Principle of Minimum Complementary Energy

* In analogy w/ the total potential energy we may define the complementary potential energy

$$\Pi' = \underbrace{A'}_{\text{Complementary internal strain energy (stress energy)}} + \underbrace{\Phi'}_{\text{Potential of prescribed displacements}}$$

Related to virtual quantities

$$\delta W_E' = \Delta \cdot \delta F$$

-> Assume Δ is given through a potential

$$\Delta = -\frac{\partial \Phi'}{\partial F}$$

* $$\delta W_E' = -\frac{\partial \Phi'}{\partial F}\delta F = -\delta \Phi'$$

-> Assume internal forces are conservative

* $$\delta W_I' = -\delta A'$$

Recall P.C.V.W

$$\delta W' = \delta W_E' + \delta W_I' = 0$$

$$= -\delta A' - \delta \Phi' = 0$$

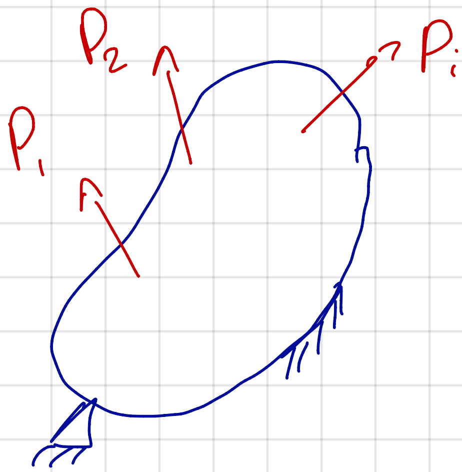
$$\delta W' = -\delta \underbrace{(A' + \Phi')}_{\Pi'} = 0$$

-> $\boxed{\delta \Pi' = 0} \longrightarrow \boxed{d\Pi' = 0}$

$\rightarrow \quad d\Pi' = 0$

* A conservative system undergoes compatible deformations if and only if the variation in the total complementary energy vanishes for all statically admissible forces.

Castigliano's 1st theorem



$$\Pi = A + \bar{\Phi}$$
$$= A - \sum_{i=1}^{N} P_i \cdot \Delta_i$$
$$\frac{\partial \Pi}{\partial \Delta_j} = \frac{\partial A}{\partial \Delta_j} - \frac{\partial}{\partial \Delta_j} \sum_{i=1}^{N} P_i \Delta_i = 0$$
$$= \frac{\partial A}{\partial \Delta_j} - P_j = 0$$

$$\boxed{P_j = \frac{\partial A}{\partial \Delta_j}}$$

* For an elastic body, the magnitude of the load at a point is equal to the derivative of A with respect to the projected displacement.

Castigliano's 2nd Theorem

$$\Pi' = A' + \Phi' = A' - \sum_{i=1}^{N} P_i \cdot \Delta_i$$

$$\frac{\partial \Pi'}{\partial P_j} = \frac{\partial A'}{\partial P_j} - \frac{\partial}{\partial P_j} \sum_{i=1}^{N} P_i \cdot \Delta_i$$

$$= \frac{\partial A'}{\partial P_j} - \Delta_j = 0$$

$$\boxed{\Delta_j = \frac{\partial A'}{\partial P_j}}$$

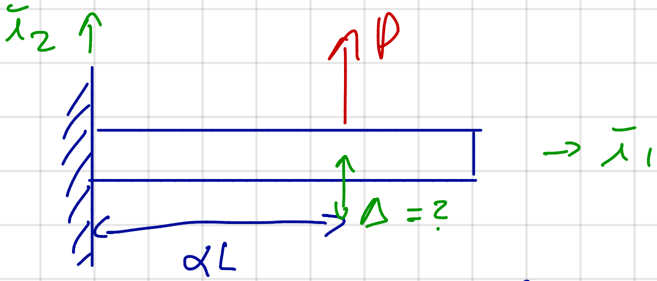
Linear Elastic $\quad \Delta_j = \dfrac{\partial A}{\partial P_j}$

$A = A'$

* For an elastic body, the deflection at a point is given by the partial derivative of the complementary strain energy with respect to the driving force!

