**CS281: Advanced ML** | **October 11, 2017**

# Lecture 11: Exact Inference: Belief Propagation

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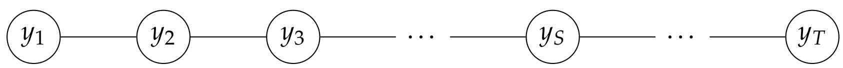
## 1 Simple Marchov Chain

For a simple Markov Chain with time limit $T$ and $V$ classes per node:

$$\begin{aligned} p(y,t) &= \exp(\sum_t \theta_t^T(y_{t-1}, y_t) + \theta_t^o(y_t) - A(\theta)) \\ &\propto \prod_t \psi_t(y_{t-1}, y_t)\psi_t(y_t) \end{aligned}$$



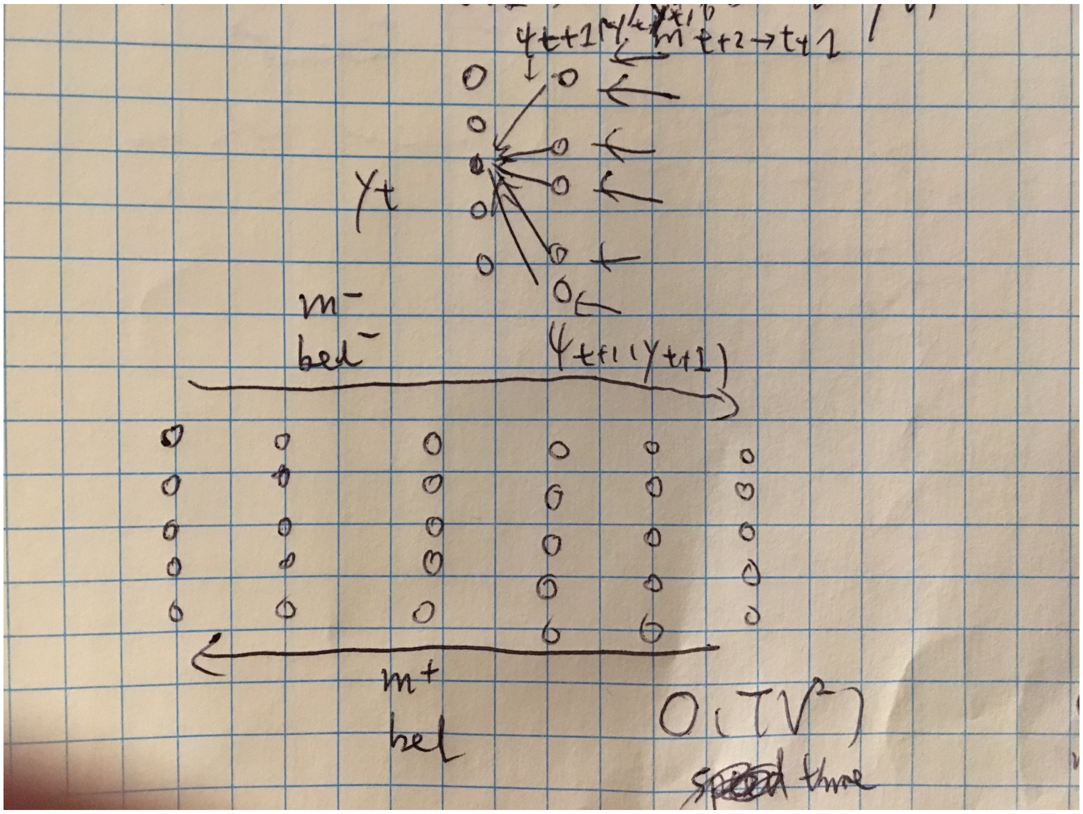
### 11.1 Distributive Property

Marginal:

$$\begin{aligned} p(y_s = v) &= \sum_{y_{1:T, y'_s = v}} \prod_t \psi_t(y'_{t-1}, y'_t)\psi(y'_t)/Z(\theta) \\ &= \sum_{y'_t} \psi_T(y'_T) \sum^{\prime}_{y_T - 1} \psi_{T-1}(y'_{t-1})\psi_T(y_{T-1}, y'_T) \sum \ldots \sum_{y'_2} \psi_2(y'_2) \sum_{y'_1} \psi_1(y'_1, y'_2)\psi_1(y'_1) \end{aligned}$$

