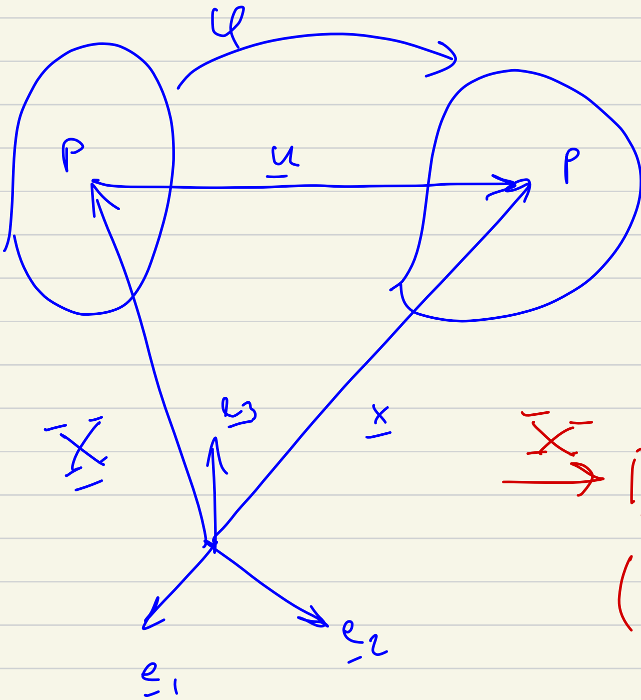
Kinematics

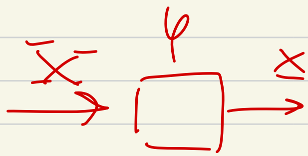
(Final)

9/15

Infinitesimal Deformation

linearize (lots of assumptions) C^{∞} continuous, etc.





($\underline{u}$: displacement)

Let the motion be defined:

- $\underline{x} = \underline{\varphi}(\underline{X}, t) = \underline{X} + \underline{u}(\underline{X}, t)$
  where $\underline{u}(\underline{X}, t)$ is the displacement of material point $\underline{X}$ at time t

- A deformation is said to be infinitesimal if:

$$\left| \frac{\partial u_i}{\partial X_j} \right| \ll 1$$

under this assumption, we can neglect higher order terms of $\partial u_i / \partial X_j$

Expressions of $\underline{\underline{F}}$ and $\underline{\underline{E}}$ in terms of $\underline{u}$

deformation gradient $\underline{\underline{F}}$:

$$\underline{\underline{F}} = F_{ij}\,\underline{e}_i\underline{e}_j \,; \qquad F_{ij} = \frac{\partial \varphi_i}{\partial X_j}$$

$$F_{ij} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial (X_i + u_i)}{\partial X_j} = \underbrace{\frac{\partial X_i}{\partial X_j}}_{\delta_{ij}} + \frac{\partial u_i}{\partial X_j}$$

$$\Rightarrow \boxed{\underline{\underline{F}} = \left(\frac{\partial u_i}{\partial X_j} + \delta_{ij}\right)\underline{e}_i\underline{e}_j}$$

$$\boxed{\underline{\underline{F}} = \nabla\underline{u} + \underline{\underline{I}}}$$ direct notation

The Lagrangian Strain Tensor is defined as:

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{F}}^T \cdot \underline{\underline{F}} - \underline{\underline{I}}\right) = \underbrace{\frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right)}_{E_{ij}}\underline{e}_i\underline{e}_j$$

$$E_{ij} = \frac{1}{2}\Big[\underbrace{\left(\frac{\partial u_k}{\partial X_i} + \delta_{ki}\right)}_{F_{ki}}\underbrace{\left(\frac{\partial u_k}{\partial X_j} + \delta_{kj}\right)}_{F_{kj}} - \delta_{ij}\Big]$$

$$= \frac{1}{2}\left(\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\delta_{kj} + \delta_{ki}\frac{\partial u_k}{\partial X_j} + \delta_{ki}\delta_{kj} - \delta_{ij}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \cancel{\delta_{ij}} - \cancel{\delta_{ij}}\right)$$

so...

$$\boxed{\underline{\underline{E}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)\underline{e_i}\,\underline{e_j}}$$

↳ so far, no simplification has been made. This expression is fully valid for linear deformations.

For infinitesimal deformations, $\left| \frac{\partial u_i}{\partial X_j} \right| << 1$,

so we can neglect $\frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j}$,

yielding to the definition of infinitesimal strain tensor:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \underline{e}_i \underline{e}_j}$$

* $\left| \frac{\partial u_i}{\partial X_j} \right| << 1$ assumption must hold *

(The undergrad strain definition.)

Stretch for infinitesimal deformations

Recall: $\lambda = \sqrt{C_{ij} N_i N_j} = \sqrt{F_{ki} F_{kj} N_i N_j}$

where $\underline{N} = N_i \underline{e}_i$ is the material direction in which we want to measure the stretch.

