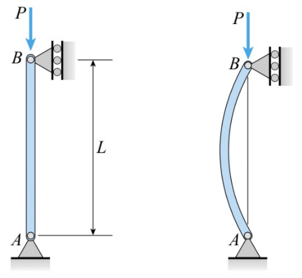
BUCKLING

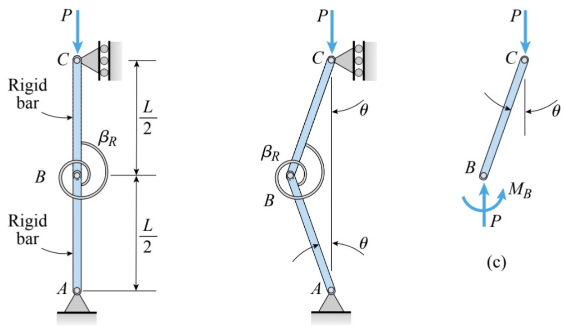
What is it?



(a) (b)

<u>Buckling is a type of failure</u>

(a) a column loaded axially in compression
(b) failure by buckling (lateral deflection)
→ column can bend throughout its length



(a) (b) (c)

<u>Idealized Structure (Buckling Model)</u>

(a) analogous to column
simple supports at ends
axial load P in compression along longitudinal axis
two bars perfectly aligned

(b) point B moves laterally (small distance)
rigid bars rotate through small angle θ
moment develops in spring
→ elasticity is "concentrated" in the rotational spring

Stable vs Unstable

<u>Stable Structure</u>

- when P is removed and structure returns to initial position

<u>Unstable Structure</u>

- when P is removed and structure collapses
- structure fails by lateral buckling

Critical Load

- denoted by P_{cr}
- value of the axial force that marks the transition between the stable and unstable conditions

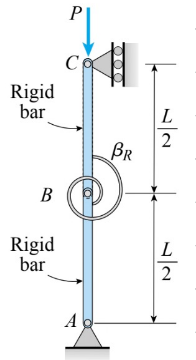
Main Points:

<u>Buckling</u> is a type of failure where an axially compressed structure fails by lateral deflection.

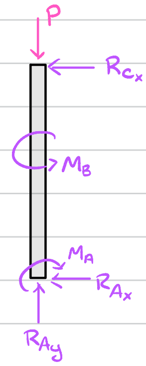
<u>Critical load</u> is the value of the axial force that occurs at the transition between the stable and unstable conditions.

BUCKLING

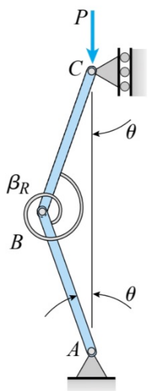
Calculating critical load



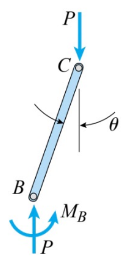
Consider the structure as a free body diagram:



$\Sigma F_x = 0 \Rightarrow$ no horizontal reaction at C



Consider BC as a free body:



$M_B = 2\beta_R\theta$ ← for translational spring: $F = \beta\delta$

β_R: rotational stiffness

2θ: angle of rotation

θ is small $\Rightarrow$ lateral displacement of B $= \frac{L}{2}\sin\theta$

↑ assumption $\simeq \frac{\theta L}{2}$

Therefore, $\Sigma M_B = 0$: $M_B - P\left(\frac{\theta L}{2}\right) = 0$

$\left(2\beta_R - \frac{PL}{2}\right)\theta = 0$

Trivial solution: $\theta = 0$

$2\beta_R - \frac{PL}{2} = 0 \Rightarrow \boxed{P_{cr} = \frac{4\beta_R}{L}}$

- At P_{cr}, structure is in equilibrium regardless of θ ← θ is small (assumption)
- P_{cr} is <u>only</u> load for which structure will be in equilibrium in the disturbed position

Stability

<u>Increase stability of structure by:</u>

- increasing stiffness
- decreasing length

Main Points:

If $0 < P < P_{cr}$: stable ; equilibrium only when $\theta = 0$

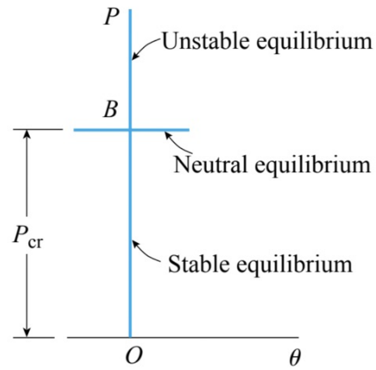
If $P = P_{cr}$: neither stable nor unstable ; equilibrium for any small angle θ

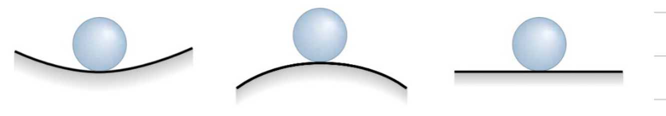
If $P > P_{cr}$: unstable ; equilibrium only when $\theta = 0$; slightest disturbance $\Rightarrow$ buckling

$P = P_{cr}$: neutral equilibrium

BUCKLING

Stability





stable   unstable   neutral equilibrium

B is the <u>bifurcation point</u>

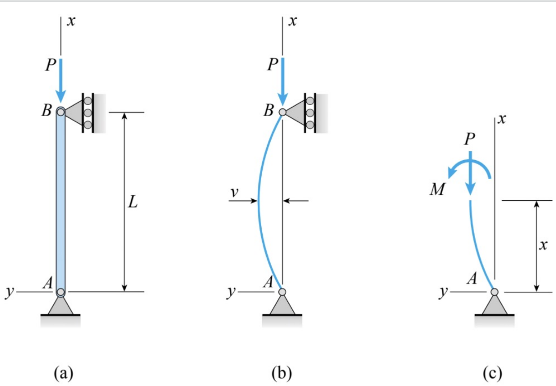
Euler buckling (Euler load)

