Supplementary Material for "A unified method for multivariate mixed-type response regression"

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Abstract

This note contains proofs and additional details for the article "A unified method for multivariate mixed-type response regression".

1 Moment calculations

We compute the moments in Example 2 that include conditionally quasi-Poisson distributed responses. We use repeatedly that the moment generating function for $\mathcal{N}(\mu, \sigma^2)$ is $M(t) = \exp(t\mu + t^2\sigma^2/2)$. First, $\mathbb{E}(Y_j) = \mathbb{E}[\mathbb{E}(Y_j \mid W_j)] = \mathbb{E}[\exp(W_j)] = \exp(X_j^\mathsf{T}\beta + \Sigma_{jj}/2)$. Similarly, for $j = 3, 4$,

$$
\begin{aligned}
\mathbb{E}[Y_j^2] &= \mathbb{E}[\mathbb{E}(Y_j^2 \mid W_j)] \\
&= \mathbb{E}[\operatorname{var}(Y_j \mid W_j) + \mathbb{E}(Y_j \mid W_j)^2] \\
&= \mathbb{E}[\psi_j \exp(W_j)] + \mathbb{E}[\exp(2W_j)] \\
&= \psi_j \exp(X_j^\mathsf{T}\beta + \Sigma_{jj}/2) + \exp(2X_j^\mathsf{T}\beta + 2\Sigma_{jj}),
\end{aligned}
$$

where we used $2W_j \sim \mathcal{N}(2X_j^\mathsf{T}\beta, 4\Sigma_{jj})$. It follows that, for $j = 3, 4$,

$$
\begin{aligned}
\operatorname{var}(Y_j) &= \mathbb{E}(Y_j^2) - \mathbb{E}(Y_j)^2 \\
&= \psi_j \exp(X_j^\mathsf{T}\beta + \Sigma_{jj}/2) + \exp(2X_j^\mathsf{T}\beta + 2\Sigma_{jj}) - \exp(2X_j^\mathsf{T}\beta + \Sigma_{jj}) \\
&= \exp(2X_j^\mathsf{T}\beta + \Sigma_{jj}) \left[\psi_j \exp(-X_j^\mathsf{T}\beta - \Sigma_{jj}/2) + \exp(\Sigma_{jj}) - 1\right].
\end{aligned}
$$

To get the covariance $\operatorname{cov}(Y_j, Y_k)$ for $j = 1, 2$ and $k = 3, 4$, observe that since Y_j and Y_k are uncorrelated given W,

$$
\begin{aligned}
\operatorname{cov}(Y_j, Y_k) &= \operatorname{cov}[\mathbb{E}(Y_j \mid W), \mathbb{E}(Y_k \mid W)] \\
&= \operatorname{cov}[W_j, \exp(W_k)] \\
&= \mathbb{E}[W_j \exp(W_k)] - X_j^\mathsf{T}\beta \exp(X_k^\mathsf{T}\beta + \Sigma_k/2),
\end{aligned}
$$

and

$$
\mathbb{E}[W_j \exp(W_k)] = \mathbb{E}\left[\frac{\partial}{\partial t_j} \exp(t_j W_j + t_k W_k) \mid_{t_j=0, t_1=1}\right].
$$

Now, for (t_j, t_k) in a neighborhood of $(0, 1)$,

$$
\left|\frac{\partial}{\partial t_j} \exp(t_j W_j + t_k W_k)\right| = |W_j \exp(t_j W_j + t_k W_k)| \le \exp(|W_j|) \exp(|W_j| + |W_k|),
$$

which has finite expectation since W_j and W_k are jointly normal. Thus, we can move the derivative outside the expectation to get

$$
\begin{aligned}
\mathbb{E}[W_j \exp(W_k)] &= \frac{\partial}{\partial t_j} \mathbb{E}\left[\exp(t_j W_j + t_k W_k)\right] \mid_{t_j=0, t_1=1} \\
&= \frac{\partial}{\partial t_j} \exp\left(t_j X_j^\mathsf{T}\beta + t_k X_k^\mathsf{T}\beta + t_j^2 \Sigma_{jj}^2/2 + t_j t_k \Sigma_{jk} + t_k^2 \Sigma_{kk}/2\right) \mid_{t_j=0, t_1=1} \\
&= (X_j^\mathsf{T}\beta + \Sigma_{jk}) \exp\left(X_k^\mathsf{T}\beta + \Sigma_{kk}/2\right)
\end{aligned}
$$

where in the second equality we used the moment generating function for

$$
t_j W_j + t_k W_k \sim \mathcal{N}\left(t_j X_j^\mathsf{T}\beta + t_k X_k^\mathsf{T}\beta, t_j^2 \Sigma_{jj}^2 + 2 t_j t_k \Sigma_{jk} + t_k^2 \Sigma_{kk}\right).
$$

