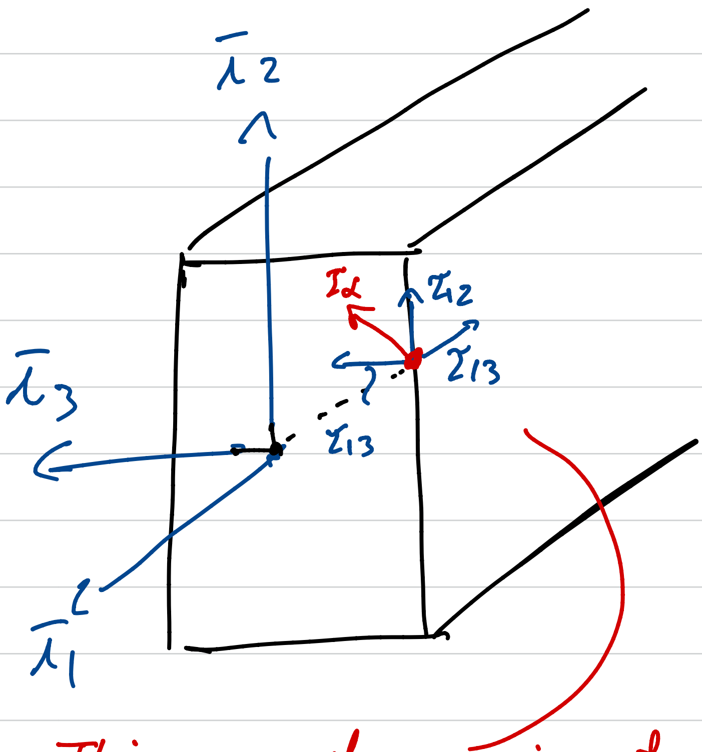
Torsion of bars w/ arbitrary cross-sections



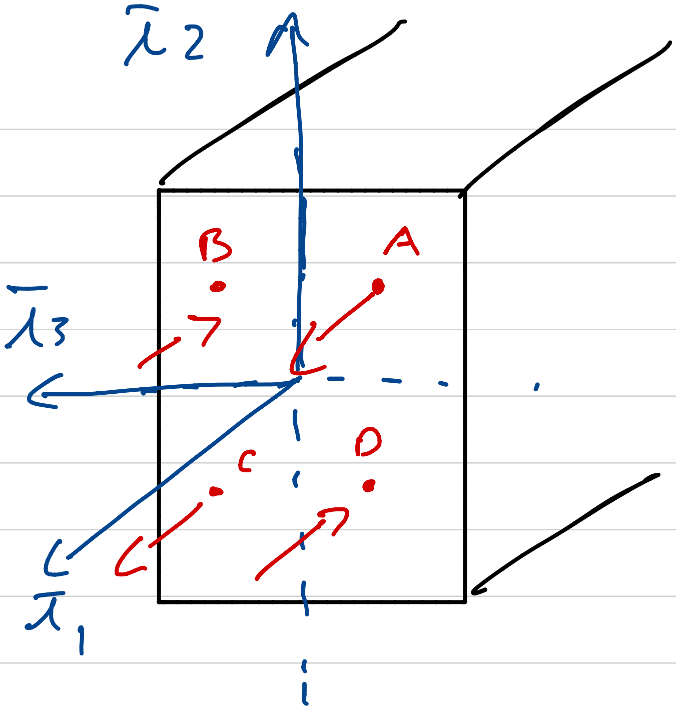
This surface is stress free, hence $\tau_{13} = 0$

If stress is of the form

$$\begin{cases} \tau_\alpha = G r K_1 \\ \tau_r = 0 \end{cases}$$

We would have a non-physical state of stress

* At any point along the edge of the bar's section, the shear stress must be tangent to the edge.



Due to antisymmetry of the loading about $(\bar{x}_2, \bar{x}_1)$ planes

$$U_1^A = -U_1^B, \quad U_1^C = -U_1^D$$

About $(\bar{x}_3, \bar{x}_1)$ plane

$$U_1^A = -U_1^D, \quad U_1^B = -U_1^C$$

Combining

$$U_1^A = -U_1^B = U_1^C = -U_1^D$$

* The section of bars with arbitrary cross-section will warp.

