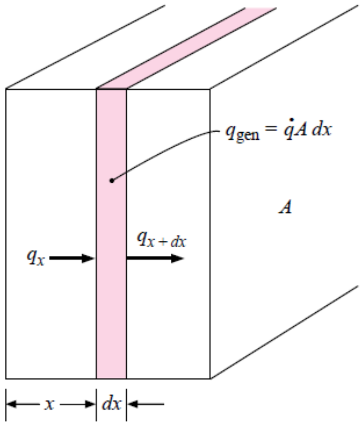
9. Introduction to Heat Transfer

Conduction: (1-1)



Energy in left face + Heat generated within element

= Change in internal energy + Energy out right face (1)

Energy out right face = $q_{x+dx} = -kA\left.\frac{\partial T}{\partial x}\right|_{x+dx}$

$$= -A\left(k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right) \quad (2)$$

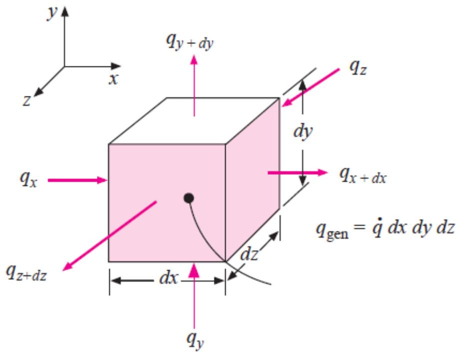
$$-kA\frac{\partial T}{\partial x} + \dot{q}Adx = \rho cA\frac{\partial T}{\partial \tau}dx - A\left(k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right) \quad (3)$$

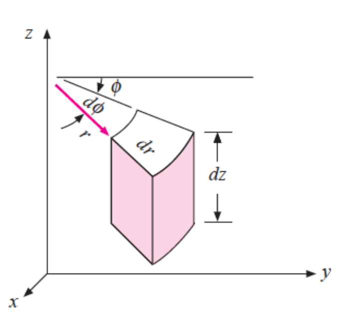
or $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho c\frac{\partial T}{\partial \tau} \quad (4)$$

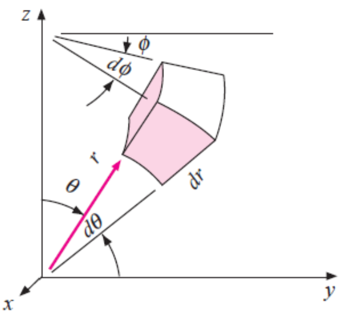
This is **1D heat-conduction equation**.

Similarly, we have: $\alpha = \frac{k}{\rho c}$ (thermal diffusivity) (5)

For constant thermal conductivity: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial^2 z} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$ (6)







Rectangular Coordinate: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial^2 z} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$ (6)

Cylindrical Coordinate: $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$ (7)

Spherical Coordinate: $\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$ (8)

Steady State 1D heat flow (with no heat generation):

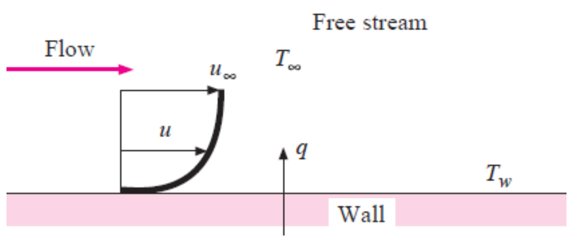
Rectangular Coordinate: $\frac{\mathrm{d}^2 T}{dx^2} = 0$ (9)

Cylindrical Coordinate: $\frac{\mathrm{d}^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0$ (10)

Steady State 1D heat flow with heat sources: $\frac{\mathrm{d}^2 T}{dx^2} + \frac{\dot{q}}{k} = 0$ (11)

Steady State 2D conduction with heat sources: $\frac{\mathrm{d}^2 T}{dx^2} + \frac{\mathrm{d}^2 T}{dy^2} = 0$ (12)

Convection: (1-3)

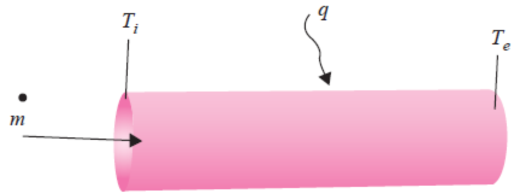


Newton's Law of Cooling: $q = hA(T_w - T_\infty)$ (13)

h = convection heat-transfer coefficient, T_w = temperature of plate, T_∞= temperature of fluid

Mode	h W/m^2 · °C	h Btu/h · ft^2 · °F
Across 2.5-cm air gap evacuated to a pressure of 10^{-6} atm and subjected to $\Delta T = 100°C - 30°C$	0.087	0.015
Free convection, $\Delta T = 30°C$		
Vertical plate 0.3 m [1 ft] high in air	4.5	0.79
Horizontal cylinder, 5-cm diameter, in air	6.5	1.14
Horizontal cylinder, 2-cm diameter, in water	890	157
Heat transfer across 1.5-cm vertical air gap with $\Delta T = 60°C$	2.64	0.46
Fine wire in air, $d = 0.02$ mm, $\Delta T = 55°C$	490	86
Forced convection		
Airflow at 2 m/s over 0.2-m square plate	12	2.1
Airflow at 35 m/s over 0.75-m square plate	75	13.2
Airflow at Mach number = 3, $p = 1/20$ atm, $T_\infty = -40°C$, across 0.2-m square plate	56	9.9
Air at 2 atm flowing in 2.5-cm-diameter tube at 10 m/s	65	11.4
Boiling water		
In a pool or container	2500–35,000	440–6200
Flowing in a tube	5000–100,000	880–17,600
Condensation of water vapor, 1 atm		
Vertical surfaces	4000–11,300	700–2000
Outside horizontal tubes	9500–25,000	1700–4400
Dropwise condensation	170,000–290,000	30,000–50,000

Convection Energy Balance in a Flow Channel:



Energy Balance: $q = \dot{m}(i_e - i_i)$ (i stands for enthalpy, to avoid confusion with h) (14)

$= \dot{m}c_p(T_e - T_i) = hA(T_{w,avg} - T_{fluid,avg})$ (15)

Mass Balance: $\dot{m} = \rho\bar{u}A_c$ (16)

for A_c = cross sectional area of tube, and A= surface area for convection.

Radiation: (1-4)

Stefan-Boltzmann Law of Thermal Radiation: $q_r = \sigma A T^4$ (17)

for $\sigma = 5.67 \times 10^{-8}\ W/m^2K^4$

The net radiant exchange of two blackbodies is $q_r = \sigma A(T_2^4 - T_1^4)$ (18)

Steady-State Conduction in 1D: (2)

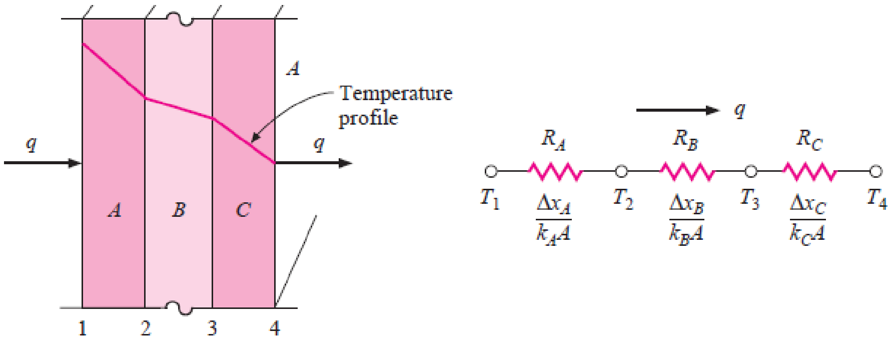
Plane Wall and Resistance Network:

Recall Fourier's Law of Conduction: $q_{cond} = -kA\frac{dT}{dx}$ (19)

With integration, assuming k= const, $q_{cond} = -\frac{kA}{\Delta x}(T_2 - T_1)$ (20)

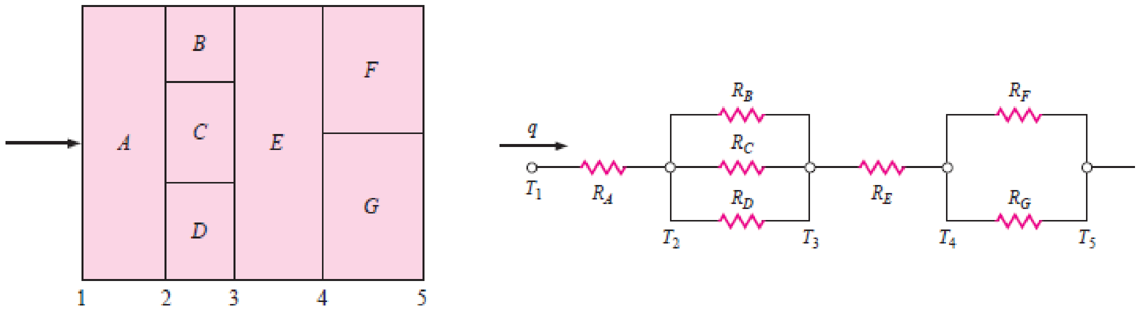
If $k = k_0(1 + \beta T)$, $q_{cond} = -\frac{k_0 A}{\Delta x}((T_2 - T_1) + \frac{\beta}{2}(T_2^2 - T_1^2)$ (21)

With KCL in multilayer materials, $q = -k_A A\frac{T_2 - T_1}{\Delta x_A} = -k_B A\frac{(T_3 - T_2)}{\Delta x_B} = -k_C A\frac{T_4 - T_3}{\Delta x_C}$ (22)

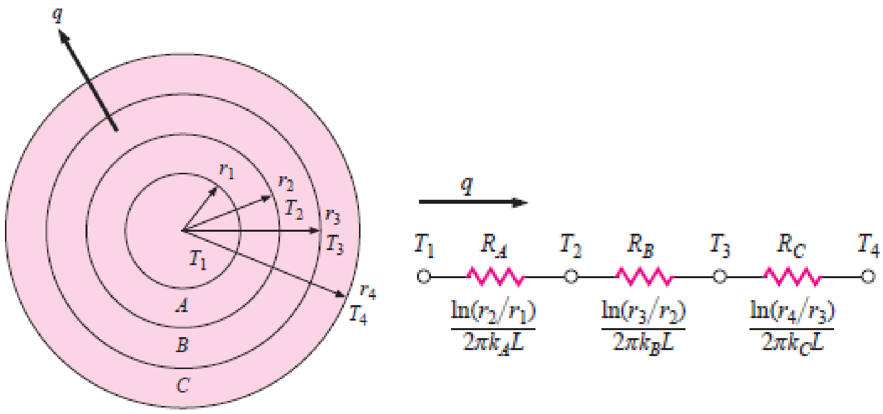


Solving (22): $$q = \frac{(T_1 - T_4)}{\frac{\Delta x_A}{k_A A} + \frac{\Delta x_B}{k_B A} + \frac{\Delta x_C}{k_C A}} \quad (23)$$

In which we can see that q~ current (I), ΔT~ voltage (V) and $\frac{l}{kA}$~ Resistance (R) and having a relationship of V = IR. Hence we can do the analysis with "thermal resistance" (R_{th}).



Cylindrical System:



For heat flow in cylindrical system, $$A_r = 2\pi r L \quad (24)$$

With Fourier Law: $$q_r = -kA_r \frac{dT}{dr} = -2\pi k r L \frac{dT}{dr} \quad (25)$$

With boundary Conditions $$T = T_i \quad \text{at } r = r_i$$

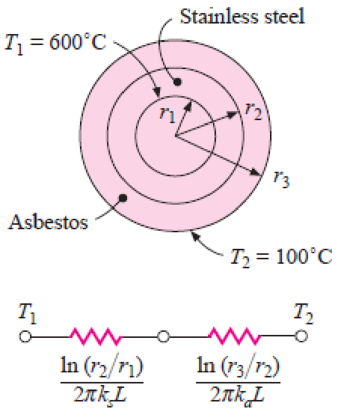
and $$T = T_o \quad \text{at } r = r_o \quad (26)$$

Solving, $$q_r = \frac{2\pi k L (T_i - T_o)}{\ln \frac{r_o}{r_i}} \quad (27)$$

Thermal Resistance in Cylindrical System: $$R_{th} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k L} \quad (28)$$

Similarly, for spherical system, $$q_s = \frac{4\pi k (T_i - T_o)}{\frac{1}{r_i} - \frac{1}{r_o}} \quad (29)$$

e.g.



A thick-walled tube of stainless steel [18% Cr, 8% Ni, $k = 19$ W/m · °C] with 2-cm inner diameter (ID) and 4-cm outer diameter (OD) is covered with a 3-cm layer of asbestos insulation [$k = 0.2$ W/m · °C]. If the inside wall temperature of the pipe is maintained at 600°C, calculate the heat loss per meter of length. Also calculate the tube–insulation interface temperature.

$$\frac{q}{L} = \frac{2\pi(T_1 - T_2)}{\ln(r_2/r_1)/k_s + \ln(r_3/r_2)/k_a} = \frac{2\pi(600 - 100)}{(\ln 2)/19 + (\ln \frac{5}{2})/0.2} = 680 \text{ W/m}$$

This heat flow may be used to calculate the interface temperature between the outside tube wall and the insulation. We have

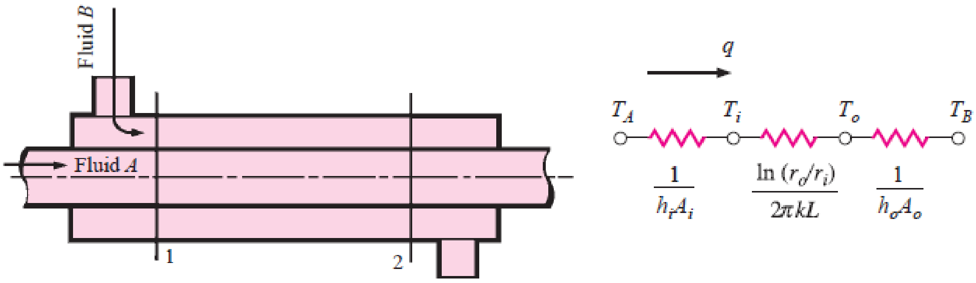
$$\frac{q}{L} = \frac{T_a - T_2}{\ln(r_3/r_2)/2\pi k_a} = 680 \text{ W/m}$$

where T_a is the interface temperature, which may be obtained as

$$T_a = 595.8°C$$

For convection, heat flow: $$q_{conv} = hA(T_w - T_\infty) \quad (13)$$

We can have: $$q_{conv} = \frac{T_w - T_\infty}{1/hA} \quad \text{with} \quad R_{th} = \frac{1}{hA} \quad (30)$$

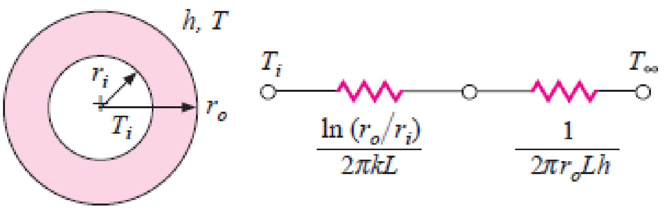


For simpler calculation, we define an overall heat-transfer coefficient U, such that

$$q = UA\Delta T_{overall} \quad (31)$$

And by observation, $$U = 1/\sum R \quad (32)$$

Critical Thickness of Insulation:



Consider a layer of insulation installed around a circular pipe with fixed inner temperature T_i and outer temperature T_∞.

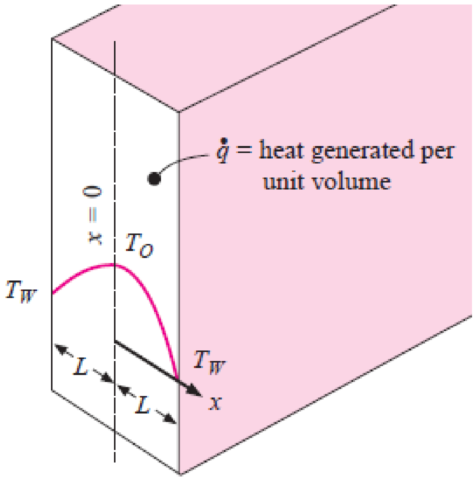
From the thermal network: $$q = \frac{2\pi L(T_i - T_\infty)}{\frac{\ln(r_o/r_i)}{k} + \frac{1}{r_o h}} \quad (33)$$

To determine the outer radius of insulation r_0, which will maximize the heat transfer, we do a differentiation: $$\frac{dq}{dr_o} = \frac{-2\pi L(T_i - T_\infty)\left(\frac{1}{kr_o} - \frac{1}{hr_o^2}\right)}{\left(\frac{\ln\left(\frac{r_o}{r_i}\right)}{k} + \frac{1}{r_o h}\right)^2} = 0 \quad (34)$$

which gives the result $$r_o = k/h \quad (35)$$

This is the critical radius of insulation concept. If $r_o < k/h$, heat transfer will be increased by adding more insulation. If $r_o > k/h$, heat transfer will be decreased by adding more insulation. It means that convective heat loss may actually increase with the addition of insulation with increased surface area.

Heat Source:



Consider a plane wall with uniformly distributed heat sources. The thickness of wall in x direction is 2L and it is assumed that all other dimension is large enough that the problem can be considered as 1D and k is const. The heat generated per unit volume is $\dot{q}$. The midplane temperature is T_o.

With the previous analysis,

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{11}$$

With boundary condition: $T = T_w$ at $x = \pm L$ (36)

The general solution is $T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$ (37)

To eliminate linear terms for symmetry, $C_1 = 0$ (38)

Hence we have $\frac{T-T_0}{T_w-T_0} = \left(\frac{x}{L}\right)^2$ (39)

At steady state condition total heat generated = heat lost at all faces.

$$2\left(-kA\frac{dT}{dx}\right)_{x=L} = \dot{q}A2L \tag{40}$$

Apply differentiation at (39): $\frac{dT}{dx}|_{x=L} = (T_w - T_o)\frac{2}{L}$ (41)

Substitute (41) back to (40): $T_0 = \frac{\dot{q}L}{2k} + T_w$ (42)

The equation for temp distribution can be: $\frac{T-T_w}{T_0-T_w} = 1 - \frac{x^2}{L^2}$ (43)

Consider a cylinder of radius R with uniformly distributed heat source, again we have

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{\dot{q}}{k} = 0 \tag{44}$$

Boundary Condition: $T = T_w$ at $r = R$ (45)

Heat generated = Heat lost at surface: $\dot{q}\pi R^2 L = -k2\pi RL\frac{dT}{dr}|_{r=R}$ (46)

Solving, we have $T - T_w = -\frac{\dot{q}}{4k}(R^2 - r^2)$ (47)

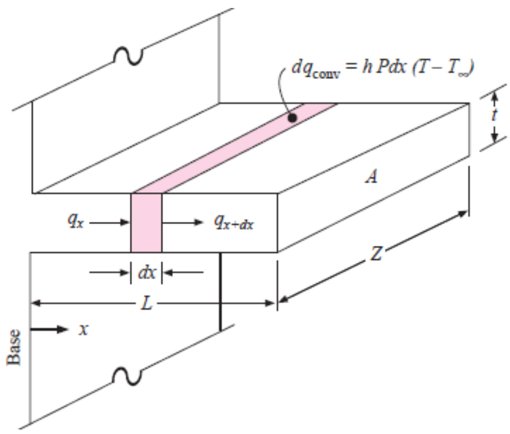
For a hollow cylinder with uniformly distributed heat sources,

Boundary Condition: $T = T_i$ at $r = r_i$ and $T = T_o$ at $r = r_o$ (48)

Solving, we have $T - T_o = \frac{\dot{q}}{4k}(r_o^2 - r^2) + C_1 \ln\frac{r}{r_o}$

Such that $C_1 = \frac{T_i - T_o + \frac{\dot{q}(r_i^2 - r_o^2)}{4k}}{\ln(r_i/r_o)}$ (49)

Conduction-Convection System: (Solid to Fluid)



Energy-in left face = Energy-out right face + energy lost by convection

Energy in left face = $q_x = -kA\frac{dT}{dx}$ (50)

Energy out right face = $q_{x+dx} = -kA\frac{dT}{dx}|_{x+dx}$

$$= -kA\left(\frac{dT}{dx} + \frac{d^2T}{dx^2}dx\right) \tag{51}$$

Energy lost by convection = $hPdx(T - T_\infty)$ (52)

(P = perimeter)

Energy Balance: $\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$ (53)

Let $\theta = T - T_\infty$: $\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0$ (54)

Boundary Conditions:

CASE 1 The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

CASE 2 The fin is of finite length and loses heat by convection from its end.

CASE 3 The end of the fin is insulated so that $dT/dx = 0$ at $x = L$.

Let $m^2 = hP/kA$: the general solution for (54) is $\theta = C_1 e^{-mx} + C_2 e^{mx}$ (55)

For Case 1: $\theta = \theta_0$ at $x = 0$ and $\theta = 0$ at $x = \infty$ (56)

Case 1 solution: $\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx}$ (57)

For Case 3: $\theta = \theta_0$ at $x = 0$ and $\frac{d\theta}{dx} = 0$ at $x = L$ (58)

Case 3 solution: $\frac{\theta}{\theta_0} = \frac{\cosh(m(L-x))}{\cosh mL}$ (59)

Case 2 solution: $\frac{\theta}{\theta_0} = \frac{\cosh m(L-x) + \left(\frac{h}{mk}\right)\sinh m(L-x)}{\cosh mL + \left(\frac{h}{mk}\right)\sinh mL}$ (60)

** As all the heat lost by the fin must be conducted into the base at x = 0, we have

$$q = -kA\frac{dT}{dx}\Big|_{x=0} \quad (61)$$

For the convective heat loss: $q = \int_0^L hP(T - T_\infty)dx = \int_0^L hP\theta dx$ (62)

Case 1: $q = \sqrt{hPkA}\,\theta_0$ (63)

Case 2: $q = \sqrt{hPkA}(T_0 - T_\infty)\frac{\left(\sinh mL + \left(\frac{h}{mk}\right)\cosh mL\right)}{\cosh mL + \left(\frac{h}{mk}\right)\sinh mL}$ (64)

Case 3: $q = \sqrt{hPkA}\,\theta_0 \tanh mL$ (65)

To indicate the effectiveness of a fin,
fin efficiency = actual heat transferred/ heat would be transferred if entire fin area were at base temp = η_f is introduced.

For Case 3: $\eta_f = \frac{\sqrt{hPkA}\,\theta_0 \tanh mL}{hPL\theta_0} = \frac{\tanh mL}{mL}$ (66)

** The fins discussed were assumed to be sufficiently long that heat flow can be analyzed as 1D.

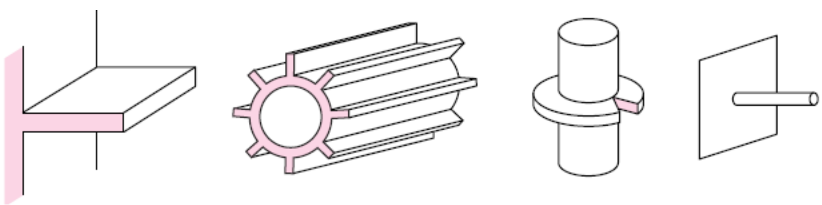
$$mL = \sqrt{\frac{hP}{kA}}L = \sqrt{\frac{h(2z+2t)}{kzt}}L \quad (67)$$

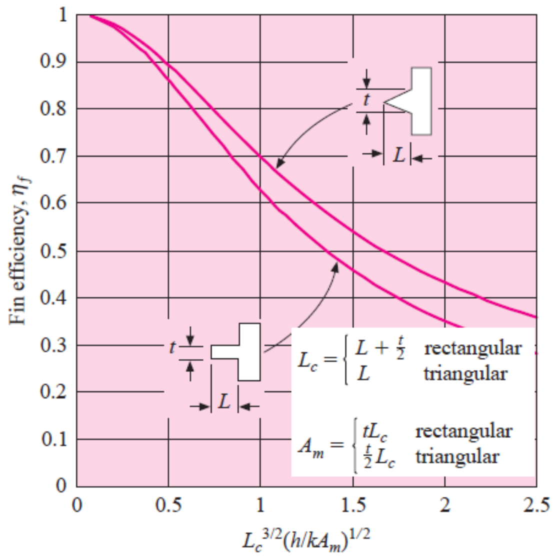
where *z* is the depth of the fin, and *t* is the thickness. If the fin is sufficiently large, 2*z* >> 2*t*.

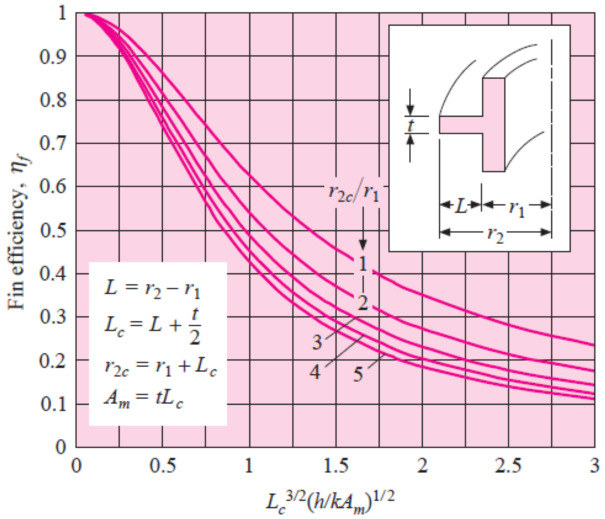
$$mL = \sqrt{\frac{2hz}{ktz}}L = \sqrt{\frac{2h}{kt}}L = \sqrt{\frac{2h}{kLt}}L^{\frac{3}{2}} = \sqrt{\frac{2h}{kA_m}}L^{\frac{3}{2}} \quad (68)$$

where A_m=Lt is the profile area of the fin.
We may therefore use the expression (68) to compute the efficiency of a fin with insulated tips as given by (66)

For Case 2, we can use L_c = L + t/2 in (66) for estimation.
For a straight cylindrical rod extending from a wall, we can use L_c = L+d/4 for estimation.







Another method for evaluating fin performance is to compare the heat transfer with fins to that without fins. i.e.

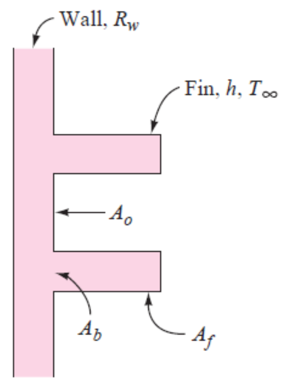
$$\frac{\text{q with fin}}{\text{q without fin}} = \frac{\eta_f A_f h\theta_0}{hA_b\theta_0} \tag{69}$$

where A_f is the total surfaced area of fin and A_b is the base area.

For the insulated-tip fins, $\quad A_f = PL \quad \text{and} \quad A_b = A \qquad (70)$

And the heat ratio:

$$\frac{\text{q with fin}}{\text{q without fin}} = \frac{\tanh mL}{\sqrt{\frac{hA}{kP}}} \tag{71}$$



Consider a fin attached to a wall. We may calculate the thermal resistance for the wall using either $R_W = \frac{l}{kA}$ or $R_W = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}$. In the absence of fins convection resistance will be 1/hA. The combined conduction-convection resistance R_f for the fins is related to the heat lost by the fin through

$$q_f = \eta_f A_f h\theta_o = \frac{\theta_o}{R_f} \tag{72}$$

The overall heat transfer through the fin-wall combination will then be

$$q_f = \frac{T_i - T_\infty}{R_{wf} + R_f} \tag{73}$$

where T_i is the inside wall temperature and R_{wf} is the wall resistance at the fin section.

This heat transfer is only for the fin portion of the wall. The open wall heat transfer is:

$$q_o = \frac{T_i - T_\infty}{R_{wo} + R_o} \tag{74}$$

where now $R_o = \frac{1}{hA_o}$ and $R_{wo} = \frac{\Delta x}{k_w A_o}$.

The total heat loss by the wall is therefore

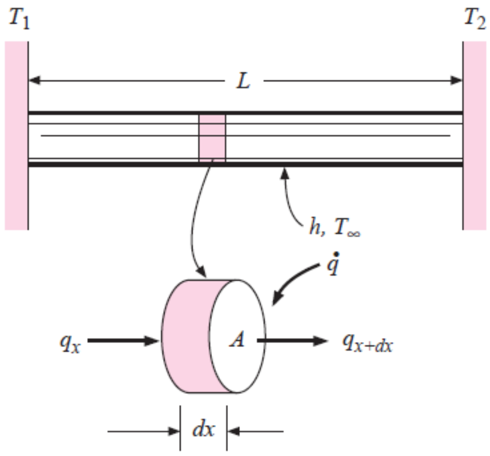
$$q_{total} = q_f + q_o \tag{75}$$

$$= (T_i - T_\infty)\left(\frac{1}{R_{wf} + R_f} + \frac{1}{R_{wo} + R_o}\right) \tag{76}$$

** Installation of fins on a heat transfer surface will not necessarily increase the heat transfer rate. If the value of h is large, as it is with high velocity fluids or boiling liquid, the fin may produce a reduction in heat transfer because the conduction resistance then represents a large impediment to the heat flow than the convection resistance.

e.g.

A rod containing uniform heat sources per unit volume $\dot{q}$ is connected to two temperatures as shown in Figure Example 2-11. The rod is also exposed to an environment with convection coefficient h and temperature T_∞. Obtain an expression for the temperature distribution in the rod.



$$-kA\frac{dT}{dx}+\dot{q}A\,dx=-kA\left(\frac{dT}{dx}+\frac{d^2T}{dx^2}dx\right)+hP\,dx\,(T-T_\infty)$$

Simplifying, we have

$$\frac{d^2T}{dx^2}-\frac{hP}{kA}(T-T_\infty)+\frac{\dot{q}}{k}=0 \qquad [a]$$

or, with $\theta=T-T_\infty$ and $m^2=hP/kA$

$$\frac{d^2\theta}{dx}-m^2\theta+\frac{\dot{q}}{k}=0 \qquad [b]$$

We can make a further variable substitution as

$$\theta'=\theta-\dot{q}/km^2$$

so that our differential equation becomes

$$\frac{d^2\theta'}{dx^2}-m^2\theta'=0 \qquad [c]$$

which has the general solution

$$\theta'=C_1e^{-mx}+C_2e^{mx} \qquad [d]$$

The two end temperatures are used to establish the boundary conditions:

$$\theta'=\theta_1'=T_1-T_\infty-\dot{q}/km^2=C_1+C_2$$
$$\theta'=\theta_2'=T_2-T_\infty-\dot{q}/km^2=C_1e^{-mL}+C_2e^{mL}$$

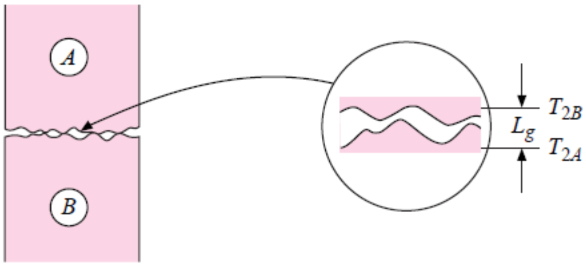
Solving for the constants C_1 and C_2 gives

$$\theta'=\frac{(\theta_1'e^{2mL}-\theta_2'e^{mL})e^{-mx}+(\theta_2'e^{mL}-\theta_1')e^{mx}}{e^{2mL}-1} \qquad [e]$$

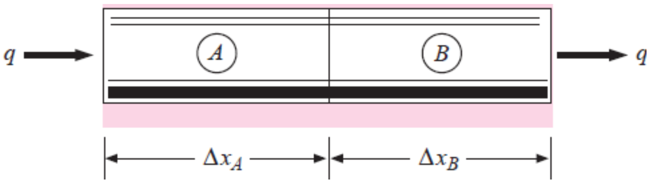
For an infinitely long heat-generating fin with the left end maintained at T_1, the temperature distribution becomes

$$\theta'/\theta_1'=e^{-mx} \qquad [f]$$

Thermal Contact Resistance: (Solid to Solid)



The actual surface roughness is believed to play a central role in determining the contact resistance. The conduction through the entrapped gases in the void spaces created by the contact is believed to represent the major resistance to heat flow.



We may write the heat flow across the joint as:

$$q = \frac{T_{2A}-T_{2B}}{\frac{L_g}{2k_A A_c}+\frac{L_g}{2k_B A_c}} + \frac{k_f A_v (T_{2A}-T_{2B})}{L_g} = \frac{T_{2A}-T_{2B}}{\frac{1}{h_c A}} \tag{77}$$

Solving h_c:

$$h_C = \frac{1}{L_g}\left(\frac{A_c}{A}\frac{2k_A k_B}{k_A+k_B} + \frac{A_v}{A}k_f\right) \tag{78}$$

where A is the total cross-sectional area of the bars, A_c is the contact area and A_v is the void area.

Exercise:

(1) Obtain an expression for the optimum thickness of a straight rectangular fin for a given profile area. Use the simplified insulated-tip solution. State the assumptions.

(2) Derive a differential equation for the temperature distribution in a straight triangular fin.

Steady-State Conduction in nD: (3)

2D Analysis, Solving PDE:

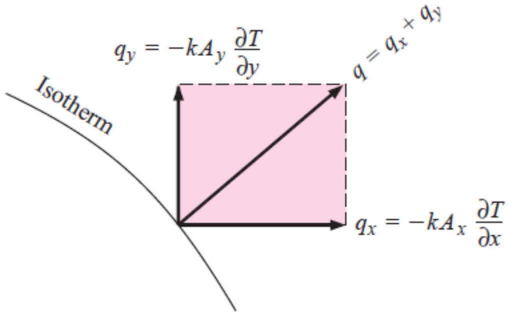
$$\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} = 0 \tag{12}$$

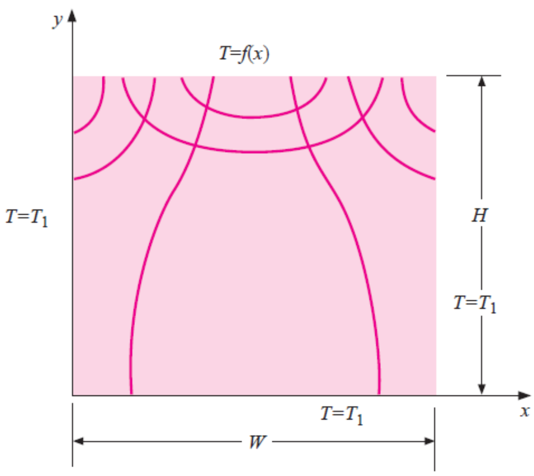
By separation of variable and the following boundary:

$$T = T_1 \text{ at } y = 0,\ x = 0,\ x = W \quad \text{and} \quad T = T_m \sin\left(\frac{\pi x}{W}\right) + T_1 \text{ at } y = H \tag{79}$$

The final solution can be expressed as:

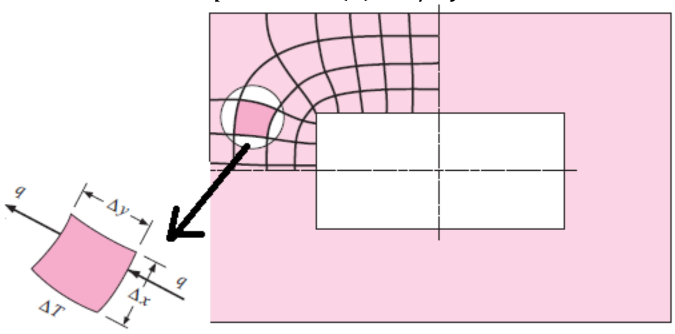
$$\frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \sin\frac{n\pi x}{W} \frac{\sinh\left(\frac{n\pi y}{W}\right)}{\sinh\left(\frac{n\pi H}{W}\right)} \tag{80}$$





Consider a 2D system with inner surface temperature T_1 and outer surface temperature T_2. Isotherms and heat flow lanes form groups of curvilinear figures like the figure below. The heat flow across this curvilinear section is given by Fourier's Law, assuming unit depth of material:

$$q = -k\Delta x(1)\,\Delta T/\Delta y \tag{81}$$



(b) (a)

The heat flow will be same through each section within this heat flow lane. If the sketch is drawn so that $\Delta x \cong \Delta y$, the heat flow is proportional to ΔT across the element. ΔT across each element must be the same within the same heat flow lane. Thus ΔT across an element is given by

$$\Delta T = \frac{\Delta T_{overall}}{N} \tag{82}$$

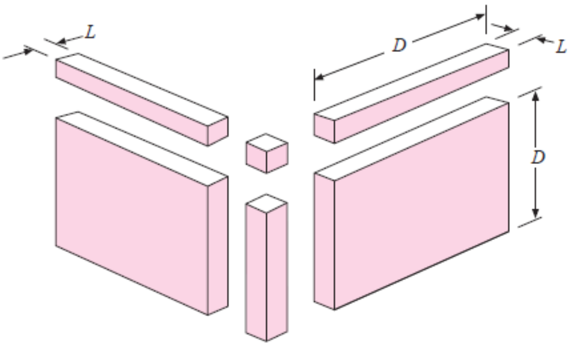
where N is the number of temperature increments between the inner and outer surfaces.

The total heat flow: $$q = \frac{M}{N}k\Delta T_{overall} = \frac{M}{N}k(T_2 - T_1) \tag{83}$$

Define a conduction shape factor S such that: $$q = kS\Delta T_{overall} \tag{84}$$

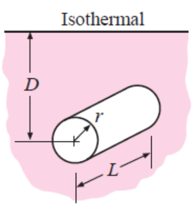
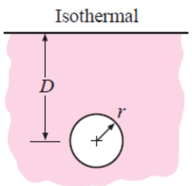
This conduction shape factor can be found as: $$S = M/N \tag{85}$$

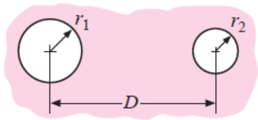
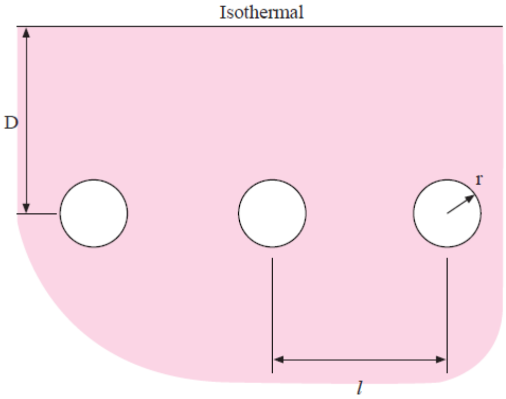
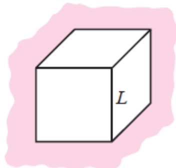
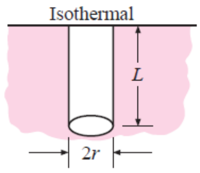
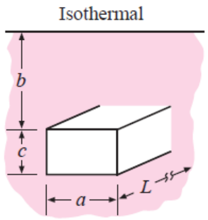
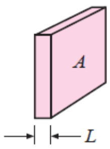
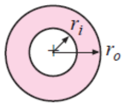
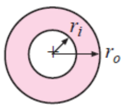
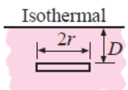
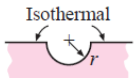
Given that $\cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1})$

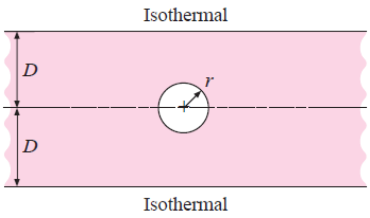
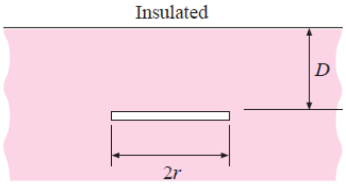


When all the interior dimensions are greater than 1/5 of wall thickness,

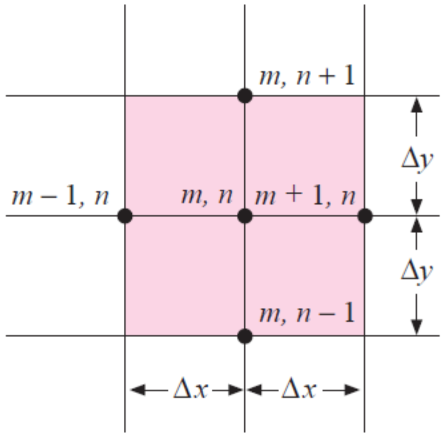
$$S_{\text{wall}} = \frac{A}{L}$$
$$S_{\text{edge}} = 0.54D$$
$$S_{corner} = 0.15L \tag{86}$$

Physical system	Schematic	Shape factor	Restrictions
Isothermal cylinder of radius r buried in semi-infinite medium having isothermal surface		$\frac{2\pi L}{\cosh^{-1}(D/r)}$	$L \gg r$
		$\frac{2\pi L}{\ln(D/r)}$	$L \gg r$ $D > 3r$
Isothermal sphere of radius r buried in semi-infinite medium having isothermal surface $\Delta T = T_{\text{surf}} - T_{\text{far field}}$		$\frac{4\pi r}{1 - r/2D}$	

Conduction between two isothermal cylinders of length L buried in infinite medium		$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2-r_1^2-r_2^2}{2r_1r_2}\right)}$	$L \gg r$ $L \gg D$
Row of horizontal cylinders of length L in semi-infinite medium with isothermal surface		$S = \dfrac{2\pi L}{\ln\left[\left(\frac{1}{\pi r}\right)\sinh(2\pi D/l)\right]}$	$D > 2r$
Buried cube in infinite medium, L on a side		$8.24L$	
Isothermal cylinder of radius r placed in semi-infinite medium as shown		$\dfrac{2\pi L}{\ln(2L/r)}$	$L \gg 2r$
Isothermal rectangular parallelepiped buried in semi-infinite medium having isothermal surface		$1.685L\left[\log\left(1+\frac{b}{a}\right)\right]^{-0.59}\left(\frac{b}{c}\right)^{-0.078}$	See Reference 7
Plane wall		$\dfrac{A}{L}$	One-dimensional heat flow
Hollow cylinder, length L		$\dfrac{2\pi L}{\ln(r_o/r_i)}$	$L \gg r$
Hollow sphere		$\dfrac{4\pi r_o r_i}{r_o - r_i}$	
Thin horizontal disk buried in semi-infinite medium with isothermal surface		$4r$ $8r$ $\dfrac{4\pi r}{\pi/2 - \tan^{-1}(r/2D)}$	$D = 0$ $D \gg 2r$ $D/2r > 1$ $\tan^{-1}(r/2D)$ in radians
Hemisphere buried in semi-infinite medium $\Delta T = T_{\text{sphere}} - T_{\text{far field}}$		$2\pi r$	

Horizontal cylinder of length L centered in infinite plate		$\dfrac{2\pi L}{\ln(4D/r)}$	
Thin horizontal disk buried in semi-infinite medium with adiabatic surface $\Delta T = T_{\text{disk}} - T_{\text{far field}}$		$\dfrac{4\pi r}{\pi/2 + \tan^{-1}(r/2D)}$	$D/2r > 1$ $\tan^{-1}(r/2D)$ in radians

Finite Element Method:



$$\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} \approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n-1/2} \approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Thus, the finite difference approximation for (12) becomes:

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} = 0 \qquad (87)$$

If $\Delta x = \Delta y$, then
$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \qquad (88)$$

For 2D heat transfer with heat generation:

$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0 \qquad (89)$$

If $\Delta x = \Delta y$, then
$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{\dot{q}}{k}\Delta x^2 - 4T_{m,n} = 0 \qquad (90)$$

Once the temperatures are determined, the heat flow may be calculated by:

$$q = \sum k\Delta x \frac{\Delta T}{\Delta y} \qquad (91)$$

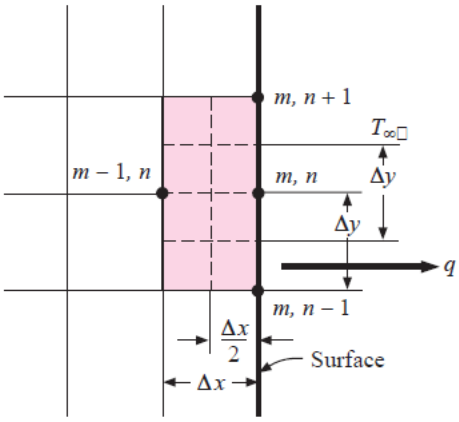
When the solid is exposed to some convection boundary condition, with the model next page, the energy balance on node (m,n) is

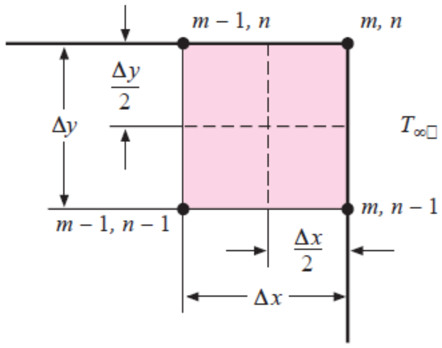
$$-k\Delta y \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h\Delta y(T_{m,n} - T_\infty) \qquad (92)$$

If $\Delta x = \Delta y$, then
$$T_{m,n}\left(\frac{h\Delta x}{k} + 2\right) - \frac{h\Delta x}{k}T_\infty - \frac{1}{2}\left(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) = 0 \qquad (93)$$

Energy Balance of corner section:

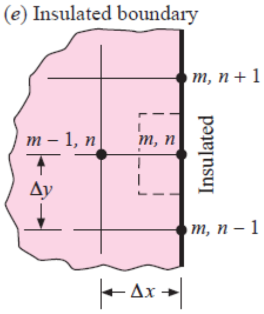
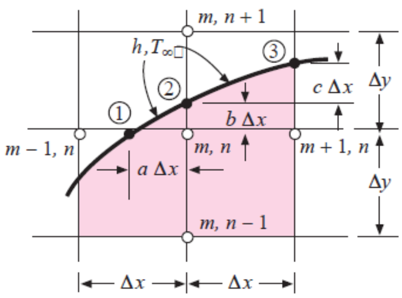
$$-k\Delta y \frac{T_{m,n} - T_{m-1,n}}{\Delta x} - k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y} = h\frac{\Delta x}{2}\left(T_{m,n} - T_\infty\right) + h\frac{\Delta y}{2}\left(T_{m,n} - T_\infty\right) \qquad (94)$$





If Δx = Δy, then $$2T_{m,n}\left(h\frac{\Delta x}{k}+1\right) - 2\frac{h\Delta x}{k}T_\infty - \left(T_{m-1,n}+T_{m,n-1}\right) = 0 \qquad (95)$$

,

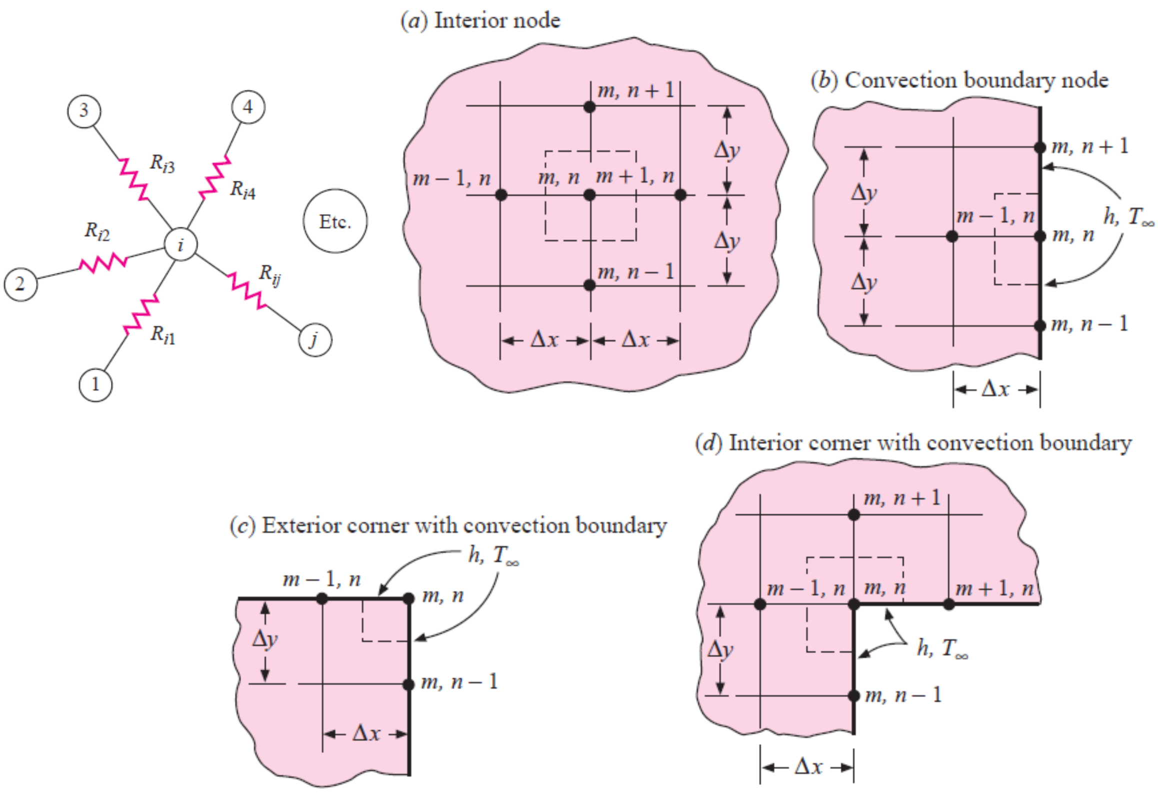
Physical situation	Nodal equation for equal increments in x and y (second equation in situation is in form for Gauss-Seidel iteration)
(e) Insulated boundary 	$0 = T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n}$ $T_{m,n} = (T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n})/4$
(f) Interior node near curved boundary‡ 	$0 = \frac{2}{b(b+1)}T_2 + \frac{2}{a+1}T_{m+1,n} + \frac{2}{b+1}T_{m,n-1} + \frac{2}{a(a+1)}T_1 - 2\left(\frac{1}{a}+\frac{1}{b}\right)T_{m,n}$
(g) Boundary node with convection along curved boundary—node 2 for (f) above§	$0 = \frac{b}{\sqrt{a^2+b^2}}T_1 + \frac{b}{\sqrt{c^2+1}}T_3 + \frac{a+1}{b}T_{m,n} + \frac{h\Delta x}{k}(\sqrt{c^2+1}+\sqrt{a^2+b^2})T_\infty$ $-\left[\frac{b}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{c^2+1}} + \frac{a+1}{b} + (\sqrt{c^2+1}+\sqrt{a^2+b^2})\frac{h\Delta x}{k}\right]T_2$

Numerical Formulation in Resistance Network and Gauss-Seidel Iteration

Another method to formulate equation for analysis is by using the resistance concept.

i.e. $$q_i + \sum_j \frac{T_j - T_i}{R_{ij}} = 0 \qquad (96)$$

for q_i = heat delivered to node i, the node of interest (node j, the adjoining node) and R_{ij} can take the form of convection boundaries, internal conduction, etc.



