Rayleigh-Ritz Results and FEM overview

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AE 3140 – Structural Analysis

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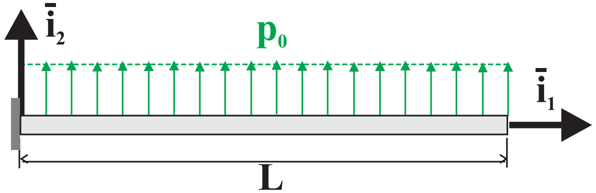
Transverse Displacement



- 2nd-order Polynomial

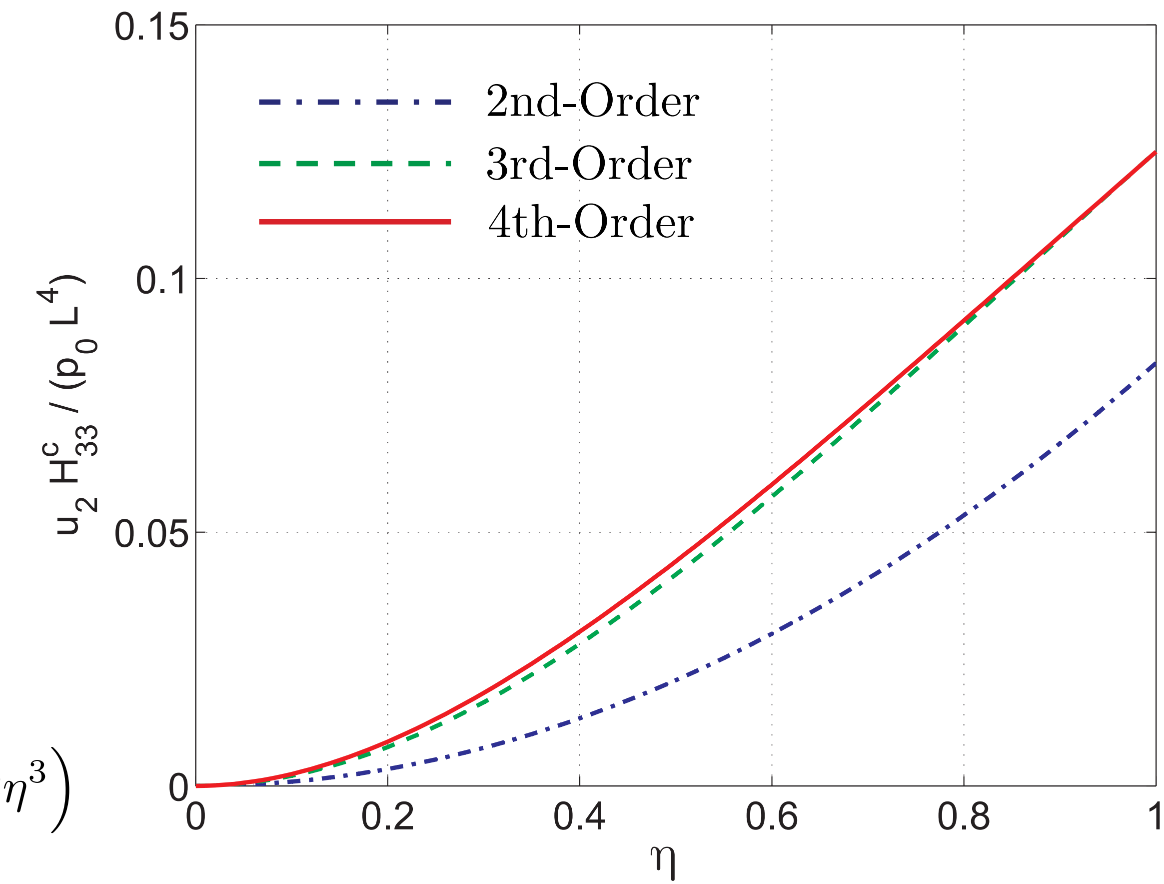
$$\bar{u}_2(\eta) = \frac{p_0 L^4}{H^c_{33}} \frac{1}{24} \left(2\eta^2 \right)$$

