Geometric Representations of Graphs

prof. RNDr. Jan Kratochvíl, CSc.

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Written by Kyrylo Karlov based on lectures in WS 2021/22

Contents

1 Introduction

1.1 Course overview

- Finite graphs
- Existence theorems
- Algorithms, NP-completeness
- Graph drawing

How much information do we need to store the drawing?
Upward drawing: each edge directed upwards.
Most of the time we talk about **Intersection graphs**.

1.2 Intersection graphs

Definition 1.1 (Intersection graph). Let $\mathcal{A}$ be a family of sets. Then **the** intersection graph is

$$IG(\mathcal{A}) = (\mathcal{A}, \{ab : a \neq b, a \cap b \neq \emptyset, a, b \in \mathcal{A}\})$$

Let $\mathcal{M}$ be a large family of sets, then G is **an** intersection graph of $\mathcal{M}$ if

$$\exists A \subseteq \mathcal{M} : G \simeq IG(A)$$

Note that in a family an element can be repeated several times.

Observation 1.2. IF $\mathcal{M}$ contains nonempty set, then $\forall$ complete graphs is in $\mathcal{IG}(\mathcal{M})$.

Proof.

$$A \in \mathcal{M}, A \neq \emptyset$$
$$V(K_n) = \{u_1, u_2, \ldots, u_m\}$$

Every vertex u is represented by A. □

Observation 1.3.

$$G \in \mathcal{IG}(\mathcal{M}) \iff \exists f : V(G) \to \mathcal{M} : \forall u \neq v \in V(G) : uv \in E(G) \iff f(n) \cap f(v) \neq \emptyset$$

Observation 1.4. $\forall \mathcal{M} : \mathcal{IG}(M)$ is **hereditary** if

$$\forall G \in \mathcal{IG}(\mathcal{M})\ \forall H \leq G : H \in IG(\mathcal{M})$$

Induced subgraph is just deleting some edges, which in sets case means forgetting edges that represent sets.

Definition 1.5 (Interval graph). Are intersection graphs of intervals on connected subsets of 1 dimensional Euclidian space.

We are interested in *arc connected* sets of the plain.

Definition 1.6 (Arc connected). Is different from topological connected.

Definition 1.7 (Circle graphs (CA)). Arcs on a circle.

Definition 1.8 (Circular arc graphs (CIR)). Chords on a circle.

Definition 1.9 (Polygon circle graphs (PC)). Polygons that can be inscribed in a circle.

Definition 1.10 (Permutation graphs (PER)). Segments connecting two parallel lines.
Formally:

$$V(G) = \{u_1, u_2, \dots, u_m\}$$
$$\exists \pi \in Sym(m) : u, v \in E \iff i < j \wedge \pi(i) > \pi(j)$$

Definition 1.11 (Function graphs (FUN)). Curves connecting two parallel lines.

Definition 1.12 (Segments graphs (SEG)). Straight-line segments in the plane.

Definition 1.13 (String graphs (STRING)). Curves in the plane.

For each 2 sets, take a point that lies in the intersection of these sets. Then connect unused dots by branching curve.
Problems we want to solve:

1. Given a graph, does it belong to the class.
2. How can we describe a representation. What size?
3. Given a graph and a representation, can we find max independent set, clique and so on. Such questions that are NP-complete in general.

Q: TODO is graph class recognition decidable?
A: Yes, but not polynomial in all cases.
INT, CA, CIR, PC, PER, FUN can be polynomially recognized. SEG, CONN are considered NP-hard. STRING is NP-complete.

$$\forall n \exists G_N \in SEG$$

but in $\forall SEG$ segment representation there is a coordinate that is at least double exponential 2^{2^n}, so even in bits is exponential.
Representation of STRING graph: make each intersection as a vertex. All curves between intersection are edges. Then graph becomes planar. Planarity can be checked in Polynomial time. But also $\forall n \exists G_n \in$ STRING $\forall$ representation requires $\geq 2^{cn}$ crossing points.

