AE6114 Assignment 1: Mathematical Preliminaries

Problem 1

Expand the following indicial expressions (all indices range from 1 to 3). Indicate the rank and the number of resulting expressions.

These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering the answers from students that were marked correct. If you find any errors, please let me know!

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1. $a_i b_i$

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

rank: 0
expressions: 1

2. $a_i b_j$

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

rank: 2
expressions: 9

3. $a_i b_i c_j$

$$a_i b_i c_j = \begin{cases} (a_1 b_1 + a_2 b_2 + a_3 b_3) c_1 \\ (a_1 b_1 + a_2 b_2 + a_3 b_3) c_2 \\ (a_1 b_1 + a_2 b_2 + a_3 b_3) c_3 \end{cases}$$

rank: 1
expressions: 3

4. $\sigma_{ik} n_k$

$$\sigma_{ik} n_k = \begin{cases} \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{cases}$$

rank: 1
expressions: 3

5. $A_{ij} x_i x_j$; $\underline{A}$ is symmetric, i.e. $A_{ij} = A_{ji}$

$$A_{ij} x_i x_j = A_{11} x_1 x_1 + A_{12} x_1 x_2 + A_{13} x_1 x_3 + A_{21} x_2 x_1 + A_{22} x_2 x_2 + A_{23} x_2 x_3 + A_{31} x_3 x_1 + A_{32} x_3 x_2 + A_{33} x_3 x_3$$

$$A_{12} = A_{21}, \quad A_{13} = A_{31}, \quad A_{23} = A_{32}$$

$$\Rightarrow A_{ij}x_ix_j = A_{11}x_1^2 + 2A_{12}x_1x_2 + 2A_{13}x_1x_3 + A_{22}x_2^2 + 2A_{23}x_2x_3 + A_{33}x_3^2$$

rank: 0
expressions: 1

Problem 2

Simplify the following indicial expressions as much as possible (all indices range from 1 to 3)

1. $\delta_{mm}\delta_{nn}$

$$\delta_{mm} = \delta_{nn} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3$$

$$\Rightarrow \delta_{mm}\delta_{nn} = (\delta_{11}+\delta_{22}+\delta_{33})(\delta_{11}+\delta_{22}+\delta_{33}) = (3)(3) = 9$$

$$\boxed{\delta_{mm}\delta_{nn} = 9}$$

2. $x_i\delta_{ik}\delta_{jk}$

$$x_i\delta_{ik}\delta_{jk} = x_k\delta_{jk} = x_j$$

$$\Rightarrow \boxed{x_i\delta_{ik}\delta_{jk} = x_j}$$

3. $B_{ij}\delta_{ij}$, $\underline{\underline{B}}$ is antisymmetric, i.e., $B_{ij} = -B_{ji}$

$\delta_{ij} = 0$ if $i \neq j$, so all we have left is

$$B_{ij}\delta_{ij} = B_{ii} = B_{jj}$$

$B_{ij}\delta_{ij} = B_{11} + B_{22} + B_{33}$, but $\underline{\underline{B}}$ is antisymmetric, so

$$B_{11} = -B_{11}, \quad B_{22} = -B_{22}, \quad B_{33} = -B_{33}$$

thus $B_{11} = B_{22} = B_{33} = 0$

$$\Rightarrow \boxed{B_{ij}\delta_{ij} = 0 \text{ if } \underline{\underline{B}} \text{ is antisymmetric}}$$

4. $(A_{ij}B_{jk} - 2A_{im}B_{mk})\delta_{ik}$

$$(A_{ij}B_{jk} - 2A_{im}B_{mk})\delta_{ik} = A_{ij}\underbrace{B_{jk}\delta_{ik}}_{B_{ji}} - 2A_{im}\underbrace{B_{mk}\delta_{ik}}_{B_{mi}} = A_{ij}B_{ji} - 2A_{im}B_{mi}$$

From the initial equation, j and m are both dummy indices, which means that they can be replaced with any symbol, as long as it doesn't interfere with the indices in the same term, and the expression will remain the same.

replacing m in the second term with j :

$$A_{ij}B_{ji} - 2A_{ij}B_{ji} = -A_{ij}B_{ji}$$

$$\Rightarrow \boxed{(A_{ij}B_{jk} - 2A_{im}B_{mk})\delta_{ik} = -A_{ij}B_{ji}}$$

