Georgia Institute of Technology
School of Aerospace Engineering
Atlanta, Georgia 30332

AE 6115 — Fundamentals of Aerospace Structural Analysis

Review of 2D Beam bending. Stress and Strain. Linear Elasticity.

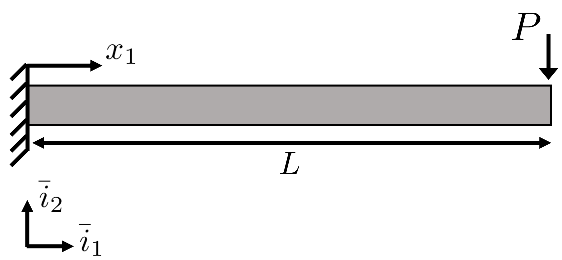
Review of Statics and Deformable Bodies.

Problem 1

Consider the simple cantilevered beam shown below.

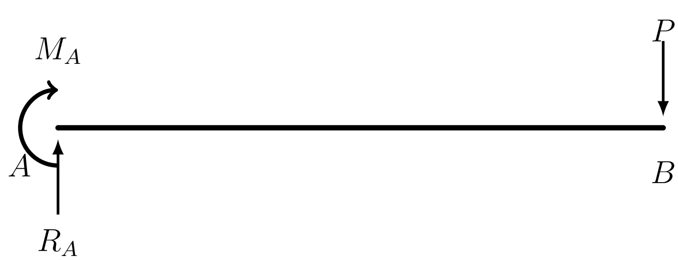
a) Draw the shear and moment diagrams for the beam.

b) Find the transverse displacement u_2 of the beam as a function of the coordinate x_1 shown in the figure below.



Solution

Shear and moment diagrams are plots of the internal shear force and moment a beam is experiencing along its length, and in order to draw them, we need to figure out all loads and their locations acting on the beam.



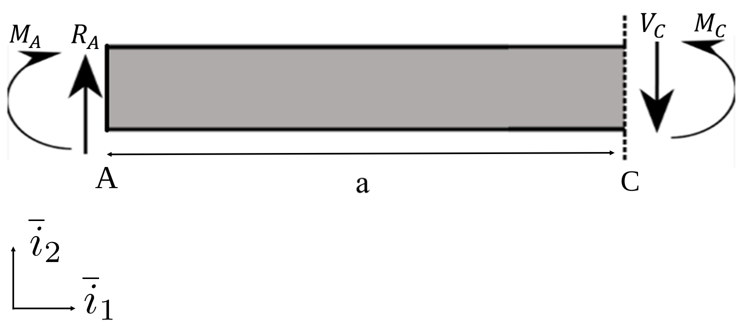
The beam is static, therefore along each axis, all moments should sum to zero and all forces should sum to zero. Analyzing the moment equilibrium at point A gives us the bending moment at A.

$$\begin{aligned} \sum M_A = M_A + PL = 0 \\ M_A = -PL \end{aligned} \tag{1}$$

Then analysis of the force equilibrium along the vertical direction yields the reaction force at A.

$$\begin{aligned} \sum F_y = R_A - P = 0 \\ R_A = P \end{aligned} \tag{2}$$

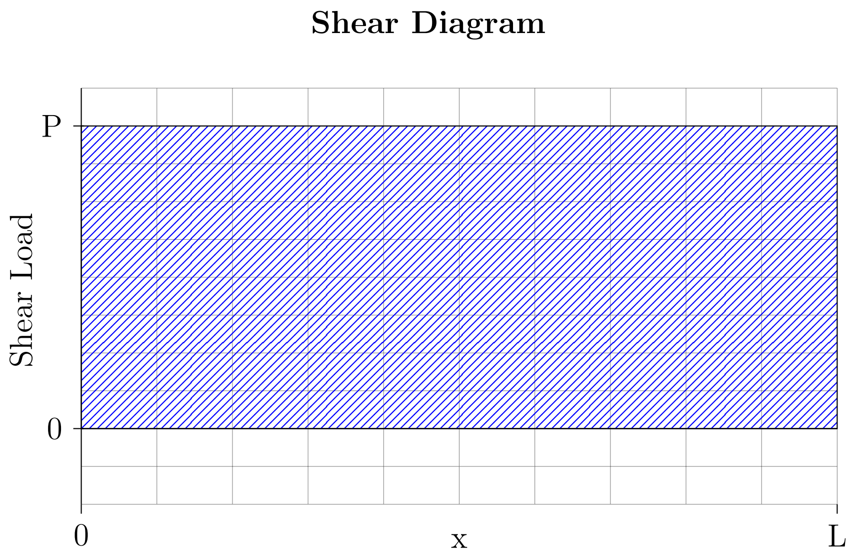
Now the shear and moment diagrams can be drawn by taking a "cut" at various locations along the beam and computing the reaction shear or reaction moment at the cut needed to maintain equilibrium.

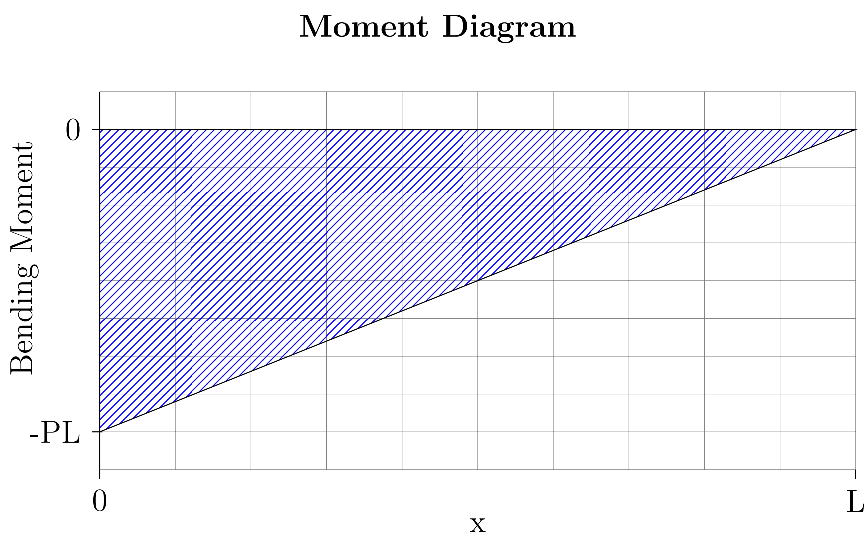


Here M_C and V_C are the moment and shear load at the arbitrary point C along the length of the beam, which can be derived by the analysis of moment equilibrium at point C, and force equilibrium along the vertical direction.

