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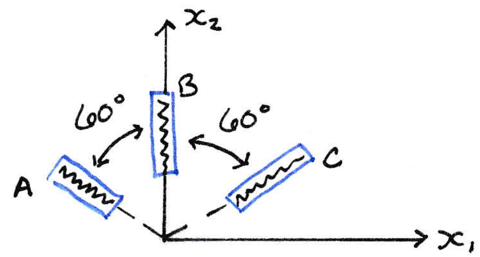
AE6114
DEC. 9, 2019

AE6114: Fundamentals of Solid Mechanics
Fall 2019
FINAL EXAM

PROBLEM 1

A strain gage rosette with 3 gages is placed at a point P on a body. One gage (B) is aligned in the x_2 direction with the other two (A and C) aligned 60° on either side of the x_2 direction. The body is then stressed and the strains from the three gages are measured to be ε_A, ε_B, and ε_C.

1. Given that a strain gage gives the normal strain in the direction of its orientation, determine all components of the infinitesimal strain tensor, assuming that the components ε_{13}, ε_{23}, and ε_{33} are zero (i.e., plane strain).



Strain gage gives normal strain in direction of its orientation.

In class we found:

$$[\sigma]' = [Q][\sigma][Q]^T$$

for $[\sigma]' = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix}$ and $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

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Which gave us the equations:

$$\sigma_{rr} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta$$

$$\sigma_{r\theta} = -(\sigma_{11} + \sigma_{22})\cos\theta\sin\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\cos\theta\sin\theta$$

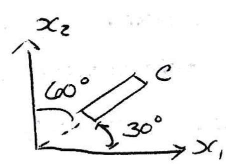
In the case of the strain gages, each gage gives ε_{rr} at that particular orientation. We also know that the transformations for stress can also be applied to transform strains.

Using this information, we can say that:

$$\varepsilon_{rr} = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta$$

Starting with gage C:

$\varepsilon_{rr} = \varepsilon_C, \quad \theta = 30°$



$$\varepsilon_C = \varepsilon_{11}\cos^2(30) + \varepsilon_{22}\sin^2(30) + 2\varepsilon_{12}\cos(30)\sin(30)$$

$$= \varepsilon_{11}\left(\frac{\sqrt{3}}{2}\right)^2 + \varepsilon_{22}\left(\frac{1}{2}\right)^2 + 2\varepsilon_{12}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

(1) $$\varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} + \frac{\sqrt{3}}{2}\varepsilon_{12}$$

Gage B:

$\varepsilon_{rr} = \varepsilon_B, \quad \theta = 90°$

$$\varepsilon_B = \varepsilon_{11}\cos^2(90) + \varepsilon_{22}\sin^2(90) + 2\varepsilon_{12}(\cos(90)\sin(90))$$

$$= \varepsilon_{11}(0) + \varepsilon_{22}(1) + 2\varepsilon_{12}(0)(1)$$

(2) $$\varepsilon_B = \varepsilon_{22}$$

Gage A:

$\varepsilon_{rr} = \varepsilon_A$, $\theta = 90 + 60 = 150°$

$$\varepsilon_A = \varepsilon_{11} \cos^2(150) + \varepsilon_{22} \sin^2(150) + 2\varepsilon_{12} \cos(150) \sin(150)$$

$$= \varepsilon_{11} \left(-\frac{\sqrt{3}}{2}\right)^2 + \varepsilon_{22} \left(\frac{1}{2}\right)^2 + 2\varepsilon_{12} \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$(3) \quad \varepsilon_A = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} - \frac{\sqrt{3}}{2}\varepsilon_{12}$$

Taking (1) - (3):

$$\varepsilon_C - \varepsilon_A = \frac{\sqrt{3}}{2}\varepsilon_{12} + \frac{\sqrt{3}}{2}\varepsilon_{12} = \sqrt{3}\,\varepsilon_{12}$$

we have $\varepsilon_{12} = \frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A)$ and $\varepsilon_{22} = \varepsilon_B$

plugging these into (1):

$$\varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_B + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A)\right)$$

$$\varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_B + \frac{1}{2}\varepsilon_C - \frac{1}{2}\varepsilon_A$$

$$\varepsilon_{11} = \left(\varepsilon_C - \frac{1}{4}\varepsilon_B - \frac{1}{2}\varepsilon_C + \frac{1}{2}\varepsilon_A\right)\frac{4}{3}$$

$$= \left(\frac{1}{2}\varepsilon_C - \frac{1}{4}\varepsilon_B + \frac{1}{2}\varepsilon_A\right)\frac{4}{3}$$

$$\varepsilon_{11} = \frac{2}{3}\varepsilon_C - \frac{1}{3}\varepsilon_B + \frac{2}{3}\varepsilon_A$$

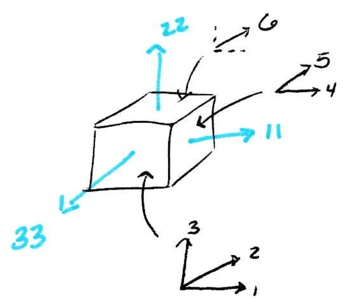
Thus, the components of the infinitesimal strain tensor are:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{2}{3}\varepsilon_C - \frac{1}{3}\varepsilon_B + \frac{2}{3}\varepsilon_A & \frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A) & 0 \\ \frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A) & \varepsilon_B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Assuming the body was not necessarily under plane strain, and you were allowed to attach a strain gage in any orientation, what is the smallest number of strain gages you would need to determine the entire strain tensor? Describe how you would place the strain gages.

we just used 3 gages to find $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}$.

