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Local Form of the balance of linear Momentum

Recall global form: $\int_B \rho(\ddot{x} - \underline{b}) = \int_{\partial B} \hat{f}\, dA$

Then, for an arbitrary subbody $E \subset B$ we have:

$$\int_E \rho(\ddot{x} - \underline{b})\, dv = \int_{\partial E} \underline{t}\, dA = \int_{\partial E} \underline{\underline{\sigma}} \cdot \underline{n}\, dA$$

$$= \int_E \operatorname{div}(\underline{\underline{\sigma}})\, dv$$

divergence theorem

$$\Rightarrow \int_E \left(\operatorname{div}(\underline{\underline{\sigma}}) + \rho \underline{b} - \rho \ddot{\underline{x}}\right) dv = 0 \quad \forall E \subset B$$

already using conservation of mass

This must be true $\forall E \subset B$, therefore the integrand must vanish:

$\partial \sigma_{ij} / \partial x_j$

$\operatorname{div}(\underline{\underline{\sigma}}) + \rho \underline{b} = \rho \ddot{\underline{x}}$; $\boxed{\sigma_{ij,j} + \rho b_i = \rho \ddot{x}_i \text{ in } \underline{x} \in B}$

Finally, for static problems:

$$\operatorname{div}(\underline{\sigma}) + \rho \underline{b} = 0 \; ; \qquad \boxed{\sigma_{ij,j} + \rho b_i = 0 \quad \text{in } \underline{x} \in B}$$

Balance of angular momentum

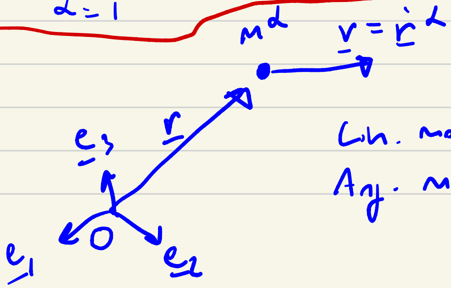
The principle of conservation of angular momentum states:

$$\frac{D}{Dt} \underline{H}_0 = \underline{M}_0^{ext} \; ; \qquad \begin{cases} \underline{H}_0 \equiv \text{angular momentum of the sys about } 0 \\ \underline{M}_0^{ext} \equiv \text{total external moment about } 0 \end{cases}$$

For a system of N particles we have:

$$\underline{H}_0 = \sum_{\alpha=1}^{N} \underline{r}^{\alpha} \times (m^{\alpha} \underline{\dot{r}}^{\alpha}) \; ;$$

$$\underline{M}_0^{ext} = \sum_{\alpha=1}^{N} \underline{r}^{\alpha} \times \underline{f}^{\alpha} \; ;$$



Lin. mom: $m^{\alpha} \underline{\dot{r}}^{\alpha}$

Ang. mom: $\underline{r}^{\alpha} \times m^{\alpha} \underline{\dot{r}}^{\alpha}$

These expressions can be generalized for a continuum.

For a subbody $E \subset B$ we have:

$$\underline{H}_o(E) = \int_E \underline{x} \times (dm\, \dot{\underline{x}}) = \int_E \underline{x} \times (\rho \dot{\underline{x}})\, dv$$

$$\underline{M}_o^{ext}(E) = \underbrace{\int_E \underline{x} \times (\rho \underline{b})\, dv}_{\text{external}} + \underbrace{\int_{\partial E} \underline{x} \times \underline{t}\, dA}_{\text{traction}}$$

Resulting into:

$$\frac{D}{Dt} \int_E \underline{x} \times (\rho \dot{\underline{x}})\, dv = \int_E \underline{x} \times \rho \underline{b}\, dv + \int_{\partial E} \underline{x} \times \underline{t}\, dA$$

In indicial notation:

$$\frac{D}{Dt} \int_E \epsilon_{ijk}\, x_j\, \dot{x}_k\, \rho\, dv = \int_E \epsilon_{ijk}\, x_j\, b_k\, \rho + \int_{\partial E} \epsilon_{ijk}\, x_j\, t_k\, dA$$

As previously done, by applying conservation of mass and $dv = J\,dV$, we can express the left term in the reference configuration, introduce the time derivative inside the integral, and then go back to the deformed configuration

Additionally, we know that $t_k = \sigma_{km} n_m$, leading to:

$$\int_E \frac{D}{Dt}(\epsilon_{ijk}\, x_j\, \dot{x}_k)\rho\, dv = \int_E \epsilon_{ijk}\, x_j\, b_k\, \rho\, dv + \int_{\partial E} \epsilon_{ijk}\, x_j\, \sigma_{km} n_m\, dA$$

