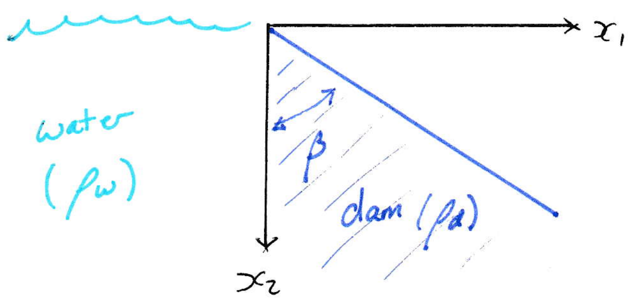
6. The axes of principal strain always coincide with the axes of principal stress.

False. The axes of principal strain always coincide with the axes of principal stresses only for isotropic materials.

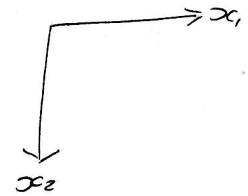
PROBLEM 5

Consider a dam represented as a wedge with two infinitely long sides (i.e., we do not consider in this problem how the dam is supported by the ground). The dam also extends infinitely in the x_3 direction. The vertical side is subjected to the pressure $\rho_w g x_2$ of water. The inclined side is traction-free. The two sides make angle β as shown. The dam is also subject to its own weight, i.e., the body force $\rho_d g$ is acting on the dam (ρ_d is the density of the dam).



1. This problem has a body force. Accounting for that fact, write down the relations between the Airy stress function and the stresses in this problem and the equation that the Airy stress function must satisfy.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_3}} + \cancelto{0}{\rho b_1} = 0$$



$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_3}} + \rho b_2 = 0$$

$$\cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_1}} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_2}} + \cancelto{0}{\frac{\partial \sigma_{33}}{\partial x_3}} + \cancelto{0}{\rho b_3} = 0$$

body force:

$$\rho \underline{b} = \begin{Bmatrix} 0 \\ \rho_d g \\ 0 \end{Bmatrix}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \rho_d g = 0$$

need a function ϕ such that

$$\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 0$$

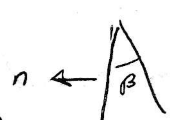
where $\quad \dfrac{\partial^2 \phi}{\partial x_2^2} = \sigma_{11}, \quad \dfrac{\partial^2 \phi}{\partial x_1^2} = \sigma_{22}, \quad -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} = \sigma_{12}$

<u>Boundary Conditions at the wall $(x_1 = 0)$:</u>

$\sigma_{12} = 0 \quad$ at $\quad x_1 = 0$

$\sigma_{11} = -\rho_w g x_2 \quad$ at $\quad x_1 = 0$

$\underline{t} = \underline{\sigma} \cdot \underline{n}$



$\rho_w g x_2 \cdot (-e_1)$

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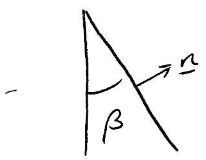
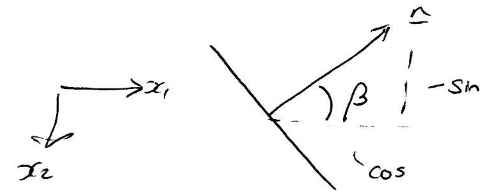
Boundary Conditions at incline:





$$\underline{n} = \begin{Bmatrix} \cos\beta \\ -\sin\beta \\ 0 \end{Bmatrix}$$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\underline{0} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{Bmatrix} \cos\beta \\ -\sin\beta \end{Bmatrix}$$

$$\sigma_{11} \cos\beta - \sigma_{12} \sin\beta = 0$$

$$\sigma_{12} \cos\beta - \sigma_{22} \sin\beta = 0$$

2. Using the Airy stress function $\phi = A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3$, find the stress distribution inside the dam.

