Problem 1

1) 13	15
2) 15	25
3) 12.5	25
4) 10.5	15
5) 20	20
6) 15	20

1.) Determine all components of the infinitesimal strain tensor.

Assume plane strain ($\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$)

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Replaced σ with ε from eqn from lecture notes: $\sigma_{rr} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta$

to get:

$$\varepsilon_A = \varepsilon_{11}\cos^2\theta_A + \varepsilon_{22}\sin^2\theta_A + 2\varepsilon_{12}\cos\theta_A\sin\theta_A, \quad \theta_A = 90^\circ + 60^\circ = 150^\circ \text{ from } x_1 \text{ axis}$$

$$= \varepsilon_{11}\cos^2(150) + \varepsilon_{22}\sin^2(150) + 2\varepsilon_{12}\cos(150)\sin(150)$$

$$\Rightarrow \varepsilon_A = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} - \frac{\sqrt{3}}{2}\varepsilon_{12} \quad [1]$$

$$\varepsilon_B = \varepsilon_{11}\cos^2\theta_B + \varepsilon_{22}\sin^2\theta_B + 2\varepsilon_{12}\cos\theta_B\sin\theta_B, \quad \theta_B = 90^\circ \text{ from } x_1 \text{ axis}$$

$$= \varepsilon_{11}\underbrace{\cos^2(90)}_{\to 0} + \varepsilon_{22}\underbrace{\sin^2(90)}_{\to 1} + 2\varepsilon_{12}\underbrace{\cos(90)}_{\to 0}\sin(90)$$

$$\Rightarrow \varepsilon_B = \varepsilon_{22} \quad [2]$$

$$\varepsilon_C = \varepsilon_{11}\cos^2\theta_C + \varepsilon_{22}\sin^2\theta_C + 2\varepsilon_{12}\cos\theta_C\sin\theta_C, \quad \theta_C = 30^\circ \text{ from } x_1 \text{ axis}$$

$$= \varepsilon_{11}\cos^2(30) + \varepsilon_{22}\sin^2(30) + 2\varepsilon_{12}\cos(30)\sin(30)$$

$$\Rightarrow \varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} + \frac{\sqrt{3}}{2}\varepsilon_{12} \quad [3]$$

3 eqns ([1], [2], and [3]) and 3 unknowns (ε_{11}, ε_{12}, and ε_{22})

Eqn [1] - Eqn [3]:

$$\varepsilon_A - \varepsilon_C = -\frac{\sqrt{3}}{2}\varepsilon_{12} - \frac{\sqrt{3}}{2}\varepsilon_{12}$$

$$\varepsilon_A - \varepsilon_C = \varepsilon_{12}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$\varepsilon_A - \varepsilon_C = -\sqrt{3}\,\varepsilon_{12}$$

$$\therefore \varepsilon_{12} = \frac{\varepsilon_A - \varepsilon_C}{-\sqrt{3}} = \frac{\varepsilon_C - \varepsilon_A}{\sqrt{3}}$$

$$\Rightarrow \varepsilon_{12} = \frac{(\varepsilon_C - \varepsilon_A)}{\sqrt{3}} \quad [4]$$

Problem 1 (continued)

Plug in Eqn [2] and Eqn [4] into Eqn [3]:

$$\varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_B + \frac{(\varepsilon_C - \varepsilon_A)}{2}$$

$$\therefore \varepsilon_{11} = \frac{4}{3}\left(\varepsilon_C - \frac{1}{4}\varepsilon_B + \frac{(\varepsilon_A - \varepsilon_C)}{2}\right)$$

$$= \frac{4}{3}\varepsilon_C - \frac{\varepsilon_B}{3} + \frac{2}{3}\varepsilon_A - \frac{2}{3}\varepsilon_C$$

$$= \frac{1}{3}(2\varepsilon_A - \varepsilon_B + 2\varepsilon_C)$$

$$\Rightarrow \varepsilon_{11} = \frac{1}{3}(2\varepsilon_A - \varepsilon_B + 2\varepsilon_C) \quad [5]$$

Components:

$$\boxed{\begin{aligned} \varepsilon_{11} &= \tfrac{1}{3}(2\varepsilon_A - \varepsilon_B + 2\varepsilon_C) \checkmark \\ \varepsilon_{22} &= \varepsilon_B \checkmark \\ \varepsilon_{12} &= \frac{(\varepsilon_C - \varepsilon_A)}{\sqrt{3}} \checkmark \end{aligned}}$$

$$[\varepsilon] = \begin{bmatrix} \frac{1}{3}(2\varepsilon_A - \varepsilon_B + 2\varepsilon_C) & \frac{(\varepsilon_C - \varepsilon_A)}{\sqrt{3}} & 0 \\ \frac{(\varepsilon_C - \varepsilon_A)}{\sqrt{3}} & \varepsilon_B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) What is the smallest number of strain gages you would need to determine the entire strain tensor? Describe how you would place the strain gages.

