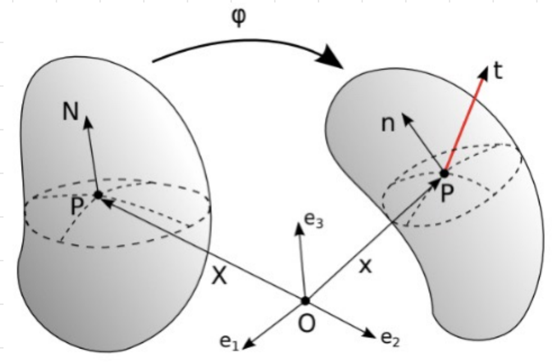
AE6114 Assignment 3: Balance Laws

Problem 1

As shown in class, the traction $\underline{t}$ on an internal surface of the body $\mathcal{B}$ with normal $\underline{n}$ is given by Cauchy's relation:

$$t_i = \sigma_{ij} n_j$$

Show that σ_{ij} are the components of a second order tensor.



To prove that σ_{ij} are the components of a second order tensor, show that σ_{ij} obeys the transformation law:

$$T'_{ijk\ldots} = \ell_{ip}\,\ell_{jq}\,\ell_{kr\ldots}\,T_{pqr\ldots}$$

Consider a change of basis:

(1) $t'_i = \ell_{ij} t_j$

(∗) $n_i = (\ell)^T_{ij} n'_j$

We know

(2) $t'_i = \sigma'_{ij} n'_j$

(3) $t_i = \sigma_{ij} n_j$

equating (1) and (2): $\ell_{ip} t_p = \sigma'_{ij} n'_j$

Change indices!

$[\ell] \Rightarrow \ell^T = \ell^{-1}$

$\det[\ell] = 1$

use (3) to substitute for t_p:

(3) $t_p = \sigma_{pq} n_q \rightarrow \ell_{ip}\sigma_{pq} n_q = \sigma'_{ij} n'_j$

use (∗) for n_q:

(∗) $n_q = (\ell)^T_{qj} n'_j \rightarrow \ell_{ip}\sigma_{pq}(\ell)^T_{qj} \cancel{n'_j} = \sigma'_{ij}\cancel{n'_j}$ $\qquad (\ell)^T_{qj} = \ell_{jq}$

$\Rightarrow \sigma'_{ij} = \ell_{ip}\ell_{jq}\sigma_{pq}$ ✓ which is the definition of a rank 2 tensor

These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering the answers from students that were marked correct. If you find any errors, please let me know!

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Another way:

$$\left.\begin{aligned} t_i &= (\ell)^T_{ij} t'_j \\ n_j &= (\ell)^T_{jk} n'_k \end{aligned}\right\} \quad t_i = (\ell)^T_{ij} t'_j = \sigma_{ij} n_j = \sigma_{ij} (\ell)^T_{jk} n'_k$$

multiplying both sides by ℓ_{pi}:

$$\underbrace{\ell_{pi} (\ell)^T_{ij}}_{\delta_{pj}} t'_j = \sigma_{ij} \ell_{pi} \underbrace{(\ell)^T_{jk}}_{(\ell)^T_{jk} = \ell_{kj}} n'_k$$

$$\underbrace{t'_j \delta_{pj}}_{t'_p} = \underbrace{\sigma_{ij} \ell_{pi} \ell_{kj}}_{\sigma'_{pk}} n'_k$$

$\hookrightarrow$ $\sigma'_{pk} = \sigma_{ij} \ell_{pi} \ell_{kj}$ satisfies the transformation law of a rank 2 tensor

Problem 2

A slab at rest occupies the following region in the deformed configuration:

$$-a \leq x_1 \leq a, \quad -a \leq x_2 \leq a, \quad -h \leq x_3 \leq h$$

It has the following stress distribution:

$$\sigma_{11} = -\frac{p(x_1^2 - x_2^2)}{a^2}, \quad \sigma_{22} = \frac{p(x_1^2 - x_2^2)}{a^2}, \quad \sigma_{12} = \sigma_{21} = \frac{2px_1x_2}{a^2}$$

with the rest of the Cauchy stress components being zero.