$$\phi = A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3$$

$$\frac{\partial \phi}{\partial x_1} = 3A_1 x_1^2 + 2A_2 x_1 x_2 + A_3 x_2^2$$

$$\frac{\partial \phi}{\partial x_2} = A_2 x_1^2 + 2A_3 x_1 x_2 + 3A_4 x_2^2$$

$$\frac{\partial^2 \phi}{\partial x_1^2} = 6A_1 x_1 + 2A_2 x_2$$

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 2A_2 x_1 + 2A_3 x_2$$

$$\frac{\partial^2 \phi}{\partial x_2^2} = 2A_3 x_1 + 6A_4 x_2$$

$$\sigma_{11} = 2A_3x_1 + 6A_4x_2$$

$$\sigma_{22} = 6A_1x_1 + 2A_2x_2$$

$$\sigma_{12} = -2A_2x_1 - 2A_3x_2$$

from the BCs:

$$\sigma_{12} = 0 \quad \text{at } x_1 = 0$$

$$\sigma_{12} = 0 = -2A_3x_2 \quad \Rightarrow \quad A_3 = 0$$

$$\sigma_{11} = -\rho_w g x_2 \quad \text{at } x_1 = 0$$

$$-\rho_w g x_2 = 6A_4x_2$$

$$6A_4 = -\rho_w g$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\cancel{2A_3}^{\,0} + \cancel{(-2A_3)}^{\,0} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = -\rho_d g$$

$$-2A_2 + 2A_2 = -\rho_d g$$

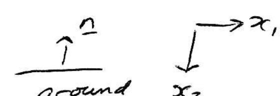
I'm missing something with the weight..

as we go down x_2, the weight of the dam increases

we have a traction vector

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\rho_d g \Rightarrow \underbrace{\rho_d g x_1}_{\sigma_{22}} \cdot (-e_2) = -\rho_d g x_1$$



density of dam:

$\downarrow \; \rho_d g x_2 x_1 \quad \leftarrow$ but this is somewhere / anywhere

$$x_2 \to \infty$$

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Wait, is this a case of $\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} + \psi \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} + \psi$?

$$\rho b_1 = -\frac{\partial \psi}{\partial x_1} \qquad \rho b_2 = -\frac{\partial \psi}{\partial x_2} = \rho_d g$$

$$\psi = -\rho_d g x_2$$

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} - \rho_d g x_2 \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} - \rho_d g x_2$$

$$\sigma_{11} = -\rho_d g x_2 + 2A_3 x_1 + 6A_4 x_2$$

$$\sigma_{22} = -\rho_d g x_2 + 6A_1 x_1 + 2A_2 x_2$$

$$\sigma_{12} = -2A_2 x_1 - 2A_3 x_2$$

from BCs:

$$\sigma_{12} = 0 \text{ at } x_1 = 0$$

$$-2A_3 x_2 = 0 \quad \Rightarrow A_3 = 0$$

$$\sigma_{11} = -\rho_w g x_2 \quad \text{at } x_1 = 0$$

$$-\rho_w g x_2 = -\rho_d g x_2 + 6A_4 x_2$$

$$\rho_d g - \rho_w g = 6A_4 \quad \Rightarrow A_4 = \frac{\rho_d g - \rho_w g}{6}$$

at the incline:

$$\sigma_{11} \cos\beta - \sigma_{12} \sin\beta = 0$$

$$\sigma_{12} \cos\beta - \sigma_{22} \sin\beta = 0$$

$$\left(\cancel{-\rho_d g x_2} + \cancel{\rho_d g x_2} - \rho_w g x_2\right)\cos\beta + 2A_2 x_1 \sin\beta = 0$$

$$-2A_2 x_1 \cos\beta + \left(\rho_d g x_2 - 6A_1 x_1 - 2A_2 x_2\right)\sin\beta = 0$$

$$A_2 = \frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta} = \frac{\rho_w g x_2}{2x_1 \tan\beta}$$

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$$-2\left(\frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta}\right) x_1 \cos\beta + \rho_d g x_2 \sin\beta - 6A_1 x_1 \sin\beta - 2\left(\frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta}\right) x_2 \sin\beta = 0$$

$$-\frac{\rho_w g x_2 \cos^2\beta}{\sin\beta} + \rho_d g x_2 \sin\beta - 6A_1 x_1 \sin\beta - \frac{\rho_w g x_2^2 \cos\beta}{x_1} = 0$$

This just doesn't look right...

The coefficients A_1, A_2, A_3, A_4 should have expressions without x_1 and x_2 ...

I keep getting $A_2 = \frac{1}{2} \rho_w g \frac{x_2}{x_1} \cot\beta$

$$A_1 = -\frac{1}{6}\left[\rho_w g \frac{x_2}{x_1} \cot^2\beta + \rho_w g \frac{x_2^2}{x_1^2} \cot\beta - \rho_d g \frac{x_2}{x_1}\right]$$

So I made a mistake somewhere