- 3rd-order Polynomial

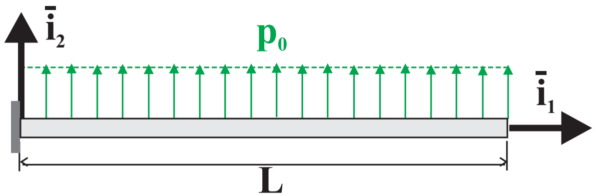
$$\bar{u}_2(x_1) = \frac{p_0 L^4}{H^c_{33}} \frac{1}{24} \left(5\eta^2 - 2\eta^3 \right)$$

- 4th-order Polynomial (exact)

$$\bar{u}_2(x_1) = \frac{p_0 L^4}{H^c_{33}} \frac{1}{24} \left(6\eta^2 - 4\eta^3 + \eta^4 \right)$$



Bending Moment and Shear?



- 2nd-order Polynomial

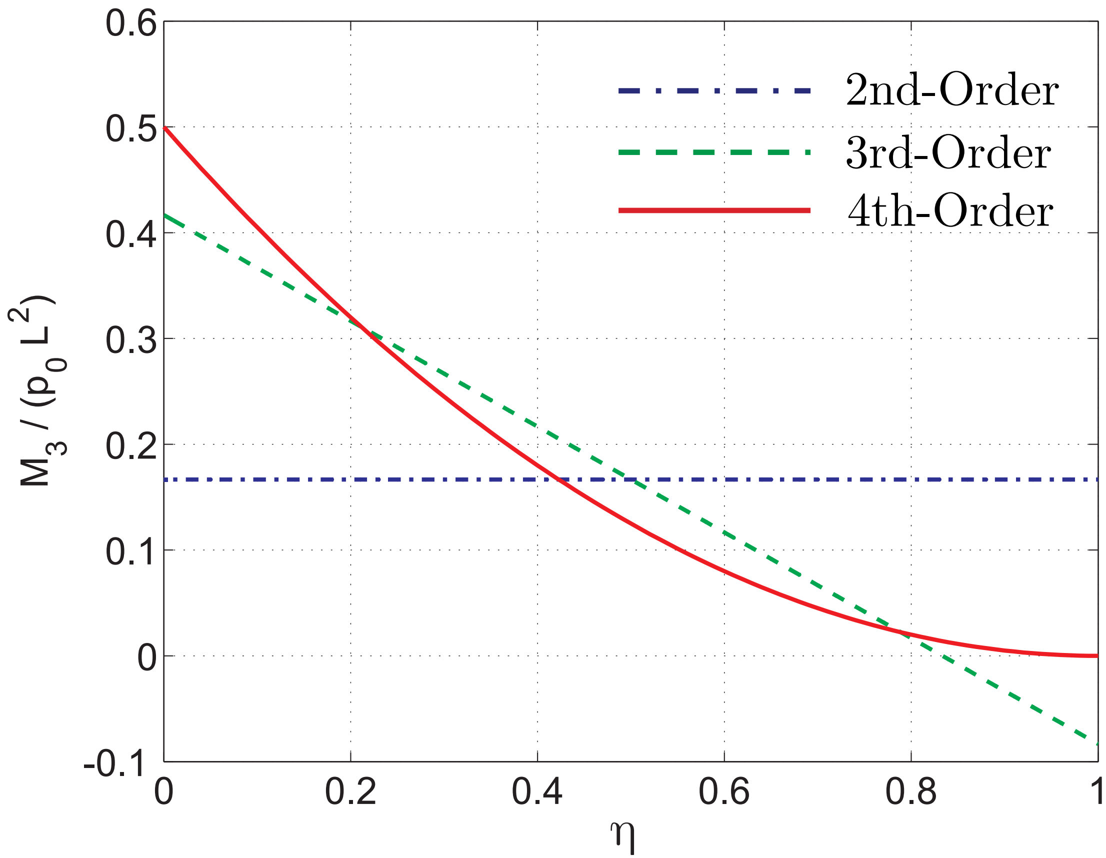
$$M_3 = H_{33}^c \frac{d^2 \bar{u}_2}{dx_1^2} = p_0 L^4 \frac{1}{24}\Big(4\Big)$$

