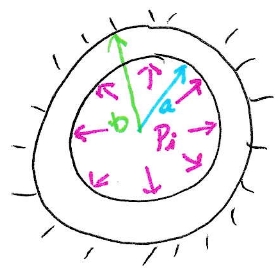
SARAH MALAK
AE6114 HW6
NOV. 26, 2019

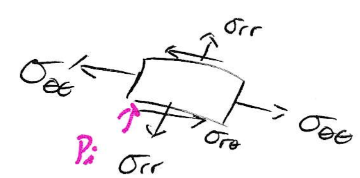
80/100

Problem 1

1.



Differential Element:



convention for positive stress

Boundary Conditions:

Traction: at $r=a$, $\sigma_{rr} = -P_i$
$\sigma_{r\theta} = 0$

at $r=b$, $\sigma_{r\theta}=0$

Displacement: $u_r(b,\theta)=0$

This problem is axisymmetric (does not depend on θ)

We want solutions to $\phi(r)$ with no body forces:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi = 0$$

The general solution is:

$$\phi(r) = A\ln(r) + Br^2\ln(r) + Cr^2 + D$$

Then, the stresses can be expressed as:

$$\sigma_{rr} = A\frac{1}{r^2} + B(1+2\ln(r)) + 2C$$

$$\sigma_{\theta\theta} = -A\frac{1}{r^2} + B(3+2\ln(r)) + 2C$$

$$\sigma_{r\theta} = 0$$

To find A, B, and C we apply the boundary conditions:

SARAH MALAK

1) as $r \to 0$, we need $B=0$

$$\sigma_{rr} = \frac{A}{r^2} + 2C$$

$$\sigma_{\theta\theta} = \frac{-A}{r^2} + 2C$$

$$\sigma_{r\theta} = 0$$

2) for $r=a$, $\sigma_{rr} = -P_i$

$$-P_i = \frac{A}{a^2} + 2C$$

we need another equation:

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta})$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})$$

$$\varepsilon_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} = 0$$

we know $u_r = \int \varepsilon_{rr}\, \partial r$

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}), \quad \nu = \frac{1}{3}$$

plane strain

$$\nu^* = \frac{\nu}{1-\nu}$$

-2

$$= \frac{1}{E}\left(\frac{A}{r^2} + 2C - \frac{1}{3}\left(-\frac{A}{r^2} + 2C\right)\right)$$

$$= \frac{1}{E}\left(\frac{A}{r^2} + 2C + \frac{A}{3r^2} - \frac{2C}{3}\right)$$

$$= \frac{1}{E}\left(\frac{4A}{3r^2} + \frac{4C}{3}\right)$$

$$u_r = \frac{1}{E}\int\left(\frac{4A}{3r^2} + \frac{4C}{3}\right)\partial r = \frac{1}{E}\left(-\frac{4A}{3r} + \frac{4Cr}{3}\right)$$

3) for $r=b$, $u_r = 0$

$$0 = \frac{1}{E}\left(-\frac{4A}{3b} + \frac{4Cb}{3}\right)$$

$$\frac{4A}{3b} = \frac{4Cb}{3} \Rightarrow A = Cb^2$$

plugging back in to (2):

$$-P_i = \frac{Cb^2}{a^2} + 2C$$

SARAH MALAIK

$$-P_i = C\left(\frac{b^2+2a^2}{a^2}\right)$$

$$C = \frac{-P_i a^2}{2a^2+b^2}$$

$$\Rightarrow A = \frac{-P_i a^2 b^2}{2a^2+b^2}$$

plugging back in to $\underline{\underline{\sigma}}$:

$$\sigma_{rr} = \frac{-P_i a^2 b^2}{(2a^2+b^2)r^2} + \frac{-2P_i a^2}{2a^2+b^2} = \frac{-P_i a^2}{2a^2+b^2}\left(\frac{b^2}{r^2}+2\right) \checkmark$$

when $\sigma_{rr} = -P_i$ at $r=a$:

$$\frac{-P_i a^2}{2a^2+b^2}\left(\frac{b^2}{a^2}+2\right) = \frac{-P_i a^2}{2a^2+b^2}\left(\frac{b^2+2a^2}{a^2}\right) = -P_i \checkmark$$

$$\sigma_{\theta\theta} = \frac{P_i a^2 b^2}{(2a^2+b^2)r^2} - \frac{2P_i a^2}{2a^2+b^2} = \frac{P_i a^2}{2a^2+b^2}\left(\frac{b^2}{r^2}-2\right)$$

$$\boxed{\begin{aligned} \sigma_{rr} &= \frac{-P_i a^2}{2a^2+b^2}\left(\frac{b^2}{r^2}+2\right) \\ \sigma_{\theta\theta} &= \frac{P_i a^2}{2a^2+b^2}\left(\frac{b^2}{r^2}-2\right) \\ \sigma_{r\theta} &= 0 \end{aligned}}$$

$$u_r = \frac{1}{E}\left(-\frac{4A}{3r} + \frac{4Cr}{3}\right)$$

$$= \frac{1}{E}\left(-\frac{4}{3r}\left(\frac{-P_i a^2 b^2}{2a^2+b^2}\right) + \frac{4r}{3}\left(\frac{-P_i a^2}{2a^2+b^2}\right)\right)$$

$$= \frac{1}{E}\left(\frac{4P_i a^2 b^2}{3(2a^2+b^2)r} - \frac{4P_i a^2 r}{3(2a^2+b^2)}\right)$$

