Geometric Representations of Graphs

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1 Introduction

1.1 Course overview

- Finite graphs
- Existence theorems
- Algorithms, NP-completeness
- Graph drawing

How much information do we need to store the drawing?
Upward drawing: each edge directed upwards.
Most of the time we talk about **Intersection graphs**.

1.2 Intersection graphs

Definition 1.1 (Intersection graph). Let $\mathcal{A}$ be a family of sets. Then **the** intersection graph is

$$IG(\mathcal{A}) = (\mathcal{A}, \{ab : a \neq b, a \cap b \neq \emptyset, a, b \in \mathcal{A}\})$$

Let $\mathcal{M}$ be a large family of sets, then G is **an** intersection graph of $\mathcal{M}$ if

$$\exists A \subseteq \mathcal{M} : G \simeq IG(A)$$

Note that in a family an element can be repeated several times.

Observation 1.2. IF $\mathcal{M}$ contains nonempty set, then $\forall$ complete graphs is in $\mathcal{IG}(\mathcal{M})$.

Proof.

$$A \in \mathcal{M}, A \neq \emptyset$$
$$V(K_n) = \{u_1, u_2, \ldots, u_m\}$$

Every vertex u is represented by A. $\square$

Observation 1.3.

$$G \in \mathcal{IG}(\mathcal{M}) \iff \exists f : V(G) \to \mathcal{M} : \forall u \neq v \in V(G) : uv \in E(G) \iff f(n) \cap f(v) \neq \emptyset$$

Observation 1.4. $\forall \mathcal{M} : \mathcal{IG}(M)$ is **hereditary** if

$$\forall G \in \mathcal{IG}(\mathcal{M}) \; \forall H \leq G : H \in IG(\mathcal{M})$$

Induced subgraph is just deleting some edges, which in sets case means forgetting edges that represent sets.

Definition 1.5 (Interval graph). Are intersection graphs of intervals on connected subsets of 1 dimensional Euclidian space.

We are interested in *arc connected* sets of the plain.

Definition 1.6 (Arc connected). Is different from topological connected.

Definition 1.7 (Circle graphs (CA)). Arcs on a circle.

Definition 1.8 (Circular arc graphs (CIR)). Chords on a circle.

Definition 1.9 (Polygon circle graphs (PC)). Polygons that can be inscribed in a circle.

Definition 1.10 (Permutation graphs (PER)). Segments connecting two parallel lines.
Formally:

$$V(G) = \{u_1, u_2, \dots, u_m\}$$
$$\exists \pi \in Sym(m) : u, v \in E \iff i < j \wedge \pi(i) > \pi(j)$$

Definition 1.11 (Function graphs (FUN)). Curves connection two parallel lines.

Definition 1.12 (Segments graphs (SEG)). Straight-line segments in the plane.

Definition 1.13 (String graphs (STRING)). Curves in the plane.

For each 2 sets, take a point that lies in the intersection of these sets. Then connect unused dots by branching curve.
Problems we want to solve:

1. Given a graph, does it belong to the class.
2. How can we describe a representation. What size?
3. Given a graph and a representation, can we find max independent set, clique and so on. Such questions that are NP-complete in general.

Q: TODO is graph class recognition decidable?
A: Yes, but not polynomial in all cases.
INT, CA, CIR, PC, PER, FUN can be polynomially recognized. SEG, CONN are considered NP-hard. STRING is NP-complete.

$$\forall n \exists G_N \in SEG$$

but in $\forall SEG$ segment representation there is a coordinate that is at least double exponential 2^{2^n}, so even in bits are exponential.
Representation of STRING graph: make each intersection as a vertex. All curves between intersection are edges. Then graph becomes planar. Planarity can be checked in Polynomial time.

$$\forall n \exists G_n \in STRING \forall \text{representation requires} \geq 2^{cn} \text{crossing points}$$

1.3 Chordal graphs

Definition 1.14 (Chordal graph). G is chordal if $\forall k \geq 4 : C_k \not\leq G$. Sometimes called triangulated graphs.