If we want $\varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33}$, it would make sense that we would need 3 more gages: 6 eqn's ↔ 6 unknowns



Gage 1: in 11 direction

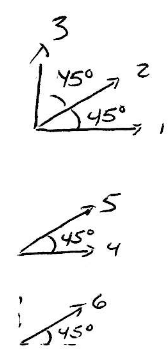
Gage 2: in 12 direction

Gage 3: in 22 direction

Gage 4: in 33 direction

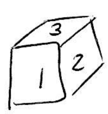
Gage 5: in 23 direction

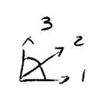
Gage 6: in 13 direction



So the question is: can we calculate all the strains with less than 6 gages?

I can't think of a way, so let's go with 6 gages:



3 on face 1: 

(45° angles

2 on face 2: 

45° angle

1 on face 3: 

45° angle

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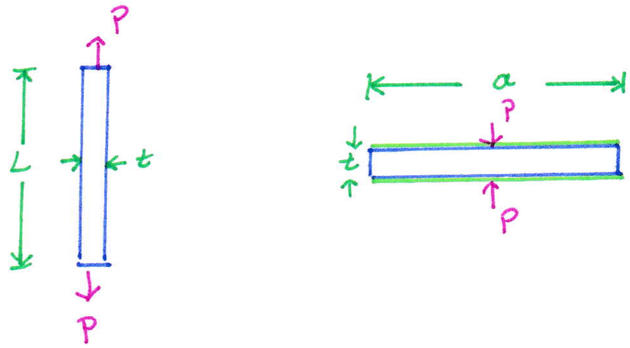
Problem 2

An engineer conducts the following two tests to find the constitutive response of a thin sheet of material of thickness t.



1. In the first test, a thin strip of material (thickness t, length L, width w, $L \gg w, t$) is subject to tension along its length
2. In the second test, the material is placed between two rigid square plates of side a ($a \gg t$) and is subject to uniform compression in the thickness direction while remaining laterally constrained.

In each of the tests, the load and displacement are measured. The engineer computes the elastic stress-strain response of the material from each of these tests and finds them to be different.

Help the engineer derive the stress-strain response of these two tests and investigate if any relation exists between these two tests. How would you compute the elastic constants from the results of these two tests?

Assumptions: linearly elastic, isotropic, no rigid body movement

E: Young's modulus

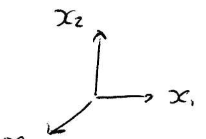
frictionless plates

ν: Poisson's ratio

Test 1

force (load) $= P$

P acting on area wt



then there is only stress in x_2 : $\sigma_{22} \neq 0$

all other components of $\underline{\sigma} = 0$

$$[\underline{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{P}{wt} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ← uniaxial stress

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\sigma_{11} = 0 = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{11}$$

$$\sigma_{22} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{22}$$

$$\sigma_{33} = 0 = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{33}$$

$$\sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \quad \Rightarrow \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

adding $\sigma_{11} + \sigma_{22} + \sigma_{33}$:

$$\sigma_{22} = 3\lambda \varepsilon_{kk} + 2\mu \underbrace{(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})}_{\varepsilon_{kk}}$$

$$\sigma_{22} = (3\lambda + 2\mu)\varepsilon_{kk} \quad \Rightarrow \quad \varepsilon_{kk} = \frac{\sigma_{22}}{3\lambda + 2\mu}$$

plugging back in to equation for σ_{22} :

$$\sigma_{22} = \lambda\left(\frac{\sigma_{22}}{3\lambda+2\mu}\right) + 2\mu\, \varepsilon_{22}$$

$$\sigma_{22} - \lambda\left(\frac{\sigma_{22}}{3\lambda+2\mu}\right) = 2\mu\, \varepsilon_{22}$$

$$\sigma_{22}\left(1 - \frac{\lambda}{3\lambda+2\mu}\right) = 2\mu\, \varepsilon_{22}$$

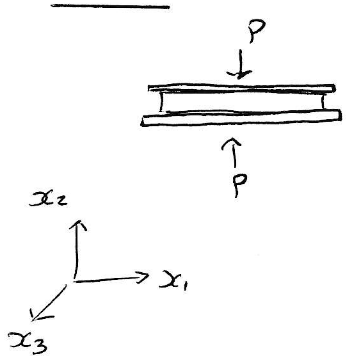
$$\sigma_{22}\left(\frac{2\lambda+2\mu}{3\lambda+2\mu}\right) = 2\mu\, \varepsilon_{22}$$

$$\sigma_{22} = 2\mu\left(\frac{3\lambda+2\mu}{2\lambda+2\mu}\right)\varepsilon_{22}$$

$$\sigma_{22} = \underbrace{\mu\left(\frac{3\lambda+2\mu}{\lambda+\mu}\right)}_{E}\varepsilon_{22}$$

$$\Rightarrow \sigma_{22} = E\, \varepsilon_{22} \quad \Rightarrow \quad \sigma_{22} = -\frac{E}{\nu}\varepsilon_{11} = -\frac{E}{\nu}\varepsilon_{33}$$

Test 2



force (load) = P

P acting on area Lw

material is laterally constrained :

$\varepsilon_{11} = \varepsilon_{33} = 0$?

or

$\varepsilon_{11} = 0, \quad \varepsilon_{33} \neq 0$?