Putting things together, we have

$$\begin{aligned}\operatorname{cov}(Y_j, Y_k) &= (X_j^\intercal \beta + \Sigma_{jk}) \exp\left(X_k^\intercal \beta + \Sigma_{kk}/2\right) - X_j^\intercal \beta \exp\left(X_k^\intercal \beta + \Sigma_{kk}/2\right) \\ &= \Sigma_{j,k} \exp\left(X_k^\intercal \beta + \Sigma_{kk}/2\right).\end{aligned}$$

Lastly, we compute

$$\begin{aligned}\operatorname{cov}(Y_3, Y_4) &= \operatorname{cov}[\exp(W_3), \exp(W_4)] \\ &= \mathbb{E}[\exp(W_3)\exp(W_4)] - \mathbb{E}[\exp(W_3)]\mathbb{E}[\exp(W_4)] \\ &= \mathbb{E}[\exp(W_3 + W_4)] - \exp(X_3^\intercal \beta + \Sigma_{33}/2 + X_4^\intercal \beta + \Sigma_{44}/2) \\ &= \exp(X_3^\intercal \beta + \Sigma_{33}/2 + X_4^\intercal \beta + \Sigma_{44}/2) \left[\exp(\Sigma_{34}) - 1\right],\end{aligned}$$

where, as before, the last step used the moment generating function for the normal variable $W_3 + W_4$.

2 Proofs

Lemma 2.1. *Let $W \sim \mathcal{N}(\mu, \sigma^2)$ and Φ denote the standard normal cumulative distribution function, then*

$$\mathbb{E}[\Phi(W)] = \Phi(\mu/\sqrt{1+\sigma^2}))$$

Proof. This is well known and is, for example, essentially Equation 10 in McCulloch (2008). □

Let $\phi_\sigma(u, v)$ be the bivariate normal density mean zero, unit variances, and covariance σ.

Lemma 2.2. *The function h defined by*

$$h(\sigma, c_1, c_2) = \frac{\partial}{\partial \sigma} \iint I(u > c_1) I(v > c_2) \phi_\sigma(u, v) \mathrm{d}u \mathrm{d}v$$

is strictly positive and continuous on $(-1, 1) \times \mathbb{R} \times \mathbb{R}$.

Proof. We first prove h is strictly positive. Let U and V denote random variables with density $\phi_\sigma(u, v)$. By using that $U \mid V \sim \mathcal{N}(\sigma V, 1 - \sigma^2)$ and letting Φ denote the standard normal cumulative distribution function,

$$\begin{aligned}
\mathbb{E}[I(U > c_1)I(V > c_2)] &= \mathbb{E}\left\{I(V > c_2)\mathbb{E}\left[I(U > c_1) \mid V\right]\right\} \\
&= \mathbb{E}\left[I(V > c_2)\mathsf{P}(U > c_1 \mid V)\right] \\
&= \mathbb{E}\left\{I(V > c_2)\left[1 - \Phi\left(\frac{c_1 - \sigma V}{\sqrt{1-\sigma^2}}\right)\right]\right\} \\
&= \mathsf{P}(V > c_2) - \mathbb{E}\left[I(V > c_2)\Phi\left(\frac{c_1 - \sigma V}{\sqrt{1-\sigma^2}}\right)\right].
\end{aligned}$$