$$\boxed{u_r = \frac{4P_i a^2}{3E(2a^2+b^2)}\left(\frac{b^2}{r} - r\right)}$$

at $r=b$, $u_r = 0$

$$\frac{b^2}{b} - b = 0 \checkmark$$

SARAH MALAK

$$\varepsilon_{\theta\theta} = \frac{1}{r}u_r + \frac{1}{r}\cancel{\frac{\partial u_\theta}{\partial \theta}}^{0}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\cancel{\frac{\partial u_r}{\partial \theta}}^{0} - \frac{1}{r}u_\theta\right) = 0 \qquad \left(\varepsilon_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} = 0\right)$$

$$= \frac{1}{2}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{2r} = 0$$

$$\frac{\partial u_\theta}{\partial r} = \frac{u_\theta}{r}$$

I want to say $u_\theta = 0$ .. I think this makes sense.

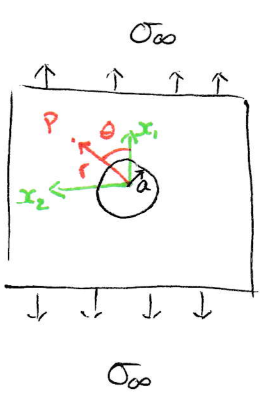
$$\boxed{u_\theta = 0}$$

2. at $r = b$

$$\boxed{\begin{aligned} \sigma_{rr} &= \frac{-P_i a^2}{2a^2+b^2}\left(\frac{b^2}{b^2}+2\right) = \frac{-3P_i a^2}{2a^2+b^2} \\ \sigma_{\theta\theta} &= \frac{P_i a^2}{2a^2+b^2}\left(\frac{b^2}{b^2}-2\right) = \frac{-P_i a^2}{2a^2+b^2} \end{aligned}}$$

Problem 2

1.



Boundary Conditions at $r = a$

$$\sigma_{rr} = 0$$
$$\sigma_{r\theta} = 0$$

SARAH MALAK

2. $r \to \infty$

$$\sigma_{11} = \sigma_\infty$$
$$\sigma_{22} = 0$$
$$\sigma_{12} = 0$$

3. Transform into polar coordinates

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_\infty \cos\theta & 0 \\ -\sigma_\infty \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_\infty \cos^2\theta & -\sigma_\infty \cos\theta \sin\theta \\ -\sigma_\infty \cos\theta \sin\theta & \sigma_\infty \sin^2\theta \end{bmatrix}$$

trig identities:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x))$$
$$\cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$$
$$\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)$$

$$\sigma_{rr} = \sigma_\infty \cos^2\theta = \tfrac{1}{2}\sigma_\infty (1+\cos 2\theta)$$
$$\sigma_{\theta\theta} = \sigma_\infty \sin^2\theta = \tfrac{1}{2}\sigma_\infty (1-\cos 2\theta) \quad \text{as } r \to \infty$$
$$\sigma_{r\theta} = -\sigma_\infty \cos\theta \sin\theta = -\tfrac{1}{2}\sigma_\infty \sin 2\theta$$

4.

$$\phi = \tfrac{1}{4}\sigma_\infty (1-\cos 2\theta) r^2 + C_1(\theta) r + C_2(\theta) \quad \text{as } r \to \infty$$

in polar coordinates: $\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2} + \cancel{\psi}^{\,0}$ no body forces

$$\Rightarrow \phi = \iint \sigma_{\theta\theta}\, \partial r\, \partial r$$

$$\sigma_{\theta\theta} = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)$$

$$\int \frac{1}{2}\sigma_\infty(1-\cos 2\theta)\,\partial r = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)r + C_1(\theta)$$

$$\int\left(\frac{1}{2}\sigma_\infty(1-\cos 2\theta)r + C_1(\theta)\right)\partial r = \frac{1}{4}\sigma_\infty(1-\cos 2\theta)r^2 + C_1(\theta)r + C_2(\theta)$$

$$\Rightarrow \phi = \frac{1}{4}\sigma_\infty(1-\cos 2\theta)r^2 + C_1(\theta)r + C_2(\theta) \quad \text{as} \quad r\to\infty \checkmark$$

5. Show that $\sigma_{rr} = \frac{1}{2}\sigma_\infty(1+\cos 2\theta) + F(C_1(\theta), C_2(\theta))$ as $r\to\infty$

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \cancel{\psi}^{\,0}$$

$$\frac{\partial\phi}{\partial r} = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)r + C_1(\theta)$$

$$\frac{\partial\phi}{\partial\theta} = \frac{1}{2}\sigma_\infty \sin 2\theta\, r^2 + C_1'(\theta)r + C_2'(\theta)$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \sigma_\infty\cos 2\theta\, r^2 + C_1''(\theta)r + C_2''(\theta)$$

$$\Rightarrow \sigma_{rr} = \sigma_\infty\cos 2\theta + \frac{C_1'(\theta)}{r} + \frac{C_2''(\theta)}{r^2} + \frac{1}{2}\sigma_\infty(1-\cos 2\theta) + \frac{C_1(\theta)}{r}$$

$$\sigma_\infty\cos 2\theta + \frac{1}{2}\sigma_\infty - \frac{1}{2}\sigma_\infty\cos 2\theta$$

$$\frac{1}{2}\sigma_\infty + \frac{1}{2}\sigma_\infty\cos 2\theta$$

$$\frac{1}{2}\sigma_\infty(1+\cos 2\theta)$$

$$F(C_1(\theta), C_2(\theta))$$

$$\Rightarrow \sigma_{rr} = \frac{1}{2}\sigma_\infty(1+\cos 2\theta) + F(C_1(\theta), C_2(\theta)) \quad \text{as} \quad r\to\infty \checkmark$$