Let's assume $\varepsilon_{11} = \varepsilon_{33} = 0$

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{P}{Lw} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

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$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -P/\ell w \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} \sigma_{12} = 0 \\ \sigma_{22} = \dfrac{-P}{\ell w} \\ \sigma_{23} = 0 \end{array} \Rightarrow \begin{array}{l} \varepsilon_{12} = 0 \\ \varepsilon_{23} = 0 \end{array}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\sigma_{11} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{11} = \lambda \varepsilon_{22}$$

$$\sigma_{22} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{22} = \lambda \varepsilon_{22} + 2\mu \varepsilon_{22} = -\frac{P}{\ell w}$$

$$\sigma_{33} = \lambda \varepsilon_{kk} + 2\mu \varepsilon_{33} = \lambda \varepsilon_{22}$$

$$\frac{\sigma_{22}}{\lambda + 2\mu} = \varepsilon_{22}$$

$$\Rightarrow \varepsilon_{22} = \frac{-P}{\ell w}\left(\frac{1}{\lambda + 2\mu}\right)$$

$$\Rightarrow \sigma_{11} = \frac{-P}{\ell w}\left(\frac{\lambda}{\lambda + 2\mu}\right)$$

$$\sigma_{33} = -\frac{P}{\ell w}\left(\frac{\lambda}{\lambda + 2\mu}\right)$$

$$\begin{bmatrix} \frac{-P}{\ell w}\left(\frac{\lambda}{\lambda+2\mu}\right) & 0 & \sigma_{13} \\ 0 & \frac{-P}{\ell w} & 0 \\ \sigma_{13} & 0 & \frac{-P}{\ell w}\left(\frac{\lambda}{\lambda+2\mu}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sigma_{13} = 0 \Rightarrow \varepsilon_{13} = 0$$

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -\frac{P}{\ell w}\left(\frac{\lambda}{\lambda+2\mu}\right) & 0 & 0 \\ 0 & \frac{-P}{\ell w} & 0 \\ 0 & 0 & \frac{-P}{\ell w}\left(\frac{\lambda}{\lambda+2\mu}\right) \end{bmatrix}$$

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-P}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from Test 1:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{P}{wt} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} -\frac{\nu}{E}\left(\frac{P}{wt}\right) & 0 & 0 \\ 0 & \frac{P}{Ewt} & 0 \\ 0 & 0 & -\frac{\nu}{E}\left(\frac{P}{wt}\right) \end{bmatrix}$$

<u>Test 1</u>

$$\sigma_{22} = E\varepsilon_{22} = -\frac{E}{\nu}\varepsilon_{11} = -\frac{E}{\nu}\varepsilon_{33}$$

<u>Test 2</u>

$$\varepsilon_{22} = \frac{\sigma_{22}}{\lambda+2\mu} = \frac{\sigma_{11}}{\lambda} = \frac{\sigma_{33}}{\lambda}$$

But, what happens if we assume that Test 1 is an infinite body along L?

Then, we don't have any changes along x_2:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\frac{\partial u_2}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\frac{\partial u_2}{\partial x_1} & 0 & \frac{1}{2}\frac{\partial u_2}{\partial x_3} \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

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$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$[\underset{=}{\sigma}] = \begin{bmatrix} (\lambda+2\mu)\dfrac{\partial u_1}{\partial x_1} + \lambda \dfrac{\partial u_3}{\partial x_3} & \mu \dfrac{\partial u_2}{\partial x_1} & \mu\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \mu \dfrac{\partial u_2}{\partial x_1} & \lambda\left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_3}{\partial x_3}\right) & \mu \dfrac{\partial u_2}{\partial x_3} \\ \mu\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) & \mu \dfrac{\partial u_2}{\partial x_3} & (\lambda+2\mu)\dfrac{\partial u_3}{\partial x_3} + \lambda \dfrac{\partial u_1}{\partial x_1} \end{bmatrix}$$

Equation of motion:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} = 0$$

$$\begin{cases} (a) \quad (\lambda+2\mu)\dfrac{\partial^2 u_1}{\partial x_1^2} + \lambda \dfrac{\partial^2 u_3}{\partial x_3 \partial x_1} + \mu\left(\dfrac{\partial^2 u_1}{\partial x_3^2} + \dfrac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) = 0 \\ (b) \quad \mu \dfrac{\partial^2 u_2}{\partial x_1^2} + \mu \dfrac{\partial^2 u_2}{\partial x_3^2} = 0 \\ (c) \quad \mu\left(\dfrac{\partial^2 u_1}{\partial x_3 \partial x_1} + \dfrac{\partial^2 u_3}{\partial x_1^2}\right) + (\lambda+2\mu)\dfrac{\partial^2 u_3}{\partial x_3^2} + \lambda \dfrac{\partial^2 u_1}{\partial x_1 \partial x_3} = 0 \end{cases}$$

$\sigma_{11} = (\lambda+2\mu)\varepsilon_{11} + \lambda\varepsilon_{33}$

$\sigma_{23} = 2\mu\varepsilon_{23}$

$\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{33})$

$\sigma_{33} = (\lambda+2\mu)\varepsilon_{33} + \lambda\varepsilon_{11}$

$\cancel{\sigma_{11} + \sigma_{33} = (\lambda+2\mu)(\varepsilon_{11}+\varepsilon_{33}) + \lambda(\varepsilon_{11}+\varepsilon_{33})}$

$\sigma_{12} = 2\mu\varepsilon_{12}$

$\cancel{\sigma_{11} + \sigma_{33} = (2\lambda+2\mu)\dfrac{\sigma_{22}}{\lambda}}$

$\sigma_{13} = 2\mu\varepsilon_{13}$

Then, we would get plane strain:

$$\sigma_{\alpha\beta} = \lambda \, \varepsilon_{\gamma\gamma} \, \delta_{\alpha\beta} + 2\mu \, \varepsilon_{\alpha\beta} \quad , \quad \alpha = 1,3 \; ; \; \beta = 1,3$$
$$\gamma = 1,3$$

(of course, this is an assumption since $\sigma_{22} \neq 0$: $\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{33})$)

