AE6114 Assignment 2: Kinematics

Problem 1

Consider a body occupying a cylinder of radius R and length L, $\Omega = \{(X_1, X_2, X_3) : X_1^2 + X_2^2 < R, \; 0 < X_3 < L\}$ in the reference configuration. It undergoes a deformation:

$$x_1 = X_1 \cos(\tau X_3) - X_2 \sin(\tau X_3)$$
$$x_2 = X_1 \sin(\tau X_3) + X_2 \cos(\tau X_3)$$
$$x_3 = X_3$$

where τ is a constant parameter with unit [1 / length].

These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering the answers from students that were marked correct. If you find any errors, please let me know!
– Sarah Malak
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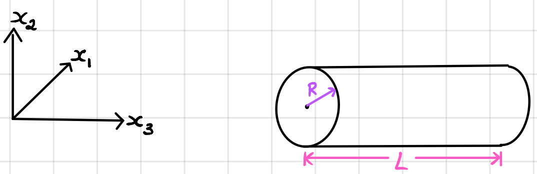
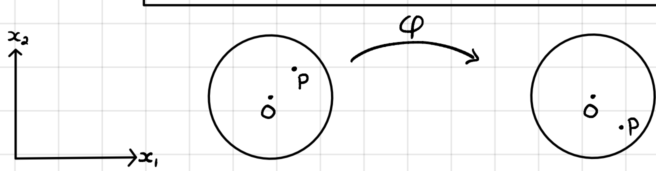
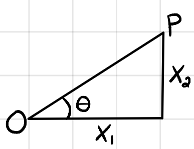
1. Describe the deformation. What is the meaning of τ?



$x_3 = X_3 \Rightarrow$ the body does not deform in the direction of L (no elongation of the bar in the x_3-direction)

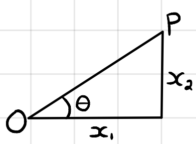


How does P change with φ?



$$\overline{OP} = \sqrt{X_1^2 + X_2^2}$$
$$X_1 = \overline{OP}\cos\theta$$
$$X_2 = \overline{OP}\sin\theta$$

Considering the deformed configuration:



$$\overline{OP} = \sqrt{x_1^2 + x_2^2}$$
$$x_1 = \overline{OP}\cos\theta = X_1 \cos(\tau X_3) - X_2 \sin(\tau X_3)$$
$$x_2 = \overline{OP}\sin\theta = X_1 \sin(\tau X_3) + X_2 \cos(\tau X_3)$$

$$\overline{OP} = \sqrt{x_1^2 + x_2^2} = \sqrt{[X_1 \cos(\tau X_3) - X_2 \sin(\tau X_3)]^2 + [X_1 \sin(\tau X_3) + X_2 \cos(\tau X_3)]^2}$$

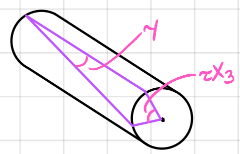
$$= \sqrt{X_1^2 \cos^2(\tau X_3) - \cancel{2X_1X_2 \sin(\tau X_3)\cos(\tau X_3)} + X_2^2 \sin^2(\tau X_3) + X_1^2 \sin^2(\tau X_3) + \cancel{2X_1X_2 \sin(\tau X_3)\cos(\tau X_3)} + X_2^2 \cos^2(\tau X_3)}$$

$$= \sqrt{X_1^2 \underbrace{\left[\cos^2(\tau X_3) + \sin^2(\tau X_3)\right]}_{1} + X_2^2 \underbrace{\left[\sin^2(\tau X_3) + \cos^2(\tau X_3)\right]}_{1}}$$

$$\overline{OP} = \sqrt{X_1^2 + X_2^2}$$

Thus, the deformation does not change the distance of a point from the origin
the radius is unchanged
a point in the reference configuration is rotated by an angle τX_3 in the deformed configuration

τ gives the twist of the bar along x_3 :



2. Find the deformation gradient $\underline{F}$ and compute the right Cauchy-Green stretch tensor $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$

$$F_{ij} = \frac{\partial \varphi_i}{\partial X_j}$$

$$F_{11} = \frac{\partial \varphi_1}{\partial X_1} = \cos(\tau X_3) \qquad F_{21} = \frac{\partial \varphi_2}{\partial X_1} = \sin(\tau X_3) \qquad F_{31} = \frac{\partial \varphi_3}{\partial X_1} = 0$$

$$F_{12} = \frac{\partial \varphi_1}{\partial X_2} = -\sin(\tau X_3) \qquad F_{22} = \frac{\partial \varphi_2}{\partial X_2} = \cos(\tau X_3) \qquad F_{32} = \frac{\partial \varphi_3}{\partial X_2} = 0$$

$$F_{13} = \frac{\partial \varphi_1}{\partial X_3} = -X_1 \tau \sin(\tau X_3) - \tau X_2 \cos(\tau X_3) \qquad F_{23} = \frac{\partial \varphi_2}{\partial X_3} = \tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3) \qquad F_{33} = \frac{\partial \varphi_3}{\partial X_3} = 1$$

$$\boxed{\underline{\underline{F}} = \begin{bmatrix} \cos(\tau X_3) & -\sin(\tau X_3) & -\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3) \\ \sin(\tau X_3) & \cos(\tau X_3) & \tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3) \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underline{\underline{F}}^T = \begin{bmatrix} \cos(\tau X_3) & \sin(\tau X_3) & 0 \\ -\sin(\tau X_3) & \cos(\tau X_3) & 0 \\ -\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3) & \tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3) & 1 \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$$

$$C_{11} = \cos(\tau X_3)\left[\cos(\tau X_3)\right] + \sin(\tau X_3)\left[\sin(\tau X_3)\right] = \cos^2(\tau X_3) + \sin^2(\tau X_3) = 1$$

$$C_{12} = \cos(\tau X_3)[-\sin(\tau X_3)] + \sin(\tau X_3)[\cos(\tau X_3)] = -\cos(\tau X_3)\sin(\tau X_3) + \cos(\tau X_3)\sin(\tau X_3) = 0$$

$$C_{13} = \cos(\tau X_3)[-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)] + \sin(\tau X_3)[\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)]$$

$$= \cancel{-\tau X_1 \cos(\tau X_3)\sin(\tau X_3)} - \tau X_2 \cos^2(\tau X_3) + \cancel{\tau X_1 \cos(\tau X_3)\sin(\tau X_3)} - \tau X_2 \sin^2(\tau X_3)$$

