Georgia Institute of Technology
School of Aerospace Engineering

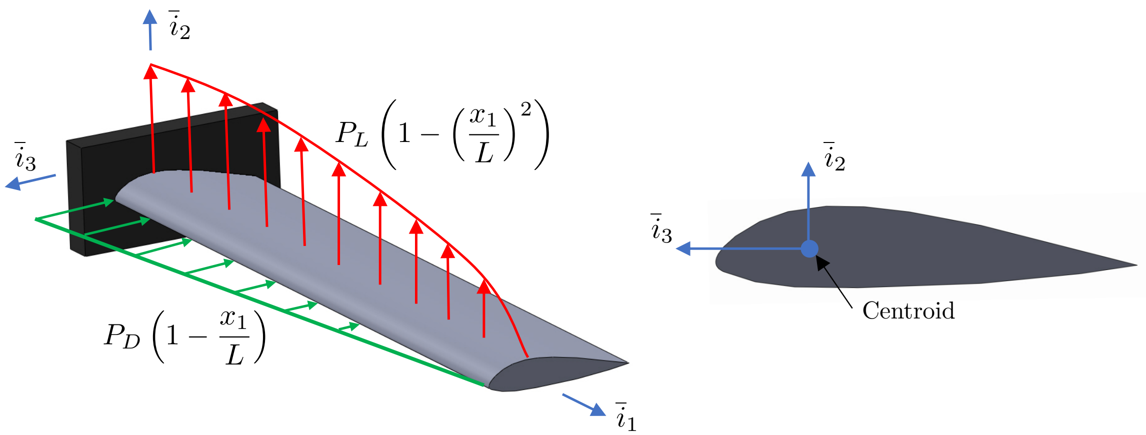
AE 6115 — Fundamentals of Aerospace Structural Analysis — Spring 2021

Quiz No.1

Consider a cantilevered wing of length L with an *asymmetric* airfoil cross-section which is constant along the long direction of the wing as shown in Fig. 1. The wing is subjected to *two distributed loads* representing the lift and drag forces which are given by

$$p_2(x_1) = P_L\left(1 - \left(\frac{x_1}{L}\right)^2\right), \quad \text{and} \quad p_3(x_1) = -P_D\left(1 - \frac{x_1}{L}\right), \tag{1}$$

with P_L and P_D known constants and with $x_1 = 0$ starting at the cantilevered end. You may assume that the wing is slender and composed of a homogeneous material of Young's modulus E.



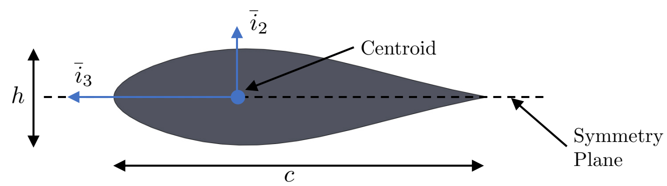
(a) (b)

Figure 1: Cantilevered wing under applied distributed loads.

a) Solve for the internal moment distributions $M_2(x_1)$ and $M_3(x_1)$ in the wing.

b) Solve for the displacement components $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$. **Important:** You may leave your answer in terms of any non-zero stiffnesses.