$$\begin{aligned} \sum M_C = M_C - M_A - R_A a = 0 \\ M_C = P(a - L) \end{aligned} \tag{3}$$

$$\begin{aligned} \sum F_y = R_A - V_C = 0 \\ V_C = P \end{aligned} \tag{4}$$





Application of the Euler-Bernoulli Equation to the first beam segment describes the relationship between the beam's deflection and the applied load as follows for a homogeneous material

$$EI\frac{d^4u_y}{d^4x} = 0 \tag{5}$$

Integrating 4 times in succession, the following relationship is obtained for transverse deflection:

$$u_y = C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \tag{6}$$

To solve for unknown constants, the following boundary conditions are applied:

Zero transverse displacement at the clamped end.

$$u_y(0) = 0 \tag{7}$$

Zero slop at the clamped end.

$$u_y'(0) = 0 \tag{8}$$

Shear load of P at the free end.

$$V(0) = EI \frac{d^3 u_y}{d^3 x}_{(x=0)} = P \tag{9}$$

No moment at the free end.

$$M(L) = EI \frac{d^2 u_y}{d^2 x}_{(x=L)} = 0 \tag{10}$$

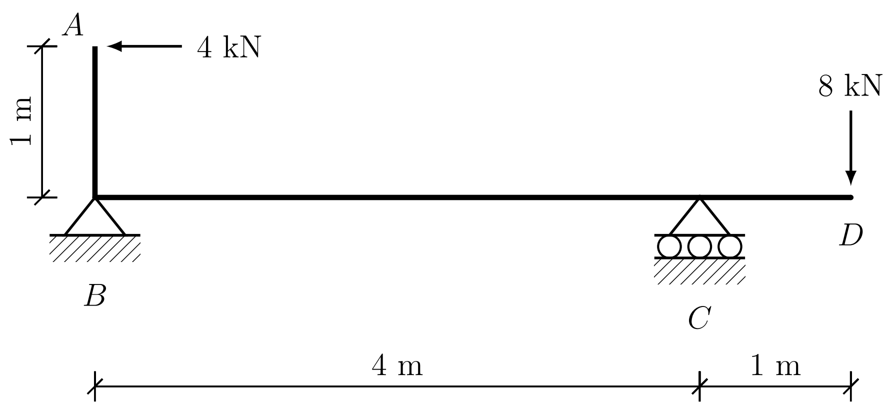
Solving the system of equations, leads to the following expression for transverse displacement of the beam:

$$u_y = \frac{P}{EI}\frac{x^3}{6} - \frac{PL}{EI}\frac{x^2}{2} \tag{11}$$

Problem 2

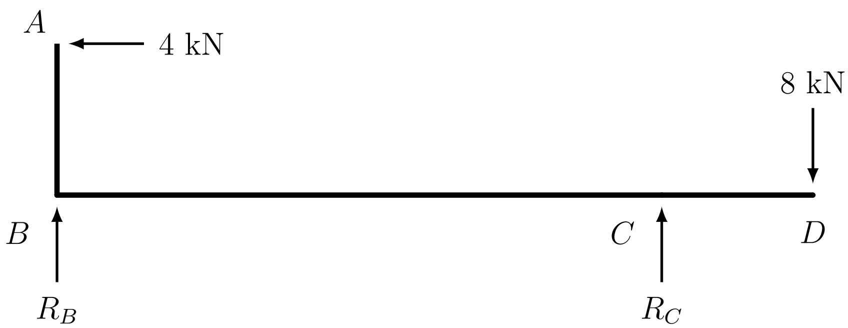
a) Consider the following beam diagram. Draw the shear and moment diagrams for the beam spanning from point B to point D.

b) Find the transverse displacement at point D. Assume a Young's modulus $E = 210\text{GPa}$, and a bending moment of inertia of $I = 128 \cdot 10^6\,\text{mm}^4$.



Solution

Shear and moment diagrams are plots of the internal shear force and moment a beam is experiencing along its length, and in order to draw them, we need to figure out all loads and their locations acting on the beam. For this beam, there are two unknown loads that need to be found first, namely R_B and R_C, the reaction forces from the supports.



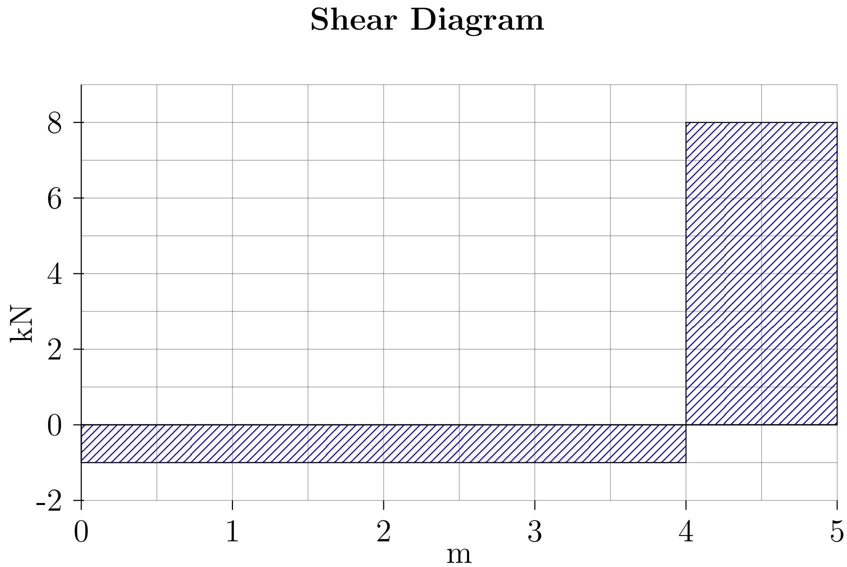
The beam is static, therefore along each axis, all moments should sum to zero and all forces should sum to zero. Analyzing the moment equilibrium at point B gives us the reaction force at C.

$$\begin{aligned} \sum M_B = (4kN)(1m) + (-8kN)(5m) + (R_C)(4m) = 0 \\ \frac{(4kNm - 40kNm)}{4m} + R_C = 0 \\ R_C = 9kN \end{aligned} \tag{12}$$

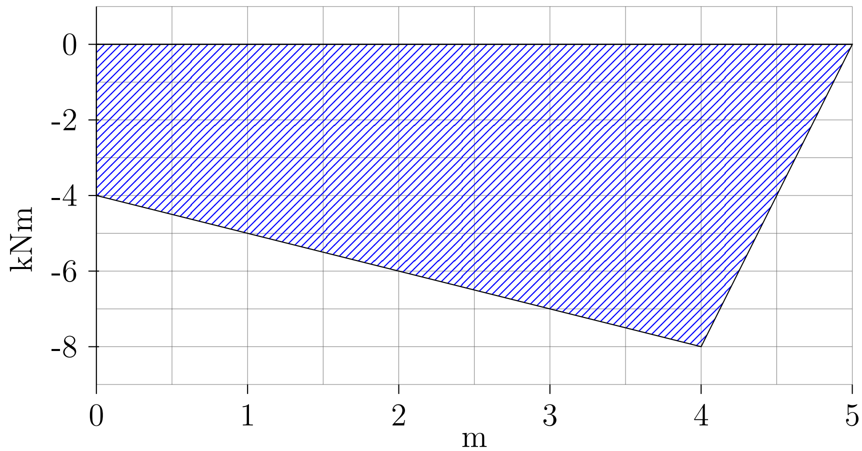
Then analysis of the force equilibrium along the vertical direction yields the reaction force at B.