2 Chordal graphs

Definition 2.1 (Chordal graph). G is chordal if $\forall k \geq 4 : C_k \not\leq_{ind} G$. Sometimes called triangulated graphs.

Definition 2.2 (Simplicial). A vertex $u \in V(G)$ is simplicial if $G[N_G(u)]$ (reduction of graph to neighborhood of u) is a complete graph. Definition is independent from taking closed (includes u) or open neighborhood.

Lemma 2.3 (1). *Every inclusion-wise minimal vertex cut in a chordal graph induces a clique.*

Proof. $G \setminus A$ has components $V_1, V_2, \dots V_k, k \geq 2$. Then

$$\forall i \forall u \in A \exists w \in V_i : uw \in E(G)$$

Pick some component V_i and some edge in A then there is an edge between them. On the contrary, if there is no edge, u can be removed from A. Which contradicts with minimality of A.
Now we take $u, v \in A$, by previous observation

$$\exists w_1, w_2 \in V_i : uw_1, vw_2 \in E(G)$$

Then take P_1 shortest path between w_1, w_2. Similarly $w_3, w_4 \in V_j$ and the shortest path P_2 between w_3, w_4.
$P_1 \cup P_2$ is an induced cycle unless $uv \in E(G)$.
Also, there is no edge between V_i and $V_j, i \neq j$ as otherwise A is not a cut.
As P_1 is shortest path $vw_1, uw_2 \notin E(G)$.
To sum up, $uv \in E(G)$. Since u, v were arbitrary, A is a complete subgraph. □

Lemma 2.4 (2). *A chordal graph is complete or it contains 2 non-adjacent simplicial vertices.*

Proof. By induction on $|V(G)|$. The first step is G is a complete graph.
Inductive step: G is not complete. Take a minimal vertex cut A. Let B be a connected component of $G \setminus A$ and $C = (V(G) \setminus A) \setminus B$.

$$G_1 = G[B \cup A]$$

$$G_2 = G[C \cup A]$$

As $|V(G_1)| < |V(G)|$ we can apply induction on it. Note that induced subgraph of chordal graph is also chordal. By induction hypothesis G_1, G_2 are either complete or have 2 simplicial vertices.
One of the simplicial vertices cannot be in A because A is complete graph and simplicial vertices are not adjacent. No edges can connect B, C therefore both of the vertices are simplicial in G. □

Corollary 2.5. *Every nonempty chordal graphs has a simplicial vertex.*

Sometimes it is easier to proof stronger statement, because we have more power in inductive step.

Definition 2.6 (PES). Perfect elimination scheme - for graph G is a *linear ordering* of its vertices.

$$V(G) = u_1, \dots, u_n$$

Such that $\forall i : u_i$ is simplicial for $G[\{u_1, \dots u_i\}]$

Lemma 2.7 (Chordal has PES). *Every chordal graph allows a PES.*

Proof. Take any simplicial vertices and move it to the right. Then delete vertex picked in previous step and repeat.
Formally: by induction on n using corollary. □

Definition 2.8 (Perfect graph). G is perfect if for every subgraph chromatic number is equal to clique number.

$$\forall H \leq G : \chi(H) = \omega(H)$$

Theorem 2.9 (Chordal is Perfect). *A chordal graph is perfect.*

Proof. Take a PES for a graph, color from left to right by colors $\in \{1,2,3,\dots\}$ by **first fit method**.
Take smallest number that was not taken by the neighbors.
If we a forced to use color k then neighbors of the vertex used $(k-1)$ colors. Which implies existence of complete graph on $(k-1)$ vertices on the left from current vertex. □

Definition 2.10 (Clique-tree decomposition). Clique-tree decomposition of a graph G is a tree T

$$T = (\mathcal{Q}, F) : V(T) = \mathcal{Q} = \{\text{maximal cliques of G}\}$$

and

$$\forall u \in V(G) : T[\{Q : u \in Q \in \mathcal{Q}\}] \text{ is connected}$$

Theorem 2.11 (Chordal equivalent statements). *For any graph G the following are equivalent:*