Saint-Venant's Solution

Kinematic Assumptions:

* Each cross-section rotates like a rigid body & warps out of plane

$$u_2 = -x_3\,\phi_1(x_1)$$

$$u_3 = x_2\,\phi_1(x_1)$$

$$u_1 = \psi(x_2, x_3)\cdot K_1(x_1)$$

Warping Function.

→ u_1 is proportional to K_1 through $\psi(x_2, x_3)$

→ We will assume that the twist rate is constant

$$K_1 = \frac{d\phi_1}{dx_1} = \text{constant}$$

Referred to as uniform torsion

Strain

$$\epsilon_1 = \frac{\partial U_1}{\partial x_1} = \psi(x_2, x_3) \cdot \frac{dK_1}{dx_1} = 0$$

$$\epsilon_2 = \epsilon_3 = 0 \quad , \quad \gamma_{23} = 0$$

$$\gamma_{12} = \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} = \left(\frac{\partial \psi}{\partial x_2} \cdot K_1 - x_3 K_1 \right)$$

$$= \left(\frac{\partial \psi}{\partial x_2} - x_3 \right) K_1$$

$$\gamma_{13} = \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} = \left(\frac{\partial \psi}{\partial x_3} K_1 + x_2 K_1 \right)$$

$$= \left(\frac{\partial \psi}{\partial x_3} + x_2 \right) K_1$$

Stress

$$\tau_{12} = G \left(\frac{\partial \psi}{\partial x_2} - x_3 \right) K_1$$

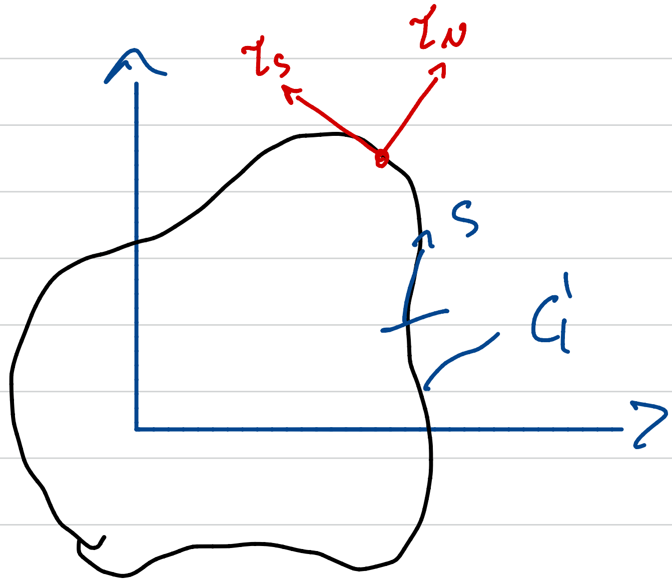
$$\tau_{13} = G \left(\frac{\partial \psi}{\partial x_3} + x_2 \right) K_1$$

Equilibrium & Boundary Conditions

$$\cancel{\frac{\partial \sigma_1}{\partial x_1}} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} \cancel{+ b_1} = 0$$

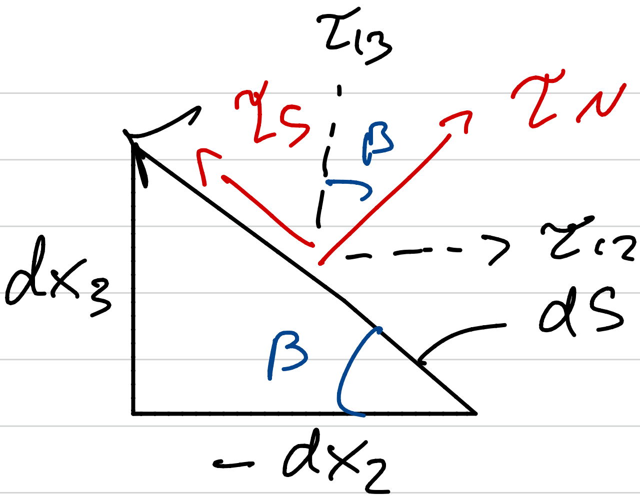
$$\frac{\partial}{\partial x_2}\left[G K_1 \left(\frac{\partial \varphi}{\partial x_2} - x_3 \right) \right] + \frac{\partial}{\partial x_3}\left[G K_1 \left(\frac{\partial \varphi}{\partial x_3} - x_2 \right) \right] = 0$$