So, for Test 2:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix}$$

← not free b/c it's laterally constrained

$\sigma_{12} = 0, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0$

$\varepsilon_{11} = 0, \quad \varepsilon_{33} = 0$

$$\varepsilon_{ij} = -\frac{\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij}$$

$$0 = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1+\nu}{E}\sigma_{11}$$

$$\varepsilon_{22} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1+\nu}{E}\sigma_{22}$$

$$0 = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1+\nu}{E}\sigma_{33}$$

$$\sigma_{\alpha\beta} = \lambda \, \varepsilon_{\gamma\gamma} \, \delta_{\alpha\beta} + 2\mu \, \varepsilon_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = \lambda \, (\varepsilon_{\gamma\gamma} + \varepsilon_{22}) \, \delta_{\alpha\beta} + 2\mu \, \varepsilon_{\alpha\beta}$$

$$\varepsilon_{11} + \varepsilon_{33} = -\frac{2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{2(1+\nu)}{E}(\sigma_{11} + \sigma_{33})$$

ok, this isn't plane stress because it's constrained

If it weren't constrained, we could assume the infinite body and say that Test 1 is plane strain and Test 2 is plane stress (unless I made a mistake somewhere).

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Going back to what we had Before:

Test 1

$\sigma_{zz} = E\varepsilon_{zz}$

Test 2

$\sigma_{zz} = (\lambda + 2\mu)\varepsilon_{zz}$

Is there a relation between these two tests?

I'm trying to find a way to get $\lambda + 2\mu$ with the constants...

I'm sure I made a mistake somewhere...

also, wouldn't you have to assume $a > L$?

Having a plate that didn't compress the entire material uniformly wouldn't make sense

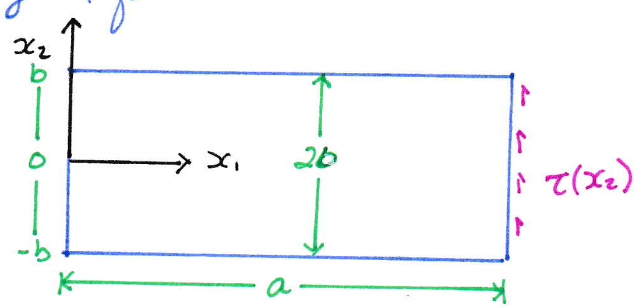
PROBLEM 3

Consider the plate $(0 < x_1 < a;\ -b < x_2 < b)$ subject to the following loads:

- $x_2 = \pm b$ are stress-free
- $x_1 = a$ subject to a force per unit length of $\tau(x_2)$ giving a total force:

$$F = \int_{-b}^{b} \tau(x_2)\, dx_2$$

- Surface $x_1 = 0$ subject to suitable loads to keep plate in equilibrium
- zero body force



Which of the following is a solution of the elasticity problem? Explain why and give correct values of A, B, C, D.

We know the plate is in equilibrium and there are no body forces, so lets begin with the equilibrium equations (balance of linear momentum). We are also assuming that $a, b >> t$ such that

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

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For equilibrium we have:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Let's check each of the possible solutions:

1. $\sigma_{11} = Ax_2 + Bx_1 x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)$

$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = Bx_2 - \frac{\pi}{2a} D\sin\left(\frac{\pi x_1}{2a}\right) \cancel{+ \frac{\pi}{b} B\cos\left(\frac{\pi x_2}{b}\right)} - Bx_2 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 + \frac{\pi}{b} B\cos\left(\frac{\pi x_2}{b}\right) = 0$$

$$\Rightarrow \quad \frac{\pi}{2a} D\sin\left(\frac{\pi x_1}{2a}\right) = 0$$

$$\frac{\pi}{b} B\cos\left(\frac{\pi x_2}{b}\right) = 0$$

2. $\sigma_{11} = Ax_2 + Bx_1 x_2$

$\sigma_{22} = D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} : \quad Bx_2 - Bx_2 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 + 0 = 0$$

$$\Rightarrow \quad 0 = 0$$

$$0 = 0$$

3. $\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2 = Ax_2 + Bx_1x_2 + \frac{B}{2}x_1^2 - Bx_1a + \frac{B}{2}a^2$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C = -\frac{1}{2}Bx_2^2 - Bx_1x_2 + Bax_2 + C$

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = Bx_2 + Bx_1 - Ba - Bx_2 - Bx_1 + Ba = 0$$

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = -Bx_2 + Bx_2 = 0$$

$0 = 0$

$0 = 0$

For (1): $\frac{\pi}{2a} D\sin\left(\frac{\pi x_1}{2a}\right) = 0$

(1) doesn't make sense

is only 0 if $D = 0$ or $\sin\left(\frac{\pi x_1}{2a}\right) = 0$

$\frac{\pi}{b} B\cos\left(\frac{\pi x_2}{b}\right) = 0$

is only 0 if $B = 0$ or $\cos\left(\frac{\pi x_2}{b}\right) = 0$

x_1 varies from 0 to a, so $\sin\left(\frac{\pi x_1}{2a}\right) = 0$ only at $x_1 = 0$

x_2 varies from $-b$ to b, so $\cos\left(\frac{\pi x_2}{b}\right) = 0$ only at $x_2 = \pm\frac{b}{2}$

Let's look at the boundary conditions:

at $x_2 = b$, $x_2 = -b$ there are no stresses:

$\sigma_{21} = 0$, $\sigma_{22} = 0$

at $x_1 = a$, we have F (assuming unit area):

$\sigma_{12}' = F$

at $x_1 = 0$, we can have loads to keep the plate in equilibrium

at $x_1 = a$, $\sigma_{11} = 0$

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$\sigma_{22} = 0$ at every x_2 and x_1