$$= -\tau X_2 \underbrace{[\cos^2(\tau X_3) + \sin^2(\tau X_3)]}_{1} = -\tau X_2$$

$$C_{21} = -\sin(\tau X_3)[\cos(\tau X_3)] + \cos(\tau X_3)[\sin(\tau X_3)] = -\cos(\tau X_3)\sin(\tau X_3) + \cos(\tau X_3)\sin(\tau X_3) = 0$$

$$C_{22} = -\sin(\tau X_3)[-\sin(\tau X_3)] + \cos(\tau X_3)[\cos(\tau X_3)] = \sin^2(\tau X_3) + \cos^2(\tau X_3) = 1$$

$$C_{23} = -\sin(\tau X_3)[-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)] + \cos(\tau X_3)[\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)]$$

$$= \tau X_1 \sin^2(\tau X_3) + \cancel{\tau X_2 \cos(\tau X_3)\sin(\tau X_3)} + \tau X_1 \cos^2(\tau X_3) - \cancel{\tau X_2 \cos(\tau X_3)\sin(\tau X_3)}$$

$$= \tau X_1 \underbrace{[\sin^2(\tau X_3) + \cos^2(\tau X_3)]}_{1} = \tau X_1$$

$$C_{31} = [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)]\cos(\tau X_3) + [\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)]\sin(\tau X_3)$$

$$= \cancel{-\tau X_1 \cos(\tau X_3)\sin(\tau X_3)} - \tau X_2 \cos^2(\tau X_3) + \cancel{\tau X_1 \cos(\tau X_3)\sin(\tau X_3)} - \tau X_2 \sin^2(\tau X_3)$$

$$= -\tau X_2 \underbrace{[\cos^2(\tau X_3) + \sin^2(\tau X_3)]}_{1} = -\tau X_2$$

$$C_{32} = [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)][-\sin(\tau X_3)] + [\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)]\cos(\tau X_3)$$

$$= \tau X_1 \sin^2(\tau X_3) + \cancel{\tau X_2 \cos(\tau X_3)\sin(\tau X_3)} + \tau X_1 \cos^2(\tau X_3) - \cancel{\tau X_2 \cos(\tau X_3)\sin(\tau X_3)}$$

$$= \tau X_1 [\sin^2(\tau X_3) + \cos^2(\tau X_3)] = \tau X_1$$

$$C_{33} = [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)][-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)]$$

$$+ [\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)][\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)] + 1$$

$$= \tau^2 X_1^2 \sin^2(\tau X_3) + \cancel{\tau^2 X_1 X_2 \cos(\tau X_3)\sin(\tau X_3)} + \cancel{\tau^2 X_1 X_2 \cos(\tau X_3)\sin(\tau X_3)} + \tau^2 X_2^2 \cos^2(\tau X_3) + \tau^2 X_1^2 \cos^2(\tau X_3)$$
$$\cancel{- \tau^2 X_1 X_2 \cos(\tau X_3)\sin(\tau X_3)} - \cancel{\tau^2 X_1 X_2 \cos(\tau X_3)\sin(\tau X_3)} + \tau^2 X_2^2 \sin^2(\tau X_3) + 1$$

$$= \tau^2 X_1^2 \underbrace{[\sin^2(\tau X_3) + \cos^2(\tau X_3)]}_{1} + \tau^2 X_2^2 \underbrace{[\cos^2(\tau X_3) + \sin^2(\tau X_3)]}_{1} + 1$$

$$= \tau^2 X_1^2 + \tau^2 X_2^2 + 1$$

$$\boxed{\underset{\approx}{C} = \begin{bmatrix} 1 & 0 & -\tau X_2 \\ 0 & 1 & \tau X_1 \\ -\tau X_2 & \tau X_1 & \tau^2 X_1^2 + \tau^2 X_2^2 + 1 \end{bmatrix}}$$

3. Calculate the stretch λ for a fiber that, in the reference configuration, is oriented parallel to the plane defined by the basis vectors $\{\underline{e}_1, \underline{e}_2\}$.

$$\lambda^2 = C_{jk} \frac{dX_j}{\|d\underline{X}\|} \frac{dX_k}{\|d\underline{X}\|} \qquad N_i = \frac{dX_i}{\|d\underline{X}\|}$$

$$\lambda(\underline{N}) = \sqrt{C_{jk} N_j N_k}$$

$$\underset{\approx}{C} = \begin{bmatrix} 1 & 0 & -\tau X_2 \\ 0 & 1 & \tau X_1 \\ -\tau X_2 & \tau X_1 & \tau^2 X_1^2 + \tau^2 X_2^2 + 1 \end{bmatrix}$$

$$\underline{N} = \cos\theta \underline{e}_1 + \sin\theta \underline{e}_2$$

$$\underline{N}^T \underset{\approx}{C} \underline{N} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\tau X_2 \\ 0 & 1 & \tau X_1 \\ -\tau X_2 & \tau X_1 & \tau^2 X_1^2 + \tau^2 X_2^2 + 1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & -\tau X_2 \cos\theta + \tau X_1 \sin\theta \end{bmatrix}$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$$\boxed{\lambda(\underline{N}) = \sqrt{1} = 1}$$

4. Calculate the local change in area for differential elements of oriented area with normal along the $\underline{e}_3$ basis vector.

$$n_p da = (F^{-1})_{op} \det(\underset{\approx}{F}) N_o dA$$

$$\underline{n} da = \underset{\approx}{F}^{-1} \underline{N} J dA \ , \quad J = \det(\underset{\approx}{F})$$

$$\underline{\underline{F}} = \begin{bmatrix} \cos(\tau X_3) & -\sin(\tau X_3) & -\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3) \\ \sin(\tau X_3) & \cos(\tau X_3) & \tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(\underline{\underline{F}}) = \cos(\tau X_3)[\cos(\tau X_3)(1) - 0] - (-\sin(\tau X_3))[\sin(\tau X_3)(1) - 0] + [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_2)](0)$$

$$= \cos^2(\tau X_3) + \sin^2(\tau X_3)$$

$\det(\underline{\underline{F}}) = 1 \Rightarrow J = 1$

$\underline{\underline{F}}^{-T}$: take the inverse, then take the transpose

To find $\underline{\underline{F}}^{-1}$:

1. $\det \underline{\underline{F}}$ -> we know $\det \underline{\underline{F}} = 1$

