\angle\delta_{ca} - \theta_{ca} = |IL_{ca}|\angle\alpha_{ca}$$

In this model, the line-to-line voltages will change during each iteration resulting in new current magnitudes and angles at the start of each iteration.

Constant Impedance Load (Z):

The "constant load impedance" is first determined from the specified complex power and line-to-line voltages:

$$Z_{ab} = \frac{|V_{ab}|^2}{S_{ab}^*} = \frac{|V_{ab}|^2}{|S_{ab}|} \angle\theta_{ab} = |Z_{ab}|\angle\theta_{ab}$$

$$Z_{bc} = \frac{|V_{bc}|^2}{S_{bc}^*} = \frac{|V_{bc}|^2}{|S_{bc}|} \angle\theta_{bc} = |Z_{bc}|\angle\theta_{bc}$$

$$Z_{ca} = \frac{|V_{ca}|^2}{S_{ca}^*} = \frac{|V_{ca}|^2}{|S_{ca}|} \angle\theta_{ca} = |Z_{ca}|\angle\theta_{ca}$$

The delta load currents as a function of the "constant load impedances" are

$$IL_{ab} = \frac{V_{ab}}{Z_{ab}} = \frac{|V_{ab}|}{|Z_{ab}|} \angle\delta_{ab} - \theta_{ab} = |IL_{ab}|\angle\alpha_{ab}$$

$$IL_{bc} = \frac{V_{bc}}{Z_{bc}} = \frac{|V_{bc}|}{|Z_{bc}|} \angle\delta_{bc} - \theta_{bc} = |IL_{bc}|\angle\alpha_{bc}$$

$$IL_{ca} = \frac{V_{ca}}{Z_{ca}} = \frac{|V_{ca}|}{|Z_{ca}|} \angle\delta_{ca} - \theta_{ca} = |IL_{ca}|\angle\alpha_{ca}$$

In this model, the line-to-line voltages will change during each iteration until convergence is achieved.

Constant Current Load (I)

$$IL_{ab} = \left(\frac{S_{ab}}{V_{ab}}\right)^* = \frac{|S_{ab}|}{|V_{ab}|}\angle\delta_{ab} - \theta_{ab} = |IL_{ab}|\angle\alpha_{ab}$$

$$IL_{bc} = \left(\frac{S_{bc}}{V_{bc}}\right)^* = \frac{|S_{bc}|}{|V_{bc}|}\angle\delta_{bc} - \theta_{bc} = |IL_{bc}|\angle\alpha_{bc}$$

$$IL_{ca} = \left(\frac{S_{ca}}{V_{ca}}\right)^* = \frac{|S_{ca}|}{|V_{ca}|}\angle\delta_{ca} - \theta_{ca} = |IL_{ca}|\angle\alpha_{ca}$$

In this model, the magnitudes of the currents are computed and then held constant while the angle of the voltage (δ) changes during each iteration. This keeps the power factor of the load constant:

$$IL_{ab} = |IL_{ab}|\angle\delta_{ab} - \theta_{ab}$$

$$IL_{bc} = |IL_{bc}|\angle\delta_{bc} - \theta_{bc}$$

$$IL_{ca} = |IL_{ca}|\angle\delta_{ca} - \theta_{ca}$$

Combination loads can be modeled by assigning a percentage of the total load to each of the three aforementioned load models. The total delta current for each load is the sum of the three components.

Line Currents Serving a Delta-Connected Loads

- The line currents entering the delta-connected load are determined by applying KCL at each of the nodes of the delta. In matrix form, the equations are

$$\begin{bmatrix} IL_a \\ IL_b \\ IL_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} IL_{ab} \\ IL_{bc} \\ IL_{ca} \end{bmatrix}$$

- Two Phase and Single Phase Load:
  In both the wye- and delta-connected loads, single-phase and two-phase loads are modeled by setting currents of missing phases to zero. The currents in the phases present are computed using the same appropriate equations for constant complex power, constant impedance, and constant current.

- Shunt Capacitance:
  - Shunt capacitor banks are commonly used in distribution systems to help in voltage regulation and to provide reactive power support.
  - The capacitor banks are modeled as constant susceptance connected in either wye or delta.
  - Similar to the load model, all capacitor banks are modeled as three-phase banks with the currents of missing phases set to zero for single-phase and two-phase banks.

Wye Connected Capacitor Bank

- The individual phase capacitor units are specified in kVAr and kV. The constant susceptance for each unit can be computed in Siemens [S]. The susceptance of a capacitor unit is computed by

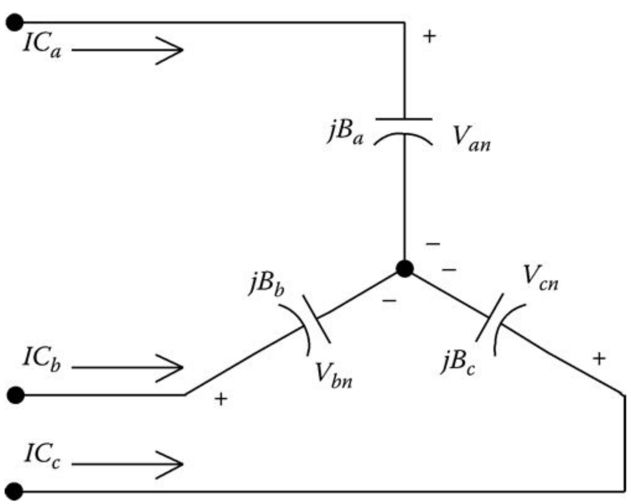
$$B_c = \frac{\mathrm{kvar}}{\mathrm{kV}_{\mathrm{LN}}^2 \cdot 1000}\ [\mathrm{S}]$$

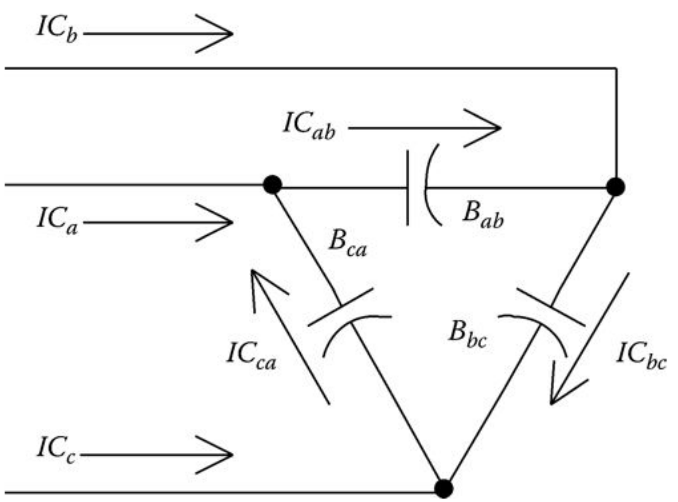
- With the susceptance computed, the line currents serving the capacitor bank are given by

$$IC_a = jB_a \cdot V_{an}$$

$$IC_b = jB_b \cdot V_{bn}$$

$$IC_c = jB_c \cdot V_{cn}$$





Delta-Connected Capacitor Bank

- The individual phase capacitor units are specified in kVAr and kV. For the delta-connected capacitors, the kV must be the line-to-line voltage. The constant susceptance for each unit can be computed in Siemens. The susceptance of a capacitor unit is computed by

$$B_c = \frac{\mathrm{kvar}}{\mathrm{kV}_{\mathrm{LL}}^2 \cdot 1000}\ [S]$$

- With the susceptance computed, the delta currents serving the capacitor bank are given by

$$IC_{ab} = jB_{ab} \cdot V_{ab}$$

$$IC_{bc} = jB_{bc} \cdot V_{bc}$$

$$IC_{ca} = jB_{ca} \cdot V_{ca}$$

- The line currents flowing into the delta-connected capacitors are computed by applying KCL at each node. In matrix form, the equations are

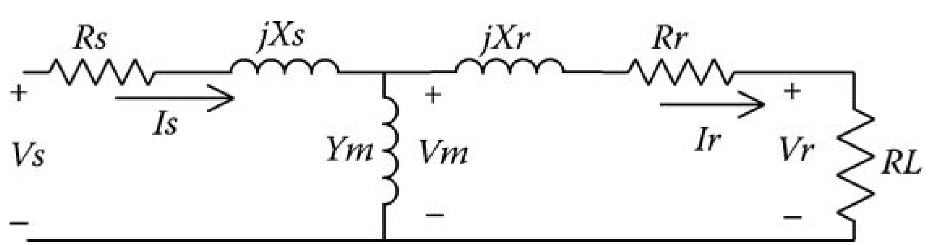
$$\begin{bmatrix} IC_a \\ IC_b \\ IC_c \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}}_{\text{Rotation Matrix}} \cdot \begin{bmatrix} IC_{ab} \\ IC_{bc} \\ IL_{ca} \end{bmatrix}$$

Induction Machines

- The analysis of an induction machine (motor or generator) when operating under unbalanced voltage conditions has traditionally been performed using symmetrical components.
- Given the sequence line-to-neutral voltages, the sequence currents are computed. The zero sequence network is not required since the machines are typically connected delta or ungrounded wye, which means that there will not be any zero sequence currents or voltages.
- The phase currents are determined by performing the transformation back to the phase line currents. The internal operating conditions are determined by the complete analysis of the sequence networks.

Procedures -

1. determine the terminal voltages and currents of the motor
2. use these values to compute the stator and rotor losses and the converted shaft power.



Induction Machines

The circuit applies to both the positive and negative sequence networks. The only difference between the two is the value of the “load resistance” RL as defined in the following:

$$RL_i = \frac{1 - s_i}{s_i} \cdot Rr_i$$

where

Positive sequence slip: $$s_i = \frac{n_s - n_r}{n_s}$$

where

n_s is the synchronous speed

n_r is the rotor speed

Negative sequence slip: $$s_2 = 2 - s_1$$

Note that the negative sequence load resistance RL_2 will be negative that will lead to a negative shaft power in the negative sequence.

If the value of positive sequence slip (s_1) is known, the input sequence impedances for the positive and negative sequence networks can be determined as

$$ZM_i = Rs_i + jXs_i + \frac{(jXm_i)(Rr_i + RL_i + jXr_i)}{Rr_i + RL_i + j(Xm_i + Xr_i)}$$

where i = 1 (positive sequence) and 2 (negative sequence).

Induction Machines

Once the input sequence impedances have been determined, the analysis of an induction machine operating with unbalanced voltages requires the following steps:

Step 1: Transform the known line-to-line voltages to sequence line-to-line voltages:

$$\begin{bmatrix} Vab_0 \\ Vab_1 \\ Vab_2 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \cdot \begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix} \quad \rightarrow [VLL_{012}] = [A]^{-1} \cdot [VLL_{abc}]$$

V_{ab0} = 0 because of Kirchhoff's voltage law (KVL).

Step 2: Compute the sequence line-to-neutral voltages from the line-to-line voltages:

$$Van_0 = Vab_0 = 0$$

It will not be true for a general case. However, for the case of the machine being connected either in delta or ungrounded wye, the zero sequence line-to-neutral voltage can be assumed to be zero:

$$Van_1 = t^* \cdot Vab_1 \qquad t = \frac{1}{\sqrt{3}} \cdot \angle 30^o$$

$$Van_2 = t^* \cdot Vab_2$$

$$Van_0 = Vab_0 = 0$$

To make it in matrix form,

$$\begin{bmatrix} Van_0 \\ Van_1 \\ Van_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & t^* & 0 \\ 0 & 0 & t \end{bmatrix} \cdot \begin{bmatrix} Vab_0 \\ Vab_1 \\ Vab_2 \end{bmatrix} \quad \rightarrow [VLN_{012}] = [T] \cdot [VLL_{012}]$$

Induction Machines

- Step 3: Compute the sequence line currents flowing into the machine:

$$Ia_0 = 0$$

$$Ia_1 = \frac{Van_1}{ZM_1}$$

$$Ia_2 = \frac{Van_2}{ZM_2}$$

- Step 4: Transform the sequence currents to phase currents:

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \cdot \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} \qquad \rightarrow [I_{abc}] = [A] \cdot [I_{012}]$$

- The four steps outlined earlier can be performed without actually computing the sequence voltages and currents. The procedure basically reverses the steps.
- Define

$$YM_i = \frac{1}{ZM_i}$$

$$I_0 = 0$$

$$I_1 = YM_1 \cdot Van_1 = YM_1 \cdot t^* \cdot Vab_1$$

$$I_2 = YM_2 \cdot Van_2 = YM_2 \cdot t^* \cdot Vab_2$$

Induction Machines

- In matrix form for current – voltage relationship,

$$I_1 = YM_1 \cdot Van_1 = YM_1 \cdot t^* \cdot Vab_1$$
$$I_2 = YM_2 \cdot Van_2 = YM_2 \cdot t^* \cdot Vab_2$$
$$I_0 = Vab_0$$

$$\rightarrow \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t^* \cdot YM_1 & 0 \\ 0 & 0 & t \cdot YM_1 \end{bmatrix} \cdot \begin{bmatrix} Vab_0 \\ Vab_1 \\ Vab_2 \end{bmatrix}$$

$$\rightarrow [I_{012}] = [YM_{012}] \cdot [VLL_{012}]$$

- Transform back to phase representation,

$$[VLL_{012}] = [A]^{-1} \cdot [VLL_{abc}]$$
$$[I_{abc}] = [A] \cdot [I_{012}]$$

$$\rightarrow [I_{abc}] = [A] \cdot [YM_{012}] \cdot [A]^{-1} \cdot [VLL_{abc}]$$

- Define Phase Admittance Matrix $[YM_{abc}] = [A] \cdot [YM_{012}] \cdot [A]^{-1}$ for induction machines.
- Current – Voltage Relationship at Phase Representation:

$$[I_{abc}] = [YM_{abc}] \cdot [VLL_{abc}]$$

- Recall that $[YM_{abc}]$ is a function of the slip of the machine so that a new matrix must be computed every time the slip changes.
- The line-to-line voltage can be solved with the inverse equation with impedance matrix.

$$[VLL_{abc}] = [ZM_{abc}] \cdot [I_{abc}]$$
$$[ZM_{abc}] = [YM_{abc}]^{-1}$$
$$[VLN_{abc}] = [W] \cdot [VLL_{abc}]$$

Induction Machines

- Define $[W] = [A] \cdot [T] \cdot [A]^{-1}$
- The matrix [W] is a very useful matrix that allows the determination of the "equivalent" line-to-neutral voltages from a knowledge of the line-to-line voltages. To define the "line-to-neutral" equation:

$$[VLN_{abc}] = \boxed{[W] \cdot [ZM_{abc}]} \cdot [I_{abc}]$$

$$[VLN_{abc}] = [ZLN_{abc}] \cdot [I_{abc}]$$

- Similarly, taking the inverse to obtain current from voltage applied,

$$[I_{abc}] = [YLN_{abc}] \cdot [VLN_{abc}]$$

$$[YLN_{abc}] = [ZLN_{abc}]^{-1}$$

- Make sure voltages used are the line-to-neutral and not the line-to-ground voltages. If only the line-to-ground voltages are known, they must first be converted to the line-to-line values and then compute the line-to-neutral voltages.

$$[VLN_{abc}] = [W] \cdot [VLL_{abc}]$$

$$[I_{abc}] = [YLN_{abc}] \cdot [VLN_{abc}]$$

- Once the machine terminal currents and line-to-neutral voltages are known, the input phase complex powers and total three-phase input complex power can be computed:

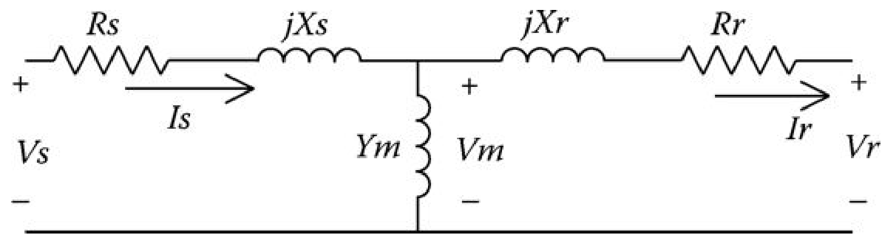
$$S_a = V_{an} \cdot (I_a)^*$$

$$S_b = V_{bn} \cdot (I_b)^* \quad \rightarrow \quad S_{Total} = S_a + S_b + S_c$$

$$S_c = V_{cn} \cdot (I_c)^*$$

Equivalent T Model for Induction Machines

- Once the terminal line-to-neutral voltages and currents are known, it is desired to analyze machine mechanism.
- Stator and rotor losses are needed in addition to the "converted" shaft power.
- The circuit can be modified by removing RL, which represents the "load resistance" in the positive and negative sequence networks, to a T circuit. The resulting networks will be modeled using A, B, C, and D parameters. This circuit can represent both the positive and negative sequence networks. The only difference (if any) will be between the numerical values of the sequence stator and rotor impedances.



$$Ym = \frac{1}{jXm}$$

- Define stator and rotor impedance –

$$Zs_i = Rs_i + jXs_i$$

$$Zr_i = Rr_i + jXr_i$$

- The [A, B, C, D] parameters for the asymmetrical T circuit is given by

$$Am_i = 1 + Ym_i \cdot Zs_i \qquad Cm_i = Ym_i$$

$$Bm_i = Zs_i + Zr_i + Ym_i \cdot Zs_i \cdot Zr_i \qquad Dm_i = 1 + Ym_i \cdot Zr_i$$

Equivalent T Model for Induction Machines

- It is noted that $\mathrm{Am_i} \cdot \mathrm{Dm_i} - \mathrm{Bm_i} \cdot \mathrm{Dm_i} = 1$
- The terminal sequence line-to-neutral voltages and currents as functions of the rotor "load voltages" (Vr) and the rotor currents are given by

$$\begin{bmatrix} Vs_i \\ Is_i \end{bmatrix} = \begin{bmatrix} Am_i & Bm_i \\ Cm_i & Dm_i \end{bmatrix} \cdot \begin{bmatrix} Vr_i \\ Ir_i \end{bmatrix} \qquad \rightarrow \begin{bmatrix} Vr_i \\ Ir_i \end{bmatrix} = \begin{bmatrix} Dm_i & -Bm_i \\ -Cm_i & Am_i \end{bmatrix} \cdot \begin{bmatrix} Vs_i \\ Is_i \end{bmatrix}$$

