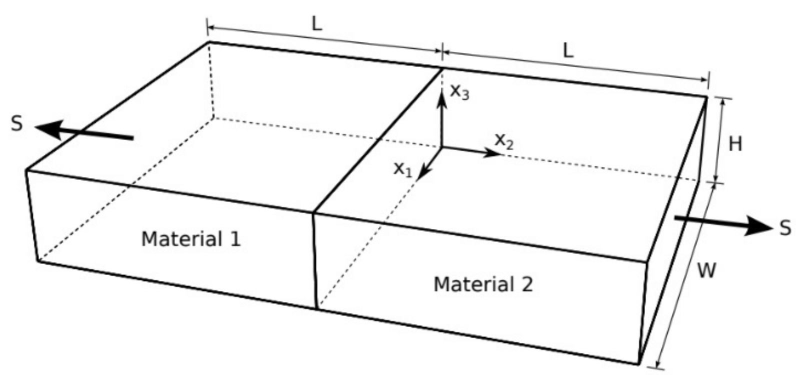
AE6114 Assignment 5: Linear Elasticity Part II

Problem 1

Consider two bricks of linear elastic materials of different properties glued together at the interface $x_2 = 0$ as shown in the figure below. The body is subject to uniaxial tension in the direction of x_2 through a uniform traction $\underline{s}$, and no body forces are present. Formulate the corresponding elastostatics boundary value problem.



These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering the answers from students that were marked correct. If you find any errors, please let me know!

– Sarah Malak

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There are no body forces:

$$\rho \underline{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad \nabla \underline{\underline{\sigma}} = \underline{0}$$

Equilibrium Equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0$$

Consitiutive Equations:

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11}$$
$$\sigma_{12} = 2\mu\varepsilon_{12}$$
$$\sigma_{13} = 2\mu\varepsilon_{13}$$
$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}$$
$$\sigma_{23} = 2\mu\varepsilon_{23}$$
$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33}$$

Compatibility Equations:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

Boundary Conditions (LHS):

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \underset{\sim}{t} \quad \Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -S \\ 0 \end{bmatrix}$$

Boundary Conditions (RHS):

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = \underset{\sim}{t} \quad \Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ S \\ 0 \end{bmatrix}$$

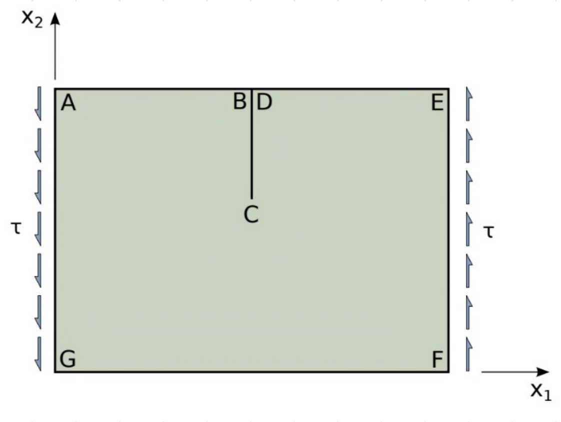
Boundary Conditions (Interface):

$$\underset{\sim}{u}(\underset{\sim}{x})_{RHS} = \underset{\sim}{u}(\underset{\sim}{x})_{LHS} \quad \text{at } x_2 = 0$$

← the displacement at the glued interface is equal for both bricks

Problem 2

Consider a 2-dimensional problem in the x_1-x_2 plane consisting of a rectangular body with a crack at the middle of one of its edges. A constant shear load is applied to two of its edges, as depicted in the figure below. Formulate the corresponding elastostatics boundary value problem.



Equilibrium Equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Compatibility Equations:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

Consitiutive Equations:

$$\sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11}$$
$$\sigma_{12} = 2\mu\varepsilon_{12}$$
$$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}$$

Boundary Conditions

$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{AB}, \quad \forall\, \underline{x} \in \partial\Omega_{DE}$$

no traction on the top faces AB and DE

$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{BC}$$

$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{CD}$$

no traction at the crack
(normal vector changes depending what side of the crack you're looking at)