2. find $Adj(\underline{\underline{F}}^T)$

$$\underline{\underline{F}}^T = \begin{bmatrix} \cos(\tau X_3) & \sin(\tau X_3) & 0 \\ -\sin(\tau X_3) & \cos(\tau X_3) & 0 \\ -\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3) & \tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3) & 1 \end{bmatrix}$$

$$Adj(\underline{\underline{F}}^T) = \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{vmatrix} \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$a' = [\cos(\tau X_3)](1) - [\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)](0) = \cos(\tau X_3)$$

$$b' = [-\sin(\tau X_3)](1) - [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)](0) = -\sin(\tau X_3)$$

$$c' = [-\sin(\tau X_3)][\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)] - [\cos(\tau X_3)][-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)]$$

$$= -\cancel{\tau X_1 \cos(\tau X_3) \sin(\tau X_3)} + \tau X_2 \sin^2(\tau X_3) + \cancel{\tau X_1 \cos(\tau X_3) \sin(\tau X_3)} + \tau X_2 \cos^2(\tau X_3)$$

$$= \tau X_2 [\cos^2(\tau X_3) + \sin^2(\tau X_3)] = \tau X_2$$

$$d' = [\sin(\tau X_3)](1) - [\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)](0) = \sin(\tau X_3)$$

$$e' = [\cos(\tau X_3)](1) - [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)](0) = \cos(\tau X_3)$$

$$f' = [\cos(\tau X_3)][\tau X_1 \cos(\tau X_3) - \tau X_2 \sin(\tau X_3)] - [-\tau X_1 \sin(\tau X_3) - \tau X_2 \cos(\tau X_3)][\sin(\tau X_3)]$$

$$= \tau X_1 \cos^2(\tau X_3) - \cancel{\tau X_2 \cos(\tau X_3)\sin(\tau X_3)} + \tau X_1 \sin^2(\tau X_3) + \cancel{\tau X_2 \cos^2(\tau X_3)\sin(\tau X_3)}$$

$$= \tau X_1 [\cos^2(\tau X_3) + \sin^2(\tau X_3)] = \tau X_1$$

$$g' = [\sin(\tau X_3)](0) - [\cos(\tau X_3)](0) = 0$$

$$h' = [\cos(\tau X_3)](0) - [-\sin(\tau X_3)](0) = 0$$

$$i' = [\cos(\tau X_3)][\cos(\tau X_3)] - [-\sin(\tau X_3)][\sin(\tau X_3)] = \cos^2(\tau X_3) + \sin^2(\tau X_3) = 1$$

$$\begin{vmatrix} \cos(\tau X_3) & -\sin(\tau X_3) & \tau X_2 \\ \sin(\tau X_3) & \cos(\tau X_3) & \tau X_1 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\mathrm{Adj}(\underline{\underline{F}}^T) = \begin{bmatrix} \cos(\tau X_3) & \sin(\tau X_3) & \tau X_2 \\ -\sin(\tau X_3) & \cos(\tau X_3) & -\tau X_1 \\ 0 & 0 & 1 \end{bmatrix}$$

3. $\underline{\underline{F}}^{-1} = \dfrac{1}{\det(\underline{\underline{F}})} \cdot \mathrm{Adj}(\underline{\underline{F}}^T)$

$$\Rightarrow \underline{\underline{F}}^{-1} = \begin{bmatrix} \cos(\tau X_3) & \sin(\tau X_3) & \tau X_2 \\ -\sin(\tau X_3) & \cos(\tau X_3) & -\tau X_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we take the transpose :

$$\underline{\underline{F}}^{-T} = \begin{bmatrix} \cos(\tau X_3) & -\sin(\tau X_3) & 0 \\ \sin(\tau X_3) & \cos(\tau X_3) & 0 \\ \tau X_2 & -\tau X_1 & 1 \end{bmatrix}$$

$$\underline{n}\,da = \underline{\underline{F}}^{-T}\underline{N} J dA$$

$\underline{N} = \langle 0 \quad 0 \quad 1 \rangle \quad (\underline{e}_3)$

$$\begin{bmatrix} \cos(\tau X_3) & -\sin(\tau X_3) & 0 \\ \sin(\tau X_3) & \cos(\tau X_3) & 0 \\ \tau X_2 & -\tau X_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This implies that there is no local change in area for differential elements of oriented area with normal along $\underline{e}_3$ basis vector.
This makes sense with what we found in Part 1: the deformation is just a rotation, so the elements would have the same area before and after the deformation.

5. Calculate the local change of differential volume.

$$dv = \det(\underline{\underline{F}})dV$$

We previously calculated $\det(\underline{\underline{F}}) = 1$, which gives $dv = dV$

This means that the volume of the differential element remains unchanged between the reference and the deformed configurations.

Problem 2

Let $d\underline{X}$ and $d\underline{Y}$ be two differential vectors in the reference configuration along the directions of the unit vectors $\underline{M}$ and $\underline{N}$ correspondingly. If the body undergoes a motion defined by the action of the deformation mapping φ, we can compute the angle θ between the corresponding deformed differential vectors $d\underline{x}$ and $d\underline{y}$ by means of the expression

$$\cos\theta = \frac{\underline{N} \cdot \underline{\underline{C}} \cdot \underline{M}}{\lambda_N \lambda_M}$$

where $\underline{\underline{C}}$ is the Cauchy-Green stretch tensor, and λ_N and λ_M the stretches in the directions of $\underline{N}$ and $\underline{M}$. For the particular case when $\underline{N} \perp \underline{M}$, we have $\cos\theta = \sin\gamma$ where γ is the change in angle between the differential elements (see Figure 1).

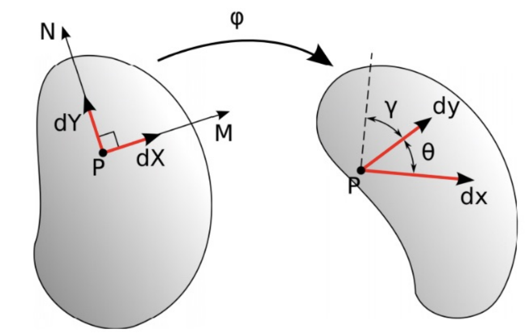


Figure 1: Schematics for Problem 2.

