3D Euler-Bernoulli Beam Theory

AE3140: Structural Analysis

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(Adapted from slides by M. Ruzzene from "Structural Analysis" by O.A. Bauchau and J.I. Craig)

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Contents

Overview

- In numerous practical applications, the beam's cross-section presents no particular symmetries and is instead of arbitrary shape.
- In addition, the applied loads may act along several distinct directions and not just in plane $(\bar{\imath}_1, \bar{\imath}_2)$

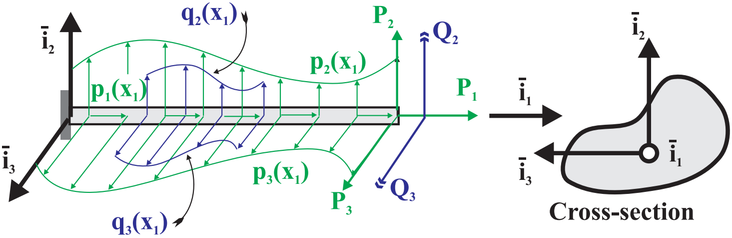

Figure: Beam with arbitrary three-dimensional loading.

Assumptions

The three-dimensional loading is general, with the following assumptions:

1. No torsional loads are applied
2. The transverse loads are assumed to be applied in such a manner that the beam will bend without twisting.
3. The cross-section of the beam is of arbitrary shape. The origin of the axes has not yet been specified, and the orientation of axes $\bar{\imath}_2$ and $\bar{\imath}_3$ within the plane of the section is arbitrary

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Kinematic description

The three-dimensional beam theory is based on the three Euler-Bernoulli assumptions.
The assumptions are of a purely kinematic nature and imply the following displacements field:

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) + x_3\Phi_2(x_1) - x_2\Phi_3(x_1), \tag{1a}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \tag{1b}$$

$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1). \tag{1c}$$

Kinematic description

The corresponding strain field is shown to be

$$\varepsilon_2 = 0; \quad \varepsilon_3 = 0; \quad \gamma_{23} = 0, \tag{2a}$$

$$\gamma_{12} = 0; \quad \gamma_{13} = 0, \tag{2b}$$

$$\varepsilon_1(x_1, x_2, x_3) = \bar{\varepsilon}_1(x_1) + x_3\ \kappa_2(x_1) - x_2\ \kappa_3(x_1). \tag{2c}$$

Sectional constitutive law

Assume that the beam is made of linearly elastic, isotropic material, the axial stress distribution is given by:

$$\sigma_1(x_1, x_2, x_3) = E\left[\bar{\varepsilon}_1(x_1) + x_3\ \kappa_2(x_1) - x_2\ \kappa_3(x_1)\right] \tag{3}$$

The axial force, N_1, is

$$\begin{aligned} N_1(x_1) =& \int_{\mathcal{A}} \sigma_1\ \mathrm{d}\mathcal{A} = \int_{\mathcal{A}} E\bar{\epsilon}_1\ \mathrm{d}\mathcal{A} + \int_{\mathcal{A}} Ex_3\kappa_2\ \mathrm{d}\mathcal{A} - \int_{\mathcal{A}} Ex_2\kappa_3\ \mathrm{d}\mathcal{A} \\ =& \left[\int_{\mathcal{A}} E\ \mathrm{d}\mathcal{A}\right]\bar{\epsilon}_1 + \left[\int_{\mathcal{A}} Ex_3\ \mathrm{d}\mathcal{A}\right]\kappa_2 - \left[\int_{\mathcal{A}} Ex_2\ \mathrm{d}\mathcal{A}\right]\kappa_3 \\ =& S\ \bar{\epsilon}_1(x_1) + S_3\ \kappa_2(x_1) - S_2\ \kappa_3(x_1), \end{aligned} \tag{4}$$

where the sectional stiffness coefficients are defined

$$S = \int_{\mathcal{A}} E \, \mathrm{d}\mathcal{A}; \quad S_2 = \int_{\mathcal{A}} E x_2 \, \mathrm{d}\mathcal{A}; \quad S_3 = \int_{\mathcal{A}} E x_3 \, \mathrm{d}\mathcal{A}. \tag{5}$$