Denote the expectation in the last line by $J_1(\sigma, c_1, c_2)$; we want to show that $\partial J_1(\sigma, c_1, c_2)/\partial\sigma < 0$. Differentiating under the integral we find

$$\int_{c_2}^{\infty} \phi\left(\frac{c_1 - \sigma v}{\sqrt{1-\sigma^2}}\right) \frac{c_1\sigma - v}{(1-\sigma^2)^{3/2}}\phi(v)\mathrm{d}v,$$

where ϕ is the standard normal probability density function. Differentiating under the integral is permissible because $\phi(\cdot)$, $1/(1-\sigma^2)^{3/2}$, and σ are all bounded on small enough neighborhoods of any $\sigma \in (-1, 1)$. Now, if $c_2 \geq \sigma c_1$ the integrand is negative on the set of integration we are done. Suppose thus $c_2 < \sigma c_1$, and note

$$\begin{aligned}
\mathbb{E}\left[\Phi\left(\frac{c_1 - \sigma V}{\sqrt{1-\sigma^2}}\right)\right] &= \mathbb{E}\left[I(V > c_2)\Phi\left(\frac{c_1 - \sigma V}{\sqrt{1-\sigma^2}}\right)\right] + \mathbb{E}\left[I(V \leq c_2)\Phi\left(\frac{c_1 - \sigma V}{\sqrt{1-\sigma^2}}\right)\right] \\
&= J_1(\sigma, c_1, c_2) + J_2(\sigma, c_1, c_2),
\end{aligned}$$

where J_2 is defined by the last equality. Lemma 2.1 says the left hand side is

$$\mathbb{E}\left[\Phi\left(\frac{c_1/\sqrt{1-\sigma^2}}{\sqrt{1+\sigma^2/(1-\sigma^2)}}\right)\right] = \Phi(c_1).$$

Thus, differentiating both sides with respect to σ gives

$$0 = \frac{\partial}{\partial\sigma}J_1(\sigma, c_1, c_2) + \frac{\partial}{\partial\sigma}J_2(\sigma, c_1, c_2),$$

so it suffices to show the last term is positive. But by argument similar to when differentiating J_1,

$$\frac{\partial}{\partial\sigma}J_2(\sigma,c_1,c_2)=\int_{-\infty}^{c_2}\phi\left(\frac{c_1-\sigma v}{\sqrt{1-\sigma^2}}\right)\frac{c_1\sigma-v}{(1-\sigma^2)^{3/2}}\phi(v)\mathrm{d}v,$$

which is positive since the integrand is positive on the set of integration. Finally, that $h(\sigma,c_1,c_2)$ is continuous follows from the dominated convergence theorem since the integrand is bounded on small enough neighborhoods around any interior point of $(-1,1)\times\mathbb{R}\times\mathbb{R}$. $\square$

Lemma 2.3. *If $f,g:\mathbb{R}\to\mathbb{R}$ are increasing and non-constant, and*

$$\iint|f(u)||g(v)|\phi_\sigma(u,v)\mathrm{d}u\mathrm{d}v<\infty$$

for all $\sigma\in(-1,1)$, then $s:(-1,1)\to\mathbb{R}$ defined by

$$s(\sigma)=\iint f(u)g(v)\phi_\sigma(u,v)\mathrm{d}u\mathrm{d}v$$

is strictly increasing.

Proof. First observe that since the marginal densities do not depend on u and v, we may replace f and g by $f-f(0)$ and $g-g(0)$; that is, we assume without loss of generality that $f(0)=g(0)=0$.

For every $n=1,2,\ldots$ and $i=0,\ldots,n2^{n+1}=m_n$, let $a_{ni}=-n+i/2^n$. Then for every n,

$$-n=a_{n0}<\cdots<a_{m_n/2}=0<\cdots<a_{m_n}=n.$$

and the distance between consecutive a_{ni} is $1/2^n$. Define

$$f_n^-(u)=f(a_{n0})+\sum_{i=1}^{m_n/2}[f(a_{ni})-f(a_{n(i-1)})]I(u\geq a_{n(i-1)}),$$

$$f_n^+(u)=\sum_{i=m_n/2+1}^{m_n}[f(a_{ni})-f(a_{n(i-1)})]I(u\geq a_{ni}),$$

and

$$f_n=f_n^-+f_n^+.$$

Note that if $u \geq -1/2^n$, then $f_n^-(u) = f(a_{m_n/2}) = 0$, and if $u < 1/2^n$, then $f_n^+(u) = 0$. Thus, $f_n(0) = 0$ for every n and at most one of f_n^- and f_n^+ are non-zero for the same u. If $u < 0$, then $f_n(u) = f_n^-(u) = f(a_{nj})$ where a_{nj}, $j = j(n)$, is the smallest a_{ni} greater than u. Since f is increasing, $0 \geq f(a_{nj}) \geq f(u)$ and if u is a point of continuity of f, $f(a_{nj}) \downarrow f(u)$. Because f is increasing, it has at most countably many points of discontinuity and hence, for Lebesgue-almost every $u < 0$, $f_n(u) \downarrow f(u)$. A similar argument shows $0 \leq f_n(u) \uparrow f(u)$ for Lebesgue-almost every $u > 0$. Thus, $|f_n| \leq |f|$ and $f_n \to f$ for Lebesgue-almost every u. For simplicity, we write

