Problem 4

Consider the half-plane subjected to the traction shown in figure 4 (as discussed in class). Show that we can obtain a solution to this problem using the Airy potential

$$\boxed{\text{Eqn 1}} \quad \phi = \frac{\tau}{\pi}\left[\frac{1}{2}x_2^2 \ln\left(x_1^2 + x_2^2\right) + x_1 x_2 \arctan\left(\frac{x_2}{x_1}\right) - x_2^2\right]$$

That is:

1. Verify that ϕ satisfies the biharmonic equation.
2. Compute the corresponding stresses and verify that they satisfy the equilibrium equation. Do you need to check for compatibility of strains? Why?
3. Verify that the stresses satisfy the boundary conditions.

TOP ($x_1 = x_1$ and $x_2 = 0$)

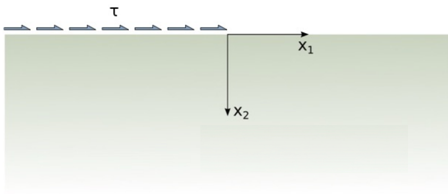

Figure 4: Schematics for problems 3.

1. Verify that ϕ satisfies the biharmonic equation.

$$\boxed{\text{Eqn 2}} \quad \frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 0 \quad \text{(biharmonic eqn)}$$

Attached Matlab Code verifies Eqn 2 with Eqn 1 Substituted in.

$\therefore$ ϕ satisfies the biharmonic eqn.

Matlab Code:

Lines 6-8: Stress from airy stress function:

$\sigma_{11} = \varphi_{,22} = \frac{\partial^2 \varphi}{\partial x_2^2}$

$\sigma_{22} = \varphi_{,11} = \frac{\partial^2 \varphi}{\partial x_1^2}$

$\sigma_{12} = \varphi_{,12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$

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Hmw6.m
1 - variables  syms x1 x2 tau
2              %Part 1
3 - Eqn 1      phi=tau/pi*(1/2*x2^2*log(x1^2+x2^2)+x1*x2*(atan(x2/x1))-x2^2);
4 - Eqn 2      simplify(diff(phi,x1,4)+diff(phi,x2,4)+2*diff(diff(phi,x2,2),x1,2))
5              %Part 2
6 -            sigma11 = simplify(diff(phi,x2,2))
7 -            sigma22 = simplify(diff(phi,x1,2))
8 -            sigma12 = simplify(-1*diff(diff(phi,x2),x1))
9 -            simplify(diff(sigma11,x1)) + simplify(diff(sigma12,x2))
10 -           simplify(diff(sigma22,x2)) + simplify(diff(sigma12,x1))
```

Lines 9-10: Stress Equilibrium Eqns

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \end{cases}$$

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Command Window
  0

  sigma11 =

  (tau*(x1^2*log(x1^2 + x2^2) + x2^2*log(x1^2 + x2^2) + x2^2))/(pi*(x1^2 + x2^2))

  sigma22 =

  -(tau*x2^2)/(pi*(x1^2 + x2^2))

  sigma12 =

  -(tau*(x1^2*atan(x2/x1) + x2^2*atan(x2/x1) + x1*x2))/(pi*(x1^2 + x2^2))

  ans =

  0

  ans =

  0
```

$$\rightarrow \sigma_{11} = \frac{\tau\left(x_1^2 \ln(x_1^2 + x_2^2) + x_2^2 \ln(x_1^2 + x_2^2) + x_2^2\right)}{\pi(x_1^2 + x_2^2)}$$

$$\rightarrow \sigma_{22} = \frac{-\tau x_2^2}{\pi(x_1^2 + x_2^2)}$$

$$\rightarrow \sigma_{12} = \frac{-\tau\left(x_1^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_2^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_1 x_2\right)}{\pi(x_1^2 + x_2^2)}$$

✓ ✓

2D stress equilibrium holds

$\therefore$ ϕ satisfies biharmonic eqn.

2. Compute the corresponding stresses and verify that they satisfy the equilibrium equation. Do you need to check for compatibility of strains? Why?

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = \frac{-\partial^2 \phi}{\partial x_1 \partial x_2}$$ Eqns 3a, 3b, and 3c.

Eqn 1 into Eqn 3a, 3b, 3c with Matlab (see code from previous page)

Gives:

$$\sigma_{11} = \frac{\tau(x_1^2 \ln(x_1^2+x_2^2) + x_2^2 \ln(x_1^2+x_2^2) + x_2^2)}{\pi(x_1^2+x_2^2)}$$

$$\sigma_{22} = \frac{-\tau x_2^2}{\pi(x_1^2+x_2^2)}$$

$$\sigma_{12} = \frac{-\tau(x_1^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_2^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_1 x_2)}{\pi(x_1^2+x_2^2)}$$ //

$\therefore$ Using Matlab: $\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$

$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$ //

$\Rightarrow$ We do not need to check for compatibility of strains because satisfies biharmonic equation $\therefore$ satisfies equilibrium equation, so there is already compatibility of strains. //

Basically: Biharmonic is satisfied $\therefore$ compatibility is satisfied

3. Verify that the stresses satisfy the boundary conditions.

Using the stresses found in the previous part:

$$\sigma_{11} = \frac{\tau(x_1^2 \ln(x_1^2+x_2^2) + x_2^2 \ln(x_1^2+x_2^2) + x_2^2)}{\pi(x_1^2+x_2^2)}$$

$$\sigma_{22} = \frac{-\tau x_2^2}{\pi(x_1^2+x_2^2)}$$

$$\sigma_{12} = \frac{-\tau(x_1^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_2^2 \tan^{-1}\left(\frac{x_2}{x_1}\right) + x_1 x_2)}{\pi(x_1^2+x_2^2)}$$

Top ($x_1 = x_1$ and $x_2 = 0$):

$$\sigma_{11} = \frac{\tau(x_1^2 \ln(x_1^2))}{\pi x_1^2} = \frac{\tau \ln(x_1^2)}{\pi}$$

$$\sigma_{22} = \frac{-\tau(0)}{\pi(x_1^2+0)} = 0$$

$$\sigma_{12} = \frac{-\tau(0+0+0)}{\pi(x_1^2+0)} = 0$$

CHECK: could use $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$ to confirm

$t_i = \sigma_{ij} n_j$

$$\begin{bmatrix} \tau \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \therefore \begin{aligned} -\sigma_{12} &= \tau \\ \sigma_{22} &= 0 \checkmark \end{aligned}$$

this should be the correct answer for σ_{12}

but I got: $\sigma_{11} = \frac{\tau \ln(x_1^2)}{\pi}$ and $\sigma_{12} = 0$

$\therefore$ I made a mistake somewhere (possible Matlab code wrong)