$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{FG}$$

← no traction on the bottom face FG

$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{EF}$$

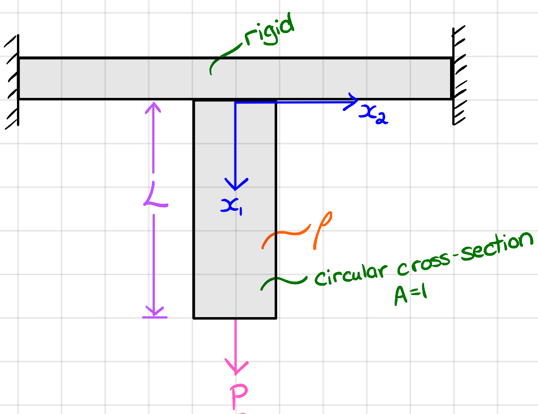
$$\underline{\underline{\sigma}} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\tau \end{bmatrix} \quad \forall\, \underline{x} \in \partial\Omega_{AG}$$

τ acts on the left and right faces (AG and EF)

Problem 3

Consider a bar of density ρ with a circular cross-section of unit area. The top of the bar is glued on the flat bottom surface of a rigid, horizontal beam. On the other end of the bar, a traction $\underline{P}$ pointing downwards is uniformly applied across the surface. The bar is deformed under its own weight and the applied traction $\underline{P}$.

1. Specify the body force acting on the bar.



$$\underline{b} = \begin{Bmatrix} \rho g \\ 0 \\ 0 \end{Bmatrix}$$

The body force is the weight of the bar

2. What are the traction boundary conditions for the bar?

$$\underset{\sim}{\sigma}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad \forall\, \underset{\sim}{x}\in\partial\Omega_{RHS}$$

$$\underset{\sim}{\sigma}\cdot\begin{bmatrix}0\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad \forall\, \underset{\sim}{x}\in\partial\Omega_{LHS}$$

sides are traction-free

Sides facing out and in to paper

$$\underset{\sim}{\sigma}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\underset{\sim}{\sigma}\cdot\begin{bmatrix}0\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\underset{\sim}{\sigma}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}P\\0\\0\end{bmatrix}\quad \forall\, \underset{\sim}{x}\in\partial\Omega_{B}$$

← bottom has applied load P

3. Assume that the problem can be treated as one-dimensional for stresses and strains, i.e., stresses and strains are only functions of x_1 and independent of x_2 and x_3. Using the traction boundary conditions and equilibrium equations, show that all components of the stress tensor are zero except σ_{11}, and that $\sigma_{11}=P+\rho g(L-x_1)$.

because of this question, x_1 must be pointing down along the length of the bar

<u>Equilibrium Equations:</u>

$$\frac{\partial\sigma_{11}}{\partial x_1}+\cancelto{0}{\frac{\partial\sigma_{12}}{\partial x_2}}+\cancelto{0}{\frac{\partial\sigma_{13}}{\partial x_3}}+\rho g=0 \qquad \frac{\partial\sigma_{11}}{\partial x_1}=-\rho g$$

$$\frac{\partial\sigma_{12}}{\partial x_1}+\cancelto{0}{\frac{\partial\sigma_{22}}{\partial x_2}}+\cancelto{0}{\frac{\partial\sigma_{23}}{\partial x_3}}=0 \qquad \frac{\partial\sigma_{12}}{\partial x_1}=0$$

$$\frac{\partial\sigma_{13}}{\partial x_1}+\cancelto{0}{\frac{\partial\sigma_{23}}{\partial x_2}}+\cancelto{0}{\frac{\partial\sigma_{33}}{\partial x_3}}=0 \qquad \frac{\partial\sigma_{13}}{\partial x_1}=0$$

$$\frac{\partial\sigma_{11}}{\partial x_1}=-\rho g \implies \partial\sigma_{11}=-\rho g\,\partial x_1 \implies \sigma_{11}=-\rho g x_1+C$$

<u>Boundary Conditions:</u>

$$\begin{bmatrix}\sigma_{11}&\sigma_{12}&\sigma_{13}\\\sigma_{12}&\sigma_{22}&\sigma_{23}\\\sigma_{13}&\sigma_{23}&\sigma_{33}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\quad \sigma_{12}+\sigma_{22}+\sigma_{23}=0$$

$$\begin{bmatrix}\sigma_{11}&\sigma_{12}&\sigma_{13}\\\sigma_{12}&\sigma_{22}&\sigma_{23}\\\sigma_{13}&\sigma_{23}&\sigma_{33}\end{bmatrix}\begin{bmatrix}0\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad -\sigma_{12}-\sigma_{22}-\sigma_{23}=0$$

