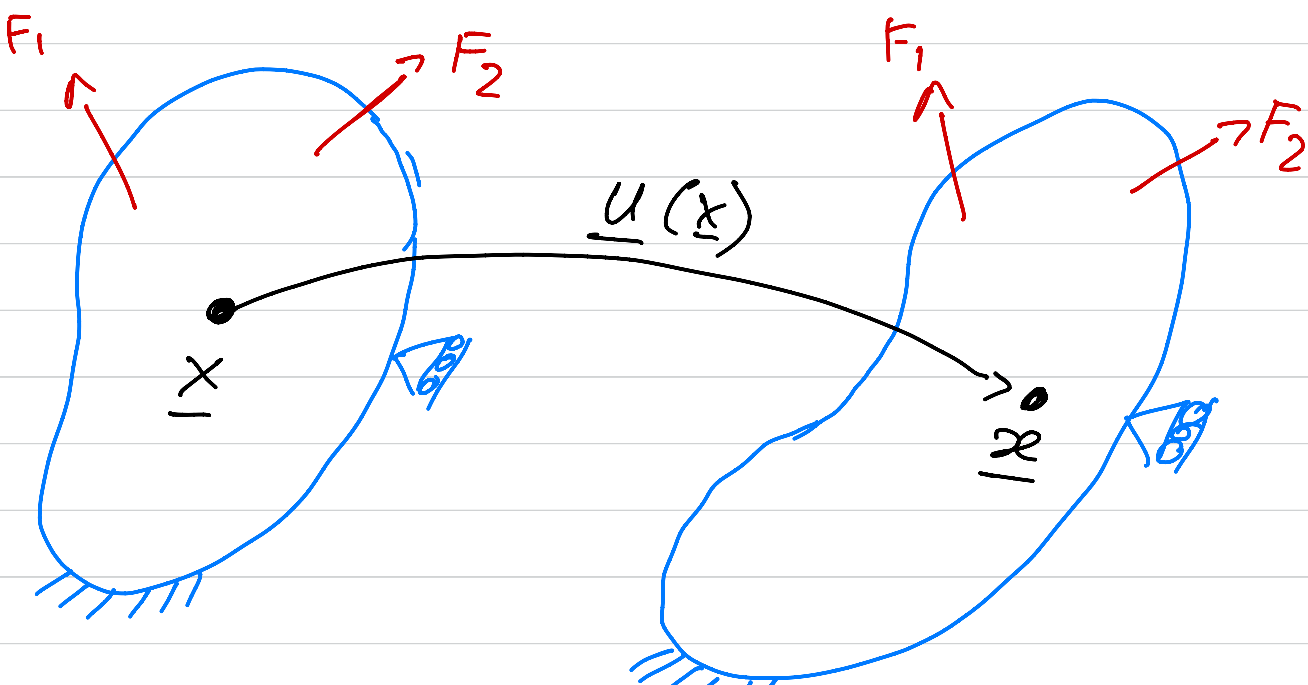
Intro & Summary of Elasticity

INTRODUCTION & SUMMARY OF THE BOUNDARY VALUE PROBLEM IN ELASTICITY



REFERENCE DEFORMED

DISPLACEMENTS → KINEMATICS

$\underline{u} \longrightarrow \underline{\epsilon}$

EQUILIBRIUM ← CONSTITUTIVE

$\sum \underline{F} = \underline{0}$ $\qquad \underline{\sigma}$

KINEMATICS (IN CARTESIAN COORDS.)

DISPLACEMENT IS A VECTOR FIELD $\underline{u}$ FUNCTION OF POSITION $\underline{x}$

$$\underline{u}(\underline{x}) = u_1(\underline{x})\,\underline{e}_1 + u_2(\underline{x})\,\underline{e}_2 + u_3(\underline{x})\,\underline{e}_3$$

STRAIN

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \qquad \epsilon_{12} = \frac{1}{2}\gamma_{12}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad \epsilon_{13} = \frac{1}{2}\gamma_{13}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \qquad \epsilon_{23} = \frac{1}{2}\gamma_{23}$$

* STRAIN IS A 2ND ORDER TENSOR

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ & \epsilon_2 & \epsilon_{23} \\ \text{sym} & & \epsilon_3 \end{bmatrix}$$