- 3rd-order Polynomial

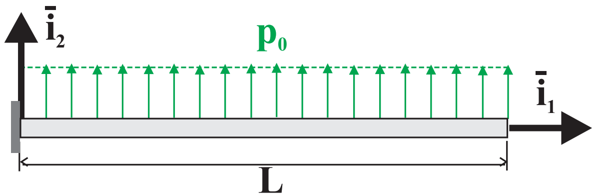
$$M_3 = p_0 L^4 \frac{1}{24}\Big(10 - 12\eta\Big)$$

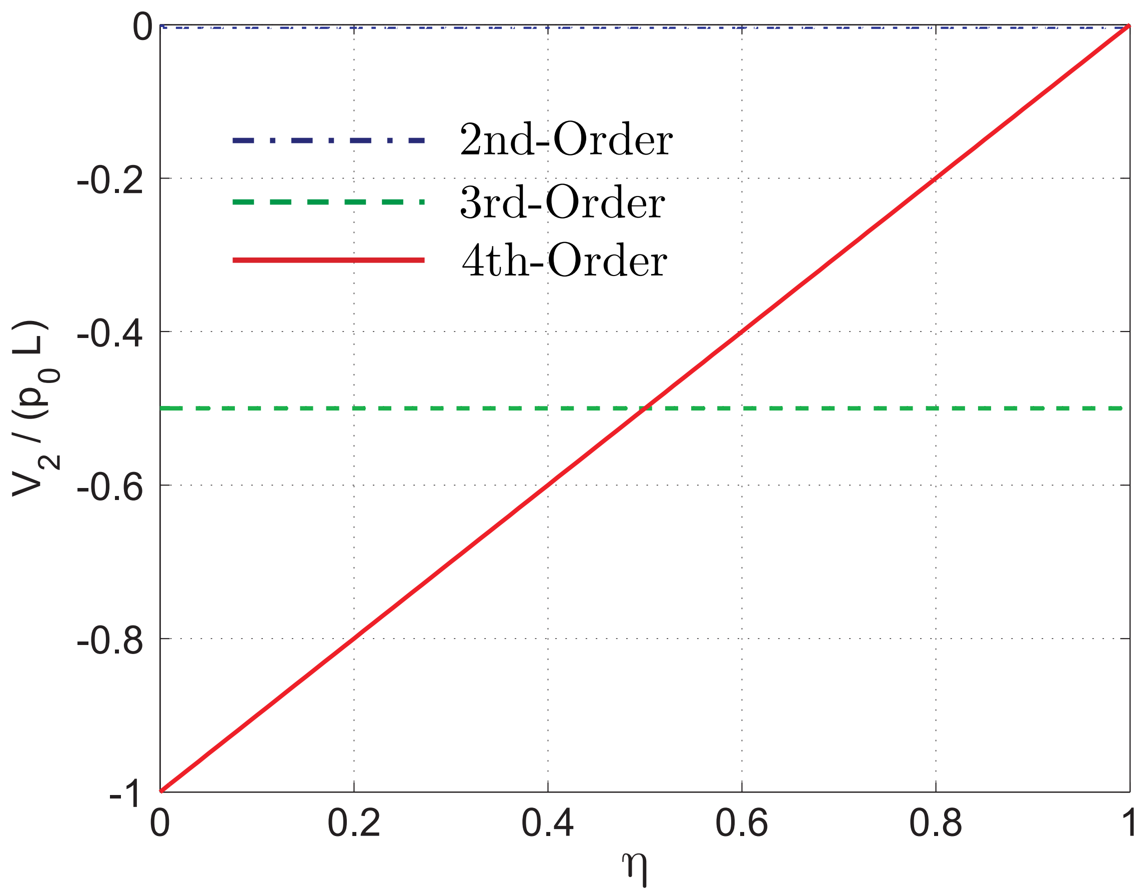
- 4th-order Polynomial (exact)

$$M_3 = p_0 L^4 \frac{1}{24}\Big(12 - 24\eta + 12\eta^2\Big)$$



Bending Moment and Shear?