$\Rightarrow$ The smallest number of strain gages I would need is 5 strain gages in total to determine the strain tensor. Given that there are already 3 strain gages in one plane (Plane A), I would add 2 more strain gages in another plane (Plane B), which lies orthogonal to Plane A that has the previous 3 strain gages. Such that Plane B makes use of one of the strain gages in Plane A. Specifically, the orientation will be such that one of the 3 strain gages in Plane A share both / are mutual to Plane A and Plane B.

✗

You need 6 since there are 6 unknowns... $\left\{ \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33} \right\}$

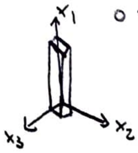
Problem 2

i) Determine the stress-strain response of the two tests and investigate if any relation exists between these two tests.

ii) How would you compute the elastic constants from the results of these two tests?

Assumptions

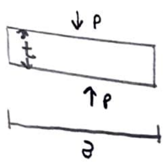
- Material is linearly elastic and isotropic with E and ν



- Test 1: thin strip of material is subject to tension along its length

Therefore, there is pulling along σ_{11} ∴ uniaxial. (only σ_{11} and all other stress components are zero).

$$\sigma_{11} = \frac{\text{Force}}{\text{area}} = \frac{P}{at}, \quad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$



- Test 2: material is placed between two rigid square plates of side a and is subject to uniform compression in the thickness direction while remaining laterally constrained

Therefore, compression along σ_{22}

$$\sigma_{22} = \frac{\text{Force}}{\text{area}} = \frac{-P}{a^2}.$$

i.) Stress-Strain Relation.

a) Solving for Test 1

From HMW 4 Problem 1:

Using the stress-strain relation for an isotropic linear elastic solid;

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

[*] $$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{kk}\delta_{ij}$$?, $\sigma_{kk} = \sigma_{11} + \cancel{\sigma_{22}}_{\to 0} + \cancel{\sigma_{33}}_{\to 0} = \sigma_{11}$ b/c uniaxial.

Since uniaxial, only σ_{11}:

For Test 1:

$$\therefore \varepsilon_{11} = \frac{1}{2\mu}\sigma_{11} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{11}(1)$$

$$= \sigma_{11}\left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\right)$$

$$= \sigma_{11}\left(\frac{2\mu+3\lambda-\lambda}{2\mu(2\mu+3\lambda)}\right)$$

$$\Rightarrow \varepsilon_{11} = \sigma_{11}\frac{(\lambda+\mu)}{\mu(2\mu+3\lambda)}$$ ✓

$$\Rightarrow \varepsilon_{22} = \varepsilon_{33} = \frac{-\lambda\sigma_{11}}{2\mu(2\mu+3\lambda)}$$

why using this expression like this?

either: $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

or $\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$

Check: $\varepsilon_{11} = \frac{\sigma_{11}}{E}$, $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ (From Lecture Notes)

$$= \frac{\sigma_{11}(\lambda+\mu)}{\mu(2\mu+3\lambda)}$$ ✓

Problem 2 (Continued)

i.) Stress - Strain Relation

b.) Solving for Test 2

laterally confined $\therefore$ $\varepsilon_{11} = \varepsilon_{33} = 0$

but there are resulting stresses σ_{11} and σ_{33} to cause this confinement.

From Eqn [*]:

$$[1] \quad \varepsilon_{11} = \frac{1}{2\mu}\sigma_{11} - \frac{\lambda \sigma_{kk}}{2\mu(2\mu+3\lambda)} = 0$$

$$[2] \quad \varepsilon_{33} = \frac{1}{2\mu}\sigma_{33} - \frac{\lambda \sigma_{kk}}{2\mu(2\mu+3\lambda)} = 0$$

$$[3] \quad \varepsilon_{22} = \frac{1}{2\mu}\sigma_{22} - \frac{\lambda \sigma_{kk}}{2\mu(2\mu+3\lambda)}$$

Eqn [1] - Eqn [2] = 0 $\quad \therefore \quad \sigma_{11} = \sigma_{33} \quad [4]$

Since $\sigma_{11} = \sigma_{33}$, I can rewrite σ_{kk} to; $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{22} + 2\sigma_{11} \quad [5]$

Plug [5] into [1]:

$$0 = \frac{1}{2\mu}\sigma_{11} - \frac{\lambda(\sigma_{22} + 2\sigma_{11})}{2\mu(2\mu+3\lambda)}$$

$$\lambda\sigma_{22} + \lambda 2\sigma_{11} = \sigma_{11}(2\mu + 3\lambda)$$

$$\lambda\sigma_{22} = \sigma_{11}(2\mu + \lambda)$$

$$\therefore \sigma_{11} = \sigma_{33} = \frac{\lambda\sigma_{22}}{2\mu+\lambda} \quad [6]$$

Plug [6] into [3]:

$$\varepsilon_{22} = \frac{\sigma_{22}}{2\mu} - \frac{\lambda\left(\sigma_{22} + \frac{2\lambda\sigma_{22}}{2\mu+\lambda}\right)}{2\mu(2\mu+3\lambda)}$$

$$= \sigma_{22}\left(\frac{1}{2\mu} - \frac{\lambda\left(\frac{2\mu+3\lambda}{2\mu+\lambda}\right)}{2\mu(2\mu+3\lambda)}\right)$$

$$= \sigma_{22}\left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(2\mu+\lambda)}\right)$$

For Test 2:

$$\Rightarrow \boxed{\varepsilon_{22} = \frac{\sigma_{22}}{2\mu+\lambda}}$$

$$\Rightarrow \boxed{\varepsilon_{11} = \varepsilon_{33} = 0}$$

Problem 2 (continued)

i) continued.

