Solutions to Optimal Control Problems for Large Scale Systems

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Paper Selected:

[1] Nazerian, K. Bhatta, and F. Sorrentino, *Exact Decomposition of Optimal Control Problems via Simultaneous Block Diagonalization of Matrices*, IEEE Open Journal of Control Systems, vol.2, pp 24-35, 2023

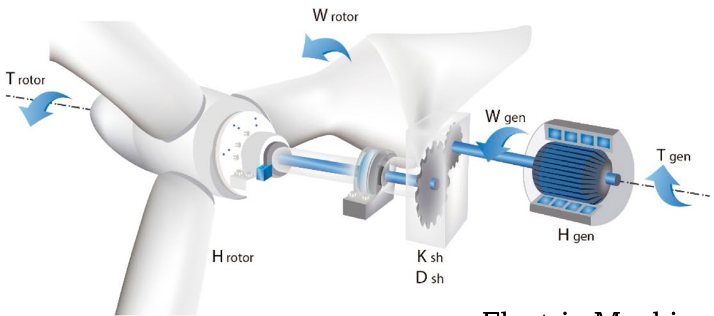
In Fulfillment of the Coursework Requirement for
ELEC8003 Linear Algebra for Signal Processing

For Instructional and Educational Use

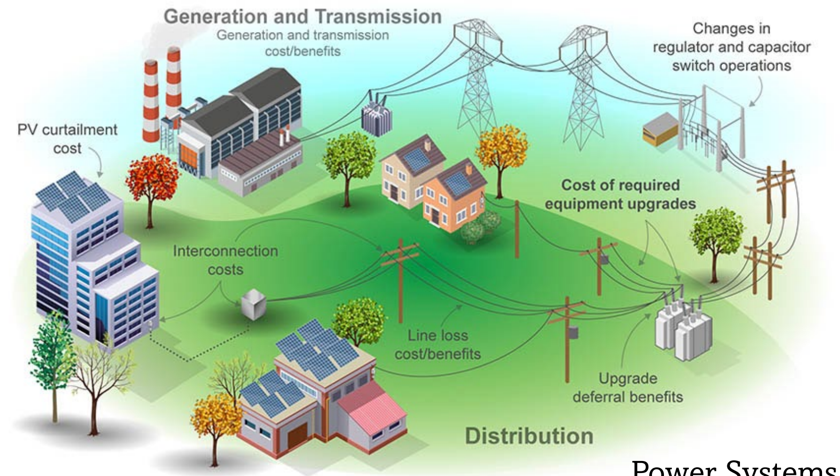
Introduction – From Control to Optimal Control

A control system is a system that uses feedback information to continuously monitor the states and adjust input to regulate states or outputs (a.k.a. state regulator or output regulator) or perform tracking.

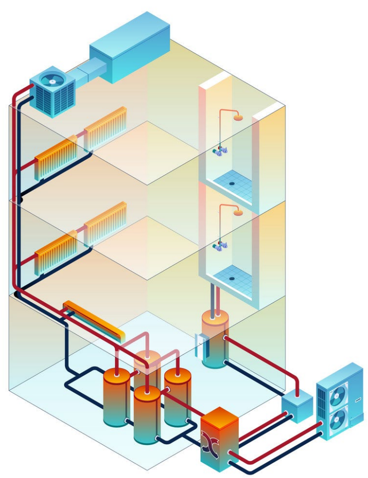
Example



Electric Machines



Power Systems



HVAC Systems

Drones & Autonomous Vehicles

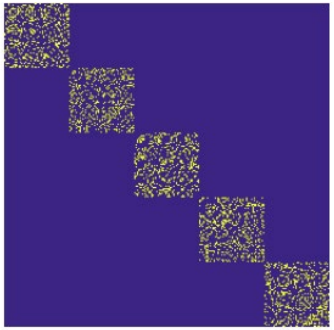
Structures

Content

1. Introduction – From Control to Optimal Control
    - Characteristics Equation and Stability
    - Similarity Transformation and Decoupled State-Space Model
    - Portryagin's Maximum Principles and Feedback Invariant
    - Solutions to Optimal Control Problems

2. Introduction – Linear Algebra
    - Properties of Similarity Transformation
    - Algorithm for Simultaneous Block Diagonalization (SBD)

3. Optimal Control Problems with SBD
    - From Sampled OCP to Transformed OCP and Decoupled OCP
    - Riccati Equation and Controllability Gramian – How does SBD help?
    - Example – 1. Mass Spring Damper System; 2. HVAC Optimal Control



4. Existing Solution and its Limitation
    - Symmetric Decomposition (Heavily depends on its Structure)
    - Model Order Reduction by Aggregation (Sub-Optimal Solution)

5. Conclusion

Introduction – From Control to Optimal Control

Plant: $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$

General Cost Function:

$$J_{LQR} = \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Qx} + \mathbf{u}^{\mathrm{T}}\mathbf{Ru} + \mathbf{2x}^{\mathrm{T}}\mathbf{Fu}\, dt = -\int_{-\infty}^{\infty} \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{Px} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}\, dt + \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Qx} + \mathbf{u}^{\mathrm{T}}\mathbf{Ru} + \mathbf{2x}^{\mathrm{T}}\mathbf{Fu} + \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{Px} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}\, dt$$

To obtain minimum cost $J_{LQR} = \underbrace{H(\mathbf{x},\mathbf{u})}_{\text{Feedback Invariant}} + \int_{-\infty}^{\infty} \underbrace{(\mathbf{u}+\mathbf{Kx})^{\mathrm{T}}\mathbf{R}(\mathbf{u}+\mathbf{Kx})}_{L(\mathbf{x},\mathbf{u})}\, dt,$

- $\mathbf{P}$ must be symmetric, and it satisfies the algebraic Riccati Equation:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} + \mathbf{Q} - (\mathbf{PB}+\mathbf{F})\mathbf{R}^{-1}(\mathbf{PB}+\mathbf{F})^{\mathrm{T}} = \mathbf{0} \longrightarrow O(n^3)$$

- The optimal input $\mathbf{u}^*(t)$ will be $\mathbf{u} = -\mathbf{Kx} = \mathbf{R}^{-1}(\mathbf{PB}+\mathbf{F})^{T}\mathbf{x}$

Hence, optimal control problem can be solved by: [Naidu, 2002 | P. 136]

