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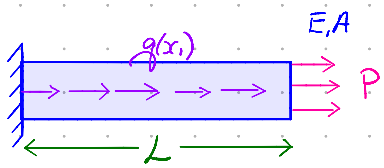
HW #7

* Graded assignments were not given back (that I know of) so these are just my answers

Problem 1

↳ If you have corrected answers or find any mistakes, let me know! :)

Consider a clamped bar of length L and cross-sectional area A, made out of an elastic material with Young's modulus E, and loaded with a distributed axial load per unit length $q(x_1)$ and a traction P at the top as depicted below.



Find the corresponding displacement field by using the principle of minimum potential energy, that is, by finding the displacement field corresponding to $\delta\Pi = 0$.

1. Assuming a uniaxial stress state ($\sigma_{11} \neq 0$, all others zero) and the full form of Hooke's law in 3D ($\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$), show that the stress reduces to $\sigma_{11} = E\varepsilon_{11}$.

Hooke's law:

$$(1) \quad \sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11}$$

$$(2) \quad 0 = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}$$

$$(3) \quad 0 = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33}$$

$$\left.\begin{aligned} 0 &= 2\mu\varepsilon_{12} \\ 0 &= 2\mu\varepsilon_{13} \\ 0 &= 2\mu\varepsilon_{23} \end{aligned}\right\} \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

adding (1), (2), and (3):

$$\begin{aligned} \sigma_{11} &= 3\lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ &= 3\lambda\varepsilon_{kk} + 2\mu\varepsilon_{kk} \\ &= (3\lambda + 2\mu)\varepsilon_{kk} \end{aligned}$$

$$\Rightarrow \varepsilon_{kk} = \frac{\sigma_{11}}{3\lambda + 2\mu}$$

plugging this back into (1):

$$\sigma_{11} = \lambda\left(\frac{\sigma_{11}}{3\lambda + 2\mu}\right) + 2\mu\varepsilon_{11}$$

$$\sigma_{11} - \lambda\left(\frac{\sigma_{11}}{3\lambda+2\mu}\right) = 2\mu\varepsilon_{11}$$

$$\sigma_{11}\left(1 - \frac{\lambda}{3\lambda+2\mu}\right) =$$

$$\sigma_{11}\left(\frac{3\lambda+2\mu-\lambda}{3\lambda+2\mu}\right) =$$

$$\sigma_{11}\left(\frac{2\lambda+2\mu}{3\lambda+2\mu}\right) = 2\mu\varepsilon_{11}$$

$$\sigma_{11} = 2\mu\left(\frac{3\lambda+2\mu}{2\lambda+2\mu}\right)\varepsilon_{11}$$

$$= \mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)\varepsilon_{11}$$

and we know that $E = \mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)$

which means

$$\sigma_{11} = E\varepsilon_{11} \quad \checkmark$$

2. Using the previous result, show that the elastic potential for this problem becomes

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 - A\int_0^L g(x_1)\, u_1\, dx_1 - AP\, u_1\Big|_{x_1=L}$$

$$\Pi = \mathcal{W} - E$$

$$\mathcal{W} = \int_\Omega \frac{1}{2}\, \nabla \underline{u} : \underline{\underline{C}} : \nabla \underline{u}\, dV$$

$$E = \int_\Omega \underline{b}\cdot\underline{u}\, dV + \int_{\partial_2\Omega} \underline{t}^* \cdot \underline{u}\, dA$$

for the linear elastic case: $W = \frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

(In the notes we had $\mathcal{W}$ and W but I didn't really understand the difference between the two since they both looked like capital W's to me ...)

$$\hookrightarrow \mathcal{W} = \int_{E_0} W\, dV$$

$$\mathcal{W} = \int_\Omega \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, dV = \int_\Omega \frac{1}{2}\sigma_{11}\varepsilon_{11}\, d\Omega \quad , \quad \sigma_{11} = E\varepsilon_{11} \text{ from before}$$

$$\mathcal{W} = \int_\Omega \frac{1}{2}\sigma_{11}\varepsilon_{11}^2\, d\Omega \quad , \quad \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

our volume is $\int_0^L$ and $\int_A$

$$W = \int_0^L \int_A \frac{1}{2} E \left(\frac{\partial u_1}{\partial x_1}\right)^2 dA\, dx_1 \quad \leftarrow \text{constant over area}$$

$$\Rightarrow W = \frac{EA}{2} \int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1$$