$$\sigma_{12}=0 \qquad \sigma_{22}=0 \qquad \sigma_{23}=0$$

$$\begin{bmatrix}\sigma_{11}&\sigma_{12}&\sigma_{13}\\\sigma_{12}&\sigma_{22}&\sigma_{23}\\\sigma_{13}&\sigma_{23}&\sigma_{33}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\sigma_{13}+\cancelto{0}{\sigma_{23}}+\sigma_{33}=0 \qquad \sigma_{13}=0,\ \sigma_{33}=0$$

at $x_1=L$, $\sigma_{11}=P$: $\quad P=-\rho g L+C \implies C=\rho g L+P$

$$\sigma_{11}(x_1)=-\rho g x_1+P+\rho g L=P+\rho g(L-x_1)\ \checkmark$$

4. Find the infinitesimal strains ε_{ij}, assuming that the bar is made of a linearly elastic material.

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

$\sigma_{11} \neq 0$ and all other $\sigma = 0$ $\Rightarrow$ $\varepsilon_{12} = 0$, $\varepsilon_{13} = 0$, $\varepsilon_{23} = 0$

$$\varepsilon_{11} = -\frac{\nu}{E}\sigma_{11} + \frac{1+\nu}{E}\sigma_{11} = \frac{\sigma_{11}}{E}$$

$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11}$$

$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11}$$

$$\boxed{\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{\sigma_{11}}{E} & 0 & 0 \\ 0 & -\frac{\nu}{E}\sigma_{11} & 0 \\ 0 & 0 & -\frac{\nu}{E}\sigma_{11} \end{bmatrix} \quad \text{where } \sigma_{11} = P + \rho g(L - x_1)}$$

5. Find the corresponding displacement field u_i. Explain why the displacements at the top surface of the bar should be identically zero. Discuss why you cannot fit this displacement boundary condition with your solution.

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\sigma_{11}}{E} = \frac{P+\rho g(L-x_1)}{E}$$

$$u_1 = \int \frac{P+\rho g(L-x_1)}{E}dx_1 = \frac{1}{E}\left(\int P dx_1 + \int \rho g L dx_1 - \int \rho g x_1 dx_1\right) = \frac{1}{E}\left(Px_1 + \rho g L x_1 - \frac{1}{2}\rho g x_1^2\right) + g(x_2, x_3)$$

$$\boxed{u_1 = \frac{(P+\rho g L)x_1}{E} - \frac{\rho g x_1^2}{2E} + g(x_2, x_3)}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -\frac{\nu}{E}\sigma_{11} = -\frac{\nu}{E}\left[P+\rho g(L-x_1)\right]$$

$$u_2 = \int -\frac{\nu}{E}\left[P+\rho g(L-x_1)\right]dx_2 = -\frac{\nu}{E}\left(Px_2 + \rho g L x_2 - \rho g x_1 x_2\right) + g(x_1, x_3)$$

$$\boxed{u_2 = \frac{-\nu\left[P+\rho g(L-x_1)\right]x_2}{E} + g(x_1, x_3)}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -\frac{\nu}{E}\sigma_{11} = -\frac{\nu}{E}\left[P+\rho g(L-x_1)\right]$$

$$u_3 = \int -\frac{\nu}{E}\left[P+\rho g(L-x_1)\right]dx_3 = -\frac{\nu}{E}\left(Px_3 + \rho g L x_3 - \rho g x_1 x_3\right) + g(x_1, x_2)$$

$$\boxed{u_3 = \frac{-\nu\left[P+\rho g(L-x_1)\right]x_3}{E} + g(x_1, x_2)}$$

The displacements at the top of the bar should be identically zero because the bar is glued to a rigid beam, thus the top surface of the bar should not have any displacement (provided the glue holds).

We cannot fit this displacement boundary condition to our solution because we cannot show that $u_1 = u_2 = u_3 = 0$ at the top surface ($x_1 = 0$) since all of our calculated equations for u_i have components that are independent of x_1.

Problem 4

1. Consider the following 2D stress distribution on a body:

$$\sigma_{11} = 12x_1^4x_2^2 \ , \quad \sigma_{22} = 12x_1^2x_2^4 \ , \quad \sigma_{12} = -16x_1^3x_2^3$$

a) Show that this stress distribution satisfies the equation of equilibrium when no body forces are present.