1. Solve the CARE: $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} + \mathbf{Q} - (\mathbf{PB}+\mathbf{F})\mathbf{R}^{-1}(\mathbf{PB}+\mathbf{F})^{\mathrm{T}} = \mathbf{0}$, where $\mathbf{P}$ is symmetric.
2. Obtain the optimal state: $\dot{\mathbf{x}}^* = \left(\mathbf{A} - \mathbf{BR}^{-1}(\mathbf{PB}+\mathbf{F})^{\mathrm{T}}\right)\mathbf{x}^*$
3. Obtain the optimal control: $\mathbf{u}^* = -\mathbf{R}^{-1}(\mathbf{PB}+\mathbf{F})^{\mathrm{T}}\mathbf{x}^*(t)$
4. Find the minimum cost: $J^* = \frac{1}{2}\mathbf{x}^{*T}(0)\mathbf{Px}^*(0)$

[2] Naidu D.S., Optimal Control Systems, CRC Press, 2002

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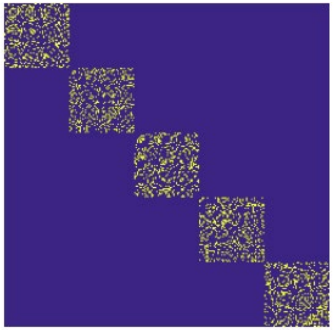

Introduction – Revisit to Linear Algebra

Properties of Similarity Transformation [Ch.6 | P. 10]

1. Definition:
   (A, B) are similar matrices if there exists an invertible P, such that $\mathbf{A} = \mathbf{PBP^{-1}}$, or $\mathbf{AP} = \mathbf{PB}$
2. Same eigenvalues:
   Given (A, B) are similar matrices.
   $$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \det(\mathbf{PBP}^{-1} - \lambda\mathbf{I}) = \det\left(\mathbf{PBP}^{-1} - \boldsymbol{\lambda}\mathbf{PP}^{-1}\right) = \det\left(\mathbf{P}(\mathbf{B} - \lambda\mathbf{I})\mathbf{P}^{-1}\right) \\ &= \det(\mathbf{P})\det(\mathbf{B} - \lambda\mathbf{I})\det(\mathbf{P}^{-1}) = \det\left(\mathbf{B} - \lambda\mathbf{I}\right).\end{aligned}$$
   Hence they share the same eigenvalue.
3. Same rank: [Ch.4 | P.20]
   Rank is unchanged upon left-multiplication or right-multiplication of some invertible matrices.
   i.e. $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{PAP}^{-1}) = \operatorname{rank}(\mathbf{B})$ if P is invertible.

Properties of Simultaneous Block Diagonalization [Nazerian | P.2]

1. Definition:
   Given a set of M square n-dimension matrices $\mathcal{S} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_\mathrm{M}\}$, an SBD transformation is an orthogonal square matrix $\mathbf{T}$ with dimension N, such that
   $$\mathbf{T}^{-1}\mathbf{A}_\mathrm{k}\mathbf{T} = \oplus_{j=1}^{l} \mathbf{B}_\mathrm{j}^\mathrm{k} \quad k = 1, 2, \dots, M$$
   where the symbol $\oplus$ is the direct sum of matrices, l is the number of block, and each blocks $\mathbf{B}_\mathrm{j}^\mathrm{k}$ is a square matrices with dimension b_j such that $\sum_{j=1}^{l} b_j = n$.

Introduction – Revisit to Linear Algebra

Properties of Simultaneous Block Diagonalization [Nazerian | P.2]

2. Finest SBD:
   The finest SBD is the SBD for which the resulting blocks cannot be further refined by any other transformation.

3. Commutative Properties:
   $(\mathbf{A}, \mathbf{B})$ is said to be simultaneous block diagonalizable if they share the same matrix with eigenvectors as its columns and they are commutative, i.e. $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = \mathbf{O}$ or $\mathbf{AB} = \mathbf{BA}$.

$$\mathbf{A} = \mathbf{UDU}^{-1} \qquad \mathbf{B} = \mathbf{UD'U}^{-1}$$

$$\mathbf{AB} = \mathbf{UDU}^{-1}\mathbf{UD'U}^{-1} = \mathbf{UDD'U}^{-1} \qquad \mathbf{BA} = \mathbf{UD'U}^{-1}\mathbf{UDU}^{-1} = \mathbf{UD'DU}^{-1}$$

   Hence, $\mathbf{D'D} = \mathbf{DD'}$ given that both $\mathbf{D}$ and $\mathbf{D'}$ are diagonal matrices and $\mathbf{AB} = \mathbf{BA}$

How to obtain the transformation $\mathbf{T} = \mathbf{SBD}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\mathrm{n})$ such that $\mathbf{T}^{-1}\mathbf{A}_\mathrm{k}\mathbf{T} = \bigoplus_{j=1}^{l} \mathbf{B}_\mathrm{j}^\mathrm{k}$?

Algorithm 1: The SBD Transformation [46].

Input: $A_1, \ A_2, \ldots, A_M \in \mathbb{R}^{n\times n}$
Output: T

$P_i = I_n \otimes A_i - A_i^\top \otimes I_n, \quad i = 1, \ldots, M$
$S = \sum_{i=1}^{M} P_i^\top P_i$
$\mathrm{vec}(U) = \mathcal{N}(S)$
Reshape $\mathrm{vec}(U)$ into $U \in \mathbb{R}^{n\times n}$ and set $U := (1/2)(U + U^\top)$
T is a matrix with eigenvectors of U as its columns
return T

1. Find a $\mathbf{U}$ such that $\mathbf{AU} = \mathbf{UA}$, or
$$\left(\mathbf{A}_{ij}^T \otimes I_{n_{pi}} - I_{n_{pi}} \otimes \mathbf{A}_{ij}\right) \mathrm{vec}(\mathrm{U}_j) = 0_{n_i, n_j}$$
2. Define $\mathbf{P} = \mathbf{A}_{ij}^T \otimes I_{n_{pi}} - I_{n_{pi}} \otimes \mathbf{A}_{ij}$ and $\mathbf{x} = \mathrm{vec}(\mathbf{U}_j)$,
   It is to find $\mathbf{Px} = 0$ or the nullspace of $N(\mathbf{P})$.
   Yet, $N(\mathbf{P}^\mathrm{T}\mathbf{P}) = N(\mathbf{P})$ as $\mathbf{Px} = \mathbf{0} \rightarrow \mathbf{P}^\mathrm{T}\mathbf{Px} = \mathbf{0}$
3. It returns such a $\mathrm{vec}(\mathbf{U})$ and hence $\mathbf{U}$, such that $\mathbf{AU} = \mathbf{UA}$.
4. Perform eigen-decomposition on $\mathbf{U} = \mathbf{T\Lambda T}^{-1}$.
5. Return $\mathbf{T}$ as the SBD transformation.

