Basic Elasticity Equations

AE3140: Structural Analysis

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Contents

Overview

- Structural analysis is concerned with the evaluation of deformations and stresses arising within a solid object under the action of applied loads.
- If time is not explicitly considered as an independent variable, the analysis is said to be static; otherwise it is referred to as structural dynamic analysis, or simply structural dynamics.
- Under the assumption of small deformations and linear elastic material behavior, three-dimensional formulations result in a set of fifteen linear first order partial differential equations involving:
  - displacement field (three components),
  - stress field (six components)
  - the strain field (six components).

Overview

- For most situations, it is not possible to develop analytical solutions to these equations.
- Consequently, structural analysis is concerned with the analysis of *structural components*, such as bars, beams, plates, or shells
- Formulations on structural components are based on geometric and kinematic assumptions about the behavior of these structural components, which considerably simply the analysis.

The state of stress at a point

The state of stress in a solid body is a measure of the intensity of forces acting within the solid.
The distribution of forces and moments that appears on the surface of a cut can be represented by an *equipollent* force, $\underline{F}$, acting at a point of the surface and a couple, $\underline{M}$.

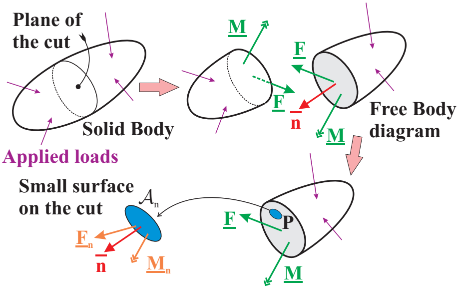


Figure: A solid body cut by a plane to isolate a free body.

Consider now a small surface of area $\mathcal{A}_n$ located at point **P** on the surface generated by the cut in the solid.
The forces and moments acting on this surface are equipollent to a force, $\underline{F}_n$, and couple, $\underline{M}_n$.
The *stress vector* is defined as

$$\underline{\tau}_n = \lim_{\mathrm{d}\mathcal{A}_n \to 0} \left(\frac{\underline{F}_n}{\mathrm{d}\mathcal{A}_n} \right). \tag{1}$$

In this limiting process, it is assumed that $\underline{M}_n \to 0$ as $\mathrm{d}\mathcal{A}_n \to 0$.
The stress vector has units of force per unit area. In the SI system, this is measured in Newtons per square meters, or Pascals (Pa).

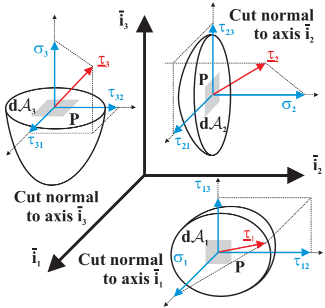


Figure: A rigid body cut at point P by three planes orthogonal to the Cartesian axes.

Three stress vectors, $\underline{\tau}_1$, $\underline{\tau}_2$ and $\underline{\tau}_3$ are acting at the same point **P**, but on three mutually orthogonal faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively.

The components of each stress vector along axes $\bar{\imath}_1$, $\bar{\imath}_2$ and $\bar{\imath}_3$ at point **P**, are defined as

$$\underline{\tau}_1 = \sigma_1 \, \bar{\imath}_1 + \tau_{12} \, \bar{\imath}_2 + \tau_{13} \, \bar{\imath}_3; \tag{2a}$$

$$\underline{\tau}_2 = \tau_{21} \, \bar{\imath}_1 + \sigma_2 \, \bar{\imath}_2 + \tau_{23} \, \bar{\imath}_3; \tag{2b}$$

$$\underline{\tau}_3 = \tau_{31} \, \bar{\imath}_1 + \tau_{32} \, \bar{\imath}_2 + \sigma_3 \, \bar{\imath}_3. \tag{2c}$$

- The stress components σ_i, are called *direct*, or *normal stresses*; they act on faces normal to axes $\bar{\imath}_i$, in directions along axes $\bar{\imath}_i$,
- The stress components τ_{ij} are called *shearing* or *shear stresses*; they act on the face normal to axis $\bar{\imath}_i$, in directions along axes $\bar{\imath}_j$

