Georgia Institute of Technology
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AE 6115 — Fundamentals of Aerospace Structural Analysis

Three-Dimensional Euler-Bernoulli Beam Bending

(Problems adapted from "Structural Analysis" by Bauchau and Craig)

Problem 1. Axial stress in a reinforced I beam

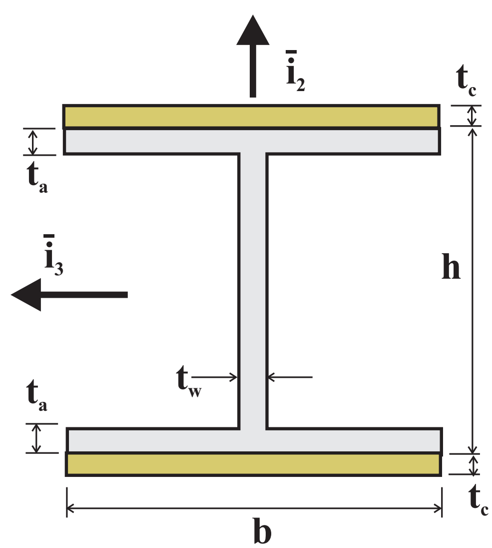

Figure 1: Cross-section of a reinforced I beam.

Figure 1 depicts an aluminum I beam of height $h = 0.25\ m$, width $b = 0.2$ m, flange thickness $t_a = 16$ mm, and web thickness $t_w = 12$ mm. The beam is reinforced by two layers of unidirectional composite material of thickness $t_c = 5$ mm. The section is subjected to an axial force $N_1 = 500$ kN. The Young's moduli for the aluminum and unidirectional composite are $E_a = 73$ GPa and $E_c = 140$ GPa, respectively.

1. Find the distribution of axial stress over the cross-section and sketch it along the $\bar{\imath}_2$ axis.

2. Find the magnitude and location of the maximum axial stress in the aluminum and composite layers.

3. Sketch the distribution of axial strain along the $\bar{\imath}_2$ axis, and describe how it varies over the entire cross-section.

Solution

1. In order to find the stress within the beam, we need to calculate the axial stiffness for the beam. To do this, begin by calculating the amount of cross sectional area each material has.

$$\begin{aligned} A_a &= 2bt_a + (h - 2t_a)t_w = 0.009 \text{ m}^2 \\ A_c &= 2bt_c = 0.002 \text{ m}^2, \end{aligned} \tag{1}$$

where subscripts a and c denote the I-beam and reinforced composite material, respectively. Now, axial stiffness can be found.

$$S = E_a A_a + E_c A_c = 9.382 \times 10^8 \text{ Pa m}^2. \tag{2}$$

Now the stress experienced by each material follows.

$$\begin{aligned} \sigma_{1,a} &= \frac{E_a N_1}{S} = 38.91 \times 10^6 \text{ Pa} \\ \sigma_{1,c} &= \frac{E_c N_1}{S} = 74.61 \times 10^6 \text{ Pa} \end{aligned} \tag{3}$$

Notice the stress is a single constant value for each material, so the sketch should look similar to the plot below.

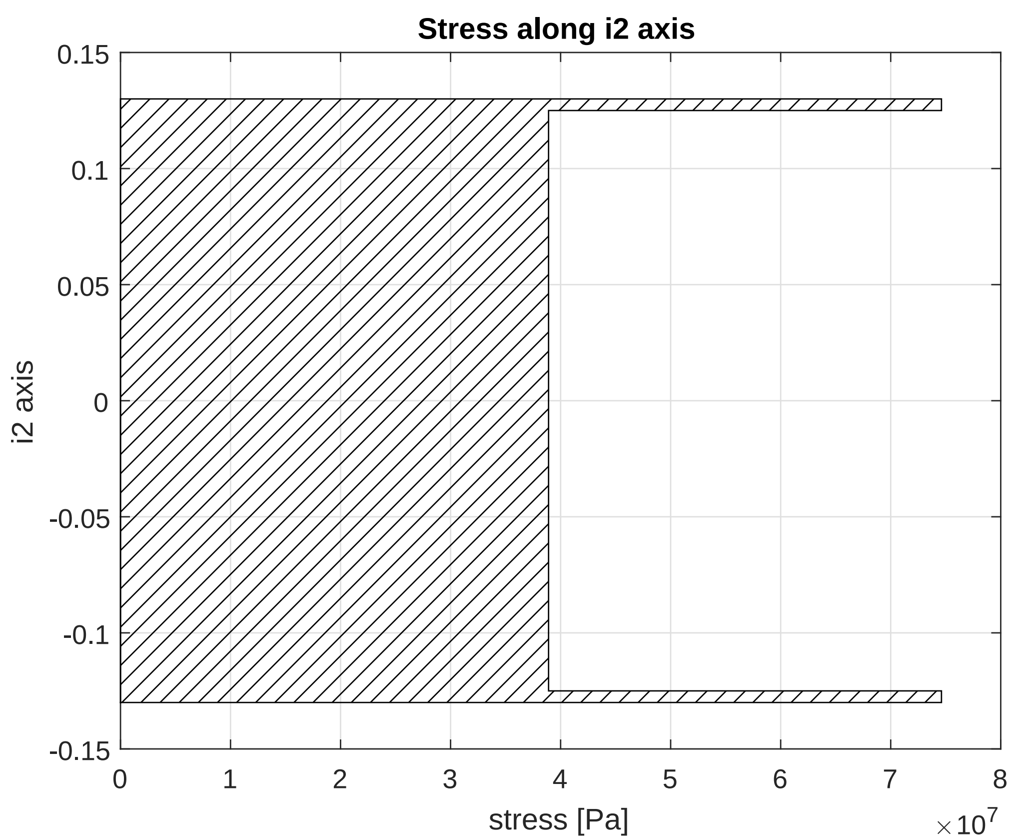

Figure 2: Plot of stress along i2 axis.

2. Since the stresses are constant within each material, the maximum stress value is simply

$$\begin{aligned} \sigma_{1,a} &= 38.91 \times 10^6 \text{ Pa} \\ \sigma_{1,c} &= 74.61 \times 10^6 \text{ Pa} \end{aligned} \tag{4}$$

3. Given that the strain is a measure of elongation and we are assuming Euler-Bernoulli kinematics, it is constant across the entire section. Therefore, the strain

$$\epsilon_1 = \frac{N_1}{S} = 0.000533 \tag{5}$$

may be plotted as below.

