Kinematics

(cont)

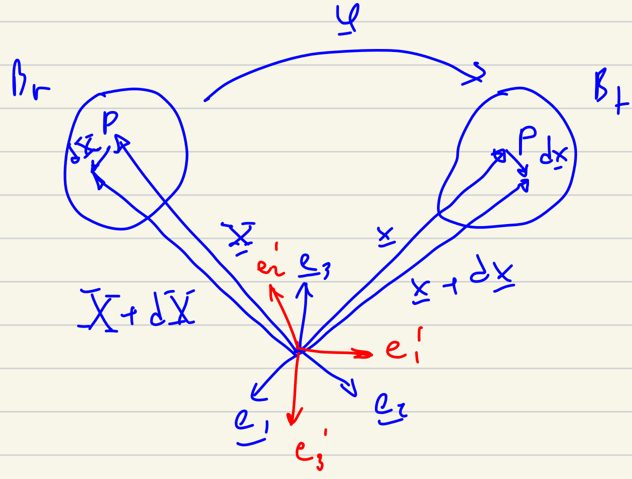
Kinematics of local deformation (cont)

Let us assume we use the same basis for the reference and deformed configurations

$dx_i = F_{ij} \, dX_j$ (dependence on X and t is assumed but dropped in notation for simplicity) $*$

where basis vectors are $\{\underline{e}_i\}$. In the previous expression dx_i and dX_j are the components of $d\underline{x}$ & $d\underline{X}$ in terms of $\{e_i\}$ respectively.

Let us express both of them in terms of the rotated frame $\{\underline{e}_i'\}$



$dx_i = F_{ij} \, dX_j$

$(*)$

$$dx'_j = \ell_{jp}\, dx_p$$

$$\Rightarrow (\ell^T)_{ij}\, dx'_j = \underbrace{(\ell^T)_{ij}\,\ell_{jp}}_{\delta_{ip}}\, dx_p$$

$$\Rightarrow dx_i = (\ell^T)_{ij}\, dx'_j$$

$$dX'_k = \ell_{kq}\, dX_q \Rightarrow \underbrace{(\ell^T)_{jk}\,\ell_{kq}}_{\delta_{jq}}\, dX_q$$

$$\Rightarrow dX_j = (\ell^T)_{jk}\, dX'_k$$

Replacing into (*): $(\ell^T)_{ij}\, dx'_j = F_{ij}\, (\ell^T)_{jk}\, dX'_k$

Multiplying by ℓ_{pi}: $\underbrace{\ell_{pi}\,(\ell^T)_{ij}}_{\delta_{pj}}\, dx'_j = \ell_{pi}\, \underbrace{(\ell^T)_{jk}}_{\ell_{kj}}\, F_{ij}\, dX'_k$

Thus, we get $\Rightarrow dx'_p = \underbrace{\ell_{pi}\,\ell_{kj}\,F_{ij}}_{F'_{pk}}\, dX'_k$

by definition of deformation gradient

Thus, $\underbrace{F'_{pk} = l_{pi} \, l_{kj} \, F_{ij}}$

The previous expression shows that the deformation gradient is a rank 2 Tensor!

$$\boxed{\underline{\underline{F}} = F_{ij} \, \underline{e}_i \, \underline{e}_j = F'_{pk} \, \underline{e}'_p \, \underline{e}'_k ; \quad \overset{\text{Related by}}{F'_{pk} = l_{pi} \, l_{kj} \, F_{ij}}}$$

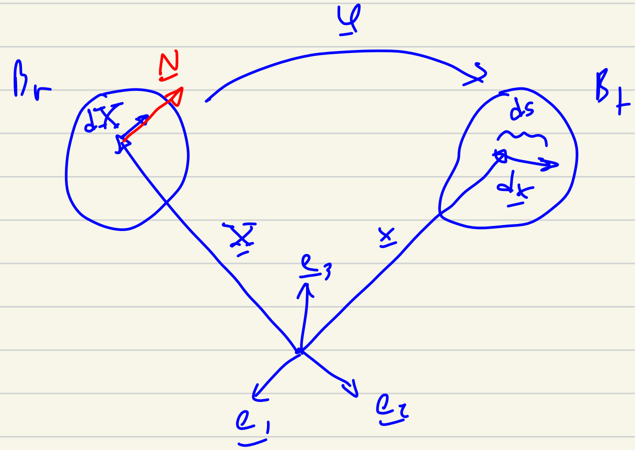
Metric Changes.

The deformation gradient encodes all needed information about deformation of infinitesimal regions around a point.

We wish to extract from it information about changes of shape of these infinitesimal neighborhood.

(ex: change of length and angles, etc)

Change in length.



$dx_i = F_{ij}\, dX_j$

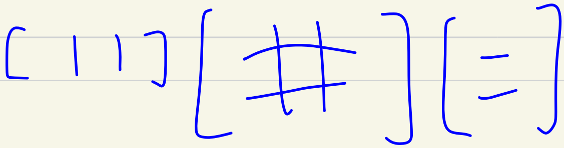
$ds^2 = dx_i\, dx_i$ (ds scalar length)

$= F_{ij}\, dX_j\, F_{ik}\, dX_k$

$= F_{ij} F_{ik}\, dX_j\, dX_k$

we define $C_{jk} = F_{ij} F_{ik}$ $\left(= F^T F\right)$ as the components of the Right Cauchy-Green deformation tensor.

$$\Rightarrow \boxed{ds^2 = C_{jk}\, dX_j\, dX_k}$$



The stretch ratio (λ) of $d\underline{X}$ is the ratio of the deformed length ds to the undeformed length $dS = \| d\underline{X} \|$

$$\lambda = \frac{ds}{dS} \Rightarrow \lambda^2 = \frac{(ds)^2}{(dS)^2} = \frac{C_{jk}\, dX_j\, dX_k}{\| d\underline{X} \|^2}$$

On the other hand, the unit vector in the direction of $d\underline{X}$ is:

$$\underline{N} = \frac{d\underline{X}}{\| d\underline{X} \|} \Rightarrow N_i = \frac{dX_i}{\| d\underline{X} \|}$$

which implies: $$\boxed{\lambda(\underline{N}) = \sqrt{C_{jk}\, N_j\, N_k}}$$

$$N^T C N$$