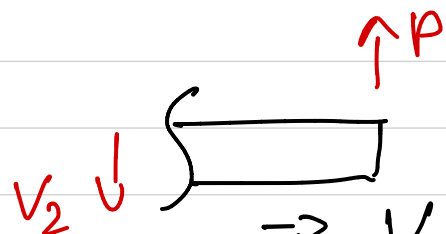
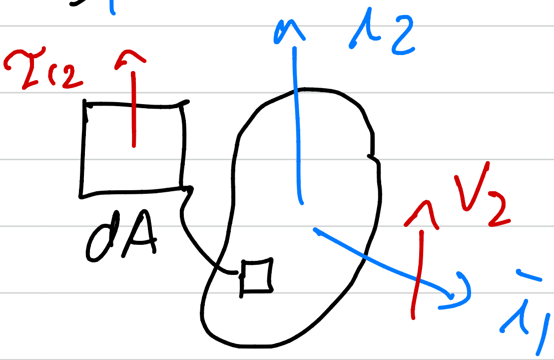
Shearing deformation in Beams. Timoshenko Beam Theory.

→ EQUILIBRIUM REQUIRES A NON-VANISHING SHEAR STRAIN



$\Rightarrow V_2 = P_2$



$$V_2 = \int_A \tau_{12}\, dA = \int_A G\gamma_{12}\, dA$$

→ EB Theory assumptions lead to a vanishing shear strain.

→ The assumption that the rotation at the section is equal to the slope at the beam

$$u_1 = \bar{u}_1(x_1) - x_2\,\Phi_3(x_1)$$

$$u_2 = \bar{u}_2(x_1)$$

$$u_3 = \bar{u}_3(x_1)$$

Slope of the Beam

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\Phi_3(x_1) + \frac{d u_2}{d x_1}$$

Rotation of the face

If $\Phi_3(x_1) = \dfrac{d u_2}{d x_1} \Rightarrow \gamma_{12} = 0$!

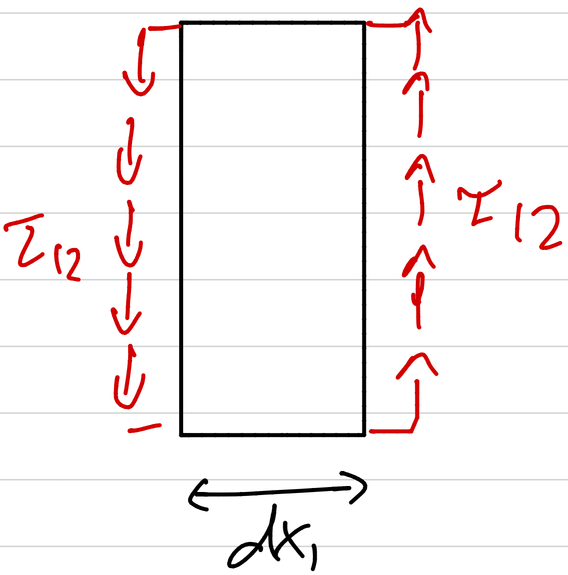
1) How do shearing deformations affect the transverse displacement?

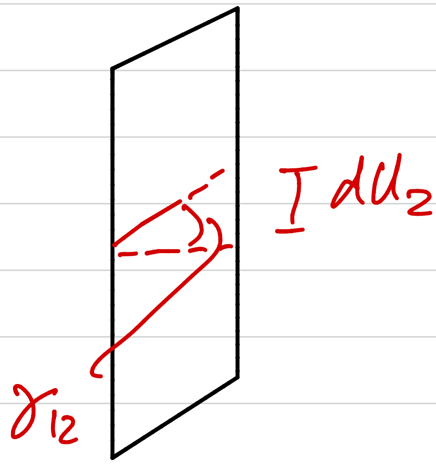
2) What is the distribution of shear stresses.

Simplified Approach

→ Assume $\phi_3(x_1) \neq du_2/dx_1$, $\phi_3(x_1)=0$

→ Assume that shear forces are uniformly distributed.





