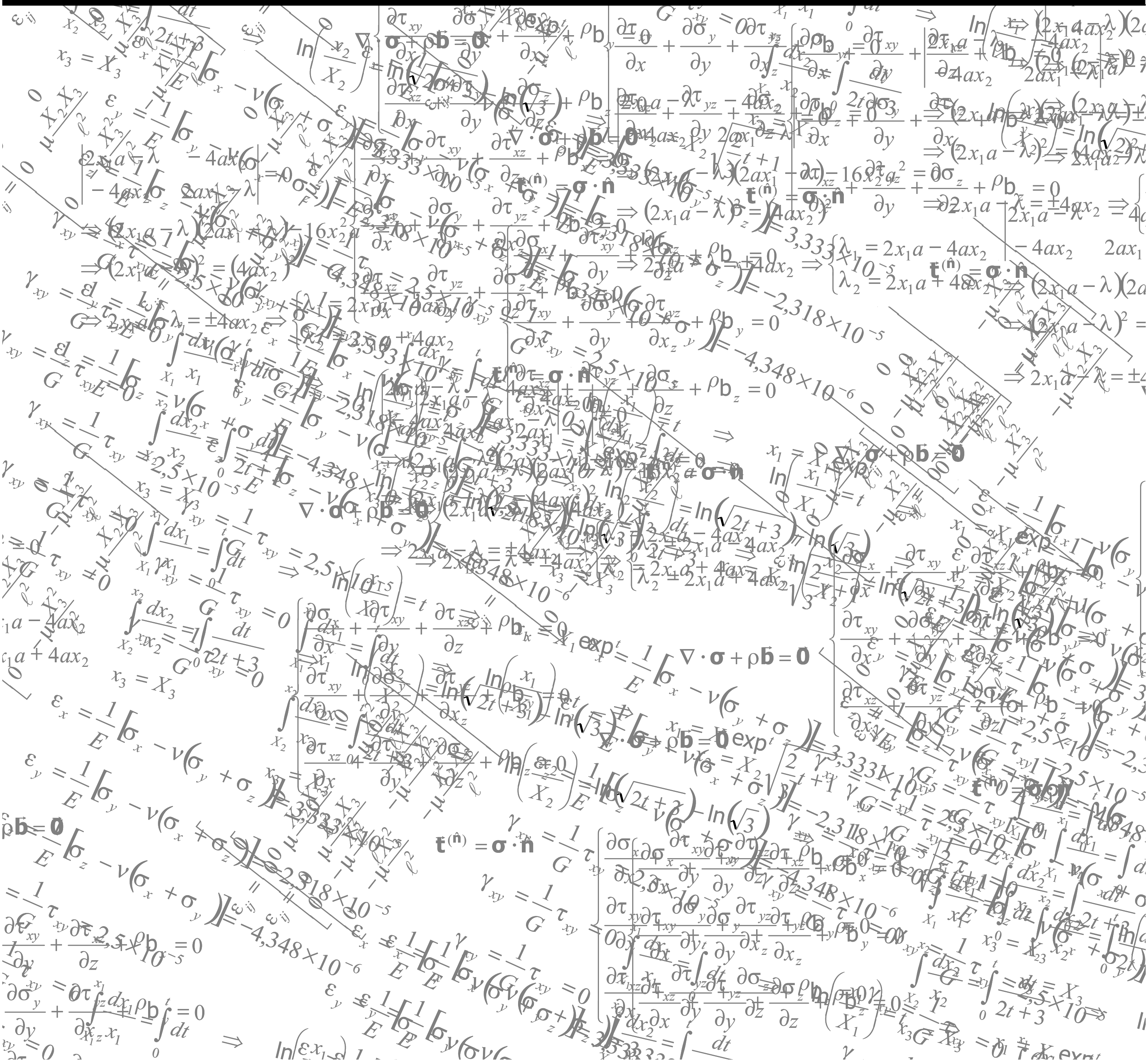
SOLVING PROBLEMS BY MEANS OF CONTINUUM MECHANICS

Eduardo W. V. Chaves



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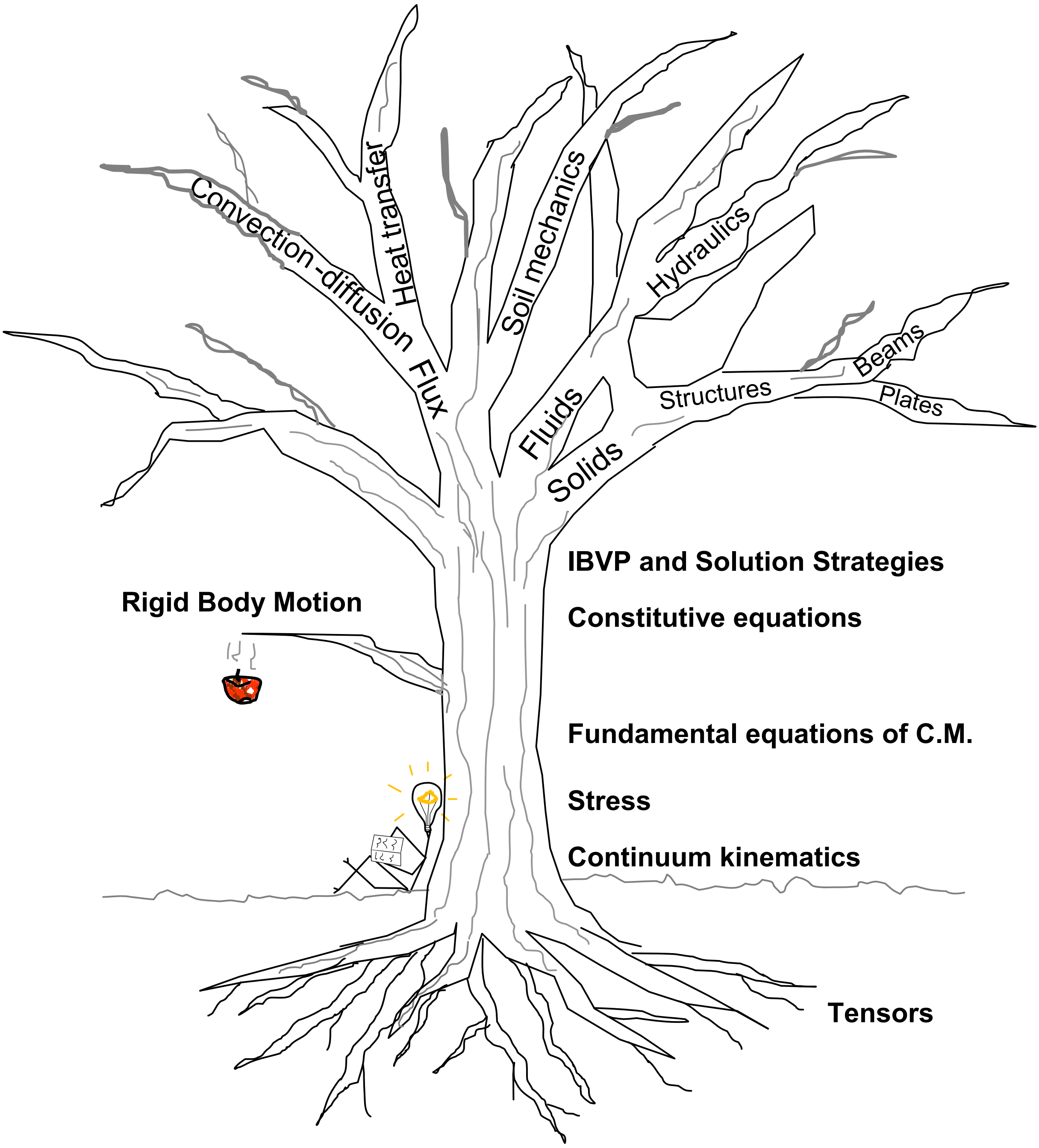
Presentation



Contents

Abbreviations

IBVP	Initial Boundary Value Problem
BVP	Boundary Value Problem
FEM	Finite Element Method
BEM	Boundary Element Method
FDM	Finite Difference Method
C.M.	Continuum Mechanics
iff	if and only if

Latin

i.e.	*id est*	that is
et al.	*et alii*	and the others
e.g.	*exempli gratia*	for example
etc.	*et cetera*	and so on
Q.E.D.	*Quod Erat Demonstrandum*	which had to be demonstrated
v., vs.	*versus*	versus
viz.	*vidilicet*	namely

Operators and Symbols

$\langle \bullet \rangle = \dfrac{\lvert\bullet\rvert + \bullet}{2}$	Macaulay bracket
$\lVert\bullet\rVert$	Euclidian norm of $\bullet$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$(\bullet)^{T}$	transpose of $(\bullet)$
$(\bullet)^{-1}$	inverse of $(\bullet)$
$(\bullet)^{-T}$	inverse of the transpose of $(\bullet)$
$(\bullet)^{sym}$	symmetric part of $(\bullet)$
$(\bullet)^{skew}$	antisymmetric (skew-symmetric) part of $(\bullet)$
$(\bullet)^{sph}$	spherical part of $(\bullet)$
$(\bullet)^{dev}$	deviatoric part of $(\bullet)$
$\lVert\bullet\rVert$	module of $\bullet$
$[\![\bullet]\!]$	jump of $\bullet$
$\cdot$	scalar product
$\det(\bullet) \equiv \lvert\bullet\rvert$	determinant of $(\bullet)$
$\dfrac{D\bullet}{Dt} \equiv \dot{\bullet}$	material time derivative of $(\bullet)$
$\mathrm{cof}(\bullet)$	cofactor of $\bullet$;
$\mathrm{Adj}(\bullet)$	adjugate of $(\bullet)$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$:$	double scalar product (or double contraction or double dot product)
∇^{2}	Scalar differential operator
$\otimes$	tensorial product
$\nabla\bullet \equiv \mathrm{grad}(\bullet)$	gradient of $\bullet$
$\nabla\cdot\bullet \equiv \mathrm{div}(\bullet)$	divergence of $\bullet$
$\wedge$	vector product (or cross product)
$I_{\bullet}$, $II_{\bullet}$, $III_{\bullet}$	first, second and third principal invariants of the tensor $\bullet$
$\vec{\bullet}$	vector
$\hat{\bullet}$	unity vector
$\mathbf{1}$	Second-order unit tensor
$\mathbb{I}$	fourth-order unit tensor
$\mathbb{I}^{sym} \equiv \mathbf{I}$	symmetric fourth-order unit tensor

SI-Units

length	m - **meter**	electric current	A - **ampere**
mass	kg - **kilogram**	amount of substance	mol - **mole**
time	s - **second**	luminous intensity	cd - **candela**
temperature	K - **kelvin**		

velocity	$\frac{m}{s}$	energy, work, heat	$J = Nm$ - Joules
acceleration	$\frac{m}{s^2}$	power	$\frac{J}{s} \equiv W$ watt
energy	$J = Nm$ - Joules		
force	N - Newton	permeability	m^2
pressure, stress	$Pa \equiv \frac{N}{m^2}$ - Pascal	dynamic viscosity	$Pa \times s$
frequency	$\frac{1}{s} \equiv Hz$ Hertz	mass flux	$\frac{kg}{m^2 s}$
thermal conductivity	$\frac{W}{mK}$	energy flux	$\frac{J}{m^2 s}$
mass density	$\frac{kg}{m^3}$	energy density	$\frac{J}{m^3}$

Prefix	Symbol	10^n	Prefix	Symbol	10^n
pico	p	10^{-12}	kilo	k	10^3
nano	η	10^{-9}	Mega	M	10^6
micro	μ	10^{-6}	Giga	G	10^9
mili	m	10^{-3}	Tera	T	10^{12}
centi	c	10^{-2}			
deci	d	10			

Physical Constants

Newtonian constant of gravitation: $G = 6.67384 \times 10^{-11} \frac{m^3}{kg\, s^2}$

Speed of light in vacuum: $c = 299\,792\,458 \frac{m}{s} \approx 300\,000\,000 \frac{m}{s}$

Absolute zero (temperature): $T = 0K = -273.15^{o}\, C$

Nomenclature

$\vec{A}(\vec{X},t) \equiv \vec{a}(\vec{X},t)$	Acceleration (reference configuration)	$\frac{m}{s^2}$
$\mathcal{A}$	Transformation matrix	
$\vec{a}(\vec{x},t)$	Acceleration (current configuration)	$\frac{m}{s^2}$
$\mathcal{B}_0$	Continuum medium in the reference configuration at $t=0$	
$\mathcal{B}$	Continuum medium in the current configuration at time t	
$\partial\mathcal{B}$	Boundary of $\mathcal{B}$	
$\vec{\mathbf{b}}(\vec{x},t)$	Body force (per unit mass)	$\frac{N}{m^3}$
$\boldsymbol{b}$	Left deformation Cauchy-Green tensor, Finger deformation tensor	
$\boldsymbol{B}$	Piola deformation tensor	
$\boldsymbol{B}$	Entropy created inside	$\frac{J}{s\,K}$
$\boldsymbol{b}$	Local entropy per unit mass per unit time	$\frac{J}{kg\,s\,K}$
$\mathbb{C}^e$	Elasticity tensor	Pa
$[\mathcal{C}]$	Elasticity matrix (Voigt notation)	Pa
$\mathbb{C}^{in}$	Inelasticity tensor	Pa
$\boldsymbol{c}$	Cauchy deformation tensor	
C_v	Calor específico a volumen constante	
C_p	Calor específico a presión constante	
c	Cohesion	Pa
c_c	Solute concentration	$\frac{mol}{m^3}$
$\boldsymbol{C}$	Right deformation Cauchy-Green tensor	
D_V	Dilation	$\frac{m}{m}$
$\mathbf{D}$	Rate-of-Deformation tensor	
$d\vec{A}$	Area element vector in the reference configuration	m^2
$d\vec{a}$	Area element vector in the current configuration	m^2
dV	Volume element	m^3

$\boldsymbol{E}$	Green-Lagrange strain tensor, or Lagrangian finite strain tensor, or Green-St_Venant strain tensor	$\frac{m}{m}$
$\boldsymbol{e}$	Almansi strain tensor, or Eulerian finite strain tensor	$\frac{m}{m}$
E	Young's modulus, or elastic modulus	Pa
$\hat{\mathbf{e}}_i$	Cartesian basis in symbolic notation	
$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	Cartesian basis	
$\boldsymbol{F}$	Deformation gradient (pseudo-tensor)	$\frac{m}{m}$
G	Shear modulus	Pa
$\boldsymbol{H}$	Biot strain tensor	
H	Total entropy	$\frac{J}{K}$
$\vec{\boldsymbol{H}}_O$	Angular momentum	$\frac{kgm^2}{s} = Js$
J	Jacobian determinant	$\frac{m^3}{m^3}$
$\boldsymbol{J}(\vec{X}, t)$	Material displacement gradient tensor	$\frac{m}{m}$
$\boldsymbol{j}(\vec{x}, t)$	Spatial displacement gradient tensor	$\frac{m}{m}$
$\vec{\mathbf{J}}$	Diffusion tensor	$\frac{mol}{m^2 s}$
$\mathbf{K}$	Thermal conductivity tensor	$\frac{W}{m\,K} = \frac{J}{s\,m\,K}$
$\mathcal{K}$	Kinetic energy	J
$\vec{L}$	Linear momentum	$\frac{kg\,m}{s}$
$\boldsymbol{\ell}$	Spatial velocity gradient	$\frac{m}{s\,m}$
m	Mass	kg
$\mathbf{M}$	Mandel stress tensor	Pa
$\hat{\mathbf{n}}$	Outward unit normal to the boundary (current configuration)	
$\hat{\mathbf{N}}$	Outward unit normal to the boundary (reference configuration)	
$\vec{\mathbf{p}}$	Body force (per unit volume)	$\frac{N}{m^3}$
$\mathbf{P}$	First Piola-Kirchhoff stress tensor	Pa
p	Thermodynamic pressure	Pa
$\vec{\mathbf{q}}(\vec{x}, t)$	Cauchy heat flux (non-convective vector)	$\frac{J}{m^2 s}$
$\mathbf{Q}$	Orthogonal tensor	
Q	Thermal work	J
$r(\vec{x}, t)$	Radiant heat constant, or heat source (per unit mass)	$\frac{J}{kg\,s}$