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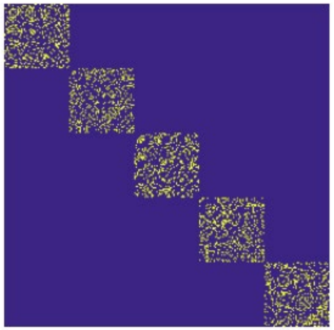


Optimal Control Problems with Simultaneous Block Diagonalization

Consider an OCP:

$$\min_{\mathbf{u}} \bar{J} = \int_0^{t_f} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \bar{\mathbf{u}}^{\mathrm{T}}\bar{\mathbf{R}}\bar{\mathbf{u}} + 2\mathbf{x}^{\mathrm{T}}\bar{\mathbf{F}}\bar{\mathbf{u}}\, dt$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \bar{\mathbf{B}}\bar{\mathbf{u}}(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(t_f) = \mathbf{x}_f$$

where

$\mathbf{x}(t) \in \mathbb{R}^n$ state vector

$\bar{\mathbf{u}}(t) \in \mathbb{R}^m$ input vector,

$\mathbf{A} \in \mathbb{R}^{n\times n}$

$\mathbf{B} \in \mathbb{R}^{n\times m}$

$\mathbf{Q} \in \mathbb{R}^{n\times n}$ $(\mathbf{Q} \geq 0)$

$\bar{\mathbf{R}} \in \mathbb{R}^{m\times m}$ $(\mathbf{R} > 0)$

$\bar{\mathbf{F}} \in \mathbb{R}^{n\times m}$ $(\mathbf{Q} - \bar{\mathbf{F}}\,\bar{\mathbf{R}}^{-1}\,\bar{\mathbf{F}}^{\mathrm{T}} \geq 0)$

$$\xrightarrow{\text{Append}}$$

$$\mathbf{u}(t) = \begin{pmatrix} \bar{\mathbf{u}} \\ \mathbf{u_0}(t) \end{pmatrix}$$

$$\mathbf{B} = [\bar{\mathbf{B}} \quad \mathbf{0}_{n\times(n-m)}]$$

$$\mathbf{R} = \bar{\mathbf{R}} \oplus \mathbf{R_0}$$

$$\mathbf{F} = [\bar{\mathbf{F}} \quad \mathbf{0}_{n\times(n-m)}]$$

Sample OCP

$$\min_{\mathbf{u}} J = \int_0^{t_f} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\, dt$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(t_f) = \mathbf{x}_f$$

where

$\mathbf{x}, \mathbf{u}(t) \in \mathbb{R}^n$

$\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{F} \in \mathbb{R}^{n\times n}$

Suppose an orthogonal similarity transformation matrix

$$\mathbf{T} = \mathrm{SBD}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{F})$$

is applied to decouple the original large problem into a sets of L lower dimensional problems by simultaneous block diagonalizing the following matrices.

$$\tilde{\mathbf{A}} = \bigoplus_{l=1}^{L} \tilde{\mathbf{A}}_{\mathbf{l}} = \mathbf{T}^{\mathrm{T}}\mathbf{A}\mathbf{T}, \quad \tilde{\mathbf{B}} = \bigoplus_{l=1}^{L} \tilde{\mathbf{B}}_{\mathbf{l}} = \mathbf{T}^{\mathrm{T}}\mathbf{B}\mathbf{T}, \quad \tilde{\mathbf{Q}} = \bigoplus_{l=1}^{L} \tilde{Q}_l = \mathbf{T}^{\mathrm{T}}\mathbf{Q}\mathbf{T}, \quad \tilde{\mathbf{R}} = \bigoplus_{l=1}^{L} \tilde{\mathbf{R}}_{\mathbf{l}} = \mathbf{T}^{\mathrm{T}}\mathbf{R}\mathbf{T},$$

Optimal Control Problems with Simultaneous Block Diagonalization

Sample OCP

$$\min_{\mathbf{u}} J = \int_0^{t_f} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\, dt$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(t_f) = \mathbf{x}_f$$

where
$\mathbf{x}, \mathbf{u}(t) \in \mathbb{R}^n$
$\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{F} \in \mathbb{R}^{n\times n}$

$$\mathbf{z}(t) = \mathbf{T}^T\mathbf{x}(t)$$
$$\mathbf{v}(t) = \mathbf{T}^T\mathbf{u}(t)$$
$$\widetilde{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T}$$
$$\widetilde{\mathbf{B}} = \mathbf{T}^T\mathbf{B}\mathbf{T}$$
$$\widetilde{\mathbf{Q}} = \mathbf{T}^T\mathbf{Q}\mathbf{T}$$
$$\widetilde{\mathbf{R}} = \mathbf{T}^T\mathbf{R}\mathbf{T}$$
$$\widetilde{\mathbf{F}} = \mathbf{T}^T\mathbf{F}\mathbf{T}$$

$\xrightarrow{\quad}$
Transform

Transformed OCP

$$\min_{\mathbf{v}} J = \int_0^{t_f} \mathbf{z}^{\mathrm{T}}\widetilde{\mathbf{Q}}\mathbf{z} + \mathbf{v}^{\mathrm{T}}\widetilde{\mathbf{R}}\mathbf{v} + 2\mathbf{z}^{\mathrm{T}}\widetilde{\mathbf{F}}\mathbf{v}\, dt$$

$$\text{s.t. } \dot{\mathbf{z}} = \widetilde{\mathbf{A}}\mathbf{z}(t) + \widetilde{\mathbf{B}}\mathbf{v}(t)$$

$$\mathbf{z}(0) = \mathbf{z}_0,\ \mathbf{z}(t_f) = \mathbf{z}_f$$

where
$\mathbf{z}, \mathbf{v}(t) \in \mathbb{R}^n$
$\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{Q}}, \widetilde{\mathbf{R}}, \widetilde{\mathbf{F}} \in \mathbb{R}^{n\times n}$

$$\min_{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_l} J_1 + J_2 + \cdots + J_L$$

$$\text{s.t. } \dot{\mathbf{z}}_l = \widetilde{\mathbf{A}}_l\mathbf{z}_l(t) + \widetilde{\mathbf{B}}_l\mathbf{v}_l(t)$$

$$\mathbf{z}_l(0) = \mathbf{z}_{l0},\ \mathbf{z}_l(t_f) = \mathbf{z}_{lf}$$

Decoupled OCP

$$\min_{\mathbf{v}_l} J = \int_0^{t_f} \mathbf{z}_l^{\mathrm{T}}\widetilde{\mathbf{Q}}_l\mathbf{z}_l + \mathbf{v}_l^{\mathrm{T}}\widetilde{\mathbf{R}}_l\mathbf{v}_l + 2\mathbf{z}_l^{\mathrm{T}}\widetilde{\mathbf{F}}_l\mathbf{v}_l\, dt$$

$$\text{s.t. } \dot{\mathbf{z}}_l = \widetilde{\mathbf{A}}_l\mathbf{z}_l(t) + \widetilde{\mathbf{B}}_l\mathbf{v}_l(t)$$

$$\mathbf{z}_l(0) = \mathbf{z}_{l0},\ \mathbf{z}_l(t_f) = \mathbf{z}_{lf}$$

where $l = 1, 2, \dots, L$
with $\mathbf{z}(t) = [\mathbf{z}_1(t)^T \quad \mathbf{z}_2(t)^T \quad \dots \quad \mathbf{z}_n(t)^T]$
$\mathbf{v}(t) = [\mathbf{v}_1(t)^T \quad \mathbf{v}_2(t)^T \quad \dots \quad \mathbf{v}_n(t)^T]$

