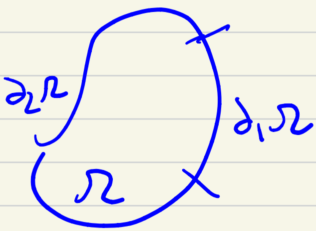
Finals Dec 3.

11/17

* When solving Any stress potential, you are <u>solving for compatibility</u>
* stress from Any stress potential <u>automatically</u> <u>satisfy conservations</u>, but not necessarily compatibility
* Traditional Boundary value problem approach
  ↳ <u>Solve pde for conservations</u>

The stresses might not be real

<u>Principle of minimum potential energy</u>



· given a body Ω, body forces $\underline{b}$, and boundary conditions:

$$\underline{u} = \underline{u}^* \text{ on } \partial_1\Omega$$
$$\underline{t} = \underline{t}^* \text{ on } \partial_2\Omega$$

We can find the solution to the BVP in elastostatics by <u>minimizing</u> the <u>functional</u> $\boxed{\Pi = W - E}$ over all displacements $\underline{u}$ that satisfy $\underline{u} = \underline{u}^*$ on $\partial_1\Omega$

- $\Pi = \Pi(\underline{u}) \equiv$ potential energy of the linear elastic solid
(scalar)

- $W = W(\underline{u}) \equiv \underbrace{\int_{\Omega} \frac{1}{2} \nabla \underline{u} : \underline{\underline{\underline{c}}} : \nabla \underline{u} \, dV}$

↳ strain energy

Analogous to
$\frac{1}{2} k (\Delta x)^2$
(Spring potential energy)

- $E = E(\underline{u}) \equiv \int_{\Omega} \underline{b} \cdot \underline{u} \, dV + \int_{\partial_2 \Omega} \underline{t}^* \cdot \underline{u} \, dA$

↑ ρ inside $\underline{b}$

$\equiv$ work done by forces $\underline{b}$ and $\underline{t}^*$

In summary,

$$\min_{\underline{u}} \Pi(\underline{u}) = \min_{\underline{u}} \frac{1}{2} \int_{\Omega} \nabla \underline{u} : \underline{\underline{\underline{c}}} : \nabla \underline{u} \, dV - \int_{\Omega} \underline{b} \cdot \underline{u} \, dV - \int_{\partial_2 \Omega} \underline{t}^* \cdot \underline{u} \, dA$$

$\underline{u} = \underline{u}^*$ on $\partial_1 \Omega$ $\qquad$ $\underline{u} = u^*$ on $\partial_1 \Omega$ $\qquad \Longleftrightarrow$ BVP in elastostatics

(Find displacement field that minimizes...)

Remarks: (Elastostatics only) (BVPE: Boundary Value problem in Elastostatics)

* The principle of minimum potential energy (PMPE) provides an alternative problem formulation to BVPE. (equivalent to)

* PMPE is a variational or weak formulation, BVPE is a differential or strong formulation

* The strong formulation contains derivatives of stress, which translate into second derivatives of displacements

* The weak formulation contains only first derivatives of displacements, and it is the starting step towards the finite element method.

Idea of proof: Let $\tilde{\underline{u}}$ be a solution for the BVPE. Then, we'd like to show that for any $\underline{u}$ that satisfies $\underline{u} = \underline{u}^*$ on $\partial_1 \Omega$, we have $\Pi(\underline{u}) - \Pi(\tilde{\underline{u}}) \geq 0$, with equality only for $\underline{u} = \tilde{\underline{u}}$

(The minimization problem is finding a function $\underline{u}$ that minimizes the functional Π)

Notes:

- The minimization of the functional $\Pi(u)$ must be carried out over all functions u over the domain Ω

- The necessary tools for performing such operation are provided by a branch of math called "Calculus of variations".

- Within the framework of variational calculus, it is minimum if and only if it is stationary, which means its variation is zero.

  we write $\delta\Pi = 0$

- Some properties of variations:

$$\delta\left(\frac{d^n v}{dx^n}\right) = \frac{d^n \delta v}{dx^n}$$

$$\delta \int v(x)\,dx = \int \delta v\,dx$$

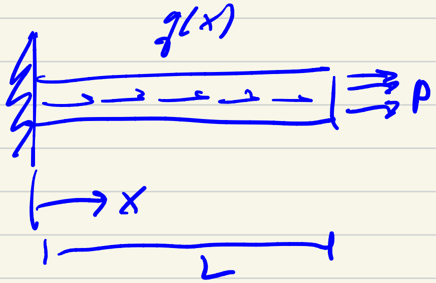
$$v = v(x_1, \ldots, x_n) \Rightarrow \delta v = \frac{\partial v}{\partial x_1}\delta x_1 + \ldots + \frac{\partial v}{\partial x_n}\delta n$$

$$\delta(v + w) = \delta v + \delta w$$

$$\delta(v \cdot w) = \delta v \cdot w + v \cdot \delta w$$

$$\delta(v^n) = n v^{n-1} \delta v$$

Example: Uniaxial bar



linear elastic material with young's modulus E, cross-sectional area A, Body force $g(x)$

Solution:
- Find Π
- Find $\delta\Pi = 0 \Rightarrow \underline{u}(x)$

$$\Pi = \frac{AE}{2}\int_0^L \left(\frac{\partial u_1}{\partial x}\right)^2 dx - A\int_0^L g(x)\,u_1\,dx - APu_1\Big|_{x=L}$$

- Now, we know the solution is given by $\delta\Pi = 0$
- Applying rules of variation, we get

$$\int_0^L \left[E\frac{\partial^2 u_1}{\partial x^2} + g(x)\right]\delta u_1\,dx = 0 \longrightarrow \left(\delta u_1 \text{ is arbitrary.}\right)$$

$$\Longrightarrow \text{implies } \underbrace{E\frac{\partial^2 u_1}{\partial x^2} + g(x) = 0}$$

(ends up w/ the strong formulation)

$$\Longrightarrow E\frac{\partial u_1}{\partial x} = P$$ (also recovered the traction BCs)