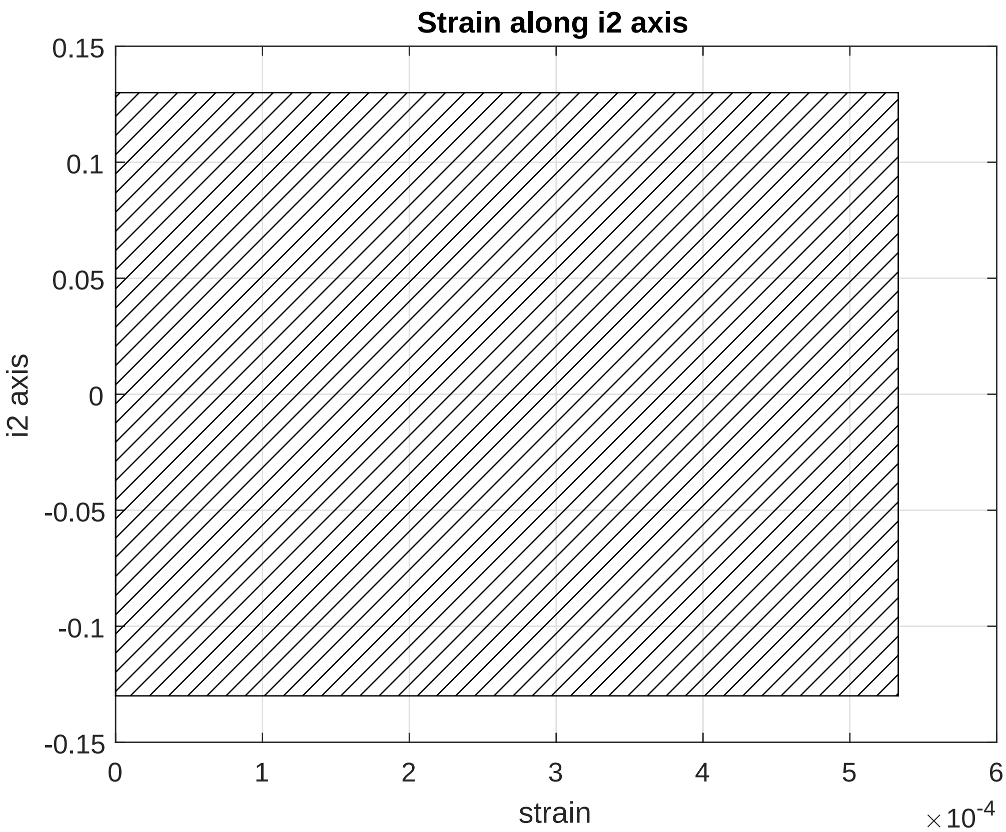

Figure 3: Plot of strain along i2 axis.

Problem 2. Short answer questions.

For each question provide a YES or NO answer **and** a justification to your answer. One paragraph (1 or 2 sentences) should suffice for your justification.

1. Is it possible to use Euler-Bernoulli assumptions for a beam bent in such a manner that the material it is made out of goes into the plastic deformation range? Why?
2. Consider a simply supported beam which has a spring supporting its mid-span, and is subjected to a uniform transverse loading p_0. Which one of the following quantities will present a discontinuity at mid-span: beam transverse deflection, beam slope, bending moment, and/or transverse shear force? Why?
3. Consider a cantilevered beam of length L, under a uniform transverse loading p_0. Does the root bending moment depend on the material Young's modulus?
4. Consider a beam of length L, cantilevered at both ends and subjected to a uniform transverse loading p_0. Does the mid-span transverse deflection depend on the material Young's modulus?
5. Consider a beam of length L, cantilevered at both ends and subjected to a uniform transverse loading p_0. Does the mid-span bending moment depend on the material Young's modulus?

Solution

1. Yes, it is possible since the Euler-Bernoulli assumptions concern only the kinematics of deformation and therefore do not rely upon the beam material possessing certain properties.
2. Transverse shear force is the only quantity which has a discontinuity because the spring creates a shear difference mid-span equal to the magnitude of the spring force.
3. No. Since the problem is statically determinant, the root moment depends only on the applied load p_0.
4. Yes. The deflection depends inversely on the bending stiffness H_{33}^c which depends directly on Young's modulus. This is true even for a hyperstatic beam configuration.
5. No. The mid-span bending moment depends only the shear and bending moments experienced by the beam.

Problem 3. Bending of reinforced solid section beam

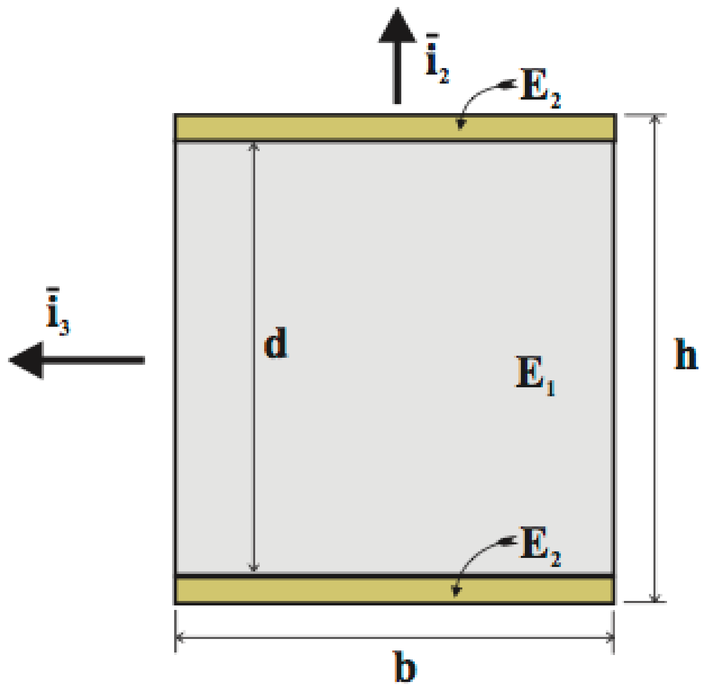

Figure 4: Reinforced rectangular cross-section.

A rectangular cross-section made of a material of Young's modulus E_1 is reinforced by thin top and bottom plates made of a material of Young's modulus E_2, as depicted in fig. 4. M_3 is the bending moment applied to the section. $E_2/E_1 = 2$; $d/h = 0.96$.

1. Plot the non-dimensional axial strain distribution $H^c_{33}\epsilon_1/(M_3 h)$ versus $2x_2/h$.
2. Plot the non-dimensional axial stress distribution $H^c_{33}\sigma_1/(M_3 E_2 h)$ versus $2x_2/h$.

Solution

1. For this pure bending moment, the sectional constitutive equation gives

$$M_3 = H^c_{33}\kappa_3$$

and the axial strain, again for pure bending around the $\bar{\imath}_3$ direction is given by

$$\epsilon_1 = -x_2\kappa_3$$

Combining these two equations and eliminating κ_3 we can express the axial strain in terms of the applied moment as

$$\epsilon_1 = -\frac{M_3 x_2}{H^c_{33}} \tag{6}$$

Multiplying both sides by $H_{33}^c/(M_3 h)$, the nondimensional strain is given by:

$$\begin{aligned} \frac{H_{33}^c \epsilon_1}{M_3 h} &= \frac{H_{33}^c}{M_3 h} \frac{-M_3 x_2}{H_{33}^c} \\ &= -\frac{x_2}{h} \end{aligned} \tag{7}$$

If we let the non-dimensionalized position along the $\bar{i}_2$ axis of the beam be $\eta = 2x_2/h$, then the non-dimensional strain becomes

$$\begin{aligned} \frac{H_{33}^c \epsilon_1}{M_3 h} &= -\frac{x_2}{h} \\ &= -\frac{\eta}{2} \end{aligned} \tag{8}$$

