Revision to State Space Control

State Space Model: $\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{cases}, \quad x(t_0) = x_0 \quad \rightarrow \quad \begin{pmatrix}\dot{\mathbf{x}} \\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ \mathbf{u}\end{pmatrix}$

Transfer Matrix:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Solution: State-Transition Matrix could be found by

1) Its definition –

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots$$

2) Inverse Laplace Transform of State Matrix Equation –

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$$

3) Eigenvalue Decomposition –

$$e^{\mathbf{A}t} = \mathbf{V}^{-1}e^{\mathbf{\Lambda}t}\mathbf{V}$$

where $\mathbf{V}$ is the eigenvector matrix, and $\mathbf{\Lambda}$ is the eigenvalue matrix.

The output can be represented by –

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\substack{\text{effect of} \\ \text{initial condition}}} + \underbrace{\int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau}_{\substack{\text{Aggregated effect of} \\ \text{input to state} \\ \mathbf{C}e^{\mathbf{A}t}\mathbf{B} * \mathbf{u}(t)}} + \underbrace{\mathbf{D}\mathbf{u}(t)}_{\substack{\text{directly acted to} \\ \text{input without} \\ \text{interacting to state}}}$$

Characteristics Equation:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

Stability holds when the solution of characteristics equation,

$$\det(s\mathbf{I} - \mathbf{A}) = 0 \rightarrow (s - p_1)(s - p_2) \ldots (s - p_n) = 0$$

which are the poles, are all at left-half plane (LHP).

Similarity Transformation: To generate special state model with nice algebraic or numerical properties –

Define a similarity transformation $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x} \rightarrow \mathbf{T}\tilde{\mathbf{x}} = \mathbf{x}$

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{cases} \rightarrow \begin{cases}\mathbf{T}\dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}\end{cases} \rightarrow \begin{cases}\dot{\tilde{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}\end{cases}$$

or in compact form, $\begin{pmatrix}\dot{\tilde{\mathbf{x}}} \\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{T}^{-1}\mathbf{A}\mathbf{T} & \mathbf{T}^{-1}\mathbf{B} \\ \mathbf{C}\mathbf{T} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}} \\ \mathbf{u}\end{pmatrix}$

Similarity transformation does not change controllability.

i.e. If (A, B) is controllable, (T A T^{-1}, T B) is controllable.

Controllability: (A, B) if for any initiate state $\mathbf{x}(0) = \mathbf{x}_0$ and $t_1 > 0$ and final state $\mathbf{x}_f$, there is a piecewise continuous input $\mathbf{u}(.)$ such that $\mathbf{x}(t_f) = \mathbf{x}_f$ if

1. The Controllability Gramian is positive definite.

$$\mathbf{W}_C(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^*e^{\mathbf{A}^*\tau}d\tau > 0, \quad \forall t \geq 0$$

2. The Controllability Matrix is in full rank.

$$C(\mathbf{A}, \mathbf{B}) = [\mathbf{B} \;\; \mathbf{A}\mathbf{B} \;\; \mathbf{A}^2\mathbf{B} \ldots \;\; \mathbf{A}^{n-1}\mathbf{B}]$$

$$\mathrm{rank}(C(\mathbf{A}, \mathbf{B})) = n$$

Pole Placement – the eigenvalue of $\mathbf{A} + \mathbf{B}\mathbf{F}$ (**F** as variable) can be freely assigned by a suitable choice of **F**.

Stability:

1. An unforced system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is stable if the eigenvalues of **A** are in the open LHP, i.e. $\mathrm{Re}\{\lambda(\mathbf{A})\} < 0$. Such matrix **A** is said to be stable, or Hurwitz.
2. The dynamical system (A, B) is stabilizable if there exists a stable feedback $\mathbf{u} = \mathbf{F}\mathbf{x}$ such that the system is stable, i.e. A + BF is stable.

(A, B) are stabilizable if

1. $[\mathbf{A} - \lambda\mathbf{I} \quad \mathbf{B}]$ has full rank for all $\mathrm{Re}\{\lambda\} \geq 0$
2. $\forall \lambda, \mathbf{x}$ such that $\mathbf{x}^*\mathbf{A} = \mathbf{x}^*\lambda$ and $\mathrm{Re}\{\lambda\} \geq 0, \mathbf{x}^*\mathbf{B} \neq \mathbf{0}$
3. There exists an F such that A+BF is Hurwitz.