- It can be expanded to show the individual sequence voltages and currents:

$$\begin{bmatrix} Vr_0 \\ Vr_1 \\ Vr_2 \\ Ir_0 \\ Ir_1 \\ Ir_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Dm_1 & 0 & 0 & -Bm_1 & 0 \\ 0 & 0 & Dm_1 & 0 & 0 & -Bm_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Cm_1 & 0 & 0 & Am_1 & 0 \\ 0 & 0 & -Cm_2 & 0 & 0 & Am_2 \end{bmatrix} \begin{bmatrix} Vs_0 \\ Vs_1 \\ Vs_2 \\ Is_0 \\ Is_1 \\ Is_2 \end{bmatrix}$$

- The matrix can be partitioned between the third and fourth rows and columns. In reduced form by incorporating the partitioning, it becomes

$$\begin{bmatrix} [Vr_{012}] \\ [Ir_{012}] \end{bmatrix} = \begin{bmatrix} [Dm_{012}] & [Bm_{012}] \\ [Cm_{012}] & [Am_{012}] \end{bmatrix} \cdot \begin{bmatrix} [Vs_{012}] \\ [Is_{012}] \end{bmatrix}$$

- Expanding,

$$[Vr_{012}] = [Dm_{012}] \cdot [Vs_{012}] + [Bm_{012}] \cdot [Is_{012}]$$
$$[Ir_{012}] = [Cm_{012}] \cdot [Vs_{012}] + [Am_{012}] \cdot [Is_{012}]$$

Equivalent T Model for Induction Machines

- Transforming back to phase representation,

$$[Vr_{abc}] = [A]\cdot\ [Vr_{012}] = [A]\cdot[Dm_{012}]\cdot[A]^{-1}\cdot[Vs_{abc}] + [A]\cdot[Bm_{012}]\cdot[A]^{-1}\cdot\ [Is_{abc}]$$

$$[Ir_{abc}] = [A]\cdot\ [Ir_{012}] = [A]\cdot[Cm_{012}]\cdot[A]^{-1}\cdot[Vs_{abc}] + [A]\cdot[Am_{012}]\cdot[A]^{-1}\cdot\ [Is_{abc}]$$

- Therefore,

$$[Vr_{abc}] = [Dm_{abc}]\cdot[Vs_{abc}] + [Bm_{abc}]\cdot\ [Is_{abc}]$$

$$[Ir_{abc}] = [Cm_{abc}]\cdot[Vs_{abc}] + [Am_{abc}]\cdot\ [Is_{abc}]$$

$$[Am_{abc}] = [A]\cdot[Am_{012}]\cdot[A]^{-1}$$

$$[Bm_{abc}] = [A]\cdot[Bm_{012}]\cdot[A]^{-1}$$

$$[Cm_{abc}] = [A]\cdot[Cm_{012}]\cdot[A]^{-1}$$

$$[Dm_{abc}] = [A]\cdot[Dm_{012}]\cdot[A]^{-1}$$

- Power converted to the shaft:

$$P_{conv} = Vr_a\cdot(Ir_a)^* + Vr_b\cdot(Ir_b)^* + Vr_c\cdot(Ir_c)^*$$

- The usable power at the shaft regardless of the rotational loss P_{FW} is:

$$P_{shaft} = P_{conv} - P_{FW}$$

- The stator and rotor copper losses are -

$$P_{rotor} = |Ir_a|^2\cdot Rr + |Ir_b|^2\cdot Rr + |Ir_c|^2\cdot Rr$$

$$P_{stator} = |Is_a|^2\cdot Rs + |Is_b|^2\cdot Rs + |Is_c|^2\cdot Rs$$

$$P_{in} = Re[Vs_a\cdot(Is_a)^* + Vs_b\cdot(Is_b)^* + Vs_c\cdot(Is_c)^*]$$

Example 2 – Induction Machines

To demonstrate the analysis of an induction motor in the phase frame, the following induction motor will be used:

- 25 hp, 240 V operating with slip = 0.035
- $Ploss_{rotation}$ = 0.75 kW
- Zs = 0.0774 + j0.1843 Ω
- Zm = 0 + j4.8384 Ω
- Zr = 0.0908 + j0.1843 Ω

The "load" resistances are

$$RL_1 = \left(\frac{1 - 0.035}{0.035}\right) \cdot 0.098 = 2.5029\ \Omega$$

$$RL_2 = \left(\frac{1 - (1.965)}{1.965}\right) \cdot 0.0908 = -0.0446\ \Omega$$

The input sequence impedance are

$$ZM_1 = Zs + \frac{Zm \cdot (Zr + RL_1)}{Zm + Zr + RL_1} = 1.9775 + j1.3431\ \Omega$$

$$ZM_2 = Zs + \frac{Zm \cdot (Zr + RL_2)}{Zm + Zr + RL_2} = 0.1203 + j0.3623\ \Omega$$

The input sequence admittance are

$$YM_1 = \frac{1}{ZM_1} = 0.3461 - j0.2350\ S$$

$$YM_2 = \frac{1}{ZM_2} = 0.8255 - j2.4863\ S$$

Example 2 – Induction Machines

The sequence admittance matrix is $[YM_{012}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3461 - j0.2350 & 0 \\ 0 & 0 & 0.8255 - j2.4863 \end{bmatrix} S$

The phase admittance matrix is

$$[YM_{abc}] = \begin{bmatrix} 0.7452 - j0.4074 & -0.0999 - j0.0923 & 0.3547 + j0.4997 \\ 0.3547 + j0.4997 & 0.7452 - j0.4074 & -0.0999 - j0.0923 \\ -0.0999 - j0.0923 & 0.3547 + j0.4997 & 0.7452 - j0.4074 \end{bmatrix} S$$

The line-to-line terminal voltages are measured to be $V_{ab} = 235\ V, V_{bc} = 240\ V, V_{ca} = 245\ V$

Since the sum of the line-to-line voltages must equal zero, the law of cosines can be used to determine the angles on the voltages. Applying the law of cosines results in

$$[VLL_{abc}] = \begin{bmatrix} 235\angle 0 \\ 240\angle - 117.9 \\ 245\angle 120.0 \end{bmatrix} V$$

The phase motor current can be computed as

$$[Is_{abc}] = [YM_{abc}] \cdot [VLL_{abc}] = \begin{bmatrix} 53.15\angle - 71.0 \\ 55.15\angle - 175.1 \\ 66.6\angle 55.6 \end{bmatrix} A$$

Current balance can be evaluated by –

$$I_{unbalance} = \left(\frac{\max\_deviation}{|I_{avg}|}\right) \cdot 100 = \left(\frac{8.3232}{58.31}\right) \cdot 100 = 14.27\%$$

Example 2 – Induction Machines

The current unbalance is approximately seven times greater than the voltage unbalance.

The equivalent line-to-neutral voltages at the motor are computed using the [W] matrix:

$$[Vs_{abc}] = [W] \cdot [VLL_{abc}]$$

$$\begin{bmatrix} V_{an} \\ V_{bn} \\ V_{cn} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 235.0\angle 0 \\ 240\angle -117.9 \\ 245\angle 120.0 \end{bmatrix} = \begin{bmatrix} 138.6\angle -30.7 \\ 135.7\angle -148.6 \\ 141.4\angle 91.4 \end{bmatrix}$$

The input complex power to the motor is

$$S_{in} = \sum_{k=1}^{3} \frac{Vs_{abc_k} \cdot I_{abc_k}}{1000} = 19.95 + j13.62$$

$$|S_{in}| = 24.15 \qquad \text{PF} = 0.83 \text{ lagging}$$

The rotor currents and voltages can be computed by first computing the equivalent A, B, C, and D matrices

$$Am_i = 1 + Ym_i \cdot Zs_i = 1.0381 - j0.0161$$

$$Bm_i = Zs_i + Zr_i + Ym_i \cdot Zs_i \cdot Zr_i = 0.1746 + j0.3742$$

$$Cm_i = Ym_i = -j0.2067 \qquad Dm_i = 1 + Ym_i \cdot Zr_i = 1.0381 - j0.0188$$

Kersting, W.H. and Phillips, W.H., Phase frame analysis of the effects of voltage unbalance on induction machines, IEEE Transactions on Industry Applications, 33, 415–420, 1997.

Example 2 – Induction Machines

- The sequence matrices

$$[Am_{012}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.0381 - j0.0161 & 0 \\ 0 & 0 & 1.0381 - j0.0161 \end{bmatrix} \qquad [Cm_{012}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & j0.2067 & 0 \\ 0 & 0 & j0.2067 \end{bmatrix}$$

$$[Bm_{012}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.1746 - j0.3742 & 0 \\ 0 & 0 & -0.1746 - j0.3742 \end{bmatrix} \qquad [Dm_{012}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.0381 - j0.0188 & 0 \\ 0 & 0 & 1.0381 - j0.0188 \end{bmatrix}$$

- Convert back to phase matrices,

$$[Am_{abc}] = [A] \cdot [Am_{012}] \cdot [A]^{-1} = \begin{bmatrix} 0.6921 - j0.0107 & -0.346 + j0.0053 & -0.346 + j0.0053 \\ -0.346 + j0.0053 & 0.6921 - j0.0107 & -0.346 + j0.0053 \\ -0.346 + j0.0053 & -0.346 + j0.0053 & 0.6921 - j0.0107 \end{bmatrix}$$

$$[Bm_{abc}] = [A] \cdot [Bm_{012}] \cdot [A]^{-1} = \begin{bmatrix} -0.1164 - j0.2494 & 0.0582 + j0.1247 & 0.0582 + j0.1247 \\ 0.0582 + j0.1247 & -0.1164 - j0.2494 & 0.0582 + j0.1247 \\ 0.0582 + j0.1247 & 0.0582 + j0.1247 & -0.1164 - j0.2494 \end{bmatrix}$$

$$[Cm_{abc}] = [A] \cdot [Cm_{012}] \cdot [A]^{-1} = \begin{bmatrix} j0.1378 & -j0.0689 & -j0.0689 \\ -j0.0689 & j0.1378 & -j0.0689 \\ -j0.0689 & -j0.0689 & j0.1378 \end{bmatrix}$$

$$[Dm_{abc}] = [A] \cdot [Dm_{012}] \cdot [A]^{-1} = \begin{bmatrix} 0.6921 - j0.0125 & -0.346 + j0.0063 & -0.346 + j0.0063 \\ -0.346 + j0.0063 & 0.6921 - j0.0125 & -0.346 + j0.0063 \\ -0.346 + j0.0063 & -0.346 + j0.0063 & 0.6921 - j0.0125 \end{bmatrix}$$

Example 2 – Induction Machines

With the matrices defined, the rotor voltages and currents can be computed:

$$[Vr_{abc}] = [Dm_{abc}] \cdot [Vs_{abc}] + [Bm_{abc}] \cdot \ [Is_{abc}] = \begin{bmatrix} 124.5\angle - 36.1 \\ 124.1\angle - 156.3 \\ 123.8\angle 83.9 \end{bmatrix}$$

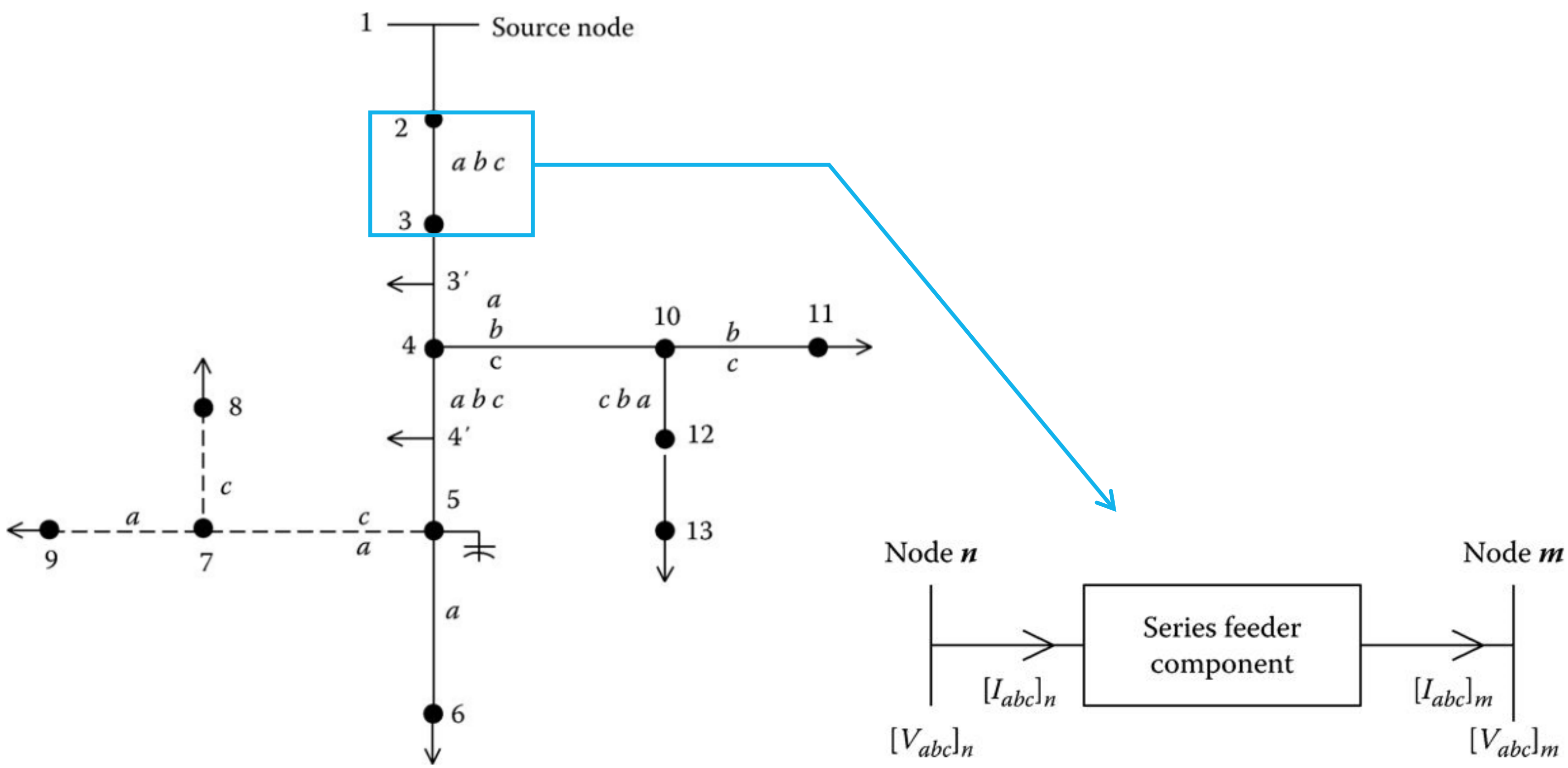
$$[Ir_{abc}] = [Cm_{abc}] \cdot [Vs_{abc}] + [Am_{abc}] \cdot \ [Is_{abc}] = \begin{bmatrix} 42.2\angle - 41.2 \\ 50.9\angle - 146.6 \\ 56.8\angle 79.1 \end{bmatrix}$$

The converted electric power to shaft power is

$$P_{convert} = \sum_{\mathrm{k}=1}^{3} \frac{\mathrm{Vs_{abc_k}} \cdot \mathrm{I^*_{abc_k}}}{1000} = 18.5\ \mathrm{kW}$$

General Feeder Modeling - Series Components

- A typical distribution feeder consists of the primary main with laterals tapped off the primary main and sub-laterals tapped off the laterals.
- A distribution feeder can be broken into the "series" components and the "shunt" components. These series components can be lines, transformers, voltage regulators...