CONSTITUTIVE

* THESE RELATIONSHIPS DESCRIBE THE MECHANICAL BEHAVIOR OF THE MATERIAL

* ASSUME LINEAR ISOTROPIC ELASTIC BEHAVIOR

$$\epsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right)$$

$$\epsilon_2 = \frac{1}{E}\left(\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right)$$

$$\epsilon_3 = \frac{1}{E}\left(\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right)$$

$$\gamma_{12} = \frac{2(1+\nu)}{E}\tau_{12}$$

$$\gamma_{23} = \frac{2(1+\nu)}{E}\tau_{23}$$

$$\gamma_{13} = \frac{2(1+\nu)}{E}\tau_{13}$$

E – YOUNG'S MODULUS

ν – POISSONS RATIO

$$G = \frac{E}{2(1+\nu)}$$ – SHEAR MODULUS

$$(\gamma_{12} = \tau_{12}/G)$$

$$K = \frac{E}{3(1-2\nu)}$$ – BULK MODULUS

EQUILIBRIUM

* THE STRESS $\underline{\underline{\sigma}}$ IS A SYMMETRIC 2ND ORDER TENSOR

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ & \sigma_2 & \tau_{23} \\ \text{Sym} & & \sigma_3 \end{bmatrix}$$

WHICH MUST SATISFY EQUILIBRIUM

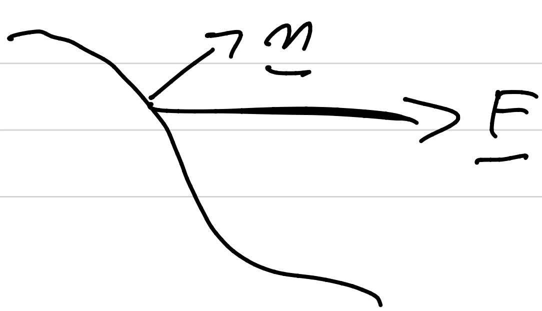
$$\operatorname{div}(\underline{\sigma}) + \underline{b} = \underline{0}$$

Body Force in $\underline{e}_1$-DIRECTION

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1 = 0$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + b_2 = 0$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 = 0$$



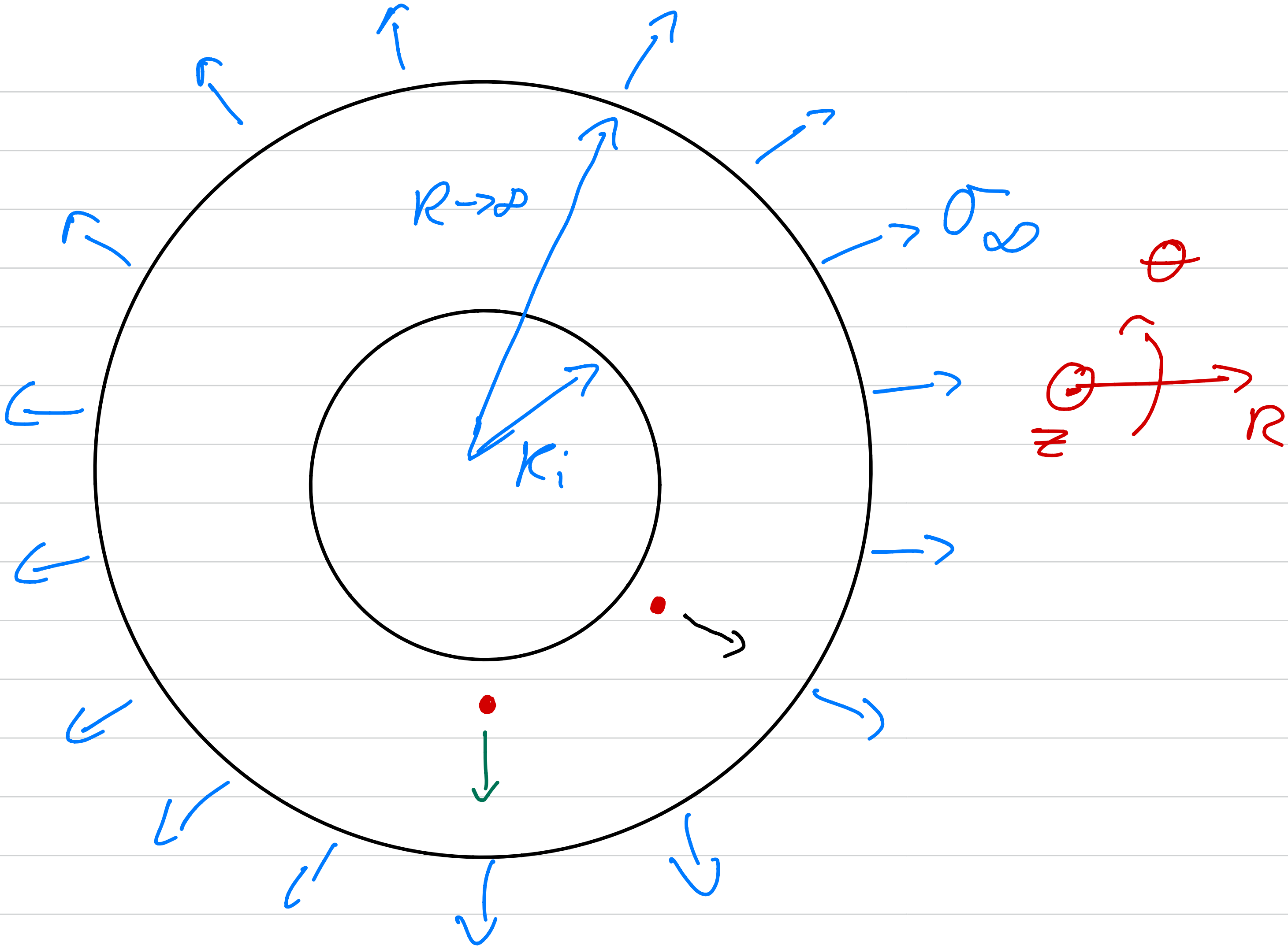
$$\underline{T}_N = \underline{\sigma} \cdot \underline{n} = \underline{F}$$