$\mathbf{R}$	Orthogonal tensor of polar decomposition	
$\mathbf{S}$	Second Piola-Kirchhoff stress tensor	Pa
$\vec{s}$	Entropy flux	$\frac{J}{kg\ s\ m^2}$
$\mathbf{T}$	Biot stress tensor	Pa
$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}(\vec{x},t,\hat{\mathbf{n}})$	Traction vector (current configuration)	Pa
$\vec{\mathbf{t}}_0^{(\hat{\mathbf{N}})}$	Traction pseudo-vector (reference configuration)	Pa
$T(\vec{x},t)$	Temperature	K
t	Time	s
$t_0 \equiv t = 0$	Initial time	s
$\dot{U}$	Rate of change of the internal energy	$\frac{J}{s} = W$
u	Specific internal energy	$\frac{J}{kg}$
$\vec{\mathbf{u}}(\vec{x},t)$	Displacement vector (Eulerian)	m
$\vec{\mathbf{u}}(\vec{X},t)$	Displacement vector (Lagrangian)	m
$\mathbf{U}(\vec{X},t)$	Right stretch tensor, or Lagrangian stretch tensor, or material stretch tensor	
$\mathbf{V}(\vec{x},t)$	Left stretch tensor, or Eulerian stretch tensor, or spatial stretch tensor	
$\vec{V}(\vec{X},t) \equiv \vec{v}(\vec{X},t)$	Velocity (reference configuration)	$\frac{m}{s}$
$\vec{v}(\vec{x},t)$	Velocity (current configuration)	$\frac{m}{s}$
$\mathbf{W}$	Spin tensor, rate-of-rotation tensor, or vorticity tensor	$\frac{m}{ms} = \frac{rad}{s}$
$\mathrm{w}_{\mathrm{int}}$	Stress power	$\frac{J}{s} = W$
$\vec{X}$	Vector position (material coordinate)	m
$\vec{x}$	Vector position (spatial coordinate)	m
α	Coefficient of thermal expansion	$\frac{1}{K}$
δ_{ij}	Kronecker delta	
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Principal strains (infinitesimal strain)	
ε	Unit Extension	$\frac{m}{m}$
ϵ_{ijk}	Permutation symbol, or Levi-Civita tensor components	
ε_V	Linear dilatation (volume ratio) (small deformation regime)	$\frac{m}{m}$
$\boldsymbol{\varepsilon}$	Infinitesimal strain tensor	$\frac{m}{m}$
η	Specific entropy	$\frac{J}{kg\ K}$
κ	Bulk modulus	Pa

κ	Thermal diffusivity	$\frac{m^2}{s}$
λ	Stretch	$\frac{m}{m}$
λ, μ	Lamé constants	Pa
ν	Poisson's ratio	
ρ_S	Solute mass density	$\frac{kg}{m^3}$
ρ_f	Fluid mass density	$\frac{kg}{m^3}$
$\rho_0(\vec{\boldsymbol{X}})$	Mass density (reference configuration)	$\frac{kg}{m^3}$
$\rho(\vec{\boldsymbol{x}}, t)$	Mass density (current configuration)	$\frac{kg}{m^3}$
$\frac{1}{\rho}$	Specific volume	$\frac{m^3}{kg}$
$\boldsymbol{\sigma}$	Cauchy stress tensor, or true stress tensor	Pa
$\vec{\boldsymbol{\sigma}}_N$	Normal traction vector	Pa
$\vec{\boldsymbol{\sigma}}_S$	Tangential traction vector	Pa
σ_m	Mean stress	Pa
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses	Pa
$\vec{\boldsymbol{\sigma}}_{oct}$	Normal octahedral vector	Pa
$\vec{\boldsymbol{\tau}}_{oct}$	Tangential octahedral vector	Pa
$\tau_{\max}$	Maximum shear stress	
$\boldsymbol{\tau}$	Kirchhoff stress tensor	Pa
ϕ	Angle of internal friction	
ψ	Helmholtz free energy, specific (per unit mass)	$\frac{J}{kg}$
Ψ	Helmholtz free energy (per unit volume)	$\frac{J}{m^3}$
$\Psi(\boldsymbol{\varepsilon}) = \Psi^e$	Strain energy density	$\frac{J}{m^3}$

Useful Formulas

Some Trigonometric Identities

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$
$$\cos(\theta \pm \phi) = \cos(\theta)\sin(\phi) \mp \sin(\theta)\sin(\phi)$$
$$\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$$
$$\sin(\theta)\sin(\phi) = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$$
$$\sin(\theta)\cos(\phi) = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)]$$
$$\cos^2(\phi) = \frac{1}{2}[1 + \cos(2\theta)]$$
$$\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$$
$$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$
$$\cos(\theta) - \cos(\phi) = 2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\phi - \theta}{2}\right)$$
$$\sin(\theta) \pm \sin(\phi) = 2\sin\left(\frac{\theta \pm \phi}{2}\right)\cos\left(\frac{\theta \mp \phi}{2}\right)$$
$$\cos^2(\theta) + \sin^2(\phi) = 1$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$
$$\sec^2(\theta) + \tan^2(\phi) = 1$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad ; \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

List of trigonometric identity

http://en.wikipedia.org/wiki/Trigonometric_identity

Some Series Expansions

$$f(x) = f(a) + \frac{\partial f}{\partial x}(x-a) + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x-a)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(x-a)^3 + \cdots \qquad \text{(Taylor's series)}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad ; \quad \left(\|x\| < 1\right) \quad \text{(binomial series)}$$

$$\exp^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$\mathrm{Ln}(1+x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots\left(\|x\| < \frac{\pi}{2}\right)$$

Some Derivatives

$$\frac{d}{dx}(\exp^x) = \exp^x \quad ; \quad \frac{d}{dx}(a^x) = \mathrm{Ln}(a)a^x \quad ; \quad \frac{d}{dx}[\mathrm{Ln}(x)] = \frac{1}{x} \quad ; \quad \frac{d}{dx}[\log_a(x)] = \frac{1}{x\mathrm{Ln}(a)}$$

$$\frac{d}{dx}[\mathrm{Ln}(f(x))] = \frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

where $\mathrm{e} \equiv \exp$ stands for exponential and Ln for natural logarithm, where it fulfils:

$$\mathrm{Ln}(\exp^x) = x \quad \text{and} \quad \exp^{\mathrm{Ln}(x)} = x$$

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad ; \quad \frac{d}{dx}[\cos(x)] = -\sin(x) \quad ; \quad \frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}} \quad ; \quad \frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad \frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

List of derivatives
http://en.wikipedia.org/wiki/List_of_derivatives

Some Integrals

$$\int \exp^x dx = \exp^x \quad ; \quad \int \frac{\partial f(x)}{\partial x} \exp^{f(x)} dx = \exp^{f(x)}$$

$$\int \frac{1}{x} dx = \mathrm{Ln}(x) \quad ; \quad \int \mathrm{Ln}(x) dx = x\mathrm{Ln}(x) - x + C$$

where $e \equiv \exp$ stands for exponential and Ln for natural logarithm.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\left\|\frac{u}{a}\right\|\right) + C$$

List of integrals
http://en.wikipedia.org/wiki/List_of_integrals

Some Function Solutions

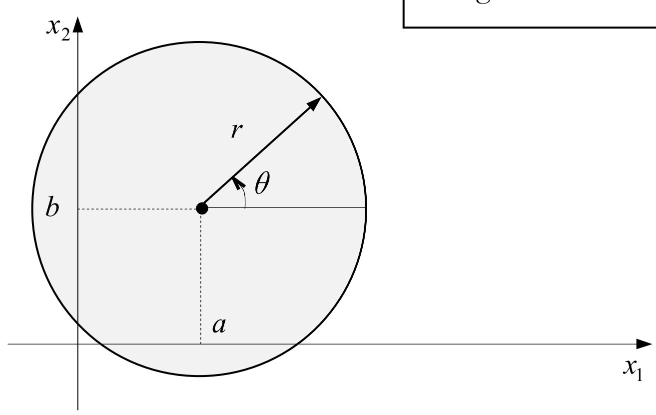
Quadratic function

$$ax^2 + bx + c = 0 \xrightarrow{solution} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (a \neq 0)$$

Ruffini's rule
http://en.wikipedia.org/wiki/Ruffini%27s_rule

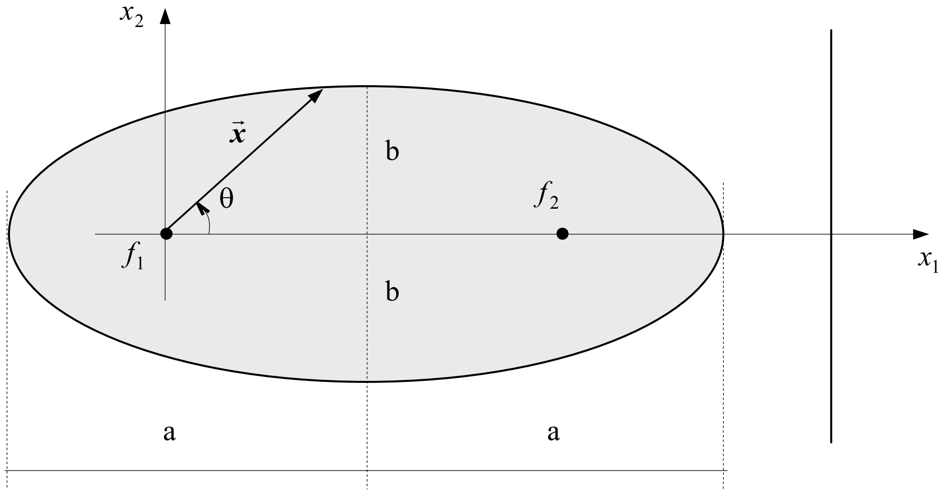
Expressions related to the circle:

Equation of the circle: $(x_1 - a)^2 + (x_2 - b)^2 \le r^2$
Area enclosed by a circumference: $A = \pi r^2$
Length of circumference: $C = 2\pi r$



The relationship $ds = r d\theta$ holds, where ds is the infinitesimal arc length.

Expressions related to the ellipse:



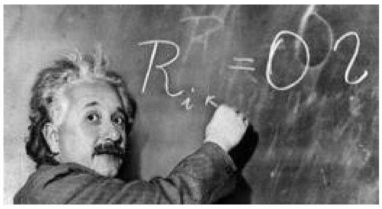
Equation of the ellipse: $\|\vec{x}\| = r = \dfrac{p}{1 + e\cos\theta}$

Eccentricity: $e = \sqrt{\dfrac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2}}$; $0 < e < 1$, where $\mathrm{a}^2 = \dfrac{p^2}{(1 - e^2)^2}$ holds.

Area enclosed by an ellipse: $A = \pi\mathrm{ab}$.

1 Tensors

The indicial notation was introduced by 'Einstein (1916, sec. 5), who later jested to a friend, "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..." (Kollros 1956; Pais 1982, p. 216). '
*Ref. (**Wolfram MathWorld** (Einstein Summation))*



1.1 Vectors, Indicial Notation

Problem 1.1

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors. Prove that the following relationship is true:

$$(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$$

Solution:

$$\begin{aligned}
(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \quad &= \left\|\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right\|^2 = \left(\left\|\vec{\mathbf{a}}\right\| \left\|\vec{\mathbf{b}}\right\| \sin\theta\right)^2 \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \sin^2\theta = \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \left(1-\cos^2\theta\right) = \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \cos^2\theta \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - \left(\left\|\vec{\mathbf{a}}\right\| \left\|\vec{\mathbf{b}}\right\| \cos\theta\right)^2 = \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2 \\
&= (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2
\end{aligned}$$

where we have taking into account that $\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = \left\|\vec{\mathbf{a}}\right\|^2$ and $\vec{\mathbf{b}} \cdot \vec{\mathbf{b}} = \left\|\vec{\mathbf{b}}\right\|^2$.

Problem 1.2

Show that: if $\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$, the module of $\vec{\mathbf{c}}$ can be expressed by means of the following relationship:

$$\left\|\vec{\mathbf{c}}\right\| = \sqrt{\left\|\vec{\mathbf{a}}\right\|^2 + 2\left\|\vec{\mathbf{a}}\right\| \left\|\vec{\mathbf{b}}\right\| \cos\beta + \left\|\vec{\mathbf{b}}\right\|^2}$$

where β is the angle formed by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, (see Figure 1.1(a)).

Solution:

Starting from the module definition of a vector it fulfills that:

$$\left\|\vec{\mathbf{a}} + \vec{\mathbf{b}}\right\|^2 = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{a}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{b}}$$

Taking into account that $\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = \left\|\vec{\mathbf{a}}\right\|^2$, $\vec{\mathbf{b}} \cdot \vec{\mathbf{b}} = \left\|\vec{\mathbf{b}}\right\|^2$ and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$ (commutative), we can conclude that:

$$\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|^2=\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\vec{\mathbf{b}}\cdot\vec{\mathbf{a}}+\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}=\left\|\vec{\mathbf{a}}\right\|^2+2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\left\|\vec{\mathbf{b}}\right\|^2=\left\|\vec{\mathbf{a}}\right\|^2+2\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\cos\beta+\left\|\vec{\mathbf{b}}\right\|^2$$

with which we prove $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|=\sqrt{\left\|\vec{\mathbf{a}}\right\|^2+2\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\cos\beta+\left\|\vec{\mathbf{b}}\right\|^2}$. Then, it is easy to show that $\left\|\vec{\mathbf{a}}-\vec{\mathbf{b}}\right\|=\sqrt{\left\|\vec{\mathbf{a}}\right\|^2-2\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\cos\beta+\left\|\vec{\mathbf{b}}\right\|^2}$. Note also that when $\beta=0^\circ\Rightarrow\cos(\beta)=1$ and the equation $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|=\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{b}}\right\|$ holds, (see Figure 1.1(b)).