Observability:

(C, A) is observable if $\forall t_1 > 0$ and $\mathbf{x}(0) = \mathbf{x}_0$, the output $y(t)$ can be determined at any time in [0, t_1], if

1. The Observability Gramian is positive definite.

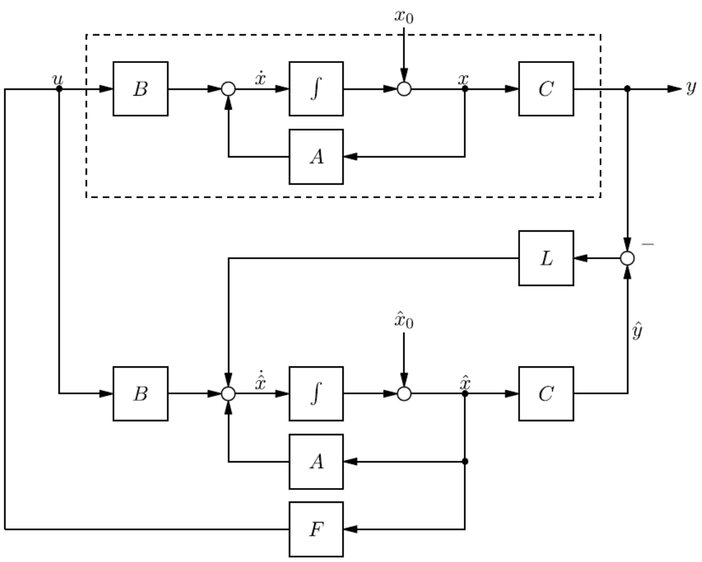
$$\mathbf{W}_O(t) = \int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{C}e^{\mathbf{A}\tau}d\tau > 0, \quad \forall t \geq 0$$

2. The Observability Matrix is in full rank.

$$O(\mathbf{C}, \mathbf{A}) = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{pmatrix}, \qquad \mathrm{rank}\big(O(\mathbf{C}, \mathbf{A})\big) = n$$

3. The matrix $(\mathbf{A} - \lambda\mathbf{I} \quad \mathbf{C})^{\mathrm{T}}$ is in full rank, $\forall \lambda \in \mathbb{C}$.
4. Let λ and y be the eigenvalue and any corresponding right eigenvector of **A**, i.e. $\mathbf{A}y = \lambda y$, then $\mathbf{C}y \neq 0$.
5. The eigenvalue of A+LC can be freely assigned by a suitable choice of L.
6. $(\mathbf{A}^*, \mathbf{C}^*)$ is controllable.

State Feedback:



Given a state-space model $\begin{cases}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{cases}$ with a simple controller $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

where **K** = (k_1 k_2 … k_n) is the feedback gain.

Close Loop Pole Placement: $\det\big(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})\big) = (s - p_1)(s - p_2) \dots (s - p_n)$

Observers and Observer-based Controllers:

Given a state space model $\begin{cases}\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du}\end{cases}$. In case some states are not detectable, observer is needed to provide any information for feedback.

An observer exists iff (C, A) is observable.

A Full-order Luenberger Observer is given by

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{L}(\underbrace{\mathbf{C}\hat{\mathbf{x}} + \mathbf{Du}}_{\substack{\text{estimated}\\ \text{output}}} - \mathbf{y})$$

where $\mathbf{L} = (l_1 \;\; l_2 \;\; \dots \; l_n)^T$ is any matrix that **A + LC** is stable.

Given state feedback gain **K** (**u = Kx**) and observer gain **L**.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} + \mathbf{Du} - \mathbf{y}) = (\mathbf{A} + \mathbf{BF} + \mathbf{LC})\hat{\mathbf{x}} - \mathbf{LCx}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{BF}\hat{\mathbf{x}}$$

In matrix form,

$$\begin{pmatrix}\dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{BF} \\ -\mathbf{LC} & \mathbf{A} + \mathbf{BF} + \mathbf{LC}\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ \hat{\mathbf{x}}\end{pmatrix}$$

Put $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$,

$$\begin{pmatrix}\mathbf{e} \\ \hat{\mathbf{x}}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ \hat{\mathbf{x}}\end{pmatrix}$$

$$\begin{pmatrix}\dot{\mathbf{e}} \\ \dot{\hat{\mathbf{x}}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} + \mathbf{LC} & \mathbf{0} \\ -\mathbf{LC} & \mathbf{A} + \mathbf{BF}\end{pmatrix}\begin{pmatrix}\mathbf{e} \\ \hat{\mathbf{x}}\end{pmatrix}$$

Given that block triangular matrix has a property that its eigenvalue is equal to the union of that of the diagonal, i.e.

$$\lambda\begin{pmatrix}\mathbf{A} + \mathbf{LC} & \mathbf{0} \\ -\mathbf{LC} & \mathbf{A} + \mathbf{BF}\end{pmatrix} = \underbrace{\lambda(\mathbf{A} + \mathbf{LC})}_{\substack{\text{Observer}\\ \text{Gain}}} \cup \underbrace{\lambda(\mathbf{A} + \mathbf{BF})}_{\substack{\text{State Feedback}\\ \text{Gain}}}$$

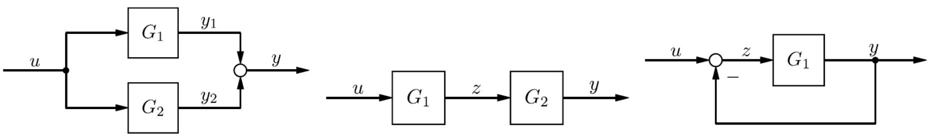
If (A, B) is controllable, and (C, A) is observable, there exists an F and L such that $\lambda_c(\mathbf{A} + \mathbf{BF})$ and $\lambda_o(\mathbf{A} + \mathbf{LC})$ can be arbitrarily placed.