EQUILIBRIUM

$$V_2 = \int_A \tau_{12}\, dA$$

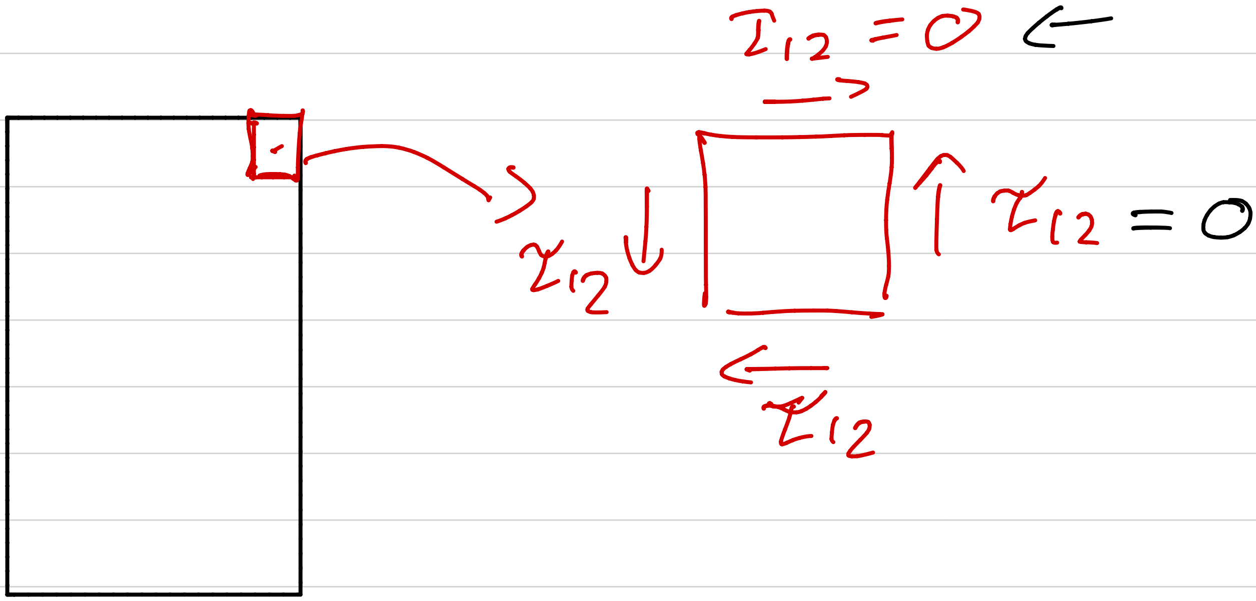
$$V_2 = \tau_{12} A$$

$$V_2 = \underbrace{G\cdot A}_{\text{SHEAR STIFFNESS}} \gamma_{12}$$

– Sectional Const. Equation.

