Torsion

Equilibrium & Governing Equations

$$\frac{dM_1}{dx_1} = -q_1(x_1)$$ ✓ Distributed Torque

$$H_{11} = \frac{M_1}{K_1}$$ ✓ TORSIONAL STIFFNESS

$$K_1 = \frac{d\phi_1}{dx_1}$$ ✓ Twist Rate

$$\frac{d}{dx_1}\left[H_{11}\frac{d\phi_1}{dx_1}\right] = -q_1(x_1)$$

Boundary Conditions

1) Fixed end, $\phi_1 = 0$

2) Free end, $M_1 = 0 \rightarrow \frac{d\phi_1}{dx_1} = 0$

3) Applied Torque, T

$$M_1 = T \qquad \frac{d\phi_1}{dx_1} = \frac{T}{H_{11}}$$

A) Circular Cylinders

i) Kinematics

$$u_2 = -x_3 \, \phi_1(x_1)$$

$$u_3 = x_2 \, \phi_1(x_1)$$

$$u_1 = 0$$

$$\gamma_{12} = -x_3 \, K_1(x_1)$$

$$\gamma_{13} = x_2 \, K_1(x_1)$$

ii) Stress

$$\tau_{12} = G\gamma_{12} = -G x_3 K_1$$

$$\tau_{13} = G\gamma_{13} = G x_2 K_1$$

$$\tau_R = 0$$

$$\tau_\alpha = G \cdot r \cdot K_1$$

iii) Torsional Stiffness

$$H_{11} = \int_A G r^2 \, dA \,, \quad H_{11} = G\frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

B) Arbitrarily Shaped Cross-Section

i) Kinematics

$$u_2 = -x_3\,\phi_1(x_1) \qquad \underline{K_1 = \text{constant}}$$

$$u_3 = x_2\,\phi_1(x_1)$$

$$u_1 = \psi(x_2, x_3)\cdot K_1$$

ii) Prandtl Stress Function, $\Phi(x_2, x_3)$

Solve $$\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = -2GK_1 \quad \text{on } A$$

with $$\frac{\partial \Phi}{\partial s} = 0 \rightarrow \Phi = 0 \quad \text{on } C_1'$$

↳ Solid bounded by a single C_1'.

iii) Sectional Eq. & Stiffness

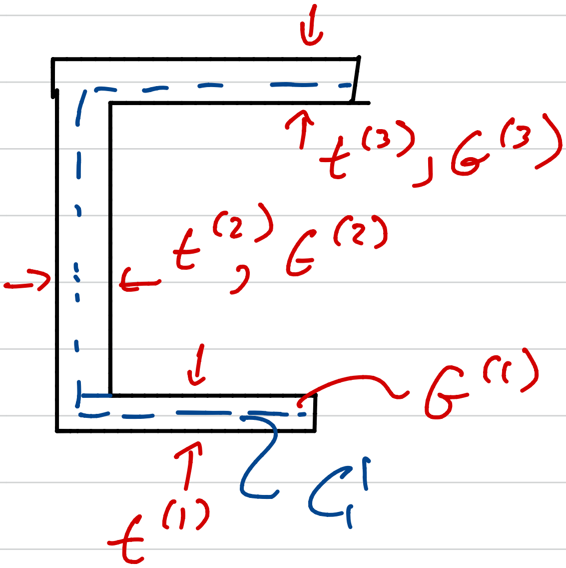
$$M_1 = 2\cdot\int_A \Phi\, dA$$

Find $H_{11} = M_1/K_1$ once Φ is known.

iv) Warping & Stress

$$\tau_{12} = \frac{\partial \Phi}{\partial x_3} = GK_1\left(\frac{\partial \psi}{\partial x_2} - x_3\right), \quad \tau_{13} = -\frac{\partial \Phi}{\partial x_2} = GK_1\left(\frac{\partial \psi}{\partial x_3} + x_2\right)$$

c) Thin-Walled Open Section



i) Stiffness

$$H_{11}^{(i)} = \frac{1}{3} G^{(i)} \cdot l^{(i)} \cdot \left(t^{(i)}\right)^3$$