* UNDERSTANDING THE DEFORMATION AND STRESS IN A 3D BODY RESULTS IN COUPLED DIFFERENTIAL EQUATIONS FOR

  - DISPLACEMENT $\underline{u}$ (3 COMPONENTS)
  - STRAIN $\underline{\underline{\epsilon}}$ (6 COMPONENTS)
  - STRESS $\underline{\underline{\sigma}}$ (6 COMPONENTS)

* IN STRUCTURAL ANALYSIS

  WE FOCUS ON STRUCTURAL COMPONENTS SUCH AS BARS, BEAMS, PLATES, OR SHELLS, FOR WHICH ANALYTICAL UNDERSTANDING CAN BE GAINED.



$$\underline{u} = u_R \, \underline{e}_R + \cancelto{0}{u_\theta} \, \underline{e}_\theta + u_z \, \underline{e}_z$$

$$u_R = \hat{u}_R(R) \, , \qquad u_z = \hat{u}_z(z)$$

$$\epsilon_R = \frac{d u_R}{dR} \, , \qquad \epsilon_\theta = \frac{u_R}{R} \, , \qquad \epsilon_z = 0$$

$$\epsilon_{R\theta} = \epsilon_{Rz} = \epsilon_{\theta z} = 0$$

$$\epsilon_R = \frac{dU_R}{dR}, \quad \epsilon_\theta = \frac{U_R}{R}, \quad \epsilon_z = 0$$

$$\sigma_R = \frac{E}{1+v}\left(\frac{dU_R}{dR} + \frac{v}{1-2v}\left(\frac{dU_R}{dR} + \frac{U_R}{R}\right)\right)$$

$$\sigma_\theta = \frac{E}{1+v}\left(\frac{U_R}{R} + \frac{v}{1-2v}\left(\frac{dU_R}{dR} + \frac{U_R}{R}\right)\right)$$

$$\frac{d\sigma_R}{dR} + \frac{1}{R}(\sigma_R - \sigma_\theta) = 0$$

$$\cancel{\frac{E}{1+v}}\left(\frac{d^2U_R}{dR^2} + \frac{v}{(1-2v)}\left(\frac{d^2U_R}{dR^2} + \frac{d}{dR}\left(\frac{U_R}{R}\right)\right)\right)$$

$$+\frac{1}{R}\cancel{\frac{E}{1+v}}\left(\frac{dU_R}{dR} - \frac{U_R}{R}\right) = 0$$

$$\frac{v}{1-2v}\left(\frac{d^2U_R}{dR^2} + \frac{1}{R}\frac{dU_R}{dR} - \frac{U_R}{R^2}\right)$$

$$+\frac{d^2U_R}{dR^2} + \frac{1}{R}\frac{dU_R}{dR} - \frac{U_R}{R^2} = 0$$

$$\frac{d^2 u_R}{dR^2} + \frac{1}{R}\frac{du_R}{dR} - \frac{u_R}{R^2} = 0$$

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}\left(R \cdot u_R\right)\right) = 0$$

$$\frac{1}{R}\frac{d}{dR}\left(R \cdot u_R\right) = A$$

$$\frac{d}{dR}\left(R \cdot u_R\right) = AR$$

$$R u_R = AR^2 + B$$

$$\boxed{u_R = AR + \frac{B}{R}}$$

2. Plug $u_R = AR + B/R$ into $\underline{\underline{\epsilon}}$ and $\underline{\underline{\sigma}}$.

3. $R \to \infty \quad \sigma_R = \sigma_\infty$

$R = R_i \quad \sigma_R = 0$

$\Big\}$ SOLVE A & B.