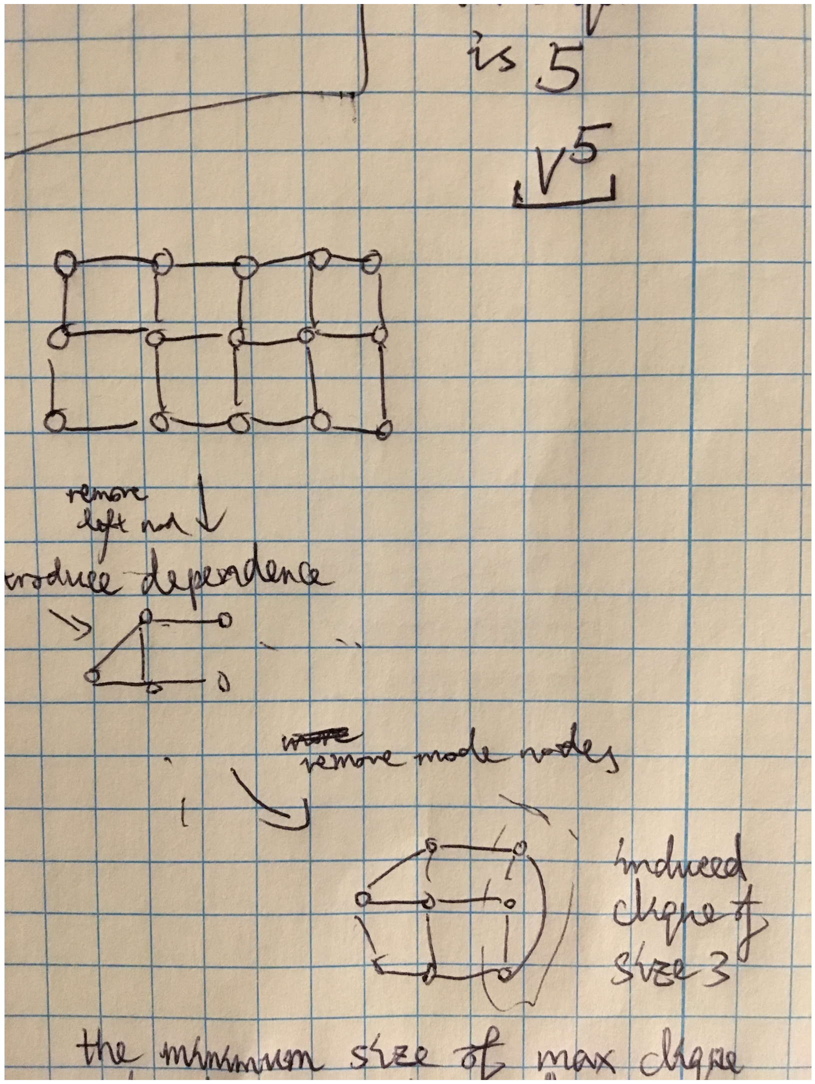
### 11.2 Compute All Marginals with Dynamic Programming

$$\begin{aligned} \mathrm{bel}_t^-(y_t) &\propto \psi_t(y_t)\mathrm{m}_{t-1\to t}^-(y_t) \\ \mathrm{m}_{t-1\to t}^- &= \sum_{y_{t-1}} \psi_{t-1}(y_{t-1}, y_t)\mathrm{bel}_{t-1}^-(y_{t-1}) \\ \mathrm{m}_{t+1\to t}^+ &= \sum_{y_{t+1}} \psi_{t+1}(y_{t+1}, y_t)\psi_{t+1}(y_{t+1})\mathrm{m}_{t+2\to t+1}^+(y_{t+1}) \\ p(y_t) = \mathrm{bel}_t(y_t) &\propto \mathrm{m}_{t+1\to t}(y_t)\mathrm{bel}_t^-(y_t) \end{aligned}$$