Note

1. $\mathrm{J_i}$ and $\mathrm{J_j}$ are independent of each other
2. $\mathrm{J_i}(\mathbf{z}_i, \mathbf{v}_i) \geq 0$ and $\mathrm{J_j}(\mathbf{z}_j, \mathbf{v}_j) \geq 0$
3. The states $\mathbf{z}_i$ and the input $\mathbf{v}_i$ are decoupled.
4. The dimension of controllable and observable subspace and the open loop poles remains the same.

Optimal Control Problems with Simultaneous Block Diagonalization

Close Form Solution of the sample OCP

Define the Hamiltonian for the OCP: $\mathcal{H} = \frac{1}{2}\left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\right) + \boldsymbol{\lambda}^{T}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^{n}$ is the time-varying costate vector.

[Ch. 3.1 | P. 6]

$$\frac{\partial \mathbf{x}^{T}\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}, \qquad \frac{\partial \mathbf{y}^{T}\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}^{T}\mathbf{y}$$

By Portryagin's Maximum Principles:

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial \lambda_i} \\ -\lambda_i &= \frac{\partial H}{\partial x_i} \\ \frac{\partial H}{\partial \mathbf{u}} &= 0 \end{aligned} \longrightarrow \left\{ \begin{aligned} \dot{\mathbf{x}}(t) &= \frac{\partial H}{\partial \boldsymbol{\lambda}(t)} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ -\boldsymbol{\lambda} &= \frac{\partial H}{\partial \mathbf{x}} = \mathbf{Q}\mathbf{x}(t) + \mathbf{F}\mathbf{u}(t) + \mathbf{A}^{T}\boldsymbol{\lambda}(t) \\ \mathbf{0} &= \frac{\partial H}{\partial \mathbf{u}} = \mathbf{F}^{\mathrm{T}}\mathbf{x}(t) + \mathbf{R}\mathbf{u}(t) + \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) \end{aligned} \right.$$

Assume $\boldsymbol{\lambda}(t) = \mathbf{P}\mathbf{x}(t) + \boldsymbol{\xi}(t)$,
Solve for $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$

Optimal Control

$$\longrightarrow \mathbf{u}^{*}(t) = -\mathbf{R}^{-1}\left(\mathbf{B}^{T}\boldsymbol{\lambda}^{*}(t) + \mathbf{F}^{T}\mathbf{x}^{*}(t)\right)$$

Assume $\boldsymbol{\lambda}(t) = \mathbf{P}\mathbf{x}(t) + \boldsymbol{\xi}(t)$,

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{Q}} & \widehat{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\xi}(t) \end{bmatrix} \longrightarrow$$

To decouple $\mathbf{x}$ and $\boldsymbol{\xi}$, we find $\mathbf{P}$ that sets $\widehat{\mathbf{Q}} = \mathbf{0}$

$$\widehat{\mathbf{Q}} = \mathbf{0}$$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \mathbf{0} & \widehat{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\xi}(t) \end{bmatrix}$$

where

$$\widehat{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}(\mathbf{F}^{T} + \mathbf{B}^{T}\mathbf{P})$$

$$\widehat{\mathbf{B}} = -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{T}$$

$$\widehat{\mathbf{Q}} = \mathbf{P}\left(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{F}^{\mathrm{T}}\right) + \left(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{F}^{\mathrm{T}}\right)\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} - \mathbf{F}\mathbf{R}^{-1}\mathbf{F}^{\mathrm{T}}$$

Optimal Control Problems with Simultaneous Block Diagonalization

To solve state vector $\mathbf{x}(t)$ and costate vector $\boldsymbol{\lambda}(t) = \mathbf{P}\mathbf{x}(t) + \boldsymbol{\xi}(t)$

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\boldsymbol{\xi}}(t)\end{bmatrix} = \begin{bmatrix}\widehat{\mathbf{A}} & \widehat{\mathbf{B}}\\ \mathbf{0} & \widehat{\mathbf{A}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \boldsymbol{\xi}(t)\end{bmatrix}$$

$$\dot{\boldsymbol{\xi}}(t) = \widehat{\mathbf{A}}^{\mathrm{T}}\boldsymbol{\xi}(t) \qquad \boldsymbol{\xi}(t) = e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - t)}\boldsymbol{\xi}_f$$

$$\dot{\mathbf{x}}(t) = \widehat{\mathbf{A}}\mathbf{x}(t) + \widehat{\mathbf{B}}\boldsymbol{\xi}(t)$$

$$= \widehat{\mathbf{A}}\mathbf{x}(t) + \widehat{\mathbf{B}}e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - t)}\boldsymbol{\xi}_f$$

$$\dot{\mathbf{x}}(t) - \widehat{\mathbf{A}}\mathbf{x}(t) = \widehat{\mathbf{B}}e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - t)}\boldsymbol{\xi}_f$$

$$e^{\widehat{\mathbf{A}}(t_f - t)}(\dot{\mathbf{x}}(t) - \widehat{\mathbf{A}}\mathbf{x}(t)) = e^{\widehat{\mathbf{A}}(t_f - t)}\widehat{\mathbf{B}}e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - t)}\boldsymbol{\xi}_f$$

$$\frac{d}{dt}\int_0^{t_f} e^{\widehat{\mathbf{A}}(t_f - t)}\mathbf{x}(t)dt = \boxed{\underbrace{\int_0^{t_f} e^{\widehat{\mathbf{A}}(t_f - \tau)}\widehat{\mathbf{B}}e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - \tau)}d\tau}_{\widehat{\mathbf{W}}(t_f)}}\boldsymbol{\xi}_f$$

Controllability Gramian
$O(n^4)$

$$\mathbf{x_f} - e^{\widehat{\mathbf{A}}t_f}\mathbf{x_0} = \widehat{\mathbf{W}}\boldsymbol{\xi}_f$$

$$\boldsymbol{\xi}_f = \widehat{\mathbf{W}}^{-1}(\mathbf{x_f} - e^{\widehat{\mathbf{A}}t_f}\mathbf{x_0})$$