- 2nd-order Polynomial

$$V_2 = -\frac{dM_3}{dx_1} = 0$$

- 3rd-order Polynomial

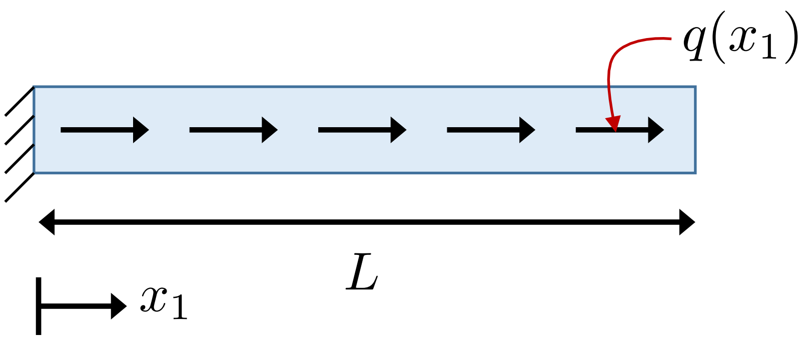
$$V_2 = p_0 L^4 \frac{1}{24}\Big(-12 \Big)$$

- 4th-order Polynomial (exact)

$$V_2 = p_0 L^4 \frac{1}{24}\Big(-24 + 24\eta \Big)$$

Finite Element Method – Example

- Beam in tension w/ distributed load.



Exact Solution:

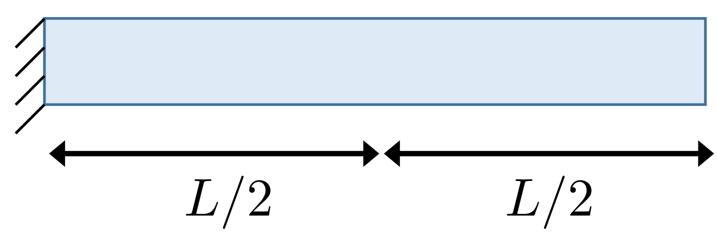
$$\bar{u}_1(x_1) = -\frac{q}{S}\left(\frac{x_1^2}{2} - Lx_1\right)$$

- Potential

$$\Pi = \int_0^L \frac{1}{2} S \left(\frac{d\bar{u}_1}{dx_1}\right)^2 dx - \int_0^L q\bar{u}_1 dx$$

- Solve using Rayleigh-Ritz w/ Piecewise Linear Functions

- Express approximate $\bar{u}_1$ in terms of approximate displacements $\hat{u}_i$ at points x_i
  - We call these points ”nodes”
  - We call the regions between these points ”elements”