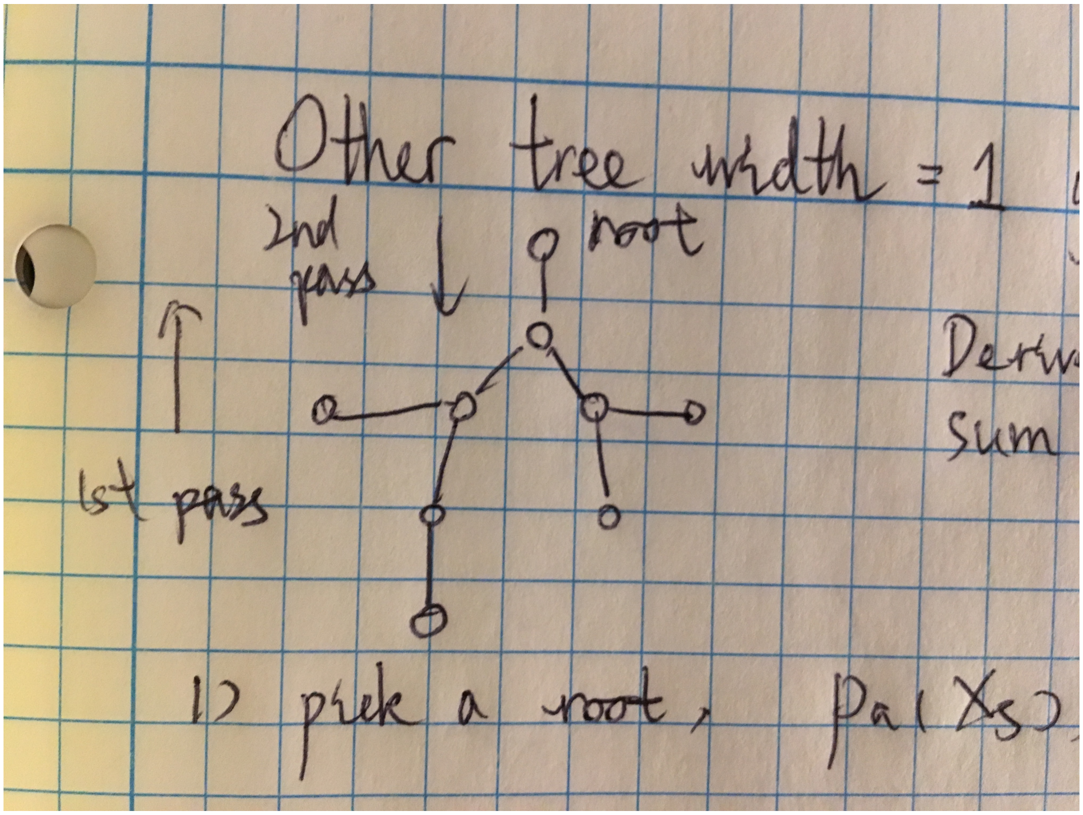

## 2 Other Graphs

$$p(y_s = v) = \sum_{y', y'_s = v} \prod_c \psi_c(y'_c)$$




Computing this sum product might be hard. Here we define the minimum size of maximum clique indcued -1 to be the *treewidth* of the graph.

### 11.1 Compute Marginals for treewidh = 1 Graph




Here we derive the generalization of the forward-backward sum-product algorithm

1. Pick a root $s$, $\mathrm{pa}(x_s)$,$\mathrm{ch}(x_s)$
2. Upward pass:

$$\mathrm{m}_{s \to t}(x_t) = \sum_{x_s} \psi_{s-t}(x_s, x_t)\mathrm{bel}_s^-(x_s)$$
$$\mathrm{bel}_t^-(x_t) \propto \psi_t(x_t) \prod_{s \in \mathrm{ch}(t)} m_{s-t}^-(x_t)$$

3. Downward pass:

$$\mathrm{bel}_s(x_s) \propto \mathrm{bel}_s^-(x_s) \prod_{t \in \mathrm{pa}(s)} \mathrm{m}_{t \to s}^+(x_s)$$
$$\mathrm{m}_{t \to s}^+(x_s) = \sum_{x_t} \psi s - t(x_s, x_t)\psi_t(x_t) \prod_{c \in \mathrm{ch}(t), c \neq s} = \mathrm{m}_t^-(x_t)$$

# 3 Parallel Protocol for Sum Product

$$\mathrm{bel}_s(x_s) \propto \psi_s(x_s) \prod_{t \in \mathrm{nbr}(s)} \mathrm{m}_{t \to s}(x_s)$$
$$\mathrm{m}_(s \to t) = \sum_{x_s} \psi_s(x_s)\psi_{s-t} \prod_{u \in \mathrm{nbr}(s), u \neq t} (x_s)$$

# 4 Final Notes

For this lecture we have been ultilizing + and $\times$ and distributive property

## 11.1 Commutative Semi-ring

$$\begin{array}{ccc} + & \text{max} & \cap \\ \times & \times & \cup \\ \vee & \vee & \vee \\ \text{marginal} & \text{argmax} & \text{satisfying assignment} \end{array}$$