AE6310: Optimization for the Design of Engineered Systems

Quiz 2
March 3rd, 2020

Briefly answer the following questions on the paper provided. Organize your work and be careful to properly answer all parts of each question.

This quiz is closed book. The length of the quiz is 40 minutes.

It may be helpful to recall the following formula:

$$||p^*(\lambda)||_2^2 = \sum_{i=1}^{n} \frac{(g^T q_i)^2}{(\lambda + \lambda_i)^2}$$

$$p = -\tau \frac{\Delta}{||g||_2} g \qquad \tau = \begin{cases} 1 & g^T B g \leq 0 \\ \min\left(1, ||g||_2^3 / \Delta g^T B g\right) & \text{otherwise} \end{cases}$$

1. (15 points) Sketch the following trust region model problems. In each case indicate the Cauchy point, the unconstrained model minimizer, if any, and the exact trust region step. Be careful to draw the contour lines and trust region constraint accurately.
   (a) A positive definite quadratic model with an inactive trust region constraint
   (b) A positive definite quadratic model with an active trust region constraint
   (c) An indefinite quadratic model
2. (35 points) The following question is based on a quadratic model function $m(p) = \frac{1}{2} p^T B p + g^T p$

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   (a) If $\Delta = 1$, find the Cauchy step for this problem
   (b) What is the maximum trust radius size Δ where the trust region constraint is still active?
   (c) If $\Delta = 1/2$, find the multiplier λ and the step p^* to the exact model minimizer
3. (15 points) State the KKT conditions for constrained optimization based on the canonical problem:

$$\min_x \quad f(x) \qquad \text{such that} \quad c(x) \leq 0$$

4. (35 points) Solve the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(x) = \frac{3}{2}(x_1^2 + x_2^2) - x_1 x_2 + x_1 + x_2 \\ \text{such that} \quad & c_1 = 2 - x_2 \leq 0 \\ & c_2 = 2 - x_1 - 2x_2 \leq 0 \end{aligned}$$