If we use (2), then $D = 0$

If we use (3), then $D = 0$ and $B = 0$, which leaves us with

$$\left.\begin{aligned} \sigma_{11} &= Ax_2 \\ \sigma_{12} &= C \end{aligned}\right\}$$ this doesn't make sense, so let's stick with (2)

$\sigma_{11} = Ax_2 + Bx_1x_2$

$\sigma_{22} = 0$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

$$\begin{pmatrix} Ax_2 + Bx_1x_2 & -\frac{1}{2}Bx_2^2 - C \\ -\frac{1}{2}Bx_2^2 - C & 0 \end{pmatrix}$$

at $x_1 = a$, $\sigma_{11} = 0$

$$\sigma_{11} = 0 = Ax_2 + Bax_2$$

$$0 = A + Ba$$

$$A = -Ba$$

$$\sigma_{11} = -Bax_2 + Bx_1x_2$$

at $x_1 = a$, $\sigma_{12} = F$

$$F = \int_{-b}^{b} \tau(x_2)\,dx_2 = \tau(x_2)\,x_2 \Big|_{-b}^{b} \quad \leftarrow ?$$

σ_{12} is constant $\Rightarrow$ $\dfrac{\partial \sigma_{12}}{\partial x_2} = 0$

$$-Bx_2 = 0 \qquad \text{hmmm...}$$

How do we incorporate F?

Can we use traction BC?

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\underline{0} = \underline{\underline{\sigma}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -Bax_2 + Bx_1x_2 & -\frac{1}{2}Bx_2^2 - C \\ -\frac{1}{2}Bx_2^2 + C & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad -\frac{1}{2}Bx_2^2 - C = 0$$

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$$C = -\frac{1}{2} B x_2^2$$

This doesn't really help

If we have $\sigma_{12} = -\frac{1}{2} B x_2^2 - C$, then even at $x_2 = 0$,
when $x_1 = a$ we still need F

$$\left.\begin{aligned} -\tfrac{1}{2} B b^2 - C &= F \\ -\tfrac{1}{2} B (0) - C &= F \end{aligned}\right\} \text{ does that make sense?}$$

$$-\frac{1}{2} B b^2 - C = \int_{-b}^{b} \tau(x_2)\, dx_2$$

force/unit length is a load

$$-C = \left[\int_{-b}^{b} \tau(x_2)\, dx_2 \right] \Bigg|_{x_2 = 0}$$

can I say $-\frac{1}{2} B x_2^2 - C = \tau(x_2)\, x_2 \Big|_{-b}^{b}$?

Can I do like a beam and say it's equal
to $\tau(x_2)$ at the middle? (or τ at the middle)?

If it can be τ in the middle, then $C = -\tau$

$$\sigma_{12} = -\frac{1}{2} B x_2^2 + \tau$$

$$\sigma_{11} = -B a x_2 + B x_1 x_2$$

$$\sigma_{11}(x_2 = b) = -Bab + B x_1 b$$

$$\sigma_{11}(x_2 = -b) = Bab - B x_1 b$$

$$\sigma_{22} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_2} = \cancel{\tfrac{1}{2}} - B x_2$$

$$F \frac{1}{d x_2} = \int_{-b}^{b} \tau(x_2)$$

I'm pretty sure (2) is the correct solution for the elasticity problem but I'm not sure what to do with F in order to solve for the constants

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Problem 4

Determine whether the following statements are true or false. Explain your answer briefly (in one to three sentences).

1. When a body is rigidly displaced, the resulting infinitesimal strain tensor is zero.

True. The strain tensor tells us the deformation of the body in terms of the relative displacement of the particles. In rigid body motion, the particles, relative to each other, are not displaced, which means there is no deformation and the strain tensor is zero.

2. For any motion, the straight line segments in the reference configuration are transformed into straight line segments in the current configuration.

False. For homogeneous deformations, straight line segments in the reference configuration are transformed into straight line segments in the current configuration, but for inhomogeneous deformations, straight line segments in the reference configuration can be transformed into curved segments.

3. The potential energy of a system satisfying static equilibrium is minimized.

True. Π is minimum if and only if it is stationary ($\delta\Pi = 0$). Principle of minimum potential energy is equivalent to the boundary value problem in elasto<u>statics</u>.

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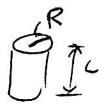
4. When the Poisson's ratio $\nu = 0$, the solid is incompressible (its volume remains constant no matter what stresses are applied).

False. $\nu = -\dfrac{\varepsilon_{lateral}}{\varepsilon_{axial}}$. If $\nu = 0$, then $\varepsilon_{lateral} = 0$, but this doesn't mean that the volume doesn't change ($\varepsilon_{axial} \neq 0$).

Taking a cylinder as an example:



$$V = \pi R^2 L$$

we want $dV = 0$

then $$dV = 2\pi R L\, dR + \pi R^2 dL = 0$$

$$2L\,dR + R\,dL = 0$$

$$2L\,dR = -R\,dL$$

$$2\,\frac{L}{R}\,\frac{dR}{dL} = -1$$

$$\Rightarrow \quad -\frac{L}{R}\,\frac{dR}{L} = \frac{1}{2}$$

$$\frac{-\varepsilon_{lateral}}{\varepsilon_{axial}} = \nu$$

So, when the Poisson's ratio $\nu = \frac{1}{2}$, the solid is incompressible.

