AE6114 - Fundamentals of Solid Mechanics

Fall 2020

Homework 2: Kinematics

Due at the indicated time on Canvas, on Tuesday, Sep 15th 2020

Problem 1

Consider a body occupying a cylinder of radius R and length L, $\Omega = \{(X_1, X_2, X_3) : X_1^2 + X_2^2 < R, 0 < X_3 < L\}$ in the reference configuration. It undergoes a deformation:

$$\begin{aligned} x_1 &= X_1 \cos(\tau X_3) - X_2 \sin(\tau X_3) \\ x_2 &= X_1 \sin(\tau X_3) + X_2 \cos(\tau X_3) \\ x_3 &= X_3 \end{aligned}$$

where τ is a constant parameter with unit $[1/\text{length}]$.

1. Describe the deformation. What is the meaning of τ?
2. Find the deformation gradient $\mathbf{F}$ and compute the right Cauchy-Green stretch tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.
3. Calculate the stretch λ for a fiber that, in the reference configuration, is oriented parallel to the plane defined by the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$.
4. Calculate the local change in area for differential elements of oriented area with normal along the $\mathbf{e}_3$ basis vector.
5. Calculate the local change of differential volume.

Problem 2

Let $\mathbf{dX}$ and $\mathbf{dY}$ be two differential vectors in the reference configuration along the directions of the unit vectors $\mathbf{M}$ and $\mathbf{N}$ correspondingly. If the body undergoes a motion defined by the action of the deformation mapping φ, we can compute the angle θ between the corresponding deformed differential vectors $\mathbf{dx}$ and $\mathbf{dy}$ by means of the expression

$$\cos\theta = \frac{\mathbf{N} \cdot \mathbf{C} \cdot \mathbf{M}}{\lambda_{\mathbf{N}} \lambda_{\mathbf{M}}}$$

where $\mathbf{C}$ is the Cauchy-Green stretch tensor, and $\lambda_{\mathbf{N}}$ and $\lambda_{\mathbf{M}}$ the stretches in the directions of $\mathbf{N}$ and $\mathbf{M}$. For the particular case when $\mathbf{N} \perp \mathbf{M}$, we have $\cos\theta = \sin\gamma$ where γ is the *change in angle* between the differential elements, see Figure 1.

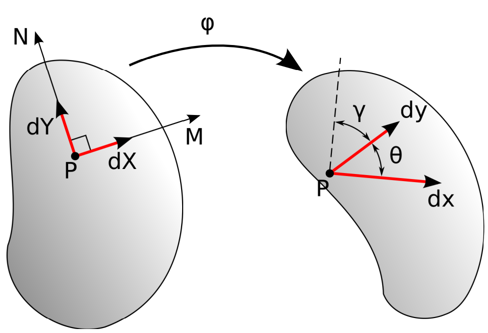

Figure 1: Schematics for Problem 2.

Show that, under the assumptions of infinitesimal deformations, the expression for γ reduces to $\gamma = 2\epsilon_{ij} N_i M_j$ where ϵ_{ij} are the components of the infinitesimal strain tensor.

Problem 3

Consider a body occupying a cube of length L, $\Omega = \{(X_1, X_2, X_3) : 0 < X_1 < L, 0 < X_2 < L, 0 < X_3 < L\}$ in the reference configuration. It undergoes a deformation:

$$\begin{aligned} x_1 &= AX_1 + BX_2 + CX_3 \\ x_2 &= CX_1 + AX_2 + BX_3 \\ x_3 &= BX_1 + CX_2 + AX_3 \end{aligned}$$

with:

$$A = \left(\frac{1}{3} + \frac{2}{3}\cos(\theta)\right), \qquad B = \left(\frac{1}{3} - \frac{1}{3}\cos(\theta) - \frac{1}{3}\sqrt{3}\sin(\theta)\right), \qquad C = \left(\frac{1}{3} - \frac{1}{3}\cos(\theta) + \frac{1}{3}\sqrt{3}\sin(\theta)\right)$$

1. Find the deformation gradient $\mathbf{F}$ and compute the Lagrangian strain tensor $\mathbf{E}$.
2. Show the expression for the displacement field $\mathbf{u}$ and compute the infinitesimal strain tensor ϵ.
3. Using both finite and infinitesimal deformations, calculate the *stretch* λ for fibers that, in the reference configuration, are oriented along the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
4. Using both finite and infinitesimal deformations, calculate the *change in angle* γ between fibers that, in the reference configuration, are oriented along the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
5. Using results from previous sections, discuss: what kind of deformation is taking place under the prescribed mapping? Why are results different? What can be inferred about infinitesimal deformations for this kind of mapping?

Problem 4

Consider a right Cauchy-Green stretch tensor $\mathbf{C}$. It has a spectral representation $\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v_i}\mathbf{v_i}$, where $\lambda_i > 0$ are the principal stretches, $|\mathbf{v}_i| = 1$, and $\mathbf{v}_i$ are mutually orthogonal. Let $\lambda_1 = \lambda_{\text{max}}$, $\lambda_2 = \lambda_{\text{min}}$, and $\lambda_{\text{max}} > \lambda_3 > \lambda_{\text{min}}$:

1. Show that the stretch λ_{max} in the direction $\mathbf{v}_1$ is larger than the stretch λ_N in any other direction $\mathbf{N}$.
2. Show that the stretch λ_{min} in the direction $\mathbf{v}_2$ is smaller than the stretch λ_N in any other direction $\mathbf{N}$.

Problem 5

A rubber block is reinforced by two sets of inextensible cables as shown in Figure 2. The block is subject to a deformation with stretching λ in the direction of $\mathbf{e}_2$. Assuming plane strain deformation in the plane defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, compute all components of the right Cauhy-Green deformation tensor $\mathbf{C}$.

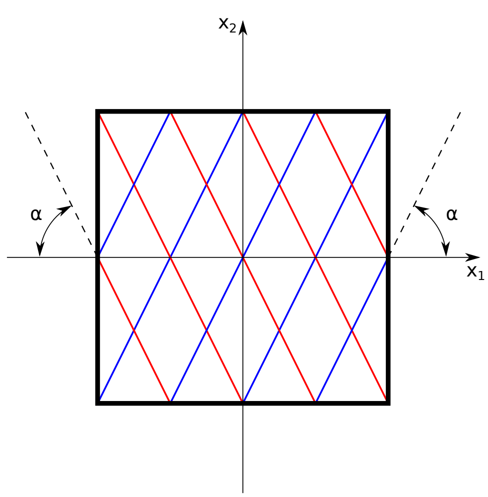

Figure 2: Schematics of reinforced rubber block. Two sets of inextensible fibers are denoted by the red and blue lines.