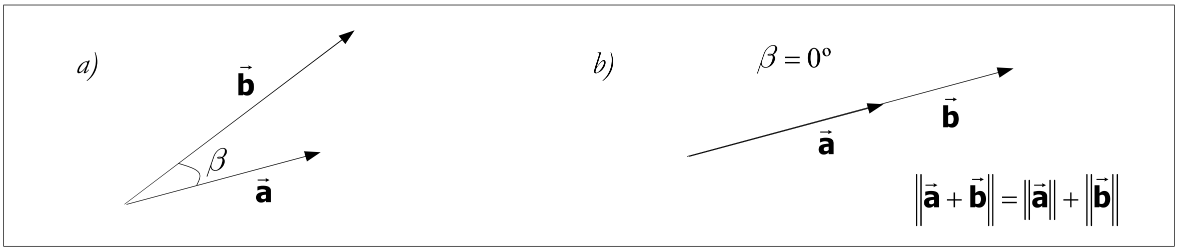


Figure 1.1

NOTE: Starting from the equation $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|^2=\left\|\vec{\mathbf{a}}\right\|^2+2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\left\|\vec{\mathbf{b}}\right\|^2$ we can conclude that the value $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|^2$ is maximum when $\beta=0^\circ$ holds, then

$$\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|^2=\left\|\vec{\mathbf{a}}\right\|^2+2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\left\|\vec{\mathbf{b}}\right\|^2=\left\|\vec{\mathbf{a}}\right\|^2+2\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|+\left\|\vec{\mathbf{b}}\right\|^2=\left(\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{b}}\right\|\right)^2$$

Then, for any value of $0^\circ<\beta\le180^\circ$ the outcome $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|$ will be less than $\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{b}}\right\|$. Then, the inequality $\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|\le\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{b}}\right\|$ holds, (see Figure 1.2):

$$\left\|\vec{\mathbf{c}}\right\|=\left\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\right\|\le\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{b}}\right\|$$

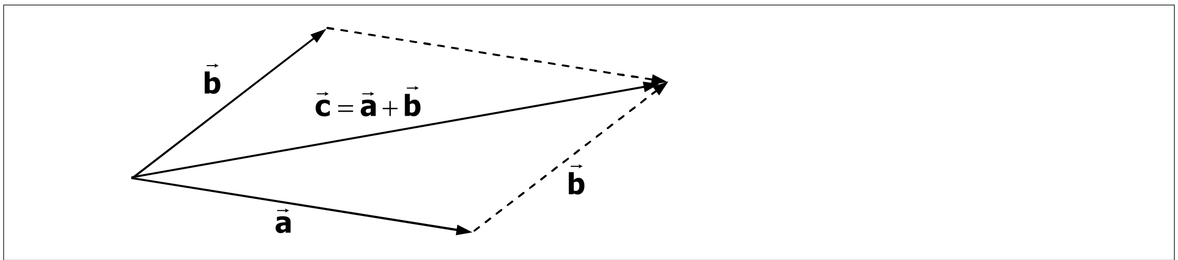


Figure 1.2

In a similar fashion we can show that $\left\|\vec{\mathbf{a}}\right\|\le\left\|\vec{\mathbf{c}}\right\|+\left\|\vec{\mathbf{b}}\right\|$ and $\left\|\vec{\mathbf{b}}\right\|\le\left\|\vec{\mathbf{a}}\right\|+\left\|\vec{\mathbf{c}}\right\|$ which is known as the triangle inequality, (see Figure 1.3).

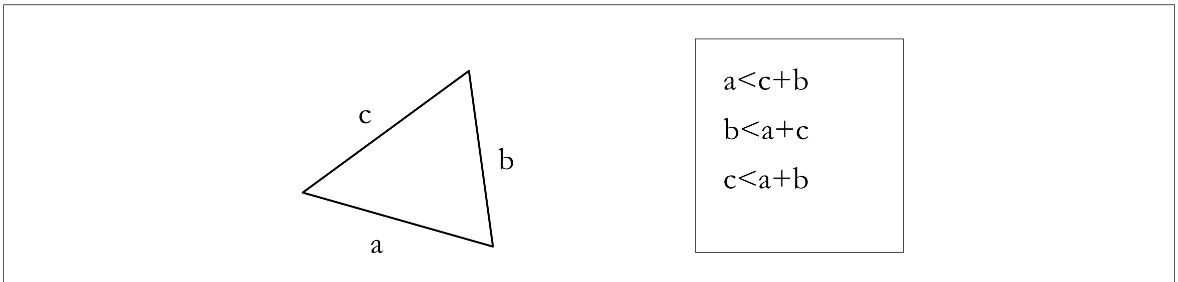


Figure 1.3

Problem 1.3

Given the following functions $\sigma(\varepsilon) = E\varepsilon$ and $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$, demonstrate whether these functions show a linear transformation or not.

Solution:

$\sigma(\varepsilon_1 + \varepsilon_2) = E[\varepsilon_1 + \varepsilon_2] = E\varepsilon_1 + E\varepsilon_2 = \sigma(\varepsilon_1) + \sigma(\varepsilon_2)$ (linear transformation)

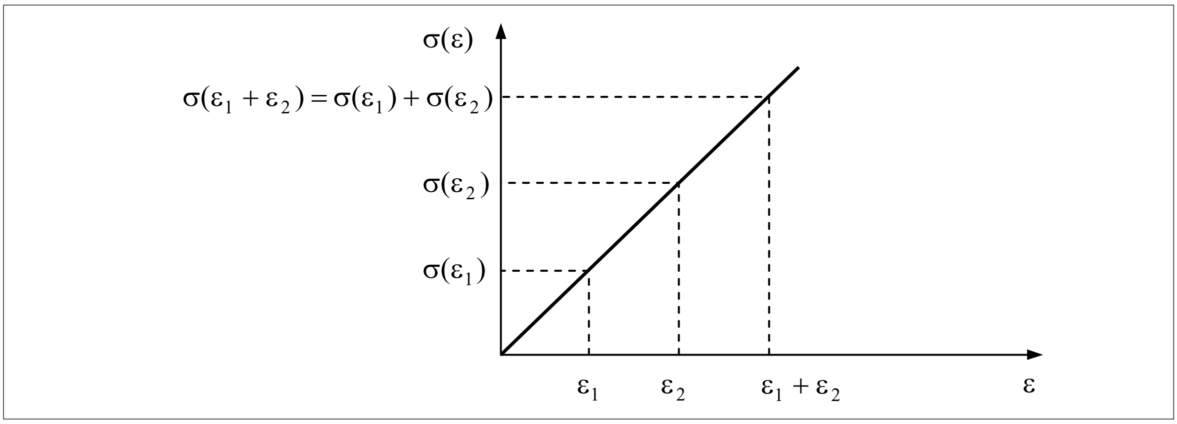


Figure 1.4

The function $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$ does not show a linear transformation because the condition $\psi(\varepsilon_1 + \varepsilon_2) = \psi(\varepsilon_1) + \psi(\varepsilon_2)$ has not been satisfied:

$$\begin{aligned}\psi(\varepsilon_1 + \varepsilon_2) &= \frac{1}{2}E[\varepsilon_1 + \varepsilon_2]^2 = \frac{1}{2}E[\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2] = \frac{1}{2}E\varepsilon_1^2 + \frac{1}{2}E\varepsilon_2^2 + \frac{1}{2}E2\varepsilon_1\varepsilon_2 \\ &= \psi(\varepsilon_1) + \psi(\varepsilon_2) + E\varepsilon_1\varepsilon_2 \neq \psi(\varepsilon_1) + \psi(\varepsilon_2)\end{aligned}$$

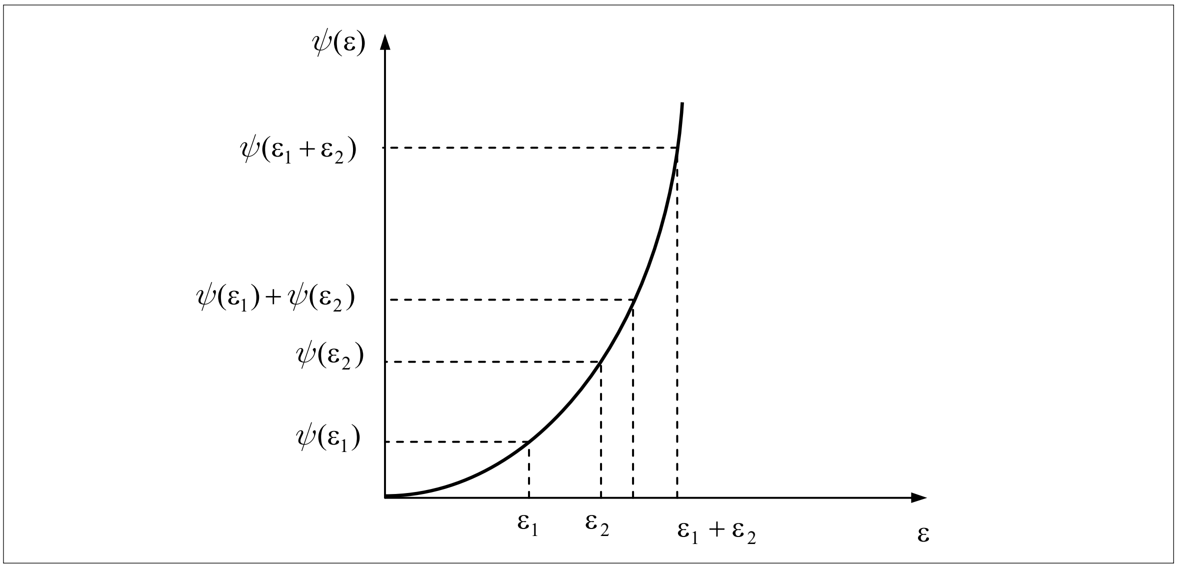


Figure 1.5

Problem 1.4

Consider the points: $A(1,3,1)$, $B(2,-1,1)$, $C(0,1,3)$ and $D(1,2,4)$, defined in the Cartesian coordinate system.

1) Find the parallelogram area defined by $\vec{AB}$ and $\vec{AC}$; 2) Find the volume of the parallelepiped defined by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$; 3) Find the projection vector of $\vec{AB}$ onto $\vec{BC}$.

Solution:

1) Firstly we calculate the vectors $\vec{AB}$ and $\vec{AC}$:

$$\vec{\mathbf{a}} = \vec{AB} = \vec{OB} - \vec{OA} = (2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) - (1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) = 1\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\vec{\mathbf{b}} = \vec{AC} = \vec{OC} - \vec{OA} = (0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) = -1\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Next, we evaluate the vector product as follows:

$$\vec{\mathbf{a}} \wedge \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -4 & 0 \\ -1 & -2 & 2 \end{vmatrix} = (-8)\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + (-6)\hat{\mathbf{k}}$$

Then, the parallelogram area can be obtained by using the following definition:

$$\mathsf{A} = \left\| \vec{\mathbf{a}} \wedge \vec{\mathbf{b}} \right\| = \sqrt{(-8)^2 + (-2)^2 + (-6)^2} = \sqrt{104}$$

2) Next, we can evaluate the vector $\vec{AD}$ as:

$$\vec{\mathbf{c}} = \vec{AD} = \vec{OD} - \vec{OA} = (1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - (1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) = 0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

we can obtain the volume of the parallelepiped as follows:

$$\mathsf{V}(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}) = \left\| \vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \right\| = \left\| (0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (-8\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) \right\| = \left\| 0 + 2 - 18 \right\| = 16$$

3) The $\vec{BC}$ vector can be calculated as:

$$\vec{BC} = \vec{OC} - \vec{OB} = (0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Hence, it is possible to evaluate the projection vector of $\vec{AB}$ onto $\vec{BC}$ as follows:

$$\begin{aligned} \overrightarrow{\mathbf{proj}}_{\vec{BC}} \vec{AB} = \frac{\vec{BC} \cdot \vec{AB}}{\underbrace{\vec{BC} \cdot \vec{BC}}_{\|\vec{BC}\|^2}} \vec{BC} &= \frac{(-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (1\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}})}{(-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})} (-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \\ &= \frac{(-2 - 8 + 0)}{(4 + 4 + 4)} (-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = \frac{5}{3}\hat{\mathbf{i}} - \frac{5}{3}\hat{\mathbf{j}} - \frac{5}{3}\hat{\mathbf{k}} \end{aligned}$$

Problem 1.5

Rewrite the following equations using indicial notation:

1) $a_1 x_1 x_3 + a_2 x_2 x_3 + a_3 x_3 x_3$

Solution: $a_i x_i x_3 \qquad (i = 1,2,3)$

2) $x_1x_1 + x_2x_2$

Solution: $x_i x_i \qquad (i = 1,2)$

3) $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_x \\ a_{21}x + a_{22}y + a_{23}z = b_y \\ a_{31}x + a_{32}y + a_{33}z = b_z \end{cases}$

Solution:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \xrightarrow[\text{index } j]{\text{dummy}} \begin{cases} a_{1j}x_j = b_1 \\ a_{2j}x_j = b_2 \\ a_{3j}x_j = b_3 \end{cases} \xrightarrow[\text{index } i]{\text{free}} \boxed{a_{ij}x_j = b_i}$$

As we can appreciate in this problem, the use of the indicial notation means that the equation becomes very concise. In many cases, if algebraic operation do not use indicial or tensorial notation they become almost impossible to deal with due to the large number of terms involved.

Problem 1.6

Show that:

a) $\delta_{3p}v_p = v_3$; b) $\delta_{3i}A_{ji} = A_{j3}$; c) $\delta_{ij}\epsilon_{ijk}$; d) $\delta_{i2}\delta_{j3}A_{ij}$.

Solution:

The Kronecker delta components are:

$$\delta_{ij} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1}$$

a) The expression ($\delta_{3p}v_p$) has no free index, then the result is a scalar:

$$\delta_{3p}v_p = \delta_{31}v_1 + \delta_{32}v_2 + \delta_{33}v_3 = v_3 \tag{1.2}$$

b) The expression $\delta_{3i}A_{ji}$ has one free index (j), then the result is a vector:

$$\delta_{3i}A_{ji} = \delta_{31}A_{j1} + \delta_{32}A_{j2} + \delta_{33}A_{j3} = A_{j3} \tag{1.3}$$

c) The expression $\delta_{ij}\epsilon_{ijk}$ has one free index (k), then the result is a vector:

$$\begin{array}{ccccccc} \delta_{ij}\epsilon_{ijk} = & \underbrace{\delta_{1j}\epsilon_{1jk}} & + & \underbrace{\delta_{2j}\epsilon_{2jk}} & + & \underbrace{\delta_{3j}\epsilon_{3jk}} \\ & \delta_{11}\epsilon_{11k} & + & \delta_{21}\epsilon_{21k} & + & \delta_{31}\epsilon_{31k} \\ & + & & + & & + \\ & \delta_{12}\epsilon_{12k} & + & \delta_{22}\epsilon_{22k} & + & \delta_{32}\epsilon_{32k} \\ & + & & + & & + \\ & \delta_{13}\epsilon_{13k} & + & \delta_{23}\epsilon_{23k} & + & \delta_{33}\epsilon_{33k} \end{array} \tag{1.4}$$

thus $\delta_{ij}\epsilon_{ijk} = 0_k$ is the null vector. Note that $\delta_{ij}\epsilon_{ijk} = \epsilon_{iik} = \epsilon_{11k} + \epsilon_{22k} + \epsilon_{33k} = 0_k$.

d)

$$\delta_{i2}\delta_{j3}A_{ij} = A_{23} \tag{1.5}$$

Problem 1.7

Expand the equation: $A_{ij}x_i x_j \qquad (i,j = 1,2,3)$

Solution: The indices i,j are dummy indices, and indicate index summation and there is no free index in the expression $A_{ij}x_i x_j$, therefore the result is a scalar. So, we expand first the dummy index i and later the index j to obtain:

$$
\begin{array}{ccccccc}
A_{ij}x_i x_j & \xrightarrow{\text{expanding } i} & \underbrace{A_{1j}x_1 x_j} & + & \underbrace{A_{2j}x_2 x_j} & + & \underbrace{A_{3j}x_3 x_j} \\
 & \Big\downarrow {\scriptstyle \text{expanding } j} & A_{11}x_1x_1 & & A_{21}x_2x_1 & & A_{31}x_3x_1 \\
 & & + & & + & & + \\
 & & A_{12}x_1x_2 & & A_{22}x_2x_2 & & A_{32}x_3x_2 \\
 & & + & & + & & + \\
 & & A_{13}x_1x_3 & & A_{23}x_2x_3 & & A_{33}x_3x_3
\end{array}
$$

Rearranging the terms we obtain:

$$
\begin{aligned}
A_{ij}x_i x_j = {} & A_{11}x_1x_1 + A_{12}x_1x_2 + A_{13}x_1x_3 + A_{21}x_2x_1 + A_{22}x_2x_2 + \\
& A_{23}x_2x_3 + A_{31}x_3x_1 + A_{32}x_3x_2 + A_{33}x_3x_3
\end{aligned}
$$

Problem 1.8

Obtain the numerical value of:

1) $\delta_{ii}\delta_{jj}$

Solution: $\quad \delta_{ii}\delta_{jj} = (\delta_{11} + \delta_{22} + \delta_{33})(\delta_{11} + \delta_{22} + \delta_{33}) = 3 \times 3 = 9$

2) $\delta_{\alpha 1}\delta_{\alpha\gamma}\delta_{\gamma 1}$

Solution: $\quad \delta_{\alpha 1}\delta_{\alpha\gamma}\delta_{\gamma 1} = \delta_{\gamma 1}\delta_{\gamma 1} = \delta_{11} = 1$

NOTE: Note that the following algebraic operation is incorrect $\delta_{\gamma 1}\delta_{\gamma 1} \neq \delta_{\gamma\gamma} = 3 \neq \delta_{11} = 1$, since what must be replaced is the repeated index, not the number.