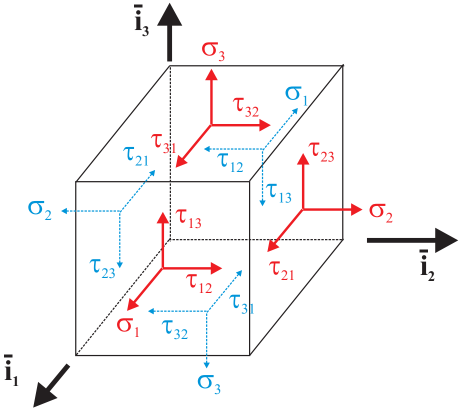


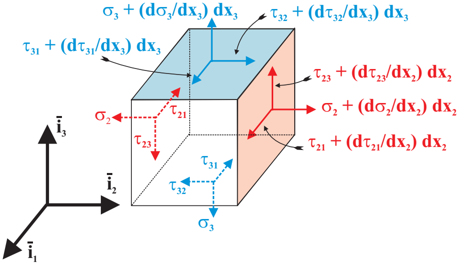
Figure: Sign conventions for the stress components acting on a differential volume element. All stress components shown here are positive. Stress components shown in solid lines act on positive faces; stress components shown in dotted lines act on negative faces

- The state of stress at point **P** is fully defined by the direct stress components σ_1, σ_2, and σ_3 and the shear stress components, τ_{12} and τ_{13}, τ_{21} and τ_{23}, and τ_{31} and τ_{32}
- A state of stress is a *second order tensor*, which requires knowledge of 9 quantities
- In contrast, a force is a vector, *i.e.*, a *first order tensor*, that is defined by 3 quantities (3 components)

Volume equilibrium equations

In general, the state of stress varies throughout a solid body, and hence, the stresses acting on two parallel faces located a small distance apart are not in general equal.

$$\sigma_2(x_2 + \mathrm{d}x_2) = \sigma_2(x_2) + \left.\frac{\partial \sigma_2}{\partial x_2}\right|_{x_2} \mathrm{d}x_2 + \ldots \text{higher order terms in } \mathrm{dx}_2.$$



Force equilibrium

Consider the differential element of volume, which is subjected to stress components acting on its six external faces and to body forces per unit volume $\underline{b} = b_1 \,\bar{\imath}_1 + b_2 \,\bar{\imath}_2 + b_3 \,\bar{\imath}_3$.
Applying Newton's law, static equilibrium requires that:

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 = 0, \tag{3a}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 = 0, \tag{3b}$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 = 0, \tag{3c}$$

which must be satisfied at all points inside the body.

Notes

- The equilibrium conditions implied by Newton's law, eqs. (3), consider a differential element *of the undeformed body*.
- The goal of the theory of elasticity is to predict the deformation of elastic bodies under load.
- The basic assumption of the linear theory of elasticity is that the displacements of the body under the applied loads are very small, and hence, the difference between the deformed and undeformed configurations of the body is very small.

Moment equilibrium

To satisfy all equilibrium requirements, the sum of all the moments acting on the differential element of volume must also vanish. This leads to the following conditions:

$$\tau_{23} = \tau_{32}; \ \tau_{13} = \tau_{31}; \ \tau_{12} = \tau_{21}. \tag{4}$$

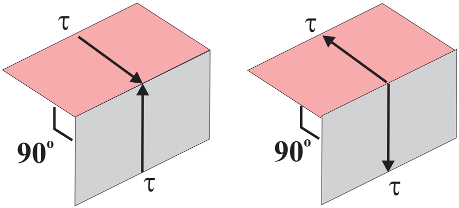


Figure: Reciprocity of the shearing stresses acting on two orthogonal faces.

Principle of reciprocity of shear stresses

Shear stresses acting in the direction normal to the common edge of two orthogonal faces must be equal in magnitude and be simultaneously oriented toward or away from the common edge.

- An implication of the reciprocity of the shearing stresses is that of the nine components of stresses, only six are independent.
- The principle of reciprocity implies the *symmetry of the stress tensor*.

$$\begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix}. \tag{5}$$

The state of plane stress

A particular state of stress of great practical importance is the *plane state of stress*. In this case:

- All stress components acting along the direction of axis $\bar{\imath}_3$ are assumed to vanish, or to be negligible
- The only non-vanishing stress components are σ_1, σ_2, and τ_{12}
- These stress components are assumed to be independent of x_3.