Note – Observer should track (10 times) faster than the controller.

$$\begin{matrix}\dot{\hat{\mathbf{x}}} = (\mathbf{A} + \mathbf{LC})\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{LDu} - \mathbf{Ly} \\ \mathbf{u} = \mathbf{F}\hat{\mathbf{x}}\end{matrix} \rightarrow \begin{pmatrix}\dot{\hat{\mathbf{x}}} \\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}\mathbf{A} + \mathbf{BF} + \mathbf{LC} + \mathbf{LDF} & -\mathbf{L} \\ \mathbf{F} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}} \\ \mathbf{y}\end{pmatrix}$$

Hence, $\mathbf{u} = \underbrace{\mathbf{C_K}(s\mathbf{I} - \mathbf{A_K})^{-1}\mathbf{B_K}}_{\mathbf{K}}\, y = \mathbf{K}y$

Operation on Systems



(a) Parallel (b) Cascade (c) Negative Feedback

(a) Parallel System: $\mathbf{G_1} = \begin{pmatrix}\mathbf{A_1} & \mathbf{B_1} \\ \mathbf{C_1} & \mathbf{D_1}\end{pmatrix}$, $\mathbf{G_2} = \begin{pmatrix}\mathbf{A_2} & \mathbf{B_2} \\ \mathbf{C_2} & \mathbf{D_2}\end{pmatrix}$, $u = u_1 = u_2;\;\; y = y_1 + y_2$

$$\mathbf{G} = \mathbf{G_1} + \mathbf{G_2} = \begin{pmatrix}\mathbf{A_1} & & \mathbf{B_1} \\ & \mathbf{A_2} & \mathbf{B_2} \\ \mathbf{C_1} & \mathbf{C_2} & \mathbf{D_1} + \mathbf{D_2}\end{pmatrix}$$

(b) Cascade System: $\mathbf{G_1} = \begin{pmatrix} \mathbf{A_1} & \mathbf{B_1} \\ \mathbf{C_1} & \mathbf{D_1} \end{pmatrix}$, $\mathbf{G_2} = \begin{pmatrix} \mathbf{A_2} & \mathbf{B_2} \\ \mathbf{C_2} & \mathbf{D_2} \end{pmatrix}$,

$u = u_1;\ y = y_2;\ \ z = y_1 = u_2$

$$\mathbf{G} = \mathbf{G_1 G_2} = \begin{pmatrix} \mathbf{A_1} & & \mathbf{B_1} \\ \mathbf{B_2 C_1} & \mathbf{A_2} & \mathbf{B_2 D_1} \\ \mathbf{C_1 D_2} & \mathbf{C_2} & \mathbf{D_1 D_2} \end{pmatrix}$$

(c) Negative Feedback System: $\mathbf{G_1} = \begin{pmatrix} \mathbf{A_1} & \mathbf{B_1} \\ \mathbf{C_1} & \mathbf{D_1} \end{pmatrix}$, $u_1 = z = u - y_1;\ \ y = y_1$

$$\mathbf{G} = \begin{pmatrix} \mathbf{A_1} - \mathbf{B_1}(\mathbf{I} + \mathbf{D_1})^{-1}\mathbf{C_1} & \mathbf{B_1}(\mathbf{I} - (\mathbf{I} + \mathbf{D_1})^{-1})\mathbf{D_1} \\ (\mathbf{I} + \mathbf{D_1})^{-1}\mathbf{C_1} & (\mathbf{I} + \mathbf{D_1})^{-1}\mathbf{D_1} \end{pmatrix}$$

Note – it requires to check if $(\mathbf{I} + \mathbf{D_1})^{-1}$ is singular or ill-conditioned.

State Space Realization:

Controllable Canonical Form: $\begin{cases} \dot{\mathbf{x}} = \mathbf{A_c x} + \mathbf{B_c u} \\ \mathbf{y} = \mathbf{C_c x} + \mathbf{D_c u} \end{cases}$

$$\mathbf{A_c} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{pmatrix} \qquad \mathbf{B_c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{C_c} = (b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad b_1 - a_1 b_0) \qquad \boldsymbol{D} = \mathbf{0}$$

Observable Canonical Form: $\begin{cases} \dot{\mathbf{x}} = \mathbf{A_o x} + \mathbf{B_o u} \\ \mathbf{y} = \mathbf{C_o x} + \mathbf{D_o u} \end{cases}$

$$\mathbf{A_o} = \begin{pmatrix} -a_1 & 1 & & & \\ -a_2 & & 1 & & \\ \vdots & & & \ddots & \\ -a_{n-1} & & & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathbf{B_o} = \begin{pmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{pmatrix}$$

$$\mathbf{C_o} = (1 \quad 0 \quad 0 \quad \cdots \quad 0) \qquad \boldsymbol{D} = \mathbf{0}$$

Optimal Control and Linear Quadratic Control

Optimal Control

Plant: $\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{u}, t)$ nonlinear, time-varying system

$\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{R}^m$.

Performance Index:

$$J(\mathbf{x}, \mathbf{u}, t) = \int_0^T \underbrace{L(\mathbf{x}(t), \mathbf{u}(t), t)dt}_{\text{Travel Cost}} + \underbrace{V(\mathbf{x}(T), T)}_{\text{Destination Cost}}$$

where [t0, T] is the time of interest.

