

Convergence Analysis of a Collapsed Gibbs Sampler for Bayesian Vector Autoregressions

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Abstract

We propose a collapsed Gibbs sampler for Bayesian vector autoregressions with predictors, or exogenous variables, and study the proposed algorithm's convergence properties. The Markov chain generated by our algorithm converges to its stationary distribution at least as fast as those of competing (non-collapsed) Gibbs samplers and is shown to be geometrically ergodic regardless of whether the number of observations in the underlying vector autoregression is small or large in comparison to the order and dimension of it. We also give conditions for when the geometric ergodicity is asymptotically stable as the number of observations tends to infinity. Specifically, the geometric convergence rate is shown to be bounded away from unity asymptotically, either almost surely or with probability tending to one, depending on what is assumed about the data generating process. Our results are among the first of their kind for practically relevant Markov chain Monte Carlo algorithms.

1 Introduction

Markov chain Monte Carlo (MCMC) is often used to explore the posterior distribution of a vector of parameters θ given data \mathcal{D} . To ensure the reliability an analysis using MCMC it is essential to understand the convergence properties of the chain in use [6, 7, 9, 10, 22, 28, 49] and, accordingly, there are numerous articles establishing such properties for different MCMC algorithms [e.g. 1, 2, 15, 17, 19, 23, 37, 41, 47]. It has been common in this literature to treat the data \mathcal{D} as fixed, or realized. Thus, the model for how the data are generated has typically been important only insofar as it determines the likelihood function based on an arbitrary realization—the stochastic properties of the data prescribed by that model have not been emphasized. This is natural since the target distribution, i.e. the posterior distribution, treats the data as fixed. On the other hand, due to the rapid growth of data available in applications, it is also desirable to understand how performance is affected as the number of observations increases. When this happens, the data are more naturally thought of as stochastic; each time the sample size increases by one, the additional observation is randomly generated. The study of how convergence properties of MCMC algorithms are affected by changes in the data is known as convergence complexity analysis [39] and it has attracted increasing attention recently [18, 36, 37, 50, 51].

Accounting for randomness in the data and letting the sample size grow leads to a more complicated analysis than when the data are fixed. In fact, to date, convergence complexity analysis has only been successfully carried out for a few practically relevant MCMC algorithms, maybe even one [37]. We propose and study a MCMC algorithm for a fundamental model in time series analysis: a Bayesian vector autoregression with predictors (VARX), or exogenous variables. Briefly, the VARX we consider assumes that $Y_t \in \mathbb{R}^r$ and $X_t \in \mathbb{R}^p$ satisfy, for $t = 1, \dots, n$,

$$Y_t = \sum_{i=1}^q \mathcal{A}_i^\top Y_{t-i} + \mathcal{B}^\top X_t + U_t \quad (1)$$

with U_1, \dots, U_n independent and multivariate normally distributed with mean zero and common covariance matrix $\Sigma \in \mathbb{S}_{++}^r$, $\mathcal{A}_i \in \mathbb{R}^{r \times r}$, $i = 1, \dots, q$, and $\mathcal{B} \in \mathbb{R}^{p \times r}$. The target distribution of our algorithm is the posterior distribution of $\theta = (\mathcal{A}_1, \dots, \mathcal{A}_q, \mathcal{B}, \Sigma)$ given $\mathcal{D} = \{(Y_1, X_1), \dots, (Y_n, X_n)\}$. More details on the model specification, priors, and resulting

posterior distribution are given in Section 2. We will consider both fixed and growing data and refer to the two settings as the small- n and large- n setting, respectively. By n being small we mean that it is fixed and possibly small in comparison to r and q , but $n > p$ throughout. Many large VARs in the literature [3, 11, 26] are covered by this setting. By n being large we mean that it is increasing and that the data are stochastic.

The algorithm we propose is a collapsed Gibbs sampler. It exploits the structure in the VARX to generate a Markov chain that converges to its stationary distribution at least as fast as those generated by competing (non-collapsed) Gibbs samplers. To discuss the more precise convergence results we establish, we require some more notation.

Let $F(\cdot|\mathcal{D})$ denote the VARX posterior distribution having density $f(\theta|\mathcal{D})$ with support on $\Theta \subseteq \mathbb{R}^d$ for some $d \geq 1$ and let K^h ($K \equiv K^1$) be the h -step transition kernel for a Markov chain with state space Θ , started at a point $\theta \in \Theta$. We assume throughout that all discussed Markov chains are irreducible, aperiodic, and Harris recurrent [34], and that sets on which measures are defined are equipped with their Borel σ -algebra. Our analysis is focused on convergence rates in total variation distance, by which we mean the rate at which $\|K^h(\theta, \cdot) - F(\cdot|\mathcal{D})\|_{TV}$ approaches zero as h tends to infinity, where $\|\cdot\|_{TV}$ denotes the total variation norm. If this convergence happens at a geometric (or exponential) rate, meaning there exist a $\rho \in [0, 1)$ and an $M : \Theta \rightarrow [0, \infty)$ such that for every $\theta \in \Theta$ and $h \in \{1, 2, \dots\}$

$$\|K^h(\theta, \cdot) - F(\cdot|\mathcal{D})\|_{TV} \leq M(\theta)\rho^h, \quad (2)$$

then the Markov chain, or the kernel K , is said to be geometrically ergodic. The geometric convergence rate ρ^* is the infimum of the set of $\rho \in [0, 1]$ such that (2) holds [37]. Since all probability measures have unit total variation norm, ρ^* is always in $[0, 1]$, and K is geometrically ergodic if and only if $\rho^* < 1$. A substantial part of the literature on convergence of MCMC algorithms is centered around geometric ergodicity, for good reasons: under moment conditions, a central limit theorem holds for functionals of geometrically ergodic Markov chains [5, 20] and the variance in the asymptotic distribution given by that CLT can be consistently estimated [8, 21, 48], allowing principled methods for ensuring reliability of the results [42, 49]. Our main result in the small- n setting gives conditions that ensure $\rho^* < 1$ when the data are fixed and K is the kernel corresponding to our proposed algorithm. Due to a well known correspondence [15, 24] between the likelihoods of the VARX and the multivariate linear regression

model when data are fixed, our small- n results also apply to certain versions of the latter.

Notice that, although it is suppressed in the notation, K , M , ρ , and, hence, ρ^* typically depend on \mathcal{D} . In the large- n setting, we are no longer considering a single dataset, but a sequence of datasets $\{\mathcal{D}_n\} := \{\mathcal{D}_1, \mathcal{D}_2, \dots\}$, where \mathcal{D}_n here denotes a dataset with n observations. Consequently, for every n there is a posterior distribution $F(\cdot|\mathcal{D}_n)$ and a Markov chain with kernel K_n that is used to explore it. To each kernel K_n there also corresponds a geometric convergence rate ρ_n^* . Since ρ_n^* depends on \mathcal{D}_n , the sequence $\{\rho_n^*\}$ is now one of random variables, ignoring possible issues with measurability. We are interested in bounding $\{\rho_n^*\}$ away from unity asymptotically, in either one of two senses: first, if there exists a sequence of random variables $\{\bar{\rho}_n\}$ such that $\rho_n^* \leq \bar{\rho}_n$ for every n and $\limsup_{n \rightarrow \infty} \bar{\rho}_n < 1$ almost surely, then we say that $\{K_n\}$ is asymptotically geometrically ergodic almost surely, or the geometric ergodicity is asymptotically stable almost surely. Secondly, if instead of the upper limit being less than unity almost surely it holds that $\lim_{n \rightarrow \infty} \mathbb{P}(\bar{\rho}_n < 1) = 1$, then we say that $\{K_n\}$ is asymptotically geometrically ergodic in probability, or that the geometric ergodicity is asymptotically stable in probability. Our main results in the large- n setting give conditions for asymptotically stable geometric ergodicity, in both of the two senses, of the Markov chain generated by our algorithm. An intuitive, albeit somewhat loose, interpretation of our main results is that the geometric ergodicity is asymptotically stable if the parameters of the VARX can be consistently estimated using maximum likelihood.

The rest of the paper is organized as follows. We begin in Section 2 by completing the specification of the model and priors. Because some of the priors may be improper we derive conditions which guarantee the posterior exists and is proper. In Section 3 we propose a collapsed Gibbs sampler for exploring the posterior. Conditions for geometric ergodicity for small n are presented in Section 4 and conditions for asymptotically stable geometric ergodicity are given in Section 5. Some concluding remarks are given in Section 6.

2 Bayesian vector autoregression with predictors

Recall the definition of the VARX in (1). To complete the specification, we assume that the starting point (Y_{-q+1}, \dots, Y_0) is non-stochastic and known and that the predictors are

strongly exogenous. By the latter we mean that $\{X_t\}$ is independent of $\{U_t\}$ and has a distribution that does not depend on the model parameters. With these assumptions the following lemma is straightforward. Its proof is provided in Appendix B for completeness. Let $Y = [Y_1, \dots, Y_n]^\top \in \mathbb{R}^{n \times r}$, $X = [X_1, \dots, X_n]^\top \in \mathbb{R}^{n \times p}$, $Z_t = [Y_{t-1}^\top, \dots, Y_{t-q}^\top]^\top \in \mathbb{R}^{qr}$, $t = 1, \dots, n$, and $Z = [Z_1, \dots, Z_n]^\top \in \mathbb{R}^{n \times qr}$. Let also $\mathcal{A} = [\mathcal{A}_1^\top, \dots, \mathcal{A}_q^\top]^\top \in \mathbb{R}^{qr \times r}$ and $\alpha = \text{vec}(\mathcal{A})$, where $\text{vec}(\cdot)$ is the vectorization operator, stacking the columns of its matrix argument.

Lemma 2.1. *The joint density for n observations in the VARX is $f(Y, X \mid \mathcal{A}, \mathcal{B}, \Sigma) = f(X)f(Y \mid X, \mathcal{A}, \mathcal{B}, \Sigma)$ with*

$$f(Y \mid X, \mathcal{A}, \mathcal{B}, \Sigma) = (2\pi)^{-nr/2} |\Sigma|^{-n/2} \text{etr} \left[-\frac{1}{2} \Sigma^{-1} (Y - Z\mathcal{A} - X\mathcal{B})^\top (Y - Z\mathcal{A} - X\mathcal{B}) \right]$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$.