For Test 1, I got: $\varepsilon_{11} = \sigma_{11} \dfrac{(\lambda+\mu)}{\mu(2\mu+3\lambda)}$ and $\varepsilon_{22} = \varepsilon_{33} = \dfrac{-\lambda\sigma_{11}}{2\mu(2\mu+3\lambda)}$

For Test 2, I got: $\varepsilon_{11} = \varepsilon_{33} = 0$ and $\varepsilon_{22} = \dfrac{\sigma_{22}}{2\mu+\lambda}$.

These relations for Test 1 and Test 2, confirm the engineer's computation that the elastic stress-strain response of the material from each test is different, There is a relation that two terms from each test are the same (i.e. $\varepsilon_{22} = \varepsilon_{33}$ for Test 1 and $\varepsilon_{11} = \varepsilon_{33}$ for Test 2),

ii) I can compute Poisson's ratio ν with ε_{22} and ε_{11} because $\nu = \dfrac{-\varepsilon_{22}}{\varepsilon_{11}}$. Both ε_{22} and ε_{11} have been solved from the two tests,

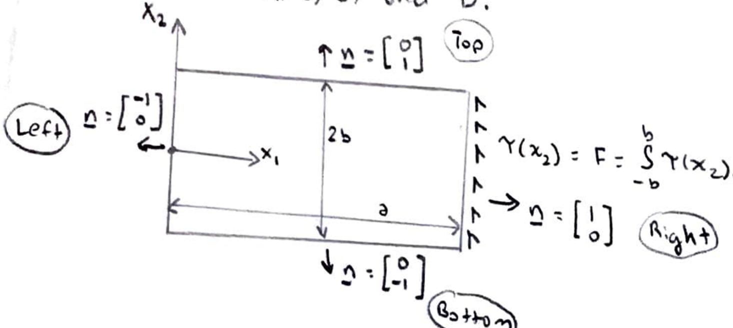
Using Test 1, I can solve for E. Since Test 1 is a uniaxial problem, E would just be the slope of σ_{11} vs. ε_{11}.

From Test 1, E can also be computed from $\varepsilon_{11} = \dfrac{\sigma_{11}}{E}$ $\therefore E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$.

$$\left.\begin{array}{l} 1)\ \sigma_{11} = E\varepsilon_{11} \rightarrow E \\ 2)\ \sigma_{22} = \dfrac{E(1-\nu)}{(1-2\nu)(1+\nu)}\varepsilon_{22}. \end{array}\right\} \rightarrow E, \nu$$

Problem 3

Which of the following (Case 1, 2, or 3) is a solution of the elasticity problem (for suitable constants A, B, C, and D)? Explain why and give correct values of A, B, C, and D.



BC's:

- **Top**: traction free, $x_2 = b$ and $x_1 = x_1$

$\underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \underline{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \therefore \sigma_{12} = 0, \quad \sigma_{22} = 0$$

- **Bottom**: traction free, $x_2 = -b$ and $x_1 = x_1$

$\underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \underline{n} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \therefore \sigma_{12} = 0, \quad \sigma_{22} = 0$$

- **Right**: traction, $x_2 = x_2$ and $x_1 = a$

$\underline{t} = \begin{bmatrix} 0 \\ F \end{bmatrix}$ $\quad \underline{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$

$$\begin{bmatrix} 0 \\ F \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore \sigma_{11} = 0, \quad \sigma_{12} = F = \int_{-b}^{b} \tau(x_2)\,dx_2$$

- **Left**: subject to suitable loads to keep the plate in equilibrium $\therefore$ traction. $x_2 = x_2$ and $x_1 = 0$

$\underline{t} = \begin{bmatrix} 0 \\ -F \end{bmatrix}$ $\quad \underline{n} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$

$$\begin{bmatrix} 0 \\ -F \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \therefore \sigma_{11} = 0, \quad -\sigma_{12} = -F, \quad \therefore \sigma_{12} = F$$

Case 1: 1.) $\sigma_{11} = Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{a}\right)$

$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

Top: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = b$ and $x_1 = x_1$

$\sigma_{11} = Ab + Bbx_1 + D\cos\left(\frac{\pi x_1}{2a}\right)$

$\sigma_{22} = 0 = B\sin\left(\frac{\pi x_2}{b}\right) \quad \therefore B = 0$

$\sigma_{12} = 0 = 0 - C \quad \therefore C = 0$

Bottom: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = -b$ and $x_1 = x_1$

$\sigma_{11} = -Ab - Bbx_1 + D\cos\left(\frac{\pi x_1}{2a}\right)$

$\sigma_{22} = 0 = B\sin\left(\frac{\pi x_2}{b}\right) \quad \therefore B = 0$

$\sigma_{12} = 0 = 0 - C \quad \therefore C = 0$

what about checking for conservation of linear momentum?