<u>Assumptions</u>

- slender column ($t, w \ll L$)
- P applied through centroid of cross section
- column is perfectly straight
- linearly elastic material that obeys Hooke's law
- no imperfections

} ideal column

- xy plane is plane of symmetry for column
- buckling occurs in xy plane



Bending moment : $EI\,v'' = M$

EI: flexural rigidity

v: lateral deflection (y-axis)

M: bending moment at any cross section

$\Sigma M_A = 0$: $M + Pv = 0 \Rightarrow EI\,v'' + Pv = 0$

homogeneous, linear, 2nd order differential equation with constant coefficients

From Diff. Equ. : if $k^2 = \frac{P}{EI}$, $v'' + k^2 v = 0 \Rightarrow v = C_1 \sin(kx) + C_2 \cos(kx)$

general solution

Boundary conditions: $v(0) = 0$ $v(L) = 0$ $\Rightarrow v = C_1 \sin(kx)$

$C_1 \sin(kL) = 0$

<u>$C_1 = 0$</u>: $v = 0 \Rightarrow$ column remains straight (trivial solution)

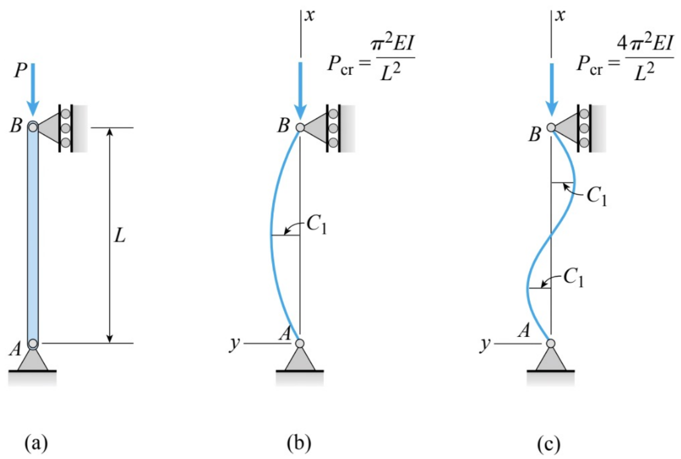
Comments: The strength of a material (represented by quantity such as proportional limit or yield stress) is not in the equation for P_{cr}. Therefore, <u>increasing the strength does not increase P_{cr}</u>

BUCKLING

Euler buckling (Euler load)

$\underline{\sin(kL)=0}$: $kL=n\pi$, $n=1,2,3,\ldots$ $\quad$ $\sin(kL)=0 \leftarrow$ buckling equation

or $P_{cr}=\dfrac{n^2\pi^2EI}{L^2}$, $n=1,2,3,\ldots$ $\leftarrow n=0 \Rightarrow kL=0 \Rightarrow P=0 \Rightarrow$ not interested



$P_{cr}=\dfrac{n^2\pi^2EI}{L^2}$, $n=1,2,3,\ldots$

(a) initially straight

(b) buckled, $n=1$

(c) buckled, $n=2$

$n=1$

The buckled shape (mode shape) is $v=C_1\sin\left(\frac{\pi x}{L}\right)$

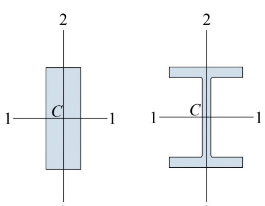
C_1: deflection at the midpoint

- Deflection at P_{cr} is undefined $\leftarrow$ must remain small
- Magnitude of P_{cr} proportional to n^2
- Number of half waves equal to n
- Column buckles when P reaches lowest critical value
  - $\Rightarrow$ buckled shapes for higher modes are of no practical interest
  - $\Rightarrow$ Only way to obtain modes higher than $n=1$ is to provide lateral support at intermediate points (as in (c) above)
- $P_{cr}\propto EI$, $P_{cr}\propto\frac{1}{L^2}$ $\quad\hookrightarrow\ \boxed{P_{cr}=\dfrac{\pi^2EI}{L^2}}$

How can we increase P_{cr}?

- Reduce length
- Provide additional lateral support
- Increase flexural rigidity (EI)
  - material with larger E $\leftarrow$ "stiffer" material
  - distributing material to increase I of cross section
    - distribute material farther from centroid of cross section (hollow tube is more economical than solid member with same cross sectional area)

Comments:



If column is supported only at ends and can buckle in any direction, then bending will occur about principal centroidal axis with smaller I (ex. I-beam will buckle in 1-1 plane $(I_1>I_2)$ but I_2 should be used in formula for P_{cr})

(square or circular cross section $\Rightarrow$ all centroidal axes have same I)

BUCKLING

Equilibrium Approach

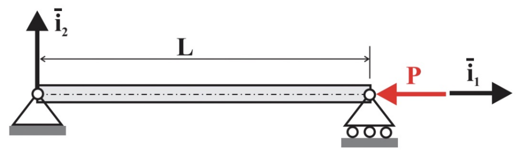


Figure: Simply-supported beam with end compressive load.

