Fall 2020 Take Home Exam 2

Due by NLT November 20, 23:55 EDT

AE 6343: Aircraft Design I
Fall 2020

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AE6343 Aircraft Design I
Fall 2020 Take Home Exam 2

Nov 13 - Nov 20, 2020

Time: Take Home
Number of questions: 4
Total number of points: 100

Exam Rules

- Read the Honor Code Statement, print your name, sign it (electronic signature is ok), and date it.
- Students have until **Friday, November 20th,** to complete it and submit all relevant information via Canvas (no later than 11:55PM EDT).
- Using word or Latex, write a report with your answers to these questions. Make sure that your report follows standard formatting guidelines, with elements such as figure and table captions, page numbers, justified margins, equations numbers, etc. Then, convert this document into a PDF for the Canvas submission.
- The questions in this exam have multiple parts, each one with some guiding questions. Make sure to, at least, address all of these questions in your answer.
- Please use the following naming convention for your file:

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- Late submissions will be allowed with a 10% penalty on the final grade for each day of delay. No exceptions.
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3 Problem 1: Thrust in steady, level flight

Problem 1 considers an aircraft in steady level flight at airspeed V. This aircraft has a parabolic drag polar given below.

$$C_D = C_{D0} + K{C_L}^2 \tag{1}$$

3.1 Problem 1A

Derive the following expression for thrust required (Tr). Note that W is the weight of the aircraft and V0 is the airspeed at minimum drag.

$$T_r = W\sqrt{C_{D0}K}\left[\left(\frac{V}{V_0}\right)^2 + \left(\frac{V}{V_0}\right)^{-2}\right] \tag{2}$$

Let us first start with a general free body diagram for an aircraft in flight.

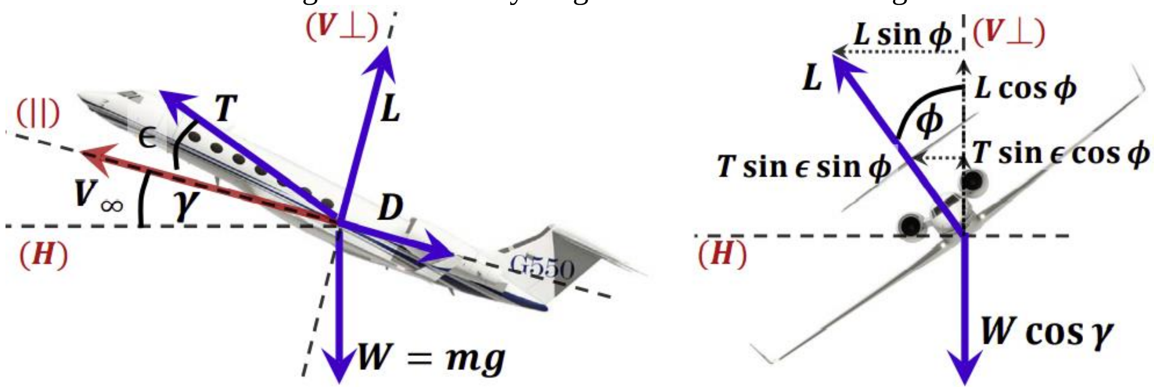


Figure 1: Free body diagram of aircraft in flight[1]

From this FBD, we can write equations for each of the three axes (parallel to flight path, normal to flight path, and orthogonal to the first two).

$$m\frac{dV_\infty}{dt} = T\cos\epsilon - D - W\sin\gamma \tag{3}$$

$$m\frac{{V_\infty}^2}{R_v} = L\cos\phi + T\sin\epsilon\cos\phi - W\cos\gamma \tag{4}$$

$$m\frac{(V_\infty\cos\gamma)^2}{R_H} = L\sin\phi + T\sin\epsilon\sin\phi \tag{5}$$

In the case of steady level flight, the following applies:

- dV/dt= 0, constant speed
- $\epsilon = \gamma = \phi = 0$

This allows us to simplify the above equations to the two conditions below.

$$T = D \tag{6}$$

$$L = W \tag{7}$$

Now we expand lift and drag to their forms with the non-dimensional coefficients.

$$L = 0.5 * \rho V^2 S C_L; \quad L_0 = 0.5 * \rho {V_0}^2 S C_{L0}; \tag{8}$$

$$D = 0.5 * \rho V^2 S C_D; \quad D_0 = 0.5 * \rho {V_0}^2 S C_{D0}; \tag{9}$$

Since we are considering a parabolic drag polar, it is useful to have an expression for CL. Recall that under this flight condition, lift is equal to weight. Then re-arranging (8) gives:

$$C_L = \frac{2W}{\rho V^2 S} \tag{10}$$

With the relevant expressions for lift and drag, we can then substitute into Equation (6).

$$T = 0.5 * \rho V^2 S C_D \tag{11}$$

Since we are given a quadratic drag polar, the equation above becomes

$$T = 0.5 * \rho V^2 S [C_{D0} + K {C_L}^2] \tag{12}$$

Then substitute the expression for CL in Equation (10).

$$T = 0.5 * \rho V^2 S \left[C_{D0} + K \left(\frac{2W}{\rho V^2 S} \right)^2 \right] \tag{13}$$

Multiplying out, then re-arranging terms yields

$$T = (0.5 \rho S C_{D0}) V^2 + (2K \frac{W^2}{\rho S}) \frac{1}{V^2} \tag{14}$$

This form is similar to the requested expression. However, we need to account for the velocity at minimum drag, so the above process is repeated using V0. This then leads to the following form (after algebraic manipulation).

$$T^2 = (0.5 \rho S C_{D0}) \left(\frac{V}{V_0} \right)^2 + (2K \frac{W^2}{\rho S}) \left(\frac{V}{V_0} \right)^{-2} = C_{D0} K W^2 [\left(\frac{V}{V_0} \right)^2 + \left(\frac{V}{V_0} \right)^{-2}]^2 \tag{15}$$

Further algebraic manipulation then leads us to the requisite form.

$$T_r = W \sqrt{C_{D0} K} \left[\left(\frac{V}{V_0} \right)^2 + \left(\frac{V}{V_0} \right)^{-2} \right] \tag{16}$$

3.2 **Problem 1B**

The second part of problem 1 asks us to derive the following expression where r is the ratio of the airspeed to the airspeed at minimum drag.

$$\frac{T_r}{T_{r,min}} = 0.5[r^2 + r^{-2}] \tag{17}$$

Let us first start from the expression for thrust in Equation (14), duplicated below.

$$T = (0.5\rho S C_{D0})V^2 + (2K\frac{W^2}{\rho S})\frac{1}{V^2} \tag{18}$$

The minimum thrust required with respect to flight velocity will occur when the derivative of the above expression is equal to zero. This is shown below.

$$\frac{dT}{dv} = (\rho S C_{D0})V - (4K\frac{W^2}{\rho S})\frac{1}{V^3} = 0 \tag{19}$$

The velocity at which the minimum thrust occurs at is then as follows.

$$V_{Tmin} = \left[\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D0}}}\right]^{1/2} \tag{20}$$

Substituting this velocity back into the thrust expression in Equation (18):

$$T_{r,min} == (0.5\rho S C_{D0})\left[\left[\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D0}}}\right]^{1/2}\right]^2 + \left(2K\frac{W^2}{\rho S}\right)\frac{1}{\left[\left[\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D0}}}\right]^{1/2}\right]^2} = 2W\sqrt{KC_{D0}} \tag{21}$$

Now take the ratio of the provided expression in part 1 with respect to the above expression. Additionally, substitute in for r (ratio of V to V0):

$$\frac{T_r}{T_{r,min}} = \frac{W\sqrt{C_{D0}K}}{2W\sqrt{KC_{D0}}}[r^2 + r^{-2}] \tag{22}$$

This then simplifies to the desired expression.

$$\frac{T_r}{T_{r,min}} = \frac{1}{2}[r^2 + r^{-2}] \tag{23}$$

4 Problem 2: Aerobatic pilot

First let us consider the free body diagram of the aircraft in the inside loop maneuver.

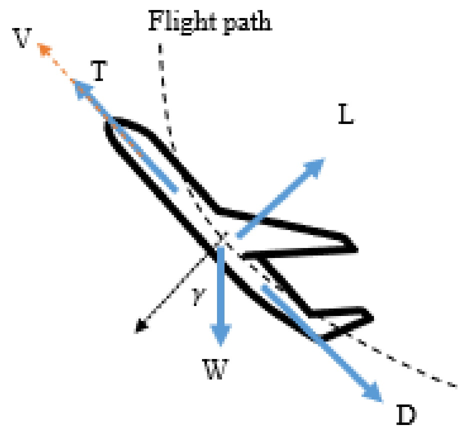


Figure 2: Free body diagram of aircraft along inside loop maneuver flight path

Using Newton's Second Law (F=m*a), two equations of motions can be derived from this FBD. The first equation (24) parallel to the flight path (perpendicular to radius). The second equation (25) can be found by applying the definition of centripetal acceleration (V^2/R) and is perpendicular to the flight path (pointing inwards to the center of the loop).

$$m\frac{dV}{dt} = T - D - W\sin\gamma \quad \textbf{(24)}$$

$$m\frac{V^2}{R} = L - W\cos\gamma \quad \textbf{(25)}$$

Recall that the load factor n is simply the ratio of lift to weight. Additionally, weight is mass times gravitation acceleration, g. Substituting these into Equation (25) yields the following:

$$n = \cos\gamma + \frac{V^2}{Rg} \quad \textbf{(26)}$$

The exam question says to assume that the maneuver is circular, and that velocity is constant. Therefore, we can infer the following:

- dV/dt= 0, constant speed
- R= constant value (since the flight path is a circle, which has a constant radius)

Therefore, the load factor depends primarily on the flight path angle γ.

