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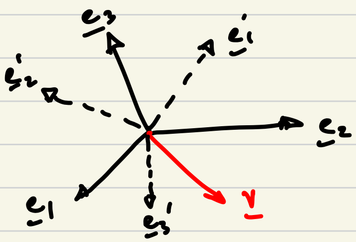
Change of Basis

- A vector is a physical entity independent on the frame we choose to represent it.
- If we know the component of a vector wrt one set of cartesian coordinates, then we know its component wrt any other set of cartesian axis.



$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$

in indicial notation,

$\underline{v} = v_i \underline{e}_i$

$\underline{v} = v_1' \underline{e}_1' + v_2' \underline{e}_2' + v_3' \underline{e}_3' = v_i' \underline{e}_i'$

How to convert from one frame to another?

$\underline{v} \cdot \underline{e}_i' = v_j' \underbrace{\underline{e}_j' \cdot \underline{e}_i'}_{\delta_{ji}}$ (This is K delta)

$= v_j' \delta_{ji} = v_i' \rightarrow$ component of v on primed coord.

$\underline{v} \cdot \underline{e}_i' = v_j \underbrace{\underline{e}_j \cdot \underline{e}_i'}_{l_{ij}} = l_{ij} v_j \rightarrow$ " — " on unprimed

This is a rank-2 tensor (not orthonormal)

$\boxed{v_i' = l_{ij} v_j}$

$\begin{bmatrix} - \\ - \end{bmatrix} = \begin{bmatrix} \# \end{bmatrix} \begin{bmatrix} - \\ - \end{bmatrix}$

change of basis or rotation matrix

we can see, by definition

$l_{ij} = \underline{e}_i' \cdot \underline{e}_j$ is the component of $\underline{e}_i'$ in the direction of $\underline{e}_j$

$$\left.\begin{aligned} \underline{e}_1' &= l_{11}\underline{e}_1 + l_{12}\underline{e}_2 + l_{13}\underline{e}_3 \\ \underline{e}_2' &= l_{21}\underline{e}_1 + l_{22}\underline{e}_2 + l_{23}\underline{e}_3 \\ \underline{e}_3' &= l_{31}\underline{e}_1 + l_{32}\underline{e}_2 + l_{33}\underline{e}_3 \end{aligned}\right\} \boxed{\underline{e}_i' = l_{ij}\,\underline{e}_j}$$

Properties of l_{ij}

Multiply $\underline{e}_i'$ by $\underline{e}_k'$ we get

orthonormal

orthonormal

$$\underbrace{\underline{e}_i' \cdot \underline{e}_k'}_{\delta_{ik}} = l_{ij}\,\underline{e}_j \cdot l_{kl}\,\underline{e}_l = l_{ij}\,l_{kl}\,\underbrace{\underline{e}_j \cdot \underline{e}_l}_{\delta_{jl}}$$

$$\delta_{ik} = l_{ij}\,l_{kl}\,\delta_{jl} \Rightarrow \delta_{ik} = l_{ij}\,\underbrace{l_{kj}}_{(l^T)_{jk}}$$

NOTE

$$[\delta_{ij}] = \begin{bmatrix} \underline{e}_1\cdot\underline{e}_1 & \underline{e}_1\cdot\underline{e}_2 & \underline{e}_1\cdot\underline{e}_3 \\ \underline{e}_2\cdot\underline{e}_1 & \underline{e}_2\cdot\underline{e}_2 & \underline{e}_2\cdot\underline{e}_3 \\ \underline{e}_3\cdot\underline{e}_1 & \underline{e}_3\cdot\underline{e}_2 & \underline{e}_3\cdot\underline{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$$[l_{ij}] = \begin{bmatrix} \underline{e}_1' \cdot \underline{e}_1 & \underline{e}_1' \cdot \underline{e}_2 & \underline{e}_1' \cdot \underline{e}_3 \\ \underline{e}_2' \cdot \underline{e}_1 & \underline{e}_2' \cdot \underline{e}_2 & \underline{e}_2' \cdot \underline{e}_3 \\ \underline{e}_3' \cdot \underline{e}_1 & \underline{e}_3' \cdot \underline{e}_2 & \underline{e}_3' \cdot \underline{e}_3 \end{bmatrix}$$

2 different basis, <u>not orthonormal</u>.

$$\delta_{ik} = l_{ij} l_{kl} \delta_{jl} \Rightarrow \delta_{ik} = l_{ij} l_{kj} \qquad [I] = [l][l]^T$$

Also, by definition: $\delta_{ik} = l_{ij} (l^{-1})_{jk}$ $\quad [I] = [l][l]^{-1}$

$$\Rightarrow \boxed{(l^T)_{jk} = (l^{-1})_{jk}} \qquad [l]^T = [l]^{-1}$$

|| That is, the inverse of l_{ij} is the transpose.

Furthermore, using properties of determinants we know: $\det([A]) = \det([A]^T)$

$$\det([A]) = \frac{1}{\det([A]^{-1})}$$

$$\det([l]) = \frac{1}{\det([l]^{-1})} = \frac{1}{\det([l]^T)} = \frac{1}{\det([l])}$$

$$\Rightarrow (\det[l])^2 = 1 \Rightarrow \det([l]) = \pm 1$$

That is, the det of the transformation matrix is either 1 or −1

rotation $\qquad$ rotation & reflection

$\underline{e}_i' = \ell_{ij}\,\underline{e}_j$

Change of basis for rank 2 tensors

$$\underline{\underline{T}} = T_{ij}'\,\overbrace{\underline{e}_i'\,\underline{e}_j'}$$

indicate where in the matrix T_{ij}' goes.

diadic product

$$= T_{ij}'\,\ell_{ik}\,\underline{e}_k\,\ell_{jl}\,\underline{e}_l = \underbrace{T_{ij}'\,\ell_{ik}\,\ell_{jl}}_{T_{kl}}\,\underline{e}_k\,\underline{e}_l$$

$\therefore\ T_{kl} = T_{ij}'\,\ell_{ik}\,\ell_{jl} \Rightarrow T_{kl} = (\ell^T)_{ki}\,(\ell^T)_{lj}\,T_{ij}'$

using the orthogonality of ℓ_{ij}: $\underline{T_{ij}' = \ell_{ik}\,\ell_{jl}\,T_{kl}}$

$[T'] = [\ell][T][\ell^T]$

(so n multiplications by ℓ for rank n tensor)

- we define Tensors based on how they transform.

Definition: In general, a tensor of rank n is a mathematical obj w/ n indices, which obeys the transformation law:

$$\underline{T_{ijk\ldots}' = \ell_{ip}\,\ell_{jq}\,\ell_{kr}\ldots T_{pqr\ldots}}$$