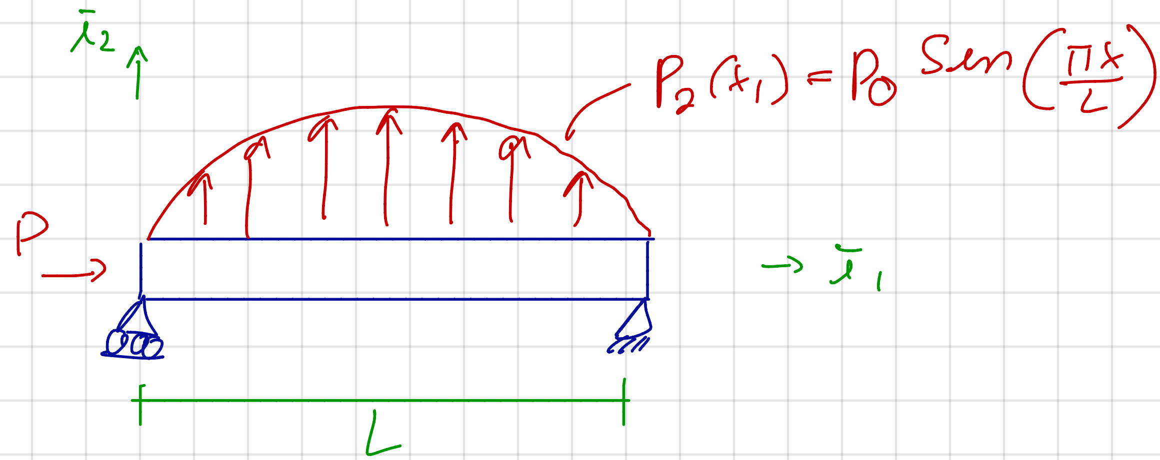
Example:

$H^c_{23} = 0$



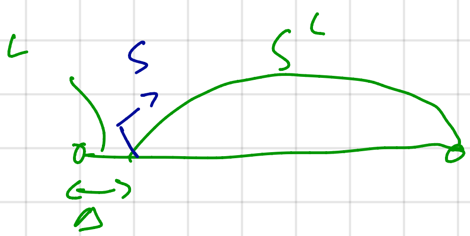
$$A = \frac{1}{2}\int_0^L \left[S\left(\frac{d\bar{u}_1}{dx_1}\right)^2 + H^c_{33}\left(\frac{d^2 u_2}{dx_1^2}\right)^2 \right] dx_1$$

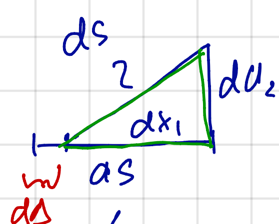
* Interested only in $u_2(x_1)$. Neglect the axial strain energy

$$A = \frac{1}{2}\int_0^L H^c_{33}\left(\frac{d^2 u_2}{dx_1^2}\right)^2 dx_1$$

$$\Phi = -\int_0^L P_2(x_1)\cdot u_2(x_1)\, dx_1 - P\cdot\Delta$$

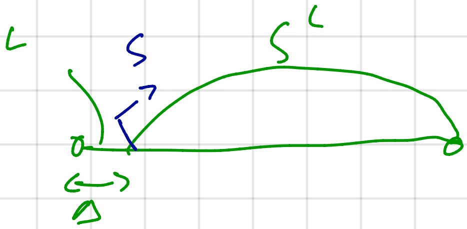
Δ - Horizontal displacement at the beam due to changes in curvature.

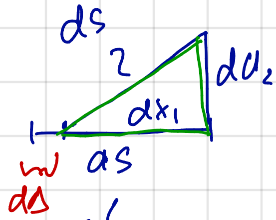




$$\Delta = \int_0^L ds - \int_0^L dx_1 = \int_0^L (ds - dx_1)$$

$$ds^2 = dx_1^2 + du_2^2$$





$$\Delta = \int_0^L ds - \int_0^L dx_1 = \int_0^L (ds - dx_1)$$

$$ds^2 = dx_1^2 + du_2^2$$

$$ds = \left(1 + \left(\frac{du_2}{dx_1}\right)^2\right)^{\frac{1}{2}} dx_1$$

$$ds = \left(1 + \frac{1}{2}\left(\frac{du_2}{dx_1}\right)^2 + \dots\right) dx_1$$

