Euler-Bernoulli Beam Theory

AE3140: Structural Analysis

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Contents

Overview

- A beam is defined as a structure having one of its dimensions much larger than the other two.
- The axis of the beam is defined along that longer dimension,
- A cross-section normal to this axis is assumed to smoothly vary along the span or length of the beam.

Overview

- The solid mechanics theory of beams is commonly referred to simply as "beam theory,"
- The theory plays an important role in structural analysis because it provides the designer with a simple tool to analyze numerous structures.
- Several beam theories have been developed based on various assumptions, and lead to different levels of accuracy.
- One of the simplest and most useful of these theories was first described by Euler and Bernoulli and is commonly called Euler-Bernoulli beam theory.

Euler-Bernoulli beam Theory: Assumptions

Assumption 1: The cross-section is infinitely rigid in its own plane.

Assumption 2: The cross-section of a beam remains plane after deformation.

Assumption 3: The cross-section remains normal to the deformed axis of the beam.

Note

Experimental measurements show that these assumptions are valid for long, slender beams made of isotropic materials with solid cross-sections. When one or more of theses conditions are not met, the predictions of Euler-Bernoulli beam theory can become inaccurate. The mathematical and physical implications of the Euler-Bernoulli assumptions will now be discussed in detail.

Reference frame and notation

- Consider a triad $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ with coordinates x_1, x_2, and x_3.
- This set of axes is attached at a point of the beam cross-section; $\bar{\imath}_1$ is along the axis of the beam and $\bar{\imath}_2$ and $\bar{\imath}_3$ define the plane of the cross-section.
- Let $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$ be the displacement of an arbitrary point of the beam along directions $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively.

Assumption 1

The cross-section is un-deformable in its own plane.
Hence, the displacement field in the plane of the cross-section consists solely of two rigid body translations $\bar{u}_2(x_1)$ and $\bar{u}_3(x_1)$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \quad u_3(x_1, x_2, x_3) = \bar{u}_3(x_1). \tag{1}$$

Assumption 2

The cross-section remains plane after deformation.
This implies an axial displacement field consisting of a rigid body translation $\bar{u}_1(x_1)$, and two rigid body rotations $\Phi_2(x_1)$ and $\Phi_3(x_1)$,

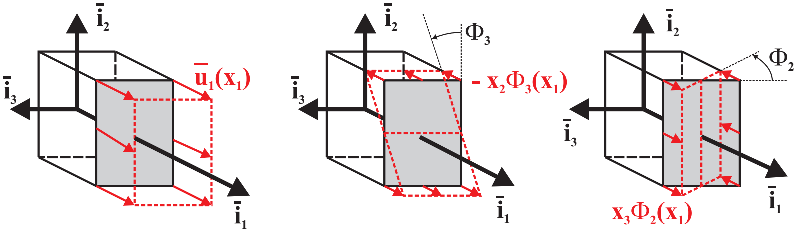

Figure: Decomposition of the axial displacement field.

The axial displacement is then

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) + x_3\Phi_2(x_1) - x_2\Phi_3(x_1), \tag{2}$$

where the location of the origin for the axis system on the cross-section is as yet undetermined.

Sign Conventions

- the rigid body translations of the cross-section $\bar{u}_1(x_1)$, $\bar{u}_2(x_1)$, and $\bar{u}_3(x_1)$ are positive in the direction of the axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$,
- the rigid body rotations of the cross-section, $\Phi_2(x_1)$ and $\Phi_3(x_1)$, are positive about axes $\bar{\imath}_2$ and $\bar{\imath}_3$,

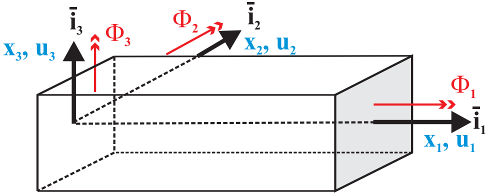

Figure: Sign convention for the displacements and rotations of a beam.

Assumption 3

The cross-section remains normal to the deformed axis of the beam.

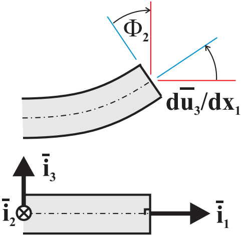

Figure: Beam slope and cross-sectional rotation.