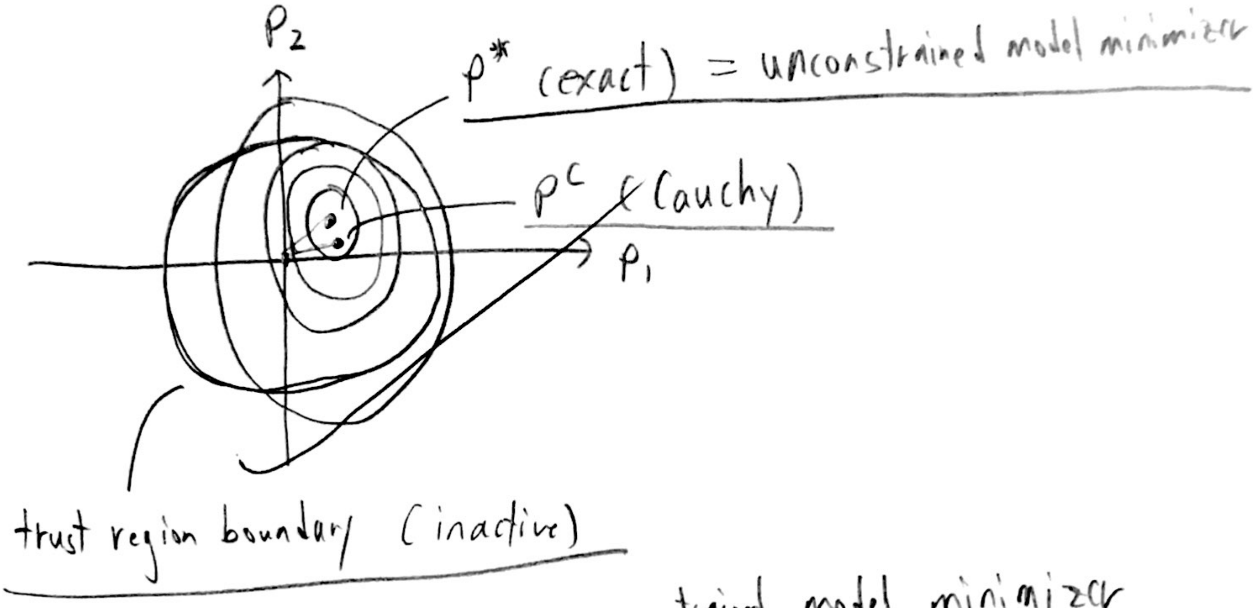
$\Rightarrow 2 - x_1 \leq 2x_2$

$1 - \frac{x_1}{2} \leq x_2$

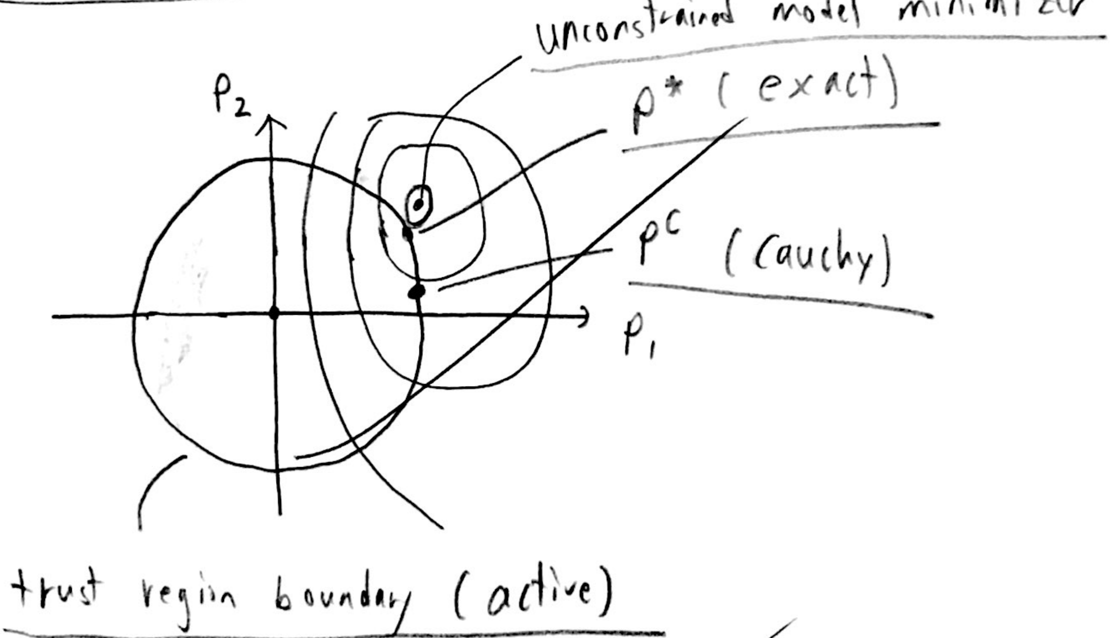
   (a) Find the unconstrained minimizer of f(x)
   (b) Show the feasible space by sketching the inequality constraints
   (c) Make an assumption about the active set of constraints and find the Lagrange multipliers. If the multipliers do not satisfy the KKT conditions, revisit your assumption about the active set.
   (d) State the design point and multipliers at the correct KKT point.
   (e) What is the correct active set at the constrained optimum?

Problem 1.