$$\begin{aligned} \sum F_y = -8kN + R_C + R_B = 0 \\ -8kN + 9kN + R_B = 0 \\ R_B = -1kN \end{aligned} \tag{13}$$

Now the shear and moment diagrams can be drawn by taking a "cut" at various locations along the beam and computing the reaction shear or reaction moment at the cut needed to maintain equilibrium.



Moment Diagram



Due to the presence of a discontinuity at point x=4m, each beam should be solved separately. First beam segment extends point B to C, while the second segment extends from C to D. Application of the Euler-Bernoulli Equation to the first beam segment describes the relationship beween the beam's deflection and the applied load as follows for a homogenous material

$$EI\frac{d^4u_y}{d^4x} = 0 \tag{14}$$

Integrating 4 times in succession, the following relationship is obtained for transverse deflection:

$$u_y = C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \tag{15}$$

To solve for unknown constants, the following boundary conditions are applied:

$$u_y(0) = 0 \tag{16}$$

$$u_y(4) = 0 \tag{17}$$

$$M(0) = EI\frac{d^2u_y}{d^2x}_{(x=0)} = -4 \text{ KNm} \tag{18}$$

$$M(4) = EI\frac{d^2u_y}{d^2x}_{(x=4)} = -8 \text{ KNm} \tag{19}$$

Solving the system of equations, leads to the following expression for transverse displacement of the beam:

$$u_y = -\frac{1000}{EI}\frac{x^3}{6} - \frac{4000}{EI}\frac{x^2}{2} + \frac{32000}{3EI}x \tag{20}$$

We can use the above relationship to calculate the slope at C-support (x= 4m) to be $\frac{-40}{3EI}$. This will become handy when evaluating the constants obtained from considering the second beam segment CD. In a similar fashion, Euler-Bernoulli theory can be applied to the second beam segment, making use of the following boundary conditions:

$$u_y(4) = 0 \tag{21}$$

$$\frac{du_y}{dx}(4) = -\frac{40}{3EI} \tag{22}$$

$$V(5) = EI\frac{d^3u_y}{d^3x}_{(x=5)} = 8 \text{ KNm} \tag{23}$$

$$M(5) = EI\frac{d^2u_y}{d^2x}_{(x=5)} = 0 \text{ KNm} \tag{24}$$

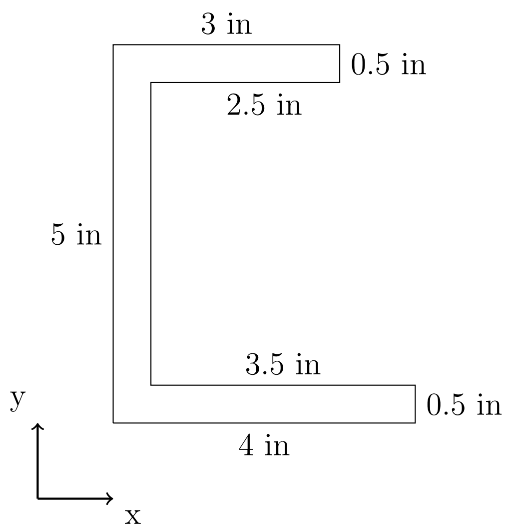
Solving the system of equations, leads to the following expression for transverse displacement of the beam:

$$u_y = \frac{4000}{EI}\frac{x^3}{3} - \frac{40000}{EI}\frac{x^2}{2} + \frac{56000}{EI}x + 10666.67 \tag{25}$$

Substituting for EI and the tip location of the beam we find $u_y = -1.59$mm

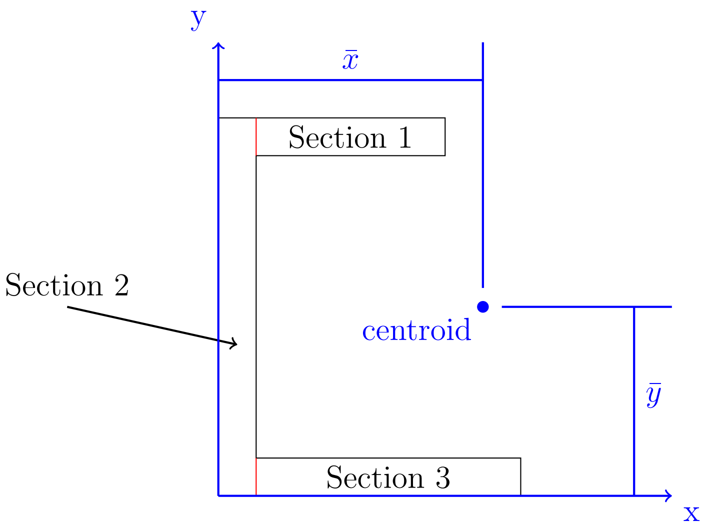
Problem 3

Consider the cross section below. Find its centroid location and second moments of area through the centroid parallel to the x and y directions (I_{xx} and I_{yy}).



Solution

Define coordinate origin (I set axes to be on bottom edge of lower flange (x axis) and left edge of the web (y axis)). Then break cross section into simpler sections (I broke it into the two flanges and one web, numbering them 1 for the top flange, 2 for the web, and 3 for the bottom flange). The centroid's position for the entire area is located at $(\bar{x}, \bar{y})$.



