3D Euler-Bernoulli Beam Bending Summary

1) Displacement

$$\left\{\begin{aligned} u_1 &= \bar{u}_1(x_1) - x_3 \frac{d u_3}{d x_1} - x_2 \frac{d u_2}{d x_1} \\ u_2 &= \bar{u}_2(x_1) \\ u_3 &= \bar{u}_3(x_1) \end{aligned}\right.$$

2) Kinematics

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) + x_3 \kappa_2(x_1) - x_2 \kappa_3(x_1)$$

$$\bar{\epsilon}_1 = \frac{d\bar{u}_1}{dx_1}, \quad \kappa_2 = -\frac{d^2 u_3}{dx_1}, \quad \kappa_3 = \frac{d^2 u_2}{dx_1^2}$$

3) Constitutive

$$\sigma_1 = E\,\epsilon_1$$

$$\sigma_1 = E\left(\bar{\epsilon}_1 + x_3 \kappa_2 - x_2 \kappa_3\right)$$

4) SECTIONAL CONSTITUTIVE EQUATIONS ABOUT THE CENTROID

$$N_1 = S\,\bar{\epsilon}_1$$

$$M_2 = H_{22}^{c}\,K_2 - H_{23}^{c}\,K_3$$

$$M_3 = -H_{23}^{c}\,K_2 + H_{33}^{c}\,K_3$$

$$x_{2c} = S_2/S = 0$$

$$x_{3c} = S_3/S = 0$$

$$\begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & H_{22}^{c} & -H_{23}^{c} \\ 0 & -H_{23}^{c} & H_{33}^{c} \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_1 \\ K_2 \\ K_3 \end{bmatrix}$$

$$\begin{vmatrix} \bar{\epsilon}_1 \\ K_2 \\ K_3 \end{vmatrix} = \begin{bmatrix} 1/S & 0 & 0 \\ 0 & H_{33}^{c}/\Delta H & H_{23}^{c}/\Delta H \\ 0 & H_{23}^{c}/\Delta H & H_{22}^{c}/\Delta H \end{bmatrix} \begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$\Delta H = H_{22}^{c} H_{33}^{c} - H_{23}^{c} H_{23}^{c}$$

$$\sigma_1 = E\left(\frac{N_1}{S} + x_3 \frac{H_{33}^c M_2 + H_{23}^c M_3}{\Delta H} - x_2 \frac{H_{23}^c M_2 + H_{22}^c M_3}{\Delta H}\right)$$

$$S = \int_A E\,dA, \quad S_2 = \int_A E x_2\,dA, \quad S_3 = \int_A E x_3\,dA$$

$$H_{22}^c = \int_A E x_3^2\,dA, \quad H_{33}^c = \int_A E x_2^2\,dA$$

$$H_{23}^c = \int_A E x_2 x_3\,dA$$

5) EQUILIBRIUM EQUATIONS

$$\frac{dN_1}{dx_1} = -p_1(x_1)$$

DISTRIBUTED AXIAL LOAD

$$\frac{dV_2}{dx_1} = -p_2(x_1)$$

$$\frac{dV_3}{dx_1} = -p_3(x_1)$$

TRANSVERSE DISTRIBUTED LOADS

$$\frac{dM_3}{dx_1} + V_2 = -q_3(x_1) + x_{2A}\, p_1(x_1)$$

$$\frac{dM_2}{dx_1} - V_3 = -q_2(x_1) - x_{3A}\, p_1(x_1)$$

DISTRIBUTED MOMENT ABOUT $\bar{\imath}_2$

DISTANCE BETWEEN $p_1(x_1)$ AND CENTROID.

6) GOVERNING PDEs

$$\frac{d}{dx_1}\left(S \frac{d\bar{u}_1}{dx_1} \right) = -p_1(x_1)$$

$$\frac{d^2}{dx_1^2}\left(H_{33}^C \frac{d^2 u_2}{dx_1^2} + H_{23}^C \frac{d^2 u_3}{dx_1^2} \right)$$

$$= p_2(x_1) + \frac{d}{dx_1}\left(x_{2A}\, p_1(x_1) - q_3(x_1) \right)$$

$$\frac{d^2}{dx_1^2}\left(H_{23}^C \frac{d^2 u_2}{dx_1^2} + H_{22}^C \frac{d^2 u_3}{dx_1^2} \right)$$

$$= p_3(x_1) + \frac{d}{dx_1}\left(x_{3A}\, p_1(x_1) + q_2(x_1) \right)$$

SOLUTION PROCESS

1) DETERMINE THE CENTROID

2) SELECT AXIS $\mathcal{I} = \{ \bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3 \}$ WITH $\bar{\imath}_1$ ALONG THE SECTIONAL CENTROID

3) COMPUTE STIFFNESSES

4) DETERMINE APPROPRIATE B.C.'s

A) STATICALLY DETERMINATE

5) COMPUTE $N_1(x_1)$, $M_2(x_1)$, $M_3(x_1)$

6) SOLVE USING THE SECTIONAL CONSTITUTIVE EQUATIONS & B.C.s

7) Find $\sigma_1(x_1, x_2, x_3)$ FROM N_1, M_2, M_3

B) STATICALCY INDETERMINATE

5) SOLVE USING GOVERNING PDES FOR $\bar{u}_1, u_2, u_3$

6) SOLVE FOR $\bar{\epsilon}_1, \kappa_2, \kappa_3$ AND SOLVE FOR $\sigma_1(x_1, x_2, x_3)$

7) SOLVE FOR N_1, M_2, M_3 USING SECTIONAL CONSTITUTIVE EQUATIONS.

* Alternatively, we may turn the problem into a statically determinate one by removing constraints.