1. Examine whether there are any body forces within the slab.

$$\underset{=}{\sigma} = \begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} & \frac{2px_1x_2}{a^2} & 0 \\ \frac{2px_1x_2}{a^2} & \frac{p(x_1^2 - x_2^2)}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{ij,j} + \rho b_i = \cancelto{0}{\rho \ddot{x}_i}$$ because body is at rest

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_3}} = -\rho b_1$$

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \cancelto{0}{\frac{\partial\sigma_{23}}{\partial x_3}} = -\rho b_2$$

$$\cancelto{0}{\frac{\partial\sigma_{13}}{\partial x_1}} + \cancelto{0}{\frac{\partial\sigma_{23}}{\partial x_2}} + \cancelto{0}{\frac{\partial\sigma_{33}}{\partial x_3}} = -\rho b_3 \quad \Rightarrow \quad b_3 = 0$$

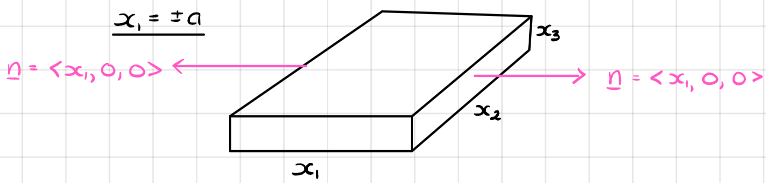
$$\left.\begin{aligned} \sigma_{11} &= -\frac{p}{a^2}x_1^2 + \frac{p}{a^2}x_2^2 \\ \sigma_{22} &= \frac{p}{a^2}x_1^2 - \frac{p}{a^2}x_2^2 \\ \sigma_{12} &= \frac{2p}{a^2}x_1x_2 \end{aligned}\right\}
\begin{aligned} &\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = -\frac{2p}{a^2}x_1 + \frac{2p}{a^2}x_1 = 0 \Rightarrow b_1 = 0 \\ &\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = \frac{2p}{a^2}x_2 - \frac{2p}{a^2}x_2 = 0 \Rightarrow b_2 = 0 \end{aligned}$$

$\Rightarrow$ No, there are no body forces within the slab. $\rho \neq 0$

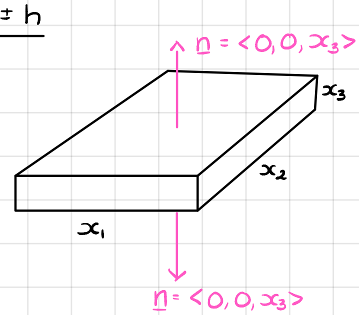
2. Calculate the tractions acting on the faces $x_1 = \pm a$, and the faces $x_3 = \pm h$.

$t_i = \sigma_{ij} n_j$

$\underline{x_1 = \pm a}$



$\underline{x_3 = \pm h}$



$\underline{x_1 = a}$

$$|\underline{n}| = \sqrt{a^2} = a$$

$$\frac{\underline{n}}{|\underline{n}|} = \frac{1}{a}\langle a, 0, 0\rangle = \langle 1, 0, 0\rangle$$ $\leftarrow$ $\underline{n}$ is a unit vector!

$$\begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} & \frac{2px_1x_2}{a^2} & 0 \\ \frac{2px_1x_2}{a^2} & \frac{p(x_1^2-x_2^2)}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} \\ \frac{2px_1x_2}{a^2} \\ 0 \end{bmatrix} = \underline{t}_{x_1=a}}$$

$\underline{x_1 = -a}$

$$|\underline{n}| = \sqrt{a^2} = a$$

$$\frac{\underline{n}}{|\underline{n}|} = \frac{1}{a} \langle -a, 0, 0 \rangle = \langle -1, 0, 0 \rangle$$ ← $\underline{n}$ is a unit vector!

