Problem Set IV

Probability ◇ MATH 305-02 ◇ Fall 2015

November 28, 2015

Due: Tuesday, December 1, 2015
Read: Chapter 6

Instructions: All solutions should be prepared carefully, recopied in a neat final form, and presented in the order given. All solution sets shall be completed on white, unlined paper and stapled. Other forms of careless presentation (e.g. notebook fringes, bad handwriting, pen scribbles, doodles, wasted space, etc) will result in a deduction of points at my discretion. Submitted work that does not demonstrate clearly the process by which one arrived at the answer will not receive credit of any kind.

Name:

Score:

1.] Let X and Y each have range $\{1, 2, 3, 4\}$. The following formula gives their joint probability mass function

$$P\{X = i, Y = j\} = \frac{i + j}{80}.$$

(a) Give the joint probability table for X and Y, with marginal distributions.

(b) Compute $P\{X = Y\}$.

(c) Compute $P\{XY = 6\}$.

(d) Compute $P\{1 \leq X \leq 2, 2 < Y \leq 4\}$.

2.] Suppose X and Y are continuous random variables with joint density function $f(x, y) = x + y$ on the unit square $(x, y) \in [0, 1] \times [0, 1]$.

(a) Let $F(a, b)$ be the joint cumulative distribution function. Compute $F(1, 1)$. Compute $F(a, b)$.

(b) Compute the probability $P\{X + Y \leq 1\}$.

(c) Compute the probability $P\{X > \sqrt{Y}\}$.

(d) Compute the marginal density functions for X and Y, $f_X(x)$ and $f_Y(y)$.

(e) Are X and Y independent? Explain briefly.

(f) Compute $E[X]$ and $E[Y]$.

3.] Chelsea leaves for cross-fit between 5 PM and 5:30 PM and takes between 40 and 50 minutes to get there, assuming she doesn't stop at TJ Maxx. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at cross-fit before 6 PM.

4.] Suppose X and Y are independent normal random variables with mean $\mu = 0$ and variance $\sigma^2 = 1$. Find $P\{X^2 + Y^2 < 1\}$.

5.] Let X_1 and X_2 be independent normal random variables with parameters mean 0 and variance σ^2. Find the joint distribution of Y_1 and Y_2, where

$$Y_1 = X_1^2 + X_2^2, \qquad \text{and} \qquad Y_2 = \frac{X_1}{\sqrt{Y_1}}.$$

6.] Prove that if the joint cumulative distribution function of X and Y satisfy

$$F(x, y) = F_X(x)F_Y(y),$$

then for any pair of intervals (a, b) and (c, d),

$$P\{a \leq X \leq b, c \leq Y \leq d\} = P\{a \leq X \leq b\}P\{c \leq Y \leq d\}.$$

7.] The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 30xy^2 & \text{if} \quad x - 1 \leq y \leq 1 - x, 0 \leq x \leq 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

(a) Show that the marginal density function of X is a beta distribution with parameters $a = 2$ and $b = 4$.

(b) Derive the marginal probability density function of Y.

(c) Derive the conditional density function of Y given $X = x$.

(d) Find $P\{Y > 0 \,|\, X = 0.75\}$.

8.] Of nine executives in a certain business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let X denote the number of married executives and Y, the number of never-married executives among the three selected. Assuming that the three are to be selected randomly, answer the following questions.

(a) Find $p(x, y)$, the joint probability distribution for X and Y.

(b) Find the marginal probability distribution of X, the number of married executives among the three selected for promotion.

(c) Find $P\{X = 1 \mid Y = 2\}$.

(d) If we let Z denote the number of divorced executives among the three selected for promotion, then $Z = 3 - X - Y$. Find $P\{Z = 1 \mid Y = 1\}$.

(e) Are X and Y independent? Justify your answer.

9.] Let X denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week, and suppose that X has a uniform distribution over the interval $0 \leq x \leq 1$. Let Y denote the amount (by weight) of this item sold by the supplier during the week, and suppose that Y has a uniform distribution over the interval $0 \leq y \leq x$, where x is a specific value of X.

(a) Find the joint density function of X and Y.

(b) If the supplier stocks half-ton of the item, what is the probability that she sells more than a quarter-ton?

(c) If it is known that the supplier sold a quarter-ton of the item, what is the probability that she had stocked more than a half-ton?

10.] If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}$ for $0 < x < \infty$, derive the joint distribution of $U = X + Y$, $V = X + Z$, and $W = Y + Z$.