We defer a discussion of exactly how n , p , r , and q compare since what is needed depends on how the prior distributions are specified. Let \mathbb{S}_+^r denote the set of $r \times r$ symmetric positive semi-definite (SPSD) matrices and, to define priors, let $m \in \mathbb{R}^{qr^2}$, $C \in \mathbb{S}_+^{qr^2}$, $D \in \mathbb{S}_+^r$, and $a \geq 0$ be hyperparameters. The prior on $\theta = (\alpha, \mathcal{B}, \Sigma) \in \Theta = \mathbb{R}^{qr^2} \times \mathbb{R}^{p \times r} \times \mathbb{S}_{++}^r$ we consider is of the form $f(\theta) = f(\alpha)f(\mathcal{B})f(\Sigma)$, with

$$f(\alpha) \propto \exp \left(-\frac{1}{2} [\alpha - m]^\top C [\alpha - m] \right),$$

$$f(\mathcal{B}) \propto 1,$$

and

$$f(\Sigma) \propto |\Sigma|^{-a/2} \text{etr} \left(-\frac{1}{2} D \Sigma^{-1} \right),$$

where $|\cdot|$ means the determinant when applied to matrices. The flat prior on \mathcal{B} is standard in multivariate scale and location problems, including in particular the multivariate regression model which is recovered when $\mathcal{A} = 0$ in the VARX. The priors on α and Σ are common in macroeconomics [25] and the prior on Σ includes the inverse Wishart ($D \in \mathbb{S}_{++}^r, a > 2r$) and Jeffreys prior ($D = 0, a = r + 1$) as special cases. The following result gives two different sets of conditions that lead to a proper posterior. Though we only consider proper normal or

flat priors for α in the rest of the paper, it may be relevant for other work to note that the proposition holds for any prior $f(\alpha)$ satisfying the conditions.

Proposition 2.2. *If either*

1. $D \in \mathbb{S}_{++}^r$, X has full column rank, $n + a > 2r + p$, and $f(\alpha)$ is proper; or
2. $[Y, Z, X] \in \mathbb{R}^{n \times (r+qr+p)}$ has full column rank, $n + a > (2 + q)r + p$, and $f(\alpha)$ is bounded,

then the posterior distribution is proper and, with $S = n^{-1}(Y - Z\mathcal{A} - X\mathcal{B})^\top(Y - Z\mathcal{A} - X\mathcal{B})$, the posterior density is characterized by

$$f(\mathcal{A}, \mathcal{B}, \Sigma \mid Y, X) \propto |\Sigma|^{-\frac{n+a}{2}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} [D + nS] - \frac{1}{2} (\alpha - m)^\top C (\alpha - m) \right). \quad (3)$$

Proof. Appendix B. □

The first set of conditions is relevant to the small n -setting. It implies that if the prior on Σ is a proper inverse Wishart density, so that $a > 2r$ and $D \in \mathbb{S}_{++}^r$, then the posterior is proper if $f(\alpha)$ is proper and X has full column rank. In particular, r or q can be arbitrarily large in comparison to n . Thus, this setting is compatible with large VARs [3, 11, 26]. The second set of conditions allows for the use of improper priors also on α and Σ when n is large in comparison to all of p , q , and r . The full column rank of $[Y, Z, X]$ is natural in large- n settings. In practice, one expects it to hold unless the least squares regression of Y on Z and X gives residuals that are identically zero.

To the best of our knowledge, the combination of an improper prior for \mathcal{B} and a proper prior for \mathcal{A} is new. In previous work, \mathcal{A} and \mathcal{B} have sometimes been grouped as $\Psi = [\mathcal{A}^\top, \mathcal{B}^\top]^\top$ and a proper multivariate normal prior assigned to $\text{vec}(\Psi)$ [27]. Treating \mathcal{A} and \mathcal{B} differently is appealing because, as indicated by point 1 in Proposition 2.2, one can then use the standard flat prior on \mathcal{B} while still allowing for large q and r . Moreover, even when n is large enough that one could use a flat prior also on \mathcal{A} , a proper prior can be preferable: many time series, in particular in economics and finance, are known to be near non-stationary in the unit root sense, and if $C \in \mathbb{S}_{++}^{qr^2}$ so that $f(\alpha)$ is proper, then m can be chosen to reflect this. For a discussion of priors in Bayesian VARs more generally we refer the reader to Karlsson [25].

As alluded to in the introduction, for fixed data the density $f(Y \mid X, \mathcal{A}, \mathcal{B}, \Sigma)$, and hence the posterior, is the same as in a multivariate linear regression with design matrix $[Z, X] \in$

$\mathbb{R}^{n \times (qr+p)}$ and coefficient matrix $[\mathcal{A}^\top, \mathcal{B}^\top]^\top$. Thus, our results with fixed data apply also to a multivariate regression model with partitioned design matrix—one part which has a flat prior for its coefficient and one which is possibly high-dimensional and has a proper prior for its coefficient. This configuration is unlike those in other work on similar models which typically assume either that $[Z, X]$ has full rank or that the prior for $[\mathcal{A}^\top, \mathcal{B}^\top]^\top$ is proper [1, 2, 11, 15, 46].

The literature on convergence properties of MCMC algorithms for Bayesian VAR(X)s is limited. An MCMC algorithm for a multivariate linear regression model has been proposed and its convergence rate studied [15]. By the preceding discussion, this includes the VARX as a special case, however, the (improper) prior used is $f(\theta) \propto |\Sigma|^{-a}$ which is not compatible with the large VARXs we allow for in the small- n setting. In the large- n setting, i.e. when doing convergence complexity analysis, the data are no longer considered fixed and, hence, the VARX is no longer a special case of multivariate linear regression. A two-component $(\mathcal{A}$ and $\Sigma)$ Gibbs sampler for Bayesian vector autoregressions without predictors has been proposed [24]. However, the analysis of it is simulation-based and as such does not provide any theoretical guarantees. Our results address this since, as we will discuss in more detail below, the algorithm we propose simplifies to this Gibbs sampler when there are no predictors.

3 A collapsed Gibbs sampler

If $C = 0$, then the VARX posterior is a normal-(inverse) Wishart for which MCMC is unnecessary. However, when $C \in \mathbb{S}_{++}^{qr^2}$ the posterior is analytically intractable and there are many potentially useful MCMC algorithms. For example, the full conditional distributions have familiar forms so it is straightforward to implement a three-component Gibbs sampler. Another sensible option is to group \mathcal{A} and \mathcal{B} and update them together. Here, we will instead make use of the particular structure the partitioned matrix $[Z, X]$ offers and devise a collapsed Gibbs sampler [29]. Well known results [30] imply that our collapsed Gibbs sampler converges to its stationary distribution at least as fast as both the three-component Gibbs sampler and the two-component sampler that groups \mathcal{A} and \mathcal{B} . For the case $C \in \mathbb{S}_{++}^{qr^2}$ but $\mathcal{B} = 0$, i.e. there are no predictors in the model, a two-component Gibbs sampler has been proposed [25]. Our algorithm specializes to this two-component Gibbs sampler when $\mathcal{B} = 0$ and, as a consequence,

our results apply almost verbatim. A formal description of one iteration of the collapsed Gibbs sampler is given in Algorithm 1.

Algorithm 1 Collapsed Gibbs sampler

- 1: *Input:* Current value $(\alpha^h, \mathcal{B}^h, \Sigma^h)$
 - 2: Draw Σ^{h+1} from the distribution of $\Sigma \mid \mathcal{A}^h, Y, X$
 - 3: Draw α^{h+1} from the distribution of $\alpha \mid \Sigma^{h+1}, Y, X$
 - 4: Draw \mathcal{B}^{h+1} from the distribution of $\mathcal{B} \mid \mathcal{A}^{h+1}, \Sigma^{h+1}, Y, X$
 - 5: Set $h = h + 1$
-

We next derive the conditional distributions necessary for its implementation. Let $\mathcal{M}(M, U, V)$ denote the matrix normal distribution with mean M and scale matrices U and V (see Definition A.1), and let $\mathcal{W}^{-1}(U, c)$ denote the inverse Wishart distribution with scale matrix U and c degrees of freedom. For any real matrix M , define P_M to be the projection onto its column space and Q_M the projection onto the orthogonal complement of its column space. Let also \otimes denote the Kronecker product and define $B = B(\Sigma) = C + \Sigma^{-1} \otimes Z^\top Q_X Z$ and $u = u(\Sigma) = B^{-1}[Cm + (\Sigma^{-1} \otimes Z^\top Q_X) \text{vec}(Q_X Y)]$.

Lemma 3.1. *If at least one of the two sets of conditions in Proposition 2.2 holds, then*

$$\begin{aligned} \Sigma \mid \mathcal{A}, Y, X &\sim \mathcal{W}^{-1}\left(D + (Y - Z\mathcal{A})^\top Q_X (Y - Z\mathcal{A}), n + a - p - r - 1\right) \\ \alpha \mid \Sigma, Y, X &\sim \mathcal{N}(u, B^{-1}), \text{ and} \\ \mathcal{B} \mid \mathcal{A}, \Sigma, Y, X &\sim \mathcal{M}\left([X^\top X]^{-1} X^\top (Y - Z\mathcal{A}), [X^\top X]^{-1}, \Sigma\right). \end{aligned}$$

Proof. Appendix B. □

The collapsed Gibbs sampler in Algorithm 1 simulates a realization from a Markov chain having the following one-step transition kernel: for any measurable $A \subseteq \Theta = \mathbb{R}^{qr^2} \times \mathbb{R}^{p \times r} \times \mathbb{S}_{++}^r$,

$$K_C(\theta', A) = \iiint I_A(\alpha, \mathcal{B}, \Sigma) f(\Sigma \mid \alpha', Y, X) f(\alpha \mid \Sigma, Y, X) f(\mathcal{B} \mid \alpha, \Sigma, Y, X) d\Sigma d\alpha d\mathcal{B},$$

where the subscript C is short for collapsed. However, instead of working directly with K_C we will use its structure to reduce the problem in a convenient way. Consider the sequence $\{(\alpha^h, \Sigma^h)\}$, $h = 1, 2, \dots$, obtained by ignoring the component for \mathcal{B} in Algorithm 1. The

sequence $\{(\alpha^h, \Sigma^h)\}$ is generated as a two-component Gibbs sampler and its transition kernel is, for any measurable $A \subseteq \mathbb{R}^{qr^2} \times \mathbb{S}_{++}^r$,

$$K_G((\alpha', \Sigma'), A) = \iint I_A(\alpha, \Sigma) f(\alpha \mid \Sigma, Y, X) f(\Sigma \mid \alpha', Y, X) d\alpha d\Sigma.$$

A routine calculation shows that since K_G , by construction, has invariant distribution $F_{\mathcal{A}, \Sigma}(\cdot \mid Y, X)$, then K_C has the VARX posterior $F(\cdot \mid Y, X)$ as its invariant distribution.