$$\begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} & \frac{2px_1x_2}{a^2} & 0 \\ \frac{2px_1x_2}{a^2} & \frac{p(x_1^2 - x_2^2)}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{p(x_1^2 - x_2^2)}{a^2} \\ \frac{2px_1x_2}{a^2} \\ 0 \end{bmatrix} = \underline{t}_{x_1 = -a}}$$

$\underline{x_1 = h}$

$$|\underline{n}| = \sqrt{h^2} = h$$

$$\frac{\underline{n}}{|\underline{n}|} = \frac{1}{h} \langle 0, 0, h \rangle = \langle 0, 0, 1 \rangle$$ ← $\underline{n}$ is a unit vector!

$$\begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} & \frac{2px_1x_2}{a^2} & 0 \\ \frac{2px_1x_2}{a^2} & \frac{p(x_1^2 - x_2^2)}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{t}_{x_3 = h}}$$

$\underline{x_1 = -h}$

$$\frac{\underline{n}}{|\underline{n}|} = \langle 0, 0, -1 \rangle$$

$$\begin{bmatrix} \frac{-p(x_1^2 - x_2^2)}{a^2} & \frac{2px_1x_2}{a^2} & 0 \\ \frac{2px_1x_2}{a^2} & \frac{p(x_1^2 - x_2^2)}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{t}_{x_3 = -h}}$$

Problem 3

The Cauchy stress tensor at a point in a solid is given by

$$\sigma_{11} = -3, \quad \sigma_{12} = 1, \quad \sigma_{13} = 2, \quad \sigma_{22} = 0, \quad \sigma_{23} = T, \quad \sigma_{33} = 0$$

where T is a constant. Find all values of T that would result in a traction-free plane through the point and, for each T, determine the orientation (or normal vector) of that plane.

$$\underset{=}{\sigma} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & T \\ 2 & T & 0 \end{bmatrix}$$

set $\det[\underset{=}{\sigma}] = 0$ and solve for T:

$$\det[\underset{=}{\sigma}] = -3(0-T^2) - 1(0-2T) + 2(T-0) = 3T^2 + 2T + 2T = 0$$

$$= 3T^2 + 4T = 0$$

$$= T(3T+4) = 0 \Rightarrow \boxed{\begin{matrix} T=0 \\ T=-\frac{4}{3} \end{matrix}}$$

$\underline{T=0}$

$t_i = \sigma_{ij} n_j = 0$ ← traction-free

$$\underline{t}_{T=0} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \left. \begin{matrix} -3n_1 + n_2 + 2n_3 = 0 \\ n_1 = 0 \\ 2n_1 = 0 \end{matrix} \right\} \begin{matrix} n_2 + 2n_3 = 0 \\ \text{or} \\ n_2 = -2n_3 \end{matrix}$$

set $n_2 = 1 \Rightarrow n_3 = -\frac{1}{2}$

$\underline{n} = \langle 0, 1, -\frac{1}{2} \rangle$ ← make this a unit vector!

$$|\underline{n}| = \sqrt{1^2 + (\tfrac{1}{2})^2} = \sqrt{1 + \tfrac{1}{4}} = \sqrt{\tfrac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\underline{n} = \frac{2}{\sqrt{5}} \langle 0, 1, -\tfrac{1}{2} \rangle \Rightarrow \boxed{\underline{n}_{T=0} = \langle 0, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}} \rangle}$$