This implies the equality of the slope of the beam and of the rotation of the section,

$$\Phi_3 = \frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1}, \quad \Phi_2 = -\frac{\mathrm{d}\bar{u}_3}{\mathrm{d}x_1}. \tag{3}$$

The minus sign in the second equation is a consequence of the sign convention for the sectional displacements and rotations.

Displacement field

The complete displacement field for Euler-Bernoulli beams is

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - x_3 \, \frac{\mathrm{d}\bar{u}_3(x_1)}{\mathrm{d}x_1} - x_2 \, \frac{\mathrm{d}\bar{u}_2(x_1)}{\mathrm{d}x_1}, \tag{4a}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \tag{4b}$$

$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1). \tag{4c}$$

This important simplification results from the Euler-Bernoulli assumptions and allows the development of a one-dimensional beam theory, *i.e.*, a theory in which the unknown displacements are functions of the span-wise coordinate, x_1, alone.

Strain field

The strain field can be evaluated from the displacement field :

$$\epsilon_2 = \frac{\partial u_2}{\partial x_2} = 0; \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3} = 0, \quad \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0, \tag{5a}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0, \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0, \tag{5b}$$

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = \frac{\mathrm{d}\bar{u}_1(x_1)}{\mathrm{d}x_1} - x_3 \frac{\mathrm{d}^2\bar{u}_3(x_1)}{\mathrm{d}x_1^2} - x_2 \frac{\mathrm{d}^2\bar{u}_2(x_1)}{\mathrm{d}x_1^2}. \tag{5c}$$

Sectional strains

At this point, it is convenient to introduce the following notation for the sectional deformations, which depend solely on the span-wise variable, x_1,

$$\bar{\epsilon}_1(x_1) = \frac{\mathrm{d}\bar{u}_1(x_1)}{\mathrm{d}x_1}, \quad \kappa_2(x_1) = -\frac{\mathrm{d}^2\bar{u}_3(x_1)}{\mathrm{d}x_1^2}, \quad \kappa_3(x_1) = \frac{\mathrm{d}^2\bar{u}_2(x_1)}{\mathrm{d}x_1^2}. \tag{6}$$

where $\bar{\epsilon}_1$ is the sectional axial strain, and κ_2 and κ_3 are the sectional curvature about the $\bar{\imath}_2$ and $\bar{\imath}_3$ axes, respectively.
The axial strain distribution over the cross-section, eq. (5c), becomes

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) + x_3\kappa_2(x_1) - x_2\kappa_3(x_1). \tag{7}$$

Notes

- The vanishing of the in-plane strain field implied by eqs. (5a) is a direct consequence of assuming the cross-section to be infinitely rigid in its own plane (assumption 1).
- The vanishing of the transverse shearing strain field implied by eqs. (5b) is a direct consequence of assuming the cross-section to remain normal to the deformed axis of the beam (assumption 3).
- the linear distribution of axial strains over the cross-section expressed by eq. (7) is a direct consequence of assuming the cross-section to remain plane (assumption 2).

Stress resultants: force resultants

Three force resultants are defined: the axial force, $N_1(x_1)$, acting along axis $\bar{\imath}_1$ of the beam, and the transverse shearing forces, $V_2(x_1)$ and $V_3(x_1)$, acting along axes $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively:

$$N_1(x_1) = \int_{\mathcal{A}} \sigma_1(x_1, x_2, x_3) \, \mathrm{d}\mathcal{A}. \tag{8}$$

$$V_2(x_1) = \int_{\mathcal{A}} \tau_{12}(x_1, x_2, x_3) \, \mathrm{d}\mathcal{A}, \quad V_3(x_1) = \int_{\mathcal{A}} \tau_{13}(x_1, x_2, x_3) \, \mathrm{d}\mathcal{A}, \tag{9}$$

where $\mathcal{A}$ is the cross-sectional area of the beam.

Stress resultants: moment resultants

Next, two moment resultants are defined: the bending moments, $M_2(x_1)$ and $M_3(x_1)$, acting about axes $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, defined as

$$M_2(x_1) = \int_{\mathcal{A}} x_3 \ \sigma_1(x_1, x_2, x_3) \ \mathrm{d}\mathcal{A}, \tag{10a}$$

$$M_3(x_1) = -\int_{\mathcal{A}} x_2 \ \sigma_1(x_1, x_2, x_3) \ \mathrm{d}\mathcal{A}. \tag{10b}$$

Sign conventions

Note: the minus sign in the definition of $M_3(x_1)$ is necessary to give a positive equipollent bending moment about axis $\bar{\imath}_3$.