In order to specifically determine the load factor at specific points during the inside loop maneuver, it is useful to identify several aircraft positions. This is shown in the diagram below and summarized in the subsequent table.

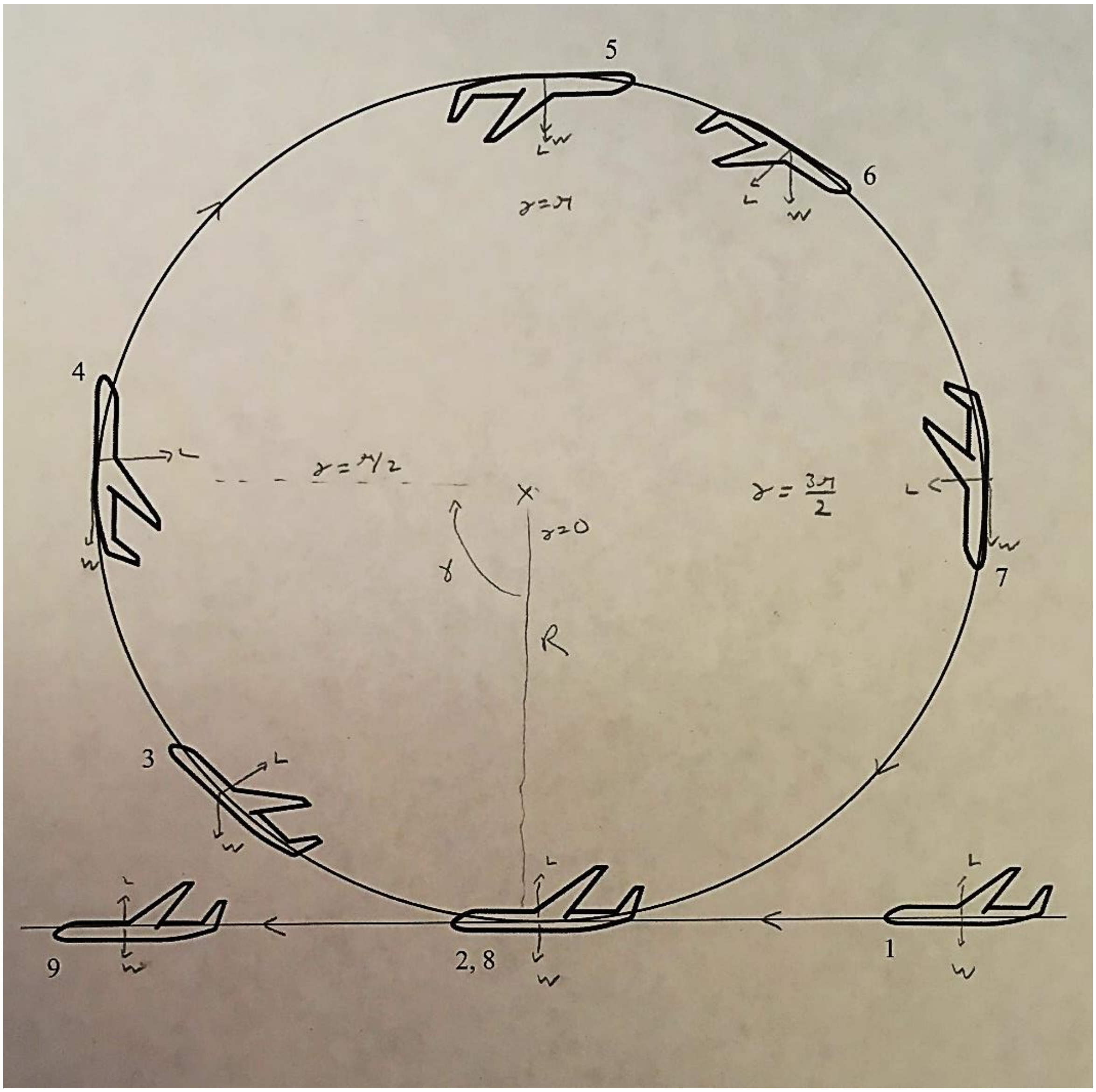


Figure 3: Aircraft positions for inside loop maneuver

Table 1: Description of aircraft positions for inside loop maneuver

Position	Flight Path angle, γ	Description
1	0	Level flight
2	0	Instantaneously level flight, entering loop
3	$0 < \gamma < \pi/2$	Pull up
4	$\pi/2$	Vertical flight (up)
5	π	Instantaneously level inverted flight (upside down)
6	$\pi < \gamma < 3\pi/2$	Pulldown/pull over
7	$3\pi/2$	Vertical flight (down)
8	0 or 2π	Instantaneously level flight, exiting loop
9	0	Level flight

As a reminder, in level flight, the load factor is 1. Inverted level flight has a load factor of -1 since lift is generated downwards. Following the logic detailed in Equation (26), the load factor will follow a cosine curve offset by some value (V^2/Rg).

Table 2: Description of aircraft positions for inside loop maneuver

Position	Flight Path angle, γ	Load factor, n	Notes
1	0	1	n= 1 in instantaneously level
2	0	1 +(V^2/Rg)	n= 1 in instantaneously level, but considering the offset this will be the maximum n during the loop (pull up)
3	$0< \gamma < \pi/2$	$\cos\gamma$ + (V^2/Rg)	$\cos\gamma$ begins to decrease from 1 (going vertical)
4	$\pi/2$	0 + (V^2/Rg)	n= offset amount (vertical up)
5	π	-1 +(V^2/Rg)	n= -1 in instantaneously inverted level, but considering the offset this will be the minimum n during the loop
6	$\pi< \gamma < 3\pi/2$	$\cos\gamma$ + (V^2/Rg)	n begins to increase from its min value towards zero (pull down)
7	$3\pi/2$	0 + (V^2/Rg)	n= offset amount (vertical downwards)
8	0 or 2π	1 +(V^2/Rg)	Same note as 2 (exiting loop)
9	0	1	Same note as 1

Essentially, the load factor has a maximum value of 1+(V^2/Rg) at the bottom of the loop and a minimum value of -1+(V^2/Rg) at the top of the loop. If you have ridden a roller coaster, you will intuitively know this as you feel weightless at the top of a loop and crushed into your seat at the bottom.

Other key takeaways include the impact of velocity and loop radius. Again, looking at Equation (26), note that increasing the velocity increases the offset from the cosine term. In other words, the maximum load factor will increase. Similarly, decreasing the loop radius also increases the offset term. Simply put, flying faster or at a tighter radius during an inside loop maneuver will increase the load factor.

Continuing this process, the radius and instantaneous pitch rates can be determined. Consider first the pull-up maneuver detailed in Figure 2. Since the flight path angle is zero, the starting load factor can be found from Equation (26), which is then solved for the radius.

$$n = 1 + \frac{V^2}{Rg} \quad \textbf{(27)}$$

$$R = \frac{V^2}{g(n-1)} \quad \textbf{(28)}$$

The instantaneous pitch rate is then just the derivative of the flight path angle with respect to time, as seen below.

$$\omega = \frac{d\gamma}{dt} = \frac{V}{R} = \frac{g(n-1)}{V} \tag{29}$$

The same process can be also repeated for the pull-down portion and vertical portions. Level turn has also been included as a point of comparison

Table 3: Description of aircraft positions for inside loop maneuver

Case	Radius, R	Instantaneous Pitch Rate, ω
Pull up	$\frac{V^2}{g(n-1)}$	$\frac{g(n-1)}{V}$
Pull down	$\frac{V^2}{g(n+1)}$	$\frac{g(n+1)}{V}$
Vertical	$\frac{V^2}{g(n)}$	$\frac{g(n)}{V}$
Level turn	$\frac{V^2}{g\sqrt{n^2-1}}$	$\frac{g\sqrt{n^2-1}}{V}$

Interestingly for cases where n is sufficiently large, all three cases simplify as follows:

$$R \approx \frac{V^2}{gn}; \quad \omega \approx \frac{gn}{v} \tag{30}$$

Furthermore, knowing the definition of lift and its relation to the load factor (L=nW), we can derive the following:

$$R_{min} = \frac{2}{\rho g}\frac{W}{S}\frac{1}{C_{Lmax}} \tag{31}$$

$$\omega_{maw} = g\sqrt{\frac{\rho}{2}\frac{C_{Lmax}n_{max}}{W/S}} \tag{32}$$

In other words, given the aerodynamics of the aircraft (CL_max) and the structural parameters (W/S, n_max), we can determine maneuvering characteristics of the aircraft.

5 Problem 3: V-n diagram

5.1 Draw V-n diagram

The V-n diagram essentially shows the "envelope" or region in the V-n space where the aircraft can fly or maneuver and where it cannot or should not. Note that the V-n diagram uses equivalent airspeed (EAS) so that the diagram is altitude independent. Equivalent airspeed can be converted to true airspeed (TAS) via the relation below.

$$V_{TAS} = V_{EAS}\sqrt{\rho_{SL}/\rho} \quad \textbf{(33)}$$

The V-n diagram is constructed using the stall limits and structural limits, seen below. Note that diagram is not symmetric about the horizontal axis as most aircraft are primarily designed to fly in an upright fashion (negative limit load factor has smaller magnitude than positive limit load factor).