5. Substitute $A_{ij} = B_{ik}C_{kj}$ into $\phi = A_{mk}C_{mk}$

$$\phi = A_{mk}C_{mk}, \quad A_{mk} = B_{m\ell}C_{\ell k}$$

$$\phi = A_{mk}C_{mk} = (B_{m\ell}C_{\ell k})C_{mk}$$

$$\boxed{= B_{m\ell}C_{\ell k}C_{mk}}$$

Problem 3

Write out the following expressions in indicial notation, if possible

1. $A_{11} + A_{22} + A_{33}$

$$\boxed{A_{11} + A_{22} + A_{33} = A_{ii}}$$

2. $\underline{\underline{A}}^T\underline{\underline{A}}$, where $\underline{\underline{A}}$ is a 3x3 matrix

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{matrix} A_{11}A_{11} + A_{21}A_{21} + A_{31}A_{31} \\ A_{11}A_{12} + A_{21}A_{22} + A_{31}A_{32} \\ A_{11}A_{13} + A_{21}A_{23} + A_{31}A_{33} \end{matrix}$$

free variables

dummy variables

one dummy variable and two free variables

$$A_{ij}A_{ik} = A_{1j}A_{1k} + A_{2j}A_{2k} + A_{3j}A_{3k}$$

j and k can be varied from 1 to 3 separately

$$\boxed{\underline{\underline{A}}^T\underline{\underline{A}} = A_{ij}A_{ik}}$$

3. $A_{11}^2 + A_{22}^2 + A_{33}^2$

the only way you could write this in indicial notation is: $\sum_{i=1}^{3} A_{ii}^2$

4. $(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$

$$\left.\begin{matrix} u_i^2 v_j^2 \leftarrow \text{NO} \\ u_i^2 v_i^2 \leftarrow \text{NO} \end{matrix}\right\} \text{needs to be like Problem 1 \#1}$$

for this problem we would have to write $u_iu_iv_jv_j$ and simplify

$$\boxed{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) = u_i u_i v_j v_j}$$

5. $A_{11} = B_{11}C_{11} + B_{12}C_{21}$
$A_{12} = B_{11}C_{12} + B_{12}C_{22}$
$A_{21} = B_{21}C_{11} + B_{22}C_{21}$
$A_{22} = B_{21}C_{12} + B_{22}C_{22}$

$$A_{11} = B_{11}C_{11} + B_{12}C_{21}$$
$$A_{12} = B_{11}C_{12} + B_{12}C_{22}$$
$$A_{22} = B_{21}C_{12} + B_{22}C_{22}$$

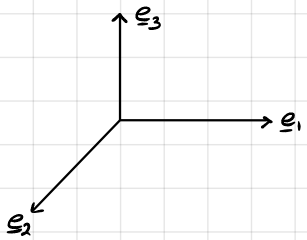
Looks similar to Problem 2 #5
$A_{ij} = B_{ik}C_{kj}$, indices go from 1 to 2

$$\boxed{A_{ij} = B_{ik}C_{kj}}$$

Problem 4

Given the right-handed orthonormal basis $\{\underline{e}_i\}$, $i = \{1,2,3\}$:

1. Show that $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$



by definition, $\|\underline{e}_1\| = \|\underline{e}_2\| = \|\underline{e}_3\| = 1$, and the three vectors are orthogonal, thus the angle between them is $\frac{\pi}{2}$

The dot product of two vectors is defined by:

$$\underline{a} \cdot \underline{b} = \|\underline{a}\|\|\underline{b}\| \cos\theta$$

Thus, if we take any two vectors in our orthonormal basis:

$$\underline{e}_1 \cdot \underline{e}_2 = \|\underline{e}_1\|\|\underline{e}_2\| \cos(\tfrac{\pi}{2}) = 0$$

and if we take the same vector:

$$\underline{e}_1 \cdot \underline{e}_1 = \|\underline{e}_1\|\|\underline{e}_1\| \cos(0) = 1$$

so we see that

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The dot product of two parallel unit vectors is 1.
The dot product of two perpendicular vectors is 0.
Given an orthonormal basis, the basis vectors are all unit vectors and are all perpendicular to each other, so we can say:

e_i, e_j are $\begin{cases} \text{parallel if } i=j \\ \text{perpendicular if } i\neq j \end{cases}$