<u>Optimal Control Problem</u>

Find an optimal $u^*(t)$ within $[t_0, T]$ that drives the plant along the trajectory $x^*(t)$ such that the cost function is minimized, i.e. $\psi(x(T), T) = 0$ for a given $\psi \in \mathbb{R}^p$.

<u>Method</u>

Use Lagrange Multiplier to adjoint the constraint into the cost function.

$\lambda(t) \in \mathbb{R}^n$ for time function constraint, and $\nu \in \mathbb{R}^p$ for final value constraint.

The augmented performance index is

$$\tilde{J} = V(\mathbf{x}(T),T) + \mathbf{v}^T\boldsymbol{\psi}(\mathbf{x}(T),T) + \int_0^T \underbrace{L(x,u,t) + \boldsymbol{\lambda}^T(\mathbf{f}(x,u,t)}_{\triangleq \text{ Hamiltonian H(x,u,t)}} - \dot{\mathbf{x}})dt$$

With the definition of Hamiltonian $H(x,u,t) = L(x,u,t) + \boldsymbol{\lambda}^T\mathbf{f}(x,u,t)$,

$$\tilde{J} = V(\mathbf{x}(T),T) + \mathbf{v}^T\boldsymbol{\psi}(\mathbf{x}(T),T) + \int_0^T (H(x,u,t) - \dot{\mathbf{x}})dt$$

Linearize the cost function around the optimal function, i.e.

$$x(t) = x^*(t) + \delta x(t), \qquad u(t) = u^*(t) + \delta u(t)$$
$$\lambda(t) = \lambda^*(t) + \delta\lambda(t), \qquad v(t) = v^*(t) + \delta v(t)$$

The incremental cost is

$$\delta\tilde{J} = \tilde{J}(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}, \boldsymbol{\lambda}^* + \delta\boldsymbol{\lambda}, \mathbf{v}^* + \delta\mathbf{v}\,) - \tilde{J}(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \mathbf{v}^*)$$

$$\approx \int_0^T \frac{\partial H}{\partial \mathbf{x}}d\mathbf{x} + \frac{\partial H}{\partial \mathbf{u}}d\mathbf{u} - \boldsymbol{\lambda}^T\delta\dot{\mathbf{x}} + \left(\frac{\partial H}{\partial \boldsymbol{\lambda}} - \dot{\mathbf{x}}^T\right)\delta\boldsymbol{\lambda}\;\,dt + \frac{\partial V}{\partial \mathbf{x}}\delta\mathbf{x}(T) + \mathbf{v}^T\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}}\delta\mathbf{x}(T) + \delta\mathbf{v}^T\boldsymbol{\psi}(\mathbf{x}(T),T)$$
$$+ \mathrm{H.O.T.}$$

If .(t) is omitted and all derivatives are evaluated along the optimal solution, we can also eliminate the dependence on $\delta\dot{\mathbf{x}}$ by integration by parts.

$$-\int_0^T \boldsymbol{\lambda}^T\delta\dot{\mathbf{x}}\;dt = -\boldsymbol{\lambda}^T(T)\delta\mathbf{x}(T) + \boldsymbol{\lambda}^T(0)\cancel{\delta\mathbf{x}(0)} + \int_0^T \dot{\boldsymbol{\lambda}}^T\delta\mathbf{x}\;dt$$

Hence, to be optimal, we have $\delta\tilde{J} = 0$ for all $\delta x, \delta u, \delta\lambda, \delta v$.

Pontryagin's Maximum Principle:

If $(\mathbf{x}^*, \mathbf{u}^*)$ is optimal, there exists a $\lambda^*(t) \in \mathbb{R}^n$ and $v^* \in \mathbb{R}^p$ such that

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial \lambda_i} \\ -\lambda_i = \dfrac{\partial H}{\partial x_i} \\ \boldsymbol{\lambda}(T) = \dfrac{\partial V}{\partial \mathbf{x}}\big(\mathbf{x}(T)\big) + \mathbf{v}^T\dfrac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}} \end{cases}$$

in which

$$\frac{\partial H}{\partial \mathbf{u}} = 0$$

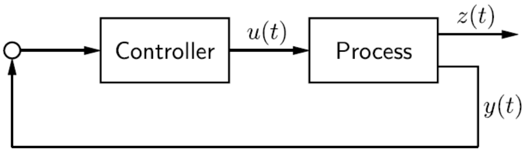
is the necessary condition for optimal input. Hence, we have

$$H(x^*, u^*, \lambda^*) \leq H(x^*, u, \lambda^*) \qquad \forall\, u \in \Omega$$

i.e. the optimal input results in optimal Hamiltonian should be smaller than that with any input.

Deterministic Linear Quadratic Regulation (LQR)

Given a state space system $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{z} = \mathbf{G}\mathbf{x} + \mathbf{H}\mathbf{u} \end{cases}$ with z(t) is the controlled output, and y(t) is the measured output for feedback.