Show that, under the assumptions of infinitesimal deformations, the expression for γ reduces to $\gamma = 2\varepsilon_{ij} N_i M_j$ where ε_{ij} are the components of the infinitesimal strain tensor.

$$\underline{\underline{C}} = 2\underline{\underline{E}} + \underline{\underline{I}} = 2E_{ij} + \delta_{ij}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \cancelto{0}{\frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}} \right)$$ assumption of infinitesimal deformations

$$\cos\theta = \frac{N_i C_{ij} M_j}{\lambda_{\underline{N}} \lambda_{\underline{M}}}, \quad \lambda_{\underline{N}} = \sqrt{C_{\ell m} N_\ell N_m} \quad \lambda_{\underline{M}} = \sqrt{C_{pq} M_p M_q}$$

$$N_i C_{ij} M_j = N_i (2\varepsilon_{ij} + \delta_{ij}) M_j = N_i\, 2\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] M_j + \underbrace{N_i \delta_{ij} M_j}_{= N_i M_i = 0}$$

$$= 2\varepsilon_{ij} N_i M_j$$

$= N_i M_i = 0$ because $\underline{N} \perp \underline{M}$

in a similar manner:

$$\lambda_{\underline{N}}^2 = N_\ell C_{\ell m} N_m = N_\ell (2\varepsilon_{\ell m} + \delta_{\ell m}) N_m = N_\ell\, 2\varepsilon_{\ell m} N_m + \underbrace{N_\ell \delta_{\ell m} N_m}_{= N_\ell N_\ell = 1}$$

$$\lambda_{\underline{M}}^2 = M_p C_{pq} M_q = M_p (2\varepsilon_{pq} + \delta_{pq}) M_q = M_p 2\varepsilon_{pq} M_q + \underbrace{M_p \delta_{pq} M_q}_{= M_p M_p = 1}$$

So we have:

$$\cos\theta = \frac{N_i\, 2\varepsilon_{ij} M_j}{\sqrt{N_\ell 2\varepsilon_{\ell m} N_m + 1}\,\sqrt{M_p 2\varepsilon_{pq} M_q + 1}}$$

now we use a trick: $(1+x)^{\alpha} \simeq (1+\alpha x)$ $\qquad 1+\alpha(1+x)|_0 x = 1+\alpha x$

$$\lambda_{\underline{N}} = (N_i 2\mathcal{E}_{ij} N_j + 1)^{\frac{1}{2}} \simeq \frac{1}{2}(N_i 2\mathcal{E}_{ij} N_j) + 1 = \underbrace{N_i \mathcal{E}_{ij} N_j}_{\mathcal{E}_{\underline{N}}} + 1$$

so $\lambda_{\underline{N}} = 1 + \mathcal{E}_{\underline{N}}$ and $\lambda_{\underline{M}} = 1 + \mathcal{E}_{\underline{M}}$

which gives us:

$$\cos\theta = \frac{N_i 2\mathcal{E}_{ij} M_j}{(1+\mathcal{E}_{\underline{N}})(1+\mathcal{E}_{\underline{M}})} = \frac{N_i 2\mathcal{E}_{ij} M_j}{\underbrace{1+\mathcal{E}_{\underline{N}}+\mathcal{E}_{\underline{M}}}_{\simeq 1} + \underbrace{\mathcal{E}_{\underline{N}}\mathcal{E}_{\underline{M}}}_{\simeq 0}}$$

for small deformations

when $\underline{N} \perp \underline{M}$, $\cos\theta = \sin\gamma$

small angle assumption: $\sin\gamma = \gamma$

and this leaves us with $\gamma = 2\mathcal{E}_{ij} N_i M_j$ ✓

Another way:

$$\cos\theta = \frac{N_i 2\mathcal{E}_{ij} M_j}{(1+\mathcal{E}_{\underline{N}})(1+\mathcal{E}_{\underline{M}})} = \frac{N_i 2\mathcal{E}_{ij} M_j}{1+\mathcal{E}_{\underline{N}}+\mathcal{E}_{\underline{M}} + \underbrace{\mathcal{E}_{\underline{N}}\mathcal{E}_{\underline{M}}}_{\simeq 0 \text{ for small deformations}}}$$

$$\frac{A}{1+x} = \frac{A}{(1+x)}\frac{(1-x)}{(1-x)} = \frac{A(1-x)}{1-x^2}$$

$$\frac{2\mathcal{E}_{ij}N_iM_j}{1+\mathcal{E}_{\underline{N}}+\mathcal{E}_{\underline{M}}} = \frac{2\mathcal{E}_{ij}N_iM_j}{1+\mathcal{E}_{\underline{N}}+\mathcal{E}_{\underline{M}}}\frac{(1-\mathcal{E}_{\underline{N}}-\mathcal{E}_{\underline{M}})}{(1-\mathcal{E}_{\underline{N}}-\mathcal{E}_{\underline{M}})} = \frac{2\mathcal{E}_{ij}N_iM_j(1-\mathcal{E}_{\underline{N}}-\mathcal{E}_{\underline{M}})}{1-\underbrace{(\mathcal{E}_{\underline{N}}+\mathcal{E}_{\underline{M}})^2}_{\simeq 0}} \simeq 2\mathcal{E}_{ij}N_iM_j$$ for small deformations ✓

$\cos\theta = \sin\gamma \simeq \gamma = 2\mathcal{E}_{ij}N_iM_j$ for small deformations when $\underline{N} \perp \underline{M}$

Another way to do this problem:

$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$ where $F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$

$$C_{jk} = F_{ij}F_{ik} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right) = \delta_{ij}\delta_{ik} + \delta_{ij}\frac{\partial u_i}{\partial x_k} + \delta_{ik}\frac{\partial u_i}{\partial x_j} + \cancelto{0}{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}}$$

$$C_{jk} = \delta_{jk} + \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} = \delta_{jk} + 2\mathcal{E}_{jk} \quad \text{since} \quad \mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

also, since the deformations are small, $\sin\gamma \simeq \gamma$

$$\gamma = \frac{(\delta_{jk} + 2\mathcal{E}_{jk})N_jM_k}{\lambda_{\underline{N}}\lambda_{\underline{M}}} = 2\mathcal{E}_{jk}N_jM_k + \underbrace{\delta_{jk}N_jM_k}_{= N_kM_k = 0} \implies \gamma = \frac{2\mathcal{E}_{jk}N_jM_k}{\lambda_{\underline{N}}\lambda_{\underline{M}}}$$

$$\lambda_{\underline{N}} = \sqrt{C_{\ell m} N_\ell N_m} = \sqrt{(\delta_{\ell m} + 2\mathcal{E}_{\ell m}) N_\ell N_m} = \sqrt{\underbrace{\delta_{\ell m} N_\ell N_m}_{= N_m N_m = 1} + 2\mathcal{E}_{\ell m} N_\ell N_m} = \sqrt{1 + 2\mathcal{E}_{\ell m} N_\ell N_m}$$

