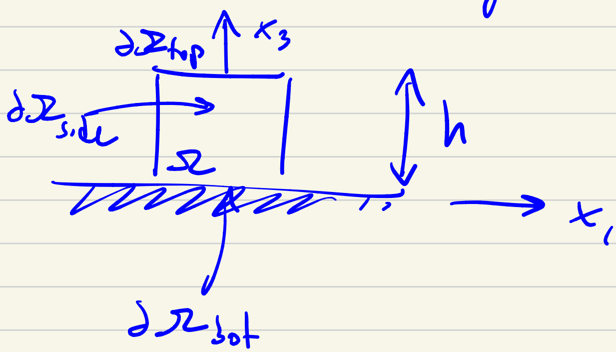
10/27 <u>Example of problem formulation</u>

- Consider a cylindrical body glued to a rigid surface and deforming under its own weight



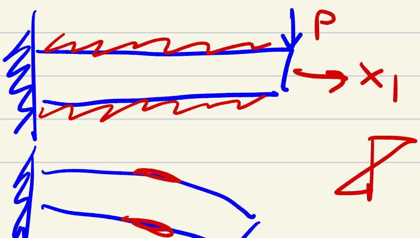
- The only body force in this problem is gravity:

$$\rho \underline{b} = \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix}$$

- Displacement boundary conditions (BCs) are applied at $\partial\Omega_{bot}$

$$\underline{u}^* = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at } x_3 = 0$$

- Traction BCs are applied at $\partial\Omega^{top}$ & $\partial\Omega^{side}$



$$\underline{\underline{\sigma}} \cdot \underline{n} = t^*$$

specify t^*, select $\underline{n}$, implies $\underline{\underline{\sigma}}$

Under these conditions, we can formulate the boundary value problem:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} - \rho g = 0$$

Balance of lin momentum

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

Linear elastic assumptions (isotropic) lead to:

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{11} \; ; \; \sigma_{12} = 2\mu \varepsilon_{12}$$

$$\sigma_{13} = 2\mu \varepsilon_{13}$$

$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{22} \; ; \; \sigma_{23} = 2\mu \varepsilon_{23}$$

$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{33}$$

strain - displ. relations:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \; ; \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \qquad \varepsilon_{21} = \varepsilon_{12}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

Finally, the BCs are:

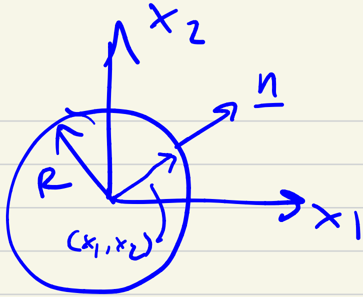
$$\boxed{u_1 = u_2 = u_3 = 0} \quad \forall\, \underline{x} \in \partial\Omega_{bot} \; (\text{for } \underline{x_3 = 0})$$

$$\boxed{\underline{\underline{\sigma}} \cdot \underline{e}_3 = \underline{0}} \quad \forall\, \underline{x} \in \partial\Omega^{top} \; (\text{for } \underline{x_3 = h})$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

traction on top surface $= 0$

$$\left.\begin{aligned} \sigma_{13} &= 0 \\ \sigma_{23} &= 0 \\ \sigma_{33} &= 0 \end{aligned}\right\} (x_3 = h)$$



$\partial\Omega_{side}: x_1^2 + x_2^2 = R^2$

$$\underline{n} = \begin{Bmatrix} x_1 \\ x_2 \\ 0 \end{Bmatrix} \cdot \frac{1}{\sqrt{x_1^2 + x_2^2}}$$

on $\partial\Omega_{side}$.

$$\underline{n} = \frac{1}{R}\begin{Bmatrix} x_1 \\ x_2 \\ 0 \end{Bmatrix} \text{ on } \partial\Omega_{side}$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0};$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} x_1/R \\ x_2/R \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left.\begin{aligned} \frac{\sigma_{11}x_1}{R} + \frac{\sigma_{12}x_2}{R} &= 0 \\ \frac{\sigma_{12}x_1}{R} + \frac{\sigma_{22}x_2}{R} &= 0 \\ \frac{\sigma_{13}x_1}{R} + \frac{\sigma_{23}x_2}{R} &= 0 \end{aligned}\right\} \text{ on } x_1^2 + x_2^2 = R^2$$