Recalling that $F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$:

$$\lambda = \sqrt{\underbrace{\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} + \delta_{ij} \right)}_{F_{ki} F_{kj}} N_i N_j}$$

$$= \sqrt{\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \delta_{ij} \right) N_i N_j}$$

$$= \sqrt{\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) N_i N_j + \underbrace{\delta_{ij} N_i N_j}_{N_i N_i = 1}}$$

$$= \sqrt{1 + \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) N_i N_j}$$

scalar of order $O\left(\frac{\partial u_i}{\partial X_j}\right) << 1$

$$\Rightarrow \lambda = \sqrt{1+\alpha} \; ;$$

$$\alpha = \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) N_i N_j \ll 1$$

Recall that

$$(1+\alpha)^n \approx 1 + n\alpha \quad \text{for } \alpha \ll 1$$

$$\Rightarrow \quad \lambda \approx 1 + \frac{1}{2}\alpha$$

$$\Rightarrow \quad \lambda \approx 1 + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right)}_{\varepsilon_{ij}} N_i N_j$$

$$\Rightarrow \boxed{\begin{aligned} \lambda &= 1 + \varepsilon_{ij} N_i N_j \\ &= 1 + \underline{N} \cdot \underline{\underline{\varepsilon}} \cdot \underline{N} = 1 + [N]^T [\varepsilon] [N] \end{aligned}}$$

* Not exact! This is an approx, valid when infinitesimal deformations *

Let us consider the basic definition of normal strain

$$\varepsilon_{\underline{N}} = \frac{\text{change in length}}{\text{original length}} = \lambda - 1$$

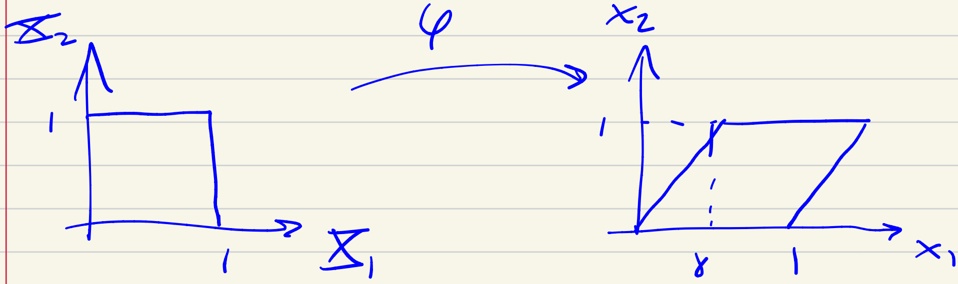
Then, for infinitesimal deformation

$$\boxed{\varepsilon_{\underline{N}} = \varepsilon_{ij} N_i N_j = \underline{N} \cdot \underline{\underline{\varepsilon}} \cdot \underline{N}}$$

NOTE: The last expression is only valid for infinitesimal deformations!

In general $\varepsilon_{\underline{N}} = \underbrace{\lambda - 1 = \sqrt{C_{ij} N_i N_j} - 1}_{\text{exact expression}}$

Let's go back to our original example:



$$\varphi : \begin{cases} x_1 = \mathbb{X}_1 + \gamma \mathbb{X}_2 \\ x_2 = \mathbb{X}_2 \\ x_3 = \mathbb{X}_3 \end{cases}$$

Note: $\underline{x} = \underline{\underline{\mathbb{X}}} + \underline{u}$

Then, the displacements are given by:

$$u_1 = \gamma \mathbb{X}_2, \quad u_2 = 0, \quad u_3 = 0$$

we can now compute the components of the inf. strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \mathbb{X}_j} + \frac{\partial u_j}{\partial \mathbb{X}_i} \right);$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial \mathbb{X}_1} = 0 \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial \mathbb{X}_2} + \frac{\partial u_2}{\partial \mathbb{X}_1} \right) = \varepsilon_{21} = \frac{1}{2}(\gamma + 0) = \frac{\gamma}{2}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial \mathbb{X}_2} = 0 \qquad \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial \mathbb{X}_3} + \frac{\partial u_3}{\partial \mathbb{X}_1} \right) = 0$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial \mathbb{X}_3} = 0$$

In matrix form:

$$[\varepsilon] = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a good measure of deformation

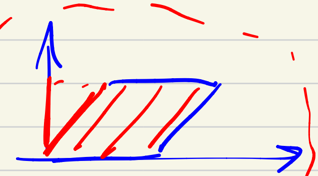
for $\left|\dfrac{\partial u_i}{\partial X_j}\right| \ll 1$ * depends on the problem *

In our case,

$\dfrac{\partial u_i}{\partial X_j} = 0$ for all, except $\dfrac{\partial u_1}{\partial X_2} = \gamma$

so good measure if $\gamma \ll 1$

Extensional strain ε in directions:



- $\underline{e_1}$ is $\varepsilon_{\underline{e_1}} = \varepsilon_{11} = 0$
- $\underline{e_2}$ is $\varepsilon_{\underline{e_2}} = \varepsilon_{22} = 0$ → This stretch is $\dfrac{\gamma}{2}$
- $\underline{N} = \left[\dfrac{\sqrt{2}}{2} \;\; \dfrac{\sqrt{2}}{2} \;\; 0\right]^T$

is $\varepsilon_N = \left[\dfrac{\sqrt{2}}{2} \;\; \dfrac{\sqrt{2}}{2} \;\; 0\right] \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \Rightarrow \varepsilon_N = \dfrac{\gamma}{2}$

Finally; $\lambda = \varepsilon_{\underline{N}} + 1 = 1 + \dfrac{\gamma}{2}$

Note:

- All quantities are consistent with those computed in finite deformation if we assume $\gamma << 1$!
- It is interesting to see that the extensional strain in the direction of $\underline{e}_z$ is 0 for the inf. def assumption while remaining finite in the exact case.

$\hookrightarrow$ 0 vs $\dfrac{\gamma^2}{2}$