11.] An environmental engineer measures the amount (by weight) of particulate pollution in the air samples (of a certain volume) collected over the smokestack of a coal-fueled power plant. Let X denote the amount of pollutant per sample when a certain cleaning device on the stack is not operating, and let Y denote the mount of pollutants per sample when the cleaning device is operating under similar environmental conditions. It is observed that X is always greater than $2Y$, and the joint probability density function of (X, Y) is given by

$$f(x,y) = \begin{cases} k & \text{if} \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1, \quad 2y \leq x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

a.) Find the value of k that makes this a probability density function.

b.) Find $P\{X \geq 3Y\}$. (In other words, find the probability that the cleaning device will reduce the amount of pollutant by $1/3$ or more.)

c.) Find the marginal density function of the amount of pollutants per sample when the cleaning device on the stack is not operating.

d.) Find the marginal density function of the amount of pollutants per sample when the cleaning device on the stack is operating.

e.) Find the conditional density function of the amount of pollutant per sample when a certain cleaning device on the stack is not operating given the amount per sample when it is working.

f.) Find the conditional density function of the amount of pollutant per sample when a certain cleaning device on the stack is working.

12.] Suppose X and Y are independent gamma distributed random variables with parameters $(\alpha, 1)$ and $(\beta, 1)$, respectively. Find the joint distribution function of the new variables $U = X + Y$ and $V = X/(X + Y)$. Show that U and V are independent, and find the distribution of each. (hint: U is gamma and V is beta.)

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Brooks Emmerich

Pts: 49

(1)

Problem Set IV Solutions

Solution 1: a.) Consider the following table:

[4 pts]

1 pt each

Table for $p(x,y) = P\{X=i, Y=j\}$

$x=i$ \ $y=j$	1	2	3	4	$P_X(x) = P\{X=i\}$
1	2/80	3/80	4/80	5/80	14/80
2	3/80	4/80	5/80	6/80	18/80
3	4/80	5/80	6/80	7/80	22/80
4	5/80	6/80	7/80	8/80	26/80
$P_Y(y)$ $P\{Y=j\}$	14/80	18/80	22/80	26/80	80/80 = 1

b.) $P\{X=Y\} = p(1,1) + p(2,2) + p(3,3) + p(4,4)$

$= \frac{2}{80} + \frac{4}{80} + \frac{6}{80} + \frac{8}{80}$

$= \boxed{\frac{1}{4}}$

c.) $P\{XY=6\} = p(2,3) + p(3,2)$

$= \frac{5}{80} + \frac{5}{80}$

$= \boxed{\frac{1}{8}}$

Solution 1 Cont:

d.) $P\{1 \le X \le 2, 2 < Y \le 4\} = p(1,3) + p(1,4) + p(2,3) + p(2,4)$

$$= \frac{4}{80} + \frac{5}{80} + \frac{5}{80} + \frac{6}{80}$$

$$= \boxed{\frac{1}{4}}$$

Solution 2: Let

[9 pts]

a–c: 2 each

d–f: 1 each

$$f(x,y) = \begin{cases} x+y & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a.) Let $F(a,b)$ be the joint cumulative distribution function then

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$= \int_0^a\int_0^b x+y\ dy\,dx$$

*$F(1,1) = 1$ obv.

$$= \int_0^a xy + \tfrac{1}{2}y^2 \Big|_0^b dx$$

$$= \int_0^a bx + \tfrac{1}{2}b^2\ dx$$

$$= \frac{b}{2}x^2 + \frac{1}{2}b^2x \Big|_0^a$$

$$= \frac{ba^2}{2} + \frac{b^2a}{2}$$

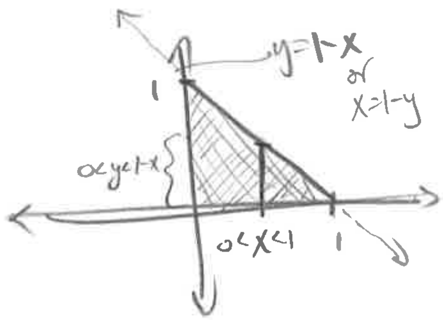
$$\boxed{F(a,b) = \frac{ab}{2}(a+b)}$$

$$\Rightarrow F(1,1) = \frac{(1)(1)}{2}(1+1)$$

$$\boxed{F(1,1) = 1}$$

Solution 2 Cont:

b.) $$P\{X+Y \le 1\} = \iint_{(x,y)\,|\,x+y\le 1} f(x,y)\,dx\,dy$$



$$= \int_0^1 \int_0^{1-x} x+y \; dy\,dx$$

$$= \int_0^1 xy + \tfrac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 x(1-x) + \tfrac{1}{2}(1-x)^2\,dx$$