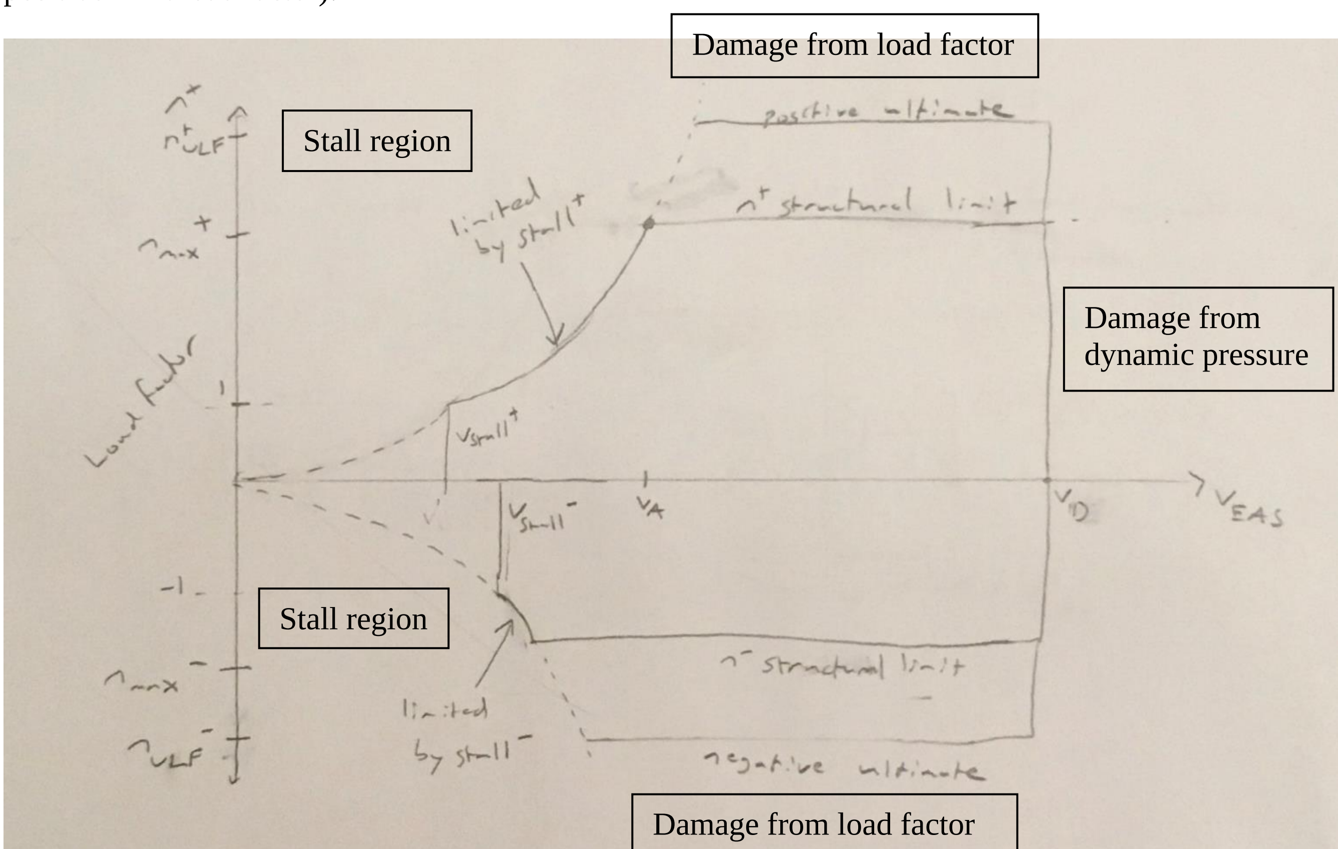


Figure 4: Notional V-n diagram

Horizontal lines: The horizontal bounds are structural limits set by n^+ and n^-. If the aircraft has a load factor outside of these bounds (i.e. outside the range $n^+ < n < n^-$), it is considered unsafe operation. Note that these load factors are called limit load factors and the structure must withstand the associated limit loads without permanent deformation.

The ultimate load factor (ULF) is simply the limit load factor multiplied by a factor of safety. The aircraft should not fly past these limits. If they do, the structure must withstand the associated ultimate loads for at least three seconds without failure.

Stall curves: The aircraft is also limited by the stall speed. If the aircraft flies as a speed less than the stall speed, the lift generated is less than the weight and the aircraft will not maintain level flight. This is why at the stall speed where lift is exactly equal to weight (n=1), a vertical line is drawn.

To find an expression for the stall load factor, recall that at the point of stall, lift equals weight. Combining this with the definition of lift and the definition of the load factor, the following expression is derived.

$$n = L/W; \quad L_{stall} = 0.5\rho S(V_{TAS})^2 C_{Lmax} \tag{34}$$

$$n_{max} = 0.5\rho_{SL}(V_{EAS})^2 \frac{C_{Lmax}}{W/S} \tag{35}$$

Corner point: The stall curve meets the structural curve at a corner point. This location is called the maneuver or corner speed. This is the speed for maximum instantaneous turn performance (max turn rate, min turn radius) without exceeding structural limits.

Since the load factor is known to be n^+ at this point, by applying the same logic as above, we can derive the corner speed (V_A).

$$V_A = \sqrt{\frac{2}{\rho_{SL}}\frac{W}{S}\frac{n^+}{C_{Lmax}}} \tag{36}$$

Right side boundary: The right side of the V-n diagram is set by dynamic pressure limits. This speed is called dive speed (V_D) or sometimes never exceed speed (V_{NE}). Flying at speeds greater than this means that the aerodynamic loads will cause damage to the structure.

There are several key take-aways from this section. Firstly, decreasing aircraft weight will reduce the maneuver speed as seen in Equation (36).

Secondly, increasing altitude will increase the corner speed. This is because density decreases with altitude, the relation between EAS and TAS is detailed in Equation (33). In other words, you need to fly faster at higher altitudes for the same "corner" performance.

Lastly, the corner point is significant in that below this point the aircraft will stall before causing structural damage. In other words, aerodynamic loads from maneuvers below this point cannot cause structural damage.

5.2 Draw how load factor changes for inside loop

This section follows the numbering format from Table 2 and the maneuver can be seen visually in Figure 3. The following assumptions are made:

- dV/dt= 0, constant speed as indicated in the Problem 2
- R= constant value (since the flight path is a circle, which has a constant radius)
- The value of V^2/Rg is less than one for convenience of calculation

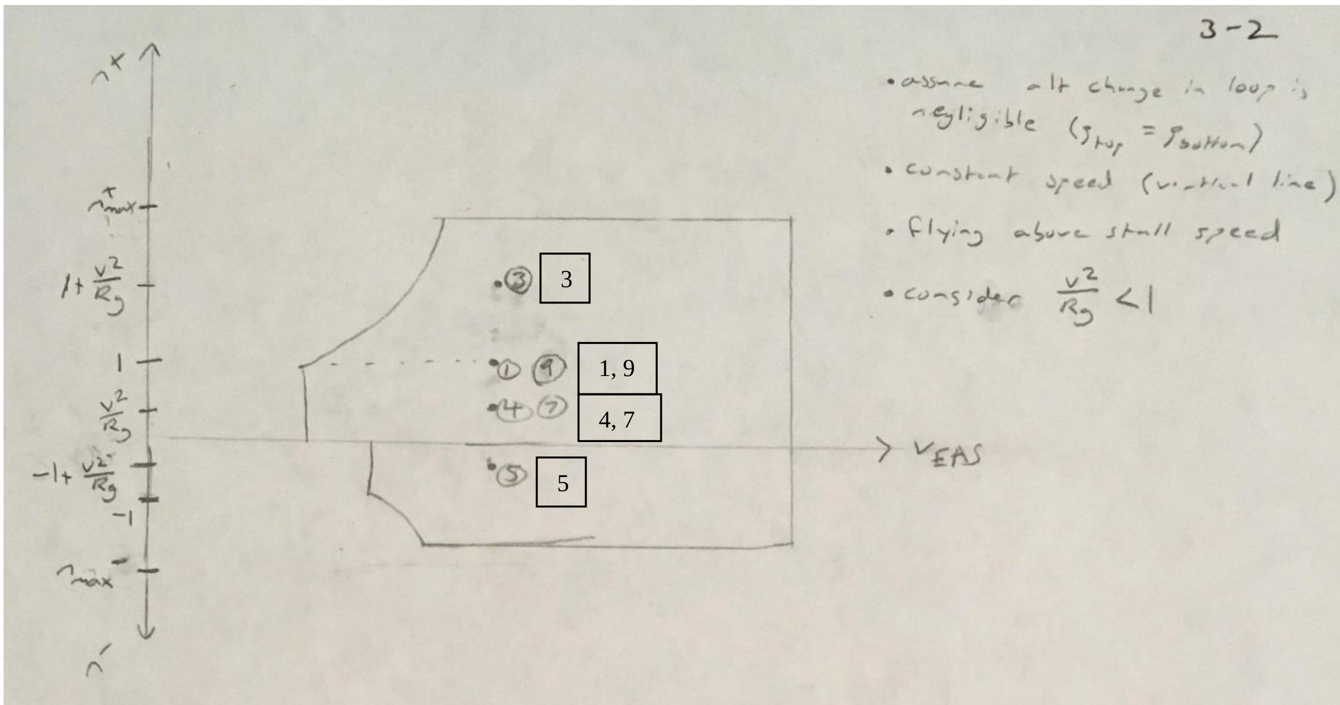


Figure 5: V-n diagram, idealized inside loop maneuver

The process for this is as follows:

1. Aircraft is flying in level flight at a position #1.
   a. Flight velocity is higher than stall speed (common safety precautions against gusts).
   b. n=1 since lift is equal to weight.
2. Start the maneuver at position #3.
   a. The instantaneous flight path angle is 0 so the cosine term is 1.
   b. Therefore, the load factor is $1+ V^2/Rg$ which is greater than 1.
   c. This is the largest load factor during the maneuver (same position as exiting loop).
3. Vertical flight upwards at position #4.
   a. Flight path angle is $\boldsymbol{\pi/2}$ making the cosine term equal to zero.
   b. The load factor is then just V^2/Rg which was set to be less than 1.
4. Inverted flight at position #5.
   a. Flight path angle is $\boldsymbol{\pi}$ making the cosine term equal to -1.
   b. The load factor is then $-1+ V^2/Rg$
   c. V^2/Rg was set to be less than 1 so the load factor is negative.
   d. This is the minimum load factor in the maneuver.