Which gives us the requested plot

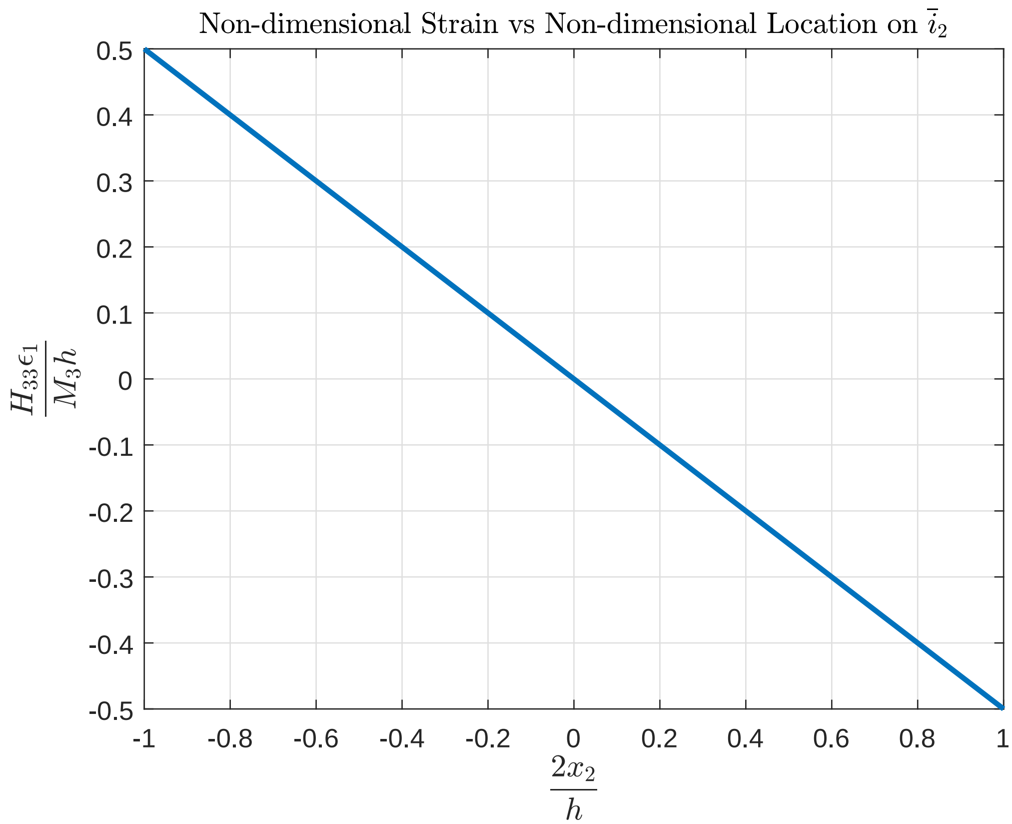

Figure 5: Plot of non-dimensional strain.

2. For non-dimensional stress, recall that for material 1

$$\sigma_1^{(1)} = E^{(1)} \epsilon_1 \tag{9}$$

and for material 2

$$\begin{aligned} \sigma_1^{(2)} &= E^{(2)} \epsilon_1 \\ &= 2E^{(1)} \epsilon_1 \end{aligned} \tag{10}$$

So then the non-dimensionalized stress for material 1 and material 2 is

$$\begin{aligned} \frac{\sigma_1^{(1)} H_{33}^c}{M_3 E^{(1)} h} &= \frac{1}{2} \frac{\sigma_1^{(1)} H_{33}^c}{M_3 E^{(2)} h} = -\frac{\eta}{4} \\ \frac{\sigma_1^{(2)} H_{33}^c}{M_3 E^{(2)} h} &= -\frac{\eta}{2} \end{aligned} \tag{11}$$

So then, in plotting the above, we have

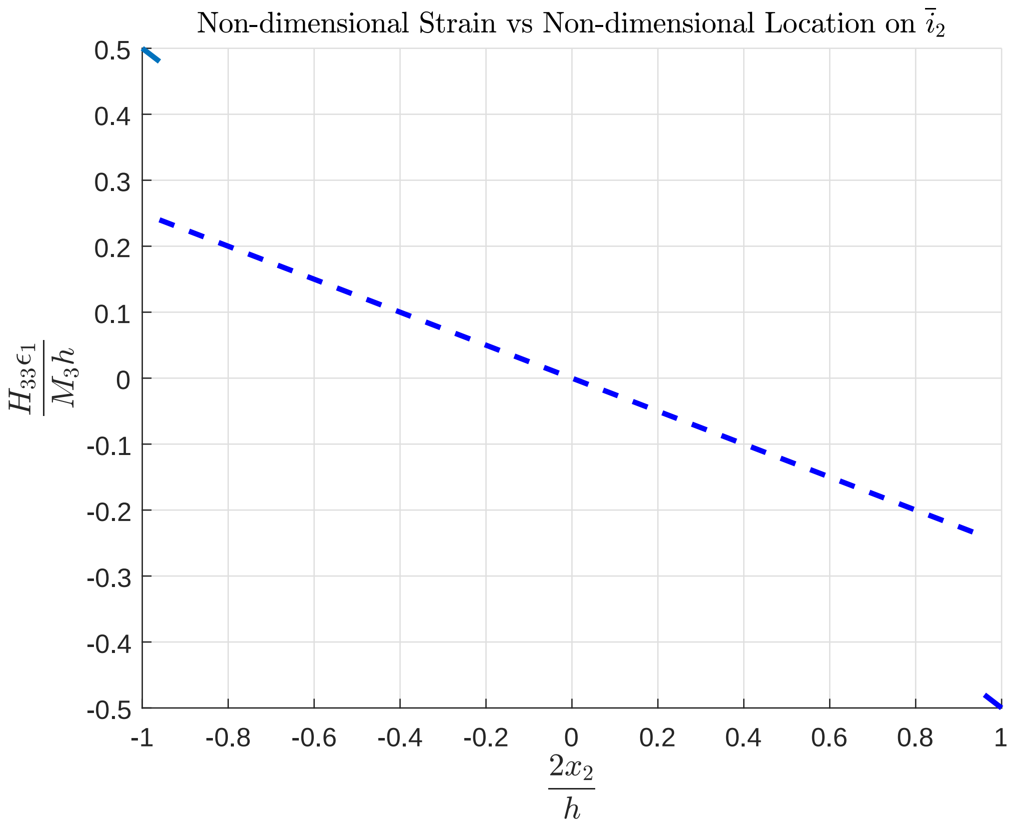

Figure 6: Plot of non-dimensional stress.

Problem 4. Short answer questions. 3D Beam Theory.

For each question provide a brief (1 or 2 sentence) answer.

1. For a particular cross-section, the centroidal bending stiffnesses have been computed as H^c_{22}, H^c_{33}, and H^c_{23}. Next, the bending stiffnesses are computed about a set of parallel axes with their origin at an arbitrary point $\mathbf{D}$ and found to be H^d_{22}, H^d_{33}, and H^d_{23}. Is it possible to find a point $\mathbf{D}$ such that $H^d_{22} < H^c_{22}$? Why?

2. For a particular cross-section, the centroidal bending stiffnesses have been computed as H^c_{22}, H^c_{33}, and H^c_{23}. Next, the bending stiffnesses are computed about a set of parallel axes with their origin at an arbitrary point $\mathbf{D}$ and found to be H^d_{22}, H^d_{33}, and H^d_{23}. Is it possible to find a point $\mathbf{D}$ such that $H^d_{23} < H^c_{23}$? Why?

