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Airy stress potential (cont) $\rightarrow (\phi)$

We can see which constraints are imposed to $\phi(x_1, x_2)$ by the compatibility equations.

Let's use Hooke's Law:

$$\varepsilon_{\alpha\beta} = \frac{-\nu^*}{E^*}\sigma_{\gamma\gamma}\,\delta_{\alpha\beta} + \frac{1+\nu^*}{E^*}\sigma_{\alpha\beta} \qquad (4)$$

where $\nu^* = \nu$ and $E^* = E$ for plane stress

and $\nu^* = \dfrac{\nu}{1-\nu}$ and $E^* = \dfrac{E}{1-\nu^2}$ for plane strain

Plugging (4) into (3)

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - \frac{2\,\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

$$= \frac{\partial^2}{\partial x_2^2}\left(-\frac{\nu^*}{E^*}(\sigma_{11}+\sigma_{22}) + \frac{1+\nu^*}{E^*}\sigma_{11}\right)$$

$$+ \frac{\partial^2}{\partial x_1^2}\left(-\frac{\nu^*}{E^*}(\sigma_{11}+\sigma_{22}) + \frac{1+\nu^*}{E^*}\sigma_{22}\right)$$

$$-2\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{1+\nu^*}{E^*}\sigma_{12}\right)$$

Next, we plug (2) into the previous expression, yielding:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

$$= \frac{1}{E^*}\left[\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2}\right] = 0$$

Hence, to satisfy compatibility, ϕ has to satisfy

$$\boxed{\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2}} = 0 \qquad (5)$$

- This is the biharmonic equation
- ϕ is called the Airy stress potential

Remarks

- we have shown that if we define stresses by (2) and that if ϕ satisfies (5) then the stresses satisfy the equilibrium equation and the corresponding strains are compatible

- It can also be shown that any stress field that satisfies the equilibrium equations and corresponds to compatible strains can be represented by a potential ϕ through (2) such that ϕ satisfies (5)

- Hence, any plane problem can be addressed by the Airy method.

The Airy stress function can be generalized to the case of conservative body forces, ie: forces that can be represented as:

$$\rho b_1 = -\frac{\partial \psi}{\partial x_1}, \quad \rho b_2 = -\frac{\partial \psi}{\partial x_2}$$

for some function $\psi(x_1, x_2)$

In this case, (2) become:

$$\boxed{\begin{aligned} \sigma_{11} &= \frac{\partial^2 \phi}{\partial x^2} + \psi \\ \sigma_{22} &= \frac{\partial^2 \phi}{\partial x_1^2} + \psi \\ \sigma_{12} &= -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \end{aligned}} \qquad (6)$$

For this, the compatibility requirement for $\phi(x_1, x_2)$ becomes:

$$\boxed{\frac{\partial^4 \phi}{\partial x_1^4} + \frac{\partial^4 \phi}{\partial x_2^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = (1-\nu^*)\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right)} \quad (7)$$

For problems in polar coordinates, the Airy stress potential is chosen as $\phi(r, \theta)$ s.t.

$$\boxed{\begin{aligned} \sigma_{rr} &= \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \psi \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} + \psi \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \end{aligned}} \quad (8)$$

where $\psi(r, \theta)$ is the potential for the body forces:

$$\boxed{\begin{aligned} \rho b_r &= -\frac{\partial \psi}{\partial r} \\ \rho b_\theta &= -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \end{aligned}} \quad (9)$$

Then, to satisfy compatibility, $\phi(r,\theta)$ must satisfy the biharmonic equation as before in polar coordinates:

$$\Delta^2 \phi = -(-1-\nu^*)\,\Delta\psi; \text{ where } \Delta = \underline{\nabla}^2$$

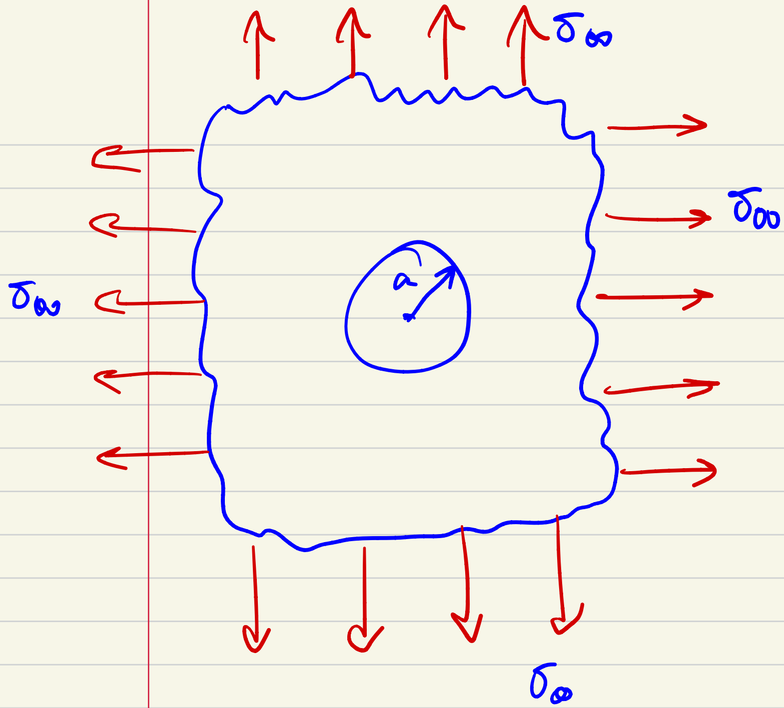
and

$$\Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

and

$$\Delta^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

<u>Note</u>: we will work with a small set of known solutions to this problem.