Assume $\bar{i}_2$ to be principal axis of bending

N_1 constant along span $\Rightarrow N_1 = -P$

axial force in beam

Governing Equation

$$H_{33}^c \frac{d^4 \bar{u}_2}{dx_1^4} + P \frac{d^2 \bar{u}_2}{dx_1^2} = 0$$

Boundary Conditions

$$\bar{u}_2 = \frac{d^2 \bar{u}_2}{dx_1^2} = 0$$

At $x_1 = 0$ and $x_1 = L$:

- governing equations and boundary conditions are homogeneous $\Rightarrow \bar{u}_2 = 0$ is trivial solution
- buckling load = lowest load for which non-trivial solution exists

Governing Equation – non-dimensional : $\eta = \dfrac{x_1}{L}$

$$\Rightarrow \bar{u}_2'''' + \lambda^2 \bar{u}_2'' = 0$$

where $\lambda^2 = \dfrac{PL^2}{H_{33}^c}$ ← non-dimensional loading parameter

Boundary Conditions – non-dimensional : $\bar{u}_2 = \bar{u}_2'' = 0$ at $\eta = 0$ and $\eta = 1$

General Solution

$$\bar{u}_2 = A + B\eta + C\cos\lambda\eta + D\sin\lambda\eta$$

B.C.'s $\Rightarrow A + C = 0$, $\lambda^2 C = 0 \Rightarrow A = C = 0$

$\Rightarrow \bar{u}_2 = B\eta + D\sin\lambda\eta$

$$\begin{bmatrix} 1 & \sin\lambda \\ 0 & \sin\lambda \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det \begin{bmatrix} 1 & \sin\lambda \\ 0 & \sin\lambda \end{bmatrix} = 0 \Rightarrow \sin\lambda = 0$$

B = D = 0 is trivial solution

$$\lambda = n\pi,\ n = 1, 2, 3, \ldots \longrightarrow P_{cr} = \frac{n^2 \pi^2 H_{33}^c}{L^2},\ n = 1, 2, 3, \ldots$$

The lowest root $(n=1)$ gives the Euler Load : $\boxed{P_{cr} = \dfrac{\pi^2 H_{33}^c}{L^2}}$

Comments: Euler Load : $P_{cr} = \dfrac{\pi^2 H_{33}^c}{L^2} = \dfrac{\pi^2 EI}{L^2}$

BUCKLING

Imperfection Approach

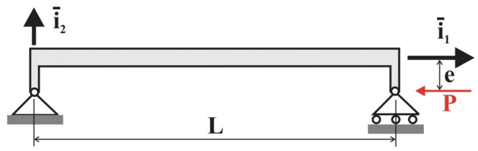


Figure: Pinned-pinned beam with eccentric end loads.

Assume $\bar{i}_2$ to be principal axis of bending

N_1 constant along span $\Rightarrow N_1 = -P$
axial force in beam

Governing Equation

$$H_{33}^c \frac{d^4 \bar{u}_2}{dx_1^4} + P \frac{d^2 \bar{u}_2}{dx_1^2} = 0$$

Boundary Conditions

$$\bar{u}_2 = 0 \quad , \quad H_{33}^c \frac{d^2 \bar{u}_2}{dx_1^2} = -Pe$$

← The problem is not homogeneous because of non-zero terms on RHS
$\Rightarrow \bar{u}_2(x_1) \equiv 0$ is not a solution

Boundary Conditions - non-dimensional : $\bar{u}_2 = 0$, $\bar{u}_2'' = -\lambda^2 e$

General Solution

$$\eta = \frac{x_1}{L} \qquad \lambda^2 = \frac{PL^2}{H_{33}^c}$$

$$\bar{u}_2 = A + B\eta + C\cos\lambda\eta + D\sin\lambda\eta$$

B.C.'s $\Rightarrow$ $A + C = 0$, $-\lambda^2 C = -\lambda^2 e$ $\Rightarrow$ $-A = C = e$

$$\Rightarrow \left.\begin{array}{l} B + D\sin\lambda = e(1-\cos\lambda) \\ -\lambda^2 D\sin\lambda = -e\lambda^2(1-\cos\lambda) \end{array}\right\} \quad D = \frac{e(1-\cos\lambda)}{\sin\lambda}, \quad B = 0$$

$$\Rightarrow \bar{u}_2 = e\left[\frac{\cos\lambda(\eta - \frac{1}{2})}{\cos\frac{\lambda}{2}} - 1\right]$$

← $\bar{u}_2 \to \infty$ when $\cos\frac{\lambda}{2} \to 0$ $(\lambda = (2n-1)\pi,\ n = 1, 2, \ldots)$

$$\longrightarrow P_{cr} = \frac{(2n-1)^2 \pi^2 H_{33}^c}{L^2}, \quad n = 1, 2, 3, \ldots$$

The lowest root $(n=1)$ gives the same Euler Load : $\boxed{P_{cr} = \frac{\pi^2 H_{33}^c}{L^2}}$

Comments:

The maximum bending moment occurs in the middle of the beam: $M_{max} = \frac{Pe}{\cos(\frac{\lambda}{2})}$
The beam deflects until the maximum axial stress reaches σ_{allow}
For structures with large imperfections, P_{allow} can be substantially lower than P_{Euler}

BUCKLING

Energy Approach

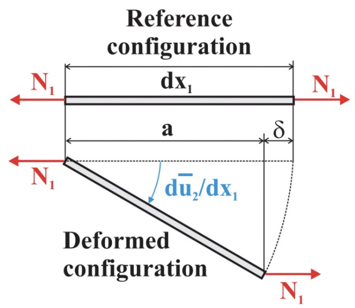


Figure: A differential element of the beam in the deformed configuration.

Cantilevered beam subject to axial force.

