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AE 6115 — Structural Analysis — Fall 2017

Exam No.1

Problem 1. Bi-material beam under centrifugal load.

Consider a bi-material beam subjected to a centrifugal load as shown in Figure 1. The beam is composed of two materials with Young's moduli E_A and E_B however the two materials have the same density ρ. The beam is rotating at an angular velocity Ω about the $\bar{i}_2$ axis which induces a centrifugal load on the beam which acts as a distributed load in the $\bar{i}_1$ direction of the form

$$p_1(x_1) = \rho A \Omega^2 x_1. \tag{1}$$

Critically, this load acts at the **center of mass** of the beam.

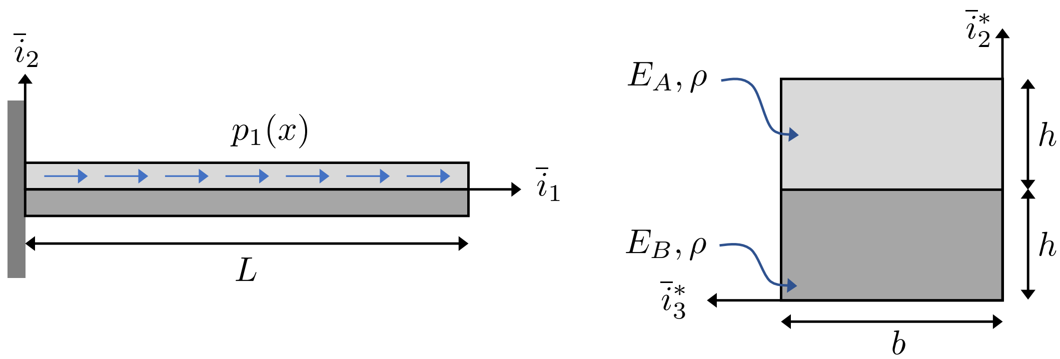


Figure 1: Schematic of a bi-material beam under centrifugal load

1) Find the location of the centroid and the center of mass.

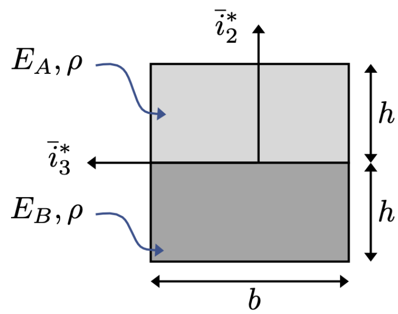
2) Find the stiffnesses of the cross section.

3) Solve for the components u_1, u_2, u_3 of the displacement field.

4) Solve for the ratio of tip displacements $[\bar{u}_1/u_2]_L$ using non-dimensional analysis (i.e. without solving the exact problem).

Solution

1) Considering the $(\bar{i}_2^*, \bar{i}_3^*)$ coordinate frame shown in the figure below



the center of mass would simply be located at the $(0, 0)$ coordinates since the density is a constant throughout the cross-section.

Since the section is symmetric about the $\bar{i}_2^*$ coordinate shown above, we know immediately that

$$x_{3c} = 0$$

To find the other coordinate we write

$$\begin{aligned} x_{2c} &= \int_A E x_2 dA \Big/ \int_A E dA \\ &= \left(E_A \left[\frac{x_2^2}{2} \right]_0^h b + E_B \left[\frac{x_2^2}{2} \right]_{-h}^0 b \right) \frac{1}{E_A(bh) + E_B(bh)} \\ &= \left(E_A \frac{1}{2} h^2 b - E_B \frac{1}{2} h^2 b \right) \frac{1}{E_A(bh) + E_B(bh)} \\ x_{2c} &= \frac{h}{2} \frac{E_A - E_B}{E_A + E_B}. \end{aligned} \tag{2}$$

Hence we have that the centroid and the center of mass are a distance x_{2c} apart from each other in the $\bar{i}_2$ direction.