3. Consider a uniform cantilevered beam subjected to a uniform transverse loading distribution $p_0\bar{n}$, where $\bar{n}$ is a unit vector perpendicular to the axis of the beam, $\bar{\imath}_1$. Under what condition will the transverse deflection of the beam be oriented in the direction of $\bar{n}$?

4. A uniform cantilevered beam is subjected to a tip axial force. The beam is made of a homogeneous material. Under what condition will the strain distribution over the cross-section be uniform?

Solution

1. Calculating H^d_{33} involves the use of the parallel axis theorem which is a procedure which only increases a value. Therefore, it is not possible to find any point $\mathbf{D}$ where $H^d_{22} < H^c_{22}$.

2. Calculating H^d_{23} involves use of the parallel axis theorem which adds an increment proportional to the product of the $\bar{i}_2$ and $\bar{i}_3$ offset distances. Since this product can be negative if one but not both offsets are negative, it is possible to find a point D where $H^d_{23} < H^c_{23}$.

3. Under conditions in which H^c_{23} for that particular frame of loading is zero.

   Specifically, if axes are principal and if the two principal bending stiffness are equal, then the displacement components in the principal axes will be equally proportional to the loading and therefore in the same direction as the load. This will also hold if $\bar{n}$ is in a principal direction and in this case the deflection will be solely in the $\bar{n}$ direction.

4. The axial strain ϵ_1 will be uniform over the cross section only if the axial force produces no bending moments about the centroidal axes. Therefore, the axial load must act through the centroid.

Problem 5. Bending stiffnesses of a bi-material "L" section

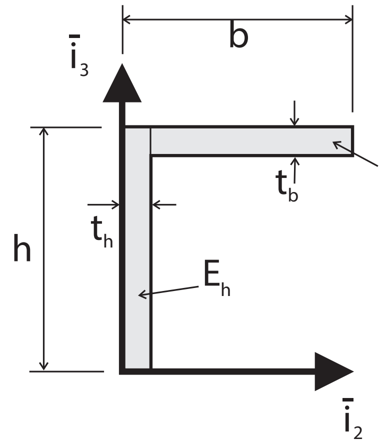

Figure 7: "L" shaped cross-section.

A beam featuring the "L" cross-section consisting of two homogeneous materials having differing Young moduli, E_b and E_h, is depicted in fig. 7.

1. With respect to the origin defined by the coordinate system shown in Fig. 7, find the location of the centroid (x_2^c, x_3^c).

2. Assuming that $E_b = 2E_h$, find the bending stiffnesses in a coordinate system parallel to that shown on the figure, but with its origin at the centroid, i.e. H_{22}^c, H_{33}^c, and H_{23}^c.

Solution

1. To find x_2^c, we need to find the areas and centroids of section b and section h. Then utilize the following equation.

$$\begin{aligned} x_2^c = \frac{S_2}{S} &= \frac{\sum_i E_i A_i x_{2(i)}^c}{\sum_i E_i A_i} \\ &= \frac{(E_h)(ht_h)(t_h/2) + (E_b)((b-t_h)t_b)(t_h + (b-t_h)/2)}{(E_h)(ht_h) + (E_b)((b-t_h)t_b)} \\ &= \frac{E_h h t_h^2 + E_b t_b (b^2 + t_h^2)}{2\left(E_h h t_h + E_b t_b (b - t_h)\right)} \end{aligned} \tag{12}$$

As for x_3^c, the same can be done.

$$\begin{aligned} x_3^c = \frac{S_3}{S} &= \frac{\sum_i E_i A_i x_{3(i)}^c}{\sum_i E_i A_i} \\ &= \frac{(E_h)(ht_h)(h/2) + (E_b)((b-t_h)t_b)(h - t_b/2)}{(E_h)(ht_h) + (E_b)((b-t_h)t_b)} \\ &= \frac{E_h h^2 t_h + E_b t_b (b - t_h)(2h - t_b)}{2\left(E_h h t_h + E_b t_b (b - t_h)\right)} \end{aligned} \tag{13}$$

2. To find the bending stiffnesses, first calculate the second moments of area about the centroid. For material b,

$$I_{22}^{c(b)} = \frac{(b-t_h)t_b^3}{12} + (b-t_h)t_b(h - \frac{t_b}{2} - x_3^c)^2 \tag{14}$$

$$I_{33}^{c(b)} = \frac{(b-t_h)^3 t_b}{12} + (b-t_h)t_b(b + \frac{t_h}{2} - x_2^c)^2 \tag{15}$$

$$I_{23}^{c(b)} = (b-t_h)t_b(b + \frac{t_h}{2} - x_2^c)(h - \frac{t_b}{2} - x_3^c) \tag{16}$$

and for material h

$$I_{22}^{c(h)} = \frac{t_h h^3}{12} + ht_h(\frac{h}{2} - x_3^c)^2 \tag{17}$$

$$I_{33}^{c(h)} = \frac{t_h^3 h}{12} + ht_h(\frac{t_h}{2} - x_2^c)^2 \tag{18}$$

$$I_{23}^{c(h)} = ht_h(\frac{t_h}{2} - x_2^c)(\frac{h}{2} - x_3^c) \tag{19}$$

Then, the bending stiffnesses are calculated simply as

$$H_{22}^c = E_b I_{22}^{c(b)} + E_h I_{22}^{c(h)} \tag{20}$$

$$H_{33}^c = E_b I_{33}^{c(b)} + E_h I_{33}^{c(h)} \tag{21}$$

$$H_{23}^c = E_b I_{23}^{c(b)} + E_h I_{23}^{c(h)} \tag{22}$$

$$\tag{23}$$

Problem 6. Bending stiffnesses of a "Z" section

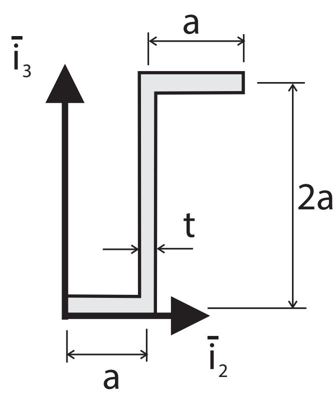

Figure 8: "Z" shaped cross-section of a beam.

A beam made of a homogeneous material features the "Z" cross-section depicted in fig. 8.

1. With respect to the origin defined by the coordinate system shown in Fig. 8, show that the location of the centroid is $(x_2^c, x_3^c) = (a, a)$.

2. Find the bending stiffnesses in a coordinate system parallel to that shown on the figure, but with its origin at the centroid, i.e. H_{22}^c, H_{33}^c, and H_{23}^c.

3. Simplify your expressions for H_{22}^c, H_{33}^c, and H_{23}^c by assuming a "thin-walled" section, that is $t/a << 1$.