6. Show that $\phi = \frac{1}{4}\sigma_\infty(1-\cos 2\theta)r^2$ also satisfies BCs for $\sigma_{r\theta}$ at $r\to\infty$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Sarah Malak

$$\frac{\partial \phi}{\partial \theta} = \frac{1}{2}\sigma_\infty \sin 2\theta\, r^2 + \cancel{c_1'(\theta) r}^{\,0} + \cancel{c_2'(\theta)}^{\,0}$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{2}\sigma_\infty \sin 2\theta\, r\right) = -\frac{1}{2}\sigma_\infty \sin 2\theta \quad \checkmark$$

as $r \to \infty$, $\sigma_{r\theta} = -\frac{1}{2}\sigma_\infty \sin 2\theta$ $\checkmark$

(from #3)

thus, the boundary conditions are satisfied

7. Mitchell's solution

$$\phi = \phi_0(r) + \phi_2(r)\cos 2\theta$$

with

$$\phi_0(r) = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0$$
$$\phi_2(r) = a_2 r^2 + b_2 r^4 + c_2 r^{-2} + d_2$$

Determine the values for the constants

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}$$
$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial}{\partial \theta}\left(a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 + a_2 r^2 \cos 2\theta + b_2 r^4 \cos 2\theta + c_2 r^{-2}\cos 2\theta + d_2 \cos 2\theta\right)$$
$$= -2a_2 r^2 \sin 2\theta - 2b_2 r^4 \sin 2\theta - 2c_2 r^{-2}\sin 2\theta - 2d_2 \sin 2\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -4a_2 r^2 \cos 2\theta - 4b_2 r^4 \cos 2\theta - 4c_2 r^{-2}\cos 2\theta - 4d_2 \cos 2\theta$$

$$\frac{\partial \phi}{\partial r} = \frac{a_0}{r} + 2b_0 r + 2c_0 r \ln r + c_0 r + 2a_2 r \cos 2\theta + 4b_2 r^3 \cos 2\theta - 2c_2 r^{-3}\cos 2\theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{-a_0}{r^2} + 2b_0 + 2c_0 \ln r + 2c_0 + c_0 + 2a_2 \cos 2\theta + 12 b_2 r^2 \cos 2\theta + 6c_2 r^{-4}\cos 2\theta$$

SARAH MALAIK

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}$$

$$= -4a_2\cos 2\theta - 4b_2 r^2\cos 2\theta - 4c_2 r^{-4}\cos 2\theta - 4d_2 r^{-2}\cos 2\theta + a_0 r^{-2} + 2b_0$$

$$+2c_0\ln(r) + c_0 + 2a_2\cos 2\theta + 4b_2 r^2\cos 2\theta - 2c_2 r^{-4}\cos 2\theta$$

$$\sigma_{rr} = -2a_2\cos 2\theta - 6c_2 r^{-4}\cos 2\theta - 4d_2 r^{-2}\cos 2\theta + a_0 r^{-2} + 2b_0 + 2c_0\ln(r) + c_0$$

as $r \to \infty$, $\boxed{c_0 = 0}$ to satisfy BCs ✓

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$= -\frac{\partial}{\partial r}\left(-2a_2 r\sin 2\theta - 2b_2 r^3\sin 2\theta - 2c_2 r^{-3}\sin 2\theta - 2d_2 r^{-1}\sin 2\theta\right)$$

$$\sigma_{r\theta} = 2a_2\sin 2\theta + 6b_2 r^2\sin 2\theta - 6c_2 r^{-4}\sin 2\theta - 2d_2 r^{-2}\sin 2\theta$$

a_2 ✓, b_2 ✓, c_2 ✓, d_2 ✓
a_0 ✓, b_0 ✓, c_0 ✓

$$\sigma_{\theta\theta} = -a_0 r^{-2} + 2b_0 + 2a_2\cos 2\theta + 12b_2 r^2\cos 2\theta + 6c_2 r^{-4}\cos 2\theta$$

$$\sigma_{rr} = -2a_2\cos 2\theta - 6c_2 r^{-4}\cos 2\theta - 4d_2 r^{-2}\cos 2\theta + a_0 r^{-2} + 2b_0$$

$r=a$: $\sigma_{rr} = 0$

$$0 = (-2a_2 - 6c_2 a^{-4} - 4d_2 a^{-2})\cos 2\theta + a_0 a^{-2} + 2b_0$$

$$\cos 2\theta \neq 0$$

$$\Rightarrow -2a_2 - 6c_2 a^{-4} - 4d_2 a^{-2} = 0$$

$$\Rightarrow a_0 = -2b_0 a^2$$

as $r \to \infty$: $\sigma_{\theta\theta} = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)$