So we can write:

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

In mathematics, we define the Kronecker delta as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

and so we can simplify our expression to

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \quad \checkmark$$

2. Using the previous result and indicial notation, show that given $\underline{a} = a_i \underline{e}_i$ and $\underline{b} = b_i \underline{e}_i$, their dot product can be expressed as $\underline{a} \cdot \underline{b} = a_i b_i$

$$\underline{a} \cdot \underline{b} = a_i \underline{e}_i \cdot b_j \underline{e}_j = a_i b_j \underline{e}_i \cdot \underline{e}_j \;, \quad \underline{e}_i \cdot \underline{e}_j = \delta_{ij} \text{ from the previous result}$$

$$= a_i b_j \delta_{ij} = a_i b_i \quad \checkmark$$

3. Show that $\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k$

The cross product of two vectors is defined by

$$\underline{a} \times \underline{b} = \|\underline{a}\| \|\underline{b}\| \sin\theta \, \underline{c}$$

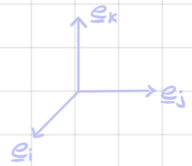
where $\underline{c}$ is a vector that is orthogonal to both $\underline{a}$ and $\underline{b}$ ($\underline{c}$ is also a unit vector)

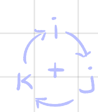
By convention, the direction of $\underline{c}$ is given by the right-hand rule

$\Rightarrow$ the cross product is anticommutative: $\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$

Given that we have a right-handed, orthonormal basis $\{\underline{e}_i\}$, $i = \{1, 2, 3\}$ we can use the definition of the cross product to see that

$$\begin{aligned} \underline{e}_i \times \underline{e}_j &= \underline{e}_k & \qquad \underline{e}_k \times \underline{e}_j &= -\underline{e}_i \\ \underline{e}_j \times \underline{e}_k &= \underline{e}_i & \qquad \underline{e}_j \times \underline{e}_i &= -\underline{e}_k \\ \underline{e}_k \times \underline{e}_i &= \underline{e}_j & \qquad \underline{e}_i \times \underline{e}_k &= -\underline{e}_j \end{aligned}$$





+ if $(i,j,k) = (1,2,3)$
$(2,3,1)$
$(3,1,2)$

as well as

$$\underline{e}_i \times \underline{e}_i = \underline{e}_j \times \underline{e}_j = \underline{e}_k \times \underline{e}_k = 0$$

− if $(i,j,k) = (3,2,1)$
$(2,1,3)$
$(1,3,2)$

The permutation symbol (Levi-Civita symbol) is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1),\ \text{or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3),\ \text{or } (1,3,2) \\ 0 & \text{if } i=j,\ j=k,\ \text{or } k=i \end{cases}$$

Putting these two definitions together, we can simplify our answer to say that

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\underline{e}_k \quad \checkmark$$

4. Using the previous result and indicial notation, show that given $\underline{a} = a_i\underline{e}_i$ and $\underline{b} = b_i\underline{e}_i$, their cross product can be expressed as $\underline{a}\times\underline{b} = \epsilon_{ijk}a_ib_j\underline{e}_k$

$$\underline{a}\times\underline{b} = a_i\underline{e}_i \times b_j\underline{e}_j = a_ib_j(\underline{e}_i\times\underline{e}_j) = a_ib_j\epsilon_{ijk}\underline{e}_k \quad \checkmark$$

$$\underline{a}\times\underline{b} = \epsilon_{ijk}a_ib_j\underline{e}_k$$

5. Using previous results and indicial notation, show that given $\underline{a} = a_i\underline{e}_i$, $\underline{b} = b_i\underline{e}_i$, and $\underline{c} = c_i\underline{e}_i$, their triple product can be expressed as $(\underline{a}\times\underline{b})\cdot\underline{c} = \epsilon_{ijk}a_ib_jc_k$

NOTE:

$$(\underline{a}\times\underline{b}) = a_1b_1\epsilon_{111}\underline{e}_1 + a_1b_1\epsilon_{112}\underline{e}_2 + \cdots$$

this is a vector!

$$(\underline{a}\times\underline{b})\cdot\underline{c} = \epsilon_{ijk}a_ib_jc_k$$

this is a scalar!

