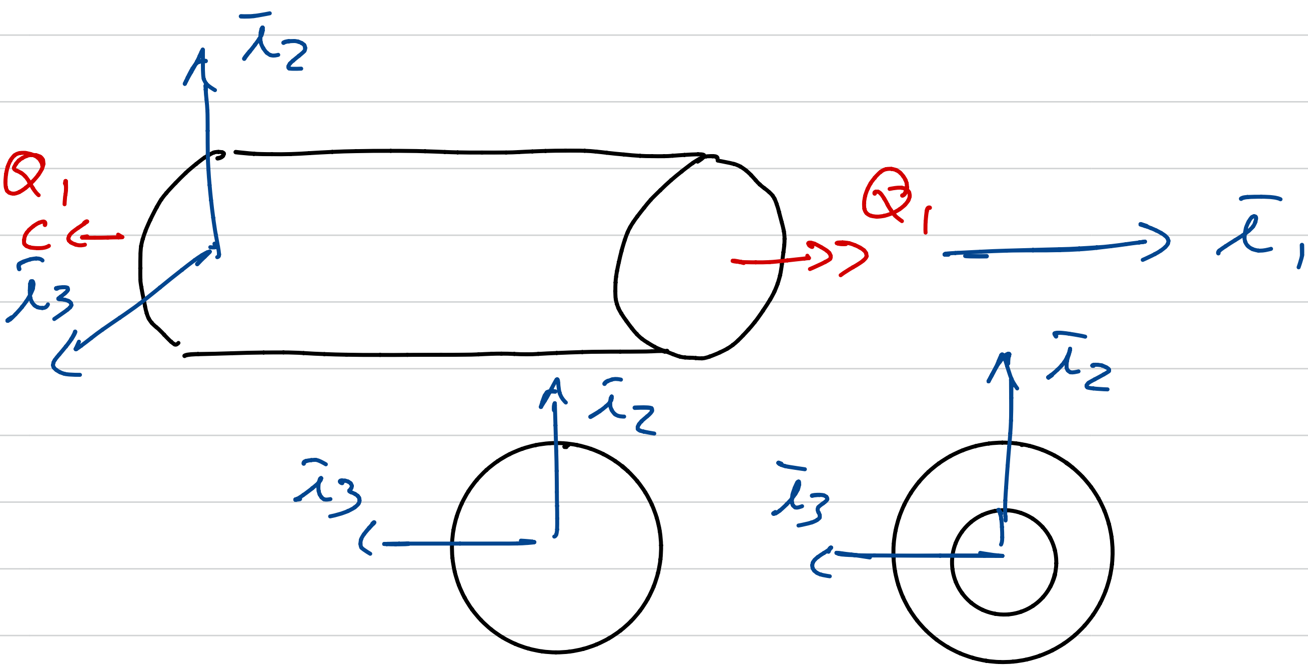
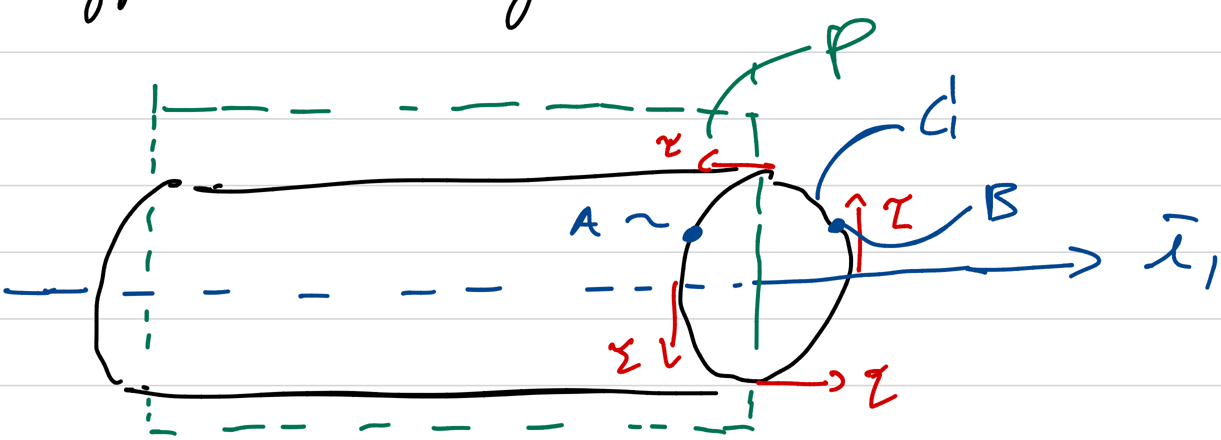
TORSION OF CIRCULAR CYLINDERS



* The problem is characterized by 2 types of symmetries.