$\sigma_{rr} = \frac{1}{2}\sigma_\infty(1+\cos 2\theta)$

$\sigma_{r\theta} = -\frac{1}{2}\sigma_\infty\sin 2\theta$

$$-\frac{1}{2}\sigma_\infty\sin 2\theta = 2a_2\sin 2\theta + 6b_2 r^2\sin 2\theta - 6c_2 r^{-4}\sin 2\theta - 2d_2 r^{-2}\sin 2\theta$$

$$-\frac{1}{2}\sigma_\infty = 2a_2 + 6b_2 \overset{r\to\infty}{r^2} - 6c_2 r^{-4} - 2d_2 r^{-2}$$

to satisfy this condition, $\boxed{b_2 = 0}$ ✓

SARAH MALAIK

at $r=a$: $\sigma_{r\theta}=0$

$0 = 2a_2 \sin 2\theta - 6c_2 a^{-4}\sin 2\theta - 2d_2 a^{-2}\sin 2\theta$ ← doesn't help

↑ wait, I was wrong

as $r\to\infty$: $\sigma_{r\theta} = -\frac{1}{2}\sigma_\infty \sin 2\theta$

as $r\to\infty$

$$-\frac{1}{2}\sigma_\infty = 2a_2 - \cancel{6c_2 r^{-4}}^{\to 0} - \cancel{2d_2 r^{-2}}^{\to 0}$$

$$\Rightarrow \boxed{a_2 = -\frac{\sigma_\infty}{4}}$$

✓

as $r\to\infty$ $\sigma_{rr} = \frac{1}{2}\sigma_\infty(1+\cos 2\theta)$

$$\frac{1}{2}\sigma_\infty(1+\cos 2\theta) = -2a_2\cos 2\theta - \cancel{6c_2 r^{-4}}^{\to 0}\cos 2\theta - \cancel{4d_2 r^{-2}}^{\to 0}\cos 2\theta + \cancel{a_0 r^{-2}}^{\to 0}$$

↑ already have a_2

as $r\to\infty$ $\sigma_{\theta\theta} = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)$

$$\frac{1}{2}\sigma_\infty(1-\cos 2\theta) = -\cancel{a_0 r^{-2}}^{\to 0} + 2\left(-\frac{\sigma_\infty}{4}\right)\cos 2\theta + \cancel{6c_2 r^{-4}}^{\to 0}\cos 2\theta + 2b_0$$

$$\frac{1}{2}\sigma_\infty - \cancel{\frac{1}{2}\sigma_\infty\cos 2\theta} + \cancel{\frac{1}{2}\sigma_\infty\cos 2\theta} = 2b_0$$

$$\Rightarrow \boxed{b_0 = \frac{\sigma_\infty}{4}} \Rightarrow \boxed{a_0 = -\frac{1}{2}\sigma_\infty a^2}$$ ✓

✓

at $r=a$: $\sigma_{r\theta}=0$

$$0 = 2a_2\sin 2\theta - 6c_2 a^{-4}\sin 2\theta - 2d_2 a^{-2}\sin 2\theta$$

$$0 = 2a_2 - 6c_2 a^{-4} - 2d_2 a^{-2} = -\frac{\sigma_\infty}{2} - 6c_2 a^{-4} - 2d_2 a^{-2}$$

$\sigma_{r\theta}=0$ at $r=a$: $\quad 0 = -\frac{\sigma_\infty}{2} - 6c_2 a^{-4} - 2d_2 a^{-2}$

$\sigma_{rr}=0$ at $r=a$: $\quad 0 = \frac{\sigma_\infty}{2} - 6c_2 a^{-4} - 4d_2 a^{-2}$

$$0 = -\sigma_\infty + 2d_2 a^{-2} \Rightarrow \boxed{d_2 = \frac{\sigma_\infty a^2}{2}}$$ ✓

$$\frac{\sigma_\infty}{2} = -6c_2 a^{-4} - \sigma_\infty \Rightarrow \frac{3\sigma_\infty}{2} = -6c_2 a^{-4} \Rightarrow \boxed{c_2 = \frac{-\sigma_\infty a^4}{4}}$$ ✓

SARAH MALAK

then ϕ_0 can be anything (constant) } we can make $\phi_0 = 0$ as well for simplicity
↳ (cannot depend on r or θ)

8. Show that the stresses are:

$$\sigma_{rr}(r,\theta) = \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\sigma_\infty}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta}(r,\theta) = \frac{\sigma_\infty}{2}\left(1-3\frac{a^4}{r^4}+2\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\phi = \phi_0(r) + \phi_2(r)\cos 2\theta$$

$$= -\frac{1}{2}\sigma_\infty a^2 \ln(r) + \frac{\sigma_\infty}{4}r^2 + \cancel{\phi_0}^{\,0 \text{ (or a constant)}} + \left(-\frac{\sigma_\infty}{4}r^2 - \frac{\sigma_\infty a^4}{4r^2} + \frac{\sigma_\infty a^2}{2}\right)\cos 2\theta$$

$$\frac{\partial \phi}{\partial \theta} = -2\left(-\frac{\sigma_\infty}{4}r^2 - \frac{\sigma_\infty a^4}{4r^2} + \frac{\sigma_\infty a^2}{2}\right)\sin 2\theta = \left(\frac{\sigma_\infty r^2}{2} + \frac{\sigma_\infty a^4}{2r^2} - \sigma_\infty a^2\right)\sin 2\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = 4\left(\frac{\sigma_\infty r^2}{4} + \frac{\sigma_\infty a^4}{4r^2} - \frac{\sigma_\infty a^2}{2}\right)\cos 2\theta = \left(\sigma_\infty r^2 + \frac{\sigma_\infty a^4}{r^2} - 2\sigma_\infty a^2\right)\cos 2\theta$$

$$\frac{\partial \phi}{\partial r} = -\frac{1}{2}\sigma_\infty a^2 \frac{1}{r} + \frac{\sigma_\infty}{2}r + \left(-\frac{\sigma_\infty}{2}r + \frac{\sigma_\infty a^4}{2r^3}\right)\cos 2\theta =$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{\sigma_\infty}{2}a^2\frac{1}{r^2} + \frac{\sigma_\infty}{2} + \left(-\frac{\sigma_\infty}{2} - \frac{3\sigma_\infty a^4}{2r^4}\right)\cos 2\theta$$