Work done by axial force that causes buckling is key

- Point of application of axial force displaces an amount δ along the line of action of the force

Work done by axial force :

$$dW = -N_1\delta \quad \text{where} \quad \delta = dx_1 - dx_1\cos\frac{d\bar{u}_2}{dx_1} = dx_1 - dx_1\left[1 - \frac{1}{2}\left(\frac{d\bar{u}_2}{dx_1}\right)^2 + \dots\right]$$

Slope of beam:

$$\delta = \frac{1}{2}(d\bar{u}_2 dx_1)^2 dx_1 \quad \leftarrow \text{assumed} \ll 1$$

Total work done by internal axial forces:

$$\phi = \frac{1}{2}\int_0^L N_1(x_1)\left(\frac{d\bar{u}_2}{dx_1}\right)^2 dx_1 \quad \leftarrow \text{integration along span of beam}$$

Total potential energy:

$$\Pi = \frac{1}{2}\int_0^L H_{33}^c\left(\frac{d^2\bar{u}_2}{dx_1^2}\right)^2 dx_1 + \frac{1}{2}\int_0^L N_1\left(\frac{d\bar{u}_2}{dx_1}\right)^2 dx_1 - \int_0^L p_2\bar{u}_2\, dx_1$$

Principle of minimum potential energy:

$$\delta\Pi = \int_0^L H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\delta\frac{d^2\bar{u}_2}{dx_1^2} dx_1 + \int_0^L N_1\frac{d\bar{u}_2}{dx_1}\delta\frac{d\bar{u}_2}{dx_1}dx_1 - \int_0^L p_2\,\delta\bar{u}_2\, dx_1 = 0 \quad \leftarrow \text{total potential energy is stationary}$$

BUCKLING

Energy Approach

$$\int_0^L \delta\bar{u}_2 \left[\frac{d^2}{dx_1^2}\left(H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2} - \frac{d}{dx_1}\left(N_1 \frac{d\bar{u}_2}{dx_1}\right) - p_2\right]dx_1 + \left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\delta\left(\frac{d\bar{u}_2}{dx_1}\right)\right]_0^L$$
$$+\left\{\left[N_1 \frac{d\bar{u}_2}{dx_1} - \frac{d}{dx_1}\left(H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right)\right]\delta\bar{u}_2\right\}_0^L = 0$$

- equation is satisfied for arbitrary variations $(\delta\bar{u}_2(x_1))$ only if integral and each boundary term vanishes independently.

Vanishing of integral leads to: $\frac{d^2}{dx_1^2}\left(H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right) - \frac{d}{dx_1}\left(N_1 \frac{d\bar{u}_2}{dx_1}\right) = p_2$

<u>Boundary conditions for cantilevered beam:</u>

$\bar{u}_2 = 0,\ \frac{d\bar{u}_2}{dx_1} = 0$ at $x_1 = 0$ ← root of beam

$H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2} = 0$ at $x_1 = L$ ← corresponds to vanishing of bending moment at tip of beam

$N_1 \frac{d\bar{u}_2}{dx_1} - \frac{d}{dx_1}\left(H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right) = 0$ at $x_1 = L$ ← corresponds to $N_1 \frac{d\bar{u}_2}{dx_1} + V_2 = 0$ at $x_1 = L$

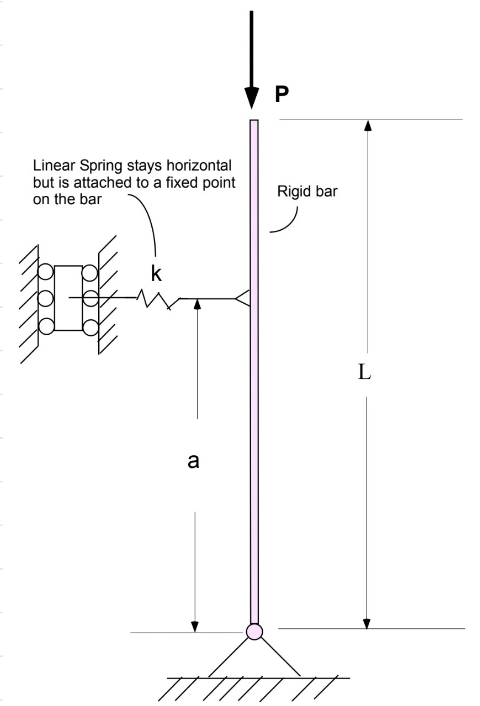
$\delta\left(\frac{d\bar{u}_2(L)}{dx_1}\right)$ and $\delta\bar{u}_2(L)$ are arbitrary

- These boundary conditions correspond to the tip equilibrium equations of the beam in its deformed configuration

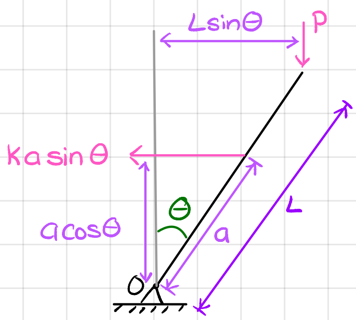
BUCKLING

Example: Example from AE8803 HW8 (Kardomateas)

Analyze the system shown using large-deflection theory. Give the load-deflection curve and the critical load.