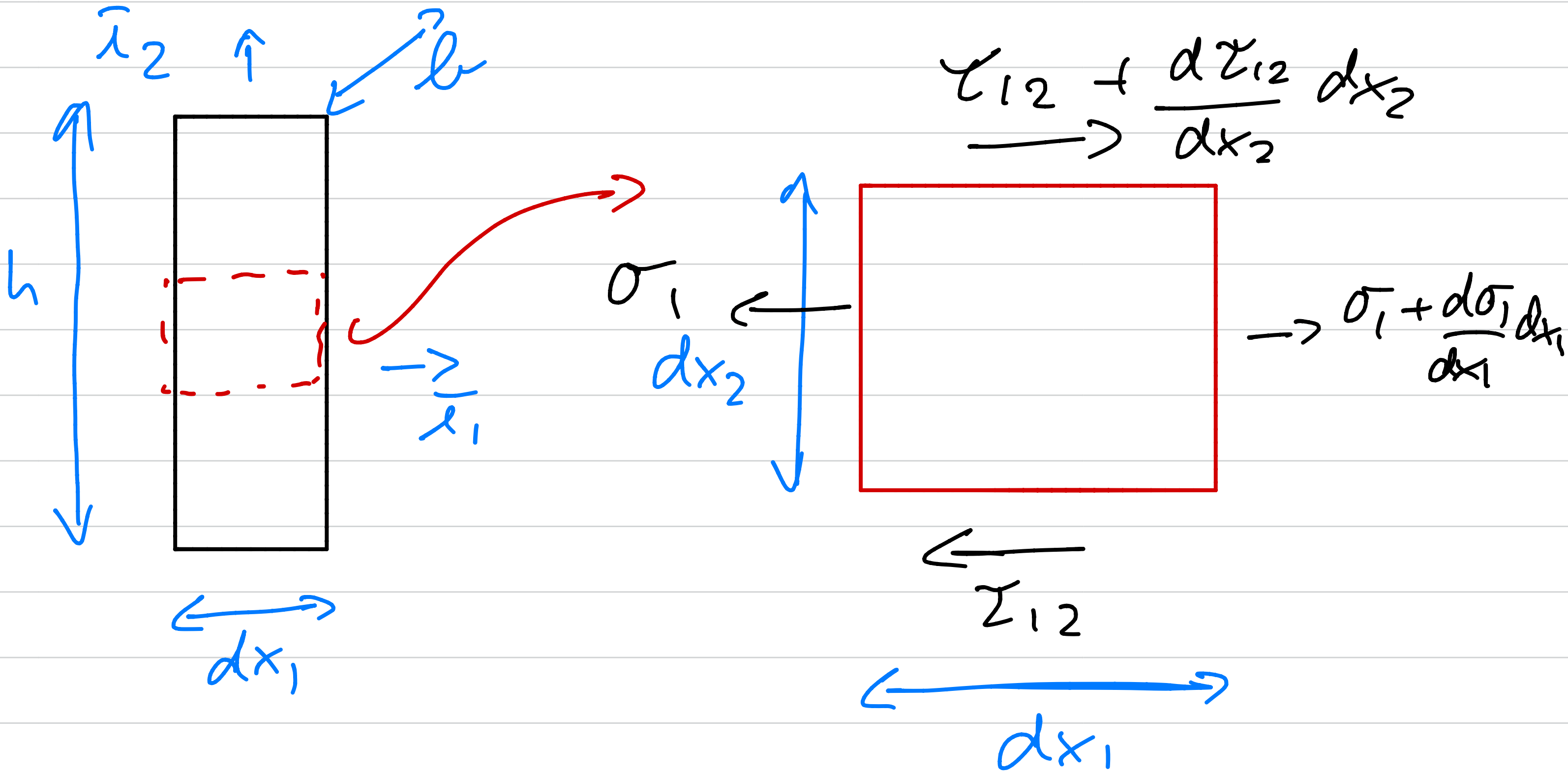
Deformation: $\gamma_{12} = \dfrac{du_2}{dx_1} = \dfrac{V_2}{G\cdot A}$

* However, we are in violation at a basic equilibrium condition



→ Shear stresses must vanish at the top and bottom faces.

EQUILIBRIUM APPROACH



Equilibrium:

$$\frac{d\sigma_1}{dx_1} + \frac{d\tau_{12}}{dx_2} = 0$$

Assume a displacement field:

$$u_1(x_1, x_2, x_3) = -x_2 \phi_3(x_1)$$

$$\epsilon_1 = -x_2 \frac{d\phi_3}{dx_1}, \quad \sigma_1 = -E x_2 \frac{d\phi_3}{dx_1}$$

$$\frac{d\tau_{12}}{dx_2} = -\frac{d\sigma_1}{dx_1} = E x_2 \frac{d^2\phi_3}{dx_1^2}$$

Integrating

$$\tau_{12} = E \frac{x_2^2}{2} \frac{d^2\phi_3}{dx_1^2} + C$$

Impose $\tau_{12} = 0$ @ $x_2 = \pm h/2$

$$* \quad \tau_{12} = \frac{E}{2}\left(x_2^2 - \frac{h^2}{4}\right)\frac{d^2\phi_3}{dx_1^2}$$

→ Parabolic distribution at shear stress.

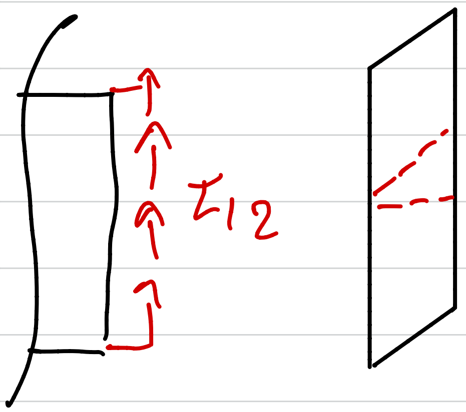
Sectional Const. Equation

$$V_2 = \int_A \tau_{12}\, dA = \int_{-h/2}^{h/2} \frac{E}{2}\left(x_2^2 - \frac{h^2}{4}\right)\frac{d^2\phi_3}{dx_1^2} \cdot b$$

$$V_2 = -\frac{E h^3 b}{12}\frac{d^2\phi_3}{dx_1^2} = -\frac{EAh^2}{12}\frac{d^2\phi_3}{dx_1^2}$$

$$* \quad \tau_{12} = -\frac{E}{2}\left(x_2^2 - \frac{h^2}{4}\right)\frac{12}{EAh^2} V_2 = -6\frac{V_2}{A}\left(\frac{1}{4} - \frac{x_2^2}{h^2}\right)$$