Solution

$$\boldsymbol{\xi}(t) = e^{\widehat{\mathbf{A}}^{\mathrm{T}}(t_f - t)}\boldsymbol{\xi}_f$$

$$\mathbf{x}(t) = \widehat{\mathbf{W}}(t)\boldsymbol{\xi}_f + e^{\widehat{\mathbf{A}}t_f}\mathbf{x_0}$$

SBD – TOCP

Performing SBD, the TOCP is still valid with the Riccati Matrix and Controllability Gramian as

$$\widetilde{\mathbf{P}} = \bigoplus_{l=1}^{L}\widetilde{\mathbf{P}}_{\mathbf{l}} = \mathbf{T}^{\mathrm{T}}\mathbf{P}\mathbf{T}, \qquad \widetilde{\mathbf{W}} = \bigoplus_{l=1}^{L}\widetilde{\mathbf{W}}_{\mathbf{l}} = \mathbf{T}^{\mathrm{T}}\mathbf{W}\mathbf{T}$$

The major task becomes solving

1) Algebraic Riccati Equation $O_R(n^3)$

$$\mathbf{0} = \widetilde{\mathbf{P}}_l(\widetilde{\mathbf{A}}_l - \widetilde{\mathbf{B}}_l\widetilde{\mathbf{R}}_l^{-1}\widetilde{\mathbf{F}}_l^{\mathrm{T}}) + (\widetilde{\mathbf{A}}_l - \widetilde{\mathbf{B}}_l\widetilde{\mathbf{R}}_l^{-1}\widetilde{\mathbf{F}}_l^{\mathrm{T}})\widetilde{\mathbf{P}}_l$$
$$-\widetilde{\mathbf{P}}_l\widetilde{\mathbf{B}}_l\widetilde{\mathbf{R}}_l^{-1}\widetilde{\mathbf{B}}_l^{\mathrm{T}}\widetilde{\mathbf{P}}_l + \widetilde{\mathbf{Q}}_l - \widetilde{\mathbf{F}}\widetilde{\mathbf{R}}_l^{-1}\widetilde{\mathbf{F}}_l^{\mathrm{T}}$$

2) Controllablility Gramian $O_G(n^4)$

$$\widehat{\mathbf{W}}_l = \int_0^{t_f} e^{\widehat{\mathbf{A}}_l(t_f - \tau)}\widehat{\mathbf{B}}_l e^{\widehat{\mathbf{A}}_l^{\mathrm{T}}(t_f - \tau)}d\tau$$

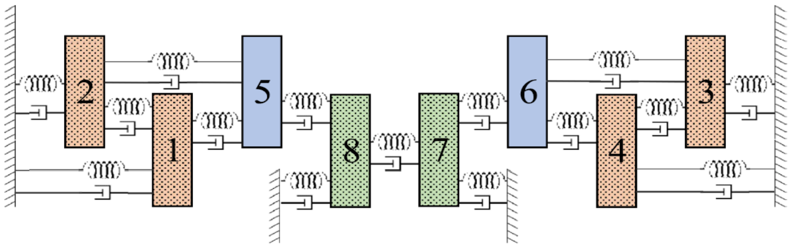
$$\boxed{\sum_{l=1}^{L} O_R(n_l^3) \le O_R(n^3), \quad \sum_{l=1}^{L} O_G(n_l^4) \le O_G(n^3)}$$

Example 1: Mass Spring Damper

Networked system =
N n-dimension interacting subsystems in S cluster

$$\dot{\mathbf{X}}(t) = \left[\sum_{s=1}^{S}(E_s \otimes A_s + GE_s \otimes H_s)\right]\mathbf{X}(t) + \left[\sum_{s=1}^{S} E_s \otimes B_s\right]\mathbf{U}(t)$$

$$\mathrm{J} = \int_0^{t_f} (\mathbf{X}^T\,(\mathbf{Q} \otimes \mathbf{W_Q})\mathbf{X} + \mathbf{U}^T(\mathbf{R} \otimes \mathbf{W_R})\mathbf{U})dt$$



$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$\xrightarrow{\quad}_{\text{SBD}}$$

$$\tilde{G} = \begin{bmatrix} -0.46 & 0.60 & -0.26 & 0 & 0 & 0 & 0 & 0 \\ 0.60 & -1.59 & -0.33 & 0 & 0 & 0 & 0 & 0 \\ -0.26 & -0.33 & 2.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.29 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.21 & 2.29 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} 1.82 & -0.24 & -0.51 & 0 & 0 & 0 & 0 & 0 \\ -0.24 & 1.67 & -0.71 & 0 & 0 & 0 & 0 & 0 \\ -0.51 & -0.71 & 0.51 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.39 & 0.92 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.92 & 0.61 \end{bmatrix}$$

SBD-TOCP Example 2: HVAC Systems

<u>HVAC Optimal Control Problem</u>

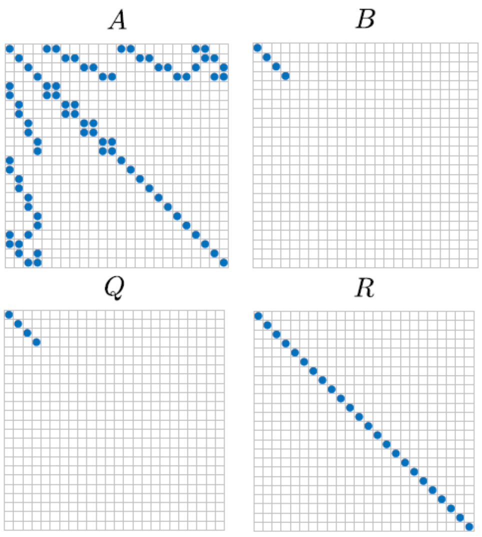
$$\min_{\mathbf{u}} \quad J = \sum_{k=0}^{\infty} \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k$$

$$\text{s.t. } \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \quad \mathbf{x}(0) = \mathbf{x}_0$$

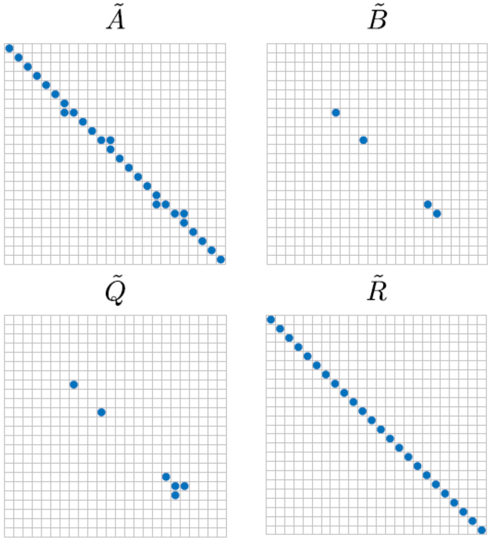
$$\mathbf{x}_k = \begin{pmatrix} T_k^{wall} \\ T_k^{zone} \end{pmatrix}$$