<u>σ_{rr}</u>|

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$= \left(\sigma_\infty + \frac{\sigma_\infty a^4}{r^4} - \frac{2\sigma_\infty a^2}{r^2}\right)\cos 2\theta - \frac{1}{2}\sigma_\infty\frac{a^2}{r^2} + \frac{\sigma_\infty a^4}{2r^4}\cos 2\theta + \frac{\sigma_\infty}{2} - \frac{\sigma_\infty}{2}\cos 2\theta$$

$$\sigma_{rr} = \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right) + \left(\sigma_\infty + \frac{\sigma_\infty a^4}{r^4} - \frac{2\sigma_\infty a^2}{r^2} + \frac{\sigma_\infty a^4}{2r^4} - \frac{\sigma_\infty}{2}\right)\cos 2\theta$$

$$\sigma_{rr} = \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta \quad \checkmark$$

Sarah Malak

$\underline{\sigma_{\theta\theta}|}$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$= \frac{1}{2}\frac{\sigma_\infty a^2}{r^2} + \frac{\sigma_\infty}{2} - \left(\frac{\sigma_\infty}{2} + \frac{3\sigma_\infty a^4}{2r^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_\infty}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \quad \checkmark$$

$\underline{\sigma_{r\theta}|}$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= -\frac{\partial}{\partial r}\left(\frac{\sigma_\infty r}{2} + \frac{\sigma_\infty a^4}{2r^3} - \frac{\sigma_\infty a^2}{r}\right)\sin 2\theta$$

$$= -\left(\frac{\sigma_\infty}{2} - \frac{3\sigma_\infty a^4}{2r^4} + \frac{\sigma_\infty a^2}{r^2}\right)\sin 2\theta$$

$$\sigma_{r\theta} = -\frac{\sigma_\infty}{2}\left(1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right)\sin 2\theta$$

↑ does this negative go away? Or is the problem statement incorrect?

9. From our BCs: as $r \to \infty$

$\sigma_{11} = \sigma_\infty$

$\Downarrow$

$\sigma_{rr} = \frac{1}{2}\sigma_\infty(1+\cos 2\theta)$	σ_∞
$\sigma_{\theta\theta} = \frac{1}{2}\sigma_\infty(1-\cos 2\theta)$	σ_∞
$\sigma_{r\theta} = -\frac{1}{2}\sigma_\infty \sin 2\theta$	$\frac{1}{2}\sigma_\infty$

at $r = a$

$\sigma_{rr} = 0$

$\sigma_{r\theta} = 0$

$$\sigma_{\theta\theta} = \frac{\sigma_\infty}{2}(1+1) - \frac{\sigma_\infty}{2}(1+3)\cos 2\theta$$

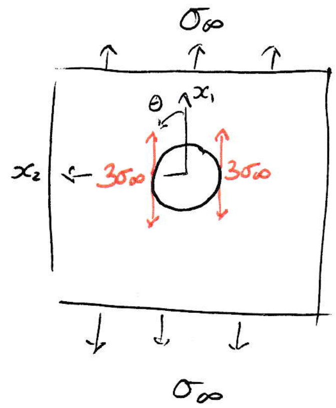
maximum value for $\cos 2\theta \Rightarrow \theta = \pm\frac{\pi}{2} \Rightarrow \cos 2\theta = -1$

maximum value for $\sin 2\theta \Rightarrow \theta = \pm\frac{\pi}{4} \Rightarrow \sin 2\theta = \pm 1$

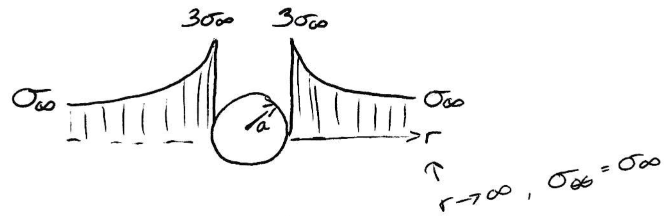
$$\sigma_{\theta\theta} = \sigma_\infty + 2\sigma_\infty = 3\sigma_\infty$$

SARAH MALAK

the stress is greatest at $\theta = \pm\frac{\pi}{2}$ at $r=a$

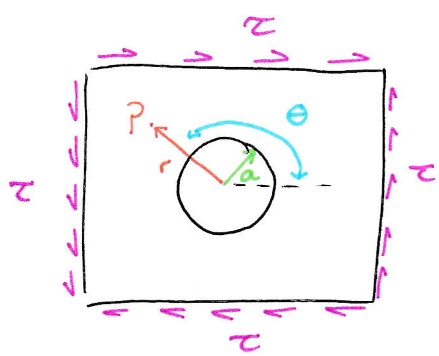


the stress concentration factor is 3.



Problem 3

1.