The sequences $\{\alpha^h\}$ and $\{\Sigma^h\}$ are also Markov chains. The transition kernel for the $\{\alpha^h\}$ sequence is, for any measurable $A \subseteq \mathbb{R}^{qr^2}$,

$$K_{\mathcal{A}}(\alpha', A) = \iint I_A(\alpha) f(\alpha \mid \Sigma, Y, X) f(\Sigma \mid \alpha', Y, X) d\Sigma d\alpha. \quad (4)$$

The transition kernel, K_{Σ} , for the $\{\Sigma^h\}$ sequence is constructed similarly. The kernel $K_{\mathcal{A}}$ satisfies detailed balance with respect to the posterior marginal $F_{\mathcal{A}}(\cdot \mid Y, X)$ and similarly for K_{Σ} and hence each has the respective posterior marginal as its invariant distribution. However, the kernels K_C and K_G do not satisfy detailed balance with respect to their invariant distributions.

In Sections 4 and 5 we will establish geometric ergodicity of K_C and study its asymptotic stability, respectively. Our approach, which is motivated by the following lemma, will be to analyze $K_{\mathcal{A}}$ in place of K_C ; the lemma says we can analyze either of K_G , $K_{\mathcal{A}}$ or K_{Σ} in place of K_C . The proof of the lemma uses only well known results about de-initializing Markov chains [40] and can be found in Appendix B.

Lemma 3.2. *For any $\theta = (\alpha, \mathcal{B}, \Sigma) \in \Theta$, and $h \in \{1, 2, \dots\}$,*

$$\|K_C^h(\theta, \cdot) - F(\cdot \mid Y, X)\|_{TV} = \|K_G^h((\alpha, \Sigma), \cdot) - F_{\mathcal{A}, \Sigma}(\cdot \mid Y, X)\|_{TV} \leq \|K_{\mathcal{A}}^{h-1}(\alpha, \cdot) - F_{\mathcal{A}}(\cdot \mid Y, X)\|_{TV}$$

The primary tool we will use for investigating both geometric ergodicity and asymptotic stability is the following well known result [43, Theorem 12], which has been specialized to the current setting. Note that the kernel $K_{\mathcal{A}}$ acts to the left on measures, that is, for a measure ν , we define

$$\nu K_{\mathcal{A}}^h(\cdot) = \int \nu(d\alpha) K_{\mathcal{A}}^h(\alpha, \cdot).$$

Theorem 3.3. *Suppose $V : \mathbb{R}^{qr^2} \rightarrow [0, \infty)$ is such that for some $\lambda < 1$ and some $L < \infty$*

$$\int V(\alpha) K_{\mathcal{A}}(\alpha', d\alpha) \leq \lambda V(\alpha') + L \quad \text{for all } \alpha'. \quad (5)$$

Also suppose there exists $\varepsilon > 0$, a measure R , and some $T > 2L/(1 - \lambda)$ such that

$$K_{\mathcal{A}}(\alpha, \cdot) \geq \varepsilon R(\cdot) \quad \text{for all } \alpha \in \{\alpha : V(\alpha) \leq T\}. \quad (6)$$

Then $K_{\mathcal{A}}$ is geometrically ergodic and, moreover, if

$$\bar{\rho} = (1 - \varepsilon)^c \vee \left(\frac{1 + 2L + \lambda T}{1 + T} \right)^{1-c} (1 + 2L + 2\lambda T)^c \quad \text{for } c \in (0, 1),$$

then, for any initial distribution ν ,

$$\|\nu K_{\mathcal{A}}^h(\cdot) - F_{\mathcal{A}}(\cdot | Y, X)\|_{TV} \leq \left(2 + \frac{L}{1 - \lambda} + \int V(\alpha) \nu(d\alpha) \right) \bar{\rho}^h. \quad (7)$$

It is common for the initial value to be chosen deterministically, in which case (7) suggests choosing a starting value to minimize V . Theorem 3.3 has been successfully employed to determine sufficient burn-in in the sense that the upper bound on the right-hand side of (7) is below some desired value [22, 23, 44], but, unfortunately, the upper bound is often so conservative as to be of little utility. However, our interest is twofold; it is easy to see that there is a $c \in (0, 1)$ such that $\bar{\rho} < 1$ and hence if $K_{\mathcal{A}}$ satisfies the conditions, then it is geometrically ergodic and, as developed and exploited in other recent research [37], the geometric convergence rate ρ^* is upper bounded by $\bar{\rho}$. Outside of toy examples, we know of no general state space Monte Carlo Markov chains for which ρ^* is known.

Consider the setting where the number of observations tends to infinity; that is, there is a sequence of data sets $\{\mathcal{D}_n\}$ and corresponding transition kernels $\{K_{\mathcal{A},n}\}$ with $n \rightarrow \infty$. If $\liminf_{n \rightarrow \infty} \bar{\rho}_n = 1$ almost surely, then we say the drift (5) and minorization (6) are asymptotically unstable in the sense that, at least asymptotically, they provide no control over ρ_n^* . On the other hand, because $\rho_n^* \leq \bar{\rho}_n$ establishing that $\limsup_{n \rightarrow \infty} \bar{\rho}_n < 1$ almost surely or that $\lim_{n \rightarrow \infty} \mathbb{P}(\bar{\rho}_n < 1) = 1$, leads to asymptotically stable geometric ergodicity as defined in the introduction.

Notice that $\bar{\rho}$ depends on the drift function V through ε , λ , and L . Thus the choice of drift function which establishes geometric ergodicity for a fixed n may not result in asymptotic stability as $n \rightarrow \infty$. Indeed, in Section 4 we use one V to show that $K_{\mathcal{A}}$ is geometrically ergodic under weak conditions when n is fixed, while in Section 5 a different drift function and slightly stronger conditions are needed to achieve asymptotically stable geometric ergodicity of $K_{\mathcal{A}}$.

4 Geometric ergodicity

In this section we consider the small- n setting. That is, n is fixed and the data Y and X observed, or realized, and hence treated as constant. Accordingly, we do not use a subscript for the sample size on the transition kernels. We next present some preliminary results that will lead to geometric ergodicity of $K_{\mathcal{A}}$, and hence K_G and K_C .

We fix some notation before stating the next result. Let $\|\cdot\|$ denote the Euclidean norm when applied to vectors and the spectral (induced) norm when applied to matrices, $\|\cdot\|_F$ denotes the Frobenius norm for matrices, and superscript $+$ denotes the Moore–Penrose pseudo-inverse. Least squares estimators of \mathcal{A} and α are denoted by $\hat{\mathcal{A}} = (Z^\top Q_X Z)^+ Z^\top Q_X Y$ and $\hat{\alpha} = \text{vec}(\hat{\mathcal{A}})$, respectively, and $y = \text{vec}(Y)$.

Lemma 4.1. *Define $V : \mathbb{R}^{rq^2} \rightarrow [0, \infty)$ by $V(\alpha) = \|\alpha\|^2$. If $C \in \mathbb{S}_{++}^{qr^2}$ and at least one of the two sets of conditions in Proposition 2.2 holds, then for any $\lambda \geq 0$ and with*

$$L = \left(\|C^{-1}\| \|Cm\| + \|C^{-1/2}\| \|C^{1/2}\hat{\alpha}\| \right)^2 + \text{tr}(C^{-1}),$$

the kernel $K_{\mathcal{A}}$ satisfies the drift condition

$$\int V(\alpha) K_{\mathcal{A}}(\alpha', d\alpha) \leq \lambda V(\alpha') + L.$$

Proof. Assume without loss of generality that $Q_X = I_n$, where I_n denotes the $n \times n$ identity matrix; the general case is recovered by replacing Z and Y by $Q_X Z$ and $Q_X Y$ everywhere. Using (4) and Fubini's Theorem yields

$$\begin{aligned} \int \|\alpha\|^2 K_{\mathcal{A}}(\alpha', d\alpha) &= \iint \|\alpha\|^2 f(\alpha \mid \Sigma, Y, X) f(\Sigma \mid \alpha', Y, X) d\Sigma d\alpha \\ &= \iint \|\alpha\|^2 f(\alpha \mid \Sigma, Y, X) f(\Sigma \mid \alpha', Y, X) d\alpha d\Sigma. \end{aligned}$$

Lemma 3.1 and standard expressions for the moments of the multivariate normal distribution [45, Theorem 10.18] gives for the inner integral that

$$\int \|\alpha\|^2 f(\alpha \mid \Sigma, Y, X) d\alpha = \|u\|^2 + \text{tr}(B^{-1}).$$

The triangle inequality gives $\|u\| \leq \|B^{-1}Cm\| + \|B^{-1}(\Sigma^{-1} \otimes Z^\top)y\|$. We work separately on the last two summands. First, since $\Sigma^{-1} \otimes Z^\top Z$ is SPSPD, we get by Lemma A.2.2 that

$$\|B^{-1}Cm\| \leq \|C^{-1}\| \|Cm\|.$$

Secondly,

$$\begin{aligned}\|B^{-1}(\Sigma^{-1} \otimes Z^{\top})y\| &= \|C^{-1/2}(I_{qr^2} + C^{-1/2}(\Sigma^{-1} \otimes Z^{\top}Z)C^{-1/2})^{-1}C^{-1/2}(\Sigma^{-1} \otimes Z^{\top})y\| \\ &\leq \|C^{-1/2}\| \|(I_{qr^2} + C^{-1/2}(\Sigma^{-1} \otimes Z^{\top}Z)C^{-1/2})^{-1}C^{-1/2}(\Sigma^{-1} \otimes Z^{\top})y\|\end{aligned}$$

Now by Lemma A.3, with $(\Sigma^{-1/2} \otimes I_n)y$ and $(\Sigma^{-1/2} \otimes Z)C^{-1/2}$ taking the roles of what is there denoted y and X , we have for any generalized inverse (denoted by superscript g) that

$$\|(I_{qr^2} + C^{-1/2}(\Sigma^{-1} \otimes Z^{\top}Z)C^{-1/2})^{-1}C^{-1/2}(\Sigma^{-1} \otimes Z^{\top})y\|$$

is upper bounded by

$$\|(C^{-1/2}(\Sigma^{-1} \otimes Z^{\top}Z)C^{-1/2})^g C^{-1/2}(\Sigma^{-1} \otimes Z^{\top})y\|. \quad (8)$$