Solution

1. Can either use an argument of symmetry or it may be calculated in the same way in which centroids were calculated in the previous homework since the entire cross section consists of a single homogeneous material (Young's modulus is constant throughout). To do so, let the upper flange be Section 1, web be Section 2, and the lower flange be Section 3. Then,

$$A_1 = A_3 = \left(a - \frac{t}{2}\right) t \tag{24}$$

$$A_2 = (2a + t)t \tag{25}$$

and

$$(\bar{x}_{2(1)}, \bar{x}_{3(1)}) = \left(\frac{a - \dfrac{t}{2}}{2}, \frac{t}{2} \right) \tag{26}$$

$$(\bar{x}_{2(2)}, \bar{x}_{3(2)}) = \left(a, a + \frac{t}{2} \right) \tag{27}$$

$$(\bar{x}_{2(3)}, \bar{x}_{3(3)}) = \left(a + t + \frac{a - \dfrac{t}{2}}{2}, 2a + \frac{t}{2} \right) \tag{28}$$

The centroid is then

$$\begin{aligned} x_2^c &= \frac{\sum_i A_i \bar{x}_{2(i)}}{\sum_i A_i} = a \\ x_3^c &= \frac{\sum_i A_i \bar{x}_{3(i)}}{\sum_i A_i} = a + \frac{t}{2} \end{aligned} \tag{29}$$

2. The centroidal bending stiffnesses are

$$\begin{aligned} H_{22}^c &= E \left(\frac{t(2a+t)^3}{12} + \left(\frac{(a-t/2)(t)^3}{12} + (a-t/2)ta^2 \right) \ldots \right. \\ &\quad \left. + \left(\frac{(a-t/2)(t)^3}{12} + (a-t/2)t(-a)^2 \right) \right) \\ &= \left(\frac{8a^3t}{3} + \frac{2at^3}{3} \right) E \\ H_{33}^c &= E \left(\frac{t^3(2a+t)}{12} + \left(\frac{(a-t/2)^3(t)}{12} + (a-t/2)t(a/2+t/4)^2 \right) \ldots \right. \\ &\quad \left. + \left(\frac{(a-t/2)^3(t)}{12} + (a-t/2)t(-a/2-t/4)^2 \right) \right) \\ &= \left(\frac{2a^3t}{3} + \frac{at^3}{6} \right) E \\ H_{23}^c &= E \left((a-t/2)t(-a/2-t/4)(-a) + (a-t/2)t(a/2+t/4)(a) \right) \\ &= \left(a^3t - \frac{at^3}{4} \right) E \end{aligned} \tag{30}$$

3. To simplify, since $t/a << 1$, then in the above terms, $at^3 << a^3t$.

$$\begin{aligned} H_{22}^c &= \frac{8}{3}a^3tE \\ H_{33}^c &= \frac{2}{3}a^3tE \\ H_{23}^c &= a^3tE \end{aligned} \tag{31}$$

Problem 7. Cantilever beam with tip rotational spring

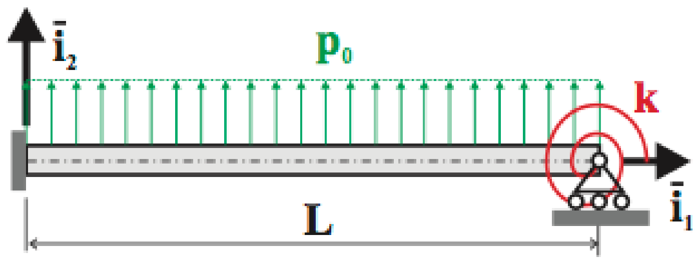

Figure 9: Cantilevered beam with tip torsional spring and distributed load

The uniform cantilevered beam of bending stiffness H_{33}^c and length L depicted in fig. 9 features a tip rotational spring of stiffness constant k and a uniform distributed load P_0. You may assume that in the undeformed (undeflected) state of the beam, the rotational spring exerts no moment.

1. Find the transverse displacement field along the length of the beam. Plot the displacement field as a function of x_1.
2. Find the distribution of moments along the beam and plot it as a function of x_1.

Solution

1. To find the transverse displacement field, recall the governing ODE $H_{33}^c \dfrac{d^4 u_2}{dx_1^4} = p_0$. Integrating this yields the following

$$H_{33}^c \frac{d^4 u_2}{dx_1^4} = p_0 \tag{32}$$

$$H_{33}^c \frac{d^3 u_2}{dx_1^3} = p_0 x_1 + C_1 \tag{33}$$

$$H_{33}^c \frac{d^2 u_2}{dx_1^2} = \frac{1}{2} p_0 x_1^2 + C_1 x_1 + C_2 \tag{34}$$

$$H_{33}^c \frac{du_2}{dx_1} = \frac{1}{6} p_0 x_1^3 + \frac{1}{2} C_1 x_1^2 + C_2 x_1 + C_3 \tag{35}$$

$$H_{33}^c u_2 = \frac{1}{24} p_0 x_1^4 + \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 + C_3 x_1 + C_4 \tag{36}$$

The boundary conditions in this problem for the cantilever support are

$$u_2(x_1 = 0) = 0 \tag{37}$$

$$\left. \frac{du_2}{dx_1} \right|_{x_1=0} = 0 \tag{38}$$

and for the roller support and torsional spring support

$$u_2(x_1 = L) = 0 \tag{39}$$

$$H_{33}^c \left. \frac{d^2 u_2}{dx_1^2} \right|_{x_1=L} = -k \left. \frac{du_2}{dx_1} \right|_{x_1=L} \tag{40}$$

$$\tag{41}$$

Utilizing the cantilever boundary conditions, we find from (36) and (35) that

$$C_3 = 0 \tag{42}$$

$$C_4 = 0 \tag{43}$$

Leaving the displacement field equation to be

$$H_{33}^c u_2 = \frac{1}{24} p_0 x_1^4 + \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 \tag{44}$$

Applying (39), the constants C_1 and C_2 can be related

$$H_{33}^c u_2(x_1 = L) = 0 \tag{45}$$

$$= \left. \left(\frac{1}{24} p_0 x_1^4 + \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 \right) \right|_{x_1=L} \tag{46}$$

$$= \frac{1}{24} p_0 L^4 + \frac{1}{6} C_1 L^3 + \frac{1}{2} C_2 L^2 \tag{47}$$

$$\Rightarrow C_2 = \frac{-1}{12} p_0 L^2 - \frac{1}{3} C_1 L \tag{48}$$

Now the last boundary condition (40) may be applied.

$$H_{33}^c \left. \frac{d^2 u_2}{dx_1^2} \right|_{x_1=L} = -k \left. \frac{du_2}{dx_1} \right|_{x_1=L} \tag{49}$$

$$\left. \left(\frac{1}{2} p_0 x_1^2 + C_1 x_1 + C_2 \right) \right|_{x_1=L} = \frac{-k}{H_{33}^c} \left. \left(\frac{1}{6} p_0 x_1^3 + \frac{1}{2} C_1 x_1^2 + C_2 x_1 \right) \right|_{x_1=L} \tag{50}$$

$$\frac{1}{2} p_0 L^2 + C_1 L + C_2 = \frac{-k}{H_{33}^c} \left(\frac{1}{6} p_0 L^3 + \frac{1}{2} C_1 L^2 + C_2 L \right) \tag{51}$$

$$\frac{5}{12} p_0 L^2 + \frac{2}{3} C_1 L = \frac{-k}{H_{33}^c} \left(\frac{5}{12} p_0 L^3 + \frac{1}{6} C_1 L^2 \right) \tag{52}$$

$$\tag{53}$$

Now the constant C_1 is the only unknown in the above equation and is found to be

$$C_1 = \frac{-k p_0 L^2 - 5 H_{33}^c p_0 L}{8 H_{33}^c + 2kL} \tag{54}$$

After substituting C_1 into the equation for C_2, it is found to be

$$C_2 = \frac{-1}{12} p_0 L + \frac{k p_0 L^3 + 5 H_{33}^c p_0 L^2}{24 H_{33}^c + 6kL} \tag{55}$$

giving us a transversal displacement field of

$$u_2(x_1) = \frac{(6H_{33}^c L^2 + kL^3)x_1^2 - (10H_{33}^c L + 2kL^2)x_1^3 + (4H_{33}^c + kL)x_1^4}{24H_{33}^c\,(4H_{33}^c + kL)} p_0 \tag{56}$$