Stress distribution in plate as fcn of $r + \theta$

Boundary Conditions

at $r=a$: $\sigma_{rr} = 0$

$\sigma_{r\theta} = 0$

as $r\to\infty$: $\sigma_{11} = 0$

$\sigma_{22} = 0$

$\sigma_{12} = \tau$

Transform into polar coordinates:

SARAH MALAK

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \tau\sin\theta & \tau\cos\theta \\ \tau\cos\theta & -\tau\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2\tau\sin\theta\cos\theta & \tau(-\sin^2\theta + \cos^2\theta) \\ \tau(-\sin^2\theta + \cos^2\theta) & -2\tau\sin\theta\cos\theta \end{bmatrix}$$

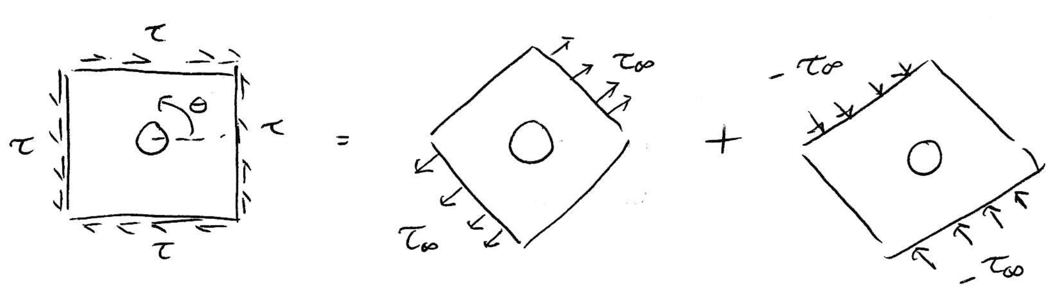
$$= \begin{bmatrix} \tau\sin 2\theta & \tau\cos 2\theta \\ \tau\cos 2\theta & -\tau\sin 2\theta \end{bmatrix}$$

$$\sigma_{rr} = \tau\sin 2\theta$$

$$\sigma_{\theta\theta} = -\tau\sin 2\theta$$

$$\sigma_{r\theta} = \tau\cos 2\theta$$

We could also use superposition to find $\underline{\sigma}(r,\theta)$:



(1) (2)

In (1): $\theta = \frac{\pi}{4}$, $\sigma_\infty = \tau$

In (2): $\theta = -\frac{\pi}{4}$, $\sigma_\infty = -\tau$

Now we can use our solution from the previous problem:

SARAH MALAK

$$\sigma_{rr} = \frac{\sigma_\infty}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_\infty}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_\infty}{2}\left(1-3\frac{a^4}{r^4}+2\frac{a^2}{r^2}\right)\sin 2\theta$$

(1): $\sigma_\infty = \tau \ , \quad \theta = \frac{\pi}{4}$

$$\sigma_{rr} = \frac{\tau}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\tau}{2}\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\sin(2\theta)$$

$$\sigma_{\theta\theta} = \frac{\tau}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{\tau}{2}\left(1+3\frac{a^4}{r^4}\right)\sin(2\theta)$$

$$\sigma_{r\theta} = \frac{\tau}{2}\left(1-3\frac{a^4}{r^4}+2\frac{a^2}{r^2}\right)\cos(2\theta)$$

(2): $\sigma_\infty = -\tau \ , \quad \theta = -\frac{\pi}{4}$

$$\sigma_{rr} = -\frac{\tau}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\tau}{2}\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\sin 2\theta$$

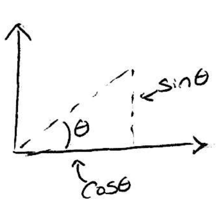
$$\sigma_{\theta\theta} = -\frac{\tau}{2}\left(1-\frac{a^2}{r^2}\right) - \frac{\tau}{2}\left(1+3\frac{a^4}{r^4}\right)\sin 2\theta$$

$$\sigma_{r\theta} = \frac{\tau}{2}\left(1-3\frac{a^4}{r^4}+2\frac{a^2}{r^2}\right)\cos 2\theta$$

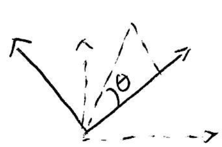
(1) + (2) =

$$\boxed{\begin{aligned} \sigma_{rr} &= \tau\left(1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right)\sin 2\theta \\ \sigma_{\theta\theta} &= -\tau\left(1+3\frac{a^4}{r^4}\right)\sin 2\theta \\ \sigma_{r\theta} &= \tau\left(1-3\frac{a^4}{r^4}+2\frac{a^2}{r^2}\right)\cos 2\theta \end{aligned}}$$

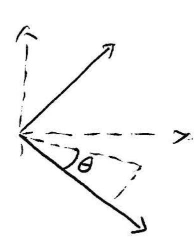
original

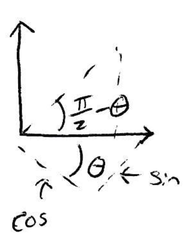


$\theta = \frac{\pi}{4}$



$\theta = -\frac{\pi}{4}$





$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

SARAH MALAK

2. Maximum $\overset{(+)}{\text{tensional}}$ and $\overset{(-)}{\text{compressional}}$ hoop stress at edge of hole

maximum value for $\sin 2\theta \Rightarrow \theta = \pm\frac{\pi}{4} \Rightarrow \sin 2\theta = \pm 1$

$$\sigma_{\theta\theta} = -\tau(1+3)\underbrace{\sin 2\theta}_{\theta = \pm\frac{\pi}{4} \Rightarrow \sin 2\theta = \pm 1}$$

maximum tensional hoop stress

$$\sigma_{\theta\theta} = 4\tau$$

maximum compressional hoop stress

$$\sigma_{\theta\theta} = -4\tau$$

$\Rightarrow$ stress concentration factor $= 4$

Problem 4

1. $$\phi = \frac{\tau}{\pi}\left[\frac{1}{2}x_2^2 \ln(x_1^2 + x_2^2) + x_1x_2\arctan\left(\frac{x_2}{x_1}\right) - x_2^2\right]$$

Biharmonic Equation

$$\frac{\partial^4\phi}{\partial x_1^4} + \frac{\partial^4\phi}{\partial x_2^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2} = 0$$

I have been trying to do this for the past two hours by hand and with Matlab. Either I don't know how to work Matlab (very possible) or this equation does not satisfy the Biharmonic Equation.