$H_{23}^{C} = 0$

Esempio



$$\eta = \frac{x_1}{L}$$

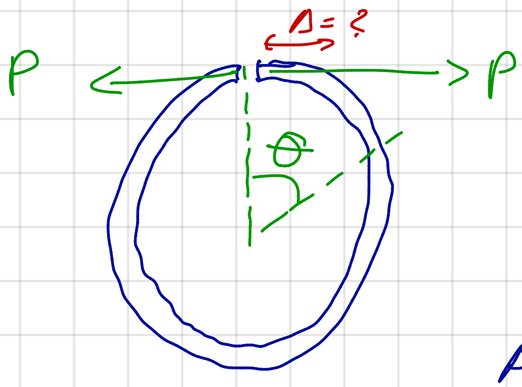
$$\mathcal{M}_3 = \begin{cases} PL(\alpha - \eta) & 0 \leq \eta \leq \alpha \\ 0 & \alpha \leq \eta < 1 \end{cases}$$

$$A' = \int_0^L \frac{1}{2} \frac{\mathcal{M}_3^2}{H_{33}^C} \, dx_1 \qquad \eta = \frac{x_1}{L} \quad d\eta L = dx_1$$

$$A' = \frac{1}{2 H_{33}^C} \int_0^\alpha (PL)^2 (\alpha - \eta)^2 \cdot L \, d\eta$$

$$A' = \frac{P^2 (\alpha L)^3}{6 \, H_{33}^C}$$

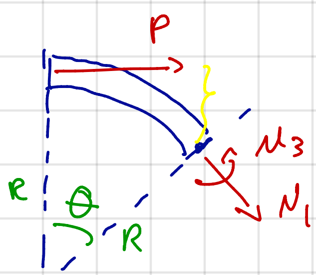
$$\Delta = \frac{\partial A'}{\partial P} = \frac{P(\alpha L)^3}{3 \, H_{33}^C}$$



Ring is subjected to both bending & axial loads

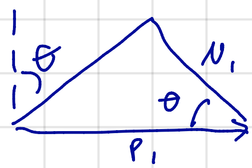
$$A' = \int_0^{2\pi} \frac{1}{2} \frac{M_3^2}{H_{33}^c} R\,d\theta + \int_0^{2\pi} \frac{1}{2} \frac{N_1^2}{S} R\,d\theta$$

$$\Delta = \frac{\partial A'}{\partial P} = \int_0^{2\pi} \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial P} R\,d\theta + \int_0^{2\pi} \frac{N_1}{S} \frac{\partial N_1}{\partial P} R\,d\theta$$



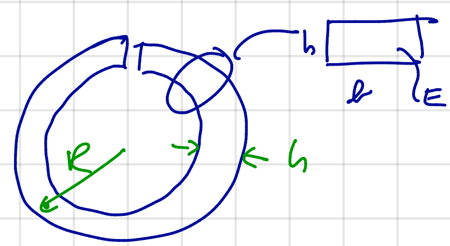
$$\Sigma M \rightarrow M_3 - P\left(R - R\cos(\theta)\right) = 0$$

$$\begin{cases} M_3 = PR(1-\cos(\theta)) \\ N_1 = P\cos(\theta) \end{cases}$$



$$\Delta = \int_0^{2\pi} \frac{PR}{H_{33}^c} (1-\cos(\theta))\, R(1-\cos(\theta)) \cdot R\,d\theta + \int_0^{2\pi} \frac{P}{S} \cos(\theta)\cos(\theta) R\,d\theta$$

$$\boxed{\Delta = \frac{PR^3}{H_{33}^c}(3\pi) + \frac{PR}{S}(\pi)}$$



$$H_{33}^c = \frac{E\,b h^3}{12}, \quad S = E\,b h$$

$$\Delta = \frac{36\pi}{E} \frac{PR^3}{b h^3} + \frac{\pi}{E} \frac{PR}{bh} = \frac{\pi}{E} \frac{PR}{bh} \Big(\underbrace{36\left(\frac{R}{h}\right)^2}_{\text{Bending}} + \underbrace{1}_{\text{Axial}} \Big) = \Delta$$

* For a thin ring, $R \gg h$, bending is the main contributor to displacement!

* Stiffness is mainly due to axial extension!