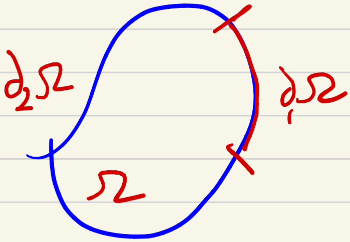
10/22

Boundary Value Problem in Elastostatics



- Let Ω be a body w/ boundary $\partial \Omega$
- Let the boundary be expressed as

$$\partial \Omega = \partial_1 \Omega \cup \partial_2 \Omega \text{ with}$$

$$\partial_1 \Omega \cap \partial_2 \Omega = \emptyset$$

- Let the body forces $\rho \underline{b}$ on Ω, the tractions $\underline{t}^*$ on $\partial_2 \Omega$, and the displacements $\underline{u}^*$ on $\partial_1 \Omega$ be given

* We call $\underline{t}^*$ Neumann (traction) boundary conditions, and $\underline{u}^*$ Dirichlet (displacement) boundary conditions

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* The boundary value problem in elastostatics consists on finding $\underline{u}$, $\underline{\underline{\varepsilon}}$, $\underline{\underline{\sigma}}$ such that:

(Momentum) $\underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$ in Ω ; $\sigma_{ij,j} + \rho b_i = 0$

(Hooke's) $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$ in Ω ; $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$

(inf. def strain) $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)$ in Ω; $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ *

Subject to boundary conditions:

$$\underline{u} = \underline{u}^* \text{ on } \partial_1 \Omega \; ; \quad u_i = u_i^* \text{ on } \partial_1 \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}^* \text{ on } \partial_2 \Omega \; ; \quad \sigma_{ij} n_j = t_i^* \text{ on } \partial_2 \Omega$$

<u>Notes</u>: (1) Body forces $\rho \underline{b}$ are given in the body

(2) One can have displacement BCs on the whole boundary, or traction BCs on the whole boundary, or a combination of both.

(3) In elastostatics, $\underline{a} = \underline{0}$ (no acceleration) and nothing depends on time. Hence, all quantities depend only on $\underline{x}$

(4) For an elastodynamics formulation, we just replace $\underline{0}$ by $\rho \frac{\partial^2 u}{\partial t^2}$. Hence, quantities also depend on time, and on initial conditions.

(5) There is a very well established mathematical theory of the well-posedness of this problem.

(6) The problem can be reduced to finding the solution for $\underline{u}(\underline{x})$

Since $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\underline{u})$

$\Rightarrow \underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}(\underline{u})) = \underline{\underline{\sigma}}(\underline{u}) \Rightarrow$ plug into the balance of linear momentum and solve for $\underline{u}$.

principle of superposition

- Let $\underline{u}^{(1)}, \underline{\underline{\varepsilon}}^{(1)}, \underline{\underline{\sigma}}^{(1)}$ be a solution to (*) with body forces $\rho \underline{b}^{(1)}$ and BCs $u^{*(1)}$ on $\partial_1 \Omega$ and $\underline{t}^{*(1)}$ on $\partial_2 \Omega$
- Let $\underline{u}^{(2)}, \underline{\underline{\varepsilon}}^{(2)}, \underline{\underline{\sigma}}^{(2)}$ be a solution to (*) with body forces $\rho \underline{b}^{(2)}$ and BCs $u^{*(2)}$ on $\partial_1 \Omega$ and $\underline{t}^{*(2)}$ on $\partial_2 \Omega$
- Then, $\underline{u} = \alpha \underline{u}^{(1)} + \beta u^{(2)}$,

$$\underline{\underline{\varepsilon}} = \alpha \underline{\underline{\varepsilon}}^{(1)} + \beta \underline{\underline{\varepsilon}}^{(2)},$$

$$\underline{\underline{\sigma}} = \alpha \underline{\underline{\sigma}}^{(1)} + \beta \underline{\underline{\sigma}}^{(2)}$$

is a solution to (*) with body forces

$$\rho \underline{b} = \alpha \rho \underline{b}^{(1)} + \beta \rho \underline{b}^{(2)} \text{ and BCs}$$

$$\underline{u}^* = \alpha \underline{u}^{*(1)} + \beta \underline{u}^{*(2)} \text{ on } \partial_1 \Omega \text{ and}$$

$$\underline{t}^* = \alpha \underline{t}^{*(1)} + \beta \underline{t}^{*(2)} \text{ on } \partial_2 \Omega$$

where α & β are real numbers.