In order to find $(\bar{x}, \bar{y})$, the areas and centroids of each section need to be found first. For each rectangular section i, let b_i represent its length of base and h_i its height, then the area of each section i is then $A_i = b_i h_i$ and the centroid of each rectangular section is at its center $(\bar{x}_i, \bar{y}_i)$. The areas of section 1, 2, and 3 are

$$A_1 = 1.25in^2; \quad A_2 = 2.5in^2; \quad A_3 = 1.75in^2; \tag{26}$$

the x-locations of each section's centroid are

$$\bar{x}_1 = 1.75in; \quad \bar{x}_2 = 0.25in; \quad \bar{x}_3 = 2.25in; \tag{27}$$

and the y-locations of each section's centroid are

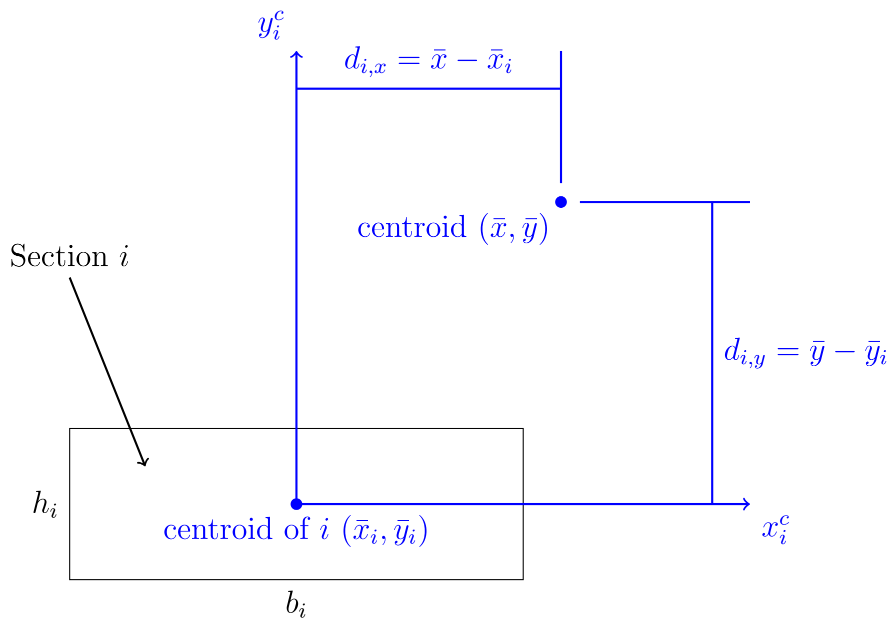
$$\bar{y}_1 = 4.75in; \quad \bar{y}_2 = 2.5in; \quad \bar{y}_3 = 0.25in. \tag{28}$$

Then the centroid of the full cross section can be calculated using

$$\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i} = 1.227in; \quad \bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = 2.295in \tag{29}$$

where i denotes the section.

Next, to find the second moments of area, find it for the individual sections and use the parallel-axis theorem to translate them about the centroid of interest found earlier. Since each section is a rectangle, note the following figure.



The second moment of area of the above rectangular section i about the x_i axis is

$$I_{xx}^{ci} = \frac{b_i h_i^3}{12} \tag{30}$$

and the second moment of area about the y_i axis is

$$I_{yy}^{ci} = \frac{h_i b_i^3}{12} \tag{31}$$

The parallel-axis theorem allows the second moment to be calculated at a different location through a simple translation. In this problem, the translation required is from the center of the rectangular sections $(\bar{x}_i, \bar{y}_i)$ to the centroid location of the cross section $(\bar{x}, \bar{y})$, described by $(d_{i,x}, d_{i,y})$. The second moment of area after the translation for section i is then

$$I_{xx}^{i} = I_{xx}^{ci} + A_i d_{i,y}^2 \tag{32}$$

along the x axis through the point $(\bar{x}, \bar{y})$ and

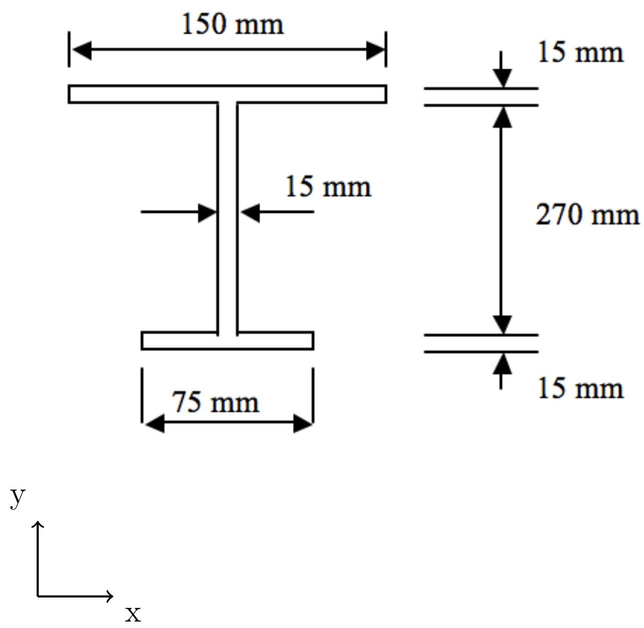
$$I_{yy}^{i} = I_{yy}^{ci} + A_i d_{i,x}^2 \tag{33}$$

along the y axis.

The resulting moments are simply added to get the total, i.e. $I_{xx} = \sum I_{xx}^i$. For this section, $I_{xx} = 20.23 in^4$ and $I_{yy} = 7.05 in^4$.

Problem 4

Consider the cross section below. Find its centroid location and second moments of area through the centroid parallel to the x and y directions (I_{xx} and I_{yy}).



Solution
Define coordinate origin (I set axes to be on bottom and left edge of lower flange (x axis)). Then break cross section into simpler sections (I broke it into the two flanges and one web, numbering them 1 for the top flange, 2 for the web, and 3 for the bottom flange). The centroid's position for the entire area is located at $(\bar{x}, \bar{y})$.

$$A_1 = 2250mm^2; \quad A_2 = 4050mm^2; \quad A_3 = 1125mm^2; \tag{34}$$

and the y-locations of each section's centroid are

$$\bar{y}_1 = 292.5mm; \quad \bar{y}_2 = 150mm; \quad \bar{y}_3 = 7.5mm. \tag{35}$$

Then the centroid of the full cross section can be calculated using

$$\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = 171.6mm \tag{36}$$

where i denotes the section.

From symmetry arguments, it can be reasoned that the x_2 centroidal coordinate is located at

$$\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i} = 37.5mm \tag{37}$$

In a similar fashion to Problem 2, the moment of inertia about the x- axis (I_{xx}) and y-axis (I_{yy}) can be calculated for each section individually and then, using parallell axis theorem gives us he final result:

$$I_{xx} = 85.9 \cdot 10^6 \text{ mm}^4 \text{ and } I_{yy} = 4.8 \cdot 10^6 \text{ mm}^4$$

Stress and Strain

Problem 5

The matrix of stress components for a 2D state of stress, relative to the cartesian basis vectors $\{\mathbf{i}_1, \mathbf{i}_2\}$, is given by

$$[\sigma] = \begin{bmatrix} 80 & 120 \\ 120 & 150 \end{bmatrix} \qquad \text{MPa.} \tag{38}$$

1. Showing all steps (i.e. without a computer), determine the principal stresses.
2. Showing all steps (i.e. without a computer), determine the principal directions. Note that the principal directions are unit vectors.