What I tried to put into Matlab:

SARAH MALAK

```
syms tau x y
Phi = (tau/pi)*((1/2)*y^2*log(x^2+y^2)+x*y*atan(y/x)-y^2);
A= diff(Phi,x,x,x,x);
B= diff(Phi,y,y,y,y);
C= diff(Phi,x,x,y,y);
A+B+2*C

ans =

-(tau*((6*y^2)/(x^2+y^2)^2-(48*x^2*y^2)/(x^2+y^2)^3
  +(48*x^4*y^2)/(x^2+y^2)^4 +(40*y^4)/(x^6*(y^2/x^2+1)^2)
  -(88*y^6)/(x^8*(y^2/x^2+1)^3)+(48*y^8)/(x^10*(y^2/x^2+1)^4)))/pi
  -12*tau*((4*x^2)/(x^2+y^2)^2-2/(x^2+y^2)...
```

this isn't zero

the matlab website said $\ln(x)$ is typed $\log(x)$

I have no idea (the idea is fine)

2.

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

$$\frac{\partial \phi}{\partial x_1} = \frac{\tau}{\pi}\left[\frac{1}{2} x_2^2 \frac{2x_1}{x_1^2+x_2^2} + x_2 \arctan\left(\frac{x_2}{x_1}\right) + x_1 x_2 \frac{-x_2}{x_1^2+x_2^2}\right]$$

$$\frac{\partial^2 \phi}{\partial x_1^2} = \frac{\tau}{\pi}\left[\frac{1}{2} x_2^2 \frac{2(x_1^2+x_2^2)-2x_1(2x_1)}{(x_1^2+x_2^2)^2} + x_2 \frac{-x_2}{x_1^2+x_2^2} + \frac{-x_2^2(x_1^2+x_2^2)-2x_1(-x_1x_2^2)}{(x_1^2+x_2^2)^2}\right]$$

$$= \frac{\tau}{\pi}\left[\frac{\cancel{x_2^2x_1^2}+\cancel{x_2^4}-\cancel{2x_1^2x_2^2}}{(x_1^2+x_2^2)^2} - \frac{x_2^2}{(x_1^2+x_2^2)} - \frac{(\cancel{x_1^2x_2^2}+\cancel{x_2^4}-\cancel{2x_1^2x_2^2})}{(x_1^2+x_2^2)^2}\right]$$

$$\frac{\partial^2 \phi}{\partial x_1^2} = \boxed{-\frac{\tau}{\pi}\frac{x_2^2}{(x_1^2+x_2^2)} = \sigma_{22}}$$

SARAH MALAK

$$\frac{\partial \phi}{\partial x_2} = \frac{\tau}{\pi}\left[\frac{1}{2}(2x_2)\ln(x_1^2+x_2^2) + \frac{1}{2}x_2^2 \frac{2x_2}{x_1^2+x_2^2} + x_1 \arctan\left(\frac{x_2}{x_1}\right) + x_1x_2 \frac{x_1}{x_1^2+x_2^2} - 2x_2\right]$$

$$\frac{\partial^2 \phi}{\partial x_2^2} = \frac{\tau}{\pi}\left[\ln(x_1^2+x_2^2) + x_2 \frac{2x_2}{x_1^2+x_2^2} + \frac{1}{2} \frac{6x_2^2(x_1^2+x_2^2) - 2x_2(2x_2^3)}{(x_1^2+x_2^2)^2} + \frac{x_1^2}{x_1^2+x_2^2}\right.$$

$$\left.\frac{x_1^2(x_1^2+x_2^2) - 2x_2(x_1^2 x_2)}{(x_1^2+x_2^2)^2} - 2\right]$$

$$= \frac{\tau}{\pi}\left[\ln(x_1^2+x_2^2) + \frac{2x_2^2 + x_1^2}{x_1^2+x_2^2} + \frac{1}{2} \frac{6\cancel{x_1^2x_2^2} + 6\cancel{x_2^4} - 4\cancel{x_2^4} + \cancel{x_1^4} + \cancel{x_1^2x_2^2} - \cancel{2x_1^2x_2^2}}{(x_1^2+x_2^2)^2}\right.$$

$$\left.-2\right]$$

$$\frac{\partial^2 \phi}{\partial x_2^2} = \frac{\tau}{\pi}\left[\ln(x_1^2+x_2^2) + \underbrace{\frac{(2x_2^2+x_1^2)(x_1^2+x_2^2)}{(x_1^2+x_2^2)^2}}_{}\right.$$

$$\cancel{2x_1^2x_2^2} + \cancel{2x_2^4} + \cancel{x_1^4} + \cancel{x_1^2x_2^2}$$

$$x_1^2x_2^2: \quad 2 + 1 + \frac{6}{2} + \frac{1}{2} - \frac{2}{2} = \frac{11}{2}$$

$$x_2^4: \quad 2 + \frac{6}{2} - \frac{4}{2} =$$

$$x_1^4: \quad 1 + \frac{1}{2} = \frac{3}{2}$$

$$\boxed{\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} = \frac{\tau}{\pi}\left[\ln(x_1^2+x_2^2 + \frac{\frac{11}{2}x_1^2 x_2^2 + x_2^4 + \frac{3}{2}x_1^4}{(x_1^2+x_2^2)^2} - 2\right]}$$

