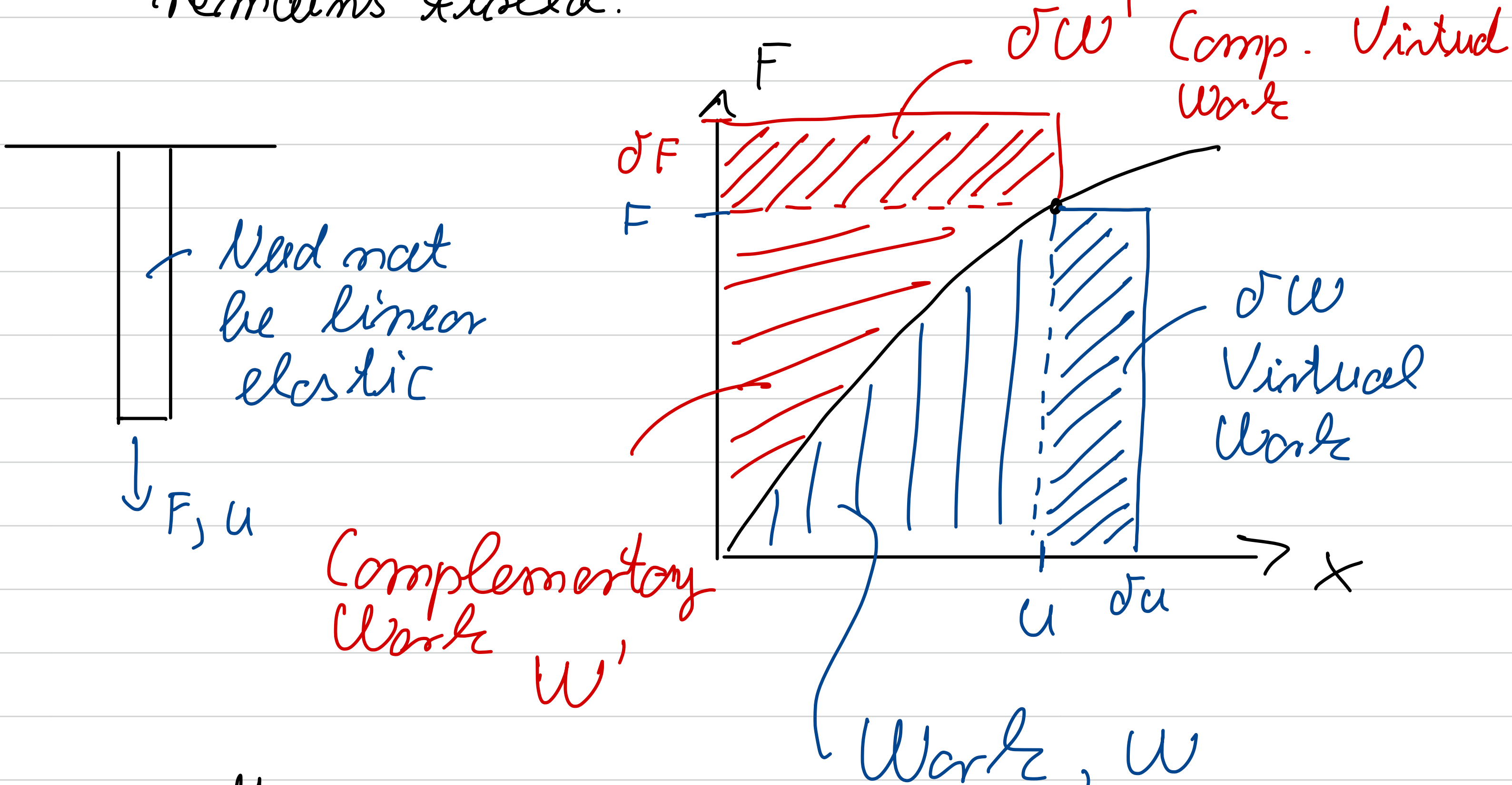
Complementary Virtual Work

* Defined as the work done by <u>virtual forces</u> acting through real displacements.

* The real quantity, i.e. the displacement remains fixed.



$$W = \int_0^u F\,du, \quad W' = \int_0^F u\,dF$$

* For a non-linear elastic material, W and W' are not equal.

* For both linear and non-linear materials

$$W + W' = F \cdot u$$

For a linear elastic material,
$F = k \cdot u$, and $W = W'$

$$W = \int_0^u F\,du = \int_0^u k u\,du = \frac{k u^2}{2} = \frac{F \cdot u}{2}$$

$$W' = \int_0^F u\,dF = \int_0^F \frac{F}{k}\,dF = \frac{F^2}{2k} = \frac{F \cdot u}{2}$$

Principle of Complementary Virtual Work (PCVW)

* The deformation in a deformable system is compatible if and only if

$$\delta W' = \delta W_E' + \delta W_I' = 0$$

for all statically admissible virtual forces.