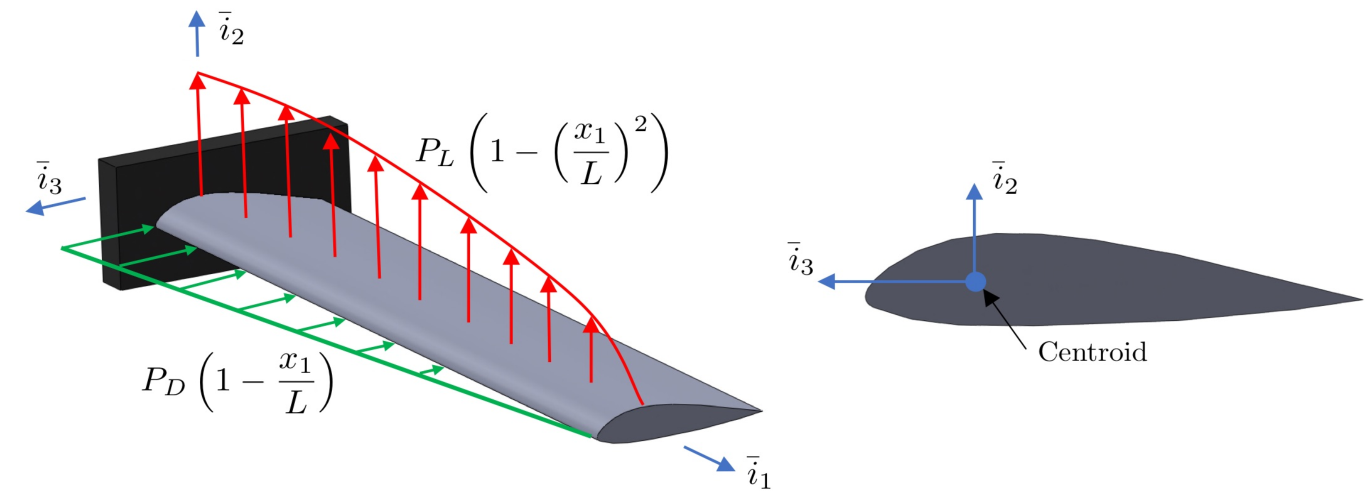
c) **Semi-Conceptual:** Consider now a *symmetric* cross-section as shown in the figure below:



For the special case of $P_L/P_D = 1$ and $c/h = 5$, find a good **estimate** for the equation of the neutral axis at the root of the beam ($x_1 = 0$). Sketch the neutral axis.

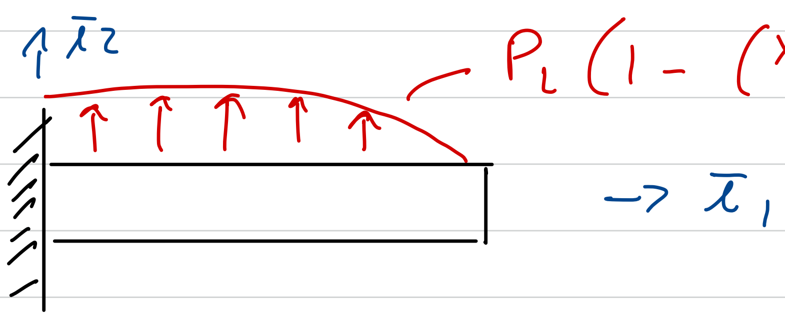
Hint: You will have to make some estimates here about the stiffnesses of this cross-section. You may also estimate any necessary results that you did not already find in parts a) and b) above.

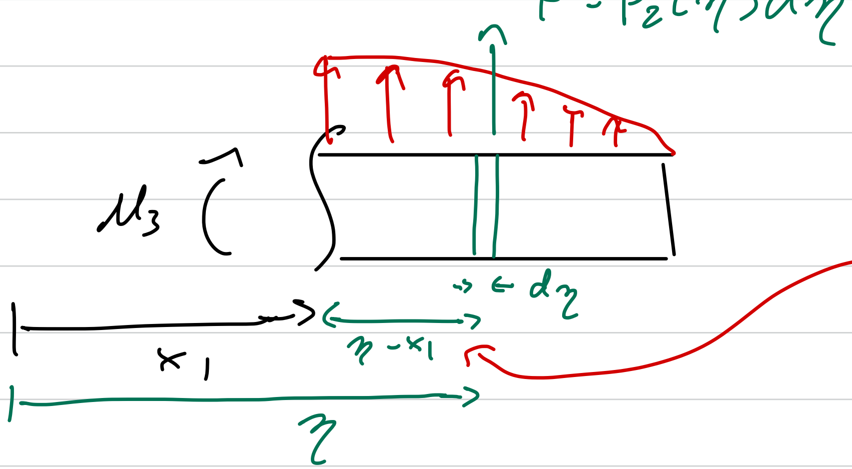
composed of a homogeneous material of Young's modulus E.