1) A cylindrical symmetry about $\bar{\imath}_1$ axis.

2) The cylindrical structure is symmetric w.r.t. any plane P passing through the $\bar{\imath}_1$ axis

Consequences:

Because of cylindrical symmetry:

a) The shear stress τ must be of constant magnitude along C_1, and tangent at all points.

b) $u_1^A = u_1^B$

The loading is antisymmetric about the plane P

This implies:

c) $u_1^A = -u_1^B$

$\Rightarrow$ The only solution consistent with this is

$\rightarrow u_1^A = u_1^B = 0$

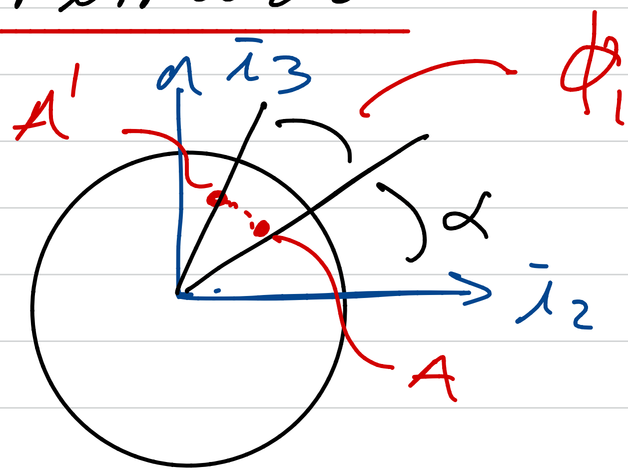
Since A, B are arbitrary

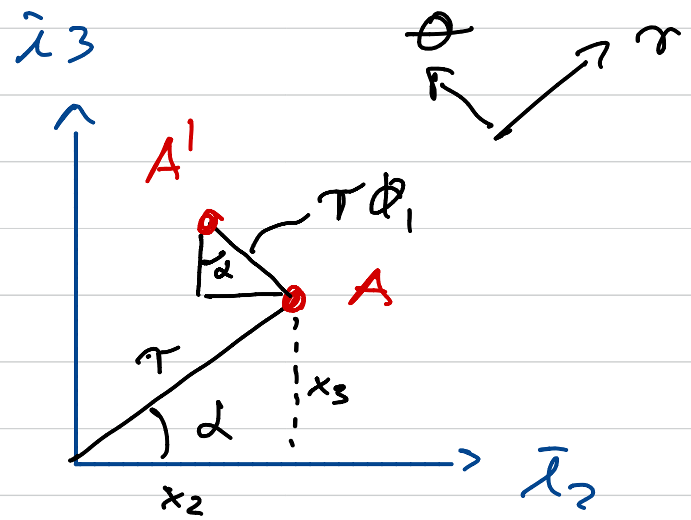
$\rightarrow u_1(x_1, x_2, x_3) = 0$

The cross-section does not warp!

* For the in-plane displacements the only solution compatible w/ the symmetries is a rigid body rotation of the cross-section about its center.

Kinematics





The motion is fully described by a rotation angle $\phi_1(x_1)$

In-plane displacements (assuming small $\phi_1(x_1)$)

$$\begin{cases} u_2 = -r\,\phi_1 \sin(\alpha) = -x_3\,\phi_1(x_1) \\ u_3 = r\,\phi_1 \cos(\alpha) = x_2\,\phi_1(x_1) \\ u_1 = 0 \end{cases}$$

Strain

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = 0, \quad \epsilon_2 = 0, \quad \epsilon_3 = 0$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -x_3 \frac{d\phi_1}{dx_1}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = x_2 \frac{d\phi_1}{dx_1}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = -\phi_1(x_1) + \phi_1(x_1) = 0$$

$$\begin{cases} \gamma_{12} = -x_3 K_1 \\ \gamma_{13} = x_2 K_1 \end{cases} \qquad K_1 = \frac{d\phi_1}{dx_1}$$