Finite Element Method – Example

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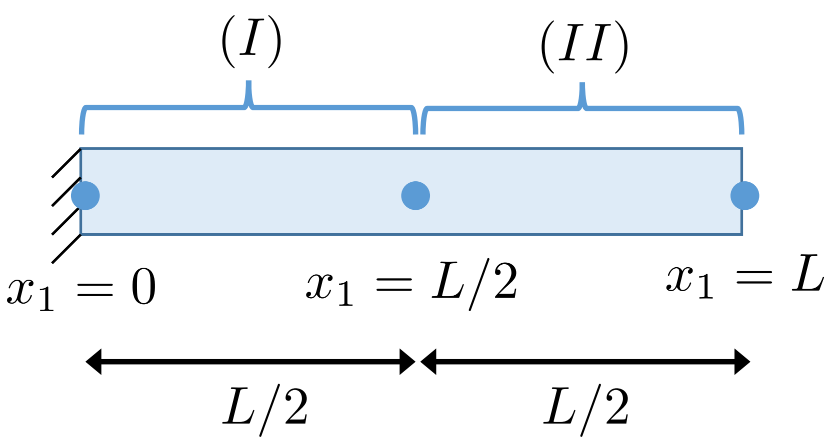
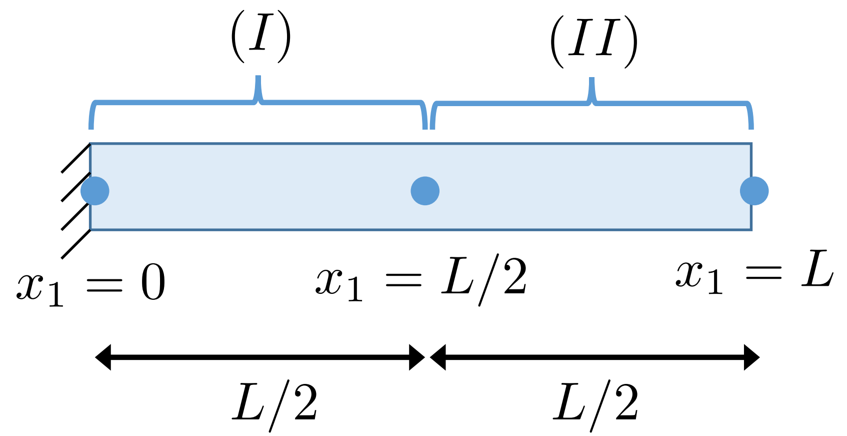


$$\hat{u}_1 = \begin{cases} u_1 + \dfrac{u_2 - u_1}{L/2} x = \hat{u}^{(I)}(x) & \text{if} \quad 0 \leq x < L/2 \\ u_2 + \dfrac{u_3 - u_2}{L/2} \left(x - \frac{L}{2}\right) = \hat{u}^{(II)}(x) & \text{if} \quad L/2 \leq x < L \end{cases}$$

Finite Element Method – Example

- Solve using Rayleigh-Ritz w/ Piece-wise Linear Functions
- Integrate and apply BCs ($u_1 = 0$)



$$\Pi = \int_0^{L/2} \frac{1}{2} S \left(\frac{d\hat{u}^{(I)}}{dx} \right)^2 dx - \int_0^{L/2} q\hat{u}^{(I)} dx$$

$$+ \int_{L/2}^{L} \frac{1}{2} S \left(\frac{d\hat{u}^{(II)}}{dx} \right)^2 dx - \int_{L/2}^{L} q\hat{u}^{(II)} dx$$

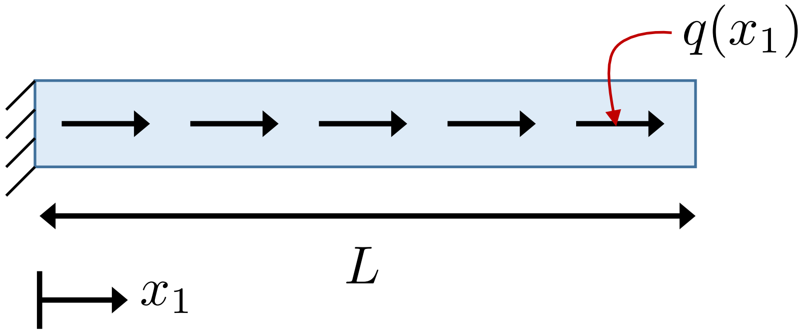
$$\Pi = \frac{1}{4L} \left(8Su_2^2 - 8Su_2u_3 + 4Su_3^2 - 2qu_2L^2 - qu_3L^2 \right)$$

$$\frac{\partial \Pi}{\partial u_2} = \frac{1}{4L}(16Su_2 - 8Su_3 - 2qL^2) = 0$$

$$\frac{\partial \Pi}{\partial u_3} = \frac{S}{4L}(-8Su_2 + 8Su_3 - qL^2) = 0$$

$$S \underbrace{\begin{bmatrix} 16 & -8 \\ -8 & 8 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 2qL^2 \\ qL^2 \end{bmatrix}}_{\mathbf{f}}$$