a) use the equilibrium (classical) approach



$\circlearrowleft \Sigma M_o = 0:$

$$Ka\sin\theta(a\cos\theta) - PL\sin\theta = 0$$

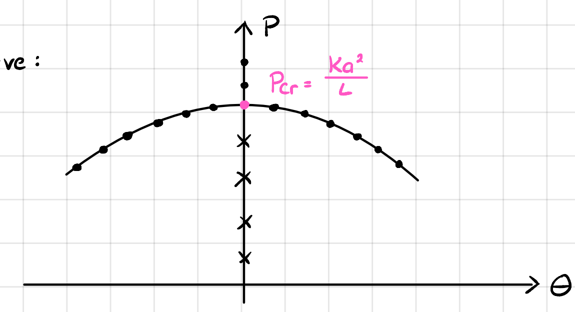
$$(Ka^2\cos\theta - PL)\sin\theta = 0 \Rightarrow$$

$\sin\theta = 0 \Rightarrow \theta = 0$ ← trivial solution

$Ka^2\cos\theta - PL = 0 \Rightarrow P = \frac{Ka^2}{L}\cos\theta$

P_{cr} is for $\theta = 0 \Rightarrow \boxed{P_{cr} = \frac{Ka^2}{L}}$

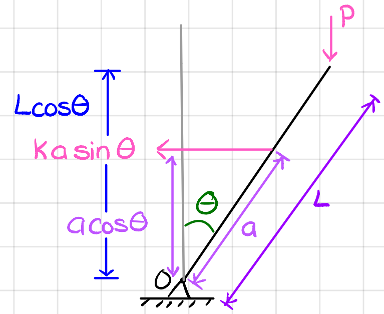
Load-deflection curve:



b) Use the energy approach

potential energy for linear spring: $U = \frac{1}{2}Kx^2$

potential energy: $U = Fx$ ← x is distance



$$U_T = \frac{1}{2}K(a\sin\theta)^2 - P(L - L\cos\theta)$$

the distance the spring has deflected is $a\sin\theta$

the distance P has moved is $L - L\cos\theta$

Take $\frac{dU_T}{d\theta} = 0$: $\quad Ka\sin\theta\,(a\cos\theta) - PL\sin\theta = 0$

$$(Ka^2\cos\theta - PL)\sin\theta = 0 \Rightarrow$$

$\sin\theta = 0$ ← trivial solution

$Ka^2\cos\theta - PL = 0 \Rightarrow P = \frac{Ka^2}{L}\cos\theta$

thus, $\boxed{P_{cr} = \frac{Ka^2}{L}}$

To check for stability: $\frac{d^2U_T}{d\theta^2} = Ka^2\cos^2\theta - Ka^2\sin^2\theta - PL\cos\theta$

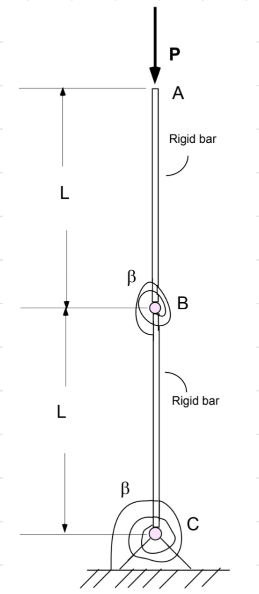
plugging in P_{cr} for P: $\quad \cancel{Ka^2\cos^2\theta} - Ka^2\sin^2\theta - \frac{Ka^2}{\cancel{L}}\cancel{L}\cos^2\theta$

$= -Ka^2\sin^2\theta$ ← this is < 0

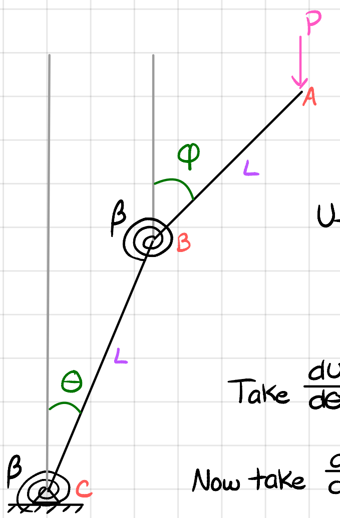
Therefore, the post-critical path is <u>unstable</u>.

Example: Example from AE8803 HW9 (Kardomateas)

Two rigid bars are connected by rotational springs to each other and to the support at C.



a) Find the critical load, P_{cr}, assuming the load remains vertical



This is a 2 DOF system

using the energy method:

$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi - \theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

Bottom spring is deflected θ

Top spring is deflected $\varphi - \theta$

Take $\frac{dU_T}{d\theta} = 0$: $\quad \beta\theta + \beta(\varphi - \theta)(-1) - PL\sin\theta = 0$

Now take $\frac{dU_T}{d\varphi} = 0$: $\quad \beta(\varphi - \theta)(1) - PL\sin\varphi = 0$

Small angle approximation: $\sin\theta \approx \theta$, $\sin\varphi \approx \varphi$

which leaves $\quad \beta\theta - \beta\varphi + \beta\theta - PL\theta = 0 \Rightarrow (2\beta - PL)\theta - \beta\varphi = 0 \quad (1)$

$\beta\varphi - \beta\theta - PL\varphi = 0 \Rightarrow -\beta\theta + (\beta - PL)\varphi = 0 \quad (2)$

