Solving Laplace's Equation in two dimensions

We'll solve Laplace's Equation in polar coordinates for the scalar field $\Phi(r,\theta)$:

$$(r\Phi_r)_r + \frac{1}{r}\left(\Phi_\theta\right)_\theta = 0$$

Assuming we can separate variables such that $\Phi(r,\theta) = R(r)\Theta(\theta)$:

$$\Phi_r = R'\Theta, \Phi_\theta = R\Theta' \Rightarrow$$

$$(rR'\Theta)_r + \frac{1}{r}(R\Theta')_\theta = 0 \Leftrightarrow$$

$$R'\Theta + rR''\Theta + \frac{1}{r}R\Theta'' = 0 \Leftrightarrow$$

$$\frac{rR' + r^2R''}{R} = \frac{-\Theta''}{\Theta} \Leftrightarrow$$

$$\frac{r(rR')'}{R} = \frac{-\Theta''}{\Theta}$$

Set both sides equal to λ^2:

$$(1) \quad \frac{r(rR')'}{R} = \lambda^2, \qquad (2) \quad \frac{-\Theta''}{\Theta} = \lambda^2$$

Solve the two differential equations separately, assuming at first that $\lambda \neq 0$. For (1), use a solution of the form $R(r) = \gamma r^\alpha$:

$$R' = \gamma\alpha r^{\alpha-1} \Rightarrow$$

$$\frac{r(r\gamma\alpha r^{\alpha-1})'}{\gamma r^\alpha} = \lambda^2 \Leftrightarrow$$

$$\alpha^2\frac{\gamma r r^{\alpha-1}}{\gamma r^\alpha} = \lambda^2 \Leftrightarrow$$

$$\alpha^2 = \lambda^2 \Rightarrow \alpha = \pm\lambda$$

For (2), we have $\Theta'' + \lambda^2\Theta = 0$, so the general form of the solution is:

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

Since $\Theta(0) = \Theta(2\pi)$ we know that $\lambda \in \mathbb{N}$ (or, more generally, that $\lambda \in \mathbb{Z}$ but due to the symmetry in cos and sin that information can be encoded into A and B). When $\lambda = 0$ we can in both cases integrate to find the solutions:

$$(1) \quad R(r) = c_1 \ln r + c_2, \qquad (2) \quad \Theta(\theta) = c_3\theta + c_4$$

Since Laplace's Equation is linear, we can add the solutions together to get the general solution set:

$$\Phi(r,\theta) = R(r)\Theta(\theta)_{\lambda=0} + R(r)\Theta(\theta)_{\lambda\in\mathbb{N}} =$$

$$(c_1\ln r + c_2)(c_3\theta + c_4) + \sum_{n\in\mathbb{N}}(\gamma_{1n}r^n + \gamma_{2n}r^{-n})\big[A\cos(n\theta) + B\sin(n\theta)\big]$$

Particular solution for infinite cylinder in $\underline{\text{E}}$ field

We will add two different solutions together to find the total potential. Let the $\underline{\text{E}}$ field be uniform in the x direction such that $\underline{E} = E_0\hat{x}$. Since $\underline{E} = -\nabla\Phi$, $\Phi = -E_0 x = -E_0 r\cos\theta$, which is in the general solution set we defined above. Our first boundary condition is then:

$$\lim_{r\to\pm\infty} \Phi(r,\theta)_{total} = -E_0 r\cos\theta$$

Next we want to find the field from the cylinder. Let the radius of the cylinder be R. The second boundary condition is that the potential on the surface of the cylinder must be constant, say V_0, so:

$$(3) \qquad \Phi(R,\theta)_{total} = -E_0 R\cos\theta + \Phi(R,\theta)_{cylinder} = V_0$$

$\Phi(r,\theta)_{cylinder}$ must also be in the general solution set defined above. We need a constant so take c_2c_4 from the first part of the general solution, but discard the $\ln r$ and θ terms to keep within the first boundary condition. From the second part we need to get an expression which lets us fulfill (3), so it is logical to take the $n = 1$ term which has $\cos\theta$:

$$\Phi(R,\theta)_{total} = -E_0 R\cos\theta + c_2c_4 + (\gamma_{11}R + \gamma_{21}R^{-1})(A\cos\theta + B\sin\theta) = V_0$$

Now B must be zero since the sin term would violate the symmetry about the x axis, and all that is left is to solve the following equation:

$$\Phi(R,\theta)_{total} = -E_0 R\cos\theta + c_2c_4 + (\gamma_{11}R + \gamma_{21}R^{-1})A\cos\theta = V_0 \Leftrightarrow$$

$$c_2c_4 - V_0 + R\cos\theta\left(A\gamma_{11} - E_0 + \frac{A\gamma_{21}}{R^2}\right) = 0$$

Hence set $c_2c_4 = V_0$, $\gamma_{11} = 0$, and $A\gamma_{21} = E_0R^2$ to fulfill (3) and get the final solution:

$$\Phi(r,\theta)_{total} = V_0 + E_0 cos\theta\left(\frac{R^2}{r} - r\right)$$