Similarly, $\lambda_{\underline{M}} = \sqrt{1 + 2\mathcal{E}_{pq} M_p M_q}$

$$\lambda_{\underline{N}} \lambda_{\underline{M}} = \sqrt{(1 + 2\mathcal{E}_{\ell m} N_\ell N_m)(1 + 2\mathcal{E}_{pq} M_p M_q)}$$

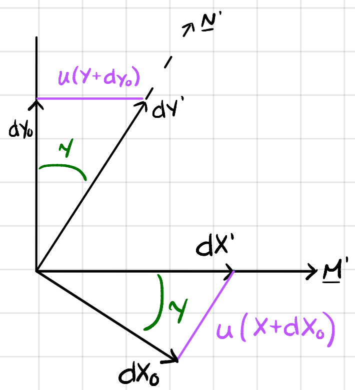
$$= \sqrt{1 + 2\mathcal{E}_{pq} M_p M_q + 2\mathcal{E}_{\ell m} N_\ell N_m + \underbrace{4\mathcal{E}_{\ell m}\mathcal{E}_{pq} N_\ell N_m M_p M_q}}$$

$$\mathcal{E}_{\ell m}\mathcal{E}_{pq} = \frac{1}{4}\left(\frac{\partial u_\ell}{\partial X_m} + \frac{\partial u_m}{\partial X_\ell}\right)\left(\frac{\partial u_p}{\partial X_q} + \frac{\partial u_q}{\partial X_p}\right) \to 0$$

$$= \sqrt{1 + 2\mathcal{E}_{\ell m} N_\ell N_m + 2\mathcal{E}_{pq} M_p M_q}$$

$$\simeq \underbrace{1 + \frac{1}{2}(2\mathcal{E}_{\ell m} N_\ell N_m + 2\mathcal{E}_{pq} M_p M_q)}_{\text{using } (1+x)^\alpha \simeq (1+\alpha x)} = 1 + \mathcal{E}_{\ell m} N_\ell N_m + \mathcal{E}_{pq} M_p M_q$$

If γ is very small and the total deformation is infinitesimal, $\cos\gamma \simeq 1$.



$$\cos\gamma = \left|\frac{dY_0}{dY'}\right| = 1$$

$$\cos\gamma = \left|\frac{dX_0}{dX'}\right| = 1$$

this shows that u is very small (the length of the fiber does not change much)

This indicates that $\lambda_{\underline{N}} \lambda_{\underline{M}} \simeq 1$, which shows that

$$\gamma = \frac{2\mathcal{E}_{ij} N_i M_j}{\lambda_{\underline{N}} \lambda_{\underline{M}}} \simeq 2\mathcal{E}_{ij} N_i M_j \quad ✓$$

Problem 3

Consider a body occupying a cube of length L, $\Omega = \{(X_1, X_2, X_3): 0 < X_1 < L,\ 0 < X_2 < L,\ 0 < X_3 < L\}$ in the reference configuration. It undergoes a deformation:

$$x_1 = AX_1 + BX_2 + CX_3$$
$$x_2 = CX_1 + AX_2 + BX_3$$
$$x_3 = BX_1 + CX_2 + AX_3$$

with $A = \left(\frac{1}{3} + \frac{2}{3}\cos\theta\right)$, $B = \left(\frac{1}{3} - \frac{1}{3}\cos\theta - \frac{\sqrt{3}}{3}\sin\theta\right)$, $C = \left(\frac{1}{3} - \frac{1}{3}\cos\theta + \frac{\sqrt{3}}{3}\sin\theta\right)$

1. Find the deformation gradient $\underline{\underline{F}}$ and compute the Lagrangian strain tensor $\underline{\underline{E}}$.

Deformation gradient $\underline{\underline{F}}$:

$$F_{11} = \frac{\partial \phi_1}{\partial X_1} = A \qquad F_{12} = \frac{\partial \phi_1}{\partial X_2} = B \qquad F_{13} = \frac{\partial \phi_1}{\partial X_3} = C$$

$$F_{21} = \frac{\partial \phi_2}{\partial X_1} = C \qquad F_{22} = \frac{\partial \phi_2}{\partial X_2} = A \qquad F_{23} = \frac{\partial \phi_2}{\partial X_3} = B$$

$$F_{31} = \frac{\partial \phi_3}{\partial X_1} = B \qquad F_{32} = \frac{\partial \phi_3}{\partial X_2} = C \qquad F_{33} = \frac{\partial \phi_3}{\partial X_3} = A$$

$$\boxed{\underline{\underline{F}} = \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} \quad \text{where} \quad \begin{aligned} A &= \tfrac{1}{3} + \tfrac{2}{3}\cos\theta \\ B &= \tfrac{1}{3} - \tfrac{1}{3}\cos\theta - \tfrac{\sqrt{3}}{3}\sin\theta \\ C &= \tfrac{1}{3} - \tfrac{1}{3}\cos\theta + \tfrac{\sqrt{3}}{3}\sin\theta \end{aligned}}$$

Lagrangian strain tensor $\underline{\underline{E}}$:

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}\right)$$

$$\underline{\underline{F}}^T = \begin{bmatrix} A & C & B \\ B & A & C \\ C & B & A \end{bmatrix}$$

$$\underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} A & C & B \\ B & A & C \\ C & B & A \end{bmatrix} \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} = \begin{bmatrix} A^2+B^2+C^2 & AB+AC+BC & AC+BC+AB \\ AB+AC+BC & A^2+B^2+C^2 & CB+AB+AC \\ AC+BC+AB & BC+AB+AC & A^2+B^2+C^2 \end{bmatrix}$$

$$A^2+B^2+C^2 = \left(\tfrac{1}{3}+\tfrac{2}{3}\cos\theta\right)\left(\tfrac{1}{3}+\tfrac{2}{3}\cos\theta\right) + \left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta - \tfrac{\sqrt{3}}{3}\sin\theta\right)\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta - \tfrac{\sqrt{3}}{3}\sin\theta\right)$$

$$+\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta+\tfrac{\sqrt{3}}{3}\sin\theta\right)\left(\tfrac{1}{3}-\tfrac{1}{3}+\tfrac{\sqrt{3}}{3}\sin\theta\right)$$