Set up a matrix
Set det = 0
Solve for P

$$\det \begin{vmatrix} 2\beta - PL & -\beta \\ -\beta & \beta - PL \end{vmatrix} = 0 \Rightarrow (2\beta - PL)(\beta - PL) - (-\beta)(-\beta) = 0$$

$$2\beta^2 - 2\beta PL - \beta PL + P^2L^2 - \beta^2 = 0$$

$$(PL)^2 - 3\beta(PL) + \beta^2 = 0$$

$$PL = \frac{3\beta \pm \sqrt{(-3\beta)^2 - 4(\beta^2)}}{2} = \frac{3\beta \pm \sqrt{5\beta^2}}{2} = \frac{\beta}{2}(3 \pm \sqrt{5})$$ ← plug values into (1)

$$(2\beta - PL)\theta - \beta\varphi = 0$$

$\frac{\beta}{2}(3+\sqrt{5})$: $\quad \left[2\beta - \frac{\beta}{2}(3+\sqrt{5})\right]\theta - \beta\varphi = 0$

$$\frac{\beta - \beta\sqrt{5}}{2}\theta = \beta\varphi \Rightarrow \varphi = \frac{1-\sqrt{5}}{2}\theta \simeq -0.62\theta$$ ← does not make sense:
- θ and φ are in the same direction
- $\varphi > \theta$

$\frac{\beta}{2}(3-\sqrt{5})$: $\quad \left[2\beta - \frac{\beta}{2}(3-\sqrt{5})\right]\theta - \beta\varphi = 0$

$$\frac{\beta + \beta\sqrt{5}}{2}\theta = \beta\varphi \Rightarrow \varphi = \frac{1+\sqrt{5}}{2}\theta \simeq 1.62\theta$$ ← makes sense
⇒ buckling mode

$$\Rightarrow \boxed{P_{cr} = \frac{\beta}{2L}(3-\sqrt{5})}$$

b) Apply the stability criterion at the primary path and prove that, for $P > P_{cr}$, the equilibrium is unstable, but it is stable for $P < P_{cr}$.

Stability test on primary path:

$$\frac{dU_T}{d\theta} : (2\beta - PL)\theta - \beta\varphi = 0$$

$$\frac{dU_T}{d\varphi} : -\beta\theta + (\beta - PL)\varphi = 0$$

$$\frac{\partial^2 U_T}{\partial\theta^2} = 2\beta - PL \qquad \frac{\partial^2 U_T}{\partial\varphi^2} = \beta - PL \qquad \frac{\partial^2 U_T}{\partial\theta\partial\varphi} = -\beta$$

$$\begin{bmatrix} 2\beta - PL & -\beta \\ -\beta & \beta - PL \end{bmatrix}$$

Taking the determinant: $\quad (2\beta - PL)(\beta - PL) - (-\beta)(-\beta) > 0$

$$2\beta^2 - 2\beta PL - \beta PL + (PL)^2 - \beta^2 > 0$$

$$(PL)^2 - 3\beta PL + \beta^2 > 0$$

↑ same as what we got in (a)

$$\Rightarrow \left[PL - \frac{\beta}{2}(3-\sqrt{5})\right]\left[PL + \frac{\beta}{2}(3+\sqrt{5})\right] > 0$$

Checking all of the possible criteria:

- From the matrix: $\beta > 0$, $PL < \beta$, $\boxed{PL < 2\beta}$ ← more constraining

  ↑ not important because we are concerned with P

$$\begin{bmatrix} 2\beta - PL & -\beta \\ -\beta & \beta - PL \end{bmatrix} > 0$$

- From the determinant: $\boxed{\left[PL - \frac{\beta}{2}(3+\sqrt{5})\right]\left[PL - \frac{\beta}{2}(3-\sqrt{5})\right] > 0}$

$$(PL - 2.62\beta)(PL - 0.38\beta) > 0$$

<u>Criteria:</u>

1) $0 < PL < 0.38\beta$
   - → fulfills $PL < 2\beta$
   - → fulfills $\underbrace{(PL - 2.62\beta)}_{(-)}\underbrace{(PL - 0.38\beta)}_{(-)} > 0$ => positive

2) $0.38\beta < PL < 2\beta$
   - → fulfills $PL < 2\beta$
   - → does not fulfill $\underbrace{(PL - 2.62\beta)}_{(-)}\underbrace{(PL - 0.38\beta)}_{(+)} > 0$ => negative

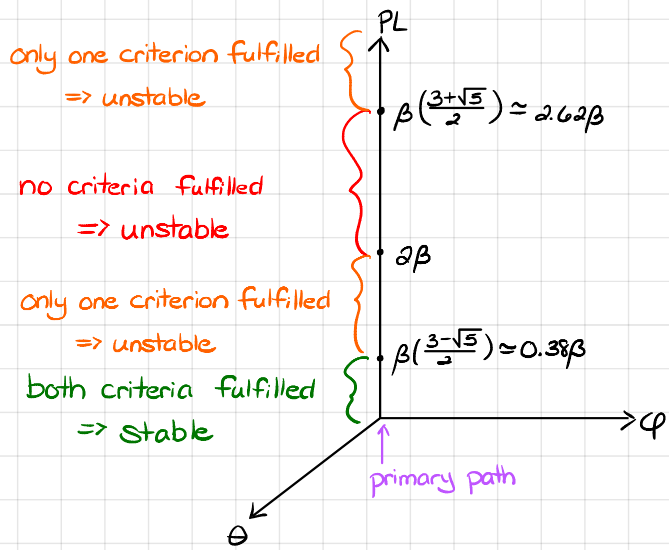
3) $2\beta < PL < 2.62\beta$
   - → does not fulfill $PL < 2\beta$
   - → does not fulfill $\underbrace{(PL - 2.62\beta)}_{(-)}\underbrace{(PL - 0.38\beta)}_{(+)} > 0$ => negative

4) $2.62\beta < PL$
   - → does not fulfill $PL < 2\beta$
   - → fulfills $\underbrace{(PL - 2.62\beta)}_{(+)}\underbrace{(PL - 0.38\beta)}_{(+)} > 0$ => positive