The equations of equilibrium reduce to

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + b_1 = 0; \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + b_2 = 0. \tag{6}$$

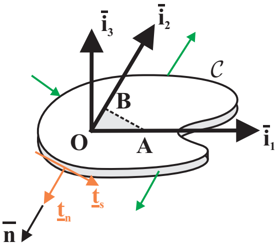


Figure: Plane stress problem.

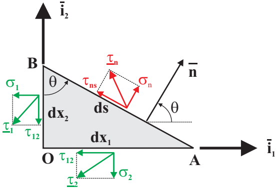


Figure: Differential element with a face at an angle θ.

Consider a differential triangle **OAB** taken from within the thin sheet with two sides cut normal to axes $\bar{\imath}_1$ and $\bar{\imath}_2$, and with the third side cut normal to a unit vector, $\bar{n} = n_1\,\bar{\imath}_1 + n_2\,\bar{\imath}_2$ at an arbitrary orientation angle θ with respect to axis $\bar{\imath}_1$. Clearly, $n_1 = \cos\theta$ and $n_2 = \sin\theta$. Equilibrium of forces acting on triangle **OAB** can be expressed as:

$$\underline{\tau}_2 \mathrm{d}x_1 + \underline{\tau}_1 \mathrm{d}x_2 = \underline{\tau}_n \mathrm{d}s + \underline{b}\ \mathrm{d}x_1 \mathrm{d}x_2/2,$$

dividing by $\mathrm{d}s$ and given that $\mathrm{d}x_1/\mathrm{d}s = \sin\theta = n_2$ and $\mathrm{d}x_2/\mathrm{d}s = \cos\theta = n_1$

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2$$

with $\underline{b}\ \mathrm{d}x_1 \mathrm{d}x_2/2\mathrm{d}s \approx 0$. Expanding the stress vectors in terms of the stress components then yields

$$\underline{\tau}_n = (\sigma_1 \bar{\imath}_1 + \tau_{12} \bar{\imath}_2)\ \cos\theta + (\tau_{21} \bar{\imath}_1 + \sigma_2 \bar{\imath}_2)\ \sin\theta. \tag{7}$$

Projecting the stress vector equation in the direction of unit vector $\bar{n}$ yields

$$\sigma_n = (\sigma_1 \cos\theta + \tau_{12} \sin\theta)\ \cos\theta + (\tau_{21} \cos\theta + \sigma_2 \sin\theta)\ \sin\theta$$

$$\sigma_n = \sigma_1\ \cos^2\theta + \sigma_2\ \sin^2\theta + 2\tau_{12}\ \cos\theta \sin\theta. \tag{8}$$

The shear stress component, τ_{ns}, is obtained from the projection along side **AB**, i.e. unit vector $\bar{s} = -\sin\theta\,\bar{\imath}_1 + \cos\theta\,\bar{\imath}_2$:

$$\tau_{ns} = (-\sigma_1 \sin\theta + \tau_{12} \cos\theta)\ \cos\theta + (-\tau_{21} \sin\theta + \sigma_2 \cos\theta)\ \sin\theta$$

$$\tau_{ns} = -\sigma_1\ \cos\theta \sin\theta + \sigma_2\ \sin\theta \cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta). \tag{9}$$

Notes

- Knowledge of the stress components σ_1, σ_2, and τ_{12} on two orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation.
- In other words, the knowledge of the stress components on two orthogonal faces fully defines the state of stress at a point.

Principal stresses

The normal stress, σ_n can be viewed as a function of θ, the orientation angle of the face it acts on. The particular orientation, θ_p, that maximizes (or minimizes) the magnitude of this stress component is determined by

$$0 = \frac{\mathrm{d}\sigma_n}{\mathrm{d}\theta} = -2\sigma_1 \cos\theta_p \sin\theta_p + 2\sigma_2 \cos\theta_p \sin\theta_p + 2\tau_{12}(\cos^2\theta_p - \sin^2\theta_p).$$

Upon manipulations this becomes:

$$\tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2}. \tag{10}$$

This equation presents two solutions θ_p and $\theta_p + \pi/2$ corresponding to two mutually orthogonal principal stress directions. The maximum axial stress is found along one direction, and the minimum is found along the other.

