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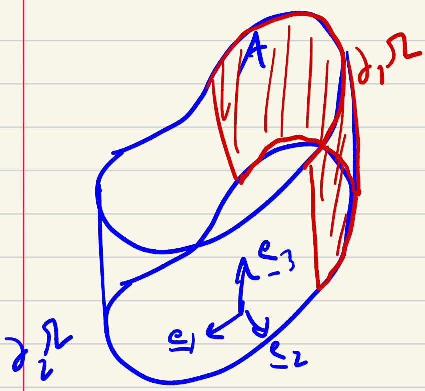
Two-dimensional problems.

In some problems, the solution does not depend on one of the spatial coordinates, let us say x_3. Then,

$$\underline{u} = \underline{u}(x_1, x_2); \quad \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(x_1, x_2); \quad \underline{\underline{\sigma}} = \underline{\underline{\sigma}}(x_1, x_2)$$



- Infinite cylinder w/ cross section A
- Coordinate system chosen so that x_3 is aligned w/ the axis of the cylinder.
- Body forces independent of x_3

- $\underline{u} = \underline{u}^*(x_1, x_2)$ on $\partial_1\Omega$ and $\underline{t} = \underline{t}^*(x_1, x_2)$ on $\partial_2\Omega$

  where $\partial_1\Omega \cup \partial_2\Omega$ = lateral surface

- Then, we expect the solution to be independent of x_3

Recall the equation of motion:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \cancel{\dfrac{\partial \sigma_{13}}{\partial x_3}} + \rho b_1 = \rho \dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \cancel{\dfrac{\partial \sigma_{23}}{\partial x_3}} + \rho b_2 = \rho \dfrac{\partial^2 u_2}{\partial t^2} \\ \dfrac{\partial \sigma_{13}}{\partial x_1} + \dfrac{\partial \sigma_{23}}{\partial x_2} + \cancel{\dfrac{\partial \sigma_{33}}{\partial x_3}} + \rho b_3 = \rho \dfrac{\partial^2 u_3}{\partial t^2} \end{cases} \quad (1)$$

In general, if $\underline{u} = \underline{u}(x_1, x_2)$ then we have:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_1} \\ & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_2} \\ \text{symm} & & 0 \end{bmatrix} \quad (2)$$

Recalling the expression for $\underline{\underline{\sigma}}$ from Hooke's Law (isotropic)

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\,\varepsilon_{11} & 2\mu\,\varepsilon_{12} & 2\mu\,\varepsilon_{13} \\ & \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\,\varepsilon_{22} & 2\mu\,\varepsilon_{23} \\ & & \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\,\varepsilon_{33} \end{bmatrix} \quad (3)$$

Plugging (2) into (3):

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} (\lambda+2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} & \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \mu\frac{\partial u_3}{\partial x_1} \\ & (\lambda+2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_1}{\partial x_1} & \mu\frac{\partial u_3}{\partial x_2} \\ \text{symm.} & & \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \end{bmatrix} \quad (4)$$

Plug (4) into (1):

$$(5)\begin{cases} (a) & (\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + \lambda\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1}\right) + \rho b_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \\ (b) & \mu\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2}\right) + (\lambda+2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + \lambda\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \rho b_2 = \rho\frac{\partial^2 u_2}{\partial t^2} \\ (c) & \mu\frac{\partial^2 u_3}{\partial x_1^2} + \mu\frac{\partial^2 u_3}{\partial x^2} + \rho b_3 = \rho\frac{\partial^2 u_3}{\partial t^2} \end{cases}$$

- The only assumption so far is that $\underline{u} = \underline{u}(x_1, x_2)$
- In elastostatic, the RHS of (5) become zero.

Note: 5a & 5b contain u_1 & u_2 only, while 5c u_3 only!

↳ decoupled into 2 systems.

- Hence, we will show that we can use prin of superposition to derive (5) by looking at two problems.
  - In plane problem
  - Anti-plane problem.

Antiplane problem:

$$u_3 = u_3(x_1, x_2) \; ; \quad u_1 = u_2 = 0$$

under these assumptions:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \text{sym} & & 0 \end{bmatrix} \quad [\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & 0 & \mu\frac{\partial u_3}{\partial x_1} \\ & 0 & \frac{\partial u_3}{\partial x_2} \\ \text{sym.} & & 0 \end{bmatrix}$$

All equations of motion reduces to 5C:

$$\boxed{\mu \frac{\partial^2 u_3}{\partial x_1^2} + \mu \frac{\partial^2 u_3}{\partial x_2^2} + \rho b_3 = \rho \frac{\partial^2 u_3}{\partial t^2}}$$

Note: - we call this problem anti-plane shear

- If $b_3 = 0$ we get

$$\boxed{\mu \frac{\partial^2 u_3}{\partial x_1^2} + \mu \frac{\partial^2 u_3}{\partial x_2^2} = \rho \frac{\partial^2 u_3}{\partial t^2}}$$ wave equation.

In plane problem

$u_1(x_1, x_2)$; $u_2(x_1, x_2)$; $u_3 = 0$

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & 0 \\ & \frac{\partial u_2}{\partial x_2} & 0 \\ \text{symm.} & & 0 \end{bmatrix}$$

$\hookrightarrow$ we call this "plane strain" problem

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

σ_{33} could be zero, but generally it is <u>NOT</u>

($\sigma_{33} \neq 0$, in general)

$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22})$ in this case.

If we plug this $\underline{\underline{\sigma}}$ into the equations of motion, we recover 5a & 5b (5c is automatically satisfied)

<u>Note</u>: In this way, it is possible to solve the general 2D problem by decomposing it into in-plane and anti-plane problems and adding up both solutions (prin. of superposition)