Physical situation	R_{m+}	R_{m-}	R_{n+}	R_{n-}	ΔV
(a) Interior node	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$(\Delta x)^2$
(b) Convection boundary	$\frac{1}{h\,\Delta x}$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{2}$
(c) Exterior corner, convection	$\frac{2}{h\,\Delta x}$	$\frac{2}{k}$	$\frac{2}{h\,\Delta x}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{4}$
(d) Interior corner, convection†	$\frac{2}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3(\Delta x)^2}{4}$
(e) Insulated boundary	∞	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{2}{k}$	$\frac{(\Delta x)^2}{2}$
(f) Interior node near curved boundary	$\frac{2}{(b+1)k}$ to node $(m+1, n)$	$\frac{2a}{(b+1)k}$ to node 1	$\frac{2b}{(a+1)k}$ to node 2	$\frac{2}{(a+1)k}$ to node $(m, n-1)$	$0.25(1+a)(1+b)(\Delta x)^2$
(g) Boundary node with curved boundary node 2 for (f) above	$R_{23} = \frac{2\sqrt{c^2+1}}{bk}$ $R_{21} = \frac{2\sqrt{a^2+b^2}}{bk}$ $R_{2-\infty} = \frac{2}{h\Delta x\left(\sqrt{c^2+1}+\sqrt{a^2+b^2}\right)}$ $R_{n-} = \frac{2b}{k(a+1)}$ to node (m, n)				$\Delta V = 0.125[(2+a)+c](\Delta x)^2$

† Also $R_\infty = 1/h\Delta x$ for convection to T_∞.

When the number of nodes is very large, Gauss-Seidel Iteration will be used/

From (96), we have $$T_i = \left(q_i + \sum_j \left(\frac{T_j}{R_{ij}}\right)\right) / \sum_j \left(\frac{1_j}{R_{ij}}\right) \qquad (97)$$

Procedures:

1. An initial set of values for T_i is assumed. For computers, T_i are usually assigned 0.
2. New value of the nodal temperature T_i are calculated according to (97), always using the most recent values of T_j.

3. The process is repeated until successive calculations differ by small amount such that

$$\left|T_{i_{n+1}} - T_{i_n}\right| \leq \delta \;\text{ and }\; \epsilon \geq \left|\frac{T_{i_{n+1}} - T_{i_n}}{T_{i_n}}\right| \qquad (98)$$

	Cartesian	Cylindrical	Spherical
Nomenclature for increments	x, m	r, m	r, m
	y, n	ϕ, n	ϕ, n
	z, k	z, k	θ, k
Volume element ΔV	$\Delta x\,\Delta y\,\Delta z$	$r_m\,\Delta r\,\Delta\phi\,\Delta z$	$r_m^2\,\sin\theta\,\Delta r\,\Delta\phi\,\Delta\theta$
R_{m+}	$\frac{\Delta x}{\Delta y\,\Delta z\,k}$	$\frac{\Delta r}{(r_m+\Delta r/2)\,\Delta\phi\,\Delta z\,k}$	$\frac{\Delta r}{(r_m+\Delta r/2)^2\,\sin\theta\,\Delta\phi\,\Delta\theta\,k}$
R_{m-}	$\frac{\Delta x}{\Delta y\,\Delta z\,k}$	$\frac{\Delta r}{(r_m-\Delta r/2)\,\Delta\phi\,\Delta z\,k}$	$\frac{\Delta r}{(r_m-\Delta r/2)^2\,\sin\theta\,\Delta\phi\,\Delta\theta\,k}$
R_{n+}	$\frac{\Delta y}{\Delta x\,\Delta z\,k}$	$\frac{r_m\,\Delta\phi}{\Delta r\,\Delta z\,k}$	$\frac{\Delta\phi\,\sin\theta}{\Delta r\,\Delta\theta\,k}$
R_{n-}	$\frac{\Delta y}{\Delta x\,\Delta z\,k}$	$\frac{r_m\,\Delta\phi}{\Delta r\,\Delta z\,k}$	$\frac{\Delta\phi\,\sin\theta}{\Delta r\,\Delta\theta\,k}$
R_{k+}	$\frac{\Delta z}{\Delta x\,\Delta y\,k}$	$\frac{\Delta z}{r_m\,\Delta\phi\,\Delta r\,k}$	$\frac{\Delta\theta}{\sin(\theta+\Delta\theta/2)\,\Delta r\,\Delta\phi\,k}$
R_{k-}	$\frac{\Delta z}{\Delta x\,\Delta y\,k}$	$\frac{\Delta z}{r_m\,\Delta\phi\,\Delta r\,k}$	$\frac{\Delta\theta}{\sin(\theta-\Delta\theta/2)\,\Delta r\,\Delta\phi\,k}$

** For nodes with $\Delta x = \Delta y$ and no heat generation, equation for convection boundaries may be converted to insulated boundaries by simply setting Bi = 0 in the respective formula.

Biot Number: $$\mathrm{Bi} = \frac{h\Delta x}{k} \qquad (99)$$

** For heat source with $\Delta x = \Delta y$, one need only to add a term q_i/k to the numerator of each of the equations, For heat source $q_i = \dot{q}\Delta V$. For interior node $\Delta V = \Delta x\Delta y$, a plane convection boundary $\Delta V = \frac{\Delta x}{2}\Delta y$, and an exterior corner $\Delta V = \frac{\Delta x}{2}\frac{\Delta y}{2}$

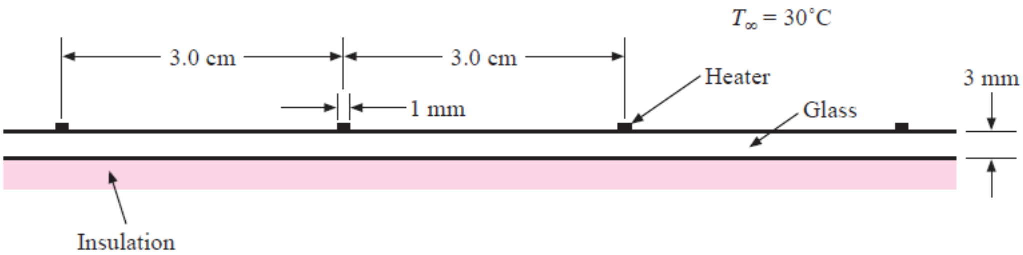
For radiation exchange at boundary node, $q_i = q_{rad,i}'' \times \Delta A$ (100)

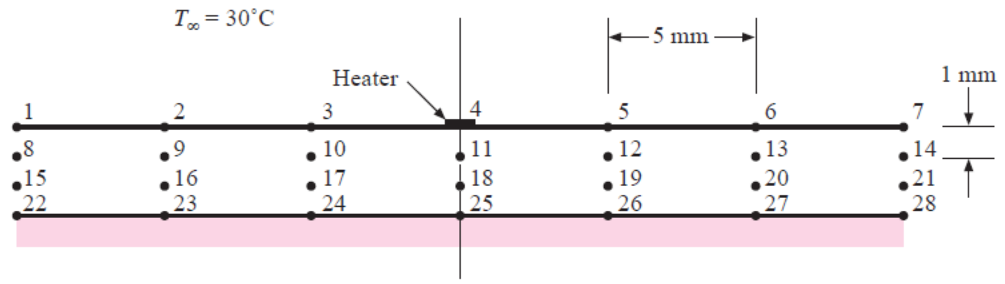
where ΔA is the surface area of the node exposed to radiation and $q_{rad,i}''$ is the net radiation transferred to node *i* per unit area as determined later.

For common case of a surface exposed to a large enclosure at radiation temperature of T_r, the net radiation to the surface per unit area is given by $q''_{rad,i} = \sigma\varepsilon_i(T_r^4 - T_i^4)$ (101)

e.g.

A layer of glass [$k = 0.8$ W/m · °C] 3 mm thick has thin 1-mm electric conducting strips attached to the upper surface, as shown in Figure Example 3-8. The bottom surface of the glass is insulated, and the top surface is exposed to a convection environment at 30°C with $h = 100$ W/m^2 · °C. The strips generate heat at the rate of 40 or 20 W per meter of length. Determine the steady-state temperature distribution in a typical glass section, using the numerical method for both heat-generation rates.





The nodal network for a typical section of the glass is shown in the figure. In this example we have *not* chosen $\Delta x = \Delta y$. Because of symmetry, $T_1 = T_7$, $T_2 = T_6$, etc., and we only need to solve for the temperatures of 16 nodes. We employ the resistance formulation. As shown, we have chosen $\Delta x = 5$ mm and $\Delta y = 1$ mm. The various resistances may now be calculated:

Nodes 1, 2, 3, 4:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k(\Delta y/2)}{\Delta x} = \frac{(0.8)(0.001/2)}{0.005} = 0.08$$

$$\frac{1}{R_{n+}} = hA = (100)(0.005) = 0.5$$

$$\frac{1}{R_{n-}} = \frac{k\Delta x}{\Delta y} = \frac{(0.8)(0.005)}{0.001} = 4.0$$

Nodes 8, 9, 10, 11, 15, 16, 17, 18:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k\Delta y}{\Delta x} = \frac{(0.8)(0.001)}{0.005} = 0.16$$

$$\frac{1}{R_{n+}} = \frac{1}{R_{n-}} = \frac{k\Delta x}{\Delta y} = 4.0$$

Nodes 22, 23, 24, 25:

$$\frac{1}{R_{m+}} = \frac{1}{R_{m-}} = \frac{k(\Delta y/2)}{\Delta x} = 0.08$$

$$\frac{1}{R_{n+}} = \frac{k\Delta x}{\Delta y} = 4.0$$

$$\frac{1}{R_{n-}} = 0 \qquad \text{(insulated surface)}$$

The nodal equations are obtained from Equation (3-31) in the general form

$$\sum(T_j/R_{ij}) + q_i - T_i\sum(1/R_{ij}) = 0$$

Only node 4 has a heat-generation term, and $q_i = 0$ for all other nodes. From the above resistances we may calculate the $\sum(1/R_{ij})$ as

Node	$\sum(1/R_{ij})$
1, 2, 3, 4	4.66
8, . . . , 18	8.32
22, 23, 24, 25	4.16

For node 4 the equation is

$$(2)(0.08)T_3 + 4.0T_5 + (0.5)(30) + q_4 - 4.66T_4 = 0$$

The factor of 2 on T_3 occurs because $T_3 = T_5$ from symmetry. When all equations are evaluated and the solution obtained, the following temperatures result:

Node temperature, °C	q/L, W/m	
	20	40
1	31.90309	33.80617
2	32.78716	35.57433
3	36.35496	42.70993
4	49.81266	69.62532
8	32.10561	34.21122
9	33.08189	36.16377
10	36.95154	43.90307
11	47.82755	65.65510
15	32.23003	34.46006
16	33.26087	36.52174
17	37.26785	44.53571
18	46.71252	63.42504
22	32.27198	34.54397
23	33.32081	36.64162
24	37.36667	44.73333
25	46.35306	62.70613

The results of the model and calculations may be checked by calculating the convection heat lost by the top surface. Because all the energy generated in the small heater strip must eventually be lost by convection (the bottom surface of the glass is insulated and thus loses no heat), we know the numerical value that the convection should have. The convection loss at the top surface is given by

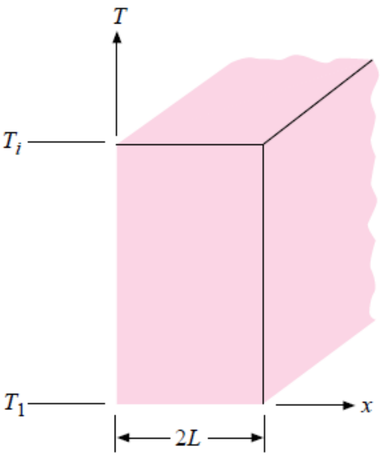
$$q_c = \sum h_i A_i (T_i - T_\infty)$$
$$= (2)(100)\left[\frac{\Delta x}{2}(T_1 - T_\infty) + \Delta x(T_2 + T_3 - 2T_\infty) + \frac{\Delta x}{2}(T_4 - T_\infty)\right]$$

The factor of 2 accounts for both sides of the section. With $T_\infty = 30°\mathrm{C}$ this calculation yields

$$q_c = 19.999995 \qquad \text{for } q/L = 20 \text{ W/m}$$
$$q_c = 40.000005 \qquad \text{for } q/L = 40 \text{ W/m}$$

Obviously, the agreement is excellent.

Unsteady-State Conduction: (4)



Consider the infinite plate of thickness 2L shown. Initially the plate is at a uniform temperature T_i and at time zero the surfaces are suddenly lowered to $T = T_1$. The differential equation is:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad (102)$$

Introducing the variable $\theta = T - T_1$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial \tau} \qquad (103)$$

Initial and Boundary Condition:

$$\theta = \theta_i = T_i - T_1 \quad \text{at } \tau = 0. \; 0 \le x \le 2L$$
$$\theta = 0 \quad \text{at } x = 0 \text{ and } x = 2L, \tau > 0$$
$$(104)$$

The final series form of the solution is therefore

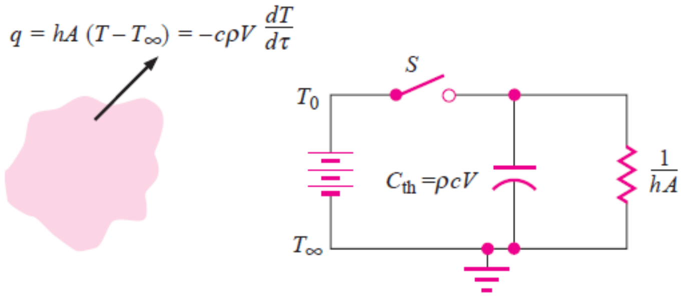
$$\theta = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{2L}\right)^2 \alpha\tau} \sin\frac{n\pi x}{2L}$$

where

$$C_n = \frac{1}{L}\int_0^{2L} \theta_i \sin\frac{n\pi x}{2L} dx = \frac{4}{n\pi}\theta_i \quad n=1,3,5,\ldots \tag{105}$$

$$\frac{\theta}{\theta_i} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{2L}\right)^2 \alpha\tau} \sin\frac{n\pi x}{2L} \quad n=1,3,5,\ldots \tag{106}$$

Lumped-Heat-Capacity System:



The convection heat loss = decrease in the internal energy of the body, i.e.

$$q = hA(T - T_\infty) = -c\rho V\frac{dT}{d\tau} \tag{107}$$

The initial condition is $T = T_0$ at $\tau = 0$ (108)

So that the solution for (107) is:

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-\left(\frac{hA}{\rho cV}\right)\tau} \tag{109}$$

In this network we notice that the thermal capacity of the system is charged initially at potential T_0 by closing the switch S. Then when the switch is opened, the energy stored in the thermal capacitance is dissipated through the resistance 1/hA. The analogy between this thermal system and RC network can be related with

$$\frac{hA}{\rho cV} = \frac{1}{R_{th}C_{th}} \qquad R_{th} = \frac{1}{hA} \quad \text{and} \quad C_{th} = \rho cV \tag{110}$$

with time constant

$$\tau = \frac{\rho cV}{hA} \tag{111}$$

At time $t = \tau$, $T - T_\infty = 36.8\% \, (T_0 - T_\infty)$.

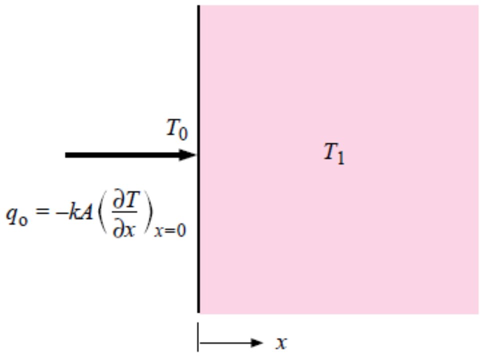
This lumped-capacity analysis assumes a uniform temperature distribution throughout the solid body, or similarly the surface convection resistance is large compared with the internal-conduction resistance. Such analysis may be expected to yield reasonable estimation within 5% when

$$\frac{h(V/A)}{k} < 0.1 \tag{112}$$

Notice that for V/A = s, a characteristic dimension of solid, and Bi = hs/k.

Physical situation	k, W/m · °C	Approximate value of h, W/m^2 · °C	$\frac{h(V/A)}{k}$
1. 3.0-cm steel cube cooling in room air	40	7.0	8.75×10^{-4}
2. 5.0-cm glass cylinder cooled by a 50-m/s airstream	0.8	180	2.81
3. Same as situation 2 but a copper cylinder	380	180	0.006
4. 3.0-cm hot copper cube submerged in water such that boiling occurs	380	10,000	0.132

Transient Heat Flow in a Semi-infinite Solid:



For constant properties (independent of time and temp), the temperature distribution $T(x, \tau)$ is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \tag{113}$$

Boundary Condition: $T(0, \tau) = T_0$

Initial Condition: $T(x, 0) = T_i$

It can be solved by Laplace Transform. The solution is

$$\frac{T(x,\tau) - T_0}{T_i - T_0} = \text{erf}\frac{x}{2\sqrt{\alpha\tau}} \tag{114}$$

where the Gauss error function is defined as:

$$\text{erf}\frac{x}{2\sqrt{\alpha\tau}} = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\alpha\tau}}} e^{-\eta^2} d\eta \tag{115}$$

The heat flow at any x position can be obtained from

$$q_x = -kA\frac{\partial T}{\partial x} \tag{116}$$

Performing Partial Differentiation with (114):

$$\frac{\partial T}{\partial x} = (T_i - T_0)\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4\alpha\tau}}\frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{\alpha\tau}}\right) = \frac{T_i - T_0}{\sqrt{\pi\alpha\tau}}e^{-\frac{x^2}{4\alpha\tau}} \tag{117}$$

At the surface (x= 0) the heat flow is:

$$q_0 = \frac{kA(T_i - T_0)}{\sqrt{\pi\alpha\tau}} \tag{118}$$

For the same uniform initial temperature distribution, we could suddenly impose a constant surface heat flux q_0/A and the boundary condition will be:

$$\frac{q_0}{A} = -k\frac{\partial T}{\partial x}\Big|_{x=0} \tag{119}$$

And the solution of this case is:

$$T - T_i = \frac{2q_0\sqrt{\frac{\alpha\tau}{\pi}}}{kA}e^{-\frac{x^2}{4\tau\alpha}} - \frac{q_0 x}{kA}\left(1 - \text{erf}\frac{x}{2\sqrt{\alpha\tau}}\right) \tag{120}$$

Eqt (120) presents the temperature response that results from a surface heat flux that remains constant with time (like a step input). A related boundary condition is a short, instantaneous impulse response with magnitude Q_0/A. The resulting temperature response is given by:

$$T - T_i = \frac{Q_0}{A\rho c(\pi\alpha\tau)^{\frac{1}{2}}}e^{-\frac{x^2}{4\alpha\tau}} \tag{121}$$

One may notice that (120) will ramp up to infinity while (121) will damp down to the initial temperature for a long period of time.

e.g.

A large block of steel [$k = 45$ W/m · °C, $\alpha = 1.4 \times 10^{-5}$ m^2/s] is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (*a*) by suddenly raising the surface temperature to 250°C and (*b*) through a constant surface heat flux of 3.2×10^5 W/m^2. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

For case *a*,

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.025}{(2)[(1.4 \times 10^{-5})(30)]^{1/2}} = 0.61$$

The error function is determined from Appendix A as

$$\text{erf}\ \frac{x}{2\sqrt{\alpha\tau}} = \text{erf}\ 0.61 = 0.61164$$

$$T(x, \tau) = T_0 + (T_i - T_0)\ \text{erf}\ \frac{x}{2\sqrt{\alpha\tau}}$$

$$= 250 + (35 - 250)(0.61164) = 118.5°\text{C}$$

For the constant-heat-flux case b, we make use of Equation (4-13a). Since q_0/A is given as 3.2×10^5 W/m^2, we can insert the numerical values to give

$$T(x, \tau) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} e^{-(0.61)^2} - \frac{(0.025)(3.2 \times 10^5)}{45}(1 - 0.61164)$$

$$= 79.3^\circ\text{C} \qquad x = 2.5 \text{ cm}, \tau = 30 \text{ s}$$

For the constant-heat-flux case the *surface* temperature after 30 s would be evaluated with $x = 0$ in Equation (4-13a). Thus,

$$T(x=0) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} = 199.4^\circ\text{C}$$

Convection Boundary Condition: (with Biot Number and Fourier Number)

Heat convected into surface = Heat conducted into surface

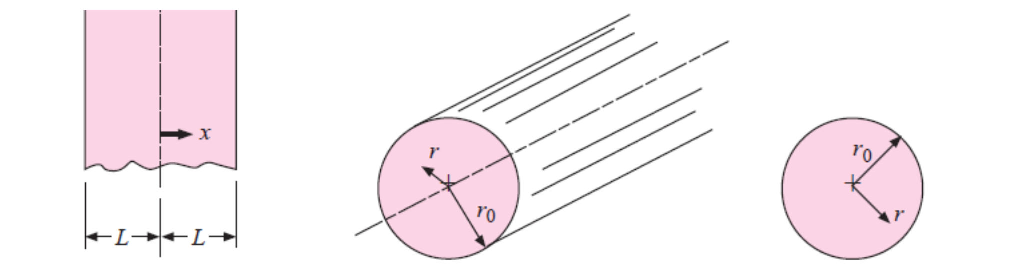
Or
$$hA(T_\infty - T)_{x=0} = -kA\frac{\partial T}{\partial x}\Big|_{x=0} \qquad (122)$$

The solution of this problem is:

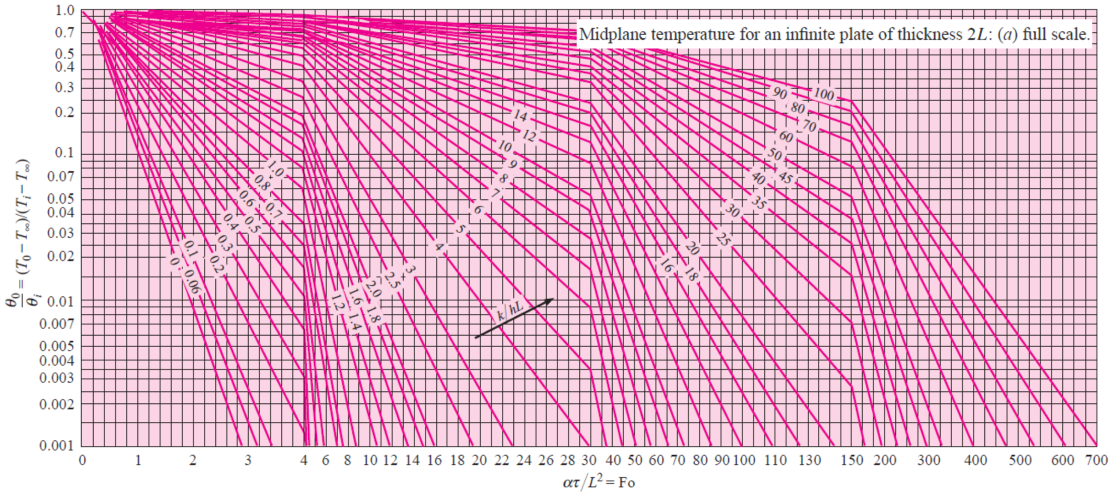
$$\frac{T - T_i}{T_\infty - T_i} = 1 - \operatorname{erf} X - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha\tau}{k^2}\right) \times \left(1 - \operatorname{erf}\left(X + \frac{h\sqrt{\alpha\tau}}{k}\right)\right) \qquad (123)$$

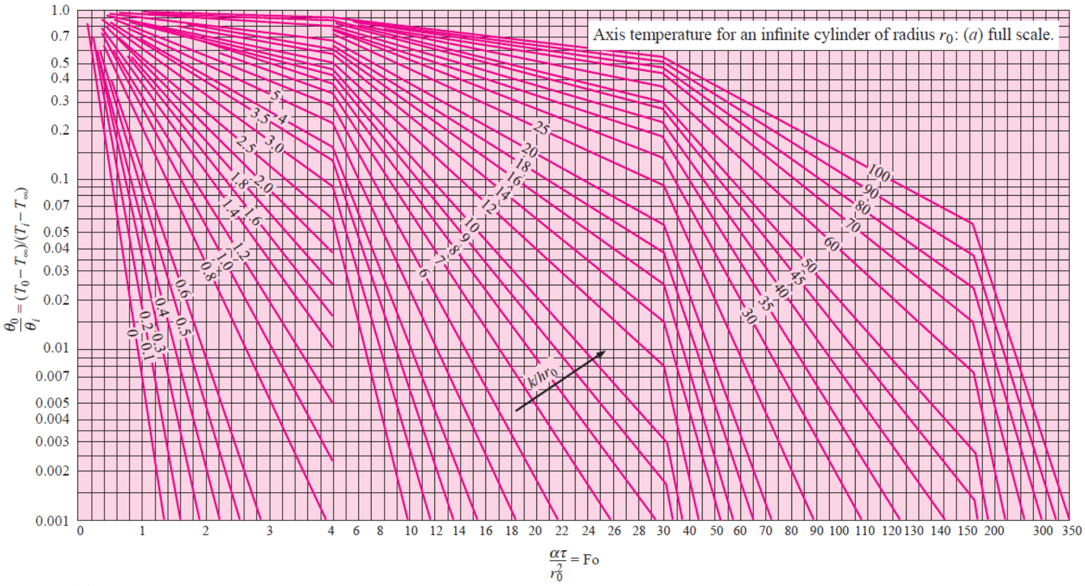
where $X = \frac{x}{2\sqrt{\alpha\tau}}$, T_i = initial temperature of solid and T_∞ = environment temperature.

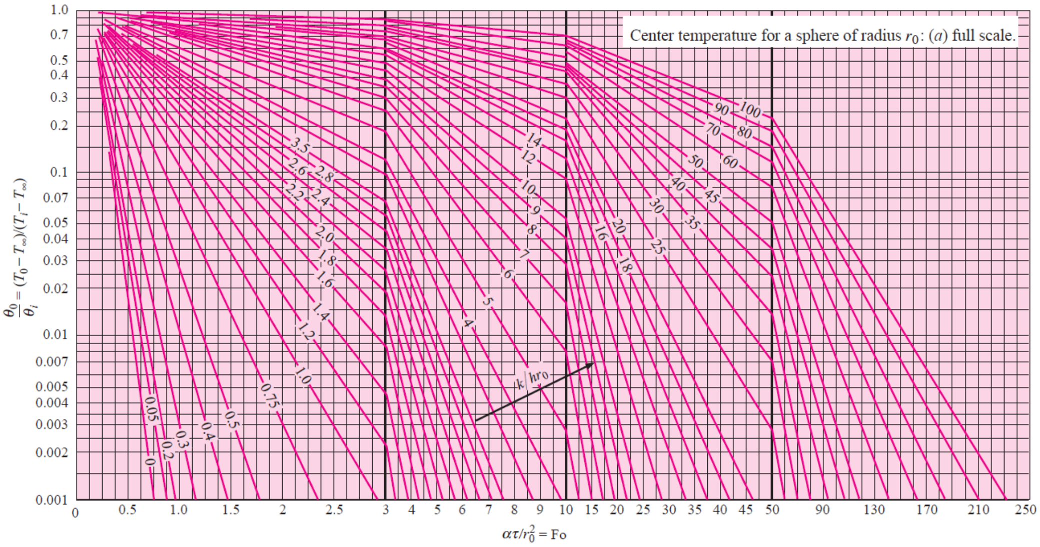
For the below geometry, one may plot out the midplane temperature.



T_0 = centerline temperature $\quad$ T_0 = centerline axis temperature $\quad$ T_0 = center temperature







A quick inspection indicates that the dimensionless temperature profile and heat flow can be expressed in Biot number = Bi = hs/k and Fourier number = Fo = $\frac{k\tau}{\rho c s^2}$. (124)

The Fourier modulus compares a characteristic body dimension with an approximate temperature-wave penetration depth for a given time τ. A low value of Biot number indicates the internal-conduction resistance is negligible in comparison with surface-convection resistance.

It is interesting to note that the Lumped-Heat Capacity System equation (109) is with the exponent

$$\frac{hA}{\rho c V}\tau = \frac{hs}{k}\frac{k\tau}{\rho c s^2} = \text{Bi Fo} \tag{125}$$

The applicability of the Heisler Chart above is Fo = $\frac{\alpha\tau}{s^2} > 0.2$.

Fourier number is the ratio of diffusion (or conductive) transport rate to the quantity storage rate.

Multidimensional System:

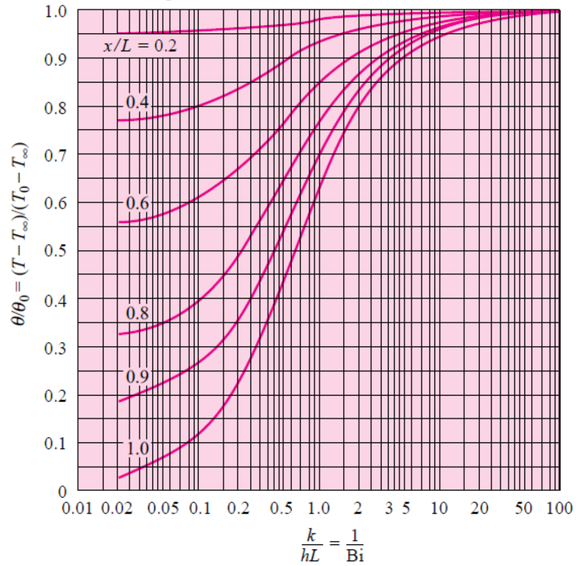
An infinite rectangular bar can be formed from two infinite plate of thickness $2L_1$ and $2L_2$. The differential equation governing is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \tag{126}$$

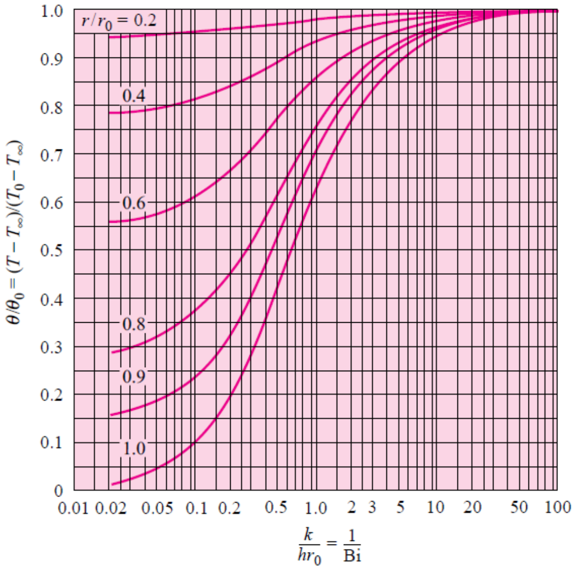
For two infinite plates the respective differential equations are:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_1}{\partial \tau} \qquad \frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \tau} \tag{127}$$

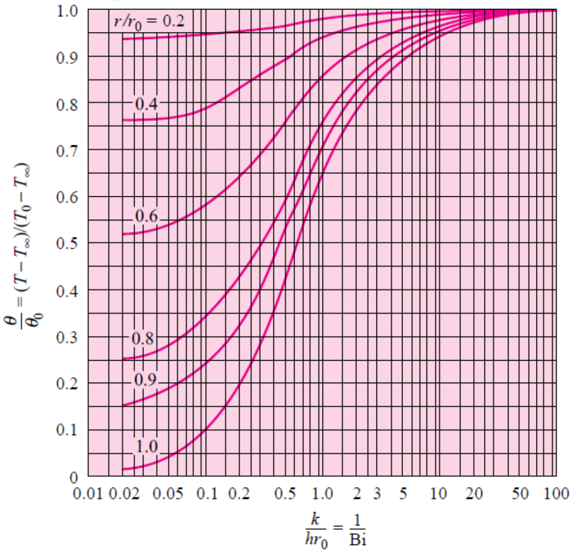
Temperature as a function of center temperature in an infinite plate of thickness $2L$, from Reference 2.



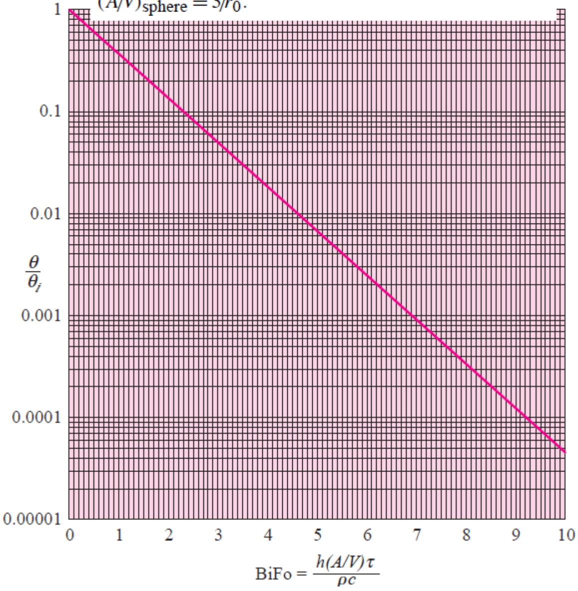
Temperature as a function of axis temperature in an infinite cylinder of radius r_0, from Reference 2.



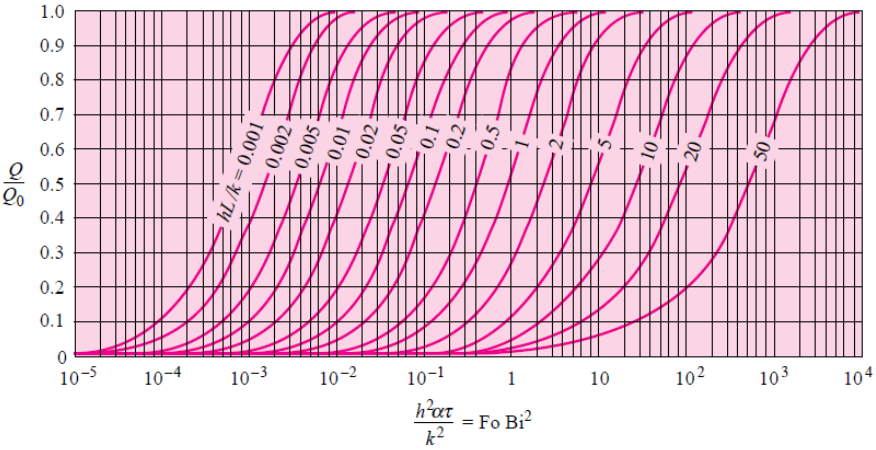
Temperature as a function of center temperature for a sphere of radius r_0, from Reference 2.



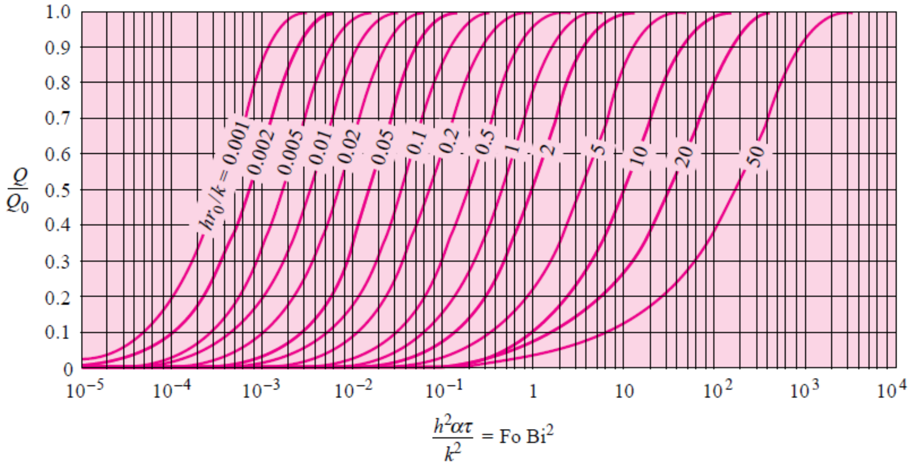
Temperature variation with time for solids that may be treated as lumped capacities: (*a*) $0 < \mathrm{BiFo} < 10$, (*b*) $0.1 < \mathrm{BiFo} < 1.0$, (*c*) $0 < \mathrm{BiFo} < 0.1$. Note: $(A/V)_{\text{inf plate}} = 1/L$, $(A/V)_{\text{inf cyl}} = 2/r_0$, $(A/V)_{\text{sphere}} = 3/r_0$.



Dimensionless heat loss Q/Q_0 of an infinite plane of thickness $2L$ with time, from Reference 6.

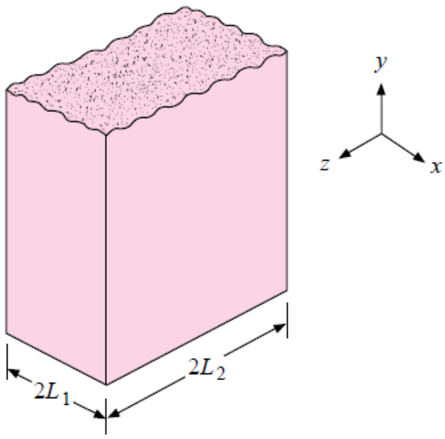


Dimensionlesss heat loss Q/Q_0 of an infinite cylinder of radius r_0 with time, from Reference 6.



By separation of variables and substitute (127) into (126), one may obtain:

$$T_2 \frac{\partial^2 T_1}{\partial x^2} + T_1 \frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\alpha}\left(\alpha T_1 \frac{\partial^2 T_2}{\partial z^2} + \alpha T_2 \frac{\partial^2 T_1}{\partial x^2}\right) \tag{128}$$



It can be shown that the dimensionless temperature distribution may be expressed as a product of the solutions for two plates problems of thickness $2L_1$ and $2L_2$:

$$\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{bar} = \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_1 plate} \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_2 plate} \tag{129}$$

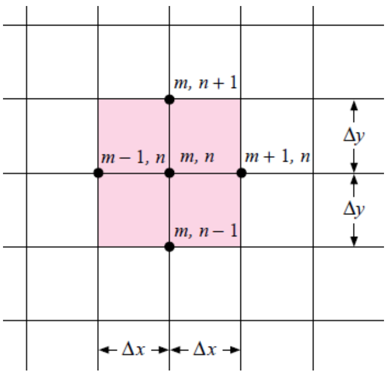
A block can be estimated as a three plates problem and the dimensionless temp distribution can be model as the product of the three plates.

In general, $$\left(\frac{\theta}{\theta_i}\right)_{combined\ solid} = \left(\frac{\theta}{\theta_i}\right)_{intersection1} \left(\frac{\theta}{\theta_i}\right)_{intersection2} \left(\frac{\theta}{\theta_i}\right)_{intersection3} \tag{130}$$

Langston has shown that it is possible to superimpose the heat loss solution for 1D bodies to obtain the heat for a nD body.

2D: $$\left(\frac{Q}{Q_0}\right)_{total} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left(1 - \left(\frac{Q}{Q_0}\right)_1\right) \tag{131}$$

3D: $$\left(\frac{Q}{Q_0}\right)_{total} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left(1 - \left(\frac{Q}{Q_0}\right)_1\right) + \left(\frac{Q}{Q_0}\right)_3 \left(1 - \left(\frac{Q}{Q_0}\right)_1\right)\left(1 - \left(\frac{Q}{Q_0}\right)_2\right) \tag{132}$$



Finite Element Method:

Consider a 2D body divided into increments with subscript *m* denoting the *x* position and *n* denoting the *y* position. Within the solid body the differential equation governing the heat flow is:

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \rho c \frac{\partial T}{\partial \tau} \tag{133}$$

Recall:
$$\frac{\partial^2 T}{\partial x^2} \approx \frac{1}{(\Delta x)^2}(T_{m+1,n} + T_{m-1,n} - 2T_{m,n})$$
$$\frac{\partial^2 T}{\partial y^2} \approx \frac{1}{(\Delta y)^2}(T_{m,n+1} + T_{m,n-1} - 2T_{m,n}) \tag{134}$$

The time derivative in eqt (133): $$\frac{\partial T}{\partial \tau} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta\tau} \quad (135)$$

Combining (134) and (135):

$$\frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{\Delta x^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{\Delta y^2} = \frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta\tau} \quad (136)$$

where the superscripts designate the time increment.

If $\Delta x = \Delta y$,

$$T_{m,n}^{p+1} = \frac{\alpha\Delta\tau}{\Delta x^2}\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + \left(1 - \frac{4\alpha\Delta\tau}{\Delta x^2}\right)T_{m,n}^{p} \quad (137)$$

If the time and distance increments are conveniently chosen so that: $$\frac{\alpha\Delta\tau}{\Delta x^2} = \frac{1}{4} \quad (138)$$

It can be seen that temperature of node (m,n) after a time increment is simple the average of the four surrounding nodal temperatures at the beginning of the time increment.

When 1D system is involved, the equation becomes:

$$T_{m}^{p+1} = \frac{\alpha\Delta\tau}{\Delta x^2}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left(1 - \frac{2\alpha\Delta\tau}{\Delta x^2}\right)T_{m}^{p} \quad (139)$$

The selection of parameter M governs the ease which we may proceed to affect the numerical solution.

For 1D system at the convection boundary, we have $$-kA\frac{\partial T}{\partial x}\Big|_{wall} = hA(T_w - T_\infty) \quad (140)$$

The finite difference approximation would be given by:

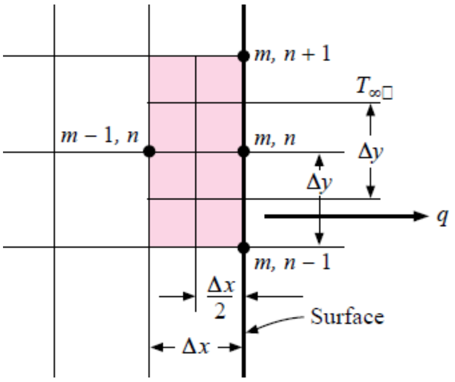
$$-k\frac{\Delta y}{\Delta x}(T_{m+1} - T_m) = h\Delta y(T_{m+1} - T_\infty) \quad (141)$$

or $$T_{m+1} = \frac{\left(T_m + \left(h\frac{\Delta x}{k}\right)T_\infty\right)}{1 + h\frac{\Delta x}{k}} \quad (142)$$

We may take the heat capacity into account in a general way by considering the 2D wall exposed to a convection transient energy balance on the node (m,n) by setting the sum of the energy conducted and convected into the node equal to the increase in the internal energy of the node.

Thus, $$k\Delta y\frac{\left(T_{m-1,n}^{p} - T_{m,n}^{p}\right)}{\Delta x} + k\frac{\Delta x}{2}\frac{\left(T_{m,n+1}^{p} - T_{m,n}^{p}\right)}{\Delta y} + k\frac{\Delta x}{2}\frac{T_{m,n-1}^{p} - T_{m,n}^{p}}{\Delta y} + h\Delta y\left(T_\infty - T_{m,n}^{p}\right)$$
$$= \rho c\frac{\Delta x}{2}\Delta y\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta\tau} \quad (143)$$

If $\Delta x = \Delta y$, $$T_{m,n}^{p+1} = \frac{\alpha\Delta\tau}{\Delta x^2}\left\{2\frac{h\Delta x}{k}T_\infty + 2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + \left(\frac{\Delta x^2}{\alpha\Delta\tau} - 2\frac{h\Delta x}{k} - 2\right)T_{m}^{p}\right\} \quad (144)$$

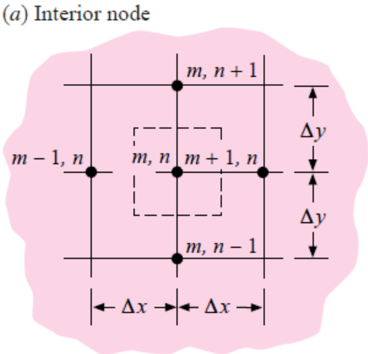


The selection of the parameter $\frac{\Delta x^2}{\alpha\Delta\tau}$ is not simple as the heat transfer coefficient influence the choice. To ensure the convergence of the numerical solution, all selection of $\frac{\Delta x^2}{\alpha\Delta\tau}$ must be restricted to

$$\frac{\Delta x^2}{\alpha\Delta\tau} \geq \begin{cases} 2\left(h\frac{\Delta x}{k} + 1\right) & 1D \\ 2\left(h\frac{\Delta x}{k} + 2\right) & 2D \end{cases} \quad (145)$$

Physical situation	Nodal equation for $\Delta x = \Delta y$	Stability requirement

(*a*) Interior node

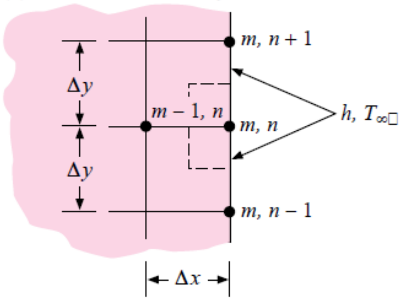


$$T_{m,n}^{p+1} = \text{Fo}\left(T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m+1,n}^{p} + T_{m,n-1}^{p}\right) + [1 - 4(\text{Fo})]T_{m,n}^{p}$$

$$T_{m,n}^{p+1} = \text{Fo}\left(T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m+1,n}^{p} + T_{m,n-1}^{p} - 4T_{m,n}^{p}\right) + T_{m,n}^{p}$$

Stability requirement: $\text{Fo} \le \frac{1}{4}$

(*b*) Convection boundary node

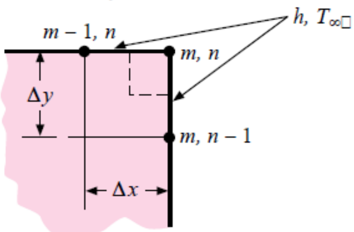


$$T_{m,n}^{p+1} = \text{Fo}\,[2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2(\text{Bi})T_{\infty}^{p}] + [1 - 4(\text{Fo}) - 2(\text{Fo})(\text{Bi})]T_{m,n}^{p}$$

$$T_{m,n}^{p+1} = \text{Fo}\,[2\text{Bi}\,(T_{\infty}^{p} - T_{m,n}^{p}) + 2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} - 4T_{m,n}^{p}] + T_{m,n}^{p}$$

Stability requirement: $\text{Fo}(2 + \text{Bi}) \le \frac{1}{2}$

(*c*) Exterior corner with convection boundary

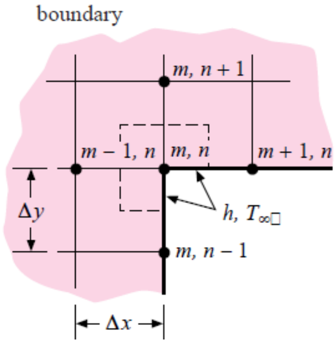


$$T_{m,n}^{p+1} = 2(\text{Fo})\,[T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2(\text{Bi})T_{\infty}^{p}] + [1 - 4(\text{Fo}) - 4(\text{Fo})(\text{Bi})]T_{m,n}^{p}$$

$$T_{m,n}^{p+1} = 2\text{Fo}\,[T_{m-1,n}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p} + 2\text{Bi}(T_{\infty}^{p} - T_{m,n}^{p})] + T_{m,n}^{p}$$

Stability requirement: $\text{Fo}(1 + \text{Bi}) \le \frac{1}{4}$

(*d*) Interior corner with convection boundary

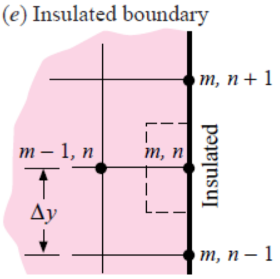


$$T_{m,n}^{p+1} = \tfrac{2}{3}(\text{Fo})\,[2T_{m,n+1}^{p} + 2T_{m+1,n}^{p} + 2T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2(\text{Bi})T_{\infty}^{p}] + [1 - 4(\text{Fo}) - \tfrac{4}{3}(\text{Fo})(\text{Bi})]T_{m,n}^{p}$$

$$T_{m,n}^{p+1} = (4/3)\text{Fo}\,[T_{m,n+1}^{p} + T_{m+1,n}^{p} + T_{m-1,n}^{p} - 3T_{m,n}^{p} + \text{Bi}\,(T_{\infty}^{p} - T_{m,n}^{p})] + T_{m,n}^{p}$$

Stability requirement: $\text{Fo}(3 + \text{Bi}) \le \frac{3}{4}$

(*e*) Insulated boundary



$$T_{m,n}^{p+1} = \text{Fo}\,[2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}] + [1 - 4(\text{Fo})]T_{m,n}^{p}$$

Stability requirement: $\text{Fo} \le \frac{1}{4}$

Thermal Resistance and Capacity Formulation:

For steady-state condition the net energy transfer into a node is zero, while for the unsteady state problems of interest is that the net energy transfer into the node must be evidenced as an increase in internal energy of the element. Each volume element behaves like a small "lumped capacity" and the interaction of all the elements determines the behavior of the solid during a transient process. If the internal energy of a node *i* can be expressed in terms of specific heat and temperature, then

$$\frac{\Delta E}{\Delta\tau} = \rho c\Delta V\,\frac{T_i^{p+1}-T_i^p}{\Delta\tau} \tag{146}$$

where ΔV is the volume element. If we define the thermal capacity as $C_i = \rho_i c_i \Delta V_i$, (147)

the general R-C formulation for the energy balance on a node is

$$q_i + \sum_j (T_j^p - T_i^p)/R_{ij} = C_i\,(T_i^{p+1} - T_i^p)/\Delta\tau \tag{148}$$

We could also write the energy balance using backward difference

$$q_i + \sum_j (T_j^{p+1} - T_i^{p+1})/R_{ij} = C_i\,(T_i^{p+1} - T_i^p)/\Delta\tau \tag{149}$$

If the solution is generated with a Gauss-Seidel iteration,

$$T_i^{p+1} = \frac{q_i + \sum_i\left(\frac{T_j^{p+1}}{R_{ij}}\right) + \frac{C_i}{\Delta\tau}T_i^p}{\sum_i\left(\frac{1}{R_{ij}}\right) + \frac{C_i}{\Delta\tau}} \tag{150}$$

Our stability requirement is 1) $T_i^p \geq 0$

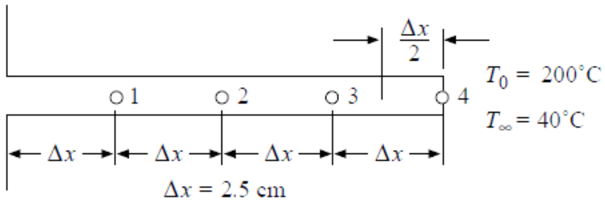
$$\text{2)}\quad 1 - \frac{\Delta\tau}{C_i}\sum_j \frac{1}{R_{ij}} \geq 0 \tag{151}$$

$$\text{i.e. } \Delta\tau \leq \min\left(\frac{C_i}{\sum_j\left(\frac{1}{R_{ij}}\right)}\right) \tag{152}$$

For the above table, if we need to include the heat source term, we can simply add $q_i = \dot{q}_i \Delta V_i$, while $\dot{q}_i$ is the heat generation per unit volume. For radiation input, $q_i = q''_{i,rad} \times \Delta A_i$, where $q''_{i,rad}$ is the net radiant energy input to the node per unit area and ΔA_i is the area of the node for radiant exchange.
(153)

e.g.