The bending moments, M_2 and M_3, acting about axes $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively, are

$$\begin{aligned}
M_2 = \int_{\mathcal{A}} x_3 \sigma_1 \, \mathrm{d}\mathcal{A} &= \int_{\mathcal{A}} x_3 E \bar{\epsilon}_1 \, \mathrm{d}\mathcal{A} + \int_{\mathcal{A}} E x_3^2 \kappa_2 \, \mathrm{d}\mathcal{A} - \int_{\mathcal{A}} E x_2 x_3 \kappa_3 \, \mathrm{d}\mathcal{A} \\
&= \left[\int_{\mathcal{A}} E x_3 \, \mathrm{d}\mathcal{A}\right] \bar{\epsilon}_1 + \left[\int_{\mathcal{A}} E x_3^2 \, \mathrm{d}\mathcal{A}\right] \kappa_2 - \left[\int_{\mathcal{A}} E x_2 x_3 \, \mathrm{d}\mathcal{A}\right] \kappa_3 \\
&= S_3 \; \bar{\epsilon}_1(x_1) + H_{22} \; \kappa_2(x_1) - H_{23} \; \kappa_3(x_1),
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
M_3 = - \int_{\mathcal{A}} x_2 \sigma_1 \, \mathrm{d}\mathcal{A} &= - \int_{\mathcal{A}} x_2 E \bar{\epsilon}_1 \, \mathrm{d}\mathcal{A} - \int_{\mathcal{A}} x_2 E x_3 \kappa_2 \, \mathrm{d}\mathcal{A} + \int_{\mathcal{A}} E x_2^2 \kappa_3 \, \mathrm{d}\mathcal{A} \\
&= - \left[\int_{\mathcal{A}} E x_2 \, \mathrm{d}\mathcal{A}\right] \bar{\epsilon}_1 - \left[\int_{\mathcal{A}} E x_2 x_3 \, \mathrm{d}\mathcal{A}\right] \kappa_2 + \left[\int_{\mathcal{A}} E x_2^2 \, \mathrm{d}\mathcal{A}\right] \kappa_3 \\
&= - S_2 \; \bar{\epsilon}_1(x_1) - H_{23} \; \kappa_2(x_1) + H_{33} \; \kappa_3(x_1),
\end{aligned} \tag{7}$$

where the following additional sectional stiffness coefficients are defined

$$H_{22} = \int_{\mathcal{A}} E\ x_3^2\ \mathrm{d}\mathcal{A}; \quad H_{33} = \int_{\mathcal{A}} E\ x_2^2\ \mathrm{d}\mathcal{A}; \tag{8}$$

$$H_{23} = \int_{\mathcal{A}} E\ x_2 x_3\ \mathrm{d}\mathcal{A}. \tag{9}$$

The sectional constitutive laws can be written in a compact matrix form

$$\left\{\begin{matrix} N_1(x_1) \\ M_2(x_1) \\ M_3(x_1) \end{matrix}\right\} = \begin{bmatrix} S & S_3 & -S_2 \\ S_3 & H_{22} & -H_{23} \\ -S_2 & -H_{23} & H_{33} \end{bmatrix} \left\{\begin{matrix} \bar{\epsilon}_1(x_1) \\ \kappa_2(x_1) \\ \kappa_3(x_1) \end{matrix}\right\}. \tag{10}$$

Notes

- The equations express a general linear relationship between the sectional stress resultants and the sectional strains.
- Thus, they are the constitutive laws for the cross-section of the beam,
- The matrix on the right hand side of eq. (10) is called the *sectional stiffness matrix*.

Notes

- A general formulation of three dimensional Euler-Bernoulli beam theory can be developed based on the constitutive laws of eq. (10).
- This leads to complex governing differential equations
- The sectional constitutive laws can be simplified by selecting the axis system appropriately.
- The formulation is developed for arbitrary origin and orientations of axes $\bar{\imath}_2$ and $\bar{\imath}_3$ within the plane of the cross-section.