Lemma A.4 says that $C^{1/2}(\Sigma^{-1} \otimes Z^{\top}Z)^+ C^{1/2}$ is one such generalized inverse. Using that the Moore–Penrose pseudo-inverse distributes over the Kronecker product [33], the middle part of this generalized inverse can be written as $(\Sigma^{-1} \otimes Z^{\top}Z)^+ = \Sigma \otimes (Z^{\top}Z)^+$. Thus, for this particular choice of generalized inverse (8) is equal to

$$\|C^{1/2}(\Sigma \otimes [Z^{\top}Z]^+)(\Sigma^{-1} \otimes Z^{\top})y\| = \|C^{1/2}(I_r \otimes [Z^{\top}Z]^+ Z^{\top})y\|.$$

Thus, using also that $\text{tr}(B^{-1}) \leq \text{tr}(C^{-1})$ by Lemma A.2 since $\Sigma^{-1} \otimes Z^{\top}Z$ SPSD,

$$\|u\|^2 + \text{tr}(B^{-1}) \leq \left(\|C^{-1}\| \|Cm\| + \|C^{-1/2}\| \|C^{1/2}(I_r \otimes [Z^{\top}Z]^+ Z^{\top})y\| \right)^2 + \text{tr}(C^{-1}).$$

Since the right-hand side does not depend on Σ , the proof is completed upon integrating both sides with respect to $f(\Sigma \mid \alpha', Y) d\Sigma$. \square

Lemma 4.2. *If at least one of the two sets of conditions in Proposition 2.2 holds, then for any $T > 0$ and α such that $\|\alpha\|^2 \leq T$, there exists a probability measure R and*

$$\varepsilon = \frac{|D + Y^{\top}Q_{[Z,X]}Y|^{(n+a-p-r-1)/2}}{|D + I_r(\|Q_X Y\| + \|Q_X Z\|\sqrt{T})^2|^{(n+a-p-r-1)/2}} > 0$$

such that

$$K_{\mathcal{A}}(\alpha, \cdot) \geq \varepsilon R(\cdot).$$

Proof. We will prove that there exists a function $g : \mathbb{S}_{++}^r \rightarrow (0, \infty)$, depending on the data and hyperparameters, such that $\int g(\Sigma) d\Sigma > 0$ and $g(\Sigma) \leq f(\Sigma \mid \mathcal{A}, Y)$ for every α such that $\|\alpha\|^2 \leq T$, or, equivalently, $\|\mathcal{A}\|_F^2 \leq T$. This suffices since if such a g exists, then we may take $\varepsilon = \int g(\Sigma) d\Sigma$ and define the distribution R by, for any Borel set $A \subseteq \mathbb{R}^{qr^2}$,

$$R(A) = \frac{1}{\varepsilon} \iint I_A(\alpha) f(\alpha \mid \Sigma, Y) g(\Sigma) d\alpha d\Sigma.$$

Let $c = n + a - p - r - 1$ and $E = Y - Z\mathcal{A}$ so that $f(\Sigma \mid \mathcal{A}, Y)$ can be written

$$\frac{|D + E^\top Q_X E|^{c/2}}{2^{cr/2} \Gamma_r(c/2)} |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} [D + E^\top Q_X E] \right).$$

To establish existence of a g with the desired properties we will lower bound the first and third term in $f(\Sigma \mid \mathcal{A}, Y)$ using two inequalities, namely

$$|D + E^\top Q_X E| \geq |D + Y^\top Q_{[Z, X]} Y|$$

and, for every \mathcal{A} such that $\|\mathcal{A}\|_F^2 \leq T$,

$$\text{tr} \left[\Sigma^{-1} E^\top Q_X E \right] \leq \text{tr} \left[\Sigma^{-1} \left(\|Q_X Y\| + \|Q_X Z\| \sqrt{T} \right)^2 \right].$$

We prove the former inequality first. Since $E^\top Q_{[Z, X]} E = Y^\top Q_{[Z, X]} Y$, it suffices to prove that $|D + E^\top Q_X E| \geq |D + E^\top Q_{[Z, X]} E|$. For this, Lemma A.2.3 says it is enough to prove that $E^\top Q_X E - E^\top Q_{[Z, X]} E$ is SPSPD. But the Frisch–Waugh–Lovell theorem [32, Section 2.4] says $E^\top Q_{[Z, X]} E = (Q_X E)^\top Q_{Q_X Z} (Q_X E)$, and therefore

$$\begin{aligned} E^\top Q_X E - E^\top Q_{[Z, X]} E &= (Q_X E)^\top (I_n - Q_{Q_X Z}) Q_X E \\ &= [(I_n - Q_{Q_X Z}) Q_X E]^\top [(I_n - Q_{Q_X Z}) Q_X E], \end{aligned}$$

which is clearly SPSPD.

For the second inequality we get, using the triangle inequality, sub-multiplicativity, and

that the Frobenius norm upper bounds the spectral norm,

$$\begin{aligned}
\|E^\top Q_X E\| &= \|(Q_X E)^\top Q_X E\| \\
&\leq \|Q_X E\|^2 \\
&\leq (\|Q_X Y\| + \|Q_X Z\| \|\mathcal{A}\|)^2 \\
&\leq (\|Q_X Y\| + \|Q_X Z\| \|\mathcal{A}\|_F)^2 \\
&\leq (\|Q_X Y\| + \|Q_X Z\| \sqrt{T})^2 \\
&=: c_1.
\end{aligned}$$

Since the spectral norm for SPSD matrices is the maximum eigenvalue, we have shown that $c_1 I_r - E^\top Q_X E$ is SPSD. Thus, $\Sigma^{-1/2}(I_r c_1 - E^\top Q_X E)\Sigma^{-1/2}$ is also SPSD and, hence, $\text{tr}(\Sigma^{-1} E^\top Q_X E) = \text{tr}(\Sigma^{-1/2} E^\top Q_X E \Sigma^{-1/2}) \leq \text{tr}(\Sigma^{-1/2} I_r c_1 \Sigma^{-1/2}) = \text{tr}(\Sigma^{-1} c_1)$, which is what we wanted to show. We have thus established

$$f(\Sigma \mid \mathcal{A}, Y) \geq \frac{|D + Y^\top Q_{[Z, X]} Y|^{c/2}}{2^{cr/2} \Gamma_r(c/2)} |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} [D + I_r c_1] \right) := g(\Sigma).$$

Finally, the stated expression for $\varepsilon = \int g(\Sigma) d\Sigma$, and that it is indeed positive, follows from that under the first set of conditions in Proposition 2.2 D is SPD, and under the second set of conditions $Y^\top Q_{[Z, X]} Y$ is SPD by Lemma A.1; in either case, both $D + Y^\top Q_{[Z, X]} Y$ and $D + I_r c_1$ are SPD and, consequently, g is proportional to an inverse Wishart density with scale matrix $D + I_r c_1$ and c degrees of freedom. \square

We are now ready for the main result of this section.

Theorem 4.3. *If $C \in \mathbb{S}_{++}^{qr^2}$ and at least one of the two sets of conditions in Proposition 2.2 holds, then the transition kernels K_C , K_G , and $K_{\mathcal{A}}$ are geometrically ergodic.*

Proof. By Lemma 3.2 it suffices to show it for $K_{\mathcal{A}}$. Lemma 4.1 establishes that a drift condition (5) holds for $K_{\mathcal{A}}$ with $V(\alpha) = \|\alpha\|^2$ and all $\lambda \in [0, 1)$, while Lemma 4.2 establishes a minorization condition (6) for $K_{\mathcal{A}}$. The claim now follows immediately from Theorem 3.3. \square

5 Asymptotic stability

We consider asymptotically stable geometric ergodicity as $n \rightarrow \infty$. Motivated by Lemma 3.2, we focus on the sequence of kernels $\{K_{\mathcal{A},n}\}$, where $K_{\mathcal{A},n}$ is the kernel $K_{\mathcal{A}}$ with the dependence on the sample size n made explicit; we continue to write $K_{\mathcal{A}}$ when n is arbitrary but fixed. Similar notation applies to the kernels K_C and K_G .

It is clear that as n changes so do the data Y and X . Treating Y and X as fixed (observed) is not appropriate unless we only want to discuss asymptotic properties holding pointwise, i.e. for particular, or all, paths of the stochastic process $\{(Y_t, X_t)\}$, which is unnecessarily restrictive. Thus, to be clear, in what follows we assume that Y_1, Y_2, \dots and X_1, X_2, \dots are defined on an underlying probability space. We also assume that, for every n , the joint distribution of Y and X is as prescribed by the VARX, for some specific, “true” $\theta \in \Theta$. Unless indicated otherwise, probability statements and expectations are with respect to the underlying probability space, or equivalently with respect to the distribution of $(X, Y) \mid \theta$, for the true $\theta \in \Theta$.

Recall that Theorem 3.3 is instrumental to our strategy: if $K_{\mathcal{A},n}$ satisfies Theorem 3.3 with some $V = V_n$, $\lambda = \lambda_n$, $L = L_n$, $\varepsilon = \varepsilon_n$, and $T = T_n$, then there exists a $\bar{\rho}_n < 1$ that upper bounds ρ_n^* . We focus on the properties of those $\bar{\rho}_n$, $n = 1, 2, \dots$, as n tends to infinity. Throughout the section we assume that the priors, and in particular the hyperparameters, are the same for every n .

Clearly, the choice of drift function V_n is important for the upper bound $\bar{\rho}_n$ one obtains. The drift function used for the small- n regime is not well suited for the asymptotic analysis in this section. Essentially, problems occur if $\lambda_n \rightarrow 1$, $L_n \rightarrow \infty$, or $\varepsilon_n \rightarrow 0$ so that the corresponding upper bounds satisfy $\lim_{n \rightarrow \infty} \bar{\rho}_n = 1$ almost surely [37, Proposition 2]. Consider Theorem 4.3. Since we can take $\lambda_n = 0$ for all n , only L_n or ε_n can lead to problems. In Appendix B.1 we show that as long as $\hat{\mathcal{A}}$ is consistent, then $L_n(Y, X) = O_{\mathbf{P}}(1)$ while if, almost surely, as $n \rightarrow \infty$,

$$n\|Q_X Z\|^2 / \|Q_{[Z,X]} Y\|^2 \rightarrow \infty,$$

then $\varepsilon_n \rightarrow 0$ almost surely. We expect this to occur in many relevant configurations of the VAR. Indeed, we expect the order of $\|Q_X Z\|$ will often be at least that of $\|Q_{[X,Z]} Y\|$. To see why, consider the case without predictors. Then $\|Q_{[X,Z]} Y\|^2 = \|Q_Z Y\|^2 = n\lambda_{\max}(Y^\top Q_Z Y/n)$, and $Y^\top Q_Z Y/n$ is the maximum likelihood estimator of Σ which is known to be consistent for

stable VARs with i.i.d. Gaussian innovations [31]. For such VARs it also holds that $Z^\top Z/n$ converges in probability to some SPD limit [31], and hence $\|Z\|^2 = n\lambda_{\max}(Z^\top Z/n) = O_{\mathbb{P}}(n)$.