Finite Element Method – Example



Exact Solution:

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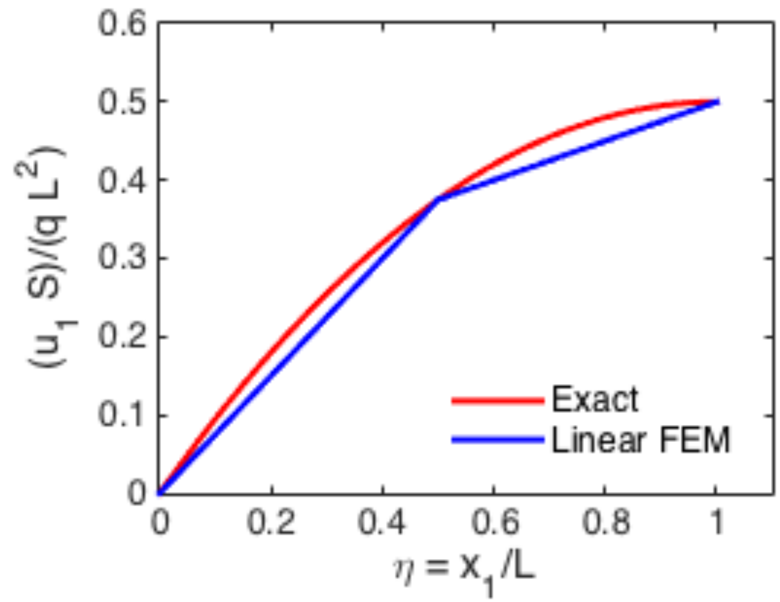
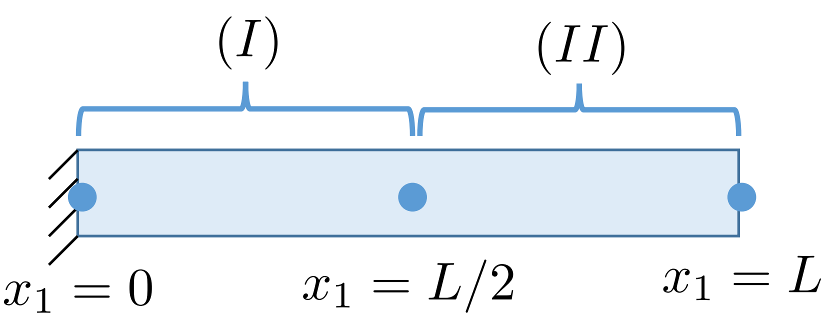


- Solved using Rayleigh-Ritz with piecewise-linear functions over two "elements"
- This solution is a basic finite element solution
- For linear elasticity, FEM is Rayleigh-Ritz!
- It can be shown that as we refine the discretization $\hat{\Pi} \to \Pi$ and $\hat{u} \to u$

