F/20

# Tensors

$$a_i \underbrace{B_{jk} c_k}_{\text{dummy}} + d_i f_j + A_{ij} + C_{ji}$$

i, j free

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## Kronecker delta

defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

(this is Rank 2)

### Notes

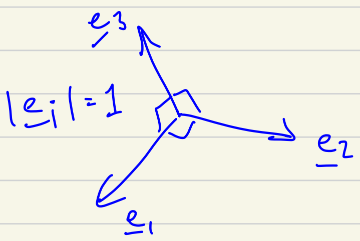
* In matrix form, $\delta_{ij}$ are the entries of the identity matrix

Basis vectors: orthogonal & orthonormal

↓ unit vectors, orthogonal

* basis $\underline{e}_i$ dotted w/ $\underline{e}_j$ gives k delta

ie: 1 if $i=j$, 0 if $i\neq j$



* Also appears when differentiating a tensor w/ its component

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

An important property of $\delta_{ij}$ is <u>index substitution</u>

(sum over i)

$$a_i \delta_{ij} = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j} \begin{cases} a_1 & \text{if } j=1 \\ a_2 & \text{if } j=2 \\ a_3 & \text{if } j=3 \end{cases}$$

$$\Rightarrow \boxed{a_i \delta_{ij} = a_j} \quad *$$

Examples (n=3)

$$\sum_{j=1}^{3} A_{jj}$$

- $A_{ij}\delta_{ij} = A_{ii} = A_{jj} = A_{11} + A_{22} + A_{33}$

as a whole, Rank = 0

- $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

Rank 0 !
(no free indices)

- $A_{ij} - A_{ik}\delta_{jk} = A_{ij} - A_{ij}$

(2 free indices, 1 dummy k) → Rank 2

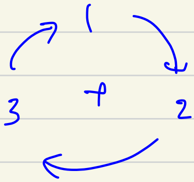
MISSED

## Permutation Symbol for $n = 3$ is defined

as:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ form an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ form an odd " ——— "} \\ 0 & \text{if } i, j, k \text{ are not a permutation of } 1, 2, 3 \end{cases}$$

**Rule**



ex:

1, 2, 3 ⊕

2, 3, 1 ⊕

3, 1, 2 ⊕

1, 3, 2 ⊖

⋮

Thus, $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1;$

$\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1;$

all other are zero.

# Operations with Tensors

**Addition** : Can only add tensors of same rank.

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}$$ (direct notation)

$$C_{ij} = A_{ij} + B_{ij}$$ (indicial)

In matrix notation :

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{bmatrix}$$

**Magnification** (multiplication by scalar)

$$\underline{\underline{B}} = \lambda \underline{\underline{A}} \implies B_{ij} = \lambda A_{ij}$$

In matrix notation:

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} \end{bmatrix}$$

## Transpose :

$$\underline{\underline{B}} = \underline{\underline{A}}^T \iff B_{ij} = A_{ji}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{21} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

## Tensor (dyadic) product

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \iff A_{ij} = a_i b_j$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

## Contracted Multiplication

Rank 2 tensor by rank 1 tensor :

$$\underline{u} = \underline{\underline{A}} \cdot \underline{v} \iff u_i = A_{ij} v_j$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} A_{11} v_1 + A_{12} v_2 + A_{13} v_3 \\ A_{21} v_1 + A_{22} v_2 + A_{23} v_3 \\ A_{31} v_1 + A_{32} v_2 + A_{33} v_3 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

(cont)

## Contracted Multiplication

Between Rank 2 tensors

$$\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}} \iff C_{ij} = A_{ik} B_{kj}$$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Trace : $tr(\underline{\underline{A}}) = A_{ii}$

$$tr(\underline{\underline{A}}) = A_{11} + A_{22} + A_{33}$$

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$$Tr(B_{ij}) = B_{ii}$$

$$Tr(A_{ij} A_{jk}) = A_{ij} A_{ji}$$