General Feeder Modeling - Series Components

The [a, b, c, d] matrix for transformers, voltage regulators and lines -

	$\triangle-$ Grounded Y Step-down	Wye-Connected Voltage Regulator	Line Segment
$[a]$	$-\frac{n_t}{3}\begin{bmatrix}0 & 2 & 1\\ 1 & 0 & 2\\ 2 & 1 & 0\end{bmatrix}$	$\begin{bmatrix}a_{R\_a} & 0 & 0\\ 0 & a_{R\_b} & 0\\ 0 & 0 & a_{R\_c}\end{bmatrix}$	$[u]$
$[b]$	$-\frac{n_t}{3}\begin{bmatrix}0 & 2Z_{t_b} & Z_{t_c}\\ Z_{t_a} & 0 & 2Z_{t_c}\\ 2Z_{t_a} & Z_{t_b} & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$[Z_{abc}]$
$[c]$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$
$[d]$	$\frac{1}{n_t}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1/a_{R\_a} & 0 & 0\\ 0 & 1/a_{R\_b} & 0\\ 0 & 0 & 1/a_{R\_c}\end{bmatrix}$	$[u]$
$[A]$	$\frac{1}{n_t}\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$	$\begin{bmatrix}1/a_{R\_a} & 0 & 0\\ 0 & 1/a_{R\_b} & 0\\ 0 & 0 & 1/a_{R\_c}\end{bmatrix}$	$[u]$
$[B]$	$\begin{bmatrix}Z_{t_a} & 0 & 0\\ 0 & Z_{t_b} & 0\\ 0 & 0 & Z_{t_c}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$[Z_{abc}]$

$$n_t = \frac{VLL_{rated\ primary}}{VLN_{rated\ secondary}}$$

$a_R = 1 \pm 0.00625 \cdot \text{Tap}$

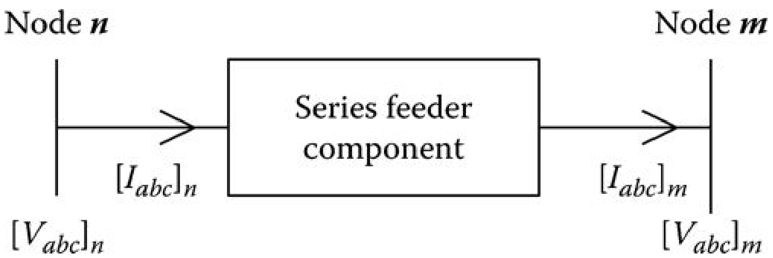
$$[VLN_{abc}] = [A][VLN_{ABC}] - [B][I_{abc}]$$

$$[VLN_{ABC}] = [a][VLN_{abc}] + [b][I_{abc}]$$

$$[I_{ABC}] = [c][VLN_{abc}] + [d][I_{abc}]$$

General Feeder Modeling - Series Components

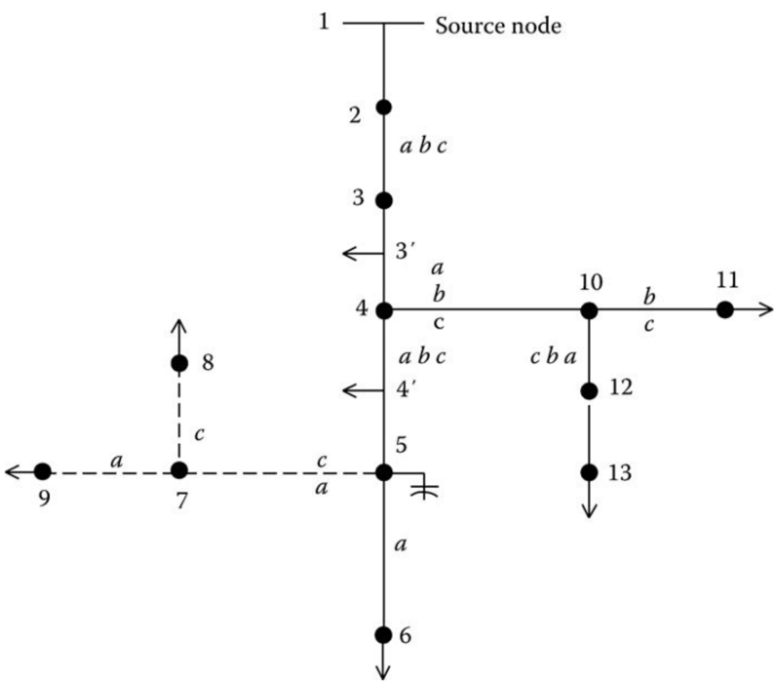
- In the series model, for any series components, they are modeled using the following two equations.
- These two equations are also known as forward and backward sweep models.
- Forward Sweep: $[VLN_{abc}]_m = [A] \cdot [VLN_{abc}]_n - [B] \cdot [I_{abc}]_n$
- Backward Sweep: $[I_{abc}]_n = [c] \cdot [VLN_{abc}]_m + [d] \cdot [I_{abc}]_m$
- For different components, the equations have the same format, and [A], [B], [c], and [d] are all 3 × 3 matrices. However, calculating the components in these matrices will be different for different components.
- For example, Carson's equations can be used for computing the line impedances for overhead and underground lines. Two-phase and single-phase lines are represented by a 3 × 3 matrix with zeros set in the rows and columns of the missing phases.
- In most cases, the [c] matrix will be zero. Long underground lines will be an exception.



General Feeder Modeling - Shunt Components

- The shunt components of a distribution feeder are
  - Spot static loads
  - Spot induction machines
  - Capacitor banks
- Spot static loads are located at a node and can be three phase, two phase, or single phase and connected in either a wye or a delta connection. The loads can be modeled as ZIP.

- To model the UDL in distribution feeders, one can model it
  - at the middle of the lines.
  - by placing one-half of the load at each end of the line.
  - by placing two-thirds of the load 25% of the way down the line from the source end. The remaining one-third of the load is connected at the receiving end node.
- This “exact” model gives the correct voltage drop down the line in addition to the correct power line power loss.



Why Modelling? Power Flow Analysis

- The developed models is used in the power-flow analysis of a distribution feeder.
- Power-flow analysis of a distribution feeder is similar to that of an interconnected transmission system. Typically, what will be known prior to the analysis will be the three-phase voltages at the substation and the complex power of all of the loads and the load model (constant complex power, constant impedance, constant current, or a combination).
- A power-flow analysis of a feeder can determine the following:
  - Voltage magnitudes and angles at all nodes of the feeder
  - Line flow in each line section specified in kW and kVAr, amps and degrees, or amps and power factor
  - Power loss in each line section
  - Total feeder input kW and kvar
  - Total feeder power losses
  - Load kW and kvar based upon the specified model for the load

Modified "Ladder" Iterative Technique

- Because a distribution feeder is radial, iterative techniques commonly used in transmission network power-flow studies are not used because of poor convergence characteristics. Instead, an iterative technique specifically designed for a radial system is used.
- Weakly linked and sparse in distribution network.
- When the source voltages are specified and the loads are specified as constant kW and kVAr (constant PQ), the system becomes nonlinear, and an iterative method will have to be used to compute the load voltages and currents.
- A simple form of that technique will be developed here in order to demonstrate how the nonlinear system can be evaluated.
- The ladder technique is composed of two parts:
    1. Forward sweep
    2. Backward sweep
- The forward sweep computes the downstream voltages from the source by:

$$[VLG_{abc}]_m = [A] \cdot [VLG_{abc}]_n - [B] \cdot [I_{abc}]$$

- To start the process, the load currents $[I_{abc}]$ are assumed to be equal to zero and the load voltages are computed. In the first iteration the load voltages will be the same as the source voltages.
- The backward sweep computes the currents from the load back to the source using the most recently computed voltages from the forward sweep. The [c, d] equation is applied for this sweep:

$$[I_{abc}]_n = [c] \cdot [VLG_{abc}]_m + [d] \cdot [I_{abc}]_m$$

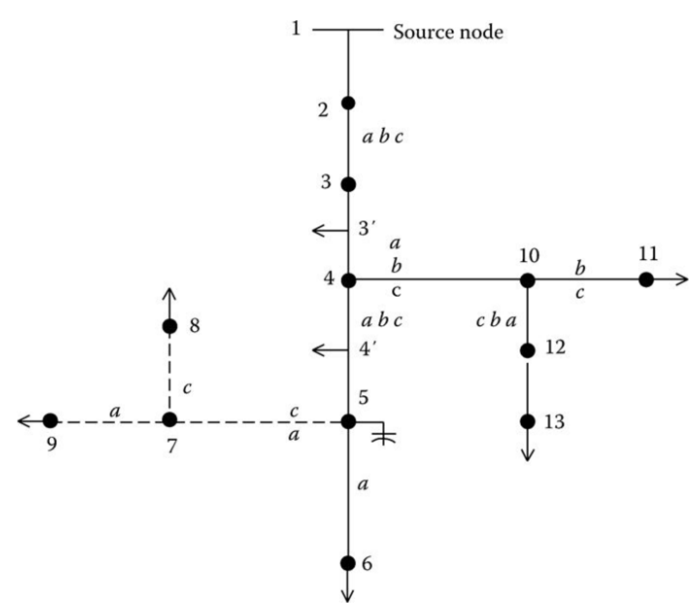
Modified "Ladder" Iterative Technique

- Recall that for all practical purposes the [c] matrix is zero so that the [c, d] equation is simplified to be

$$[I_{abc}]_n=[d]\cdot[I_{abc}]_m$$

- After the first forward and backward sweeps, the new load voltages are computed using the most recent currents. Also, load/shunt device currents should be updated using the most recent voltages before moving to backward sweep. The forward and backward sweeps continue until the error between the new and previous load voltages is within a specified tolerance.

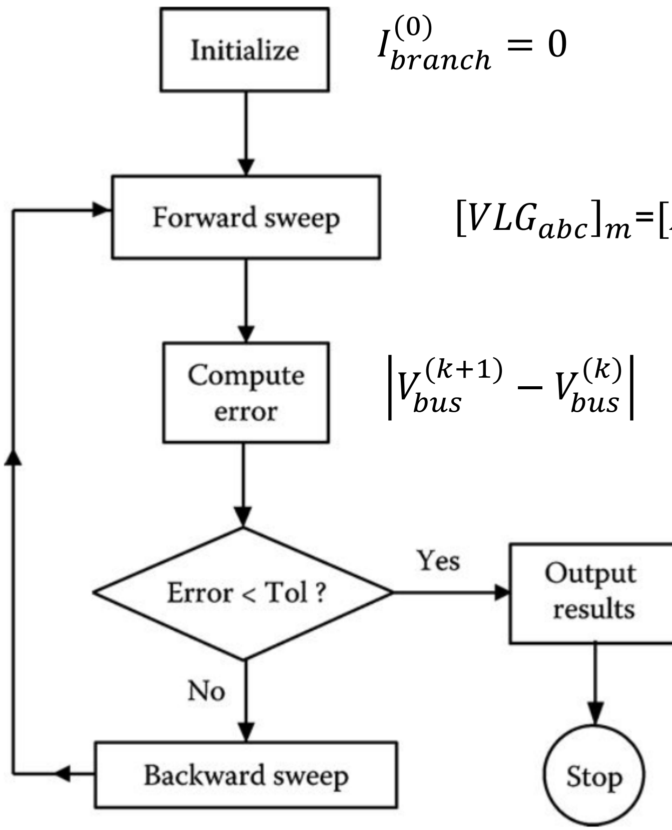
- Nodes 4, 10, 5, and 7 are referred to as "junction nodes." In both the forward and backward sweeps, the junction nodes must be recognized. In the forward sweep, the voltages at all nodes down the lines from the junction nodes must be computed. In the backward sweeps, the currents at the junction nodes must be summed before proceeding toward the source.
- The "node" currents may be three phase, two phase, or single phase and consist of the sum of the spot load currents and one-half of the distributed load currents (if any) at the node plus the capacitor current (if any) at the node.
- It is possible that at a given node the distributed load can be one-half of the distributed load in the "from" segment plus one-half of the distributed load in the "to" segment.



Modified “Ladder” Iterative Technique

- The modified ‘ladder’ iterative technique can be summarized in the flowchart.

$$I_{branch}^{(0)} = 0 \qquad V_{bus}^{(0)} = 1$$



$$[VLG_{abc}]_m = [A] \cdot [VLG_{abc}]_n - [B] \cdot [I_{abc}]$$

$$\left| V_{bus}^{(k+1)} - V_{bus}^{(k)} \right|$$

$$[I_{abc}]_n = d] \cdot [I_{abc}]_m$$

General Feeder Modelling

- How to calculate [A], [B], [c], [d] matrices for different components (lines, transformers, voltage regulators)?
- How to model shunt devices (loads, capacitor banks)?
- Let's start with lines (including overhead lines and underground cables)

Distribution Feeder Analysis

- The analysis of a distribution feeder will typically consist of a study of the feeder under normal steady-state operating conditions (power-flow analysis) and a study of the feeder under short-circuit conditions (short-circuit analysis).
- Both analyses are performed in phase frame.
- Models of all of the components of a distribution feeder have been developed in previous chapters. These models will be applied for the analysis under steady-state and short-circuit conditions.

Consideration in Power Flow Studies

- Line impedances should be computed using the exact spacings and phasing.
- Because of the unbalanced loading and resulting unbalanced line currents, the voltage drops due to the mutual coupling of the lines become very important.
- It is not unusual to observe a voltage rise on a lightly loaded phase of a line segment that has an extreme current unbalance.

Applying the Modified Ladder Iterative Technique

Power loss Computation

First method: compute the power loss in each phase by I^2R

- Care must be taken to not use the resistance value from the phase impedance matrix. The actual phase resistance that was used in Carson's equations must be used. It requires that storing the conductor resistance in the active data base for each line segment.
- It does not give the total power loss as the power loss in the neutral conductor and ground are not included.

Second method: compute power loss as $P_{IN} - P_{OUT}$

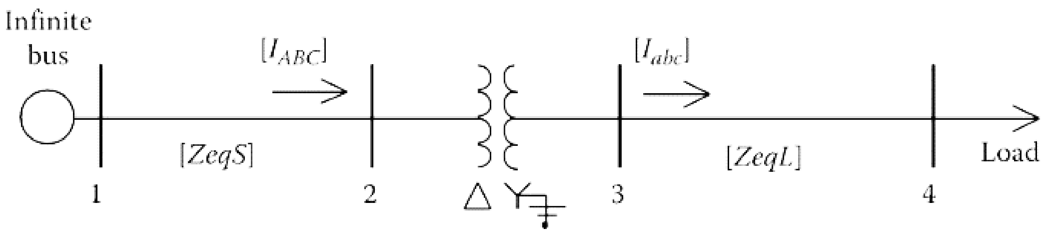
- Because the effects of the neutral conductor and ground are included in the phase impedance matrix of the total power loss, this method will give the same results as mentioned earlier where the neutral and ground power losses are computed separately.
- It is possible to compute what appears to be a negative phase power loss. This is a direct result of the inaccurate modeling of the mutual coupling between phases. Remember that the effect of the neutral conductor and the ground resistance is included in Carson's equations.
- In reality, there cannot be a negative phase power loss. Using this method, the algebraic sum of the line power losses will equal the total three-phase power loss that were computed using I^2R for the phase and neutral conductors along with ground current.

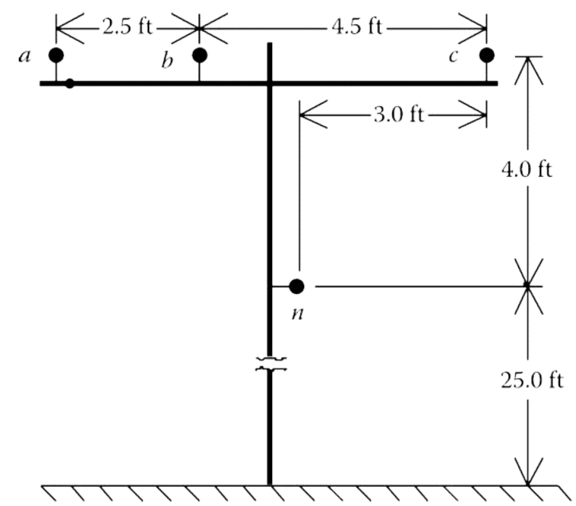
Example 3 – Forward – Backward Sweep

For the distribution feeder in the figure, the infinite bus voltages are balanced three phase of 12.47 kV_{LL}. The "source" line segment from node 1 to node 2 is a three-wire delta 2000 ft long line and is constructed on the pole configuration of as shown without the neutral. The "load" line segment from node 3 to node 4 is 2500 ft long and also is constructed on the pole configuration as shown but is a four-wire wye so the neutral is included. Both line segments use 336,400 26/7 ACSR phase conductors and the neutral conductor on the four-wire wye line is 4/0 6/1 ACSR. Since the lines are short, the shunt admittance will be neglected. The 25°C resistance is used for the phase and neutral conductors:

$$\text{336,400 26/7 ACSR: resistance at } 25°\text{C} = 0.278\ \Omega/\text{mile}$$

$$4/0\ 6/1 \quad \text{ACSR: resistance at } 25°\text{C} = 0.445\ \Omega/\text{mile}$$





Example 3 – Forward – Backward Sweep e 3

The phase impedance matrices for the two line segments are

$$[ZeqS_{ABC}] = \begin{bmatrix} 0.1414 + j0.5353 & 0.0361 + j0.3225 & 0.0361 + j0.2572 \\ 0.0361 + j0.3225 & 0.1414 + j0.5353 & 0.0361 + j0.2955 \\ 0.0361 + j0.2572 & 0.0361 + j0.2955 & 0.1414 + j0.5353 \end{bmatrix} \Omega$$

$$[ZeqL_{abc}] = \begin{bmatrix} 0.1907 + j0.5035 & 0.0607 + j0.2302 & 0.0598 + j0.1751 \\ 0.0607 + j0.2302 & 0.1939 + j0.4885 & 0.0614 + j0.1931 \\ 0.0598 + j0.1751 & 0.0614 + j0.1931 & 0.1921 + j0.4970 \end{bmatrix} \Omega$$

The transformer bank is connected delta–grounded wye and consists of three single-phase transformers each rated

$$2000 \text{ kVA}, 12.47 - 2.4 \text{ kV}, \text{Z} = 1.0 + \text{j}6.0\%$$

The feeder serves an unbalanced three-phase wye-connected constant PQ load of

$$\text{S}_\text{a} = 750 \text{ kVA at } 0.85 \text{ lagging power factor}$$

$$\text{S}_\text{b} = 1000 \text{ kVA at } 0.90 \text{ lagging power factor}$$

$$\text{S}_\text{c} = 1230 \text{ kVA at } 0.95 \text{ lagging power factor}$$

Before starting the iterative solution, the forward and backward sweep matrices must be computed for each series element. The modified ladder method is going to be employed so only the [A], [B], and [d] matrices need to be computed.

Source line segment with shunt admittance neglected.