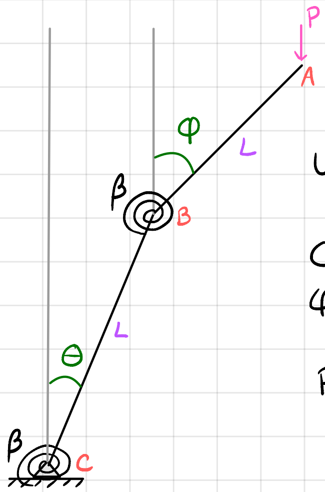
Thus it is proven that $P_{cr} = \frac{\beta}{2L}(3-\sqrt{5})$ since:

for $PL < \frac{\beta(3-\sqrt{5})}{2}$: stability criteria are fulfilled

for $PL > \frac{\beta(3-\sqrt{5})}{2}$: stability criteria are not fulfilled



Just out of curiosity, let's take a look at some other buckling configurations:

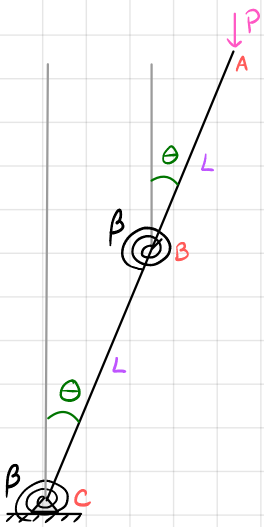


$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi-\theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

$\varphi > \theta$

φ and θ in same direction

$$P_{cr} = \frac{\beta}{2L}(3-\sqrt{5})$$

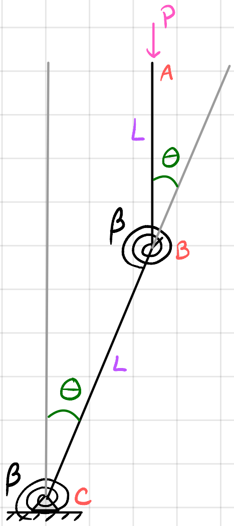


$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi-\theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

$$\varphi = \theta \quad \Rightarrow \quad U_T = \frac{1}{2}\beta\theta^2 - P(2L - 2L\cos\theta)$$

$$\frac{\partial U_T}{\partial\theta} = 0 = \beta\theta - 2PL\sin\theta \simeq \beta\theta - 2PL\theta$$

$$P_{cr} = \frac{\beta}{2L}$$

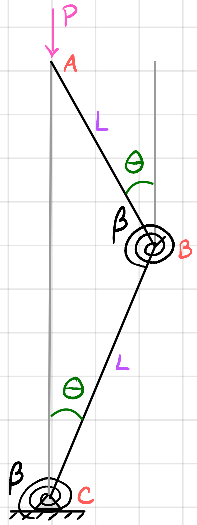


$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi-\theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

$$\varphi = 0 \quad \Rightarrow \quad U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta\theta^2 - P(2L - L\cos\theta - L) = \beta\theta^2 - PL(1-\cos\theta)$$

$$\frac{\partial U_T}{\partial\theta} = 2\beta\theta - PL\sin\theta \simeq 2\beta\theta - PL\theta = 0$$

$$P_{cr} = \frac{2\beta}{L}$$

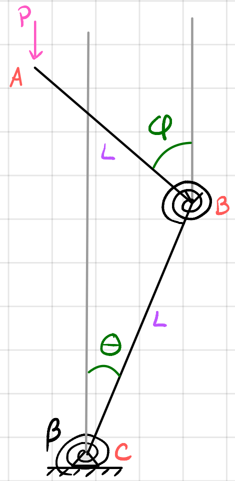


$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi-\theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

$$\varphi = -\theta \Rightarrow U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(2\theta)^2 - P(2L - 2L\cos\theta) = \frac{3}{2}\beta\theta^2 - 2PL(1-\cos\theta)$$

$$\frac{\partial U_T}{\partial\theta} = 3\beta\theta - 2PL\sin\theta \simeq 3\beta\theta - 2PL\theta = 0$$

$$P_{cr} = \frac{3\beta}{2L}$$



$$U_T = \frac{1}{2}\beta\theta^2 + \frac{1}{2}\beta(\varphi-\theta)^2 - P(2L - L\cos\theta - L\cos\varphi)$$

$\varphi > \theta$

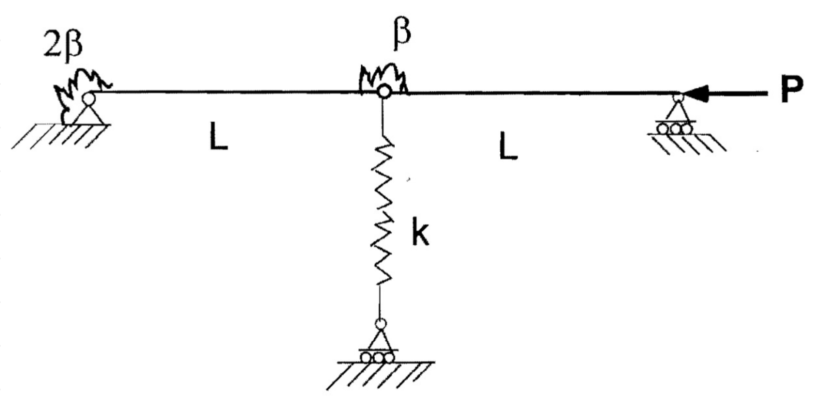
φ and θ in different directions

This would give the same eigenvalues as we got previously.
Even though φ and θ are in different directions, $\varphi = -0.68\theta$ still does not make sense because $\varphi > \theta$

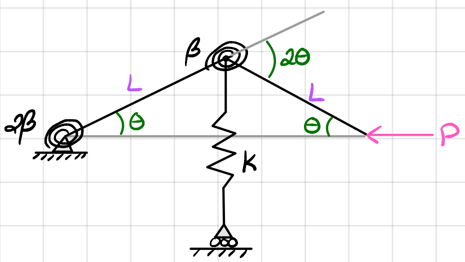
$$\Rightarrow P_{cr} = \frac{\beta}{2L}(3-\sqrt{5})$$

Example: Example from AE8803 Exam 2 April 2016 (Kardomateas)

In the system shown below, two rigid bars of length L are connected together with a rotational spring of stiffness β and a linear horizontal spring of stiffness k at the common joint, whereas one end of the system is connected to a rotational spring of stiffness 2β.



a) Find the critical load P_{cr}.