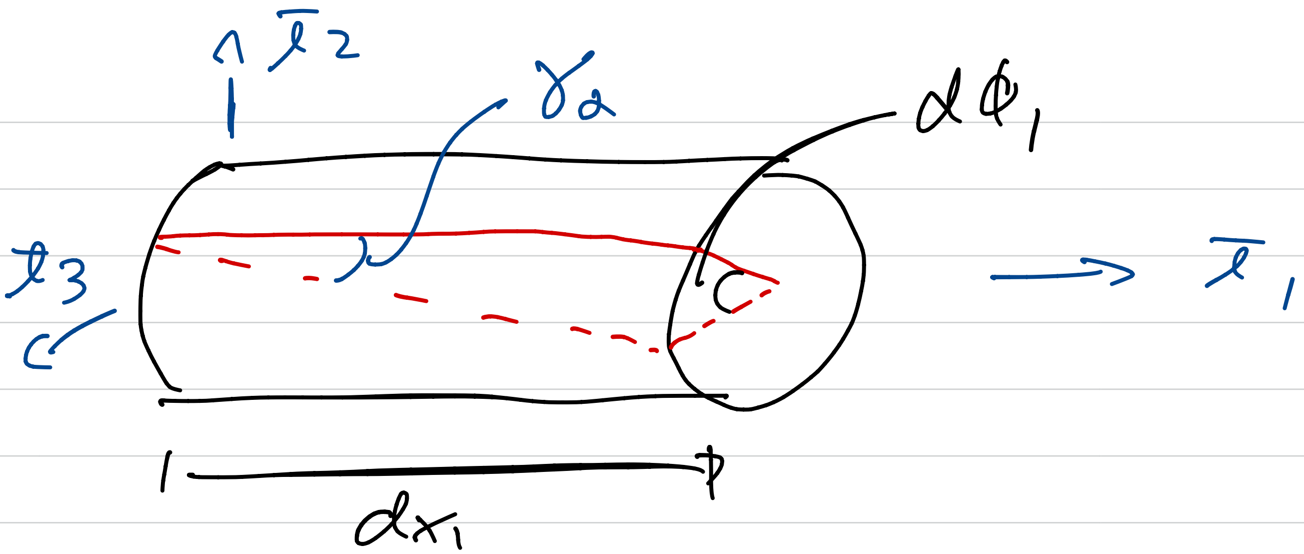
– Sectional Twist Rate

-> The sectional twist rate K_1 measures the deformation

-> A constant twist angle

$$\phi_1 = \text{constant} \rightarrow K_1 = 0$$

implies a rigid body rotation.

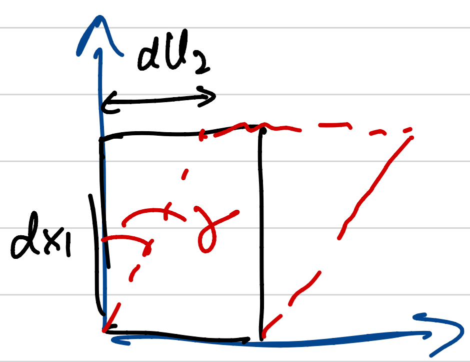


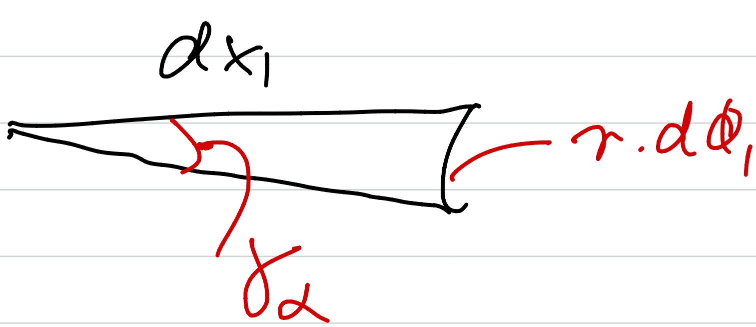
Polar

$$\gamma_r = \gamma_{12}\,\mathrm{Cos}(\alpha) + \gamma_{13}\,\mathrm{Sen}(\alpha)$$

$$\gamma_\alpha = -\gamma_{12}\,\mathrm{Sen}(\alpha) + \gamma_{13}\,\mathrm{Cos}(\alpha)$$

$$\begin{cases} \gamma_r = 0 \\ \gamma_\alpha(x_1, r, \alpha) = r\,K_1(x_1) = r\,\dfrac{d\phi_1}{dx_1} \end{cases}$$





$$\gamma_{12} \sim \frac{du_2}{dx_1} \qquad \gamma_\alpha \sim r\,\frac{d\phi_1}{dx_1}$$