Example 3 – Forward – Backward Sweep

Source Line Segment

$$[A_1] = [U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[B_1] = [ZeqS_{ABC}] = \begin{bmatrix} 0.1414 + j0.5353 & 0.0361 + j0.3225 & 0.0361 + j0.2572 \\ 0.0361 + j0.3225 & 0.1414 + j0.5353 & 0.0361 + j0.2955 \\ 0.0361 + j0.2572 & 0.0361 + j0.2955 & 0.1414 + j0.5353 \end{bmatrix}$$

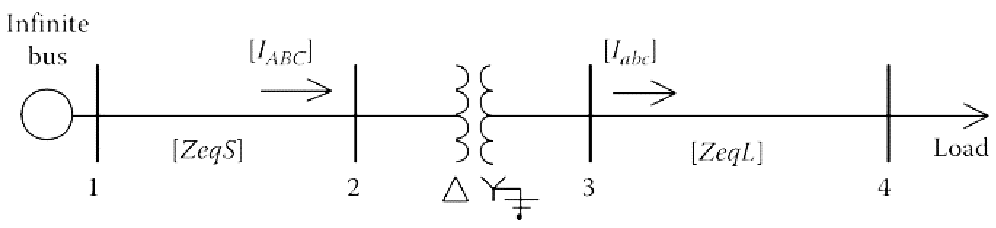
$$[d_1] = [U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Load Line Segment

$$[A_2] = [U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[B_2] = [ZeqL_{abc}] = \begin{bmatrix} 0.1907 + j0.5035 & 0.0607 + j0.2302 & 0.0598 + j0.1751 \\ 0.0607 + j0.2302 & 0.1939 + j0.4885 & 0.0614 + j0.1931 \\ 0.0598 + j0.1751 & 0.0614 + j0.1931 & 0.1921 + j0.4970 \end{bmatrix}$$

$$[d_2] = [U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example 3 – Forward – Backward Sweep

Transformer

- The transformer impedance must be converted to actual value in Ohms referenced to the low-voltage windings.

$$Z_{base} = \frac{2.4^2 \cdot 1000}{2000} = 2.88\ \Omega$$

$$Zt_{low} = (0.01 + j0.06) \cdot 2.88 = 0.0288 + j0.1728\ \Omega$$

$$[Zt_{abc}] = \begin{bmatrix} 0.0228 + j0.1728 & 0 & 0 \\ 0 & 0.0228 + j0.1728 & 0 \\ 0 & 0 & 0.0228 + j0.1728 \end{bmatrix}$$

- The "turns" ratio: n_t = 12.47/2.4 = 5.1958

$$[A_t] = \frac{1}{n_t} \cdot \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.1925 & 0 & -0.1925 \\ -0.1925 & 0.1925 & 0 \\ 0 & -0.1925 & 0.1925 \end{bmatrix}$$

$$[B_t] = [Zt_{abc}] = \begin{bmatrix} 0.0228 + j0.1728 & 0 & 0 \\ 0 & 0.0228 + j0.1728 & 0 \\ 0 & 0 & 0.0228 + j0.1728 \end{bmatrix}$$

$$[d_t] = \frac{1}{n_t} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1925 & -0.1925 & 0 \\ 0 & 0.1925 & -0.1925 \\ -0.1925 & 0 & 0.1925 \end{bmatrix}$$

Example 3 – Forward – Backward Sweep

Define the node 4 loads:

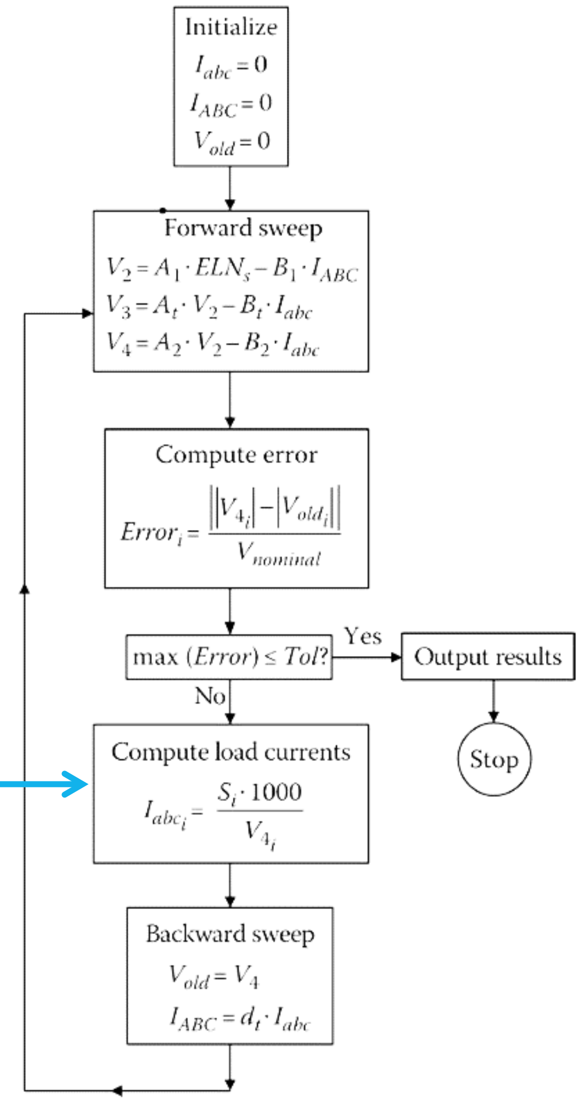
$$[S4] = \begin{bmatrix} 750\angle \mathrm{acos}(0.85) \\ 1000\angle \mathrm{acos}(0.90) \\ 1250\angle \mathrm{acos}(0.95) \end{bmatrix} = \begin{bmatrix} 750\angle 31.79 \\ 1000\angle 25.84 \\ 1250\angle 18.19 \end{bmatrix} kVA$$

Define infinite bus line-to-line and line-to-neutral voltages:

$$[ELL_s] = \begin{bmatrix} 12{,}470\angle 30 \\ 12{,}470\angle -90 \\ 12{,}470\angle 150 \end{bmatrix} V \qquad [ELN_s] = \begin{bmatrix} 7199.6\angle 0 \\ 7199.6\angle -120 \\ 7199.6\angle 120 \end{bmatrix} V$$

After 8 iteration on forward-backward sweep,

$$[V4_{120}] = \begin{bmatrix} 113.9 \\ 110.0 \\ 110.6 \end{bmatrix} V$$



Update nodal injection currents
(loads, capacitors, ...)
before moving to backward sweep.

Example 4 – Forward – Backward Sweep with Voltage Regulator

The voltages at node 4 are below the desired 120 V. These low voltages can be corrected with the installation of three step-voltage regulators connected in wye on the secondary bus (node 3) of the transformer. The new configuration of the feeder is shown.

For the regulator, the potential transformer ratio will be 2400–120 V (N_{pt} = 20) and the CT ratio is selected to carry the rated current of the transformer bank. The rated current is

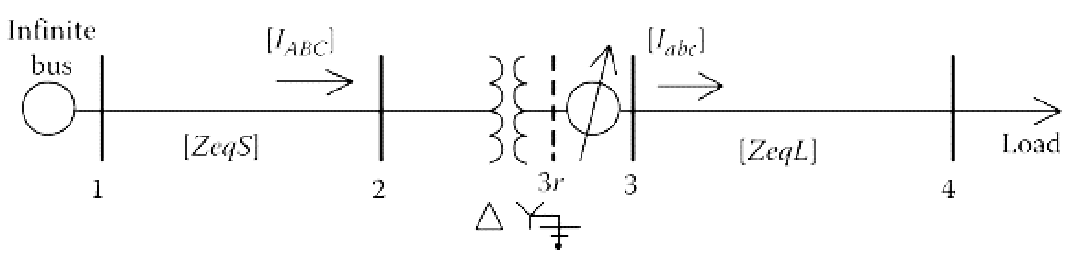
$$I_{rated} = \frac{6000}{\sqrt{3} \cdot 2.4} = 832.7$$

The CT ratio is selected to be 1000 : 5 = CT = 200.

$$Zeq_i = \frac{V3_i - V4_i}{I3_i} = \begin{bmatrix} 0.1414 + j0.1830 \\ 0.2079 + j0.2827 \\ 0.0889 + j0.3833 \end{bmatrix} \Omega$$

The three regulators are to have the same R and X compensator settings. The average value of the computed impedances will be used:

$$Z_{avg} = \frac{1}{3} \cdot \sum_{k=1}^{3} Zeq_k = 0.1451 + j0.2830 \ \Omega$$



Example 4 – Forward – Backward Sweep with Voltage Regulator

The value of the compensator impedance in volts is given as

$$R' + jX' = (0.1451 + j0.2830) \cdot \frac{1000}{20} = 7.3 + j14.2\ V$$

The value of the compensator settings in Ohms is

$$R_\Omega + jX_\Omega = \frac{7.3 + j14.2}{5} = 1.46 + j2.84\Omega$$

With the regulator in the neutral position, the voltages being input to the compensator circuit for the given conditions are

$$Vreg_i = \frac{V3_i}{PT} = \begin{bmatrix} 117.5\angle -31.2 \\ 117.1\angle -151.7 \\ 116.7\angle 87.8 \end{bmatrix} V$$

The compensator currents are

$$Icomp_i = \frac{Iabc_i}{CT} = \begin{bmatrix} 1.6460\angle -63.6 \\ 2.2727\angle -179.4 \\ 2.8264\angle 64.9 \end{bmatrix} A$$

With the input voltages and compensator currents, the voltages across the voltage relays in the compensator circuit are computed to be

$$[V_{relay}] = [V_{reg}] - [Z_{comp}] \cdot [I_{comp}] = \begin{bmatrix} 113.0\angle -32.5 \\ 111.3\angle -153.8 \\ 109.0\angle 84.5 \end{bmatrix} V$$

Example

Assume that the voltage level has been set at 121 V with a bandwidth of 2 V. In the real world, the regulators on each phase will change taps one at a time until the relay on that phase reaches 120 V. To model this system, the flowchart is slightly modified in the forward and backward sweeps.

Forward Sweep:

$$[VLN_2] = [A_1] \cdot [E_s] - [B_1] \cdot [I_{ABC}]$$

$$[VLN_{3r}] = [A_t] \cdot [VLN_2] - [B_t] \cdot [I_{in}]$$

$$[VLN_3] = \left[A_{reg}\right] \cdot [VLN_3] - \left[B_{reg}\right] \cdot [I_{abc}]$$

$$[VLN_4] = [A_2] \cdot [VLN_3] - [B_2] \cdot [I_{abc}]$$

Backward Sweep:

$$[V_{old}] = [VLN_4]$$

$$[I_{in}] = \left[d_{reg}\right] \cdot [I_{abc}]$$

$$[I_{ABC}] = [d_t] \cdot [I_{in}]$$

$$
\begin{aligned}
Y := \quad & \text{for } i \in 1 \,..\, 3 \\
& \qquad V_{reg_i} \leftarrow \frac{VLN_{3_i}}{N_{pt}} \\
& \qquad I_{reg_i} \leftarrow \frac{I_{abc_i}}{CT} \\
& V_{relay} \leftarrow V_{reg} - Z_{comp} \cdot I_{reg} \\
& Tap_1 \leftarrow Tap_1 + 1 \text{ if } \left|V_{relay_1}\right| < 120 \\
& Tap_2 \leftarrow Tap_2 + 1 \text{ if } \left|V_{relay_2}\right| < 120 \\
& Tap_3 \leftarrow Tap_3 + 1 \text{ if } \left|V_{relay_3}\right| < 120 \\
& Out_1 \leftarrow Tap \\
& Out
\end{aligned}
$$

After the analysis routine has converged, a new routine will compute whether tap changes need to be made. The Mathcad routine for computing the new taps is shown in the flowchart.

Example

The computational sequence for the determination of the final tap settings and convergence of the system is shown in the flowchart.

The tap changing routine changes individual regulators one step at a time. The final tap settings are

$$[\mathrm{Tap}] = \begin{bmatrix} 9 \\ 11 \\ 12 \end{bmatrix}$$

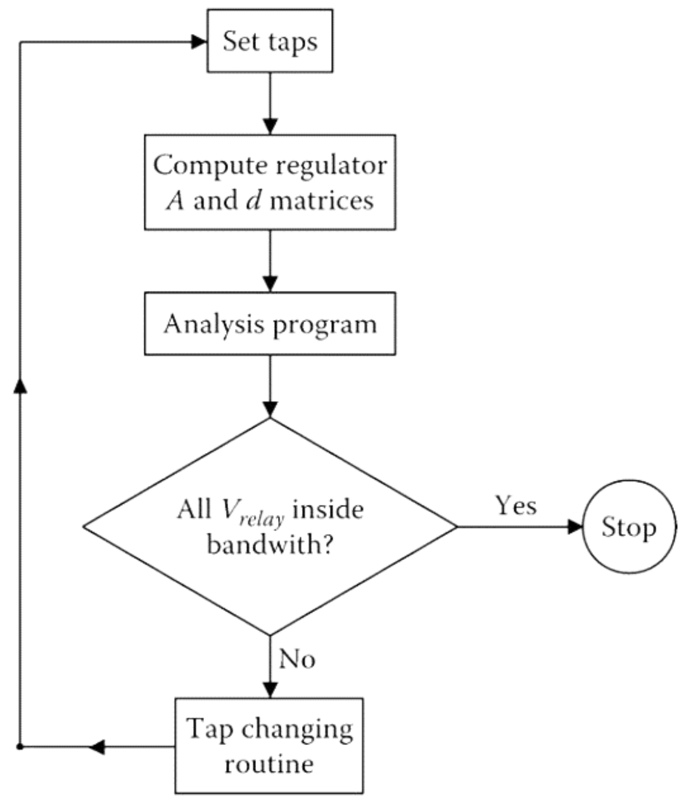
The final relay voltages are

$$[V_{relay}] = \begin{bmatrix} 120.3 \\ 120.4 \\ 120.1 \end{bmatrix}$$

The final voltages on a 120 V base at the load center (node 4) are

$$[VLN_{4_{120}}] = \begin{bmatrix} 121.0 \\ 119.3 \\ 120.7 \end{bmatrix}$$

Unlike the previous example, the compensator relay voltages and the actual load center voltages are very close to each other.



Load Allocation

- kW and kVAr to a feeder is often known with metering at substation.
- It is desirable to force the computed input complex power to the feeder matching the metered input. This can be accomplished (following a converged iterative solution) by computing the ratio of the metered input to the computed input.
- The phase loads can now be modified by multiplying the loads by this ratio. Because the losses of the feeder will change when the loads are changed, it is necessary to go through the ladder iterative process to determine a new computed input to the feeder. This new computed input will be closer to the metered input but most likely not within a specified tolerance. The ratio can be determined, and the value of the loads modified. This process is repeated until the computed input is within a specified tolerance of the metered input.
- Load allocation does not have to be limited to matching metered readings just at the substation. The same process can be performed at any point on the feeder where metered data is available. The only difference is that now only the “downstream” nodes from the metered point will be modified.

Per-unit Analysis of Power-Flow Studies

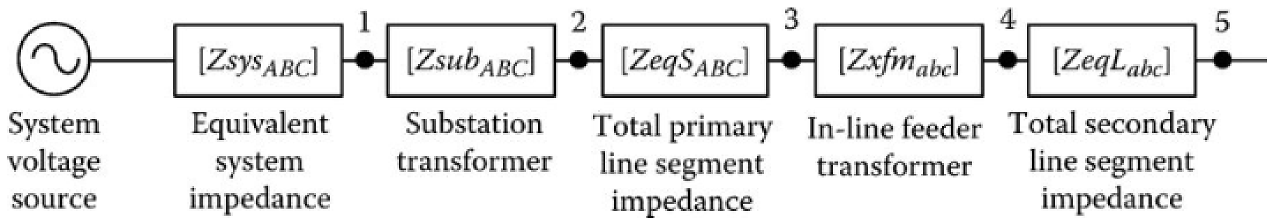
- The development of Backward-Forward Sweep Model and the examples uses actual values of voltage, current, impedance, and complex power.
- The reason why per unit analysis is not applied is that the calculating system is unbalanced, and per unit analysis required standardization of voltage base (phase-to-neutral or line-to-line). A conversion factor $\sqrt{3}$ can convert phase quantities to line quantities.
- When this is done, for example, the per-unit positive and negative sequence voltages will be the $\sqrt{3}$ times the per-unit positive and negative sequence line-to-neutral voltages. Similarly, the positive and negative sequence per-unit line currents will be $\sqrt{3}$ times the positive and negative sequence per-unit delta currents. By using just one base voltage and one base current, the per-unit generalized matrices for all system models can be determined.

Short Circuit Analysis at Distribution Network

- Computation of short-circuit currents for unbalanced faults has traditionally been accomplished by symmetrical network.
- However, this method is not well suited to a distribution feeder that is inherently unbalanced. The unequal mutual coupling between phases leads to mutual coupling between sequence networks. When this happens, there is no advantage in using symmetrical components.
- Another reason for not using symmetrical components is that the phases between which faults occur is limited. For example, using symmetrical components, line-to-ground faults are limited to phase a to ground. What happens if a single-phase lateral is connected to phase b or c and the short-circuit current is needed?
- This section develops a method for short-circuit analysis of an unbalanced three-phase distribution feeder using the phase frame.

Short Circuit Analysis at Distribution Network – General Theory

- Short circuits can occur at any one of the five points.
- Point 1 is the high-voltage bus of the distribution substation transformer. The values of the short-circuit currents at point 1 are normally determined from a transmission system short-circuit study. (This means if point 1 has a fault, it should be analyzed as a part of transmission systems using sequence analysis.) The results of these studies are supplied in terms of the three-phase and single-phase short-circuit MVAs. Using the short-circuit MVAs, the positive and zero sequence impedances of the equivalent system can be determined. These values are needed for the short-circuit studies at the other four points.



- Given the three-phase short-circuit MVA magnitude and angle, the positive sequence equivalent system impedance in Ohms is determined by

$$Z_+ = \frac{\mathrm{KV}_{\mathrm{LL}}^2}{(MVA_{3\phi})^*}$$

- Given the single-phase short-circuit MVA magnitude and angle, the zero sequence equivalent system impedance in Ohms is determined by

$$Z_0 = \frac{3KV_{LL}^2}{(MVA_{1\phi})^*} - 2Z_+$$

Short Circuit Analysis at Distribution Network – General Theory

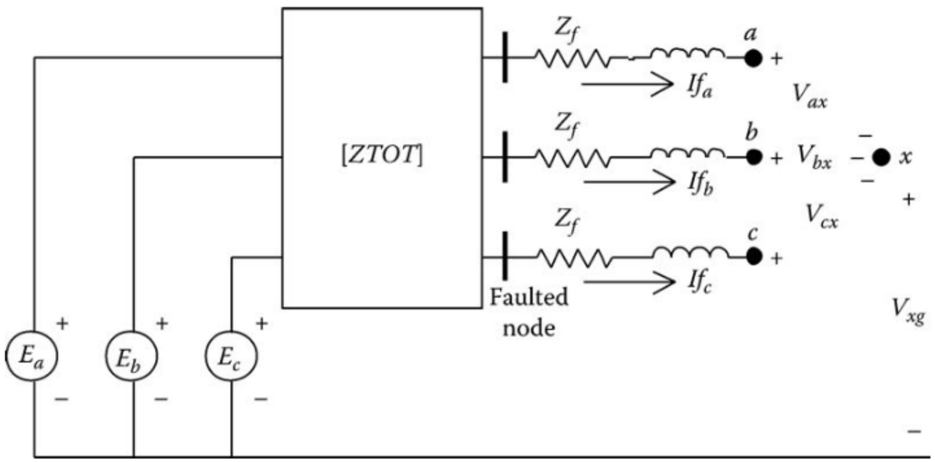
- In the equations, kV_{LL} is the nominal line-to-line voltage in kV of the transmission system.
- The computed positive and zero sequence impedances need to be converted into the phase impedance matrix using the symmetrical component transformation matrix defined as follows:

$$[Z_{012}] = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix} \qquad [Z_{abc}] = [A_s] \cdot [Z_{012}] \cdot [A_s]^{-1}$$

- For short circuits at points 2 through 5 (these are in distribution systems), it is going to be necessary to compute the Thevenin equivalent three-phase circuit at the short-circuit point.
- The Thevenin equivalent voltages will be the nominal line-to-ground voltages with the appropriate angles. For example, the equivalent system line-to-ground voltages are balanced three phase of nominal voltage with the phase a voltage at 0°. (This is to assume the infinite bus balanced L-G voltages). The Thevenin equivalent voltages at points 2 and 3 will be computed by multiplying the system voltages by the generalized transformer matrix $[A_t]$ of the substation transformer (here the transformer internal impendences are rounded into the Thevenin equivalent impedance as shown below). Carrying this further, the Thevenin equivalent voltages at points 4 and 5 will be the voltages at node 3 multiplied by the generalized matrix $[A_t]$ for the in-line transformer.
- The Thevenin equivalent phase impedance matrices will be the sum of the phase impedance matrices of each device between the system voltage source and the point of fault. Step-voltage regulators are assumed to be set in the neutral position, so they do not enter the short-circuit calculations. Anytime that a three-phase transformer is encountered, the total phase impedance matrix on the primary side of the transformer must be referred to the secondary side.