(a) (b)

a) Approach #1: Take a cut





"Force" due to distributed load

Moment arm is $\eta - x_1$

$$M_3 = \int_{x_1}^{L} (\eta - x_1)\, P_L\left(1-\left(\frac{\eta}{L}\right)^2\right) d\eta$$

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$$M_3 = P_L \int_{x_1}^{L} \left(\eta - \frac{\eta^3}{L^2} - x_1 + \frac{x_1 \eta^2}{L^2}\right) d\eta$$

$$M_3 = P_L \left[\frac{\eta^2}{2} - \frac{\eta^4}{4L^2} - x_1 \eta + \frac{x_1 \eta^3}{3L^2} \right]_{x_1}^{L}$$

$$= P_L \left[\frac{L^2}{2} - \frac{L^2}{4} - x_1 L + \frac{1}{3} x_1 L \right.$$

$$\left. - \frac{x_1^2}{2} + \frac{x_1^4}{4L^2} + x_1^2 - \frac{x_1^4}{3L^2} \right]$$

$$\boxed{M_3 = P_L \left(\frac{L^2}{4} - \frac{2}{3} x_1 L + \frac{1}{2} x_1^2 - \frac{1}{12} \frac{x_1^4}{L^2} \right)}$$

SIMILARELY

$$M_2 = \int_{x_1}^{L} (\eta - x_1)\, P_D \left(1 - \frac{\eta}{L}\right) d\eta$$

$$M_2 = P_D \int_{x_1}^{L} \left(\eta - \frac{\eta^2}{L} - x_1 + \frac{x_1 \eta}{L}\right) d\eta$$

$$M_2 = P_D \int_{x_1}^{L} \left(\eta - \frac{\eta^2}{L} - x_1 + \frac{x_1 \eta}{L} \right) d\eta$$

$$M_2 = P_D \left[\frac{\eta^2}{2} - \frac{\eta^3}{3L} - x_1 \eta + \frac{x_1 \eta^2}{2L} \right]_{x_1}^{L}$$

$$M_2 = P_D \left(\frac{L^2}{2} - \frac{L^2}{3} - x_1 L + \frac{x_1 L}{2} - \frac{x_1^2}{2} + \frac{1}{3}\frac{x_1^3}{L} + x_1^2 - \frac{1}{2}\frac{x_1^3}{L} \right)$$

$$\boxed{M_2 = P_D \left(\frac{1}{6} L^2 - \frac{1}{2} x_1 L + \frac{x_1^2}{2} - \frac{1}{6}\frac{x_1^3}{L} \right)}$$

APPROACH #2: USE BALANCE LAWS

$$\frac{dM_3}{dx_1} + V_2 = -\cancel{q_3}(x_1) + x_{2A}\cancel{p_1}(x_1)$$

$$\frac{dV_2}{dx_1} = -p_2(x_1)$$

COMBINING

$$\frac{d^2 M_3}{dx_1^2} = p_2(x_1) = p_L\left(1 - \left(\frac{x_1}{L}\right)^2\right)$$

$\underline{\text{B.C.'s}}$ $\quad @\, x_1 = L \qquad V_2 = \dfrac{dM_3}{dx_1} = 0 \qquad (1)$

$$M_3 = 0 \qquad (2)$$

INTEGRATING:

$$\frac{dM_3}{dx_1} = p_L\left(x_1 - \frac{x_1^3}{3L^2}\right) + C$$

APPLY (1)

$$0 = p_L\left(L - \frac{L}{3}\right) + C \qquad C = -p_L \frac{2}{3} L$$

$$\frac{dM_3}{dx_1} = p_L\left(x_1 - \frac{x_1^3}{3L^2} - \frac{2}{3} L\right)$$

$$\frac{dM_3}{dx_1} = P_L\left(x_1 - \frac{x_1^3}{3L^2} - \frac{2}{3}L\right)$$

$$M_3 = P_L\left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3}Lx_1\right) + C$$

Apply (2)

$$0 = P_L\left(\frac{L^2}{2} - \frac{L^2}{12} - \frac{2}{3}L^2\right) + C$$

$$C = \frac{1}{4}L^2 P_L$$

$$\boxed{M_3 = P_L\left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3}Lx_1 + \frac{1}{4}L^2\right)}$$

→ MATCHES THE SAME AS BEFORE.