A steel rod [$k = 50$ W/m · °C] 3 mm in diameter and 10 cm long is initially at a uniform temperature of 200°C. At time zero it is suddenly immersed in a fluid having $h = 50$ W/m² · °C and $T_\infty = 40$°C while one end is maintained at 200°C. Determine the temperature distribution in the rod after 100 s. The properties of steel are $\rho = 7800$ kg/m³ and $c = 0.47$ kJ/kg · °C.



The selection of increments on the rod is as shown in the Figure Example 4-11. The cross-sectional area of the rod is $A = \pi(1.5)^2 = 7.069$ mm². The volume element for nodes 1, 2, and 3 is

$$\Delta V = A\Delta x = (7.069)(25) = 176.725 \text{ mm}^3$$

Node 4 has a ΔV of half this value, or 88.36 mm³. We can now tabulate the various resistances and capacities for use in an explicit formulation. For nodes 1, 2, and 3 we have

$$R_{m+} = R_{m-} = \frac{\Delta x}{kA} = \frac{0.025}{(50)(7.069 \times 10^{-6})} = 70.731\text{°C/W}$$

and

$$R_\infty = \frac{1}{h(\pi d\,\Delta x)} = \frac{1}{(50)\pi(3 \times 10^{-3})(0.025)} = 84.883\text{°C/W}$$

$$C = \rho c\, \Delta V = (7800)(470)(1.7673 \times 10^{-7}) = 0.6479\ \mathrm{J/°C}$$

For node 4 we have

$$R_{m+} = \frac{1}{hA} = 2829\ \mathrm{°C/W} \qquad R_{m-} = \frac{\Delta x}{kA} = 70.731\ \mathrm{°C/W}$$

$$C = \frac{\rho c\, \Delta V}{2} = 0.3240\ \mathrm{J/°C} \qquad R_\infty = \frac{2}{h\pi d \Delta x} = 169.77\ \mathrm{°C/W}$$

To determine the stability requirement we form the following table:

Node	$\sum(1/R_{ij})$	C_i	$\dfrac{C_i}{\sum(1/R_{ij})}$, s
1	0.04006	0.6479	16.173
2	0.04006	0.6479	16.173
3	0.04006	0.6479	16.173
4	0.02038	0.3240	15.897

Thus node 4 is the most restrictive, and we must select $\Delta\tau < 15.9$ s. Since we wish to find the temperature distribution at 100 s, let us use $\Delta\tau = 10$ s and make the calculation for 10 time increments using Equation (4-47) for the computation. We note, of course, that $q_i = 0$ because there is no heat generation. The calculations are shown in the following table.

Time increment	Node temperature			
	T_1	T_2	T_3	T_4
0	200	200	200	200
1	170.87	170.87	170.87	169.19
2	153.40	147.04	146.68	145.05
3	141.54	128.86	126.98	125.54
4	133.04	115.04	111.24	109.70
5	126.79	104.48	98.76	96.96
6	122.10	96.36	88.92	86.78
7	118.53	90.09	81.17	78.71
8	115.80	85.23	75.08	72.34
9	113.70	81.45	70.31	67.31
10	112.08	78.51	66.57	63.37

We can calculate the heat-transfer *rate* at the end of 100 s by summing the convection heat losses on the surface of the rod. Thus

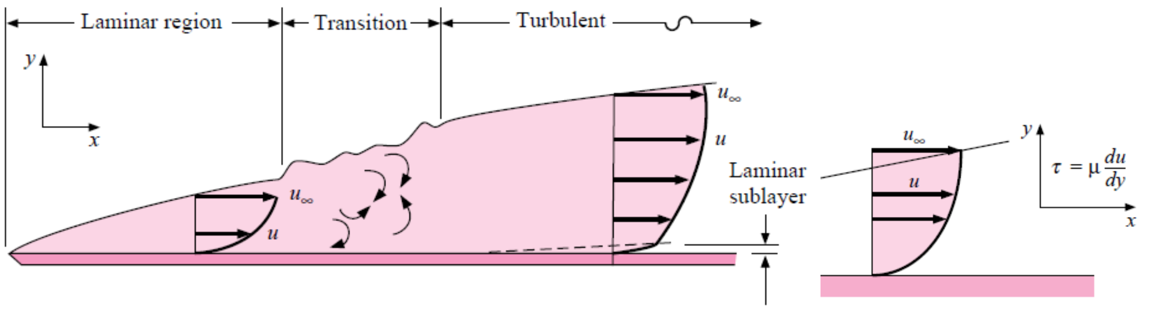
$$q = \sum_i \frac{T_i - T_\infty}{R_{i\infty}}$$

and

$$q = \frac{200 - 40}{(2)(84.883)} + \frac{112.08 + 78.51 + 66.57 - (3)(40)}{84.883} + \left(\frac{1}{169.77} + \frac{1}{2829}\right)(63.37 - 40)$$
$$= 2.704\ \mathrm{W}$$

Convection: (5)

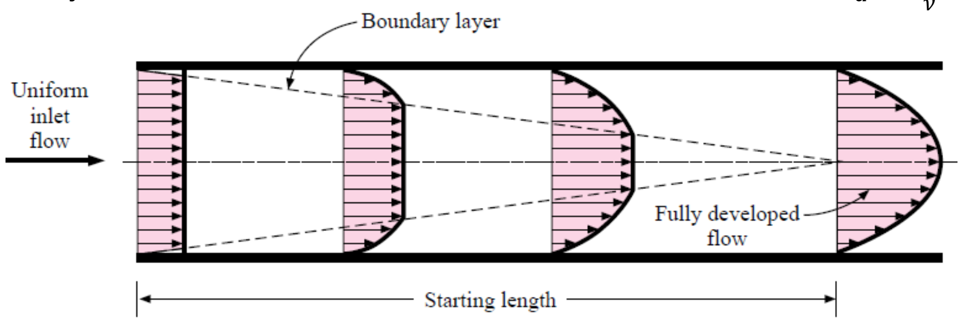
Viscous forces are described under a shear stress $\tau = \mu \frac{du}{dy}$ (154)

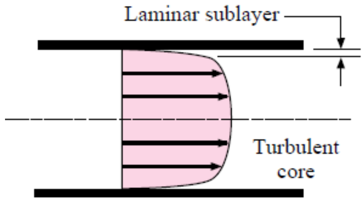


The region of flow that develops from the leading edge of a plate in which the effects of viscosity are observed is called the **boundary layer**. The end of boundary layer is usually chosen as the y coordinate where the velocity becomes 99% of the free-stream value.

The transition from laminar to turbulent flow occurs when $\text{Re} = \frac{u_\infty x}{\nu} > 5 \times 10^5$ (155)

Consider the flow in a tube, when the flow is turbulent, a blunter profile instead of a parabolic one is experienced, Reynold number will be used to determine if the flow is turbulent $\text{Re}_d = \frac{u_m d}{\nu} > 2300$





Continuity: $\dot{m} = \rho u_m A$ (156)

Define **mass velocity** as $G = \frac{\dot{m}}{A} = \rho u_m$ (157)

Consider the energy balance in the incompressible flow along a stream line,

Bernoulli Equation: $\frac{P}{\rho} + \frac{1}{2}\frac{V^2}{g} = const.$ (158)

or in differential form: $\frac{dp}{\rho} + \frac{VdV}{g} = 0$ (159)

When the fluid is compressible, we can write an energy equation as:

$$i_1 + \frac{V_1^2}{2g} + Q = i_2 + \frac{V_2^2}{2g} + W_k \quad (160)$$

where i = enthalpy = u+Pv.

To calculate the pressure drop in compressible flow, it's necessary to specify the equation of state

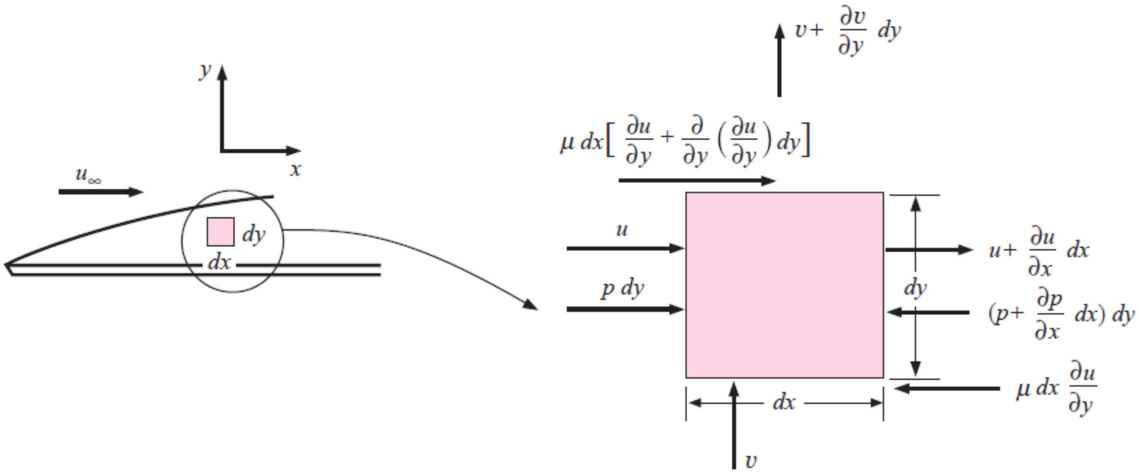
$$P = \rho RT \qquad \Delta u = c_v \Delta T \qquad \Delta i = c_p \Delta T \quad (161)$$

For air, R_{air} = 287 J/kgK, $c_{p,air}$ = 1.005 kJ/kg°C and $c_{v,air}$ = 0.718 kJ/kg°C

To solve a particular problem, we must also specify the process like reversible adiabatic flow through nozzle yields properties related to Mach number and the stagnation properties.

e.g. $\frac{T_0}{T} = 1 + \frac{k-1}{2} Ma^2 \quad \frac{P_0}{P} = \left(1 + \frac{k-1}{2} Ma^2\right)^{\frac{k}{k-1}} \quad \frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} Ma^2\right)^{\frac{1}{k-1}} \quad c = \sqrt{kRT}$ (162)

Laminar Boundary Layer of a Flat Plane:



Mass Balance: $\rho u dy + \rho v dx = \rho\left(u + \frac{\partial u}{\partial x} dx\right) dy + \rho\left(v + \frac{\partial v}{\partial y} dy\right) dx$ (163)

or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (164)

Momentum Balance: Momentum flux in *x* direction entering bottom face = $\rho v u dx$

Momentum flux in x direction leaving top face = $\rho\left(v + \frac{\partial v}{\partial y} dy\right)\left(u + \frac{\partial u}{\partial y} dy\right) dx$

Pressure force on the left face = $p\, dy$

Pressure forces on the right = $-\left(p + \frac{\partial p}{\partial x} dx\right) dy$

Viscous shear force on bottom face = $-\mu \frac{\partial u}{\partial y} dx$

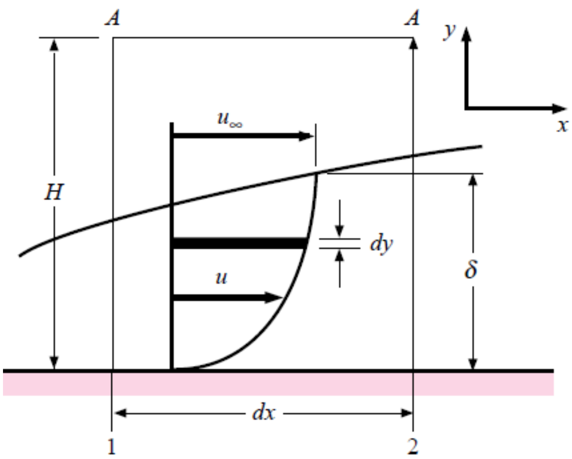
Viscous shear force on the top = $\mu dx\left(\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) dy\right)$ (165)

Summing: $\mu \frac{\partial^2 u}{\partial y^2} dx dy - \frac{\partial p}{\partial x} dx dy = \rho\left(u + \frac{\partial u}{\partial x} dx\right)^2 dy - \rho u^2 dy$

$+\rho\left(v + \frac{\partial v}{\partial y} dy\right)\left(u + \frac{\partial u}{\partial y} dy\right) dx - \rho v u dx$ (166)

$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}$ (167)

Approximate Method due to von Karman:



Mass flow$_1$ = $\int_0^H \rho u dy$

Momentum flow$_1$ = $\int_0^H \rho u^2 dy$

Mass flow$_2$ = $\int_0^H \rho u dy + \frac{d}{dx}\left(\int_0^H \rho u dy\right) dx$

Momentum flow$_2$

$= \int_0^H \rho u^2 dy + \frac{d}{dx}\left(\int_0^H \rho u^2 dy\right) dx$

(168)

Mass flow$_{A\text{-}A}$ = Mass flow$_2$ – Mass flow$_1$

$= u_\infty \frac{d}{dx}\left(\int_0^H \rho u dy\right) dx$ (169)

Net momentum flow

$= \frac{d}{dx}\left(\int_0^H \rho u^2 dy\right) dx - u_\infty \frac{d}{dx}\left(\int_0^H \rho u dy\right) dx$

With basic differential relationship:

$$u_\infty \frac{d}{dx}\left(\int_0^H \rho u dy\right) dx = \frac{d}{dx}\left(u_\infty \int_0^H \rho u dy\right) dx - \frac{du_\infty}{dx}\left(\int_0^H \rho u dy\right) dx$$
$$= \frac{\mathrm{d}}{\mathrm{dx}}\left(\int_0^H \rho u u_\infty dy\right) dx - \frac{du_\infty}{dx}\left(\int_0^H \rho u dy\right) dx$$

Force on plane 1: Pressure force = pH

Force on plane 2: Pressure force = $\left(\mathrm{p} + \frac{\mathrm{dp}}{\mathrm{dx}} dx\right) H$

Shear force at the wall= $-\tau_w dx = -\mu dx \frac{\partial u}{\partial y}|_{y=0}$ (170)

Setting the forces on the element = net increase in momentum

$$-\tau_w - \frac{dp}{dx} H = -\rho \frac{d}{dx}\int_0^H (u_\infty - u) u dy + \frac{du_\infty}{dx}\int_0^H \rho u dy \qquad (171)$$

This is the integral momentum equation of the boundary layer. If the pressure is constant, $\frac{dp}{dx} = 0 = -\rho u_\infty \frac{du_\infty}{dx}$:

$$\rho \frac{\mathrm{d}}{\mathrm{dx}}\int_0^\delta (u_\infty - u) u dy = \tau_w = \mu \frac{\partial u}{\partial y}|_{y=0} \qquad (172)$$

The upper limit is changed to δ as the integrand is zero for $y > \delta$ since $u_\infty = u$ for $y > \delta$.

If the velocity profile were known, the appropriate function should be inserted into (172) to obtain an expression for the boundary layer thickness. Here are the conditions that the velocity function must satisfy.

$$\mathrm{u} = 0 \quad \text{at} \quad \mathrm{y} = 0$$
$$\mathrm{u} = u_\infty \quad \text{at} \quad \mathrm{y} = \delta$$
$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad \mathrm{y} = \delta$$
$$\text{(Constant pressure condition):} \ \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{at} \quad \mathrm{y} = 0$$

With polynomial function $u = C_1 + C_2 y + C_3 y^2 + C_4 y^3$,

one may solve that $\frac{\mathrm{u}}{\mathrm{u}_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ (173)

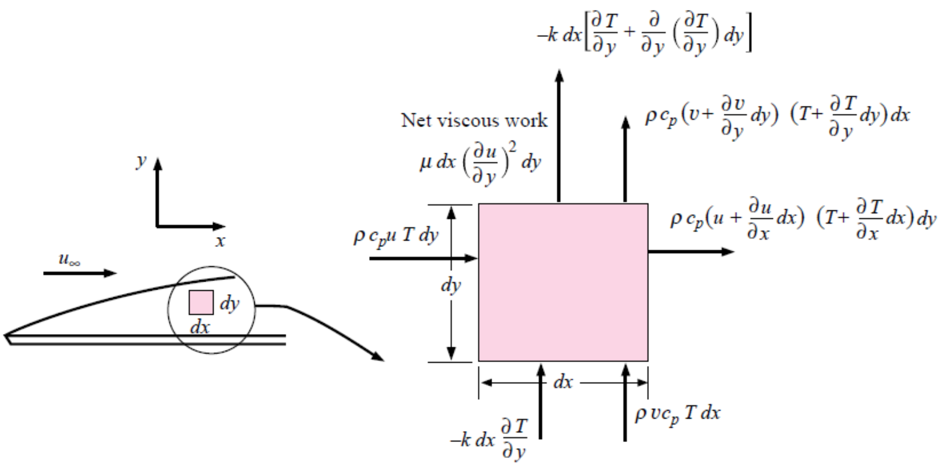
Inserting the expression into (172),

$$\frac{\mathrm{d}}{\mathrm{dx}}\left\{\rho u_\infty^2 \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) dy\right\} = \mu \frac{\partial u}{\partial y}|_{y=0} = \frac{\frac{3}{2}\mu u_\infty}{\delta} \qquad (174)$$

$$\delta \mathrm{d}\delta = \frac{140}{13}\frac{\mu}{\rho u_\infty} + const. \qquad (175)$$

With initial condition: $\frac{\delta}{\mathrm{x}} = \frac{4.64}{Re_x^{1/2}}$ (176)

Energy Equation of the Boundary Layer:



Consider the elemental control volume shown in last page. To simplify the analysis, we assume:

1. Incompressible steady flow
2. Constant viscosity, thermal conductivity, and specific heat
3. Negligible heat conduction in the direction of flow (*x* direction), i.e. $\frac{\partial \mathrm{T}}{\partial \mathrm{x}} \ll \frac{\partial T}{\partial y}$

Energy convected in left face + energy convected in bottom face + heat conducted in bottom face + net viscous work done on element
= energy convected out right face + energy convected out top face + heat conducted out top face

Viscous shear force = $\mu \frac{\partial \mathrm{u}}{\partial \mathrm{y}} dx$ Distance moved = $\frac{\partial \mathrm{u}}{\partial \mathrm{y}} dy$

Viscous energy delivered to the element = $\mu \left(\frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right)^2 dxdy$

Writing energy balance with assumption unit depth in the *z* direction:

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + T \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) dxdy = k \frac{\partial^2 T}{\partial y^2} dxdy + \mu \left(\frac{\partial u}{\partial y} \right)^2 dxdy \qquad (177)$$

Dividing ρc_p:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 \qquad (178)$$

This is the energy equation of the laminar boundary layer. The left side represents the net transport of energy into the control volume, and the right side represents the sum of the net heat conducted out of the control volume and the net viscous work done on the element.

Order of magnitude analysis:

$$u \sim u_\infty \text{ and } y \sim \delta$$

$$\alpha \frac{\partial^2 T}{\partial y^2} \sim \frac{T}{\delta^2}$$

So that

$$\frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 \sim \frac{\mu}{\rho c_p} \frac{u_\infty^2}{\delta^2} \qquad (179)$$

If the ratio of the quantities is small,

$$\frac{\mu}{\rho c_p} \frac{u_\infty^2}{T} \ll 1 \qquad (180)$$

then the viscous dissipation is small in comparison with the conduction term.

Define Prandtl number:

$$\mathrm{Pr} = \frac{v}{\alpha} = \frac{c_p \mu}{k} \qquad (181)$$

Criterion in (180):

$$\mathrm{Pr} \frac{u_\infty^2}{c_p T} \ll 1 \qquad (182)$$

Consider a flow of air $u_\infty = \frac{70m}{s}$ $\quad T = 20^o C = 293K \quad p = 1atm \quad c_p = 1.005 kJ/kg.^o C$

$$\mathrm{Pr} = 0.1 \quad \mathrm{Pr} \frac{u_\infty^2}{c_p T} = 0.012 \ll 1.0$$

Indicating that viscous dissipation is small for even this rather large flow velocity of 70m/s.

For low-velocity incompressible flow:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (183)$$

Recall momentum equation:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = v \frac{\partial^2 T}{\partial y^2} \qquad (184)$$

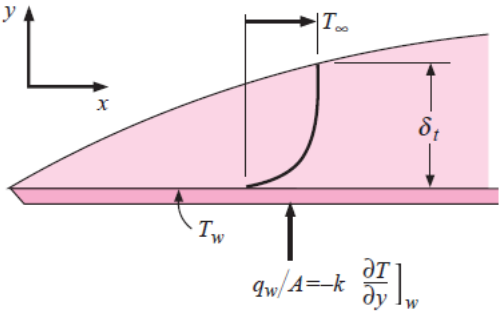
The solution of the two equations will have the same form when $\alpha = v$. We should expect that the relative magnitudes of the thermal diffusivity and kinematic viscosity would have influence on convective heat transfer since these magnitudes relate the velocity distribution and temperature distribution.

Thermal Boundary Layer:

Let the thickness of the thermal boundary layer is designated as δ_t. At the wall the velocity is zero. The heat transfer into the fluid takes place by conduction, rather than viscosity.

The local heat flux per unit area, q'': $\frac{q}{A} = q'' = -k\frac{\partial T}{\partial y}|_{wall}$ (185)

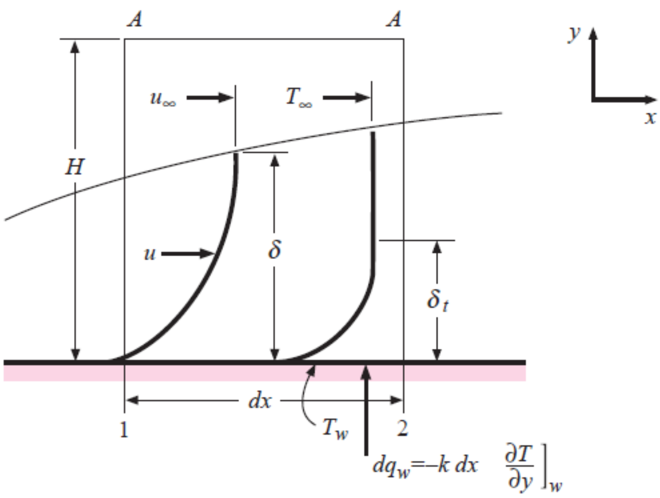
From Newton's Law of cooling, we have: $h = -\frac{k\frac{\partial T}{\partial y}|_{wall}}{T_w - T_\infty}$ (186)



Conditions that temperature distribution must satisfy are
1) $T = T_w$ at $y = 0$
2) $\frac{\partial T}{\partial y} = 0$ at $y = \delta_t$
3) $T = T_\infty$ at $y = \delta_t$
4) $\frac{\partial^t T}{\partial y^2} = 0$ at $y = 0$ (187)

The condition may be fitted to cubic polynomial to obtain: $\frac{\theta}{\theta_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ (188)

where $\theta = T - T_w$.



Energy Balance: (189)

Energy convected in + viscous work within element + heat transfer at wall = Energy convected out

Energy convected in plane 1: $\rho c_p \int_0^H uT\,dy$

Energy convected out plane 2: $\rho c_p \int_0^H uT\,dy + \frac{d}{dx}\left(\rho c_p \int_0^H uT\,dy\right) dx$

Mass flow through A-A: $\frac{d}{dx}(\int_0^H \rho u dy)dx$

Energy carried through A-A: $c_p T_\infty \frac{d}{dx}(\int_0^H \rho u dy)dx$

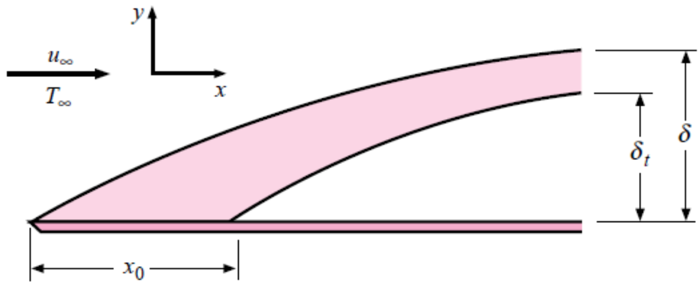
Viscous Work done within: $\mu\left(\int_0^H \left(\frac{du}{dy}\right)^2 dy\right) dx$ (190)

Combining: $\frac{d}{dx}\left(\int_0^H (T_\infty - T)u dy\right) + \frac{\mu}{\rho c_p}\left(\int_0^H \left(\frac{du}{dy}\right)^2 dy\right) = \alpha \frac{\partial T}{\partial y}|_w$ (191)

Inserting (188) into (191) and neglecting the viscous-dissipation term, we have

$$\frac{d}{dx}\left(\int_0^H (T_\infty - T)u\,dy\right) = \frac{3\alpha\theta_\infty}{2\delta_t} \qquad (192)$$

Hydrodynamic and thermal boundary layers on a flat plate. Heating starts at $x = x_0$.



Let us assume that the thermal boundary layer is thinner than the hydrodynamic boundary layer. We only need to carry the integration to $y = \delta_t$ since the integrand is zero for $y > \delta_t$. Substitute $\zeta = \frac{\delta_t}{\delta}$,

$$\theta_\infty u_\infty \frac{d}{dx}\left(\delta\left(\frac{3}{20}\zeta^2 - \frac{3}{280}\zeta^4\right)\right) = \frac{3}{2}\frac{\alpha\theta_\infty}{\delta\zeta} \qquad (193)$$

Because $\delta_t < \delta, \zeta < 1$ and term involving ζ^4 term and write

$$\frac{3}{20}\theta_\infty u_\infty \frac{d}{dx}(\delta\zeta^2) = \frac{3}{2}\frac{\alpha\theta_\infty}{\delta\zeta} \qquad (194)$$

Performing differentiation, $$\frac{1}{10}u_\infty\left(2\delta\zeta\frac{d\zeta}{dx} + \zeta^2\frac{d\delta}{dx}\right) = \frac{\alpha}{\delta\zeta} \qquad (195)$$

And $$\delta d\delta = \frac{140}{13}\frac{v}{u_\infty}dx \qquad (196)$$

So that we have $$\zeta^3 + 4x\zeta^2\frac{d\zeta}{dx} = \frac{13}{14}\frac{\alpha}{\upsilon} \qquad (197)$$

Solving the differential equation, $$\zeta^3 = Cx^{-\frac{3}{4}} + \frac{13}{14}\frac{\alpha}{\upsilon} \qquad (198)$$

With Boundary Condition: $$\delta_t = 0 \text{ } at \text{ } x = x_0 \quad \text{and} \quad \zeta = 0 \text{ } at \text{ } x = x_0 \qquad (199)$$

The final solution becomes $$\zeta = \frac{\delta_t}{\delta} = \frac{1}{1.026}\mathrm{Pr}^{-1/3}\left(1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right)^{\frac{1}{3}} \qquad (200)$$

Recall Prandtl Number $$\mathrm{Pr} = \frac{v}{\alpha} = \frac{\mu/\rho}{k/\rho c_p} = \frac{c_p\mu}{k} \qquad (201)$$

Return to the analysis, we have $$h = -\frac{k\frac{\partial T}{\partial y}|_{wall}}{T_w - T_\infty} = \frac{3}{2}\frac{k}{\delta_t} = \frac{3}{2}\frac{k}{\zeta\delta} \qquad (202)$$

Substitute (200) into (176), $$h_x = 0.332\,k\,\mathrm{Pr}^{\frac{1}{3}}\left(\frac{u_\infty}{\nu x}\right)^{\frac{1}{2}}\left(1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right)^{-\frac{1}{3}} \qquad (203)$$

Multiplying both sides with x/k and producing Nusselt Number: $$\mathrm{Nu_x} = \frac{h_x x}{k} \qquad (204)$$

Finally, we have $$\mathrm{Nu_x} = 0.332\,Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}}\left(1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right)^{-\frac{1}{3}} \qquad (205)$$

For a plate heated over its entire length, $x_0 = 0$ and

$$\mathrm{Nu_x} = 0.332\,Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}} \qquad (206)$$

Eqts (203), (205), (206) express the local values of the heat-transfer coefficient in terms of the distance from the leading edge of the plate and the fluid properties. For the case where $x_0 = 0$ the average heat-transfer coefficient and Nusselt number may be obtained by integrating over length of the plate:

$$\bar{h} = \int_0^L h_x dx \,/ \int_0^L dx = 2\mathrm{h_{x=L}} \qquad (207)$$

For a plate where heating starts at $x=x_0$, the average heat transfer coefficient can be expressed as:

$$\frac{\bar{h}_{x_0-L}}{h_{x=L}} = \frac{2L\left(1-\left(\frac{x_0}{L}\right)^{\frac{3}{4}}\right)}{L-x_0} \tag{208}$$

The total heat transfer for the plate would be

$$q_{total} = \bar{h}_{x_0-L}(L-x_0)(T_w - T_\infty) \tag{209}$$

For the plate heated over the entired length,

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = 2\mathrm{Nu}_{x=L} \tag{210}$$

or

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = 0.664\, Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}} \tag{211}$$

where

$$\mathrm{Re}_L = \frac{\rho u_\infty L}{\mu} \tag{212}$$

The above analysis has consider the laminar heat transfer from an isothermal surface. In many practical problems the surface *heat flux* is essentially constant. For the constant heat flux case, it can be shown that the local Nusselt Number is given by:

$$\mathrm{Nu}_x = \frac{hx}{k} = 0.453 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} \tag{213}$$

which can also be expressed in terms of wall heat flux and temperature difference as

$$\mathrm{Nu}_x = \frac{q_w x}{k(T_w - T_\infty)} \tag{214}$$

The average temperature different along the plate, for constant heat flux condition, may be obtained by performing the integration

$$\overline{T_w - T_\infty} = \frac{1}{L}\int_0^L (T_w - T_\infty)dx = \frac{1}{L}\int_0^L \frac{q_w x}{Nu_x} dx = \frac{\frac{q_x L}{k}}{0.6795 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}} \tag{215}$$

Or

$$q_w = \frac{3}{2} h_{x=L}(\overline{T_w - T_\infty}) \tag{216}$$

Friction and Heat Transfer:

We have already discussed that the temperature and flow fields are related. Now we seek an expression whereby the frictional resistance may be directly related to heat transfer.

The shear stress at the wall may be expressed in terms of a friction coefficient C_f:

$$\tau_w = C_f \frac{\rho u_\infty^2}{2} = \mu \frac{\partial u}{\partial y}|_w \tag{217}$$

Using the velocity distribution derived before, we have

$$\tau_w = \frac{3}{2}\frac{\mu u_\infty}{\delta} \tag{218}$$

Making use of the relation for the boundary layer thickness,

$$\tau_w = \frac{3}{2}\frac{\mu u_\infty}{4.64}\left(\frac{u_\infty}{\nu x}\right)^{\frac{1}{2}} \tag{219}$$

Combining (217) and (219)

$$\frac{C_{fx}}{2} = \frac{3}{2}\frac{\mu u_\infty}{4.64}\left(\frac{u_\infty}{\nu x}\right)^{\frac{1}{2}}\frac{1}{\rho u_\infty^2} = 0.323\, Re_x^{-\frac{1}{2}} \tag{220}$$

For equation (206), it can be rewritten as

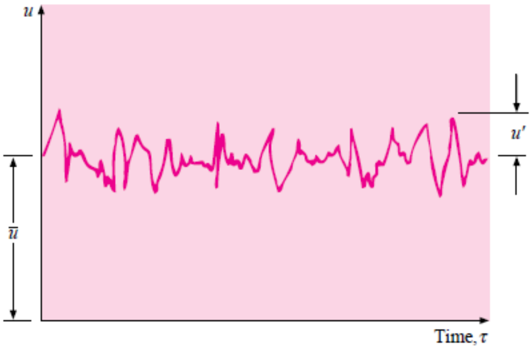
$$\frac{\mathrm{Nu}_x}{Re_x Pr} = \frac{h_x}{\rho c_p u_\infty^2} = 0.332 Pr^{-2/3} Re_x^{-1/2} = St_x \tag{221}$$

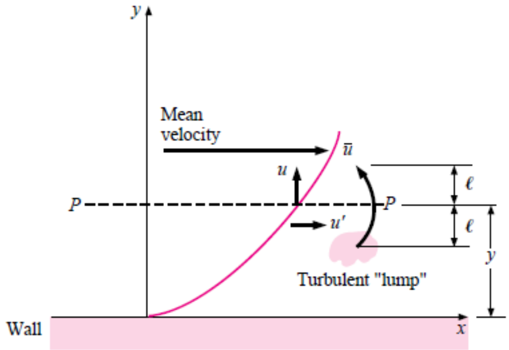
This is Stanton Number, which is the ratio of thermal diffusivity and viscosity.

Turbulence Boundary Layer Heat Transfer:

Consider a turbulence flow with $u = \bar{u} + u'$ and $v = \bar{v} + v'$ with ū as average velocity and u' as the fluctuation from the mean.

For a unit area of the plane P-P, the instantaneous turbulent mass-transport rate across the plane is $\rho v'$. The net momentum flux per unit area, in the x-direction, represents the turbulent shear stress at the plane P-P or $\rho v' u'$





The average turbulent-shear stress will be given as $\tau_t = -\rho\overline{v'u'}$ (222)

The eddy viscosity or eddy diffusivity for momentum ϵ_M such that $\tau_t = -\rho\overline{v'u'} = \rho\epsilon_M\frac{\partial u}{\partial y}$ (223)

Notice that with Taylor series, one may approximate that $u' \approx l\frac{\partial u}{\partial y}$ (224)

The distance l is called the Prandtl mixing length. Prandtl also postulated that v' would be in same order of magnitude as u' so that the turbulent shear stress of (223) can be written as:

$$\tau_t = -\rho\overline{v'u'} = \rho l^2\left(\frac{\partial u}{\partial y}\right)^2 = \rho\epsilon_M\frac{\partial u}{\partial y} \tag{225}$$

The eddy viscosity ϵ_M then becomes $\epsilon_M = l^2\frac{\partial u}{\partial y}$ (226)

Prandtl's hypothesis was that the mixing length is proportional to distance from the wall, or $l=Ky$.

Apply integration on (225) we have: $u = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\ln y + C$ (227)

Let us quantify our earlier qualitative description of a turbulent boundary layer by expressing the shear stress as the sum of a molecular and turbulent part, i.e. $\frac{\tau}{\rho} = (\nu + \epsilon_M)\frac{\partial u}{\partial y}$ (228)

The so-called universal velocity profile is obtained by introducing two nondimensional coordinates:

$$u^+ = \frac{u}{\sqrt{\frac{\tau_w}{\rho}}} \quad \text{and} \quad y^+ = \frac{\sqrt{\frac{\tau_w}{\rho y}}}{\nu} \tag{229}$$

We can rewrite (228) as $du^+ = \frac{dy^+}{1+\frac{\epsilon_M}{\upsilon}}$ (230)

As what we have discussed before, the laminar sublayer is the region where $\epsilon_M \sim 0$, the buffer layer has $\epsilon_M \sim \upsilon$ and the turbulent layer has $\epsilon_M \gg \upsilon$. Therefore, taking $\epsilon_M = 0$, integration yields

$$u^+ = y^+ \tag{231}$$

It is the velocity relation (a linear one) for the laminar sublayer. In the fully turbulent region, $\epsilon_M \gg \upsilon$, (227) becomes $\frac{\partial u}{\partial y} = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\frac{1}{y}$. Substitute this in (226), we have $\epsilon_M = K\sqrt{\frac{\tau_w}{\rho}}y$ (232)

or $\frac{\epsilon_M}{\upsilon} = Ky^+$ (233)

Substitute (233) into (230) and we have $u^+ = \frac{1}{K}\ln y^+ + C$ (234)

Experimental Result shows that:

Laminar sublayer: $0 < y^+ < 5$	$u^+ = y^+$
Buffer layer: $5 < y^+ < 30$	$u^+ = 5.0\ln y^+ - 3.05$
Turbulent layer: $30 < y^+ < 400$	$u^+ = 2.5\ln y^+ + 5.5$

Turbulent heat transfer: $\left(\frac{q}{A}\right)_{turb} = -\rho c_p\epsilon_H\frac{\partial T}{\partial y}$ (235)

Or for the region where both molecular and turbulent energy transport are important,

$$\frac{q}{A} = -\rho c_p(\alpha + \epsilon_H)\frac{\partial T}{\partial y} \tag{226}$$

In the turbulent-flow region, turbulent Prandtl number $Pr_t = \frac{\epsilon_M}{\epsilon_H}$, the local skin-friction coefficient is given by $C_{fx} = 0.0592\, Re_x^{-\frac{1}{5}}$ or $C_{fx} = 0.370(\log Re_x)^{-2.584}$ for low Reynolds number and high Reynold number. A simpler formula can be obtained for low Reynolds for average-friction coefficient is $\overline{C_f} = \frac{0.074}{\mathrm{Re}_L^{0.2}} - \frac{A}{Re_L}$, for $\mathrm{Re}_L < 10^7$. (227)

Applying the fluid-friction analogy: St $\mathrm{Pr}^{2/3} = \frac{C_f}{2}$, we can obtain the local turbulent heat transfer:

$$\mathrm{St_x}Pr^{2/3} = 0.0296\, Re_x^{-0.2} \qquad 5 \times 10^5 < Re_x < 10^7$$

$$\mathrm{St_x}Pr^{2/3} = 0.185(\log Re_x)^{-2.584} \qquad 10^7 < Re_x < 10^9 \tag{228}$$

The average heat-transfer coefficient can be obtained by integrating the local values over the entire length of the plate. Thus,

$$h = \frac{1}{L}\left(\int_0^{x_{crit}} h_{lam}dx + \int_{x_{crit}}^{L} h_{turb}dx\right) \tag{229}$$

If we apply 1/7 power relation:

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{230}$$

We can obtain

$$\tau_w = 0.0296\, Re_x^{1/5}\rho u_\infty^2 \tag{231}$$

Applying the integral momentum equation for zero pressure gradient,

$$\frac{d}{dx}\int_0^\delta \left(1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right)\left(\frac{y}{\delta}\right)^{\frac{1}{7}} dy = 00296 Re_x^{1/5} \tag{232}$$

$$\frac{d\delta}{dx} = \frac{72}{7}(0.0296)\left(\frac{\nu}{u_\infty}\right)^{\frac{1}{5}} x^{\frac{1}{5}} \tag{233}$$

We shall integrate this equation for two physical situations:

1. The boundary layer is fully turbulent form the leading edge of the plate
2. The boundary layer follows a laminar growth pattern up to $\mathrm{Re_{crit}} = 5 \times 10^5$ and a turbulent growth thereafter.

For the first case, we integrate (231) with $\delta = 0$ at x = 0 to obtain:

$$\frac{\delta}{x} = 0.381 Re_x^{1/5} \tag{234}$$

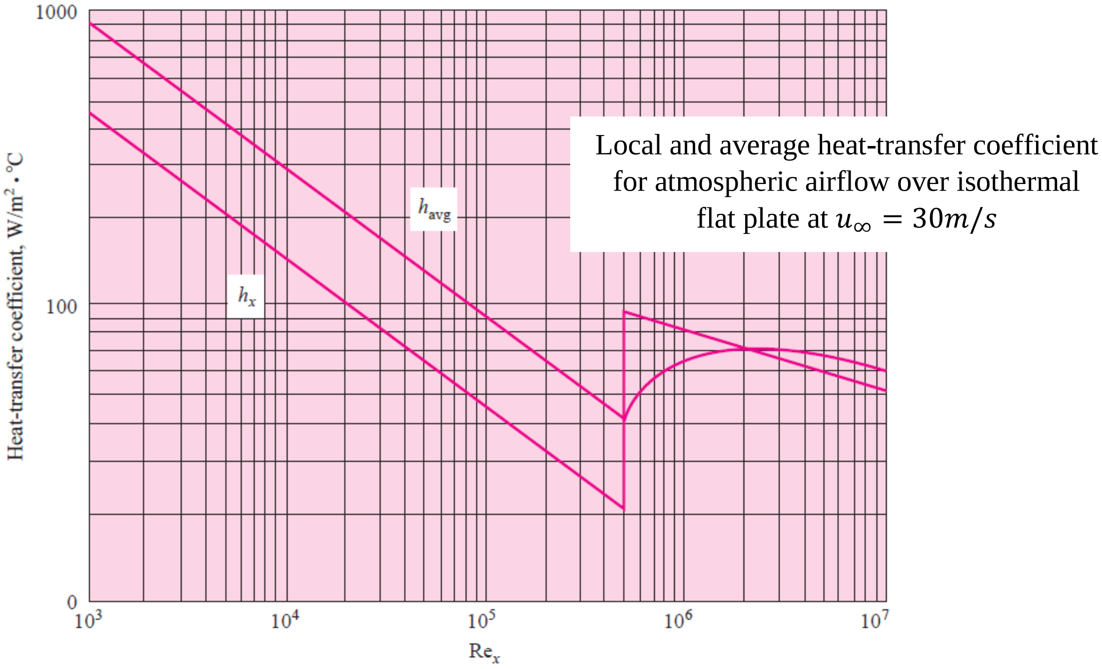
For case 2 we have the condition $\delta = \delta_{\mathrm{lam}}\ at\ x_{crit} = 5 \times 10^5 \frac{\nu}{u_\infty}$

with $\delta_{\mathrm{lam}} = 5.0\, \mathrm{x_{crit}}(5 \times 10^5)\frac{\nu}{u_\infty}$ and integrate (231):

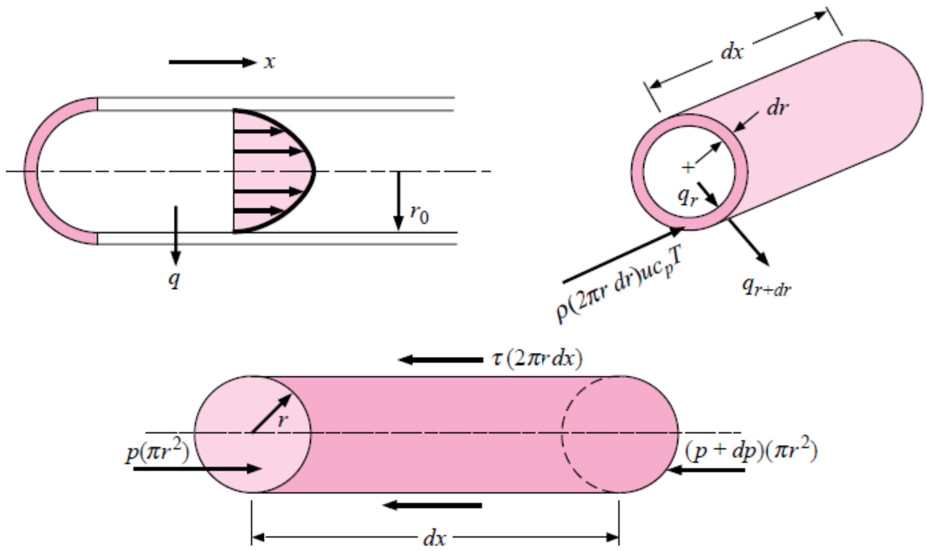
$$\delta - \delta_{\mathrm{lam}} = \frac{72}{7}\ 0.0296\ \left(\frac{\nu}{u_\infty}\right)^{\frac{1}{5}} \frac{5}{4}\left(x^{\frac{4}{5}} - x_{crit}^{\frac{4}{5}}\right) \tag{235}$$

$$\frac{\delta}{x} = 0.381\, Re_x^{-1/5} - 10256\, Re_x^{-1} \tag{236}$$

This relation applies only for the region $5 \times 10^5 < \mathrm{Re_x} < 10^7$.



Heat Transfer in Laminar and Turbulence Tube Flow:



The pressure forces are balanced by the viscous shear forces so that:

$$\pi r^2 dp = \tau 2\pi r dx = 2\pi r\mu\, dx \frac{du}{dr} \tag{237}$$

$$\text{or} \quad du = \frac{1}{2\mu} r \frac{dp}{dx} dr \tag{238}$$

With boundary condition u=0 at r=r_0:

$$u = \frac{1}{4\mu}\frac{dp}{dx}(r^2 - r_0^2) \tag{239}$$

Velocity at the center of the tube:

$$u_0 = -\frac{r_0^2}{4\mu}\frac{dp}{dx} \tag{240}$$

The velocity distribution can be written as:

$$\frac{\mathrm{u}}{\mathrm{u}_0} = 1 - \left(\frac{r}{r_0}\right)^2 \tag{241}$$

To simplify the analysis, we assume that there is a constant heat flux at the tube wall: $\frac{dq_w}{dx} = 0$.

The heat flow conducted into the annular element is: $dq_r = -k2\pi r dx \frac{\partial T}{\partial r}$

The heat conducted out the annular element is: $dq_r = -k2\pi(r+dr)dx\left(\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}dr\right)$

The net heat convected out of the element is: $dq_0 = 2\pi r dr\, \rho c_p u \frac{\partial T}{\partial x}dx$ (242)

Energy Balance: $$r\rho c_p u \frac{\partial T}{\partial x} dxdr = k\left(\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right)dxdr \quad (243)$$

or $$\frac{1}{\mathrm{ur}}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x} \quad (244)$$

We assume that the heat flux at the wall is constant, so that the average fluid temperature must increase linearly with *x*, or $\frac{\partial \mathrm{T}}{\partial \mathrm{x}} = const.$

Hence we have the Boundary Condition: $\frac{\partial \mathrm{T}}{\partial \mathrm{x}} = 0$ at $r = 0$ and $k\frac{\partial T}{\partial r}_{r=r_0} = q_w = const$ (245)

Substitute the velocity profile: $$\frac{1}{\mathrm{r}}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_0\left(1-\frac{r^2}{r_0^2}\right)r \quad (246)$$

Integration yields: $$T - T_c = \frac{1}{\alpha}\frac{\partial T}{\partial x}\frac{u_0 r_0^2}{4}\left(\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4\right) \quad (247)$$

where T = T_c at r = 0.

In tube flow, the convection heat-transfer coefficient is usually defined by

Local heat flux: $$q'' = h(T_w - T_b) \quad (248)$$

T_b is so-called *bulk temperature*: $$\mathrm{T_b} = \frac{\int_0^{r_0}\rho 2\pi r\, dr\, u c_p T}{\int_0^{r_0}\rho 2\pi r\, dr\, u c_p} \quad (249)$$

Solving (249) with (247), $$\mathrm{T_b} = \mathrm{T_c} + \frac{7}{96}\frac{u_0 r_0^2}{\alpha}\frac{\partial T}{\partial x} \quad (250)$$

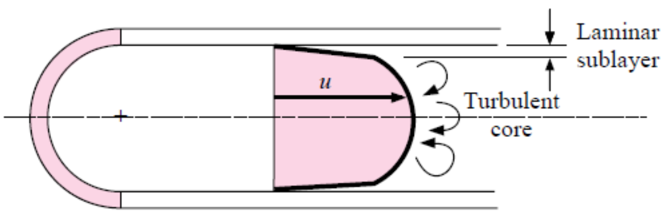
And for wall temperature: $$\mathrm{T_w} = T_c + \frac{3}{16}\frac{u_0 r_0^2}{\alpha}\frac{\partial T}{\partial x} \quad (251)$$

The heat-transfer coefficient is calculated from $$q = hA(T_w - T_b) = kA\left(\frac{\partial T}{\partial r}\right)_{r=r_0} \quad (252)$$

$$h = \frac{k\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_w - T_b} \quad (253)$$

The temperature gradient is given by

$$\frac{\partial T}{\partial r}_{r=r_0} = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\left(\frac{r}{2} - \frac{r^3}{4r_0^2}\right)_{r=r_0} = \frac{u_0 r_0}{4\alpha}\frac{\partial T}{\partial x} \quad (254)$$



Again, for turbulence flow we have: $$\frac{q}{\rho c_p A} = -(\alpha + \epsilon_H)\frac{dT}{dy} \quad (255)$$

where ϵ_H is eddy diffusivity of heat.

In similar fashion, shear stress in turbulent flow: $$\frac{\tau}{\rho} = \left(\frac{\mu}{\rho} + \epsilon_M\right)\frac{du}{dy} = (\upsilon + \epsilon_M)\frac{du}{dy} \quad (256)$$

where ϵ_M is eddy diffusivity of momentum.

Assuming heat and momentum are transported at the same rate, i.e. $\epsilon_H = \epsilon_M$ and $\nu = \alpha$ (or Pr=1)

Dividing (255)/(256): $$\frac{q}{c_p A\tau}\, du = -dT \quad (257)$$

Assume also ratio of heat transfer per unit area to shear stress is uniform across the flow field.

i.e. $\frac{q}{A\tau} = const = \frac{q_w}{A_w\tau_w}$

Then integrating (257) between wall conditions and mean bulk condition gives:

$\frac{q_w}{A_w\tau_w c_p}\int_0^{u_m} du = -\int_{T_w}^{T_b} dT$: $\quad \frac{hA_w(T_w-T_b)u_m}{A_w\tau_w c_p} = T_w - T_b$ (258)

The shear stress may be calculated from: $\quad \tau_w = \frac{\Delta p(\pi d_0^2)}{4\pi d_0 L}$ (259)

The pressure drop can be expressed in terms of a friction factor f by $\quad \Delta p = f\frac{L}{d}\frac{\rho u_m^2}{2}$ (260)

Therefore, $\quad \tau_w = \frac{f}{8}\rho u_m^2$ (261)

Substitute (261) into (258), $\quad St = \frac{h}{\rho c_p u_m} = \frac{Nu_d}{Re_d \mathrm{Pr}} = \frac{f}{8}$ (262)

(262) is the Reynolds analogy for tube flow. It relates heat-transfer rate to friction loss in tube flow.

