Final Exam - Problem 3

Problem 3

Consider the plate ($0 < x_1 < a$; $-b < x_2 < b$) shown in Figure 3 and subject to the following loads:

- The surfaces $x_2 = \pm b$ are stress-free
- The surface $x_1 = a$ is subject to a force per unit length of $\tau(x_2)$ pointed in the x_2 direction, giving a total force equal to F, i.e.

$$F = \int_{-b}^{b} \tau(x_2) dx_2$$

- The surface $x_1 = 0$ is subject to suitable loads to keep the plate in equilibrium.
- Zero body force.

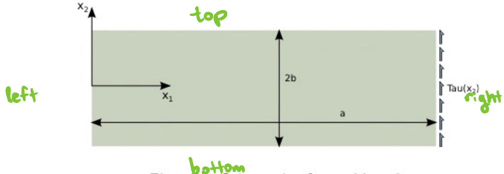

Figure 3: Schematics for problem 3.

Which of the following is a solution of the elasticity problem (for suitable constants A, B, C, and D)? Explain why and give the correct values of A, B, C, and D.

✗ 1. $\sigma_{11} = Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)$
$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$
$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

✓ 2. $\sigma_{11} = Ax_2 + Bx_1x_2$
$\sigma_{22} = D$
$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

✗ 3. $\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2$
$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$
$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

Each case need to check the following to identify if it is a solution of the elasticity problem:

- Balance of linear momentum
- Boundary conditions

Case 1 - is it the solution? Let's Check!

1. $\sigma_{11} = Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)$

$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

- Balance of linear momentum: $\sigma_{ij} + \rho b_i = 0$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancel{\frac{\partial \sigma_{13}}{\partial x_3}}_{\to 0} + \cancel{\rho b_1}_{\to 0} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \cancel{\frac{\partial \sigma_{23}}{\partial x_3}}_{\to 0} + \cancel{\rho b_2}_{\to 0} = 0$$

$$\cancel{\frac{\partial \sigma_{13}}{\partial x_1}}_{\to 0} + \cancel{\frac{\partial \sigma_{32}}{\partial x_2}}_{\to 0} + \cancel{\frac{\partial \sigma_{33}}{\partial x_3}}_{\to 0} + \cancel{\rho b_3}_{\to 0} = 0$$

$\therefore$ Simplifies to:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Check if this case satisfies balance of linear momentum:

$$\frac{\partial\left(Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)\right)}{\partial x_1} + \frac{\partial\left(-\frac{1}{2}Bx_2^2 - C\right)}{\partial x_2} = 0$$

$$Bx_2 - \frac{D\pi}{2a}\sin\left(\frac{\pi x_1}{2a}\right) - Bx_2 = 0$$

$$\therefore \quad \frac{-D\pi}{2a}\sin\left(\frac{\pi x_1}{2a}\right) = 0$$

$$\sin\left(\frac{\pi x_1}{2a}\right) = 0 \quad \times$$

Doesn't always satisfy balance of linear momentum
$\therefore$ Case 1 is not the solution.

Case 2 - Is it the solution? Let's check!

2. $\sigma_{11} = Ax_2 + Bx_1x_2$
$\sigma_{22} = D$
$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

- Balance of linear momentum:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial (Ax_2 + Bx_1x_2)}{\partial x_1} + \frac{\partial\left(-\frac{1}{2}Bx_2^2 - C\right)}{\partial x_2} = 0$$

$$Bx_2 - Bx_2 = 0$$

$$0 = 0 \quad \checkmark$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

$$\frac{\partial\left(-\frac{1}{2}Bx_2^2 - C\right)}{\partial x_1} + \frac{\partial (D)}{\partial x_2} = 0$$

$$0 = 0 \quad \checkmark$$

$\therefore$ Case 2 satisfies balance of linear momentum, so now check B.C.s.

- B.C.s for Case 2:

<u>Right side:</u>

$$\sigma_{11} = 0$$
$$\sigma_{12} = \tau(x_2)$$
$$x_2 = x_2$$
$$x_1 = a$$

$$\sigma_{11} = Ax_2 + Bx_1x_2 = 0 \quad \therefore \; A = -Ba$$

$$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C = \tau(x_2) \quad \therefore \; C = -\frac{1}{2}Bx_2^2 - \tau(x_2)$$

<u>Left side:</u>

$$\sigma_{11} = 0$$
$$\sigma_{12} = -\tau(x_2) \qquad = +\tau(x_2)$$
$$x_2 = x_2$$
$$x_1 = 0$$

$$\sigma_{11} = Ax_2 + B\underbrace{\cancel{x_1}}_{\to 0}x_2 = 0 \quad \therefore \; A = 0$$

trivial solutions!!!