1. *G is chordal*
2. *G has a PES*
3. *G allows a Clique-tree decomposition*
4. *G is an intersection graph of subtrees of a tree*

Proof. $1 \Rightarrow 2$ By induction on the number of vertices, using lemma 2.4. Pick simplicial vertex, put it at the end of PES. Remove vertex from graph and continue.
$2 \Rightarrow 3$ let we have a PES $\{u_1, \dots u_n\}$. G has maximal cliques: $Q = \{Q_1, \dots, Q_k\}$, $T = (\mathcal{Q}, E(T))$. Remove last vertex in PES from graph

$$G^{'} = G \setminus u_n$$

It has cliques: $Q^{'} = \{Q^{'}_1, \dots, Q^{'}_k\} \Rightarrow$ by i.h

$$\exists T^{'} = (\mathcal{Q}^{'}, E(T^{'})$$

Consider 2 cases: $N_G(u_n) \cup \{u_n\}$ is a maximal clique. Then is a maximal clique in $G^{'}$.
O/w $N_G(u_n)$ is not a maximal clique of $G^{'}$. $\Rightarrow \exists Q^{'}_i \in \mathcal{Q}^{'} : N_G(u_n) \subsetneq Q^{'}_i$. We connect $N_G(u_n)$ to $Q^{'}_i$.
$3 \Rightarrow 4$ we want to find an intersection graph of subtrees in tree.

$$G \simeq IF(\{T_u : u \in V(G)\})$$

such that

$$V(T_u) = \{Q_i : y \in Q_i\} \subseteq \mathcal{Q}$$

Proving the equivalence:

$$uv \in E(G) \Rightarrow \exists Q_i \in \mathcal{Q} : u, v \in Q_i \Rightarrow Q_i \in V(T_u) \cap V(T_v) \neq \emptyset$$

On the other hand

$$V(T_u) \cap V(T_v) \neq \emptyset \Rightarrow \exists Q_i \in V(T_u) \cap V(T_v) \Rightarrow u, v \in Q_i \Rightarrow uv \in E(G)$$

$4 \Rightarrow 1$ Let we have a tree T with a collection of subtrees.

$$V_u \subseteq V(T), u \in V(G) : T[V_u] \text{ is connected} : \forall u \neq v \in V(G) : uv \in E(G) \iff V_u \cap V_v \neq \emptyset$$

Assume by contradiction G is not chordal. By definition $\exists k \geq 4 \in \mathbb{N} : C_k \leq_{ind} G$. Let the cycle be $\{u_1, u_2, \ldots, u_k\}$. $u_k u_1 \in E(G)$ by the assumption, take

$$T_1 = T[V_{u_1}], T_2 = T[V_{u_2}], T_3 = T[V_{u_3}]$$

T_1 should cross T_2, also T_3 should cross T_2 but not T_1.

$$V_{u_1} \cap V_{u_2} \neq \emptyset \wedge V_{u_1} \cap V_{u_3} = \emptyset$$

Therefore

$$\exists e \in E(T_2) : e \notin E(T_1), E(T_2)$$

Removing 1 edge makes tree disconnect:

$$T \setminus e = T_a \dot{\cup} T_b, V_{u_1} \subseteq V(T_a), V_{u_3} \subseteq V(T_b)$$

Proceed by induction

$$\forall j, j = 3 \ldots k : V(T_j) \subseteq T_b$$

inductive step

$$V_{u_j} \subseteq V(T_b), V_{u_{j+1}} \cap V(T_b) \neq \emptyset, V_{u_{j+1}} \cap V_{u_2} \neq \emptyset \Rightarrow V_{u_{j+1}} \subseteq V(T_b)$$

Therefore

$$V_{u_k} \cap V_{u_1} = \emptyset$$

which contradicts $u_k u_1 \in E(g)$.

□

3 Interval graphs

If there is an interval that is *open* on one side, we can replace it by smaller one but closed. This does not change the graph. Therefore WLOG we may assume that all intervals are closed.
Works because we have finite number of intervals.