Centroidal frame

The origin of the axis system can be selected to coincide with the centroid of the section:

$$x_{2c} = \frac{1}{S}\int_{\mathcal{A}} E\ x_2\ \mathrm{d}\mathcal{A} = \frac{S_2}{S} = 0; \quad x_{3c} = \frac{1}{S}\int_{\mathcal{A}} E\ x_3\ \mathrm{d}\mathcal{A} = \frac{S_3}{S} = 0, \quad (11)$$

The sectional constitutive laws, eq. (10), reduce to

$$\left\{\begin{matrix} N_1(x_1) \\ M_2(x_1) \\ M_3(x_1) \end{matrix}\right\} = \begin{bmatrix} S & 0 & 0 \\ 0 & H_{22}^c & -H_{23}^c \\ 0 & -H_{23}^c & H_{33}^c \end{bmatrix} \left\{\begin{matrix} \bar{\epsilon}_1(x_1) \\ \kappa_2(x_1) \\ \kappa_3(x_1) \end{matrix}\right\}. \quad (12)$$

Notes

- These partially uncoupled equations show that the axial force N_1 is now related to only the axial strain $\bar{\epsilon}_1$
- The bending moments are related to the curvatures κ_2 and κ_3 only
- This decoupling of the axial and bending behavior results from locating the origin of the axis system the centroid of the cross-section
- The two bending moments and corresponding curvatures are still coupled due to the presence of the stiffness coefficient, H_{23}^c

Strain distribution

The sectional constitutive equations, eqs. (12), can be inverted and solved for the sectional strain, $\bar{\epsilon}_1$, and curvatures, κ_2 and κ_3, in terms of stress resultants, N_1, M_2 and M_3

$$\left\{\begin{matrix} \bar{\epsilon}_1(x_1) \\ \kappa_2(x_1) \\ \kappa_3(x_1) \end{matrix}\right\} = \begin{bmatrix} 1/S & 0 & 0 \\ 0 & H^c_{33}/\Delta_H & H^c_{23}/\Delta_H \\ 0 & H^c_{23}/\Delta_H & H^c_{22}/\Delta_H \end{bmatrix} \left\{\begin{matrix} N_1(x_1) \\ M_2(x_1) \\ M_3(x_1) \end{matrix}\right\}, \qquad (13)$$

where $\Delta_H = H^c_{22}H^c_{33} - H^c_{23}H^c_{23}$.

Stress distribution

The axial stress is given by

$$\sigma_1 = E\left[\frac{N_1}{S} + x_3\ \frac{H_{33}^c M_2 + H_{23}^c M_3}{\Delta_H} - x_2\ \frac{H_{23}^c M_2 + H_{22}^c M_3}{\Delta_H}\right], \qquad (14)$$

or

$$\sigma_1 = E\left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}\ M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H}\ M_3\right]. \qquad (15)$$

This is a key result because it relates the axial stress distribution to the stress resultants which are, in turn, functions of the applied loads.

Sectional equilibrium equations

Consider an infinitesimal slice of the beam of length $\mathrm{d}x_1$

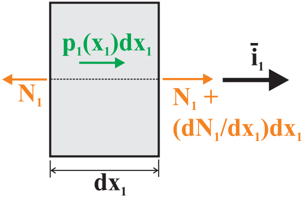

Figure: Free body diagram for the axial forces.

Summing all the forces in the axial direction yields the axial equilibrium equation

$$\frac{\mathrm{d}N_1}{\mathrm{d}x_1} = -p_1(x_1). \tag{16}$$

Sectional equilibrium equations

Similarly, for transverse loads:

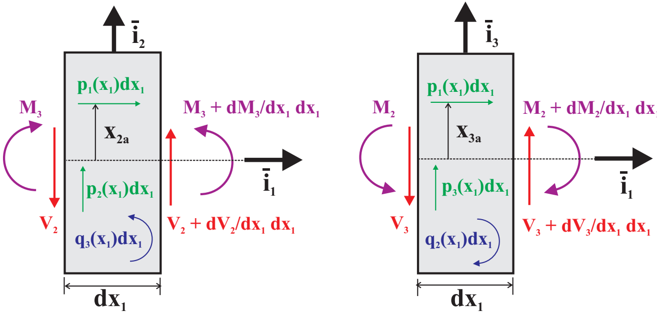

Figure: Free body diagram for the transverse shear forces and bending moments. Left figure: view of the $(\bar{\imath}_1, \bar{\imath}_2)$ plane; right figure: view of the $(\bar{\imath}_1, \bar{\imath}_3)$ plane;

Summation of the forces acting along axis $\bar{\imath}_2$ gives the transverse equilibrium equation

$$\frac{\mathrm{d}V_2}{\mathrm{d}x_1} = -p_2(x_1). \tag{17}$$

Summation of the moments taken about the centroidal axis $\bar{\imath}_3$ yields

$$\frac{\mathrm{d}M_3}{\mathrm{d}x_1} + V_2 = -q_3(x_1) + x_{2a}p_1(x_1), \tag{18}$$

Summing the forces along axis $\bar{\imath}_3$ gives the second transverse equilibrium equation

$$\frac{\mathrm{d}V_3}{\mathrm{d}x_1} = -p_3(x_1), \tag{19}$$

and summing the moments about the centroidal axis $\bar{\imath}_2$ leads to

$$\frac{\mathrm{d}M_2}{\mathrm{d}x_1} - V_3 = -q_2(x_1) - x_{3a}p_1(x_1), \tag{20}$$

where x_{3a} defines the location at which the axial force p_1 acts on the cross-section.

Sectional equilibrium equations

The shear forces, V_2 and V_3, can be eliminated from the equilibrium equations by taking a derivative of eqs. (20) and (18), then introducing eqs. (19) and (17), respectively, to yield the equilibrium equations

$$\frac{\mathrm{d}^2 M_2}{\mathrm{d}x_1^2} = -p_3(x_1) - \frac{\mathrm{d}}{\mathrm{d}x_1}[x_{3a}p_1(x_1) + q_2(x_1)], \tag{21a}$$

$$\frac{\mathrm{d}^2 M_3}{\mathrm{d}x_1^2} = \quad p_2(x_1) + \frac{\mathrm{d}}{\mathrm{d}x_1}[x_{2a}p_1(x_1) - q_3(x_1)]. \tag{21b}$$

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Governing equations

The governing equations for the beam transverse displacement field can be formulated as second order differential equations through the sectional constitutive laws

$$\begin{aligned} H_{23}^c \frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}x_1^2} + H_{22}^c \frac{\mathrm{d}^2 \bar{u}_3}{\mathrm{d}x_1^2} &= -M_2(x_1), \\ H_{33}^c \frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}x_1^2} + H_{23}^c \frac{\mathrm{d}^2 \bar{u}_3}{\mathrm{d}x_1^2} &= \quad M_3(x_1). \end{aligned} \tag{22}$$

These differential equations can be used to solve for the beam transverse displacement field when the bending moments, $M_2(x_1)$ and $M_3(x_1)$, are known.

Governing equations

Fourth order differential equations are obtained by introducing the sectional constitutive laws, eqs. (12), into the equilibrium equations, eqs. (16), (21a), and (21b), and then using the definition of the sectional strains to find

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[S\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1}\right] = -p_1, \tag{23a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left[H_{33}^c\frac{d^2\bar{u}_2}{\mathrm{d}x_1^2} + H_{23}^c\frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}x_1^2}\right] = p_2 + \frac{\mathrm{d}}{\mathrm{d}x_1}[x_{2a}p_1 - q_3], \tag{23b}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left[H_{23}^c\frac{d^2\bar{u}_2}{\mathrm{d}x_1^2} + H_{22}^c\frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}x_1^2}\right] = p_3 + \frac{\mathrm{d}}{\mathrm{d}x_1}[x_{3a}p_1 + q_2]. \tag{23c}$$

Decoupling the three-dimensional problem

- The theory developed in the previous section requires the *axis system to be centroidal*, that is that axis $\bar{\imath}_1$ passes through the centroid of the cross-section.
- This choice decouples the axial behavior from bending
- If an axial force is not applied at the centroid, it will contribute to the bending problem
- An important case of axial forces not applied at the centroid is found in air vehicles such as helicopters. The large centrifugal force generated by the rotation of the blade is an axial force *applied at the sectional center of mass*, which is, in general, distinct from its centroid.