0

Problem 3 (continued)

Case 1 (continued):

Right: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = a$

$\sigma_{11} = 0 = Ax_2 + \cancel{B_0}x_2 + D\cancel{\cos(\pi/2)}$
(→0, →0)

$\therefore A = 0$

$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$

$\sigma_{12} = F = -\frac{1}{2}Bx_2^2 - C$

Left: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = 0$

$\sigma_{11} = 0 = Ax_2 + B(0)x_1 + D\cos(0)$

$0 = Ax_2 + D$

$\therefore D = -Ax_2 = 0$ since $A = 0$

$\therefore D = 0$

$\sigma_{12} = B\sin\left(\frac{\pi x_2}{b}\right)$

$\sigma_{12} = F = -\frac{1}{2}Bx_2^2 - C$

Therefore, for Case 1, the solution is:

$A = 0$, $B = 0$, $C = 0$, $D = 0$

$\sigma_{11} = 0$, $\sigma_{22} = 0$, $\sigma_{12} = 0$

Case 1 is not a suitable solution of this elasticity problem because $[\sigma] = [0]$, which based on the schematics of the problem is contradicting given the loads. ✓

Case 2:

2.) $\sigma_{11} = Ax_2 + Bx_1x_2$

$\sigma_{22} = D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

Top: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = b$ and $x_1 = x_1$

$\sigma_{11} = Ab + Bbx_1$

$\sigma_{22} = 0 = D \quad \therefore D = 0$ ✓

$\sigma_{12} = 0 = -\frac{1}{2}Bx_2^2 - C$

$0 = -\frac{1}{2}Bb^2 - C$

$\therefore C = -\frac{1}{2}Bb^2$ ✓

Bottom: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = -b$ and $x_1 = x_1$

$\sigma_{11} = -Ab - Bx_1b$

$\sigma_{12} = -\frac{1}{2}Bb^2 - C = 0$

$\therefore C = -\frac{1}{2}Bb^2$

$\sigma_{22} = 0 = D$

Problem 3 (continued)

Case 2 (continued):

Right: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = a$

$\sigma_{11} : 0 = Ax_2 + Bax_2$

$\therefore A = -Ba$

$\sigma_{12} : F = -\frac{1}{2}Bx_2^2 - C$

$-2F = Bx_2^2 - C, \; C = -\frac{1}{2}Bb^2$

$-2F = Bx_2^2 + \frac{1}{2}Bb^2$

$-2F = B(x_2^2 + 0.5b^2)$

$\therefore B = \dfrac{-2F}{x_2^2 + 0.5b^2}$

$\therefore$ with $C = -\frac{1}{2}Bb^2$

$= -\frac{1}{2}\left[\dfrac{-2F}{x_2^2+0.5b^2}\right]$

$C = \dfrac{F}{x_2^2+0.5b^2}$

Left: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = 0$

$\sigma_{11} = 0 = Ax_2 + B\cancel{x_1}x_2$

$\therefore A = 0$

$\sigma_{12} = F = -\frac{1}{2}Bx_2^2 - C$

$$\int_{-b}^{b} \tau(x_2)\,dx_2 = \underbrace{\int_{-b}^{b} \sigma_{12}\Big|_{x_1=a} dx_2 = F}_{B.}$$

Therefore, for Case 2, the solution is:

$A = 0$

$B = \dfrac{-2F}{x_2^2 + 0.5b^2}$

$C = \dfrac{F}{x_2^2 + 0.5b^2}$

$D = 0$

$\sigma_{11} = \dfrac{-2F}{x_2^2+0.5b^2}x_1x_2$

$\sigma_{22} = 0$

$\sigma_{12} = \dfrac{Fx_2^2}{x_2^2+0.5b^2} - \dfrac{F}{x_2^2+0.5b^2} = \dfrac{F(x_2^2-1)}{x_2^2+0.5b^2}$

Case 2 is a suitable solution of this elasticity problem because $[\sigma] \neq 0$ and the derived values of A, B, C and D do not contradict each other for the top, bottom, left, and right sides.

Problem 3 (continued)

Case 3: 3.) $\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

Top: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = b$ and $x_1 = x_1$

$\sigma_{11} = Ab + Bbx_1 + \frac{B}{2}(x_1 - a)^2$

$\sigma_{11} : Ab + B(bx_1 + \frac{(x_1-a)^2}{2})$

$\sigma_{22} = 0 : \frac{1}{2}Bx_2^2 + D$

$\therefore D = -\frac{1}{2}Bb^2$

$\sigma_{12} = 0 = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

$0 = B(-0.5x_2^2 - x_1x_2 + ax_2) + C$

$C = -B(-0.5b^2 - x_1b + ab)$

Bottom: $\sigma_{12} = 0$ and $\sigma_{22} = 0$

$x_2 = -b$ and $x_1 = x_1$

$\sigma_{11} = -Ab - Bbx_1 + \frac{B}{2}(x_1 - a)^2$

$\sigma_{11} = -Ab - B(bx_1 - \frac{(x_1 - a)^2}{2})$

$\sigma_{22} = 0 = \frac{1}{2}Bb^2 + D$

$\therefore D = -\frac{1}{2}Bb^2$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - B(x_1 - a)x_2 + C$