5. Vertical flight downwards at position #7
    a. Flight path angle is $3\boldsymbol{\pi}/\mathbf{2}$ making the cosine term equal to zero.
    b. The load factor is then just V^2/Rg which was set to be less than 1.
6. Return to level flight at position #9
    a. Return to original velocity
    b. Lift is again equal to weight so load factor is 1.

In actuality, it is unlikely that an inside loop would be carried out at constant speed. Position #4 (vertical flight upwards) would likely be less than the level flight speed since thrust is directly countering weight rather than drag. Conversely, the speed at position #7 (vertical flight downwards) would likely be higher since weight is acting in the same direction as thrust. This would mean that the points are not on a single line- #4 should be to the left (lower speed) and #7 should be to the right (higher speed).

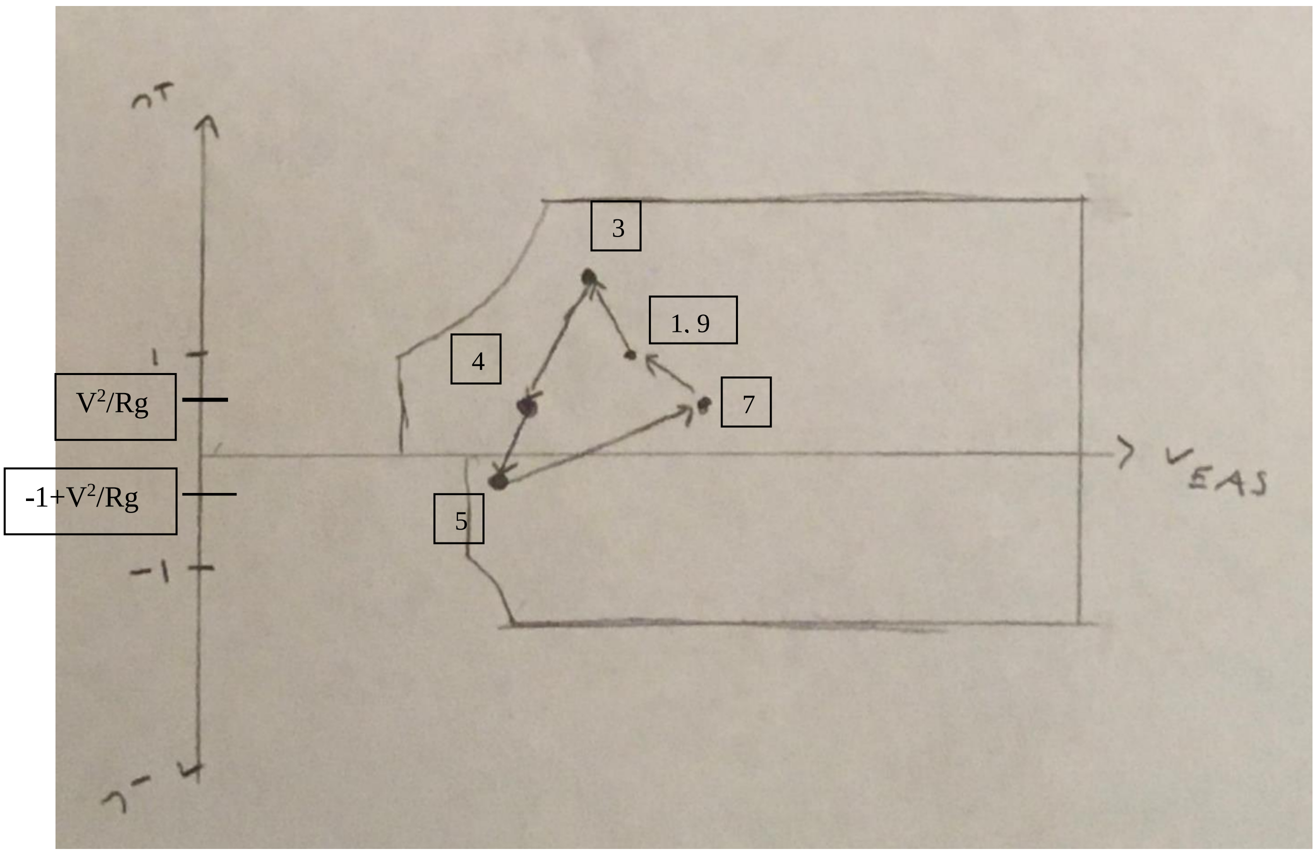


Figure 6: V-n diagram, speed consideration for inside loop maneuver

5.3 Consider a bomber aircraft (before/after payload)

After a bomber aircraft drops its payload, the largest change is that of aircraft weight which is reduced significantly. If this payload were attached to the wings, there would also be in increase in the lift producing potential of the wing and related surfaces. However, the following explanations only assume a decrease in aircraft weight.

Firstly, consider the structural limits n^+ and n^-, which are fixed values. As a reminder, the load factor is the ratio of lift to weight. It is assumed lift has not changed, but weight has decreased. Basically, for the same lift, load factor is higher since weight has been reduced. It is important to know that the sudden decrease in payload weight will increase the load factor so pilots must be account for this to avoid hitting the structural limit after the drop.

However, the structural limits are values fixed by the airframe, so unless they are set differently for the no-payload aircraft, they do not change.

Now consider the stall boundary, which is defined in Equation (35) modified below.

$$n_{max} = 0.5\rho_{SL}(V_{EAS})^2 C_{Lmax}\frac{S}{W} \quad \textbf{(37)}$$

Since weight has decreased, this curve should become steeper (all else being equal). Additionally, the stall speed decreases since weight has decreased. This can also be found via taking the ratio of the lift equation for the two weights but is summarized below.

$$\sqrt{\frac{new\ weight}{known\ weight}} \times (known\ V_s) = new\ V_s \quad \textbf{(38)}$$

It is also important to look at the corner speed. As mentioned in 5.1, a decrease in weight reduces the maneuver speed, as seen in Equation (36) which is duplicated below.

$$V_A = \sqrt{\frac{2}{\rho_{SL}}\frac{W}{S}\frac{n^+}{C_{Lmax}}} \quad \textbf{(39)}$$

Therefore, the changes after dropping the payload include a lower maneuver speed and a steeper stall boundary curve.

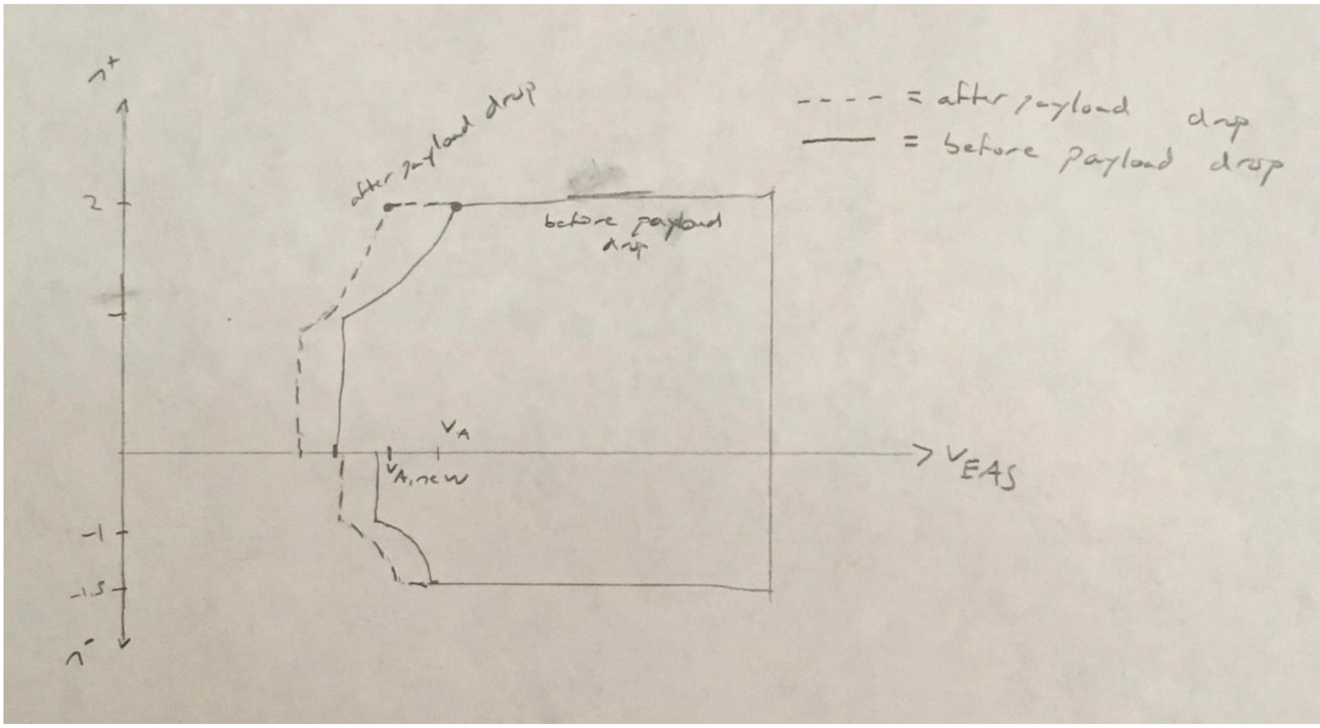


Figure 7: V-n diagram, bomber aircraft

6 Problem 4: Propulsion Cycle & Efficiency

6.1 Propulsive efficiency expression (turbojet)

Propulsive efficiency is defined by the following expression

$$\eta_p = \frac{Useful\ power\ available}{Total\ power\ generated} \tag{40}$$

Let us start first with useful power available. The purpose of an engine is to produce thrust. The air experience a force opposite to the thrust in occurrence to Newton's third law. Specifically, the air is accelerated to from the freestream velocity to an exit velocity Vj.