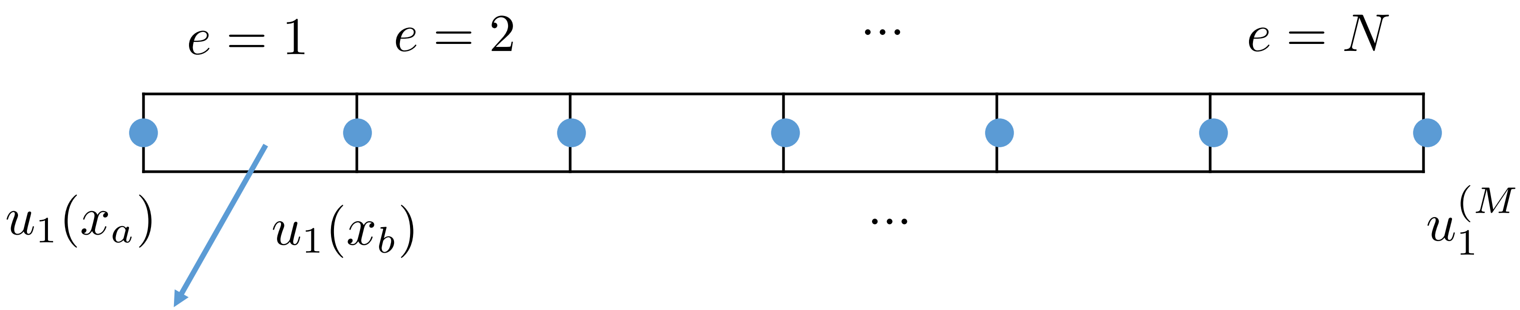


$$\hat{u}_1 = \begin{cases} \dfrac{3}{4}\dfrac{qL}{S}x & \text{if} \quad 0 \leq x < L/2 \\[2ex] \dfrac{1}{4}\dfrac{qL^2}{S} + \dfrac{1}{4}\dfrac{qL}{S}x & \text{if} \quad L/2 \leq x < L \end{cases}$$

Finite Element Method – High Level Overview

- Consider the total energy of the system as given by a sum of the energies over multiple smaller ”elements”

$$\Pi = \sum_{e=1}^{N} \Pi^e \qquad \Pi^e = \int_{\Omega^e} \left[\frac{1}{2} S(x) \left(\frac{du^e}{dx} \right)^2 - p(x) u^e(x) \right] dx - u^e(x_a) F_A - u^e(x_b) F_B$$



Each element is approximated with **simple polynomial shape functions**

$$u_1^e = \sum_{j=1}^{2} h_j u_1^{(j)}$$

$$u_1^e(x) = \mathbf{H}(x) \mathbf{u}^e$$

$$\frac{du_1^e(x)}{dx} = \mathbf{B}(x) \mathbf{u}^e$$

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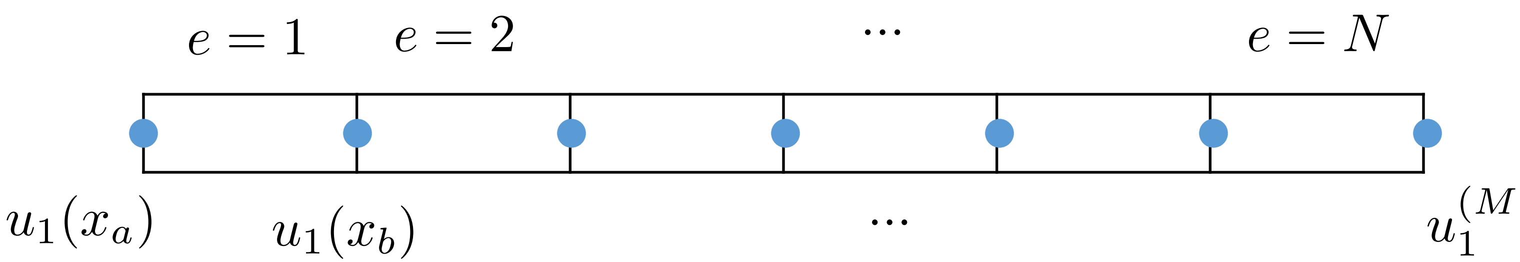

$$\Pi^e = \frac{1}{2} \mathbf{u}^{e,T} \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^{e,T} \mathbf{f}^e \qquad \mathbf{K}^e = \int_{\Omega^e} S(x) \mathbf{B}(x)^T \mathbf{B}(x) dx$$

$$\delta \Pi = \sum_e \delta \Pi^e = 0 \qquad \mathbf{f}^e = \int_{\Omega^e} \mathbf{H}^T(x) q(x) dx + \sum_i \mathbf{H}^T(x_i) \mathbf{F}_i$$

$$\boxed{\mathbf{K}\mathbf{u} = \mathbf{f}}$$

Solve system of linear equations with M unknowns (minus boundary conditions)!