Short Circuit Analysis at Distribution Network – General Theory

- The figure illustrates Thevenin equivalent circuit at the faulted node.
- The voltage sources Ea, Eb, and Ec represent the Thevenin equivalent line-to-ground voltages at the faulted node; the matrix $[Z_{TOT}]$ represents the Thevenin equivalent phase impedance matrix at the faulted node; and Z_f represents the fault impedance.



- KVL in matrix form can be applied to the circuit.

$$\begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \cdot \begin{bmatrix} If_a \\ If_b \\ If_c \end{bmatrix} - \begin{bmatrix} Z_f & 0 & 0 \\ 0 & Z_f & 0 \\ 0 & 0 & Z_f \end{bmatrix} \cdot \begin{bmatrix} If_a \\ If_b \\ If_c \end{bmatrix} + \begin{bmatrix} V_{ax} \\ V_{bx} \\ V_{cx} \end{bmatrix} + \begin{bmatrix} V_{xg} \\ V_{xg} \\ V_{xg} \end{bmatrix}$$

$$[E_{abc}] = [ZTOT] \cdot [If_{abc}] - [ZF] \cdot [If_{abc}] + [V_{abcx}] + [V_{xg}]$$

[3] Kersting, W.H. and Phillips, W.H., Distribution system short-circuit analysis, 25th Intersociety Energy Conversion Engineering Conference, Reno, NV, August 12–17, 1990

Short Circuit Analysis at Distribution Network – General Theory

- Combining the terms in the equation and put it in admittance form,

$$[E_{abc}] = [ZTOT] \cdot [If_{abc}] - [ZF] \cdot [If_{abc}] + [V_{abcx}] + [V_{xg}]$$

$$[E_{abc}] = [ZEQ] \cdot [If_{abc}] + [V_{abcx}] + [V_{xg}] \qquad [ZEQ] = [ZTOT] + [ZF]$$

$$[If_{abc}] = [Y] \cdot [E_{abc}] - [Y] \cdot [V_{abcx}] - [Y] \cdot [V_{xg}] \qquad [Y] = [ZEQ]^{-1}$$

- Since the matrices [Y] and [E_{abc}] are known, define

$$[IP_{abc}] = [Y] \cdot [E_{abc}]$$

$$[IP_{abc}] = [If_{abc}] + [Y] \cdot [V_{abcx}] + [Y] \cdot [V_{xg}]$$

- Expanding,

$$\begin{bmatrix} IP_a \\ IP_b \\ IP_c \end{bmatrix} = \begin{bmatrix} If_a \\ If_b \\ If_c \end{bmatrix} + \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix} \cdot \begin{bmatrix} V_{ax} \\ V_{bx} \\ V_{cx} \end{bmatrix} + \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix} \cdot \begin{bmatrix} V_{xg} \\ V_{xg} \\ V_{xg} \end{bmatrix}$$

- Define the equivalent admittance in each phase,

$$\begin{aligned} Y_{S_a} &= Y_{aa} + Y_{ab} + Y_{ac} \\ Y_{S_b} &= Y_{ba} + Y_{bb} + Y_{bc} \\ Y_{S_c} &= Y_{ca} + Y_{cb} + Y_{cc} \end{aligned} \longrightarrow \begin{aligned} IP_a &= If_a + (Y_{aa} \cdot V_{ax} + Y_{ab} \cdot V_{bx} + Y_{ac} \cdot V_{cx}) + Y_{S_a} \cdot V_{xg} \\ IP_b &= If_b + (Y_{ba} \cdot V_{ax} + Y_{bb} \cdot V_{bx} + Y_{bc} \cdot V_{cx}) + Y_{S_b} \cdot V_{xg} \\ IP_c &= If_c + (Y_{ca} \cdot V_{ax} + Y_{cb} \cdot V_{bx} + Y_{cc} \cdot V_{cx}) + Y_{S_c} \cdot V_{xg} \end{aligned}$$

Short Circuit Analysis at Distribution Network – General Theory

$$IP_a = If_a + (Y_{aa} \cdot V_{ax} + Y_{ab} \cdot V_{bx} + Y_{ac} \cdot V_{cx}) + Y_{S_a} \cdot V_{xg}$$
$$IP_b = If_b + (Y_{ba} \cdot V_{ax} + Y_{bb} \cdot V_{bx} + Y_{bc} \cdot V_{cx}) + Y_{S_b} \cdot V_{xg}$$
$$IP_c = If_c + (Y_{ca} \cdot V_{ax} + Y_{cb} \cdot V_{bx} + Y_{cc} \cdot V_{cx}) + Y_{S_c} \cdot V_{xg}$$

- It become the general equations that are used to simulate all types of short circuits. Basically, there are three equations and seven unknowns (If_a, If_b, If_c, V_{ax}, V_{bx}, V_{cx}, and V_{xg}). The other three variables in the equations (IP_a, IP_b, and IP_c) are functions of the total impedance and the Thevenin voltages and are therefore known.
- To solve the general equations, it will be necessary to specify four additional independent equations. These equations are functions of fault type being simulated. The additional required four equations for various types of faults are given in the following.

Three-phase faults: $V_{ax} = V_{bx} = V_{cx} = 0$ $\quad I_a + I_b + I_c = 0$

Three-phase-to-ground faults: $V_{ax} = V_{bx} = V_{cx} = V_{xg} = 0$

Line-to-line faults (assume i–j fault with phase k unfaulted):

$$V_{ix} = V_{jx} = 0 \qquad If_a = 0 \qquad If_i + If_j = 0$$

Line-to-line-to-ground faults (assume i–j–g fault with phase k unfaulted):

$$V_{ix} = V_{jx} = 0 \qquad V_{xg} = 0 \qquad I_k = 0$$

Line-to-ground faults (assume phase k fault with phases i and j unfaulted):

$$V_{kx} = V_{xg} = 0 \qquad If_i = If_j = 0$$

Short Circuit Analysis at Distribution Network – General Theory

- A good way to solve the seven equations is to set them up in matrix form:

$$\begin{bmatrix} IP_a \\ IP_a \\ IP_a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & Y_{1,1} & Y_{1,2} & Y_{1,3} & Y_{S_1} \\ 0 & 1 & 0 & Y_{2,1} & Y_{2,2} & Y_{2,3} & Y_{S_2} \\ 0 & 0 & 1 & Y_{3,1} & Y_{3,2} & Y_{3,3} & Y_{S_3} \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix} \square \begin{bmatrix} If_a \\ If_b \\ If_c \\ V_{ax} \\ V_{bx} \\ V_{cx} \\ V_{xg} \end{bmatrix}$$

- In condensed form:

$$[IP_s] = [C] \cdot [X] \rightarrow [X] = [C]^{-1} \cdot [IP_s]$$

- The blanks in the last four rows of the coefficient matrix are filled in with the known variables depending upon fault type. For example, the elements in the [C] matrix simulating a three-phase fault would be

$$C_{4,4} = C_{5,5} = C_{6,6} = 1$$

$$C_{7,1} = C_{7,2} = C_{7,3} = 1$$

- All of the other elements in the last four rows will be set to zero.

Power Flow Analysis Outline

- Conventional power flow calculations in transmission systems
  - Gauss-Seidel method
  - Newton-Raphson method

- Features of electrical distribution networks
  - Ill-conditioned Jacobian matrix in Newton-Raphson method

- Power flow calculations in distribution systems
  - Forward/Backward sweep method
    - Kirchhoff's formulation
      - BIBC & BCBV matrices
    - Dist-flow formulation
      - Linearized Dist-flow formulation
      - Extension to three-phase systems
  - Modified Newton-Raphson method

Power Flow Analysis Outline

Power flow analysis of power system is used to determine the steady state solution for a given set of bus loading condition.

$$P_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j)\right)$$
$$Q_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\sin(\theta_i - \theta_j) - B_{ij}\cos(\theta_i - \theta_j)\right)$$

Parameters

- Network topology $j \in N_i$
- Line conductance and susceptance G_{ij}, B_{ij}

Variables

- Bus voltage magnitudes and bus phase angles V_i, θ_i
- Line active and reactive power flow P_{ij}, Q_{ij}

Usually, two of the four variables are known for each bus:

- PQ bus (load) at which P and Q are fixed; iteration solves for V and θ .
- PV bus (generator) at which P and V are fixed; iteration solves for θ and Q.
- Slack bus at which the V and θ are fixed; iteration solves for P and Q.

Conventional power flow calculation in transmission systems

Gauss-Seidel method

- It needs to rewrite the equations in an implicit form: $x = h(x)$
- It starts with initial guess: x^0
- Then we update the solution using the following form: $x^{n+1} = h(x^n)$
- It repeats the procedure until converged

- It needs to put the equation in the correct form:

$$S_i = V_i I_i^* = V_i \left(\sum_{j=1}^{n} Y_{ij} V_j \right)^* = V_i \sum_{j=1}^{n} Y_{ij}^* V_j^*$$

$$S_i^* = V_i^* I_i = V_i^* \sum_{j=1}^{n} Y_{ij} V_j$$

$$\frac{S_i^*}{V_i^*} = \sum_{j=1}^{n} Y_{ij}^* V_j^* = Y_{ii} V_i + \sum_{j=1, j \neq i}^{n} Y_{ij} V_j$$

- The update rule for each bus voltage:

$$\boxed{V_i^{n+1} = \frac{1}{Y_{ii}} \left(\frac{S_i^*}{V_i^{n*}} - \sum_{j=1, j \neq i}^{n} Y_{ij} V_j^n \right) = h_i(V_1^n, V_2^n, \dots, V_I^n)}$$

Conventional power flow calculation in transmission systems

Newton-Raphson method

- x is the vector of θ and V for all the buses, except for the slack bus.
- Active and reactive power balance equation: $f(x)$

$$\boxed{x^{n+1} = x^n - J^{-1}(x^n) f(x^n)}$$

where

$$x = \begin{bmatrix} \theta \\ V \end{bmatrix} \qquad f(x) = \begin{bmatrix} P(x) \\ Q(x) \end{bmatrix} \qquad J(x) = \begin{bmatrix} \frac{\partial P}{\partial \theta} & \frac{\partial P}{\partial V} \\ \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial V} \end{bmatrix}$$

Compared to Gauss-Seidel method, Newton-Raphson method has a faster convergence rate, but each iteration takes much longer time. Also, Newton-Raphson is more complicated to code.

Accelerated (convergence) Gauss-Seidel method

- Previously, it calculates each value x as: $x^{n+1} = h(x^n)$
- To accelerate convergence, the above equation can be rewritten as: $\boxed{x^{n+1} = x^n + h(x^n) - x^n}$
- Acceleration parameter α: $x^{n+1} = x^n + \alpha(h(x^n) - x^n)$
- Larger value of α **may** result in faster convergence

Conventional power flow calculation in transmission systems

Decoupled Newton-Raphson method

- Approximation of the Jacobian matrix is used to decouple the real and reaction power equations.
- Assume $(\theta_i - \theta_j) \approx 0$, thus $\sin(\theta_i - \theta_j) \approx 0$

$$\frac{\partial P_i}{\partial V_j} = |V_i| B_{ij} \sin(\theta_i - \theta_j) \approx 0$$

$$\frac{\partial Q_i}{\partial \theta_j} = -|V_i||V_j| B_{ij} \sin(\theta_i - \theta_j) \approx 0$$

$$\longrightarrow \left[\begin{aligned} J(x) &= \begin{bmatrix} \frac{\partial P}{\partial \theta} & 0 \\ 0 & \frac{\partial Q}{\partial V} \end{bmatrix} \\ \theta^{n+1} &= \theta^n - \left[\frac{\partial P}{\partial \theta}\right]^{-1} P(x^n) \\ V^{n+1} &= V^n - \left[\frac{\partial Q}{\partial V}\right]^{-1} Q(x^n) \end{aligned}\right.$$

- Traditional Gauss-Seidel method and Newton-Raphson method need the calculation of the **Y matrix** and **Jacobian matrix**.
- When solving large power systems, the most difficult computation task is inverting the **Y matrix** and **Jacobian matrix**:
  - Inverting a full matrix needs an order of n^3 operation, meaning the amount of computation increases with the cube of the size n.
  - This amount of computation can be decreased substantially by recognizing both Y matrix and Jacobian matrix are sparse matrices.
  - Using sparse matrix methods results in a computational order of about $n^{1.5}$.

Features of electrical distribution networks

- Because of the following special features in **distribution network**, the Y matrix and Jacobian matrix ceases to be diagonally dominant and convergence problems arise in power flow solutions that rely on its inverse.
    - Radial or near radial structure
    - High R/X rations
    - Un-transposed lines
    - Multi-phase, unbalanced, grounded or ungrounded operation
    - Multi-phase, multi-mode control distribution equipment
    - Unbalanced distributed load
    - Extremely large number of branches/nodes
- Thus, traditional Gauss-Seidel method and Newton-Raphson method have lost popularity due to their poor convergence in distribution system studies.

[1] C. S. Cheng and D. Shirmohammadi, "A three-phase power flow method for real-time distribution system analysis," in IEEE Transactions on Power Systems, vol. 10, no. 2, pp. 671-679, May 1995.

Convergence of Newton's method for distribution systems

Condition number defines the condition of a matrix with respect to the computing problem. $\lambda_{max}(J)$ and $\lambda_{min}(J)$ are maximum and minimum eigenvalues of matrix J.

$$k(J) = \frac{\lambda_{max}(J)}{\lambda_{min}(J)}$$

A very high value of the condition number of matrix J indicates that:

- The system is ill-conditioned, the computed values are very sensitive to small changes in input values.
- The matrix J is invertible.

Type of System	No. of Buses	Maximum Eigenvalue	Minimum Eigenvalue	Condition Number (k)*	Remarks
Well-conditioned system	30	$.1087\times10^{3}$	$.2322\times10^{0}$	$.4681\times10^{3}$	Moderately well-conditioned (fair k)
Ill-conditioned system	11	$.1222\times10^{3}$	$.1126\times10^{0}$	$.1086\times10^{4}$	Ill-conditioned (bad k)
	13	$.2905\times10^{2}$	$.1442\times10^{-1}$	$.2014\times10^{4}$	Ill-conditioned (bad k)
	43	$.2426\times10^{4}$	$.9476\times10^{-1}$	$.2560\times10^{5}$	Ill-conditioned (bad k)

* Ideal value of condition number: k = 1

[2] S. C. Tripathy and G. S. S. S. K. Purge Prasad, "Load flow solution for ill-conditioned power systems by quadratically convergent Newton-like method," in IEE Proceedings C - Generation, Transmission and Distribution, vol. 127, no. 5, pp. 273-280, September 1980.

Forward/Backward Sweep-based Algorithm

- Methods developed for the solution of ill-conditioned radial distribution systems may be divided into two categories:
  - Forward and/or backward sweep
    - Kirchhoff's formulation
      - BIBC & BCBV
    - Quadratic equation-based algorithm
      - Dist-Flow
  - Modification of existing methods
    - Modified N-R method
- Forward/backward sweep-based power flow algorithm generally takes advantage of the radial network topology and consists of forward and backward sweep processes.
  - The forward sweep is mainly the node voltage calculation from the sending end to the far end of the lines.
  - The backward sweep is primarily the branch current or power summation from the far end to the sending end of the lines.