Problem 1.9

a) Prove the following is true $\epsilon_{ijk}\epsilon_{pjk} = 2\delta_{ip}$, $\epsilon_{ijk}\epsilon_{ijk} = 6$ and $\epsilon_{ijk}a_j a_k = 0_i$. b) Obtain the numerical value of $\epsilon_{ijk}\delta_{2j}\delta_{3k}\delta_{1i}$.

Solution: a) Using the equation $\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$, and by substituting q for j, we obtain:

$$\epsilon_{ijk}\epsilon_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = \delta_{ip}3 - \delta_{ip} = 2\delta_{ip}$$

Based on the above result, it is straight forward to check that:

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6$$

Note that $\epsilon_{ijk} = -\epsilon_{ikj}$, i.e. it is antisymmetric in jk and also note that $a_j a_k$ is a symmetric second-order tensor. So, as we know, the double scalar product between a symmetric and an antisymmetric second-order tensors is zero, thus:

$$\epsilon_{ijk} a_j a_k = \epsilon_{ijk} (\vec{a} \otimes \vec{a})_{jk} = 0_i = (\vec{a} \wedge \vec{a})_i = 0_i$$

b) $\epsilon_{ijk} \delta_{2j} \delta_{3k} \delta_{1i} = \epsilon_{123} = 1$

Problem 1.10

Get the value of the following expressions:

a) $\epsilon_{ijk} \delta_{i1} \delta_{j2} \delta_{3k}$

b) $\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$ for the following cases:

b.1) $i = 1, j = q = 2, p = 3$

b.2) $i = q = 1, j = p = 2$

c) $(\epsilon_{ijk} \mathsf{A}_{jp} \mathsf{c}_p \mathsf{A}_{kq} \mathsf{c}_q + \delta_{i1})(\epsilon_{ist} \mathsf{A}_{sa} \mathsf{c}_a \mathsf{A}_{tb} \mathsf{c}_b + \delta_{i1})$

where ϵ_{ijk} is the permutation symbol, (see Figure 1.6), and δ_{ij} is the Kronecker delta.

Reminder: Permutation symbol

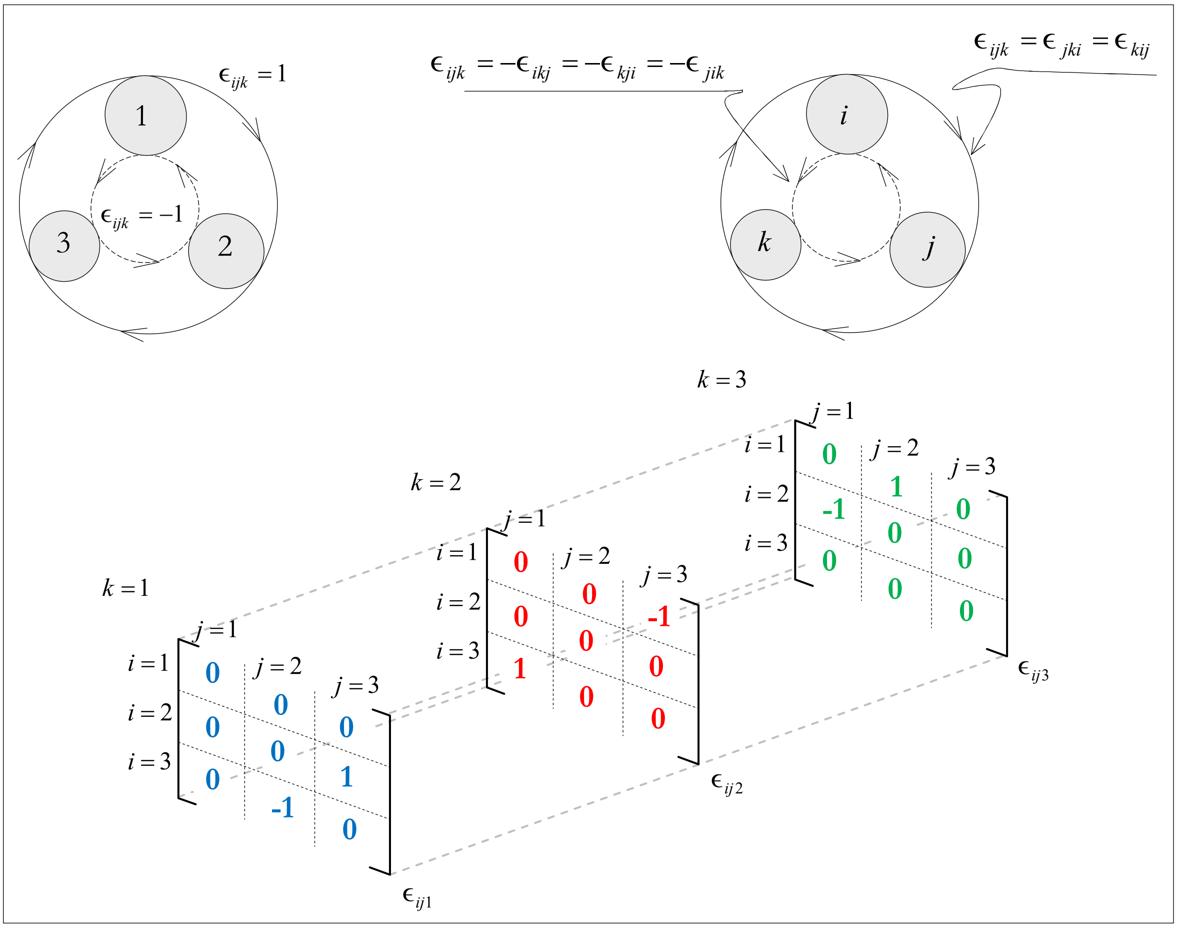


Figure 1.6

Solution:

a) $\epsilon_{ijk} \delta_{i1} \delta_{j2} \delta_{3k} = \epsilon_{123} = 1$;

b.1) $\epsilon_{12k} \epsilon_{32k} = \epsilon_{121} \epsilon_{321} + \epsilon_{122} \epsilon_{322} + \epsilon_{123} \epsilon_{323} = 0 \times (-1) + 0 \times 0 + 0 \times 0 = 0$

b.2) $\epsilon_{12k} \epsilon_{21k} = \epsilon_{121} \epsilon_{211} + \epsilon_{122} \epsilon_{212} + \epsilon_{123} \epsilon_{213} = 0 \times 0 + 0 \times 0 + 1 \times (-1) = -1$

c) Note that the result of $\mathsf{A}_{jp}\mathsf{c}_p = \mathsf{b}_j$ is a vector, and also note that the following is true $\epsilon_{ijk}\mathsf{A}_{jp}\mathsf{c}_p\mathsf{A}_{kq}\mathsf{c}_q = \left[(\mathbf{A}\cdot\vec{\mathbf{c}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]_i = (\vec{\mathbf{b}}\wedge\vec{\mathbf{b}})_i = 0_i$, with which we can obtain:

$$(\epsilon_{ijk}\mathsf{A}_{jp}\mathsf{c}_p\mathsf{A}_{kq}\mathsf{c}_q + \delta_{i1})(\epsilon_{ist}\mathsf{A}_{sa}\mathsf{c}_a\mathsf{A}_{tb}\mathsf{c}_b + \delta_{i1}) = \delta_{i1}\delta_{i1} = \delta_{11} = 1$$

NOTE 1: The second-order tensor $\epsilon_{ijk}w_k$ can easily be obtained as follows:

$$\begin{aligned}\epsilon_{ijk}w_k &= \epsilon_{ij1}w_1 + \epsilon_{ij2}w_2 + \epsilon_{ij3}w_3\\ &= w_1\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix} + w_2\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix} + w_3\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}0&w_3&-w_2\\-w_3&0&w_1\\w_2&-w_1&0\end{bmatrix}\end{aligned}$$

NOTE 2: The relationship $\hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_j$ can also be expressed in terms of ϵ_{ijk}, since:

$$\begin{aligned}\hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_j &= \begin{bmatrix}\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_3\wedge\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3\wedge\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\wedge\hat{\mathbf{e}}_3\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{0}} & \hat{\mathbf{e}}_3 & -\hat{\mathbf{e}}_2\\ -\hat{\mathbf{e}}_3 & \vec{\mathbf{0}} & \hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2 & -\hat{\mathbf{e}}_1 & \vec{\mathbf{0}}\end{bmatrix}\\ &= \begin{bmatrix}\vec{\mathbf{0}}&\vec{\mathbf{0}}&\vec{\mathbf{0}}\\ \vec{\mathbf{0}}&\vec{\mathbf{0}}&\hat{\mathbf{e}}_1\\ \vec{\mathbf{0}}&-\hat{\mathbf{e}}_1&\vec{\mathbf{0}}\end{bmatrix} + \begin{bmatrix}\vec{\mathbf{0}}&\vec{\mathbf{0}}&-\hat{\mathbf{e}}_2\\ \vec{\mathbf{0}}&\vec{\mathbf{0}}&\vec{\mathbf{0}}\\ \hat{\mathbf{e}}_2&\vec{\mathbf{0}}&\vec{\mathbf{0}}\end{bmatrix} + \begin{bmatrix}\vec{\mathbf{0}}&\hat{\mathbf{e}}_3&\vec{\mathbf{0}}\\ -\hat{\mathbf{e}}_3&\vec{\mathbf{0}}&\vec{\mathbf{0}}\\ \vec{\mathbf{0}}&\vec{\mathbf{0}}&\vec{\mathbf{0}}\end{bmatrix}\\ &= \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\hat{\mathbf{e}}_1 + \begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}\hat{\mathbf{e}}_2 + \begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\hat{\mathbf{e}}_3\\ &= \epsilon_{ij1}\hat{\mathbf{e}}_1 + \epsilon_{ij2}\hat{\mathbf{e}}_2 + \epsilon_{ij3}\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_j &= \epsilon_{ijk}\hat{\mathbf{e}}_k = \epsilon_{kij}\hat{\mathbf{e}}_k\end{aligned}$$

Note also that ϵ_{ijk} is antisymmetric tensor in ij, since $\epsilon_{ijk} = -\epsilon_{jik}$, (see Figure 1.6).

Problem 1.11

Write in indicial notation: a) the modulus of the vector $\vec{\mathbf{a}}$; b) $\cos\theta$, where θ is the angle formed by the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution:

$$\|\vec{\mathbf{a}}\|^2 = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \mathsf{a}_i\overbrace{\hat{\mathbf{e}}_i\cdot\mathsf{a}_j\hat{\mathbf{e}}_j}^{\delta_{ij}} = \mathsf{a}_i\mathsf{a}_j\delta_{ij} = \mathsf{a}_i\mathsf{a}_i = \mathsf{a}_j\mathsf{a}_j \quad\Rightarrow\quad \|\vec{\mathbf{a}}\| = \sqrt{\mathsf{a}_i\mathsf{a}_i}$$

thus, it is also true that $\|\vec{\mathbf{b}}\| = \sqrt{\mathsf{b}_i\mathsf{b}_i}$.

By definition $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\cos\theta$ where:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \mathsf{a}_i\hat{\mathbf{e}}_i\cdot\mathsf{b}_j\hat{\mathbf{e}}_j = \mathsf{a}_i\mathsf{b}_j\delta_{ij} = \mathsf{a}_i\mathsf{b}_i = \mathsf{a}_j\mathsf{b}_j$$

Taking into account that the index cannot appear more than twice in a term of the expression, we can express $\cos\theta$ as follows:

$$\cos\theta = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|} = \frac{\mathsf{a}_j\mathsf{b}_j}{\sqrt{\mathsf{a}_i\mathsf{a}_i}\sqrt{\mathsf{b}_k\mathsf{b}_k}}$$

Problem 1.12

Show the Schwarz inequality:

$$\boxed{\|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\| \le \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|} \quad \textit{Schwarz inequality} \tag{1.6}$$

Solution:

Let us consider a scalar α, then the following is true:

$$\underbrace{\|\vec{\mathbf{a}}\alpha-\vec{\mathbf{b}}\|^2}_{\ge 0} = (\vec{\mathbf{a}}\alpha-\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}\alpha-\vec{\mathbf{b}}) = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\alpha^2 - \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\alpha - \vec{\mathbf{b}}\cdot\vec{\mathbf{a}}\alpha + \vec{\mathbf{b}}\cdot\vec{\mathbf{b}} \ge 0$$

$$= \|\vec{\mathbf{a}}\|^2\alpha^2 - 2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\alpha + \|\vec{\mathbf{b}}\|^2 \ge 0$$

where we define $f(\alpha) = \|\vec{\mathbf{a}}\|^2\alpha^2 - 2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\alpha + \|\vec{\mathbf{b}}\|^2 \ge 0$. Now, for the case when $\alpha = \dfrac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{a}}\|^2}$ we can obtain:

$$f\left(\alpha = \frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}{\|\vec{\mathbf{a}}\|^2}\right) = \|\vec{\mathbf{a}}\|^2\left(\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}{\|\vec{\mathbf{a}}\|^2}\right)^2 - 2(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}{\|\vec{\mathbf{a}}\|^2} + \|\vec{\mathbf{b}}\|^2 \ge 0$$

$$= \|\vec{\mathbf{a}}\|^2\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2}{\|\vec{\mathbf{a}}\|^4} - 2(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}{\|\vec{\mathbf{a}}\|^2} + \|\vec{\mathbf{b}}\|^2 = \frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2}{\|\vec{\mathbf{a}}\|^2} - 2\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2}{\|\vec{\mathbf{a}}\|^2} + \|\vec{\mathbf{b}}\|^2 \ge 0$$

$$= -\frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2}{\|\vec{\mathbf{a}}\|^2} + \|\vec{\mathbf{b}}\|^2 \ge 0$$

$$\Rightarrow \|\vec{\mathbf{b}}\|^2 \ge \frac{(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2}{\|\vec{\mathbf{a}}\|^2} \quad \Rightarrow \quad \|\vec{\mathbf{a}}\|^2\|\vec{\mathbf{b}}\|^2 \ge (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2 \quad \Rightarrow \quad \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \ge \|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\|$$

Q.E.D.

Alternative solution

Taking in account that $0 \le \|\cos\theta\| \le 1$ we obtain $\|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\| = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\|\cos\theta\| \le \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|$, thus we can conclude that $\|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\| \le \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|$.