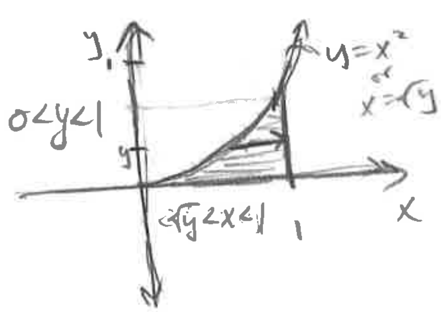
$$= \int_0^1 x - x^2 + \tfrac{1}{2}(1-x)^2\,dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{6}$$

$$= \boxed{1/3}$$

c.) $$P\{X > \sqrt{Y}\} = \iint_{(x,y)\,|\,x<\sqrt{y}} f(x,y)\,dx\,dy$$



$$= \int_0^1 \int_{\sqrt{y}}^1 x+y \; dx\,dy$$

$$= \int_0^1 \tfrac{1}{2}x^2 + xy \Big|_{\sqrt{y}}^1 dy$$

$$= \int_0^1 \left(\tfrac{1}{2} + y - \tfrac{1}{2}y - y^{3/2}\right) dy$$

$$= \tfrac{1}{2}y + \tfrac{1}{4}y^2 - \tfrac{2}{5}y^{5/2} \Big|_0^1$$

$$= \boxed{7/20}$$

Solution 2 Cont:

d.) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_0^1 x+y\,dx$

$= \frac{1}{2}x^2 + xy\Big|_0^1 = \boxed{\frac{1}{2}+y}$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_0^1 x+y\,dy$

$= xy + \frac{1}{2}y^2\Big|_0^1 = \boxed{\frac{1}{2}+x}$

e.) No, $f(x,y) = x+y$ cannot be split into functions $h(x)$ and $g(y)$ such that $h(x)g(y) = f(x,y)$. In short, they're not separable.

f.) Using part d.), we obtain

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\left(\tfrac{1}{2}+x\right)dx = \frac{1}{4}x^2 + \frac{1}{3}x^3\Big|_0^1 = \boxed{\frac{7}{12}}$$

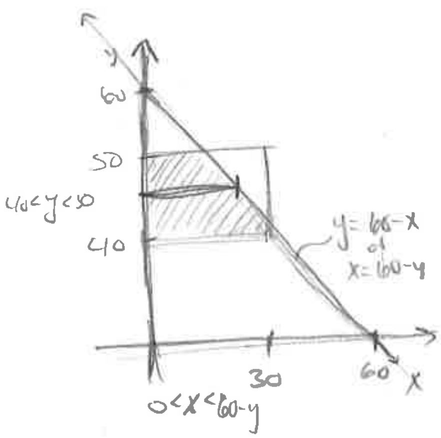
$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y\left(\tfrac{1}{2}+y\right)dy = \frac{1}{4}y^2 + \frac{1}{3}y^3\Big|_0^1 = \boxed{\frac{7}{12}}$$

Solution 3: Let X be the time of departure and let Y be the travel time. Then

[3 pts]

$$f_X(x) = \begin{cases} 1/30 & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1/10 & 40 \le y \le 50 \\ 0 & \text{otherwise} \end{cases}$$

and the joint cumulative distribution function is $f(x,y) = f_X(x) f_Y(y) = 1/300$ for $x \in [0,30]$, $y \in [40,50]$ as each distribution is independent of the other. We seek the probability $P\{X + Y < 60\}$. Hence,



$$P\{X+Y<60\} = \iint_{(x,y)\,|\,x+y<60} f(x,y)\,dx\,dy$$

$$= \int_{40}^{50}\int_{0}^{60-y} \frac{1}{300}\,dx\,dy$$

$$= \frac{1}{300}\int_{40}^{50} (60-y)\,dy$$

$$= \frac{1}{300}\left(60y - \frac{1}{2}y^2\right)\Big|_{40}^{50}$$

$$= \frac{1}{300}\left[3000 - \frac{1}{2}50^2 - 2400 + \frac{1}{2}40^2\right]$$

$$= \frac{1}{300}\left[600 - \frac{1}{2}(50^2 - 40^2)\right]$$

$$= 2 - \frac{1}{6}(5^2 - 4^2)$$

$$= 2 - \frac{3}{2}$$

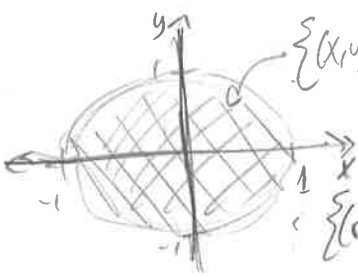
$$= \boxed{\frac{1}{2}}$$

Solution 4: [2 pts] If X and Y are independent standard normal random variables then the joint pdf for X & Y is

$$f(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)$$
$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

The probability $P\{X^2+Y^2<1\}$ is given by the following integral, which is evaluated using polar coordinates:

$$P\{X^2+Y^2<1\} = \iint_{(x,y)\,|\,x^2+y^2<1} f(x,y)\,dx\,dy$$



$\{(x,y)\,|\,x^2+y^2<1\}$ or $\{(r,\theta)\,|\,0\le\theta\le 2\pi,\ 0\le r\le 1\}$

$$= \iint_{(x,y)\,|\,x^2+y^2<1} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\,dx\,dy$$

Let $x = r\cos\theta$ and $y = r\sin\theta$, then $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

and

$$dx\,dy = \begin{vmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial r}(r\sin\theta) \\ \frac{\partial}{\partial\theta}(r\cos\theta) & \frac{\partial}{\partial\theta}(r\sin\theta) \end{vmatrix} dr\,d\theta$$
$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$
$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta$$
$$= r\,dr\,d\theta$$