$$= \underline{\tfrac{1}{9}} + \cancel{\tfrac{2}{9}\cos\theta} + \cancel{\tfrac{2}{9}\cos\theta} + \underline{\tfrac{4}{9}\cos^2\theta} + \underline{\tfrac{1}{9}} - \cancel{\tfrac{1}{9}\cos\theta} - \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} - \cancel{\tfrac{1}{9}\cos\theta} + \underline{\tfrac{1}{9}\cos^2\theta} + \cancel{\tfrac{\sqrt{3}}{9}\cos\theta\sin\theta}$$

$$- \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} + \cancel{\tfrac{\sqrt{3}}{9}\cos\theta\sin\theta} + \underline{\tfrac{1}{3}\sin^2\theta} + \underline{\tfrac{1}{9}} - \cancel{\tfrac{1}{9}\cos\theta} + \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} - \cancel{\tfrac{1}{9}\cos\theta} + \underline{\tfrac{1}{9}\cos^2\theta} - \cancel{\tfrac{\sqrt{3}}{9}\cos\theta\sin\theta}$$

$$+ \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} - \cancel{\tfrac{\sqrt{3}}{9}\cos\theta\sin\theta} + \underline{\tfrac{1}{3}\sin^2\theta}$$

$$A^2+B^2+C^2 = \frac{1}{3} + \frac{2}{3}\cos^2\theta + \frac{2}{3}\sin^2\theta = \frac{1}{3} + \frac{2}{3} = 1$$

$$AB+AC+BC = \left(\tfrac{1}{3}+\tfrac{2}{3}\cos\theta\right)\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta - \tfrac{\sqrt{3}}{3}\sin\theta\right) + \left(\tfrac{1}{3}+\tfrac{2}{3}\cos\theta\right)\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta+\tfrac{\sqrt{3}}{3}\sin\theta\right)$$

$$+\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta-\tfrac{\sqrt{3}}{3}\sin\theta\right)\left(\tfrac{1}{3}-\tfrac{1}{3}\cos\theta+\tfrac{\sqrt{3}}{3}\sin\theta\right)$$

$$= \underline{\tfrac{1}{9}} - \cancel{\tfrac{1}{9}\cos\theta} - \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} + \cancel{\tfrac{2}{9}\cos\theta} - \underline{\tfrac{2}{9}\cos^2\theta} - \cancel{\tfrac{2\sqrt{3}}{9}\cos\theta\sin\theta} + \underline{\tfrac{1}{9}} - \cancel{\tfrac{1}{9}\cos\theta} + \cancel{\tfrac{\sqrt{3}}{9}\sin\theta}$$

$$+ \cancel{\tfrac{2}{9}\cos\theta} - \underline{\tfrac{2}{9}\cos^2\theta} + \cancel{\tfrac{2\sqrt{3}}{9}\cos\theta\sin\theta} + \underline{\tfrac{1}{9}} - \cancel{\tfrac{1}{9}\cos\theta} + \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} - \cancel{\tfrac{1}{9}\cos\theta} + \underline{\tfrac{1}{9}\cos^2\theta}$$

$$- \cancel{\tfrac{\sqrt{3}}{9}\sin\theta\cos\theta} - \cancel{\tfrac{\sqrt{3}}{9}\sin\theta} + \cancel{\tfrac{\sqrt{3}}{9}\cos\theta\sin\theta} - \underline{\tfrac{1}{3}\sin^2\theta}$$

$$AB + AC + BC = \frac{1}{3} - \frac{1}{3}\cos^2\theta - \frac{1}{3}\sin^2\theta = \frac{1}{3} - \frac{1}{3} = 0$$

$$\underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\underline{\underline{E}} = [\underline{\underline{0}}]}$$

=> the body is strain-free

2. Show the expression for the displacement field $\underline{u}$ and compute the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$.

$$x = \phi(\underline{X}, t) = \underline{X} + \underline{u}(\underline{X}, t)$$

$$\Rightarrow \boxed{\underline{u}(\underline{X}, t) = \phi(\underline{X}, t) - \underline{X}}$$

$u_i(X_i, t) = \phi(X_i, t) - X_i$

So then

$$u_1(X_1, t) = \phi(X_1, t) - X_1$$
$$u_2(X_2, t) = \phi(X_2, t) - X_2$$
$$u_3(X_3, t) = \phi(X_3, t) - X_3$$

$$\boxed{\begin{aligned} u_1 &= (A-1)X_1 + BX_2 + CX_3 \\ u_2 &= CX_1 + (A-1)X_2 + BX_3 \\ u_3 &= BX_1 + CX_2 + (A-1)X_3 \end{aligned}}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\underline{e}_i \underline{e}_j$$

$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = A-1$ $\qquad$ $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(B+C) = \varepsilon_{21}$

$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = A-1$ $\qquad$ $\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}(C+B) = \varepsilon_{31}$

$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = A-1$ $\qquad$ $\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}(B+C) = \varepsilon_{32}$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} A-1 & \frac{1}{2}(B+C) & \frac{1}{2}(B+C) \\ \frac{1}{2}(B+C) & A-1 & \frac{1}{2}(B+C) \\ \frac{1}{2}(B+C) & \frac{1}{2}(B+C) & A-1 \end{bmatrix}$$

$$\boxed{\underline{\underline{\varepsilon}} = \begin{bmatrix} -\frac{2}{3}+\frac{2}{3}\cos\theta & \frac{1}{3}-\frac{1}{3}\cos\theta & \frac{1}{3}-\frac{1}{3}\cos\theta \\ \frac{1}{3}-\frac{1}{3}\cos\theta & -\frac{2}{3}+\frac{2}{3}\cos\theta & \frac{1}{3}-\frac{1}{3}\cos\theta \\ \frac{1}{3}-\frac{1}{3}\cos\theta & \frac{1}{3}-\frac{1}{3}\cos\theta & -\frac{2}{3}+\frac{2}{3}\cos\theta \end{bmatrix}}$$

3. Using both finite and infinitesimal deformations, calculate the stretch λ for fibers that, in the reference configuration, are oriented along the basis vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