Problem 1.13

Rewrite the expression $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})$ without using the vector product symbol.

Solution: The vector product $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$ can be expressed as $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathrm{a}_j\hat{\mathbf{e}}_j \wedge \mathrm{b}_k\hat{\mathbf{e}}_k = \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\hat{\mathbf{e}}_i$. Likewise, it is possible to express $(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})$ as $(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) = \epsilon_{nlm}\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_n$, thus:

$$\begin{aligned}(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) &= (\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\hat{\mathbf{e}}_i)\cdot(\epsilon_{nlm}\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_n) = \epsilon_{ijk}\epsilon_{nlm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_n \\ &= \epsilon_{ijk}\epsilon_{nlm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m\delta_{in} = \epsilon_{ijk}\epsilon_{ilm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m\end{aligned}$$

Taking into account that $\epsilon_{ijk}\epsilon_{ilm} = \epsilon_{jki}\epsilon_{lmi}$ and by applying the equation $\epsilon_{jki}\epsilon_{lmi} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = \epsilon_{jki}\epsilon_{ilm}$, we obtain:

$$\epsilon_{ijk}\epsilon_{ilm}\mathsf{a}_j\mathsf{b}_k\mathsf{c}_l\mathsf{d}_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\,\mathsf{a}_j\mathsf{b}_k\mathsf{c}_l\mathsf{d}_m = \mathsf{a}_l\mathsf{b}_m\mathsf{c}_l\mathsf{d}_m - \mathsf{a}_m\mathsf{b}_l\mathsf{c}_l\mathsf{d}_m$$

Since $\mathsf{a}_l\mathsf{c}_l = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})$ and $\mathsf{b}_m\mathsf{d}_m = (\vec{\mathbf{b}}\cdot\vec{\mathbf{d}})$ holds true, we can conclude that:

$$(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{d}}) - (\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})$$

Therefore, it is also valid when $\vec{\mathbf{a}} = \vec{\mathbf{c}}$ and $\vec{\mathbf{b}} = \vec{\mathbf{d}}$, thus:

$$(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2 = (\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}) - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{a}}) = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2 - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2$$

which is the same equation obtained in **Problem 1.1**.

NOTE: We can start from the above equation to show $\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| = \left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\left\|\sin\theta\right\|$, i.e.:

$$\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2 = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2 - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2 = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2 - \left(\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\cos\theta\right)^2 = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2(1-\cos^2\theta) = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2\sin^2\theta$$

$$\Rightarrow \left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| = \left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\left\|\sin\theta\right\|$$

Note that $0 \le \left\|\sin\theta\right\| \le 1$, with that we can prove that $\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| = \left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\left\|\sin\theta\right\| \le \left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|$, thus

$$\boxed{\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| \le \left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|}$$

Problem 1.14

Show that:

a) $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_k = 0$;

b) $\epsilon_{ijk}\left(\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3}\right) = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$;

c) $A_{ij}A_{ji}$ is an invariant.

Solution:

a) $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_k = \epsilon_{ij1}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1 + \epsilon_{ij2}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_2 + \epsilon_{ij3}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_3$. The term $\epsilon_{ij1}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1$ can be evaluated as follows:

$$\begin{aligned}
\epsilon_{ij1}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1 &= \epsilon_{1j1}\mathsf{a}_1\mathsf{a}_j\mathsf{b}_1 + \epsilon_{2j1}\mathsf{a}_2\mathsf{a}_j\mathsf{b}_1 + \epsilon_{3j1}\mathsf{a}_3\mathsf{a}_j\mathsf{b}_1 \\
&= \epsilon_{111}\mathsf{a}_1\mathsf{a}_1\mathsf{b}_1 + \epsilon_{211}\mathsf{a}_2\mathsf{a}_1\mathsf{b}_1 + \epsilon_{311}\mathsf{a}_3\mathsf{a}_1\mathsf{b}_1 + \\
&\quad + \epsilon_{121}\mathsf{a}_1\mathsf{a}_2\mathsf{b}_1 + \epsilon_{221}\mathsf{a}_2\mathsf{a}_2\mathsf{b}_1 + \epsilon_{321}\mathsf{a}_3\mathsf{a}_2\mathsf{b}_1 + \\
&\quad + \epsilon_{131}\mathsf{a}_1\mathsf{a}_3\mathsf{b}_1 + \epsilon_{231}\mathsf{a}_2\mathsf{a}_3\mathsf{b}_1 + \epsilon_{331}\mathsf{a}_3\mathsf{a}_3\mathsf{b}_1 \\
&= \epsilon_{321}\mathsf{a}_3\mathsf{a}_2\mathsf{b}_1 + \epsilon_{231}\mathsf{a}_2\mathsf{a}_3\mathsf{b}_1 = -\mathsf{a}_3\mathsf{a}_2\mathsf{b}_1 + \mathsf{a}_2\mathsf{a}_3\mathsf{b}_1 \\
&= 0
\end{aligned}$$

In the same way we can obtain $\epsilon_{ij2}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_2 = \epsilon_{ij3}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_3 = 0$.

b)

$$\begin{aligned}
&\epsilon_{ijk}\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \epsilon_{ijk}\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \epsilon_{ijk}\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3} = \\
&\qquad \epsilon_{12k}\mathsf{a}_k\mathsf{b}_3 + \epsilon_{1j3}\mathsf{a}_j\mathsf{b}_2 + \epsilon_{i23}\mathsf{a}_i\mathsf{b}_1 = \mathsf{a}_3\mathsf{b}_3 + \mathsf{a}_2\mathsf{b}_2 + \mathsf{a}_1\mathsf{b}_1 = \mathsf{a}_i\mathsf{b}_i = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}
\end{aligned}$$

Problem 1.15

Prove that $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}}) = \vec{\mathbf{c}}[\vec{\mathbf{d}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})] - \vec{\mathbf{d}}[\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]$

Solution: Expressing the correct equality term in indicial notation we obtain:

$$\left\{\vec{\mathbf{c}}[\vec{\mathbf{d}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})] - \vec{\mathbf{d}}[\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]\right\}_p = \mathrm{c}_p[\mathrm{d}_i(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)] - \mathrm{d}_p[\mathrm{c}_i(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)]$$

$$\Rightarrow \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_p\mathrm{d}_i - \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_i\mathrm{d}_p \quad \Rightarrow \quad \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k(\mathrm{c}_p\mathrm{d}_i - \mathrm{c}_i\mathrm{d}_p)$$

Using the Kronecker delta the above equation becomes:

$$\Rightarrow \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k(\delta_{pm}\mathrm{c}_m\mathrm{d}_n\delta_{ni} - \delta_{im}\mathrm{c}_m\mathrm{d}_n\delta_{np}) \quad \Rightarrow \quad (\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)\,\mathrm{c}_m\mathrm{d}_n(\delta_{pm}\delta_{ni} - \delta_{im}\delta_{np})$$

and by applying the equation $\delta_{pm}\delta_{ni} - \delta_{im}\delta_{np} = \epsilon_{pil}\epsilon_{mnl}$, the above equation can be rewritten as follows:

$$\Rightarrow (\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)\,\mathrm{c}_m\mathrm{d}_n(\epsilon_{pil}\epsilon_{mnl}) \quad \Rightarrow \quad \epsilon_{pil}[(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)(\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n)]$$

Since $\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k$ and $\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n$ represent the components of $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ and $(\vec{\mathbf{c}} \wedge \vec{\mathbf{d}})$, respectively, we can conclude that:

$$\epsilon_{pil}[(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)(\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n)] = [(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}})]_p$$

Problem 1.16

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be linearly independent vectors, and $\vec{\mathbf{v}}$ be a vector, demonstrate that:

$$\vec{\mathbf{v}} = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}} + \gamma\vec{\mathbf{c}} \neq \vec{\mathbf{0}} \xrightarrow{\text{components}} \mathrm{v}_i = \alpha\mathrm{a}_i + \beta\mathrm{b}_i + \gamma\mathrm{c}_i \neq 0_i$$

where the scalars α, β, γ are given by:

$$\alpha = \frac{\epsilon_{ijk}\mathrm{v}_i\mathrm{b}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r} \quad ; \quad \beta = \frac{\epsilon_{ijk}\mathrm{a}_i\mathrm{v}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r} \quad ; \quad \gamma = \frac{\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{v}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}$$

b) Given three linearly independent vectors, show that: when interchanging two rows or two columns the sign of the determinant $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})$ changes.

Solution: a) The scalar product made up of $\vec{\mathbf{v}}$ and $(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})$ becomes:

$$\vec{\mathbf{v}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = \alpha\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) + \beta\underbrace{\vec{\mathbf{b}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}_{=0} + \gamma\underbrace{\vec{\mathbf{c}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}_{=0} \quad \Rightarrow \quad \alpha = \frac{\vec{\mathbf{v}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}{\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}$$

which is the same as:

$$\alpha = \frac{\begin{vmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}}{\begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}} = \frac{\begin{vmatrix} \mathrm{v}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{v}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{v}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}}{\begin{vmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}} = \frac{\epsilon_{ijk}\mathrm{v}_i\mathrm{b}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}$$

One can obtain the parameters β and γ in a similar fashion, i.e.:

$$\vec{\mathbf{v}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})=\alpha\underbrace{\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})}_{=0}+\beta\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})+\gamma\underbrace{\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})}_{=0}$$

$$\Rightarrow\beta=\frac{\vec{\mathbf{v}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})}{\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})}=\frac{\epsilon_{ijk}\mathsf{v}_i\mathsf{a}_j\mathsf{c}_k}{\epsilon_{pqr}\mathsf{b}_p\mathsf{a}_q\mathsf{c}_r}=\frac{-\epsilon_{jik}\mathsf{a}_j\mathsf{v}_i\mathsf{c}_k}{-\epsilon_{qpr}\mathsf{a}_q\mathsf{b}_p\mathsf{c}_r}=\frac{\vec{\mathbf{a}}\cdot(\vec{\mathbf{v}}\wedge\vec{\mathbf{c}})}{\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}$$

$$\vec{\mathbf{v}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=\alpha\underbrace{\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})}_{=0}+\beta\underbrace{\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})}_{=0}+\gamma\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$$

$$\Rightarrow\gamma=\frac{\vec{\mathbf{v}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})}{\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})}=\frac{\epsilon_{ijk}\mathsf{v}_i\mathsf{a}_j\mathsf{b}_k}{\epsilon_{pqr}\mathsf{c}_p\mathsf{a}_q\mathsf{b}_r}=\frac{\epsilon_{jki}\mathsf{a}_j\mathsf{b}_k\mathsf{v}_i}{\epsilon_{qrp}\mathsf{a}_q\mathsf{b}_r\mathsf{c}_p}=\frac{\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{v}})}{\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}$$

NOTE 1: We can restructure the $\vec{\mathbf{v}}$-components as follows:

$$\mathsf{v}_i=\begin{Bmatrix}\mathsf{v}_1\\\mathsf{v}_2\\\mathsf{v}_3\end{Bmatrix}=\begin{bmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{c}_3\end{bmatrix}\begin{Bmatrix}\alpha\\\beta\\\gamma\end{Bmatrix}=\begin{bmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{c}_3\end{bmatrix}\begin{Bmatrix}\mathsf{z}_1\\\mathsf{z}_2\\\mathsf{z}_3\end{Bmatrix}=\mathsf{B}_{ij}\mathsf{z}_j$$

where we have denoted by $\mathsf{z}_1=\alpha$, $\mathsf{z}_2=\beta$, $\mathsf{z}_3=\gamma$, in which:

$$\alpha=\mathsf{z}_1=\frac{\epsilon_{ijk}\mathsf{v}_i\mathsf{b}_j\mathsf{c}_k}{\epsilon_{pqr}\mathsf{a}_p\mathsf{b}_q\mathsf{c}_r}=\frac{\begin{vmatrix}\mathsf{v}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{v}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{v}_3&\mathsf{b}_3&\mathsf{c}_3\end{vmatrix}}{\begin{vmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{c}_3\end{vmatrix}}=\frac{\left|\overline{\mathbf{B}}^{(1)}\right|}{|\mathbf{B}|};\quad \beta=\mathsf{z}_2=\frac{\epsilon_{ijk}\mathsf{a}_i\mathsf{v}_j\mathsf{c}_k}{\epsilon_{pqr}\mathsf{a}_p\mathsf{b}_q\mathsf{c}_r}=\frac{\begin{vmatrix}\mathsf{a}_1&\mathsf{v}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{v}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{v}_3&\mathsf{c}_3\end{vmatrix}}{\begin{vmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{c}_3\end{vmatrix}}=\frac{\left|\overline{\mathbf{B}}^{(2)}\right|}{|\mathbf{B}|}$$

$$\gamma=\mathsf{z}_3=\frac{\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{v}_k}{\epsilon_{pqr}\mathsf{a}_p\mathsf{b}_q\mathsf{c}_r}=\frac{\begin{vmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{v}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{v}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{v}_3\end{vmatrix}}{\begin{vmatrix}\mathsf{a}_1&\mathsf{b}_1&\mathsf{c}_1\\\mathsf{a}_2&\mathsf{b}_2&\mathsf{c}_2\\\mathsf{a}_3&\mathsf{b}_3&\mathsf{c}_3\end{vmatrix}}=\frac{\left|\overline{\mathbf{B}}^{(3)}\right|}{|\mathbf{B}|}$$

where $\left|\overline{\mathbf{B}}^{(i)}\right|$ is the determinant of the resulting matrix by replacing the column (i) of the matrix $\mathbf{B}$ by the $\vec{\mathbf{v}}$-components. With that we can state that:

$$\text{Given } \mathsf{v}_i=\mathsf{B}_{ij}\mathsf{z}_j \quad\Rightarrow\quad \mathsf{z}_i=\frac{\left|\overline{\mathbf{B}}^{(i)}\right|}{|\mathbf{B}|}\quad \textit{Cramer's rule}$$

NOTE 2: Although we have demonstrated for 3×3 matrix, this procedure is also valid for matrices of *n*-dimensions, which is known, in the literature, as Cramer's Rule.

NOTE 3: The solution (z_i) is possible if $|\mathbf{B}|\neq0$.

NOTE 4: If $\mathsf{v}_i=\mathsf{0}_i$ we have $\mathsf{B}_{ij}\mathsf{z}_j=\mathsf{0}_i$ and $\left|\overline{\mathbf{B}}^{(i)}\right|=0_i$, with that according to Cramer's rule we have:

$$\mathsf{z}_i|\mathbf{B}|=\left|\overline{\mathbf{B}}^{(i)}\right|=\mathsf{0}_i$$

Note that the non-trivial solution $\mathsf{z}_i\neq\mathsf{0}_i$ is only possible if and only if $|\mathbf{B}|=0$, (see **Problem 1.50**).

b) The determinant defined by $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$ in indicial notation becomes $\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k$, and by taking into account the permutation symbol properties, (see Figure 1.7), we can conclude that:

$$\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = \begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}$$

$$-\epsilon_{ikj}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = -\begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \end{vmatrix}$$

$$\epsilon_{jki}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = \begin{vmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \\ \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{vmatrix}$$

$$\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] = -\epsilon_{ikj}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = -[\vec{\mathbf{a}},\vec{\mathbf{c}},\vec{\mathbf{b}}] = \epsilon_{jki}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = [\vec{\mathbf{b}},\vec{\mathbf{c}},\vec{\mathbf{a}}]$$

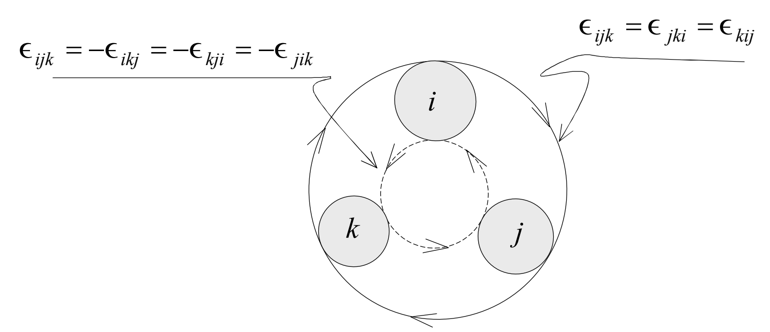


Figure 1.7

Problem 1.17

a) Show that

$$\boxed{\begin{aligned} &\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\,\vec{\mathbf{b}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\,\vec{\mathbf{c}} = (\vec{\mathbf{b}}\otimes\vec{\mathbf{c}} - \vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}} \\ &\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}) = [(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}}\otimes\vec{\mathbf{a}}]\cdot\vec{\mathbf{b}} \end{aligned}}$$

b) Obtain the explicit component of the tensor $[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}}\otimes\vec{\mathbf{a}}]$.