The plot of the dimensionless displacement ($U_2 = u_2 H_{33}^c/(p_0 L^4)$) vs. dimensionless position along beam ($\eta = x_1/L$) is then

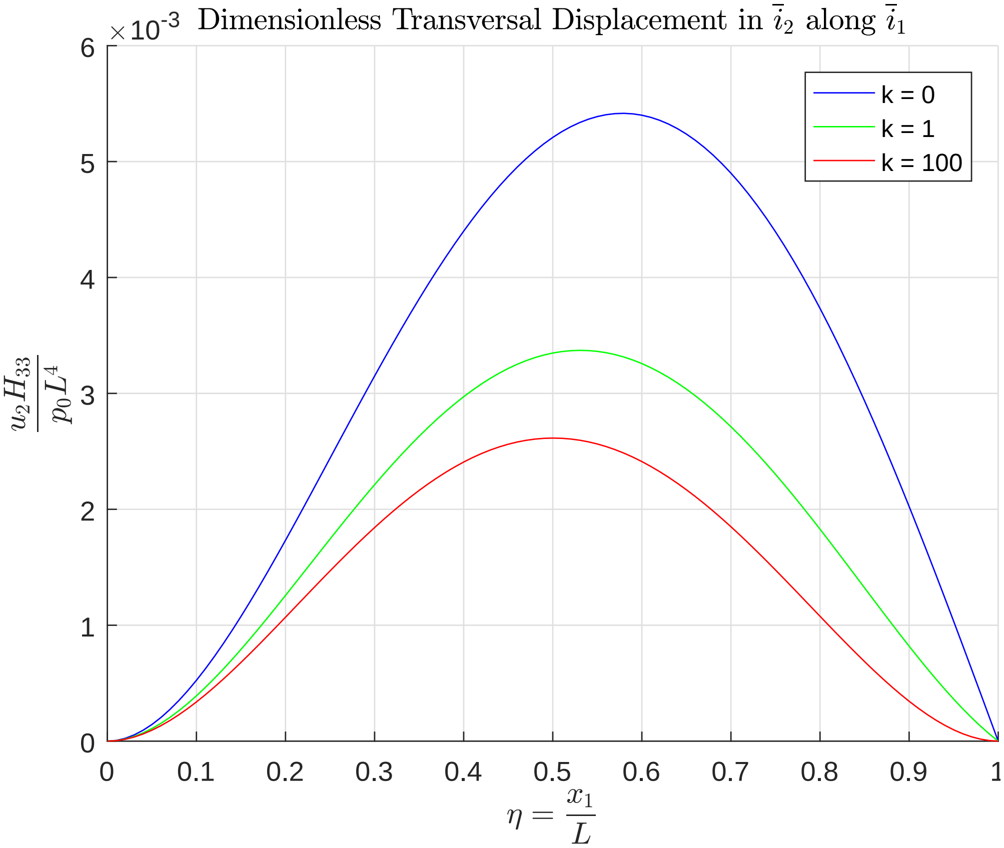

Figure 10: Plot of non-dimensional transversal displacement.

2. As for the moments, simply recognise that $M_3(x_1) = H_{33}^c \dfrac{d^2 u_2(x_1)}{dx_1^2}$ which yields

$$M_3(x_1) = \frac{6H_{33}^c L^2 + kL^3 - (30H_{33}^c L + 6kL^2)x_1 + (24H_{33}^c + 6kL)x_1^2}{12(4H_{33}^c + kL)} p_0 \tag{57}$$

Plot of the non-dimensional moment ($M_n = M_3/(p_0 L^2)$) vs. non-dimensional position along the beam is then

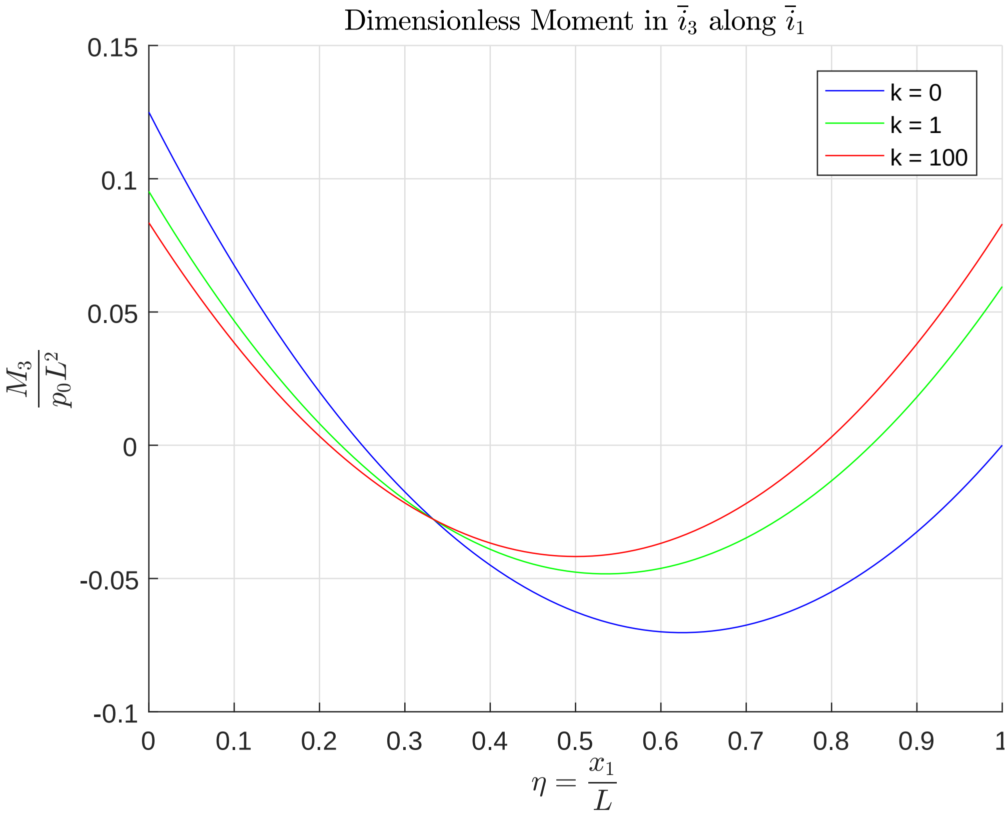

Figure 11: Plot of non-dimensional moment.

Problem 8. Cantilever beam with tip rotational spring and point load.