$$f_n(u) = f(-n) + \sum_{i=1}^{m_n} d_{ni}^f I(u \geq c_{ni}),$$

where $d_{ni} = f(a_{ni}) - f(a_{n(i-1)})$ and $c_{ni} = a_{n(i-1)}$ for $i = 1, \ldots, m_{n/2}$ and $c_{ni} = a_{ni}$ for $i = m_n/2 + 1, \ldots, m_n$. Note that the d_{ni}^f are non-negative since f is increasing.

Define h_n as f_n but with g playing the role of f, so that

$$h_n(v) = g(-n) + \sum_{i=1}^{m_n} d_{ni}^g I(v \geq c_{ni}).$$

Now with $s_n(\sigma) = \iint f_n(u)h_n(v)\phi_\sigma(u,v)\mathrm{d}u\mathrm{d}v$ and $\sigma_1 > \sigma_2$,

$$s_n(\sigma_1) - s_n(\sigma_2) = \sum_{i=1}^{m_n}\sum_{j=1}^{m_n} d_{ni}^f d_{nj}^g \iint I(u \geq c_{ni})I(v \geq c_{nj})[\phi_{\sigma_1}(u,v) - \phi_{\sigma_2}(u,v)]\mathrm{d}u\mathrm{d}v$$

Lemma 2.2 implies all summands are non-negative; to show some summands are strictly positive, note that since f is non-constant, we can find $-\infty < l_f < u_f < -\infty$ such that

$$\lim_{u\uparrow l_f} f(u) \leq \lim_{u\downarrow l_f} f(u) < \lim_{u\uparrow u_f} f(u) \leq \lim_{u\downarrow u_f} f(u).$$

Similarly, we can find $l_g < u_g$ with the same property for g. Now, since all summands are non-negative, the sum is made no smaller by only retaining some summands. Specifically, let us retain only those i for which both a_{ni} and $a_{n(i-1)}$ are in $[l_f, u_f]$ and those j for which both a_{nj} and $a_{n(j-1)}$ are in $[l_g, u_g]$.

For such summands, by the mean value theorem, applicable owing to Lemma 2.2,

$$\iint I(u \geq c_{ni})I(v \geq c_{nj})[\phi_{\sigma_1}(u,v) - \phi_{\sigma_2}(u,v)]\mathrm{d}u\mathrm{d}v = h(\tilde{\sigma}, c_{ni}, c_{nj})$$

for some $\tilde{\sigma}$ between σ_1 and σ_2. By Lemma 2.2, h is continuous and strictly positive on the compact $[\sigma_1, \sigma_2] \times [l_f, u_f] \times [l_g, u_g]$, and hence attains a strictly positive infimum there, say $\epsilon > 0$. Thus,

$$s_n(\sigma_1) - s_n(\sigma_2) \geq \epsilon \sum_i \sum_j d^f_{ni} d^g_{nj} = \epsilon \left[\sum_i d^f_{ni} \right] \left[\sum_j d^g_{nj} \right],$$

where the sums are over the retained indexes, which are consecutive. Consider the first sum: it is the sum of jumps of f_n in $[l_f, u_f]$, and hence it tends to $\lim_{u \uparrow u_f} f(u) - \lim_{u \downarrow l_f} f(u) > 0$. Similarly, the second sum tends to $\lim_{v \uparrow u_g} g(v) - \lim_{v \downarrow l_g} g(v) > 0$. Thus, we can find a $c > 0$ such that for all n large enough, $s_n(\sigma_1) - s_n(\sigma_2) \geq c$, and the proof is completed by sending n to infinity and applying the dominated convergence theorem – the dominating function can be $|fg|\phi_{\sigma_i} \geq |f_n||h_n|\phi_{\sigma_i}$, $i = 1, 2$. $\square$