$(a_ib_j\epsilon_{ijk}\underline{e}_k)\cdot c_k\underline{e}_k$ ✗ $\qquad$ $(a_ib_j\epsilon_{ijk}\underline{e}_k)\cdot c_m\underline{e}_m$ $\checkmark$

too many k's!! $\qquad$ dummy variable so we can choose anything

$$(\underline{a}\times\underline{b})\cdot\underline{c} = (a_i\underline{e}_i \times b_j\underline{e}_j)\cdot c_k\underline{e}_k = [a_ib_j(\underline{e}_i\times\underline{e}_j)]\cdot c_k\underline{e}_k$$

$$= (a_ib_j\epsilon_{ijm}\underline{e}_m)\cdot c_k\underline{e}_k = a_ib_jc_k\epsilon_{ijm}\underbrace{(\underline{e}_m\cdot\underline{e}_k)}_{\delta_{mk}}$$

$$= a_ib_jc_k\epsilon_{ijm}\delta_{mk} = a_ib_jc_k\epsilon_{ijk} \quad \checkmark$$

6. Show that the permutation symbol and the Kroneker delta are related through the expression $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

One way to do this problem:

The permutation symbol and the Kronecker delta are related through the following relationship:

$$\epsilon_{ijk}\epsilon_{lmn} = \det\begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

expanding the determinant:

$$\epsilon_{ijk}\epsilon_{\ell mn} = \delta_{i\ell}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}) + \delta_{in}(\delta_{j\ell}\delta_{km} - \delta_{k\ell}\delta_{jm})$$

if we let $\ell = i$:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}) + \delta_{in}(\delta_{ji}\delta_{km} - \delta_{ki}\delta_{jm})$$

$$= 3(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}\delta_{ji}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{in}\delta_{ki}\delta_{jm}$$

$$= \underline{3\delta_{jm}\delta_{kn}} - \underline{3\delta_{jn}\delta_{km}} - \underline{\delta_{jm}\delta_{kn}} + \underline{\delta_{km}\delta_{jn}} + \underline{\delta_{jn}\delta_{km} - \delta_{kn}\delta_{jm}}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad \checkmark$$

Another way to do this problem:

We can use the definition of the triple vector product:

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

Looking at the left-hand side:

$$\underline{b} \times \underline{c} = b_j \underline{e}_j \times c_k \underline{e}_k = \epsilon_{jki} b_j c_k \underline{e}_i$$

let $d_i \underline{e}_i = \epsilon_{jki} b_j c_k \underline{e}_i$, then $\underline{a} \times \underline{d} = a_n \underline{e}_n \times d_i \underline{e}_i = \epsilon_{nim} a_n d_i \underline{e}_m$

plugging in for $\underline{d}$:

$$\underline{a} \times (\underline{b} \times \underline{c}) = \epsilon_{nim} a_n (\epsilon_{jki} b_j c_k) \underline{e}_m$$

$$= \epsilon_{ijk}\epsilon_{imn} a_n b_j c_k \underline{e}_m$$

note that $\epsilon_{nim} = \epsilon_{imn} = \epsilon_{mni}$
$\epsilon_{jki} = \epsilon_{kij} = \epsilon_{ijk}$

Looking at the right-hand side:

$$(\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} = (a_i \underline{e}_i \cdot c_k \underline{e}_k) b_j \underline{e}_j - (a_i \underline{e}_i \cdot b_j \underline{e}_j) c_k \underline{e}_k$$

$$= (a_i c_k \delta_{ik}) b_j \underline{e}_j - (a_i b_j \delta_{ij}) c_k \underline{e}_k$$

$$= (a_i c_i) b_j \underline{e}_j - (a_i b_i) c_k \underline{e}_k$$

Let's change the indices so that they match what we got on the left-hand side:

$$(a_n c_n) b_m \underline{e}_m - (a_n b_n) c_k \underline{e}_k$$

$$a_n c_n b_m \underline{e}_m = \delta_{nk}\delta_{mj} a_n c_k b_j \underline{e}_m \qquad a_n b_n c_k \underline{e}_k = \delta_{nj}\delta_{km} a_n b_j c_m \underline{e}_k$$