The intuition as to why the drift function that works in the small- n regime is not suitable for convergence complexity analysis is that the drift function should be centered (minimized) at a point the chain in question can be expected to visit often [37]. The function defined by $V(\alpha) = \|\alpha\|^2$ is minimized when $\alpha = 0$, but there is in general no reason to believe the α -component of the chain will visit a neighborhood of the origin often. On the other hand, if the number of observations grows fast enough in comparison to other quantities, then we expect the marginal posterior density of \mathcal{A} to concentrate around the true \mathcal{A} , i.e. the \mathcal{A} according to which the data is generated. We also expect that for large n the least squares and maximum likelihood estimator $\hat{\mathcal{A}} = (Z^\top Q_X Z)^+ Z^\top Q_X Y$ is close to the true \mathcal{A} . Thus, intuitively, the α -component of the chain should visit the vicinity of $\hat{\alpha} = \text{vec}(\hat{\mathcal{A}})$ often. Formalizing this intuition leads to the main result of the section.

Let us re-define $V : \mathbb{R}^{qr^2} \rightarrow [0, \infty)$ by $V(\alpha) = \|Q_X Z \mathcal{A} - Q_X Z \hat{\mathcal{A}}\|_F^2 = \|(I_r \otimes Q_X Z)(\alpha - \hat{\alpha})\|^2$. The following lemma establishes a result that will lead to verification of the drift condition in (9) for all large enough n and almost all sample paths of the VAR under appropriate conditions. Notice, however, that the λ given here need not be less than unity for a fixed n or particular sample path of the VARX.

Lemma 5.1. *If $[Z, X]$ has full column rank, $C \in \mathbb{S}_{++}^{qr^2}$, at least one of the two sets of conditions in Proposition 2.2 holds,*

$$\lambda = \frac{qr + \left(\|C\|^{1/2} \|\hat{\mathcal{A}}\|_F + \|C^{-1}\|^{1/2} \|Cm\| \right)^2}{n + a - 2r - p - 2}, \quad \text{and} \quad L = \lambda \text{tr}(D) + \lambda \|Q_{[Z, X]} Y\|_F^2,$$

then

$$\int V(\alpha) K_{\mathcal{A}}(\alpha', d\alpha) \leq \lambda V(\alpha') + L.$$

Proof. Suppose first that $Q_X = I_n$ and notice that Z has full column rank, and hence $(Z^\top Z)^{-1}$ exists. Since $f(\alpha \mid \Sigma, Y)$ is a multivariate normal density, standard expressions for the moments of the multivariate normal distribution gives

$$\int V(\alpha, \Sigma) f(\alpha \mid \Sigma, Y) d\alpha = \|(I_r \otimes Z)(u - \hat{\alpha})\|^2 + \text{tr}((I_r \otimes Z) B^{-1} (I_r \otimes Z)^\top). \quad (9)$$

For the second term we use cyclical invariance of the trace to write

$$\begin{aligned}\text{tr} \left((I_r \otimes Z) B^{-1} (I_r \otimes Z)^\top \right) &= \text{tr} \left[(I_r \otimes Z^\top Z) (C + \Sigma^{-1} \otimes Z^\top Z)^{-1} \right] \\ &= \text{tr} \left[(I_r \otimes Z^\top Z)^{1/2} (C + \Sigma^{-1} \otimes Z^\top Z)^{-1} (I_r \otimes Z^\top Z)^{1/2} \right].\end{aligned}$$

Since C and $\Sigma^{-1} \otimes Z^\top Z$ are both SPD, the last expression is in the form required by Lemma A.2, and hence

$$\begin{aligned}\text{tr} \left((I_r \otimes Z) B^{-1} (I_r \otimes Z)^\top \right) &\leq \text{tr} \left[(I_r \otimes Z^\top Z)^{1/2} (\Sigma^{-1} \otimes Z^\top Z)^{-1} (I_r \otimes Z^\top Z)^{1/2} \right] \\ &= \text{tr} \left[\Sigma \otimes Z (Z^\top Z)^{-1} Z^\top \right] \\ &= \text{tr}(\Sigma) \text{tr}[Z (Z^\top Z)^{-1} Z^\top] \\ &= \text{tr}(\Sigma) q r,\end{aligned}$$

where the last line uses that the trace of a projection matrix is the dimension of the space onto which it is projecting. Focusing now on the first term on the right hand side in (9) we have, defining $H = \Sigma^{-1} \otimes Z^\top Z$ and using $\hat{\alpha} = H^{-1}(\Sigma^{-1} \otimes Z^\top)y$, that

$$\|(I_r \otimes Z)(u - \hat{\alpha})\| = \|(I_r \otimes Z)(\hat{\alpha} - B^{-1}(Cm + [\Sigma^{-1} \otimes Z^\top]y))\|$$

is upper bounded by

$$\|(I_r \otimes Z)(H^{-1} - B^{-1})(\Sigma^{-1} \otimes Z^\top)y\| + \|(I_r \otimes Z)B^{-1}Cm\|. \quad (10)$$

Moreover, since $B = C + H$ the Woodbury identity gives $H^{-1} - B^{-1} = H^{-1}(C^{-1} + H^{-1})^{-1}H^{-1}$ so that the first term in (10) can be upper bounded as follows:

$$\begin{aligned}\|(I_r \otimes Z)(H^{-1} - B^{-1})(\Sigma^{-1} \otimes Z^\top)y\| &= \|(I_r \otimes Z)H^{-1}(H^{-1} + C^{-1})^{-1}H^{-1}(\Sigma^{-1} \otimes Z^\top)y\| \\ &= \|(I_r \otimes Z)H^{-1}(H^{-1} + C^{-1})^{-1}\hat{\alpha}\| \\ &\leq \|(I_r \otimes Z)H^{-1/2}\| \|H^{-1/2}(H^{-1} + C^{-1})^{-1}\| \|\hat{\alpha}\|.\end{aligned}$$

Here, the power G^t , $t \in \mathbb{R}$, for a SPD matrix G is defined by taking the spectral decomposition $G = U_G \text{diag}(\lambda_{\max}(G), \dots, \lambda_{\min}(G))U_G^\top$, where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalues, respectively, and setting

$$G^t = U_G \text{diag}(\lambda_{\max}^t(G), \dots, \lambda_{\min}^t(G))U_G^\top.$$

Now by standard properties of eigenvalues and eigenvectors of Kronecker products [16, Theorem 4.2.12] we get

$$\|(I_r \otimes Z)H^{-1/2}\| = \|(I_r \otimes Z)(\Sigma^{1/2} \otimes [Z^\top Z]^{-1/2})\| = \|\Sigma^{1/2}\| \|Z(Z^\top Z)^{-1/2}\| = \|\Sigma^{1/2}\|.$$

In addition, using Lemma A.2.2,

$$\begin{aligned} \|H^{-1/2}(H^{-1} + C^{-1})^{-1}\| &= \lambda_{\max}^{1/2}((H^{-1} + C^{-1})^{-1}H^{-1}(H^{-1} + C^{-1})^{-1}) \\ &\leq \lambda_{\max}^{1/2}((H^{-1} + C^{-1})^{-1}(H^{-1} + C^{-1})(H^{-1} + C^{-1})^{-1}) \\ &= \lambda_{\max}^{1/2}((H^{-1} + C^{-1})^{-1}) \\ &\leq \lambda_{\max}^{1/2}(C) \\ &= \|C\|^{1/2}. \end{aligned}$$

It remains to deal with the second term in (10). Using a similar technique as with the previous term, applying sub-multiplicativity and Lemma A.2.2 twice, we have

$$\begin{aligned} \|(I_r \otimes Z)B^{-1}Cm\| &= \|(\Sigma^{1/2} \otimes I_n)(\Sigma^{-1/2} \otimes I_n)(I_r \otimes Z)B^{-1}Cm\| \\ &\leq \|\Sigma^{1/2}\| \|(\Sigma^{-1/2} \otimes Z)(C + \Sigma^{-1} \otimes Z^\top Z)^{-1}\| \|Cm\| \\ &= \|\Sigma^{1/2}\| \lambda_{\max}^{1/2}([C + \Sigma^{-1} \otimes Z^\top Z]^{-1}[\Sigma^{-1} \otimes Z^\top Z][C + \Sigma^{-1} \otimes Z^\top Z]^{-1}) \|Cm\| \\ &\leq \|\Sigma^{1/2}\| \lambda_{\max}^{1/2}([C + \Sigma^{-1} \otimes Z^\top Z]^{-1}) \|Cm\| \\ &\leq \|\Sigma^{1/2}\| \|C^{-1}\|^{1/2} \|Cm\|. \end{aligned}$$

Putting things together we have shown that, for any Σ ,

$$\|(I_r \otimes Z)(\hat{\alpha} - u)\| \leq \|\Sigma^{1/2}\| \left(\|C\|^{1/2} \|\hat{\alpha}\| + \|C^{-1}\|^{1/2} \|Cm\| \right),$$

and hence we get from (9)

$$\int V(\alpha) f(\alpha \mid \Sigma, Y, X) d\alpha \leq \|\Sigma\| \left(\|C\|^{1/2} \|\hat{\alpha}\| + \|C^{-1}\|^{1/2} \|Cm\| \right)^2 + qr \operatorname{tr}(\Sigma).$$

The proof for the case $Q_X = I_n$ is completed by upper bounding $\|\Sigma\| \leq \operatorname{tr}(\Sigma)$, integrating both sides with respect to $f(\Sigma \mid \alpha', Y) d\Sigma$, and noting that

$$\begin{aligned} \int \operatorname{tr}(\Sigma) f(\Sigma \mid \alpha', Y, X) d\Sigma &= \frac{1}{n + a - 2r - p - 2} \operatorname{tr} \left(D + (Y - Z\mathcal{A}')^\top (Y - Z\mathcal{A}') \right) \\ &\leq \frac{1}{n + a - 2r - p - 2} \left(\operatorname{tr}(D) + \|Q_Z Y\|_F^2 + \|Z\hat{\mathcal{A}} - Z\mathcal{A}'\|_F^2 \right), \end{aligned}$$

where we have used that $(Y - Z\mathcal{A}')^\top(Y - Z\mathcal{A}') = (Y - Z\mathcal{A}')^\top P_Z(Y - Z\mathcal{A}') + (Y - Z\mathcal{A}')^\top Q_Z(Y - Z\mathcal{A}')$, and that $P_Z Y = Z\hat{\mathcal{A}}$. The general case is recovered by replacing Z and Y by $Q_X Z$ and $Q_X Z$ everywhere and invoking Lemma A.1. That $Z^\top Q_X Z$ is invertible also in the general case follows from the same lemma. \square

Lemma 5.2. *If at least one of the two sets of conditions in Proposition 2.2 holds, then for any $T > 0$ and $\alpha = \text{vec}(\mathcal{A})$ such that $\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|_F^2 \leq T$, there exists a probability measure R and*

$$\varepsilon = \left(\frac{|D + Y^\top Q_Z Y|}{|D + Y^\top Q_Z Y + I_r T|} \right)^{(n+a-p-r-1)/2} > 0$$

such that

$$K_{\mathcal{A}}(\alpha, \cdot) \geq \varepsilon R(\cdot).$$

Proof. The proof idea is similar to that for Lemma 4.2. We prove that there exists a $g : \mathbb{S}_{++}^r \rightarrow [0, \infty)$, depending on the data and the hyperparameters, such that $\int g(\Sigma) d\Sigma = \varepsilon > 0$ and $g(\Sigma) \leq f(\Sigma \mid \mathcal{A}, Y, X)$ for every \mathcal{A} such that $\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|_F^2 \leq T$.