Solution:

a) Taking into account that $(\vec{\mathbf{d}})_i = (\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})_i = \epsilon_{ijk}\mathrm{b}_j\mathrm{c}_k$ and that $(\vec{\mathbf{a}}\wedge\vec{\mathbf{d}})_q = \epsilon_{qjk}\mathrm{b}_j\mathrm{c}_k$, we obtain:

$$\begin{aligned}
[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]_r &= \epsilon_{rsi}\mathrm{a}_s(\epsilon_{ijk}\mathrm{b}_j\mathrm{c}_k) \\
&= \epsilon_{rsi}\epsilon_{ijk}\mathrm{a}_s\mathrm{b}_j\mathrm{c}_k = \epsilon_{rsi}\epsilon_{jki}\mathrm{a}_s\mathrm{b}_j\mathrm{c}_k \\
&= (\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj})\,\mathrm{a}_s\mathrm{b}_j\mathrm{c}_k \\
&= \delta_{rj}\delta_{sk}\mathrm{a}_s\mathrm{b}_j\mathrm{c}_k - \delta_{rk}\delta_{sj}\mathrm{a}_s\mathrm{b}_j\mathrm{c}_k && = \mathrm{a}_s\mathrm{b}_r\mathrm{c}_s - \mathrm{a}_s\mathrm{b}_s\mathrm{c}_r \\
&= \mathrm{a}_k\mathrm{b}_r\mathrm{c}_k - \mathrm{a}_j\mathrm{b}_j\mathrm{c}_r && = (\mathrm{b}_r\mathrm{c}_s - \mathrm{b}_s\mathrm{c}_r)\mathrm{a}_s \\
&= \mathrm{b}_r(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}) - \mathrm{c}_r(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) && = [(\vec{\mathbf{b}}\otimes\vec{\mathbf{c}} - \vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}}]_r \\
&= [\vec{\mathbf{b}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})]_r
\end{aligned}$$

With that we conclude that:

$$\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\,\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\,\vec{\mathbf{c}} = (\vec{\mathbf{b}} \otimes \vec{\mathbf{c}} - \vec{\mathbf{c}} \otimes \vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}$$

Note that it is also true that:

$$\boxed{\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{b}} \otimes \vec{\mathbf{c}} - \vec{\mathbf{c}} \otimes \vec{\mathbf{b}}) \cdot \vec{\mathbf{a}} = [(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\mathbf{1} - \vec{\mathbf{c}} \otimes \vec{\mathbf{a}}] \cdot \vec{\mathbf{b}} = [\vec{\mathbf{b}} \otimes \vec{\mathbf{a}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\mathbf{1}] \cdot \vec{\mathbf{c}}}$$

In the particular case when $\vec{\mathbf{a}} = \vec{\mathbf{c}}$ we obtain:

$$\begin{aligned}
[\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{a}})]_r &= (a_k a_k) b_r - (a_j b_j) a_r = (a_j a_j) b_p \delta_{rp} - (a_j b_p \delta_{jp}) a_r \\
&= [(a_j a_j)\delta_{rp} - (a_j \delta_{jp}) a_r] b_p = [(a_j a_j)\delta_{rp} - a_p a_r] b_p \\
&= \left\{ [(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}} \otimes \vec{\mathbf{a}}] \cdot \vec{\mathbf{b}} \right\}_r
\end{aligned}$$

b) The components of $[(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}} \otimes \vec{\mathbf{a}}]$ can be obtained as follows:

$$\begin{aligned}
[(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}} \otimes \vec{\mathbf{a}}]_{ij} &= (a_k a_k)\delta_{ij} - a_i a_j = (a_1^2 + a_2^2 + a_3^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2 a_2 & a_1 a_3 \\ a_1 a_3 & a_1 a_3 & a_3 a_3 \end{bmatrix} \\
&= \begin{bmatrix} (a_2^2 + a_3^2) & -a_1 a_2 & -a_1 a_3 \\ -a_1 a_2 & (a_1^2 + a_3^2) & -a_1 a_3 \\ -a_1 a_3 & -a_1 a_3 & (a_1^2 + a_2^2) \end{bmatrix}
\end{aligned}$$

Problem 1.18

Show the *Jacobi identity:*

$$\boxed{\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) + \vec{\mathbf{b}} \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{a}}) + \vec{\mathbf{c}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \vec{\mathbf{0}}}$$

Solution: By means of **Problem 1.17** in which $\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\,\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\,\vec{\mathbf{c}}$ was proven, we can obtain that:

$$\vec{\mathbf{b}} \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{a}}) = (\vec{\mathbf{b}} \cdot \vec{\mathbf{a}})\,\vec{\mathbf{c}} - (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})\,\vec{\mathbf{a}} \quad ; \quad \vec{\mathbf{c}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{b}})\,\vec{\mathbf{a}} - (\vec{\mathbf{c}} \cdot \vec{\mathbf{a}})\,\vec{\mathbf{b}}$$

Then, by considering that the dot product is commutative, i.e. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{a}})$, $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) = (\vec{\mathbf{b}} \cdot \vec{\mathbf{a}})$, $(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{b}})$, we can conclude that:

$$\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) + \vec{\mathbf{b}} \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{a}}) + \vec{\mathbf{c}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \begin{matrix} \cancel{(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})}\vec{\mathbf{b}} - \cancel{(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})}\vec{\mathbf{c}} \\ + \\ \cancel{(\vec{\mathbf{b}} \cdot \vec{\mathbf{a}})}\vec{\mathbf{c}} - \cancel{(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})}\vec{\mathbf{a}} \\ + \\ \cancel{(\vec{\mathbf{c}} \cdot \vec{\mathbf{b}})}\vec{\mathbf{a}} - \cancel{(\vec{\mathbf{c}} \cdot \vec{\mathbf{a}})}\vec{\mathbf{b}} \end{matrix} = \vec{\mathbf{0}}$$

1.2 Algebraic Operations with Higher Order Tensors

Problem 1.19

Define the order of the tensors represented by their Cartesian components: v_i, Φ_{ijk}, F_{ijj}, ε_{ij}, $\mathbb{C}_{ijkl}$, σ_{ij}. Determine the number of components in tensor $\mathbb{C}$.

Solution: The order of the tensor is given by the number of free indices, so it follows that:

First-order tensor (vector): $\vec{v}$, $\vec{\mathbf{F}}$; Second-order tensor: $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$; Third-order tensor: Φ; Fourth-order tensor: $\mathbb{C}$

The number of tensor components is given by the maximum index range value, i.e. $i,j,k,l=1,2,3$, to the power of the number of free indices which is equal to 4 in the case of $\mathbb{C}_{ijkl}$. Thus, the number of independent components in $\mathbb{C}$ is given by:

$$3^4=(i=3)\times(j=3)\times(k=3)\times(l=3)=81$$

The fourth-order tensor $\mathbb{C}_{ijkl}$ has 81 components.

Problem 1.20

Show that a) $(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}=(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\,\vec{\mathbf{a}}$; b) $(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})=(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\,\vec{\mathbf{a}}\otimes\vec{\mathbf{d}}$

Solution:

a) We express the vector in the Cartesian basis and be aware that we cannot repeat index more than twice:

$$(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}=(\mathrm{a}_i\hat{\mathbf{e}}_i\otimes\mathrm{b}_j\hat{\mathbf{e}}_j)\cdot\mathrm{c}_k\hat{\mathbf{e}}_k=\mathrm{a}_i\hat{\mathbf{e}}_i\mathrm{b}_j\mathrm{c}_k\delta_{jk}=(\mathrm{b}_k\mathrm{c}_k)\mathrm{a}_i\hat{\mathbf{e}}_i=(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\vec{\mathbf{a}}\equiv(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\otimes\vec{\mathbf{a}}$$

b) The expression $(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$, which is a second-order tensor, can be expressed in indicial notation as follows:

$$\begin{aligned}[(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})]_{ij}&=(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})_{ik}(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})_{kj}=(\mathrm{a}_i\mathrm{b}_k)(\mathrm{c}_k\mathrm{d}_j)=\mathrm{a}_i\mathrm{b}_k\mathrm{c}_k\mathrm{d}_j=\mathrm{b}_k\mathrm{c}_k\mathrm{a}_i\mathrm{d}_j\\&=\underbrace{(\mathrm{b}_k\mathrm{c}_k)}_{scalar}(\mathrm{a}_i\mathrm{d}_j)=(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})(\vec{\mathbf{a}}\otimes\mathbf{d})_{ij}\end{aligned}$$

Problem 1.21

Expand and simplify the expression $\mathrm{A}_{ij}x_ix_j$ when a) $\mathrm{A}_{ij}=\mathrm{A}_{ji}$; b) $\mathrm{A}_{ij}=-\mathrm{A}_{ji}$.

Solution:

By expanding $\mathrm{A}_{ij}x_ix_j$ (scalar) we can obtain:

$$\begin{aligned}\mathrm{A}_{ij}x_ix_j&=\mathrm{A}_{1j}x_1x_j+\mathrm{A}_{2j}x_2x_j+\mathrm{A}_{3j}x_3x_j=\\&=\mathrm{A}_{11}x_1x_1+\mathrm{A}_{21}x_2x_1+\mathrm{A}_{31}x_3x_1+\\&\quad\mathrm{A}_{12}x_1x_2+\mathrm{A}_{22}x_2x_2+\mathrm{A}_{32}x_3x_2+\\&\quad\mathrm{A}_{13}x_1x_3+\mathrm{A}_{23}x_2x_3+\mathrm{A}_{33}x_3x_3\end{aligned}\tag{1.7}$$

a) If $\mathrm{A}_{ij}=\mathrm{A}_{ji}$ (symmetry) we have

$$\mathrm{A}_{ij}x_ix_j=\mathrm{A}_{11}x_1^2+2\mathrm{A}_{12}x_1x_2+2\mathrm{A}_{13}x_1x_3+\mathrm{A}_{22}x_2^2+2\mathrm{A}_{23}x_2x_3+\mathrm{A}_{33}x_3^2\tag{1.8}$$

b) If $\mathrm{A}_{ij}=-\mathrm{A}_{ji}$ we have $\mathrm{A}_{11}=-\mathrm{A}_{11}=0$, $\mathrm{A}_{22}=-\mathrm{A}_{22}=0$, $\mathrm{A}_{33}=-\mathrm{A}_{33}=0$, $\mathrm{A}_{12}=-\mathrm{A}_{21}$, $\mathrm{A}_{13}=-\mathrm{A}_{31}$, $\mathrm{A}_{23}=-\mathrm{A}_{32}$, with that the equation (1.7) becomes

$$\mathrm{A}_{ij}x_ix_j=0\tag{1.9}$$

NOTE: As we will see later, when $\mathrm{A}_{ij}=-\mathrm{A}_{ji}$ holds we said that the tensor is antisymmetric. And also note that for the case (b) the following is true:

$$\mathsf{A}_{ij} x_i x_j = \vec{x} \cdot \mathbf{A} \cdot \vec{x} = \mathbf{A} : (\vec{x} \otimes \vec{x}) = \mathbf{A}^{skew} : (\vec{x} \otimes \vec{x})^{sym} = 0 \tag{1.10}$$

That is, if $\mathbf{A} = \mathbf{A}^{skew}$ is an antisymmetric and $(\vec{x} \otimes \vec{x}) = (\vec{x} \otimes \vec{x})^{sym}$ is a symmetric tensor, the double scalar product between them is always equal to zero.

Problem 1.22

Let $\boldsymbol{\varepsilon}$ and $\mathbf{T}$ be second-order tensors, whose Cartesian components are:

$$\varepsilon_{ij} = \begin{bmatrix} 5 & 2 & 4 \\ -1 & 2 & 1 \\ 4 & 3 & 6 \end{bmatrix} \quad ; \quad T_{ij} = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 1 \\ 1 & 3 & 8 \end{bmatrix} \tag{1.11}$$

Obtain $\mathbf{T} : \boldsymbol{\varepsilon}$.

Solution:

$$\mathbf{T} : \boldsymbol{\varepsilon} = \mathsf{T}_{ij} \varepsilon_{ij} \tag{1.12}$$

$$\begin{array}{ccccccc} \mathsf{T}_{ij}\varepsilon_{ij} = & \underbrace{\mathsf{T}_{1j}\varepsilon_{1j}} & + & \underbrace{\mathsf{T}_{2j}\varepsilon_{2j}} & + & \underbrace{\mathsf{T}_{3j}\varepsilon_{3j}} \\ & \mathsf{T}_{11}\varepsilon_{11} & + & \mathsf{T}_{21}\varepsilon_{21} & + & \mathsf{T}_{31}\varepsilon_{31} \\ & + & & + & & + \\ & \mathsf{T}_{12}\varepsilon_{12} & + & \mathsf{T}_{22}\varepsilon_{22} & + & \mathsf{T}_{32}\varepsilon_{32} \\ & + & & + & & + \\ & \mathsf{T}_{13}\varepsilon_{13} & + & \mathsf{T}_{23}\varepsilon_{23} & + & \mathsf{T}_{33}\varepsilon_{33} \end{array} \tag{1.13}$$

thus,

$$\mathsf{T}_{ij}\varepsilon_{ij} = 5\times 3 + 2\times 1 + 4\times 2 + (-1)\times 4 + 2\times 2 + 1\times 1 + 4\times 1 + 3\times 3 + 6\times 8 = 87 \tag{1.14}$$

Problem 1.23

Given the $\mathbf{B}$ tensor components:

$$\mathsf{B}_{ij} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ 5 & 7 & 9 \end{bmatrix} \tag{1.15}$$

Obtain:

a) $\mathsf{C}_{ij} = \mathsf{B}_{ik}\mathsf{B}_{kj}$; b) $\mathsf{D}_{ij} = \mathsf{B}_{ik}\mathsf{B}_{jk}$; c) $\mathsf{E}_{ij} = \mathsf{B}_{ki}\mathsf{B}_{kj}$; d) C_{ii}, D_{ii}, E_{ii}