Neglecting the pressure forces on the control volume (selected to be around the engine), we can then apply Newton's second law to determine the thrust. Recall that this law states that the force on an object is equal to the time rate of change of momentum of that object.

In this instance, the object is the air. Recalling that momentum is mass times velocity, we can apply the time rate of change concept to yield a mass flow rate times a velocity difference across the engine, as seen below.

$$T = (\dot{m}_a + \dot{m}_f)V_j - \dot{m}_a V_\infty \tag{41}$$

If we assume the mass flow rate of fuel is much smaller than the mass flow rate of the air (i.e. the fuel to air ratio, FAR << 1), then thrust simplifies to just the air mass flow rate times the velocity difference across the engine.

$$T = \dot{m}_a (V_j - V_\infty) \tag{42}$$

Power is simply force times velocity so multiplying by the free stream yields the useful power available.

$$useful\ power\ available = TV_\infty = \dot{m}_a (V_j - V_\infty) V_\infty \tag{43}$$

Now we consider the total power generated, which is the sum of the useful power available and the wasted energy. In this case we assume that energy is only "wasted" via kinetic means, i.e. the "wasted" energy is imparted to the air as shown by a kinetic energy difference. Note that power is energy per unit time, so we consider mass flow rate instead of just mass:

$$power\ "wasted" = \frac{(\dot{m}_a + \dot{m}_f){V_j}^2}{2} - \frac{(\dot{m}_a){V_\infty}^2}{2} \tag{44}$$

Again, we assume the mass flow rate of fuel is much smaller than the mass flow rate of the air (i.e. the fuel to air ratio, FAR << 1) reducing the above expression.

$$power\ "wasted" = \frac{(\dot{m}_a)(V_j - V_\infty)^2}{2} \tag{45}$$

Substituting Equations (43) and (45) into (40) yields the following

$$\eta_p = \frac{\dot{m}_a (V_j - V_\infty) V_\infty}{\dot{m}_a (V_j - V_\infty) V_\infty + \frac{(\dot{m}_a)(V_j - V_\infty)^2}{2}} \quad \textbf{(46)}$$

Then through algebraic manipulation, the above expression is simplified.

$$\eta_p = \frac{1}{1 + \frac{(V_j - V_\infty)}{2}} = \frac{2}{1 + \frac{V_j}{V_\infty}} \quad \textbf{(47)}$$

As a reminder, this expression assumes no pressure forces and that the fuel mass flow rate is significantly less than the air mass flow rate.

A key thing to note from this is that propulsive efficiency is maximized when $V_j = V_\infty$ as seen in Equation (47). However, if this is the case, then no thrust is produced as seen in Equation (42). Essentially to produce the most thrust without sacrificing propulsive efficiency, you want to have a large airmass flow rate with minimum velocity change (indeed this is what propellors and fans do).

6.2 Propulsive efficiency tradeoffs

As discussed previously, propulsive efficiency can be thought of as the ratio between useful power available and the total power generated. This term is dependent on the term $\frac{V_j}{V_\infty}$ and reaches its maximum when $V_j = V_\infty$.

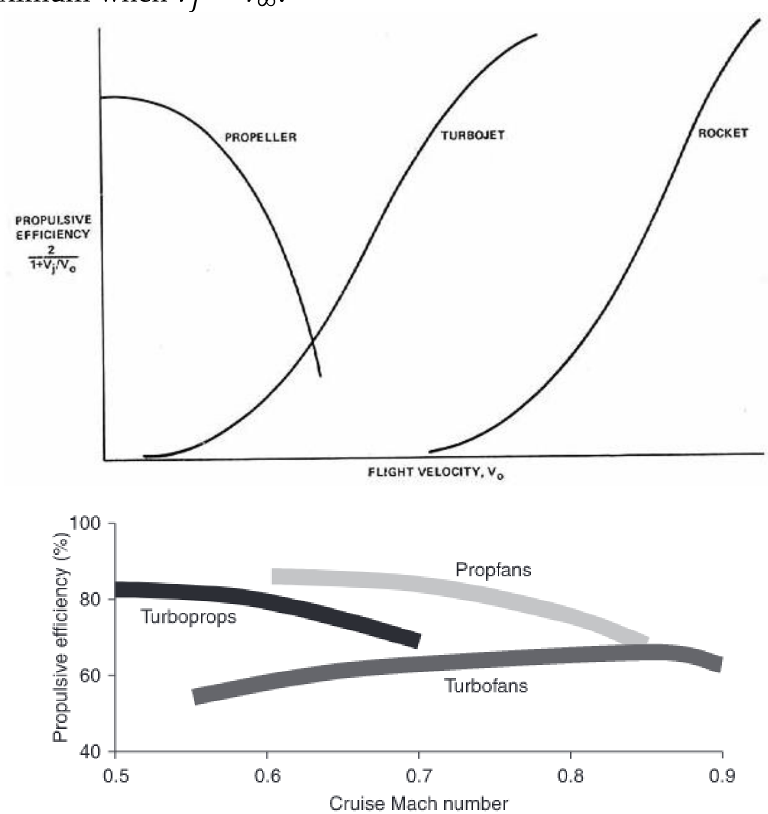


Figure 8: Propulsive efficiency for different types of propulsion systems[2][3]

Propulsive efficiency is heavily influenced by flight velocity as the exit velocity of the engine is a limitation of its components (you can only make it go so fast). Therefore, you see a shift in architectures as speeds increase in order to maintain propulsive efficiency. However, there are still other considerations when selecting a propulsion system type. At a given design flight velocity a turbojet might have a higher propulsive efficiency, but a propellor might be selected due to a lower weight and maintenance cost.

As discussed earlier, to produce the most thrust without sacrificing propulsive efficiency, you want to have a large airmass flow rate with minimum velocity change. This is what fans and propellors accomplish- they accelerate a large capture area of air by a small velocity increment. However, at higher speeds the large capture area produces a large amount of ram drag. Therefore, higher speed aircraft tend to shift more towards turbojets.

As a side note, propellers are limited to low flight speeds by compressibility effects since a larger diameter propeller has a high tip Mach number. By ducting the propeller (like in the case of a turbofan), the flow entering the fan can be diffused. This in turn delays the compressibility effects to much higher flight speeds.

However, reaching ultimate propulsive efficiency is not desirable. Propulsive efficiency is maximized when $V_j = V_\infty$ as seen in Equation (47). However, if this is the case, then no thrust is produced as seen in Equation (42).

6.3 Thermal efficiency expression (turbojet)

To consider a turbojet as an ideal Brayton cycle, we make the following assumptions:

- Working fluid is just air in a closed loop and behaves like an ideal gas
- Combustion process is replaced with heat addition from an external source
- Exhaust process is replaced with heat rejection (to the atmosphere) which restores the gas to its initial state.

The ideal Brayton cycle can then be represented via a h-s (enthalpy- entropy) diagram.

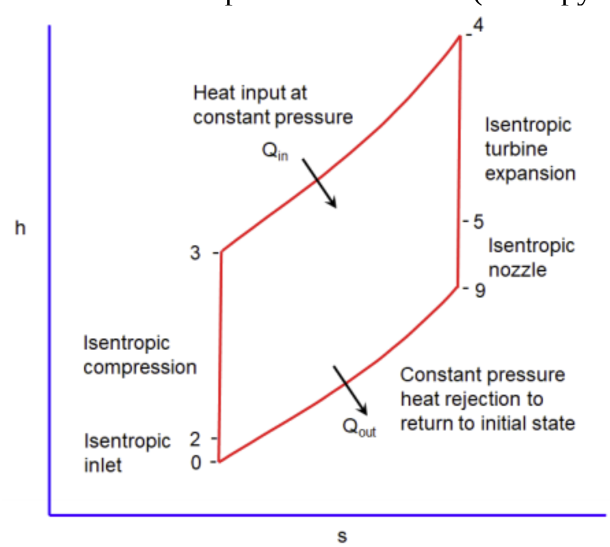


Figure 9: Ideal Brayton cycle[4]

In this case station numbering is defined as follows:

- Station 0= freestream
- Station 2= entering compressor
- Station 3= exiting compressor
- Station 4= entering turbine
- Station 5= exiting turbine
- Station 9= exiting nozzle

First define thermal efficiency.

$$\eta_{th} = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}} \tag{48}$$

Then define Qin and Qout. Static properties are used at station 0 and 9 since the freestream air is assumed to be stationary. Therefore, static and stagnation properties are equivalent.

$$Q_{in} = \dot{m}c_p(T_{t4} - T_{t3}) \tag{49}$$

$$Q_{out} = \dot{m}c_p(T_{s9} - T_0) \tag{50}$$

Therefore, we can substitute Qin and Qout.

$$\eta_{th} = 1 - \frac{T_0(\frac{T_{s9}}{T_0} - 1)}{T_{t3}(\frac{T_{t4}}{t3} - 1)} \tag{51}$$

If we assume isentropic compression (stations 2 to 3) and isentropic expansion (stations 4 to 5), then we can apply the isentropic relations.

$$\frac{T_{t3}}{T_0} = (\frac{P_{t3}}{P_0})^{\frac{\gamma-1}{\gamma}}; \quad \frac{T_{t4}}{T_{s9}} = (\frac{P_{t4}}{P_{s9}})^{\frac{\gamma-1}{\gamma}} \tag{52}$$

Assuming constant pressure heat input (combustion, Pt4= Pt3) and constant pressure heat rejection (Ps9= P0), then the following is true.

$$\frac{T_{t3}}{T_0} = (\frac{P_{t3}}{P_0})^{\frac{\gamma-1}{\gamma}} = (\frac{P_{t4}}{P_{s9}})^{\frac{\gamma-1}{\gamma}} = \frac{T_{t4}}{T_{s9}}; \qquad \frac{T_{s9}}{T_0} = \frac{T_{t4}}{T_{t3}} \tag{53}$$

Now substituting back into Equation (51) yields

$$\eta_{th} = 1 - \frac{T_{t4}}{T_{t3}} = 1 - \frac{T_0}{T_{t2}}\frac{T_{t2}}{T_{t3}} \tag{54}$$

We can then apply the following Mach stagnation relations and isentropic temperature-pressure relations to introduce the flight Mach number and pressure ratio:

$$\frac{T_{t2}}{T_0} = (1 + \frac{\gamma - 1}{2} {M_0}^2) \quad \textbf{(55)}$$

$$\frac{T_{t3}}{T_{t2}} = (\frac{P_{t3}}{P_{t2}})^{\frac{\gamma-1}{\gamma}} = PR^{\frac{\gamma-1}{\gamma}} \quad \textbf{(56)}$$

Then substituting into Equation (54) gives

$$\eta_{th} = 1 - \frac{1}{(1 + \frac{\gamma - 1}{2} {M_0}^2)} \frac{1}{PR^{\frac{\gamma-1}{\gamma}}} \quad \textbf{(57)}$$

The above expression for an ideal turbojet thermal efficiency does not include any losses. The key takeaways from the above equation is that thermal efficiency of an ideal turbojet increases with pressure ratio and flight Mach number. In actuality, there is a maximum thermal efficiency which is dependent on the fuel to air ratio and heating value of the fuel.