* Statically admissible virtual forces satisfy equilibrium.

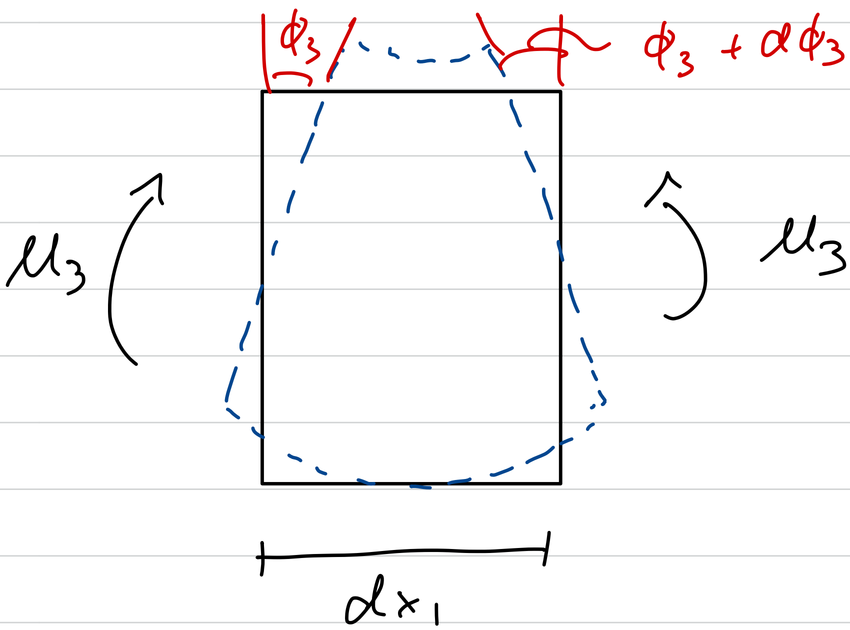
* PCVW is equivalent to compatibility

* PVW is equivalent to equilibrium.

For elastic bodies, the points within a body can move and hence the internal forces can do work

-> Need to compute δW_I and $\delta W_I'$ for elastic beams.

Beam in Bending



$$dW_I = -M_3 \phi_3 + M_3(\phi_3 + d\phi_3)$$

$$dW_I = M_3 \left(\frac{d\phi_3}{dx_1}\right) dx_1$$

$$W_I = -\int_0^L M_3 \frac{d\phi_3}{dx_1} dx_1 = -\int_0^L M_3 K_3 \, dx_1$$

Virtual Quantities.

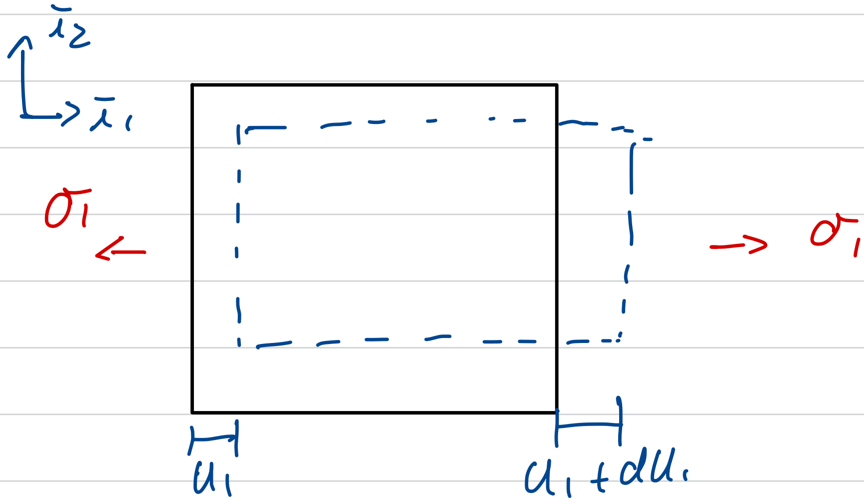
$$\delta W_I = -\int_0^L M_3 \cdot \delta K_3 \cdot dx_1$$