An empirical formula for the turbulent-friction factor up to Reynold numbers of about 2×10^5 for the flow in smooth tubes is $f = \frac{0.316}{Re_d^{1/4}}$. Insert this into (261), we get $\quad \mathrm{Nu_d} = 0.0395\, Re_d^{3/4}$ (263)

Heat Transfer in High Speed Flow:

Define stagnation enthalpy = $i_0 = i_\infty + \frac{1}{2g}u_\infty^2$. This equation can be written as $c_p(T_0 - T_\infty) = \frac{1}{2g}u_\infty^2$, for T_0 is the stagnation temperature and T_∞ is the static free-stream temperature. Expressing with Ma, we have $\frac{T_0}{T_\infty} = 1 + \frac{k-1}{2}Ma_\infty^2$, for $Ma_\infty = \frac{u_\infty}{c}$ and $c = \sqrt{kRT}$. In the actual case of boundary layer flow, the flow is irreversible for viscous lost. To take into account for the irreversibilities, we have a recovery factor is defined as $r = \frac{T_{aw}-T_\infty}{T_0 - T_\infty}$, where T_{aw} is the actual adiabatic wall temperature and T_∞ is the static temperature of the free stream. The boundary layer energy equation $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$. The high speed heat transfer analysis can be done with the same relations used for low-speed incompressible flow when average heat transfer coefficient can be redefined with $q = \bar{h}A(T_w - T_{aw})$. For laminar flow and turbulent flow, r can be approximated as $Pr^{1/2}$ and $Pr^{1/3}$ respectively. In high velocity boundary layers substantial temperature gradients may occurs and properties will have large variation in the boundary layer. The constant-property heat transfer equations may still be used if a reference temperature is introduced. $T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22(T_{aw} - T_\infty)$. The analogy between heat transfer and fluid friction may be used when the friction coefficient is known. (264)

To summarize,

Laminar Boundary Layer ($Re_x < 5 \times 10^5$): $\quad St_x^*\, Pr^{*2/3} = 0.332\, Re_x^{*-1/2}$

Turbulent Boundary Layer ($5 \times 10^5 < Re_x < 10^7$): $\quad St_x^* Pr^{*2/3} = 0.0296\, Re_x^{*-1/5}$

($10^7 < Re_x$): $\quad St_x^* Pr^{*2/3} = 0.185(\log Re_x^*)^{-2.584}$ (265)

To obtain the average heat transfer coefficient the above expressions must be integrated over the length of the plate. When very high flow velocities are encountered, the adiabatic wall temperature may become so high that dissociation of gas will take place and there will be a wide range for the properties. These problems should be treated on the basis of a heat transfer coefficient defined in terms of enthalpy difference $q = h_i A(i_w - i_{aw})$ and the enthalpy recovery factor is defined as $r_i = \frac{i_{aw}-i_\infty}{i_0 - i_\infty}$, where i_{aw} is the enthalpy at the adiabatic wall conditions. A reference enthalpy i^* is given by $i^* = i_\infty + 0.5(i_w - i_\infty) + 0.22(i_{aw} - i_\infty)$ and Stanton number is redefined as $St_i = \frac{h_i}{\rho u_\infty}$. (266)

e.g.

A flat plate 70 cm long and 1.0 m wide is placed in a wind tunnel where the flow conditions are $M = 3$, $p = \frac{1}{20}$ atm, and $T = -40°C$. How much cooling must be used to maintain the plate temperature at 35°C?

We must consider the laminar and turbulent portions of the boundary layer separately because the recovery factors, and hence the adiabatic wall temperatures, used to establish the heat flow will be different for each flow regime. It turns out that the difference is rather small in this problem, but we shall follow a procedure that would be used if the difference were appreciable, so that the general method of solution may be indicated. The free-stream acoustic velocity is calculated from

$$a = \sqrt{\gamma g_c R T_\infty} = [(1.4)(1.0)(287)(233)]^{1/2} = 306 \text{ m/s} \quad [1003 \text{ ft/s}]$$

so that the free-stream velocity is

$$u_\infty = (3)(306) = 918 \text{ m/s} \quad [3012 \text{ ft/s}]$$

The maximum Reynolds number is estimated by making a computation based on properties evaluated at free-stream conditions:

$$\rho_\infty = \frac{(1.0132 \times 10^5)(\frac{1}{20})}{(287)(233)} = 0.0758 \text{ kg/m}^3 \quad [4.73 \times 10^{-3} \text{ lb}_m/\text{ft}^3]$$

$$\mu_\infty = 1.434 \times 10^{-5} \text{ kg/m} \cdot \text{s} \quad [0.0347 \text{ lb}_m/\text{h} \cdot \text{ft}]$$

$$\text{Re}_{L,\infty} = \frac{(0.0758)(918)(0.70)}{1.434 \times 10^{-5}} = 3.395 \times 10^6$$

Thus we conclude that both laminar and turbulent-boundary-layer heat transfer must be considered. We first determine the reference temperatures for the two regimes and then evaluate properties at these temperatures.

Laminar portion

$$T_0 = T_\infty \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) = (233)[1 + (0.2)(3)^2] = 652 \text{ K}$$

Assuming a Prandtl number of about 0.7, we have

$$r = \text{Pr}^{1/2} = (0.7)^{1/2} = 0.837$$

$$r = \frac{T_{aw} - T_\infty}{T_0 - T_\infty} = \frac{T_{aw} - 233}{652 - 233}$$

and $T_{aw} = 584 \text{ K} = 311°\text{C}$ [592°F]. Then the reference temperature from Equation (5-123) is

$$T^* = 233 + (0.5)(308 - 233) + (0.22)(584 - 233) = 347.8 \text{ K}$$

Checking the Prandtl number at this temperature, we have

$$\text{Pr}^* = 0.697$$

so that the calculation is valid. If there were an appreciable difference between the value of Pr* and the value used to determine the recovery factor, the calculation would have to be repeated until agreement was reached.

The other properties to be used in the laminar heat-transfer analysis are

$$\rho^* = \frac{(1.0132 \times 10^5)(1/20)}{(287)(347.8)} = 0.0508 \text{ kg/m}^3$$

$$\mu^* = 2.07 \times 10^{-5} \text{ kg/m} \cdot \text{s}$$

$$k^* = 0.03 \text{ W/m} \cdot °\text{C} \quad [0.0173 \text{ Btu/h} \cdot \text{ft} \cdot °\text{F}]$$

$$c_p^* = 1.009 \text{ kJ/kg} \cdot °\text{C}$$

Turbulent portion

Assuming $\mathrm{Pr} = 0.7$ gives

$$r = \mathrm{Pr}^{1/3} = 0.888 = \frac{T_{aw} - T_\infty}{T_0 - T_\infty} = \frac{T_{aw} - 233}{652 - 233}$$

$$T_{aw} = 605\ \mathrm{K} = 332°\mathrm{C}$$

$$T^* = 233 + (0.5)(308 - 233) + (0.22)(605 - 233) = 352.3\ \mathrm{K}$$

$$\mathrm{Pr}^* = 0.695$$

The agreement between Pr* and the assumed value is sufficiently close. The other properties to be used in the turbulent heat-transfer analysis are

$$\rho^* = \frac{(1.0132 \times 10^5)(1/20)}{(287)(352.3)} = 0.0501\ \mathrm{kg/m^3}$$

$$\mu^* = 2.09 \times 10^{-5}\ \mathrm{kg/m \cdot s}$$

$$k^* = 0.0302\ \mathrm{W/m \cdot °C} \qquad c_p^* = 1.009\ \mathrm{kJ/kg \cdot °C}$$

Laminar heat transfer

We assume

$$\mathrm{Re}^*_{\mathrm{crit}} = 5 \times 10^5 = \frac{\rho^* u_\infty x_c}{\mu^*}$$

$$x_c = \frac{(5 \times 10^5)(2.07 \times 10^{-5})}{(0.0508)(918)} = 0.222\ \mathrm{m}$$

$$\overline{\mathrm{Nu}^*} = \frac{\overline{h} x_c}{k^*} = 0.664 \left(\mathrm{Re}^*_{\mathrm{crit}}\right)^{1/2}\ \mathrm{Pr}^{*1/3}$$

$$= (0.664)(5 \times 10^5)^{1/2}(0.697)^{1/3} = 416.3$$

$$\overline{h} = \frac{(416.3)(0.03)}{0.222} = 56.25\ \mathrm{W/m^2 \cdot °C} \quad [9.91\ \mathrm{Btu/h \cdot ft^2 \cdot °F}]$$

This is the average heat-transfer coefficient for the laminar portion of the boundary layer, and the heat transfer is calculated from

$$q = \overline{h} A (T_w - T_{aw})$$

$$= (56.26)(0.222)(308 - 584)$$

$$= -3445\ \mathrm{W} \quad [-11{,}750\ \mathrm{Btu/h}]$$

so that 3445 W of cooling is required in the laminar region of the plate per meter of depth in the z direction.

Turbulent heat transfer

To determine the turbulent heat transfer we must obtain an expression for the local heat-transfer coefficient from

$$\mathrm{St}^*_x\ \mathrm{Pr}^{*2/3} = 0.0296\ \mathrm{Re}_x^{*-1/5}$$

and then integrate from $x = 0.222$ m to $x = 0.7$ m to determine the total heat transfer:

$$h_x = \mathrm{Pr}^{*-2/3} \rho^* u_\infty c_p (0.0296) \left(\frac{\rho^* u_\infty x}{\mu^*}\right)^{-1/5}$$

Inserting the numerical values for the properties gives

$$h_x = 94.34 x^{-1/5}$$

The average heat-transfer coefficient in the turbulent region is determined from

$$\bar{h} = \frac{\int_{0.222}^{0.7} h_x\, dx}{\int_{0.222}^{0.7} dx} = 111.46 \text{ W/m}^2 \cdot {}^\circ\text{C} \quad [19.6 \text{ Btu/h} \cdot \text{ft}^2 \cdot {}^\circ\text{F}]$$

Using this value we may calculate the heat transfer in the turbulent region of the flat plate:

$$\begin{aligned} q &= \bar{h} A (T_w - T_{aw}) \\ &= (111.46)(0.7 - 0.222)(308 - 605) \\ &= -15{,}823 \text{ W} \quad [-54{,}006 \text{ Btu/h}] \end{aligned}$$

The total amount of cooling required is the sum of the heat transfers for the laminar and turbulent portions:

$$\text{Total cooling} = 3445 + 15{,}823 = 19{,}268 \text{ W} \quad [65{,}761 \text{ Btu/h}]$$

These calculations assume unit depth of 1 m in the z direction.

Forced Convection for Heat Transfer: (6)

Flow regime	Restrictions	Equation
		Heat transfer
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.332\, \text{Pr}^{1/3} \text{Re}_x^{1/2}$
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $\text{Re}_x\, \text{Pr} > 100$	$\text{Nu}_x = \dfrac{0.3387\, \text{Re}_x^{1/2}\, \text{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0468}{\text{Pr}}\right)^{2/3}\right]^{1/4}}$
Laminar, local	$q_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.453\, \text{Re}_x^{1/2}\, \text{Pr}^{1/3}$
Laminar, local	$q_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$	$\text{Nu}_x = \dfrac{0.4637\, \text{Re}_x^{1/2}\, \text{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0207}{\text{Pr}}\right)^{2/3}\right]^{1/4}}$
Laminar, average	$\text{Re}_L < 5 \times 10^5$, $T_w = \text{const}$	$\overline{\text{Nu}}_L = 2\, \text{Nu}_{x=L} = 0.664\, \text{Re}_L^{1/2}\, \text{Pr}^{1/3}$
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $\text{Pr} \ll 1$ (liquid metals)	$\text{Nu}_x = 0.564 (\text{Re}_x\, \text{Pr})^{1/2}$
Laminar, local	$T_w = \text{const}$, starting at $x = x_0$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.332\, \text{Pr}^{1/3}\, \text{Re}_x^{1/2} \left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3}$
Turbulent, local	$T_w = \text{const}$, $5 \times 10^5 < \text{Re}_x < 10^7$	$\text{St}_x\, \text{Pr}^{2/3} = 0.0296\, \text{Re}_x^{-0.2}$
Turbulent, local	$T_w = \text{const}$, $10^7 < \text{Re}_x < 10^9$	$\text{St}_x\, \text{Pr}^{2/3} = 0.185 (\log \text{Re}_x)^{-2.584}$
Turbulent, local	$q_w = \text{const}$, $5 \times 10^5 < \text{Re}_x < 10^7$	$\text{Nu}_x = 1.04\, \text{Nu}_{x\,T_w = \text{const}}$
Laminar-turbulent, average	$T_w = \text{const}$, $\text{Re}_x < 10^7$, $\text{Re}_{\text{crit}} = 5 \times 10^5$	$\overline{\text{St}}\, \text{Pr}^{2/3} = 0.037\, \text{Re}_L^{-0.2} - 871\, \text{Re}_L^{-1}$ $\overline{\text{Nu}}_L = \text{Pr}^{1/3} (0.037\, \text{Re}_L^{0.8} - 871)$
Laminar-turbulent, average	$T_w = \text{const}$, $\text{Re}_x < 10^7$, liquids, μ at T_∞, μ_w at T_w	$\overline{\text{Nu}}_L = 0.036\, \text{Pr}^{0.43} (\text{Re}_L^{0.8} - 9200) \left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$
High-speed flow	$T_w = \text{const}$, $q = h A (T_w - T_{aw})$ $r = (T_{aw} - T_\infty)/(T_o - T_\infty)$ = recovery factor $= \text{Pr}^{1/2}$ (laminar) $= \text{Pr}^{1/3}$ (turbulent)	Same as for low-speed flow with properties evaluated at $T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22(T_{aw} - T_\infty)$

		Boundary-layer thickness
Laminar	$\mathrm{Re}_x < 5 \times 10^5$	$\frac{\delta}{x} = 5.0\ \mathrm{Re}_x^{-1/2}$
Turbulent	$\mathrm{Re}_x < 10^7$, $\delta = 0$ at $x = 0$	$\frac{\delta}{x} = 0.381\ \mathrm{Re}_x^{-1/5}$
Turbulent	$5 \times 10^5 < \mathrm{Re}_x < 10^7$, $\mathrm{Re}_{crit} = 5 \times 10^5$, $\delta = \delta_{lam}$ at Re_{crit}	$\frac{\delta}{x} = 0.381\ \mathrm{Re}_x^{-1/5} - 10{,}256\ \mathrm{Re}_x^{-1}$
		Friction coefficients
Laminar, local	$\mathrm{Re}_x < 5 \times 10^5$	$C_{fx} = 0.332\ \mathrm{Re}_x^{-1/2}$
Turbulent, local	$5 \times 10^5 < \mathrm{Re}_x < 10^7$	$C_{fx} = 0.0592\ \mathrm{Re}_x^{-1/5}$
Turbulent, local	$10^7 < \mathrm{Re}_x < 10^9$	$C_{fx} = 0.37(\log \mathrm{Re}_x)^{-2.584}$
Turbulent, average	$\mathrm{Re}_{crit} < \mathrm{Re}_x < 10^9$	$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re}_L)^{2.584}} - \frac{A}{\mathrm{Re}_L}$, A from Table 5-1

Forced Convection: (6)

Subscripts: b = bulk temperature, f = film temperature, ∞ = free stream temperature, w = wall temperature

Geometry	Equation	Restrictions
Tube flow	$\mathrm{Nu}_d = 0.023\ \mathrm{Re}_d^{0.8}\mathrm{Pr}^n$	Fully developed turbulent flow, $n = 0.4$ for heating, $n = 0.3$ for cooling, $0.6 < \mathrm{Pr} < 100$, $2500 < \mathrm{Re}_d < 1.25 \times 10^5$
Tube flow	$\mathrm{Nu}_d = 0.0214(\mathrm{Re}_d^{0.8} - 100)\mathrm{Pr}^{0.4}$	$0.5 < \mathrm{Pr} < 1.5$, $10^4 < \mathrm{Re}_d < 5 \times 10^6$
	$\mathrm{Nu}_d = 0.012(\mathrm{Re}_d^{0.87} - 280)\mathrm{Pr}^{0.4}$	$1.5 < \mathrm{Pr} < 500$, $3000 < \mathrm{Re}_d < 10^6$
Tube flow	$\mathrm{Nu}_d = 0.027\ \mathrm{Re}_d^{0.8}\mathrm{Pr}^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$	Fully developed turbulent flow
Tube flow, entrance region	$\mathrm{Nu}_d = 0.036\ \mathrm{Re}_d^{0.8}\mathrm{Pr}^{1/3}\left(\frac{d}{L}\right)^{0.055}$	Turbulent flow
	See also Figures 6-5 and 6-6	$10 < \frac{L}{d} < 400$
Tube flow	Petukov relation	Fully developed turbulent flow, $0.5 < \mathrm{Pr} < 2000$, $10^4 < \mathrm{Re}_d < 5 \times 10^6$, $0 < \frac{\mu_b}{\mu_w} < 40$
Tube flow	$\mathrm{Nu}_d = 3.66 + \frac{0.0668(d/L)\ \mathrm{Re}_d\mathrm{Pr}}{1 + 0.04[(d/L)\ \mathrm{Re}_d\mathrm{Pr}]^{2/3}}$	Laminar, $T_w = \text{const.}$
Tube flow	$\mathrm{Nu}_d = 1.86(\mathrm{Re}_d\mathrm{Pr})^{1/3}\left(\frac{d}{L}\right)^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$	Fully developed laminar flow, $T_w = \text{const.}$ $\mathrm{Re}_d\mathrm{Pr}\frac{d}{L} > 10$
Rough tubes	$\mathrm{St}_b\mathrm{Pr}_f^{2/3} = \frac{f}{8}$ or Equation (6-7)	Fully developed turbulent flow
Noncircular ducts	Reynolds number evaluated on basis of hydraulic diameter $D_H = \frac{4A}{P}$, A = flow cross-section area, P = wetted perimeter	Same as particular equation for tube flow

Flow across cylinders	$\mathrm{Nu}_f = C\,\mathrm{Re}_{df}^n\,\mathrm{Pr}^{1/3}$ C and n from Table 6-2	$0.4 < \mathrm{Re}_{df} < 400{,}000$
Flow across cylinders	$\mathrm{Nu}_{df} = 0.3 + \dfrac{0.62\,\mathrm{Re}_f^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(\dfrac{0.4}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}}\left[1+\left(\dfrac{\mathrm{Re}_f}{282{,}000}\right)^{5/8}\right]^{4/5}$	$10^2 < \mathrm{Re}_f < 10^7$, $\mathrm{Pe} > 0.2$
Flow across cylinders		See text
Flow across noncircular cylinders	$\mathrm{Nu} = C\,\mathrm{Re}_{df}^n\,\mathrm{Pr}^{1/3}$ See Table 6-3 for values of C and n.	
Flow across spheres	$\mathrm{Nu}_{df} = 0.37\,\mathrm{Re}_{df}^{0.6}$	$\mathrm{Pr} \sim 0.7$ (gases), $17 < \mathrm{Re} < 70{,}000$
	$\mathrm{Nu}_d\mathrm{Pr}^{-0.3}(\mu_w/\mu)^{0.25} = 1.2 + 0.53\,\mathrm{Re}_d^{0.54}$	Water and oils $1 < \mathrm{Re} < 200{,}000$ Properties at T_∞
	$\mathrm{Nu}_d = 2 + \left(0.4\,\mathrm{Re}_d^{1/2} + 0.06\,\mathrm{Re}_d^{2/3}\right)\mathrm{Pr}^{0.4}(\mu_\infty/\mu_w)^{1/4}$	$0.7 < \mathrm{Pr} < 380$, $3.5 < \mathrm{Re}_d < 80{,}000$, Properties at T_∞
Flow across tube banks	$\mathrm{Nu}_f = C\mathrm{Re}_{f,\max}^n\,\mathrm{Pr}_f^{1/3}$ C and n from Table 6-4	See text
Flow across tube banks	$\mathrm{Nu}_d = C\mathrm{Re}_{d,\max}^n\,\mathrm{Pr}^{0.36}\left(\dfrac{\mathrm{Pr}}{\mathrm{Pr}_w}\right)^{1/4}$	$0.7 < \mathrm{Pr} < 500$, $10 < \mathrm{Re}_{d,\max} < 10^6$
Liquid metals		See text
Friction factor	$\Delta p = f(L/d)\rho u_m^2/2g_c$, $u_m = \dot m/\rho A_c$	

Order of magnitude analysis: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (267)$$

$$u \sim u_\infty,\ y \sim \delta,\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\ \frac{u_\infty}{x} + \frac{v}{\delta} \approx 0,\ v \sim \frac{u_\infty \delta}{x} \quad (268)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2},\ u_\infty\frac{u_\infty}{x} + \frac{u_\infty \delta}{x}\frac{u_\infty}{\delta} \approx \upsilon\frac{u_\infty}{\delta^2} \quad (269)$$

$$\delta^2 \sim \frac{\upsilon x}{u_\infty},\ \delta \sim \sqrt{\frac{\upsilon x}{u_\infty}} \quad (270)$$

$$\frac{\delta}{x} \sim \sqrt{\frac{\upsilon}{u_\infty x}} = \frac{1}{\sqrt{Re_x}} \quad (271)$$

Pipe Flow and Tube Flow:

$$q = mc_p(T_{b_2} - T_{b_1}) \quad (272)$$

for ΔT_b= bulk temperature difference, provided that c_p = const. over the length. In differential length *dx* the heat added *dq* can be expressed either in terms of bulk temperature difference or in terms of heat transfer coefficient.

$$dq = \dot m c_p dT_b = h(2\pi r)dx(T_w - T_b) \quad (273)$$

The total heat transfer can be expressed as: $$q = hA(T_w - T_b)_{av} \quad (274)$$

A traditional expression for calculation of heat transfer in fully developed turbulent flow in smooth tubes is that $$\mathrm{Nu_d} = 0.023\,Re_d^{0.8}Pr^n \quad (275)$$

for n = 0.4 for heating and 0.3 for cooling. This is true only for Pr = 0.6 ~ 100.

Other equations:

$$\mathrm{Nu} = 0.0214(\mathrm{Re}^{0.8} - 100)\mathrm{Pr}^{0.4} \qquad 0.5 < \mathrm{Pr} < 1.5 \text{ and } 10^4 < \mathrm{Re} < 5 \times 10^6$$

$$\mathrm{Nu} = 0.012(\mathrm{Re}^{0.87} - 280)\mathrm{Pr}^{0.4} \qquad 1.5 < \mathrm{Pr} < 500 \text{ and } 3000 < \mathrm{Re} < 10^6 \quad (276)$$

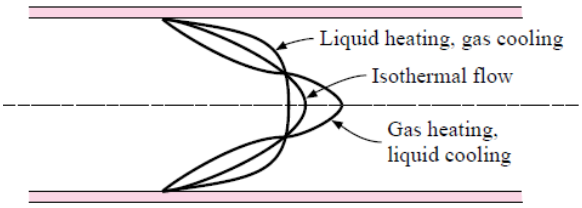
A power relation is used: $$\mathrm{Nu_d} = C\,Re_d^m Pr^n \quad (277)$$

Experimental Results: $$\mathrm{Nu_d} = 0.027\,Re_d^{0.8}Pr^{\frac{1}{3}}\left(\frac{\mu}{\mu_w}\right)^{0.14} \text{ for fully developed turbulence flow}$$

$$\mathrm{Nu_d} = 0.036\,Re_d^{0.8}Pr^{\frac{1}{3}}\left(\frac{d}{L}\right)^{0.055} \text{ for entrance flow.} \quad (278)$$

A more accurate relation: $$\mathrm{Nu_d} = \frac{(f/8)Re_d Pr}{1.07+12.7(f/8)^{\frac{1}{2}}\left(Pr^{\frac{2}{3}}-1\right)}\left(\frac{\mu_b}{\mu_w}\right)^n \quad (279)$$

where n = 0.11 for $T_w > T_b$, n = 0.25 for $T_w < T_b$, n=0 for constant heat flux or for gases.
All properties are evaluated at $T_f = \frac{T_w+T_b}{2}$ except for μ_b and μ_w. Friction factor may be contained in Moody Chart or $f = (1.82 \log Re_d - 1.64)^{-2}$. (280)



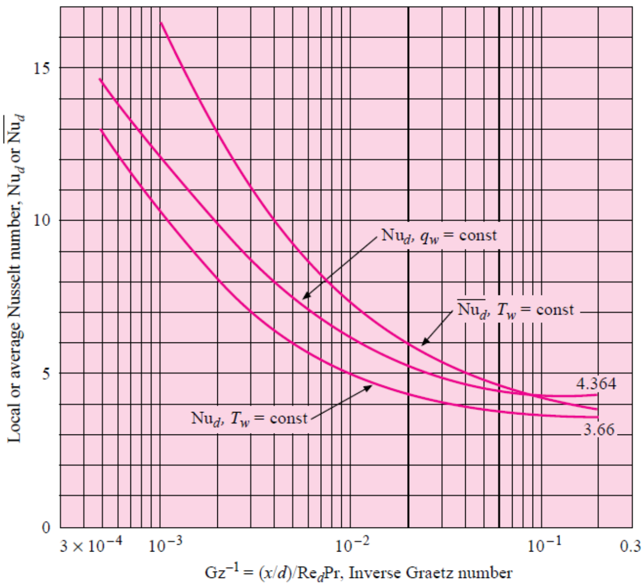
For fully developed laminar flow in tubes at constant wall temperature:
$$\overline{\mathrm{Nu}}_\mathrm{d} = 1.86(Re_d Pr)^{\frac{1}{3}}\left(\frac{d}{L}\right)^{\frac{1}{3}}\left(\frac{\mu}{\mu_w}\right)^{0.14} \quad (281)$$
In this formula the average heat-transfer coefficient is based on mean of the inlet and outlet temperature difference and all fluid properties are evaluated at the mean bulk temperature of the fluid, except μ_w evaluated at the wall temperature.

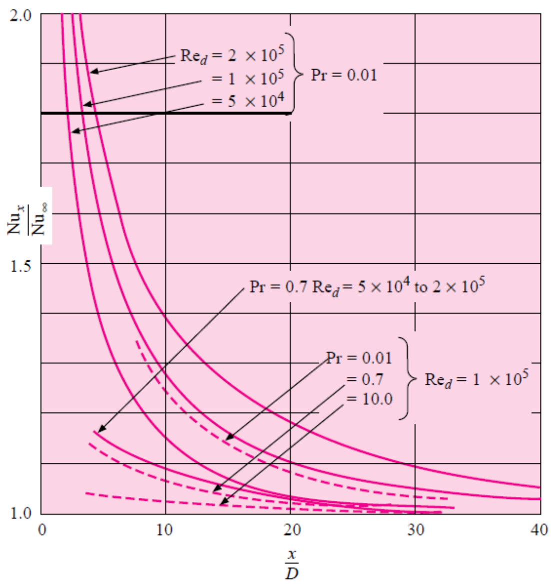
(281) cannot be used for extremely long tubes since it would yield a zero heat transfer coefficient.
(281) is only valid for $Re_d Pr\, d/L > 10$.

For rough tubes, $$\mathrm{St_b Pr}_f^{2/3} = \frac{f}{8} \quad \text{and} \quad f = \frac{1.325}{\left(\ln\left(\frac{\varepsilon}{3.7d}\right)+\frac{5.74}{Re_d^{0.9}}\right)^2} \quad (282)$$

for $10^{-6} < \frac{\varepsilon}{d} < 10^{-3}$ and $5000 < Re_d < 10^8$.
If the channel is not circular, we may adopt hydraulic diameter D_H $$\mathrm{D_H} = \frac{4A}{P} \quad (283)$$

For entrance flow, Graetz Number is used to determine if it is entrance or fully developed.
$$\mathrm{Gz} = \mathrm{Re\,Pr}\frac{d}{x} \quad (284)$$





e.g.

Air at 2 atm and 200°C is heated as it flows through a tube with a diameter of 1 in (2.54 cm) at a velocity of 10 m/s. Calculate the heat transfer per unit length of tube if a constant-heat-flux condition is maintained at the wall and the wall temperature is 20°C above the air temperature, all along the length of the tube. How much would the bulk temperature increase over a 3-m length of the tube?

We first calculate the Reynolds number to determine if the flow is laminar or turbulent, and then select the appropriate empirical correlation to calculate the heat transfer. The properties of air at a bulk temperature of 200°C are

$$\rho = \frac{p}{RT} = \frac{(2)(1.0132\times 10^5)}{(287)(473)} = 1.493\ \mathrm{kg/m^3} \qquad [0.0932\ \mathrm{lb}_m/\mathrm{ft}^3]$$

$$\mathrm{Pr} = 0.681$$

$$\mu = 2.57\times 10^{-5}\ \mathrm{kg/m\cdot s} \qquad [0.0622\ \mathrm{lb}_m/\mathrm{h\cdot ft}]$$

$$k = 0.0386\ \mathrm{W/m\cdot {}^\circ C} \qquad [0.0223\ \mathrm{Btu/h\cdot ft\cdot {}^\circ F}]$$

$$c_p = 1.025\ \mathrm{kJ/kg\cdot {}^\circ C}$$

$$\mathrm{Re}_d = \frac{\rho u_m d}{\mu} = \frac{(1.493)(10)(0.0254)}{2.57\times 10^{-5}} = 14{,}756$$

so that the flow is turbulent. We therefore use Equation (6-4a) to calculate the heat-transfer coefficient.

$$\mathrm{Nu}_d = \frac{hd}{k} = 0.023\,\mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{0.4} = (0.023)(14{,}756)^{0.8}(0.681)^{0.4} = 42.67$$

$$h = \frac{k}{d}\mathrm{Nu}_d = \frac{(0.0386)(42.67)}{0.0254} = 64.85\ \mathrm{W/m^2\cdot {}^\circ C} \qquad [11.42\ \mathrm{Btu/h\cdot ft^2\cdot {}^\circ F}]$$

The heat flow per unit length is then

$$\frac{q}{L} = h\pi d(T_w - T_b) = (64.85)\pi(0.0254)(20) = 103.5\ \mathrm{W/m} \qquad [107.7\ \mathrm{Btu/ft}]$$

We can now make an energy balance to calculate the increase in bulk temperature in a 3.0-m length of tube:

$$q = \dot m c_p \Delta T_b = L\left(\frac{q}{L}\right)$$

We also have

$$\dot m = \rho u_m \frac{\pi d^2}{4} = (1.493)(10)\pi\frac{(0.0254)^2}{4}$$

$$= 7.565\times 10^{-3}\ \mathrm{kg/s} \qquad [0.0167\ \mathrm{lb}_m/\mathrm{s}]$$

so that we insert the numerical values in the energy balance to obtain

$$(7.565\times 10^{-3})(1025)\Delta T_b = (3.0)(103.5)$$

and

$$\Delta T_b = 40.04^\circ\mathrm{C} \qquad [104.07^\circ\mathrm{F}]$$

e.g.

A 2.0-cm-diameter tube having a relative roughness of 0.001 is maintained at a constant wall temperature of 90°C. Water enters the tube at 40°C and leaves at 60°C. If the entering velocity is 3 m/s, calculate the length of tube necessary to accomplish the heating.

We first calculate the heat transfer from

$$q = \dot{m} c_p \Delta T_b = (989)(3.0)\pi(0.01)^2(4174)(60-40) = 77{,}812 \text{ W}$$

For the rough-tube condition, we may employ the Petukhov relation, Equation (6-7). The mean film temperature is

$$T_f = \frac{90+50}{2} = 70°\text{C}$$

and the fluid properties are

$$\rho = 978 \text{ kg/m}^3 \qquad \mu = 4.0 \times 10^{-4} \text{ kg/m} \cdot \text{s}$$
$$k = 0.664 \text{ W/m} \cdot °\text{C} \qquad \text{Pr} = 2.54$$

Also,

$$\mu_b = 5.55 \times 10^{-4} \text{ kg/m} \cdot \text{s}$$
$$\mu_w = 2.81 \times 10^{-4} \text{ kg/m} \cdot \text{s}$$

The Reynolds number is thus

$$\text{Re}_d = \frac{(978)(3)(0.02)}{4 \times 10^{-4}} = 146{,}700$$

Consulting Figure 6-4, we find the friction factor as

$$f = 0.0218 \qquad f/8 = 0.002725$$

Because $T_w > T_b$, we take $n = 0.11$ and obtain

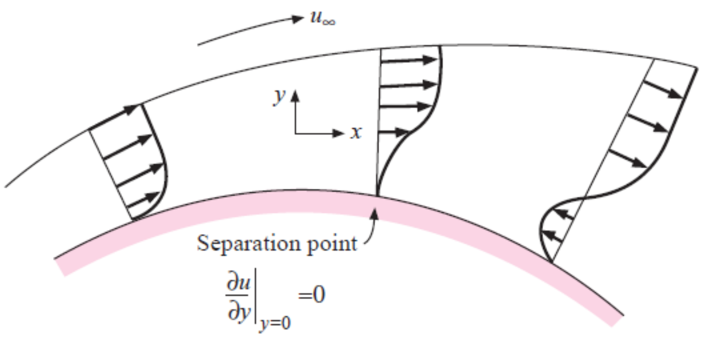
$$\text{Nu}_d = \frac{(0.002725)(146{,}700)(2.54)}{1.07 + (12.7)(0.002725)^{1/2}(2.54^{2/3} - 1)} \left(\frac{5.55}{2.81}\right)^{0.11}$$
$$= 666.8$$

$$h = \frac{(666.8)(0.664)}{0.02} = 22138 \text{ W/m}^2 \cdot °\text{C}$$

The tube length is then obtained from the energy balance

$$q = \bar{h}\pi d L(\overline{T_w} - \overline{T_b}) = 77{,}812 \text{ W}$$
$$L = 1.40 \text{ m}$$

Flow across Cylinders and Spheres:



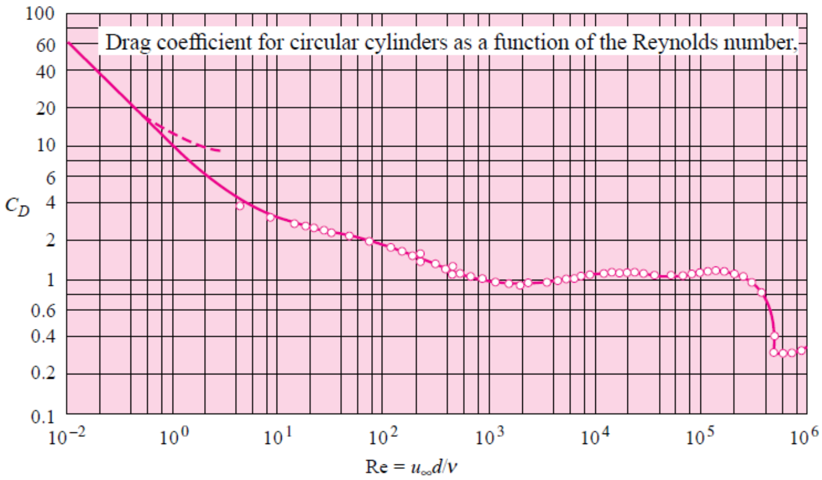
The pressure increase and reduction in velocity are related through the Bernoulli equation written along a streamline. $\frac{dp}{\rho} = -d(\frac{u^2}{2g})$ (285)

When the velocity gradient at the surface becomes zero, the flow is said to have reacted a separation, i.e. Separation point at $\frac{\partial u}{\partial y}|_{y=0} = 0$ (286)

As the flow proceeds past the separate point, reverse-flow phenomena occur and eventually separation-flow region on the back side of the cylinder becomes turbulent and random in motion.

The drag coefficient for bluff bodies is defined as: $$\mathrm{F_D} = \mathrm{C_D} A \frac{\rho u_\infty^2}{2g} \qquad (287)$$

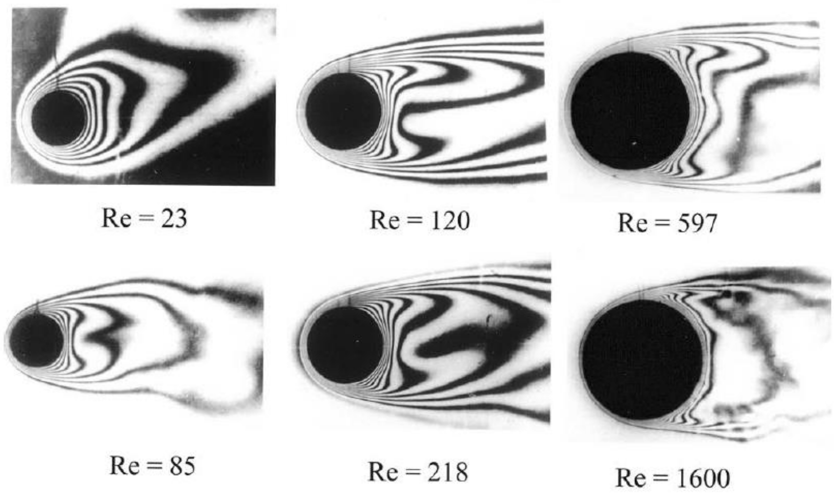
where C_D is the drag coefficient and A is the frontal area of the body exposed to the flow.



The resulting correlation for average heat transfer coefficient in cross flow over cylinders is

$$\mathrm{Nu_{df}} = \frac{hd}{k_f} = C\left(\frac{u_\infty d}{\nu_f}\right)^n Pr_f^{1/3} \qquad (288)$$

Re_{df}	C	n
0.4–4	0.989	0.330
4–40	0.911	0.385
40–4000	0.683	0.466
4000–40,000	0.193	0.618
40,000–400,000	0.0266	0.805



For heat transfer from tubes in cross flow,

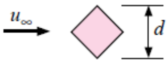
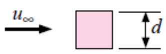
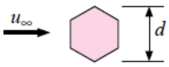
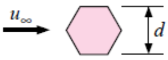
$$\mathrm{Nu} = (0.43 + 0.50\,\mathrm{Re}^{0.5}) Pr^{0.38}\left(\frac{Pr_f}{Pr_w}\right)^{0.25} \qquad \text{for } 1 < \mathrm{Re} < 10^3$$

$$\mathrm{Nu} = 0.25\,\mathrm{Re}^{0.6} Pr^{0.38}\left(\frac{Pr_f}{Pr_w}\right)^{0.25} \qquad \text{for } 10^3 < \mathrm{Re} < 2 \times 10^5 \qquad (289)$$

Or $$\mathrm{Nu_d} = 0.3 + \frac{0.62\,Re^{0.5}Pr^{\frac{1}{3}}}{\left(1+\left(\frac{0.4}{\mathrm{Pr}}\right)^{\frac{2}{3}}\right)^{\frac{1}{4}}}\left(1+\left(\frac{Re}{282000}\right)^{\frac{5}{8}}\right)^{\frac{4}{5}} \qquad (290)$$

for $10^2 < \mathrm{Re_d} < 10^7$; $\mathrm{Pe_d} > 0.2$.

For non-circular cylinder, we may substitute the following into (288)

Geometry	Re_{df}	C	n
	$5 \times 10^3 - 10^5$	0.246	0.588
	$5 \times 10^3 - 10^5$	0.102	0.675
	$5 \times 10^3 - 1.95 \times 10^4$	0.160	0.638
	$1.95 \times 10^4 - 10^5$	0.0385	0.782
	$5 \times 10^3 - 10^5$	0.153	0.638
	$4 \times 10^3 - 1.5 \times 10^4$	0.228	0.731

For Sphere, McAdams recommends to a flowing gases: $$\frac{\mathrm{hd}}{\mathrm{k_f}} = 0.37\left(\frac{u_\infty d}{v_f}\right)^{0.6} \qquad (291)$$

Whitaker recommends for gases and liquids:

$$\mathrm{Nu} = 2 + \left(0.4\mathrm{Re_d^{0.5}} + 0.06Re_d^{\frac{2}{3}}\right)Pr^{0.4}\left(\frac{\mu_\infty}{\mu_w}\right)^{0.25} \qquad (292)$$

e.g.

Air at 1 atm and 35°C flows across a 5.0-cm-diameter cylinder at a velocity of 50 m/s. The cylinder surface is maintained at a temperature of 150°C. Calculate the heat loss per unit length of the cylinder.

$$T_f = \frac{T_w + T_\infty}{2} = \frac{150+35}{2} = 92.5°\mathrm{C} = 365.5\ \mathrm{K}$$

$$\rho_f = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(365.5)} = 0.966\ \mathrm{kg/m^3} \quad [0.0603\ \mathrm{lb}_m/\mathrm{ft}^3]$$

$$\mu_f = 2.14 \times 10^{-5}\ \mathrm{kg/m \cdot s} \quad [0.0486\ \mathrm{lb}_m/\mathrm{h \cdot ft}]$$

$$k_f = 0.0312\ \mathrm{W/m \cdot °C} \quad [0.018\ \mathrm{Btu/h \cdot ft \cdot °F}]$$

$$\mathrm{Pr}_f = 0.695$$

$$\mathrm{Re}_f = \frac{\rho u_\infty d}{\mu} = \frac{(0.966)(50)(0.05)}{2.14 \times 10^{-5}} = 1.129 \times 10^5$$

From Table 6-2

$$C = 0.0266 \qquad n = 0.805$$

so from Equation (6-17)

$$\frac{hd}{k_f} = (0.0266)(1.129 \times 10^5)^{0.805}(0.695)^{1/3} = 275.1$$

$$h = \frac{(275.1)(0.0312)}{0.05} = 171.7\ \mathrm{W/m^2 \cdot °C} \qquad [30.2\ \mathrm{Btu/h \cdot ft^2 \cdot °F}]$$

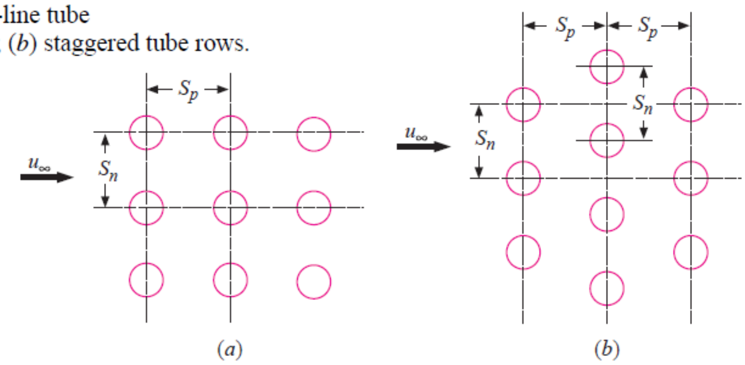
The heat transfer per unit length is therefore

$$\begin{aligned}\frac{q}{L} &= h\pi d(T_w - T_\infty)\\ &= (171.7)\pi(0.05)(150-35)\\ &= 3100\ \mathrm{W/m} \quad [3226\ \mathrm{Btu/ft}]\end{aligned}$$

Flow across Tube Banks:

For flow in heat exchanger, passing through multiple rows of tubes, the minimum frontal area ($S_n – d$) constrained the maximum flow velocity. $\quad u_{max} = u_\infty\left(\frac{S_n}{S_n - d}\right)$ (in line arrangement) (293)

(*a*) in-line tube rows; (*b*) staggered tube rows.



$\frac{S_p}{d}$	$\frac{S_n}{d}$ = 1.25 C	1.25 n	1.5 C	1.5 n	2.0 C	2.0 n	3.0 C	3.0 n
In line								
1.25	0.386	0.592	0.305	0.608	0.111	0.704	0.0703	0.752
1.5	0.407	0.586	0.278	0.620	0.112	0.702	0.0753	0.744
2.0	0.464	0.570	0.332	0.602	0.254	0.632	0.220	0.648
3.0	0.322	0.601	0.396	0.584	0.415	0.581	0.317	0.608
Staggered								
0.6	—	—	—	—	—	—	0.236	0.636
0.9	—	—	—	—	0.495	0.571	0.445	0.581
1.0	—	—	0.552	0.558	—	—	—	—
1.125	—	—	—	—	0.531	0.565	0.575	0.560
1.25	0.575	0.556	0.561	0.554	0.576	0.556	0.579	0.562
1.5	0.501	0.568	0.511	0.562	0.502	0.568	0.542	0.568
2.0	0.448	0.572	0.462	0.568	0.535	0.556	0.498	0.570
3.0	0.344	0.592	0.395	0.580	0.488	0.562	0.467	0.574

N	1	2	3	4	5	6	7	8	9	10
Ratio for staggered tubes	0.68	0.75	0.83	0.89	0.92	0.95	0.97	0.98	0.99	1.0
Ratio for in-line tubes	0.64	0.80	0.87	0.90	0.92	0.94	0.96	0.98	0.99	1.0

(ratio of *h* for *N* rows deep to that for 10 rows deep)

Recall equation (288): $\quad Nu_{df} = \frac{hd}{k_f} = C\left(\frac{u_\infty d}{\nu_f}\right)^n Pr_f^{1/3}$

For the staggered case, flow enters the tube bank through the area $S_n – d$ and then splits into two $((S_n/2)^2 + S_p^2)^{1/2} – d$. If the sum of these two areas is less than $S_n – d$, then they will represent the minimum flow area and the maximum velocity in the tube bank will be

$$u_{max} = \frac{\left(u_\infty\left(\frac{S_n}{2}\right)\right)}{\left(\left(\frac{S_n}{2}\right)^2 + S_p^2\right)^{\frac{1}{2}} - d} \tag{294}$$

Pressure drop for flow of gases over a tube bank may be calculated with

$$\Delta p = \frac{2f' G_{max}^2 N}{\rho}\left(\frac{\mu_w}{\mu_b}\right)^{0.14} \tag{295}$$

where G_{max} = mass velocity at minimum flow area and μ_b = average free-stream viscosity.

The empirical friction factor $$f' = \left\{0.25 + \frac{0.118}{\left(\frac{S_n - d}{d}\right)^{1.08}}\right\} Re_{max}^{-0.16} \quad \text{for staggered tube} \tag{296}$$

and $$f' = \left\{0.044 + \frac{\frac{0.08 S_p}{d}}{\left(\frac{(S_n - d)}{d}\right)^{0.43} + \frac{1.13d}{S_p}}\right\} Re_{max}^{-0.15} \quad \text{for in line tube} \tag{297}$$

The correlating equations for tube banks with 20 rows or more:

$$\mathrm{Nu} = \frac{\bar{\mathrm{h}}\mathrm{d}}{\mathrm{k}} = C\, Re_{d,max}^{n} Pr^{0.36} \left(\frac{Pr}{Pr_w}\right)^{\frac{1}{4}} \tag{298}$$

Geometry	$Re_{d,max}$	C	n
In-line	10–100	0.8	0.4
	$100–10^3$	Treat as individual tubes	
	$10^3 - 2 \times 10^5$	0.27	0.63
	$> 2 \times 10^5$	0.21	0.84
Staggered	10–100	0.9	0.4
	$100–10^3$	Treat as individual tubes	
	$10^3 - 2 \times 10^5$	$0.35\left(\frac{S_n}{S_L}\right)^{0.2}$ for $\frac{S_n}{S_L} < 2$	0.60
	$10^3 - 2 \times 10^5$	0.40 for $\frac{S_n}{S_L} > 2$	0.60
	$> 2 \times 10^5$	0.022	0.84

Ratio of h for N rows deep to that for 20 rows deep

N	2	3	4	5	6	8	10	16	20
Staggered	0.77	0.84	0.89	0.92	0.94	0.97	0.98	0.99	1.0
In-line	0.70	0.80	0.90	0.92	0.94	0.97	0.98	0.99	1.0

e.g.

Air at 1 atm and 10°C flows across a bank of tubes 15 rows high and 5 rows deep at a velocity of 7 m/s measured at a point in the flow before the air enters the tube bank. The surfaces of the tubes are maintained at 65°C. The diameter of the tubes is 1 in [2.54 cm]; they are arranged in an in-line manner so that the spacing in both the normal and parallel directions to the flow is 1.5 in [3.81 cm]. Calculate the total heat transfer per unit length for the tube bank and the exit air temperature.

$$\frac{S_p}{d} = \frac{3.81}{2.54} = 1.5 \qquad \frac{S_n}{d} = \frac{3.81}{2.54} = 1.5$$

so that

$$C = 0.278 \qquad n = 0.620$$

The properties of air are evaluated at the film temperature, which at entrance to the tube bank is

$$T_{f_1} = \frac{T_w + T_\infty}{2} = \frac{65 + 10}{2} = 37.5°\mathrm{C} = 310.5\ \mathrm{K} \quad [558.9°\mathrm{R}]$$

Then

$$\rho_f = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(310.5)} = 1.137\ \mathrm{kg/m^3}$$
$$\mu_f = 1.894 \times 10^{-5}\ \mathrm{kg/m \cdot s}$$
$$k_f = 0.027\ \mathrm{W/m \cdot °C} \quad [0.0156\ \mathrm{Btu/h \cdot ft \cdot °F}]$$
$$c_p = 1007\ \mathrm{J/kg \cdot °C} \quad [0.24\ \mathrm{Btu/lb_m \cdot °F}]$$
$$\mathrm{Pr} = 0.706$$

The maximum velocity is thus

$$u_{\max} = u_\infty \frac{S_n}{S_n - d} = \frac{(7)(3.81)}{3.81 - 2.54} = 21 \text{ m/s} \quad [68.9 \text{ ft/s}] \qquad [a]$$

where u_∞ is the incoming velocity before entrance to the tube bank. The Reynolds number is computed by using the maximum velocity.

$$\mathrm{Re} = \frac{\rho u_{\max} d}{\mu} = \frac{(1.137)(21)(0.0254)}{1.894 \times 10^{-5}} = 32{,}020 \qquad [b]$$

$$\frac{hd}{k_f} = (0.278)(32{,}020)^{0.62}(0.706)^{1/3} = 153.8 \qquad [c]$$

$$h = \frac{(153.8)(0.027)}{0.0254} = 164 \text{ W/m}^2 \cdot {}^\circ\text{C} \quad [28.8 \text{ Btu/h} \cdot \text{ft}^2 \cdot {}^\circ\text{F}] \qquad [d]$$

This is the heat-transfer coefficient that would be obtained if there were 10 rows of tubes in the direction of the flow. Because there are only 5 rows, this value must be multiplied by the factor 0.92, as determined from Table 6-5.

The total surface area for heat transfer, considering unit length of tubes, is

$$A = N\pi d(1) = (15)(5)\pi(0.0254) = 5.985 \text{ m}^2/\text{m}$$

where N is the total number of tubes.