Similarly for M_2

$$\frac{dM_2}{dx_1} - V_3 = 0 \;,\quad \frac{dV_3}{dx_1} = -P_3(x_1)$$

$$\frac{d^2 M_2}{dx_1^2} = -p_3(x_1) = P_D\left(1 - \frac{x_1}{L}\right)$$

B.C.'s $\quad @\ x_1 = L \qquad V_3 = \frac{dM_2}{dx_1} = 0$

$$M_2 = 0$$

$$\frac{dM_2}{dx_1} = P_D\left(x_1 - \frac{x_1^2}{2L}\right) + C$$

$$0 = P_D\left(L - \frac{L}{2}\right) + C_1 \rightarrow \quad C_1 = -P_D\frac{L}{2}$$

$$\frac{dM_2}{dx_1} = P_D\left(x_1 - \frac{x_1^2}{2L} - \frac{L}{2}\right)$$

$$M_2 = P_D\left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L}{2}x_1\right) + C_2$$

$$0 = P_D\left(\frac{L^2}{2} - \frac{L^2}{6} - \frac{L^2}{2}\right) + C_2$$

$$\rightarrow \quad C_2 = \frac{L^2}{6}P_D$$

$$\boxed{M_2 = P_D\left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L}{2}x_1 + \frac{L^2}{6}\right)}$$

-> MATCHES BEFORE.

APPROACH #3: SOLVE GOVERNING EQUATIONS.

$$H_{33}^{c}\frac{d^4 u_2}{dx_1^4} + H_{23}^{c}\frac{d^4 u_3}{dx_1^4} = p_2(x_1)$$

$$H_{23}^{c}\frac{d^4 u_2}{dx_1^4} + H_{22}^{c}\frac{d^4 u_3}{dx_1^4} = p_3(x_1)$$

COMBINING TO SOLVE FOR d^4u_2/dx_1^4 & d^4u_3/dx_1^4 YIELDS

$$\frac{d^4 u_2}{dx_1^4} = \frac{H_{22}^{c}}{\Delta H}p_2(x_1) - \frac{H_{23}^{c}}{\Delta H}p_3(x_1)$$

$$\frac{d^4 u_3}{dx_1^4} = \frac{H_{33}^{c}}{\Delta H}p_3(x_1) - \frac{H_{23}^{c}}{\Delta H}p_2(x_1)$$

<u>B.Cs</u> @ $x_1 = 0$

$$u_2 = u_3 = 0$$

$$\frac{du_2}{dx_1} = \frac{du_3}{dx_1} = 0$$

@ $x_1 = L$

$$\frac{d^2 u_2}{dx_1^2} = \frac{d^2 u_3}{dx_1^2} = 0$$

$$\frac{d^3 u_2}{dx_1^2} = \frac{d^3 u_3}{dx_1^3} = 0$$

$$\frac{d^4 u_2}{dx_1^4} = \frac{H_{22}^c}{\Delta H} P_L \left(1 - \left(\frac{x_1}{2}\right)^2\right) + \frac{H_{23}^c}{\Delta H} P_D \left(1 - \frac{x_1}{L}\right)$$

$$\frac{d^3 u_2}{dx_1^3} = \frac{H_{22}^c}{\Delta H} P_L \left(x_1 - \frac{x_1^3}{3L^2}\right) + \frac{H_{23}^c}{\Delta H} P_D \left(x_1 - \frac{x_1^2}{2L}\right) + C_1$$

$$0 = \frac{H_{22}^c}{\Delta H} P_L \left(\frac{2}{3} L\right) + \frac{H_{23}^c}{\Delta H} P_D \left(\frac{1}{2} L\right) + C_1$$