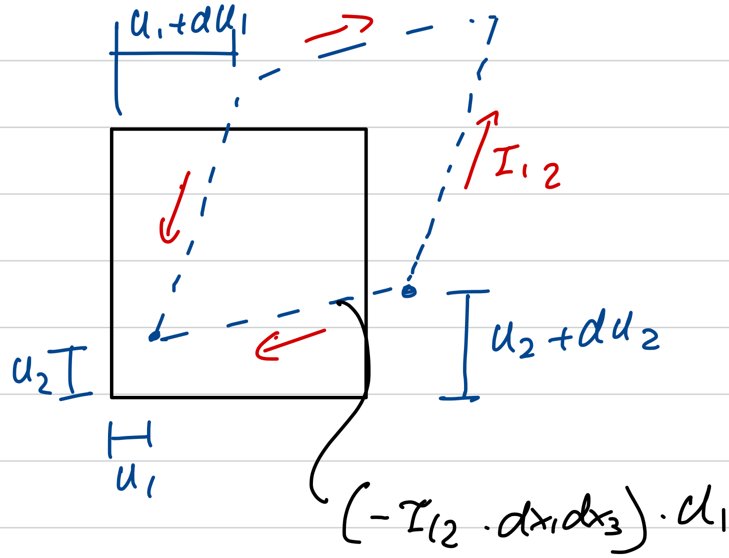
Virtual Curvature

$$\delta W_I' = -\int_0^L \delta M_3 \cdot K_3 \, dx_1$$

Virtual Moment

3D - Solid





Axial

$$W_I = -\int_V \sigma_1 \cdot \left(\frac{\partial u_1}{\partial x_1}\right) dx_1\, dx_2\, dx_3$$

$$= -\int_V \sigma_1\, \epsilon_1\, dV$$

Shear

$$\begin{aligned} dW_I = & -(\tau_{12}\, dx_1\, dx_3)\, u_1 \\ & (\tau_{12}\, dx_1\, dx_3)(u_1 + du_1) \\ & -(\tau_{12}\, dx_2\, dx_3)\, u_2 \\ & (\tau_{12}\, dx_2\, dx_3)(u_2 + du_2) \end{aligned}$$

$$dW_I = -\tau_{12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) dx_1\, dx_2\, dx_3$$

$$W_I = -\int_V \tau_{12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) dV$$

$$= -\int_V \tau_{12}\,\gamma_{12}\, dV$$

Generally

$$W_I = -\int_V \left(\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3 + \tau_{12}\gamma_{12} + \tau_{13}\gamma_{13} + \tau_{23}\gamma_{23}\right) dV$$

$$W_I = -\int_V \underline{\sigma}^T \underline{\epsilon}\, dV$$

Stress array

Strain array.

Euler-Bernoulli Beams

$$W_I = -\int_V \sigma_1 \epsilon_1 \, dV$$

$$= -\int_V \sigma_1 (\bar{\epsilon}_1 + x_3 \kappa_2 - x_2 \kappa_3) \, dV$$

$$W_I = -\int_0^L \Big[\bar{\epsilon}_1 \underbrace{\int_A \sigma_1 \, dA}_{N_1} + \kappa_2 \underbrace{\int_A \sigma_1 x_3 \, dA}_{M_2} + \kappa_3 \underbrace{\int_A -\sigma_1 x_2 \, dA}_{M_3} \Big] dx_1$$

$$W_I = -\int_0^L (N_1 \bar{\epsilon}_1 + M_2 \kappa_2 + M_3 \kappa_3) \, dx_1$$

Virtual Work

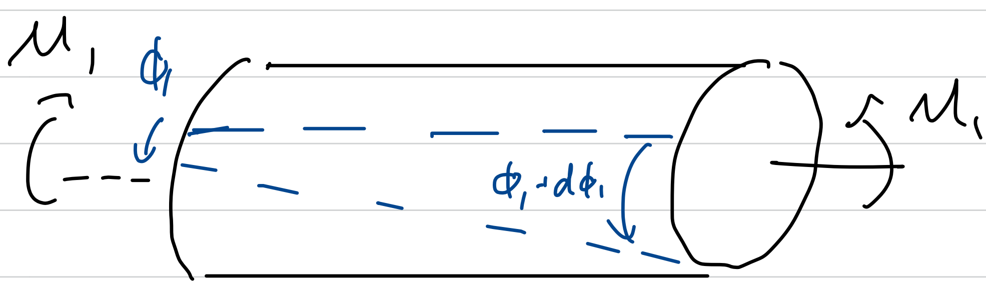
$$\delta W_I = -\int_0^L \left(N_1 \cdot \delta \bar{\epsilon}_1 + M_2 \delta \kappa_2 + M_3 \delta \kappa_3 \right) dx_1$$

Complementary Virtual Work

$$\delta W_I' = -\int_0^L \left(\bar{\epsilon}_1 \cdot \delta N_1 + \kappa_2 \cdot \delta M_2 + \kappa_3 \cdot \delta M_3 \right) dx_1$$

Torsion

$$W_I = -\int_V \tau_s \cdot \gamma_s \, dV = -\int_0^l M_1 \cdot \kappa_1 \, dx_1$$



$$\delta W_I = -\int_0^L M_1 \cdot \delta \kappa_1 \, dx_1$$

$$\delta W_I' = -\int_0^L \kappa_1 \cdot \delta M_1 \cdot dx_1$$