$\underline{T = -\frac{4}{3}}$

$$\underline{t}_{T=-\frac{4}{3}} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -\frac{4}{3} \\ 2 & -\frac{4}{3} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \left. \begin{matrix} -3n_1 + n_2 + 2n_3 = 0 \ (*) \\ n_1 - \frac{4}{3} n_3 = 0 \\ 2n_1 - \frac{4}{3} n_2 = 0 \end{matrix} \right\} \begin{matrix} n_1 = \frac{4}{3} n_3 \quad (1) \\ 2n_1 = \frac{4}{3} n_2 \\ n_1 = \frac{2}{3} n_2 \quad (2) \end{matrix}$$

plugging (1) into (2) and solving for n_2:

$$\frac{4}{3}n_3 = \frac{2}{3}n_2 \Rightarrow n_2 = 2n_3 \quad (3)$$

plugging in (1) and (3) into (*):

$$-3\left(\frac{4}{3}n_3\right) + (2n_3) + 2n_3 = 0$$

$$-4n_3 + 4n_3 = 0 \Rightarrow \text{let } n_3 = 1 \Rightarrow n_2 = 2, \quad n_1 = \frac{4}{3}$$

$\underline{n} = \langle \frac{4}{3}, 2, 1 \rangle$ ← make this a unit vector!

$$|\underline{n}| = \sqrt{\left(\frac{4}{3}\right)^2 + (2)^2 + (1)^2} = \sqrt{\frac{16}{9} + 4 + 1} = \sqrt{\frac{16+36+9}{9}} = \sqrt{\frac{61}{9}} = \frac{\sqrt{61}}{3}$$

$$\underline{n} = \frac{3}{\sqrt{61}} \langle \frac{4}{3}, 2, 1 \rangle \Rightarrow \boxed{\underline{n}_{T=-\frac{4}{3}} = \langle \frac{4}{\sqrt{61}}, \frac{6}{\sqrt{61}}, \frac{3}{\sqrt{61}} \rangle}$$

Problem 4

The components of the Cauchy stress tensor at a point in a solid are given, in the basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, by:

$$[\sigma] = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

1. Calculate the stress vector (traction) $\underline{t}$ on a surface element with normal $[0,-1,1]^T$

$$t_i = \sigma_{ij} n_j$$

$\underline{n} = \langle 0, -1, 1 \rangle$ ← must be a unit vector!

$$|\underline{n}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \Rightarrow \underline{n} = \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} + 2/\sqrt{2} \\ -2/\sqrt{2} + 2/\sqrt{2} \end{bmatrix} = \boxed{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \underline{t}}$$

2. Give the normal and shear components of this stress vector (indicating both magnitude and direction of the components).

$$t_n = \sigma_{ij} n_i n_j = t_i n_i$$

$$= \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right] \cdot \left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] = 0 + \left(\frac{-1}{2}\right) + 0 = -\frac{1}{2}$$

$$\underline{t}_n = t_n \underline{n} = -\frac{1}{2}\left[0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T = \left[0, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right]^T$$

normal component of stress vector:

$$|\underline{t}_n| = -\frac{1}{2}$$

$$\underline{t}_n = \left[0, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right]^T$$

$$|\underline{t}_s| = \sqrt{|\underline{t}|^2 - |\underline{t}_n|^2} \qquad |\underline{t}|^2 = \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$= \sqrt{1 - \left(-\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$\underline{t}_s = \underline{t} - \underline{t}_n = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right] - \left[0, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right] = \left[-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]^T$$

Shear component of stress vector:

$$|\underline{t}_s| = \frac{\sqrt{3}}{2}$$

$$\underline{t}_s = \left[-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]^T$$

3. Find the principal stresses and principal directions of the stress tensor.

$$\det|\underline{\sigma} - \lambda \underline{\underline{I}}| = 0$$

$$\det\begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 1-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = (3-\lambda)\left[(1-\lambda)(2-\lambda) - (2)(2)\right] - 1\left[(1)(2-\lambda) - (2)(0)\right] + 0 = 0$$

$$= (3-\lambda)\left[\underbrace{2 - 3\lambda + \lambda^2 - 4}_{\lambda^2 - 3\lambda - 2}\right] - (2-\lambda) = 0$$