Solution:

$$\mathbf{C} = \mathbf{B}\cdot\mathbf{B} \quad \Rightarrow \quad \mathsf{C}_{ij} = \mathsf{B}_{ik}\mathsf{B}_{kj} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ 5 & 7 & 9 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ 5 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 31 & 44 & 54 \\ 23 & 48 & 46 \\ 67 & 108 & 122 \end{bmatrix} \tag{1.16}$$

$$\mathbf{D} = \mathbf{B}\cdot\mathbf{B}^T \quad \Rightarrow \quad \mathsf{D}_{ij} = \mathsf{B}_{ik}\mathsf{B}_{jk} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ 5 & 7 & 9 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ 5 & 7 & 9 \end{bmatrix}^T = \begin{bmatrix} 29 & 25 & 65 \\ 25 & 35 & 67 \\ 65 & 67 & 155 \end{bmatrix} \tag{1.17}$$

$$\mathbf{E}=\mathbf{B}^T\cdot\mathbf{B} \quad\Rightarrow\quad \mathsf{E}_{ij}=\mathsf{B}_{ki}\mathsf{B}_{kj}=\begin{bmatrix}3&2&4\\1&5&3\\5&7&9\end{bmatrix}^T\begin{bmatrix}3&2&4\\1&5&3\\5&7&9\end{bmatrix}=\begin{bmatrix}35&46&60\\46&78&86\\60&86&106\end{bmatrix} \tag{1.18}$$

Then:

$$\begin{aligned}\mathrm{Tr}(\mathbf{C})&=\mathrm{Tr}(\mathbf{B}\cdot\mathbf{B})=\mathsf{C}_{ii}=\mathsf{C}_{11}+\mathsf{C}_{22}+\mathsf{C}_{33}=31+48+122=201\\ \mathrm{Tr}(\mathbf{D})&=\mathrm{Tr}(\mathbf{B}\cdot\mathbf{B}^T)=\mathsf{D}_{ii}=\mathsf{D}_{11}+\mathsf{D}_{22}+\mathsf{D}_{33}=29+35+155=219\\ \mathrm{Tr}(\mathbf{E})&=\mathrm{Tr}(\mathbf{B}^T\cdot\mathbf{B})=\mathsf{E}_{ii}=\mathsf{E}_{11}+\mathsf{E}_{22}+\mathsf{E}_{33}=35+78+106=219\end{aligned} \tag{1.19}$$

NOTE: The numerical value for $\mathbf{B}:\mathbf{B}$ is:

$$\mathbf{B}:\mathbf{B}=\mathsf{B}_{ij}\mathsf{B}_{ij}=\mathsf{B}_{11}\mathsf{B}_{11}+\mathsf{B}_{21}\mathsf{B}_{21}+\mathsf{B}_{31}\mathsf{B}_{31}+\mathsf{B}_{12}\mathsf{B}_{12}+\mathsf{B}_{22}\mathsf{B}_{22}+\mathsf{B}_{32}\mathsf{B}_{32}+\mathsf{B}_{13}\mathsf{B}_{13}+\mathsf{B}_{23}\mathsf{B}_{23}+\mathsf{B}_{33}\mathsf{B}_{33}=219$$

With that we can verify the following is true: $\mathrm{Tr}(\mathbf{B}\cdot\mathbf{B}^T)=\mathrm{Tr}(\mathbf{B}^T\cdot\mathbf{B})=\mathbf{B}:\mathbf{B}=219$, which is a trace property.

Problem 1.24

Given the $\mathbf{B}$ second-order tensor components:

$$\mathsf{B}_{ij}=\begin{bmatrix}1&0&2\\0&1&2\\3&0&3\end{bmatrix}$$

Obtain: a) B_{kk} b) $\mathsf{B}_{ij}\mathsf{B}_{ij}$ c) $\mathsf{B}_{jk}\mathsf{B}_{kj}$

Solution:

a) $\mathsf{B}_{kk}=\mathsf{B}_{11}+\mathsf{B}_{22}+\mathsf{B}_{33}=1+1+3=5$

b)
$$\begin{array}{cccccc}\mathsf{B}_{ij}\mathsf{B}_{ij}= & \underbrace{\mathsf{B}_{1j}\mathsf{B}_{1j}} & + & \underbrace{\mathsf{B}_{2j}\mathsf{B}_{2j}} & + & \underbrace{\mathsf{B}_{3j}\mathsf{B}_{3j}}\\ & \mathsf{B}_{11}\mathsf{B}_{11} & + & \mathsf{B}_{21}\mathsf{B}_{21} & + & \mathsf{B}_{31}\mathsf{B}_{31}\\ & + & & + & & +\\ & \mathsf{B}_{12}\mathsf{B}_{12} & + & \mathsf{B}_{22}\mathsf{B}_{22} & + & \mathsf{B}_{32}\mathsf{B}_{32}\\ & + & & + & & +\\ & \mathsf{B}_{13}\mathsf{B}_{13} & + & \mathsf{B}_{23}\mathsf{B}_{23} & + & \mathsf{B}_{33}\mathsf{B}_{33}\end{array}$$

which the result is:

$$\mathsf{B}_{ij}\mathsf{B}_{ij}=1\times1+0\times0+2\times2+0\times0+1\times1+2\times2+3\times3+0\times0+3\times3=28$$

c)
$$\begin{array}{cccccc}\mathsf{B}_{jk}\mathsf{B}_{kj}= & \underbrace{\mathsf{B}_{1k}\mathsf{B}_{k1}} & + & \underbrace{\mathsf{B}_{2k}\mathsf{B}_{k2}} & + & \underbrace{\mathsf{B}_{3k}\mathsf{B}_{k3}}\\ & \mathsf{B}_{11}\mathsf{B}_{11} & + & \mathsf{B}_{21}\mathsf{B}_{12} & + & \mathsf{B}_{31}\mathsf{B}_{13}\\ & + & & + & & +\\ & \mathsf{B}_{12}\mathsf{B}_{21} & + & \mathsf{B}_{22}\mathsf{B}_{22} & + & \mathsf{B}_{32}\mathsf{B}_{23}\\ & + & & + & & +\\ & \mathsf{B}_{13}\mathsf{B}_{31} & + & \mathsf{B}_{23}\mathsf{B}_{32} & + & \mathsf{B}_{33}\mathsf{B}_{33}\end{array}$$

$$\mathsf{B}_{jk}\mathsf{B}_{kj} = \mathsf{B}_{11}\mathsf{B}_{11} + \mathsf{B}_{22}\mathsf{B}_{22} + \mathsf{B}_{33}\mathsf{B}_{33} + 2\mathsf{B}_{21}\mathsf{B}_{12} + 2\mathsf{B}_{31}\mathsf{B}_{13} + 2\mathsf{B}_{32}\mathsf{B}_{23}$$
$$= 1\times 1 + 1\times 1 + 3\times 3 + 2(0\times 0) + 2(3\times 2) + 2(0\times 2) = 23$$

Problem 1.25

The $\boldsymbol{D}$ tensor is given by the algebraic operation $\boldsymbol{D} = \boldsymbol{A} : \boldsymbol{B}$. Obtain the order of the tensor $\boldsymbol{D}$ and its components for the following cases:

a) *when* $A_{ij} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix}$; $B_{ij} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

b) *when* $A_{ik}B_{kj} = \begin{bmatrix} 7 & 13 & 14 \\ 11 & 18 & 11 \\ 16 & 27 & 31 \end{bmatrix}$; $A_{ik}B_{jk} = \begin{bmatrix} 13 & 9 & 17 \\ 15 & 9 & 13 \\ 18 & 12 & 32 \end{bmatrix}$

Solution:

a) Note that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are second-order tensors, the $\boldsymbol{D} = \boldsymbol{A} : \boldsymbol{B}$ is a scalar (zeroth-order tensor), so, we only have one component:

$$\boldsymbol{A} : \boldsymbol{B} = 2\times 2 + 3\times 3 + 2\times 1 + 4\times 1 + 1\times 2 + 1\times 1 + 1\times 1 + 1\times 2 + 5\times 5 = 50$$

b) Taking into account that $\mathrm{Tr}(\boldsymbol{A}\cdot\boldsymbol{B}^T) = \mathrm{Tr}(\boldsymbol{A}^T\cdot\boldsymbol{B}) = \boldsymbol{A} : \boldsymbol{B}$ and $A_{ik}B_{jk} = \boldsymbol{A}\cdot\boldsymbol{B}^T$, we can conclude that $\boldsymbol{A} : \boldsymbol{B} = \mathrm{Tr}(\boldsymbol{A}\cdot\boldsymbol{B}^T) = 13 + 9 + 32 = 54$.

Problem 1.26

Let us consider the following second-order tensor $\boldsymbol{T} = \mathrm{Tr}(\boldsymbol{E})\mathbf{1} + (\boldsymbol{F} : \boldsymbol{E})\boldsymbol{E}$ which in indicial notation is $T_{ij} = E_{kk}\delta_{ij} + (F_{kp}E_{kp})E_{ij}$. If the components of $\boldsymbol{E}$ and $\boldsymbol{F}$ are given by:

$$E_{ij} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad ; \quad F_{ij} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

a) Obtain the $\boldsymbol{T}$ tensor components. b) Are $\boldsymbol{T}$ and $\boldsymbol{E}$ coaxial tensors? Prove it.

Solution:

Next, we obtain the following scalars:

$\mathrm{Tr}(\boldsymbol{E}) = 2 + 5 + 1 = 8$

$\boldsymbol{F} : \boldsymbol{E} = 2\times 4 + 1\times 3 + 4\times 1 + 1\times 2 + 5\times 0 + 0\times 3 + 2\times 2 + 0\times 0 + 1\times 0 = 21$

Then

$$T_{ij} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 21\begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 50 & 21 & 84 \\ 21 & 113 & 0 \\ 42 & 0 & 29 \end{bmatrix}$$

Two tensors are coaxial when they have the same eigenvectors or when the relationship $\boldsymbol{T}\cdot\boldsymbol{E} = \boldsymbol{E}\cdot\boldsymbol{T}$ holds:

$$(\boldsymbol{T}\cdot\boldsymbol{E})_{ij}=T_{ik}E_{kj}=\begin{bmatrix}50 & 21 & 84\\21 & 113 & 0\\42 & 0 & 29\end{bmatrix}\begin{bmatrix}2 & 1 & 4\\1 & 5 & 0\\2 & 0 & 1\end{bmatrix}=\begin{bmatrix}289 & 155 & 284\\155 & 586 & 84\\142 & 42 & 197\end{bmatrix}$$

$$(\boldsymbol{E}\cdot\boldsymbol{T})_{ij}=E_{ik}T_{kj}=\begin{bmatrix}2 & 1 & 4\\1 & 5 & 0\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}50 & 21 & 84\\21 & 113 & 0\\42 & 0 & 29\end{bmatrix}=\begin{bmatrix}289 & 155 & 284\\155 & 586 & 84\\142 & 42 & 197\end{bmatrix}$$

with that we can conclude that $\boldsymbol{T}$ and $\boldsymbol{E}$ are coaxial tensors.

Problem 1.27

Obtain the result of the following algebraic operations: $\mathbb{I}:\mathbb{I}$, $\bar{\mathbb{I}}:\bar{\mathbb{I}}$, $\bar{\bar{\mathbb{I}}}:\bar{\bar{\mathbb{I}}}$, $\bar{\mathbb{I}}:\mathbb{I}$, $\mathbb{I}:\bar{\mathbb{I}}$, $\mathbb{I}:\bar{\bar{\mathbb{I}}}$, $\bar{\mathbb{I}}:\bar{\bar{\mathbb{I}}}$, $\bar{\bar{\mathbb{I}}}:\mathbb{I}$, $\mathbb{I}^{sym}:\mathbb{I}^{sym}$, $\mathbb{I}^{sym}:\bar{\bar{\mathbb{I}}}$, $\bar{\bar{\mathbb{I}}}:\mathbb{I}^{sym}$, where

$$\mathbb{I}=\boldsymbol{1}\overline{\otimes}\boldsymbol{1}=\mathbb{I}_{ijkl}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l \qquad where \qquad \mathbb{I}_{ijkl}=\delta_{ik}\delta_{jl} \tag{1.20}$$

$$\bar{\mathbb{I}}=\boldsymbol{1}\underline{\otimes}\boldsymbol{1}=\bar{\mathbb{I}}_{ijkl}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l \qquad where \qquad \bar{\mathbb{I}}_{ijkl}=\delta_{il}\delta_{jk} \tag{1.21}$$

$$\bar{\bar{\mathbb{I}}}=\boldsymbol{1}\otimes\boldsymbol{1}=\bar{\bar{\mathbb{I}}}_{ijkl}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l \qquad where \qquad \bar{\bar{\mathbb{I}}}_{ijkl}=\delta_{ij}\delta_{kl} \tag{1.22}$$

Solution:

$$(\mathbb{I}:\mathbb{I})_{ijkl}=\mathbb{I}_{ijpq}\mathbb{I}_{pqkl}=\delta_{ip}\delta_{jq}\delta_{pk}\delta_{ql}=\delta_{ik}\delta_{jl}=\mathbb{I}_{ijkl}$$

$$(\bar{\mathbb{I}}:\bar{\mathbb{I}})_{ijkl}=\bar{\mathbb{I}}_{ijpq}\bar{\mathbb{I}}_{pqkl}=\delta_{iq}\delta_{jp}\delta_{pl}\delta_{qk}=\delta_{ik}\delta_{jl}=\mathbb{I}_{ijkl}$$

$$(\bar{\bar{\mathbb{I}}}:\bar{\bar{\mathbb{I}}})_{ijkl}=\bar{\bar{\mathbb{I}}}_{ijpq}\bar{\bar{\mathbb{I}}}_{pqkl}=\delta_{ij}\delta_{pq}\delta_{pq}\delta_{kl}=\delta_{qq}\delta_{ij}\delta_{kl}=3\bar{\bar{\mathbb{I}}}_{ijkl}$$

$$(\bar{\mathbb{I}}:\mathbb{I})_{ijkl}=\bar{\mathbb{I}}_{ijpq}\mathbb{I}_{pqkl}=\delta_{iq}\delta_{jp}\delta_{pk}\delta_{ql}=\delta_{il}\delta_{jk}=\bar{\mathbb{I}}_{ijkl}$$

$$(\mathbb{I}:\bar{\mathbb{I}})_{ijkl}=\mathbb{I}_{ijpq}\bar{\mathbb{I}}_{pqkl}=\delta_{ip}\delta_{jq}\delta_{pl}\delta_{qk}=\delta_{il}\delta_{jk}=\bar{\mathbb{I}}_{ijkl}$$

$$(\mathbb{I}:\bar{\bar{\mathbb{I}}})_{ijkl}=\mathbb{I}_{ijpq}\bar{\bar{\mathbb{I}}}_{pqkl}=\delta_{ip}\delta_{jq}\delta_{pq}\delta_{kl}=\delta_{iq}\delta_{jq}\delta_{kl}=\delta_{ij}\delta_{kl}=\bar{\bar{\mathbb{I}}}_{ijkl}$$

$$(\bar{\mathbb{I}}:\bar{\bar{\mathbb{I}}})_{ijkl}=\bar{\mathbb{I}}_{ijpq}\bar{\bar{\mathbb{I}}}_{pqkl}=\delta_{iq}\delta_{jp}\delta_{pq}\delta_{kl}=\delta_{iq}\delta_{jq}\delta_{kl}=\delta_{ij}\delta_{kl}=\bar{\bar{\mathbb{I}}}_{ijkl}$$