$$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C = \overset{+}{-}\tau(x_2) \quad \therefore \; C = -\frac{1}{2}Bx_2^2 \overset{-}{+}\tau(x_2) \quad [1]$$

<u>Bottom:</u>

$$\sigma_{12} = 0$$
$$\sigma_{22} = 0$$
$$x_2 = -b$$
$$x_1 = x_1$$

$$\sigma_{11} = -Ab - Bx_1b$$
$$\sigma_{12} = -\frac{1}{2}Bb^2 - C = 0 \quad \therefore \; C = -\frac{1}{2}Bb^2 \quad [2]$$
$$\sigma_{22} = D = 0 \quad \therefore \; D = 0$$

<u>Set [1] = [2]</u>

$$-\frac{1}{2}Bx_2^2 + \tau(x_2) = -\frac{1}{2}Bb^2$$
$$\tau(x_2) = \frac{B}{2}(x_2^2 - b^2)$$
$$\therefore \; B = \frac{2\tau(x_2)}{(x_2^2 - b^2)}$$

<u>Top:</u>

$$\sigma_{12} = 0$$
$$\sigma_{22} = 0$$
$$x_2 = b$$
$$x_1 = x_1$$

$$\sigma_{11} = Ab + Bbx_1, \qquad \sigma_{22} = D = 0 \quad \therefore \; D = 0$$
$$\sigma_{12} = -\frac{1}{2}Bb^2 - C \quad \therefore \; C = -\frac{1}{2}Bb^2$$

$$A = -Ba$$
$$\therefore \quad D = 0$$
$$C = -\tfrac{1}{2}Bb^2$$
$$B = \frac{2\tau(x_2)}{(x_2^2 - b^2)} \quad [3] \qquad \text{know: } F = \int_{-b}^{b} \tau(x_2)\,dx_2 \quad [4]$$

Solve [3] for $\tau(x_2)$ and plug into [4] to get B in terms of F.

From [3]: $\tau(x_2) = \dfrac{B(x_2^2 - b^2)}{2}$

Plug eqn for $\tau(x_2)$ into [4]:

$$F = \int_{-b}^{b} \frac{B}{2}(x_2^2 - b^2)\,dx_2$$
$$= \frac{B}{2}\int_{-b}^{b} x_2^2 - b^2\,dx_2$$
$$= \frac{B}{2}\left[\frac{x_2^3}{3} - b^2 x_2\right]\Bigg|_{-b}^{b}$$
$$= \frac{B}{2}\left[\left(\frac{b^3}{3} - b^3\right) - \left(\frac{-b^3}{3} + b^3\right)\right]$$
$$= \frac{B}{2}\left[\frac{b^3}{3} - b^3 + \frac{b^3}{3} - b^3\right]$$
$$F = \frac{B}{2}\cdot\frac{-4}{3}b^3$$
$$F = B\left(\frac{-2b^3}{3}\right)$$
$$\therefore B = \frac{-3F}{2b^3}$$

$$\therefore A = -Ba = \frac{3Fa}{2b^3} \quad \text{(right side)}$$

$$B = \frac{-3F}{2b^3}$$

$$C = -\tfrac{1}{2}Bb^2 = \frac{3F}{4b}$$

$$D = 0$$

$\therefore \sigma_{11} = Ax_2 + Bx_1x_2$

$\Rightarrow \sigma_{11} = \frac{3F_a}{2b^3}x_2 - \frac{3F}{2b^3}x_1x_2$ ✓ Satisfies σ_{11} at each side

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

$= -\frac{1}{2}\left(\frac{-3F}{2b^3}\right)x_2^2 + \frac{3F}{4b}$

$= \frac{3F}{4b^3}x_2^2 + \frac{3F}{4b}$

$\Rightarrow \sigma_{12} = \frac{3}{4}F\left(\frac{x_2^2}{b^3} + \frac{1}{b}\right)$ ✓ Satisfies σ_{12} at each side

$\sigma_{22} = D$

$\Rightarrow \sigma_{22} = 0$ ✓ Satisfies σ_{22} at each side.

$\therefore$ Case 2 is a suitable solution of this elasticity problem because balance of linear momentum is satisfied and has suitable constants for A, B, C, and D.

Case 3 - Is it a solution? Let's check!

3. $\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

- Balance of linear momentum

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial\left(Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1-a)^2\right)}{\partial x_1} + \frac{\partial\left(-\frac{1}{2}Bx_2^2 - B(x_1-a)x_2 + C\right)}{\partial x_2} = 0$$

$$Bx_2 + Bx_1 - 2 - Bx_2 - Bx_1 + Ba = 0$$

$$-2 + Ba = 0$$

$B_2 = 2$ ✗ even if $B = 0$,
we wouldn't satisfy
balance of linear momentum.

Doesn't satisfy balance of linear momentum
$\therefore$ Case 3 is not the solution.