Before calculating the heat transfer, we must recognize that the air temperature increases as the air flows through the tube bank. Therefore, this must be taken into account when using

$$q = hA(T_w - T_\infty) \qquad [e]$$

As a good approximation, we can use an arithmetic average value of T_∞ and write for the energy balance*

$$q = hA\left(T_w - \frac{T_{\infty,1} + T_{\infty,2}}{2}\right) = \dot{m}c_p(T_{\infty,2} - T_{\infty,1}) \qquad [f]$$

where now the subscripts 1 and 2 designate entrance and exit to the tube bank. The mass flow at entrance to the 15 tubes is

$$\dot{m} = \rho_\infty u_\infty (15) S_n$$

$$\rho_\infty = \frac{p}{RT_\infty} = \frac{1.0132 \times 10^5}{(287)(283)} = 1.246 \text{ kg/m}^3 \qquad [g]$$

$$\dot{m} = (1.246)(7)(15)(0.0381) = 4.99 \text{ kg/s} \quad [11.0 \text{ lb}_m/\text{s}]$$

so that Equation (f) becomes

$$(0.92)(164)(5.985)\left(65 - \frac{10 + T_{\infty,2}}{2}\right) = (4.99)(1006)(T_{\infty,2} - 10)$$

that may be solved to give

$$T_{\infty,2} = 19.08^\circ\text{C}$$

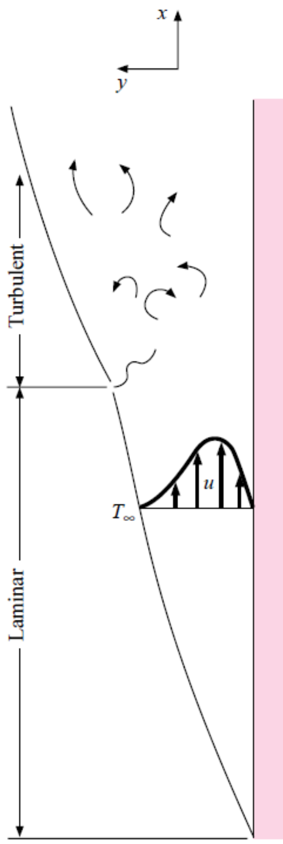
The heat transfer is then obtained from the right side of Equation (f):

$$q = (4.99)(1006)(19.08 - 10) = 45.6 \text{ kW/m}$$

Natural Convection: (7)

Geometry	Equation	Restrictions
A variety of isothermal surfaces	$\mathrm{Nu}_f = C(\mathrm{Gr}_f\,\mathrm{Pr}_f)^m$ C and m from Table 7-1	See Table 7-1
Vertical isothermal surface	$\overline{\mathrm{Nu}}^{1/2} = 0.825 + \frac{0.387\,\mathrm{Ra}^{1/6}}{[1+(0.492/\mathrm{Pr})^{9/16}]^{8/27}}$	$10^{-1} < \mathrm{Ra}_L < 10^{12}$ Also see Fig. 7-5
Vertical surface, constant heat flux, local h	$\mathrm{Nu}_{xf} = C(\mathrm{Gr}_x^*\mathrm{Pr}_f)^m$	$C = 0.60$, $m = \frac{1}{5}$ for $10^5 < \mathrm{Gr}_x^*\mathrm{Pr} < 10^{11}$ $C = 0.17$, $m = \frac{1}{4}$ for $2\times10^{13} < \mathrm{Gr}^*\mathrm{Pr} < 10^{16}$
Isothermal horizontal cylinders	$\overline{\mathrm{Nu}}^{1/2} = 0.60 + 0.387\left\{\frac{\mathrm{Gr\,Pr}}{[1+(0.559/\mathrm{Pr})^{9/16}]^{16/9}}\right\}^{1/6}$	$10^{-5} < \mathrm{Gr\,Pr} < 10^{13}$ Also see Fig. 7-6
Horizontal surface, constant heat flux		See text
Inclined surfaces	Section 7-7	See text
Spheres	$\mathrm{Nu} = 2 + 0.43(\mathrm{Gr\,Pr})^{1/4}$ $\mathrm{Nu} = 2 + 0.5(\mathrm{Gr\,Pr})^{1/4}$ $\mathrm{Nu} = 2 + \frac{0.589(\mathrm{Gr\,Pr})^{1/4}}{[1+(0.469/\mathrm{Pr})^{9/16}]^{4/9}}$	$1 < \mathrm{Gr\,Pr} < 10^5$ water, $3\times10^5 < \mathrm{Gr\,Pr} < 8\times10^8$ $0.5 < \mathrm{Pr}$ $\mathrm{Gr\,Pr} < 10^{11}$
Enclosed spaces	$q = k_e A(\Delta T/\delta)$ $\frac{k_e}{k} = C(\mathrm{Gr}_\delta\,\mathrm{Pr})^n\,(L/\delta)^m$	Constants C, m, and n from Table 7-3 Pure conduction for $\mathrm{Gr}_\delta\,\mathrm{Pr} < 2000$
Across evacuated spaces	Most transfer is by radiation	

The movement of fluid in free convection results form the buoyancy forces imposed on the fluid when its density in the proximity of the heat-transfer surface is decreased as a result of heating process. Gravity or centrifugal forces field can also form free-convection currents. The buoyancy forces that give rise to the free-convection currents are called body forces.



The momentum flux through the control volume dxdy is:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - \rho g + \mu\frac{\partial^2 u}{\partial y^2} \qquad (299)$$

The pressure gradient in the x direction results from the change in elevation up the plate. Thus, $\frac{\partial p}{\partial x} = -\rho_\infty g$ (300)

In other words, the change in pressure over a height dx is equal to the weight per unit area of the fluid element. Substituting (300) into (299):

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = g(\rho_\infty - \rho) + \mu\frac{\partial^2 u}{\partial y^2} \qquad (301)$$

The density difference $\rho_\infty - \rho$ may be expressed in terms of the volume expansivity β, defined by $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{V_\infty}\frac{V - V_\infty}{T - T_\infty} = \frac{\rho_\infty - \rho}{\rho(T - T_\infty)}$ (302)

so that

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = g\rho\beta(T - T_\infty) + \mu\frac{\partial^2 u}{\partial y^2} \qquad (303)$$

The energy equation for the free-convection system is the same as that for a forced convection system at a low velocity:

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} \qquad (304)$$

For ideal gases, $\beta = \frac{1}{T}$. (305)

For a free-convection system, the integral momentum equation becomes:

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy = -\tau_w + \int_0^\delta \rho g\beta(T - T_\infty)dy = -\mu\frac{\partial u}{\partial y}|_{y=0} + \int_0^\delta \rho g\beta(T - T_\infty)dy \qquad (306)$$

The following conditions apply for the temperature distribution:

$$\mathrm{T} = \mathrm{T_w} \text{ at } y = 0,\ T = T_\infty \text{ at } y = \delta,\ \frac{\partial T}{\partial y} = 0 \text{ at } y = \delta \qquad (307)$$

We can obtain for the temp distribution: $\frac{\mathrm{T}-\mathrm{T}_\infty}{\mathrm{T_w}-\mathrm{T}_\infty} = \left(1 - \frac{y}{\delta}\right)^2$ (308)

For velocity profile, the following condition should be satisfied. (309)

$$\mathrm{u} = 0 \text{ at } y = 0,\ u = 0 \text{ at } y = \delta,\ \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta \text{ and } \frac{\partial^2 u}{\partial y^2} = -g\beta\frac{T_w - T_\infty}{\nu} \text{ at } y = 0$$

We have: $\frac{\mathrm{u}}{\mathrm{u_x}} = \frac{\beta\delta^2 g(T_w - T_\infty)}{4u_x\nu}\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2$ (310)

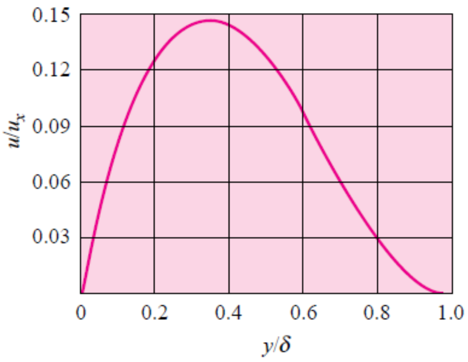
Absorbing the terms into u_x, we have $\frac{\mathrm{u}}{\mathrm{u_x}} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2$ (311)

Substitute (311) and (308) into (306), performing integration and differentiation,

$$\frac{1}{105}\frac{d}{dx}(u_x^2\delta) = \frac{1}{3}g\beta(T_w - T_\infty)\delta - \nu\frac{u_x}{\delta} \qquad (312)$$

The integral form of the energy equation for the free-convection system is:

$$\frac{d}{dx}\int_0^\delta u(T - T_\infty)dy = -\alpha\frac{dT}{dy}|_{y=0} \qquad (313)$$



When assumed velocity and temperature distributions are inserted into this equation, there results

$$\frac{1}{30}(T_w - T_\infty)\frac{d}{dx}(u_x\delta) = 2\alpha\frac{T_w - T_\infty}{\delta} \qquad (314)$$

Again, $u_x \sim \delta$, (311): $\delta \sim x^{1/4}$, then $u_x = C_1 x^{1/2}$ and $\delta = C_2 x^{1/4}$. Substituting into (312) and (314),

$$\frac{5}{420}C_1^2 C_2 x^{1/4} = g\beta(T_w - T_\infty)\frac{C_2}{3}\ x^{1/4} - \frac{C_1}{C_2}\ \nu x^{1/4} \qquad (315)$$

and

$$\frac{1}{40}C_1 C_2 x^{-1/4} = \frac{2\alpha}{C_2}x^{-1/4} \qquad (316)$$

Solving,

$$\frac{\delta}{x} = 3.93\ Pr^{-\frac{1}{2}}(0.952 + Pr)^{\frac{1}{4}}\ Gr_x^{-\frac{1}{4}} \qquad (317)$$

$$u_x\frac{x}{\upsilon} = 5.17(0.952 + Pr)^{-\frac{1}{2}}\ Gr_x^{\frac{1}{2}} \qquad (318)$$

Velocity profile reaches its maximum value at $\frac{y}{\delta} = \frac{1}{3}$, giving $\mathrm{u_{max}} = \frac{4}{27}u_x$. (319)

Mass flow through the boundary layer at any *x* position can be found by evaluating the integral

$$\dot{m} = \int \rho u dy = \int_0^\delta \rho u_x\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 dy = \frac{1}{12}\rho u_x\delta = \frac{9}{16}\rho u_{max}\delta \qquad (320)$$

The value of δ and u_x determined in (317) and (318) can be substituted into (320) to get mass flow.

Grashof number = ratio of buoyancy to viscous force: $\mathrm{Gr_x} = \frac{g\beta(T_w - T_\infty)x^3}{\upsilon^2}$ (321)

Using the temperature distribution of Eqt (308): $h = \frac{2k}{\delta}$ or $\frac{hx}{k} = Nu_x = 2\frac{x}{\delta}$ (322)

The dimensionless equation for heat-transfer coefficient becomes:

$$\mathrm{Nu_x} = 0.508\,\mathrm{Pr}^{\frac{1}{2}}(0.952 + \mathrm{Pr})^{-\frac{1}{4}}\mathrm{Gr_x^{\frac{1}{4}}} \quad (323)$$

The average heat-transfer coefficient may be obtained by performing the integration:

$$\bar{h} = \frac{1}{L}\int_0^L h_x dx \quad (324)$$

For a variation given by (323), the average coefficient is $\bar{h} = \frac{4}{3}h_{x=L}$. (325)

Average free-convection heat-transfer coefficients can be represented in: $\mathrm{Nu_f} = C\left(Gr_f Pr_f\right)^m$ (326)

where the subscript *f* indicates that the properties in the dimensionless group are evaluated at the film temperature $\mathrm{T_f} = \frac{T_\infty + T_w}{2}$. The produce of Grashof and Prandtl numbers is called Rayleigh number, i.e. Ra = Gr Pr.

Geometry	**$Gr_f\,Pr_f$**	***C***	***m***
Vertical planes and cylinders	10^{-1}–10^4	Use Fig. 7-5	Use Fig. 7-5
	10^4–10^9	0.59	$\frac{1}{4}$
	10^9–10^{13}	0.021	$\frac{2}{5}$
	10^9–10^{13}	0.10	$\frac{1}{3}$
Horizontal cylinders	0–10^{-5}	0.4	0
	10^{-5}–10^4	Use Fig. 7-6	Use Fig. 7-6
	10^4–10^9	0.53	$\frac{1}{4}$
	10^9–10^{12}	0.13	$\frac{1}{3}$
	10^{-10}–10^{-2}	0.675	0.058
	10^{-2}–10^2	1.02	0.148
	10^2–10^4	0.850	0.188
	10^4–10^7	0.480	$\frac{1}{4}$
	10^7–10^{12}	0.125	$\frac{1}{3}$
Upper surface of heated plates or lower surface of cooled plates	2×10^4–8×10^6	0.54	$\frac{1}{4}$
Upper surface of heated plates or lower surface of cooled plates	8×10^6–10^{11}	0.15	$\frac{1}{3}$
Lower surface of heated plates or upper surface of cooled plates	10^5–10^{11}	0.27	$\frac{1}{4}$
Vertical cylinder, height = diameter characteristic length = diameter	10^4–10^6	0.775	0.21
Irregular solids, characteristic length = distance fluid particle travels in boundary layer	10^4–10^9	0.52	$\frac{1}{4}$

Free Convection from Vertical Planes and Cylinders:

The general criterion for a vertical cylinder treated as a vertical flat plate when $\frac{\mathrm{D}}{\mathrm{L}} \geq \frac{35}{Gr_L^{1/4}}$ (327)

The flat plate results for the average heat-transfer coefficient should be multiplied by a factor F to account for the curvature, where $\mathrm{F} = 1.3\left(\frac{\frac{\mathrm{L}}{\mathrm{D}}}{\mathrm{Gr_D}}\right)^{\frac{1}{4}} + 1.0$ (328)

Churchill and Chu suggest: $\overline{\mathrm{Nu}}^{1/2} = 0.825 + \frac{0.387\mathrm{Ra}^{1/6}}{\left(1+\left(\frac{0.492}{\mathrm{Pr}}\right)^{9/16}\right)^{8/27}}$ for $10^{-1} < Ra_L < 10^{12}$ (329)

The above is a satisfactory representation for isothermal plates.

For Constant heat flux surface, the results are presented in terms of a modified Grashof number, Gr*:

$$\mathrm{Gr}_x^* = Gr_x Nu_x = \frac{g\beta q_w x^4}{k\nu} \quad (330)$$

where $q_w = q/A$ is the heat flux per unit area, and it is assumed constant over the entire plane.

The local heat-transfer coefficients were correlated by: $\mathrm{Nu}_{xf} = \frac{hx}{k_f} = 0.6\left(Gr_x^* Pr_f\right)^{1/5}$ (331)

It should be noted that the criterion for laminar flow expressed in terms of Gr_x^* is not the same as expressed in terms of Gr_x. (331) also assumes laminar flow. Boundary layer transition is observed to begin between $Gr_x^*Pr = 3 \times 10^{12}$ and 4×10^{13} and to end between 2×10^{13} and 10^{14}. Fully developed turbulent flow was present by $Gr_x^*Pr = 10^{14}$.

For the turbulent region, local heat transfer coefficient is: $Nu_x = 0.17(Gr_x^* Pr)^{1/4}$ (332)

For laminar region, using (331) to evaluate h_x and $\bar{h} = \frac{1}{L}\int_0^L h_x dx = \frac{5}{4}h_{x=L} \quad q_w = const.$ (333)

Writing Eqt (326) as a local heat-transfer form gives $Nu_x = C\,(Gr_x\,Pr)^m$ (334)

Inserting $Gr_x = Gr_x^* / Nu_x$ gives $Nu_x = C^{1/(1+m)}\,(Gr_x^*\,Pr)^{m/(1+m)}$ (335)

m = 1/4 for laminar ($h_x \sim \frac{1}{x}(x^3)^{1/4} = x^{-1/4}$) and m = 1/3 for turbulent ($h_x \sim \frac{1}{x}(x^3)^{1/3} = const$).

e.g.

In a plant location near a furnace, a net radiant energy flux of 800 W/m^2 is incident on a vertical metal surface 3.5 m high and 2 m wide. The metal is insulated on the back side and painted black so that all the incoming radiation is lost by free convection to the surrounding air at 30°C. What average temperature will be attained by the plate?

$$\Delta T = \frac{q_w}{h} \approx \frac{800}{10} = 80°\mathrm{C}$$

Then

$$T_f \approx \frac{80}{2} + 30 = 70°\mathrm{C} = 343\ \mathrm{K}$$

At 70°C the properties of air are

$$\nu = 2.043 \times 10^{-5}\ \mathrm{m^2/s} \qquad \beta = \frac{1}{T_f} = 2.92 \times 10^{-3}\ \mathrm{K^{-1}}$$

$$k = 0.0295\ \mathrm{W/m \cdot °C} \qquad \mathrm{Pr} = 0.7$$

From Equation (7-30), with $x = 3.5$ m,

$$\mathrm{Gr}_x^* = \frac{g\beta q_w x^4}{k\nu^2} = \frac{(9.8)(2.92 \times 10^{-3})(800)(3.5)^4}{(0.0295)(2.043 \times 10^{-5})^2} = 2.79 \times 10^{14}$$

We may therefore use Equation (7-32) to evaluate h_x:

$$\begin{aligned} h_x &= \frac{k}{x}(0.17)(\mathrm{Gr}_x^*\,\mathrm{Pr})^{1/4} \\ &= \frac{0.0295}{3.5}(0.17)(2.79 \times 10^{14} \times 0.7)^{1/4} \\ &= 5.36\ \mathrm{W/m^2 \cdot °C}\ [0.944\ \mathrm{Btu/h \cdot ft^2 \cdot °F}] \end{aligned}$$

In the turbulent heat transfer governed by Equation (7-32), we note that

$$\mathrm{Nu}_x = \frac{hx}{k} \sim (\mathrm{Gr}_x^*)^{1/4} \sim (x^4)^{1/4}$$

or h_x does not vary with x, and we may take this as the average value. The value of $h = 5.41\ \mathrm{W/m^2 \cdot {}^\circ C}$ is less than the approximate value we used to estimate T_f. Recalculating ΔT, we obtain

$$\Delta T = \frac{q_w}{h} = \frac{800}{5.36} = 149^\circ\mathrm{C}$$

Our new film temperature would be

$$T_f = 30 + \frac{149}{2} = 104.5^\circ\mathrm{C}$$

At 104.5°C the properties of air are

$$\nu = 2.354 \times 10^{-5}\ \mathrm{m^2/s} \qquad \beta = \frac{1}{T_f} = 2.65 \times 10^{-3}/\mathrm{K}$$

$$k = 0.0320\ \mathrm{W/m \cdot {}^\circ C} \qquad \mathrm{Pr} = 0.695$$

Then

$$\mathrm{Gr}_x^* = \frac{(9.8)(2.65 \times 10^{-3})(800)(3.5)^4}{(0.0320)(2.354 \times 10^{-5})^2} = 1.75 \times 10^{14}$$

and h_x is calculated from

$$\begin{aligned} h_x &= \frac{k}{x}(0.17)(\mathrm{Gr}_x^*\,\mathrm{Pr})^{1/4} \\ &= \frac{(0.0320)(0.17)}{3.5}[(1.758 \times 10^{14})(0.695)]^{1/4} \\ &= 5.17\ \mathrm{W/m^2 \cdot {}^\circ C}\ [-0.91\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}] \end{aligned}$$

Our new temperature difference is calculated as

$$\Delta T = (T_w - T_\infty)_{\mathrm{av}} = \frac{q_w}{h} = \frac{800}{5.17} = 155^\circ\mathrm{C}$$

The average wall temperature is therefore

$$T_{w,\mathrm{av}} = 155 + 30 = 185^\circ\mathrm{C}$$

Another iteration on the value of T_f is not warranted by the improved accuracy that would result.

Free Convection from Horizontal Planes and Cylinders:

$$\overline{\mathrm{Nu}}^{1/2} = 0.6 + 0.387\frac{(GrPr)^{1/6}}{\left(1+\left(\frac{0.559}{Pr}\right)^{\frac{9}{16}}\right)^{2/27}} \quad \text{for } 10^{-5} < \mathrm{Gr\,Pr} < 10^{12} \qquad (336)$$

e.g.

A fine wire having a diameter of 0.02 mm is maintained at a constant temperature of 54°C by an electric current. The wire is exposed to air at 1 atm and 0°C. Calculate the electric power necessary to maintain the wire temperature if the length is 50 cm.

The film temperature is $T_f = (54 + 0)/2 = 27^\circ\mathrm{C} = 300\ \mathrm{K}$, so the properties are

$$\beta = 1/300 = 0.00333 \qquad \nu = 15.69 \times 10^{-6}\ \mathrm{m^2/s}$$

$$k = 0.02624\ \mathrm{W/m \cdot {}^\circ C} \qquad \mathrm{Pr} = 0.708$$

The Gr Pr product is then calculated as

$$\mathrm{Gr\,Pr} = \frac{(9.8)(0.00333)(54-0)(0.02\times10^{-3})^3}{(15.69\times10^{-6})^2}(0.708) = 4.05\times10^{-5}$$

From Table 7-1 we find $C = 0.675$ and $m = 0.058$ so that

$$\overline{\mathrm{Nu}} = (0.675)(4.05\times10^{-5})^{0.058} = 0.375$$

and

$$\bar{h} = \overline{\mathrm{Nu}}\left(\frac{k}{d}\right) = \frac{(0.375)(0.02624)}{0.02\times10^{-3}} = 492.6\ \mathrm{W/m^2\cdot{}^\circ C}$$

The heat transfer or power required is then

$$q = \bar{h}A\,(T_w - T_\infty) = (492.6)\pi(0.02\times10^{-3})(0.5)(54-0) = 0.836\ \mathrm{W}$$

For horizontal plates under constant heat flux:

$$L = \frac{A}{P} \qquad \overline{Nu}_{\mathrm{L}} = 0.13(Gr_L Pr)^{\frac{1}{3}} \qquad \text{for } \mathrm{Gr_L\,Pr} < 2\times10^8$$

$$\overline{Nu}_{\mathrm{L}} = 0.16(Gr_L Pr)^{\frac{1}{3}} \qquad \text{for } 2\times10^8 < \mathrm{Gr_L\,Pr} < 10^{11}$$

For the heated surface facing downward,

$$\overline{Nu}_{\mathrm{L}} = 0.58(Gr_L Pr)^{\frac{1}{5}} \qquad \text{for } 10^6 < \mathrm{Gr_L\,Pr} < 10^{11} \tag{337}$$

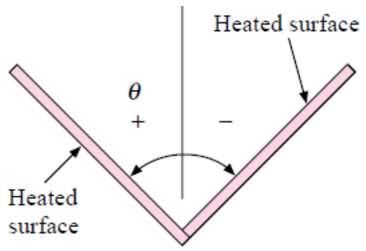
In these equations all properties except β are evaluated at a temperature T_e defined by

$$\mathrm{T_e} = T_w - 0.25(T_w - T_\infty) \tag{338}$$

T_w is the average wall temperature related to the heat flux by $\bar{h} = \frac{q_w}{T_w - T_\infty}$ (339)

The Nusselt number is formed as before, $\overline{Nu}_{\mathrm{L}} = \frac{\bar{h}L}{k} = \frac{q_w L}{(T_w - T_\infty)k}$ (340)

Free Convection from Inclined Surfaces:



For heated surface facing downward:

$$\overline{\mathrm{Nu}_{\mathrm{e}}} = 0.56(Gr_e Pr_e \cos\theta)^{\frac{1}{4}}$$

$$\theta < 88^o;\ 10^5 < Gr_e Pr_e \cos\theta < 10^{11} \tag{341}$$

In (341) all properties except β are evaluated at a reference temperature T_e defined by

$$\mathrm{T_e} = T_w - 0.25(T_w - T_\infty) \tag{342}$$

Where T_w is the mean wall temperature and T_∞ is the free-stream temperature. β is evaluated at a temperature $T_\infty + 0.50(T_w - T_\infty)$

For heated surface facing upward, the empirical correlation become more complicated. For -15° ~ -75°

$$\overline{\mathrm{Nu}_{\mathrm{e}}} = 0.14(\mathrm{Gr}_e Pr_e)^{\frac{1}{3}} - (\mathrm{Gr}_c Pr_e)\hat{\ }(\tfrac{1}{3})) + 0.56(Gr_e Pr_e \cos\theta)^{\frac{1}{4}} \tag{343}$$

where $\mathrm{Gr_c}$ is a critical Grashof number indicating when the Nusselt number starts to separate from the laminar relation of (341) and is given the following tabulation:

θ, degrees	Gr_c
−15	5×10^9
−30	2×10^9
−60	10^8
−75	10^6

For $\mathrm{Gr_e} < \mathrm{Gr_c}$, the first term of (343) is dropped out. (342)(343) can be applied to isothermal surface.

For constant-heat-flux surface, similar to (331), $$\mathrm{Nu_{xf}} = \frac{hx}{k_f} = 0.6\left(Gr_x^* \cos\theta\, Pr_f\right)^{1/5} \quad (344)$$

for both upward and downward facing heated surface. In the turbulent region with air, the following empirical correlation was obtained. $$\mathrm{Nu_x} = 0.17(Gr_x^* Pr)^{\frac{1}{4}} \quad 10^{10} < Gr_x Pr < 10^{15} \quad (345)$$

For inclined cylinders, the laminar heat transfer under constant-heat-flux conditions:

$$\mathrm{Nu_L} = (0.6 - 0.488\sin^{1.03}\theta)(Gr_L Pr)^{\frac{1}{4}+\frac{1}{12}\sin^{1.75}\theta} \quad (346)$$

Note that the above relation is useless when we consider **Non-Newtonian Fluid**.

Simplified equations for free-convection from various surfaces to air at 1 atm.

Surface	Laminar, $10^4 < Gr_f Pr_f < 10^9$	Turbulent, $Gr_f Pr_f > 10^9$
Vertical plane or cylinder	$h = 1.42\left(\frac{\Delta T}{L}\right)^{1/4}$	$h = 1.31(\Delta T)^{1/3}$
Horizontal cylinder	$h = 1.32\left(\frac{\Delta T}{d}\right)^{1/4}$	$h = 1.24(\Delta T)^{1/3}$
Horizontal plate:		
Heated plate facing upward or cooled plate facing downward	$h = 1.32\left(\frac{\Delta T}{L}\right)^{1/4}$	$h = 1.52(\Delta T)^{1/3}$
Heated plate facing downward or cooled plate facing upward	$h = 0.59\left(\frac{\Delta T}{L}\right)^{1/4}$	
Heated cube; L = length of side, Area $= 6L^2$	$h = 1.052\left(\frac{\Delta T}{L}\right)^{1/4}$	

where h = heat-transfer coefficient, W/m$^2 \cdot$ °C
$\Delta T = T_w - T_\infty$, °C
L = vertical or horizontal dimension, m
d = diameter, m

Free Convection from Spheres:
For sphere in air, free-convection heat transfer:

$$\mathrm{Nu_f} = \frac{\bar{h}d}{k_f} = 2 + 0.392 Gr_f^{\frac{1}{4}} = 2 + 0.43\left(Gr_f Pr_f\right)^{\frac{1}{4}} \quad \text{for } 1 < \mathrm{Gr_f} < 10^5 \quad (347)$$

For a wider range, from Churchill,

$$\mathrm{Nu} = \frac{2 + 0.589\mathrm{Ra_d}^{\frac{1}{4}}}{\left(1+\left(\frac{0.469}{Pr}\right)^{\frac{9}{16}}\right)^{\frac{4}{9}}} \quad \text{for } \mathrm{Ra_d} < 10^{11} \text{ and } \mathrm{Pr} > 0.5. \quad (348)$$

Free Convection in Enclosed Spaces:

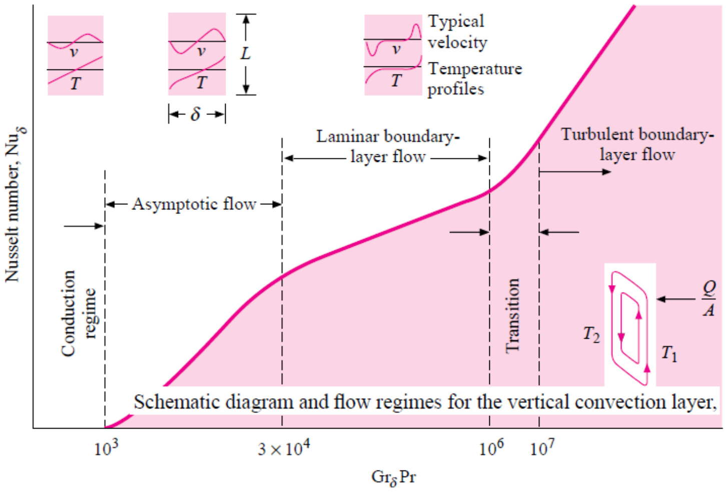
In the figure next page, Grashof number is calculated as: $$\mathrm{Gr_\delta} = \frac{g\beta(T_1 - T_2)\delta^3}{\upsilon^2} \quad (349)$$

At very low Grashofs, there are very minute free-convection current and the heat transfer occurs mainly by conduction. As the Grashof number is increased, different flow regimes are encountered, with progressively increasing heat transfer as expressed through Nusselt: $$\mathrm{Nu_\delta} = \frac{h\delta}{k} \quad (350)$$

For two plates constant heat flux condition:

$$\mathrm{Nu_\delta} = 0.42(\mathrm{Gr_\delta Pr})^{0.25}(\mathrm{Pr})^{0.012}\left(\frac{L}{\delta}\right)^{-0.30} \quad 10^4 < \mathrm{Gr_\delta Pr} < 10^7,\ 1 < \mathrm{Pr} < 20000,\ 10 < \frac{L}{\delta} < 40. \quad (351)$$

$$\mathrm{Nu_\delta} = 0.46(\mathrm{Gr_\delta Pr})^{\frac{1}{3}} \quad 10^6 < \mathrm{Gr_\delta Pr} < 10^9,\ 1 < \mathrm{Pr} < 20,\ 1 < \frac{L}{\delta} < 40 \quad (352)$$



Schematic diagram and flow regimes for the vertical convection layer,

The heat flux is calculated as: $\frac{\mathrm{q}}{\mathrm{A}} = q_w = h(T_1 - T_2) = Nu_\delta \frac{k}{\delta}(T_1 - T_2)$ (353)

The results are sometimes expressed in the alternate form of effective thermal conductivity k_e.

$$\frac{\mathrm{q}}{\mathrm{A}} = \frac{k_e(T_1 - T_2)}{\delta} \quad (354)$$

By comparing (353) and (354): $\mathrm{Nu}_\delta = \frac{k_e}{k}$ (355)

In building industry, heat transfer across an air gap is expressed in $\frac{q}{A} = \frac{\Delta T}{R}$ s.t. $R = \frac{\delta}{k_e}$ (356)

Transient natural convection heating/ cooling in closed vertical or horizontal cylindrical enclosure:

$$\mathrm{Nu_f} = 0.55\left(Gr_f Pr_f\right)^{\frac{1}{4}} \quad \text{for } 0.75 < \mathrm{L/d} < 2.0 \quad (357)$$

The effective thermal conductivity for fluids between concentric spheres:

$$\frac{k_e}{k} = 0.228\,(Gr_\delta Pr)^{0.226}, \quad \delta = \mathrm{r_0} - r_i \quad \text{such that } q = \frac{4\pi k_e r_i r_o \Delta T}{r_o - r_i} \quad (358)$$

Properties are evaluated at a volume mean temperature $\mathrm{T_m}$ defined as:

$$\mathrm{T}_{,} = \frac{(r_m^3 - r_i^3)T_i + (r_o^3 - r_m^3)T_o}{r_o^3 - r_i^3} \quad \text{where } r_m = \frac{r_i + r_o}{2} \quad (359)$$

Experimental results for free enclosures are not always in agreement, but we can express it in a general form: $\frac{k_e}{k} = C(Gr_\delta Pr)^n \left(\frac{L}{\delta}\right)^m$ (360)

Fluid	Geometry	Gr_δ Pr	Pr	$\frac{L}{\delta}$	C	n	m
Gas	Vertical plate, isothermal	< 2000	$k_e/k = 1.0$				
		6000–200,000	0.5–2	11–42	0.197	$\frac{1}{4}$	$-\frac{1}{9}$
		$200{,}000–1.1 \times 10^7$	0.5–2	11–42	0.073	$\frac{1}{3}$	$-\frac{1}{9}$
	Horizontal plate, isothermal heated from below	< 1700	$k_e/k = 1.0$				
		1700–7000	0.5–2	—	0.059	0.4	0
		$7000–3.2 \times 10^5$	0.5–2	—	0.212	$\frac{1}{4}$	0
		$> 3.2 \times 10^5$	0.5–2	—	0.061	$\frac{1}{3}$	0
Liquid	Vertical plate, constant heat flux or isothermal	< 2000	$k_e/k = 1.0$				
		$10^4–10^7$	1–20,000	10–40	Eq. 7-52	—	—
		$10^6–10^9$	1–20	1–40	0.046	$\frac{1}{3}$	0
	Horizontal plate, isothermal, heated from below	< 1700	$k_e/k = 1.0$	—			
		1700–6000	1–5000	—	0.012	0.6	0
		6000–37,000	1–5000	—	0.375	0.2	0
		$37{,}000–10^8$	1–20		0.13	0.3	0
		$> 10^8$	1–20		0.057	$\frac{1}{3}$	0
Gas or liquid	Vertical annulus	Same as vertical plates					
	Horizontal annulus, isothermal	$6000–10^6$	1–5000	—	0.11	0.29	0
		$10^6–10^8$	1–5000	—	0.40	0.20	0
	Spherical annulus	$120–1.1 \times 10^9$	0.7–4000	—	0.228	0.226	0

Radiation R-Value for a Gap: $\frac{q}{A} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}-1} = \frac{\Delta T}{R_{rad}}$ $\quad R_{conv} = \frac{\delta}{k_e}$ $\quad \frac{1}{R_{tot}} = \frac{1}{R_{rad}} + \frac{1}{R_{conv}}$ (361)

e.g.

Air at atmospheric pressure is contained between two 0.5-m-square vertical plates separated by a distance of 15 mm. The temperatures of the plates are 100 and 40°C, respectively. Calculate the free-convection heat transfer across the air space. Also calculate the radiation heat transfer across the air space if both surfaces have $\epsilon = 0.2$.

We evaluate the air properties at the mean temperature between the two plates:

$$T_f = \frac{100+40}{2} = 70°\text{C} = 343 \text{ K}$$

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(343)} = 1.029 \text{ kg/m}^3$$

$$\beta = \frac{1}{T_f} = \frac{1}{343} = 2.915 \times 10^{-3} \text{ K}^{-1}$$

$$\mu = 2.043 \times 10^{-5} \text{ kg/m} \cdot \text{s} \qquad k = 0.0295 \text{ W/m} \cdot °\text{C} \qquad \text{Pr} = 0.7$$

The Grashof-Prandtl number product is now calculated as

$$\text{Gr}_\delta \text{Pr} = \frac{(9.8)(1.029)^2(2.915 \times 10^{-3})(100-40)(15 \times 10^{-3})^3}{(2.043 \times 10^{-5})^2} 0.7$$

$$= 1.027 \times 10^4$$

We may now use Equation (7-64) to calculate the effective thermal conductivity, with $L = 0.5$ m, $\delta = 0.015$ m, and the constants taken from Table 7-3:

$$\frac{k_e}{k} = (0.197)(1.027 \times 10^4)^{1/4} \left(\frac{0.5}{0.015}\right)^{-1/9} = 1.343$$

The heat transfer may now be calculated with Equation (7-54). The area is $(0.5)^2 = 0.25$ m^2, so that

$$q = \frac{(1.343)(0.0295)(0.25)(100-40)}{0.015} = 39.62 \text{ W} \qquad [135.2 \text{ Btu/h}]$$

The radiation heat flux is calculated with Equation (7-67), taking $T_1 = 373$ K, $T_2 = 313$ K, and $\epsilon_1 = \epsilon_2 = 0.2$. Thus, with $\sigma = 5.669 \times 10^{-8}$ W/m$^2 \cdot$ K^4,

$$(q/A)_{\text{rad}} = \frac{(5.669 \times 10^{-8})(373^4 - 313^4)}{[1/0.2 + 1/0.2 - 1]} = 61.47 \text{ W/m}^2$$

and

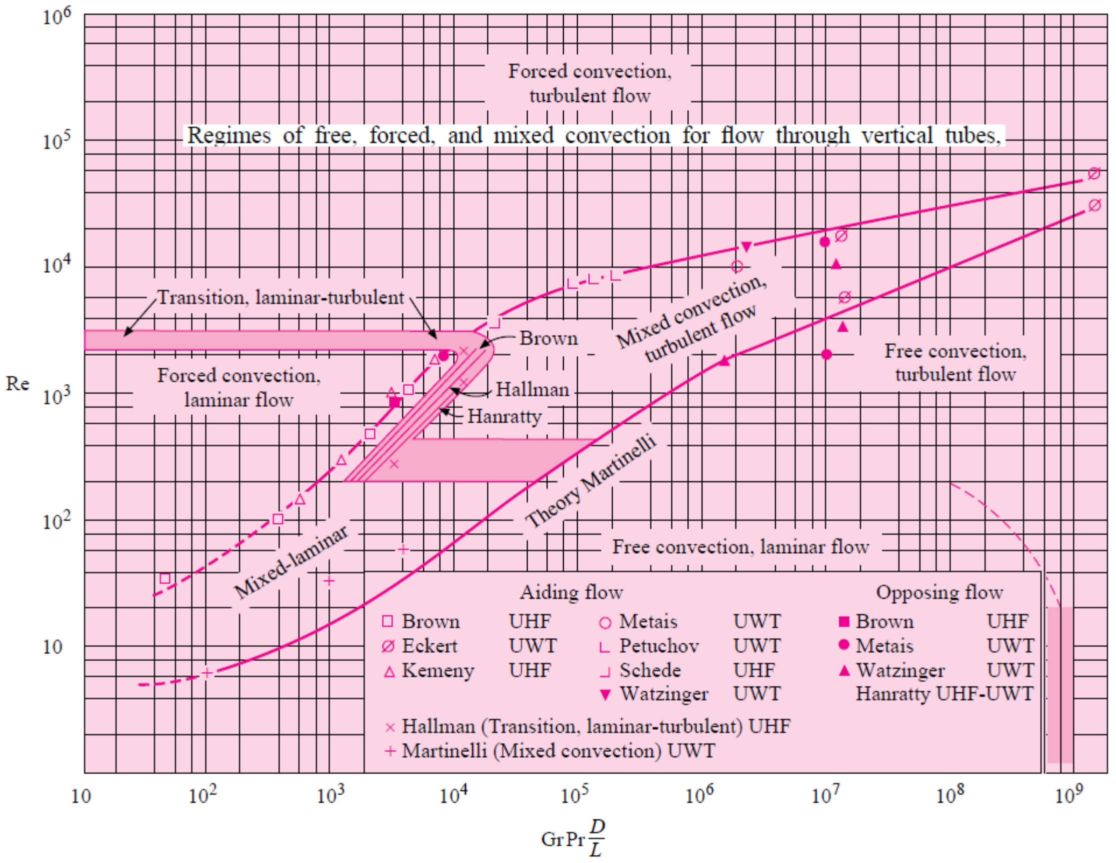
$$q_{\text{rad}} = (0.5)^2(61.47) = 15.37 \text{ W}$$

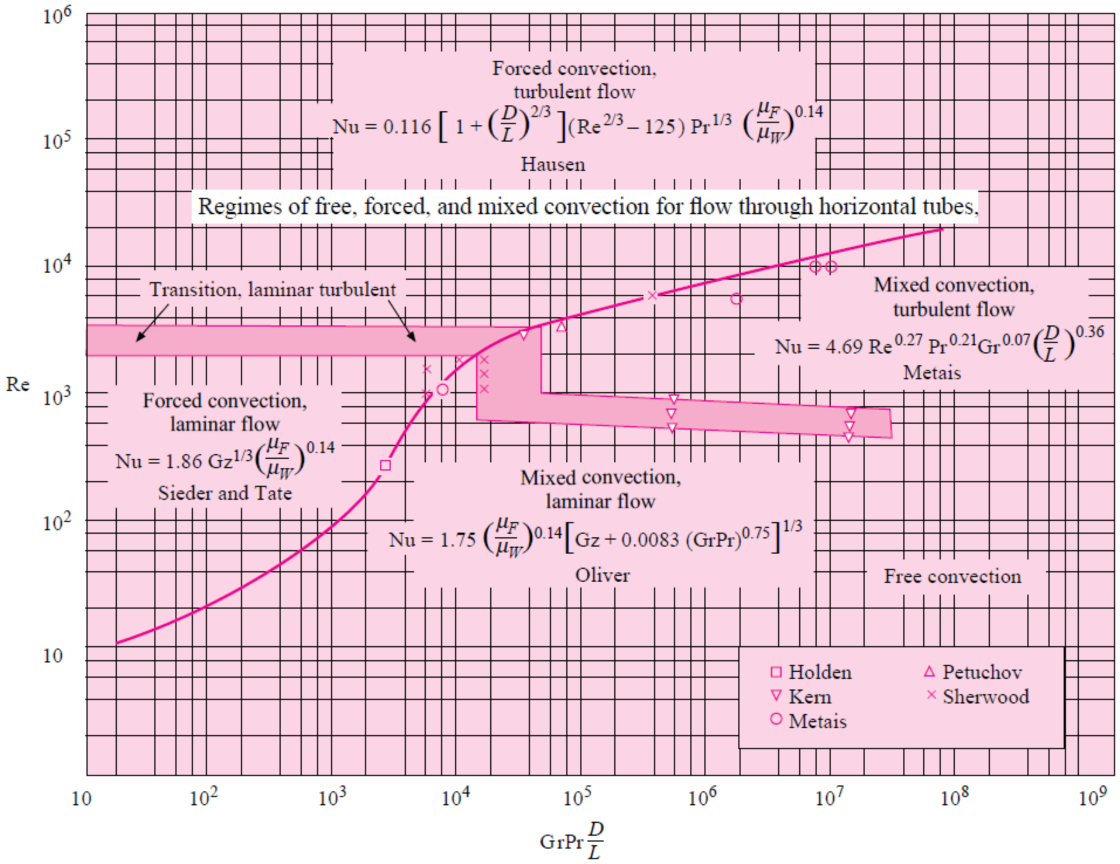
or about half the value of the convection transfer across the space. Further calculation would show that for a smaller value of $\epsilon = 0.05$, the radiation transfer is reduced to 3.55 W or, for a larger value of $\epsilon = 0.8$, the transfer is 92.2 W. In any event, radiation heat transfer can be an important factor in such problems.

Combined Free and Forced Convection:

1. *Aiding flow* means forced and free convection currents are in same direction while *opposing flow* means that they are in the opposite direction.
2. *UWT* stands for uniform wall temperature and *UHF* stands for uniform heat flux.
3. A large Reynolds implies a large forced-flow velocity, hence less influence of free-convection currents. A larger Grashof-Prandtl product implies a free convection effect to prevail.

Define Graetz number: Gz = Re Pr d/L (362)





The applicable range of the above figures is for $10^{-2} < Pr\,(d/L) < 1$.

For mixed convection Laminar flow:

$$\mathrm{Nu} = 1.75\left(\frac{\mu_b}{\mu_w}\right)^{0.14}\left(Gz + 0.012\left(Gz\, Gr^{\frac{1}{3}}\right)^{\frac{4}{3}}\right)^{\frac{1}{3}} \tag{363}$$

Criterion for Free or Force Convection: Gr / Re^2 > 10 when free convection is of more important.

e.g.

Air at 1 atm and 27°C is forced through a horizontal 25-mm-diameter tube at an average velocity of 30 cm/s. The tube wall is maintained at a constant temperature of 140°C. Calculate the heat-transfer coefficient for this situation if the tube is 0.4 m long.

For this calculation we evaluate properties at the film temperature:

$$T_f = \frac{140+27}{2} = 83.5°\mathrm{C} = 356.5\ \mathrm{K}$$

$$\rho_f = \frac{p}{RT} = \frac{1.0132\times 10^5}{(287)(356.5)} = 0.99\ \mathrm{kg/m^3}$$

$$\beta = \frac{1}{T_f} = 2.805\times 10^{-3}\ \mathrm{K^{-1}} \qquad \mu_w = 2.337\times 10^{-5}\ \mathrm{kg/m\cdot s}$$

$$\mu_f = 2.102\times 10^{-5}\ \mathrm{kg/m\cdot s} \qquad k_f = 0.0305\ \mathrm{W/m\cdot °C} \qquad \mathrm{Pr} = 0.695$$

Let us take the bulk temperature as 27°C for evaluating μ_b; then

$$\mu_b = 1.8462\times 10^{-5}\ \mathrm{kg/m\cdot s}$$

The significant parameters are calculated as

$$\mathrm{Re}_f = \frac{\rho u d}{\mu} = \frac{(0.99)(0.3)(0.025)}{2.102\times 10^{-5}} = 3.53$$

$$\mathrm{Gr} = \frac{\rho^2 g\beta(T_w - T_b)d^3}{\mu^2} = \frac{(0.99)^2(9.8)(2.805\times 10^{-3})(140-27)(0.025)^3}{(2.102\times 10^{-5})^2}$$

$$= 1.007\times 10^5$$

$$\mathrm{Gr\,Pr}\frac{d}{L} = (1.077\times 10^5)(0.695)\frac{0.025}{0.4} = 4677$$

According to Figure 7-14, the mixed-convection-flow regime is encountered. Thus we must use Equation (7-77). The Graetz number is calculated as

$$\mathrm{Gz} = \mathrm{Re\,Pr}\frac{d}{L} = \frac{(353)(0.695)(0.025)}{0.4} = 15.33$$

and the numerical calculation for Equation (7-77) becomes

$$\mathrm{Nu} = 1.75\left(\frac{1.8462}{2.337}\right)^{0.14}\{15.33 + (0.012)[(15.33)(1.077\times 10^5)^{1/3}]^{4/3}\}^{1/3}$$
$$= 7.70$$

The average heat-transfer coefficient is then calculated as

$$\bar{h} = \frac{k}{d}\mathrm{Nu} = \frac{(0.0305)(7.70)}{0.025} = 9.40\ \mathrm{W/m^2\cdot °C} \quad [1.67\ \mathrm{Btu/h\cdot ft^2\cdot °F}]$$

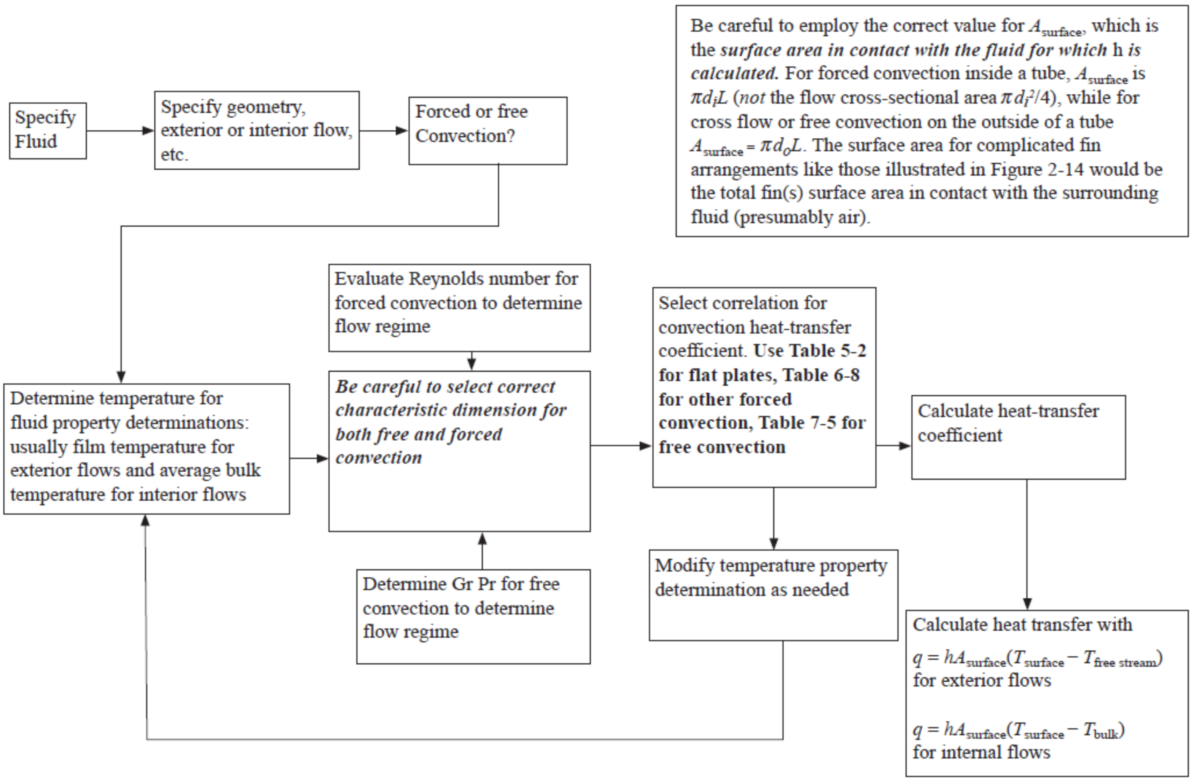
It is interesting to compare this value with that which would be obtained for strictly laminar forced convection. The Sieder-Tate relation [Equation (6-10)] applies, so that

$$\mathrm{Nu} = 1.86(\mathrm{Re\,Pr})^{1/3}\left(\frac{\mu_f}{\mu_w}\right)^{0.14}\left(\frac{d}{L}\right)^{1/3}$$
$$= 1.86\ \mathrm{Gz}^{1/3}\left(\frac{\mu_f}{\mu_w}\right)^{0.14}$$
$$= (1.86)(15.33)^{1/3}\left(\frac{2.102}{2.337}\right)^{0.14}$$
$$= 4.55$$

and

$$\bar{h} = \frac{(4.55)(0.0305)}{0.025} = 5.55\ \mathrm{W/m^2 \cdot {}^\circ C} \quad [0.977\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

Thus there would be an error of −41 percent if the calculation were made strictly on the basis of laminar forced convection.