Solution

To find the principal stresses σ_{p1} and σ_{p2} and principal directions $\mathbf{i}_{p1}$ and $\mathbf{i}_{p2}$ for the given stress state, we need to solve the eigenvalue/eigenvector problem:

$$\begin{aligned} [\sigma]\mathbf{i}_{pj} &= \sigma_{pj}\mathbf{i}_{pj} \\ \left([\sigma] - \sigma_{pj}[\mathbf{I}]\right)\mathbf{i}_{pj} &= 0 \end{aligned} \tag{39}$$

1. For $\mathbf{i}_{pj}$ to be a nontrivial solution, we must have

$$\begin{aligned} \det\left([\sigma] - \sigma_{pj}[\mathbf{I}]\right) &= 0 \\ \det\left(\begin{bmatrix} 80 & 120 \\ 120 & 150 \end{bmatrix} - \sigma_{pj}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 80 - \sigma_{pj} & 120 \\ 120 & 150 - \sigma_{pj} \end{bmatrix}\right) &= 0 \\ (80 - \sigma_{pj})(150 - \sigma_{pj}) - (120)^2 &= 0 \\ 12000 - 230\sigma_{pj} + \sigma_{pj}^2 - 14400 &= 0 \\ \sigma_{pj}^2 - 230\sigma_{pj} - 2400 &= 0 \\ (\sigma_{pj} - 240)(\sigma_{pj} + 10) &= 0 \end{aligned}$$

   In class the principal stresses were ordered: $\sigma_{p1} > \sigma_{p2}$. So, the principal stresses are $\boxed{\sigma_{p1} = 240 \text{ MPa} \text{ and } \sigma_{p2} = -10 \text{ MPa.}}$

2. Then to find the principal directions, we need to solve (39) for the unit vectors $\mathbf{i}_{pj}$ corresponding to each principal stress σ_{pj}:

$$\left(\begin{bmatrix} 80 & 120 \\ 120 & 150 \end{bmatrix} - \sigma_{pj}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} i_{pj,1} \\ i_{pj,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\sigma_{p1} = 240$:

$$\begin{bmatrix} 80-240 & 120 \\ 120 & 150-240 \end{bmatrix} \begin{bmatrix} i_{p1,1} \\ i_{p1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -160 & 120 \\ 120 & -90 \end{bmatrix} \begin{bmatrix} i_{p1,1} \\ i_{p1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} i_{p1,1} \\ i_{p1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4i_{p1,1} - 3i_{p1,2} = 0$$

For $\sigma_{p2} = -10$:

$$\begin{bmatrix} 80-(-10) & 120 \\ 120 & 150-(-10) \end{bmatrix} \begin{bmatrix} i_{p2,1} \\ i_{p2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 90 & 120 \\ 120 & 160 \end{bmatrix} \begin{bmatrix} i_{p2,1} \\ i_{p2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} i_{p2,1} \\ i_{p2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3i_{p2,1} + 4i_{p2,2} = 0$$

So we can see that $\mathbf{i}_{p1} = [3, 4]^T$ and $\mathbf{i}_{p2} = [4, -3]^T$ will satisfy these equations, but because the principal directions are unit vectors, we need to normalize to get the principal directions:

For $\sigma_{p1} = 240$:

$$\mathbf{i}_{p1} = \begin{bmatrix} \frac{3}{\sqrt{3^2+4^2}} \\ \frac{4}{\sqrt{3^2+4^2}} \end{bmatrix}$$

$$\boxed{\mathbf{i}_{p1} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}$$

For $\sigma_2 = -10$:

$$\mathbf{i}_{p2} = \begin{bmatrix} \frac{4}{\sqrt{(-3)^2+4^2}} \\ \frac{-3}{\sqrt{(-3)^2+4^2}} \end{bmatrix}$$

$$\boxed{\mathbf{i}_{p2} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}}$$

Problem 6

Direct stresses of $\sigma_x = 160\text{MPa}$ (tension) and $\sigma_y = -120\text{MPa}$ (compression) are applied at a particular point in an elastic material on two mutually perpendicular planes. One of the components of the principal stress in the material is $\sigma_{p1} = 200\text{MPa}$.

1. Calculate the value of shear stress at the point on the given planes. That is calculate τ_{xy}.
2. Determine the value of the other principal stress and the maximum value of shear stress at the point.

Solution

1. The principal values are determined by solving the eigenvalue problem

$$\det \begin{bmatrix} \sigma_x - \sigma_p & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_p \end{bmatrix} = 0$$

which yields the equation

$$(\sigma_x - \sigma_p)(\sigma_y - \sigma_p) - \tau_{xy}^2 = 0$$

We are given the values for $\sigma_x = 160\text{MPa}$, $\sigma_y = -120\text{MPa}$ and one of the principal stresses $\sigma_{p1} = 200\,\text{MPa}$. Plugging in the values gives

$$(160 - 200)(-120 - 200) = \tau_{xy}^2$$

and we get

$$\tau_{xy} = \pm\sqrt{12,800}\,\text{MPa} \approx \pm 113\,\text{MPa}.$$

2. We may solve this in multiple ways. The simplest is to use the fact that

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

is a stress invariant, and hence

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_{p1} + \sigma_{p2} + \sigma_{p3}.$$

Since $\sigma_z = \sigma_{p3} = 0$ we simply have

$$\sigma_x + \sigma_y = \sigma_{p1} + \sigma_{p2}$$

and with knowledge of σ_x, σ_y, and σ_{p1} we may simply solve for

$$\sigma_{p2} = \sigma_x + \sigma_y - \sigma_{p1} = (160 - 120 - 200)\,\text{MPa} = -160\,\text{MPa}.$$

Alternatively, we can solve the principal stress by using the characteristic equation from above

$$(\sigma_x - \sigma_p)(\sigma_y - \sigma_p) - \tau_{xy}^2 = 0$$

and solving for σ_p which yields

$$\sigma_p = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}.$$

Plugging in values of $\sigma_x = 160\text{MPa}$, $\sigma_y = -120\text{MPa}$, and $\tau_{xy}^2 = 12,800\,\text{MPa}$ yields

$$\sigma_p = \frac{160 - 120}{2} \pm \left[\left(\frac{160 + 120}{2}\right)^2 + 12800\right]^{1/2} \text{MPa}$$
$$\sigma_p = 20 \pm 180\,\text{MPa}$$

Hence, the second eigenvalue is $\sigma_{p3} = -160\,\text{MPa}$.

Now to calculate the max shear stress τ_{max}, we can use

$$\tau_{max} = \frac{|\sigma_{p1} - \sigma_{p3}|}{2}$$

since, due to the ordering $\sigma_{p1} > \sigma_{p2} > \sigma_{p3}$, the largest shear stress is achieved by the drastic stress difference between σ_{p1} and σ_{p3}. The max shear is then $\tau_{max} = 180\,\text{MPa}$.