$$= 3\lambda^2 - 9\lambda - 6 - \lambda^3 + 3\lambda^2 + 2\lambda - 2 + \lambda = 0$$

$$= -\lambda^3 + 6\lambda^2 - 6\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda^2 - 2\lambda - 2) = 0 \qquad \frac{2 \pm \sqrt{4 - 4(-2)}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$$\Rightarrow \lambda = 4,\ 1+\sqrt{3},\ 1-\sqrt{3}$$

principal stresses: $\sigma_1 = 4$, $\sigma_2 = 1+\sqrt{3}$, $\sigma_3 = 1-\sqrt{3}$

← Convention: $\sigma_1 > \sigma_2 > \sigma_3$

$\underline{\lambda = 4}$

$$\begin{bmatrix} 3-4 & 1 & 0 \\ 1 & 1-4 & 2 \\ 0 & 2 & 2-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}$$

$t_i = \sigma_{ij} n_j = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{c} -n_1 + n_2 = 0 \\ n_1 - 3n_2 + 2n_3 = 0 \\ 2n_2 - 2n_3 = 0 \end{array}\right\} \quad \left.\begin{array}{l} n_1 = n_2 \\ n_2 = n_3 \end{array}\right\} \text{ let } n_1 = n_2 = n_3 = 1$$

$\underline{n} = \langle 1, 1, 1 \rangle$ ← make this a unit vector!

$|\underline{n}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\Rightarrow \underline{v}_1 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T$

$\underline{\lambda = 1 + \sqrt{3}}$

$$\begin{bmatrix} 3-(1+\sqrt{3}) & 1 & 0 \\ 1 & 1-(1+\sqrt{3}) & 2 \\ 0 & 2 & 2-(1+\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 2-\sqrt{3} & 1 & 0 \\ 1 & -\sqrt{3} & 2 \\ 0 & 2 & 1-\sqrt{3} \end{bmatrix}$$

$t_i = \sigma_{ij} n_j = 0$

$$\begin{bmatrix} 2-\sqrt{3} & 1 & 0 \\ 1 & -\sqrt{3} & 2 \\ 0 & 2 & 1-\sqrt{3} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{c} (2-\sqrt{3})n_1 + n_2 = 0 \\ n_1 - \sqrt{3}n_2 + 2n_3 = 0 \\ 2n_2 + (1-\sqrt{3})n_3 = 0 \end{array}\right\} \quad \begin{array}{l} n_1 = \frac{-1}{2-\sqrt{3}} n_2 \simeq -3.732 n_2 \\ n_3 = \frac{-2}{1-\sqrt{3}} n_2 \simeq -2.732 n_2 \end{array}$$

$$\frac{-1}{2-\sqrt{3}} n_2 - \sqrt{3} n_2 - \frac{2(2)}{1-\sqrt{3}} n_2 = 0$$

$$0 n_2 = 0 \Rightarrow \text{let } n_2 = 1$$

$\underline{n} = \left\langle \frac{-1}{2-\sqrt{3}}, 1, \frac{-2}{1-\sqrt{3}} \right\rangle$ ← make this a unit vector!

$|\underline{n}| = \sqrt{\left(\frac{-1}{2-\sqrt{3}}\right)^2 + 1^2 + \left(\frac{-2}{1-\sqrt{3}}\right)^2} = \sqrt{(3+4\sqrt{3}+4) + 1 + 2\sqrt{3} + 4} = \sqrt{6\sqrt{3}+12} = 3+\sqrt{3}$

$$\underline{n} = \left[\frac{-1}{\underbrace{(2-\sqrt{3})(3+\sqrt{3})}_{\substack{6-\sqrt{3}-3 \\ = 3-\sqrt{3}}}}, \frac{1}{3+\sqrt{3}}, \frac{-2}{\underbrace{(1-\sqrt{3})(3+\sqrt{3})}_{\substack{3-2\sqrt{3}-3 \\ = -2\sqrt{3}}}}\right]^T$$