System Dimension N_{sys}

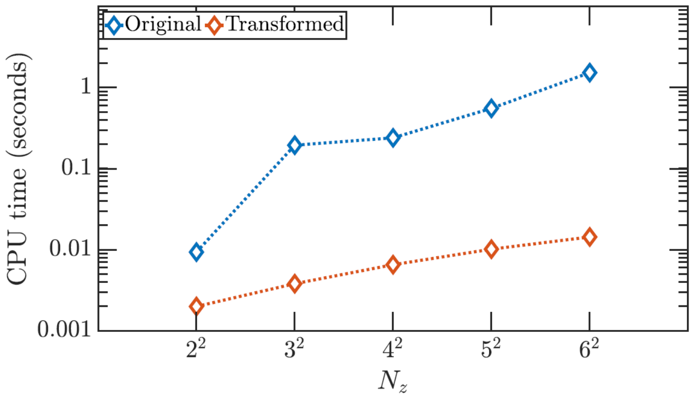
N_z	N_{sys}	$\tilde{N}$	$\tilde{N}_1$	$\tilde{N}_2$
2^2	24	20	16	4
3^2	51	42	33	9
4^2	88	72	56	16
5^2	135	110	85	25
6^2	192	156	120	36

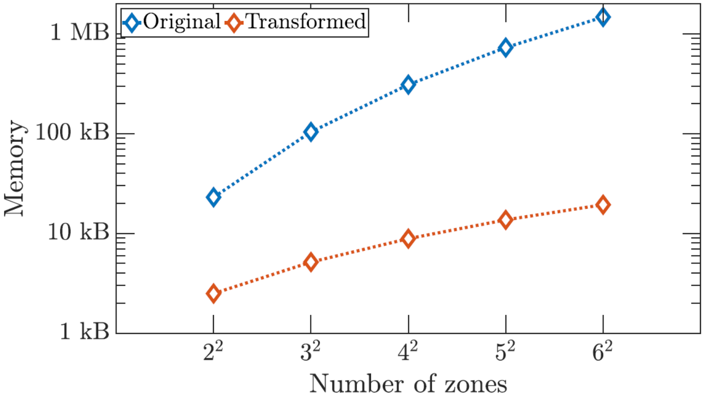


(a) Original System



(b) SBD Transformed System





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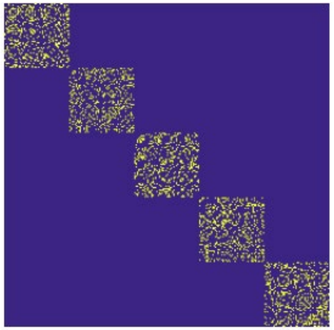


Existing Methods – Symmetric Decomposition [Danielson, 2020]

Given a Constrained Finite-Time Optimal Control (CFTOC) problem

$$\min_{\mathbf{u}} \ \frac{1}{2}\sum_{k=0}^{N-1} \begin{bmatrix}\mathbf{x}_k\\ \mathbf{u}_k\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^{\mathrm{T}} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\mathbf{x}_k\\ \mathbf{u}_k\end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix}\mathbf{x}_{k+1}\\ \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{x}_k\\ \mathbf{u}_k\end{bmatrix}, \quad \underline{y} \le y \le \bar{y}$$

$\longrightarrow$

Requirement – Rotational Symmetric

Rotation Matrix

$$\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^{\mathrm{T}} & \mathbf{R}\end{bmatrix} = \begin{bmatrix}\mathbf{\Theta}^x & \\ & \mathbf{\Theta}^u\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^{\mathrm{T}} & \mathbf{R}\end{bmatrix} \boxed{\begin{bmatrix}\mathbf{\Theta}^x & \\ & \mathbf{\Theta}^u\end{bmatrix}}$$

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{\Theta}^x & \\ & \mathbf{\Theta}^u\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} \boxed{\begin{bmatrix}\mathbf{\Theta}^x & \\ & \mathbf{\Theta}^u\end{bmatrix}}$$

$$\mathbf{\Theta}^{u,x,y} = \begin{bmatrix}\boxed{\mathbf{\Pi}\otimes\mathbf{I}_{n_1}} & \\ & \mathbf{I}_{n_{m+1}}\end{bmatrix}$$

Symmetry permutes the cost and state matrices block-wise

<u>Symmetric Decomposition</u>

$$\begin{bmatrix}\mathbf{\Phi}_i^x & \\ & \mathbf{\Phi}_i^u\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^{\mathrm{T}} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\mathbf{\Phi}_i^x & \\ & \mathbf{\Phi}_i^u\end{bmatrix} = \begin{cases}\begin{bmatrix}\hat{\mathbf{Q}}_{ii} & \hat{\mathbf{S}}_{ii}\\ \hat{\mathbf{S}}_{ii}^T & \hat{\mathbf{R}}_{ii}\end{bmatrix} & \text{if } \ i = j\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } \ i \neq j\end{cases}$$

$$\begin{bmatrix}\mathbf{\Phi}_i^x & \\ & \mathbf{\Phi}_i^u\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} \begin{bmatrix}\mathbf{\Phi}_i^x & \\ & \mathbf{\Phi}_i^u\end{bmatrix} = \begin{cases}\begin{bmatrix}\hat{\mathbf{A}}_{ii} & \hat{\mathbf{B}}_{ii}\\ \hat{\mathbf{C}}_{ii}^T & \hat{\mathbf{D}}_{ii}\end{bmatrix} & \text{if } \ i = j\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } \ i \neq j\end{cases}$$

Transformation

$$\mathbf{\Phi}^{u,y,x} = \begin{bmatrix}\mathbf{\Phi}\otimes\mathbf{I}_{n_1} & \\ & \mathbf{I}_{n_{m+1}}\end{bmatrix}$$

$$\boxed{\mathbf{\Phi} = \begin{bmatrix}1 & 1 & 1 & 1 & \cdots & 1\\ -1 & 1 & 1 & 1 & \cdots & 1\\ & -2 & 1 & 1 & \cdots & 1\\ & & \ddots & \vdots & & \vdots\\ & & & \ddots & \ddots & \vdots\\ & & & & 1-m & 1\end{bmatrix}\mathbf{\Lambda}}$$

where $\mathbf{\Lambda} \in \mathbb{R}^{m\times m}$ is a diagonal matrix, with element $\lambda_{ii} = 1/\sqrt{i^2 + i}$ for $i = 1, 2, \ldots, m-1$ and