$\underline{b} = \underline{0}$ assume body is static

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \cancel{\frac{\partial\sigma_{13}}{\partial x_3}}^{\,0} = 0 \qquad\qquad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \cancel{\frac{\partial\sigma_{23}}{\partial x_3}}^{\,0} = 0$$

$$48x_1^3x_2^2 - 48x_1^3x_2^2 + 0 = 0 \ \checkmark \qquad\qquad -48x_1^2x_2^3 + 48x_1^2x_2^3 + 0 = 0 \ \checkmark$$

$$\cancel{\frac{\partial\sigma_{13}}{\partial x_1}}^{\,0} + \cancel{\frac{\partial\sigma_{23}}{\partial x_2}}^{\,0} + \cancel{\frac{\partial\sigma_{33}}{\partial x_3}}^{\,0} = 0 \ \checkmark$$

b) Assuming isotropic linear elastic material, find the corresponding 2D strains $\varepsilon_{11}, \varepsilon_{12}$, and ε_{22}.

$$\varepsilon_{11} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) + \frac{1+\nu}{E}\sigma_{11} = -\frac{\nu}{E}(12x_1^4x_2^2 + 12x_1^2x_2^4) + \frac{1+\nu}{E}(12x_1^4x_2^2)$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+\nu}{E}(-16x_1^3x_2^3)$$

$$\varepsilon_{22} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) + \frac{1+\nu}{E}\sigma_{22} = -\frac{\nu}{E}(12x_1^4x_2^2 + 12x_1^2x_2^4) + \frac{1+\nu}{E}(12x_1^2x_2^4)$$

$$\boxed{\begin{aligned} \varepsilon_{11} &= \frac{12}{E}x_1^2x_2^2\left(x_1^2 - \nu x_2^2\right) \\ \varepsilon_{12} &= -\frac{(1+\nu)}{E}16x_1^3x_2^3 \\ \varepsilon_{22} &= \frac{12}{E}x_1^2x_2^2\left(x_2^2 - \nu x_1^2\right) \end{aligned}}$$

c) Show that these strains do not satisfy the compatibility equation:

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$$

$$\varepsilon_{11} = \frac{1}{E}(12x_1^4x_2^2) - \frac{\nu}{E}(12x_1^2x_2^4) \rightarrow \frac{\partial\varepsilon_{11}}{\partial x_2} = \frac{2}{E}(12x_1^4x_2) - \frac{4\nu}{E}(12x_1^2x_2^3) \rightarrow \frac{\partial^2\varepsilon_{11}}{\partial x_2^2} = \frac{2}{E}(12x_1^4) - \frac{12\nu}{E}(12x_1^2x_2^2)$$

$$\varepsilon_{22} = \frac{1}{E}(12x_1^2x_2^4) - \frac{\nu}{E}(12x_1^4x_2^2) \rightarrow \frac{\partial\varepsilon_{22}}{\partial x_1} = \frac{2}{E}(12x_1x_2^4) - \frac{4\nu}{E}(12x_1^3x_2^2) \rightarrow \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} = \frac{2}{E}(12x_2^4) - \frac{12\nu}{E}(12x_1^2x_2^2)$$

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_1^2} = \frac{2}{E}(12x_1^4 + 12x_2^4) - \frac{24\nu}{E}(12x_1^2x_2^2)$$

$$\varepsilon_{12} = -\frac{(1+\nu)}{E} 16x_1^3 x_2^3 \rightarrow \frac{\partial \varepsilon_{12}}{\partial x_1} = -\frac{3(1+\nu)}{E} 16x_1^2 x_2^3 \rightarrow \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = -\frac{9(1+\nu)}{E} 16x_1^2 x_2^2$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{24x_1^4 + 24x_2^4 - 288\nu x_1^2 x_2^2}{E} \neq \frac{-144x_1^2x_2^2 - 144\nu x_1^2 x_2^2}{E}$$

$\Rightarrow$ these strains do not satisfy the compatibility equation ✓

d) Integrate the ε_{11} strains to obtain an expression for the displacement component u_1 and the ε_{22} strains to obtain an expression for the displacement component u_2. Show that it is impossible to obtain the prescribed strain ε_{12} from the obtained expressions for u_1 and u_2.