← this doesn't seem right

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\tau}{\pi}\cancel{\left[\frac{1}{2}(2x_2)\frac{2x_1}{x_1^2+x_2^2}\right.} + \frac{1}{2}x_2^2 \qquad \frac{x_1x_2^2}{x_1^2+x_2^2} \qquad -x_1x_2^2$$

$$= \frac{\tau}{\pi}\left[\frac{2x_2x_1(x_1^2+x_2^2) - 2x_2(x_1x_2^2)}{(x_1^2+x_2^2)^2} + \arctan\left(\frac{x_2}{x_1}\right) + x_2 \frac{x_1}{x_1^2+x_2^2}\right.$$

$$- \frac{2x_1x_2(x_1^2+x_2^2) - (2x_2)(-x_1x_2^2)}{(x_1^2+x_2^2)^4}$$

SARAH MALAK

$$= \frac{\tau}{\pi}\left[\frac{2x_1^3x_2+2x_1x_2^3-2x_1x_2^3}{(x_1^2+x_2^2)^2} + \arctan\left(\frac{x_2}{x_1}\right) + \frac{x_1x_2}{x_1^2+x_2^2} - \frac{2x_1^3x_2-2x_1x_2^3+2x_1x_2^3}{(x_1^2+x_2^2)^4}\right]$$

$$= \frac{\tau}{\pi}\Big[$$

It's almost midnight and I'm exhausted.

You really need a program for this

-8

If the Biharmonic Equation is satisfied, compatibility is satisfied.

trying again:

SARAH MALAK

$$\frac{\partial \phi}{\partial x_2} = \frac{\tau}{\pi}\Big[\underbrace{\tfrac{1}{2}(2x_2)\ln(x_1^2+x_2^2)}_{x_2\ln(x_1^2+x_2^2)} + \underbrace{\tfrac{1}{2}x_2^2\,\frac{2x_2}{x_1^2+x_2^2}}_{\frac{x_2^3}{x_1^2+x_2^2}} + x_1\arctan\left(\frac{x_2}{x_1}\right) + \underbrace{x_1x_2\,\frac{x_1}{x_1^2+x_2^2}}_{\frac{x_1^2x_2}{x_1^2+x_2^2}} - 2x_2 \Big]$$

$$\frac{\partial^2 \phi}{\partial x_2^2} = \frac{\tau}{\pi}\Big[\ln(x_1^2+x_2^2) + x_2\,\frac{2x_2}{x_1^2+x_2^2} + \frac{3x_2^2(x_1^2+x_2^2) - 2x_2(x_2^3)}{(x_1^2+x_2^2)^2} + x_1\,\frac{x_1}{x_1^2+x_2^2}$$

$$+ \frac{x_1^2(x_1^2+x_2^2) - 2x_2(x_1^2x_2)}{(x_1^2+x_2^2)^2} - 2\Big]$$

$$= \frac{\tau}{\pi}\Big[\ln(x_1^2+x_2^2) + \frac{2x_2^2}{x_1^2+x_2^2} + \frac{3x_1^2x_2^2 - 3x_2^4 + 2x_2^4}{(x_1^2+x_2^2)^2} + \frac{x_1^2}{x_1^2+x_2^2}$$

$$+ \frac{x_1^4 + x_1^2x_2^2 - 2x_1^2x_2^2}{(x_1^2+x_2^2)^2} - 2\Big]$$

$$= \frac{\tau}{\pi}\Big[\ln(x_1^2+x_2^2) + \frac{2x_2^2+x_1^2}{x_1^2+x_2^2} + \frac{2x_1^2x_2^2 - x_2^4 + x_1^4}{(x_1^2+x_2^2)^2} - 2\Big]$$

$$(2x_2^2+x_1^2)(x_1^2+x_2^2) = 2x_1^2x_2^2 + 2x_2^4 + x_1^4 + x_1^2x_2^2$$

$$2x_2^4 + x_1^4 + 3x_1^2x_2^2$$

$$2x_2^4 + x_1^4 + 3x_1^2x_2^2 + 2x_1^2x_2^2 - x_2^4 + x_1^4$$

$$2x_1^4 + x_2^4 + 5x_1^2x_2^2 \quad - 2(x_1^2+x_2^2)^2$$

$$- 2(x_1^2+x_2^2)(x_1^2+x_2^2)$$

$$- 2(x_1^4 + 2x_1^2x_2^2 + x_2^4)$$

$$- 2x_1^4 - 4x_1^2x_2^2 - 2x_2^4$$

$$x_1^2x_2^2 - x_1^4$$

$$\boxed{\sigma_{11} = \frac{\tau}{\pi}\Big[\ln(x_1^2+x_2^2) + \frac{x_1^2x_2^2 - x_1^4}{(x_1^2+x_2^2)^2} \Big]}$$

← still doesn't seem right... I wish I could get matlab to work

I give up

Last point missing.

OK

-5