2) The beams sectional stiffness S is given by

$$S = \int_A E dA = E_A(bh) + E_B(bh) = bh(E_A + E_B). \tag{3}$$

For the computation of the bending stiffnesses we must use the parallel axis theorem since centroid of the beam does not necessarily coincide with the centroid of the individual segments. The stiffness H_{33}^c is given by

$$H_{33}^c = E_A \left(\frac{bh^3}{12} + (bh)(\delta - h/2)^2 \right) + E_B \left(\frac{bh^3}{12} + (bh)(\delta + h/2)^2 \right) \tag{4}$$

The cross-bending stiffness H_{23}^c is equal to zero. This can be shown as follows. The parallel axes theorem for the cross-bending stiffness is given by zero

$$H_{23}^c = H_{23}^{c,\text{local}} + S\delta_2\delta_3 \tag{5}$$

where δ_2 and δ_3 are the distances from the local centroid of each segment to the beam (global) centroid. For each of the two rectangular portions of the beam, we can show that the term H_{23}^c is zero. Further, since the beam centroid is only displaced by an amount δ only in the $\bar{\imath}_2$-direction, the term $\delta_3 = 0$ and hence the second term of (5) is also zero. For completeness, for a rectangular beam of width b and height h, we have that

$$\begin{aligned} H_{23}^{c,\text{local}} &= \int_A E x_2 x_3 dA \\ &= E\left[\frac{x_2^2}{2}\right]_{-h/2}^{h/2}\left[\frac{x_3^2}{2}\right]_{-b/2}^{b/2} \\ &= R\left(\frac{h^2}{8}-\frac{h^2}{8}\right)\left(\frac{b^2}{8}-\frac{b^2}{8}\right) = 0 \end{aligned} \tag{6}$$

3) We can recognize from the geometry of the cross-section and the direction of the applied loading that this is only a two-dimensional bending problem in the $\bar{\imath}_1 - \bar{\imath}_2$ plane. First the governing equation for the axial deformation is given by

$$\frac{d}{dx_1}\left[S\frac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1). \tag{7}$$

Noting that the sectional stiffness S is not a function of x_1, and applying the centrifugal load yields the simplified form as

$$S\frac{d^2\bar{u}_1}{dx_1^2} = -\rho A\Omega^2 x_1. \tag{8}$$

The second order differential equation (8) for the axial displacement $\bar{u}_1$ requires two boundary conditions. These are

$$\begin{aligned} \bar{u}_1 = 0 &\quad \text{at} \quad x_1 = 0, \\ N_1 = 0 &\quad \text{at} \quad x_1 = L. \end{aligned} \tag{9}$$

The second boundary condition come from the fact that there are no point loads on the tip of the beam. This boundary condition can be converted to a boundary condition on the displacement using the sectional constitutive equation and yields

$$\begin{aligned} \bar{u}_1 = 0 &\quad \text{at} \quad x_1 = 0, \\ \frac{d\bar{u}_1}{dx_1} = 0 &\quad \text{at} \quad x_1 = L. \end{aligned} \tag{10}$$

Integrating (8) once yields

$$S\frac{d\bar{u}_1}{dx_1} = -\rho A\Omega^2\frac{x_1^2}{2} + C_1. \tag{11}$$

Applying the boundary condition (10), that is $d\bar{u}_1/dx_1 = 0$ at $x_1 = L$ yields

$$0 = -\rho A\Omega^2\frac{L^2}{2} + C_1 \quad \rightarrow \quad C_1 = \rho A\Omega^2\frac{L^2}{2} \tag{12}$$

Hence we may write

$$S\frac{d\bar{u}_1}{dx_1} = -\rho A\Omega^2\left(\frac{x_1^2}{2} - \frac{L^2}{2}\right). \tag{13}$$

Integrating again yields

$$S\bar{u}_1 = -\rho A\Omega^2\left(\frac{x_1^3}{6} - \frac{L^2 x_1}{2}\right) + B. \tag{14}$$

Using the boundary condition (10), that is $\bar{u}_1 = 0$ at $x_1 = 0$ simply yields that $B = 0$. Hence,

$$\bar{u}_1 = -\frac{\rho A\Omega^2}{S}\left(\frac{x_1^3}{6} - \frac{L^2 x_1}{2}\right). \tag{15}$$