Hence, our integral becomes

Solution 4 Cont:

$$P\{X^2+Y^2<1\} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 e^{-r^2/2}\, r\,dr\,d\theta$$

$$= \frac{1}{2\pi}(2\pi)\int_0^1 re^{-r^2/2}\,dr$$

$$= -e^{-r^2/2}\Big|_0^1$$

$$= \boxed{1-e^{-1/2}}$$

Solution 5: [3 pts] Since X_1 & X_2 are independent normal random variables with parameters $\mu=0$ and $\sigma^2=\sigma^2$, we have the joint pdf as

$$f_{X_1X_2}(x_1,x_2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x_1^2/2\sigma^2}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x_2^2/2\sigma^2}\right)$$

$$= \frac{1}{2\pi\sigma^2}e^{-(x_1^2+x_2^2)/2\sigma^2}$$

Now, define the following functions for X_1 and X_2:

$$y_1 = g_1(x_1,x_2) = x_1^2+x_2^2$$

$$y_2 = g_2(x_1,x_2) = \frac{x_1}{\sqrt{y_1}} = \frac{x_1}{\sqrt{x_1^2+x_2^2}}$$

We can solve these equations for X_1 and X_2:

$$x_1 = \sqrt{y_1}\,y_2$$

$$x_2 = \sqrt{y_1}\sqrt{1-y_2^2}$$

Solution 5 Cont: Next, we find the Jacobian determinant:

$$J(x_1,x_2) = \begin{vmatrix} \partial g_1/\partial x_1 & \partial g_2/\partial x_1 \\ \partial g_1/\partial x_2 & \partial g_2/\partial x_2 \end{vmatrix}$$

$$= \begin{vmatrix} 2x_1 & x_2^2/(x_1^2+x_2^2)^{3/2} \\ 2x_2 & -x_1x_2/(x_1^2+x_2^2)^{3/2} \end{vmatrix}$$

$$= \frac{1}{(x_1^2+x_2^2)^{3/2}}\left[-2x_1^2x_2 - 2x_2^3\right]$$

$$= \frac{-2x_2(x_1^2+x_2^2)}{(x_1^2+x_2^2)^{3/2}}$$

$$= \frac{-2x_2}{\sqrt{x_1^2+x_2^2}}$$

Putting this in terms of y_1 and y_2 gives.

$$J(y_1,y_2) = -\frac{2\sqrt{y_1}\sqrt{1-y_2^2}}{\sqrt{y_1}} = -2\sqrt{1-y_2^2}$$

therefore, our new joint pdf for Y_1 and Y_2 is

$$f_{Y_1Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2}e^{-y_1/2\sigma^2}\cdot\frac{1}{\left|-2\sqrt{1-y_2^2}\right|}$$

$$= \boxed{\frac{e^{-y_1/2\sigma^2}}{4\pi\sigma^2\sqrt{1-y_2^2}}}$$

$y_1 \geq 0$

$-1 \leq y_2 \leq 1$

Solution 6: It follows from the properties of probabilities:

[2 pts]

$$P\{a\le X\le b,\ c\le Y\le d\} = P\{X\le b,\ c\le Y\le d\} - P\{X\le a,\ c\le Y\le d\}$$

$$= \Big[P\{X\le b,\ Y\le d\} - P\{X\le b,\ Y\le c\}\Big] - \Big[P\{X\le a,\ Y\le d\} - P\{X\le a,\ Y\le c\}\Big]$$

$$= F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

where the last step follows from the definition of the cdf. Applying the hypothesis that $F(x,y) = F_X(x)F_Y(y)$, we have

$$P\{a\le X\le b,\ c\le Y\le d\} = F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c)$$

$$= \big[F_X(b) - F_X(a)\big]\big[F_Y(d) - F_Y(c)\big]$$

$$= \big[P\{X\le b\} - P\{X\le a\}\big]\big[P\{Y\le d\} - P\{Y\le c\}\big]$$

$$= P\{a\le X\le b\}\,P\{c\le Y\le d\}$$

as desired.

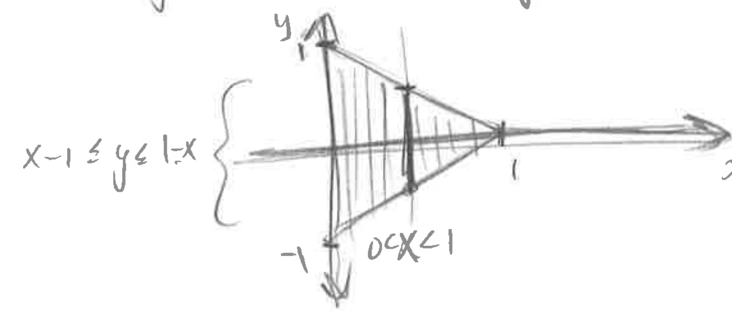
Solution 7: Let the joint pdf of X & Y be

[6 pts]

2 pt each a,b
1 pt each c,d.

$$f(x,y) = \begin{cases} 30xy^2 & \text{if } x-1\le y\le 1-x,\ 0\le x\le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution 7 Cont: a.) Assume X is fixed, then we wish to integrate out y. Consider the region of integration:



[Mow the grass vertically!]