Using the energy method:

$$U_T = \frac{1}{2}(2\beta)\theta^2 + \frac{1}{2}\beta(2\theta)^2 + \frac{1}{2}K(L\sin\theta)^2 - P(2L - 2L\cos\theta)$$

$$= \beta\theta^2 + 2\beta\theta^2 + \frac{1}{2}KL^2\sin^2\theta - 2PL(1-\cos\theta)$$

Take $\frac{dU_T}{d\theta}=0$: $6\beta\theta + KL^2\sin\theta\cos\theta - 2PL\sin\theta = 0$

$\downarrow$

P_{eql}: $2PL\sin\theta = 6\beta\theta + KL^2\sin\theta\cos\theta$

$$P_{eql} = 3\frac{\beta\theta}{L\sin\theta} + \frac{KL\cos\theta}{2}$$

$$P_{cr} = \lim_{\theta\to 0} P_{eql}$$

Small angle approx: $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$\Rightarrow 6\beta\theta + KL^2\theta - 2PL\theta = 0$

$\Rightarrow PL = \frac{6\beta + KL^2}{2} \Rightarrow \boxed{P_{cr} = 3\frac{\beta}{L} + \frac{KL}{2}}$

b) If $\beta = KL^2$, is the post-critical path stable?

$$\frac{d^2U_T}{d\theta^2} = 6\beta + \underbrace{KL^2\cos^2\theta - KL^2\sin^2\theta}_{KL^2(\cos^2\theta - \sin^2\theta) = KL^2\cos 2\theta} - 2PL\cos\theta = 6\beta + KL^2\cos 2\theta - 2PL\cos\theta$$

To evaluate post-buckling, substitute P_{eql} into $\frac{d^2U_T}{d\theta^2}$:

$$\frac{d^2U_T}{d\theta^2} = 6\beta + KL^2\cos 2\theta - 2\left(\frac{3\beta\theta}{L\sin\theta} + \frac{KL\cos\theta}{2}\right)L\cos\theta$$

$$= 6\beta + KL^2\cos 2\theta - 6\beta\frac{\theta\cos\theta}{\sin\theta} - KL^2\cos^2\theta$$

$$= 6\beta\left(\underbrace{1-\frac{\theta\cos\theta}{\sin\theta}}_{1-\frac{\theta}{\tan\theta}}\right) + KL^2\left(\overbrace{\cos^2\theta-\sin^2\theta}^{\cos 2\theta}-\cos^2\theta\right)$$

$$= 6\beta\left(1-\frac{\theta}{\tan\theta}\right) - KL^2\sin^2\theta$$ ← Taylor series expansion for $\tan\theta \simeq \theta+\frac{\theta^3}{3}$, $\sin\theta \simeq \theta$

$$\simeq 6\beta\left(1-\frac{\cancel{\theta}}{\cancel{\theta}+\frac{\theta^{\cancel{3}\,2}}{3}}\right) - KL^2\theta^2$$

$$\simeq 6\beta\left[1-\left(\frac{1}{1+\frac{\theta^2}{3}}\right)\right] - KL^2\theta^2$$ ← $\frac{1}{1+\frac{\theta^2}{3}}\left[\frac{1-\frac{\theta^2}{3}}{1-\frac{\theta^2}{3}}\right] = \frac{1-\frac{\theta^2}{3}}{1-\frac{\theta^4}{9}} \simeq 1-\frac{\theta^2}{3}$

↑ θ is small $\Rightarrow \theta^4 \simeq 0$

$$\simeq 6\beta\left[1-\left(1-\frac{\theta^2}{3}\right)\right] - KL^2\theta^2$$

$$\simeq (2\beta - KL^2)\theta^2$$

If $\beta = KL^2$: $(2KL^2 - KL^2)\theta^2 = KL^2\theta^2$ ← >0 $\Rightarrow$ $\boxed{\text{post-critical path is \underline{stable} if } \beta = KL^2}$

b) If $\beta = \frac{1}{4}KL^2$, is the post-critical path stable?

If $\beta = \frac{KL^2}{4}$: $\left(2\left(\frac{KL^2}{4}\right) - KL^2\right)\theta^2 = -\frac{1}{2}KL^2\theta^2$ ← <0 $\Rightarrow$ $\boxed{\text{post-critical path is \underline{unstable} if } \beta = \frac{1}{4}KL^2}$

Another way to find P_{cr}:

$$\left.\frac{d^2U_T}{d\theta^2}\right|_{\theta=0} = 0 \; : \; 6\beta + KL^2\underbrace{\cos 2\theta}_{=1} - 2PL\underbrace{\cos\theta}_{=1} \Rightarrow 6\beta + KL^2 - 2PL = 0$$

$$\Rightarrow P_{cr} = 3\frac{\beta}{L} + \frac{KL}{2}$$