$\delta_{nk}\delta_{mj}a_n b_j c_k \underline{e}_m - \delta_{nj}\delta_{km}a_n b_j c_m \underline{e}_k$

$\delta_{ij} = \delta_{ji}$

$\delta_{jm}\delta_{km}a_n b_j c_k \underline{e}_m - \delta_{jn}\delta_{km}a_n b_j c_k \underline{e}_m$

$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_n b_j c_k \underline{e}_m$

Putting everything together:

$$\underline{a} \times (\underline{b} \times \underline{c}) = \epsilon_{ijk}\epsilon_{imn}a_n b_j c_k \underline{e}_m = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_n b_j c_k \underline{e}_m = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$$

$$\Rightarrow \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad \checkmark$$

Problem 5

Considering that for a second order tensor $\underline{A} = A_{ij}\underline{e}_i\underline{e}_j$ the partial derivatives with respect to its components are given by $\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}$

1. Show that $\frac{\partial Tr(\underline{A})}{\partial A_{ij}} = \delta_{ij}$

$$\frac{\partial Tr(\underline{A})}{\partial A_{ij}} \rightarrow Tr(\underline{A}) = A_{kk} \rightarrow \frac{\partial A_{kk}}{\partial A_{ij}} = \delta_{ki}\delta_{kj} = \delta_{ij} \quad \checkmark$$

you could also say:

$$\frac{\partial Tr(\underline{A})}{\partial A_{ij}} = \underbrace{\frac{\partial A_{11}}{\partial A_{ij}} + \frac{\partial A_{22}}{\partial A_{ij}} + \frac{\partial A_{33}}{\partial A_{ij}}}$$

if $i \neq j$, all of these terms $= 0$

if $i = j$, one of these terms will $= 1$, the rest $= 0$

so $\frac{\partial Tr(\underline{A})}{\partial A_{ij}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ which is the definition of δ_{ij}

$$\Rightarrow \frac{\partial Tr(\underline{A})}{\partial A_{ij}} = \delta_{ij} \quad \checkmark$$

2. Show that $\frac{\partial Tr(\underline{A}\cdot\underline{A})}{\partial A_{ij}} = 2(A^T)_{ij}$

$$\frac{\partial Tr(\underline{A}\cdot\underline{A})}{\partial A_{ij}} = \frac{\partial (A_{mn}A_{nm})}{\partial A_{ij}} = \frac{\partial A_{mn}}{\partial A_{ij}}A_{nm} + A_{mn}\frac{\partial A_{nm}}{\partial A_{ij}}$$

$$= \delta_{mi}\delta_{nj}A_{nm} + A_{mn}\delta_{ni}\delta_{mj}$$

$$= \delta_{mi}A_{jm} + A_{mi}\delta_{mj}$$

$$= A_{ji} + A_{ji} = 2A_{ji} = 2(A^T)_{ij} \quad \checkmark$$

Note:

$\underline{A} : \underline{A} = A_{ij}A_{ji} \leftarrow$ scalar

$\underline{A} \cdot\cdot \underline{A} = A_{ij}A_{ij}$

3. Prove that $\frac{\partial (A^{-1})_{kl}}{\partial A_{ij}} = -(A^{-1})_{ki}(A^{-1})_{jl}$

The identity is: $A^{-1}_{km} A_{mn} = \delta_{kn}$

Taking the derivative of both sides:

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} + A^{-1}_{km} \frac{\partial A_{mn}}{\partial A_{ij}} = 0$$

using $\frac{\partial A_{mn}}{\partial A_{ij}} = \delta_{mi}\delta_{nj}$:

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} + \underbrace{A^{-1}_{km} \delta_{mi}\delta_{nj}}_{\text{move to other side}} = 0$$

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} = -A^{-1}_{km} \delta_{mi}\delta_{nj} = -A^{-1}_{ki}\delta_{nj}$$

multiply both sides by A^{-1}_{nl} :

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} \underbrace{A_{mn} A^{-1}_{nl}}_{\delta_{ml}} = -A^{-1}_{ki}\delta_{nj} A^{-1}_{nl}$$

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} \delta_{ml} = -A^{-1}_{ki} A^{-1}_{jl}$$

$$\frac{\partial (A^{-1})_{kl}}{\partial A_{ij}} = -(A^{-1})_{ki}(A^{-1})_{jl} \quad \checkmark$$