Definition 3.1 (Interval graph(INT)).

$$INT = IG(\{\text{interval on a line}\}) = IG(\{\text{closed intervals on a line}\})$$

Theorem 3.2 (Chordal equivalent statements). *For any graph G the following are equivalent:*

1. $G \in INT$
2. *G allows a Clique-path decomposition*

3. $G \in IG(\{subpaths\ of\ a\ path\})$

Proof. $1 \iff 3$
Let Q be a maximal clique of G. All intervals that are represented by Q intersect.

$$\forall u, v \in Q : I_u \cap I_v \neq \emptyset$$

Intervals on a line have a Helly 2-property. E.g. convex sets in the plain have Helly 3-property.
Therefore

$$\bigcap_{u \in Q} I_n \neq \emptyset$$

Let x be the rightmost of the endpoints of $I_u, u \in Q$. Then $x \in \bigcap_{u \in Q} I_n$ As otherwise

$$\exists v \in Q : x \notin I_v$$

Also

$$\exists w \in Q : w = [x, \ldots] \Rightarrow I_v \cap I_w = \emptyset$$

and otherwise x was not the rightmost point.
G has maximal cliques: $Q = \{Q_1, \ldots, Q_k\}$. Let $p_1, \ldots, p_k$ be path such that

$$p_i \in \bigcap_{u \in Q_i} I_u$$

It cannot happen that $p_i = p_{i+1}$ as it means $Q_i \cap Q_{i+1} = p_i$. Combining the two will make a larger clique. Therefore

$$p_1 < \ldots < p_k$$

Then

$$P = (\mathcal{Q}, \{Q_i, Q_{i+1} | i = 1, 2, \ldots, k-1\})$$

is a clique-path decomposition.
Having

$$u \in Q_i, Q_h : i < h$$

We want for $i < j < h \Rightarrow p_i, p_g \in I_u \Rightarrow p_j \in I_u \Rightarrow q \in Q_j$.
$1 \iff 2$.
"$\Rightarrow$"

$$uv \in E(G) \Rightarrow \exists i : u, v \in Q_i \Rightarrow Q_i \in V_u \cap V_v \neq \emptyset$$

"$\Leftarrow$"

$$V_u \cap V_v \in Q_i \Rightarrow u, v \in Q_i \Rightarrow uv \in E(G)$$

□

Definition 3.3 (Comparability graphs (CO)). POSET is $\mathcal{P} = (P, \leq)$, $\leq$ is

1. antireflexive $x \not\leq x$ - no loops
2. antisymmetric $x \leq y \Rightarrow y \not\leq x$ - no oriented 2 cycles
3. transitive $x \leq y \wedge y \leq z \Rightarrow x \leq z$ - edge between 2 connected

G is comparability graph $\iff \exists \mathcal{P} = (P, \leq)$, such that

$$G \simeq (P, \{xy : x \leq y \vee y \leq x\}) := C_G(\mathcal{P})$$

Comparing to Hasse diagram, comparability graph has more edges as in Hasse diagram we do not depict edges that come from transitivity.

Observation 3.4. How to check if graph is comparability? Orient every edge from the smaller to larger element. $G \in CO \iff$ edges of G can be transitively oriented.
Every edge gets only 1 orientation.
Note that such orientation preserves antireflexive and antisymmetric property.

Notation 3.5 (Graph Complement).

$$-G = \left(V(G), \binom{V(G)}{2} \setminus E(G) \right)$$

If $\mathcal{A}$ is a class of graphs, then $co - \mathcal{A} = \{-G : G \in \mathcal{A}\}$.

Theorem 3.6 (Complements). .