Assume first that $Q_X = I_n$ and let $c = n + a - r - p - 1$ be the degrees of freedom in the full conditional distribution for Σ . Using that $(Y - Z\mathcal{A})^\top(Y - Z\mathcal{A}) - (Y - Z\mathcal{A})^\top Q_Z(Y - Z\mathcal{A})$ is SPSPD and that $Q_Z Z = 0$, we get by Lemma A.2 that

$$|D + (Y - Z\mathcal{A})^\top(Y - Z\mathcal{A})| \geq |D + Y^\top Q_Z Y|.$$

Moreover, for any \mathcal{A} such that $\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|_F^2 \leq T$,

$$\begin{aligned} \text{tr} \left[\Sigma^{-1}(Y - Z\mathcal{A})^\top(Y - Z\mathcal{A}) \right] &= \text{tr} \left[\Sigma^{-1}Y^\top Q_Z Y + \Sigma^{-1}(Y - Z\mathcal{A})^\top P_Z(Y - Z\mathcal{A}) \right] \\ &= \text{tr} \left[\Sigma^{-1}Y^\top Q_Z Y + \Sigma^{-1}(Z\hat{\mathcal{A}} - Z\mathcal{A})^\top(Z\hat{\mathcal{A}} - Z\mathcal{A}) \right] \\ &\leq \text{tr} \left[\Sigma^{-1}Y^\top Q_Z Y + \Sigma^{-1}\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|^2 \right] \\ &\leq \text{tr} \left[\Sigma^{-1}Y^\top Q_Z Y + \Sigma^{-1}T \right], \end{aligned}$$

where the first inequality follows from that $\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|^2 = \|(Z\hat{\mathcal{A}} - Z\mathcal{A})^\top(Z\hat{\mathcal{A}} - Z\mathcal{A})\|$ and that, therefore, $I_r\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|^2 - (Z\hat{\mathcal{A}} - Z\mathcal{A})^\top(Z\hat{\mathcal{A}} - Z\mathcal{A})$ is SPSPD, and the second inequality follows from that the Frobenius norm upper bounds the spectral norm, so that $\|Z\hat{\mathcal{A}} - Z\mathcal{A}\|^2 \leq \|Z\hat{\mathcal{A}} - Z\mathcal{A}\|_F^2 \leq T$.

With the determinant and trace inequalities just established, we have that $f(\Sigma \mid \mathcal{A}, Y, X)$ is, for any \mathcal{A} satisfying the hypotheses, lower bounded by

$$g(\Sigma) := \frac{|D + Y^\top Q_Z Y|^{c/2}}{2^{cr/2} \Gamma_r(c/2)} |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} [D + Y^\top Q_Z Y + I_r T] \right).$$

Noticing that g so defined is proportional to an inverse Wishart density and using well known expression for its normalizing constant finishes the proof for the case where $Q_X = I_n$. The general case is recovered upon replacing Z and Y by $Q_X Z$ and $Q_Z Y$ everywhere and invoking Lemma A.1. \square

We are ready to state the main result of the section.

Theorem 5.3. *If*

- (a) $C \in \mathbb{S}_{++}^{qr^2}$,
- (b) $[Y, Z, X]$ has full column rank for all large enough n almost surely,
- (c) $\|\hat{\alpha}\|^2 = O(1)$ almost surely as $n \rightarrow \infty$, and
- (d) there exists a random variable $M : \Omega \rightarrow (0, \infty)$ such that, almost surely,

$$M^{-1} \leq \liminf_{n \rightarrow \infty} n^{-1} \lambda_{\min}(Y^\top Q_{[Z, X]} Y) \leq \limsup_{n \rightarrow \infty} n^{-1} \lambda_{\max}(Y^\top Q_{[Z, X]} Y) \leq M,$$

then $\{K_{C,n}\}, \{K_{G,n}\}$, and $\{K_{\mathcal{A},n}\}$ are asymptotically geometrically ergodic almost surely.

Proof. By Lemma 3.2, it is enough to prove that $\limsup_{n \rightarrow \infty} \bar{\rho}_n < 1$ almost surely for the $\bar{\rho}_n$ corresponding to $K_{\mathcal{A},n}$. Inspecting the definition of $\bar{\rho}_n$ in Theorem 3.3 one sees that it suffices to show that Lemmas 5.1 and 5.2 apply and that the $\lambda = \lambda_n$, $L = L_n$, $T = T_n$, and $\varepsilon = \varepsilon_n$ they give almost surely satisfy, respectively: (i) $\limsup_{n \rightarrow \infty} \lambda_n < 1$, (ii) $\limsup_{n \rightarrow \infty} L_n < \infty$, (iii) $\limsup_{n \rightarrow \infty} T_n < \infty$, and (iv) $\liminf_{n \rightarrow \infty} \varepsilon_n > 0$. Assumption (a) and (b) ensure Lemma 5.1 applies and assumption (c) gives $\lambda_n = O(1/n)$, so (i) holds. That $\lambda_n = O(1/n)$ and assumption (d) give $L_n = O(1)$, i.e. (ii) holds, and hence we can pick a sequence $T_n > 2L_n(1 - \lambda_n)$, $n = 1, 2, \dots$, such that (iii) holds. For (iv), notice that (iii), assumption (d), and that $[r(n + a - p - r - 1)/2]/n \rightarrow r/2$ imply that we can find random variables $K_1, K_2 > 0$ such that, almost surely,

$$\varepsilon_n \geq (1 + K_1/n)^{-K_2 n} \rightarrow e^{-K_1 K_2} > 0, \quad n \rightarrow \infty,$$

which gives (iv). \square

Assumption (b) is relatively weak in the large n setting we are currently considering. Assumption (c) holds if, for example, the least squares estimator $\hat{\alpha}$ is strongly consistent, conditions for which are known in both the case with deterministic predictors [35] and the case with stochastic predictors [12, 13]. Assumption (d) holds if, for example, the MLE $n^{-1}Y^\top Q_{[Z,X]}Y$ of Σ is strongly consistent, or more generally if it converges to a positive definite matrix almost surely.

If some of the assumptions in Theorem 5.3 are relaxed to hold in probability, or with probability tending to one, instead of almost surely, then the conclusion can be weakened accordingly to give the following corollary.

Corollary 5.1. *If*

$$(a) \ C \in \mathbb{S}_{++}^{qr^2}$$

and, as $n \rightarrow \infty$,

$$(b) \ [Y, Z, X] \text{ has full column rank with probability tending to one,}$$

$$(c) \ \|\hat{\alpha}\|^2 = O_{\mathbb{P}}(1), \text{ and}$$

$$(d) \ \text{there exists a constant } M > 0 \text{ such that, with probability tending to one,}$$

$$M^{-1} \leq n^{-1}\lambda_{\min}(Y^\top Q_{[Z,X]}Y) \leq n^{-1}\lambda_{\max}(Y^\top Q_{[Z,X]}Y) \leq M,$$

then $\{K_{C,n}\}$, $\{K_{G,n}\}$, and $\{K_{\mathcal{A},n}\}$ are asymptotically geometrically ergodic in probability.

6 Discussion

Markov chain Monte Carlo is used in a wide range of problems, including but not limited to the Bayesian settings considered here. However, the theoretical properties of algorithms used by practitioners are not always well understood. We have focused on the case of Bayesian vector autoregressions with predictors. This is one of the most common models in time series, and in particular in the analysis and forecasting of macroeconomic time series. The Gibbs sampler has been suggested for exploring the posterior distribution of the parameters \mathcal{A} and Σ when there are no predictors [25], but there has been a lack of theoretical support. We have addressed this by proposing a collapsed Gibbs sampler that handles predictors and studying

its convergence properties. Since our algorithm simplifies to the usual Gibbs sampler when there are no predictors, our results apply also in that setting.

We have proven that our algorithm generates a geometrically ergodic Markov chain under reasonable assumptions (Theorem 4.3). This result is applicable both in classical settings where the sample size is large (but fixed) in comparison to the number of parameters, and in large VARXs where the dimension of the process or the lag length is large in comparison to the number of observations. Thus, with the algorithm we propose, characteristics of the posterior distribution can be reasonably estimated using principled approaches to ensuring the simulation results are trustworthy [7, 22, 49]. Our asymptotic analysis, or convergence complexity analysis, indicates our algorithm should perform well in large samples; we have proven that, as the sample size tends to infinity, the geometric ergodicity of the sequence of transition kernels corresponding to our algorithm is asymptotically stable. This result is one of the first of its kind for practically relevant MCMC algorithms.

Avenues for future research include convergence complexity analysis of cases where the dimension of the process or the lag length tends to infinity, either together with the sample size or for a fixed sample size. By inspecting the proof of Theorem 5.3 one sees that the same proof idea can work also if the dimension of the process or the lag length changes, as long as the sample size grows fast enough. However, the proof relies on formalizing the intuition that as the sample size increases, the posterior mode of the α -chain and the maximum likelihood estimator of α are close—if the sample size is fixed or grows slowly in comparison to other quantities, then we do not expect this to be the case. For such settings one would likely have to use a different drift function than the one used in Theorem 5.3 or move to an approach that avoids the use of the minorization condition [36, 38].