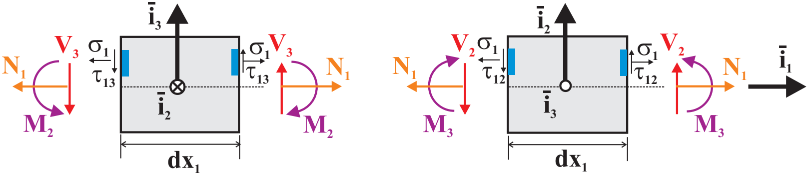

Figure: Sign convention for the sectional stress resultants.

Beams subjected to axial loads

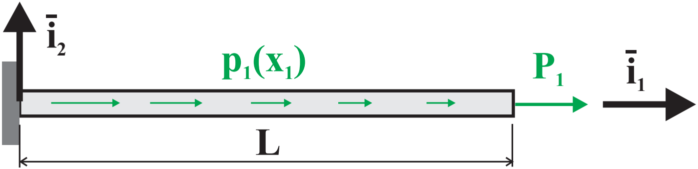

Figure: Beam subjected to axial loads.

Kinematic description

Assume that axial loads cause only axial displacement of the section. The general displacement field described by eq. (4) then reduces to

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1), \tag{11a}$$

$$u_2(x_1, x_2, x_3) = 0, \tag{11b}$$

$$u_3(x_1, x_2, x_3) = 0, \tag{11c}$$

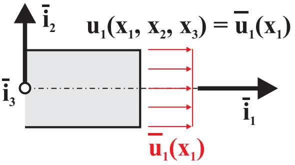

Figure: Axial displacement distribution.

the corresponding axial strain field is

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1). \tag{12}$$

The axial strain is uniform over the cross-section of the beam.

Sectional constitutive

Assumptions:

- The beam is assumed to be made of a linearly elastic, isotropic material that obeys Hooke's law
- The stresses acting in the plane of the cross-section, σ_2 and σ_3, are much smaller than the axial stress component, σ_1: $\sigma_2 \ll \sigma_1$ and $\sigma_3 \ll \sigma_1$
- Consequently: $\sigma_2 \approx 0$ and $\sigma_3 \approx 0$.

The generalized Hooke's law reduces to:

$$\sigma_1(x_1, x_2, x_3) = E\ \epsilon_1(x_1, x_2, x_3). \tag{13}$$

Notes

- When describing the beam's kinematics, it is assumed that the cross-section does not deform in its own plane, and the strains in the plane of the cross-section vanish.
- The transverse stress components are also assumed to vanish. This is an inconsistency in Euler-Bernoulli beam theory that uses two contradictory assumptions
- In view of Hooke's law, these two sets of quantities cannot vanish simultaneously. Indeed, if $\sigma_2 = \sigma_3 = 0$, from Hooke's law results in $\epsilon_2 = -\nu\sigma_1/E$ and $\epsilon_3 = -\nu\sigma_1/E$, which implies that the in-plane strains do not vanish due to Poisson's effect.
- Because this effect is very small, assuming the vanishing of these in-plane strain components when describing the beam's kinematics does not cause significant errors for most problems.

The axial stress distribution over the cross-section is:

$$\sigma_1(x_1, x_2, x_3) = E\ \bar{\epsilon}_1(x_1). \tag{14}$$

The axial force in the beam is given by

$$N_1(x_1) = \int_{\mathcal{A}} \sigma_1(x_1, x_2, x_3)\ \mathrm{d}\mathcal{A} = \left[\int_{\mathcal{A}} E\ \mathrm{d}\mathcal{A}\right] \bar{\epsilon}_1(x_1) = S\ \bar{\epsilon_1}(x_1). \tag{15}$$

Since the sectional axial strain $\bar{\epsilon}_1(x_1)$ varies only along the span of the beam, it can be factored out of the integral over the section. The *axial stiffness*, S, of the beam is then defined as

$$S = \int_{\mathcal{A}} E\ \mathrm{d}\mathcal{A}. \tag{16}$$

If the section is made of a homogeneous material of Young's modulus E, the axial stiffness of the section becomes $S = E\ \int_{\mathcal{A}} \mathrm{d}\mathcal{A} = E\mathcal{A}$.

Note

Relationship (15) is the constitutive law for the axial behavior of the beam. It expresses the proportionality between the axial force and the sectional axial strain, with a constant of proportionality called the axial stiffness. This constitutive law is written at the sectional level, whereas Hooke's law, eq. (13), is written at the local, infinitesimal level.