$$\frac{d^3 u_2}{dx_1^3} = \frac{H_{22}^c}{\Delta H} P_L \left(x_1 - \frac{x_1^3}{3L^2} - \frac{2}{3} L\right) + \frac{H_{23}^c}{\Delta H} P_D \left(x_1 - \frac{x_1^2}{2L} - \frac{L}{2}\right)$$

$$\frac{d^2 u_2}{dx_1^2} = \frac{H_{22}^c}{\Delta H} P_L \left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3} L x_1\right)$$

$$+ \frac{H_{23}^c}{\Delta H} P_D \left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L x_1}{2}\right) + C_2$$

$$0 = \frac{H_{22}^c}{\Delta H} P_L \left(-\frac{1}{4} L^2\right) + \frac{H_{23}^c}{\Delta H} P_D \left(-\frac{1}{6} L^2\right) + C_2$$

$$\frac{d^2 u_2}{dx_1^2} = \frac{H_{22}^c}{\Delta H} P_L \left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3} L x_1 + \frac{1}{4} L^2\right)$$

$$+ \frac{H_{23}^c}{\Delta H} P_D \left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L x_1}{2} + \frac{1}{6} L^2\right)$$

$$\frac{d^2 u_2}{dx_1^2} = \frac{H_{22}^c}{\Delta H} P_L \left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3} L x_1 + \frac{1}{4} L^2 \right)$$

$$+ \frac{H_{23}^c}{\Delta H} P_D \left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L x_1}{2} + \frac{1}{6} L^2 \right)$$

Recall the sectional constitutive equation

$$\frac{d^2 u_2}{dx_1^2} = \frac{H_{23}^c}{\Delta H} M_2 + \frac{H_{22}^c}{\Delta H} M_3$$

HENCE WE HAVE ALSO FOUND

$$\boxed{\begin{aligned} M_2 &= P_D \left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L x_1}{2} + \frac{1}{6} L^2 \right) \\ M_3 &= P_L \left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3} L x_1 + \frac{1}{4} L^2 \right) \end{aligned}}$$

$\rightarrow$ MATCHES PREVIOUS ANSWERS!

PART (b) SOLVE $\underline{u}$.

→ RECOGNIZE $H_{23}^{c} \neq 0$!

→ WITH M_2 AND M_3 KNOWN WE MAY SOLVE USING THE SECTIONAL CONSTITUTIVE EQUATIONS.

$$K_2 = \frac{H_{33}^{c}}{\Delta H} M_2 + \frac{H_{23}^{c}}{\Delta H} M_3 = -\frac{d^2 u_3}{dx_1^2}$$

$$K_3 = \frac{H_{23}^{c}}{\Delta H} M_2 + \frac{H_{22}^{c}}{\Delta H} M_3 = \frac{d^2 u_2}{dx_1^2}$$

B.C.s

@ $x_1 = 0$

$u_2 = u_3 = 0$

$\frac{du_2}{dx_1} = \frac{du_3}{dx_1} = 0$

$$\frac{d^2 u_2}{dx_1^2} = \frac{H_{23}^{c}}{\Delta H} P_D \left(\frac{x_1^2}{2} - \frac{x_1^3}{6L} - \frac{L}{2} x_1 + \frac{L^2}{6} \right) + \frac{H_{22}^{c}}{\Delta H} P_L \left(\frac{x_1^2}{2} - \frac{x_1^4}{12L^2} - \frac{2}{3} L x_1 + \frac{L^2}{4} \right)$$

$$\frac{du_2}{dx_1} = \frac{H_{23}^{c}}{\Delta H} P_D \left(\frac{x_1^3}{6} - \frac{x_1^4}{24L} - \frac{L x_1^2}{4} + \frac{L^2 x_1}{6} \right) + \frac{H_{22}^{c}}{\Delta H} P_L \left(\frac{x_1^3}{6} - \frac{x_1^5}{60L^2} - \frac{2}{6} L x_1^2 + \frac{L^2 x_1}{4} \right) + \cancelto{=0}{C_1}$$

$$u_2 = \frac{H_{23}^{c}}{\Delta H} P_D \left(\frac{x_1^4}{24} - \frac{x_1^5}{120L} - \frac{L x_1^3}{12} + \frac{L^2 x_1^2}{12} \right) + \frac{H_{22}^{c}}{\Delta H} P_L \left(\frac{x_1^4}{24} - \frac{x_1^6}{360L^2} - \frac{1}{9} L x_1^3 + \frac{L^2 x_1^2}{8} \right) + \cancelto{=0}{C_2}$$