$$\varepsilon_{11} = \frac{1}{E}(12x_1^4 x_2^2) - \frac{\nu}{E}(12x_1^2 x_2^4) = \frac{\partial u_1}{\partial x_1}$$

$$u_1 = \frac{1}{E}\int 12x_1^4 x_2^2 \, \partial x_1 - \frac{\nu}{E}\int 12x_1^2 x_2^4 \, \partial x_1$$

$$\rightarrow u_1 = \frac{12x_1^5 x_2^2}{5E} - \frac{\cancel{12}^{\,4}\nu x_1^3 x_2^4}{\cancel{3}E} + g_1(x_2)$$

$$\varepsilon_{22} = \frac{1}{E}(12x_1^2 x_2^4) - \frac{\nu}{E}(12x_1^4 x_2^2) = \frac{\partial u_2}{\partial x_2}$$

$$u_2 = \frac{1}{E}\int 12x_1^2 x_2^4 \, \partial x_2 - \frac{\nu}{E}\int 12x_1^4 x_2^2 \, \partial x_2$$

$$\rightarrow u_2 = \frac{12x_1^2 x_2^5}{5E} - \frac{\cancel{12}^{\,4}\nu x_1^4 x_2^3}{\cancel{3}E} + g_2(x_1)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -\frac{(1+\nu)}{E} 16x_1^3 x_2^3$$

$$\frac{\partial u_1}{\partial x_2} = \frac{24x_1^5 x_2}{5E} - \frac{16\nu x_1^3 x_2^3}{E} + \frac{\partial g_1(x_2)}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{24x_1 x_2^5}{5E} - \frac{16\nu x_1^3 x_2^3}{E} + \frac{\partial g_2(x_1)}{\partial x_1}$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{12x_1^5 x_2}{5E} + \frac{12x_1 x_2^5}{5E} - \frac{8\nu x_1^3 x_2^3}{E} - \frac{8\nu x_1^3 x_2^3}{E} + \frac{1}{2}\left(\frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right)$$

$$= \frac{24x_1 x_2}{5E}(x_1^4 + x_2^4) - \frac{16\nu x_1^3 x_2^3}{E} + \frac{1}{2}\left(\frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right)$$

Setting this equal to the ε_{12} we calculated before:

$$\frac{-16x_1^3 x_2^3}{E} - \cancel{\frac{16\nu x_1^3 x_2^3}{E}} = \frac{24x_1 x_2}{5E}(x_1^4 + x_2^4) - \cancel{\frac{16\nu x_1^3 x_2^3}{E}} + \frac{1}{2}\left(\frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right)$$

we're left with:

$$-\frac{16x_1 x_2}{E}\left[2x_1^2 x_2^2 + \frac{3}{5}(x_1^4 + x_2^4)\right] = \frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}$$

This expression cannot be satisfied because the addition of g_1 and g_2 will never result in a product of x_1 and x_2.

2. Consider the following 2D stress distribution on a body:

$$\sigma_{11} = -e^{x_1}\sin x_2, \quad \sigma_{22} = e^{x_1}\sin x_2, \quad \sigma_{12} = -e^{x_1}\cos x_2$$

a) Show that this stress distribution satisfies the equation of equilibrium when no body forces are present.

$$\underline{b} = \underline{0}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = -e^{x_1}\sin x_2 + e^{x_1}\sin x_2 = 0 \ \checkmark$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = -e^{x_1}\cos x_2 + e^{x_1}\cos x_2 = 0 \ \checkmark$$

b) Assuming isotropic linear elastic material, find the corresponding 2D strains $\mathcal{E}_{11}$, $\mathcal{E}_{12}$, and $\mathcal{E}_{22}$.

$$\mathcal{E}_{11} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) + \frac{1+\nu}{E}\sigma_{11} = -\frac{(1+\nu)}{E}e^{x_1}\sin x_2$$

$$\mathcal{E}_{12} = \frac{1+\nu}{E}\sigma_{12} = -\frac{(1+\nu)}{E}e^{x_1}\cos x_2$$

$$\mathcal{E}_{22} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) + \frac{1+\nu}{E}\sigma_{22} = \frac{1+\nu}{E}e^{x_1}\sin x_2$$

c) Show that these strains do satisfy the compatibility equation:

$$\frac{\partial^2 \mathcal{E}_{11}}{\partial x_2^2} + \frac{\partial^2 \mathcal{E}_{22}}{\partial x_1^2} = 2\frac{\partial^2 \mathcal{E}_{12}}{\partial x_1 \partial x_2}$$

$$\frac{\partial \mathcal{E}_{11}}{\partial x_2} = -\frac{(1+\nu)}{E}e^{x_1}\cos x_2 \rightarrow \frac{\partial^2 \mathcal{E}_{11}}{\partial x_2^2} = \frac{(1+\nu)}{E}e^{x_1}\sin x_2$$

$$\frac{\partial \mathcal{E}_{22}}{\partial x_1} = \frac{(1+\nu)}{E}e^{x_1}\sin x_2 \rightarrow \frac{\partial^2 \mathcal{E}_{22}}{\partial x_1^2} = \frac{(1+\nu)}{E}e^{x_1}\sin x_2$$

$$\frac{\partial \mathcal{E}_{12}}{\partial x_1} = \frac{-(1+\nu)}{E}e^{x_1}\cos x_2 \rightarrow \frac{\partial^2 \mathcal{E}_{12}}{\partial x_1 \partial x_2} = \frac{(1+\nu)}{E}e^{x_1}\sin x_2$$