Equilibrium equations

The equilibrium of an infinitesimal slice of the beam of length $\mathrm{d}x_1$ is considered:

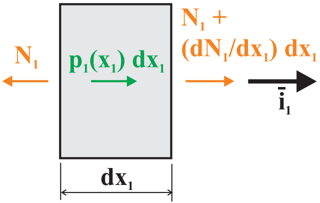

Figure: Axial forces acting on an infinitesimal slice of the beam.

Equilibrium equation

$$\frac{\mathrm{d}N_1}{\mathrm{d}x_1} = -p_1. \tag{17}$$

Governing equations

T he governing equation of the problem is found by introducing the axial force, eq. (15), into the equilibrium, eq. (17), and recalling the definition of the sectional axial strain, eq. (6),

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[S\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1}\right] = -p_1(x_1). \tag{18}$$

This second order differential equation can be solved for the axial displacement field, $\bar{u}_1(x_1)$, given the axial load distribution, $p_1(x_1)$.

Two boundary conditions are required for the solution of eq. (18), one at each end of the beam. Typical boundary conditions are:

1. A fixed (or clamped) end allows no axial displacement, *i.e.*,

$$\bar{u}_1 = 0;$$

2. A free (unloaded) end corresponds to $N_1 = 0$; using eq. (15), then leads to

$$\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} = 0;$$

3. Finally, if the end of the beam is subjected to a concentrated load P_1, the boundary condition is $N_1 = P_1$, which implies

$$S \, \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} = P_1.$$

The axial stress distribution

The local axial stress, σ_1, for a given axial load, p_1, is given by:

$$\sigma_1(x_1, x_2, x_3) = \frac{E}{S} N_1(x_1) \tag{19}$$

For a beam made of a homogeneous material

$$\sigma_1(x_1, x_2, x_3) = \frac{N_1(x_1)}{\mathcal{A}}. \tag{20}$$

The axial stress is uniformly distributed over the section, and its value is independent of Young's modulus.

In contrast, the axial stress distribution for sections made of layers presenting different stiffness moduli will vary from layer to layer.

$$\sigma_1^{[i]}(x_1, x_2, x_3) = E^{[i]} \frac{N_1(x_1)}{S} \tag{21}$$

where $\sigma_1^{[i]}$ indicates the axial stress in layer i. The axial stress in layer i is proportional to the modulus of that layer.

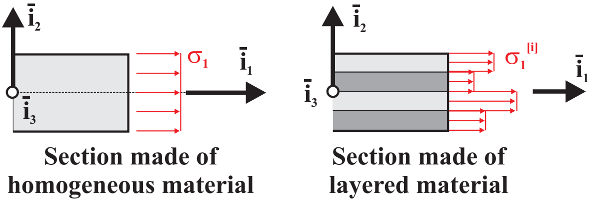

Figure: Axial stress distribution for sections made of homogeneous and layered materials.

Beams under transverse loads

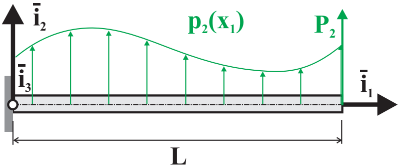

Figure: Beam subjected to transverse loads.

Assumptions:

- Plane ($\bar{\imath}_1$, $\bar{\imath}_2$) is a plane of symmetry of the structure.
- Loads are applied in this plane of symmetry, so that the response of the beam will be entirely contained in that plane.
- Transverse loads only cause transverse displacement and curvature of the section.

Kinematic relations

The general displacement field reduces to

$$u_1(x_1, x_2, x_3) = -x_2 \frac{\mathrm{d}\bar{u}_2(x_1)}{\mathrm{d}x_1}, \tag{22a}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \tag{22b}$$

$$u_3(x_1, x_2, x_3) = 0. \tag{22c}$$

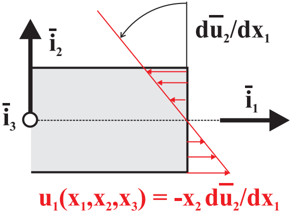

Figure: Axial displacement distribution on cross-section.

The only non-vanishing strain component from eq. (7) is

$$\epsilon_1(x_1, x_2, x_3) = -x_2 \kappa_3(x_1). \tag{23}$$

This describes a linear distribution of the axial strain over the cross-section.