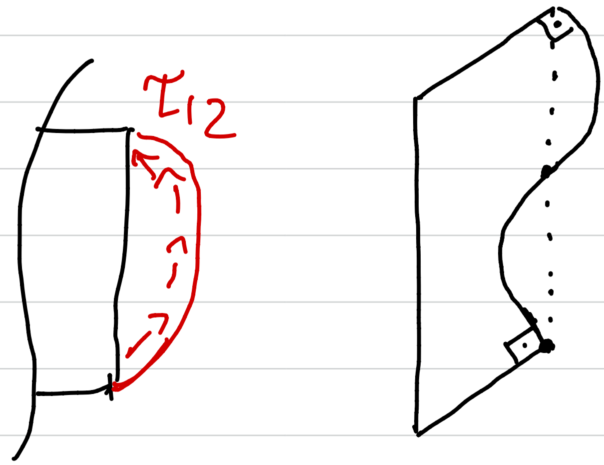
SIMPLIFIED



$$\tau_{12}^{MAX} = V_2 / A$$

* VIOLATES EQUILIBRIUM

EQUILIBRIUM



$$\tau_{12}^{MAX} = \frac{3}{2} \frac{V_2}{A}$$

→ 50% Higher

* Violates "plane sections remain plane". (i.e. Warping)

EQUIVALENT SHEAR DEFORMATION MODEL

* We can ensure that the strain energy at the two models is equivalent

$$dE = \int_A \tau_{12} \gamma_{12} \, dx_1$$ – Strain energy at a differential segment.

Simplified Model

$$dE = \frac{\tau_{12}^2}{G} \cdot A \, dx_1 = \frac{V_2^2}{GA} \, dx_1$$

Equilibrium Model

$$dE = \frac{1}{2} \int_{-h/2}^{h/2} \left[-\frac{6 V_2}{GA} \left(\frac{1}{2} - \frac{x_2^2}{h^2} \right) \right]^2 dx_2 \, dx_1 \cdot b$$

$$dE = \frac{1}{2} \frac{6}{5} \frac{V_2^2}{GA^2} (hb) \, dx_1 = \frac{1}{2} \frac{6}{5} \frac{V_2^2}{GA} \, dx_1$$

Define the shear stiffness K_{22}

$$dE = \frac{1}{2} \frac{V_2^2}{K_{22}} \rightarrow \boxed{K_{22} = \frac{5}{6} G \cdot A}$$

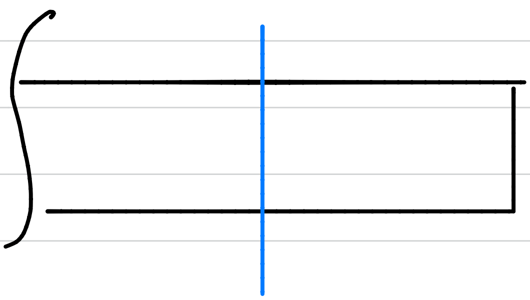
→ For a homogeneous rectangular bar.

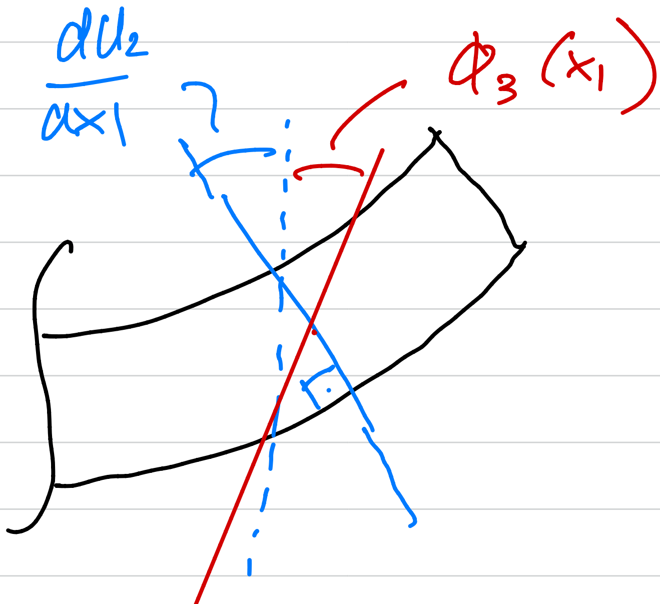
* The two models can be made equivalent in an energy sense if we substitute GA w/ the equilibrium stiffness K_{22}.

