AE6114 - Fundamentals of Solid Mechanics

Fall 2020

Homework 1: Mathematical Preliminaries

Due at the indicated time on Canvas, on Thursday, August 27^{th} 2020

Problem 1

Expand the following indicial expressions (all indices range from 1 to 3). Indicate the **rank** and the **number of resulting expressions**.

1. $a_i b_i$
2. $a_i b_j$
3. $a_i b_i c_j$
4. $\sigma_{ik} n_k$
5. $A_{ij} x_i x_j$; $\mathbf{A}$ is symmetric, i.e. $A_{ij} = A_{ji}$.

Problem 2

Simplify the following indicial expressions as much as possible (all indices range from 1 to 3)

1. $\delta_{mm}\delta_{nn}$
2. $x_i \delta_{ik} \delta_{jk}$
3. $B_{ij}\delta_{ij}$; $\mathbf{B}$ is antisymmetric, i.e. $B_{ij} = -B_{ji}$.
4. $(A_{ij}B_{jk} - 2A_{im}B_{mk})\delta_{ik}$
5. Substitute $A_{ij} = B_{ik}C_{kj}$ into $\phi = A_{mk}C_{mk}$

Problem 3

Write out the following expressions in indicial notation, if possible:

1. $A_{11} + A_{22} + A_{33}$
2. $\mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is a 3×3 matrix.
3. $A_{11}^2 + A_{22}^2 + A_{33}^2$
4. $(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$
5. $A_{11} = B_{11}C_{11} + B_{12}C_{21}$; $A_{12} = B_{11}C_{12} + B_{12}C_{22}$
   $A_{21} = B_{21}C_{11} + B_{22}C_{21}$; $A_{22} = B_{21}C_{12} + B_{22}C_{22}$

Problem 4

Given the right-handed orthonormal basis $\{\mathbf{e}_i\}$, $i = \{1, 2, 3\}$:

1. Show that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.
2. Using the previous result and indicial notation, show that given $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, their dot product can be expressed as $\mathbf{a} \cdot \mathbf{b} = a_ib_i$.
3. Show that $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$.
4. Using the previous result and indicial notation, show that given $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, their cross product can be expressed as $\mathbf{a} \times \mathbf{b} = \epsilon_{ijk}a_ib_j\mathbf{e}_k$.
5. Using previous results and indicial notation, show that given $\mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = b_i\mathbf{e}_i$, and $\mathbf{c} = c_i\mathbf{e}_i$ their triple product can be expressed as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk}a_ib_jc_k$.
6. Show that the permutation symbol and the Kronecker delta are related through the expression $e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

Problem 5

In solid mechanics, measures of deformation (e.g. strains) and measures of stress are related through constitutive laws. The simplest possible constitutive law, called linear elasticity, corresponds to a linear relationship between stresses and strains. In addition, every time a body is deformed by external forces, work is done on the body. Under linear elasticity assumptions, this recoverable work is stored as *strain energy density* and is expressed as:

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{1}$$

where ε_{ij} and σ_{ij} are the components of the strain and stress tensor respectively. According to linear elasticity theory, the stress can be expressed as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

where C_{ijkl} is the elasticity tensor that depends on the (known) material constants λ and μ for the isotropic case as follows:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3}$$

With this information,

1. Find a simplified expression for the stress tensor σ_{ij} assuming that the strain tensor is symmetric, i.e., $\varepsilon_{ij} = \varepsilon_{ji}$. This expression is known as *Hooke's Law*.
2. Find a simplified expression for the strain energy density w such that it is only written in terms of the strains $\underline{\underline{\varepsilon}}$ (or ε_{ij}).
3. What are the ranks of σ_{ij}, ϵ_{kl}, C_{ijkl} and w?

Problem 6

Considering that for a second order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j$ the partial derivatives with respect to its components are given by $\dfrac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}$

1. Show that $\dfrac{\partial Tr(\mathbf{A})}{\partial A_{ij}} = \delta_{ij}$
2. Show that $\dfrac{\partial Tr(\mathbf{A} \cdot \mathbf{A})}{\partial A_{ij}} = 2(A^T)_{ij}$
3. Prove that $\dfrac{\partial (A^{-1})_{kl}}{\partial A_{ij}} = -(A^{-1})_{ki}(A^{-1})_{jl}$