In general, $\varepsilon_{\underline{N}} = \lambda - 1$

$$\lambda = \sqrt{\left(\frac{\partial u_K}{\partial x_i} + \delta_{Ki}\right)\left(\frac{\partial u_K}{\partial x_j} + \delta_{Kj}\right) N_i N_j} \qquad (1)$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{I}} \cdot \underline{\underline{I}} = \underline{\underline{I}}$$

then $\lambda(\underline{e}_1) = \lambda(\underline{e}_2) = \lambda(\underline{e}_3) = 1$

using (1):

$$\lambda(\underline{e}_1) = \sqrt{\left(\frac{\partial u_1}{\partial x_1}+1\right)\left(\frac{\partial u_1}{\partial x_1}+1\right) + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} = \sqrt{A^2 + C^2 + B^2}$$

$$\lambda(\underline{e}_2) = \sqrt{\left(\frac{\partial u_2}{\partial x_2}+1\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2} = \sqrt{A^2 + B^2 + C^2}$$

$$\lambda(\underline{e}_3) = \sqrt{\left(\frac{\partial u_3}{\partial x_3}+1\right)^2 + \left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2} = \sqrt{A^2 + C^2 + B^2}$$

only $\neq 0$ when $i=j$

$$\boxed{\begin{array}{l}\text{then } \lambda(\underline{e}_1) = \lambda(\underline{e}_2) = \lambda(\underline{e}_3) = \sqrt{A^2+B^2+C^2} = 1 \\ \text{for finite deformations}\end{array}}$$

for infinitesimal deformations: $\lambda = 1 + \varepsilon_{ij} N_i N_j$

again, when $i \neq j \Rightarrow \varepsilon_{ij} N_i N_j = 0$ (when $i \neq j$, $\lambda = 1$)

$$\left.\begin{array}{l} \lambda(\underline{e}_1) = 1 + \varepsilon_{11} \\ \lambda(\underline{e}_2) = 1 + \varepsilon_{22} \\ \lambda(\underline{e}_3) = 1 + \varepsilon_{33} \end{array}\right\} \qquad \boxed{\begin{array}{l}\lambda(\underline{e}_1) = \lambda(\underline{e}_2) = \lambda(\underline{e}_3) = A = \frac{1}{3} + \frac{2}{3}\cos\theta \\ \text{for infinitesimal deformations}\end{array}}$$

4. Using both finite and infinitesimal deformations, calculate the change in angle γ for fibers that, in the reference configuration, are oriented along the basis vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

Because the basis vectors in the reference configuration are orthogonal,

$$\cos\theta = \sin\gamma$$

in Part 1 we calculated $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left.\begin{aligned} \sin\gamma(\underline{e}_1, \underline{e}_2) &= \frac{C_{12}}{\lambda(\underline{e}_1)\lambda(\underline{e}_2)} = 0 \\ \sin\gamma(\underline{e}_2, \underline{e}_3) &= \frac{C_{23}}{\lambda(\underline{e}_2)\lambda(\underline{e}_3)} = 0 \\ \sin\gamma(\underline{e}_1, \underline{e}_3) &= \frac{C_{13}}{\lambda(\underline{e}_1)\lambda(\underline{e}_3)} = 0 \end{aligned}\right\} \quad \sin^{-1}(0) = 0$$

$\Rightarrow$ $\boxed{\gamma(\underline{e}_1, \underline{e}_2) = \gamma(\underline{e}_1, \underline{e}_3) = \gamma(\underline{e}_2, \underline{e}_3) = 0 \text{ for finite deformations}}$

for infinitesimal deformations:

$$\left.\begin{aligned} \gamma(\underline{e}_1, \underline{e}_2) &= 2\mathcal{E}_{12} \\ \gamma(\underline{e}_1, \underline{e}_3) &= 2\mathcal{E}_{13} \\ \gamma(\underline{e}_2, \underline{e}_3) &= 2\mathcal{E}_{23} \end{aligned}\right\} \text{ if fibers are aligned with basis vectors}$$

$$\boxed{\begin{gathered}\gamma(\underline{e}_1, \underline{e}_2) = \gamma(\underline{e}_1, \underline{e}_3) = \gamma(\underline{e}_2, \underline{e}_3) = B + C = \tfrac{2}{3} - \tfrac{2}{3}\cos\theta \\ \text{for infinitesimal deformations}\end{gathered}}$$

5. Using results from previous sections, discuss:

What kind of deformation is taking place under the prescribed mapping?

Because the body is not under any strain, the body is undergoing a rigid body rotation.

Why are results different?

The results are different because they are not calculated using the same assumptions. (the linearization of the strain tensor does not hold for large deformations)

What can be inferred about infinitesimal deformations for this kind of mapping?

When θ is very small, the finite and infinite deformations will be approximately equal, so the infinitesimal deformation calculations hold for very small rotations.

Problem 4

Consider a right Cauchy-Green stretch tensor $\underline{\underline{C}}$. It has a spectral representation

$$\underline{\underline{C}} = \sum_{i=1}^{3} \lambda_i^2 \underline{v}_i \underline{v}_i$$

where $\lambda_i > 0$ are the principal stretches, $|\underline{v}_i| = 1$, and $\underline{v}_i$ are mutually orthogonal.

Let $\lambda_1 = \lambda_{max}$, $\lambda_2 = \lambda_{min}$, and $\lambda_{max} > \lambda_3 > \lambda_{min}$:

1. Show that the stretch λ_{max} in the direction $\underline{v}_1$ is larger than the stretch λ_N in any other direction $\underline{N}$.

$$\underline{\underline{C}} = \sum_{i=1}^{3} \lambda_i^2 \underline{v}_i \underline{v}_i = \lambda_1^2 \underline{v}_1 \underline{v}_1 + \lambda_2^2 \underline{v}_2 \underline{v}_2 + \lambda_3^2 \underline{v}_3 \underline{v}_3 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} \lambda_{max}^2 & 0 & 0 \\ 0 & \lambda_{min}^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$\underline{N} = n_1 \underline{v}_1 + n_2 \underline{v}_2 + n_3 \underline{v}_3 \quad \text{where } n_1^2 + n_2^2 + n_3^2 = 1^2$$

$$\lambda^2 = \underline{N}\,\underline{\underline{C}}\,\underline{N} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_{max}^2 & 0 & 0 \\ 0 & \lambda_{min}^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\lambda^2 = \lambda_{max}^2 n_1^2 + \lambda_{min}^2 n_2^2 + \lambda_3^2 n_3^2$$

in the $\underline{v}_1$ direction: $\underline{N} = 1\underline{v}_1 + 0\underline{v}_2 + 0\underline{v}_3$

$$\lambda^2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{max}^2 & 0 & 0 \\ 0 & \lambda_{min}^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_{max}^2$$

in any other direction $\underline{N}$, $\lambda^2 = \lambda_{max}^2 n_1^2 + \lambda_{min}^2 n_2^2 + \lambda_3^2 n_3^2$, which will always be less than $\lambda^2(\underline{N}_1) = \lambda_{max}^2$ because $\sqrt{n_1^2 + n_2^2 + n_3^2} = 1$

2. Show that the stretch λ_{min} in the direction $\underline{v}_2$ is smaller than the stretch λ_N in any other direction $\underline{N}$.