We have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x-1}^{1-x} 30xy^2\,dy$$

$$= 10xy^3\Big|_{x-1}^{1-x}$$

$$= 10x\left[(1-x)^3 - (x-1)^3\right]$$

$$\boxed{f_X(x) = 20x(1-x)^3 \quad \text{for } 0<x<1}$$

The Beta distribution with parameters a & b is given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$$

where $B(a,b)$ is known to be

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

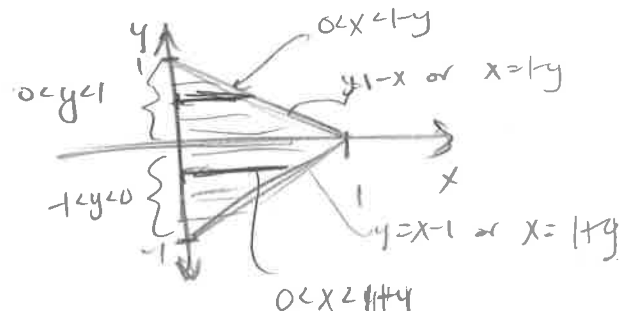
Hence, if $a=2$ and $b=4$, we have

Solution 7 Cont:

$$f(x) = \frac{x^{2-1}(1-x)^{4-1}}{B(2,4)} = \frac{\Gamma(2+4)}{\Gamma(2)\Gamma(4)} x(1-x)^3$$
$$= \frac{5!}{1!\,3!} x(1-x)^3$$
$$= 20x(1-x)^3 = f_X(x).$$

Well, I'll be damned! They are the same!

b.) Here, we fix y and integrate out the x variable to find the marginal density function in y. Consider the region of integration:



The bounds switch depending on where y is located!

Hence, we'll have a piecewise function for $f_Y(y)$:

$$f_Y(y) = \begin{cases} \int_0^{1-y} f(x,y)\,dx & \text{if } 0 \le y \le 1 \\ \int_0^{1+y} f(x,y)\,dx & \text{if } -1 \le y < 0 \end{cases}$$

This gives the following:

Solution 7 Cont:

$$\boxed{f_Y(y) = \begin{cases} 15y^2(1-y)^2 & \text{if } 0 \le y \le 1 \\ 15y^2(1+y)^2 & \text{if } -1 \le y < 0 \end{cases}}$$

c.) We know the conditional density function of Y given X is by definition,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{30xy^2}{20x(1-x)^3}$$

$$\Rightarrow \boxed{f_{Y|X}(y|x) = \frac{3}{2}\frac{y^2}{(1-x)^3}} \quad \text{for } x-1 < y \le 1-x$$

d.) We have

$$P\{Y>0 \mid X=0.75\} = \int_0^\infty f_{Y|X}(y|0.75)\,dy$$

$$= \int_0^{1-0.75} \frac{3}{2}\frac{y^2}{(1-0.75)^3}\,dy$$

$$= \frac{3}{2}\frac{1}{(\frac{1}{4})^3}\int_0^{1/4} y^2\,dy$$

$$= \frac{3(4^3)}{2}\frac{1}{3}\,y^3\Big|_0^{1/4}$$

$$= \boxed{\frac{1}{2}}$$

Solution 8: [5 pts] 1 pt each

a.) The joint density mass function can be found as

$$p(x,y) = \frac{\binom{4}{x}\binom{3}{y}\binom{2}{3-x-y}}{\binom{9}{3}}, \quad \left.\begin{array}{l} x=0,1,2,3, \\ y\geq 0 \\ 1\leq x+y\leq 3 \end{array}\right\} \text{ x and y, are integers}$$

since there are 4 married executives to choose x from, 3 never-married executives to choose y from, and 2 divorced executives to choose the remaining 3-x-y spots from. The denominator is the total number of possible groups of size 3 from 9.

b.) To determine $P_X(x)$, we sum over the y-values, assuming x is fixed. Hence,

$x=0$: $y=1,2,3$: $P_X(0) = \frac{\binom{4}{0}}{\binom{9}{3}} \sum_{y=1}^{3} \binom{3}{y}\binom{2}{3-y} = 10\frac{\binom{4}{0}}{\binom{9}{3}}$

$x=1$: $y=0,1,2$: $P_X(1) = \frac{\binom{4}{1}}{\binom{9}{3}} \sum_{y=0}^{2} \binom{3}{y}\binom{2}{2-y} = 10\frac{\binom{4}{1}}{\binom{9}{3}}$