6.4 Propulsion system cycle parameters

FPR= Fan Pressure Ratio, the pressure across the fan component
As discussed earlier, to produce the most thrust without sacrificing propulsive efficiency, you want to have a large airmass flow rate with minimum velocity change. This is what fans and propellors accomplish- they accelerate a large capture area of air by a small velocity increment.
The fan also occurs before the compressor. In terms of thermal efficiency, a higher-pressure ratio leads to a higher efficiency. Therefore, a higher FPR means a lower required compressor PR (often a lighter compressor), or a higher OPR. However, this is a limit to this as FPR impacts BPR as well as thrust and thermal efficiency. In other words, increasing FPR too much will actually reduce critical cycle values.

OPR= Overall Pressure Ratio, the pressure across the entire engine
The thrust equation actually has a pressure component, as seen below.[4]

$$T = \dot{m}_e V_e - \dot{m}_a V_\infty + (p_e - p_\infty) A_e \quad \textbf{(58)}$$

Therefore, a higher-pressure ratio leads to both a higher thrust and higher thermal efficiency (see Equation (57)).

BPR= Bypass Ratio, ratio between bypass air stream and core air stream
Generally, the largest effect of increasing bypass ratio is the increase in propulsive efficiency. This is because to produce the most thrust without sacrificing propulsive efficiency, you want to have a large airmass flow rate with minimum velocity change (i.e. a bypass air stream through a fan). This large mass flow rate has a similar effect of increasing thrust. Thermal efficiency is usually measured as a function of the core stream. As such, BPR is not considered to affect thermal efficiency if it is calculated this way.

TSFC= Thrust Specific Fuel Consumption
TSFC is the ratio of fuel burned per hour to the amount of thrust produced by the engines. In other words, you want this value to be low which means you are burning less fuel per unit thrust. Typically, TSFC for a turbojet/turbofan is defined as the (air mass flow rate entering the combustor *FAR) divided by the thrust.

T4max= Max temperature exiting the combustor
A higher T4max means a higher thermal efficiency engine. Conceptually this can be thought of as increasing the area contained in the curve on the h-s diagram, Figure 9. In other words, for the same expansion enthalpy drop, you now have more available energy for the nozzle component, i.e. more thrust.

7 References

[1] Mavris, D. "09 - FWD 1 - Performance Lecture 1 and 2 - Fall 2020" AE 6343, 2020. Daniel Guggenheim School of Aerospace Engineering, Georgia Institute of Technology.

[2] "Liquid Hydrogen As A Propulsion Fuel,1945-1959", Appendix B. NASA History, https://history.nasa.gov/SP-4404/app-b4.htm#f65

[3] R. Singh, G. Ameyugo, F. Noppel, 4 - Jet engine design drivers: past, present and future, Innovation in Aeronautics,Woodhead Publishing, 2012, Pages 56-82, https://doi.org/10.1533/9780857096098.1.56.

[4] Tai, J. "Lecture07-08 Thermodynamic Cycle v4" AE6310, 2020. Daniel Guggenheim School of Aerospace Engineering, Georgia Institute of Technology.

8 Written Work

Combined into singular PDF starting on next page.
This work is simply the initial written version of the above content. Graphs are inserted in the context of how I thought about each problem.

The start of each problem is labeled as such and follows the page number format of "pg. problem- pg#".

AE6343 Aircraft Design I
Fall 2020 Take Home Exam 2

Nov 13 - Nov 20, 2020

Time: Take Home
Number of questions: 4
Total number of points: 100

Exam Rules

- Read the Honor Code Statement, print your name, sign it (electronic signature is ok), and date it.
- Students have until **Friday, November 20th,** to complete it and submit all relevant information via Canvas (no later than 11:55PM EDT).
- Using word or Latex, write a report with your answers to these questions. Make sure that your report follows standard formatting guidelines, with elements such as figure and table captions, page numbers, justified margins, equations numbers, etc. Then, convert this document into a PDF for the Canvas submission.
- The questions in this exam have multiple parts, each one with some guiding questions. Make sure to, at least, address all of these questions in your answer.
- Please use the following naming convention for your file:

  AE6343_Exam2_Fall2020_LastName_FirstName.pdf

- Late submissions will be allowed with a 10% penalty on the final grade for each day of delay. No exceptions.
- This exam is meant to be completed individually. No collaboration among students is allowed.
- Students might ask the Teaching Assistants for clarification on the deliverables, but not about the content itself.

Honor Code Statement

"I understand and accept my responsibility as a member of the Georgia Tech Community to uphold the Academic Honor Code at all times. In addition, I understand my options for reporting honor violations as detailed in the code. To the best of my knowledge, I have not used materials or information which can provide me with an unfair advantage over my peers."

Full Name: KARL ROUSH

Signature: [signature] Date: NOV 18, 2020

1. (20 points) For an aircraft with a parabolic drag polar given by $C_D = C_{D0} + KC_L^2$ in steady level flight at airspeed V, show that:

    1. The thrust required (T_r) is given by,

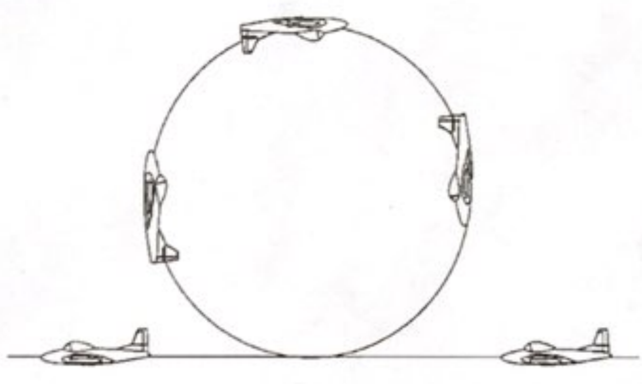
    $$T_r = W\sqrt{C_{D0}K}\left[\left(\frac{V}{V_0}\right)^2 + \left(\frac{V}{V_0}\right)^{-2}\right] \tag{1}$$

    Where W is the weight of the aircraft and V_0 is the airspeed when drag is minimum.

    2. If r denotes the ratio of the airspeed to the airspeed at minimum drag, show that:

    $$\frac{T_r}{T_{r,\min}} = \frac{1}{2}(r^2 + r^{-2}) \tag{2}$$

2. (20 points) An aerobatic pilot performs a full inside loop maneuver. Assume that the aircraft trajectory is circular and that velocity is constant. Describe how load factor changes throughout this maneuver. Make sure to justify your answers using performance equations and derivations. Qualitative descriptions are not sufficient. You may or may not make assumptions for some quantities to be used in this analysis. If you do, make sure that these values are realistic. However, this is not necessary. Hint: identify at least 6 "key" points (aircraft positions) in this maneuver to perform your calculations and analysis.



3. The V-n diagram:

    (a) (10 points) Draw a V-n diagram. Make sure to include all axis labels. Describe and derive expressions for each one of its relevant sections and curves.

    (b) (10 points) For the inside loop maneuver in the previous question, draw how the load factor changes throughout the maneuver. Start with the aircraft flying at cruise represented by a single point, and then show how this point moves in the diagram through the identified key points.

(c) (10 points) Now, assume your V-n diagram represents a bomber aircraft. At some point during its mission, this aircraft drops its payload. Show how the curves in this diagram would change after having dropped its payload. Include both curves in your diagram and clearly label them. Justify your reasoning behind these changes. Just drawing a new diagram without any reasoning will not grant any credit.

4. Propulsion cycle design and efficiency:

(a) (10 points) Given that the propulsive efficiency is defined by:

$$\eta_p = \frac{Useful\ Power\ Available}{Total\ Power\ Generated}$$

Derive an expression for propulsive efficiency of a turbojet that only has constants V_j and V_∞. Show steps, not just a final answer. Also make sure to state and discuss the extent of your assumptions.

(b) (5 points) Discuss the propulsive efficiency trade-offs when it comes to the different types of propulsion systems available (e.g. turbofan, turbojet, etc.). Also, discuss why reaching the ultimate propulsive efficiency is not desirable.

(c) (5 points) Thermal efficiency: assume a turbojet is analyzed as an ideal Brayton cycle without losses. Then, derive an expression for thermal efficiency (η_{th}). Final result should, at least, show a relationship with respect to a pressure ratio. Then, discuss the implications that the assumptions used in this derivation have, and explain how a real engine's efficiency might be different from the trends shown in here.