Problem 7

A cylindrical pressure vessel of radius R and thickness t is subjected to an internal pressure p_i, see Fig. 1. At any point in the cylindrical portion of vessel wall, two stress components are acting: the hoop stress, $\sigma_h = Rp_i/t$ and the axial stress, $\sigma_a = Rp_i/(2t)$. The radial stress, acting in the direction perpendicular to the wall, is very small, $\sigma_r \approx 0$. The pressure vessel features a weld line at a 45 degree angle with respect to the axis of the cylinder, as shown below.

1. Find the direct stress acting in the direction perpendicular to the weld line.

2. Find the shear stress acting along the weld line.

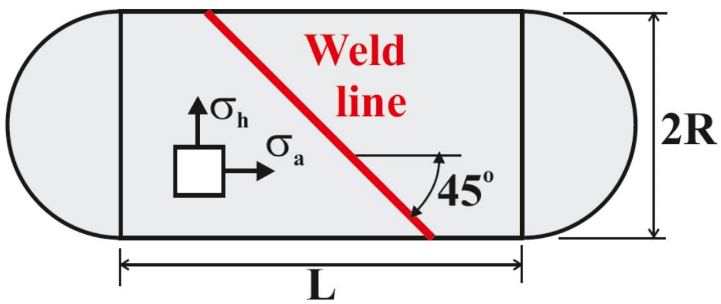


Figure 1: Capped cylindrical pressure

Solution

1. To find the stress acting across the weld line, recall the stress transformation equation

$$\sigma_1^* = \sigma(\theta) = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos(2\theta) + \tau_{12}\sin(2\theta)$$

   Letting $\sigma_1 = \sigma_a$ and $\sigma_2 = \sigma_h$, we observe that the weld line is at an angle of $\theta = -45°$; therefore, let $\theta = 45°$ since this angle is perpendicular to the weld line. Also notice that $\tau_{12} = 0$. Solving the above equation gives the direct stress across the weld to be

$$\begin{aligned}\sigma_1^* = \sigma(45°) &= \frac{Rp_i/(2t) + Rp_i/t}{2} + \frac{Rp_i/(2t) - Rp_i/t}{2}\cos(90°) + 0\sin(90°) \\ &= \frac{3Rp_i}{4t}\end{aligned}$$

2. To find the shear stress acting along the weld line, recall the shear stress transformation equation

$$\tau_{12}^* = \tau(\theta) = -\frac{\sigma_1 - \sigma_2}{2}\sin(2\theta) + \tau_{12}\cos(2\theta)$$

Letting $\theta = 45°$ and solving for the shear stress along the weld line gives

$$\begin{aligned}\tau_1^* = \tau(45°) &= -\frac{Rp_i/(2t) - Rp_i/t}{2}\sin(90°) + 0\cos(90°)\\ &= \frac{Rp_i}{4t}\end{aligned}$$

Problem 8

Suppose, that by means of careful measurement of the displacement of many points in a body we find that

$$u_1 = Ax_1^3 + Bx_2 \sin\frac{x_3}{L}, \quad u_2 = C\, x_1 x_2 x_3, \quad u_3 = e^{-(x_3/L)}\cos\frac{2x_2}{y_0},$$

represent the displacement field in a body, where, $\{A, B, C, L, y_0\}$ are constants chosen to fit the displacement data.

1. Calculate the strain field.
2. Calculate the strain at $(x_1 = 0, x_2 = y_0, x_3 = L)$.

Solution

1. Given the entire displacement field, we may easily calculate the individual strain components:

$$\begin{aligned}\epsilon_1 &= \frac{\partial u_1}{\partial x_1} = 3Ax_1^2\\ \epsilon_2 &= \frac{\partial u_2}{\partial x_2} = Cx_1x_3\\ \epsilon_3 &= \frac{\partial u_3}{\partial x_3} = -\frac{1}{L}e^{(-x_3/L)}\cos\frac{2x_2}{y_0}\\ \gamma_{12} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \frac{1}{2}\left(B\sin\frac{x_3}{L} + Cx_2x_3\right)\\ \gamma_{23} &= \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] = \frac{1}{2}\left(Cx_1x_2 - \frac{2}{y_0}e^{(-x_3/L)}\sin\frac{2x_2}{y_0}\right)\\ \gamma_{13} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \frac{1}{2}\left(\frac{B}{L}x_2\cos\frac{x_3}{L}\right)\end{aligned}$$

2. Evaluating at $(x_1 = 0, x_2 = y_0, x_3 = L)$,

$$
\begin{aligned}
\epsilon_1 &= 0 \\
\epsilon_2 &= 0 \\
\epsilon_3 &= -\frac{1}{L} e^{(-1)} \cos 2 \\
\gamma_{12} &= \frac{1}{2} \left(B \sin 1 + C y_0 L \right) \\
\gamma_{23} &= -\frac{1}{2} \left(\frac{2}{y_0} e^{(-1)} \sin 2 \right) \\
\gamma_{13} &= \frac{1}{2} \left(\frac{B}{L} y_0 \cos 1 \right)
\end{aligned}
$$

Linear Elasticity

Problem 9

Consider a rectangular steel plate with thickness $t = 5$mm subjected to uniform normal stresses σ_x and σ_y. You may assume that the plate is under a state of plane-stress with the z-direction the out of plane direction.

Strain gages oriented along the x and y directions give strain readings of $\epsilon_x = 0.0012$ (elongation) and $\epsilon_y = -0.0005$ (shortening). Given that $E = 210$ GPa and $\nu = 0.3$, determine the stresses σ_x and σ_y and the change in thickness of the plate.

Solution

The strains σ_x, σ_y, and the change in plate thickness Δt can all be found using the generalized Hooke's Law.

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \\
\epsilon_z &= \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y))
\end{aligned}$$

Combining the equations for ϵ_x and ϵ_y and using $\sigma_z = 0$, a direct solution for σ_x can be found.

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\
&= \frac{210 \text{ GPA}}{1-0.09}(0.0012 - 0.3 * 0.0005) \\
&= 0.2423 \text{ GPa}
\end{aligned}$$

Allowing us to calculate now σ_y.

$$\begin{aligned}
\sigma_y &= E\epsilon_y + \nu\sigma_x \\
&= (210 \text{ GPA})(-0.0005) + (0.3)(0.2423 \text{ GPa}) \\
&= -0.0323 \text{ GPa}
\end{aligned}$$

Using the fact that $\Delta t/t = \epsilon_z$, the change in thickness of the plate is then

$$\begin{aligned}
\Delta t &= t\epsilon_z \\
&= \frac{t\nu}{E}(\sigma_x + \sigma_y) \\
&= \frac{(5 \text{ mm})(0.3)}{210 \text{ GPa}}(0.2423 \text{ GPa} - 0.0323 \text{ GPa}) \\
&= -0.0015 \text{ mm (thinner)}
\end{aligned}$$

Problem 10

A rectangular element in a linearly elastic, isotropic material, with $E = 200$ GPa and $\nu = 0.3$, is subjected to tensile in-plane stresses of 83 MPa and 65 MPa on mutually perpendicular planes.