The maximum and minimum axial stresses, called the *principal stresses*, are evaluated by introducing eq. (10) into eq. (8) to find

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + \left[\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + (\tau_{12})^2 \right]^{1/2} . \qquad (11)$$

$$\sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - \left[\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + (\tau_{12})^2 \right]^{1/2} . \qquad (12)$$

The shearing stress acting on the faces normal to the principal stress directions is obtained by introducing eq. (10) into eq. (9), which gives:

$$\tau_{ns} = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta_p + \tau_{12} \cos 2\theta_p = 0.$$

Similarly, the orientation, θ_s, of the face on which the maximum shear stress acts satisfies the following extremal condition

$$\frac{\mathrm{d}\tau_{ns}}{\mathrm{d}\theta} = -\frac{\sigma_1 - \sigma_2}{2} 2 \cos 2\theta_s - \tau_{12}\; 2 \sin 2\theta_s = 0, \tag{13}$$

or

$$\tan 2\theta_s = -\frac{\sigma_1 - \sigma_2}{2\tau_{12}} = -\frac{1}{\tan 2\theta_p}, \tag{14}$$

Since $\tan 2\theta_s = -1/\tan 2\theta_p$, trigonometric identities reveal that

$$\theta_s = \theta_p - \frac{\pi}{4}. \tag{15}$$

This means that the faces on which the maximum shear stresses occur are inclined at a 45° angle with respect to the principal stress directions. Finally, the axial stresses acting on these faces are

$$\sigma_{1s} = \sigma_{2s} = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{p1} + \sigma_{p2}}{2}. \tag{16}$$

Rotation of stresses

- Faces were cut in planes normal to the two axes of an orthonormal basis $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$, and the stress vectors were resolved into stress components along the same directions.
- The orientation of this basis is entirely arbitrary: an orthonormal basis $\mathcal{I}^* = (\bar{\imath}_1^*, \bar{\imath}_2^*)$ could have been selected, and an analysis identical to that of the previous sections would have led to the definition of axial stresses σ_1^* and σ_2^*, and shearing stress τ_{12}^*.
- Given two distinct orthonormal bases, $\mathcal{I}$ and $\mathcal{I}^*$, the relationship between stress components in the two basis is of interest and can be easily found.

Consider the stress component σ_1^* acting on the face normal to axis $\bar{\imath}_1^*$. Let θ be angle between unit vector $\bar{\imath}_1^*$ and axis $\bar{\imath}_1$. Equation (8) can now be used to express the stress component σ_1^* in terms of the stress components resolved in axis system $\mathcal{I}$ to find

$$\sigma_1^* = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12} \sin\theta \cos\theta. \tag{17}$$

A similar equation can be derived to express σ_2^* in terms of the stress components resolved in axis system $\mathcal{I}$ by replacing angle θ by $\theta + \pi/2$ (angle between unit vector $\bar{\imath}_2^*$ and axis $\bar{\imath}_1$). Finally, the shear stress component can be computed from eq. (9) as

$$\tau_{12}^* = -\sigma_1 \sin\theta \cos\theta + \sigma_2 \sin\theta \cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta). \tag{18}$$

These results can be combined into a compact matrix form as

$$\begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}. \qquad (19)$$

This relationship can be easily inverted (replacing θ by $-\theta$) to find

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{Bmatrix}. \qquad (20)$$

An alternative form using trigonometric transformations is:

$$\sigma_1^* = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \ \cos 2\theta + \tau_{12} \ \sin 2\theta, \tag{21a}$$

$$\sigma_2^* = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \ \cos 2\theta - \tau_{12} \ \sin 2\theta, \tag{21b}$$

$$\tau_{12}^* = \qquad - \frac{\sigma_1 - \sigma_2}{2} \ \sin 2\theta + \tau_{12} \ \cos 2\theta. \tag{21c}$$

Note

Knowledge of the stress components σ_1, σ_2, and τ_{12} on two orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation.