1. $FUN = co - CO$
2. $PER = CO \cap co - CO$
3. $INT = CHOR \cap co - CO$

Proof. $FUN \subseteq co - CO$
Let $G \in FUN$ and $\{c_u, u \in V(G)\}$ is a FUN representation. Having

$$u \neq v : uv \neq E(G) \Rightarrow c_u \cap c_v = \emptyset$$

From the Jordan path theorem, plane is divided into 2 parts by the line. $-G$ can be oriented such that $u \to v \iff c_u$ lies below c_v.
Therefore either $u \to v \vee v \to u$.
Orientation is transitive from
$co - CO \subseteq FUN^* \subseteq FUN$ Having $G \in co - CO$ we can transitively orient non edges $\binom{V(G)}{2} \setminus E(G)$. Which corresponds to some partial order.

Definition 3.7 (PO dimension). Dimension of partial order is minimal number of linear orders k such that

$$\exists L_1, \ldots, L_k : \mathcal{P} = \bigcap L_i$$

Take k vertical lines and make n points on each of the line. Points on the line correspond to $V(G)$. Put vertices in the order of linear order L_i for line i. Connect the occurrences of vertex on each line.
Take $u, v \in V(G)$ and check their lines

$$\begin{gathered} uv \in E(G) \iff uv \notin E(-G) \Rightarrow uv \notin P \wedge vu \notin P \\ \Rightarrow \exists i \neq j uv \in L_i \wedge vu \in L_j \Rightarrow c_u \cap c_v \neq \emptyset \end{gathered}$$

On the other hand

$$\begin{gathered} wv \notin E(G) \Rightarrow uv \in E(-G) \Rightarrow uv \in \mathcal{P} \vee vu \notin \mathcal{P} \\ \Rightarrow \forall i : uw \in L_i \Rightarrow c_u \cap c_w = \emptyset \end{gathered}$$

Such piecewise continuous lines is the representation of the graph as a FUN graph.

Lemma 3.8.
$$co-PER \subseteq PER$$

Proof. Given a representation, swap order of the right side.
If $uv \in E(G)$ then after transformation edges would not cross. Otherwise $uv \notin E(G)$ then edges do not cross but after the transformation edges cross. □

Consequence 3.9. *Complements of the permutation graphs are permutation graphs*

Proof. if $co-PER \subseteq PER \Rightarrow$:

$$PER = co-(co-PER) \subseteq co-PER$$

Therefore
$$PER = co-(co-PER)$$

□

Proof of 2. "$PER \subseteq CO \cap co-CO$" then using lemma and 1. we get

$$PER \subseteq FUN = co-CO$$

$$PER = co-PER \subseteq co-(co-CO) = CO$$

"$CO \cap co-CO \subseteq PER$" Let G is a graph, $G \in CO \cap co-CO$. $E(G)$ can be transitively oriented, let E_1 be such orientation. Also, edges of the complements can be transitively oriented, let E_2 be such orientation. Taking union $E_1 \cup E_2$ of both orientations as binary relations gives orientation on $K_{V(G)}$. New relation is *antisymmetric*, *antireflexive* and also **transitive**. Transitivity can be proved by checking all combinations of 2 edges $u_1, u_2 \in E_1, E_2$. It is also a linear order, because of the complete graph.

Observation 3.10 (Reverse transitive orientation). Note that reverse of the transitive orientation is also transitive.

Draw a linear order $E_1 \cup E_2$ on left line and $E_1^{-1} \cup E_2$ of the right. If $uv \in E(G)$ then uv is oriented in E_1, on the right line the order is reversed therefore lines cross.

$$S_u \cap S_v \neq \emptyset$$

If $wu \notin E(G)$ it is oriented in E_2. Therefore lines do not cross.

$$S_u \cap S_2 = \emptyset$$

3. Interval graphs are chordal as they are intersection graphs of subpath (connected subpaths) on a path. Which is a subclass if connected subgraphs of trees = Chordal.
"$INT \subseteq co-Co$". Let $G \in INT$. $-G$ is a graph of disjoint intervals. Natural transitive orientation of disjoint intervals is "being to the left/right".
"$CHOR \cap co-Co \subseteq INT$". Suppose $G \in CHOR \cap co-Co$ and E_2 is transitive orientation of non-edges of G. Let $\mathcal{Q}$ be a set of all maximal cliques of G. Define $Q_i < Q_j \in \mathcal{Q} : \exists u \in Q_i, v \in Q_j : (u \to v) \in E_2$. Properties of relation on cliques:

- antireflexive: as clique is a complete graph and $u \to v \wedge v \to u$ cannot happen.