1. Determine the strain in the direction of each stress (in-plane components of strain) and in the direction perpendicular to both stresses (out-of-plane component of strain).

2. Determine the principal strains, maximum in-plane shear stress, maximum in-plane shear strain, and their directions.

3. If you consider the full three-dimensional stress state (including the zero out of plan stress), what is the maximum shear stress and what is its direction.

Solution

1. We can let $\sigma_1 = 83$ MPa and $\sigma_2 = 65$ MPa since they act on along mutually perpendicular axes. In addition, we also know that $\sigma_3 = 0$ and $\tau_{12} = 0$. In order to find strains, we can use Hooke's Law.

$$\epsilon_1 = \frac{1}{E}(\sigma_1 - \nu(\sigma_2 + \sigma_3)) = \frac{1}{200 \text{ GPa}}(83 \text{ MPa} - (0.3)(65 \text{ MPa}) = 0.0003175$$
$$\epsilon_2 = \frac{1}{E}(\sigma_2 - \nu(\sigma_1 + \sigma_3)) = \frac{1}{200 \text{ GPa}}(65 \text{ MPa} - (0.3)(83 \text{ MPa})) = 0.0002005$$
$$\epsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1 + \sigma_2)) = \frac{-0.3}{200 \text{ GPa}}(83 \text{ MPa} + 65 \text{ MPa}) = -0.000222$$

2. In this case, since there are no shear stresses acting on the planes 1 and 2, i.e. $\tau_{12} = 0$, the stresses σ_1 and σ_2 are principal stresses and that ϵ_1 and ϵ_2 are also the principal stresses, so their direction is in the same direction as σ_1 and σ_2. To see this, recall that in plane-stress shear stress and shear strain are related by the equation

$$\tau_{12} = \frac{E}{2(1+\nu)}\gamma_{12}$$

in which case $\tau_{12} = 0$ indicates $\gamma_{12} = 0$.

Now to find max in-plane shear stress, can just calculate

$$\tau_{12,max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{83 \text{ MPa} - 65 \text{ MPa}}{2} = 9 \text{ MPa}$$

which acts at $\theta = 45°$ in the 1-2 plane. For max in-plane shear strain, we may rewrite (2) as

$$\gamma_{12,max} = \frac{2(1+\nu)}{E}\tau_{12,max} = \frac{2(1+0.3)}{200 \text{ GPa}} 9 \text{ MPa}$$

So $\gamma_{12,max} = 0.000117$ and acts in the same direction as $\tau_{12,max}$.

3. If we then consider the 3-dimensional stress state, then we may now include $\sigma_3 = 0$, leading the max shear stress to be

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{83 \text{ MPa}}{2} = 41.5 \text{ MPa}$$

acting at $\theta = 45°$ in the 1-3 plane. The max shear strain then becomes

$$\gamma_{max} = \frac{2(1+\nu)}{E}\tau_{max} = \frac{2(1+0.3)}{200 \text{ GPa}} 41.5 \text{ MPa} = 0.0005395$$

γ_{max} also acts in the same direction as τ_{max}.

Problem 11

A cube of material of dimensions L x L x L is fixed to a rigid wall on its bottom side. As depicted in Figure 2, the top face of the cube is displaced in the $\mathbf{i_2}$ direction a distance δL, and then in the $\mathbf{i_1}$ direction a distance γL. There are no displacement in the $\mathbf{i_3}$ direction.

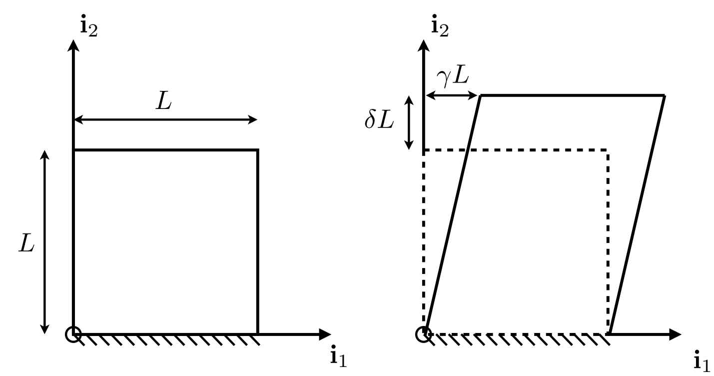


Figure 2: Deformation of the cube.

The values of γ and δ are given by

$$\gamma = 800 \times 10^{-6} \qquad \delta = 600 \times 10^{-6}$$

1. Write the components of the displacement field, u_1, u_2, and u_3, using the origin indicated on Figure 2.
2. Calculate the components of the strain field, $\boldsymbol{\epsilon}$, corresponding to this displacement field.
3. Assume the material obeys a linear elastic stress-strain relation with Young's modulus $E = 210\,\text{GPa}$ and poisson's ratio $\nu = 0.3$. Calculate all components of the stress tensor $\boldsymbol{\sigma}$. Give your answer in MPa.
4. Determine the principal values and principal directions of this state of stress. What is the maximum tensile stress?

5. For a ductile material with $\sigma_y = 175\,\mathrm{MPa}$, will yield occur? You may use the Mises yield condition.

Solution

1. The components of the displacement field are given by

$$u_1 = \gamma x_2 = 800 \times 10^{-6} x_2, \quad u_2 = \delta x_2 = 600 \times 10^{-6} x_2, \quad \text{and} \quad u_3 = 0$$

2. The strain components are given by

$$\begin{aligned}
\epsilon_1 &= \frac{\partial u_1}{\partial x_1} = 0, \\
\epsilon_2 &= \frac{\partial u_2}{\partial x_2} = \delta = 600 \times 10^{-6}, \\
\epsilon_{12} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \frac{\gamma_{12}}{2} = \frac{\gamma}{2} = 400 \times 10^{-6} \\
\epsilon_3 &= \epsilon_{13} = \epsilon_{23} = 0.
\end{aligned}$$

Thus, the strain field is given by

$$[\epsilon] = \begin{bmatrix} 0 & 400 & 0 \\ 400 & 600 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-6}.$$

3. The stress can be found by resolving Hooke's Law resulting in the isotropic linear elastic stress-strain relations known as the generalized inverse Hooke's law as follows