Proof of Lemma 2.1. By a change of variables, the fist integral is

$$\int g(\mu_1 + \sigma_1 u)\phi(u)\mathrm{d}u$$

where ϕ is the standard normal density. For $\mu_1 > \mu_1'$,

$$\begin{aligned} \int g(\mu_1 + \sigma_1 u)\phi(u)\mathrm{d}u - \int g(\mu_1' + \sigma_1 u)\phi(u)\mathrm{d}u &= \int [g(\mu_1 + \sigma_1 u) - g(\mu_1' + \sigma_1 u)]\phi(u)\mathrm{d}u \\ &\geq 0 \end{aligned}$$

since the integrand is non-negative due to g being increasing. Moreover, equality holds if and only if $g(\mu_1 + \sigma_1 u) = g(\mu_1' + \sigma_1 u)$ for Lebesgue-almost every u. But since g is increasing and non-constant, we can find a point s such that g is strictly greater on (s, ∞) than on $(-\infty, s)$. Thus, for all u such that $\mu_1' + \sigma_1 u < s < \mu_1 + \sigma_1 u$, which is a set of positive Lebesgue measure since $\mu_1 > \mu_1'$, it holds that $g(\mu_1 + \sigma_1 u) > g(\mu_1' + \sigma_1 u)$, and this proves the first claim.

To prove the second claim, make another change of variables to get that the integral is

$$\int g(\mu_1 + \sigma_1 u_1)h(\mu_2 + \sigma_2 u_2)\phi_C(u)\,\mathrm{d}u,$$

where ϕ_C is the bivariate normal density with the covariance matrix C that has ones on the diagonal and $\rho = \sigma/(\sigma_1\sigma_2)$ on the off-diagonal; that is, C is the correlation matrix corresponding to Σ. Since

$u_1 \mapsto g(\mu_1 + \sigma_1 u_1)$ and $u_2 \mapsto h(\mu_2 + \sigma_2 u_2)$ are increasing and non-constant because g and h are, Lemma 2.3 says the integral in the last display is strictly increasing in ρ, and from this the claim follows since σ_1 and σ_2 are strictly positive. $\square$

Proof of 2.2. We first show that distinct parameters give distinct first and second moments of the elements of $\mathcal{Y}$. To this end, recall from Example 2 that $\mathbb{E}(Y_{i,j}) = X_{i,j}^{\mathsf{T}}\beta$ and $\mathrm{var}(Y_{i,j}) = \psi_j + \Sigma_{jj}$ if $Y_{i,j}$ is normal; and if it is conditionally Poisson, then $\mathbb{E}(Y_{i,j}) = \exp(X_{i,j}^{\mathsf{T}}\beta + \Sigma_{jj}/2)$ and

$$\begin{aligned}
\mathbb{E}(Y_{i,j}^2) &= \mathbb{E}[\mathbb{E}(Y_{i,j}^2 \mid W_i)] \\
&= \mathbb{E}[\mathrm{var}(Y_{i,j} \mid W_i) + \mathbb{E}(Y_{i,j} \mid W_i)^2] \\
&= \mathbb{E}[\exp(W_i)] + \mathbb{E}[\exp(2W_i)] \\
&= \exp(X_{i,j}^{\mathsf{T}}\beta + \Sigma_{jj}/2) + \exp(2X_{i,j}^{\mathsf{T}}\beta + 2\Sigma_{jj}).
\end{aligned}$$

Recall also from Example 3 that, owing to Lemma 2.1, $\mathbb{E}(Y_{i,j})$ is strictly increasing in $X_{i,j}^{\mathsf{T}}\beta$. Thus, the first and second moments of the elements of $\mathcal{Y}$ corresponding to pairs (β, Σ) and (β_*, Σ_*) are the same only if

$$X_{i,j}^{\mathsf{T}}\beta = X_{i,j}^{\mathsf{T}}\beta_* \text{ and } \psi_j + \Sigma_{jj} = \psi_j + \Sigma_{*jj}$$

for every i and j corresponding to normal responses;

