Unit Load Method (ULM) for Beams

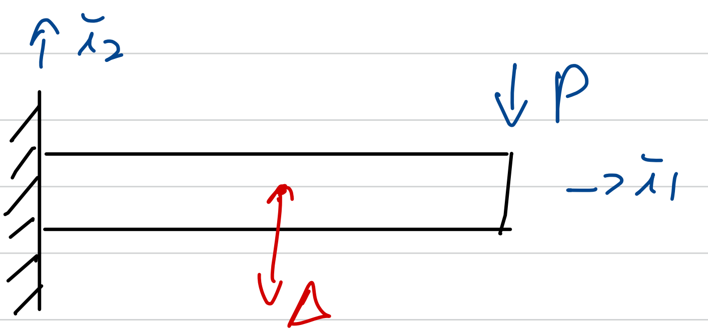
* Direct consequence of the PCVW.

* Convenient to provide the displacement of a structure at a given point and direction.

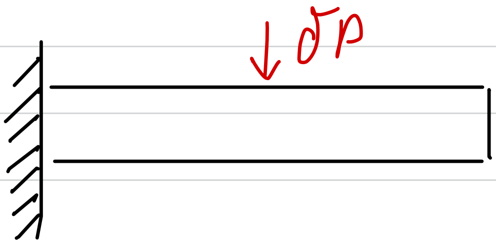
* Useful for statically indeterminate structures.

PCVW: $\delta W_I' + \delta W_E' = 0$

for all statically admissible virtual forces.



Real System



Unit Load System

$$\delta W_E' = \Delta \cdot \delta P$$

$$\Delta \cdot \delta P + \delta W_I' = 0$$

$$\delta W_I' = -\int_0^L \delta M_3 \cdot K_3 \, dx_1$$

Let $\hat{M}_3 = \delta M_3$ be a statically admissible virtual force

$$\Delta \cdot \delta P = \int_0^L \hat{M}_3 \cdot K_3 \, dx_1$$

$\hat{M}_3$ must satisfy equilibrium and balance with δP.

Statics (taking a cut)

$$\hat{M}_3 = \begin{cases} -\delta P (L/2 - x_1) & x_1 < L/2 \\ 0 & x_1 > L/2 \end{cases}$$ Unit Load System

$$K_3 = \frac{M_3}{H_{33}^c}$$

$$M_3 = -P(L - x_1)$$

Real System

$$\Delta \cdot \delta P = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} \, dx_1$$

$$\Delta \cdot \delta P = \int_0^{L/2} -\delta P(L/2 - x_1) \cdot \frac{-P(L-x_1)}{H_{33}^c} dx_1$$

$$+ \int_{L/2}^{L} (0) \frac{-P(L-x_1)}{H_{33}^c} dx_1$$

Since δP is arbitrary. Choose $\delta P = 1$

$$\Delta = \int_0^{L/2} (L/2 - x_1) \cdot \frac{(L-x_1)}{H_{33}^c} \cdot P \, dx_1$$

$$\boxed{\Delta = \frac{5}{48} \frac{PL^3}{H_{33}^c}}$$

→ Δ is defined positive in the direction of the applied virtual force.

More Generally for Beams

$$\Delta \cdot \delta P = -\delta W_I'$$

$$\Delta \cdot \delta P = \int_0^L \left(\bar{\varepsilon}_1 \delta N_1 + \kappa_2 \delta M_2 + \kappa_3 \delta M_3 \right) dx_1$$

Chooses $\delta P = 1$, and $\hat{N}_1 = \delta N_1$, $\hat{M}_2 = \delta M_2$, $\hat{M}_3 = \delta M_3$ are the resulting statically admissible virtual forces and moments.

$$\Delta = \int_0^L \left(\bar{\varepsilon}_1 \hat{N}_1 + \kappa_2 \hat{M}_2 + \kappa_3 \hat{M}_3 \right) dx_1$$

$$\left\{ \begin{aligned} \kappa_2 &= \left(H_{33}^c M_2 + H_{23}^c M_3 \right) / \Delta H \\ \kappa_3 &= \left(H_{23}^c M_2 + H_{22}^c M_3 \right) / \Delta H \end{aligned} \right.$$

$$\bar{\varepsilon}_1 = N_1 / S_1$$

PROCEDURE

1) Find the actual (real) force and moment distribution in the beam

2) Apply a unit load (or moment) in the direction at the desired displacement (or rotation) component.

3) Evaluate the corresponding virtual stress and moment distribution. This is called the unit load system.

4) Solve for the displacement (or rotation) desired.