[3] U. Eminoglu & M. H. Hocaoglu, "Distribution Systems Forward/ Backward Sweep-based Power Flow Algorithms: A Review and Comparison Study', in Electric Power Components and Systems, 37:1, 91-110, 2008

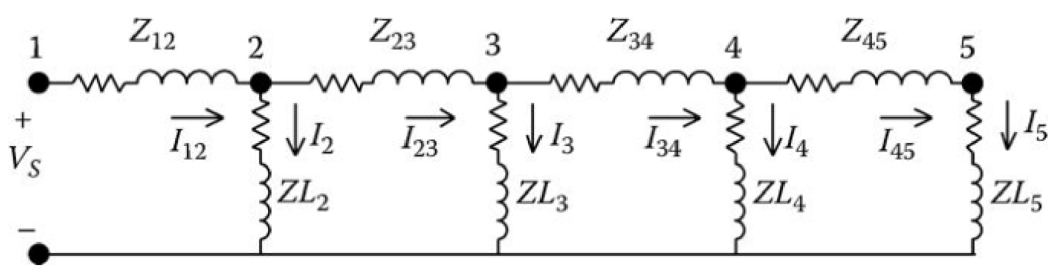
Forward/Backward Sweep-based Algorithm

- For the ladder network, it is assumed that all of the line impedances and load impedances are known along with the voltage (V_S) at the source.
- The solution for this network is to perform the “forward” sweep by calculating the voltage at node 5 (V_5) under a no-load condition.
- With no load currents there are no line currents, so the computed voltage at node 5 will equal that of the specified voltage at the source.
- The “backward” sweep commences by computing the load current at node 5.
- The load current I_5 is

$$I_5 = \frac{V_5}{ZL_5}$$

- For this “end node” case, the line current I_{45} is equal to the load current I_5. The “backward” sweep continues by applying Kirchhoff's voltage law (KVL) to calculate the voltage at node 4:

$$I_{45} = I_5 \qquad\qquad V_4 = V_5 + Z_{45} \cdot I_{45}$$



Forward/Backward Sweep-based Algorithm

- The load current I_4 can be determined and then Kirchhoff's current law (KCL) applied to determine the line current I_{34}:

$$I_4 = \frac{V_4}{ZL_4} \qquad I_{34} = I_{45} + I_4$$

- KVL is applied to determine the node voltage V_3. The backward sweep continues until a voltage (V_1) has been computed at the source.

$$V_3 = V_4 + Z_{34} \cdot I_{34}$$

$$\dots$$

$$\Downarrow$$

$$V_1$$

- The computed voltage V_1 is compared to the specified voltage V_S. There will be a difference between these two voltages. The ratio of the specified voltage to the compute voltage can be determined as

$$\text{Ratio} = \frac{V_S}{V_1}$$

- Since the network is linear, all of the line and load currents and node voltages in the network can be multiplied by the ratio for the final solution to the network.

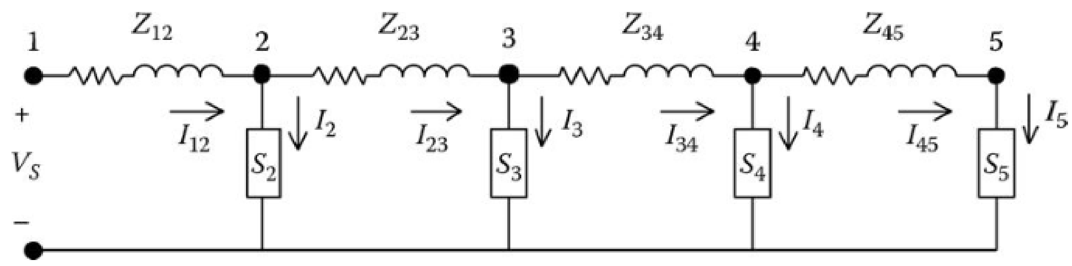
Forward/Backward Sweep-based Algorithm

- Linear network is modified to a nonlinear network by replacing all the constant load impedances by constant complex power loads.
- As with the linear network, the "forward" sweep computes the voltage at node 5 assuming no load. As before, the node 5 (end node) voltage will equal that of the specified source voltage. In general, the load current at each node is computed by

$$I_n = \left(\frac{S_n}{V_n}\right)^*$$

- The "backward" sweep will determine a computed source voltage V_1.
  - As in the linear case, this first "iteration" will produce a voltage that is not equal to the specified source voltage VS. Because the network is nonlinear, multiplying currents and voltages by the ratio of the specified voltage to the computed voltage will not give the solution.
  - The most direct modification using the ladder network theory is to perform a "forward" sweep. The forward sweep commences by using the specified source voltage and the line currents from the previous "backward" sweep. KVL is used to compute the voltage at node 2 by

$$V_2 = V_s - Z_{12} \cdot I_{12}$$



Ladder Network
with Nonlinear Load

Forward/Backward Sweep-based Algorithm

- This procedure is repeated for each line segment until a “new” voltage is determined at node 5.
  - Using the “new” voltage at node 5, a second backward sweep is started that will lead to a “new” computed voltage at the source.
  - The procedure shown earlier works but, in general, will require more time to converge. A modified version is to perform the “forward” sweep calculating all of the node voltages using the line currents from the previous “backward” sweep.
  - The new “backward” sweep will use the node voltages from the previous “forward” sweep to compute the new load and line currents.
  - In general, this modification will require more iterations but less time to converge. In this modified version of the ladder technique, convergence is determined by computing the ratio of difference between the voltages at the n − 1 and n iterations and the nominal line-to-neutral voltage. Convergence is achieved when all of the phase voltages at all nodes satisfy

$$\frac{\left||V_n| - |V_{n-1}|\right|}{V_{nomial}} \leq \text{Specified tolerance}$$

Example

Apply Modified Ladder Method to the single-phase lateral in the figure compute the load voltage. The line impedance is $z = 0.3 + j0.6\ \Omega/\text{mile}$.

The load connected are $S_2 = 1500 + \text{j}750\ [\text{kVAr}], S_3 = 900 + j500\ [\text{kVAr}]$

The source voltage at node 1 is 7200V.

Solution

Impedance of Segment 1-2: $Z_{12} = (0.3 + j0.6) \cdot \frac{3000}{5280} = 0.1705 + j0.3409\ \Omega$

Impedance of Segment 2-3: $Z_{23} = (0.3 + j0.6) \cdot \frac{4000}{5280} = 0.2273 + j0.4545\ \Omega$

Set initial condition: $I_{12} = I_{23} = 0 \qquad V_{old} = 0 \qquad \text{Tol.} = 0.0001$

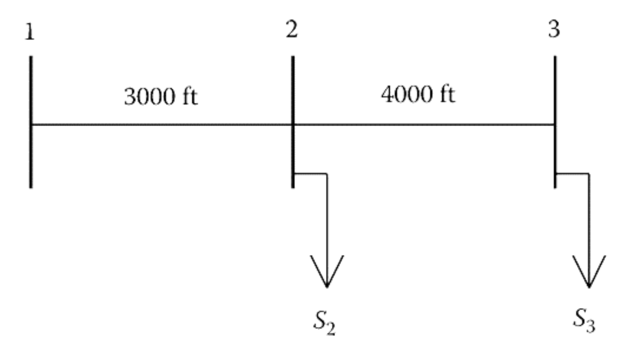
The First Forward Sweep

$$V_2 = V_s - Z_{12} \cdot I_{12} = 7200\angle 0$$

$$V_3 = V_2 - Z_{23} \cdot I_{23} = 7200\angle 0$$

$$\text{Error} = \frac{\left||\text{V}_3| - |\text{V}_{\text{old}}|\right|}{7200} = 1 > \text{Tol.}$$

$$V_{old} = V_3$$



Example

The first backward sweep:

$$I_3 = \left(\frac{(900 + j500) \cdot 1000}{7200\angle 0}\right)^* = 143.0\angle - 29.0\ A$$

The current flowing in the line segment 2–3 is $I_{23} = I_3 = 143.0\angle - 29.0\ A$

The load current at node 2 is

$$I_2 = \left(\frac{(1500 + j750) \cdot 1000}{7200\angle 0}\right)^* = 232.9\angle - 27.5\ A$$

The current in line segment 1–2 is $I_{12} = I_{23} + I_2 = 373.8\angle - 27.5\ A$

The second forward sweep**:**

$$V_2 = V_2 - Z_{12} \cdot I_{12} = 7084.5\angle - 0.7$$

$$V_3 = V_2 - Z_{23} \cdot I_{23} = 7025.1\angle - 1.0$$

$$Error = \frac{\left||V_3| - |V_{old}|\right|}{7200} = \frac{|7084.5 - 7200|}{7200} = 0.0243 > \text{Tol.}$$

$$V_{old} = V_3$$

- At this point, the second backward sweep is used to compute the new line currents. Then it is followed by the third forward sweep.

Forward/Backward Sweep-based Algorithm

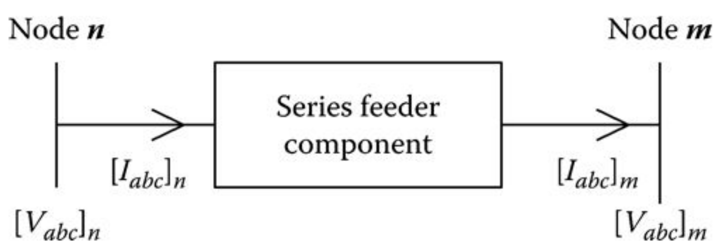
Forward and backward sweep equations are

Forward sweep: $[VLN_{abc}]_m = [A] \cdot [VLN_{abc}]_n - [B] \cdot [I_{abc}]_n$

Backward sweep: $[I_{abc}]_n = [c] \cdot [VLN_{abc}]_m + [d] \cdot [I_{abc}]_m$

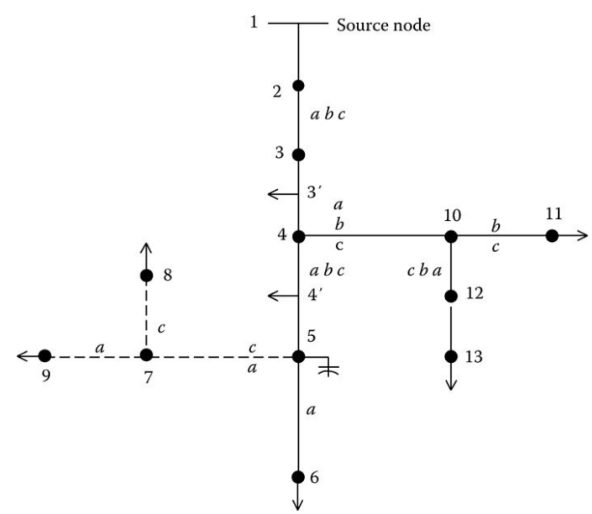
It was also shown that for the grounded wye–delta transformer bank, the backward sweep equation is

$$[I_{abc}]_n = [x_t] \cdot [VLN_{abc}]_n + [d] \cdot [I_{abc}]_m$$



Nodes 4, 10, 5, and 7 are referred to as "junction nodes."

- In the forward sweep, the voltages at all nodes down the lines from the junction nodes must be computed.
- In the backward sweeps, the currents at the junction nodes must be summed before proceeding toward the source.
- In developing a program to apply the modified ladder method, it is necessary for the ordering of the lines and nodes to be such that all node voltages in the forward sweep are computed and all currents in the backward sweep are computed.



BIBC matrix and BCBV matrix

There are two **matrices** can be used to improve computational efficiency, which takes advantages of the topological characteristics of distribution systems and solves the distribution load flow [5]:

- Bus Injection to Bus Current (BIBC) matrix: relationship between the bus current injections and branch currents
- Branch current to Bus Voltage (BCBV) matrix: relationship between the branch currents and bus voltages

The reason why the BIBC and BCBV are applied:

- In conventional forward/backward sweep method, the bus voltages and line currents are calculated segments by segments (with topological characteristics) in each iteration.
- While by using the BIBC and BCBV, the two matrices are calculated only once and they have already included all topological information. BIBC/BCBV will not be updated in each iteration. Only voltage drop and branch currents will be updated.

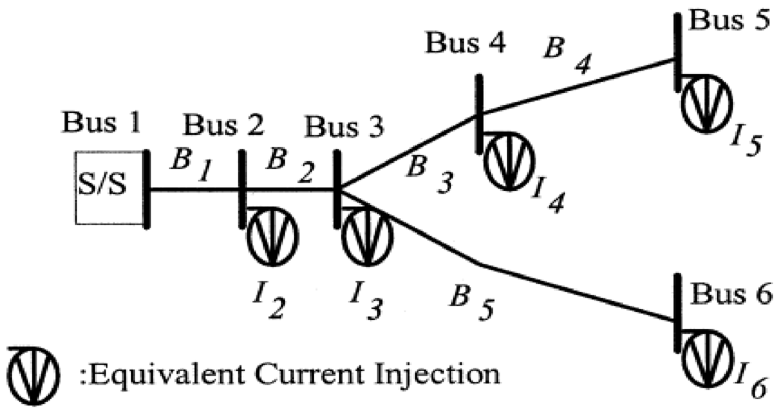
[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in IEEE Transactions on Power Delivery, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

By using the KCL

$$B_1 = I_2+I_3+I_4+I_5+I_6$$
$$B_3 = I_4+I_5$$
$$B_5 = I_6$$

$$\longrightarrow \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}.$$



$$[B] = [\text{BIBC}][I]$$

The constant BIBC matrix is an upper triangular matrix and contains values of 0 and 1 only.

It represents the direct connectivity of nodes.

B is branch current
I is bus current injection

To build BIBC matrix,

Step.1 For a distribution system with m-branch section and n-bus, the dimension of the BIBC matrix is $m \times (n-1)$.
Step.2 If a line section is located between bus i and bus j, copy the column of the i-th bus of the BIBC matrix to the column of the j-th bus and fill a +1 to the position of the k-th row and the j-th bus column
Step.3 Repeat Step.2 until all line sections are included in the BIBC matrix

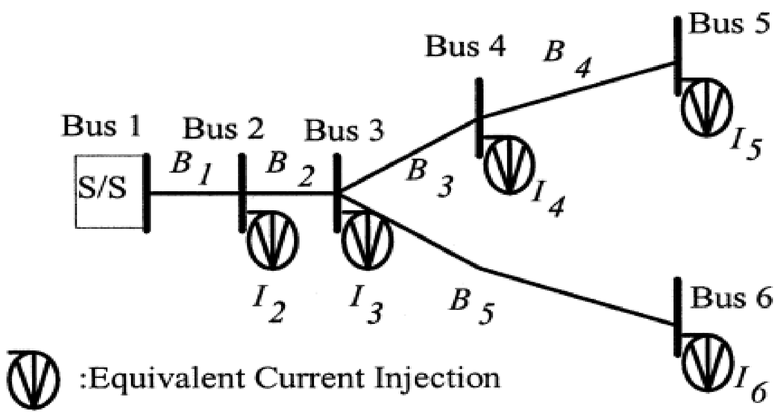
BIBC matrix and BCBV matrix

By using the KVL

$$V_2 = V_1 - B_1 Z_{12}$$
$$V_3 = V_2 - B_2 Z_{23}$$
$$V_4 = V_3 - B_3 Z_{34}$$
$$V_4 = V_1 - B_1 Z_{12} - B_2 Z_{23} - B_3 Z_{34}$$

$$\rightarrow \begin{bmatrix} V_1 \\ V_1 \\ V_1 \\ V_1 \\ V_1 \end{bmatrix} - \begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} Z_{12} & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & Z_{45} & 0 \\ Z_{12} & Z_{23} & 0 & 0 & Z_{36} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

$$[\Delta V] = [\mathrm{BCBV}][B]$$



The constant BIBV matrix is a lower triangular matrix and contains values of 0 and line impedance only.

B is branch current
I is bus current injection

To build BIBV matrix,

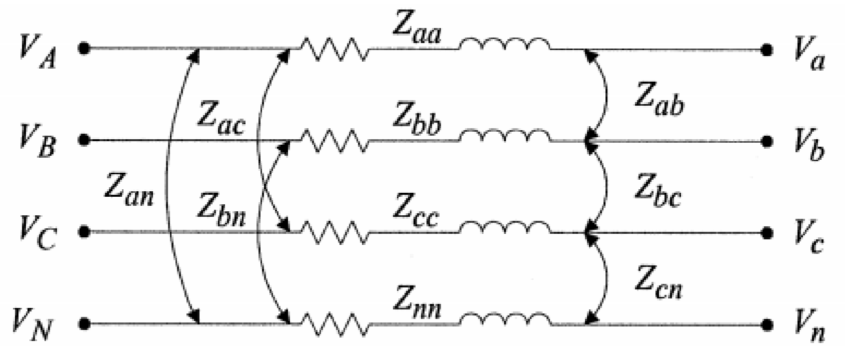
Step.1 For a distribution system with m-branch section and n-bus, the dimension of the BCBV matrix is $\mathrm{n} \times (m-1)$.

Step.2 If a line section is located between bus i and bus j, copy the column of the i-th bus of the BCBV matrix to the column of the j-th bus and fill the line impedance Z_{ij} to the position of the j-th row and the k-th bus column.

Step.3 Repeat Step.2 until all line sections are included in the BCBV matrix.

Three-phase BCBV matrix

- The algorithm can easily be expanded to a multiphase line section or bus.
- For example, if the line section between bus i and bus j is a three-phase line section, the corresponding branch current B_i will be a 3×1 vector and the in the BIBC matrix will be a 3×3 identity matrix.
- Similarly, if the line section between bus i and bus j is a three-phase line section, the Z_{ij} in the BCBV matrix is a 3×3 impedance matrix as follows.



$$[Z_{abc}] = \begin{bmatrix} Z_{aa-n} & Z_{ab-n} & Z_{ac-n} \\ Z_{ba-n} & Z_{bb-n} & Z_{bc-n} \\ Z_{ca-n} & Z_{cb-n} & Z_{cc-n} \end{bmatrix}$$

- Distribution Load Flow (**DLF**) matrix is a multiplication matrix of BCBV and BIBC matrices.

$$[B] = [BIBC][I] \qquad\qquad [\Delta V] = [BCBV][B]$$

Combine two steps into one

$$[\Delta V] = [BCBV][B] = [BCBV][BIBC][I] = [\boldsymbol{DLF}][I]$$

Three-phase BCBV matrix

- The solution for distribution load flow can be updated and obtained iteratively as follows:

$$I_i^k = I_i^k V_i^k + jI_i^k V_i^k = \left(\frac{(P_i + jQ_i)}{V_i^k}\right)^*$$

$$\left[\Delta V_i^{k+1}\right] = [DLF]\left[I_i^k\right]$$

$$\left[V_i^{k+1}\right] = [V^0] + \left[\Delta V_i^{k+1}\right]$$

- The voltage drop on each branch is computed using the DLF and **load currents**.
- The **node voltages** are computed by using the source bus voltage and **voltage drops**.