Radiation Heat Transfer: (8)

Using relativistic relation between mass and energy ($E=mc^2$), and the principles of quantum-statistical thermodynamics, one may derive the energy density of radiation per unit volume and per unit wavelength as:

$$u_\lambda = \frac{8\pi hc\lambda^{-5}}{e^{\frac{hc}{\lambda kT}}-1} \qquad (364)$$

for k is Boltzmann's constant, 1.38066 x 10^{-23} J/ molecule K and Planck's constant h= 6.625 x 10^{-34} Js

When energy density is integrated over all wavelengths, the total energy emitted is proportional to the absolute temperature to the fourth power: $\mathrm{E_b} = \sigma T^4$ (365)

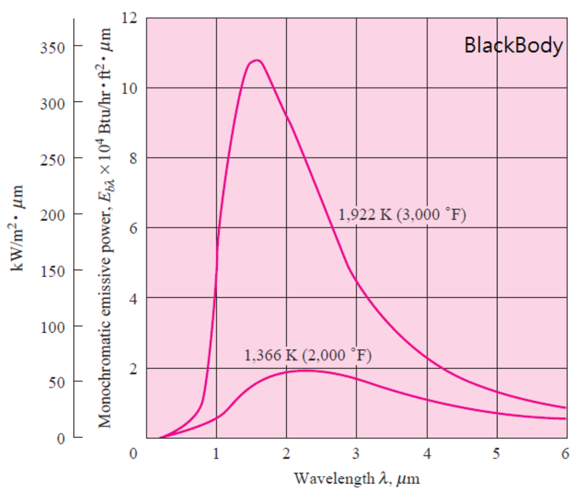
for $\sigma = 5.67 \times 10^{-8}$, the Stefan-Boltzmann constant. The subscript b denotes blackbody radiation.

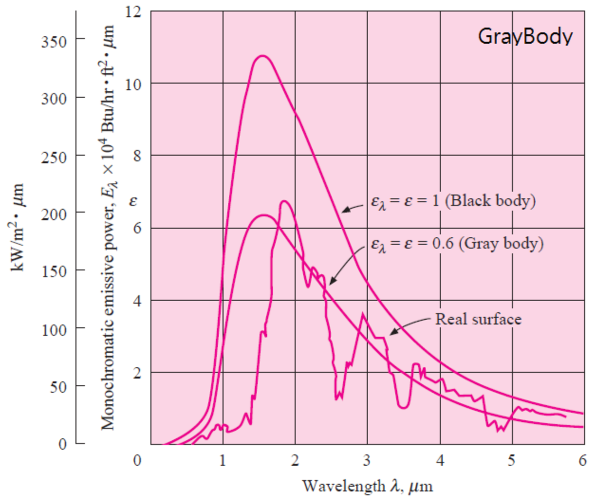
Radiation properties: $\rho + \alpha + \tau = 1$ ρ = reflectivity, α =absorptivity, τ =transmissivity

If angle of incident = angle of reflection, the reflection is called specular. If not, it is called diffusion.

The functional relation for $\mathrm{E_{b\lambda}}$, the energy density of radiation, was derived by Planck by introducing the quantum concept for electromagnetic energy. The derivation is usually performed by method of statistical thermodynamics. It's related by $\mathrm{E_{b\lambda}} = \frac{u_\lambda c}{4}$ or $\mathrm{E_{b\lambda}} = \frac{C_1\lambda^{-5}}{e^{\frac{C_2}{\lambda T}}-1}$ where C_1 = 3.743 x 10^8, C_2 =1.4387 x 10^8, λ=wavelength (um) and T= temperature (K). (366)

The maximum point of $\mathrm{E_{b\lambda}}$ Vs λ is related by Wien's Law: $\lambda_{max}\mathrm{T} = 2897.6$ umK (367)





We are frequently interested in energy radiated from blackbody in a certain specific wavelength range. The fraction of the total energy radiated between 0 and λ is given by:

$$\frac{E_{b0-\lambda}}{E_{b0-\infty}} = \frac{\int_0^\lambda E_{b\lambda} d\lambda}{\int_0^\infty E_{b\lambda} d\lambda} = \frac{\int_0^\lambda E_{b\lambda} d\lambda}{\sigma T^4} \tag{368}$$

Eqt (366) may be rewritten by dividing both sides by T^5.

$$\frac{E_{b\lambda}}{T^5} = \frac{C_1}{(\lambda T)^5 (e^{\frac{C_2}{\lambda T}} - 1)} \tag{369}$$

λT $\mu m \cdot K$	$E_{b\lambda}/T^5$ $\frac{W}{m^2 \cdot K^5 \cdot \mu m} \times 10^{11}$	$\frac{E_{b_0-\lambda T}}{\sigma T^4}$	λT $\mu m \cdot K$	$E_{b\lambda}/T^5$ $\frac{W}{m^2 \cdot K^5 \cdot \mu m} \times 10^{11}$	$\frac{E_{b_0-\lambda T}}{\sigma T^4}$
1000	0.02110	0.00032	6300	0.42760	0.76180
1100	0.04846	0.00091	6400	0.41128	0.76920
1200	0.09329	0.00213	6500	0.39564	0.77631
1300	0.15724	0.00432	6600	0.38066	0.78316
1400	0.23932	0.00779	6700	0.36631	0.78975
1500	0.33631	0.01285	6800	0.35256	0.79609
1600	0.44359	0.01972	6900	0.33940	0.80219
1700	0.55603	0.02853	7000	0.32679	0.80807
1800	0.66872	0.03934	7100	0.31471	0.81373
1900	0.77736	0.05210	7200	0.30315	0.81918
2000	0.87858	0.06672	7300	0.29207	0.82443
2100	0.96994	0.08305	7400	0.28146	0.82949
2200	1.04990	0.10088	7500	0.27129	0.83436
2300	1.11768	0.12002	7600	0.26155	0.83906
2400	1.17314	0.14025	7700	0.25221	0.84359
2500	1.21659	0.16135	7800	0.24326	0.84796
2600	1.24868	0.18311	7900	0.23468	0.85218
2700	1.27029	0.20535	8000	0.22646	0.85625
2800	1.28242	0.22788	8100	0.21857	0.86017
2900	1.28612	0.25055	8200	0.21101	0.86396
3000	1.28245	0.27322	8300	0.20375	0.86762
3100	1.27242	0.29576	8400	0.19679	0.87115
3200	1.25702	0.31809	8500	0.19011	0.87456
3300	1.23711	0.34009	8600	0.18370	0.87786
3400	1.21352	0.36172	8700	0.17755	0.88105
3500	1.18695	0.38290	8800	0.17164	0.88413
3600	1.15806	0.40359	8900	0.16596	0.88711
3700	1.12739	0.42375	9000	0.16051	0.88999
3800	1.09544	0.44336	9100	0.15527	0.89277
3900	1.06261	0.46240	9200	0.15024	0.89547
4000	1.02927	0.48085	9300	0.14540	0.89807
4100	0.99571	0.49872	9400	0.14075	0.90060
4200	0.96220	0.51599	9500	0.13627	0.90304
4300	0.92892	0.53267	9600	0.13197	0.90541
4400	0.89607	0.54877	9700	0.12783	0.90770
4500	0.86376	0.56429	9800	0.12384	0.90992
4600	0.83212	0.57925	9900	0.12001	0.91207
4700	0.80124	0.59366	10,000	0.11632	0.91415
4800	0.77117	0.60753	10,200	0.10934	0.91813
4900	0.74197	0.62088	10,400	0.10287	0.92188
5000	0.71366	0.63372	10,600	0.09685	0.92540
5100	0.68628	0.64606	10,800	0.09126	0.92872
5200	0.65983	0.65794	11,000	0.08606	0.93184
5300	0.63432	0.66935	11,200	0.08121	0.93479
5400	0.60974	0.68033	11,400	0.07670	0.93758
5500	0.58608	0.69087	11,600	0.07249	0.94021
5600	0.56332	0.70101	11,800	0.06856	0.94270
5700	0.54146	0.71076	12,000	0.06488	0.94505
5800	0.52046	0.72012	12,200	0.06145	0.94728
5900	0.50030	0.72913	12,400	0.05823	0.94939
6000	0.48096	0.73778	12,600	0.05522	0.95139
6100	0.46242	0.74610	12,800	0.05240	0.95329
6200	0.44464	0.75410	13,000	0.04976	0.95509

e.g.

A glass plate 30 cm square is used to view radiation from a furnace. The transmissivity of the glass is 0.5 from 0.2 to 3.5 μm. The emissivity may be assumed to be 0.3 up to 3.5 μm and 0.9 above that. The transmissivity of the glass is zero, except in the range from 0.2 to 3.5 μm. If the furnace is a blackbody at 2000◦C, calculate the energy absorbed in the glass and the energy transmitted.

$$T = 2000°\mathrm{C} = 2273\ \mathrm{K}$$
$$\lambda_1 T = (0.2)(2273) = 454.6\ \mu\mathrm{m}\cdot\mathrm{K}$$
$$\lambda_2 T = (3.5)(2273) = 7955.5\ \mu\mathrm{m}\cdot\mathrm{K}$$
$$A = (0.3)^2 = 0.09\ \mathrm{m}^2$$

$$\frac{E_{b0-\lambda 1}}{\sigma T^4} = 0 \qquad \frac{E_{b0-\lambda 2}}{\sigma T^4} = 0.85443$$

$$\sigma T^4 = (5.669 \times 10^{-8})(2273)^4 = 1513.3\ \mathrm{kW/m^2}$$

$$0.2\ \mu m < \lambda < 3.5\ \mu m$$
$$= (1.5133 \times 10^6)(0.85443 - 0)(0.3)^2$$
$$= 116.4\ \mathrm{kW} \quad [3.97 \times 10^5\ \mathrm{Btu/h}]$$

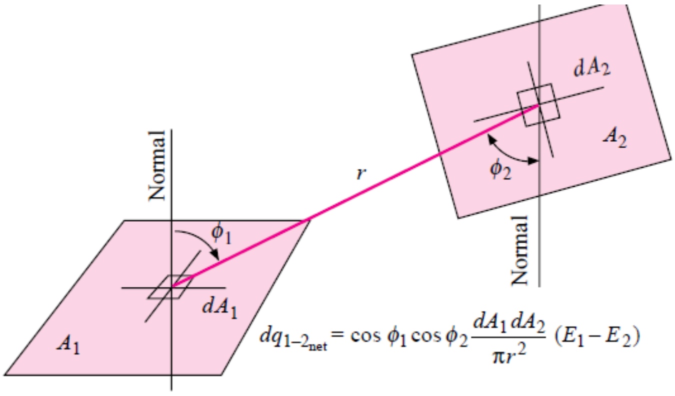
Total radiation transmitted

$$= (0.5)(116.4) = 58.2\ \mathrm{kW}$$

$$\begin{array}{l}\text{Radiation} \\ \text{absorbed}\end{array} = \begin{cases} (0.3)(116.4) = 34.92 \text{ kW} & \text{for } 0 < \lambda < 3.5\ \mu\text{m} \\ (0.9)(1 - 0.85443)(1513.3)(0.09) = 17.84 \text{ kW} & \text{for } 3.5\ \mu\text{m} < \lambda < \infty \end{cases}$$

$$\text{Total radiation absorbed } = 34.92 + 17.84 = 52.76 \text{ kW} \quad [180{,}000 \text{ Btu/h}]$$

Shape Factor:



Energy leaving surface 1 and arriving surface 2 = $E_{b1}A_1F_{12}$

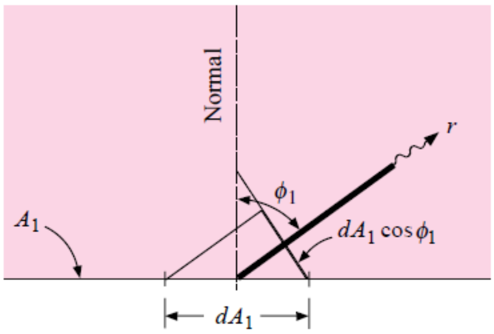
Energy leaving surface 2 and arriving surface 1 = $E_{b2}A_2F_{21}$

Assume blackbody, all incident radiation will be absorbed:

$$E_{b1}A_1F_{12} - E_{b2}A_2F_{21} = Q_{1\text{-}2}$$

If both surface are at the same temp,

$$Q_{1\text{-}2} = 0$$



As $E_{b1} = E_{b2}$, we have $A_1 F_{12} = A_2 F_{21}$.
(F_{12} = energy leaving surface 1 reaching surface 2)
In general way, $A_i F_{ij} = A_j F_{ji}$.

For a differential element, projection of dA_1 on the line between center = $dA_1 \cos\phi_1$
The energy leaving dA_1 in direction ϕ_1 is

$$I_b dA_1 \cos\phi_1$$

for I_b is the blackbody intensity,

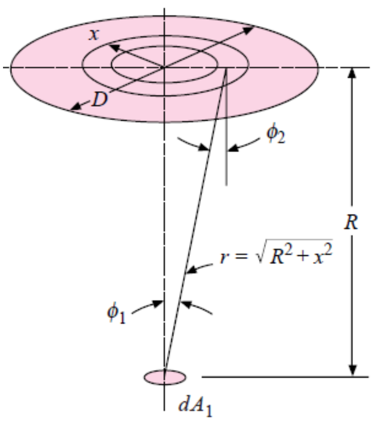
The radiation arriving at some area element dA_n at a distance r from A_1 would be $I_b dA_1 \cos\phi_1 \frac{dA_n}{r^2}$
In spherical coordinate, $dA_n = r^2 \sin\phi\, d\psi d\phi$
Then $E_b dA_1 = I_b dA_1 \int_0^{2\pi}\int_0^{\pi/2} \cos\phi \sin\phi\, d\phi d\psi = \pi I_b dA_1$ so that $E_b = \pi I_b$.

The element dA_n is given by $dA_n = \cos\phi_2\, dA_2$ so that energy leaving dA_2 and arriving at dA_1 is

$$dq_{1-2} = E_{b1}\cos\phi_1 \cos\phi_2 \frac{dA_1 dA_2}{\pi r^2}$$

The energy leaving dA_2 and arriving at dA_1 is $\quad dq_{2-1} = E_{b2}\cos\phi_1\cos\phi_2 \frac{dA_1 dA_2}{\pi r^2}$

And the net energy exchange is $\quad q_{net\,1-2} = (E_{b1} - E_{b2})\int_{A_2}\int_{A_1} \cos\phi_1 \cos\phi_2 \frac{dA_1 dA_2}{\pi r^2} \qquad (370)$



Consider the radiation from small area dA_1 to the flat disk A_2, as shown in the figure, $\quad dA_2 = 2\pi x\, dx$

$\phi_1 = \phi_2$: $\quad dA_1 F_{dA_1 - A_2} = dA_1 \int_{A_2} \cos^2\phi_1 \frac{2\pi x dx}{\pi r^2}$

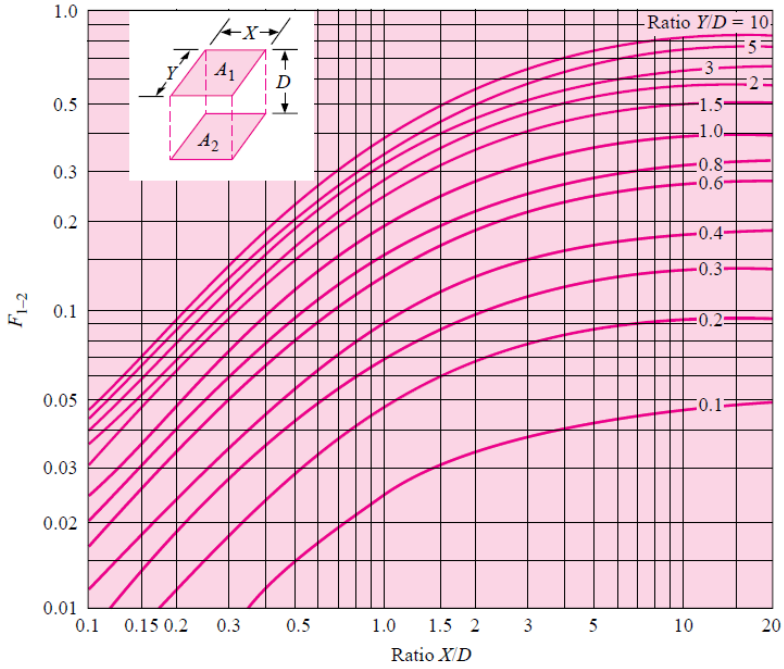
Making the substitution: $\quad r = (R^2 + x^2)^{1/2} \quad \cos\phi_1 = \frac{R}{(R^2+x^2)^{\frac{1}{2}}}$

We have $\quad dA_1 F_{dA_1 - A_2} = dA_1 \int_0^{D/2} \frac{2R^2 x dx}{(R^2+r^2)^2}$

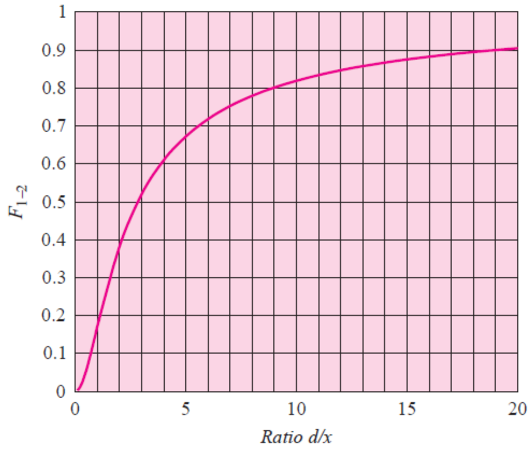
Performing the integration gives:

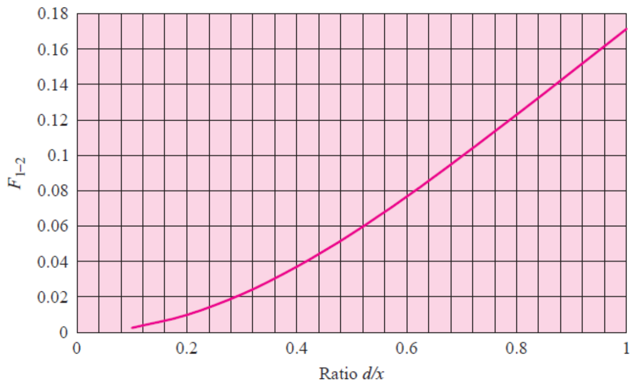
$$dA_1 F_{dA_1 - A_2} = dA_1 \frac{D^2}{4R^2 + D^2} \qquad F_{dA_1 - A_2} = \frac{D^2}{4R^2+D^2} \qquad (371)$$

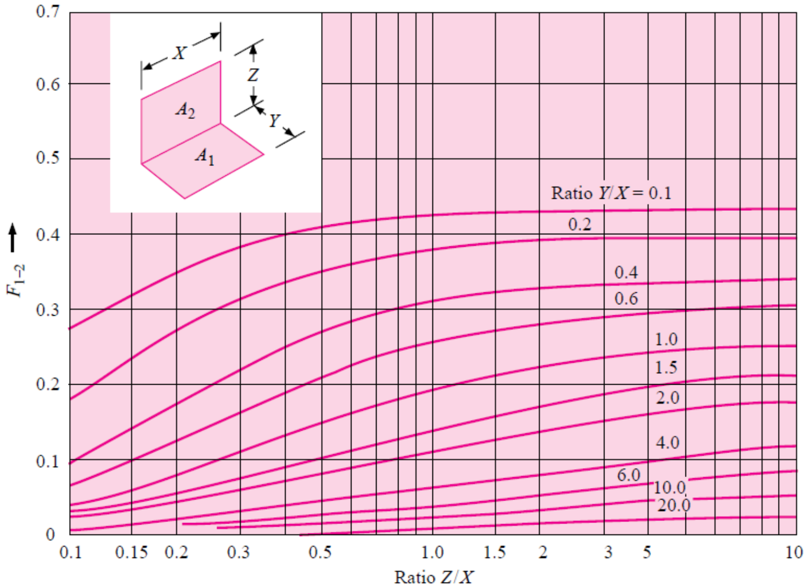
Radiation shape factor:



Radiation shape factor for radiation between parallel equal coaxial disks.





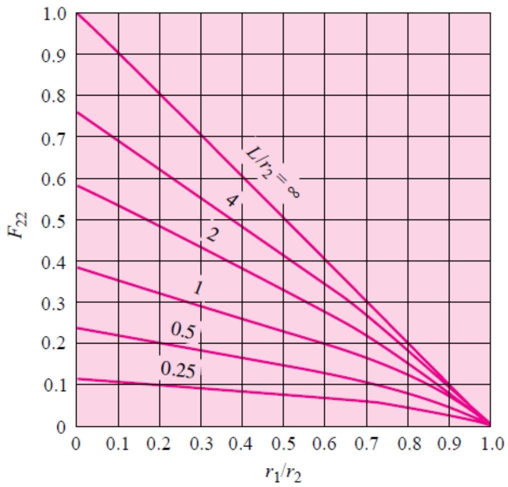


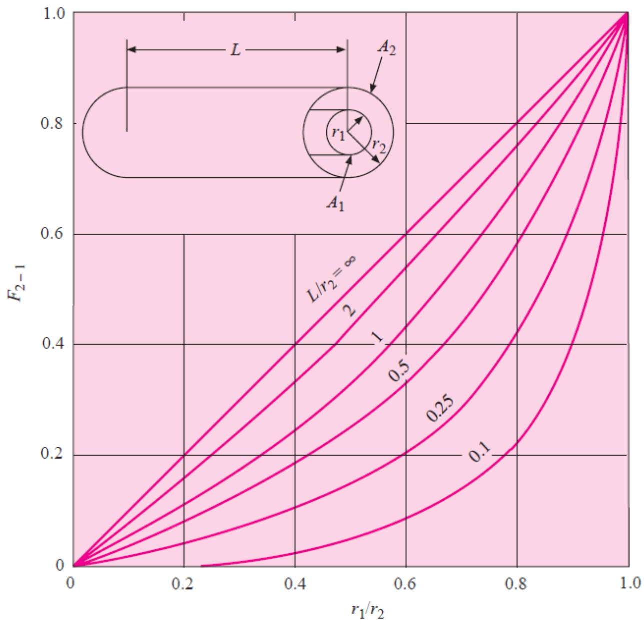
The total absorptivity will be given: $\alpha = \int_0^\infty \alpha_\lambda G_\lambda d\lambda \,/\, \int_0^\infty G_\lambda d\lambda$ (372)

The total incident radiation on surface per unit time, per unit area, per unit wavelength = G_λ. The total absorptivity will be given as the ratio of the total energy absorbed to the total energy incident on the surface.

If we are fortunate energy to have a gray body such that $\epsilon_\lambda = \epsilon = const$. It may be shown with KCL that for monochromatic radiation as $\epsilon_\lambda = \alpha_\lambda$. Therefore, for a gray body, $\alpha_\lambda = const$.

Radiant shape factors: (a)The outer cylinder to itself (b) outer cylinder to inner cylinder.





Geometry	Shape factor
1. Parallel, equal rectangles (Fig. 8-12) $x = X/D,\ y = Y/D$	$F_{1-2} = (2/\pi xy)\{\ln[(1+x^2)(1+y^2)/(1+x^2+y^2)]^{1/2} + x(1+y^2)^{1/2}\tan^{-1}[x/(1+y^2)^{1/2}] + y(1+x^2)^{1/2}\tan^{-1}[y/(1+x^2)^{1/2}] - x\tan^{-1}x - y\tan^{-1}y\}$
2. Parallel, equal, coaxial disks (Fig. 8-13) $R = d/2x,\ X = (2R^2+1)/R^2$	$F_{1-2} = [X - (X^2-4)^{1/2}]/2$
3. Perpendicular rectangles with a common edge (Fig. 8-14) $H = Z/X,\ W = Y/X$	$F_{1-2} = (1/\pi W)\big(W\tan^{-1}(1/W) + H\tan^{-1}(1/H) - (H^2+W^2)^{1/2}\tan^{-1}[1/(H^2+W^2)^{1/2}] + (1/4)\ln\{[(1+W^2)(1+H^2)/(1+W^2+H^2)] \times [W^2(1+W^2+H^2)/(1+W^2)(W^2+H^2)]^{W^2} \times [H^2(1+H^2+W^2)/(1+H^2)(H^2+W^2)]^{H^2}\}\big)$
4. Finite, coaxial cylinders (Fig. 8-15) $X = r_2/r_1,\ Y = L/r_1$ $A = X^2+Y^2-1$ $B = Y^2-X^2+1$	$F_{2-1} = (1/X) - (1/\pi X)\{\cos^{-1}(B/A) - (1/2Y)[(A^2+4A-4X^2+4)^{1/2}\cos^{-1}(B/XA) + B\sin^{-1}(1/X) - \pi A/2]\}$ $F_{2-2} = 1 - (1/X) + (2/\pi X)\tan^{-1}[2(X^2-1)^{1/2}/Y] - (Y/2\pi X)\{[\sqrt{(4X^2+Y^2)}/Y]\sin^{-1}\{[4(X^2-1)+(Y/X)^2(X^2-2)]/[Y^2+4(X^2-1)]\} - \sin^{-1}[(X^2-2)/X^2] + (\pi/2)[(4X^2+Y^2)^{1/2}/Y - 1]\}$
5. Parallel, coaxial disks (Fig. 8-16) $R_1 = r_1/L$ $R_2 = r_2/L$ $X = 1+(1+R_2^2)/R_1^2$	$F_{1-2} = \{X - [X^2 - 4(R_2/R_1)^2]^{1/2}\}/2$

e.g.

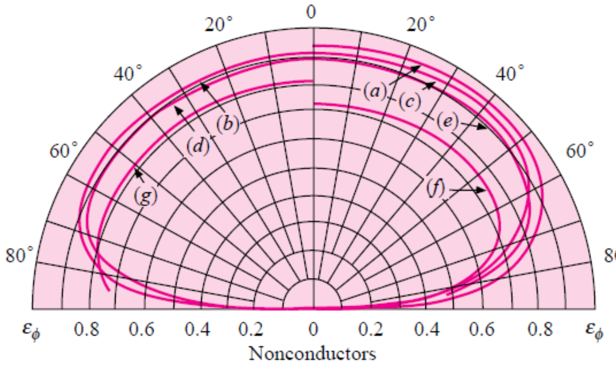
Two parallel black plates 0.5 by 1.0 m are spaced 0.5 m apart. One plate is maintained at 1000°C and the other at 500°C. What is the net radiant heat exchange between the two plates?

The ratios for use with Figure 8-12 are

$$\frac{Y}{D} = \frac{0.5}{0.5} = 1.0 \qquad \frac{X}{D} = \frac{1.0}{0.5} = 2.0$$

so that $F_{12} = 0.285$. The heat transfer is calculated from

$$q = A_1 F_{12}(E_{b1} - E_{b2}) = \sigma A_1 F_{12}(T_1^4 - T_2^4)$$
$$= (5.669 \times 10^{-8})(0.5)(0.285)(1273^4 - 773^4)$$
$$= 18.33 \text{ kW} \quad [62{,}540 \text{ Btu/h}]$$



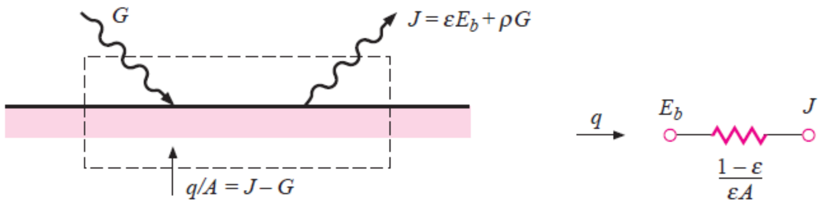
Typical directional behavior of emissivity for conductors and nonconductors. ϵ_ϕ is emissivity at angle ϕ measured from normal to surface. Nonconductor curves are for (*a*) wet ice, (*b*) wood, (*c*) glass, (*d*) paper, (*e*) clay, (*f*) copper oxide, and (*g*) aluminum oxide.

In the foregoing discussion the assumption has been made that the various bodies do not see themselves. i.e. $F_{ii} = 0$. To be general, $\sum_{j=1}^{n} F_{ij} = 1.0$, where F_{ij} is the fraction of total energy leaving surface *i* that arrive at surface *j*.

For non-blackbodies:

G = irradiation = total radiation incident upon a surface per unit time and unit area

J = radiosity = total radiation that leaves a surface per unit time and per unit area



The radiosity is the sum of energy emitted and energy reflected when no energy is transmitted. i.e.

$$J = \epsilon E_b + \rho G \tag{373}$$

where ϵ is the emissivity and E_b is the blackbody emissive power. Since the transmissivity is assumed to be zero, the reflectivity may be expressed as $\rho = 1 - \alpha = 1 - \epsilon$ so that

$$J = \epsilon E_b + (1 - \epsilon)G \tag{374}$$

$$\frac{q}{A} = J - G = \epsilon E_b + (1 - \epsilon)G - G$$

$$q = \frac{\epsilon A}{1-\epsilon}(E_b - J) \tag{375}$$

Consider the exchange of radiant energy by two surfaces, A_1 and A_2.

Of the total radiation leaving surface 1, the amount that reaches surface 2 is $J_1 A_1 F_{12}$

Of the total radiation leaving surface 2, the amount that reaches surface 1 is $J_2 A_2 F_{21}$.

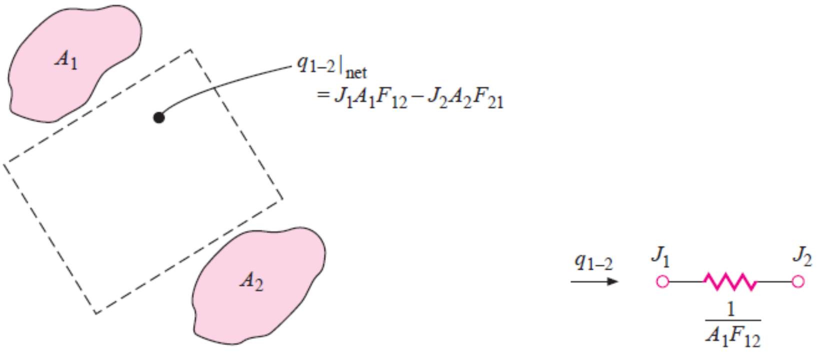
The net interchange between two surfaces $= q_{1-2} = J_1 A_1 F_{12} - J_2 A_2 F_{21}$. But $A_1F_{12} = A_2F_{21}$.

So that

$$q_{1-2} = \frac{J_1 - J_2}{1/A_1 F_{12}} \tag{376}$$

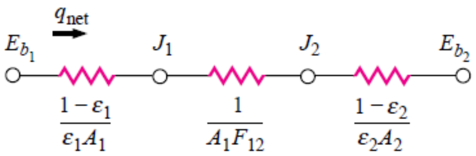
Therefore the space resistance is

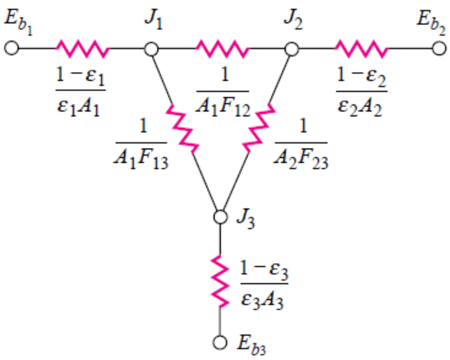
$$\mathrm{R} = \frac{1}{\mathrm{A}_1 F_{12}} \tag{377}$$



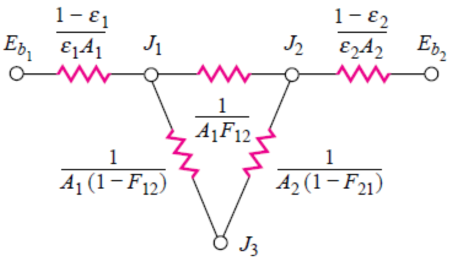
(a) (b)

The net heat transfer can be analyzed with Radiation Network.



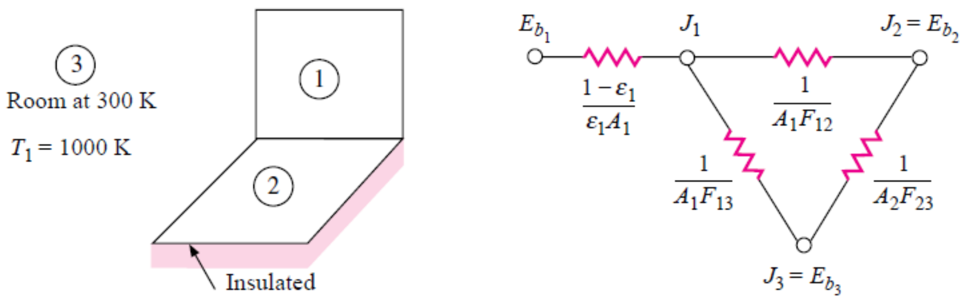


For the radiation network with two planes enclosed by third surface that is nonconducting but re-radiating (insulated).



e.g.

Two rectangles 50 by 50 cm are placed perpendicularly with a common edge. One surface has $T_1 = 1000$ K, $\epsilon_1 = 0.6$, while the other surface is insulated and in radiant balance with a large surrounding room at 300 K. Determine the temperature of the insulated surface and the heat lost by the surface at 1000 K.



(a) (b)

Although this problem involves two surfaces that exchange heat and one that is insulated or re-radiating, Equation (8-41) may not be used for the calculation because one of the heat-exchanging surfaces (the room) is not convex. The radiation network is shown in Figure Example 8-7 where surface 3 is the room and surface 2 is the insulated surface. Note that $J_3 = E_{b_3}$ because the room is large and $(1-\epsilon_3)/\epsilon_3 A_3$ approaches zero. Because surface 2 is insulated it has zero heat transfer and $J_2 = E_{b_2}$. J_2 "floats" in the network and is determined from the overall radiant balance. From Figure 8-14 the shape factors are

$$F_{12} = 0.2 = F_{21}$$

Because $F_{11} = 0$ and $F_{22} = 0$ we have

$$F_{12} + F_{13} = 1.0 \qquad \text{and} \qquad F_{13} = 1 - 0.2 = 0.8 = F_{23}$$

$$A_1 = A_2 = (0.5)^2 = 0.25 \text{ m}^2$$

The resistances are

$$\frac{1-\epsilon_1}{\epsilon_1 A_1} = \frac{0.4}{(0.6)(0.25)} = 2.667$$

$$\frac{1}{A_1 F_{13}} = \frac{1}{A_2 F_{23}} = \frac{1}{(0.25)(0.8)} = 5.0$$

$$\frac{1}{A_1 F_{12}} = \frac{1}{(0.25)(0.2)} = 20.0$$

We also have

$$E_{b_1} = (5.669 \times 10^{-8})(1000)^4 = 5.669 \times 10^4 \text{ W/m}^2$$

$$J_3 = E_{b_3} = (5.669 \times 10^{-8})(300)^4 = 459.2 \text{ W/m}^2$$

The overall circuit is a series-parallel arrangement and the heat transfer is

$$q = \frac{E_{b_1} - E_{b_3}}{R_{\text{equiv}}}$$

We have

$$R_{\text{equiv}} = 2.667 + \frac{1}{\frac{1}{5} + 1/(20+5)} = 6.833$$

and

$$q = \frac{56{,}690 - 459.2}{6.833} = 8.229 \text{ kW} \quad [28{,}086 \text{ Btu/h}]$$

This heat transfer can also be written

$$q = \frac{E_{b_1} - J_1}{(1-\epsilon_1)/\epsilon_1 A_1}$$

Inserting the values we obtain

$$J_1 = 34{,}745 \text{ W/m}^2$$

The value of J_2 is determined from proportioning the resistances between J_1 and J_3, so that

$$\frac{J_1 - J_2}{20} = \frac{J_1 - J_3}{20+5}$$

and

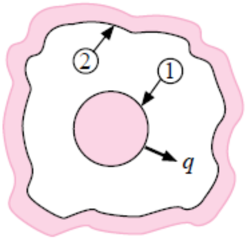
$$J_2 = 7316 = E_{b_2} = \sigma T_2^4$$

Finally, we obtain the temperature of the insulated surface as

$$T_2 = \left(\frac{7316}{5.669 \times 10^{-8}}\right)^{1/4} = 599.4 \text{ K} \quad [619°\text{F}]$$

For radiant heat transfer between simple two body diffuse, gray surface with $F_{12} = 1.0$

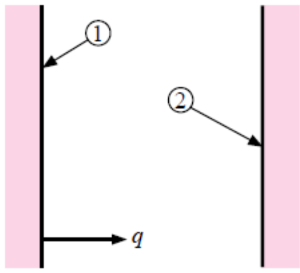
Small convex object in large enclosure



$q = A_1\epsilon_1\sigma(T_1^4 - T_2^4)$

for $A_1/A_2 \to 0$

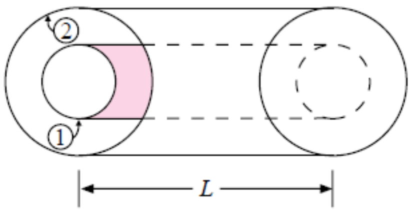
Infinite parallel planes



$(q/A) = \frac{\sigma(T_1^4 - T_2^4)}{1/\epsilon_1 + 1/\epsilon_2 - 1}$

with $A_1 = A_2$

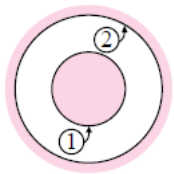
Infinite concentric cylinders



$q = \frac{\sigma A_1(T_1^4 - T_2^4)}{1/\epsilon_1 + (1/\epsilon_2 - 1)(r_1/r_2)}$

with $A_1/A_2 = r_1/r_2;\quad r_1/L \to 0$

Concentric spheres

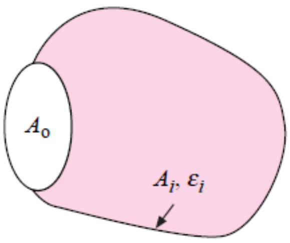


$q = \frac{\sigma A_1(T_1^4 - T_2^4)}{1/\epsilon_1 + (1/\epsilon_2 - 1)(r_1/r_2)^2}$

for $A_1/A_2 = (r_1/r_2)^2$

For radiation energy loss from a hot object in a large room, we have $q = \sigma A_1\epsilon_1(T_1^4 - T_2^4)$ (378)

Consider an apparent emissivity of cavity with surrounding temp T_s and $A_s >> A_0$.



For an imaginary surface A_o covering the opening, and exchange heat with A_i, we have $\quad F_{oi} = 1.0 \quad$ (379)

From $A_oF_{oi} = A_iF_{io}$ and $F_{io} = F_{is}$, $\quad A_iF_{is} = A_o \quad$ (380)

The net radiant exchange of surface A_i with the large enclosure A_s is given by:

$$q_{i-s} = (E_{bi} - E_{bs})/\left(\frac{1-\epsilon_i}{\epsilon_i A_i} + \frac{1}{A_i F_{is}}\right) \quad (381)$$

The net radiant heat exchange of A_0 having apparent ϵ_a with large surrounding is given by:

$$q_{o-s} = \epsilon_a A_o(E_{bi} - E_{bs}) \quad (382)$$

Comparing: $$\epsilon_a = \epsilon_i A_i/(A_o + \epsilon_i(A_i - A_o)) \quad (383)$$

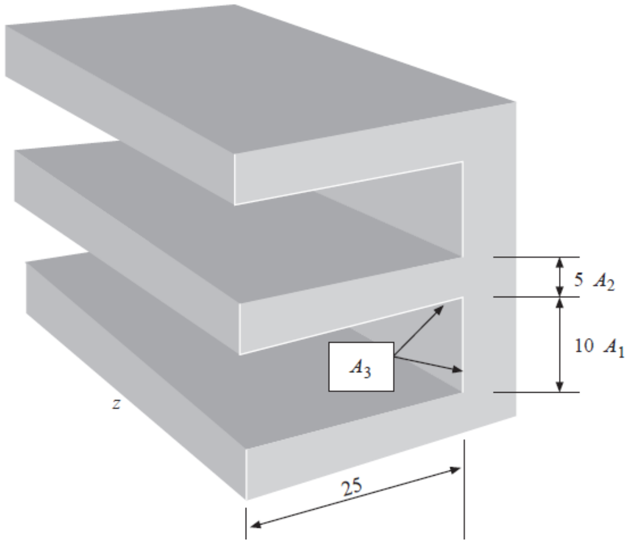
For limiting case, $\epsilon_a = \epsilon_i$ for $A_o = A_i$ or no cavity at all or $\epsilon_a \to 1.0$ for $A_i \gg A_o$

e.g.

A repeating finned surface having the relative dimensions shown in Figure Example 8-9 is utilized to produce a higher effective emissivity than that for a flat surface alone. Calculate the effective emissivity of the combination of fin tip and open cavity for surface emissivities of 0.2, 0.5, and 0.8.

For unit depth in the z-dimension we have

$$A_1 = 10,\ A_2 = 5,\ A_3 = (2)(25) + 10 = 60$$



The apparent emissivity of the open cavity area A_1 is given by Equation (8-47) as

$$\epsilon_{a1} = \epsilon A_3/[A_1 + \epsilon(A_3 - A_1)] = 60\epsilon/(10+50\epsilon) \qquad [a]$$

For constant *surface emissivity* the emitted energy from the total erea $A_1 + A_2$ is

$$(\epsilon_{a1} A_1 + \epsilon A_2) E_b \qquad [b]$$

and the energy emitted per unit area for that total area is

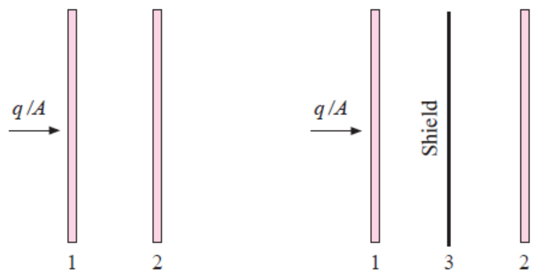
$$[(\epsilon a_1 A_1 + \epsilon A_2)/(A_1 + A_2)] E_b \qquad [c]$$

The coefficient of E_b is the effective emissivity, ϵ_{eff} of the combination of the flat surface and open cavity. Inserting Equation (*a*) in (*c*) gives the following numerical values:

For $\epsilon = 0.2$	$\epsilon_{\text{eff}} = 0.4667$
For $\epsilon = 0.5$	$\epsilon_{\text{eff}} = 0.738$
For $\epsilon = 0.8$	$\epsilon_{\text{eff}} = 0.907$

One could employ these effective values to calculate the radiation performance of such a finned surface in conjunction with applicable radiation properties of surrounding surfaces.

Radiation Shields:

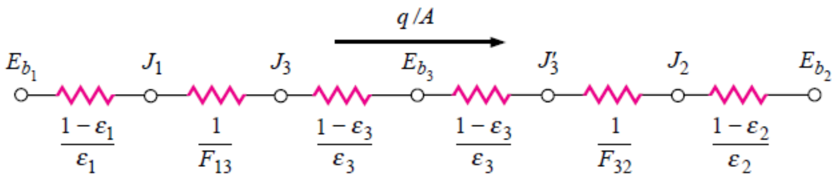


For Radiation shield, $$\left(\frac{q}{A}\right)_{1-3} = \left(\frac{q}{A}\right)_{3-2} = \left(\frac{q}{A}\right) = \frac{\sigma(T_1^4 - T_3^4)}{\frac{1}{\epsilon_1}+\frac{1}{\epsilon_3}-1} = \frac{\sigma(T_3^4 - T_2^4)}{\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2}-1} \qquad (384)$$

If the emissivity of all three surfaces are the same, i.e. $\epsilon_1 = \epsilon_2 = \epsilon_3$, we obtain $T_3^4 = \frac{1}{2}(T_1^4 + T_2^4)$ and the heat transfer is

$$\frac{q}{A} = \frac{\frac{1}{2}\sigma(T_1^4 - T_2^4)}{\frac{1}{\epsilon_1}+\frac{1}{\epsilon_3}-1} \qquad (385)$$

For n-shields, $$\mathrm{R(n\ shields)} = (2n+2)\frac{1-\epsilon}{\epsilon} + (n+1)(1) = (n+1)\left(\frac{2}{\epsilon} - 1\right) \qquad (386)$$

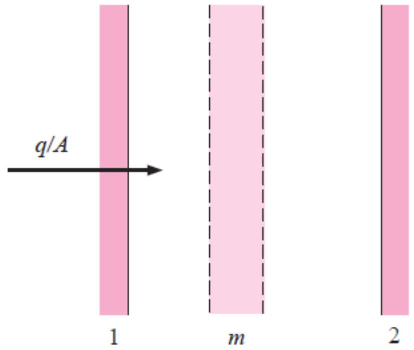


For no shields, $\quad R(\text{no shield}) = \frac{1}{\epsilon} + \frac{1}{\epsilon} - 1 = \frac{2}{\epsilon} - 1 \quad (387)$

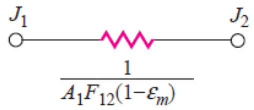
We note that the resistance with the shields in place is n+1 times as large as when the shield is absent.

i.e. $\quad \left(\frac{q}{A}\right)_{with\ shield} = \frac{1}{n+1}\left(\frac{q}{A}\right)_{without\ shield} \quad (388)$

Radiation Network between an Absorbing and Transmitting Medium:



$$q_{1-2_{transmitted}} = A_1F_{12}\tau_m(J_1 - J_2) = \frac{(J_1-J_2)}{1/A_1F_{12}\tau_m} \quad (389)$$



Consider the exchange process between surface 1 and transmitting medium. Since we assume that this medium is not reflecting, energy leaving medium = energy emitted by medium

i.e. $\quad J_m = \epsilon_m E_{bm}$

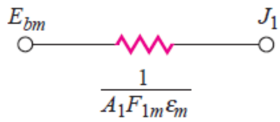
Energy leaving the medium reaching surface 1: $\quad A_mF_{m1}J_m = A_mF_{m1}\epsilon_m E_{bm}$

Energy leaving surface 1 reaching transparent medium *m*: $\quad J_1A_1F_{1m}\alpha_m = J_1A_1F_{1m}\epsilon_m$

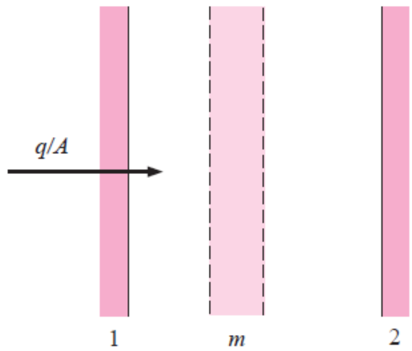
The net energy exchange between the medium and surface 1:

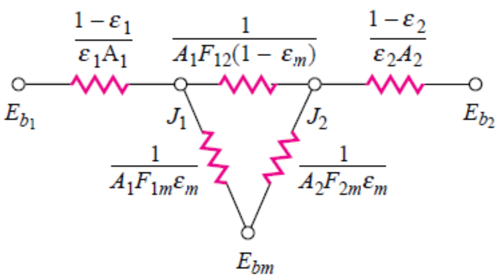
$$q_{m-1_{net}} = A_mF_{m1}\epsilon_m E_{bm} - J_1A_1F_{1m}\epsilon_m \quad (390)$$

With $A_1F_{1m} = A_m F_{m1}$: $\quad q_{m-1_{net}} = \frac{E_{bm}-J_1}{1/A_1F_{1m}\epsilon_m} \quad (391)$



The total radiation network for system can be modelled as the following resistance network.

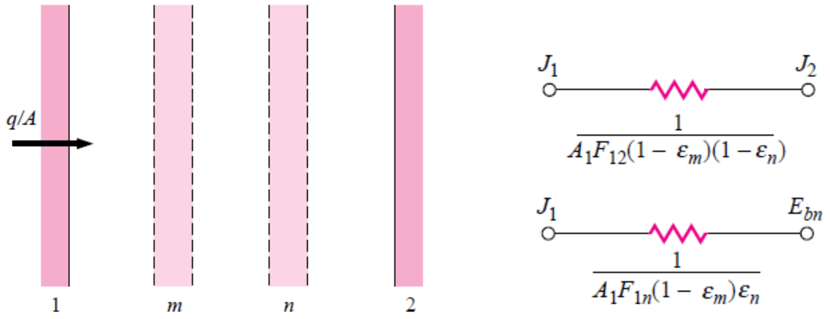


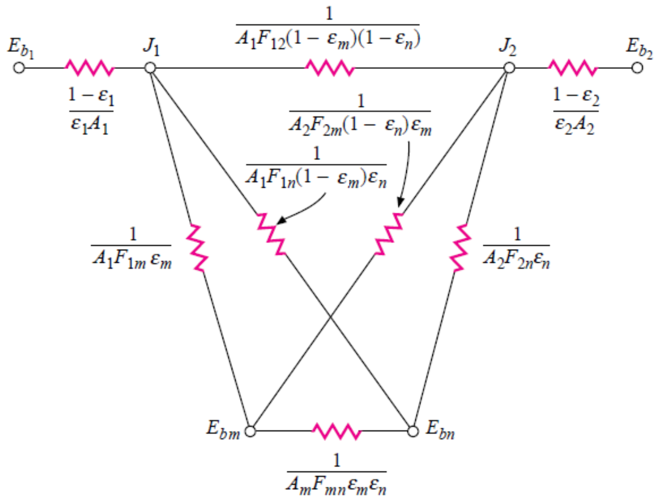


Similarly, for radiation system consisting of two transmitting layers:

$$q_{1-2_{transmitted}} = A_1F_{12}(1-\epsilon_m)(1-\epsilon_n)(J_1 - J_2) = \frac{(J_1-J_2)}{1/A_1F_{12}(1-\epsilon_m)(1-\epsilon_n)} \quad (392)$$

$$q_{1-n_{net}} = A_1F_{1n}\tau_m\epsilon_n(J_1 - E_{bn}) = A_1F_{1n}(1-\epsilon_m)\epsilon_n(J_1 - E_{bn}) = \frac{J_1-E_{bn}}{1/A_1F_{1n}(1-\epsilon_m)\epsilon_n} \quad (393)$$

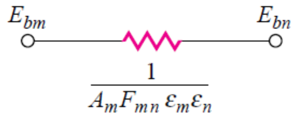




The net energy exchange between m and n:

$$q_{m-n_{net}} = A_m F_{mn} \epsilon_m \epsilon_n (E_{bm} - E_{bn})$$

$$= \frac{E_{bm} - E_{bn}}{1/A_m F_{mn} \epsilon_m \epsilon_n} \tag{394}$$



Radiation Exchange with Specular Surface:

Reflectivity = sum of specular component and diffuse component: $\rho = \rho_s + \rho_D$ (395)

By KCL: $\epsilon = \alpha = 1 - \rho.$ (396)

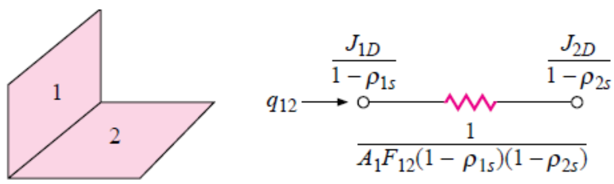
The net heat lost by a surface = energy emitted – energy absorbed:

$$q = A(\epsilon E_b - \alpha G) \tag{397}$$

With diffuse radiosity J_D as the total diffuse energy leaving the surface per unit area per unit time,

$$J_D = \epsilon E_b + \rho_D G \tag{398}$$

$$q = \frac{\epsilon A}{\rho D}(E_b(\epsilon + \rho_D) - J_D) = \frac{E_b - \frac{J_D}{1-\rho_s}}{\frac{\rho_D}{\epsilon A(1-\rho_s)}} \tag{399}$$

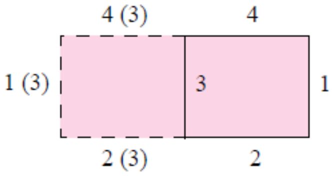


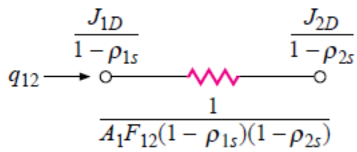
For the surface perpendicular to each other, the diffuse exchange is

$$q_{1\to 2} = J_{1D} A_1 F_{12}(1 - \rho_{2s}) \quad q_{2\to 1} = J_{2D} A_2 F_{21}(1 - \rho_{1s}) \tag{400}$$

The net exchange is given by the difference between the two equations, i.e.

$$q_{12} = \frac{\frac{J_{1D}}{1-\rho_{1s}} - \frac{J_{2D}}{1-\rho_{2s}}}{\frac{1}{A_1 F_{12}(1-\rho_{1s})(1-\rho_{2s})}} \tag{401}$$





Consider the enclosure with four long surfaces as shown above. The enclosure has specular surface 3 and diffusive surface 1, 2, 4.