5. In a homogeneous, isotropic material, the hydrostatic pressure may depend on the deviatoric strain.

Hydrostatic pressure $= \frac{1}{3}\sigma_{ii}$

Deviatoric strain $\Rightarrow$ $tr(\hat{\underline{\underline{\varepsilon}}}) = \underline{\underline{0}}$

$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$, but if hydrostatic pressure $= \frac{1}{3}\sigma_{kk}$, then it will <u>not</u> depend on the deviatoric strain

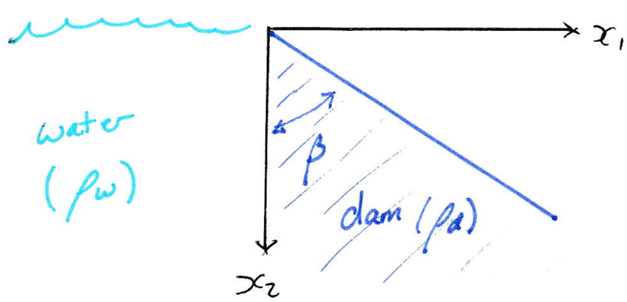
$\Rightarrow$ False.

6. The axes of principal strain always coincide with the axes of principal stress.

False. The axes of principal strain always coincide with the axes of principal stresses only for isotropic materials.

Problem 5

Consider a dam represented as a wedge with two infinitely long sides (i.e., we do not consider in this problem how the dam is supported by the ground). The dam also extends infinitely in the x_3 direction. The vertical side is subjected to the pressure $\rho_w g x_2$ of water. The inclined side is traction-free. The two sides make angle β as shown. The dam is also subject to its own weight, i.e., the body force $\rho_d g$ is acting on the dam (ρ_d is the density of the dam).



1. This problem has a body force. Accounting for that fact, write down the relations between the Airy stress function and the stresses in this problem and the equation that the Airy stress function must satisfy.

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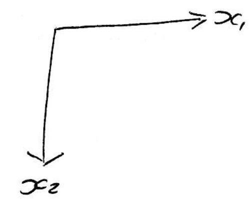
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_3}} + \cancelto{0}{\rho b_1} = 0$$



$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_3}} + \rho b_2 = 0$$

$$\cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_1}} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_2}} + \cancelto{0}{\frac{\partial \sigma_{33}}{\partial x_3}} + \cancelto{0}{\rho b_3} = 0$$

body force:

$$\rho \underline{b} = \begin{Bmatrix} 0 \\ \rho_d g \\ 0 \end{Bmatrix}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \rho_d g = 0$$

need a function ϕ such that

$$\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 0$$

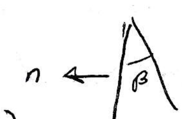
where $\frac{\partial^2 \phi}{\partial x_2^2} = \sigma_{11}$, $\frac{\partial^2 \phi}{\partial x_1^2} = \sigma_{22}$, $-\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \sigma_{12}$

<u>Boundary Conditions at the wall ($x_1 = 0$):</u>

$\sigma_{12} = 0$ at $x_1 = 0$

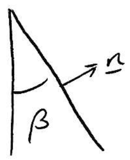
$\sigma_{11} = -\rho_w g x_2$ at $x_1 = 0$

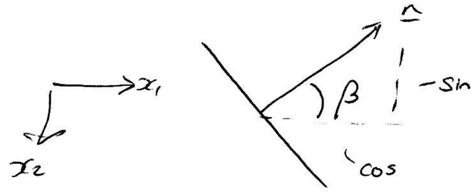
$\underline{t} = \underline{\sigma} \cdot \underline{n}$



$\rho_w g x_2 \cdot (-e_1)$

Boundary Conditions at incline:





$$\underline{n} = \begin{Bmatrix} \cos\beta \\ -\sin\beta \\ 0 \end{Bmatrix}$$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\underline{0} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{Bmatrix} \cos\beta \\ -\sin\beta \end{Bmatrix}$$

$$\sigma_{11}\cos\beta - \sigma_{12}\sin\beta = 0$$

$$\sigma_{12}\cos\beta - \sigma_{22}\sin\beta = 0$$

2. Using the Airy stress function $\phi = A_1x_1^3 + A_2x_1^2x_2 + A_3x_1x_2^2 + A_4x_2^3$, find the stress distribution inside the dam.

$$\phi = A_1x_1^3 + A_2x_1^2x_2 + A_3x_1x_2^2 + A_4x_2^3$$

$$\frac{\partial\phi}{\partial x_1} = 3A_1x_1^2 + 2A_2x_1x_2 + A_3x_2^2$$

$$\frac{\partial\phi}{\partial x_2} = A_2x_1^2 + 2A_3x_1x_2 + 3A_4x_2^2$$

$$\frac{\partial^2\phi}{\partial x_1^2} = 6A_1x_1 + 2A_2x_2 \qquad \frac{\partial^2\phi}{\partial x_1\partial x_2} = 2A_2x_1 + 2A_3x_2$$

$$\frac{\partial^2\phi}{\partial x_2^2} = 2A_3x_1 + 6A_4x_2$$

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$$\sigma_{11} = 2A_3x_1 + 6A_4x_2$$

$$\sigma_{22} = 6A_1x_1 + 2A_2x_2$$

$$\sigma_{12} = -2A_2x_1 - 2A_3x_2$$

from the BCs:

$$\sigma_{12} = 0 \quad \text{at } x_1 = 0$$

$$\sigma_{12} = 0 = -2A_3x_2 \quad \Rightarrow A_3 = 0$$

$$\sigma_{11} = -\rho_w g x_2 \quad \text{at } x_1 = 0$$

$$-\rho_w g x_2 = 6A_4x_2$$

$$6A_4 = -\rho_w g$$

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = 0$$

$$\cancel{2A_3}^{\,0} + \cancel{(-2A_3)}^{\,0} = 0$$

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = -\rho_d g$$

$$-2A_2 + 2A_2 = -\rho_d g$$

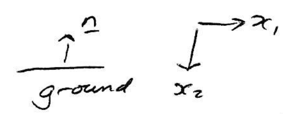
I'm missing something with the weight..