$$\boxed{\frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0}$$



On the outer edge (curve C_1') we want

$$\tau_n = 0$$



$$\tau_N = \tau_{12}\,\sin(\beta) + \tau_{13}\cos(\beta)$$

$$= \tau_{12}\left(\frac{dx_3}{ds}\right) + \tau_{13}\left(-\frac{dx_2}{ds}\right) = 0$$

$$\underbrace{GK_1\left[\left(\frac{\partial\varphi}{\partial x_2} - x_3\right)\left(\frac{dx_3}{ds}\right) - \left(\frac{\partial\varphi}{\partial x_3} + x_2\right)\left(\frac{dx_2}{ds}\right)\right]}_{=0} = 0$$

Solve

$$0 = \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}, \quad \text{on } A$$

$$\left(\frac{\partial\varphi}{\partial x_2} - x_3\right)\left(\frac{dx_3}{ds}\right) - \left(\frac{\partial\varphi}{\partial x_3} + x_2\right)\left(\frac{dx_2}{ds}\right) = 0$$

on C

Prandtl Stress Function

Let $\Phi(x_2, x_3)$ be a function define as

$$\tau_{12} = \frac{\partial \Phi}{\partial x_3}, \quad \tau_{13} = -\frac{\partial \Phi}{\partial x_2}$$

→ Automatically satisfies equilibrium

$$\frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = 0$$

$$\frac{\partial^2 \Phi}{\partial x_2 \partial x_3} - \frac{\partial^2 \Phi}{\partial x_3 \partial x_2} = 0 \checkmark$$

Relates to $\psi(x_2, x_3)$ through

$$\tau_{12} = G\kappa_1 \left(\frac{\partial \psi}{\partial x_2} - x_3 \right) = \frac{\partial \Phi}{\partial x_3}$$

$$\tau_{13} = G\kappa_1 \left(\frac{\partial \psi}{\partial x_3} + x_2 \right) = \frac{\partial \Phi}{\partial x_2}$$

$$\frac{\partial \tau_{12}}{\partial x_3} = GK_1\left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} - 1\right) = \frac{\partial^2 \Phi}{\partial x_3^2}$$

$$\frac{\partial \tau_{13}}{\partial x_2} = GK_1\left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} + 1\right) = -\frac{\partial^2 \Phi}{\partial x_2^2}$$

Subtracting

$$\boxed{-2GK_1 = \frac{\partial^2 \Phi}{\partial x_3^2} + \frac{\partial^2 \Phi}{\partial x_2^2}}$$

→ Governing Equation for Φ

Boundary Condition

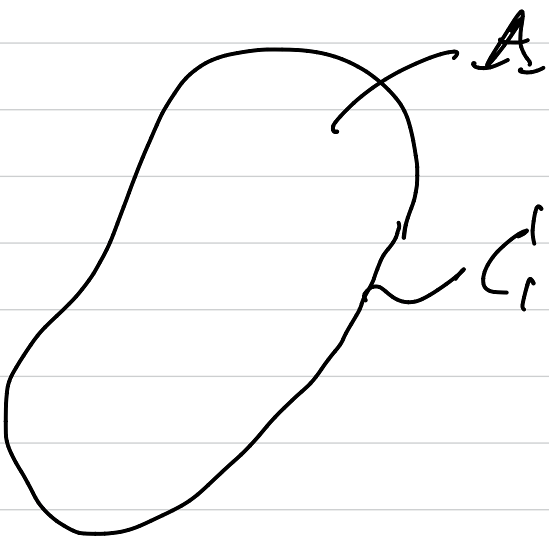
$$\tau_N = \tau_{12}\frac{dx_3}{ds} + \tau_{13}\left(-\frac{dx_2}{ds}\right) = 0$$

$$\tau_N = \frac{\partial \Phi}{\partial x_3}\frac{dx_3}{ds} + \frac{\partial \Phi}{\partial x_2}\frac{dx_2}{ds} = 0$$