Stress

$$\tau_{12} = -G\, x_3\, K_1(x_1)$$

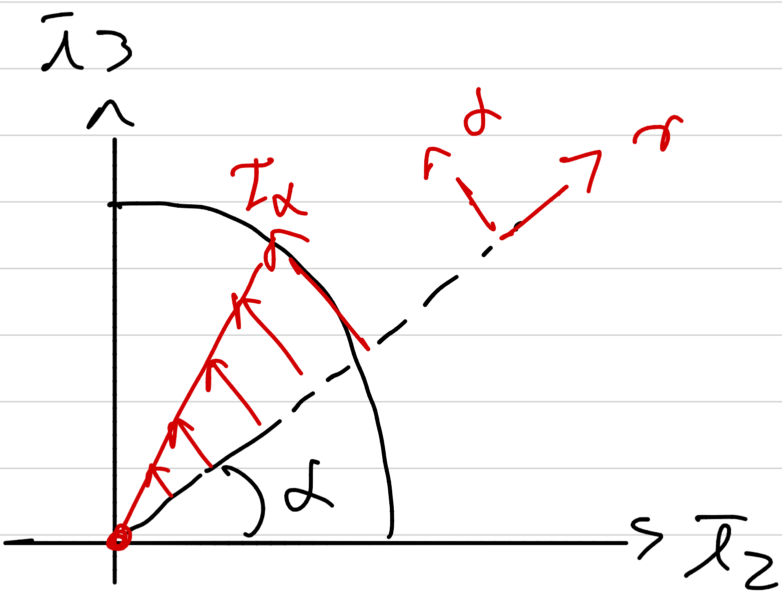
$$\tau_{13} = G\, x_2\, K_1(x_1)$$

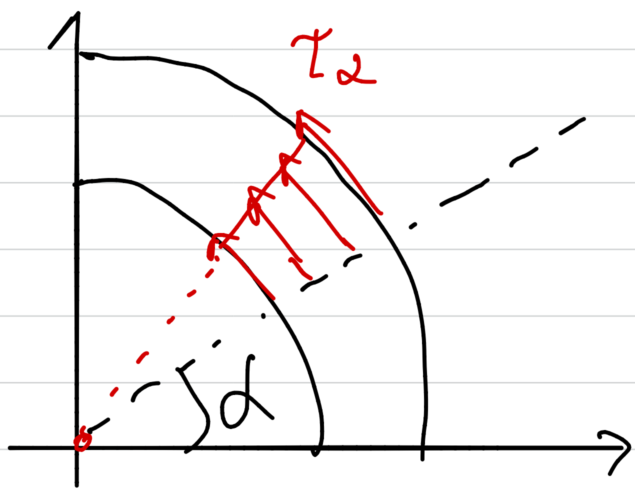
All other are zero

G — Shear Modulus

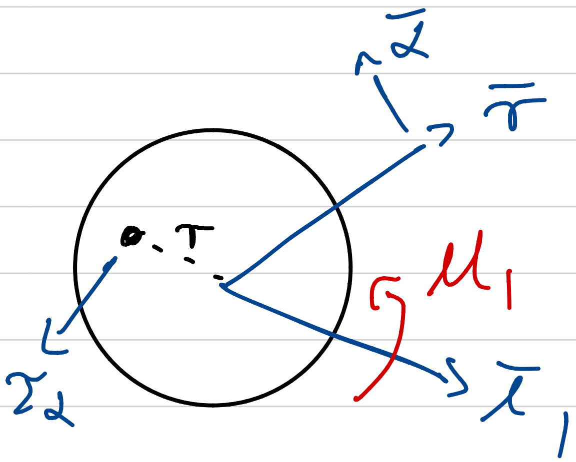
$\tau_r = 0$ — Radial Shear Stress

$\tau_\alpha = G\, r\, K_1(x_1)$ — Circumferential Shear Stress





Sectional Constitutive Law



$$M_1(x_1) = \int_A \tau_\alpha \cdot r \, dA$$

$$M_1(x_1) = \int_A \left(G r K_1(x_1) \right) r \, dA$$

$$M_1(x_1) = K_1 \int_A G r^2 \, dA$$

Define

$$H_{11} = \frac{M_1(x_1)}{K_1(x_1)} = \int_A G r^2 \, dA$$

→ TORSIONAL STIFFNESS

* For a circular cross-section only

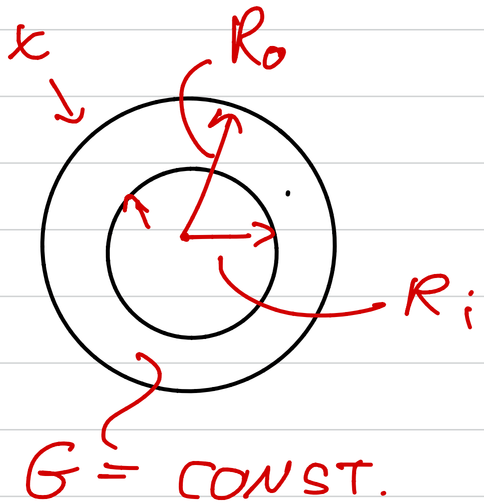
$$H_{11} = \int_A G r^2 \, dA$$

* For all problems one may define

$$H_{11} = \frac{M_1}{K_1}$$

→ If G = constant, $H_{11} = GJ$

J - Polar Moment.