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A Preliminaries

Definition A.1. We say that $X \in \mathbb{R}^{n \times m}$ has a matrix normal distribution with mean $M \in \mathbb{R}^{n \times m}$ and scale matrices $U \in \mathbb{S}_+^n$ and $V \in \mathbb{S}_+^m$ if $\text{vec}(X) \sim \mathcal{N}(\text{vec}(M), V \otimes U)$. We write $X \sim \mathcal{M}(M, U, V)$.

Lemma A.1. If $X_i \in \mathbb{R}^{n \times m_i}$, $m_i \in \{1, 2, \dots\}$, $i = 1, 2, 3$, and $X = [X_1, X_2, X_3] \in \mathbb{R}^{n \times (m_1 + m_2 + m_3)}$ has full column rank, then with $\tilde{X}_i = Q_{X_2} X_i$, $i = 1, 2, 3$,

1. $X_1^\top Q_{X_2} X_1$ is invertible,
2. $X_1^\top Q_{[X_2, X_3]} X_1$ is invertible, and
3. $X_1^\top Q_{[X_2, X_3]} X_1 = \tilde{X}_1^\top Q_{\tilde{X}_3} \tilde{X}_1$.

Proof. We start with 1. Suppose for contradiction that there exists $v \in \mathbb{R}^{m_1} \setminus \{0\}$ such that $X_1^\top Q_{X_2} X_1 v = 0$, which is equivalent to $Q_{X_2} X_1 v = 0$. This can happen either if $X_1 v = 0$, which contradicts the full column rank of X , or if $w = X_1 v$ is a non-zero vector in the column space of X_1 that also lies in the column space of X_2 , which again contradicts the full column rank of X . The proof for 2 is exactly the same as that of 1 but with $[X_2, X_3]$ in place of

X_2 . Point 3 is an immediate consequence the Frisch–Waugh–Lovell theorem [32, Section 2.4], which says among other things that $Q_{\tilde{X}_3}\tilde{X}_1 = Q_{[X_2, X_3]}X_1$. \square

Lemma A.2. *For any $A \in \mathbb{S}_{++}^n$, $B \in \mathbb{S}_+^n$, and invertible $C \in \mathbb{R}^{n \times n}$,*

1. $\text{tr}(C^\top[A + B]^{-1}C) \leq \text{tr}(C^\top A^{-1}C)$,
2. $\|C^\top(A + B)^{-1}C\| \leq \|C^\top A^{-1}C\|$,
3. $|C^\top(A + B)C| \geq |C^\top AC|$, and
4. $|C^\top(A + B)^{-1}C| \leq |C^\top A^{-1}C|$.

Proof. All claims can be reduced to the case where $C = I_n$ by either writing $C^\top(A + B)^{-1}C = (C^{-1}AC^{-\top} + C^{-1}BC^{-\top})^{-1}$ and replacing A and B by $C^{-1}AC^{-\top}$ and $C^{-1}BC^{-\top}$, respectively, or by writing $C^\top(A + B)C = C^\top AC + C^\top BC$ and replacing A and B by $C^\top AC$ and $C^\top BC$, respectively. Assume thus that $C = I_n$. Since $A + B$ is SPD, the eigenvalues of $(A + B)^{-1}$ are the reciprocals of those of $A + B$. But, letting $\lambda_i(A)$ denote the i th eigenvalue in, say, decreasing order, Weyl's inequalities [4, Corollary III.2.2] say $\lambda_i(A + B) \geq \lambda_i(A) + \lambda_n(B) = \lambda_i(A) + \lambda_{\min}(B) \geq \lambda_i(A)$, and hence $\text{tr}([A + B]^{-1}) = \sum_{i=1}^n 1/\lambda_i(A + B) \leq \sum_{i=1}^n 1/\lambda_i(A) = \text{tr}(A^{-1})$, which proves the first claim. The remaining claims follow similarly since the spectral norm is the maximum eigenvalue for SPSP matrices and the determinant is the product of eigenvalues. \square

Lemma A.3. *For any $X \in \mathbb{R}^{n \times p}$, $y \in \mathbb{R}^n$, and $c > 0$,*

$$\|(I_p c + X^\top X)^{-1} X^\top y\| \leq \|(X^\top X)^g X^\top y\|,$$

where superscript g denotes an arbitrary generalized inverse.

Proof. Consider the optimization problem of minimizing $g_c : \mathbb{R}^p \rightarrow [0, \infty)$ defined by

$$g_c(b) := \|y - Xb\|^2 + c\|b\|^2.$$

If $c = 0$, then any b such that $X^\top Xb = X^\top y$ is a solution. Thus, for any generalized inverse, $b_1 = (X^\top X)^g X^\top y$ solves the problem [14, Theorem 9.1.2]. On the other hand, if $c > 0$ then since $Ic + X^\top X$ has full rank, the unique solution is $b_2 = (cI + X^\top X)^{-1} X^\top y$. Now a contradiction arises if for some $c > 0$, $\|b_1\| < \|b_2\|$, which finishes the proof. \square

Lemma A.4. For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{S}_{++}^n$, we have that $B^{-1}A^gB^{-1}$ is a generalized inverse of BAB , where superscript g indicates a generalized inverse.

Proof. We check the definition, namely that $BAB B^{-1}A^gB^{-1}BAB = BAB$. Indeed, using that $AA^gA = A$, $BAB B^{-1}A^gB^{-1}BAB = BAA^gAB = BAB$. \square

B Main results

Proof Lemma 2.1. Let us suppress conditioning on the parameters for simplicity. We have

$$\begin{aligned} f(Y, X) &= f(X)f(Y \mid X) \\ &= f(X)f(y_1, \dots, y_n \mid X) \\ &= f(X)f(y_n \mid y_1, \dots, y_{n-1}, X)f(y_1, \dots, y_{n-1} \mid X) \\ &= \vdots \\ &= f(X) \prod_{t=1}^n f(y_t \mid y_1, \dots, y_{t-1}, X). \end{aligned}$$

Consider an arbitrary term in the product. We have $Y_t = \mathcal{A}^\top Z_t + \mathcal{B}^\top X_t + U_t$. Since Z_t is a function of Y_1, \dots, Y_{t-1} , both Z_t and X_t are fixed when conditioning on X and Y_1, \dots, Y_{t-1} . Thus, the distribution of $Y_t \mid X, Y_1, \dots, Y_{t-1}$ is determined by that of $U_t \mid X, Y_1, \dots, Y_{t-1}$. But Y_1, \dots, Y_{t-1} are functions of X_1, \dots, X_{t-1} and U_1, \dots, U_{t-1} , and $\{U_t\}$ is an i.i.d. sequence independent of $\{X_t\}$, and hence X . Thus, $U_t \mid X, Y_1, \dots, Y_{t-1} \sim \mathcal{N}(0, \Sigma)$, and, consequently, $Y_t \mid X, Y_1, \dots, Y_{t-1} \sim \mathcal{N}(\mathcal{A}^\top Z_t + \mathcal{B}^\top X_t, \Sigma)$. Now the result follows by straightforward algebra and the fact that the distribution of $\{X_t\}$ does not depend on the model parameters. \square

Proof Proposition 2.2. Assuming the posterior is proper, the given expression for the density, up to scaling, follows from routine calculations. We prove the posterior is indeed proper under either of the two sets of conditions. Since

$$f(Y, X \mid \mathcal{A}, \mathcal{B}, \Sigma) = f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X)f(X \mid \mathcal{A}, \mathcal{B}, \Sigma) = f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X)f(X),$$

only the conditional density $f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X)$ matters when deriving the posterior. Under either of the two sets of conditions, X has full column rank so $X^\top X$ is invertible and we may

define $H_X = (X^\top X)^{-1} X^\top$, $P_X = X H_X$, and $Q_X = I_n - P_X$. Let also $E = Y - Z\mathcal{A}$ and use $Q_X + P_X = I_n$ to write

$$\begin{aligned}
& f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X) \\
& \propto |\Sigma|^{-\frac{n}{2}} \text{etr} \left(-\frac{1}{2} [X\mathcal{B} - E]^\top [X\mathcal{B} - E] \Sigma^{-1} \right) \\
& = |\Sigma|^{-\frac{n}{2}} \text{etr} \left(-\frac{1}{2} [X\mathcal{B} - E]^\top (Q_X + P_X) [X\mathcal{B} - E] \Sigma^{-1} \right) \\
& = |\Sigma|^{-\frac{n}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right) \text{etr} \left(-\frac{1}{2} [\mathcal{B} - H_X E]^\top X^\top X [\mathcal{B} - H_X E] \Sigma^{-1} \right). \quad (11)
\end{aligned}$$

The right-most term is a kernel of a matrix normal density for \mathcal{B} with mean $H_X E$ and scale matrices $(X^\top X)^{-1}$ and Σ . Thus, integrating with respect to \mathcal{B} gives,

$$\begin{aligned}
\int f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X) d\mathcal{B} & \propto |\Sigma|^{-\frac{n}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right) (2\pi)^{rp/2} |X^\top X|^{-r} |\Sigma|^p \\
& \propto |\Sigma|^{-\frac{n-p}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right).
\end{aligned}$$

Thus, to show that $f(Y \mid \mathcal{A}, \mathcal{B}, \Sigma, X) f(\alpha) f(\Sigma)$ can be normalized to a proper posterior, we need only show that

$$\iint |\Sigma|^{-\frac{n-p}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right) f(\alpha) f(\Sigma) d\alpha d\Sigma < \infty. \quad (12)$$

Let us consider the two sets of conditions separately, starting with the first. Since

$$\text{tr}(E^\top Q_X E \Sigma^{-1}) = \text{tr}(\Sigma^{-1/2} E^\top Q_X E \Sigma^{-1/2}) \geq 0,$$

we can upper bound the integrand in (12) by

$$|\Sigma|^{-\frac{n-p}{2}} f(\alpha) f(\Sigma) = |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} D \right) f(\alpha),$$

which since we are assuming that $n - p + a - r - 1 > r - 1$, i.e. that $n + a > 2r + p$ and that D is SPD, is the product of a proper inverse Wishart and a proper density for α . This finishes the proof for the first set of conditions.