(d) (10 points) Propulsion system cycle parameters: provide a description for each one of the following parameters. Also, describe how they can affect (or be a measure of) an engine's propulsive and thermal efficiency.

- FPR
- OPR
- BPR → Fan to core [illegible] KE penalty, [illegible]
- TSFC
- $T_{4_{max}}$

PROBLEM 1

ⓐ Prove thrust required $(T_r) = w\sqrt{C_{D_0} K\left[\left(\frac{V}{V_0}\right)^2 + \left(\frac{V}{V_0}\right)^{-2}\right]}$

* V_0 = airspeed @ min drag

@ steady level flight, airspeed V, $C_D = C_{D0} + KC_L^2$

- - - - - - - - - - - - -

In steady level, we know the following

$$m\frac{dV_\infty}{dt} = T\cos(\varepsilon) - D - W\sin(\gamma) \implies T = D$$

(V_∞ = const, $\varepsilon = 0$, $=1$, $\gamma = 0$)

$$m\frac{V_\infty^2}{r_c} = L\cos\phi + T\sin(\varepsilon)\cos\phi - W\cos(\gamma) \implies L = W$$

($r_c = \infty$; $1, \phi = 0$; $0, \varepsilon = 0$; $1, \gamma = 0$)

therefore we now have $T = D$ and $L = W$

$L = \frac{1}{2}\rho V_\infty^2 S C_L \quad \Rightarrow \quad L_0 = \frac{1}{2}\rho V_0^2 S C_L$

$D = \frac{1}{2}\rho V_\infty^2 S C_D \qquad D_0 = \frac{1}{2}\rho V_0^2 S C_D$

$C_L = \frac{2w}{\rho V^2}$

$T = D \Rightarrow \quad T_r = \frac{1}{2}\rho V^2 S C_D$

$= 0.5\rho V^2 S\left[C_{D0} + KC_L^2\right]$

$L = W \Rightarrow w = 0.5\rho V^2 S C_L$

$V = \sqrt{\frac{2W}{\rho S C_L}}$

in other words, we need to combine the following:

- quadratic drag polar
- Thrust to drag relation
- C_L relation to L and W

$T = 0.5\rho V^2 S C_{D0} + 2K\frac{w^2}{\rho S V^2} \quad \leftarrow \quad = 0.5\rho V^2 S\left[C_{D0} + K\left(\frac{2w}{\rho V^2}\right)^2\right]$

$= (0.5\rho S C_{D0})V^2 + \left(\frac{2Kw^2}{\rho S}\right)\frac{1}{V^2}$

$= C_{D0} K w^2\left[V^2 + \frac{1}{V^2}\right] \quad \dashrightarrow \quad w\sqrt{C_{D0}K\left[V^2 + V^{-2}\right]}$

(5) now convert to a ratio for $V_{\text{min drag}}$ (ratio is r)

$$T = 0.5\rho V^2 S C_{D_0} + 2K \frac{W^2}{\rho S V^2}$$

$r = \frac{1}{2}\rho S V^2 C_D$

$\frac{dT}{dD} = 1$!

$$\frac{dT}{dV} = \rho S V S C_{D_0} - 4K \frac{W^2}{\rho S V^3} = 0$$

$$V_{w,T_{min}} = \left[\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_0}}}\right]^{1/2}$$

so now sub back into T

$$T_{req,min} = 0.5\rho[\,\cdot\,]^2 S C_{D_0} + 2K\frac{W^2}{\rho S[\,\cdot\,]^2}$$

$$= 2W\sqrt{K C_{D_0}}$$

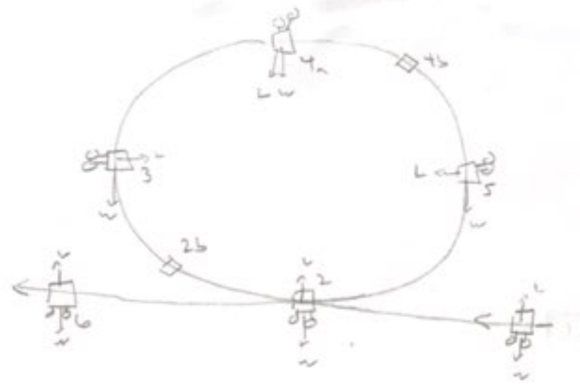
since we assume $T \approx D$ then $T_{req,min}$ occurs at min drag

now divide expression previously given

$$T_r = W\sqrt{C_{D_0} K}\left[\left(\frac{V}{V_0}\right)^2 + \left(\frac{V}{V_0}\right)^{-2}\right] \implies \frac{T_r}{T_{min}} = \frac{1}{2}\left[r^2 + r^{-2}\right]$$

$T_{req,min} = 2W\sqrt{K C_{D_0}}$; $r = V/V_0$

~~PROBLEM 2~~ PROBLEM 2



1, 6 = level flight
2a = level flight
2b = pull up

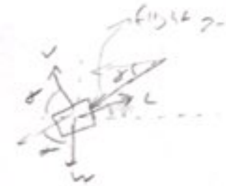
3, 5 = vertical
4a = inverted level
4b = Pull down / Pull over

so basically immelmann + split S
but without the rolls
⇒ specifically "inside loop" maneuver

$n = L/W$ Case 1, 6 (level flight) = in level flight $L = W$, therefore $n = 1$

Case 4 (upsidedown) = Flying upside down has L acting in same direction as W, excluding centripetal then $n = -1$

⇒ therefore $n = f(\cos\gamma)$



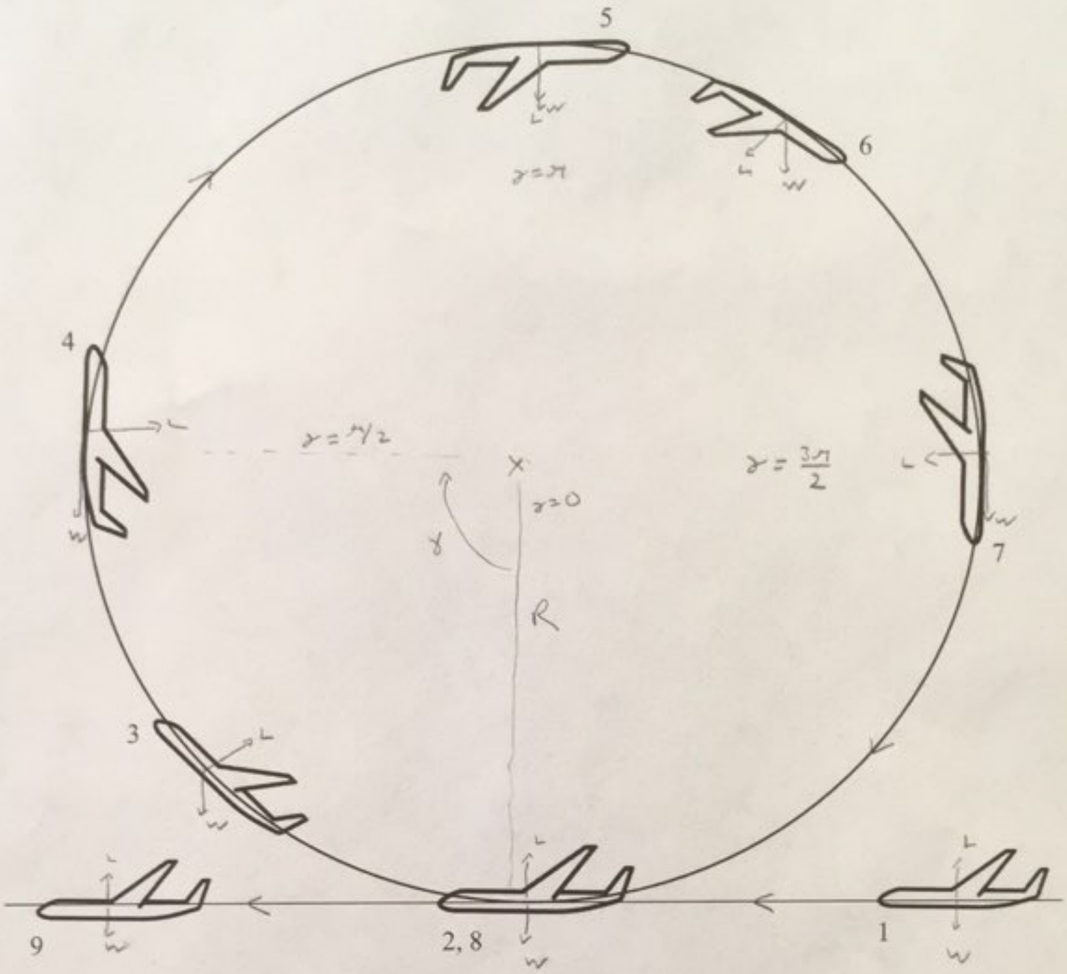
via $$m\frac{V^2}{R} = L - W\cos\gamma$$

since $W = mg \Rightarrow n = \cos\gamma + \frac{V_\infty^2}{Rg}$

We assume circular loop @ fixed velocity so V_∞, R, g are same at all points during the inside loop. * $R = \inf$ if level