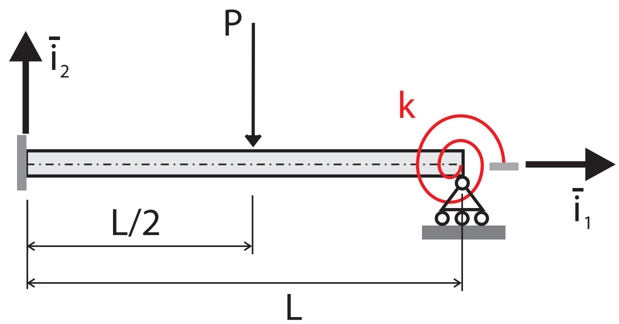

Figure 12: Cantilevered beam with tip torsional spring and point load.

The uniform cantilevered beam of bending stiffness H_{33}^c and length L depicted in fig. 15 features a tip rotational spring of stiffness constant k and a point load P applied in the middle of the beam. You may assume that in the undeformed (undeflected) state of the beam, the rotational spring exerts no moment.

- Find the transverse displacement field along the length of the beam. Plot the displacement field as a function of x_1.
- Find the distribution of moments along the beam and plot it as a function of x_1.

Solution

1. First, since the beam is hyperstatic, replace some of the supports by their equivalent reaction forces or moments until the beam is simply supported. Here, it will be chosen such that the beam should be a simply-supported cantilever beam with the roller support and rotational spring on the right end of the beam replaced by an unknown reaction force and reaction moment, respectively, as depicted below.

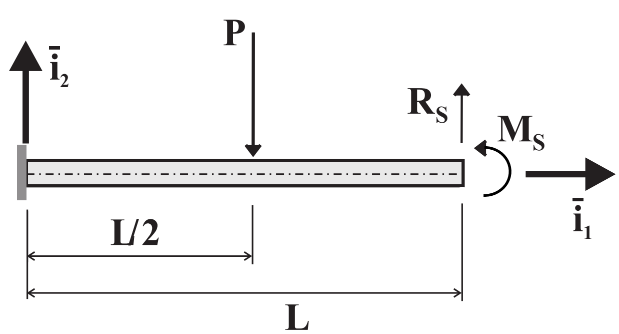

Figure 13: Cantilevered beam with point load and end supports replaced by reaction force and moment.

Now, find the reaction moment and force of the cantilever support on the left.

$$R_c = P - R_s \tag{58}$$

$$M_c = -P\frac{L}{2} + R_s L + M_s \tag{59}$$

To deal with the point load P, the beam is broken apart at the point load creating two sub-problems. Since there is no distributed loading and moments can be found on either ends of the beam, we will rely on the equation $H_{33}^c \frac{d^2u_2}{dx_1^2} = M$ to be the governing ODE for the sub-problems.

Consider first the section of beam where $0 \le x_1 \le \frac{L}{2}$.

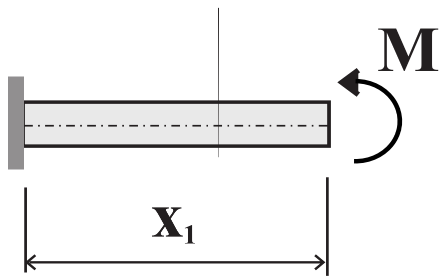

Figure 14: Beam section to the left of the point load.

Because we know the reaction force and moment at the cantilever wall, we can find the moment in the beam and its integrals, the slope and transversal displacement of the beam. For now, we will only find the moment.

$$H_{33}^c \frac{d^2u_2}{dx_1^2} = M = -\frac{1}{2}PL + R_s L + M_s + (P - R_s)x_1 \tag{60}$$

As for the right half of the beam, the moment takes the form below.

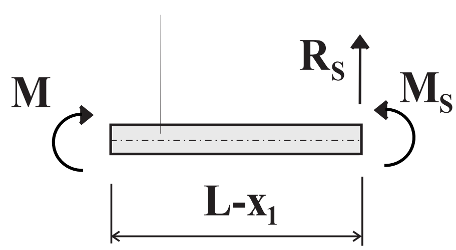

Figure 15: Beam section to the right of the point load.

$$H_{33}^c \frac{d^2u_2}{dx_1^2} = M = M_s + R_s L - R_s x_1 \tag{61}$$

Now we can write the equation for the change in slope along the beam

$$\frac{d^2u_2}{dx_1^2} = \frac{1}{H_{33}^c}\begin{cases} -\frac{1}{2}PL + R_sL + M_s + (P - R_s)x_1 & , \quad 0 \leq x_1 \leq L/2 \\ M_s + R_sL - R_sx_1 & , \quad L/2 \leq x_1 \leq L \end{cases} \tag{62}$$

the equation for the slope along the beam

$$\frac{du_2}{dx_1} = \frac{1}{H_{33}^c}\begin{cases} \left(-\frac{1}{2}PL + R_sL + M_s\right)x_1 + \frac{1}{2}(P - R_s)x_1^2 + C_1 & , \quad 0 \leq x_1 \leq L/2 \\ \left(M_s + R_sL\right)x_1 - \frac{1}{2}R_sx_1^2 + C_3 & , \quad L/2 \leq x_1 \leq L \end{cases} \tag{63}$$

and the equation for the displacement along the beam

$$u_2 = \frac{1}{H_{33}^c}\begin{cases} \frac{1}{2}\left(-\frac{1}{2}PL + R_sL + M_s\right)x_1^2 + \frac{1}{6}(P - R_s)x_1^3 + C_1x_1 + C_2 & , \quad 0 \leq x_1 \leq L/2 \\ \frac{1}{2}\left(M_s + R_sL\right)x_1^2 - \frac{1}{6}R_sx_1^3 + C_3x_2 + C_4 & , \quad L/2 \leq x_1 \leq L \end{cases} \tag{64}$$

Notice now that there are six unknowns: C_1, C_2, C_3, C_4, R_s, and M_s. Ignoring the unknown force and moment, the four other integration constants can be found using the four boundary conditions

$$u_2(x_1 = 0) = 0 \tag{65}$$

$$\left.\frac{du_2}{dx_1}\right|_{x_1=0} = 0 \tag{66}$$

$$u_2(x_1 = L/2^-) = u_2(x_1 = L/2^+) \tag{67}$$

$$\left.\frac{du_2}{dx_1}\right|_{x_1=L/2^-} = \left.\frac{du_2}{dx_1}\right|_{x_1=L/2^+} \tag{68}$$

where the last two conditions come from the fact that the beam at $L/2$ must possess the same displacement and same slope at either side of the point load. The first two boundary conditions indicate $C_1 = C_2 = 0$. From the second two, we find

$$C_3 = \frac{-1}{8}PL^2 \tag{69}$$

$$C_4 = \frac{1}{48}PL^3 \tag{70}$$

The transversal displacement is now

$$u_2 = \frac{1}{H_{33}^c}\begin{cases} \frac{1}{2}\left(-\frac{1}{2}PL + R_sL + M_s\right)x_1^2 + \frac{1}{6}(P - R_s)x_1^3 & , \quad 0 \leq x_1 \leq L/2 \\ \frac{1}{2}\left(M_s + R_sL\right)x_1^2 - \frac{1}{6}R_sx_1^3 - \frac{1}{8}PL^2x_2 + \frac{1}{48}PL^3 & , \quad L/2 \leq x_1 \leq L \end{cases} \tag{71}$$