TIMOSHENKO BEAM THEORY

→ Assume that $\phi_3(x_1) \neq du_2/dx_1$ with $\phi_3(x_1)$ an unknown rotation which must be solved for.

→ Use the equilibrium shear stiffness K_{22}





Kinematics / Strain

$$u_1 = -x_2\, \phi_3(x_1)$$

$$u_2 = \bar{u}_2(x_1)$$

$$\epsilon_1 = -x_2 \frac{d\phi_3}{dx_1}$$

$$\gamma_{12} = -\phi_3(x_1) + \frac{du_2}{dx_1}$$

Function only at x_1. Strain will be uniformly distributed.

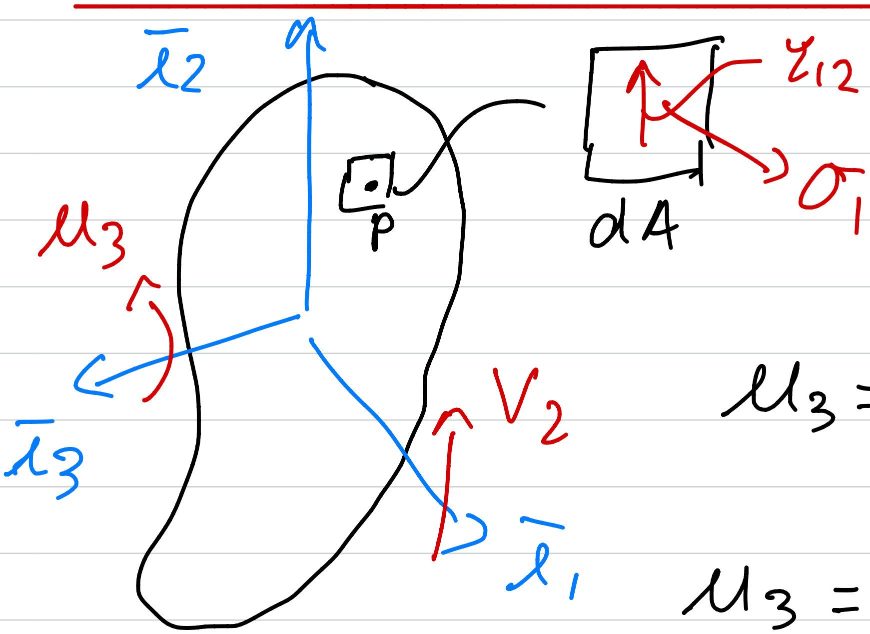
Stress

→ Assume $\sigma_2 \approx 0$

$$\sigma_1 = E\,\epsilon_1 = -E\,x_2 \frac{d\Phi_3}{dx_1}$$

$$\tau_{12} = G\,\gamma_{12} = G\left(\frac{du_2}{dx_1} - \Phi_3(x_1)\right)$$

Sectional Equilibrium



$$M_3 = -\int_A \sigma_1\, x_2\, dA$$

$$M_3 = \int E\, x_2^2 \frac{d\Phi_3}{dx_1}\, dA$$

$$\boxed{M_3 = H_{33}^{c} \frac{d\Phi_3}{dx_1}}$$

$$V_2 = \int_A \tau_{12}\, dA = \int G \cdot \left(\frac{du_2}{dx_1} - \phi_3\right) dx_1$$