Now we turn our attention to the bending the deformation. For the bending deformation the governing equations are given by

$$\begin{aligned}
\frac{d^2}{dx_1^2}\left[H_{33}^c\frac{d^2u_2}{dx_1^2} + H_{23}^c\frac{d^2u_3}{dx_1^2}\right] &= p_2 + \frac{d}{dx_1}\left[x_{2a}p_1 - q_3\right] \\
\frac{d^2}{dx_1^2}\left[H_{23}^c\frac{d^2u_2}{dx_1^2} + H_{22}^c\frac{d^2u_3}{dx_1^2}\right] &= p_3 + \frac{d}{dx_1}\left[x_{3a}p_1 + q_2\right]
\end{aligned} \tag{16}$$

As mentioned above, due to the symmetry of the beam ($H_{23}^c = 0$) and the fact that $x_{3a} = 0$, the second equation can be neglected and we are left with an equation governing the transverse displacement $\bar{u}_2$ which can be simplified to

$$\begin{aligned}
H_{33}^c\frac{d^4u_2}{dx_1^4} &= \frac{d}{dx_1}\left[-\delta\rho A\Omega^2 x_1\right] \\
H_{33}^c\frac{d^4u_2}{dx_1^4} &= -\delta\rho A\Omega^2.
\end{aligned} \tag{17}$$

Note: The quantity $x_{2a} = -\delta$ since the center of mass is below (in the negative $\bar{\imath}_2$-direction) the centroid of the beam.

The bending problem requires four boundary conditions. On the left hand side we have a cantilevered beam and hence that

$$\bar{u}_2 = 0 \quad \text{and} \quad \frac{d\bar{u}_2}{dx_1} = 0, \quad \text{at} \quad x_1 = 0. \tag{18}$$

On the right hand side we have no applied moments or vertical loads such that

$$M_3 = 0 \quad \text{and} \quad V_2 = 0, \quad \text{at} \quad x_1 = L. \tag{19}$$

however **care must be exercised here** in converting this to boundary conditions for u_2. First, let us recall the balance equation derived from a sum of the moments in the $\bar{i}_3$ direction

$$\frac{dM_3}{dx_1} + V_2 = -q_3(x_1) + x_{2a}p_1(x_1), \tag{20}$$

and let us also recall the sectional constitutive equation which now may be written as

$$M_3 = H^c_{33}\frac{d^2u_2}{dx_1^2}. \tag{21}$$

Using $M_3 = 0$ from (19) in the sectional constitutive equation (20) yields the boundary condition

$$H^c_{33}\frac{d^2u_2}{dx_1^2} = 0 \quad \text{at} \quad x_1 = L. \tag{22}$$

Finally, using $V_2 = 0$ form (19) in the balance equation (20) along with the sectional constitutive equation (21) yields

$$\frac{d}{dx_1}\left(H^c_{33}\frac{d^2u_2}{dx_1^2}\right) + 0 = [x_{3a}p_1(x_1)]_L = -x_{2c}\rho A\Omega^2 L, \tag{23}$$

which yields the boundary condition

$$H^c_{33}\frac{d^3u_2}{dx_1^3} = -x_{2c}\rho A\Omega^2 L \tag{24}$$

for u_2.

Integrating the governing equation for the transverse displacement (17) once yields

$$H^c_{33}\frac{d^3\bar{u}_2}{dx_1^3} = -\delta\rho A\Omega^2 x_1 + C_1. \tag{25}$$

We may now use one of the boundary condition at $x_1 = L$ from (24) involving the third derivative of the transverse displacement. This yields

$$-\delta\rho A\Omega^2 L = -\delta\rho A\Omega^2 L + C_1 \quad \rightarrow \quad C_1 = 0. \tag{26}$$

Integrating (25) with $C_1 = 0$ then yields

$$H^c_{33}\frac{d^2\bar{u}_2}{dx_1^2} = -\delta\rho A\Omega^2\frac{x_1^2}{2} + C_2. \tag{27}$$