$$H_{11} = G\int_0^{2\pi}\int_{R_1}^{R_o} r^2\, r\,dr\,d\theta$$

$$H_{11} = G\frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

For a thin-walled tube

$R_m = (R_o + R_2)/2 \qquad t/R_m \ll 1$

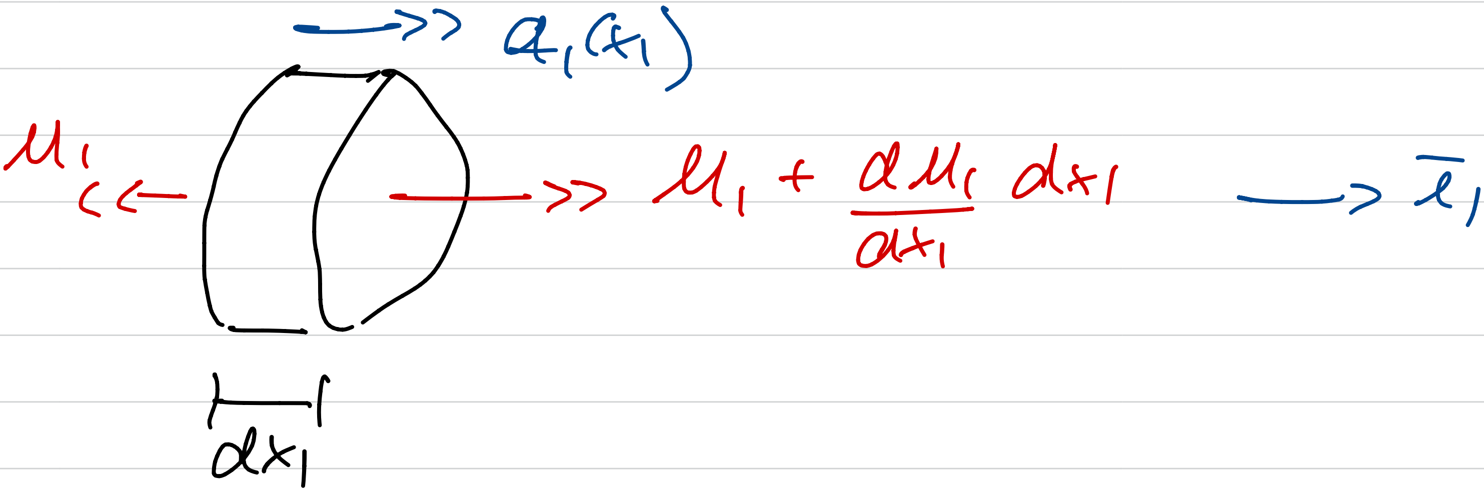
$$H_{11} = 2\pi G R^3 t$$

Aside: $H_{11} = G\frac{\pi}{2}\left((R+t)^4 - R^4\right)$

$$= G\frac{\pi}{2}\left(R^4 + 4R^3t + \cancel{6R^2t^2} + \cancel{4Rt^3} \cancel{+t^4} - R^4\right)$$

$$H_{11} = 2GR^3t$$

Equilibrium (General):



$$M_1 + \frac{dM_1}{dx_1} dx_1 + q_1(x_1) \cdot dx_1 - M_1 = 0$$

$$\boxed{\frac{dM_1}{dx_1} = -q_1(x_1)}$$

Governing Eq.

$$M_1 = H_{11} K_1 , \quad K_1 = \frac{d\phi_1}{dx_1}$$

$$\boxed{\frac{d}{dx_1}\left[H_{11} \frac{d\phi_1}{dx_1}\right] = -q_1(x_1)}$$

B.C.'s : 1) Fixed $\phi_1 = 0$

2) Free $M_1 = 0 \rightarrow d\phi_1/dx_1 = 0$

3) Applied Torque $T \rightarrow M_1 = T$

$d\phi_1/dx_1 = T/H_{11}$