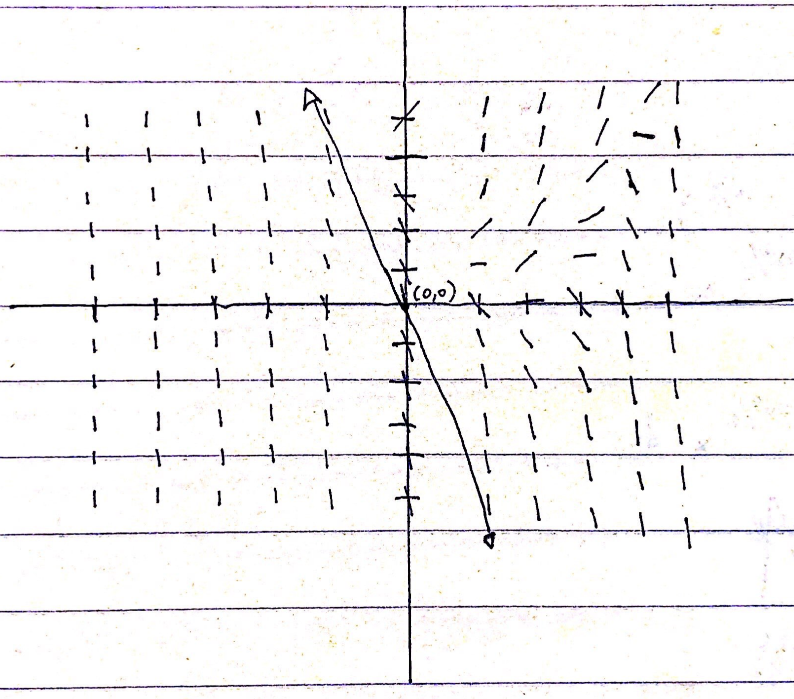
1.) $y' = y - (x-2)^2$ over the region $[-5, 5] \times [-5, 5]$

$\Rightarrow f(x, y) = c$

$\Rightarrow y - (x-2)^2 = c$



2.) $y' - 3xy - xe^{-x^2} = 0$

<u>integrating factors</u> :

* $y' + p(x)\,y = q(x)$

$\Rightarrow\; y' - 3xy = xe^{-x^2}$

$\Rightarrow\; \mu(x) = e^{\int -3x\,dx} = e^{\frac{-3x^2}{2}}$ //

$\Rightarrow\; e^{\frac{-3x^2}{2}} y' - 3e^{\frac{-3x^2}{2}} xy = xe^{-x^2}\left(e^{\frac{-3x^2}{2}}\right)$

$\Rightarrow\; e^{\frac{-3x^2}{2}} y' - 3e^{\frac{-3x^2}{2}} xy = e^{\frac{-5x^2}{2}} x$

$\Rightarrow\; \int \left(e^{\frac{-3x^2}{2}} y\right)' dx = \int e^{\frac{-5x^2}{2}} x \, dx$

u-sub: $\Rightarrow\; \int e^{\frac{-5x^2}{2}} x \, dx \qquad u = \frac{-5x^2}{2}$

$\Rightarrow\; -\frac{1}{5}\int e^{u}\,du \;\Rightarrow\; -\frac{1}{5}e^{u}$

$\Rightarrow\; -\frac{1}{5} e^{\frac{-5x^2}{2}} + C$ //

$\Rightarrow\; \frac{\cancel{e^{\frac{-3x^2}{2}}}\, y}{\cancel{e^{\frac{-3x^2}{2}}}} = \frac{-\frac{1}{5}e^{\frac{-5x^2}{2}} + C}{e^{\frac{-3x^2}{2}}}$

$$\boxed{\Rightarrow\; y = -\frac{1}{5e^{x^2}} + Ce^{\frac{3x^2}{2}}}$$

3.) $y^{(4)} + 3y'' - 4y = 0$,

$y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 10$

$r^4 + 3r^2 - 4 = (r+1)(r-1)(r^2+4) = 0$

$r = 1, \ -1, \ 2i, \ -2i$

* $y = C_1 e^x + C_2 e^{-x} + C_3 \cos(2x) + C_4 \sin(2x)$

$y' = C_1 e^x - C_2 e^{-x} - 2C_3 \sin(2x) + 2C_4 \cos(2x)$

$y'' = C_1 e^x + C_2 e^{-x} - C_3 4\cos(2x) - C_4 4\sin(2x)$

$y''' = C_1 e^x - C_2 e^{-x} + C_3 8\sin(2x) - C_4 8\cos(2x)$

Plug in initial values:

$C_1 + C_2 + C_3 + 0 = 0$

$C_1 - C_2 - 0 + 2C_4 = 0$

$C_1 + C_2 - 4C_3 - 0 = 0$

$C_1 - C_2 + 0 - 8C_4 = 10$

Solve coefficients:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 1 & -4 & 0 \\ 1 & -1 & 0 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