Existing Methods – Symmetric Decomposition

Example – Mass-Spring-Dashpot System

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & A_2 \\ 0 & A_1 & A_2 \\ A_2 & A_2 & A_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

$$\mathbf{x}_i = \begin{pmatrix} y_i \\ \dot{y}_i \end{pmatrix} \begin{matrix} = \text{position} \\ = \text{velocity} \end{matrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2K/M & -2B/M \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ K/M & B/M \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

It was observed that the system is reflective symmetric, i.e.

$$\Theta^u = \Theta^y = \begin{bmatrix} \Pi_2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

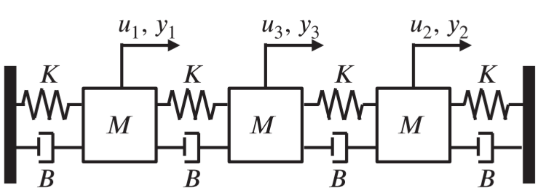
$$\mathbf{\Phi} \otimes \mathbf{I}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

The symmetric decomposition matrix is $\mathbf{\Phi} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow$

$$\mathbf{\Phi}^x = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ -\frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ & & \mathbf{I} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ -\frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ & & \mathbf{I} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_1 & & \mathbf{A}_2 \\ & \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ -\frac{1}{\sqrt{2}}\mathbf{I} & \frac{1}{\sqrt{2}}\mathbf{I} & \\ & & \mathbf{I} \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{A}_1 & & \\ \hline & \mathbf{A}_1 & \sqrt{2}\mathbf{A}_2 \\ & \sqrt{2}\mathbf{A}_2 & \mathbf{A}_1 \end{array}\right]$$



Existing Methods – Model Order Reduction (MOR – Aggregation)

Model Order Reduction (MOR) by Truncation [Obinata, 2001]

Assumption: The given system (A, B, C, D) are asymptotically stable, controllable and observable.

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \mathbf{u}(t) \qquad \longrightarrow \qquad \dot{\mathbf{x}}(t) = \mathbf{A}_{11}\mathbf{x}(t) + \mathbf{B}_1\mathbf{u}(t)$$

$$\mathbf{y}(t) = (\mathbf{C}_1 \quad \mathbf{C}_2)\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad\qquad \mathbf{y}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t)$$

Original System: $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Reduced System: $\mathbf{G}_\mathbf{r}(s) = \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1 + \mathbf{D}$

Error: $\mathbf{G}(s) - \mathbf{G}_\mathbf{r}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$

$$= (\mathbf{C}_1 \quad \mathbf{C}_2)\left(s\mathbf{I} - \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}\right)^{-1} \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} + \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

(Block Matrices Inverse)

$$= (\mathbf{C}_1 \quad \mathbf{C}_2)\left[\begin{pmatrix} (s\mathbf{I} - \mathbf{A}_{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12} \\ \mathbf{I} \end{pmatrix} \mathbf{S}^{-1}\left((s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12} \quad \mathbf{I}\right)\right]\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}$$

$$+ \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

where $\mathbf{S}^{-1} = (s\mathrm{I} - \mathrm{A}_{22} - \mathrm{A}_{21}(s\mathrm{I} - \mathrm{A}_{11})^{-1}\mathrm{A}_{12})^{-1}$ (Schur Complement)

[Ch.4 | P. 41]

where $= \tilde{\mathbf{C}}(s)\Delta^{-1}(s)\tilde{\mathbf{B}}(s)$

$\tilde{\mathbf{C}}(s) = \mathbf{C}_1\boldsymbol{\phi}(s)\mathbf{A}_{12} + \mathbf{C}_2$

$\Delta(s) = s\mathbf{I} - (\mathbf{A}_{22} - \mathbf{A}_{21}\boldsymbol{\phi}(s)\mathbf{A}_{12})^{-1}$

$\tilde{\mathbf{B}}(s) = \boldsymbol{\phi}(s)\mathbf{A}_{12}\mathbf{B}_1 + \mathbf{B}_2$ $\qquad \boldsymbol{\phi}(s) = (s\mathbf{I} - \mathbf{A}_{11})^{-1}$

1. Error is state coordinate dependent
2. $\mathrm{G}(jw) = \mathrm{G_r}(jw)|_{w\to\infty}$
3. $\mathrm{G}(jw) \neq \mathrm{G_r}(jw)|_{w\to 0}$

Existing Methods – Model Order Reduction (MOR – Aggregation)

Model Order Reduction (MOR) by Aggregation – [Aoki, 1968]

$$\underline{\dot{\mathbf{x}}} = (\mathbf{A})_{n\times n}\underline{\mathbf{x}}_{n\times 1} + (\mathbf{B})_{n\times r}\underline{\mathbf{u}}_{r\times 1}$$

Aggregated state vector: $\underline{\mathbf{z}}_{l\times 1} = (\mathbf{C})_{l\times n}\underline{\mathbf{x}}_{n\times 1}$

$$\mathbf{z} = \boxed{\mathbf{C}}\,\mathbf{x}$$

rank($\mathbf{C}$) = l

$$\mathbf{z} = \begin{pmatrix} \text{---}\ \mathbf{c_1}\ \text{---} \\ \text{---}\ \mathbf{c_2}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{c_l}\ \text{---} \end{pmatrix} \mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad \rightarrow \quad \mathbf{C}\dot{\mathbf{x}} = \mathbf{CAx} + \mathbf{CBu}$$

$$\mathbf{C}\dot{\mathbf{x}} = \mathbf{FCx} + \mathbf{Gu} \quad \rightarrow \quad \dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Gu}$$

$z_i = <\mathbf{c}_i, \mathbf{x}>$ **Projection of $\mathbf{c_i}$ on x**

Goal: Design an aggregation matrix C, such that

1. $\mathbf{C} \in \mathbb{R}^{l\times n}$ is a fat matrix with rank($\mathbf{C}$) = l.
2. $\mathbf{C}$ satisfies $\mathbf{CA} = \mathbf{FC}$ and $\mathbf{CB} = \mathbf{G}$.

$$\mathbf{FC} = \mathbf{CA} \quad \rightarrow \quad \mathbf{F}\left(\mathbf{CC}^{\mathrm{T}}\right) = \mathbf{CAC}^{\mathrm{T}}$$

$$\rightarrow \quad \mathbf{F} = \mathbf{CA}\boxed{\mathbf{C}^{\mathrm{T}}\left(\mathbf{CC}^{\mathrm{T}}\right)^{-1}}$$

Right Pseudo-Inverse $\mathbf{C}^{\dagger}$ [Ch. 3 | P.34]

$\mathbf{F}$ retains some characteristics of $\mathbf{A} = \mathbf{U\Lambda U}^{\mathrm{T}}$.

