Kinematics

(cont)

Ex problems book

- Example problems for continuum mechanics of solids (Lallit Anand)

Other Measures of Deformations

* As previously seen, the deformation gradient $\underline{\underline{F}}$ encodes all the necessary information about the kinematics of local deformation.

* we also introduced $\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}}$ which proved to be essential for the computation of metric changes:

stretch $\quad \lambda(\underline{N}) = \sqrt{C_{jk} N_j N_k} = \sqrt{\underline{N} \cdot \underline{\underline{C}} \cdot \underline{N}}$

angle $\quad \cos\theta(\underline{M}, \underline{N}) = \dfrac{C_{ij} N_i M_j}{\sqrt{C_{lm} N_l N_m}\sqrt{C_{pq} M_p M_q}}$

* From a traditional engineering approach, extensional strain in the direction of $\underline{N}$ is usually defined as:

$$\varepsilon_{\underline{N}} = \frac{\text{change of length}}{\text{original length}} = \frac{|d\underline{x}| - |d\underline{X}|}{|d\underline{X}|} = \lambda - 1$$

↳ one direction $\quad \left(\dfrac{|d\underline{x}|}{|d\underline{X}|} = \lambda \right)$

* However, to better represent large deformation cases, another (more convenient) measure of strain was introduced:

$$\varepsilon^*_{\underline{N}} = \frac{\frac{1}{2}(\text{def length})^2 - (\text{orig. length})^2}{(\text{original length})^2}$$

$$= \frac{1}{2}(\lambda^2 - 1)$$

* In terms of $\underline{\underline{C}}$ we can write:

$$\varepsilon^*_{\underline{N}} = \frac{1}{2}(\underline{N} \cdot \underline{\underline{C}} \cdot \underline{N} - 1)$$

* on the other hand;

$$\underline{N} \cdot \underline{\underline{I}} \cdot \underline{N} = N_i \delta_{ij} N_j = N_i N_i = \underbrace{N_1^2 + N_2^2 + N_3^2}_{|N|^2} = 1$$

$$\uparrow$$

$$|N| = 1$$

$$\Rightarrow \varepsilon^*_{\underline{N}} = \frac{1}{2}(\underline{N} \cdot \underline{\underline{C}} \cdot \underline{N} - \underline{N} \cdot \underline{\underline{I}} \cdot \underline{N})$$

$$= \frac{1}{2} \underline{N} \cdot (\underline{\underline{C}} - \underline{\underline{I}}) \cdot \underline{N}$$

$$= \underline{N} \cdot \underbrace{\frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}_{\text{denoted by } \underline{\underline{E}}} \cdot \underline{N}$$

Non linear measure of strain in the direction of $\underline{N}$

$$\boxed{\varepsilon^{*}_{\underline{N}} = \underline{N} \cdot \underline{\underline{E}} \cdot \underline{N} \; ; \quad \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \frac{1}{2}(\underline{\underline{F}}^{T} \cdot \underline{\underline{F}} - \underline{\underline{I}})}$$

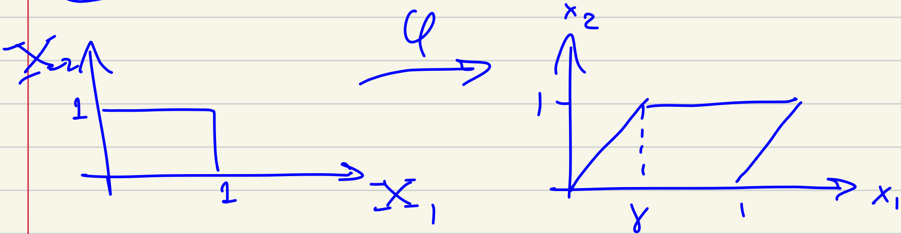
$\underline{\underline{E}}$ = Lagrangian strain tensor (also called Green strain tensor)

* By using $\underline{\underline{C}} = 2\underline{\underline{E}} + \underline{\underline{I}}$ in previously derived equations we can write them in terms of $\underline{\underline{E}}$ instead of $\underline{\underline{C}}$

* In summary: - Diagonal components of $\underline{\underline{E}}$ describe extensional strain in the direction of the basis vectors

- Off diagonal components enter the expressions for change of angles between fibers in the direction of the basis vectors.

Example: Simple shear (deck of cards)



how to define deformation mapping?

$$\varphi : \begin{cases} x_1 = X_1 + \gamma X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

- bottom not shifted
- Top shifted by γ

does not shift upward.

how to get deformation gradient?

$$F_{ij} = \frac{\partial \varphi_i}{\partial x_j} \qquad [F] = \begin{bmatrix} \partial x_1/\partial X_1 & \partial x_1/\partial X_2 & \partial x_1/\partial X_3 \\ \partial x_2/\partial X_1 & - & \cdot \\ - & \cdot & \partial x_3/\partial X_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

how to get C?

$$[C] = [F^T][F] = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & \gamma^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

how to get E?

$$\underline{\underline{E}} = \frac{[C]-[I]}{2} = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & \gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Compute ε^* in given directions, for example:

- Direction $\underline{e}_1$: $\varepsilon^*_{\underline{e}_1} = \underline{e}_1 \cdot \underline{\underline{E}} \cdot \underline{e}_1 = E_{11} = 0 \Rightarrow \boxed{\varepsilon^*_{\underline{e}_1} = 0}$

- Direction $\underline{e}_2$: $\varepsilon^*_{\underline{e}_2} = \underline{e}_2 \cdot \underline{\underline{E}} \cdot \underline{e}_2 = E_{22} = \frac{\gamma^2}{2}$

$$\Rightarrow \boxed{\varepsilon^*_{\underline{e}_2} = \frac{\gamma^2}{2}}$$

- Direction $\underline{N} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T$

$$\varepsilon^*_{\underline{N}} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & \gamma^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \varepsilon^*_{\underline{N}} = \frac{1}{2}(\gamma + \gamma^2)$$

since by definition, $\varepsilon^*_N = \frac{1}{2}(\lambda^2 - 1)$ we can compute stretches as $\lambda = \sqrt{2\varepsilon^*_N + 1}$

$$\lambda(\underline{e}_1) = 1$$

$$\lambda(\underline{e}_2) = \sqrt{\gamma^2 + 1}$$

$$\lambda(\underline{N}) = \sqrt{\gamma^2 + \gamma + 1}$$

Note: could have used $\underline{\underline{C}}$ instead of $\underline{\underline{E}}$ as well.