$x=2$: $y=0,1$: $P_X(2) = \frac{\binom{4}{2}}{\binom{9}{3}} \sum_{y=0}^{1} \binom{3}{y}\binom{2}{1-y} = 5\frac{\binom{4}{2}}{\binom{9}{3}}$

$x=3$: $y=0$: $P_X(3) = \frac{\binom{4}{3}}{\binom{9}{3}} \sum_{y=0} \binom{3}{y}\binom{2}{0-y} = \frac{\binom{4}{3}}{\binom{9}{3}}$

Writing this marginal density function in a different way gives

Solution 8 Cont: We note here that we can write

$$P_X(x) = \frac{\binom{4}{x}\binom{5}{3-x}}{\binom{9}{3}}, \quad \text{for } x = 0,1,2,3$$

which is a hypergeometric random variable with parameters $N=9$, $m=4$, and $n=3$.

c.) We seek $P\{X=1 \mid Y=2\}$. We have

$$P_{X|Y}(x|y) = \frac{p(x,y)}{P_Y(y)} \Rightarrow P_{X|Y}(1|2) = \frac{p(1,2)}{P_Y(2)}$$

Assume $y=2$, then

$$\underline{y=2}: \ x=0,1 \quad P_Y(2) = \frac{\binom{3}{2}}{\binom{9}{3}} \sum_{x=0}^{1} \binom{4}{x}\binom{2}{1-x}$$

$$= \frac{3!\,6!\,3!}{2!\,1!\,9!}\left[\binom{4}{0}\binom{2}{1} + \binom{4}{1}\binom{2}{0}\right]$$

$$= \frac{3\cdot 3\cdot 2}{9\cdot 8\cdot 7}\left[2+4\right]$$

$$= \frac{3}{14}$$

Also, we have

$$p(1,2) = \frac{\binom{4}{1}\binom{3}{2}\binom{2}{0}}{\binom{9}{3}}$$

$$= \frac{4\cdot 3\cdot 3\cdot 2}{9\cdot 8\cdot 7}$$

$$= \frac{1}{7}$$

Therefore, we have $P_{X|Y}(1|2) = \frac{1/7}{3/14} = \boxed{2/3}$

Solution 8 Cont:

d.) We note that the probability $P\{Z=1 \mid Y=1\}$ can be found by noting that if $y=1$, then $z=2-x$. Therefore, $z=1$ when $x=1$ so that

$$P\{Z=1 \mid Y=1\} = P\{X=1 \mid Y=1\}$$

$$= p_{X|Y}(1,1)$$

$$= \frac{p(1,1)}{p_Y(1)}$$

Hence, we find that when $y=1$

$y=1$: $x=0,1,2$: $$p_Y(1) = \frac{\binom{3}{1}}{\binom{9}{3}} \sum_{x=0}^{2} \binom{4}{x}\binom{2}{2-x}$$

$$= \frac{3 \cdot 3 \cdot 2}{9 \cdot 8 \cdot 7}\left[\binom{4}{0}\binom{2}{2} + \binom{4}{1}\binom{2}{1} + \binom{4}{2}\binom{2}{0}\right]$$

$$= \frac{1}{28}\left[1+8+6\right]$$

$$= \frac{15}{28}$$

Also,
$$p(1,1) = \frac{\binom{4}{1}\binom{3}{1}\binom{2}{1}}{\binom{9}{3}}$$

$$= \frac{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2}{9 \cdot 8 \cdot 7}$$

$$= \frac{2}{7}$$

Therefore, $$P\{Z=1 \mid Y=1\} = \frac{2/7}{15/28} = \boxed{8/15}$$

e.) No because $P(1,2) = \frac{1}{7} \neq \frac{5}{49} = \left(\frac{10}{21}\right)\left(\frac{3}{14}\right) = P_X(1)P_Y(2)$.

Solution 9: Here, we are given two distributions: $f_X(x)$ is uniform over $0 \le x \le 1$
[4 pts] and $f_{Y|X}(y|x)$ which is uniform over $0 \le y \le x$, given some x
1 for a, b value. Hence, we know the following:
2 for c

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} 1/x & \text{if } 0 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

a.) The joint density function, $f(x,y)$, is given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$\Rightarrow f(x,y) = f_X(x)\, f_{Y|X}(y|x)$$

$$\Rightarrow \boxed{f(x,y) = \begin{cases} 1/x & \text{if } 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{otherwise.} \end{cases}}$$

b.) We seek the probability $P\{Y > \frac{1}{4} \mid X = \frac{1}{2}\}$, which is given by

$$P\{Y > \tfrac{1}{4} \mid X = \tfrac{1}{2}\} = \int_{1/4}^{\infty} f_{Y|X}(y|\tfrac{1}{2})\, dy$$

$$= \int_{1/4}^{1/2} \frac{1}{1/2}\, dy$$

$$= 2\, y \Big|_{1/4}^{1/2}$$

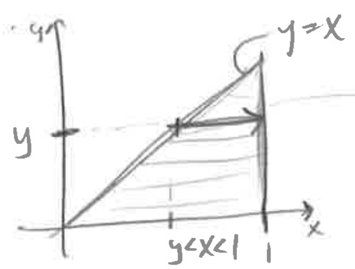
$$= \boxed{1/2}$$

Solution 9 Cont:

(c) We seek the probability $P\{X > \frac{1}{2} \mid Y = \frac{1}{4}\}$. We need the conditional pdf $f_{X|Y}(x|y)$, given by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

The marginal distributions for y is given by



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_y^1 \frac{1}{x}\,dx$$

$$= \ln(x)\Big|_y^1$$

$$= -\ln(y)$$

$$f_Y(y) = \ln\left(\frac{1}{y}\right)$$

Therefore, we obtain

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{x\ln(1/y)} & y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$P\{X > \tfrac{1}{2} \mid Y = \tfrac{1}{4}\} = \int_{\frac{1}{2}}^{\infty} f_{X|Y}(x|\tfrac{1}{4})\,dx$$

$$= \int_{1/2}^{1} \frac{1}{x\ln(4)}\,dx$$

$$= \frac{1}{\ln(4)} \ln(x)\Big|_{1/2}^{1}$$

$$= \frac{\ln(2)}{\ln(4)} = \boxed{\frac{1}{2}}$$

Solution 10: [2 pts] Suppose X, Y, and Z have the same pdfs given by

$$f_X(x) = e^{-x}, \quad f_Y(y) = e^{-y}, \quad f_Z(z) = e^{-z}$$

where each variable takes on values $0 < x, y, z < \infty$.

Now, consider the new variables defined by

$$U = X + Y \Rightarrow u = x + y =: g_1(x,y,z)$$
$$V = X + Z \Rightarrow v = x + z =: g_2(x,y,z)$$
$$W = Y + Z \Rightarrow w = y + z =: g_3(x,y,z).$$

Then we can solve for x,y,z in terms of u,v,w:

$$\begin{aligned} x + y \quad &= u \\ x \quad + z &= v \\ y + z &= w \end{aligned}$$

$$\Rightarrow \begin{aligned} x+y &= u \\ x-y &= v-w \end{aligned} \Rightarrow 2x = u+v-w \Rightarrow \boxed{x = \frac{u+v-w}{2}}$$

$$\Rightarrow y = u - x \Rightarrow \boxed{y = \frac{u-v+w}{2}}$$

$$\Rightarrow z = v - x \Rightarrow \boxed{v = \frac{-u+v+w}{2}}$$

Now that we have solved for x,y,z, we can find the Jacobian:

$$J(x,y,z) = \begin{vmatrix} \partial g_1/\partial x & \partial g_1/\partial y & \partial g_1/\partial z \\ \partial g_2/\partial x & \partial g_2/\partial y & \partial g_2/\partial z \\ \partial g_3/\partial x & \partial g_3/\partial y & \partial g_3/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (1)\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -1 - 1 = -2$$

Solution 10 Cont:

Finally, since each X, Y, Z are independent, we know the joint pdf is

$$f_{X,Y,Z}(x,y,z) = e^{-x}e^{-y}e^{-z}$$

The new joint pdf is given by

$$f_{U,V,W}(u,v,w) = \frac{e^{-\left(\frac{u+v-w}{2}\right)} e^{-\left(\frac{u-v+w}{2}\right)} e^{-\left(\frac{-u+v+w}{2}\right)}}{|-2|}$$

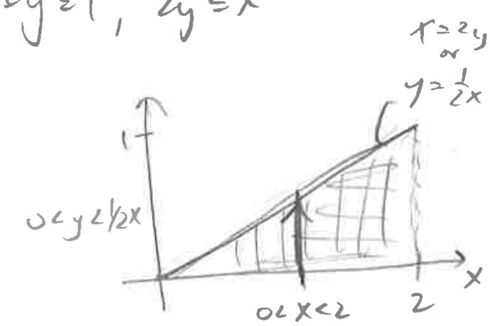
$$\Rightarrow f_{U,V,W}(u,v,w) = \frac{1}{2}\exp\left[-\frac{u}{2}-\frac{v}{2}-\frac{w}{2}\right]$$

$$= \frac{1}{2}e^{-\left(\frac{u+v+w}{2}\right)}$$

Solution 11: Let X = amount of pollutant when not working and Y = amount of pollutant when working. Then
[6 pts]

$$f(x,y) = \begin{cases} k & 0<x<2,\ 0<y<1,\ 2y\leq x \\ 0 & \text{otherwise.}\end{cases}$$