LQR Problem – find the control input $u(t), t \in [0, \infty)$ that makes the following criterion as small as possible.

$$J_{LQR1} = \int_0^{\infty} \underbrace{\left|\left|z(t)\right|\right|^2}_{\substack{\text{energy of} \\ \text{controlled output}}} + \underbrace{\rho}_{\substack{\text{penalty} \\ \text{factor}}} \underbrace{\left|\left|u(t)\right|\right|^2}_{\substack{\text{energy of} \\ \text{controlled input}}} dt = \int_0^{\infty} \mathbf{z}^{\mathrm{T}}\mathbf{z} + \mathbf{u}^{\mathrm{T}}\mathbf{u}\, dt$$

A general cost function will be

$$J_{LQR} = \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\, dt$$

Feedback Invariant System:

Given a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^k$.

If H(x(t), u(t)) only depends on the initial value x(0) and NOT depends on the input. It is feedback invariant.

For any symmetric P, the functional

$$H\big(\mathbf{x}(t), \mathbf{u}(t)\big) = -\int_{-\infty}^{\infty} (\dot{x}^T P x + x^T P \dot{x}) dt$$

$$= -\int_{-\infty}^{\infty} \frac{d\mathbf{x}^T\mathbf{P}\mathbf{x}}{dt}\, dt = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) - \lim_{t\to\infty} \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t) = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$$

if $\lim_{t\to\infty} \mathbf{x}(t) = 0$, i.e. the system is stable.

A cost function J is minimized by an appropriate choice of input u(.) in the form of

$$J = \underbrace{H\big(\mathbf{x}(t), \mathbf{u}(t)\big)}_{\substack{\text{feedback} \\ \text{invariant}}} + \underbrace{\int_0^{\infty} L\big(\mathbf{x}(t), \mathbf{u}(t)\big) dt}_{\text{Property: } \min_{\mathbf{u}\in\mathbb{R}^k} L(\mathbf{x},\mathbf{u}) = 0}$$

The optimal control input

$$u^*(t) = \arg\min_{\mathbf{u}\in\mathbb{R}^k} L\big(\mathbf{x}(t), \mathbf{u}(t)\big)$$

Minimized the criterion J, and the optimal value J is the feedback invariant $J^* = H\big(\mathbf{x}(t), \mathbf{u}(t)\big)$

Optimal State Feedback:

$$J_{LQR} = \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\, dt$$

$$= -\int_{-\infty}^{\infty} \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{P}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}\, dt + \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{P}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}}\, dt$$

$$= \underbrace{H(\mathbf{x}, \mathbf{u})}_{\substack{\text{Feedback} \\ \text{Invariant}}} + \int_{-\infty}^{\infty} \Big(\mathbf{u} + \underbrace{\mathbf{R}^{-1}(\mathbf{P}\mathbf{B} + \mathbf{F})^{\mathrm{T}}}_{\mathbf{K}}\mathbf{x}\Big)^{\mathrm{T}} \mathbf{R} \Big(\mathbf{u} + \underbrace{\mathbf{R}^{-1}(\mathbf{P}\mathbf{B} + \mathbf{F})^{\mathrm{T}}}_{\mathbf{K}}\mathbf{x}\Big) + \mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}\, dt$$

where $\mathbf{M}$ is the product after completing square and

$$\mathbf{M} = \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - (\mathbf{P}\mathbf{B} + \mathbf{F})^{-1}\mathbf{R}(\mathbf{P}\mathbf{B} + \mathbf{F})$$

Hence, to obtain the minimum cost,

$$J_{LQR} = \underbrace{H(\mathbf{x}, \mathbf{u})}_{\substack{\text{Feedback} \\ \text{Invariant}}} + \int_{-\infty}^{\infty} \underbrace{(\mathbf{u} + \mathbf{K}\mathbf{x})^{\mathrm{T}}\mathbf{R}(\mathbf{u} + \mathbf{K}\mathbf{x})}_{L(\mathbf{x},\mathbf{u})}\, dt$$

1. P must be symmetric, and it satisfies the algebraic Riccati Equation:
$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - (\mathbf{P}\mathbf{B} + \mathbf{F})^{-1}\mathbf{R}(\mathbf{P}\mathbf{B} + \mathbf{F}) = \mathbf{0}$$
2. The optimal input will be
$$\mathbf{u} = -\mathbf{K}\mathbf{x};\ \mathbf{K} = -\mathbf{R}^{-1}(\mathbf{P}\mathbf{B} + \mathbf{F})^T \quad \text{such that} \quad \min L(\mathbf{x}, \mathbf{u}) = 0$$

Algebraic Riccati Equation:

Assume there exists a symmetric P such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - (\mathbf{P}\mathbf{B} + \mathbf{F})^{-1}\mathbf{R}(\mathbf{P}\mathbf{B} + \mathbf{F}) = \mathbf{0}$$

Then the feedback law $\mathbf{u} = -\mathbf{K}\mathbf{x};\ \mathbf{K} = -\mathbf{R}^{-1}(\mathbf{P}\mathbf{B} + \mathbf{F})^{T}$ minimize the LQR criterion and lead to

$$J_{LQR} = \int_{-\infty}^{\infty} \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + 2\mathbf{x}^{\mathrm{T}}\mathbf{F}\mathbf{u}\, dt = \mathbf{x}^{\mathrm{T}}(0)\mathbf{P}\mathbf{x}(0)$$

Optimal State Feedback with Reference Input

Assume there is an equilibrium point at $(\mathbf{x}_d, \mathbf{u}_d)$.