$\det A = -100$

$$C_1 = \frac{\begin{vmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & -4 & 0 \\ 10 & -1 & 0 & -8 \end{vmatrix}}{-100} = \frac{-100}{-100} = 1$$

$$C_2 = \frac{\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & -4 & 0 \\ 1 & 10 & 0 & -8 \end{vmatrix}}{-100} = \frac{100}{-100} = -1$$

$$C_3 = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 10 & -8 \end{vmatrix}}{-100} = \frac{0}{-100} = 0$$

$$C_4 = \frac{\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -4 & 0 \\ 1 & -1 & 0 & 10 \end{vmatrix}}{-100} = \frac{100}{-100} = -1$$

Particular Sol'n:

$$\boxed{y = e^x - e^{-x} - \sin 2x}$$

4.) $\dfrac{dP}{dt} = rP + r_{in} - r_{out}$

$\Rightarrow \dfrac{dP}{dt} = rP - 5$, ~~$P(0) = 100$~~

Solve for r :

* $\dfrac{dP}{dt} = rP$, $P = P_0 e^{rt}$

$\Rightarrow 2\cancel{P_0} = \cancel{P_0} e^{7r}$

$\Rightarrow \ln(2) = \ln(e^{7r})$

$\Rightarrow \ln 2 = 7r \Rightarrow r = \dfrac{\ln 2}{7}$

Integrating Factors :

$\Rightarrow \dfrac{dP}{dt} \underbrace{- rP}_{p(t)} = -5$

$\mu(t) = e^{\int -r\,dt} = e^{-rt} = e^{-\frac{\ln 2}{7}t}$ //

$\Rightarrow P' - rP = -5$

$\Rightarrow e^{-rt}(P' - rP) = -5e^{-rt}$

$\Rightarrow \int [Pe^{-rt}]'\,dt = \int -5e^{-rt}\,dt$

$\Rightarrow Pe^{-rt} = \dfrac{-5e^{-rt}}{-r} + C$

$\Rightarrow \cancel{e^{rt}}(P\cancel{e^{-rt}}) = \left(\dfrac{-5e^{-rt}}{-r} + C\right)e^{rt}$

$\Rightarrow P = \dfrac{5}{r} + Ce^{rt}$, $r = \dfrac{\ln 2}{7}$

Solve for C :

$\Rightarrow P(0) = \dfrac{5}{r} + Ce^{0r} = 100$

$\Rightarrow \dfrac{5}{r} + C = 100$

$\Rightarrow C = 100 - \dfrac{5}{r}$, $r = \dfrac{\ln 2}{7}$

Will the population survive ?

$\Rightarrow \lim_{t\to\infty}\left(\dfrac{5}{r} + Ce^{rt}\right)$

$\Rightarrow \lim_{t\to\infty}\left(\dfrac{5}{\ln 2/7} + \left(100 - \dfrac{5}{\ln 2/7}\right)e^{\ln 2/7\, t}\right)$

$\Rightarrow +\infty$ $\therefore$ the population will survive

When will the population reach 500 ?

$\Rightarrow 500 = \dfrac{5}{\frac{\ln 2}{7}} + \left(100 - \dfrac{5}{\frac{\ln 2}{7}}\right)e^{\ln 2/7\, t}$

$\Rightarrow \dfrac{35}{\ln 2} + \left(100 - \dfrac{35}{\ln 2}\right)e^{\frac{\ln 2}{7}t} = 500$

$\Rightarrow \cancel{-\dfrac{35}{\ln 2} + \dfrac{35}{\ln 2}} + \left(100 - \dfrac{35}{\ln 2}\right)e^{\frac{\ln 2}{7}t} = 500 - \dfrac{35}{\ln 2}$

$\Rightarrow \dfrac{\cancel{\left(100 - \frac{35}{\ln 2}\right)}e^{\frac{\ln 2}{7}t}}{\cancel{100 - \frac{35}{\ln 2}}} = \dfrac{500 - \frac{35}{\ln 2}}{100 - \frac{35}{\ln 2}}$

$\Rightarrow e^{\ln 2/7\, t} = \dfrac{500 \ln 2 - 35}{100 \ln 2 - 35}$

$\Rightarrow \ln\left(e^{\frac{\ln 2}{7}t}\right) = \ln\left(\dfrac{500 \ln 2 - 35}{100 \ln 2 - 35}\right)$

$\Rightarrow \dfrac{\ln 2}{7}t = \ln\left(\dfrac{500 \ln 2 - 35}{100 \ln 2 - 35}\right)$

$\Rightarrow t = \dfrac{\ln\left(\frac{500 \ln 2 - 35}{100 \ln 2 - 35}\right)}{\frac{\ln 2}{7}}$

$\Rightarrow t = \dfrac{7\ln\left(\frac{500 \ln 2 - 35}{100 \ln 2 - 35}\right)}{\ln 2}$

$\boxed{\approx 22.2787 \text{ days} \approx 22 \text{ days}}$