$0 = B(-0.5b^2 - x_1b + ab) + C$

$C = -B(-0.5b^2 - x_1b + ab)$

Different (for σ_{11}); Same (for D); Same (for C)

Right: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = a$

$\sigma_{11} = 0 = Ax_2 + Bax_2 + \frac{B}{2}\underbrace{(a-a)^2}_{\to 0}$

$\therefore A = -Ba$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = F = -\frac{1}{2}Bx_2^2 - B\underbrace{(a-a)}_{\to 0}x_2 + C$

$\therefore F = -\frac{1}{2}Bx_2^2 + C$

Left: $\sigma_{11} = 0$ and $\sigma_{12} = F$

$x_2 = x_2$ and $x_1 = 0$

$\sigma_{11} = Ax_2 + 0 + \frac{B}{2}a^2 = 0$

$\therefore A = -\frac{Ba^2}{2x_2}$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = F = -\frac{1}{2}Bx_2^2 - B(-a)x_2 + C$

$F = -\frac{1}{2}Bx_2^2 + Bax_2 + C$

$F = B(-0.5x_2^2 + ax_2 + C)$

Different (for A)

Case 3 is not a suitable solution for this elasticity problem because the constants are indeterminate and contradicting each other.

$\Rightarrow$ Therefore, the solution for this Problem is Case 2 with the following values:

$A = 0$, $B = \frac{-2F}{x_2^2 + 0.5b^2}$, $C = \frac{F}{x_2^2 + 0.5b^2}$, and $D = 0$

Problem 4

1.) True

The infinitesimal strain tensor $\underline{\underline{\varepsilon}} = [0]$ because there is rigid body motion; therefore, there is no relative displacement between the deformed and undeformed configuration. X

what about rotations?

2.5

2.) False ✓

2

This statement is not true because shear strain can cause rotation. X

3.) True

This statement is true because the governing differential equation is found by minimizing the energy. ✓

4.) False

If $\nu = \frac{\varepsilon_y}{\varepsilon_x} = 0$, then this implies that deformation in one axis would not lead to deformation in an orthogonal axis; therefore, the volume would not be conserved. Hence, $\nu = 0$ doesn't necessarily mean the solid is incompressible. ✓

5.) False

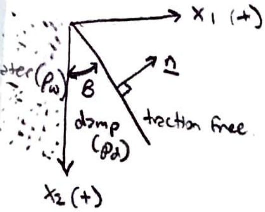
Hydrostatic pressure may depend on devietonic stress, not strain because $\underline{\underline{\sigma}} = P\underline{\underline{I}} + \hat{\underline{\underline{\sigma}}}$. ✓

6.) False (True for linearly elastic isotropic materials)

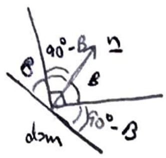
This statement is not always true for anisotropic materials, but the statement is true always for linearly elastic isotropic materials. ✓

Problem 5

1.) This problem has a body force. Accounting for that fact, write down the relations between the Airy stress function and stresses in this problem and the equation that the Airy stress function must satisfy.



Dam extends infinitely in the x_3 direction $\therefore$ plane strain 2D problem



$\underline{n} = \cos B - \sin B$

- traction along $x_2 = 0$
- traction along $x_1 = P_{water}$

Due to the body force, the Airy stress function must satisfy the following equation:

$$\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = -(1-\nu^*)\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right)$$

To get relations:

$$\cancel{\rho b_1} = -\frac{\partial \psi}{\partial x_1} \Rightarrow \psi = f(x_2) \quad [1]$$

$\rho b_1 \to 0$

$$\rho b_2 = -\frac{\partial \psi}{\partial x_2} \Rightarrow \psi = -\rho_d g x_2 + f(x_1) \quad [2]$$

Since Eqn [1] does not depend on x_1, $f(x_1) = 0$.

Therefore Eqn [2] becomes $\psi = -\rho_d g x_2$.

For this case:

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} + \psi = \frac{\partial^2 \phi}{\partial x_2^2} - \rho_d g x_2$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} + \psi = \frac{\partial^2 \phi}{\partial x_1^2} - \rho_d g x_2$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

Problem 5 (Continued)

2) Using the Airy stress function $\phi = A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3$, find the stress distribution inside the dam.

$$\sigma_{11} = \frac{\partial^2}{\partial x_2^2}\left(A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3\right) - \rho_d g x_2$$

$$= \frac{\partial}{\partial x_2}\left(A_2 x_1^2 + 2A_3 x_1 x_2 + 3A_4 x_2^2\right) - \rho_d g x_2$$

$$\Rightarrow \sigma_{11} = 2A_3 x_1 + 6A_4 x_2 - \rho_d g x_2 \quad [*]$$

$$\sigma_{22} = \frac{\partial^2}{\partial x_1^2}\left(A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3\right) - \rho_d g x_2$$

$$= \frac{\partial}{\partial x_1}\left(3A_1 x_1^2 + 2A_2 x_1 x_2 + A_3 x_2^2\right) - \rho_d g x_2$$