For the second set of conditions, notice that for (12) it suffices, since D is SPSD, and hence $f(\Sigma)$ and $f(\alpha)$ both bounded, to show that

$$\iint |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right) d\alpha d\Sigma < \infty.$$

Let $\tilde{Y} = Q_X Y$ and $\tilde{Z} = Q_X Z$ so that $Q_X E = \tilde{Y} - \tilde{Z} \mathcal{A}$. Using the same decomposition as before we have for the last integrand

$$\begin{aligned} & |\Sigma|^{-\frac{n+a-p}{2}} \text{etr} \left(-\frac{1}{2} E^\top Q_X E \Sigma^{-1} \right) \\ &= |\Sigma|^{-\frac{n+a-p-qr}{2}} \text{etr} \left(-\frac{1}{2} \tilde{Y}^\top Q_{\tilde{Z}} \tilde{Y} \Sigma^{-1} \right) |\Sigma|^{-\frac{qr}{2}} \text{etr} \left(-\frac{1}{2} [\mathcal{A} - H_{\tilde{Z}} \tilde{Y}]^\top \tilde{Z}^\top \tilde{Z} [\mathcal{A} - H_{\tilde{Z}} \tilde{Y}] \Sigma^{-1} \right). \end{aligned}$$

Under the second set of assumptions, the last line is proportional to the product of an inverse Wishart density for Σ with scale matrix $\tilde{Y}^\top Q_{\tilde{Z}} \tilde{Y}$ and $n + a - p - qr - r - 1$ degrees of freedom and a matrix normal density for \mathcal{A} with mean $H_{\tilde{Z}} \tilde{Y}$ and scale matrices $(\tilde{Z}^\top \tilde{Z})^{-1}$ and Σ , and hence integrable. The assumption that $[Y, X, Z]$ has full column ensures that, by Lemma A.1, $\tilde{Y}^\top Q_{\tilde{Z}} \tilde{Y}$ and $\tilde{Z}^\top \tilde{Z}$ are positive definite matrices. \square

Proof of Lemma 3.1. The full conditional distribution of \mathcal{B} is immediate from dropping terms not depending on \mathcal{B} in (11). Consider next the integrand in (12). The first term in the exponential is $\text{tr}([Y - Z\mathcal{A}]^\top Q_X [Y - Z\mathcal{A}] \Sigma^{-1}) = \|Q_X(Y - Z\mathcal{A})\Sigma^{-1/2}\|_F^2 = \|(\Sigma^{-1/2} \otimes I_n)(\text{vec}(Q_X Y) - \text{vec}(Q_X Z\mathcal{A}))\|^2 = \|(\Sigma^{-1/2} \otimes I_n)(\text{vec}(Q_X Y) - [I_r \otimes Q_X Z]\alpha)\|^2$. Thus, the log of the integrand is quadratic as a function of α , with Hessian $-B = -\Sigma^{-1} \otimes Z^\top Q_X Z - C$ and gradient $-(\Sigma^{-1} \otimes Z^\top Q_X) \text{vec}(Q_X Y) - Cm$, which implies the desired distribution for $\alpha \mid \Sigma, Y$. Finally, the distribution of $\Sigma \mid \alpha, Y$ is immediate from dropping terms in the integrand in (12) not depending on Σ . \square

Proof Lemma 3.2. Assume $\alpha^h, \mathcal{B}^h, \Sigma^h$, $h = 1, 2, \dots$ are generated by the collapsed Gibbs sampler in Algorithm 1 started at some point $\theta^0 \in \Theta$. The equality follows from showing that $\xi^h = (\alpha^h, \Sigma^h)$ and θ^h are co-de-initializing Markov chains [40, Corollary 1]. That they are both Markov chains is clear from the construction of the updates in Algorithm 1. That θ^h is de-initializing for ξ^h , i.e. that the distribution of $\xi^h \mid \theta^h, \xi^0$ does not depend on ξ^0 , is immediate from that ξ^h is a function (coordinate projection) of θ^h . The other direction, that ξ^h is de-initializing for θ^h , is by construction of the algorithm: since ξ^h is a coordinate projection of θ^h , the distribution of $\theta^h \mid \xi^h, \theta^0$ is determined by that of $\mathcal{B}^h \mid \xi^h, \theta^0$, and the distribution from which this value is drawn (line 4, Algorithm 1) does not depend on θ^0 . Similarly, notice that the distribution of $\xi^h \mid \xi^{h-1}$ is the same as $\xi^h \mid \alpha^{h-1}$ by construction of the algorithm. Thus, α^{h-1} is de-initializing for ξ^h and the inequality follows [40, Theorem 1]. \square

B.1 Inadequacy of Drift Function in Theorem 4.3

Proposition B.1. *For the $L = L_n(Y, X)$ defined in Lemma 4.1 it holds for some $c_1 > 0$ and $c_2 > 0$, depending on the hyperparameters but not the data, that*

$$\|\hat{\mathcal{A}}\|_F^2 \leq L_n(Y, X) \leq c_1 \|\hat{\mathcal{A}}\|_F^2 + c_2.$$

In particular, if $\hat{\mathcal{A}}$ is consistent, then $L_n(Y, X) = O_{\mathbb{P}}(1)$.

Proof. Since C^{-1} is SPD, the term $\text{tr}(C^{-1})$ is positive, and so dropping it and the term $\|C^{-1}\| \|Cm\|$ in the expression for $L_n(Y, X)$ gives

$$L_n(Y, X) > \|C^{-1/2}\|^2 \|C^{1/2}\hat{\alpha}\|^2 = \lambda_{\min}(C)^{-1} \|C^{1/2}\hat{\alpha}\|^2.$$

On the other hand, using that $(r_1 + r_2)^2 \leq 2r_1^2 + 2r_2^2$ for any real numbers r_1 and r_2 ,

$$L_n(Y, X) \leq 2\|C^{-1}\|^2 \|Cm\|^2 + 2\|C^{-1}\| \|C^{1/2}\hat{\alpha}\|^2 + \text{tr}(C^{-1}).$$

Now notice that $C - \lambda_{\min}(C)I_{qr^2}$ and $\lambda_{\max}(C)I_{qr^2} - C$ are both SPSPD, and therefore

$$\lambda_{\min}(C) \|\hat{\alpha}\|^2 \leq \hat{\alpha}^\top C \hat{\alpha} \leq \lambda_{\max}(C) \|\hat{\alpha}\|^2.$$

Thus, since $0 < \lambda_{\min}(C) < \lambda_{\max}(C) < \infty$ and $\hat{\alpha} = \text{vec}(\hat{\mathcal{A}})$, we are done. \square

Proposition B.2. *If, almost surely as $n \rightarrow \infty$,*

$$n\|Q_X Z\|^2 / \|Q_{[Z, X]} Y\|^2 \rightarrow \infty,$$

then the $\varepsilon = \varepsilon_n$ in Theorem 4.3 tends to zero almost surely as $n \rightarrow \infty$. In particular, $\varepsilon_n \rightarrow 0$ almost surely if $n^{-1}Y^\top Q_{[Z, X]} Y$ and $n^{-1}Z^\top Q_X Z$ have positive definite limits almost surely.

Proof. Recall from Lemma 4.2 the definition of $\varepsilon = \varepsilon_n$, $c = n + a - p - r - 1$, and $c_1 = (\|Q_X Y\| + \|Q_X Z\| \sqrt{T})^2$. It suffices to show that $\zeta_n := \varepsilon_n^{2n/c} \rightarrow 0$ almost surely since $2n/c \rightarrow 2$.

We have

$$\zeta_n = \left[\frac{|D + Y^\top Q_{[Z, X]} Y|}{|D + I_r c_1|} \right]^n.$$

By arguments similar to those in the proof of Lemma 4.2, using the Frisch–Waugh–Lovell theorem [32, Section 2.4] and that the maximum eigenvalue of a SPSPD matrix is its spectral

norm, we have that $\|Q_{[Z,X]}Y\|^2 I_r - Y^\top Q_{[Z,X]}Y$ is SPSD, and hence Lemma A.2.3 gives $|D + Y^\top Q_{[Z,X]}Y| \leq |D + I_r \|Q_{[Z,X]}Y\|^2|$. Thus,

$$\zeta_n \leq \left[\frac{|D + I_r \|Q_{[Z,X]}Y\|^2|}{|D + I_r c_1|} \right]^n,$$

and subsequently, using that $c_1 \geq \|Q_X Y\|^2 + \|Q_X Z\|^2 T$ in another application of Lemma A.2.3,

$$\zeta_n \leq \left[\frac{|D + I_r \|Q_X Y\|^2|}{|D + I_r \|Q_X Y\|^2 + I_r \|Q_X Z\|^2 T|} \right]^n.$$

Let $\kappa_1 \geq \kappa_2 \cdots \geq \kappa_r$ denote the eigenvalues of D , then the last upper bound can be written

$$\begin{aligned} \left(\prod_{i=1}^r \frac{\kappa_i + \|Q_{[Z,X]}Y\|^2}{\kappa_i + \|Q_X Y\|^2 + \|Q_X Z\|^2 T} \right)^n &= \left(\prod_{i=1}^r \left[\frac{\kappa_i + \|Q_X Y\|^2}{\kappa_i + \|Q_{[Z,X]}Y\|^2} + \frac{\|Q_X Z\|^2 T}{\kappa_i + \|Q_{[Z,X]}Y\|^2} \right] \right)^{-n} \\ &\leq \left(\prod_{i=1}^r \left[1 + \frac{\|Q_X Z\|^2 T}{\kappa_i + \|Q_{[Z,X]}Y\|^2} \right] \right)^{-n} \\ &\leq \left(1 + \frac{\|Q_X Z\|^2 T}{\kappa_1 + \|Q_{[Z,X]}Y\|^2} \right)^{-rn}, \end{aligned}$$

where the penultimate step uses that, again by an application of the Frisch–Waugh–Lovell theorem [32, Section 2.4], $Y^\top Q_X Y - Y^\top Q_{[Z,X]}Y$ is SPSD, which in turn implies $\|Q_X Y\| \geq \|Q_{[Z,X]}Y\|$, and the last step uses that the i th term in the product in the second line is made no larger by replacing κ_i by κ_1 .

Now, for any n we have that the $L = L_n(Y, X)$ in Lemma 4.2 is lower bounded by $\text{tr}(C^{-1}) > 0$, and hence every $T = T_n$ greater than $2L/(1 - \lambda)$ is also lower bounded by $\text{tr}(C^{-1})$, for every n . Thus,

$$\zeta_n \leq \left(1 + \frac{1}{n} \left[\frac{\|Q_X Z\|^2 \text{tr}(C^{-1})}{\kappa_1/n + \|Q_{[Z,X]}Y\|^2/n} \right] \right)^{-rn},$$

and consequently, since $\lim_{n \rightarrow \infty} (1 + r_1/n)^{-rn} \rightarrow e^{-r_1 r}$ for any real number r_1 , $\zeta_n \rightarrow 0$ if $n\|Q_X Z\|^2/\|Q_{[Z,X]}Y\|^2 \rightarrow \infty$. \square