$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\mathbf{U} = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$

$\mathbf{FCv_i} = \mathbf{CAv_i} = \mathbf{C}(\lambda_i\mathbf{v_i}) = \lambda_i\mathbf{Cv_i}$ if $\mathbf{Cv_i} \neq 0$

3. Additional Requirement: a) at least one entry in each column
   b) $\mathbf{c}_i$ are orthogonal to each other

Hence, $\mathbf{Cv_i} \neq 0 \quad 1 \leq i \leq l;$
$\mathbf{Cv_i} = 0 \quad l+1 \leq i \leq n$

This method designs Aggregation Matrix **C** by eigen-decomposition of A to obtain v_i and require F to <u>inherit</u> the eigenvalue of A.

Consider $A = \begin{pmatrix} \mathbf{\Lambda_1} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{\Lambda_2} \end{pmatrix}$ such that $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n1})$, $\Lambda_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n2})$
and n1 + n2 = n

Define $r = \max|\lambda_i|$ and $R = \min|\mu_i|$. If $r/R << 1$, it is considered as weak coupled

Existing Methods – Model Order Reduction (MOR – Aggregation)

With the spectral representation of A [Ch.6 | P.19] $\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{w}_i^T$ $\quad \begin{matrix} \mathbf{C}\mathbf{v}_\mathrm{i} \neq 0 & 1 \leq i \leq l \\ \mathbf{C}\mathbf{v}_\mathrm{i} = 0 & l+1 \leq i \leq n \end{matrix}$

It is to separate the eigenvalue λ_i into two sets $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ and $\{\lambda_{l+1}, \lambda_{l+2}, \dots, \lambda_n\}$.

To estimate $\mathbf{x}(t)$ with $\mathbf{z}(t)$, system requirements are:

1. $|\lambda_i|$ must be large with Re $\lambda_i < 0$ (stability)
2. $< \mathbf{w}_\mathrm{i}, \mathbf{b}_\mathrm{i} >$ must be small for $l + 1 \leq i \leq n$. (steady state error).

The mentioned method only guarantee acceptable system performance.

For the optimal control part, recall the aggregation requirement

$$\begin{cases} \mathbf{FC} = \mathbf{CA} \\ \mathbf{G} = \mathbf{CB} \end{cases} \qquad \mathbf{F} = \mathbf{CAC}^\mathrm{T}\left(\mathbf{CC}^\mathrm{T}\right)^{-1}, \qquad \mathbf{CA} = \mathbf{FC} = \mathbf{CAC}^\mathrm{T}\left(\mathbf{CC}^\mathrm{T}\right)^{-1}\mathbf{C}$$

For Riccati Equation,

$$\mathbf{O} = \mathbf{A}^\mathrm{T}\mathbf{T}^* + \mathbf{T}^*\mathbf{A} - \mathbf{T}^*\mathbf{B}\boxed{\mathbf{R}^{-1}\mathbf{B}\mathbf{T}^*} + \mathbf{Q}$$ - Optimal Riccati T* with Full model

$$\mathbf{O} = \mathbf{F}^\mathrm{T}\mathbf{P} + \mathbf{PF} - \mathbf{PGR}^{-1}\mathbf{G}^\mathrm{T}\mathbf{P} + \mathbf{Q}_\mathbf{M}$$ - Optimal Riccati P with MOR model

$$\mathbf{O} = \mathbf{A}^\mathrm{T}\mathbf{C}^\mathrm{T}\mathbf{PC} + \mathbf{C}^\mathrm{T}\mathbf{PCA} - \mathbf{C}^\mathrm{T}\mathbf{PCB}\boxed{\mathbf{R}^{-1}\mathbf{B}^\mathrm{T}\mathbf{C}^\mathrm{T}\mathbf{PC}} + \mathbf{C}^\mathrm{T}\mathbf{Q}_\mathbf{M}\mathbf{C}$$

Hence, it induces $\quad \mathbf{C}^\mathrm{T}\mathbf{PC} \rightarrow \mathbf{T}^*$ and $\underbrace{\mathbf{C}^\mathrm{T}\mathbf{Q}_\mathbf{M}\mathbf{C}}_{\text{Rank } l} \rightarrow \underbrace{\mathbf{Q}}_{\text{Rank } n}$ - Sub-Optimal Solution for Full model

Optimal Cost Matrix: $\mathbf{Q}_\mathbf{M} = \left(\mathbf{CC}^\mathrm{T}\right)^{-1}\mathbf{C}\,\mathbf{Q}\,\mathbf{C}^\mathrm{T}\left(\mathbf{CC}^\mathrm{T}\right)^{-1}$

Conclusion

Methods for Optimal Control Problem Computation

Methods	Description	Advantages	Disadvantages
Symmetric Decomposition	1. Determine the (geometric or combinatoric) symmetric of matrix ($\mathbf{A, B, C, D}$) and ($\mathbf{Q, S, R, S}^{\mathrm{T}}$). 2. Use the given basis to perform diagonalization. ❖ Problem Specific to Symmetric Properties	• Simple to design if fulfilled symmetric properties	• Strict symmetric properties to fulfill • Require permutation to transform to required symmetric
Model Order Reduction (Truncation / Aggregation)	1. Find the aggregation matrix $\mathbf{C}$ to capture the "effective" eigenvalues of $\mathbf{A}$ and represent $\mathbf{z} = \mathbf{Cx}$. 2. Try to obtain the cost matrix $\mathbf{Q_M}$ for the sub-optimal problem. ❖ Suboptimal Solution but Quick Method	• Simple to design and quick to process in case some information is useless in cost optimization and system dynamics	• Sub-optimal
Simultaneous Block Diagonalization (SBD)	1. Determine the commutant matrix $\mathbf{A_i P} = \mathbf{P A_i}$ such that both all $\mathbf{A_i}$ and $\mathbf{P}$ are commutant. 2. Perform eigen-decomposition on $\mathbf{P}$ to obtain such random transformation $\mathbf{T}$ with $\mathbf{P} = \mathbf{T \Lambda T}^{-1}$ ❖ Assume there is a Random Matrix $\mathbf{T}$ to Block Diagonalize Given Sets of Matrices (e.g. $\mathbf{A, B, C, D, Q, R, F}$)	• Exact and Optimal (Preserves all information including the cost and control)	• Error prone (e.g. measurement error, truncation error)

Note:
Properties are generally valid for the computation algorithm. There are researches to fill in the gaps or further improve the performance.

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