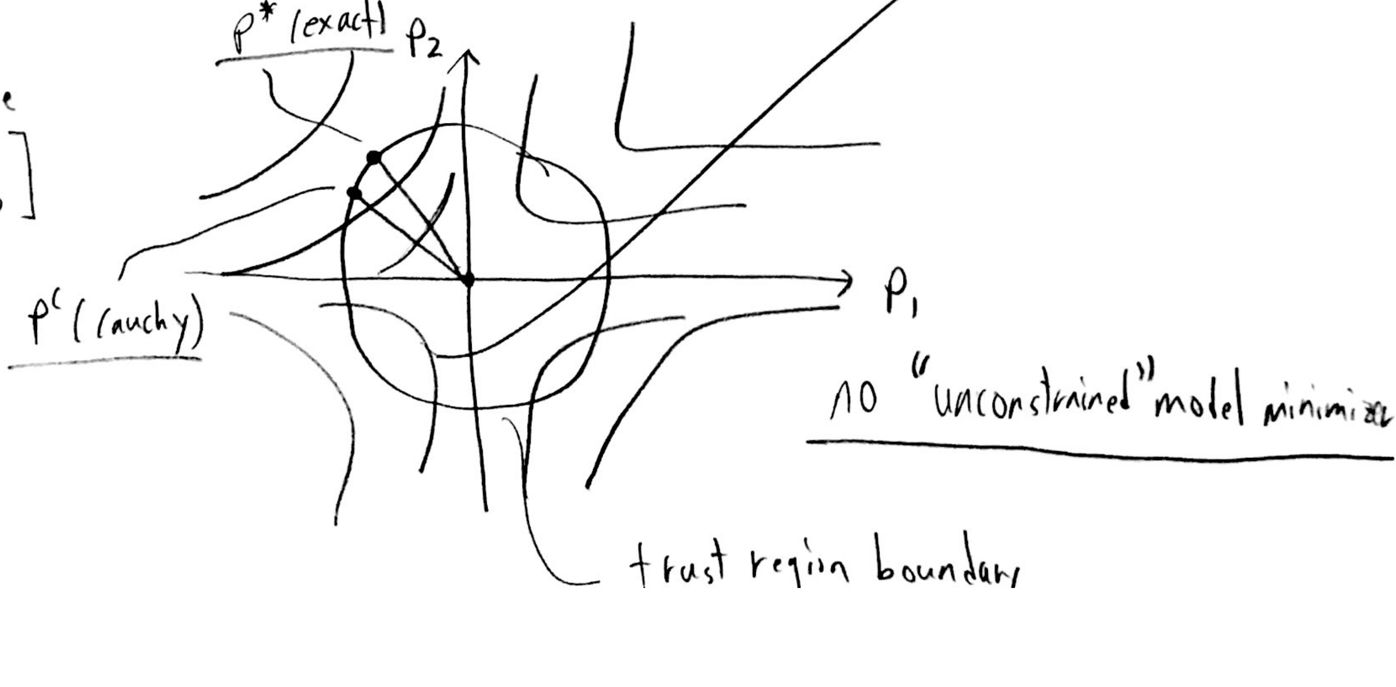
(a) Assume $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



(b) Assume $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



(c) Assume $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



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$$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 2 \text{ and } \lambda_2 = 4$$

$$q_1 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix} \text{ and } q_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

B is positive definite

(a) $p^c = - \tau \frac{\Delta}{\|g\|_2} g$ where $\tau = \begin{cases} 1, & g^T B g \le 0 \\ \min(1, \|g\|_2^3 / \Delta g^T B g, & \text{otherwise} \end{cases}$

$$g^T B g = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4 > 0$$

$$\therefore \tau = \min(1, \|g\|_2^3 / \Delta g^T B g)$$

$$\|g\|_2 = \sqrt{g^T g} = \sqrt{2}$$

$$\frac{\|g\|_2^3}{\Delta g^T B g} = \frac{2\sqrt{2}}{1 \cdot 4} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \tau = \frac{\sqrt{2}}{2} \qquad \Rightarrow p^c = -\frac{\sqrt{2}}{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \boxed{p^c = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}}$$

(b) $\dfrac{\|g\|_2^3}{\Delta_{max}\, g^T B g} = 1$

Solving for Δ_{max}, $\dfrac{\|g\|_2^3}{g^T B g} = \Delta_{max}$

$$\therefore \boxed{\Delta_{max} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}}$$