⇒ therefore n is solely a function of γ

($V\uparrow$) Flying faster will increase max n
OR
($R\downarrow$) tighter turn


5
6
4
7
3
9
2, 8
1
R

PROBLEM 3

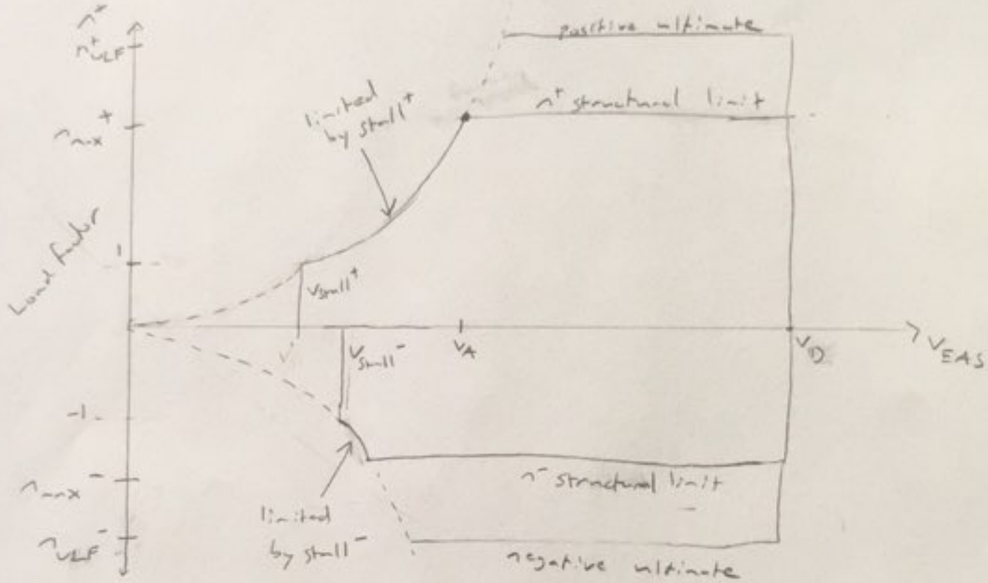
(A) v-n diagram essentially shows structural limits when EAS to be altitude indep. $q \propto V_{EAS}^2$

$n^- < n < n^+$ — limit load factor, structure must withstand loads without permanent deformation

several limits: n^+/n^- set by structure

stall: what is n @ stall speed

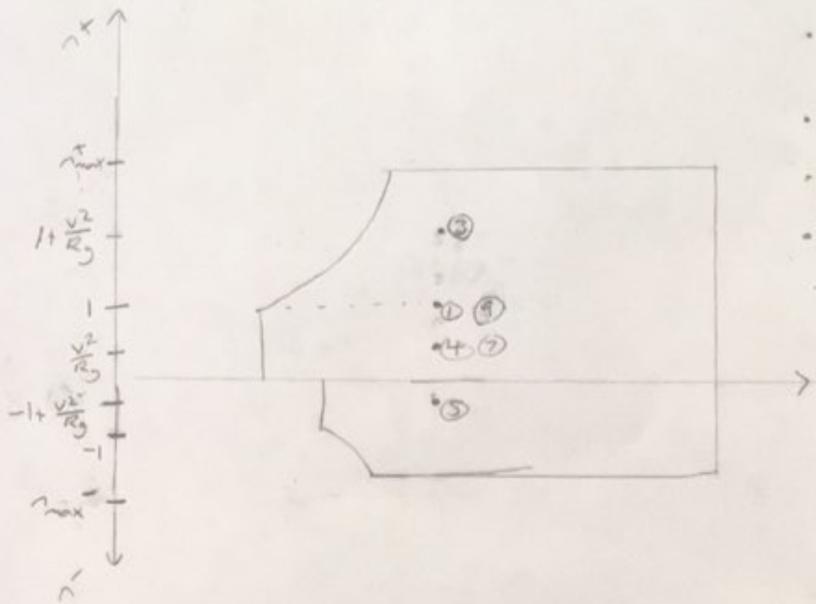
V_D: flying faster means aero loads cause damage



can find corner point & stall via lift relation

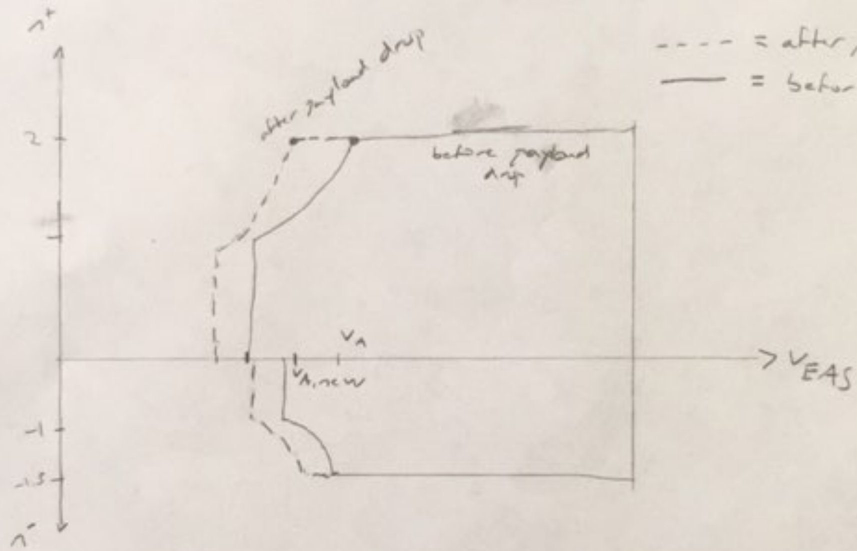
$$n_{max} = 0.5\rho_{SL} V_E^2 \frac{C_{Lmax}}{W/S} \qquad V_A = \sqrt{\frac{2}{\rho_{SL}} \frac{W}{S} \frac{n^+}{C_{Lmax}}}$$

(B)

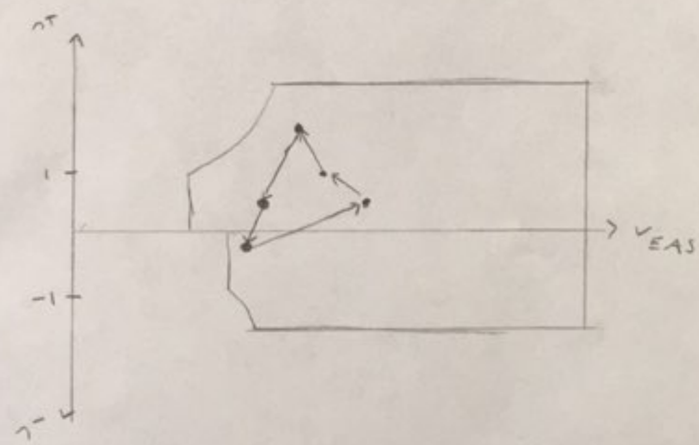


- assume alt change in loop is negligible ($\rho_{top} = \rho_{bottom}$)
- constant speed (vertical line)
- flying above stall speed
- consider $\frac{V^2}{Rg} < 1$

(C)



- - - - = after payload drop
——— = before payload drop



PROBLEM 4

(A) define η_p

$$\eta_p = \frac{\text{Useful power available}}{\text{total power generated}}$$

useful power ~ available as thrust $\cdot V_\infty$ $\qquad V_\infty =$

total power is $\Delta KE = \frac{\dot{m}_{out} V_j^2}{2} - \frac{\dot{m}_{in} V_\infty^2}{2}$

$$\eta_p = \frac{F_n V_j}{\frac{1}{2}(\dot{m}_a + \dot{m}_f) V_j^2 - \frac{1}{2}(\dot{m}_a) V_\infty^2}$$

assume $FAR \ll 1$ and $FAR = \frac{\dot{m}_f}{\dot{m}_a}$

$$\eta_p = \frac{F_n V_j}{(1+FAR)\frac{V_j^2}{2} - \frac{V_\infty^2}{2}}$$

thrust is found via $F = \frac{d(mv)}{dt} = \dot{m}(V_j - V_\infty)$

$$= \dot{m}_a((1+FAR)V_j - V_\infty)$$

$$\eta_p = \frac{\dot{m}_a[(1+FAR)V_j - V_\infty]}{(1+FAR)\frac{V_j^2}{2} - \frac{V_\infty^2}{2}} \quad \text{apply } FAR \ll 1$$

$$\eta_p = \frac{2V_\infty}{V_j + V_\infty} = \frac{2}{1 + V_j/V_\infty}$$

alternatively:

useful power available: Thrust $\times V_j \Rightarrow$ Thrust is a force so apply Newton's law #2

$T = \dot{m}(V_j - V_\infty)$ $\qquad$ momentum $= \dot{m}V$

total generated = available + wasted

since kinetic energy is imparted to air consider ΔKE

$$\Delta E = \frac{1}{2} m (V_j - V_\infty)^2$$

Power is energy/time so $m \to \dot{m} \Rightarrow \frac{1}{2}\dot{m}(V_j - V_\infty)^2$

substitute
$$\eta_p = \frac{\dot{m}(V_j - V_\infty)V_\infty}{\dot{m}(V_j - V_\infty)V_\infty + \frac{1}{2}\dot{m}(V_j - V_\infty)^2}$$

algebraic manipulation
$$\eta_p = \frac{1}{1 + \frac{1}{2}(V_j - V_\infty)} = \frac{1}{\frac{1}{2}\left(1 + \frac{V_j}{V_\infty}\right)} = \boxed{\frac{2}{1 + \frac{V_j}{V_\infty}} = \eta_p}$$

assumes Fuel input is negligible, ignores pressure forces

(B)

Balancing Thrust, TSFC, weight (complexity), size, noise

Basically pick best η_p for V_∞

(C)

2 = before compressor
3 = after compressor
0 = freestream

4 = combustor exit
5 = after turbine
9 = exit of nozzle

$Q_{in} = \dot{m} c_p (T_{t4} - T_{t3})$ $\quad Q_{out} = \dot{m} c_p (T_{9} - T_{0})$

apply isentropic relations for 2-3; 4-5

$$\frac{T_{t4}}{T_{5}} = \left(\frac{P_{t4}}{P_{5}}\right)^{\frac{\gamma-1}{\gamma}} \quad \text{also} \quad \frac{T_{t2}}{T_0} = \left(1 + \frac{\gamma-1}{2} M_0^2\right)$$

(D) FPR = fan pressure ratio

BPR = Bypass ratio

OPR = overall pressure ratio

TSFC = thrust specific fuel consumption

T_{4max} = max combustor exit temp (don't melt turbine blades)