Therefore,

$$\frac{\partial^2 \mathcal{E}_{11}}{\partial x_2^2} + \frac{\partial^2 \mathcal{E}_{22}}{\partial x_1^2} = \frac{(1+\nu)}{E}e^{x_1}\sin x_2 + \frac{(1+\nu)}{E}e^{x_1}\sin x_2 = 2\frac{(1+\nu)}{E}e^{x_1}\sin x_2 = 2\frac{\partial^2 \mathcal{E}_{12}}{\partial x_1 \partial x_2} \quad \checkmark$$

$\Rightarrow$ these strains satisfy the compatibility equation

d) Integrate the $\mathcal{E}_{11}$ strains to obtain an expression for the displacement component u_1 and the $\mathcal{E}_{22}$ strains to obtain the prescribed strain $\mathcal{E}_{12}$ from the obtained expressions for u_1 and u_2.

$$\mathcal{E}_{11} = -\frac{(1+\nu)}{E}e^{x_1}\sin x_2 = \frac{\partial u_1}{\partial x_1}$$

$$u_1 = -\frac{(1+\nu)}{E}\int e^{x_1}\sin x_2 \, \partial x_1 = -\frac{(1+\nu)}{E}e^{x_1}\sin x_2 + g_1(x_2)$$

$$\mathcal{E}_{22} = \frac{1+\nu}{E} e^{x_1} \sin x_2 = \frac{\partial u_2}{\partial x_2}$$

$$u_2 = \frac{1+\nu}{E} \int e^{x_1} \sin x_2 \, \partial x_2 = -\frac{(1+\nu)}{E} e^{x_1} \cos x_2 + g_2(x_1)$$

$$\mathcal{E}_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\frac{\partial u_1}{\partial x_2} = -\frac{(1+\nu)}{E} e^{x_1} \cos x_2 + \frac{\partial g_1(x_2)}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{-(1+\nu)}{E} e^{x_1} \cos x_2 + \frac{\partial g_2(x_1)}{\partial x_1}$$

$$\mathcal{E}_{12} = \frac{1}{2}\left(-2\frac{(1+\nu)}{E} e^{x_1} \cos x_2 + \frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right)$$

Setting this equal to the $\mathcal{E}_{12}$ we calculated before:

$$-\frac{(1+\nu)}{E} \cancel{e^{x_1}} \cos x_2 = -\frac{(1+\nu)}{E} \cancel{e^{x_1}} \cos x_2 + \frac{1}{2}\left(\frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right)$$

we're left with:

$$\frac{1}{2}\left(\frac{\partial g_1(x_2)}{\partial x_2} + \frac{\partial g_2(x_1)}{\partial x_1}\right) = 0$$

This expression will be true if g_1 and g_2 are constants or if $g_1(x_2) = C_1 x_2$ and $g_1(x_1) = -C_1 x_1$

3. Comment on the differences between cases 1 and 2. Does a stress field that satisfies conservation of linear and angular momentum necessarily make physical sense? Please elaborate.

A stress field that satisfies the conservation of linear and angular momentum may not make physical sense but will still produce a possible displacement field. This is because the conservations do not rely on the material properties or on how the material behaves. The relationship between stresses and displacements, however, do depend on the material properties and its behavior. This means that a material can only create a certain set of stress fields based on how it behaves. Therefore, a given stress field may require an object to undergo what would be considered an impossible deformation in order to produce that stress field.

The stress field for Case 1 was unable to satisfy the compatibility equations for infinitesimal strains. This would indicate that the strains were either not infinitesimal or not linear (the stress field given could not be produced by a linearly elastic material undergoing infinitesimal strain). It could be possible, however, for this stress field to have been produced by a different type of material.

For Case 2, the material is linearly elastic and undergoing infinitesimal strains, so the displacement field is physically possible and could produce the given stress field.