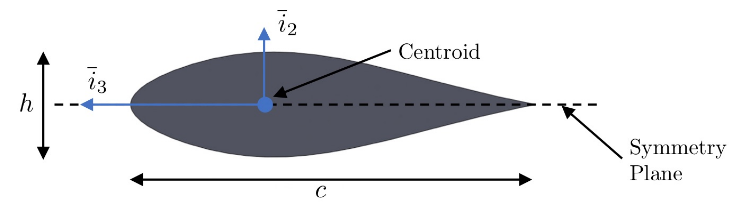
$$u_2 = \frac{H_{23}^c}{\Delta H} P_D \left(\frac{x_1^4}{24} - \frac{x_1^5}{120L} - \frac{L x_1^3}{12} + \frac{L^2 x_1^2}{12} \right) + \frac{H_{22}^c}{\Delta H} P_L \left(\frac{x_1^4}{24} - \frac{x_1^6}{360L^2} - \frac{1}{9} L x_1^3 + \frac{L^2 x_1^2}{8} \right)$$

By inspection

$$u_3 = -\frac{H_{33}^c}{\Delta H} P_D \left(\frac{x_1^4}{24} - \frac{x_1^5}{120L} - \frac{L x_1^3}{12} + \frac{L^2 x_1^2}{12} \right) - \frac{H_{23}^c}{\Delta H} P_L \left(\frac{x_1^4}{24} - \frac{x_1^6}{360L^2} - \frac{1}{9} L x_1^3 + \frac{L^2 x_1^2}{8} \right)$$

$$u_1 = -x_3 \frac{du_3}{dx_1} - x_2 \frac{du_2}{dx_1}$$

PART (c).



For the special case of $P_L/P_D = 1$ and $c/h = 5$, find a good **estimate** for the equation of the neutral axis at the root of the beam ($x_1 = 0$). Sketch the neutral axis.

Hint: You will have to make some estimates here about the stiffnesses of this cross-section. You may also estimate any necessary results that you did not already find in parts a) and b) above.

WE ARE LOOKING FOR $\sigma_1 = 0$ @ $x_1 = 0$.

$$\sigma_1 = E\left(\cancel{\frac{N_1}{S}} + x_3 \frac{H_{33}^c M_2 + \cancel{H_{23}^c} M_3}{\Delta H} - x_2 \frac{\cancel{H_{23}^c} M_2 + H_{22}^c M_3}{\Delta H} \right)$$

SINCE THE AIRFOIL IS SYMMETRIC,
$H_{23}^c = 0$

$$\sigma_1 = E\left(\frac{M_2}{H_{22}^c} x_3 - \frac{M_3}{H_{33}^c} x_2 \right)$$

NOW WE ESTIMATE $M_2, M_3, H_{22}^c, H_{33}^c$
(ASSUME WE DID NOT SOLVE (a) or (b))

$$M_3 \sim P_L \cdot L^2, \quad M_2 \sim P_D L^2$$

$$H_{22}^c \sim E h c^3, \quad H_{33}^c \sim E c h^3,$$

PLUG IN

$$\sigma_1 = E\left(\frac{P_D L^2}{E h c^3} x_3 - \frac{P_L L^2}{E h^3 c} x_2\right)$$

$$\sigma_1 = \frac{P_L L^2}{h^3 c}\underbrace{\left(\frac{P_D}{P_L}\left(\frac{h}{c}\right)^2 x_3 - x_2\right)}_{=0} = 0$$

The neutral axis is given by

$$* \quad x_2 = \frac{P_L}{P_D}\left(\frac{h}{c}\right)^2 x_3$$

Using the given numbers

$$* \quad x_3 = \frac{1}{25} x_2$$