now for E

$$E = \int_\Omega \underline{b} \cdot \underline{u}\, dV + \int_{\partial_2 \Omega} \underline{t}^* \cdot \underline{u}\, dA$$

$$\int_\Omega \underline{b} \cdot \underline{u}\, dV = \int_0^L \int_A g(x_1) u_1\, dA\, dx_1 \quad \leftarrow \text{also constant over area}$$

$$\Rightarrow A \int_0^L g(x_1) u_1\, dx_1$$

$$\int_{\partial_2 \Omega} \underline{t}^* \cdot \underline{u}\, dA \leftarrow \text{only at end of bar } (\underline{t}^* = P \text{ and } x_1 = L)$$

$$\Rightarrow A P u_1 \Big|_{x_1 = L}$$

putting these together:

$$\Pi = \underbrace{\frac{EA}{2} \int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1}_{W} \underbrace{- A \int_0^L g(x_1) u_1\, dx_1 - A P u_1 \Big|_{x_1 = L}}_{- \quad E} \quad \checkmark$$

3. Using the properties of variations discussed in class, show that

$$\delta \Pi = AE \int_0^L \frac{\partial u_1}{\partial x_1} \frac{\partial \delta u_1}{\partial x_1} dx_1 - A \int_0^L g(x) \delta u_1\, dx_1 - A P \delta u_1 \Big|_{x_1 = L}$$

$$\Pi = \frac{EA}{2} \int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 - A \int_0^L g(x_1) u_1\, dx_1 - A P u_1 \Big|_{x_1 = L}$$

$$\delta \Pi = \delta \left[\frac{EA}{2} \int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 \right] - \delta \left[A \int_0^L g(x_1) u_1\, dx_1 \right] - \delta \left[A P u_1 |_{x_1 = L} \right]$$

applying properties of variations:

$$\delta \int v\, dx = \int \delta v\, dx$$

$$\delta \Pi = \frac{EA}{2} \int_0^L \underbrace{\delta \left(\frac{\partial u_1}{\partial x_1}\right)^2}_{\delta(v^n) = n v^{n-1} \delta v} dx_1 - A \int_0^L \underbrace{\delta (g(x_1) u_1)}_{g(x_1) \delta u_1} dx_1 - A P \delta u_1 \Big|_{x_1 = L}$$

$$\delta\Pi = \frac{EA}{2}\int_0^L 2\left(\frac{\partial u_1}{\partial x_1}\right)\underbrace{\delta\left(\frac{\partial u_1}{\partial x_1}\right)}_{\frac{\partial \delta u_1}{\partial x_1}} dx_1 - A\int_0^L g(x_1)\delta u_1 dx_1 - AP\delta u_1\Big|_{x_1=L}$$

$$\Rightarrow \delta\Pi = EA\int_0^L \frac{\partial u_1}{\partial x_1}\frac{\partial \delta u_1}{\partial x_1} dx_1 - A\int_0^L g(x_1)\delta u_1 dx_1 - AP\delta u_1\Big|_{x_1=L} \quad \checkmark$$

4. Show that, after integration by parts and considering that $\delta u_1 = 0$ at $x_1 = 0$, the first integral in the expression above becomes

$$\int_0^L \frac{\partial u_1}{\partial x_1}\frac{\partial \delta u_1}{\partial x_1} dx_1 = \frac{\partial u_1}{\partial x_1}\delta u_1\Big|_{x_1=L} - \int_0^L \frac{\partial^2 u_1}{\partial x_1^2}\delta u_1 dx_1$$

first integral: $AE\int_0^L \frac{\partial u_1}{\partial x_1}\frac{\partial \delta u_1}{\partial x_1} dx_1$

integration by parts: $\int u\,dv = uv - \int v\,du$

here, we let $u = \frac{\partial u_1}{\partial x_1}$ $\Rightarrow$ $du = \frac{\partial^2 u_1}{\partial x_1^2} dx_1$

$dv = \frac{\partial \delta u_1}{\partial x_1} dx_1$ $\qquad v = \delta u_1$

$$\int u\,dv = \int_0^L \frac{\partial u_1}{\partial x_1}\frac{\partial \delta u_1}{\partial x_1} dx_1 = uv - \int v\,du = \underbrace{\frac{\partial u_1}{\partial x_1}\delta u_1\Big|_0^L}_{\delta u_1 = 0 \text{ at } x_1 = 0} - \int_0^L \delta u_1 \frac{\partial^2 u_1}{\partial x_1^2} dx_1$$

$$= \frac{\partial u_1}{\partial x_1}\delta u_1\Big|_{x_1=L} - \int_0^L \frac{\partial^2 u_1}{\partial x_1^2}\delta u_1 dx_1 \quad \checkmark$$