$$\Delta \simeq \int_0^L \frac{1}{2}\left(\frac{du_2}{dx_1}\right)^2 dx_1$$

$$A = \frac{1}{2}\int_0^L H_{33}^c \left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1$$

$$\Phi = -\int_0^L P_2(x_1)\cdot u_2(x_1)\, dx_1 - P\cdot\Delta$$

$$\Phi = -\int_0^L \left[P_2(x_1)\, u_2(x_1) + \frac{P}{2}\left(\frac{du_2}{dx_1}\right)^2\right] dx_1$$

$$\Pi = A + \Phi$$

$$\Pi = \int_0^L \left[\frac{H_{33}^c}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 - \frac{P}{2}\left(\frac{du_2}{dx_1}\right)^2 - P_2(x_1)\, u_2(x_1)\right] dx_1$$

$$\Pi = \int_0^L \left[\frac{H_{33}^c}{2} \left(\frac{d^2 u_2}{dx_1^2} \right)^2 - \frac{P}{2} \left(\frac{d u_2}{d x_1} \right)^2 - P_2(x_1)\, u_2(x_1) \right] dx_1$$

* The solution is at the form

$$u_2 = C \cdot \sin\left(\frac{\pi x}{L}\right) \quad -1 \text{ Unknown, } C.$$

$\rightarrow$ Satisfies Kinematic B.C.'s!

$$\Pi = \int_0^L \left[\frac{H_{33}^c}{2} \left(C \cdot \overbrace{\sin\left(\frac{\pi x}{L}\right)}^{L/2} \frac{\pi^2}{L^2} \right)^2 - \frac{P}{2} \left(C \cdot \cos\left(\frac{\pi x}{L}\right) \frac{\pi}{L} \right)^2 - P_0 \cdot C \cdot \sin\left(\frac{\pi x}{L}\right)^2 \right] dx_1$$

$$\Pi = \left[\frac{H_{33}^c}{2} \left(C^2 \frac{L}{2} \frac{\pi^4}{L^4} \right) - \frac{P}{2} C^2 \frac{L}{2} \frac{\pi^2}{L^2} - P_0 C \frac{L}{2} \right]$$

$$\Pi = \frac{L}{4} \left(\frac{C^2 \pi^2}{L^2} \left(H_{33}^c \frac{\pi^2}{L^2} - P \right) - 2 P_0 C \right]$$

For the system to be in equilibrium

$$d\Pi = \frac{\partial \Pi}{\partial C} = 0$$

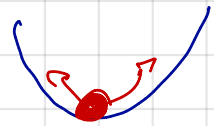
$$\frac{\partial \Pi}{\partial C} = \frac{L}{4} \underbrace{\left(\frac{2 C \pi^2}{L^2} \left(H_{33}^c \frac{\pi^2}{L^2} - P \right) - 2 P_0 \right)}_{=0} = 0$$

$$\boxed{C = \frac{P_0 L^4}{\pi^2 (H_{33}^c \pi^2 - P L^2)}} \Rightarrow u_2 = \frac{P_0 L^4}{\pi^2 (H_{33}^c \pi^2 - P L^2)} \cdot \sin\left(\frac{\pi x}{L}\right)$$

$$\frac{\partial \Pi}{\partial C} = \frac{L}{4}\left(\frac{2C\pi^2}{L^2}\left(H_{33}^{C}\frac{\pi^2}{L^2} - P\right) - 2P_0 \right) = 0$$

Is the equilibrium stable?

$$\frac{d^2\Pi}{dC^2} = \frac{L}{4}\cdot\frac{2\pi^2}{L^2}\underbrace{\left(H_{33}^{C}\frac{\pi^2}{L^2} - P\right)}_{>0} > 0$$



$$H_{33}^{C}\frac{\pi^2}{L^2} - P > 0$$

$$P < H_{33}^{C}\frac{\pi^2}{L^2}$$

$$P_{CR} = H_{33}^{C}\frac{\pi^2}{L^2}$$ Unstable!

→ Euler Buckling Load!