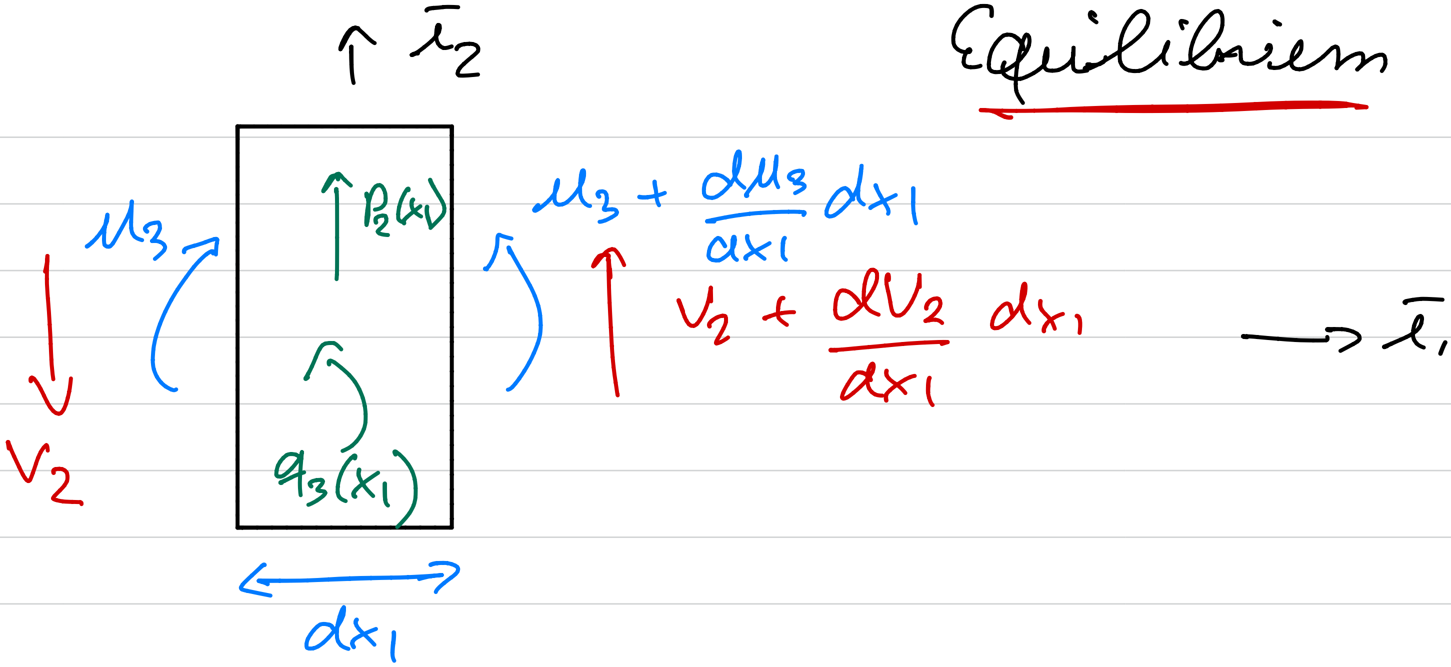
$$V_2 = GA \left(\frac{du_2}{dx_1} - \phi_3(x_1)\right)$$

* To make this model "strain energy equivalent" w/ an equilibrium model we write

$$\boxed{V_2 = K_{22} \left(\frac{du_2}{dx_1} - \phi_3(x_1)\right)}$$

w/ K_{22} the shear stiffness computed from the strain energy at an equilibrium model

Equilibrium



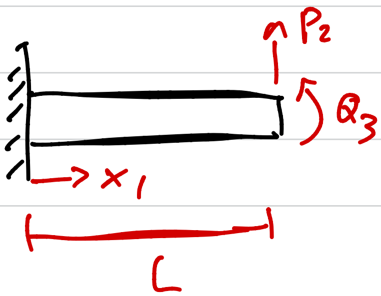
$$\frac{d\mathcal{M}_3}{dx_1} + V_2 = -q_3(x_1), \quad \frac{dV_2}{dx_1} = -p_2(x_1)$$

Governing Equations

$$\frac{d}{dx_1}\left(H^c_{33} \frac{d\Phi_3}{dx_1}\right) + K_{22}\left(\frac{du_2}{dx_1} - \Phi_3\right) = -q_3(x_1)$$

$$\frac{d}{dx_1}\left(K_{22}\left(\frac{du_2}{dx_1} - \phi_3\right)\right) = -p_2(x_1)$$

Boundary Conditions



@ $x_1 = 0$	@ $x_1 = L$
$u_2 = 0$	$V_2 = P_2$
$\phi_3 = 0$	$\mathcal{M}_3 = Q_3$