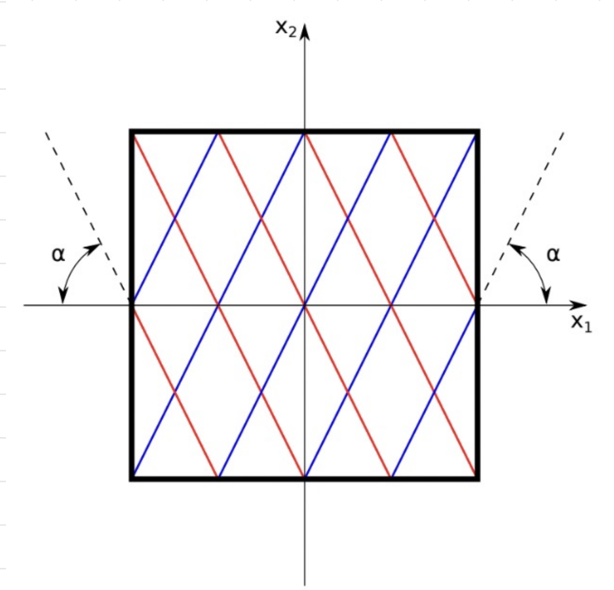
in the $\underline{v}_2$ direction: $\underline{N} = 0\underline{v}_1 + 1\underline{v}_2 + 0\underline{v}_3$

$$\lambda^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{max}^2 & 0 & 0 \\ 0 & \lambda_{min}^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \lambda_{min}^2$$

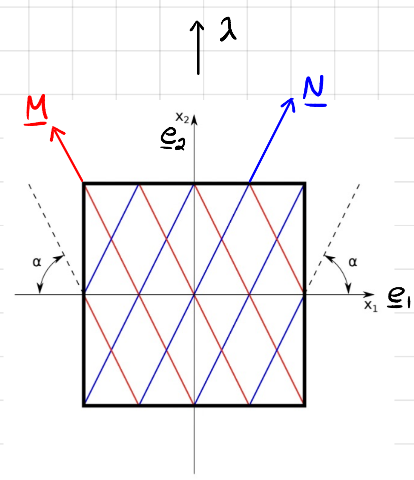
in any other direction $\underline{N}$, $\lambda^2 = \lambda_{max}^2 n_1^2 + \lambda_{min}^2 n_2^2 + \lambda_3^2 n_3^2$, which will always be greater than $\lambda^2(\underline{N}_2) = \lambda_{min}^2$ because $\sqrt{n_1^2 + n_2^2 + n_3^2} = 1$

Problem 5

A rubber block is reinforced by two sets of inextensible cables as shown in the figure below. The block is subject to a deformation with stretching λ in the direction of $\underline{e}_2$. Assuming plane strain deformation in the plane defined by $\underline{e}_1$ and $\underline{e}_2$, compute all components of the right Cauchy-Green deformation tensor $\underline{\underline{C}}$.



Schematics of reinforced rubber block.
Two sets of inextensible fibers are denoted by the red and blue lines.



Cables are inextensible
compute $\underline{\underline{C}}$

$\lambda^2 = \underline{N}\,\underline{\underline{C}}\,\underline{N}$ ← $\underline{N}$ is in the reference configuration

$\lambda^2_{Cables} = 1$

$N_\alpha = [\cos(\alpha) \quad \sin(\alpha) \quad 0]$

unit vector!

$$\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

9 unknowns
↓
$\underline{\underline{C}}$ is symmetric
↓
6 unknowns
↓
plane strain $\Rightarrow$ $C_{13} = C_{23} = C_{31} = C_{32} = 0$
↓
$C_{33} = 1 \qquad (\underline{\underline{\varepsilon}} = \varepsilon_{i3} = 0)$

$$\lambda^2 = 1 = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{bmatrix} = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} + 2\sin(\alpha)\cos(\alpha)C_{12} \qquad (1)$$

→ 3 unknowns

$$M_\alpha = \begin{bmatrix} -\cos(\alpha) & \sin(\alpha) & 0 \end{bmatrix}$$

$$\lambda^2 = 1 = \begin{bmatrix} -\cos(\alpha) & \sin(\alpha) & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\cos(\alpha) \\ \sin(\alpha) \\ 0 \end{bmatrix} = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} - 2\sin(\alpha)\cos(\alpha)C_{12} \qquad (2)$$

we need one more equation:

$$\lambda^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = C_{22} \qquad (3)$$

given

adding (1) and (2):

$$\begin{array}{rl} & 1 = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} + 2\cos(\alpha)\sin(\alpha)C_{12} \\ + & 1 = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} - 2\cos(\alpha)\sin(\alpha)C_{12} \\ \hline & 2 = 2\cos^2(\alpha)C_{11} + 2\sin^2(\alpha)C_{22} \end{array}$$

Simplifying:

$$1 = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22}$$

Substituting (3):

$$1 = \cos^2(\alpha)C_{11} + \lambda^2\sin^2(\alpha) \quad \Rightarrow \quad C_{11} = \frac{1 - \lambda^2\sin^2(\alpha)}{\cos^2(\alpha)}$$

Subtracting (1) and (2):

$$
\begin{array}{rl}
 & 1 = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} + 2\cos(\alpha)\sin(\alpha)\,C_{12} \\
- & 1 = \cos^2(\alpha)C_{11} + \sin^2(\alpha)C_{22} - 2\cos(\alpha)\sin(\alpha)C_{12} \\
\hline
\end{array}
$$

$$0 = 4\cos(\alpha)\sin(\alpha)\,C_{12}$$

$\Rightarrow$ C_{12} must be 0

thus,

$$\boxed{\underset{\equiv}{C} = \begin{bmatrix} \frac{1-\lambda^2\sin^2(\alpha)}{\cos^2(\alpha)} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$