- Consider an infinite plate w/ a hole of radius "a" subject to uniform hydrostatic stress σ_∞ at infinity
- Let the surface of the hole be traction free
- This problem is straightforward using Airy stress functions.

BC:

- As $r \to \infty$, $\sigma_{rr} = \sigma_{\theta\theta} \to \sigma_\infty$ and $\sigma_{r\theta} \to 0$
- At $r = a$, $\underline{t}^* = \underline{0}$ $\quad \underline{\underline{\sigma}} \cdot \underline{n} = 0$

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\to$ $\therefore$ at $r = a$: $\sigma_{rr} = \sigma_{r\theta} = 0$

Note that the problem is axissymmetric so the solution does not have any dependence on θ

$\Rightarrow$ we look for solutions $\phi(r)$, with no body forces, the equation for ϕ becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi = 0$$

The general solution for this ODE is:

$$\phi(r) = A \ln r + B r^2 \ln r + C r^2 + D$$

Using this definition (8) we can find the stresses:

$$\sigma_{rr} = A\frac{1}{r^2} + B(1 + 2\ln r) + 2C$$

$$\sigma_{\theta\theta} = -A\frac{1}{r^2} + B(3 + 2\ln r) + 2C$$

$$\sigma_{r\theta} = 0$$

To find A, B, C we apply the BCs.

a) as $r \to \infty$, $\sigma_{rr} = \sigma_{\theta\theta} \to \sigma_{00}$; $\sigma_{r\theta} = 0$

The first observation is that stresses remain finite

$\to B = 0$ (since $\ln r \to \infty$ as $r \to \infty$)

$$\begin{cases} \sigma_{rr} = A \frac{1}{r^2} + 2C \\ \sigma_{\theta\theta} = -A \frac{1}{r^2} + 2C \\ \sigma_{r\theta} = 0 \end{cases}$$

As $r \to \infty$, $\sigma_{rr} = \sigma_{\theta\theta} \to 2C \Rightarrow 2C = \sigma_{00} \Rightarrow C = \frac{\sigma_{00}}{2}$

The condition on $\sigma_{r\theta}$ is automatically satisfied

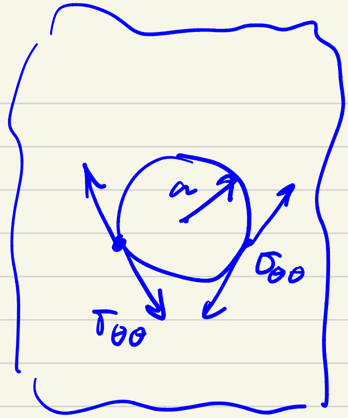
b) At $r = a$, $\sigma_{rr} = \sigma_{r\theta} = 0$:

$$\frac{A}{a^2} + \sigma_{00} = 0 \Rightarrow A = -\sigma_{00} a^2$$

So we obtain:

$$\begin{cases} \sigma_{rr} = \sigma_{00}\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_{\theta\theta} = \sigma_{00}\left(1 + \frac{a^2}{r^2}\right) \\ \sigma_{r\theta} = 0 \end{cases}$$

- In a plate w/o hole, $\sigma_{\theta\theta} = \sigma_{\infty}$ everywhere, but in this case $\sigma_{\theta\theta} = 2\sigma_{\infty}$ at $r = a$

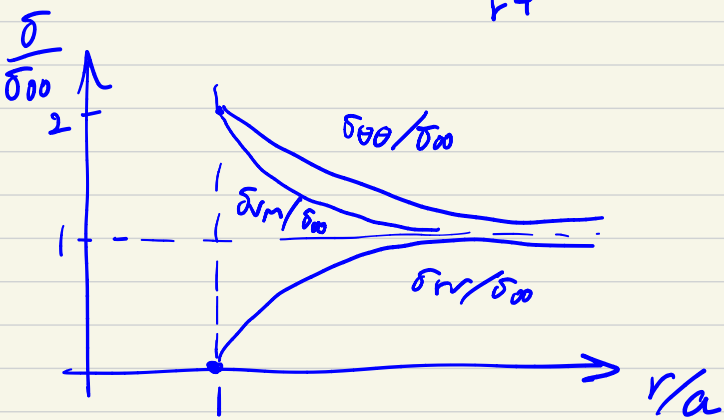


↳ stress concentration

Von Misses:

$$\sigma_{VM} = \sqrt{\sigma_{rr}^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2 + 3\sigma_{r\theta}^2}$$

$$= \sigma_{\infty}\sqrt{1 + \frac{3a^4}{r^4}}$$



Remarks

SCF

(1) The stress concentration factor for this problem is 2.

Each point along the hole experiences this stress concentration on the hoop stress $\sigma_{\theta\theta}$

As a result, we expect that the plate will fail at the hole.

(2) Note that in this problem the stress concentration factor does not depend on the size of the hole! Any circular hole will result in a SCF of value 2.

(3) The stress state does not depend on material properties

The same SCF would arise for any isotropic elastic material.