$$\exp(X_{i,j}^{\mathsf{T}}\beta + \Sigma_{jj}/2) = \exp(X_{i,j}^{\mathsf{T}}\beta_* + \Sigma_{*jj}/2) \text{ and } \exp(2X_{i,j}^{\mathsf{T}}\beta + 2\Sigma_{jj}) = \exp(2X_{i,j}^{\mathsf{T}}\beta_* + 2\Sigma_{*jj})$$

for every i and j corresponding to conditionally Poisson responses; and $X_{i,j}^{\mathsf{T}}\beta = X_{i,j}^{\mathsf{T}}\beta_*$ for every i and j corresponding to Bernoulli responses. Since the exponential function is invertible, if $\mathcal{X} = [X_1^{\mathsf{T}}, \ldots, X_n^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{rn \times p}$ has full column rank, this can happen only if $\beta = \beta_*$ and $\Sigma_{jj} = \Sigma_{*jj}$ for every j. Finally, the off-diagonal elements of Σ are identifiable by Lemma 2.1 since the link functions are strictly increasing. $\square$

3 Comparison to existing software

Suppose there are r conditionally Poisson-distributed responses, each with its own intercept. Specifically, for $j = 1, \dots, r$ and independently for $i = 1, \dots, n$,

$$Y_{i,j} \mid W_i \overset{indep.}{\sim} \mathrm{Poi}(W_{i,j}),\;\; W_i \sim \mathcal{N}(\beta, \Sigma),\;\; (\beta, \Sigma) \in \mathbb{R}^q \times \mathbb{S}^r_{++}.$$

This model is equivalent to a generalized linear mixed model for $[Y_{1,1}, Y_{1,2}, \dots, Y_{n,r}]^\mathsf{T} \in \mathbb{R}^{rn}$, the vector of all responses, with linear predictor

$$\eta = (1_n \otimes I_r)\beta + U,$$

where the random effects vector $U \sim \mathcal{N}(0, I_n \otimes \Sigma)$. Even with these simplifications of the model, it is not clear that common software can fit it: the Kronecker structure is supported by neither the `GLIMMIX` procedure in SAS (Schabenberger, 2005) nor any of the R functions `glmer` from the package `lme4` (Bates et al., 2015), `glmmPQL` from the package `MASS` (Venables and Ripley, 2002), `glmmTMB` from the package with the same name (Brooks et al., 2017), or `glmm` from the package with the same name (Knudson et al., 2020). Some of the packages can fit this model if Σ is constrained to be diagonal since that corresponds to including a separate random effect for each of the observed rn responses and then constraining some of the variances of those random effects to be equal. However, a diagonal Σ is equivalent to assuming all responses are independent, and hence is typically not an interesting alternative. An arguably more reasonable alternative when faced with these data, one which all of the mentioned software packages support, is to treat $Y_{i,1}, \dots, Y_{i,r}$ as observations from the same cluster and model within-cluster dependence by including a shared random effect. That is, by considering the linear predictor

$$\eta = (1_n \otimes I_r)\beta + (I_n \otimes 1_r)U,$$

where $U \sim \mathcal{N}(0, \sigma^2 I_n)$. This implies the covariance

$$\mathrm{cov}(\eta) = I_n \otimes \sigma^2 1_r 1_r^\mathsf{T},$$

which is equivalent to taking $\Sigma = \sigma^2 1_r 1_r^\mathsf{T}$ in our model. We expect that if this structure is correct, then our method should give coefficient estimates similar to those of `glmmPQL`.

4 A quasi-Poisson distribution

Recall, we say a response Y_j has conditional quasi-Poisson moments if $\mathbb{E}(Y_j \mid W) = \exp(W_j)$ and $\mathrm{var}(Y_j \mid W) = \psi_j \exp(W_j)$ for $\psi_j > 0$. To generate such responses, notice that if $\tilde{Y}_j \mid W$ is Poisson with parameter $\psi_j^{-1} \exp(W_j)$, then $Y_j = \psi_j \tilde{Y}_j$ satisfies $\mathbb{E}(Y_j \mid W) = \exp(W_j)$ and

$$\mathrm{var}(Y_j \mid W) = \mathrm{var}(\psi_j \tilde{Y}_j \mid W) = \psi_j^2 \psi_j^{-1} \exp(W_j) = \psi_j \exp(W_j),$$

as desired. That is, conditionally quasi-Poisson responses can be generated by scaling Poisson responses. Notably, the quasi-Poisson responses will in general not be integer-valued.

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