Special states of stress: Hydrostatic stress state

State of stress were the principal stresses are equal, *i.e.*, $\sigma_{p1} = \sigma_{p2} = p$, where p is the *hydrostatic pressure*.
The stresses acting on a face with an arbitrary orientation are

$$\sigma_1 = \sigma_2 = p; \quad \tau_{12} = 0. \tag{22}$$

Special states of stress: Pure shear state

State of stress characterized by principal stresses of equal magnitude but opposite signs, *i.e.*, $\sigma_{p2} = -\sigma_{p1}$ On faces oriented at 45° angles with respect to the principal stress directions, the direct and shear stresses are

$$\tau_{12}^* = -\sigma_{p1}; \quad \sigma_1^* = \sigma_2^* = 0. \tag{23}$$

i.e., the direct stresses vanish and the shear has a maximum value, equal in magnitude to the common magnitudes of the two principal stresses.

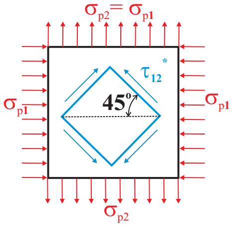


Figure: A differential plane stress element in a state of pure shear.

The concept of strain

The state of strain at a point characterizes the deformation in the neighborhood of the material point in a solid[1].

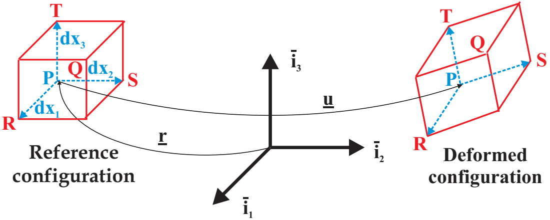


Figure: The neighborhood of point **P** in the reference and deformed configurations.

[1] This quantity is more complicated than stress state, and the presence of nonlinear terms is much more obvious.

- The position vector $\underline{r} = x_1\, \bar{\imath}_1 + x_2\, \bar{\imath}_2 + x_3\, \bar{\imath}_3$ locates a material point **P**
- The displacement vector is a measure of how much a material point moves from the reference to the deformed configuration:

$$\underline{u}(x_1, x_2, x_3) = u_1(x_1, x_2, x_3)\, \bar{\imath}_1 + u_2(x_1, x_2, x_3)\, \bar{\imath}_2 + u_3(x_1, x_2, x_3)\, \bar{\imath}_3. \tag{24}$$

- The displacement field consists of two parts: a rigid body motion and a *deformation* or *straining* of the solid. The rigid body motion itself consists of two parts: a rigid body translation and a rigid body rotation.
- By definition, a rigid body motion does not produce strain in the body.
- Consequently, equations for the strain must extract from the displacement field the information that describes only the deformation of the body and not the rigid body motion.

The concept of strain:

The state of strain at a point

Consider all the material particles forming material lines **PR**, **PS**, **PT**. Due to the differential nature of this segment, they are assumed to remain straight in the deformed configuration.
The motion to the deformed configuration causes: a change in orientation and a change in length.

- The stretching or *relative elongations* of material lines **PR**, **PS** and **PT** are denoted as ϵ_1, ϵ_2 and ϵ_3, respectively.
- The *angular distortions* between segments **PS** and **PT**, **PR** and **PT**, and **PR** and **PS** will be denoted γ_{23}, γ_{13}, and γ_{12}, respectively.

The concept of strain:

Relative elongations or extensional strains

The relative elongation, ϵ_1, of material line **PR** is defined as

$$\epsilon_1 = \frac{\|\mathbf{PR}\|_{\rm def} - \|\mathbf{PR}\|_{\rm ref}}{\|\mathbf{PR}\|_{\rm ref}}, \tag{25}$$

where

$$\|\mathbf{PR}\|_{\rm ref} = \|\mathrm{d}x_1\ \bar{\imath}_1\| = \mathrm{d}x_1; \tag{26}$$

and

$$\begin{aligned}
\|\mathbf{PR}\|_{\rm def} =& \|\mathrm{d}x_1\bar{\imath}_1 + \underline{u}(x_1+\mathrm{d}x_1) - \underline{u}(x_1)\| \\
=& \|\mathrm{d}x_1\bar{\imath}_1 + \underline{u}(x_1) + \frac{\partial \underline{u}}{\partial x_1}\mathrm{d}x_1 - \underline{u}(x_1)\| = \|\mathrm{d}x_1\bar{\imath}_1 + \frac{\partial \underline{u}}{\partial x_1}\ \mathrm{d}x_1\| \\
=& \|\bar{\imath}_1\mathrm{d}x_1 + \left(\frac{\partial u_1}{\partial x_1}\bar{\imath}_1 + \frac{\partial u_2}{\partial x_1}\bar{\imath}_2 + \frac{\partial u_3}{\partial x_1}\bar{\imath}_3\right)\mathrm{d}x_1\| \\
=& \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\ \mathrm{d}x_1,
\end{aligned}$$