(c) $\Delta = \frac{1}{2}$ $\quad p_{min} = -B^{-1} g = -\dfrac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\therefore \sqrt{p_{min}^T p_{min}} = \frac{\sqrt{2}}{2} > \frac{1}{2}$$

$(B + \lambda I) p^* = -g \Rightarrow \begin{bmatrix} 3+\lambda & 1 \\ 1 & 3+\lambda \end{bmatrix} p^* = -g$

$$p^* = -\frac{1}{(3+\lambda)^2 - 1}\begin{bmatrix} 3+\lambda & -1 \\ -1 & 3+\lambda \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{\lambda^2 + 6\lambda + 8}\begin{bmatrix} \lambda+4 \\ -(\lambda+4) \end{bmatrix}$$

$$= -\frac{1}{(\lambda+2)(\lambda+4)}\begin{bmatrix} \lambda+4 \\ -(\lambda+4) \end{bmatrix}$$

$$= \frac{1}{(\lambda+2)(\lambda+4)}\begin{bmatrix} -(\lambda+4) \\ \lambda+4 \end{bmatrix}$$

$$\therefore p^*(\lambda) = \begin{bmatrix} -\frac{1}{\lambda+2} \\ \frac{1}{\lambda+2} \end{bmatrix}$$

solve for λ

$\|p^*(\lambda)\|_2^2$ ...

p ...

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solve for λ,

$$\|p^*(\lambda)\|_2^2 = \Delta^2 \Rightarrow \frac{(q_1^T g)^2}{(\lambda_1+\lambda)^2} + \frac{(q_2^T g)^2}{(\lambda_2+\lambda)^2} = \frac{1}{4}$$

Plugging all parameters,

$$\left.\begin{aligned}(q_1^T g) &= -1.4142\\ (q_2^T g) &= 0\\ \lambda_1 &= 2\\ \lambda_2 &= 4\end{aligned}\right\} \quad \frac{2}{(2+\lambda)^2} = \frac{1}{4}$$

$$\therefore \lambda = -2 \pm 2\sqrt{2}$$

since $\lambda \geq 0$,

$$\therefore \lambda = -2 + 2\sqrt{2}$$

$$p^*(\lambda) = \begin{bmatrix} -\frac{1}{\lambda+2} \\ \frac{1}{\lambda+2} \end{bmatrix}$$

$$\therefore \boxed{p^* = \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}}$$