$\Rightarrow \underline{v}_2 = \left[\frac{-1}{3-\sqrt{3}}, \frac{1}{3+\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T \simeq [-0.789, 0.211, 0.577]^T$

$\underline{\lambda = 1-\sqrt{3}}$

$$\begin{bmatrix} 3-(1-\sqrt{3}) & 1 & 0 \\ 1 & 1-(1-\sqrt{3}) & 2 \\ 0 & 2 & 2-(1-\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1 & 0 \\ 1 & +\sqrt{3} & 2 \\ 0 & 2 & 1+\sqrt{3} \end{bmatrix}$$

$$t_i = \sigma_{ij} n_j = 0$$

$$\begin{bmatrix} 2+\sqrt{3} & 1 & 0 \\ 1 & +\sqrt{3} & 2 \\ 0 & 2 & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} (2+\sqrt{3})n_1 + n_2 &= 0 \\ n_1 + \sqrt{3}n_2 + 2n_3 &= 0 \\ 2n_2 + (1+\sqrt{3})n_3 &= 0 \end{aligned}\right\} \begin{aligned} n_1 &= \frac{-1}{2+\sqrt{3}} n_2 \simeq -0.268 n_2 \\ n_3 &= \frac{-2}{1+\sqrt{3}} n_2 \simeq -0.732 n_2 \end{aligned}$$

$$\frac{-1}{2+\sqrt{3}} n_2 + \sqrt{3} n_2 - \frac{2(2)}{1+\sqrt{3}} n_2 = 0$$

$$0 n_2 = 0 \Rightarrow \text{let } n_2 = 1$$

$$\underline{n} = \left< \frac{-1}{2+\sqrt{3}}, 1, \frac{-2}{1+\sqrt{3}} \right>$$ ← make this a unit vector!

$$|\underline{n}| = \sqrt{\left(\frac{-1}{2+\sqrt{3}}\right)^2 + 1^2 + \left(\frac{-2}{1+\sqrt{3}}\right)^2} = \sqrt{7-4\sqrt{3}+1+3-2\sqrt{3}+1} = \sqrt{-6\sqrt{3}+12} = 3-\sqrt{3}$$

$$\underline{n} = \left[\frac{-1}{\underbrace{(2+\sqrt{3})(3-\sqrt{3})}_{\substack{6+\sqrt{3}-3 \\ = 3+\sqrt{3}}}}, \frac{1}{3-\sqrt{3}}, \frac{-2}{\underbrace{(1+\sqrt{3})(3-\sqrt{3})}_{\substack{3+2\sqrt{3}-3 \\ = 2\sqrt{3}}}} \right]^T$$

$$\Rightarrow \underline{v}_2 = \left[\frac{-1}{3+\sqrt{3}}, \frac{1}{3-\sqrt{3}}, \frac{-1}{\sqrt{3}} \right]^T \simeq [-0.211, 0.789, -0.577]^T$$

principal directions:

$$\underline{v}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1/(3-\sqrt{3}) \\ 1/(3+\sqrt{3}) \\ 1/\sqrt{3} \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -1/(3+\sqrt{3}) \\ 1/(3-\sqrt{3}) \\ -1/\sqrt{3} \end{bmatrix}$$

← $\underline{v}_1$ corresponds to σ_1
$\underline{v}_2$ corresponds to σ_2
$\underline{v}_3$ corresponds to σ_3

4. Write the matrix representation of the stress tensor $\underline{\underline{\sigma}}$ with respect to the eigenbasis (basis that consists of principal directions).

$$\underline{\underline{\sigma}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{bmatrix}$$

Problem 5

The components of the Cauchy stress tensor in a solid <u>at rest</u> are given, in the basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, by:

$$[\underline{\underline{\sigma}}] = \frac{1}{4}\rho\omega^2 \begin{bmatrix} x_1^2 & 2x_1x_2 & 0 \\ 2x_1x_2 & x_2^2 & 0 \\ 0 & 0 & 2(x_1^2+x_2^2) \end{bmatrix}$$

where ρ is the density (constant) and ω is a constant. Find the body force $\underline{b}$ that must be acting on this body.