Principal axes of bending

The governing equations can be further simplified by a judicious choice of the orientation of the centroidal axis system.

The *principal centroidal axes of bending* are defined as a set of axes with their origin at the centroid of the section and for which

$$H_{23}^{c} = \int_{\mathcal{A}} E x_2 x_3 \, \mathrm{d}\mathcal{A} = 0. \tag{24}$$

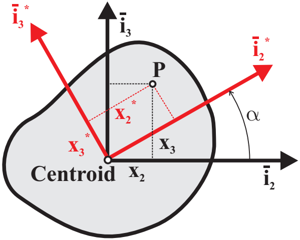


Figure: Rotation of the axes of the cross-section.

The centroidal bending stiffnesses in system $\mathcal{I}^*$ can be computed using eq. (8) to find

$$H_{22}^{c*} = \int_{\mathcal{A}} E\ (-x_2 \sin\alpha + x_3 \cos\alpha)^2\ \mathrm{d}\mathcal{A},$$
$$H_{33}^{c*} = \int_{\mathcal{A}} E\ (x_2 \cos\alpha + x_3 \sin\alpha)^2\ \mathrm{d}\mathcal{A},$$
$$H_{23}^{c*} = \int_{\mathcal{A}} E\ (x_2 \cos\alpha + x_3 \sin\alpha)(-x_2 \sin\alpha + x_3 \cos\alpha)\ \mathrm{d}\mathcal{A}.$$

Expanding these expressions, and noting that centroidal axes are being used, gives

$$H_{22}^{c*} = H_{22}^{c} \cos^2\alpha + H_{33}^{c} \sin^2\alpha - 2H_{23}^{c} \sin\alpha\cos\alpha, \tag{25a}$$
$$H_{33}^{c*} = H_{22}^{c} \sin^2\alpha + H_{33}^{c} \cos^2\alpha + 2H_{23}^{c} \sin\alpha\cos\alpha, \tag{25b}$$
$$H_{23}^{c*} = (H_{22}^{c} - H_{33}^{c}) \sin\alpha\cos\alpha + H_{23}^{c}(\cos^2\alpha - \sin^2\alpha). \tag{25c}$$

These expressions can be rewritten as

$$H_{22}^{c*} = \frac{H_{22}^c + H_{33}^c}{2} + \frac{H_{22}^c - H_{33}^c}{2}\cos 2\alpha - H_{23}^c \sin 2\alpha; \tag{26a}$$

$$H_{33}^{c*} = \frac{H_{22}^c + H_{33}^c}{2} - \frac{H_{22}^c - H_{33}^c}{2}\cos 2\alpha + H_{23}^c \sin 2\alpha; \tag{26b}$$

$$H_{23}^{c*} = + \frac{H_{22}^c - H_{33}^c}{2}\sin 2\alpha + H_{23}^c \cos 2\alpha. \tag{26c}$$

In the *principal* frame of reference, the sectional constitutive laws are fully decoupled

$$\bar{\epsilon}_1^* = \frac{N_1^*}{S^*}, \quad \kappa_2^* = \frac{M_2^*}{H_{22}^{c*}}, \quad \kappa_3^* = \frac{M_3^*}{H_{33}^{c*}}. \tag{27}$$

The corresponding axial stress distribution becomes

$$\sigma_1^* = E\left[\frac{N_1^*}{S^*} + x_3^* \, \frac{M_2^*}{H_{22}^{c*}} - x_2^* \, \frac{M_3^*}{H_{33}^{c*}}\right], \tag{28}$$