Step.1 Input the radial system topology data

Step.2 Form the BIBC matrix $[BIBC] = [I]/[B]$

Step.3 Form the BCBV matrix $[BCBV] = [B]\ /[\Delta V]$

Step.4 Calculate DLF matrix and set iteration k=0 $[\boldsymbol{DLF}] = [BCBV][BIBC]$

Step.5 Update voltage and iteration k=k+1

$$I_i^k = I_i^k V_i^k + jI_i^k V_i^k = \left(\frac{(P_i + jQ_i)}{V_i^k}\right)^*$$

$$\left[\Delta V_i^{k+1}\right] = [DLF]\left[I_i^k\right]$$

$$\left[V_i^{k+1}\right] = [V^0] + \left[\Delta V_i^{k+1}\right]$$

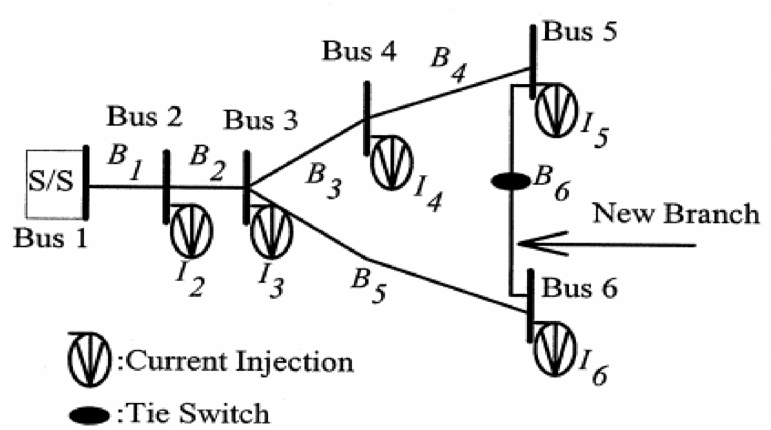
Step.6 If $\left(\left|V_i^{k+1}\right| - \left|V_i^k\right|\right) <$ tolerance, go to next step; else, go back to Step. 5

Step.7 Calculate line flows and power losses using final voltage

BIBC matrix and BCBV matrix

- Some distribution feeders serve high-density load areas and contain loops. The proposed method introduced before can be extended for "weakly-meshed" distribution feeders.

Modification for BIBC matrix:



Taking the new branch current into account, the current injections of bus 5 and bus 6 will be:

$$I_5' = I_5 + B_6$$

$$I_6' = I_6 - B_6$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ B_6 \end{bmatrix}.$$

$$\boxed{\begin{bmatrix} B \\ B_{new} \end{bmatrix} = [BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix}}$$

Modification for BCBV matrix:

Considering the loop, KVL:

$$B_3 Z_{34} + B_4 Z_{45} + B_6 Z_{56} - B_5 Z_{36} = 0$$

$$\begin{bmatrix} V_1 \\ V_1 \\ V_1 \\ V_1 \\ V_1 \\ 0 \end{bmatrix} - \begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{12} & 0 & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & Z_{45} & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & Z_{36} & 0 \\ 0 & 0 & Z_{34} & Z_{45} & -Z_{36} & Z_{56} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix}.$$

$$\boxed{\begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV] \begin{bmatrix} B \\ B_{new} \end{bmatrix}}$$

BIBC matrix and BCBV matrix

Modification for solution techniques:

$$\begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV] \begin{bmatrix} B \\ B_{new} \end{bmatrix}$$

$$\begin{bmatrix} B \\ B_{new} \end{bmatrix} = [BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV][BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix} = \begin{bmatrix} A & M^T \\ M & N \end{bmatrix} \begin{bmatrix} I \\ B_{new} \end{bmatrix}$$

The modified algorithm for weakly meshed networks can be expressed as

$$[\Delta V] = [A - M^T N^{-1} M][I] = [DLF][I]$$

Except for some modifications needed to be done for the BIBC, BCBV, and DLF matrices, the proposed solution techniques require no modification.

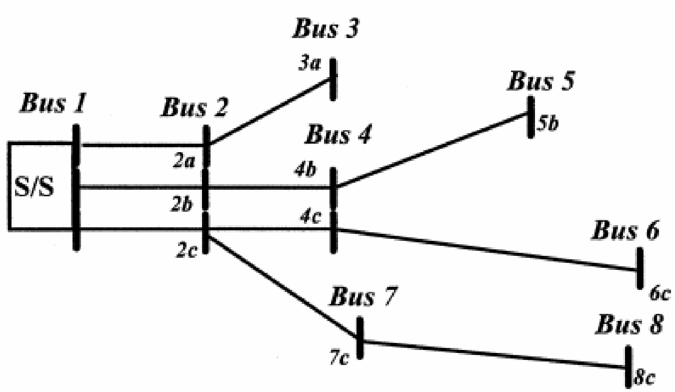
The proposed three-phase load flow algorithm was implemented on an 8-bus distribution system. Two methods are used for tests and the convergence tolerance is set at 0.001.

- Method 1: The Gauss implicit Z-matrix method
- Method 2: The proposed algorithm with BIBC and BCBV

BIBC matrix and BCBV matrix

The final voltage solutions of method 1 and method 2 are shown in the Table.

- From the Table, the final converged voltage solutions of method 2 are very close to the solution of method 1.
- It can be seen that method 2 is more efficient, especially when the network size increases. It is because the time-consuming processes such as LU factorization and forward/backward substitution of Y-bus matrix are not necessary for method 2.



Feeder No.	No. of Nodes	Length
1	45	1.5 km
2	90	2.5 km
3	135	3.2 km
4	180	4.0 km
5	270	7.4 km

Bus Number	Method 1 |V|(pu)	Method 1 Ang. (Rad.)	Method 2 |V|(pu)	Method 2 Ang (Rad.).	Phase
1	1.0000	0.0000	1.0000	0.0000	A
1	1.0000	-2.0944	1.0000	-2.0944	B
1	1.0000	2.0944	1.0000	2.0944	C
2	0.9840	0.0032	0.9839	0.0032	A
2	0.9714	-2.0902	0.9712	-2.0902	B
2	0.9699	2.0939	0.9697	2.0939	C
3	0.9833	0.0031	0.9832	0.0031	A
4	0.9653	-2.0897	0.9652	-2.0897	B
4	0.9672	2.0932	0.9669	2.0932	C
5	0.9644	-2.0898	0.9640	-2.0898	B
6	0.9652	2.0930	0.9650	2.0930	C
7	0.9686	2.0937	0.9683	2.0937	C
8	0.9674	2.0936	0.9671	2.0936	C

Feeder No.	Method 1		Method 2	
	NET	IT	NET	IT
1	2.6229	3	1.0000	3
2	14.426	3	2.1639	3
3	52.131	4	5.4098	4
4	131.15	4	9.0164	4
5	432.79	4	18.033	4

(1) NET means the Normalized Execution Time.
(2) IT means the Number of Iteration.
(3) Performance 1.0 is set in Method 2 for Feeder 1.

Node voltage calculations (Quadratic Equation)

- The quadratic equation relates the voltage magnitude at the receiving end to the branch power and the voltage at the sending end.
- Let us consider a distribution line model as below, the real and reactive power at the receiving end can be written as

$$P_r = \frac{V_s V_r}{Z}\cos(\theta_z - \delta_s + \delta_r) - \frac{V_r^2}{Z}\cos(\theta_z)$$

$$Q_r = \frac{V_s V_r}{Z}\sin(\theta_z - \delta_s + \delta_r) - \frac{V_r^2}{Z}\sin(\theta_z)$$

$$\cos(\theta_z - \delta_s + \delta_r) = \frac{P_r Z}{V_s V_r} + \frac{V_s V_r}{V_s}\cos(\theta_z)$$

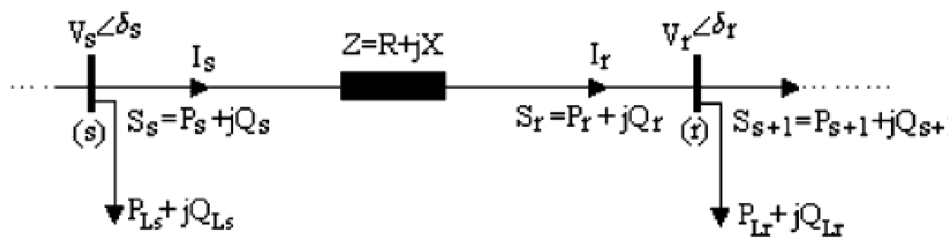
$$\sin(\theta_z - \delta_s + \delta_r) = \frac{Q_r Z}{V_s V_r} + \frac{V_s V_r}{V_s}\sin(\theta_z)$$

$$cos^2(\theta_z - \delta_s + \delta_r) + sin^2(\theta_z - \delta_s + \delta_r) = 1$$

$$V_r^4 + 2V_r^2(P_r\mathrm{R} + Q_r\mathrm{X}) - V_s^2 V_r^2 + (P_r^2 + Q_r^2)Z^2 = 0$$

Node voltages are calculated by solving quadratic equation

$$\Rightarrow \quad V_r^2 = \sqrt{V_s^2 - 2(P_r\mathrm{R} + Q_r\mathrm{X}) + \frac{(P_r^2 + Q_r^2)Z^2}{V_s^2}}$$



[3] U. Eminoglu & M. H. Hocaoglu, "Distribution Systems Forward/ Backward Sweep-based Power Flow Algorithms: A Review and Comparison Study', in Electric Power Components and Systems, 37:1, 91-110, 2008

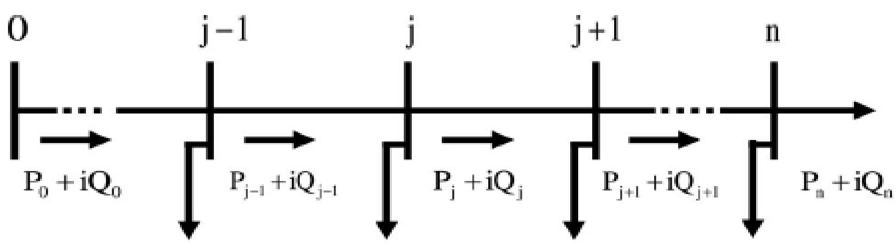
Dist-Flow method (Single Phase)

Nodal Voltage (V_{j+1}) on Bus j+1: $$V_{j+1}^2 = V_j^2 - 2(r_j P_j + x_j Q_j) + (\mathrm{r}_j^2 + \mathrm{x}_j^2)\frac{\mathrm{P}_j^2+\mathrm{Q}_j^2}{\mathrm{V}_j^2}$$

Active (P_{j+1}) and Reactive (Q_{j+1}) Branch Power Flow from Bus j to Bus j+1:

$$P_{j+1} = P_j - p_{j+1} - \mathrm{r}_j\frac{\mathrm{P}_j^2+\mathrm{Q}_j^2}{\mathrm{V}_j^2} \qquad Q_{j+1} = Q_j - q_{j+1} - \mathrm{x}_j\frac{\mathrm{P}_j^2+\mathrm{Q}_j^2}{\mathrm{V}_j^2}$$

$$p_{j+1} = p_{j+1}^{(c)} - p_{j+1}^{(g)} \qquad q_{j+1} = q_{j+1}^{(c)} - q_{j+1}^{(g)}$$



- $p_{j+1}^{(c)}, q_{j+1}^{(c)}$: Power consumptions at Bus j+1
- $p_{j+1}^{(g)}, q_{j+1}^{(g)}$: Power generations at Bus j+1
- r_j, x_j: Complex impedance of the line between Bus j to Bus j+1
- $\mathrm{r}_j\frac{\mathrm{P}_j^2+\mathrm{Q}_j^2}{\mathrm{V}_j^2}, \mathrm{x}_j\frac{\mathrm{P}_j^2+\mathrm{Q}_j^2}{\mathrm{V}_j^2}$: Active and reactive power losses of the line between Bus j to Bus j+1

[6] Q. Zhang, K. Dehghanpour and Z. Wang, "Distributed CVR in Unbalanced Distribution Systems With PV Penetration," in IEEE Transactions on Smart Grid, vol. 10, no. 5, pp. 5308-5319, Sept. 2019.

Dist-Flow method (Single Phase)

- Forward nodal voltage calculation:

$$V_{j+1}^2 = V_j^2 - 2\left(r_j P_j + x_j Q_j\right) + \left(r_j^2 + x_j^2\right)\frac{P_j^2 + Q_j^2}{V_j^2}$$

- Backward branch power flow and branch power loss calculation:

$$P_{j+1} = P_j - p_{j+1} - r_j \frac{P_j^2 + Q_j^2}{V_j^2}$$

$$Q_{j+1} = Q_j - q_{j+1} - x_j \frac{P_j^2 + Q_j^2}{V_j^2}$$

- The calculation is ended when certain values (for example, bus voltages or the system's total active and reactive power loss mismatches) are lower than a specified error value.

Linearized Dist-Flow method (single phase)

The branch power flow and voltage constraints in Dist-Flow method have power loss terms

$$\frac{\mathrm{P_i^2 + Q_i^2}}{\mathrm{V_i^2}}$$

that make the problem nonlinear.

There are several methods to linearize the nonlinear power loss terms:

(1) **Neglect the nonlinear terms**

The linearization is based on the fact that the nonlinear terms are much smaller than the linear terms. So that the nonlinear terms are neglected for the sake of developing efficient solution algorithms.

However, it is noted that such linearization neglects an accurate calculation of power loss.

$$P_{i+1} = P_i - p_{i+1} - r_i \frac{P_i^2 + Q_i^2}{V_i^2}$$

$$Q_{i+1} = Q_i - q_{i+1} - x_i \frac{P_i^2 + Q_i^2}{V_i^2}$$

$$V_{i+1}^2 = V_i^2 - 2(r_i P_i + x_i Q_i) + \left(r_i^2 + x_i^2\right) \frac{P_i^2 + Q_i^2}{V_i^2}$$

$$\Longrightarrow$$

$$P_{i+1} = P_i - p_{i+1}$$

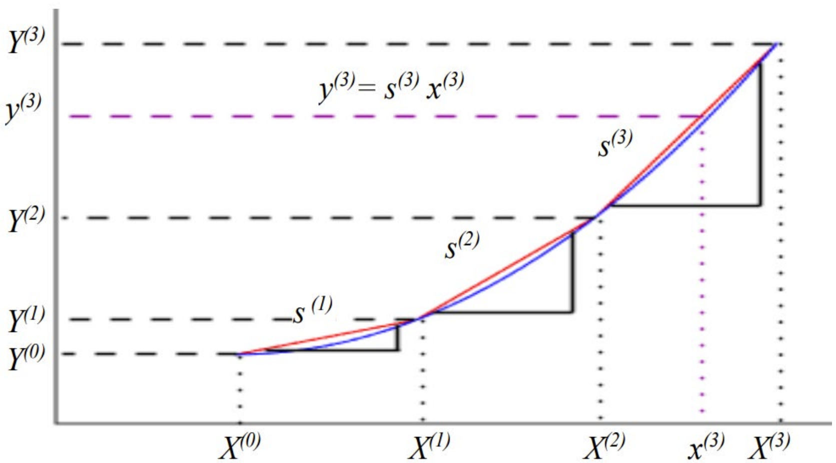
$$Q_{i+1} = Q_i - q_{i+1}$$

$$V_{i+1}^2 = V_i^2 - 2(r_i P_i + x_i Q_i)$$

Linearized Dist-Flow method (single phase)

(2) **Piecewise linear formulation (more accurate)**

The quadratic calculation of branch active power losses ($P_i^{loss} = \mathrm{r_i}\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$) and reactive power losses ($Q_i^{loss} = \mathrm{x_i}\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$) can be linearized through piecewise linear formulation.



$$f_i(x) = \frac{R_i}{V_0^2}x^2$$

$$g_i(x) = \frac{X_i}{V_0^2}x^2$$

Linear equation slopes

$$a_{ik} = \frac{f_i\left(P_i^{(k)}\right) - f_i\left(P_i^{(k-1)}\right)}{P_i^{(k)} - P_i^{(k-1)}}$$

$$b_{il} = \frac{f_i\left(Q_i^{(l)}\right) - f_i\left(Q_i^{(l-1)}\right)}{Q_i^{(l)} - Q_i^{(l-1)}}$$

$$c_{ik} = \frac{g_i\left(P_i^{(k)}\right) - g_i\left(P_i^{(k-1)}\right)}{P_i^{(k)} - P_i^{(k-1)}}$$

$$d_{il} = \frac{g_i\left(Q_i^{(l)}\right) - g_i\left(Q_i^{(l-1)}\right)}{Q_i^{(l)} - Q_i^{(l-1)}}$$

[7] C. Zhang, Y. Xu, Z. Dong and J. Ravishankar, "Three-Stage Robust Inverter-Based Voltage/Var Control for Distribution Networks With High-Level PV," in IEEE Transactions on Smart Grid, vol. 10, no. 1, pp. 782-793, Jan. 2019.