as we go down x_2, the weight of the dam increases

we have a traction vector

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\rho_d g \Rightarrow \underbrace{\rho_d g x_1}_{\sigma_{22}} \cdot (-e_2) = -\rho_d g x_1$$



density of dam:

$\downarrow \quad \rho_d g \, x_2 x_1 \quad \leftarrow$ but this is somewhere/anywhere

$$x_2 \to \infty$$

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Wait, is this a case of $\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} + \psi$ $\quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} + \psi$?

$$\rho b_1 = -\frac{\partial \psi}{\partial x_1} \qquad \rho b_2 = -\frac{\partial \psi}{\partial x_2} = \rho_d g$$

$$\psi = -\rho_d g x_2$$

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} - \rho_d g x_2 \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} - \rho_d g x_2$$

$$\sigma_{11} = -\rho_d g x_2 + 2A_3 x_1 + 6A_4 x_2$$

$$\sigma_{22} = -\rho_d g x_2 + 6A_1 x_1 + 2A_2 x_2$$

$$\sigma_{12} = -2A_2 x_1 - 2A_3 x_2$$

from BCs:

$$\sigma_{12} = 0 \text{ at } x_1 = 0$$

$$-2A_3 x_2 = 0 \quad \Rightarrow A_3 = 0$$

$$\sigma_{11} = -\rho_w g x_2 \quad \text{at } x_1 = 0$$

$$-\rho_w g x_2 = -\rho_d g x_2 + 6A_4 x_2$$

$$\rho_d g - \rho_w g = 6A_4 \quad \Rightarrow A_4 = \frac{\rho_d g - \rho_w g}{6}$$

at the incline:

$$\sigma_{11}\cos\beta - \sigma_{12}\sin\beta = 0$$

$$\sigma_{12}\cos\beta - \sigma_{22}\sin\beta = 0$$

$$\left(\cancel{-\rho_d g x_2} + \cancel{\rho_d g x_2} - \rho_w g x_2\right)\cos\beta + 2A_2 x_1 \overset{\sin\beta}{\vee} = 0$$

$$-2A_2 x_1 \cos\beta + \left(\rho_d g x_2 - 6A_1 x_1 - 2A_2 x_2\right)\sin\beta = 0$$

$$A_2 = \frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta} = \frac{\rho_w g x_2}{2x_1 \tan\beta}$$

$$-2\left(\frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta}\right)x_1\cos\beta + \rho_d g x_2 \sin\beta - 6A_1 x_1 \sin\beta - 2\left(\frac{\rho_w g x_2 \cos\beta}{2x_1 \sin\beta}\right)x_2 \sin\beta = 0$$

$$-\frac{\rho_w g x_2 \cos^2\beta}{\sin\beta} + \rho_d g x_2 \sin\beta - 6A_1 x_1 \sin\beta - \frac{\rho_w g x_2^2 \cos\beta}{x_1} = 0$$

This just doesn't look right...

The coefficients A_1, A_2, A_3, A_4 should have expressions without x_1 and x_2 ...

I keep getting $A_2 = \frac{1}{2}\rho_w g \frac{x_2}{x_1}\cot\beta$

$$A_1 = -\frac{1}{6}\left[\rho_w g \frac{x_2}{x_1}\cot^2\beta + \rho_w g \frac{x_2^2}{x_1^2}\cot\beta - \rho_d g \frac{x_2}{x_1}\right]$$

So I made a mistake somewhere

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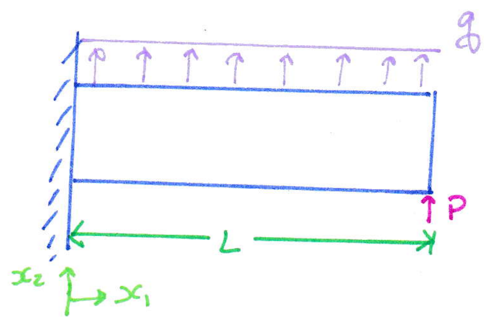
BONUS PROBLEM

Derive the strong formulation by using the principle of minimum potential energy.



let u_2 = vertical displacement of beam

Strain energy for beam in bending: $\frac{M^2 L}{2EI}$

so for this beam: $\int_0^L \frac{M^2}{2EI} dx_1$

then we have q: $A\int_0^L q u_2 \, dx_1$ $\qquad M = EI \frac{d^2 u_2}{dx_1^2}$

and finally P: $APu_2\big|_{x_1 = L}$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - \int_0^L Aq u_2 \, dx_1 - APu_2\big|_{x_1=L}$$

$$\delta\Pi = \delta\left[\int_0^L \frac{EI}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1\right] - \delta\left[\int_0^L Aqu_2 \, dx_1\right] - \delta\left[APu_2\big|_{x_1=L}\right]$$

$$= \frac{EI}{2}\int_0^L 2\left(\frac{d^2u_2}{dx_1^2}\right)\delta\left(\frac{d^2u_2}{dx_1^2}\right)dx_1 - A\int_0^L q\,\delta u_2 \, dx_1 - AP\delta u_2\big|_{x_1=L}$$

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$$\delta\Pi = EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1 - Aq\int_0^L \delta u_2\, dx_1 - AP\delta u_2\Big|_{x_1=L}$$