The optimal input should include the setpoint, i.e. $\mathbf{u} = \mathbf{u}_d - \mathbf{K}(\mathbf{x} - \mathbf{x}_d)$

To achieve the equilibrium point based on reference r, we need to relate r with $(\mathbf{x}_d, \mathbf{u}_d)$, i.e.

$$\begin{cases} \dot{\mathbf{x}} = 0 = \mathbf{A}\mathbf{x}_{\mathbf{d}} + \mathbf{B}\mathbf{u}_{\mathbf{d}} \\ \mathbf{y} = \mathbf{r} = \mathbf{C}\mathbf{x}_{\mathbf{d}} + \mathbf{D}\mathbf{u}_{\mathbf{d}} \end{cases}$$

With such linear equation, we could have

$$\begin{pmatrix} \mathbf{x}_{\mathbf{d}} \\ \mathbf{u}_{\mathbf{d}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathrm{N_x} \\ \mathrm{N_u} \end{pmatrix} \mathbf{r}$$

then

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{N}_{\mathbf{x}}\mathbf{r}) + \mathbf{N}_{\mathbf{u}}\mathbf{r} = -\mathbf{K}\mathbf{x} + \mathbf{N}\mathbf{r}$$

where $\mathbf{N} = \mathbf{N}_{\mathbf{u}} + \mathbf{K}\mathbf{N}_{\mathbf{x}}$ and hence $\mathbf{u} = \underbrace{-\mathbf{K}\mathbf{x}}_{\substack{\text{Feedback}\\\text{Gain}}} \underbrace{+\mathbf{N}\mathbf{r}}_{\substack{\text{Feedforward}\\\text{Gain}}}$.

The closed loop system will be $\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{N}\mathbf{r} \\ \mathbf{y} = (\mathbf{C} - \mathbf{D}\mathbf{K})\mathbf{x} + \mathbf{D}\mathbf{N}\mathbf{r} \end{cases}$

It is noted that the system is not robust as the system model could be erroneous and it creates large steady state error.

Integral Action for Optimal State Feedback with Reference Input

To reduce such steady state error, integral action will be used.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \dot{\mathbf{x}}_i = \mathbf{e} = \mathbf{r} - \mathbf{C}\mathbf{x} \end{cases} \rightarrow \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_{\mathbf{i}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}_{\mathbf{i}} \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix} \mathbf{u} + \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix} \mathbf{r}$$

Design a control law where r = 0 to get state feedback.

$$\mathbf{u} = -(\mathbf{K} \quad \mathbf{K}_{\mathbf{I}}) \begin{pmatrix} \mathbf{x} - \mathbf{x}_{\mathbf{d}} \\ \mathbf{x}_{\mathbf{i}} \end{pmatrix} + \mathbf{u}_{\mathbf{d}}$$

Optimal Regulator Problem

Regulator Problem:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_{\mathbf{0}}$$

$$V(\mathbf{x}(t_0), \mathbf{u}(.), t_0) = \int_{t_0}^{T} (\mathbf{u}^{\mathbf{T}}\mathbf{R}\mathbf{u} + \mathbf{x}^{\mathbf{T}}\mathbf{Q}\mathbf{x})dt + \mathbf{x}^{T}(T)\mathbf{M}\mathbf{x}(T)$$

given that $Q(t) \geq 0$ (positive semi-definite) and $R(t) > 0$ (positive definite and symmetric).

To solve such problem:

1. If $V^*(\mathbf{x}(t), t)$ exists, it must be in form of $\mathbf{x}^{T}(t)\mathbf{P}(t)\mathbf{x}(t)$ with $\mathbf{P}(t) \geq 0$.

   Note – if P(t) is not symmetric, one can replace it with $(\mathbf{P} + \mathbf{P}^{T})/2$

2. If $V^*(\mathbf{x}(t), t)$ exists, it must satisfy Riccati Differential Equation.

$$\frac{\partial V^*}{\partial t}(\mathbf{x}(t), t) = -\min_{\mathbf{u}(t)} \left(l(\mathbf{x}, \mathbf{u}, t) + \left[\frac{\partial V^*}{\partial \mathbf{x}}\right]^{T} \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right)$$

For a linear regulator problem, we have

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ V = \int_{t_0}^{T} (\mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{Q}\mathbf{x})dt + \mathbf{x}^T(T)\mathbf{M}\mathbf{x}(T) \end{cases}$$

Given that $V^* = \mathbf{x}^T(t)\mathbf{P}(t)\mathbf{x}(t)$, we have

$$\frac{\partial V^*}{\partial t} = 2\mathbf{x}^T(t)\mathbf{P}(t) \text{ and } \frac{\partial V^*}{\partial \mathbf{x}} = \mathbf{x}^T(t)\dot{\mathbf{P}}(t)\mathbf{x}(t)$$