But $\epsilon_{ijk}\, x_j\, \sigma_{km} \equiv \Gamma_{im}$ (from free indices)

so the divergence theorem gives

$$\int_{\delta E} \Gamma_{im} n_m\, dA = \int_E (\Gamma_{im})_{,m}\, dA$$

$$\int_E \epsilon_{ijk}(\dot{x}_j\, \dot{x}_k + x_j\, \ddot{x}_k)\rho\, dv = \int_E \epsilon_{ijk}\, x_j\, b_k\, \rho\, dv + \int_E [\epsilon_{ijk}\, x_j\, \sigma_{km}]_{,m}\, dv$$

$$\epsilon_{ijk}\, \dot{x}_j\, \dot{x}_k = (\dot{\underline{x}} \times \dot{\underline{x}})_i = 0$$

$$(\epsilon_{ijk}\, x_j\, \sigma_{km})_{,m} = \epsilon_{ijk} \underbrace{\frac{\partial x_j}{\partial x_m}}_{\delta_{jm}} \sigma_{km} + \epsilon_{ijk}\, x_j \frac{\partial \sigma_{km}}{\partial x_m}$$

$$= \epsilon_{ijk}\, \sigma_{kj} + \epsilon_{ijk}\, x_j\, \sigma_{km,m}$$

Resulting into:

$$\int_E \epsilon_{ijk} x_j \underbrace{(\rho \ddot{x}_k - \rho b_k - \sigma_{km,m})}_{\substack{=0 \\ \text{(balance of linear momentum)}}} dv = \int_E \epsilon_{ijk} \sigma_{kj} \, dv = 0 \quad \forall E \subset B$$

Since the last equation must hold $\forall E \subset B$, then the integrand must vanish pointwise:

$$\epsilon_{ijk} \sigma_{kj} = 0$$

Expanding this expression, we get:

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{13} = \sigma_{31}$$

That is, balance of linear momentum implies that the cauchy stress tensor is symmetric: $\sigma_{ij} = \sigma_{ji}$

The symmetry of $\underline{\underline{\sigma}}$ has the following implications:

(1) $\underline{\underline{\sigma}}$ has three real eigenvalues, which are called principal stresses: $\sigma_1 \geq \sigma_2 \geq \sigma_3$

(2) one can find a basis consisting entirely of eigenvectors of $\underline{\underline{\sigma}}$ $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ with $\underline{v}_i \cdot \underline{v}_j = \delta_{ij}$

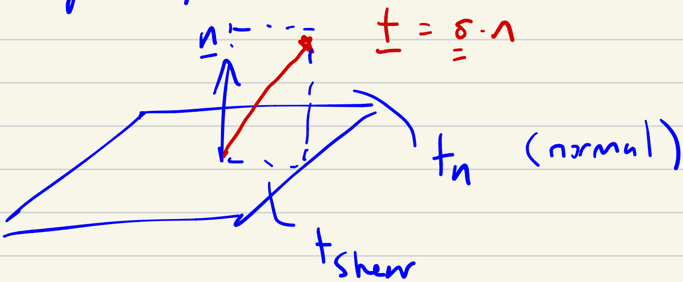
so:

$$\underline{\underline{\sigma}} \cdot \underline{v}_1 = \sigma_1 \underline{v}_1$$

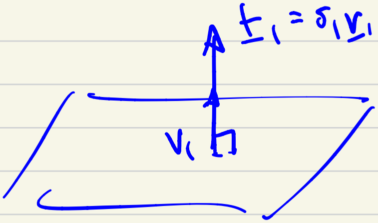
$$\underline{\underline{\sigma}} \cdot \underline{v}_2 = \sigma_2 \underline{v}_2$$

$$\underline{\underline{\sigma}} \cdot \underline{v}_3 = \sigma_3 \underline{v}_3$$

In general,



but, if we pick a face w/ normal $\underline{v}$



no shear.

(traction only normal)

That is, for surfaces $\underline{v}_i$, the corresponding traction is purely normal.

(3) In the basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ the components of $\underline{\underline{\sigma}}$ form a diagonal matrix:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

σ_i gives the normal stress in the direction $\underline{v}_i$, there are no shear stresses in those planes.

(4) The largest shear stress is $\frac{1}{2}(\sigma_1 - \sigma_3)$ and corresponds to directions $\frac{1}{\sqrt{2}}(\underline{v}_1 + \underline{v}_3)$

and

$\frac{1}{\sqrt{2}}(\underline{v}_1 - \underline{v}_3)$

(5) If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ then, in any basis we have $\underline{\underline{\sigma}} = \sigma \underline{\underline{I}}$ (no shear in any direction)