→ $\tau_N = \frac{\partial \Phi}{\partial s} = 0$

→ Φ must be a constant along the edge!

→ For a solid cross-section bounded by a single curve, we may chose $\Phi = 0$ on the boundary



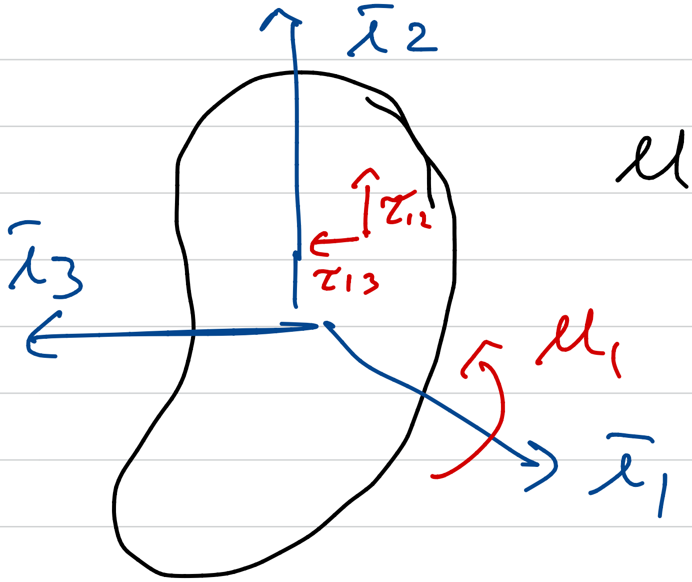
Solve

$$-2GK_1 = \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2}$$

on $\underline{A}$

$\Phi = 0$ on C_1'

Sectional Equilibrium



$$M_1 = \int_A \left(x_2 \tau_{13} - x_3 \tau_{12} \right) dA$$

For a solid cross-section bounded by a single curve

$$\boxed{M_1 = 2 \cdot \int_A \Phi \, dA}$$

Summary

Kinematics

$$u_2 = - x_3 \, \phi_1 (x_1) \qquad \kappa_1 = \text{const!}$$

$$u_3 = x_2 \, \phi_1 (x_1)$$

$$u_1 = \psi (x_2, x_3) \cdot \kappa_1$$

$\Phi (x_2, x_3)$ — Prandtl Stress Function

$$\tau_{12} = \frac{\partial \Phi}{\partial x_3} , \quad \tau_{13} = - \frac{\partial \Phi}{\partial x_2}$$

Solve: $\dfrac{\partial^2 \Phi}{\partial x_2^2} + \dfrac{\partial^2 \Phi}{\partial x_3^2} = - 2 \cdot G \cdot \kappa_1$ on A

with $\dfrac{\partial \Phi}{\partial s} = 0$ (Φ = const.) on C_1

Find: $H_{11} = \dfrac{M_1}{\kappa_1}$ once Φ is known

using $M_1 = 2 \cdot \int_A \Phi \, dA$

Knowing H_{11} we may solve $\phi_1(x_1)$ using the governing equation.

$$\frac{d}{dx_1}\left[H_{11}\frac{d\phi_1}{dx_1}\right] = -q_1(x_1)$$

Solve for $\psi(x_2, x_3)$ from

$$\tau_{12} = \frac{\partial \Phi}{\partial x_3} = G K_1\left(\frac{\partial \psi}{\partial x_2} - x_3\right)$$

$$\tau_{13} = \frac{\partial \Phi}{\partial x_2} = G K_1\left(\frac{\partial \psi}{\partial x_3} + x_2\right)$$