Decoupled governing equations: the axial problem

The axial problem is governed by eq. (23a), which now takes on the following form

$$\frac{\mathrm{d}}{\mathrm{d}x_1^*}\left[S^*\frac{\mathrm{d}\bar{u}_1^*}{\mathrm{d}x_1^*}\right] = -p_1^*. \tag{29}$$

Decoupled governing equations: the first bending problem

The bending problem reduces to two independent equations. The first of these takes the following form

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^{*2}}\left[H_{33}^{c*}\frac{\mathrm{d}^2\bar{u}_2^*}{\mathrm{d}x_1^{*2}}\right] = p_2^* + \frac{\mathrm{d}}{\mathrm{d}x_1^*}[x_{2a}^* p_1^* - q_3^*], \tag{30}$$

which describes bending in plane $(\bar{\imath}_1^*, \bar{\imath}_2^*)$.

Decoupled governing equations: The second bending problem

Bending in the plane $(\bar{\imath}_1, \bar{\imath}_3^*)$ is described by

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^{*2}}\left[H_{22}^{c*}\frac{\mathrm{d}^2\bar{u}_3^*}{\mathrm{d}x_1^{*2}}\right] = p_3^* + \frac{\mathrm{d}}{\mathrm{d}x_1^*}[x_{3a}^* p_1^* + q_2^*], \tag{31}$$

Principal centroidal bending stiffness

In summary, the orientation of the principal centroidal axes of bending is obtained according to the following procedure.

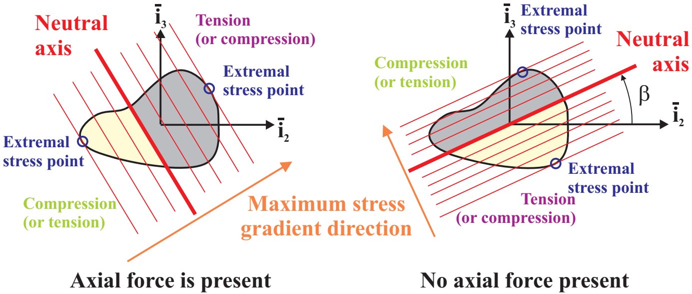
1. Compute the centroid of the section using the definition;
2. Compute the bending stiffnesses in this axis system;
3. Compute the orientation of the principal axes of bending;
4. Compute the principal bending stiffnesses.

The neutral axis

For sections made of homogeneous material, three distinct types of the axial stress distribution are possible over the cross-section.

1. If the *axial force, N_1, has a sufficiently large tensile (positive) value*, the axial stress is tensile over the entire cross-section.
2. If the *axial force, N_1, has a sufficiently large compressive (negative) value*, the axial stress is compressive over the entire cross-section.
3. If the *axial force, N_1, assumes an intermediate value or vanishes*, the axial stress will vanish along a straight line intersecting the boundaries of the cross-section; the axial stress will be tensile on one side of this line and compressive on the other. The locus of zero axial stress is a straight line called the *neutral axis*.

The neutral axis



The neutral axis

The neutral axis is a straight line, and its equation is readily found by imposing the vanishing of the axial stress in eq. (14) to find

$$\frac{N_1}{S} + \frac{H_{33}^c M_2 + H_{23}^c M_3}{\Delta_H}\ x_3 - \frac{H_{23}^c M_2 + H_{22}^c M_3}{\Delta_H}\ x_2 = 0. \tag{32}$$

The slope of this line is found as

$$\tan\beta = \frac{x_3}{x_2} = \frac{H_{23}^c M_2 + H_{22}^c M_3}{H_{33}^c M_2 + H_{23}^c M_3}. \tag{33}$$

The neutral axis

In the principal centroidal axes of bending frame, the equation of the neutral axis is found by imposing the vanishing of the axial stress in eq. (28) to find

$$\frac{N_1^*}{S^*} + x_3^* \ \frac{M_2^*}{H_{22}^{c*}} - x_2^* \ \frac{M_3^*}{H_{33}^{c*}} = 0. \tag{34}$$

The slope of the neutral axis is simply

$$\tan\beta^* = x_3^*/x_2^* = (H_{22}^{c*} M_3^*)/(H_{33}^{c*} M_2^*)$$