1 each



(a) The value of k is found to be

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= \int_0^2\int_0^{\frac{1}{2}x} k\,dy\,dx$$

$$= \int_0^2 k\tfrac{1}{2}x\,dx$$

$$= k\tfrac{1}{4}x^2\Big|_0^2$$

$$\boxed{1 = k}$$

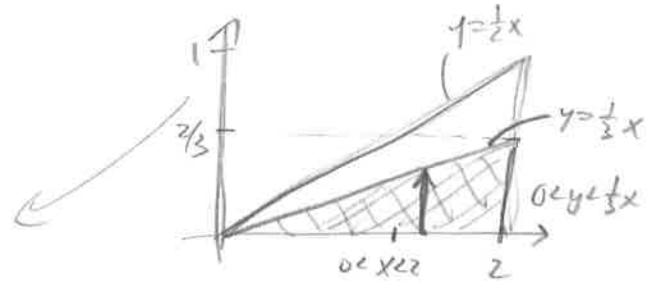
Solution 11 Cont:

b.) $$P\{X \geq 3Y\} = \iint_{(x,y)\,|\,x>3y} f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^{\frac{1}{3}x} 1 \, dy\,dx$$

$$= \int_0^2 \frac{1}{3}x \, dx$$

$$= \frac{1}{6}x^2 \Big|_0^2 = \boxed{\frac{2}{3}}$$

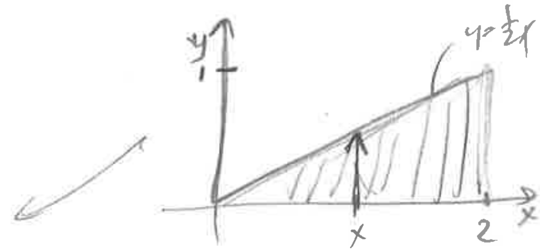


c.) we seek $f_X(x)$, which is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\frac{1}{2}x} 1\,dy$$

$$= \boxed{\frac{1}{2}x} \qquad 0 \leq x \leq 2.$$

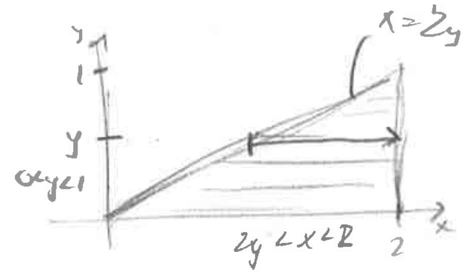


d.) we seek $f_Y(y)$, which is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{2y}^{2} dx$$

$$= 2-2y = \boxed{2(1-y)} \qquad 0 \leq y \leq 1$$



e.) we seek $f_{X|Y}(x|y)$, which is given by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \boxed{\frac{1}{2(1-y)}, \quad 2y<x<2}$$

Solution 11 Cont:

f) we seek $f_{Y|X}(y|x)$, which is given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \boxed{\frac{2}{x} \quad 0<y\le \tfrac{1}{2}x}$$

Solution 12: [3 pts] Let X & Y be independent gamma variables with parameters $(\alpha,1)$ and $(\beta,1)$, respectively. Then we have

$$f_X(x) = \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}, \quad f_Y(y) = \frac{e^{-y}y^{\beta-1}}{\Gamma(\beta)}$$

which implies

$$f_{X,Y}(x,y) = \frac{e^{-(x+y)}}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}y^{\beta-1}.$$

Consider the new variables given by

$$U = X+Y \quad \Rightarrow \quad u = x+y =: g_1(x,y)$$

$$V = \frac{X}{X+Y} \quad \Rightarrow \quad v = \frac{x}{x+y} =: g_2(x,y).$$

Then we can solve for x and y in terms of u and v:

$$x = uv \qquad y = u(1-v)$$

Furthermore, we obtain the following partial derivatives:

$$\frac{\partial g_1}{\partial x} = 1 \qquad \frac{\partial g_2}{\partial x} = \frac{y}{(x+y)^2}$$

$$\frac{\partial g_1}{\partial y} = 1 \qquad \frac{\partial g_2}{\partial y} = \frac{-x}{(x+y)^2}$$

Solution 12 Cont:

Thus, the Jacobian is

$$J(x,y) = \begin{vmatrix} 1 & \frac{y}{(x+y)^2} \\ 1 & \frac{-x}{(x+y)^2} \end{vmatrix} = \frac{-(x+y)}{(x+y)^2} = -\frac{1}{x+y} = -\frac{1}{u}$$

Therefore, the new joint pdf for u and v is given by:

$$f_{U,V}(u,v) = \frac{1}{\left|-\frac{1}{u}\right|} f_{X,Y}(uv, u(1-v))$$

$$= \frac{u}{\Gamma(\alpha)\Gamma(\beta)} e^{-(uv+u(1-v))} (uv)^{\alpha-1}[u(1-v)]^{\beta-1}$$

$$= \frac{u^{\alpha+\beta-1} e^{-u} v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\boxed{f_{U,V}(u,v) = \underbrace{\frac{e^{-u} u^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}}_{\text{Gamma Dist}} \cdot \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1}}_{\text{Beta Dist}}}$$

Note: the term $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is $\frac{1}{B(\alpha,\beta)}$ as defined in the book.

Integrating out u and v will give the desired distributions for u and v, respectively.