$$\sigma_{ij,j} + \rho b_i = \cancel{\rho \ddot{x}_i}^{\to 0}$$ because body is at rest

$$\Rightarrow \quad \sigma_{ij,j} = -\rho b_i$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = -\rho b_1 \qquad \sigma_{11} = \frac{1}{4}\rho\omega^2 x_1^2, \quad \sigma_{12} = \frac{1}{2}\rho\omega^2 x_1 x_2, \quad \sigma_{13} = 0$$

$$\frac{1}{2}\cancel{\rho}\omega^2 x_1 + \frac{1}{2}\cancel{\rho}\omega^2 x_1 + 0 = -\cancel{\rho} b_1 \quad \Rightarrow \quad \omega^2 x_1 = -b_1$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = -\rho b_2 \qquad \sigma_{22} = \frac{1}{4}\rho\omega^2 x_2^2, \quad \sigma_{12} = \frac{1}{2}\rho\omega^2 x_1 x_2, \quad \sigma_{23} = 0$$

$$\frac{1}{2}\cancel{\rho}\omega^2 x_2 + \frac{1}{2}\cancel{\rho}\omega^2 x_2 + 0 = -\cancel{\rho} b_2 \quad \Rightarrow \quad \omega^2 x_2 = -b_2$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = -\rho b_3 \qquad \sigma_{33} = \frac{1}{2}\rho\omega^2 x_1^2 + \frac{1}{2}\rho\omega^2 x_2^2, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0$$

$$0 + 0 + 0 = -\rho b_3 \quad \Rightarrow \quad 0 = b_3$$

$$\boxed{\underline{b} = \begin{bmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \\ 0 \end{bmatrix}}$$

Problem 6

Consider a general stress state at a point given by the Cauchy stress tensor $\underline{\underline{\sigma}}$. Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be the eigenvalues of $\underline{\underline{\sigma}}$ and $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ the corresponding eigenvectors. Show that the largest shear stress is $\frac{1}{2}(\sigma_1 - \sigma_3)$ and it corresponds to directions $\frac{1}{\sqrt{2}}(\underline{v}_1 + \underline{v}_3)$ and $\frac{1}{\sqrt{2}}(\underline{v}_1 - \underline{v}_3)$.

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \Rightarrow \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 v_1 \\ \sigma_2 v_2 \\ \sigma_3 v_3 \end{bmatrix}$$

Shear stress: $|\underline{t}_s|^2 = |\underline{t}|^2 - |\underline{t}_n|^2$

$$= \sigma_{ij} n_j \sigma_{ik} n_k - (\sigma_{ij} n_i n_j)^2$$

then

$$t_s^2 = (\sigma_1^2 v_1^2 + \sigma_2^2 v_2^2 + \sigma_3^2 v_3^2) - \underbrace{(\sigma_1 v_1^2 + \sigma_2 v_2^2 + \sigma_3 v_3^2)}_{\text{call this } A}{}^2$$

to find the maximum shear, take the derivative of t_s^2 with respect to the normal directions and set them equal to zero:

$$\frac{\partial t_s^2}{\partial v_1} = \cancel{2}\sigma_1^2 \cancel{v_1} - \cancel{2}A(2\sigma_1 \cancel{v_1}) = 0 \qquad \sigma_1^2 - 2\sigma_1 A = 0 \quad (1)$$

$$\frac{\partial t_s^2}{\partial v_2} = \cancel{2}\sigma_2^2 \cancel{v_2} - \cancel{2}A(2\sigma_2 \cancel{v_2}) = 0 \quad \Rightarrow \quad \sigma_2^2 - 2\sigma_2 A = 0 \quad (2)$$

$$\frac{\partial t_s^2}{\partial v_3} = \cancel{2}\sigma_3^2 \cancel{v_3} - \cancel{2}A(2\sigma_3 \cancel{v_3}) = 0 \qquad \sigma_3^2 - 2\sigma_3 A = 0 \quad (3)$$

because $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum will be $\sigma_1 - \sigma_3$:

(1) − (3):

$$\sigma_1^2 - \sigma_3^2 - 2\sigma_1 A - 2\sigma_3 A = 0$$

$$\cancel{(\sigma_1 - \sigma_3)}(\sigma_1 + \sigma_3) - 2A\cancel{(\sigma_1 - \sigma_3)} = 0$$

replacing A in the equation above:

$$\sigma_1 + \sigma_3 - 2(\sigma_1 v_1^2 + \sigma_2 v_2^2 + \sigma_3 v_3^2) = 0$$

$$\sigma_1 + \sigma_3 - 2\sigma_1 v_1^2 - 2\sigma_2 v_2^2 - 2\sigma_3 v_3^2) = 0$$

→ we know that $v_1^2 + v_2^2 + v_3^2 = 1 \Rightarrow v_3^2 = 1 - v_1^2 - v_2^2$

$$\sigma_1 + \sigma_3 - 2\sigma_1 v_1^2 - 2\sigma_2 v_2^2 - 2\sigma_3(1 - v_1^2 - v_2^2) = 0$$

$$\sigma_1 + \sigma_3 - 2\sigma_1 v_1^2 - 2\sigma_2 v_2^2 - 2\sigma_3 + 2\sigma_3 v_1^2 + 2\sigma_3 v_2^2 = 0$$

because we are looking at the 1-3 plane, $v_2 = 0$:

$$\sigma_1 + \sigma_3 - 2\sigma_1 v_1^2 - 2\sigma_3 + 2\sigma_3 v_1^2 = 0$$

$$\sigma_1 - \sigma_3 - 2\sigma_1 v_1^2 + 2\sigma_3 v_1^2 = 0$$

$$\underbrace{(\sigma_1 - \sigma_3)}_{\neq 0}(1 - 2v_1^2) = 0$$

$$\downarrow$$

$$1 - 2v_1^2 = 0 \implies v_1^2 = \frac{1}{2} \implies v_1 = \pm\frac{1}{\sqrt{2}}$$

if $v_1^2 + v_2^2 + v_3^2 = 1$ and $v_2 = 0$:

$$\frac{1}{2} + 0 + v_3 = 1 \implies v_3^2 = \frac{1}{2} \implies v_3 = \pm\frac{1}{\sqrt{2}}$$

plugging these values back in to t_s^2:

$$t_s^2 = \sigma_1^2\left(\tfrac{1}{2}\right) + \sigma_2^2(0) + \sigma_3^2\left(\tfrac{1}{2}\right) - \left(\sigma_1\left(\tfrac{1}{2}\right) + \sigma_2(0) + \sigma_3\left(\tfrac{1}{2}\right)\right)^2$$

$$= \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} - \left(\frac{\sigma_1}{2} + \frac{\sigma_3}{2}\right)^2$$

$$= \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} - \left(\frac{\sigma_1^2}{4} + \frac{\sigma_1\sigma_3}{2} + \frac{\sigma_3^2}{4}\right)$$

$$t_s^2 = \frac{\sigma_1^2}{4} - \frac{\sigma_1\sigma_3}{2} + \frac{\sigma_3^2}{4} = \frac{1}{4}(\sigma_1 - \sigma_3)^2$$

$$t_s = \sqrt{\frac{1}{4}(\sigma_1 - \sigma_3)^2}$$

$$\implies t_{s_{max}} = \frac{1}{2}(\sigma_1 - \sigma_3) \text{ corresponding to directions } \frac{1}{\sqrt{2}}(\underline{v}_1 \pm \underline{v}_3) \quad \checkmark$$