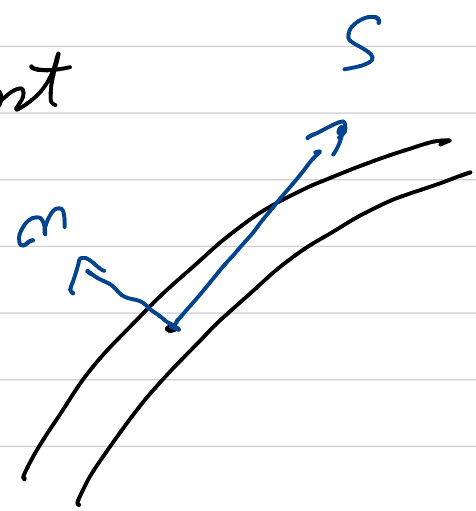
$$H_{11} = \sum_i H_{11}^{(i)}$$

i-th Segments

- $l^{(i)}$ – length at the segment
- $t^{(i)}$ – thickness
- $G^{(i)}$ – Shear Modulus

ii) Stress

In each Segment

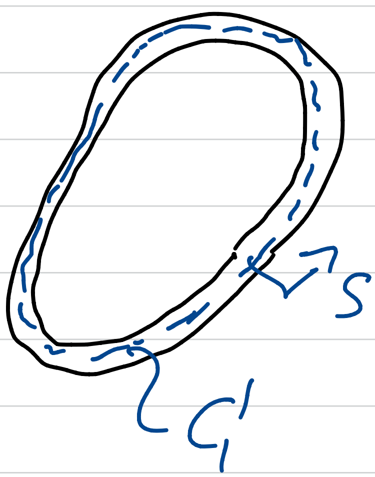


$$\tau_s^{(i)} = 2 \cdot G^{(i)} \cdot K_1 \cdot n$$

$$\tau_s^{(i)} = 2\, G^{(i)} \frac{M_1}{H_{11}}\, n$$

$$\tau_{s,max}^{(i)} = G^{(i)} \frac{M_1}{H_{11}}\, t^{(i)}$$

D) Thin-Walled Closed Sections



* Shear flow is constant

$$f = \tau_s \cdot t(s) = \text{constant}$$

i) Stiffness & Twist Rate

$$K_1 = \frac{M_1}{4 \cdot A_c^2} \cdot \int_{C_1'} \frac{ds}{G \cdot t}$$

$$H_{11} = \frac{4 \cdot A_c^2}{\int_{C_1'} \frac{ds}{Gt}}$$

If G & t are constant

$$H_{11} = \frac{4 A_c^2 G \cdot t}{l}$$

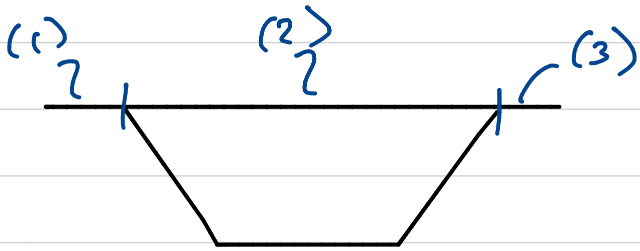
$l = \int_{C_1'} ds$ - length at C_1'

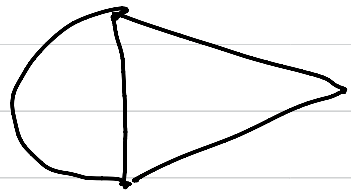
A_c - Area enclosed by C_1'

ii) Moment & Stress

$$M_1 = 2 A_c \cdot f \, , \quad \tau_s(s) = \frac{M_1}{2 A_c t(s)}$$

Torsion of Multi-Component & Multi-Cellular Structures





$\rightarrow \quad M_1 = M_1^{(1)} + M_1^{(2)} + \dots + M_1^{(i)}$

$\rightarrow \quad K_1 = K_1^{(1)} = K_1^{(2)} = \dots = K_1^{(i)}$