Fully Linearized Dist-Flow method (single phase)

- Linear calculation for the complex power loss:

$$P_i^{loss} = \sum_{k \in K_i} a_{i,k}\left(P_{i,k} - P_{i,k}^{*}\right) + \sum_{l \in L_i} b_{i,l}\left(Q_{i,l} - Q_{i,l}^{*}\right)$$

$$Q_i^{loss} = \sum_{k \in K_i} c_{i,k}\left(P_{i,k} - P_{i,k}^{*}\right) + \sum_{l \in L_i} d_{i,l}\left(Q_{i,l} - Q_{i,l}^{*}\right)$$

- Piecewise power flow variable can vary only within its corresponding interval:

$$0 \leq P_{i,k} \leq P_i^{(k)} - P_i^{(k-1)} \qquad\qquad 0 \leq Q_{i,k} \leq Q_i^{(l)} - Q_i^{(l-1)}$$

$$P_i^{(k-1)} - P_i^{(k)} \leq P_{i,k}^{*} \leq 0 \qquad\qquad Q_i^{(l-1)} - Q_i^{(k)} \leq Q_{i,l}^{*} \leq 0$$

- Based on the piecewise linear formulation, the fully linearized Dist-Flow with power loss is developed as:

$$P_{i+1} = P_i - p_{i+1} - r_i \frac{P_i^2 + Q_i^2}{V_i^2} \qquad\Longrightarrow\qquad P_{i+1} = P_i - p_{i+1} - P_i^{loss}$$

$$Q_{i+1} = Q_i - q_{i+1} - x_i \frac{P_i^2 + Q_i^2}{V_i^2} \qquad\qquad Q_{i+1} = Q_i - q_{i+1} - Q_i^{loss}$$

Extension to unbalanced three-phase systems

- Up to this point, it has only considered the single phase; however, distribution networks are inherently three-phase unbalanced.
- Also, the coupling between phases for the **system voltages requires additional approximations to simplify the unbalanced case.**

- Formulations are developed by L. Gan and S. Low at Caltech (Patent number: US20150346753A1) [8]:

$$\begin{bmatrix} \left(V_j^a\right)^2 \\ \left(V_j^b\right)^2 \\ \left(V_j^c\right)^2 \end{bmatrix} - \begin{bmatrix} (V_i^a)^2 \\ \left(V_i^b\right)^2 \\ (V_i^c)^2 \end{bmatrix} + 2 \begin{bmatrix} P_{ij}^a \tilde{R}_{aa} + P_{ij}^b \tilde{R}_{ab} + P_{ij}^c \tilde{R}_{ac} + Q_{ij}^a \tilde{X}_{aa} + Q_{ij}^b \tilde{X}_{ab} + Q_{ij}^c \tilde{X}_{ac} \\ P_{ij}^a \tilde{R}_{ba} + P_{ij}^b \tilde{R}_{bb} + P_{ij}^c \tilde{R}_{bc} + Q_{ij}^a \tilde{X}_{ba} + Q_{ij}^b \tilde{X}_{bb} + Q_{ij}^c \tilde{X}_{bc} \\ P_{ij}^a \tilde{R}_{ca} + P_{ij}^b \tilde{R}_{cb} + P_{ij}^c \tilde{R}_{cc} + Q_{ij}^a \tilde{X}_{ca} + Q_{ij}^b \tilde{X}_{cb} + Q_{ij}^c \tilde{X}_{cc} \end{bmatrix} = 0$$

- In single-phase distribution system, it has $V_j = V_i - z_{ik} \dfrac{P_{ij} - jQ_{ij}}{V_i^*}$

- Extend to three-phase system, it has $V_j^\phi = V_i^\phi - z_{ij}^\phi \left[S_{ij}^{\phi^*} \oslash V_i^{\phi^*} \right] \quad S_{ij}^{\phi^*} = P_{ij}^\phi - jQ_{ij}^\phi$

  where

$$V_i^\phi = \left[V_i^a, V_i^b, V_i^c\right]^T, V_j^\phi = \left[V_j^a, V_j^b, V_j^c\right]^T, P_{ij}^\phi = \left[P_{ij}^a, P_{ij}^b, P_{ij}^c\right]^T, Q_{ij}^\phi = \left[Q_{ij}^a, Q_{ij}^b, Q_{ij}^c\right]^T, z_{ij}^\phi \in \mathbb{C}^{3\times 3}$$

$\oslash$ and $\odot$ denote the element-wise division multiplication, respectively.

[8] Gan, Lingwen, and Steven H. Low. "Systems and Methods for Convex Relaxations and Linear Approximations for Optimal Power Flow in Multiphase Radial Networks." U.S. Patent Application No. 14/724,757.

Extension to unbalanced three-phase systems

- Unlike the per-phase equivalent case, multiplying by the complex conjugate of both side of the three-phase formulation will not remove the dependence on θ.
- This is due to the fact that there is a coupling between the phase at bus i that arises from the cross-products of the three-phase equations for the phase voltage and line current.
- To address this problem, it has observed that the voltage magnitude between the phases are similar, i.e., $|V_i^a| \approx |V_i^b| \approx |V_i^c|$, and that the phase unbalances on each bus are not very severe, so it assumes that the voltages are nearly balanced. Thus, it can approximate the phase different at bus i as:

$$\cos(\theta_i^a - \theta_i^b) = \cos\left(\frac{2}{3}\pi + \theta^*\right) = -\frac{1}{2}\cos(\theta^*) - \frac{\sqrt{3}}{2}\sin(\theta^*) \approx -\frac{1}{2}$$

$$\sin(\theta_i^a - \theta_i^b) = \sin\left(\frac{2}{3}\pi + \theta^*\right) = \frac{1}{2}\cos(\theta^*) + \frac{\sqrt{3}}{2}\sin(\theta^*) \approx \frac{\sqrt{3}}{2}$$

where θ^*represents the relative phase unbalance, which is sufficiently small.

- Therefore, the nearly balanced voltages are

$$\frac{V_i^a}{V_i^b} \approx \frac{V_i^b}{V_i^c} \approx \frac{V_i^c}{V_i^a} \approx e^{j2\pi/3}$$

[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

Extension to unbalanced three-phase systems

- It can update the **voltage magnitude** in **Dist-Flow method** for the unbalanced case with

$$\begin{bmatrix} \left|V_j^a\right|^2 \\ \left|V_j^b\right|^2 \\ \left|V_j^c\right|^2 \end{bmatrix} = \begin{bmatrix} |V_i^a|^2 \\ \left|V_i^b\right|^2 \\ |V_i^c|^2 \end{bmatrix} - \tilde{z}_{ij} S_{ij}^* - \tilde{z}_{ij}^* S_{ij}$$

where

$$\tilde{z}_{ij} = \alpha \odot z_{ij} = \begin{bmatrix} 1 & e^{-j2\pi/3} & e^{j2\pi/3} \\ e^{j2\pi/3} & 1 & e^{-j2\pi/3} \\ e^{-j2\pi/3} & e^{j2\pi/3} & 1 \end{bmatrix} \odot \begin{bmatrix} z_{ij}^{aa} & z_{ij}^{ab} & z_{ij}^{ac} \\ z_{ij}^{ba} & z_{ij}^{bb} & z_{ij}^{bc} \\ z_{ij}^{ca} & z_{ij}^{cb} & z_{ij}^{cc} \end{bmatrix}$$

α is phase shift matrix

- Apply above equations to Dist-Flow formulation to for the extension to unbalanced three-phase systems.

$$\begin{bmatrix} \left|V_j^a\right|^2 \\ \left|V_j^b\right|^2 \\ \left|V_j^c\right|^2 \end{bmatrix} = \begin{bmatrix} |V_i^a|^2 \\ \left|V_i^b\right|^2 \\ |V_i^c|^2 \end{bmatrix} - \begin{bmatrix} z_{ij}^{aa} + e^{-j2\pi/3} z_{ij}^{ba} + e^{j2\pi/3} z_{ij}^{ca} & z_{ij}^{ab} + e^{-j2\pi/3} z_{ij}^{bb} + e^{j2\pi/3} z_{ij}^{cb} & z_{ij}^{ac} + e^{-j2\pi/3} z_{ij}^{bc} + e^{j2\pi/3} z_{ij}^{cc} \\ e^{j2\pi/3} z_{ij}^{aa} + z_{ij}^{ba} + e^{-j2\pi/3} z_{ij}^{ca} & e^{j2\pi/3} z_{ij}^{ab} + z_{ij}^{bb} + e^{-j2\pi/3} z_{ij}^{cb} & e^{j2\pi/3} z_{ij}^{ac} + z_{ij}^{bc} + e^{-j2\pi/3} z_{ij}^{cc} \\ e^{-j2\pi/3} z_{ij}^{aa} + e^{j2\pi/3} z_{ij}^{ba} + z_{ij}^{ca} & e^{-j2\pi/3} z_{ij}^{ab} + e^{j2\pi/3} z_{ij}^{bb} + z_{ij}^{cb} & e^{-j2\pi/3} z_{ij}^{ac} + e^{j2\pi/3} z_{ij}^{bc} + z_{ij}^{cc} \end{bmatrix} \begin{bmatrix} P_{ij}^a - jQ_{ij}^a \\ P_{ij}^b - jQ_{ij}^b \\ P_{ij}^c - jQ_{ij}^c \end{bmatrix}$$
$$- \begin{bmatrix} z_{ij}^{aa^*} + z_{ij}^{ba^*}(e^{-j2\pi/3})^* + z_{ij}^{ca^*}(e^{j2\pi/3})^* & z_{ij}^{ab^*} + z_{ij}^{bb^*}(e^{-j2\pi/3})^* + z_{ij}^{cb^*}(e^{j2\pi/3})^* & z_{ij}^{ac^*} + z_{ij}^{bc^*}(e^{-j2\pi/3})^* + z_{ij}^{cc^*}(e^{j2\pi/3})^* \\ z_{ij}^{aa^*}(e^{j2\pi/3})^* + z_{ij}^{ba^*} + z_{ij}^{ca^*}(e^{-j2\pi/3})^* & z_{ij}^{ab^*}(e^{j2\pi/3})^* + z_{ij}^{bb^*} + z_{ij}^{cb^*}(e^{-j2\pi/3})^* & z_{ij}^{ac^*}(e^{j2\pi/3})^* + z_{ij}^{bc^*} + z_{ij}^{cc^*}(e^{-j2\pi/3})^* \\ z_{ij}^{aa^*}(e^{-j2\pi/3})^* + z_{ij}^{ba^*}(e^{j2\pi/3})^* + z_{ij}^{ca^*} & z_{ij}^{ab^*}(e^{-j2\pi/3})^* + z_{ij}^{bb^*}(e^{j2\pi/3})^* + z_{ij}^{cb^*} & z_{ij}^{ac^*}(e^{-j2\pi/3})^* + z_{ij}^{bc^*}(e^{j2\pi/3})^* + z_{ij}^{cc^*} \end{bmatrix} \begin{bmatrix} P_{ij}^a + jQ_{ij}^a \\ P_{ij}^b + jQ_{ij}^b \\ P_{ij}^c + jQ_{ij}^c \end{bmatrix}$$

$$P_j^\phi = P_i^\phi - p_i^\phi - r_{ij}^\phi \frac{\left(P_i^\phi\right)^2 + \left(Q_i^\phi\right)^2}{\left(V_i^\phi\right)^2} \qquad Q_j^\phi = Q_i^\phi - q_i^\phi - z_{ij}^\phi \frac{\left(P_i^\phi\right)^2 + \left(Q_i^\phi\right)^2}{\left(V_i^\phi\right)^2}$$

r_{ij}^ϕ = Re($\alpha \odot z_{ij}$)

z_{ij}^ϕ = Im($\alpha \odot z_{ij}$)

Modified Newton-Raphson method

- A modified Newton method for radial distribution system is derived in which the Jacobian matrix is in UDU^T form, where U is a constant upper triangular matrix depending only on system topology and D is a block diagonal matrix.
- With this formulation, the conventional steps of forming the Jacobian matrix are replaced by back/forward sweeps on radial feeders with equivalent impedances.
- In conventional Newton-Raphson method, the power flow problem is to solve

$$\begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta V/V \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}$$

where

$$H_{ij} = -V_i V_j \left(G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij} \right), j \neq i \qquad H_{ii} = V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij} \right)$$

$$N_{ij} = -V_i V_j \left(G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij} \right), j \neq i \qquad N_{ii} = -V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij} \right) - 2V_i^2 G_{ii}$$

$$J_{ij} = V_i V_j \left(G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij} \right), j \neq i \qquad J_{ii} = -V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij} \right)$$

$$L_{ij} = -V_i V_j \left(G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij} \right), j \neq i \qquad L_{ii} = -V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij} \right) + 2V_i^2 B_{ii}$$

[11] F. Zhang and C. S. Cheng, "A Modified Newton Method for Radial Distribution System Power Flow Analysis," in IEEE Transactions on Power System, vol. 12, no. 1, pp. 882-887, Feb 1997.

Modified Newton-Raphson method

Assumption 1: small voltage difference between two adjacent nodes (typical distribution lines are short and line flows are not high).

Assumption 2: no shunt branches (all the shunt branches can be converted to node power injections using initial and updated node voltages).

- Therefore, the Jacobian matrix can be approximated as:

$$H_{ij} \approx V_i V_j \left(B_{ij} \cos \theta_{ij}\right), j \neq i$$

$$N_{ij} \approx -V_i V_j \left(G_{ij} \cos \theta_{ij}\right), j \neq i$$

$$J_{ij} \approx V_i V_j \left(G_{ij} \cos \theta_{ij}\right), j \neq i$$

$$L_{ij} \approx V_i V_j \left(B_{ij} \cos \theta_{ij}\right), j \neq i$$

$$H_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(B_{ij} \cos \theta_{ij}\right)$$

$$N_{ii} \approx V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos \theta_{ij}\right)$$

$$J_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos \theta_{ij}\right)$$

$$L_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(B_{ij} \cos \theta_{ij}\right)$$

- The matrices H, N, J and L all have the same properties (symmetry, sparsity pattern) as the Nodal Admittance Matrix.

Modified Newton-Raphson method

- Hence, the matrices H, N, J and L can be formed as:

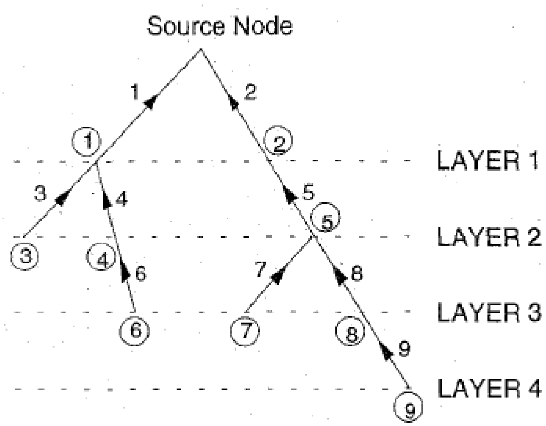
$$H = L = A_{n-1} D_B A_{n-1}^T$$

$$J = -N = A_{n-1} D_G A_{n-1}^T$$

where D_B and D_B are diagonal matrices with diagonal entries to be $V_i V_j B_{ij} \cos\theta_{ij}$ and $V_i V_j G_{ij} \cos\theta_{ij}$, respectively. Therefore, the conventional Newton Raphson can be rewritten as:

$$\begin{bmatrix} A_{n-1} & 0 \\ 0 & A_{n-1} \end{bmatrix} \begin{bmatrix} D_B & -D_G \\ D_G & D_B \end{bmatrix} \begin{bmatrix} A_{n-1}^T & 0 \\ 0 & A_{n-1}^T \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta V/V \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}$$

- A_{n-1} is an upper triangular matrix with all diagonal entries to be 1 and all non-zero off-diagonal entries to be -1.



$$A_{n-1} = \begin{bmatrix} 1 & & -1 & -1 & & & & & \\ & 1 & & & -1 & & & & \\ & & 1 & & & & & & \\ & & & 1 & & -1 & & & \\ & & & & 1 & & -1 & -1 & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & -1 \\ & & & & & & & & 1 \end{bmatrix}$$

Modified Newton-Raphson method

- By now it has shown that the Jacobian matrix can be formed as the product of three square matrices. It can be solved by back/forward sweeps as well. It defines:

$$\dot{E} = \Delta\theta + j\,\Delta V/V$$

$$\dot{S} = \Delta P + j\Delta Q$$

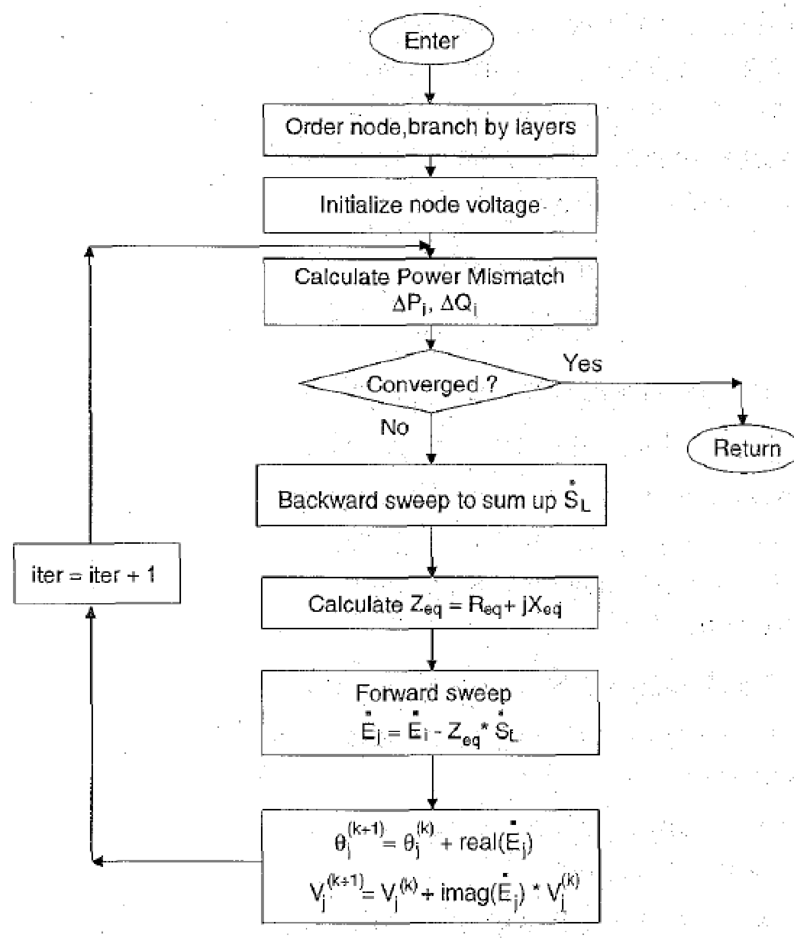
$$\dot{W} = D_B + jD_G$$

- Therefore, the formulations in Newton-Raphson method can be modified:

$$A_{n-1}\dot{W}A_{n-1}^{T}\dot{E}=\dot{S}_L \longrightarrow \begin{aligned} A_{n-1}\dot{S}_L &= \dot{S} \\ \dot{W}A_{n-1}^{T}\dot{E} &= \dot{S}_L \end{aligned}$$

- When solving $\dot{E}$, the diagonal matrix $\dot{W}$ can be inverted for each line. The diagonal in $\dot{W}^{-1}$ is denoted as the equivalent line impedance:

$$Z_{eq}=R_{eq} + jX_{eq}$$



$A_{n-1}\dot{S}_L = \dot{S}$ Backward sweep

$WA_{n-1}^{T}\dot{E} = \dot{S}_L$ Forward sweep