$$\Rightarrow \sigma_{22} = 6A_1 x_1 + 2A_2 x_2 - \rho_d g x_2 \quad [**]$$

$$\sigma_{12} = \frac{-\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{-\partial^2}{\partial x_1 \partial x_2}\left(A_1 x_1^3 + A_2 x_1^2 x_2 + A_3 x_1 x_2^2 + A_4 x_2^3\right)$$

$$= \frac{-\partial}{\partial x_1}\left(A_2 x_1^2 + 2A_3 x_1 x_2 + 3A_4 x_2^2\right)$$

$$\Rightarrow \sigma_{12} = -2A_2 x_1 - 2A_3 x_2 \quad [***]$$

To get constants A_1, A_2, A_3, and A_4, need to use conditions:

i) vertical side

$$\underline{t} = \begin{bmatrix} \rho_w g x_2 \\ 0 \end{bmatrix} \qquad \underline{n} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad x_1 = 0 \text{ and } x_2 = x_2$$

ii) inclined side

$$\underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ b/c no tractions} \qquad \underline{n} = \begin{bmatrix} \cos\beta \\ -\sin\beta \end{bmatrix}, \quad \tan\beta = \frac{x_1}{x_2} \therefore \frac{x_2}{x_1} = \cot\beta$$

$$x_1 = x_2 \tan\beta$$

i) Vertical side

$$\underline{t} = \underline{\sigma} \cdot \underline{n}$$

$$\begin{bmatrix} \rho_w g x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2A_3 x_1 + 6A_4 x_2 - \rho_d g x_2) & (-2A_2 x_1 - 2A_3 x_2) \\ (-2A_2 x_1 - 2A_3 x_2) & (6A_1 x_1 + 2A_2 x_2 - \rho_d g x_2) \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \rho_w g x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2A_3 x_1 - 6A_4 x_2 + \rho_d g x_2 \\ 2A_2 x_1 + 2A_3 x_2 \end{bmatrix}$$

[1] $\rho_w g x_2 = -2A_3 x_1 - 6A_4 x_2 + \rho_d g x_2$

[2] $0 = 2A_2 x_1 + 2A_3 x_2$

[2] with B.C. $x_1 = 0$ and $x_2 = x_2 \Rightarrow A_3 = 0$ [3]

Problem 5 (continued)

Eqn [1] and [3]:

$$[4] \quad \rho_w g x_2 = -6A_4 x_2 + \rho_d g x_2$$

$$6A_4 x_2 = g x_2(\rho_d - \rho_w)$$

$$\Rightarrow A_4 = \frac{g(\rho_d - \rho_w)}{6} \quad [5]$$

ii) inclined side / slope.

$$\underline{t} = \underline{\sigma} \cdot \underline{n}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos B \\ -\sin B \end{bmatrix} = \begin{bmatrix} (2A_3 x_1 + 6A_4 x_2 - \rho_d g x_2) & (-2A_2 x_1 - 2A_3 x_2) \\ (-2A_2 x_1 - 2A_3 x_2) & (6A_1 x_1 + 2A_2 x_2 - \rho_d g x_2) \end{bmatrix} \begin{bmatrix} \cos B \\ -\sin B \end{bmatrix}$$

$$[6] \quad 0 = (2A_3 x_1 + 6A_4 x_2 - \rho_d g x_2)\cos B + (2A_2 x_1 + 2A_3 x_2)\sin B$$

$$[7] \quad 0 = (-2A_2 x_1 - 2A_3 x_2)\cos B - (6A_1 x_1 + 2A_2 x_2 - \rho_d g x_2)\sin B$$

[6] with $A_3 = 0$:

$$0 = (6A_4 x_2 - \rho_d g x_2)\cos B + 2A_2 x_1 \sin B \quad [8]$$

[7] with $A_3 = 0$:

$$0 = -2A_2 x_1 \cos B - (6A_1 x_1 + 2A_2 x_2 - \rho_d g x_2)\sin B \quad [9]$$

[8] with [5]:

$$0 = \left[6x_2\left(\frac{g(\rho_d - \rho_w)}{6}\right) - \rho_d g x_2\right]\cos B + 2A_2 x_1 \sin B$$

$$0 = (g x_2(\rho_d - \rho_w) - \rho_d g x_2)\cos B + 2A_2 x_1 \sin B$$

$$0 = -g x_2 \rho_w \cos B + 2A_2 x_1 \sin B$$

$$A_2 = \frac{g x_2 \rho_w \cos B}{2 x_1 \sin B} \qquad \frac{x_2}{x_1} = \cot B, \quad \frac{\cos B}{\sin B} = \cot B$$

$$\Rightarrow A_2 = \frac{g \rho_w}{2}\cot^2 B \quad [10]$$

[9] with [10]:

$$0 = -2\left(\frac{g\rho_w}{2}\cot^2 B\right)x_1 \cos B - \left[6A_1 x_1 + 2\left(\frac{g\rho_w}{2}\cot^2 B\right)x_2 - \rho_d g x_2\right]\sin B$$

$$0 = -g\rho_w \cot^2 B x_1 \cos B - 6A_1 x_1 \sin B - g\rho_w \cot^2 B x_2 \sin B + \rho_d g x_2 \sin B$$

$$A_1 = \frac{-g\rho_w \cot^2 B x_1 \cos B - g\rho_w \cot^2 B x_2 \sin B + \rho_d g x_2 \sin B}{6 x_1 \sin B}$$