5. Using previous results, and considering that they must apply to all compatible variations δu_1 (in particular for which $\delta u_1\big|_{x_1=L} = 0$), show that $\delta\Pi = 0$ necessarily implies:

$$E\frac{\partial^2 u_1}{\partial x_1^2} + g(x_1) = 0 \quad \text{and} \quad E\frac{\partial u_1}{\partial x_1} = P \text{ at } x_1 = L$$

from our previous results we have:

$$\delta\Pi = AE\frac{\partial u_1}{\partial x_1}\delta u_1\Big|_{x_1=L} - AE\int_0^L \frac{\partial^2 u_1}{\partial x_1^2}\delta u_1 dx_1 - A\int_0^L g(x_1)\,\delta u_1\, dx_1 - AP\,\delta u_1\Big|_{x_1=L} = 0$$

taking the case where $x_1 = L$:

$$0 = AE\frac{\partial u_1}{\partial x_1}\delta u_1\Big|_{x_1=L} \underbrace{- AE\int_0^L \frac{\partial^2 u_1}{\partial x_1^2}\delta u_1\, dx_1}_{-AE\frac{\partial^2 u_1}{\partial x_1^2}\delta u_1\big|_{x_1=L}} \underbrace{- A\int_0^L g(x_1)\,\delta u_1\, dx_1}_{-Ag(x_1)\delta u_1\big|_{x_1=L}} - AP\delta u_1\Big|_{x_1=L}$$

Considering the case where $\delta u_1\big|_{x_1=L} = 0$

$\Rightarrow$ these are equal to zero

(I'm not sure if this logic makes sense, because I guess I would also have to say that the other two terms are also zero, but let's go with it for now)

$$0 = \cancel{A}E\frac{\partial u_1}{\partial x_1}\cancel{\delta u_1} - \cancel{A}P\cancel{\delta u_1}$$

$$E\frac{\partial u_1}{\partial x_1} = P \quad \checkmark$$

Taking the case where $x_1 = 0$:

$$0 = -AE\int_0^L \frac{\partial^2 u_1}{\partial x_1^2}\delta u_1\, dx_1 - A\int_0^L g(x_1)\,\delta u_1\, dx_1$$

$$0 = \cancel{+}\cancel{A}E\frac{\partial^2 u_1}{\partial x_1^2}\cancel{\delta u_1} + \cancel{A}g(x_1)\cancel{\delta u_1} \quad \leftarrow \text{only looking at } x_1 = 0$$

$$E\frac{\partial^2 u_1}{\partial x_1^2} + g(x_1) = 0 \quad \checkmark$$

6. Comment on the previous result. What do these equations represent?

As we found in class, $E\frac{\partial u_1}{\partial x_1} = P$ at $x_1 = L$ comes from our traction boundary conditions. At $x_1 = L$, $\sigma_{11} = E\varepsilon_{11}$ $\left(\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}\right)$ and thus $\sigma_{11} = P$. this is our basic stress-strain relationship at the end of the bar.

$E\frac{\partial^2 u_1}{\partial x_1^2} + g(x_1) = 0$ comes from the conservation of linear momentum. Looking closer at this equation,

$E \cdot \frac{\partial^2 u_1}{\partial x_1^2} = E\frac{\partial \varepsilon_{11}}{\partial x_1} = \frac{\partial \sigma_{11}}{\partial x_1}$ and $g(x_1)$ are the body forces, so essentially this equation is our equation of static equilibrium:

$\sigma_{ij,j} + \rho b_i = 0$ for linear elastic body

$$E\frac{\partial^2 u_1}{\partial x_1^2} + g(x_1) = 0$$

7. Solve for u_1 using the equations shown in (5) above and assuming $g(x_1) = \text{constant} = g$. Plot your results.

from (5): $E\frac{\partial u_1}{\partial x_1} = P$ at $x_1 = L$

$$E\frac{\partial^2 u_1}{\partial x_1^2} + g_1(x_1) = 0$$

$$E\frac{\partial^2 u_1}{\partial x_1^2} + g = 0 \Rightarrow \frac{\partial^2 u_1}{\partial x_1^2} = -\frac{g}{E}$$