Hence, the Hamilton-Jacobi-Bellman (HJB) equation could be simplified to

$$\mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} = -\min_{\mathbf{u}(t)} (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\mathbf{x}^T\mathbf{P}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}))$$

Completing the square,

$$\mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} = -\min_{\mathbf{u}(t)} ((\mathbf{u} + \underbrace{\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}}_{\triangleq -\bar{\mathbf{u}}})\mathbf{R}(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}) + \mathbf{x}^T(\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{x})$$

If $\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\dot{\mathbf{P}}$ is satisfied,

$$\mathbf{x}^T\dot{\mathbf{P}}\mathbf{x} = -\min_{\mathbf{u}(t)} \left((\mathbf{u} - \bar{\mathbf{u}})\mathbf{R}(\mathbf{u} - \bar{\mathbf{u}}) + \mathbf{x}^T\dot{\mathbf{P}}\mathbf{x}\right) \rightarrow \min_{\mathbf{u}(t)} (\mathbf{u} - \bar{\mathbf{u}})\mathbf{R}(\mathbf{u} - \bar{\mathbf{u}}) = 0$$

The equation holds for all x if the Matrix Riccati Equation is true.

Hence, the problem is satisfied with a symmetric P such that

$$\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\dot{\mathbf{P}}$$

and $\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t)$. linear, time varying feedback

A boundary condition for the Riccati Equation follows the boundary of the HJB Equation, i.e.

$$V^*(\mathbf{x}(T), T) = \mathbf{x}^T(T)\mathbf{P}(T)\mathbf{x}(T) = \mathbf{x}^T(T)\mathbf{S}\mathbf{x}(T) \rightarrow \mathbf{P}(T) = \mathbf{S}$$

Solution of Riccati Equation

To solve the Riccati Equation,

$$-\dot{\mathbf{P}} = \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P}, \quad \mathbf{P}(T) = \mathbf{S}$$

Define $\mathbf{P}(t) = \mathbf{Y}(t)\mathbf{X}^{-1}(t)$

Given the linear system $\begin{pmatrix} \dot{\mathbf{X}}(t) \\ \dot{\mathbf{Y}}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{pmatrix}}_{\mathbf{H}=\text{Hamiltonian Matrix}} \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}(T) \\ \mathbf{Y}(T) \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ \mathbf{S} \end{pmatrix}$

$$\frac{d\mathbf{P}}{dt} = \frac{d\mathbf{Y}\mathbf{X}^{-1}}{dt} = -\mathbf{Y}\mathbf{X}^{-1}\frac{d\mathbf{X}}{dt}\mathbf{X}^{-1} + \frac{d\mathbf{Y}}{dt}\mathbf{X}^{-1} = -\underbrace{\mathbf{Y}\mathbf{X}^{-1}}_{\mathbf{P}}\mathbf{A} + \mathbf{Y}\mathbf{X}^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{Y}\mathbf{X}^{-1} - \mathbf{Q} - \mathbf{A}^T\mathbf{Y}\mathbf{X}^{-1}$$

which is the same as the Riccati Equation, with P = YX^{-1}.

If Hamiltonian Matrix $\mathbf{H} = \begin{pmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{pmatrix}$ has no eigenvalue on the imaginary axis (or $\mathrm{Re}\{\lambda(\mathbf{H})\} \neq 0$), there exists a non-singular transformation U such that eigenvalue decomposition can be executed as

$$\mathbf{U}^{-1}\mathbf{H}\mathbf{U} = \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_s & \\ & \mathbf{\Lambda}_u \end{pmatrix}$$

where $\mathbf{\Lambda}$ and $\mathbf{U}$ are the eigenvalue and eigenvector matrix respectively.

Partition $\mathbf{U} = \begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix}$, $\quad \begin{pmatrix} \dot{\mathbf{X}} \\ \dot{\mathbf{Y}} \end{pmatrix} = \mathbf{H}\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \rightarrow \begin{pmatrix} \dot{\tilde{\mathbf{X}}} \\ \dot{\tilde{\mathbf{Y}}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_s & \\ & \mathbf{\Lambda}_u \end{pmatrix}\begin{pmatrix} \tilde{\mathbf{X}} \\ \tilde{\mathbf{Y}} \end{pmatrix}$

With $\begin{cases} \tilde{\mathbf{X}}(T) = e^{\mathbf{\Lambda}_s(T-t)}\tilde{\mathbf{X}}(t) \\ \tilde{\mathbf{Y}}(T) = e^{\mathbf{\Lambda}_u(T-t)}\tilde{\mathbf{Y}}(t) \end{cases}$, and $\begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix}\begin{pmatrix} \tilde{\mathbf{X}}(T) \\ \tilde{\mathbf{Y}}(T) \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ \mathbf{S} \end{pmatrix}$

$$\tilde{\mathbf{Y}}(T) = -(\mathbf{U}_{22} - \mathbf{S}\mathbf{U}_{12})^{-1}(\mathbf{U}_{21} - \mathbf{S}\mathbf{U}_{11})\tilde{\mathbf{X}}(T) = \mathbf{G}\tilde{\mathbf{X}}(T)$$

$$\begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{X}}(t) \\ \widetilde{\mathbf{Y}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix} \to \begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix} \begin{pmatrix} e^{-\boldsymbol{\Lambda}_s(T-t)}\widetilde{\mathbf{X}}(T) \\ e^{-\boldsymbol{\Lambda}_u(T-t)}\mathbf{G}\widetilde{\mathbf{X}}(T) \end{pmatrix} = \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix}$$