The relative elongation now becomes

$$\epsilon_1 = \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} - 1. \qquad (27)$$

If all displacement components are very small, *all second order terms can be neglected*.
Using the binomial expansion[2], gives

$$\epsilon_1 \approx 1 + \frac{\partial u_1}{\partial x_1} - 1 = \frac{\partial u_1}{\partial x_1}. \qquad (28)$$

Similarly, expressions for other material lines (**PS** and **PT**) give:

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}; \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}; \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3}. \qquad (29)$$

[2]When $|a| \ll 1$, it is possible to expand $(1 \pm a)^n \approx 1 \pm na$.

The concept of strain:

Angular distortions or shear strains

The angular distortion, γ_{23}, between two material lines **PT** and **PS** is defined as the change of the initially right angle

$$\gamma_{23} = \langle TPS \rangle_{\mathrm{ref}} - \langle TPS \rangle_{\mathrm{def}} = \frac{\pi}{2} - \langle TPS \rangle_{\mathrm{def}}, \tag{30}$$

To eliminate the difference between the two angles, the basic properties of the sine function are used: the sine of the angular distortion becomes

$$\sin\gamma_{23} = \sin\left(\frac{\pi}{2} - \langle TPS \rangle_{\mathrm{def}}\right) = \cos\langle TPS \rangle_{\mathrm{def}}. \tag{31}$$

Exploiting the law of cosines:

$$\|\mathbf{TS}\|^2_{\mathrm{def}} = \|\mathbf{PT}\|^2_{\mathrm{def}} + \|\mathbf{PS}\|^2_{\mathrm{def}} - 2\cos\langle TPS \rangle_{\mathrm{def}} \|\mathbf{PT}\|_{\mathrm{def}} \|\mathbf{PS}\|_{\mathrm{def}}. \tag{32}$$

The angular distortion thus becomes

$$\gamma_{23} = \arcsin\frac{\|\mathbf{PT}\|^2_{\mathrm{def}} + \|\mathbf{PS}\|^2_{\mathrm{def}} - \|\mathbf{TS}\|^2_{\mathrm{def}}}{2\|\mathbf{PT}\|_{\mathrm{def}}\|\mathbf{PS}\|_{\mathrm{def}}}. \tag{33}$$

Upon manipulations:

$$\gamma_{23} \approx \frac{\partial u_2/\partial x_3 + \partial u_3/\partial x_2}{1 + \partial u_2/\partial x_2 + \partial u_3/\partial x_3} \approx \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}. \tag{34}$$

and

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}; \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}; \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \tag{35}$$

The concept of strain:

Strain-displacement relations

- The relative elongations, eq. (29), and angular distortions, eq. (35) characterize the state of deformation at a point.
- The relative elongations are also called *direct strains* or *axial strains*, and they are given by:

$$\epsilon_i = \frac{\partial u_i}{\partial x_i}$$

- The angular distortions are called *shearing strains* or simply *shear strains*, and they are given by:

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

The expressions above define the *strain-displacement relations* under the assumption of small displacements and strains.

Notes

- Similarly to stresses, a strain tensor can be defined to characterized the strain state at a point.
- The strain tensor is defined in terms of a slightly modified definition for the shear strain terms, which are called *tensor shear strain components*, and are defined as:

$$\epsilon_{23} = \frac{\gamma_{23}}{2}, \quad \epsilon_{13} = \frac{\gamma_{13}}{2}, \quad \epsilon_{12} = \frac{\gamma_{12}}{2}. \tag{36}$$

- The shearing strain components γ_{23}, γ_{13} and γ_{12} are called the *engineering shear strain components*,

The state of plane strain

- A particular state of strain of great practical importance is the *plane state of strain*.
- In this case, the displacement component along the direction of axis $\bar{\imath}_3$ is assumed to vanish, or to be negligible compared to the displacement components in the other two directions.
- Only non-vanishing strain components are ϵ_1, ϵ_2, and γ_{12},
- Furthermore, these strain components are assumed to be independent of x_3.
- The strain-displacement relations reduce to:

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \tag{37}$$

Notes

- Consider the strain tensor and its components in a plane strain state, allows deriving rotation expressions for strain that are analogous to those introduced for stress.
- Principal strain directions and values can be identified in an analogous manners
- Directions and values of maximum shear strain can be also computed through formulas that have the same form as those for stresses
- The directions of principal strain do not correspond to the directions of principal stress, in general

Measurement of strains

- Measurement of the state of strain is itself not an entirely straightforward process.
- Most measurement methods measure the strain on an external surface of the body, which provides information on a state of plane strain.
- Measurement of angular changes associated with shear strain is very difficult, but measurement of the extensional strain on a surface is surprisingly easy.
- The relative elongation at the surface of a body can be measured with the help of what are called electrical resistance strain gauges, or more simply *strain gauges*. This device consists of a very thin electric wire, or an etched foil pattern, which is glued to the surface of the solid.

The complete state of strain at the surface is evaluated from the measurement of relative elongation in three distinct directions on the surface.

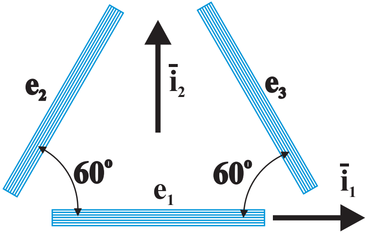


Figure: Three strain gauges in a *strain rosette* on the surface of a solid.

Let e_1, e_2, and e_3 be the experimentally measured relative elongations in the three gauge directions.
With the help of strain rotation equations, these measurements can be related to the strain components along the axes $\bar{\imath}_1$ and $\bar{\imath}_2$ as follows

$$
\begin{aligned}
e_1 &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2};\\
e_2 &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2}\cos(+2\times 60^\circ) + \frac{\gamma_{12}}{2}\sin(+2\times 60^\circ);\\
e_3 &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2}\cos(-2\times 60^\circ) + \frac{\gamma_{12}}{2}\sin(-2\times 60^\circ).
\end{aligned}
$$

These relationships can be inverted to yield the strain components in terms of the measured axial strains

$$
\epsilon_1 = e_1; \quad \epsilon_2 = \frac{2}{3}\left(e_2 + e_3 - \frac{e_1}{2}\right); \quad \gamma_{12} = \frac{2}{\sqrt{3}}\left(e_2 - e_3\right). \tag{38}
$$

The principal strain directions then follow.

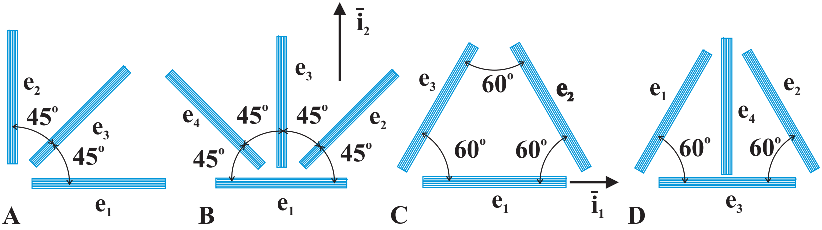


Figure: Various commonly used strain gauge arrangements.

Strain compatibility equations

- The displacement field uniquely defines the deformation of a solid body. However, six strain components have been defined to characterize the state of deformation at a point.
- Hence, the strain components are not independent and must satisfy a set of relationships called *strain compatibility equations*.

Consider the following derivatives of the shear strain components

$$\frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2}{\partial x_2 \partial x_3}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{\partial^3 u_2}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u_3}{\partial^2 x_2 \partial x_3} = \frac{\partial^2 \epsilon_2}{\partial x_3^2} + \frac{\partial^2 \epsilon_3}{\partial x_2^2}.$$

This clearly implies that the shear and axial strain components are not independent. Consider now a different set of derivatives

$$\frac{\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2 \partial x_3}; \quad \frac{\partial \gamma_{23}}{\partial x_1} = \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2};$$

$$\frac{\partial \gamma_{13}}{\partial x_2} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2}; \quad \frac{\partial \gamma_{12}}{\partial x_3} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3},$$