Radiation leaving 2 arriving 1: $(q_{2\to1})_{direct\ diffuse} = J_2A_2F_{21}$

Part of the diffuse radiation from 2 reflected in 3 striking 1: $(q_{2\to1})_{reflectd} = J_2A_{2(3)}F_{2(3)1}\rho_{3s}$

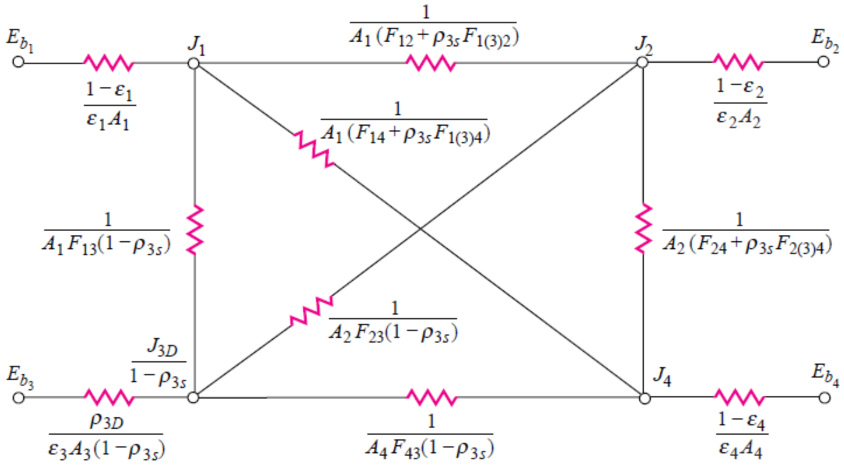
where $F_{2(3)1}$ is the one between surface 2(3) and surface 1. The reflectivity ρ_{3s} is inserted because only this fraction of radiation gets to 1. With $A_2 = A_{2(3)}$, $q_{2\to1} = J_2A_2(F_{21} + \rho_{3s}F_{2(3)1})$ (402)

Similarly, we have $q_{1\to2} = J_1A_1(F_{12} + \rho_{3s}F_{1(3)2})$ (403)

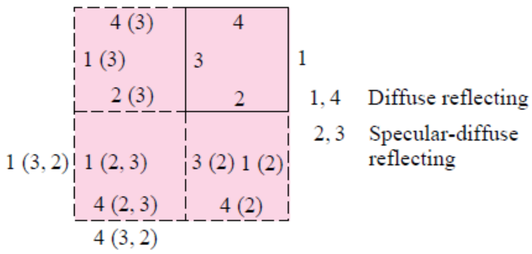
Combining (402) and (403) and making use of $A_1F_{12} = A_2F_{21}$:

$$q_{12} = \frac{J_1 - J_2}{1/(A_1(F_{12} + \rho_{3s}F_{1(3)2})} \tag{404}$$

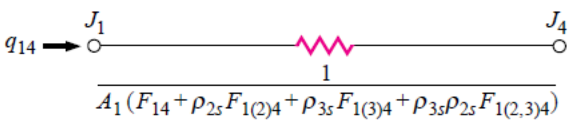
Note that $\rho_{1s} = \rho_{2s} = \rho_{4s} = 0$ and surface 3 is completely specular ($J_{3D} = \epsilon_3E_{b3}$). We can obtain the following resistance network.

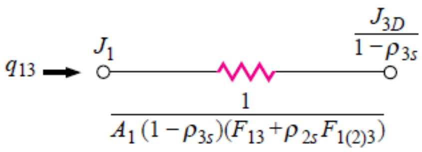


Similarly, for two-specular diffuse surfaces 2 and 3:



direct:	$J_1A_1F_{14}$
reflection in 2 only:	$J_1A_1F_{1(2)4}\rho_{2s}$
reflection in 3 only:	$J_1A_1F_{1(3)4}\rho_{3s}$
reflection first in 2 and then in 3:	$J_1A_1\rho_{3s}\rho_{2s}F_{1(2,3)4}$
reflection first in 3 and then in 2:	$J_1A_1\rho_{2s}\rho_{3s}F_{1(3,2)4}$



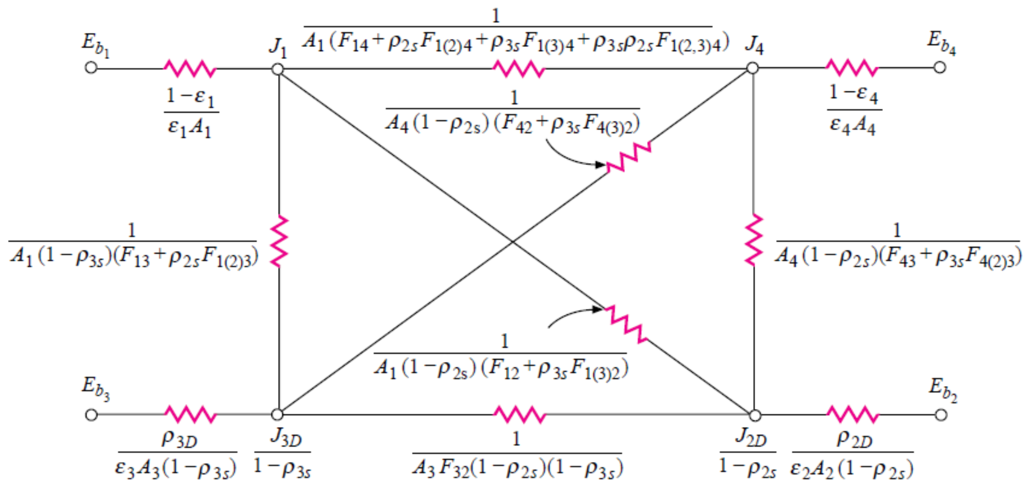


Summing the above factor: $q_{1\to4} = J_1A_1(F_{14} + \rho_{2s}F_{1(2)4} + \rho_{3s}F_{1(3)4} + \rho_{2s}\rho_{3s}F_{1(2,3)4}$ (405)

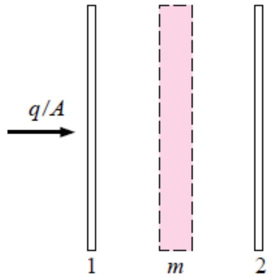
$$q_{1\to3} = J_1A_1F_{13}(1-\rho_{3s}) + J_1A_1\rho_{2s}F_{1(2)3}(1-\rho_{3s}) \tag{406}$$

The diffuse energy going from 3 to 1: $q_{3\to1} = J_{3D}A_3F_{31} + J_{3D}A_3\rho_{2s}F_{3(2)1}$ (407)

From (405) to (407) we can obtain the above two resistors.



Radiation Exchange with Transmitting, Reflecting and Absorbing Media:



Assume surface of transmitting medium have both specular and diffuse component of reflection.

For transmitting medium m we have:

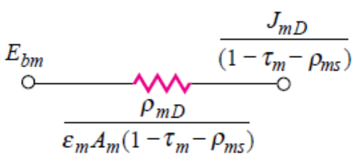
$$\alpha_m + \rho_{mD} + \rho_{ms} + \tau_m = 1 \qquad (408)$$

Also, $\epsilon_m = \alpha_m$.

The diffuse radiosity of a surface is defined by:

$$J_{mD} = \epsilon_m E_{bm} + \rho_{mD} G \qquad (409)$$

Note that J_{mD} no longer represents the total diffuse energy leaving the surface. It represents only emission and diffuse reflection. The transmitted energy will be analyzed with addition terms.



$$q = A(\epsilon E_b - \alpha G) \qquad (410)$$

Solving for G in (409) and making use of (408), we have

$$q = \frac{E_{bm} - \frac{J_{mD}}{1-\tau_m-\rho_{ms}}}{\frac{\rho_{mD}}{\epsilon_m A_m(1-\tau_m-\rho_{ms})}} \qquad (411)$$

The transmitted heat exchange between surface 1 and 2 is same as before:

$$q = \frac{J_1 - J_2}{1/A_1F_{12}\tau_m} \qquad (412)$$

The heat exchange between surface 1 and m is computed as below:

Energy leaving surface 1 arriving m, contributes to diffuse radiosity of m:

$$q_{1\to m} = J_1 A_1 F_{1m}(1-\tau_m-\rho_{ms}) \qquad (413)$$

Diffuse energy leaving m arriving 1: $\quad q_{m\to 1} = J_{mD} A_m F_{m1} \qquad (414)$

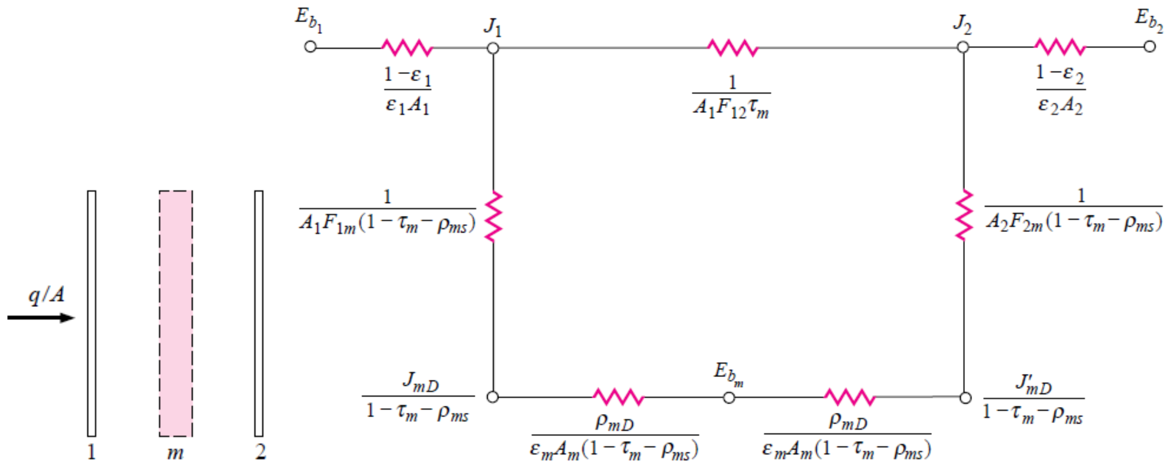
Solving (413) and (414), using the reciprocity relation $A_1F_{1m} = A_m F_{m1}$:

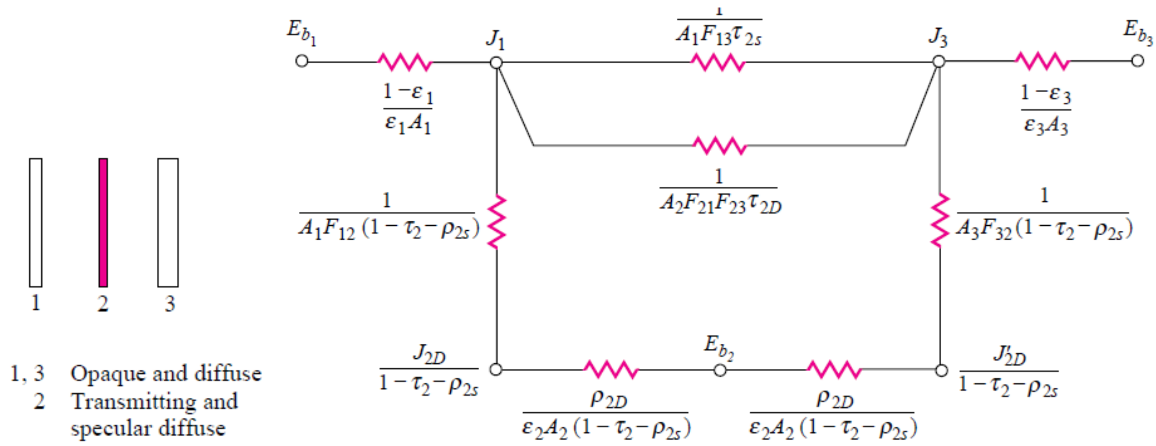
$$q = \frac{J_1 - \frac{J_{mD}}{1-\tau_m-\rho_{ms}}}{\frac{1}{A_1F_{1m}(1-\tau_m-\rho_{ms})}} \qquad (415)$$

Similar to reflected energy, transmissivity may be represented with specular and diffuse component. i.e. $\tau = \tau_S + \tau_D$. For radiation network for infinite parallel planes separated by a transmitting specular diffuse plane, we have $(q_{13})_{specular} = \frac{J_1 - J_3}{1/A_1F_{13}\tau_{2s}}$ $\quad (q_{13})_{Diffuse} = J_1A_1F_{12}\tau_{2D}F_{23} \qquad (416)$

$$(q_{31})_{Diffuse} = J_3A_3F_{32}\tau_{2D}F_{21}$$

Hence, $\quad (q_{13})_{net\ Diffuse} = \frac{J_1 - J_3}{1/A_1F_{21}F_{23}\tau_{2D}} \qquad (417)$





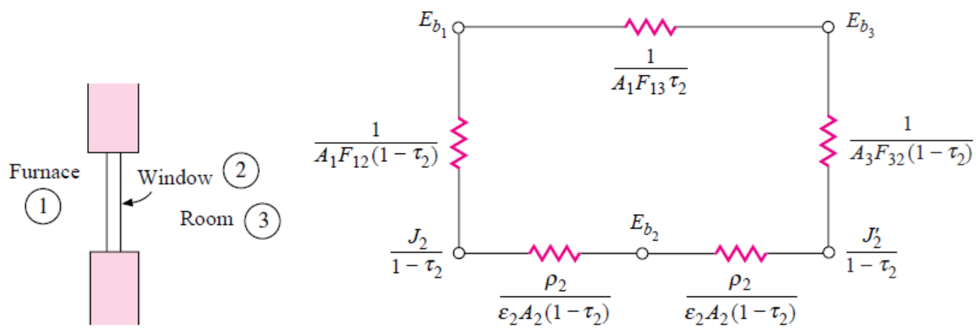
e.g.

A furnace at 1000°C has a small opening in the side that is covered with a quartz window having the following properties:

$$0 < \lambda < 4\ \mu\text{m} \qquad \tau = 0.9 \qquad \epsilon = 0.1 \qquad \rho = 0$$

$$4 < \lambda < \infty \qquad \tau = 0 \qquad \epsilon = 0.8 \qquad \rho = 0.2$$

The interior of the furnace may be treated as a blackbody. Calculate the radiation lost through the quartz window to a room at 30°C. Diffuse surface behavior is assumed.



$$A_1 = A_2 = A_3 = 1.0$$

$$F_{12} = 1.0 \qquad F_{13} = 1.0 \qquad F_{32} = 1.0$$

The total emissive powers are

$$E_{b_1} = (5.669 \times 10^{-8})(1273)^4 = 1.4887 \times 10^5 \text{ W/m}^2$$
$$E_{b_3} = (5.669 \times 10^{-8})(303)^4 = 477.8 \text{ W/m}^2$$

To determine the fraction of radiation in each wavelength band, we calculate

$$\lambda T_1 = (4)(1273) = 5092 \ \mu\text{m} \cdot \text{K}$$
$$\lambda T_3 = (4)(303) = 1212 \ \mu\text{m} \cdot \text{K}$$

Consulting Table 8-1, we find

$$E_{b_1}(0-4\ \mu\text{m}) = 0.6450 E_{b_1} = 96{,}021 \text{ W/m}^2$$
$$E_{b_3}(0-4\ \mu\text{m}) = 0.00235 E_{b_3} = 1.123 \text{ W/m}^2$$
$$E_{b_1}(4-\infty) = (1-0.6450)E_{b_1} = 52{,}849 \text{ W/m}^2$$
$$E_{b_3}(4-\infty) = (1-0.00235)E_{b_3} = 476.7 \text{ W/m}^2$$

We now apply these numbers to the network for the two wavelength bands, with unit areas. $0 < \lambda < 4\ \mu\text{m}$ *band:*

$$\frac{1}{F_{13}\tau_2} = \frac{1}{0.9} \qquad \frac{1}{F_{32}(1-\tau_2)} = \frac{1}{0.1} = \frac{1}{F_{12}(1-\tau_2)}$$
$$\frac{\rho_2}{\epsilon_2(1-\tau_2)} = 0$$

The net heat transfer from the network is then

$$q = \frac{E_{b_1} - E_{b_3}}{R_{\text{equiv}}} = \frac{96{,}021 - 1.123}{1.0526} = 91{,}219 \text{ W/m}^2 \qquad 0 < \lambda < 4\ \mu\text{m}$$

$4\ \mu\text{m} < \lambda < +\infty$ *band:*

$$\frac{1}{F_{13}\tau_2} = \infty \qquad \frac{1}{F_{32}(1-\tau_2)} = \frac{1}{F_{12}(1-\tau_2)} = 1.0$$
$$\frac{\rho_2}{\epsilon_2(1-\tau_2)} = \frac{0.2}{0.8} = 0.25$$

The net heat transfer from the network is

$$q = \frac{E_{b_1} - E_{b_3}}{1 + 0.25 + 0.25 + 1} = \frac{52{,}849 - 476.7}{2.5} = 20{,}949 \text{ W/m}^2 \qquad 4 < \lambda < \infty$$

The total heat loss is then

$$q_{\text{total}} = 91{,}219 + 20{,}949 = 112{,}168 \text{ W/m}^2 \qquad [35{,}560 \text{ Btu/h} \cdot \text{ft}^2]$$

With no window at all, the heat transfer would have been the difference in blackbody emissive powers,

$$q - E_{b_1} - E_{b_3} = 1.4887 \times 10^5 - 477.8 = 1.4839 \times 10^5 \text{ W/m}^2 \qquad [47{,}040 \text{ Btu/h} \cdot \text{ft}^2]$$

Numerical Method:

For $\epsilon = \alpha$, $$\frac{q_i}{A_i} = \epsilon_i E_b - \alpha G = \epsilon_i\left(E_{b_i} - \sum_j G_j\right) \qquad (418)$$

$A_j J_j F_{ji} = G_j A_i$ and $A_j F_{ji} = A_i F_{ij}$: $$\frac{q_i}{A_i} = \epsilon_i\left(E_{B_i} - \sum_j F_{ij} J_j\right) = J_i - \sum_j F_{ij} J_j \qquad (419)$$

The nodal analysis for the radiosities may be derived from the nodes in the network formation. At each J_i node, an energy balance gives: $$\frac{\epsilon_i}{1-\epsilon_i}\left(E_{B_i} - J_i\right) + \sum_j F_{ij}\left(J_j - J_i\right) = 0 \qquad (420)$$

* For insulated surface (i.e. one for which there is no net heat transfer, $E_{bi} = J_i$.
* For practical point of view, its better to use Gauss-Seidel Scheme to solve the above equations and it the equations must be organized in explicit form for J_j. Solving for J_i,

$$J_i = (1-\epsilon_i)\sum_{j\neq i} F_{ij}J_j + (1-\epsilon_i)F_{ii}J_i + \epsilon_i E\_b_i \tag{421}$$

$$J_i = \frac{1}{1-F_{ii}(1-\epsilon_i)}\left((1-\epsilon_i)\sum_{i\neq j} F_{ij}J_j + \epsilon_i E_{b_i}\right) \tag{422}$$

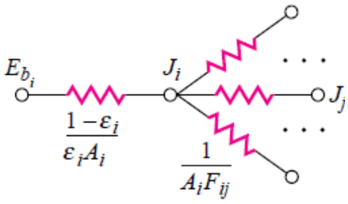
* For a surface in radiant equilibrium, $q_i / A_i = 0$ and $J_i = E_{bi}$ may be substituted into (422) to obtain

$$J_i = \frac{1}{1-F_{ii}}\sum_{j\neq i} F_{ij}J_j \tag{423}$$

For a specific heat flux q_i / A_i at one of the *i* th surfaces,

$$E_{b_i} = J_i + \frac{1-\epsilon_i}{\epsilon_i}\frac{q_i}{A_i} \tag{424}$$

Substituting (422), we can obtain $$J_i = \frac{1}{1-F_{ii}}\left(\sum_{i\neq j} F_{ij}J_j + \frac{q_i}{A_i}\right) \tag{425}$$



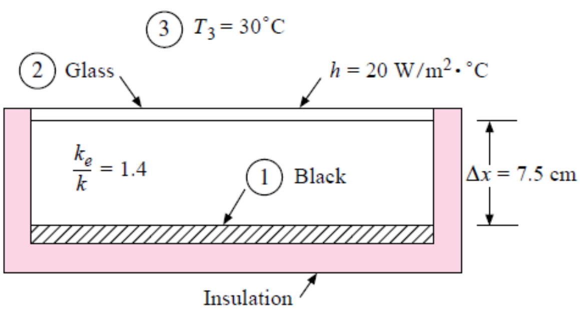
$$\frac{\varepsilon_i}{1-\varepsilon_i}(E_{b_i} - J_i) + \sum_j F_{ij}(J_j - J_i) = 0$$

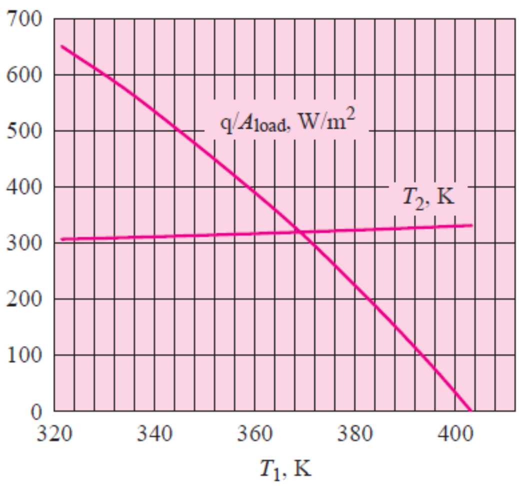
For steady state:

Heat conducted into surface + heat convected into surface = radiant heat lost from surface.

e.g.

A flat-plate solar collector is constructed as shown in Figure Example 8-22a. A glass plate covers the blackened surface, which is insulated. Solar energy at the rate of 750 W/m^2 is transmitted through the glass and absorbed in the blackened surface. The surface heats up and radiates to the glass and also loses heat by convection across the air gap, which has $k_e/k = 1.4$. The outside surface of the glass loses heat by radiation and convection to the environment at 30°C with $h = 20$ W/m$^2 \cdot$°C. It is assumed that the glass does not transmit any of the thermal radiation and has $\epsilon = 0.9$. The blackened surface is assumed to have $\epsilon = 1.0$ for all radiation. Determine the temperatures of the glass and inside surface.





This is an interesting example of combined radiation and convection heat-transfer analysis. We designate the black plate as surface 1, the glass as surface 2, and the surroundings as surface 3. We assume no absorption of the solar energy in the glass.

For the black plate

$$J_1 = E_{b1}$$

The convection and solar energy delivered to surface 1 is

$$\left.\frac{q}{A}\right]_1 = \frac{k_e}{\Delta x}(T_2 - T_1) + \left.\frac{q}{A}\right]_s \qquad [a]$$

If Equation (8-109) is now applied, we have

$$E_{b1} - F_{12}J_{2i} = \frac{k_e}{\Delta x}(T_2 - T_1) + \left.\frac{q}{A}\right]_s \qquad [b]$$

where J_{2i} is the inside radiosity for the glass. The overall energy balance for surface 2 is

$$\frac{\epsilon_2}{1-\epsilon_2}(2E_{b2} - J_{2i} - J_{2o}) = \frac{k_e}{\Delta x}(T_1 - T_2) + h(T_3 - T_2) \qquad [c]$$

where, now, J_{2o} is the outside radiosity of the glass.

For the overall system, the solar energy absorbed must eventually be lost by convection and radiation from the outside surface of the glass. Thus,

$$\left.\frac{q}{A}\right]_s = h(T_2 - T_3) + \epsilon_2(E_{b2} - E_{b3}) \qquad [d]$$

Finally, the radiation lost from the outside of the glass can be written two ways:

$$\left.\frac{q}{A}\right]_{\text{rad}} = \epsilon_2(E_{b2} - E_{b3}) = (E_{b2} - J_{2o})\frac{\epsilon_2}{1-\epsilon_2} \qquad [e]$$

The area of the collector is very large compared to the spacing so $F_{12} \approx 1.0$. We now have four equations and four unknowns: E_{b1}, E_{b2}, J_{2i}, and J_{2o}. Of course, T_1 and T_2 are expressed in terms of E_{b1} and E_{b2}.

The above equations may be rearranged algebraically into the set

$$E_{b1} = \sigma T_1^4$$

$$E_{b2} = \sigma T_2^4$$

$$\epsilon_2(E_{b1} + E_{b3} - 2E_{b2}) + \frac{k_e}{\Delta x}(T_1 - T_2) + h(T_3 - T_2) = 0$$

$$\epsilon_2(E_{b2} - E_{b1}) + \frac{k_e}{\Delta x}(T_2 - T_1) + \frac{q}{A}_{\text{solar}} = 0$$

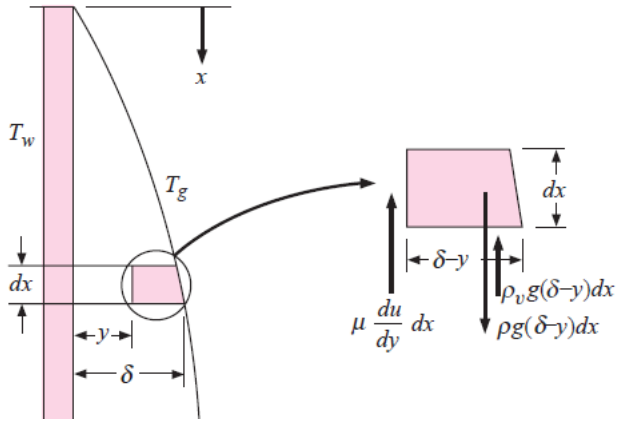
This set may be solved to yield

$$T_1 = 401.4\,\text{K}$$

$$T_2 = 331.3\,\text{K}$$

A performance chart for the collector may be calculated very simply by reducing the solar input ($750\ \text{W/m}^2$) by the amount of extracted energy and recomputing the values of T_1 and T_2 under the new condition. The results are shown in Figure Example 8-22b. Note that the adiabatic condition, that is, zero load, produces a collector temperature of 401.4 K. Extensive information on solar collectors is given in Reference 31.

Condensation and Boiling: (9)



The weight of fluid element of thickness *dx* between y and δ

= Viscous shear force at *y* + Buoyancy force due to displaced vapor

i.e. $$\rho g(\delta - y)dx = \mu \frac{du}{dy} dx + \rho_v g(\delta - y)dx \qquad (426)$$

Integrating and using the boundary condition u=0 at y=0:

$$u = \frac{(\rho - \rho_v)g}{\mu}\left(\delta y - \frac{1}{2}y^2\right) \qquad (427)$$

Mass flow of condensate through any x position:

$$\dot{m} = \int_0^\delta \rho \frac{(\rho-\rho_v)g}{\mu}\left(\delta y - \frac{1}{2}y^2\right)dy = \frac{\rho(\rho-\rho_v)g\delta^3}{3\mu} \qquad (428)$$

where unit depth is assumed. The heat transfer at the wall in the area *dx* is

$$q_x = -kdx \frac{\partial T}{\partial y}\Big|_{y=0} = kdx \frac{(T_g - T_w)}{\delta} \qquad (429)$$

As the flow proceeds from *x* to *x*+*dx*, the film grows from δ to $\delta + d\delta$ as a result of the influx of additional condensate. The amount of condensate added between *x* and *x* + *dx* is

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\rho(\rho-\rho_v)g\delta^3}{3\mu}\right)dx = \frac{d}{d\delta}\left(\frac{\rho(\rho-\rho_v)g\delta^3}{3\mu}\right)\frac{d\delta}{dx}dx = \frac{\rho(\rho-\rho_v)g\delta^2 d\delta}{3\mu} \qquad (430)$$

The heat removed by the wall must equal the incremental mass x its latent heat of condensation.

$$\frac{\rho(\rho-\rho_v)g\delta^2 d\delta}{3\mu} h_{fg} = kdx \frac{(T_g - T_w)}{\delta} \qquad (431)$$

The heat-transfer coefficient is now written:

$$h\, dx(T_w - T_g) = -k\, dx\, \frac{(T_g - T_w)}{\delta} \quad \text{or} \quad h = \frac{k}{\delta} \qquad (432)$$

so that $$h_x = \left(\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{4\mu x(T_g - T_w)}\right)^{\frac{1}{4}} \qquad (433)$$

Expressed in dimensionless form in terms of Nusselt number,

$$\mathrm{Nu_x} = \frac{hx}{k} = \left(\frac{\rho(\rho-\rho_v)gh_{fg}x^3}{4\mu k(T_g - T_w)}\right)^{\frac{1}{4}} \qquad (434)$$

Average value of the heat transfer coefficient is obtained by integrating over the length of the plate:

$$\bar{h} = \frac{1}{L}\int_0^L h_x dx = \frac{4}{3}h_{x=L} \qquad (435)$$

or $$\bar{h} = 0.943\left(\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{L\mu_f(T_g - T_w)}\right)^{\frac{1}{4}} \qquad (436)$$

Nonlinearity of temperature profile and addition energy used to cool the film below saturation temp needs fine toning of h_{fg}: $$h'_{fg} = h_{fg} + 0.68\, c\, (T_g - T_w) \qquad (437)$$

where *c* is the specific heat of the liquid. Otherwise, properties in (433) and (436) should be evaluated at the film temperature $T_f = \frac{T_g + T_w}{2}$. (436) can be used for vertical plates and cylinders with fluid Pr > 0.5 and cT/h_{fg} <=1.0.

For laminar film condensation on horizontal tubes, Nusselt obtained the relation

$$h = 0.725\left(\frac{\rho(\rho-\rho_v)gh_{fg}k_f^3}{\mu_f d(T_g-T_w)}\right)^{\frac{1}{4}} \tag{438}$$

In forced convection flow problems, the criterion for determining whether the flow is turbulent or laminar is the Reynolds number, and for the condensation system, it is defined as

$$\mathrm{Re_f} = \frac{D_H\rho V}{\mu_f} = \frac{4A\rho V}{P\mu_f} \tag{439}$$

Reynolds is sometimes expressed in terms of mass flow per unit depth of plate Γ, so that

$$\mathrm{Re_f} = \frac{4\Gamma}{\mu_f} \tag{440}$$

With

$$q = \bar{h}A(T_{sat} - T_w) = \dot{m}\mathrm{h_{fg}} \tag{441}$$

$$\dot{m} = \frac{q}{h_{fg}} = \frac{\bar{h}A(T_{sat}-T_w)}{h_{fg}} \tag{442}$$

Substitute back to (439),

$$\mathrm{Re_f} = \frac{4A\bar{h}A(T_{sat}-T_w)}{P\mu_f h_{fg}} \tag{443}$$

For (436), it is a conservative approach that provides a safety factor in design problems. If one wishes to employ 20% higher coefficient, the resulting equation for vertical plates is:

$$\bar{h} = 1.13\left(\frac{\rho(\rho-\rho_v)gh_{fg}k^3}{L\mu_f(T_g-T_w)}\right)^{\frac{1}{4}} \tag{444}$$

For inclined surfaces, the net effect on the above analysis is to replace g' = g sin ϕ. (445)

From (441), we can solve for $T_g - T_w$ as

$$T_g - T_w = \frac{\dot{m}\mathrm{h_{fg}}}{\bar{h}A} \tag{446}$$

Substitute (446) into (445)

$$\bar{h}^{\frac{3}{4}} = C\left(\frac{\rho(\rho-\rho_v)g\sin\phi k^3\ A/L}{\mu\dot{m}}\right)^{\frac{1}{4}} \tag{447}$$

We may decompose $\bar{h}$ as

$$\bar{h} = C^{\frac{4}{3}}\left(\frac{\rho(\rho-\rho_v)gk^3}{\mu^2}\frac{\mu P}{4\dot{m}}\frac{4\sin\phi\frac{\mathrm{A}}{\mathrm{P}}}{\mathrm{L}}\right)^{\frac{1}{3}} \tag{448}$$

Define a new dimensionless group, the *condensation number* Co, as

$$\mathrm{Co} = \bar{h}\left(\frac{\mu^2}{k^3\rho(\rho-\rho_v)g}\right)^{\frac{1}{3}} \tag{449}$$

So (448) can be expressed as

$$\mathrm{Co} = \mathrm{C}^{\frac{4}{3}}\left(\frac{4\sin\phi\frac{\mathrm{A}}{\mathrm{P}}}{\mathrm{L}}\right)^{\frac{1}{3}} Re_f^{-\frac{1}{3}} \tag{450}$$

For a vertical plates A/PL = 1.0: $\mathrm{Co} = 1.47\ \mathrm{Re_f^{-1/3}}$ $\quad \mathrm{Re_f} < 1800$

For a horizontal cylinder A/PL = π: $\mathrm{Co} = 1.514\ \mathrm{Re_f^{-1/3}}$ $\quad \mathrm{Re_f} < 1800$ (451)

For turbulence countered in films, $\mathrm{Co} = 0.0077\ \mathrm{Re_f^{-1/3}}$ $\quad \mathrm{Re_f} > 1800$ (452)

Film condensation inside horizontal tubes:

For condensation of refrigerants at low vapor velocity inside horizontal tubes:

$$\bar{h} = 0.555\left(\frac{\rho(\rho-\rho_v)gk^3h'_{fg}}{\mu d(T_g-T_w)}\right)^{\frac{1}{4}} \tag{453}$$

where the modified enthalpy of vaporization is given by $h'_{fg} = h_{fg} + 0.375\ c_{p,l}(T_g - T_w)$. This equation is restricted to low vapor Reynolds such that $\mathrm{Re_v} = \frac{dG_v}{\mu_v} < 35000$, where $\mathrm{Re_v}$ is evaluated at inlet condition to the tube.

For higher flow rates, $$\frac{\bar{h}d}{k_f} = 0.026 Pr_f^{1/3} Re_m^{0.8} \qquad (454)$$

where now Re_m is a mixture Reynolds, defined as $$Re_m = \frac{d}{\mu_f}\left(G_f + G_v\left(\frac{\rho_f}{\rho_v}\right)^{\frac{1}{2}}\right) \qquad (456)$$

e.g.

One hundred tubes of 0.50-in (1.27-cm) diameter are arranged in a square array and exposed to atmospheric steam. Calculate the mass of steam condensed per unit length of tubes for a tube wall temperature of 98°C.

The condensate properties are obtained from Example 9-1. We employ Equation (9-12) for the solution, replacing d by nd, where $n = 10$. Thus,

$$\bar{h} = 0.725\left[\frac{\rho_f^2 g h_{fg} k_f^3}{\mu_f n d(T_g - T_w)}\right]^{1/4}$$

$$= 0.725\left[\frac{(960)^2(9.8)(2.255\times 10^6)(0.68)^3}{(2.82\times 10^{-4})(10)(0.0127)(100-98)}\right]^{1/4}$$

$$= 12{,}540\ \text{W/m}^2\cdot{}^\circ\text{C} \quad [2209\ \text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F}]$$

The total surface area is

$$\frac{A}{L} = n\pi d = (100)\pi(0.0127) = 3.99\ \text{m}^2/\text{m}$$

so the heat transfer is

$$\frac{q}{L} = h\frac{A}{L}(T_g - T_w)$$
$$= (12{,}540)(3.99)(100-98) = 100.07\ \text{kW/m}$$

The total mass flow of condensate is then

$$\frac{\dot{m}}{L} = \frac{q/L}{h_{fg}} = \frac{1.0007\times 10^5}{2.255\times 10^6} = 0.0444\ \text{kg/s} = 159.7\ \text{kg/h} \quad [352\ \text{lb}_m/\text{h}]$$

e.g.

A vertical square plate, 30 by 30 cm, is exposed to steam at atmospheric pressure. The plate temperature is 98°C. Calculate the heat transfer and the mass of steam condensed per hour.

Properties are evaluated at the film temperature:

$$T_f = \frac{100+98}{2} = 99^\circ\text{C} \qquad \rho_f = 960\ \text{kg/m}^3$$
$$\mu_f = 2.82\times 10^{-4}\ \text{kg/m}\cdot\text{s} \qquad k_f = 0.68\ \text{W/m}\cdot{}^\circ\text{C}$$

For this problem the density of the vapor is very small in comparison with that of the liquid, and we are justified in making the substitution

$$\rho_f(\rho_f - \rho_v) \approx \rho_f^2$$

the heat-transfer coefficient, and then check the Reynolds number to see if our assumption was correct. Let us assume laminar film condensation. At atmospheric pressure we have

$$T_{\text{sat}} = 100^\circ\text{C} \qquad h_{fg} = 2255\ \text{kJ/kg}$$

$$\bar{h} = 0.943\left[\frac{\rho_f^2 g h_{fg} k_f^3}{L\mu_f(T_g - T_w)}\right]^{1/4}$$

$$= 0.943\left[\frac{(960)^2(9.8)(2.255\times10^6)(0.68)^3}{(0.3)(2.82\times10^{-4})(100-98)}\right]^{1/4}$$

$$= 13{,}150 \text{ W/m}^2\cdot{}^\circ\text{C} \quad [2316 \text{ Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F}]$$

Checking the Reynolds number with Equation (9-17), we have

$$\text{Re}_f = \frac{4\bar{h}L(T_{\text{sat}} - T_w)}{h_{fg}\mu f}$$

$$= \frac{(4)(13{,}150)(0.3)(100-98)}{(2.255\times10^6)(2.82\times10^{-4})} = 49.6$$

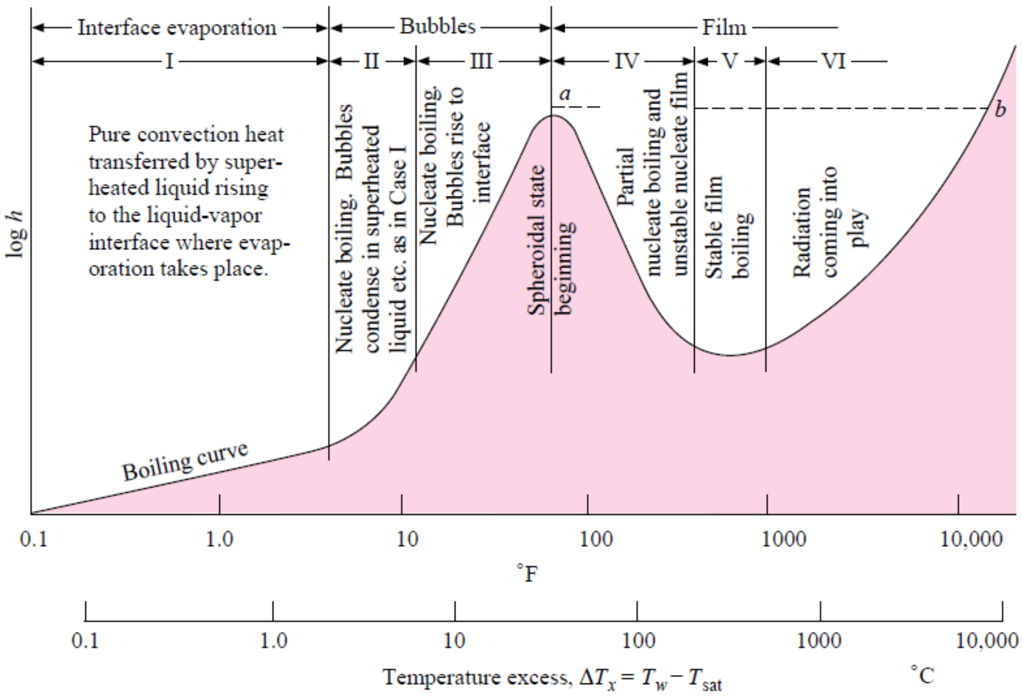
so that the laminar assumption was correct. The heat transfer is now calculated from

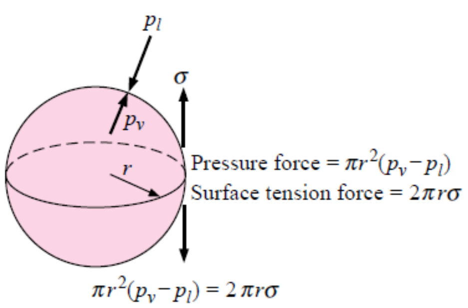
$$q = \bar{h}A(T_{\text{sat}} - T_w) = (13{,}150)(0.3)^2(100-98) = 2367 \text{ W} \quad [8079 \text{ Btu/h}]$$

The total mass flow of condensate is

$$\dot{m} = \frac{q}{h_{fg}} = \frac{2367}{2.255\times10^6} = 1.05\times10^{-3} \text{ kg/s} = 3.78 \text{ kg/h} \quad [8.33 \text{ lb}_m/\text{h}]$$

For heating,





Force balance: $\pi r^2(p_v - p_l) = 2\pi r\sigma$

Or $$p_v - p_l = \frac{2\sigma}{r} \quad (457)$$

For boiling,

$$\frac{c_l \Delta T_x}{h_{fg} Pr_l^s} = C_{sf}\left(\frac{\frac{q}{A}}{\mu_l h_{fg}}\sqrt{\frac{g_c\sigma}{g(\rho_l - \rho_v)}}\right)^{0.33} \quad (458)$$

$\Delta T_x = T_w - T_{sat}$ s = 1.0 for water and 1.7 for others.

Fluid-heating-surface combination	C_{sf}
Water-copper [11][†]	0.013
Water-platinum [12]	0.013
Water-brass [13]	0.0060
Water–emery-polished copper [29]	0.0128
Water–ground and polished stainless steel [29]	0.0080
Water–chemically etched stainless steel [29]	0.0133
Water–mechanically polished stainless steel [29]	0.0132
Water–emery-polished and paraffin-treated copper [29]	0.0147
Water–scored copper [29]	0.0068
Water–Teflon pitted stainless steel [29]	0.0058
Carbon tetrachloride–copper [11]	0.013
Carbon tetrachloride–emery-polished copper [29]	0.0070
Benzene-chromium [14]	0.010
n-Butyl alcohol–copper [11]	0.00305
Ethyl alcohol–chromium [14]	0.027
Isopropyl alcohol–copper [11]	0.00225
n-Pentane–chromium [14]	0.015
n-Pentane–emery-polished copper [29]	0.0154
n-Pentane–emery-polished nickel [29]	0.0127
n-Pentane–lapped copper [29]	0.0049
n-Pentane–emery-rubbed copper [29]	0.0074
35% K_2CO_3–copper [11]	0.0054
50% K_2CO_3–copper [11]	0.0027

Saturation temperature		Surface tension	
°F	°C	σ, mN/m	$\sigma \times 10^4$, lb_f/ft
32	0	75.6	51.8
60	15.56	73.3	50.2
100	37.78	69.8	47.8
140	60	66.0	45.2
200	93.33	60.1	41.2
212	100	58.8	40.3
320	160	46.1	31.6
440	226.67	32.0	21.9
560	293.33	16.2	11.1
680	360	1.46	1.0
705.4	374.1	0	0

For heat flow, boiling is also a kind of forced convection effect

$$\left(\frac{q}{A}\right)_{total} = \left(\frac{q}{A}\right)_{boil} + \left(\frac{q}{A}\right)_{forced\ conv} \tag{459}$$

McAdams suggests $\frac{q}{A} = 2.253(\Delta T_x)^{3.96}$ for low pressure boiling water $0.2 < p < 0.7$

and $\frac{q}{A} = 283.2p^{4/3}(\Delta T_x)^3$ for high pressure boiling water $0.7 < p < 14$ (460)

Zuber has developed an analytical expression for peak heat flux in nucleate boiling by considering the stability requirements of the interface between the vapor film and liquid:

$$\left(\frac{q}{A}\right)_{max} = \frac{\pi}{24}\, h_{fg}\rho_v \left(\frac{\sigma g(\rho_l-\rho_v)}{\rho_v^2}\right)^{\frac{1}{4}} \left(1+\frac{\rho_v}{\rho_l}\right)^{\frac{1}{2}} \tag{461}$$

Bromley suggests the following relation for calculation of heat-transfer coefficient in stable film boiling region on a horizontal tube:

$$h_b = 0.62\left(\frac{k_g\rho_v(\rho_l-\rho_v)g\left(h_{fg}+0.4c_{pv}\Delta T_x\right)}{d\mu_v\Delta T_x}\right)^{\frac{1}{4}} \tag{462}$$

This heat transfer coefficient considers only the conduction through the film and does not include the effects of radiation. The total heat transfer coefficient may be calculated from this empirical formula:

$$h = h_b\left(\frac{h_b}{h}\right)^{\frac{1}{3}} + \frac{\sigma\epsilon\left(T_w^4-T_{sat}^4\right)}{T_w-T_{sat}} \tag{463}$$

Here are the relations for boiling heat transfer with water:

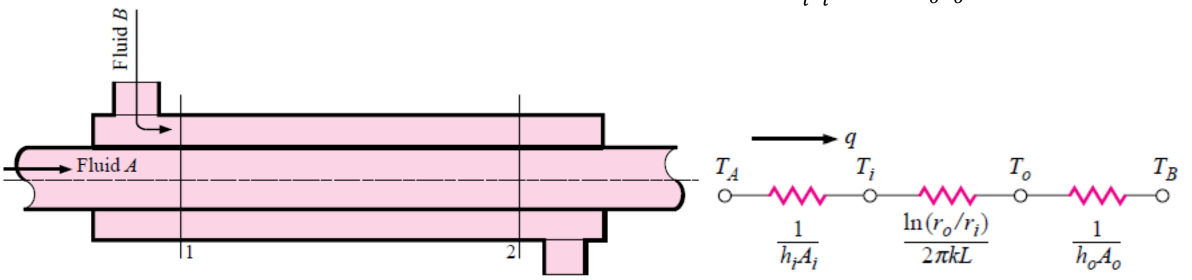
Surface	$\frac{q}{A}$, kW/m^2	h, W/m$^2\cdot$°C	Approximate range of ΔT, °C	Approximate range of h, W/m$^2\cdot$°C
Horizontal	$\frac{q}{A} < 16$	$1042(\Delta T_x)^{1/3}$	0–7.76	0–2060
	$16 < \frac{q}{A} < 240$	$5.56(\Delta T_x)^3$	7.32–14.4	2180–16,600
Vertical	$\frac{q}{A} < 3$	$537(\Delta T_{ax})^{1/7}$	0–4.51	0–670
	$3 < \frac{q}{A} < 63$	$7.96(\Delta T_x)^3$	4.41–9.43	680–6680

For forced convection local boiling inside vertical tubes is:

$$h = 2.54(\Delta T_x)^3 e^{p/1.551} \tag{464}$$

Heat Exchangers: (10)

Recall the heat flow of double-pipe heat exchange: $$q = \frac{(T_A - T_B)}{\frac{1}{h_i A_i} + \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k L} + \frac{1}{h_o A_o}} \qquad (465)$$

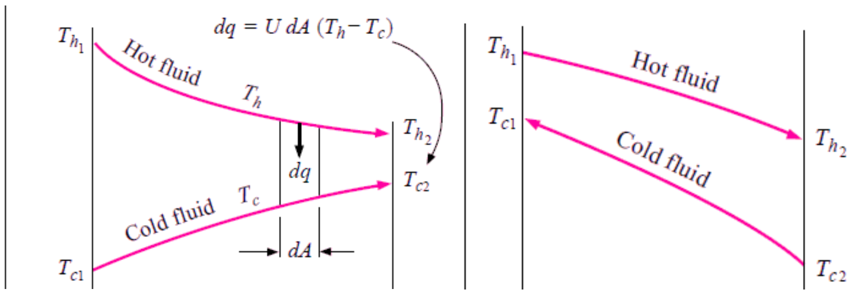


Physical situation	U: Btu/h · ft^2 · °F	U: W/m^2 · °C
Brick exterior wall, plaster interior, uninsulated	0.45	2.55
Frame exterior wall, plaster interior: uninsulated	0.25	1.42
with rock-wool insulation	0.07	0.4
Plate-glass window	1.10	6.2
Double plate-glass window	0.40	2.3
Steam condenser	200–1000	1100–5600
Feedwater heater	200–1500	1100–8500
Freon-12 condenser with water coolant	50–150	280–850
Water-to-water heat exchanger	150–300	850–1700
Finned-tube heat exchanger, water in tubes, air across tubes	5–10	25–55
Water-to-oil heat exchanger	20–60	110–350
Steam to light fuel oil	30–60	170–340
Steam to heavy fuel oil	10–30	56–170
Steam to kerosone or gasoline	50–200	280–1140
Finned-tube heat exchanger, steam in tubes, air over tubes	5–50	28–280
Ammonia condenser, water in tubes	150–250	850–1400
Alcohol condenser, water in tubes	45–120	255–680
Gas-to-gas heat exchanger	2–8	10–40

The fouling factor is thus defined as $$R_f = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}} \qquad (466)$$

Type of fluid	Fouling factor, h · ft^2 · °F/Btu	m^2 · °C/W
Seawater, below 125°F	0.0005	0.00009
Above 125°F	0.001	0.002
Treated boiler feedwater above 125°F	0.001	0.0002
Fuel oil	0.005	0.0009
Quenching oil	0.004	0.0007
Alcohol vapors	0.0005	0.00009
Steam, non-oil-bearing	0.0005	0.00009
Industrial air	0.002	0.0004
Refrigerating liquid	0.001	0.0002

Temperature Profile:



Log Mean Temperature Difference:

With the simplest equation in heat transfer: $$q = UA\Delta T_m \tag{467}$$

For the heat exchanger shown in the last page, the heat transferred through an element dA may be written as $$dq = -\dot{m}_h c_h dT_h = \dot{m}_c c_c dT_c \tag{468}$$

where subscripts h and c designate for hot and cold fluids, respectively.