We may now use the second boundary condition at $x_1 = L$ from (22) which gives

$$0 = -\delta\rho A\Omega^2\frac{L^2}{2} + C_2. \quad \rightarrow \quad C_2 = \delta\rho A\Omega^2\frac{L^2}{2}. \tag{28}$$

Hence we may write (27) as

$$H_{33}^c \frac{d^2 u_2}{dx_1^2} = -\delta\rho A\Omega^2 \left(\frac{x_1^2}{2} - \frac{L^2}{2}\right). \tag{29}$$

Integrating again yields

$$H_{33}^c \frac{du_2}{dx_1} = -\delta\rho A\Omega^2 \left(\frac{x_1^3}{6} - \frac{L^2 x_1}{2}\right) + C_3, \tag{30}$$

however using the fixed boundary condition at $x_1 = 0$ for $d\bar{u}_2/dx_1$ yields that $C_3 = 0$. Integrating one last time yields

$$H_{33}^c u_2 = -\delta\rho A\Omega^2 \left(\frac{x_1^4}{24} - \frac{L^2 x_1^2}{4}\right) + C_4, \tag{31}$$

and again we have that $C_4 = 0$ since $\bar{u}_2 = 0$ at $x_1 = 0$. Thus our final answer is

$$u_2 = -\frac{\delta\rho A\Omega^2}{H_{33}^c} \left(\frac{x_1^4}{24} - \frac{L^2 x_1^2}{4}\right). \tag{32}$$

Finally, by inspection of the symmetry of the problem and the fact that $H_{23}^c = 0$ we may simply write

$$u_3 = 0. \tag{33}$$

Finally, the axial displacement is computed as

$$u_1 = \bar{u}_1 - x_2 \frac{du_2}{dx_1}, \tag{34}$$

where we may use our solution for $\bar{u}_1$ from (15) and our solution for u_2 from (32)

4) We now wish to find the ratio of tip displacements $[\bar{u}_1/u_2]_L$ using non-dimensional analysis. Since the ratio is a non-dimensionless constant, we write that it is given by

$$\left[\frac{\bar{u}_2}{\bar{u}_1}\right]_L \propto \frac{S x_{2c} L}{H_{33}^c}. \tag{35}$$

where α is an unknown constant. This, and it's inverse, is the only non-dimensional group that can be formed of the relevant quantities in the problem. This solution makes intuitive sense

- If $\delta = 0$ we have no transverse displacement, and the ratio is zero.
- If the sectional stiffness S is reduced, we have more axial to transverse displacement. Conversely if the bending stiffness H_{33}^c is reduced we have more transverse to axial displacement.
- Making the beam longer increases the amount of transverse displacement

There are a few way to approach finding the above. One is to first think "intuitively" about how each of the two displacements should scale, for example we would write

$$\bar{u}_1 \propto L^2 \frac{1}{S} p_1 \tag{36}$$

we write this because:

- we expect that if the beam is longer the tip will extend more (it will be more "flimsy");
- if the stiffness of the beam increases, we expect the extension to decrease;
- the extension should increase with the applied load.

Note that we add the square term over L in order to make sure the units match! We may repeat the same process for u_2 and write

$$u_2 =\propto L^3 \frac{1}{H_{33}^c} (p_1 x_{3c}) \tag{37}$$

where we follow the same thought process as above with the addition that we understand the applied bending moment (applied load) is proportal to $(p_1 \cdot x_{2c})$ and the x_{2c} quantity must be included. Note that again in (37) the cubic power on L is added to ensure the units match! Combining (35) and (36) will yield the non-dimensional quantity shown in (37).

Finally, since we have solved the displacements exactly in part 3), we may also compute the exact ratio which is given by

$$\begin{aligned} \left[\frac{\bar{u}_2}{\bar{u}_1}\right]_L &= \left(\frac{\delta \rho A \Omega^2}{H_{33}^c}\right) L^4 \frac{5}{24} \left(\frac{S}{\rho A \Omega^2}\right) \frac{1}{L^3} \frac{3}{1} \\ &= \frac{5}{8} \frac{S \delta L}{H_{33}^c} \end{aligned} \tag{38}$$