Now that we know the displacement as a function of the unknown support force and moment, we can find what they are by imposing the two boundary conditions the original supports at $x_1 = L$ enforced, that is,

$$u_2(x_1 = L) = 0 \tag{72}$$

$$H_{33}^c \left.\frac{d^2u_2}{dx_1^2}\right|_{x_1=L} = -k \left.\frac{du_2}{dx_1}\right|_{x_1=L} \tag{73}$$

leading to a reaction force and moment of

$$R_s = \frac{5H_{33}^c + 2kL}{16H_{33}^c + 4kL}P \tag{74}$$

$$M_s = \frac{P - 4R_s}{8(H_{33}^c + kL)}kL^2 \tag{75}$$

The plot of the dimensionless displacement ($U_2 = u_2H_{33}^c/(PL^4)$) vs. dimensionless position along beam ($\eta = x_1/L$) is then

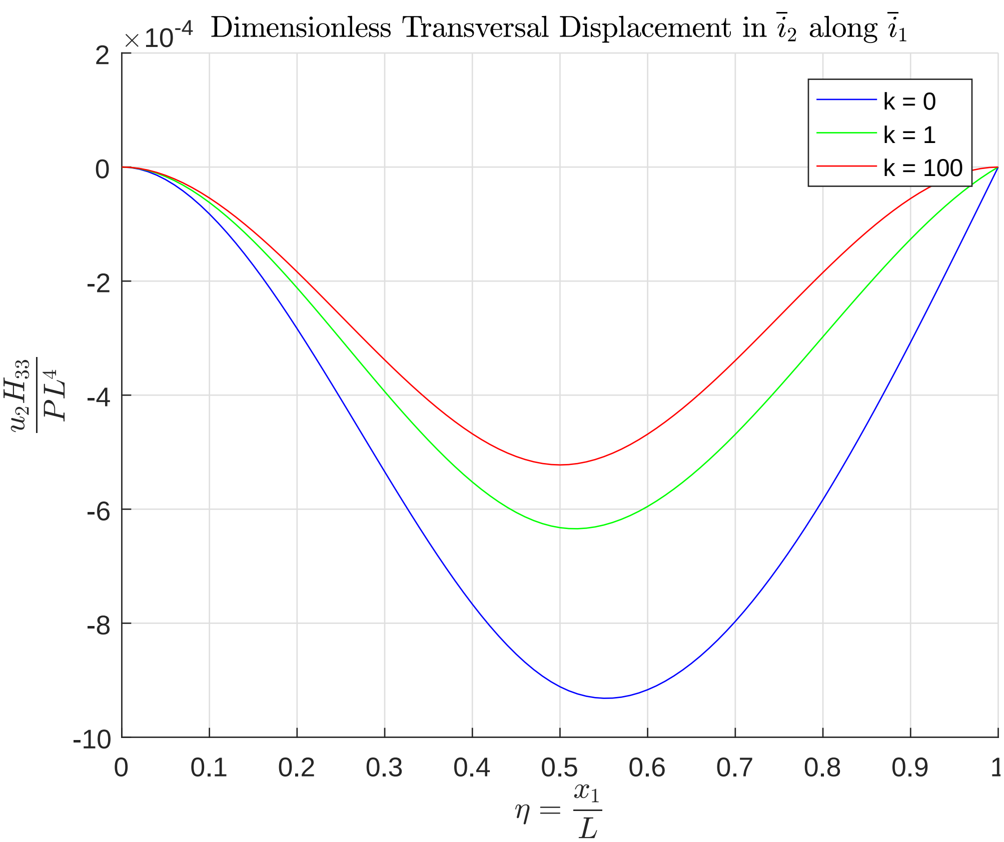

Figure 16: Plot of non-dimensional transversal displacement.

As for the moments, simply recognise that $M_3(x_1) = H_{33}^c \dfrac{d^2u_2(x_1)}{dx_1^2}$ which yields

$$M_3(x_1) = \begin{cases} -\dfrac{1}{2}PL + R_sL + M_s + (P - R_s)x_1 & , \quad 0 \le x_1 \le L/2 \\ M_s + R_sL - R_sx_1 & , \quad L/2 \le x_1 \le L \end{cases} \tag{76}$$

Plot of the non-dimensional moment ($M_n = M_3/(PL^2)$) vs. non-dimensional position along the beam is then

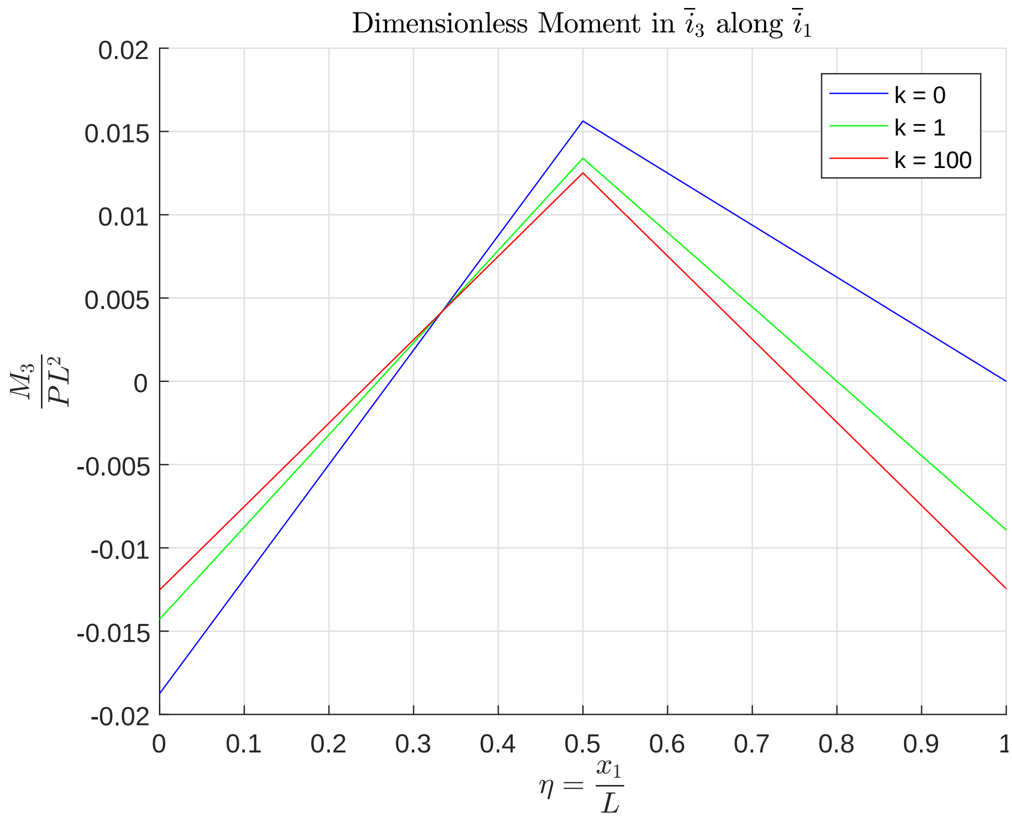

Figure 17: Plot of non-dimensional moment.