The heat transfer could also be expressed as $$dq = U(T_h - T_c)dA \tag{469}$$

From expression (468), $$dT_h = -\frac{dq}{\dot{m}_h c_h} = \frac{dq}{\dot{m}_c c_c} \tag{470}$$

Thus, $$dT_h - dT_c = d(T_h - T_c) = -dq\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) \tag{471}$$

Solving for dq from (469) and substitute into (471),

$$\frac{d(T_h - T_c)}{T_h - T_c} = -U\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) dA \tag{472}$$

Integrating both sides,

$$\ln\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = -UA\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) \tag{473}$$

The product $\dot{m}_h c_h$ and $\dot{m}_c c_c$ can be expressed in terms of the total heat transfer q and overall temperature difference of the hot and cold fluids, i.e.

$$\dot{m}_h c_h = \frac{q}{T_{h1} - T_{h2}} \quad \text{and} \quad \dot{m}_c c_c = \frac{q}{T_{c2} - T_{c1}} \tag{474}$$

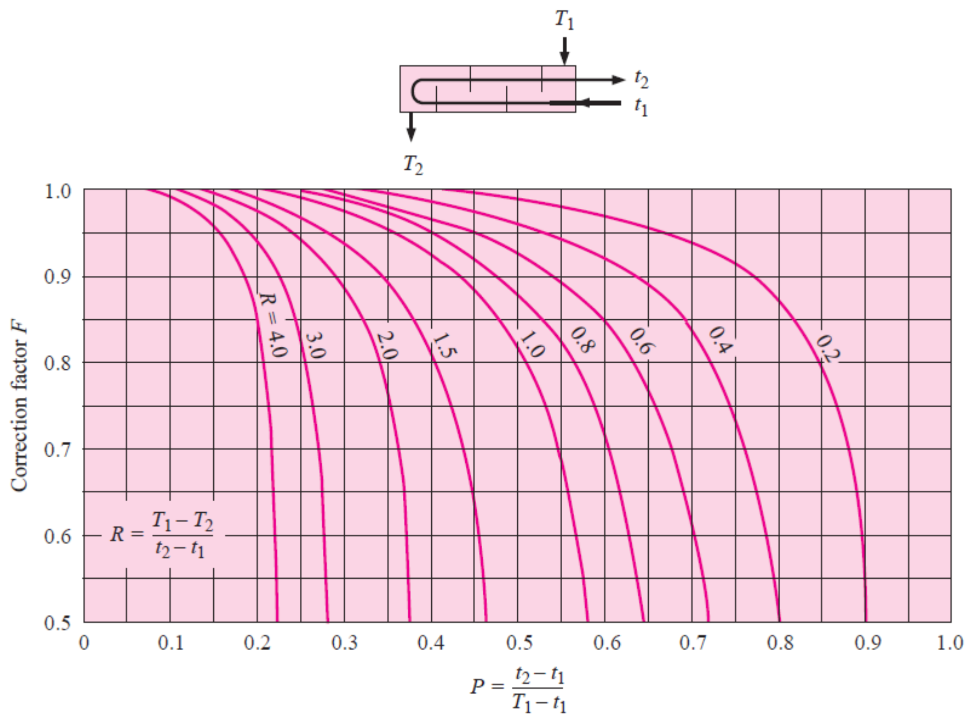
Substitute (474) into (473):

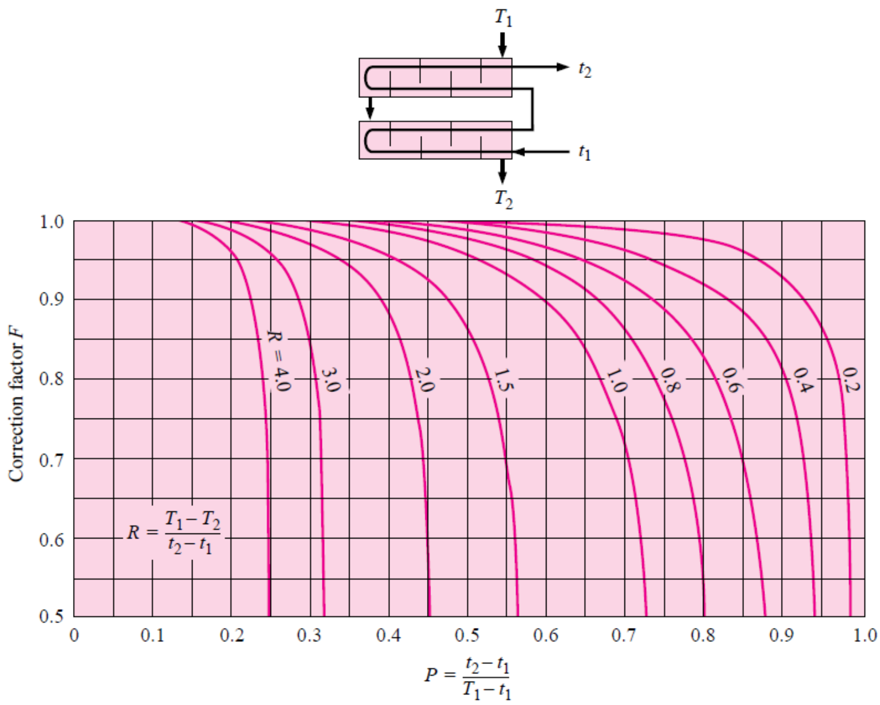
$$q = \frac{UA\big((T_{h2} - T_{c2}) - (T_{h1} - T_{c1})\big)}{\ln\frac{(T_{h2} - T_{c2})}{(T_{h1} - T_{c1})}} \tag{475}$$

Comparing (475) with (467), we discover that

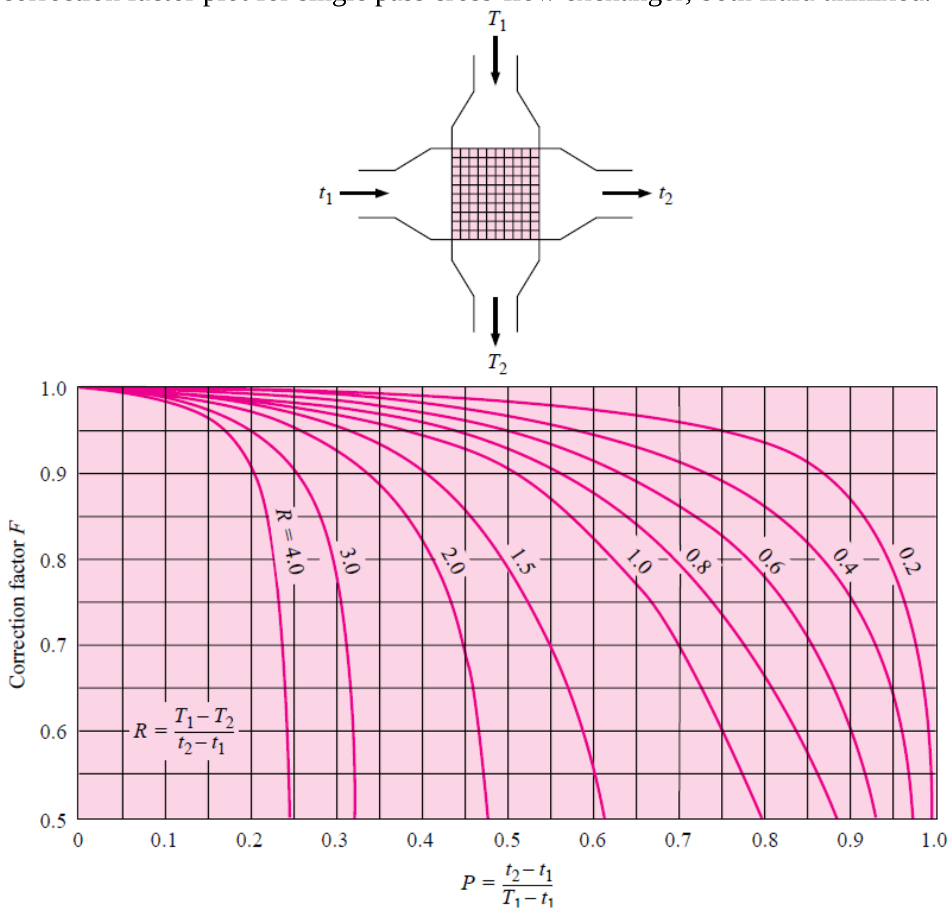
$$\Delta T_m = \frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{\ln\frac{(T_{h2} - T_{c2})}{(T_{h1} - T_{c1})}} \tag{476}$$

This temperature difference is called the log mean temperature difference (LMTD). If a heat exchanger other than the double-pipe type is used, the heat transfer is calculated by using a correction factor applied to the LMTD for a **counterflow double-pipe arrangement with the same hot and cold fluid temperature**. i.e. $$q = UAF\Delta T_m \tag{477}$$





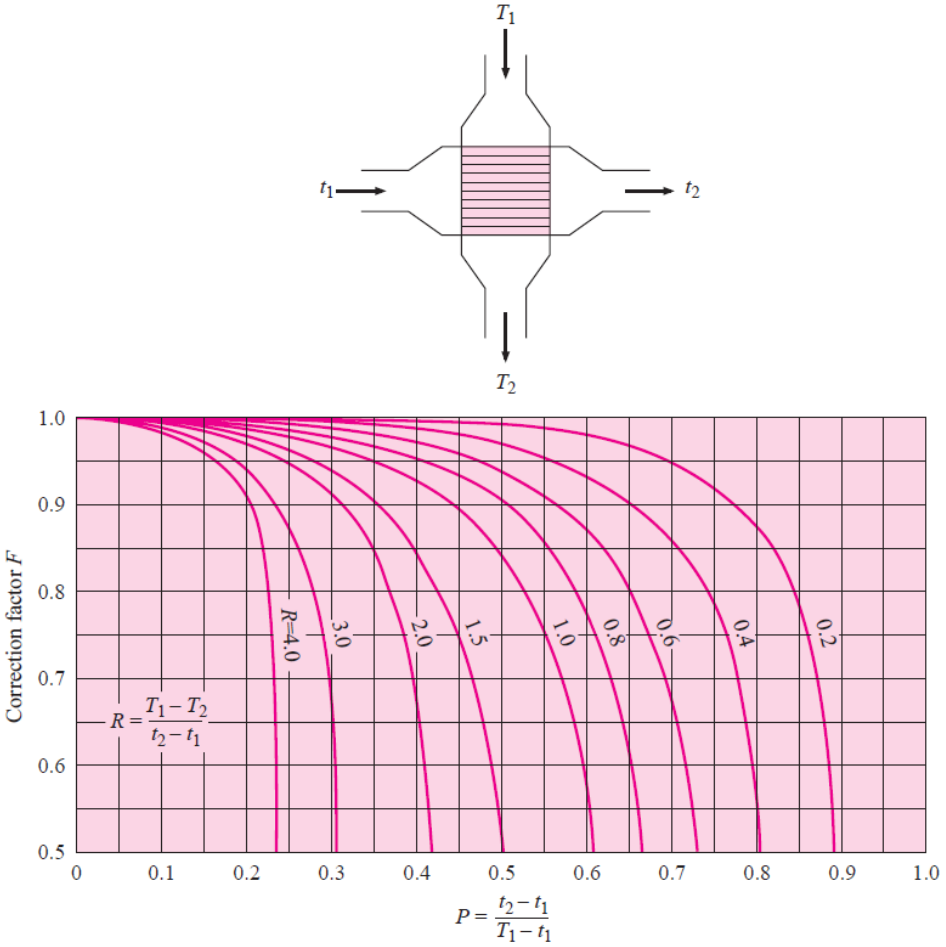
For the correction factor plot for single pass cross-flow exchanger, both fluid unmixed.



When a phase change is involved, as in condensation or boiling (evaporation), the fluid normally remains at essentially constant temperature and the relations are simplified, i.e. P or R = 0, we obtain

F = 1.0 for boiling or condensation

For the correction factor plot for single pass cross-flow exchanger, one fluid mixed, another unmixed.



e.g.

Water at the rate of 68 kg/min is heated from 35 to 75°C by an oil having a specific heat of 1.9 kJ/kg · °C. The fluids are used in a counterflow double-pipe heat exchanger, and the oil enters the exchanger at 110°C and leaves at 75°C. The overall heat-transfer coefficient is 320 W/m^2 · °C. Calculate the heat-exchanger area.

The total heat transfer is determined from the energy absorbed by the water:

$$q = \dot{m}_w c_w \Delta T_w = (68)(4180)(75 - 35) = 11.37 \text{ MJ/min} \qquad [a]$$

$$= 189.5 \text{ kW} \qquad [6.47 \times 10^5 \text{ Btu/h}]$$

Since all the fluid temperatures are known, the LMTD can be calculated by using the temperature scheme in Figure 10-7*b*:

$$\Delta T_m = \frac{(110 - 75) - (75 - 35)}{\ln[(110 - 75)/(75 - 35)]} = 37.44°\text{C} \qquad [b]$$

Then, since $q = UA\,\Delta T_m$,

$$A = \frac{1.895 \times 10^5}{(320)(37.44)} = 15.82 \text{ m}^2 \qquad [170 \text{ ft}^2]$$

e.g.

Water at the rate of 30,000 lb_m/h [3.783 kg/s] is heated from 100 to 130°F [37.78 to 54.44°C] in a shell-and-tube heat exchanger. On the shell side one pass is used with water as the heating fluid, 15,000 lb_m/h [1.892 kg/s], entering the exchanger at 200°F [93.33°C]. The overall heat-transfer coefficient is 250 $Btu/h \cdot ft^2 \cdot °F$ [1419 $W/m^2 \cdot °C$], and the average water velocity in the $\frac{3}{4}$-in [1.905-cm] diameter tubes is 1.2 ft/s [0.366 m/s]. Because of space limitations, the tube length must not be longer than 8 ft [2.438 m]. Calculate the number of tube passes, the number of tubes per pass, and the length of the tubes, consistent with this restriction.

We first assume one tube pass and check to see if it satisfies the conditions of this problem. The exit temperature of the hot water is calculated from

$$q = \dot{m}_c c_c \, \Delta T_c = \dot{m}_h c_h \, \Delta T_h$$

$$\Delta T_h = \frac{(30{,}000)(1)(130-100)}{(15{,}000)(1)} = 60°\mathrm{F} = 33.33°\mathrm{C} \qquad [a]$$

so

$$T_{h,\mathrm{exit}} = 93.33 - 33.33 = 60°\mathrm{C}$$

The total required heat transfer is obtained from Equation (a) for the cold fluid:

$$q = (3.783)(4182)(54.44 - 37.78) = 263.6\ \mathrm{kW} \quad [8.08 \times 10^5\ \mathrm{Btu/h}]$$

For a counterflow exchanger, with the required temperature

$$\mathrm{LMTD} = \Delta T_m = \frac{(93.33 - 54.44) - (60 - 37.78)}{\ln[(93.33 - 54.44)/(60 - 37.78)]} = 29.78°\mathrm{C}$$

$$q = UA\,\Delta T_m$$

$$A = \frac{2.636 \times 10^5}{(1419)(29.78)} = 6.238\ \mathrm{m}^2 \quad [67.1\ \mathrm{ft}^2] \qquad [b]$$

Using the average water velocity in the tubes and the flow rate, we calculate the total flow area with

$$\dot{m}_c = \rho A u$$

$$A = \frac{3.783}{(1000)(0.366)} = 0.01034\ \mathrm{m}^2 \qquad [c]$$

This area is the product of the number of tubes and the flow area per tube:

$$0.01034 = n\frac{\pi d^2}{4}$$

$$n = \frac{(0.01034)(4)}{\pi(0.01905)^2} = 36.3$$

or $n = 36$ tubes. The surface area per tube per meter of length is

$$\pi d = \pi(0.01905) = 0.0598\ \mathrm{m}^2/\mathrm{tube} \cdot \mathrm{m}$$

We recall that the total surface area required for a one-tube-pass exchanger was calculated in Equation (b) as 6.238 m^2. We may thus compute the length of tube for this type of exchanger from

$$n\pi d\ L = 6.238$$

$$L = \frac{6.238}{(36)(0.0598)} = 2.898\ \mathrm{m}$$

This length is greater than the allowable 2.438 m, so we must use more than one tube pass. When we increase the number of passes, we correspondingly increase the total surface area required because of the reduction in LMTD caused by the correction factor F. We next try two tube passes. From Figure 10-8, $F = 0.88$, and thus

$$A_{\text{total}} = \frac{q}{UF\Delta T_m} = \frac{2.636 \times 10^5}{(1419)(0.88)(29.78)} = 7.089 \text{ m}^2$$

The number of tubes per pass is still 36 because of the velocity requirement. For the two-tube-pass exchanger the total surface area is now related to the length by

$$A_{\text{total}} = 2n\pi d\ L$$

so that

$$L = \frac{7.089}{(2)(36)(0.0598)} = 1.646 \text{ m} \quad [5.4 \text{ ft}]$$

This length is within the 2.438-m requirement, so the final design choice is

Number of tubes per pass = 36

Number of passes = 2

Length of tube per pass = 1.646 m [5.4 ft]

e.g.

A heat exchanger like that shown in Figure 10-4 is used to heat an oil in the tubes ($c = 1.9$ kJ/kg · °C) from 15°C to 85°C. Blowing across the outside of the tubes is steam that enters at 130°C and leaves at 110°C with a mass flow of 5.2 kg/sec. The overall heat-transfer coefficient is 275 W/m^2 · °C and c for steam is 1.86 kJ/kg · °C. Calculate the surface area of the heat exchanger.

The total heat transfer may be obtained from an energy balance on the steam

$$q = \dot{m}_s c_s\, \Delta T_s = (5.2)(1.86)(130 - 110) = 193 \text{ kW}$$

We can solve for the area from Equation (10-13). The value of ΔT_m is calculated *as if the exchanger were counterflow double pipe* (i.e., as shown in Figure Example 10-7). Thus,

$$\Delta T_m = \frac{(130 - 85) - (110 - 15)}{\ln\left(\dfrac{130 - 85}{110 - 15}\right)} = 66.9°\text{C}$$

Now, from Figure 10-11, t_1 and t_2 will represent the unmixed fluid (the oil) and T_1 and T_2 will represent the mixed fluid (the steam) so that

$$T_1 = 130 \qquad T_2 = 110 \qquad t_1 = 15 \qquad t_2 = 85°\text{C}$$

and we calculate

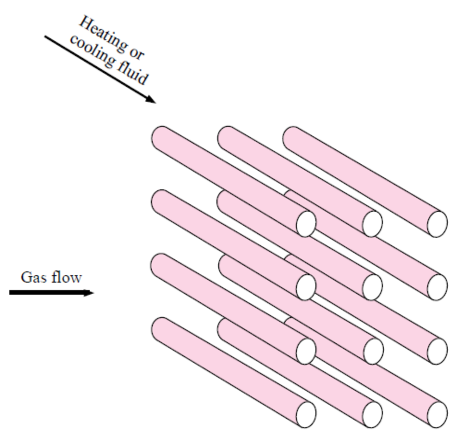
$$R = \frac{130 - 110}{85 - 15} = 0.286 \qquad P = \frac{85 - 15}{130 - 15} = 0.609$$

Consulting Figure 10-11 we find

$$F = 0.97$$

so the area is calculated from

$$A = \frac{q}{UF\,\Delta T_m} = \frac{193{,}000}{(275)(0.97)(66.9)} = 10.82 \text{ m}^2$$



Effectiveness NTU Method:

We define the heat-exchanger effectiveness as: $$\epsilon = \frac{\text{actual heat transfer}}{\text{maximum possible heat transfer}} \tag{478}$$

For parallel-flow exchanger: $$q = \dot{m}_h c_h (T_{h1} - T_{h2}) = \dot{m}_c c_c (T_{c2} - T_{c1})$$

For counter-flow exchanger: $$q = \dot{m}_h c_h (T_{h1} - T_{h2}) = \dot{m}_c c_c (T_{c1} - T_{c2}) \tag{479}$$

The fluid that might undergo the maximum temperature difference is the one having the minimum value of $\dot{m}c$ because energy balance requires that the energy received by one fluid be equal to given up by another fluid. If we let the fluid with the larger value of $\dot{m}c$ go through the maximum temperature, this would require that the other fluid undergo a temperature difference greater than the maximum, and this is impossible.

Maximum possible heat transfer is expressed as: $$q_{max} = (\dot{m}c)_{min}(T_{h_{inlet}} - T_{c_{inlet}}) \tag{480}$$

The minimum fluid may be either the hot or cold fluid, depending on the mass flow rates and specific heat.

For parallel flow: $$\epsilon_h = \frac{\dot{m}_h c_h (T_{h1}-T_{h2})}{\dot{m}_h c_h (T_{h1}-T_{c2})} = \frac{T_{h1}-T_{h2}}{T_{h1}-T_{c1}} \qquad \epsilon_c = \frac{\dot{m}_c c_c (T_{c2}-T_{c1})}{\dot{m}_c c_c (T_{h1}-T_{c1})} = \frac{T_{c2}-T_{c1}}{T_{h1}-T_{c1}} \tag{481}$$

For counter flow: $$\epsilon_h = \frac{\dot{m}_h c_h (T_{h1}-T_{h2})}{\dot{m}_h c_h (T_{h1}-T_{c2})} = \frac{T_{h1}-T_{h2}}{T_{h1}-T_{c2}} \qquad \epsilon_c = \frac{\dot{m}_c c_c (T_{c2}-T_{c1})}{\dot{m}_c c_c (T_{h1}-T_{c1})} = \frac{T_{c1}-T_{c2}}{T_{h1}-T_{c2}} \tag{482}$$

The minimum fluid is always the one experiencing the larger temperature difference in the heat exchanger, the maximum temperature difference in the heat exchanger is always the difference in inlet temperatures of the hot and cold fluids.

In a general way the effectiveness is expressed as:

$$\epsilon = \frac{\Delta\text{T(min fluid)}}{\text{Maximum temperature difference in heat exchanger}} \tag{483}$$

Rewriting (473): $$\ln\frac{T_{h2}-T_{c2}}{T_{h1}-T_{c1}} = -UA\left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c}\right) = -\frac{UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right) \tag{484}$$

or $$\frac{T_{h2}-T_{c2}}{T_{h1}-T_{c1}} = \exp\left(-\frac{UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\right) \tag{485}$$

If the cold fluid is minimum fluid, $\epsilon = \frac{T_{c2}-T_{c1}}{T_{h1}-T_{c1}}$.

Rewriting the temperature ratio in (485), we have

$$\frac{T_{h2}-T_{c2}}{T_{h1}-T_{c1}} = \frac{T_{h1} + \left(\frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)(T_{c1}-T_{c2}) - T_{c2}}{T_{h1}-T_{c1}} \tag{486}$$

$$= 1 - \left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\epsilon \tag{487}$$

Inserting this relation back, we have

$$\epsilon = \left(1 - \exp\left(-\frac{UA}{\dot{m}_c c_c}\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right)\right)\right)/\left(1 + \frac{\dot{m}_c c_c}{\dot{m}_h c_h}\right) \tag{488}$$

As a consequence, the effectiveness is usually written

$$\epsilon = \frac{1-\exp\left(-\frac{UA}{C_{min}}\left(1+\frac{C_{min}}{C_{max}}\right)\right)}{1+\frac{C_{min}}{C_{max}}} \tag{489}$$

for $C = \dot{m}c$ is defined as capacity rate.

Similar analysis may be applied to counterflow case, the following relation can be obtained.

$$\epsilon = \frac{1-\exp\left(-\frac{UA}{C_{min}}\left(1+\frac{C_{min}}{C_{max}}\right)\right)}{1-\frac{C_{min}}{C_{max}}\exp\left(-\frac{UA}{C_{min}}\left(1+\frac{C_{min}}{C_{max}}\right)\right)} \tag{490}$$

The grouping of terms UA/C_{min} is called the number of transfer units (NTU) since it is indicative of the size of the heat exchanger.

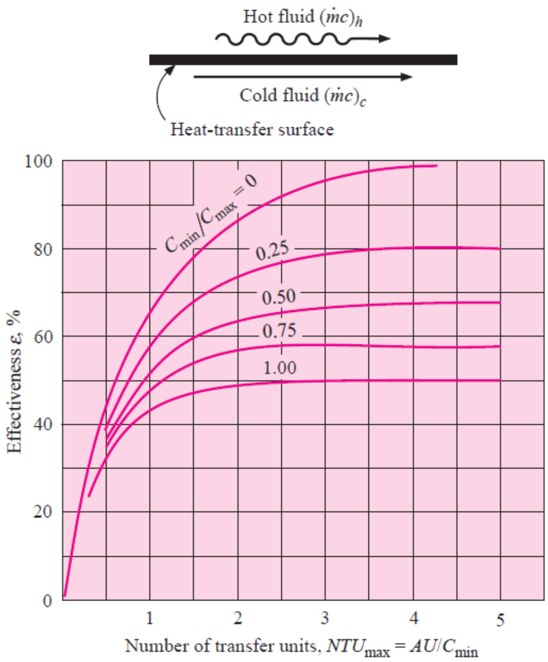
For boilers and condensers, as in boiling and condensation process the fluid temperature stays essentially constant and the fluid acts as if it had infinite specific heat. In these cases $\frac{C_{\min}}{C_{max}} \to 0$ and all the heat-exchanger effectivenss relations approach a single simple equation: $\epsilon = 1 - e^{-NTU}$ (491)

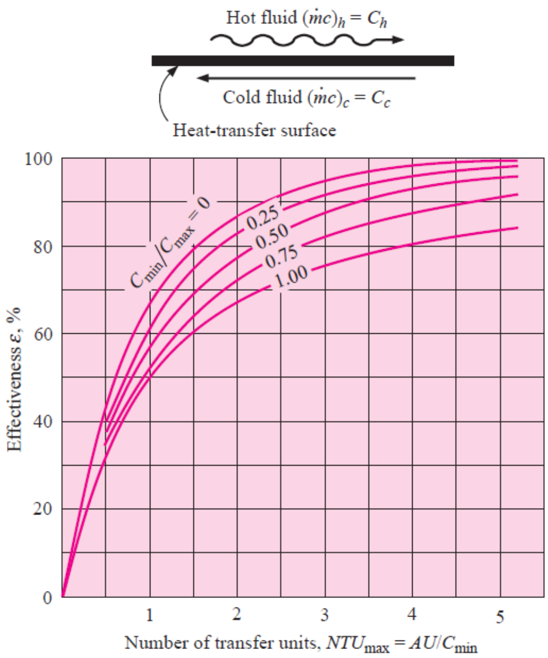
For this case: $$q = C_{min}(T_{h,inlet} - T_{c,inlet})(1 - e^{-NTU}) \quad (492)$$

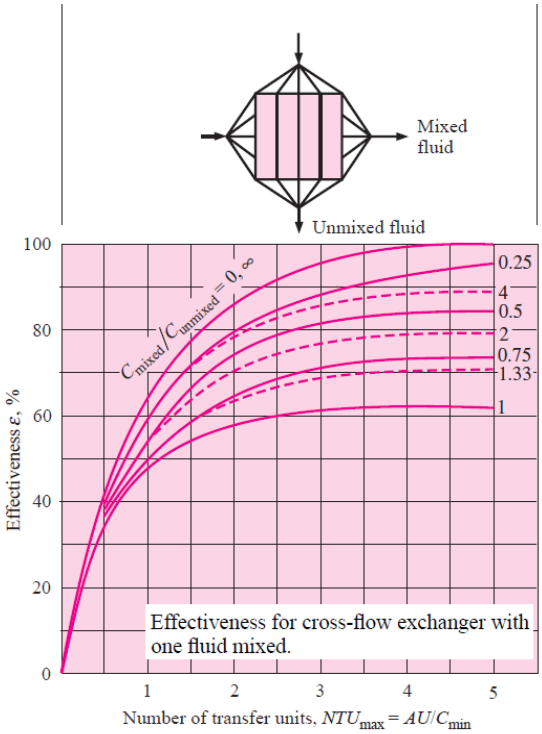
where $C_{\min} = \dot{m}_c c_c$ for a condenser (condensing fluid is *losing* heat)

$= \dot{m}_h c_h$ for a boiler (boiling fluid is *gaining* heat) (493)

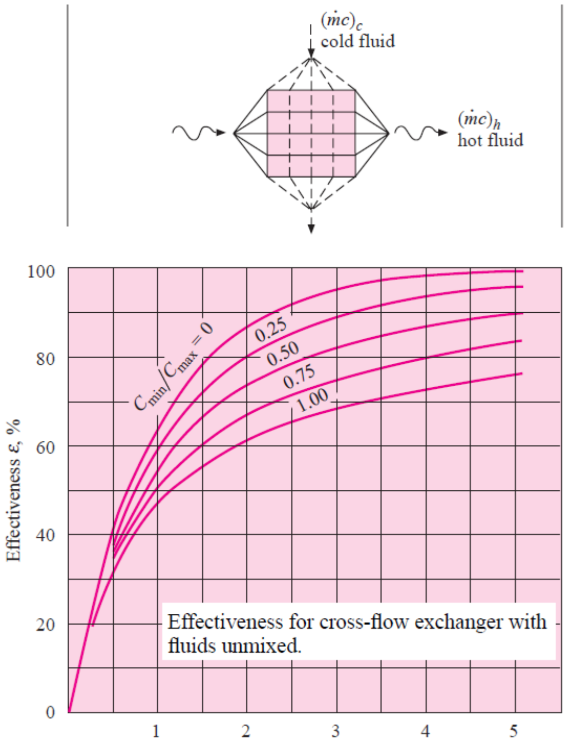
** A heat exchanger has a high *effectiveness* at a certain flow condition does not mean that it will have a higher heat-transfer *rate* than at some low effectiveness condition.



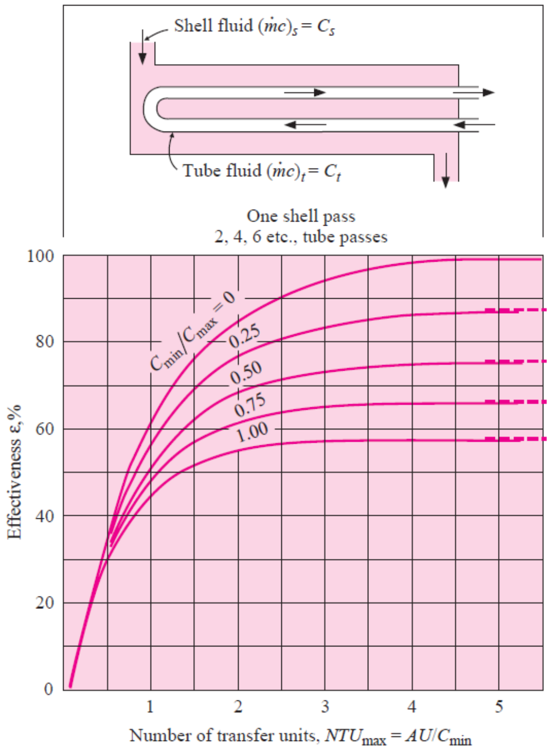


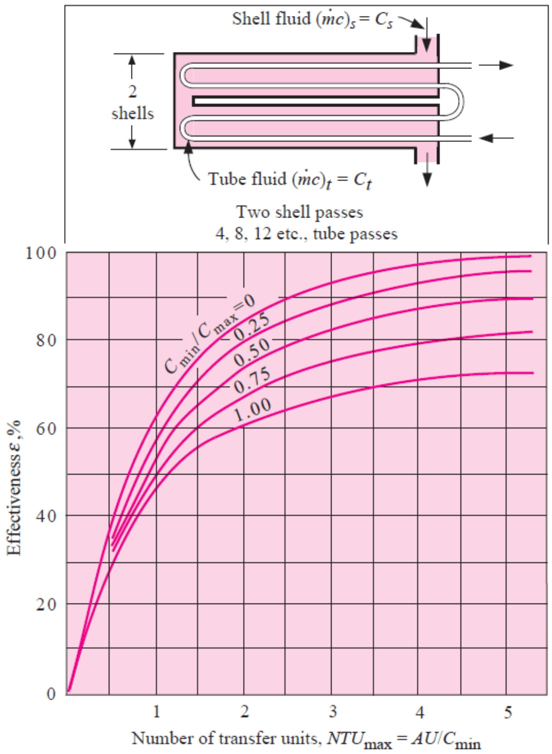


Effectiveness for cross-flow exchanger with one fluid mixed.



Effectiveness for cross-flow exchanger with fluids unmixed.





$N = \mathrm{NTU} = \dfrac{UA}{C_{min}}$ $\qquad C = \dfrac{C_{min}}{C_{max}}$

Flow geometry	Relation
Double pipe:	
Parallel flow	$\epsilon = \dfrac{1 - \exp[-N(1+C)]}{1+C}$
Counterflow	$\epsilon = \dfrac{1 - \exp[-N(1-C)]}{1 - C\exp[-N(1-C)]}$
Counterflow, $C = 1$	$\epsilon = \dfrac{N}{N+1}$
Cross flow:	
Both fluids unmixed	$\epsilon = 1 - \exp\left[\dfrac{\exp(-NCn) - 1}{Cn}\right]$ where $n = N^{-0.22}$
Both fluids mixed	$\epsilon = \left[\dfrac{1}{1-\exp(-N)} + \dfrac{C}{1-\exp(-NC)} - \dfrac{1}{N}\right]^{-1}$
C_{max} mixed, C_{min} unmixed	$\epsilon = (1/C)\{1 - \exp[-C(1 - e^{-N})]\}$
C_{max} unmixed, C_{min} mixed	$\epsilon = 1 - \exp\{-(1/C)[1 - \exp(-NC)]\}$
Shell and tube:	
One shell pass, 2, 4, 6, tube passes	$\epsilon = 2\left\{1 + C + (1+C^2)^{1/2} \times \dfrac{1+\exp[-N(1+C^2)^{1/2}]}{1-\exp[-N(1+C^2)^{1/2}]}\right\}^{-1}$
Multiple shell passes, $2n$, $4n$, $6n$ tube passes ($\epsilon_p =$ effectiveness of each shell pass, $n =$ number of shell passes)	$\epsilon = \dfrac{[(1-\epsilon_p C)/(1-\epsilon_p)]^n - 1}{[(1-\epsilon_p C)/(1-\epsilon_p)]^n - C}$
Special case for $C = 1$	$\epsilon = \dfrac{n\epsilon_p}{1 + (n-1)\epsilon_p}$
All exchangers with $C = 0$	$\epsilon = 1 - e^{-N}$

$C = C_{min}/C_{max}$	$\epsilon =$ **effectiveness**	$N = \mathrm{NTU} = UA/C_{min}$
Flow geometry		**Relation**
Double pipe:		
Parallel flow		$N = \dfrac{-\ln[1-(1+C)\epsilon]}{1+C}$
Counterflow		$N = \dfrac{1}{C-1}\ln\left(\dfrac{\epsilon-1}{C\epsilon-1}\right)$
Counterflow, $C = 1$		$N = \dfrac{\epsilon}{1-\epsilon}$
Cross flow:		
C_{max} mixed, C_{min} unmixed		$N = -\ln\left[1+\dfrac{1}{C}\ln(1-C\epsilon)\right]$
C_{max} unmixed, C_{min} mixed		$N = \dfrac{-1}{C}\ln[1+C\ln(1-\epsilon)]$
Shell and tube:		
One shell pass, 2, 4, 6, tube passes		$N = -(1+C^2)^{-1/2} \times \ln\left[\dfrac{2/\epsilon-1-C-(1+C^2)^{1/2}}{2/\epsilon-1-C+(1+C^2)^{1/2}}\right]$
All exchangers, $C = 0$		$N = -\ln(1-\epsilon)$

e.g.

A finned-tube heat exchanger like that shown in Figure 10-5 is used to heat 5000 ft^3/min [2.36 m^3/s] of air at 1 atm from 60 to 85°F (15.55 to 29.44°C). Hot water enters the tubes at 180°F [82.22°C], and the air flows across the tubes, producing an average overall heat-transfer coefficient of 40 Btu/h · ft^2 · °F [227 W/m^2 · °C]. The total surface area of the exchanger is 100 ft^2 [9.29 m^2]. Calculate the exit water temperature and the heat-transfer rate.

$$\dot{m}_c = (2.36)(1.223) = 2.887 \text{ kg/s}$$

The heat transfer is then

$$q = \dot{m}_c c_c \,\Delta T_c = (2.887)(1006)(29.44 - 15.55)$$
$$= 40.34 \text{ kW} \quad [1.38 \times 10^5 \text{ Btu/h}] \qquad [a]$$

From the statement of the problem we do not know whether the air or water is the minimum fluid. If the air is the minimum fluid, we may immediately calculate NTU and use Figure 10-15 to determine the water-flow rate and hence the exit water temperature. If the water is the minimum fluid, a trial-and-error procedure must be used with Figure 10-15 or Table 10-3. We assume that the air is the minimum fluid and then check our assumption. Then

$$\dot{m}_c c_c = (2.887)(1006) = 2904 \text{ W/°C}$$

and

$$\mathrm{NTU}_{max} = \frac{UA}{C_{min}} = \frac{(227)(9.29)}{2904} = 0.726$$

and the effectiveness based on the air as the minimum fluid is

$$\epsilon \frac{\Delta T_{air}}{\Delta T_{max}} = \frac{29.44 - 15.55}{82.22 - 15.55} = 0.208 \qquad [b]$$

Entering Figure 10-15, we are unable to match these quantities with the curves. This requires that the hot fluid be the minimum. We must therefore assume values for the water-flow rate until we are able to match the performance as given by Figure 10-15 or Table 10-3. We first note that

$$C_{max} = \dot{m}_c c_c = 2904 \text{ W/°C} \qquad [c]$$

$$\mathrm{NTU}_{max} = \frac{UA}{C_{min}} \qquad [d]$$

$$\epsilon = \frac{\Delta T_h}{\Delta T_{max}} = \frac{\Delta T_h}{82.22 - 15.55} \qquad [e]$$

$$\Delta T_h = \frac{4.034 \times 10^4}{C_{min}} = \frac{4.034 \times 10^4}{C_h} \qquad [f]$$

The iterations are:

				ϵ	
$\frac{C_{min}}{C_{max}}$	$C_{min} = \dot{m}_h c_h$	NTU_{max}	ΔT_h	From Figure 10-15 or Table 10-3	Calculated from Equation (*e*)
0.5	1452	1.452	27.78	0.65	0.417
0.25	726	2.905	55.56	0.89	0.833
0.22	639	3.301	63.13	0.92	0.947

We thus estimate the water-flow rate as about

$$\dot{m}_h c_h = 660 \text{ W/°C}$$

and

$$\dot{m}_h = \frac{660}{4180} = 0.158 \text{ kg/s}$$

The exit water temperature is accordingly

$$T_{w,exit} = 82.22 - \frac{4.034 \times 10^4}{660} = 21.1°\text{C}$$

Alternatively, Equations (*c, d, e, f*) may be rearranged to give

$$N = 0.7762/C \qquad [g]$$

$$\epsilon = 0.22084/C \qquad [h]$$

where N and C are defined as in Table 10-3. The appropriate effectiveness equation from Table 10-3 (cross flow, both fluids unmixed) is

$$\epsilon = 1 - \exp\{[\exp(-NCn) - 1]/Cn\} \qquad [i]$$

where $n = N^{-0.22}$

Substituting Equations (g) and (h) in Equation (i) gives a single equation in terms of the capacity ratio C, which may be solved numerically to yield

$$C = 0.23$$

The value of C_{min} is then

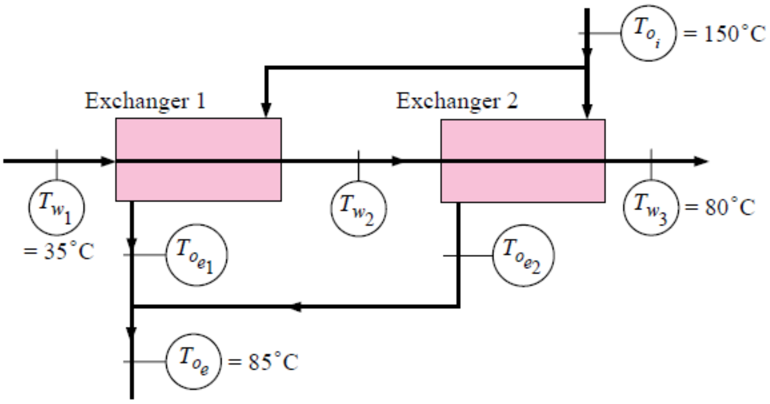
$$C_{min} = 2904 \times C = (2904)(0.23) = 668 \text{ W/°C}$$

Or, a slightly different value from the above iteration. The resulting exit water temperature is thus

$$T_{w,exit} = 82.22 - 40,340/668 = 21.8°\text{C}$$

e.g.

A counterflow double-pipe heat exchanger is used to heat 1.25 kg/s of water from 35 to 80°C by cooling an oil [$c_p = 2.0$ kJ/kg · °C] from 150 to 85°C. The overall heat-transfer coefficient is 150 Btu/h · ft^2 · °F. A similar arrangement is to be built at another plant location, but it is desired to compare the performance of the single counterflow heat exchanger with two smaller counterflow heat exchangers connected in series on the water side and in parallel on the oil side, as shown in Figure Example 10-12. The oil flow is split equally between the two exchangers, and it may be assumed that the overall heat-transfer coefficient for the smaller exchangers is the same as for the large exchanger. If the smaller exchangers cost 20 percent more per unit surface area, which would be the most economical arrangement—the one large exchanger or two equal-sized small exchangers?



We calculate the surface area required for both alternatives and then compare costs. For the one large exchanger

$$
\begin{aligned}
q &= \dot{m}_c c_c \, \Delta T_c = \dot{m}_h c_h \, \Delta T_h \\
&= (1.25)(4180)(80 - 35) = \dot{m}_h c_h (150 - 85) \\
&= 2.351 \times 10^5 \text{ W} \quad [8.02 \times 10^5 \text{ Btu/h}]
\end{aligned}
$$

$$\dot{m}_c c_c = 5225 \text{ W/°C} \qquad \dot{m}_h c_h = 3617 \text{ W/°C}$$

so that the oil is the minimum fluid:

$$\epsilon_h = \frac{\Delta T_h}{150 - 35} = \frac{150 - 85}{150 - 35} = 0.565$$

$$\frac{C_{\min}}{C_{\max}} = \frac{3617}{5225} = 0.692$$

From Figure 10-13 or Table 10-4, $\text{NTU}_{\max} = 1.09$, so that

$$A = \text{NTU}_{\max} \frac{C_{\min}}{U} = \frac{(1.09)(3617)}{850} = 4.649 \text{ m}^2 \quad [50.04 \text{ ft}^2]$$

We now wish to calculate the surface-area requirement for the two small exchangers shown in the sketch. We have

$$\dot{m}_h c_h = \frac{3617}{2} = 1809 \text{ W/°C}$$

$$\dot{m}_c c_c = 5225 \text{ W/°C}$$

$$\frac{C_{\min}}{C_{\max}} = \frac{1809}{5225} = 0.347$$

The number of transfer units is the same for each heat exchanger because UA and $C_{\min}$ are the same for each exchanger. This requires that the effectiveness be the same for each exchanger.

Thus,

$$\epsilon_1 = \frac{T_{oi} - T_{oe,1}}{T_{oi} - T_{w,1}} = \epsilon_2 = \frac{T_{oi} - T_{oe,2}}{T_{oi} - T_{w,2}}$$

$$\epsilon_1 = \frac{150 - T_{oe,1}}{150 - 35} = \epsilon_2 = \frac{150 - T_{oe,2}}{150 - T_{w,2}} \qquad [a]$$

where the nomenclature for the temperatures is indicated in the sketch. Because the oil flow is the same in each exchanger and the average exit oil temperature must be 85°C, we may write

$$\frac{T_{oe,1} + T_{oe,2}}{2} = 85 \qquad [b]$$

An energy balance on the second heat exchanger gives

$$(5225)(T_{w3} - T_{w2}) = (1809)(T_{oi} - T_{oe,2})$$

$$(5225)(80 - T_{w2}) = (1809)(150 - T_{oe,2}) \qquad [c]$$

We now have the three equations (*a*), (*b*), and (*c*) that may be solved for the three unknowns $T_{oe,1}$, $T_{oe,2}$, and T_{w2}. The solutions are

$$T_{oe,1} = 76.98°\text{C}$$
$$T_{oe,2} = 93.02°\text{C}$$
$$T_{w2} = 60.26°\text{C}$$

The effectiveness can then be calculated as

$$\epsilon_1 = \epsilon_2 = \frac{150 - 76.98}{150 - 35} = 0.635$$

From Figure 10-13 or Table 10-4, we obtain $\text{NTU}_{\text{max}} = 1.16$, so that

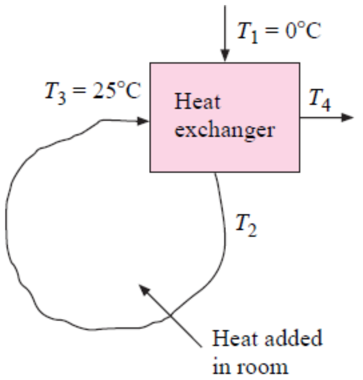
$$A = \text{NTU}_{\text{max}} \frac{C_{\text{min}}}{U} = \frac{(1.16)(1809)}{850} = 2.47 \text{ m}^2$$

We thus find that 2.47 m^2 of area is required for each of the small exchangers, or a total of 4.94 m^2. This is greater than the 4.649 m^2 required in the one larger exchanger; in addition, the cost per unit area is greater so that the most economical choice would be the single larger exchanger. It may be noted, however, that the pumping costs for the oil would probably be less with the two smaller exchangers, so that this could precipitate a decision in favor of the smaller exchangers if pumping costs represented a sizable economic factor.

e.g.

A recuperator used as an energy conservation measure employs a cross-flow heat exchanger with both fluids unmixed as shown in Figure Example 10-15. The exchanger is designed to remove 210 kW from 1200 kg/min of atmospheric air entering at 25°C. This energy is used to preheat the same quantity of air that enters from outdoor conditions at 0°C before being used for a building heating application. The design value of U for this flow condition is 30 $\text{W/m}^2 \cdot °\text{C}$. The following calculations are desired:

1. The design value for the area of the heat exchanger
2. The percent reduction in heat-transfer rate if the flow rate is reduced by 50 percent while keeping the inlet temperatures and value of U constant
3. The percent reduction in heat-transfer rate if the flow rate is reduced by 50 percent and the value of U varies as mass flow to the 0.8 power, with the same inlet temperature conditions



1. The hot and cold fluids have the same flow rate with

$$\dot{m}_h = \dot{m}_c = 1200/60 = 20\,\text{kg/s}$$

and

$$C_h = C_c = (20)(1005) = 20{,}100\ \text{W/°C};\ C_{\min}/C_{\max} = 1.0 = C \text{ for use in Table 10-3}$$

The energy balance gives

$$q = 210{,}000 = C_h\,\Delta\,T_h = C_c\,\Delta\,T_c$$

and

$$\Delta\,T_h = \Delta\,T_c = 210{,}000/20{,}100 = 10.45°\text{C}$$

The heat-exchanger effectiveness is

$$\varepsilon = \Delta T_{\text{min fluid}}/\Delta T_{\text{max HX}} = 10.45/(25-0) = 0.4179 \qquad [a]$$

Consulting Table 10-3 for a cross-flow exchanger with both fluids unmixed, and inserting the value $C = 1.0$, we have

$$\epsilon = 1 - \exp\{N^{0.22}[\exp(-N^{0.78}) - 1]\} \qquad [b]$$

Inserting $\varepsilon = 0.4179$, Equation (b) may be solved for N to yield

$$N = 0.8 = UA/C_{\min} = (30)A/20{,}100$$

and

$$A = 536\,\text{m}^2$$

This is the *design value* for the area of the heat exchanger.

2. We now examine the effect of reducing the flow rate by half, while keeping the inlet temperatures and value of U the same. Note that the flow rate of *both* fluids is reduced because they are physically the same fluid. This means that the value of $C_{\min}/C_{\max}$ will remain the same at a value of 1.0, and Equation (b) above may still be used for the calculation of effectiveness. The new value of $C_{\min}$ is

$$C_{\min} = (1/2)(20{,}100) = 10{,}050\ \text{W/°C}$$

so that

$$\text{NTU} = N = UA/C_{\min} = (30)(536)/10050 = 1.6$$

Inserting this value in Equation (b) above gives

$$\epsilon = 0.5713$$

The temperature difference for each fluid is then

$$\Delta T = \epsilon \Delta T_{\text{max HX}} = (0.5713)(25-0) = 14.28°\text{C}$$

The resulting heat transfer is then

$$q = \dot{m}c\Delta T = (10{,}050)(14.28) = 143.5\,\text{kW}$$

or a reduction of 32 percent for a reduction in flow rate of 50 percent.

3. Finally, we examine the effect of reducing the flow rate by 50 percent coupled with reduction in overall heat-transfer coefficient under the assumption that

$$U \text{ varies as } \dot{m}^{0.8} \text{ or, correspondingly, as } C_{\text{min}}^{0.8}$$

Still keeping the area constant, we would find that NTU varies as

$$\text{NTU} = N = UA/C_{\text{min}} \approx C^{0.8} \times C^{-1} = C^{-0.2}$$

Our new value of N under these conditions would be

$$N = (0.8)(10{,}050/20{,}100)^{-0.2} = 0.919$$

Inserting this value in Equation (b) above gives for the effectiveness

$$\epsilon = 0.4494$$

The corresponding temperature difference in each fluid is

$$\Delta T = \epsilon\Delta T_{\text{max HX}} = (0.4494)(25-0) = 11.23°\text{C}$$

The heat transfer is calculated as

$$q = \dot{m}c\Delta T = (10{,}050)(11.23) = 112.9\,\text{kW}$$

This is a reduction of 46 percent from the 210 kW design value at full flow. Again, we note the rather pronounced effect because *both* the hot and cold fluid flow rates are reduced, coupled with an anticipated decrease in the overall heat-transfer coefficient that may accompany the lower mass flows.

THE END