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1 + \nu\epsilon_2 + \nu\epsilon_3\right], \tag{40a}$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_1 + (1-\nu)\epsilon_2 + \nu\epsilon_3\right], \tag{40b}$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_1 + \nu\epsilon_2 + (1-\nu)\epsilon_3\right], \tag{40c}$$

and

$$\tau_{12} = \frac{E}{2(1+v)}\gamma_{12}, \quad \tau_{23} = \frac{E}{2(1+v)}\gamma_{23}, \quad \tau_{13} = \frac{E}{2(1+v)}\gamma_{13}. \tag{41}$$

Using these equation, the stresses can all be solved for.

$$\begin{aligned}
\sigma_1 &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_2\right] \\
&= \frac{210000}{(1+.3)(1-.6)}\left[(.3)(0.0006)\right] \\
&= 72.69\,\mathrm{MPa}
\end{aligned}$$

$$
\begin{aligned}
\sigma_2 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_2\right] \\
&= \frac{210000}{(1+.3)(1-.6)}\left[(1-.3)(0.0006)\right] \\
&= 169.62\,\text{MPa}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\epsilon_2\right] \\
&= \frac{210000}{(1+.3)(1-.6)}\left[(.3)(0.0006)\right] \\
&= 72.69\,\text{MPa}
\end{aligned}
$$

$$
\begin{aligned}
\tau_{12} &= \frac{E}{2(1+v)}\gamma_{12} = \frac{E}{1+v}\epsilon_{12} = \frac{210000}{1.3}(0.0004) = 64.62\,\text{MPa} \\
\tau_{23} &= 0 \\
\tau_{13} &= 0.
\end{aligned}
$$

Now the resulting stress tensor can be expressed in the matrix form

$$
[\sigma] = \begin{bmatrix} 72.69 & 64.62 & 0 \\ 64.62 & 169.62 & 0 \\ 0 & 0 & 72.69 \end{bmatrix}\ \text{MPa}
$$

4. We can immediately recognize from the structure of $[\sigma]$ that $\sigma_3 = 72.69\,\text{MPa}$ is a principal stress with corresponding principal direction $\mathbf{i}_{p3} = \mathbf{i}_3$. This reduces the problem to finding the principal stresses and directions of the following 2x2 matrix of stress:

$$
[\sigma] = \begin{bmatrix} 72.69 & 64.62 \\ 64.62 & 169.62 \end{bmatrix}\ \text{MPa}. \tag{42}
$$

Solving the eigenvalue problem on this matrix:

$$
\begin{aligned}
&\det\left([\sigma] - \sigma_p[1]\right) = 0, \\
&\det\left(\begin{bmatrix} 72.69 & 64.62 \\ 64.62 & 169.62 \end{bmatrix} - \sigma_p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0, \\
&\det\begin{bmatrix} 72.69 - \sigma_p & 64.62 \\ 64.62 & 169.62 - \sigma_p \end{bmatrix} = 0, \\
&(72.69 - \sigma_p)(169.62 - \sigma_p) - (64.62)^2 = 0, \\
&\sigma_p^2 - 242.3\sigma_p + 8154 = 0, \\
&(\sigma_p - 40.38)(\sigma_p - 201.9) = 0.
\end{aligned}
$$

We order the principal stresses as $\sigma_{p1} > \sigma_{p2} > \sigma_{p3}$. Thus, the principal stresses are

$$
\begin{aligned}
\sigma_{p1} &= 201.9\,\text{MPa} \\
\sigma_{p2} &= 72.69\,\text{MPa} \\
\sigma_{p3} &= 40.38\,\text{MPa}.
\end{aligned}
$$

Then to find the principal directions, we need to solve for the unit vectors $\mathbf{i}_{pj}$ corresponding to each principal stress σ_{pj}:

$$\left(\begin{bmatrix}72.69 & 64.62\\64.62 & 169.62\end{bmatrix} - \sigma_{pj}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}\mathbf{i}_{pj,1}\\\mathbf{i}_{pj,2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

For $\sigma_{p1} = 201.9\,\text{MPa}$:

$$\begin{bmatrix}72.69-201.9 & 64.621\\64.62 & 169.62-201.9\end{bmatrix}\begin{bmatrix}\mathbf{i}_{p1,1}\\\mathbf{i}_{p1,2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}-129.21 & 64.62\\64.62 & -32.28\end{bmatrix}\begin{bmatrix}\mathbf{i}_{p1,1}\\\mathbf{i}_{p1,2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow -2\mathbf{i}_{p1,1} + \mathbf{i}_{p1,2} = 0$$

For $\sigma_{p3} = 40.38\,\text{MPa}$:

$$\begin{bmatrix}72.69-40.38 & 64.62\\64.62 & 169.62-40.38\end{bmatrix}\begin{bmatrix}\mathbf{i}_{p3,1}\\\mathbf{i}_{p3,2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}32.31 & 64.62\\64.62 & 129.24\end{bmatrix}\begin{bmatrix}\mathbf{i}_{p3,1}\\\mathbf{i}_{p3,2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \mathbf{i}_{p3,1} + 2\mathbf{i}_{p3,2} = 0$$

We can see that $\mathbf{i}_{p1} = \mathbf{i}_1 + 2\mathbf{i}_2$ and $\mathbf{i}_{p2} = 2\mathbf{i}_1 - \mathbf{i}_2$ will satisfy these equations, but because the principal directions are unit vectors, we need to normalize to get the principal directions:

For $\sigma_{p1} = 201.9\,\text{MPa}$:

$$\mathbf{i}_{p1} = \frac{1}{\sqrt{5}}\mathbf{i}_1 + \frac{2}{\sqrt{5}}\mathbf{i}_2.$$

For $\sigma_{p2} = 40.38\,\text{MPa}$:

$$\mathbf{i}_{p3} = \frac{2}{\sqrt{5}}\mathbf{i}_1 - \frac{1}{\sqrt{5}}\mathbf{i}_2.$$

It can be easily verified that this forms a right handed coordinate system, i.e. $\mathbf{i}_{p1} \times \mathbf{i}_{p2} = \mathbf{i}_{p3}$.

The maximum tensile stress is $\sigma_{p1} = 201.9\,\text{MPa}$.

5. Von Mises stress is

$$\begin{aligned}\sigma_{EQ} &= \left(\frac{1}{2}\left((\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right) + 3\left(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2\right)\right)^{1/2}\\ &= \left(\frac{1}{2}\left((72.69-169.62)^2 + (72.69-72.69)^2 + (169.62-72.69)^2\right) + 3\left(64.62^2\right)\right)^{1/2}\\ &= 148.06\text{ MPa}\end{aligned}$$

Since $\sigma_{EQ} = 148.06\,\text{MPa} < \sigma_y = 175\,\text{MPa}$, a ductile material *would not* yield.