Sectional constitutive law

Axial stress distribution:

$$\sigma_1(x_1, x_2, x_3) = -E x_2 \kappa_3(x_1). \tag{24}$$

The sectional axial force, must vanish under the assumptions made:

$$N_1(x_1) = \int_{\mathcal{A}} \sigma_1(x_1, x_2, x_3) \, \mathrm{d}\mathcal{A} = -\left[\int_{\mathcal{A}} E \; x_2 \; \mathrm{d}\mathcal{A}\right] \kappa_3(x_1). \tag{25}$$

This requirement can be written as

$$x_{2c} = \frac{1}{S} \int_{\mathcal{A}} E \; x_2 \; \mathrm{d}\mathcal{A} = \frac{S_2}{S} = 0, \tag{26}$$

where x_{2c} is the location of the *modulus-weighted centroid* of the cross-section.

Sectional constitutive law

The bending moment defined in eq. (10) is evaluated by introducing the axial stress distribution, eq. (24), to find

$$M_3(x_1) = \left[\int_{\mathcal{A}} E\ x_2^2 \mathrm{d}\mathcal{A}\right] \kappa_3(x_1) = H_{33}^c\ \kappa_3(x_1), \tag{27}$$

The *centroidal bending stiffness* about axis $\bar{\imath}_3$ is defined as

$$H_{33}^c = \int_{\mathcal{A}} E\ x_2^2\ \mathrm{d}\mathcal{A}. \tag{28}$$

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Equilibrium equations

Equilibrium equations are now derived to complete the formulation; an infinitesimal slice of the beam of length $\mathrm{d}x_1$ is considered for equilibrium

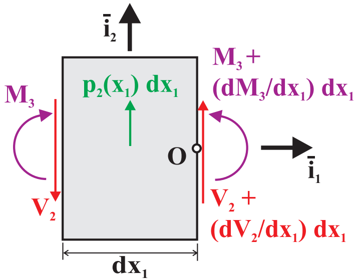

Figure: Equilibrium of an infinitesimal slice of the beam.

Equilibrium equations

$$\frac{\mathrm{d}V_2}{\mathrm{d}x_1} = -p_2(x_1), \tag{29a}$$

$$\frac{\mathrm{d}M_3}{\mathrm{d}x_1} + V_2 = 0, \tag{29b}$$

where the first equation expresses vertical force equilibrium and the second expresses moment equilibrium about point **O**.
The transverse shearing force, V_2, is readily eliminated from these two equilibrium equations to obtain a single equilibrium equation,

$$\frac{\mathrm{d}^2 M_3}{\mathrm{d}x_1^2} = p_2(x_1). \tag{30}$$

Governing equations

The governing equation for the transverse deflection of the beam are found by introducing the moment-curvature relation, eq. (**??**), into the equation of equilibrium, eq. (30), and recalling the expression for the curvature, eq. (6), to yield

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left[H_{33}^c \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right] = p_2(x_1). \tag{31}$$

Four boundary conditions are required for the solution of eq. (31), two at each end of the beam.

Typical boundary conditions are listed here.

1. *Clamped end*:
$$\bar{u}_2 = 0, \quad \frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1} = 0.$$
2. *Simply supported* (or pinned) end :
$$\bar{u}_2 = 0, \quad \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0.$$
3. *Free (or unloaded) end*
$$\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0, \quad -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[H_{33}^c \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right] = 0.$$
4. At an *end subjected to a concentrated transverse load*, P_2,
$$\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0, \quad -\frac{\mathrm{d}}{\mathrm{d}x_1}\left[H_{33}^c \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right] = P_2.$$
5. *Rectilinear springs*, with stiffness constant is denoted k
$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[H_{33}^c \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right]_{x_1=L} - k\ \bar{u}_2(L) = 0, \quad \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} = 0,$$

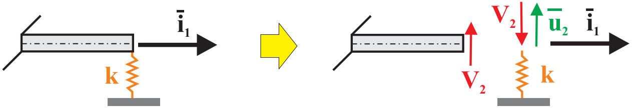

Figure: Free body diagram for the beam end linear spring of stiffness constant k.

In other cases, a *rotational spring* may be acting at the tip of the beam:

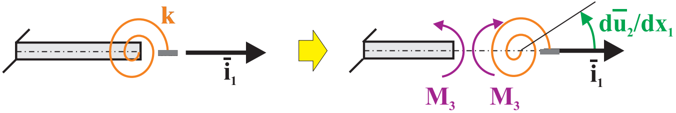

Figure: Free body diagram for a beam with end rotational spring of stiffness constant k.