em 3,

$$\min_{x} \; f(x)$$

$$\text{s.t.} \quad c(x) \le 0$$

$$\nabla f(x^*) = -A(x^*)\lambda^*$$

$$c(x^*) \le 0$$

$$c_i(x^*)\lambda_i = 0$$

$$\lambda \ge 0$$

where $A(x) = \nabla c(x)$

KKT conditions

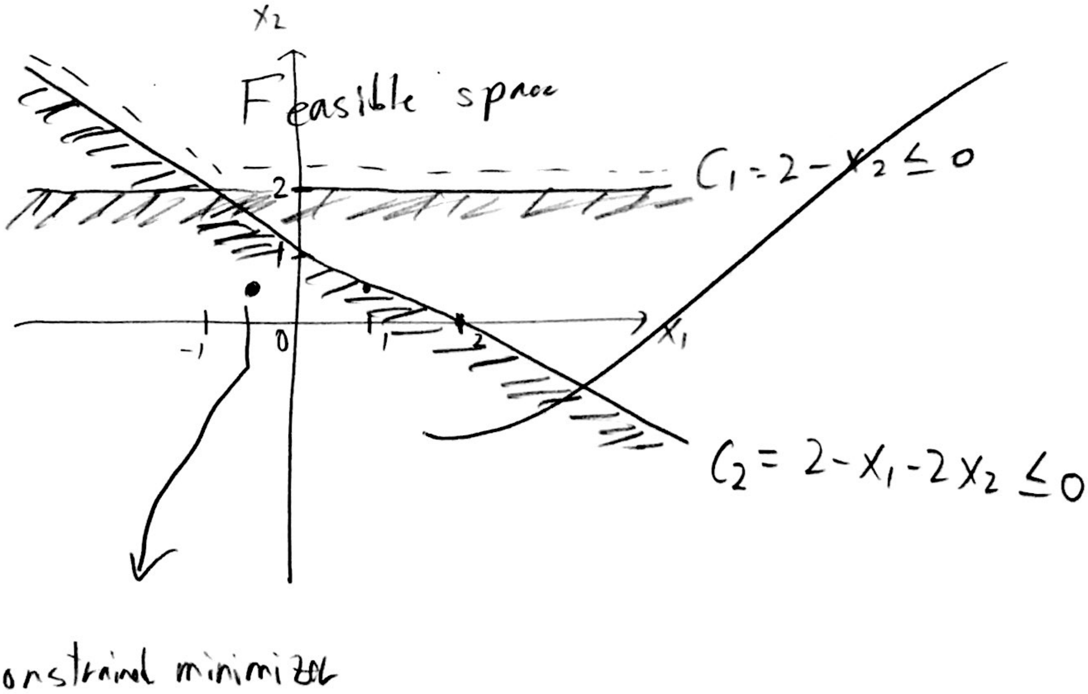
oblem 4.

(a) $\nabla f(x) = \begin{bmatrix} 3x_1 - x_2 + 1 \\ 3x_2 - x_1 + 1 \end{bmatrix}$

Set $\nabla f(x) = 0$ and solve for x^*

$$\left.\begin{aligned} 3x_1 - x_2 + 1 &= 0 \\ 3x_2 - x_1 + 1 &= 0 \end{aligned}\right\} \quad \boxed{\begin{aligned} x_1^* &= -\tfrac{1}{2} \\ x_2^* &= -\tfrac{1}{2} \end{aligned}}$$

(b)



It seems logical that C_1 is active and C_2 is inactive (dominated by C_1)

(c) Assume that C_1 is active and C_2 is inactive,

Then the problem becomes,

$$3x_1 - x_2 + 1 = 0$$

$$3x_2 - x_1 + 1 - \lambda_1 = 0$$

$$\lambda_1 (2 - x_2) = 0$$

$$2 - x_2 \leq 0$$

$$\lambda_1 \geq 0$$

Solving this system of equations,

$$\boxed{\begin{aligned} x_1^* &= \frac{1}{3} \\ x_2^* &= 2 \\ \lambda_1 &= \frac{20}{3} \end{aligned}}$$

$\Rightarrow$ satisfies active C_1 constraint

inactive C_2 constraint

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$\boxed{\text{inactive } C_2 \text{ constraint implies } \lambda_2 = 0}$

KKT conditions

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$$\boxed{\begin{aligned} &3x_1 - x_2 + 1 - \lambda_2 = 0 \\ &3x_2 - x_1 + 1 - \lambda_1 - 2\lambda_2 = 0 \\ &2 - x_2 \le 0 \\ &2 - x_1 - 2x_2 \le 0 \\ &\lambda_1(2 - x_2) = 0 \\ &\lambda_2(2 - x_1 - 2x_2) = 0 \\ &\lambda_1 \ge 0 \\ &\lambda_2 \ge 0 \end{aligned}}$$

Solve for the KKT point,

We get the following $\boxed{\begin{aligned} x_1^* &= \frac{1}{3} \\ x_2^* &= 2 \\ \lambda_1 &= \frac{20}{3} \\ \lambda_2 &= 0 \end{aligned}}$

This is consistent with part (c)!

(e) Since $\lambda_2 = 0$,

C_2 is inactive

since $\lambda_1 > 0$, C_1 is active