Problem 5 (continued)

$$\therefore A_1 = \frac{-g\rho_w}{6}\cot^3\beta - \frac{g\rho_w x_2}{6x_1}\cot^2\beta + \frac{\rho_d g x_2}{6x_1}, \quad \frac{x_2}{x_1} = \cot\beta$$

$$= \frac{-g\rho_w}{6}\cot^3\beta - \frac{g\rho_w\cot^3\beta}{6} + \frac{g\rho_d}{6}\cot\beta$$

$$= \frac{-g\rho_w\cot^3\beta}{3} + \frac{g\rho_d\cot\beta}{6}$$

$$\Rightarrow A_1 = \frac{g}{6}\left(-2\rho_w\cot^3\beta + \rho_d\cot\beta\right) \quad [11]$$

Therefore with known A_1, A_2, A_3, and A_4 constants, I can plug them into σ_{11}, σ_{22}, and σ_{12} to find the stress distribution inside the dam.

[*] $\sigma_{11} = 2A_3x_1 + 6A_4x_2 - \rho_d g x_2$, $A_3 = 0$ and $A_4 = \frac{g(\rho_d - \rho_w)}{6}$

$$= 6\left(\frac{g(\rho_d - \rho_w)}{6}\right)x_2 - \rho_d g x_2$$

$$= gx_2(\rho_d - \rho_w - \rho_d)$$

$$\Rightarrow \sigma_{11} = -g\rho_w x_2$$

[**] $\sigma_{22} = 6A_1x_1 + 2A_2x_2 - \rho_d g x_2$, $A_1 = \frac{g}{6}(-2\rho_w\cot^3\beta + \rho_d\cot\beta)$, $A_2 = \frac{g\rho_w}{2}\cot^2\beta$

$$= gx_1(-2\rho_w\cot^3\beta + \rho_d\cot\beta) + g\rho_w x_2\cot^2\beta - \rho_d g x_2$$

$$\Rightarrow \sigma_{22} = gx_1(-2\rho_w\cot^3\beta + \rho_d\cot\beta) + gx_2(\rho_w\cot^2\beta - \rho_d)$$

[***] $\sigma_{12} = -2A_2x_1 - 2A_3x_2$, $A_3 = 0$ and $A_2 = \frac{g\rho_w}{2}\cot^2\beta$

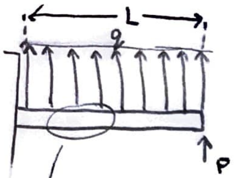
$$= -2\left(\frac{g\rho_w}{2}\cot^2\beta\right)x_1 - 0$$

$$\Rightarrow \sigma_{12} = -gx_1\rho_w\cot^2\beta$$

Bonus Problem

Euler - Bernoulli beam

Derive the strong formulation of the problem (differential equation and traction BC.) by using the principle of minimum potential energy.



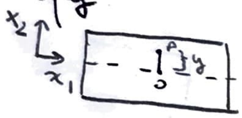
Young's modulus E

moment of inertia I

$\varepsilon_{22} = 0$

$\varepsilon_{12} = 0$

i) Before Bending



ii) After Bending

$O, P, x_2 \rightarrow O', P', x_2'$ and small $\measuredangle$ change

Assumptions: $u_3 = 0$, $u_1 = u_1(x_1)$, and $u_2 = u_2(x_2)$

After bending:

$$u_1(P') = u_1(O') - x_2 \sin\theta$$

$$[*] \quad u_1(P') = u_1(O') - x_2\theta \quad (\text{if } \theta \ll 1 \therefore \sin\theta = \theta)$$

Given $\varepsilon_{12} = 0$

$$\therefore \varepsilon_{12} = 0 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\therefore \frac{\partial u_1}{\partial x_2} = -\theta$$

$$\therefore \frac{\partial u_2}{\partial x_1} - \theta = 0 \Rightarrow \frac{\partial u_2}{\partial x_1} = \theta \quad [**]$$

With $[*]$ and $[**]$:

$$u_1(P') = u_1(O') - x_2 \frac{\partial u_2}{\partial x_1}$$

$$\Rightarrow u_1 = \underbrace{\cancel{u_1(x_1)}}_{\to 0} - x_2 \frac{\partial u_2}{\partial x_1}$$

$\to 0$ b/c assume no extension.

$$\therefore u_1 = -x_2 \frac{\partial u_2}{\partial x_1}$$

✓

Bonus Problem (continued)