The boundary conditions at the tip of the beam now become

$$H_{33}^c \left.\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right|_{x_1=L} + k \left.\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1}\right|_{x_1=L} = 0, \quad -\frac{\mathrm{d}}{\mathrm{d}_1}\left[H_{33}^c \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right] = 0,$$

Notes on sectional bending stiffness

If the beam is made of a homogeneous material Young's modulus can be factored out of the definition

$$H_{33}^c = E\ I_{33}^c, \tag{32}$$

where

$$I_{33}^c = \int_{\mathcal{A}} x_2^2 \, \mathrm{d}\mathcal{A}. \tag{33}$$

I_{33}^c is a purely geometric quantity known as the area second moment of the section computed about the area center.

Notes on sectional bending stiffness

An important case is that of a rectangular section of width b made of layered materials of different stiffnesses:

$$H_{33}^c = \int_{\mathcal{A}} E x_2^2 \, \mathrm{d}\mathcal{A} = \sum_{i=1}^{n} E^{[i]} \int_{\mathcal{A}^{[i]}} x_2^2 \, \mathrm{d}\mathcal{A}^{[i]}.$$

For the rectangular areas, this expression reduces to

$$H_{33}^c = \frac{b}{3} \sum_{i=1}^{n} E^{[i]} \left[(x_2^{[i+1]})^3 - (x_2^{[i]})^3 \right]. \tag{34}$$

The bending stiffness is a *weighted average* of the Young's moduli of the various layers. The weighting factor, strongly biases the average in favor of the outermost layers, whereas the layers near the centroid, contribute little to the overall bending stiffness.

The axial stress distribution

The axial stress can be readily in terms of curvature

$$\sigma_1(x_1, x_2, x_3) = -E\ x_2 \frac{M_3(x_1)}{H_{33}^c}. \tag{35}$$

Notes

- For a homogeneous material, the stress reduces to

$$\sigma_1(x_1, x_2, x_3) = -x_2 \frac{M_3(x_1)}{I_{33}}. \tag{36}$$

  The axial stress is linearly distributed over the section, and is independent of Young's modulus.
- The maximum tensile axial stress is found at the point of the section the farthest below the centroid,
- The axial stress distribution for layered sections is linear only within each layer and will present a discontinuity at the interfaces.

$$\sigma_1^{[i]}(x_1, x_2, x_3) = -E^{[i]} x_2 \frac{M_3(x_1)}{H_{33}^c} \tag{37}$$

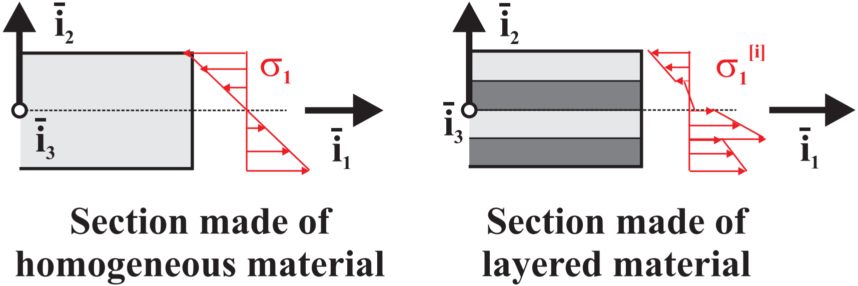

Figure: Axial stress distributions in homogeneous and layered sections.

Neutral axis

Definition

The axial stress clearly vanishes anywhere along axis $\bar{\imath}_3$ of the beam, which passes through the section's centroid. This line on the cross-section is called the *neutral axis* of the beam.

- The material located near the neutral axis carries almost no stresses and contributes little to the overall load carrying capability of the beam.
- The rational design of a beam under bending calls for the removal of the material located at and near the neutral axis and its relocation away from it

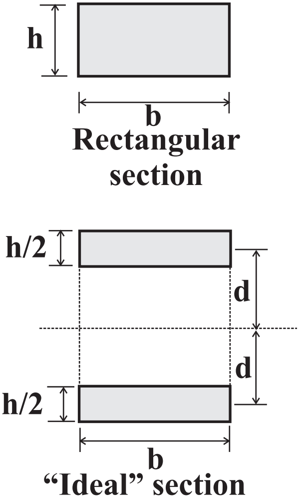

Figure: A rectangular section, and the ideal section.