Definition 1.15 (Simplicial). A vertex $u \in V(G)$ is simplicial if $G[N_G(u)]$ (reduction of graph to is a complete graph. Definition is independent from taking closed (includes u) or open neighborhood.

Lemma 1.16 (1). *Every inclusion-vise minimal vertex cut in a chordal graph induces a clique.*

Proof. $G \setminus A$ has components $V_1, V_2, \dots V_k, k \geq 2$. Then

$$\forall i \forall u \in A \exists w \in V_i : uw \in E(G)$$

Pick some component V_i and some edge in A then there is an edge between them. On the contrary, if there is no edge, u can be removed from A. Which contradicts with minimality of A.
Now we take $u, v \in A$, by previous observation

$$\exists w_1, w_2 \in V_i : uw_1, vw_2 \in E(G)$$

Then take P_1 shortest path between w_1, w_2. Similarly $w_3, w_4 \in V_j$ and the shortest path P_2 between w_3, w_4.
$P_1 \cup P_2$ is an induced cycle unless $uv \in E(G)$.
Also, there is no edge between $V_i, V_j, i \neq j$ as o/w A is not a cut.
As P_1 is shortest path $vw_1, uw_2 \notin E(G)$.
To sum up, $uv \in E(G)$. Since u, v were arbitrary, A is a complete subgraph. ☐

Lemma 1.17 (2). *A chordal graph is complete or it contains 2 non-adjacent simplicial vertices.*

Proof. By induction on $|V(G)|$. The first step is G is a complete graph.
Inductive step: G is not complete. Take a minimal vertex cut A. Let B be a connected component of $G \setminus A$ and $C = (V(G) \setminus A) \setminus B$.

$$G_1 = G[B \cup A]$$

$$G_2 = G[C \cup A]$$

As $|V(G_1)| < |V(G)|$ we can apply induction on it. Note that induced subgraph of chordal graph is also chordal. By induction hypothesis G_1, G_2 are either complete or have 2 simplicial vertices.
One of the simplicial vertices cannot be in A because A is complete graph and simplicial vertices are not adjacent. No edges can connect B, C therefore both of the vertices are simplicial in G. ☐

Corollary 1.18. *Every nonempty chordal graphs has a simplicial vertex.*

Sometimes it is easier to proof stronger statement, because we have more power in inductive step.

Definition 1.19 (PES). Perfect elimination scheme - for graph G is a *linear ordering* of its vertices.

$$V(G) = u_1, \dots, u_n$$

Such that $\forall i : u_i$ is simplicial for $G[\{u_1, \dots u_i\}]$

Note 1.20. Every chordal graph allows a PES.

Proof. Take any simplicial vertices and move it to the right. Then delete vertex picked in previous step and repeat.
Formally: by induction on n using corollary. ☐

Definition 1.21 (Perfect graph). G is perfect if for every subgraph chromatic number is equal to clique number.

$$\forall H \leq G : \chi(H) = \omega(H)$$

Theorem 1.22 (Chordal is Perfect). *A chordal graph is perfect.*

Proof. Take a PES for a graph, color from left to right by colors which a numbers

$$\{1, 2, 3, \dots\}$$

by **first fit method**.
Take smallest number that was not taken by the neighbors.
If we a forced to use color k then neighbors of the vertex used $(k-1)$ colors. Meaning that on the left there is complete graph on $(k-1$ vertices. □

Definition 1.23 (Clique-tree decompositon). Clique-tree decomposition of a graph G is a tree T

$$T = (\mathcal{Q}, F) : V(T) = \mathcal{Q} = \{\text{maximal cliques of G}\}$$

and

$$\forall u \in V(G) : T[\{Q : u \in Q \in \mathcal{Q}\}] \text{ is connected}$$

Theorem 1.24 (Chordal equivalent statements). *For any graph G the following are equivalent:*

1. *G is chordal*
2. *G has a PES*
3. *G allows a Clique-tree decomposition*
4. *G is an intersection graph of subtrees of a tree*

Proof. $1 \Rightarrow 2$ By induction on the number of vertices, using lemma 1.17.

□

List of Theorems

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