Given: $\varepsilon_{22} = 0$ and $\varepsilon_{12} = 0$

but $\varepsilon_{11} \neq 0$

$$\therefore \varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right)$$

$$= \frac{\partial u}{\partial x_1}$$

$$= -x_2 \frac{\partial^2 u}{\partial x_1^2}$$

From Lecture Notes (week 14 Lecture 2): $W = \int_\Omega \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, dV$

$$W = \frac{1}{2}\int \sigma_{ij} \varepsilon_{ij} \, dV$$

$$= \frac{1}{2}\int \sigma_{11} \varepsilon_{11} \, dV \quad , \quad \sigma_{11} = E\varepsilon_{11} = E\left(-x_2 \frac{d^2 u_2}{\partial x_1^2}\right) = -E x_2 \frac{\partial^2 u_2}{\partial x_1^2}$$

$$[1] \quad \therefore W = \frac{1}{2}\int \left(-E x_2 \frac{\partial^2 u_2}{\partial x_1^2}\right)\left(-x_2 \frac{\partial^2 u_2}{\partial x_1^2}\right) dV$$

$$E = \int_\Omega \cancel{b_i} u_i \, dV + \int_{\partial_2 \Omega} t_i u_i \, dA = \int t_2 u_2 \, dA \; ,$$

$\rightarrow 0$ b/c no body force

i.) Traction due to P at $x = L$: $t_2 = \frac{\text{Force}}{\text{Area}} = \frac{P}{A}$

ii) Traction due to q (distributed load) : $t_2 = q$

$$\therefore E = \int t_2 u_2 \, dA = \int \left(\frac{P}{A} + q\right) u_2 \, dA = \frac{1}{A}\cancel{\int dA} \cdot P(u_2)\Big|_{x=\ell} + \int q(u_2) \, dA$$

$\rightarrow 1$

$$= P u_2\Big|_{x=\ell} + \int q u_2 \, dA \, ,$$

$$[2] \quad E = P u_2\Big|_{x=\ell} + \int q u_2 \, dx_1 \, , \quad dA \Rightarrow dx_1 \text{ because it is a linear load.}$$

$\Pi = W - E$ with [1] and [2]

$$= \frac{1}{2} E \int x_2^2 \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 dV - \left(\int q u_2 \, dx_1 + P u_2\big|_{x=\ell}\right) \, , \quad dV = dA \, dx$$

$$= \frac{1}{2} E \int_0^\ell x_2^2 \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 dx_1 \, dA - \int_0^\ell q u_2 \, dx_1 - P u_2\big|_{x=\ell}$$

Bonus Problem (continued)

$$\delta\Pi = \frac{E}{2}\,\delta\left[\int_0^l x_2^2\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 dx_1 dA\right] - \delta\left[\int_0^l q\,u_2\,dx_1\right] - \delta\left[P u_2|_{x_1=l}\right]$$

Moment of inertia: $I = \int_{dA} x_2^2\, dA$

$$\delta\Pi = \underbrace{\frac{EI}{\cancel{2}}\int_0^l \cancel{2}\,\frac{\partial^2 u_2}{\partial x_1^2}\,\frac{\partial^2(\delta u_2)}{\partial x_1^2}\,dx_1}_{(I)} - \int_0^l q(\delta u_2)\,dx_1 - P\,\delta u_2\Big|_{x_1=l}$$

$$u = \frac{\partial^2 u_2}{dx_1^2}\,,\quad \frac{dv}{dx_1} = \frac{d^2\delta u_2}{dx_1^2}$$

$$\therefore \frac{du}{dx_1} = \frac{d^3 u_2}{dx_1^3}\qquad v = \frac{d\,\delta u_2}{dx_1}$$

$$(I) = EI\left[\left(\frac{\partial\delta u_2}{\partial x_1}\cdot\frac{d^2u_2}{dx_1^2}\right]\Big|_0^l - \int_0^l \frac{d\delta u_2}{dx_1}\,\frac{d^3u_2}{dx_1^3}\,dx_1\right]$$

$$u = \frac{d^3u_2}{dx_1^3}\,,\quad \frac{dv}{dx} = \frac{d\,\delta u_2}{dx_1}$$

$$\therefore \frac{du}{dx_1} = \frac{d^4u_2}{dx_1^4}\qquad v = \delta u_2$$

$$\therefore (I) = EI\left[\left(\frac{d\,\delta u_2}{dx_1}\,\frac{d^2u_2}{dx_1^2}\right)\Big|_0^l - \left[\left(\frac{d^3u_2}{dx_1^3}\,\delta u_2\right)\Big|_0^l - \underbrace{\int_0^l \delta u_2\,\frac{d^4u_2}{dx_1^4}\,dx_1}_{(II)}\right]\right]$$

$$\Rightarrow \delta\Pi = (I) - \underbrace{\int_0^l q\,\delta u_2\,dx_1}_{(III)} - P(\delta u_2)|_{x_1=l} = 0$$

Using (II) and (III): $EI\,\dfrac{d^4u_2}{dx_1^4} - q = 0$

BC's: $+EI\,\dfrac{d^3u_2}{dx_1^3}\Big|_{x_1=l} = -P$

$\dfrac{d^3u_2}{dx_1^3}\Big|_{x_1=0} = 0$ X $\quad u_2\big|_{x_1=0} = 0 \quad \dfrac{du_2}{dx_1}\Big|_{x_1=0} = 0$

$\dfrac{d^2u_2}{dx_1^2}\Big|_{x_1=l} = 0$ ✓ and $\dfrac{d^2u_2}{dx_1^2}\Big|_{x_1=0} = 0$ X