1.) $y'' - 2y' + 2y = \cos x \quad , \quad y(0) = 0$

Laplace Transform:

$$\Rightarrow \mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{\cos x\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - 2sY(s) + y(0) + 2Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow s^2 Y(s) - 2s Y(s) + 2Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow (s^2 - 2s + 2) Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+1)(s^2-2s+2)}$$

Partial Fraction:

$$\Rightarrow \frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+b}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

$$\Rightarrow s = As + B(s^2 - 2s + 2) + Cs - D(s^2+1)$$

$$\Rightarrow s = As^3 + Bs^2 - 2As^2 - 2Bs + 2As + 2B + Cs^3 + Ds^2 + Cs + D$$

$$\Rightarrow s = (A+C)s^3 + (B - 2A + D)s^2 + (-2B + 2A + C)s + (2B + D)$$

$0 = 2B + D$ $\qquad A = \frac{1}{5}$ $\qquad C = -\frac{1}{5}$

$1 = -2B + 2A + C$ $\qquad B = -\frac{2}{5}$ $\qquad D = \frac{4}{5}$

$0 = B - 2A + D \quad \Rightarrow$

$0 = A + C$

$$\Rightarrow \frac{\frac{1}{5}s + (-\frac{2}{5})}{s^2+1} + \frac{-\frac{1}{5}s + \frac{4}{5}}{s^2 - 2s + 2} \Rightarrow \frac{s - \frac{2}{5}}{s^2+1} + \frac{-s+4}{5(s^2-2s+2)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s-\frac{2}{5}}{s^2+1} + \frac{-s+4}{5(s^2-2s+2)}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s-\frac{2}{5}}{s^2+1} - \frac{1}{5}\left(\frac{s-1}{(s-1)^2+1}\right) + \frac{3}{5}\left(\frac{1}{(s-1)^2+1}\right)\right\}$$

$$\boxed{\Rightarrow y(t) = \frac{1}{5}(\cos t - 2\sin t) - \frac{1}{5}e^t \cos t + \frac{3}{5}e^t \sin t}$$

2.) $(1-x)\,y'' + xy' - y = 0$ about $x_0 = 0$

$$\Rightarrow (1-x)\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1)n\,a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)\,a_{n+2} - (n+1)n\,a_{n+1} + n a_n - a_n\right]x^n = 0$$

$2a_2 - a_0 = 0$

$a_2 = \dfrac{a_0}{2}$

$$a_{n+2} = \frac{n(n+1)\,a_{n+1} - n a_n + a_n}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

$n=0: \quad a_2 = \dfrac{a_0}{2}$

$n=1: \quad a_3 = \dfrac{2a_2 - a_1 + a_1}{3(2)} = \dfrac{2\left(\frac{a_0}{2}\right) - a_1 + a_1}{3(2)} = \dfrac{a_0 - a_1 + a_1}{3(2)} = \dfrac{a_0}{6}$

$n=2: \quad a_4 = \dfrac{2(3)\,a_3 - 2(a_2) + a_2}{4(3)} = \dfrac{2(3)\left(\frac{a_0}{2(3)}\right) - 2\left(\frac{a_0}{2}\right) + \frac{a_0}{2}}{4(3)} = \dfrac{a_0 - a_0 + \frac{a_0}{2}}{4(3)} = \dfrac{\frac{a_0}{2}}{4(3)} = \dfrac{a_0}{24}$

$n=3: \quad a_5 = \dfrac{3(4)\,a_4 - 3a_3 + a_3}{5(4)} = \dfrac{3(4)\left(\frac{a_0/2}{4(3)}\right) - 3\left(\frac{a_0}{3(2)}\right) + \frac{a_0}{3(2)}}{5(4)} = \dfrac{\frac{a_0}{2} - \frac{a_0}{2} + \frac{a_0}{3(2)}}{5(4)} = \dfrac{\frac{a_0}{3(2)}}{5(4)} = \dfrac{a_0}{120}$

$n=4: \quad a_6 = \dfrac{4(5)\,a_5 - 4a_4 + a_4}{6(5)} = \dfrac{4(5)\left(\frac{a_0/3(2)}{5(4)}\right) - 4\left(\frac{a_0/2}{4(3)}\right) \frac{a_0/2}{4(3)}}{6(5)} = \dfrac{\frac{a_0}{3(2)} - \frac{a_0/2}{3} + \frac{a_0/2}{4(3)}}{6(5)} = \dfrac{a_0}{24}$

$$\boxed{\therefore\; y(x) = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 + \frac{a_0}{24}x^4 + \frac{a_0}{120}x^5 + \frac{a_0}{24}x^6 + \ldots}$$