Hence,

$$\begin{cases} \mathbf{X}(t) = \mathbf{U}_{11}e^{-\boldsymbol{\Lambda}_s(T-t)}\widetilde{\mathbf{X}}(T) + \mathbf{U}_{12}e^{-\boldsymbol{\Lambda}_u(T-t)}\mathbf{G}\widetilde{\mathbf{X}}(T) \\ \mathbf{Y}(t) = \mathbf{U}_{21}e^{-\boldsymbol{\Lambda}_s(T-t)}\widetilde{\mathbf{X}}(T) + \mathbf{U}_{22}e^{-\boldsymbol{\Lambda}_u(T-t)}\mathbf{G}\widetilde{\mathbf{X}}(T) \end{cases}$$

$$\to \mathbf{P}(t) = \mathbf{Y}(t)\mathbf{X}(t)^{-1} = \left(\mathbf{U}_{21} + \mathbf{U}_{22}e^{-\boldsymbol{\Lambda}_u(T-t)}\mathbf{G}e^{\boldsymbol{\Lambda}_s(T-t)}\widetilde{\mathbf{X}}\right)\left(\mathbf{U}_{11} + \mathbf{U}_{12}e^{-\boldsymbol{\Lambda}_u(T-t)}\mathbf{G}e^{\boldsymbol{\Lambda}_s(T-t)}\right)^{-1}$$

Properties of Solution:

Existence:

P(t), as the solution of the algebraic Riccati Equation for the infinite time regulator problem exists, if the model **(A, B) is controllable**.

Controllability of (A, B)^C ensure the existence of **bounded** P to the algebraic Riccati Equation and of a state feedback gain to minimize the performance index.

Stability:

A bounded solution P guarantees stability if **all modes** are reflected in V.

Stability of the closed loop system is guaranteed if **(A, C) are observable**.

The system $\dot{\mathbf{x}} = \mathbf{Ax}$ is stable if there exists a Lyapunov Function $V = \mathbf{x}^T\mathbf{Px}$ such that $P > 0$ with $V > 0$ and $\dot{V} < 0$ and where $\dot{V} \equiv 0$ implies $\mathbf{x}(t) \equiv 0$.

1. The observability of (A, C) and P > 0 implies V > 0.
   Let $\bar{\mathbf{A}} = \mathbf{A} + \mathbf{BF}^* = \mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P}$.
   Assume $P \geq 0$, there exists a non-zero initial state $\mathbf{x_0} \neq \mathbf{0}$ such that
   $$V = \mathbf{x}_0^{\mathrm{T}}\mathbf{P}\mathbf{x_0} = \int_0^\infty (\mathbf{Cx})^{\mathrm{T}}(\mathbf{Cx}) + \mathbf{u}^{\mathrm{T}}\mathbf{Ru}\,\mathbf{dt} = 0$$
   It implies $\mathbf{Cx}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x_0} = 0$ for $t \in [0, \infty)$. But the observability of (A, C) also implies $\mathbf{x_0} = \mathbf{0}$. Hence, observability of (A, C) and P > 0 leads to a V > 0.

2. The observability of (A, C) implies $\dot{V} < 0$.
   $$\dot{V} = \frac{d\mathbf{x}^{\mathrm{T}}\mathbf{Px}}{dt} = \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{Px} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}} = [(\mathbf{A} + \mathbf{BF}^*)\mathbf{x}]^{\mathrm{T}}\mathbf{Px} + \mathbf{x}^{\mathrm{T}}\mathbf{P}[(\mathbf{A} + \mathbf{BF}^*)\mathbf{x}]$$
   $$= \mathbf{x}^{\mathrm{T}}(\bar{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{A}})\mathbf{x} = \mathbf{x}^{\mathrm{T}}(-(\mathbf{PB})\mathbf{R}^{-1}(\mathbf{PB})^{\mathrm{T}} - \mathbf{C}^{\mathrm{T}}\mathbf{C})\mathbf{x} \leq 0$$
   $\dot{V} \equiv 0$ is true if Cx(t) = 0, which by observability implies $x_0 = 0$ and x(t) = 0.

Closed Loop Eigenvalue:

$$\mathbf{T}^{-1}\mathbf{HT} = \widetilde{\mathbf{H}},\ \mathbf{T} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{I} \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} \mathbf{A} & -\mathbf{BR}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{pmatrix}$$

$$\widetilde{\mathbf{H}} = \mathbf{T}^{-1}\mathbf{HT} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & -\mathbf{BR}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{A}} & -\mathbf{BR}^{-1}\mathbf{B}^T \\ \mathbf{0} & -\mathbf{A}^T \end{pmatrix}$$

Hence, eigenvalue of H is the union of eigenvalue of $\bar{\mathbf{A}}$ and $-\bar{\mathbf{A}}$.

THE END