$\delta u_2 = 0$ at $x_1 = 0$

$$EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1$$

integration by parts: $\int u\,dv = uv - \int v\,du$

$$u = \frac{\partial^2 u_2}{\partial x_1^2} \qquad du = \frac{\partial^3 u_2}{\partial x_1^3}dx_1$$

$$dv = \frac{\partial^2 \delta u_2}{\partial x_1^2} \qquad v = \frac{\partial \delta u_2}{\partial x_1}$$

$$EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1 = EI\left[\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_0^L - \int_0^L \frac{\partial \delta u_2}{\partial x_1}\frac{\partial^3 u_2}{\partial x_1^3}dx_1\right]$$

$$EI\int_0^L \frac{\partial \delta u_2}{\partial x_1}\frac{\partial^3 u_2}{\partial x_1^3}dx_1$$

$$u = \frac{\partial^3 u_2}{\partial x_1^3} \qquad du = \frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$dv = \frac{\partial \delta u_2}{\partial x_1} \qquad v = \delta u_2$$

$$EI\,\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_{x_1=L} - \frac{\partial^3 u_2}{\partial x_1^3}\delta u_2\Big|_{x_1=L} + \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$\delta\Pi = EI\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_{x_1=L} - EI\frac{\partial^3 u_2}{\partial x_1^3}\delta u_2\Big|_{x_1=L} + EI\int_0^L \delta u_2\frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$- Aq\int_0^L \delta u_2\, dx_1 - AP\delta u_2\Big|_{x_1=L}$$

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Let's try taking the case $\delta u_2 |_{x_1=L} = 0$

$$0 = EI \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial \delta u_2}{\partial x_1} - EI \frac{\partial^3 u_2}{\partial x_1^3} \delta u_2 - AP\delta u_2$$

oh, wait, why am I using AP?

$$0 = EI \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial \delta u_2}{\partial x_1} - EI \frac{\partial^3 u_2}{\partial x_1^3} \delta u_2 - P\delta u_2$$

$x_1 = 0$:

$$0 = EI \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} dx_1 - Aq \int_0^L \delta u_2 \, dx_1$$

crap, I'm using Aq too...

$$0 = EI \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} dx_1 - q \int_0^L \delta u_2 \, dx_1$$

$$0 = EI \, \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} - q \, \delta u_2$$

$$EI \frac{\partial^4 u_2}{\partial x_1^4} - q = 0$$

$$EI \frac{\partial^3 u_2}{\partial x_1^3} = P$$

now we want to solve for u_2

$$\frac{\partial^4 u_2}{\partial x_1^4} = \frac{q}{EI}$$

$$\int \frac{\partial^4 u_2}{\partial x_1^4} dx_1 = \int \frac{q}{EI} dx_1 \Rightarrow \frac{\partial^3 u_2}{\partial x_1^3} = \frac{q}{EI} x_1 + C$$

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$$\int \frac{\partial^3 u_2}{\partial x_1^3}\, dx_1 = \int \left(\frac{q}{EI} x_1 + C\right) dx_1$$

$$\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{2}\frac{q}{EI} x_1^2 + C_1 x_1 + C_2$$

$$\int \frac{\partial^2 u_2}{\partial x_1^2} = \int \left(\frac{1}{2}\frac{q}{EI} x_1^2 + C_1 x_1 + C_2\right) dx_1$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{6}\frac{q}{EI} x_1^3 + \frac{1}{2} C_1 x_1^2 + C_2 x_1 + C_3$$

$$\int \frac{\partial u_2}{\partial x_1} dx_1 = \frac{1}{24}\frac{q}{EI} x_1^4 + \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 + C_3 x_1 + C_4 = u_2(x_1)$$

$$u_2(0) = 0 \Rightarrow C_4 = 0$$

at $x_1 = L$:

$$EI \frac{\partial^3 u_2}{\partial x_1^3} = P$$

$$\frac{\partial^3 u_2}{\partial x_1^3} = \frac{P}{EI} = \frac{q}{EI} x_1 + C_1$$

$$\frac{P}{EI} = \frac{qL}{EI} + C_1 \Rightarrow C_1 = \frac{P - qL}{EI}$$

$$\int \frac{\partial^3 u_2}{\partial x_1^3} dx_1 = \frac{P}{EI} x_1 = \frac{1}{2}\frac{q}{EI} x_1^2 + \frac{P - qL}{EI} x_1 + C_2$$

$$C_2 = \frac{PL}{EI} - \frac{1}{2}\frac{qL^2}{EI} - \frac{PL + qL^2}{EI} = -\frac{1}{2}\frac{qL^2}{EI} + \frac{qL^2}{EI} = \frac{qL^2}{2EI}$$

$$\int \frac{\partial^2 u_2}{\partial x_1^2} dx_1 = \frac{P}{2EI} x_1^2 = \frac{1}{6}\frac{q}{EI} x_1^3 + \frac{P - qL}{2EI} x_1^2 + \frac{qL^2}{2EI} x_1 + C_3$$

$$C_3 = \cancel{\frac{PL^2}{2EI}} - \frac{qL^3}{6EI} - \cancel{\frac{PL^2}{2EI}} + \cancel{\frac{qL^3}{2EI}} - \cancel{\frac{qL^3}{2EI}}$$

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$$u_z(x_1) = \frac{1}{24}\frac{q}{EI}x_1^4 + \frac{1}{6}\frac{P-qL}{EI}x_1^3 + \frac{1}{2}\frac{qL^2}{2EI}x_1^2 - \frac{qL^3}{6EI}x_1$$

$$u_z(0) = 0$$

$$u_z(L) = \frac{q}{24EI}L^4 + \frac{PL^3}{6EI} - \frac{qL^4}{6EI} + \frac{qL^4}{4EI} - \frac{qL^4}{6EI}$$

$$= \frac{qL^4}{24EI} - \frac{4qL^4}{24EI} + \frac{6qL^4}{24EI} - \frac{4qL^4}{24EI} + \frac{PL^3}{6EI}$$

$$= \frac{-qL^4}{24EI} + \frac{PL^3}{6EI}$$

I have no idea if this is correct but I've been working on this exam for over 24 hours straight now and I can't think any more

Thank you for a wonderful Semester!