$$\int \frac{\partial^2 u_1}{\partial x_1^2}\,dx_1 = \int -\frac{g}{E}\,dx_1 \Rightarrow \frac{\partial u_1}{\partial x_1} = -\frac{g}{E}x_1 + C_1$$

$$\int \frac{\partial u_1}{\partial x_1}\,dx_1 = \int -\frac{g}{E}x_1\,dx_1 + \int C_1\,dx_1 \Rightarrow u_1(x_1) = -\frac{1}{2}\frac{gx_1^2}{E} + C_1x_1 + C_2$$

$$u_1(0) = 0 \Rightarrow C_2 = 0$$

at $x_1 = L$, $E\frac{\partial u_1}{\partial x_1} = P \Rightarrow \frac{\partial u_1}{\partial x_1} = \frac{P}{E}$

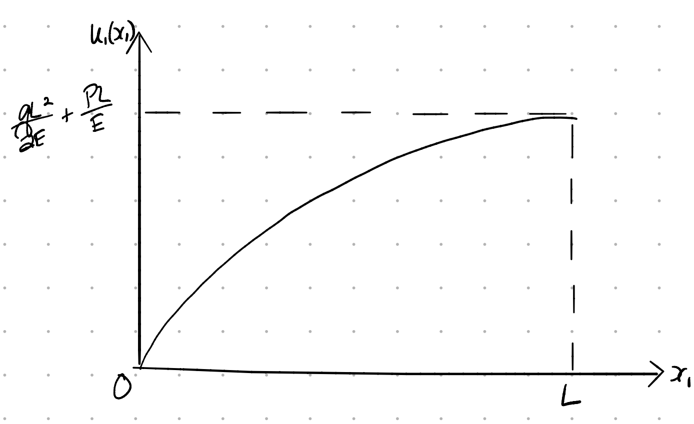
$\frac{P}{E} = \frac{\partial u_1}{\partial x_1} = -\frac{gx_1}{E} + C_1$ at $x_1 = L$: $\frac{\partial u_1}{\partial x_1} = \frac{P}{E} = -\frac{gL}{E} + C_1 \Rightarrow C_1 = \frac{gL}{E} + \frac{P}{E}$

plugging back in:

$$u_1(x_1) = -\frac{1}{2}\frac{gx_1^2}{E} + \left(\frac{gL + P}{E}\right)x_1$$

$$u_1(0) = 0$$

$$u_1(L) = -\frac{1}{2}\frac{gL^2}{E} + \frac{gL^2}{E} + \frac{PL}{E} = \frac{gL^2}{2E} + \frac{PL}{E}$$



8. If you were to solve this problem using the boundary value problem in elastostatics formulation, what would have been different? Elaborate on the pros and cons of each approach, if any.

Finding the solution using the principle of minimum potential energy is equivalent to finding the solution using the boundary value problem.

For the BVP, we want to find $\underline{u}$, $\underline{\varepsilon}$, and $\underline{\sigma}$ such that

$$\sigma_{ij,j} + \rho b_i = 0$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega$$

subject to
$$\underline{u} = \underline{u}^* \quad \text{on } \partial_1\Omega$$
$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}^* \quad \text{on } \partial_2\Omega$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\underline{u}) \Rightarrow \underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\varepsilon(\underline{u})) \Rightarrow \underline{\underline{\sigma}}(\underline{u})$$

↳ plug into balance of linear momentum and solve for $\underline{u}$

we get in-plane and anti-plane and then we use superposition

The BVP is called the strong formulation or differential. This means that stresses are related to strains, which gives us second derivatives of stress. Sometimes, however, the solutions to these equations are difficult to obtain, so we turn to the weak formulation or variational form: principle of minimum potential energy.

The weak form makes it easier to obtain a solution and is a starting point for FE problems. A disadvantage to the weak form is that its conditions are met on average over the body, whereas the strong form (BVP) has its conditions met at every material point.

BVP (strong)	MPE (weak)
- exact solution - 2^{nd} order diff. equ's - might be harder or impossible to solve - more complicated geometry => more complicated BVP	- approximate (or could be exact) solution - 1^{st} order diff. equ's - easier to solve - useful for complicated geometries

To solve using BVP we would have needed the displacement and traction BCs on all parts of the beam. We would then use the balance of linear momentum to get our equilibrium equations and relate our stresses to our strains. Using our BCs we would then solve for the stresses and strains and then integrate to get our displacements.