- antisymmetric
  Suppose $Q_i < Q_j \Rightarrow \exists u \in Q_i, v \in Q_j : (u \to v) \in E_2$ and if $Q_j < Q_i$ by contradiction.

  $$Q_j < Q_i \exists x : Q_i, y \in Q_k : yx \in E_2$$

  Cases:

  1. $y = v, x \neq v$
  2. $y \neq v, x = v$ same as 1.

- transitive
  $Q_i < Q_j < Q_h \Rightarrow Q_i < Q_h$

  $$\exists x \in Q_i, u, v \in Q_j : \exists y \in Q_H : xu, vy \in E_2$$

  Cases

  1. $u = v \Rightarrow x \to u \to y \Rightarrow x \to y$ by transitivity.
  2. $u \neq v$, subcases
     (a) $xv \notin E(G) \Rightarrow xv \in E_2 \Rightarrow x \to y$. By transitivity of E_2
     (b) $uy \notin E(G)$ similar to a)
     (c) $xv, uy \in E(G)$, subcases
        i. $xv \notin E(G) \Rightarrow yx \in E_2 \overset{\text{transitivity}}{\to} yu \in E_2 \Rightarrow uy \notin E(G)$ contradiction Or $xy \in E_2 \Rightarrow Q_i < Q_h$.
        ii. $xy \in E(G) \Rightarrow C_4 \leq_I G$ contradiction with Chordal.

- transitive

It is also a total order

$$\forall i \neq j : Q_i < Q_j \vee Q_j < Q_i$$

As $Q_i \neq Q_j \Rightarrow u \in Q_i \setminus Q_j$.

$$u \notin Q_j \Rightarrow v \in Q_j : uv \notin E(G)$$

As otherwise, clique is not maximal. Then uv is oriented one way or another.
Take Q_i as vertices and make edges $Q_i Qi - 1$. We claim that this graph is a clique path for G.
It remains to prove:

$$\forall i < j < h \forall u \in V(G) : u \in Q_i \wedge Q_h \Rightarrow u \in Q_j$$

Suppose $u \notin Q_j \Rightarrow \exists v \in Q_j : uv \notin E(G)$

$$Q_i < Q_j \Rightarrow uv \in E_2$$
$$Q_j < Q_h \Rightarrow vu \in E_2$$

Contradiction with antisymmetric relation. $\square$

3.1 Interval Filament graphs

Definition 3.11 (Interval Filament graph(IFA)).

$$IFA = \mathcal{IG}(\text{interval filaments on the half plane})$$

Were introduced by F.Gavril (2000) as an attempt to generalize graphs for which we can find maximal clique in polynomial time.

Observation 3.12.

- $INT \subseteq IFA$ we can draw interval as a half rectangle.
- $CHOR \subseteq IFA$
- $PER \subseteq IFA$
- $FUN \subseteq IFA$ extend the intervals to new line placed below every point of the representation.
- $CIR \subseteq IFA$
- $CA \subseteq IFA$
- $PC \subseteq IFA$

Exercise 3.13. Prove that $CHO \subseteq PC \Rightarrow CHO \subseteq IFA$

Definition 3.14 ($\mathcal{A}$-mixed graphs). If $\mathcal{A}$ is a class of graphs, $\mathcal{A}$-mixed graph if it allowes partition $E = E_1 \dot{\cup} E_2$ and a transitive orientation such that

1. $(V, E) \in \mathcal{A}$
2. $\forall u, v, w \in uv \in E_2 \wedge vw \in E_1 \Rightarrow uw \in E_1$

Theorem 3.15 (Complements).

$$co - IFA = (co - INT) - mixed$$

Proof. $\square$

List of Theorems

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