- We can use this principle to split a given problem into simpler ones.

- The principle of superposition is a consequence of the linearity of the problem.

- To understand this, let us assume we have a displacement field that is the result of linearly combining solutions

$\underline{u}^{(1)}$ & $\underline{u}^{(2)}$

$$u_i = \alpha u_i^{(1)} + \beta u_i^{(2)} \quad \text{on } \Omega, \text{ w/ } \alpha, \beta \in \mathbb{R}$$

The corresponding strains are the following:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\alpha u_i^{(1)} + \beta u_i^{(2)}\right) + \frac{\partial}{\partial x_i}\left(\alpha u_j^{(1)} + \beta u_j^{(2)}\right) \right)$$

$$= \underbrace{\alpha \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial x_j} + \frac{\partial u_j^{(1)}}{\partial x_i} \right)}_{\varepsilon_{ij}^{(1)}} + \underbrace{\beta \frac{1}{2}\left(\frac{\partial u_i^{(2)}}{\partial x_j} + \frac{\partial u_j^{(2)}}{\partial x_i} \right)}_{\varepsilon_{ij}^{(2)}}$$

$$\Rightarrow \varepsilon_{ij} = \alpha \varepsilon_{ij}^{(1)} + \beta \varepsilon_{ij}^{(2)} \quad \text{on } \Omega$$

which implies the following:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = C_{ijkl}\left(\alpha\,\varepsilon_{kl}^{(1)} + \beta\,\varepsilon_{kl}^{(2)}\right)$$

$$\Rightarrow \sigma_{ij} = \alpha \underbrace{C_{ijkl}\,\varepsilon_{kl}^{(1)}}_{\sigma_{ij}^{(1)}} + \beta \underbrace{C_{ijkl}\,\varepsilon_{kl}^{(2)}}_{\sigma_{ij}^{(2)}}$$

$$\Rightarrow \underbrace{\sigma_{ij} = \alpha\,\sigma_{ij}^{(1)} + \beta\,\sigma_{ij}^{(2)}}$$

That is, if $u_i = \alpha u_i^{(1)} + \beta u_i^{(2)}$

then necessarily:

$$\begin{cases} \varepsilon_{ij} = \alpha\,\varepsilon_{ij}^{(1)} + \beta\,\varepsilon_{ij}^{(2)} \\ \sigma_{ij} = \alpha\,\sigma_{ij}^{(1)} + \beta\,\sigma_{ij}^{(2)} \end{cases}$$

But is this a solution? If so, to which problem?

we plug σ_{ij} into $(*)$:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \frac{\partial}{\partial x_j}\left(\alpha\,\sigma_{ij}^{(1)} + \beta\,\sigma_{ij}^{(2)}\right) + \alpha \rho b_i^{(1)} + \beta \rho b_i^{(2)}$$

$$= \alpha \underbrace{\left(\frac{\partial \sigma_{ij}^{(1)}}{\partial x_j} + \rho b_i^{(1)}\right)}_{=0} + \beta \underbrace{\left(\frac{\partial \sigma_{ij}^{(2)}}{\partial x_j} + \rho b_i^{(2)}\right)}_{=0}$$

since $\sigma_{ij}^{(1)}$ is a sol. since $\sigma_{ij}^{(2)}$ is a sol.

That is, the equation of equilibrium is automatically satisfied.

so $\sigma_{ij} = \alpha\,\sigma_{ij}^{(1)} + \beta\,\sigma_{ij}^{(2)}$ is a solution on Ω

But, what are the BCs for this problem?

$$\sigma_{ij} n_j = \left(\alpha\,\sigma_{ij}^{(1)} + \beta\,\sigma_{ij}^{(2)}\right) n_j$$

$$= \underbrace{\alpha\,\sigma_{ij}^{(1)} n_j}_{t_i^{*(1)}} + \underbrace{\beta\,\sigma_{ij}^{(2)} n_j}_{t_i^{*(2)}}$$

$$\Rightarrow \sigma_{ij} n_j = \alpha\, t_i^{*(1)} + \beta\, t_i^{*(2)} \text{ on } \partial_2 \Omega$$

And for the displacements, by definition:

$$u_i^* = \alpha\, u_i^{*(1)} + \beta\, u_i^{*(2)} \text{ on } \partial_1 \Omega$$

* we assumed C_{ijkl} same for both problems
* partition between $\partial_1\Omega$ & $\partial_2\Omega$ are the same for both problem.

* Valid for linear elastic material.
(components of stress tensors are linear comb.)