We summarize the above in tensorial notation as follows:

$$\mathbb{I}:\mathbb{I}=(\boldsymbol{1}\overline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\overline{\otimes}\boldsymbol{1})=\boldsymbol{1}\overline{\otimes}\boldsymbol{1}=\mathbb{I}$$

$$\bar{\mathbb{I}}:\bar{\mathbb{I}}=(\boldsymbol{1}\underline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\underline{\otimes}\boldsymbol{1})=\boldsymbol{1}\overline{\otimes}\boldsymbol{1}=\mathbb{I}$$

$$\bar{\bar{\mathbb{I}}}:\bar{\bar{\mathbb{I}}}=(\boldsymbol{1}\otimes\boldsymbol{1}):(\boldsymbol{1}\otimes\boldsymbol{1})=3(\boldsymbol{1}\otimes\boldsymbol{1})=3\bar{\bar{\mathbb{I}}}$$

$$\bar{\mathbb{I}}:\mathbb{I}=(\boldsymbol{1}\underline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\overline{\otimes}\boldsymbol{1})=\boldsymbol{1}\underline{\otimes}\boldsymbol{1}=\bar{\mathbb{I}}$$

$$\mathbb{I}:\bar{\mathbb{I}}=(\boldsymbol{1}\overline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\underline{\otimes}\boldsymbol{1})=\boldsymbol{1}\underline{\otimes}\boldsymbol{1}=\bar{\mathbb{I}}$$

$$\mathbb{I}:\bar{\bar{\mathbb{I}}}=(\boldsymbol{1}\overline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\otimes\boldsymbol{1})=\boldsymbol{1}\otimes\boldsymbol{1}=\bar{\bar{\mathbb{I}}}$$

$$\bar{\mathbb{I}}:\bar{\bar{\mathbb{I}}}=(\boldsymbol{1}\underline{\otimes}\boldsymbol{1}):(\boldsymbol{1}\otimes\boldsymbol{1})=\boldsymbol{1}\otimes\boldsymbol{1}=\bar{\bar{\mathbb{I}}}$$

Taking into account the definition $\mathbb{I}^{sym}=\frac{1}{2}\left(\mathbb{I}+\bar{\mathbb{I}}\right)=\frac{1}{2}\left(\boldsymbol{1}\overline{\otimes}\boldsymbol{1}+\boldsymbol{1}\underline{\otimes}\boldsymbol{1}\right)$, we can conclude that:

$$
\begin{aligned}
\mathbb{I}^{sym} : \mathbb{I}^{sym} &= \frac{1}{4}\left(\mathbf{1}\overline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1}\right) : \left(\mathbf{1}\overline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1}\right) \\
&= \frac{1}{4}\left[\left(\mathbf{1}\overline{\otimes}\mathbf{1} : \mathbf{1}\overline{\otimes}\mathbf{1}\right) + \left(\mathbf{1}\overline{\otimes}\mathbf{1} : \mathbf{1}\underline{\otimes}\mathbf{1}\right) + \left(\mathbf{1}\underline{\otimes}\mathbf{1} : \mathbf{1}\overline{\otimes}\mathbf{1}\right) + \left(\mathbf{1}\underline{\otimes}\mathbf{1} : \mathbf{1}\underline{\otimes}\mathbf{1}\right)\right] \\
&= \frac{1}{4}\left[\mathbf{1}\overline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1} + \mathbf{1}\overline{\otimes}\mathbf{1}\right] \\
&= \frac{1}{2}\left(\mathbf{1}\overline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1}\right) \\
&= \mathbb{I}^{sym}
\end{aligned}
$$

$$
\mathbb{I}^{sym} : (\mathbf{1}\otimes\mathbf{1}) = \mathbb{I}^{sym} : \bar{\bar{\mathbb{I}}} = \frac{1}{2}\left(\mathbb{I} + \bar{\mathbb{I}}\right) : \bar{\bar{\mathbb{I}}} = \frac{1}{2}\left(\mathbb{I} : \bar{\bar{\mathbb{I}}} + \bar{\mathbb{I}} : \bar{\bar{\mathbb{I}}}\right) = \frac{1}{2}\left(\bar{\bar{\mathbb{I}}} + \bar{\bar{\mathbb{I}}}\right) = \bar{\bar{\mathbb{I}}} = \mathbf{1}\otimes\mathbf{1}
$$

$$
(\mathbf{1}\otimes\mathbf{1}) : \mathbb{I}^{sym} = \bar{\bar{\mathbb{I}}} : \mathbb{I}^{sym} = \frac{1}{2}\bar{\bar{\mathbb{I}}} : \left(\mathbb{I} + \bar{\mathbb{I}}\right) = \frac{1}{2}\left(\bar{\bar{\mathbb{I}}} : \mathbb{I} + \bar{\bar{\mathbb{I}}} : \bar{\mathbb{I}}\right) = \frac{1}{2}\left(\bar{\bar{\mathbb{I}}} + \bar{\bar{\mathbb{I}}}\right) = \bar{\bar{\mathbb{I}}} = \mathbf{1}\otimes\mathbf{1}
$$

1.3 Tensor Transpose

Problem 1.28

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be arbitrary second-order tensors. Show that:

$$
\mathbf{A} : (\mathbf{B}\cdot\mathbf{C}) = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C} = (\mathbf{A}\cdot\mathbf{C}^T) : \mathbf{B}
$$

Solution: Expressing the term $\mathbf{A} : (\mathbf{B}\cdot\mathbf{C})$ in indicial notation we obtain:

$$
\mathbf{A} : (\mathbf{B}\cdot\mathbf{C}) = \mathsf{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j : (\mathsf{B}_{lk}\hat{\mathbf{e}}_l \otimes \overbrace{\hat{\mathbf{e}}_k \cdot \mathsf{C}_{pq}\hat{\mathbf{e}}_p}^{\delta_{kp}} \otimes \hat{\mathbf{e}}_q) = \mathsf{A}_{ij}\mathsf{B}_{lk}\mathsf{C}_{pq}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j : (\delta_{kp}\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_q)
$$

$$
\mathbf{A} : (\mathbf{B}\cdot\mathbf{C}) = \mathsf{A}_{ij}\mathsf{B}_{lk}\mathsf{C}_{pq}\delta_{kp}\overbrace{\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j : (\hat{\mathbf{e}}_l}^{\delta_{il}} \otimes \hat{\mathbf{e}}_q)
$$

$$
\underbrace{\hat{\mathbf{e}}_j : (\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_q)}_{\delta_{jq}}
$$

$$
\mathbf{A} : (\mathbf{B}\cdot\mathbf{C}) = \mathsf{A}_{ij}\mathsf{B}_{lk}\mathsf{C}_{pq}\delta_{kp}\delta_{il}\delta_{jq} = \mathsf{A}_{ij}\mathsf{B}_{ik}\mathsf{C}_{kj}
$$

Note that, when we are dealing with indicial notation the position of the terms does not matter, i.e.:

$$
\mathsf{A}_{ij}\mathsf{B}_{ik}\mathsf{C}_{kj} = \mathsf{B}_{ik}\mathsf{A}_{ij}\mathsf{C}_{kj} = \mathsf{A}_{ij}\mathsf{C}_{kj}\mathsf{B}_{ik}
$$

We can now observe that the algebraic operation $\mathsf{B}_{ik}\mathsf{A}_{ij}$ is equivalent to the components of the second-order tensor $(\mathbf{B}^T \cdot \mathbf{A})_{kj}$, thus,

$$
\mathsf{B}_{ik}\mathsf{A}_{ij}\mathsf{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A})_{kj}\mathsf{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C}.
$$

Likewise, we can state that $\mathsf{A}_{ij}\mathsf{C}_{kj}\mathsf{B}_{ik} = (\mathbf{A}\cdot\mathbf{C}^T) : \mathbf{B}$.

Problem 1.29

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ be vectors and $\mathbf{A}$ be a second-order tensor. Show that the following relationship holds:

$$\vec{\mathbf{u}}\cdot\mathbf{A}^T\cdot\vec{\mathbf{v}}=\vec{\mathbf{v}}\cdot\mathbf{A}\cdot\vec{\mathbf{u}}$$

Solution:

$$\begin{aligned}
\vec{\mathbf{u}}\cdot\mathbf{A}^T\cdot\vec{\mathbf{v}} &= \vec{\mathbf{v}}\cdot\mathbf{A}\cdot\vec{\mathbf{u}}\\
\mathsf{u}_i\hat{\mathbf{e}}_i\cdot\mathsf{A}_{jl}\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_j\cdot\mathsf{v}_k\hat{\mathbf{e}}_k &= \mathsf{v}_k\hat{\mathbf{e}}_k\cdot\mathsf{A}_{jl}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_l\cdot\mathsf{u}_i\hat{\mathbf{e}}_i\\
\mathsf{u}_i\mathsf{A}_{jl}\delta_{il}\mathsf{v}_k\delta_{jk} &= \mathsf{v}_k\delta_{kj}\mathsf{A}_{jl}\mathsf{u}_i\delta_{il}\\
\mathsf{u}_l\mathsf{A}_{jl}\mathsf{v}_j &= \mathsf{v}_j\mathsf{A}_{jl}\mathsf{u}_l
\end{aligned}$$

1.4 Symmetry and Antisymmetry

Problem 1.30

Show that $\boldsymbol{\sigma}:\mathbf{W}=0$ is always true when $\boldsymbol{\sigma}$ is a symmetric second-order tensor and $\mathbf{W}$ is an antisymmetric second-order tensor.

Solution:

$$\boldsymbol{\sigma}:\mathbf{W}=\sigma_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j):\mathsf{W}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)=\sigma_{ij}\mathsf{W}_{lk}\delta_{il}\delta_{jk}=\sigma_{ij}\mathsf{W}_{ij}\quad\text{(scalar)}$$

Thus,

$$\sigma_{ij}\mathsf{W}_{ij}=\underbrace{\sigma_{1j}\mathsf{W}_{1j}}_{\substack{\sigma_{11}\mathsf{W}_{11}\\+\\\sigma_{12}\mathsf{W}_{12}\\+\\\sigma_{13}\mathsf{W}_{13}}}+\underbrace{\sigma_{2j}\mathsf{W}_{2j}}_{\substack{\sigma_{21}\mathsf{W}_{21}\\+\\\sigma_{22}\mathsf{W}_{22}\\+\\\sigma_{23}\mathsf{W}_{23}}}+\underbrace{\sigma_{3j}\mathsf{W}_{3j}}_{\substack{\sigma_{31}\mathsf{W}_{31}\\+\\\sigma_{32}\mathsf{W}_{32}\\+\\\sigma_{33}\mathsf{W}_{33}}}$$

Taking into account the characteristics of a symmetric and an antisymmetric tensor, i.e. $\sigma_{12}=\sigma_{21}$, $\sigma_{31}=\sigma_{13}$, $\sigma_{32}=\sigma_{23}$, and $\mathsf{W}_{11}=\mathsf{W}_{22}=\mathsf{W}_{33}=0$, $\mathsf{W}_{21}=-\mathsf{W}_{12}$, $\mathsf{W}_{31}=-\mathsf{W}_{13}$, $\mathsf{W}_{32}=-\mathsf{W}_{23}$, the equation above becomes:

$$\boldsymbol{\sigma}:\mathbf{W}=0$$

Q.E.D.

Problem 1.31

Show that a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}$; b) $\mathbf{A}:\mathbf{B}=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\mathbf{A}^{skew}:\mathbf{B}^{skew}$ where $\vec{\mathbf{M}}$ is a vector, and $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{B}$ are arbitrary second-order tensors; c) Show that the relationship $\epsilon_{ijk}\mathsf{T}_{jk}=0_i$ holds, where $\mathbf{T}$ is symmetric, i.e. $\mathsf{T}_{ij}=\mathsf{T}_{ji}$.

Solution:

a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot(\mathbf{Q}^{sym}+\mathbf{Q}^{skew})\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}+\vec{\mathbf{M}}\cdot\mathbf{Q}^{skew}\cdot\vec{\mathbf{M}}$

Since the relation $\vec{\mathbf{M}} \cdot \mathbf{Q}^{skew} \cdot \vec{\mathbf{M}} = \mathbf{Q}^{skew} : \underbrace{(\vec{\mathbf{M}} \otimes \vec{\mathbf{M}})}_{symmetric\ tensor} = 0$ holds, it follows that:

$$\vec{\mathbf{M}} \cdot \mathbf{Q} \cdot \vec{\mathbf{M}} = \vec{\mathbf{M}} \cdot \mathbf{Q}^{sym} \cdot \vec{\mathbf{M}}$$

NOTE: We can make the geometric interpretation of $\vec{\mathbf{M}} \cdot \mathbf{Q}^{skew} \cdot \vec{\mathbf{M}} = 0$ as follows. Note that the algebraic operation $\mathbf{Q}^{skew} \cdot \vec{\mathbf{M}} = \vec{\mathbf{q}}^{(\vec{\mathbf{M}})}$ is a vector, thus $\vec{\mathbf{M}} \cdot \mathbf{Q}^{skew} \cdot \vec{\mathbf{M}} = \vec{\mathbf{M}} \cdot \vec{\mathbf{q}}^{(\vec{\mathbf{M}})} = 0$, which implies that $\vec{\mathbf{M}}$ and $\vec{\mathbf{q}}^{(\vec{\mathbf{M}})}$ are orthogonal vectors. With that we conclude that: the projection of an antisymmetric second-order tensor according to the direction ($\vec{\mathbf{M}}$) is the vector ($\vec{\mathbf{q}}^{(\vec{\mathbf{M}})}$) which is orthogonal to $\vec{\mathbf{M}}$, (see Figure 1.8).

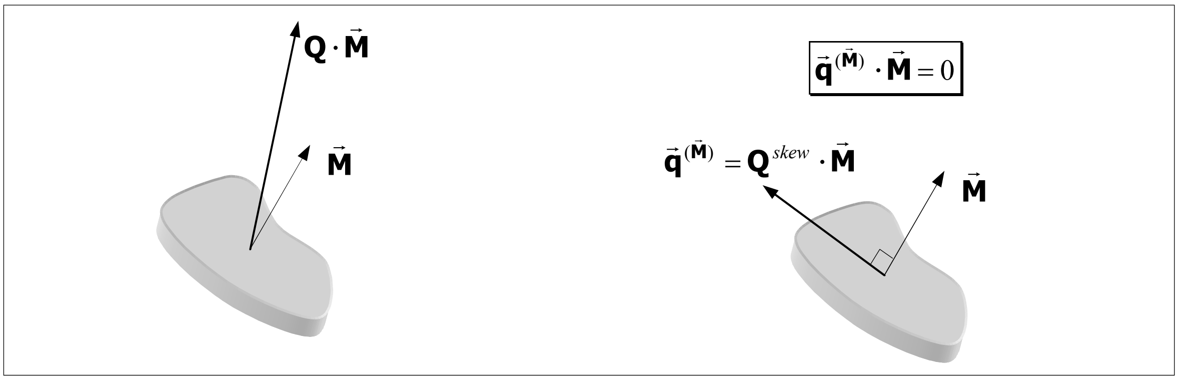


Figure 1.8

b)

$$\begin{aligned}
\mathbf{A} : \mathbf{B} &= (\mathbf{A}^{sym} + \mathbf{A}^{skew}) : (\mathbf{B}^{sym} + \mathbf{B}^{skew}) \\
&= \mathbf{A}^{sym} : \mathbf{B}^{sym} + \underbrace{\mathbf{A}^{sym} : \mathbf{B}^{skew}}_{=0} + \underbrace{\mathbf{A}^{skew} : \mathbf{B}^{sym}}_{=0} + \mathbf{A}^{skew} : \mathbf{B}^{skew} \\
&= \mathbf{A}^{sym} : \mathbf{B}^{sym} + \mathbf{A}^{skew} : \mathbf{B}^{skew}
\end{aligned}$$

Then, it is also valid that:

$$\mathbf{A} : \mathbf{B}^{sym} = \mathbf{A}^{sym} : \mathbf{B}^{sym} \quad ; \quad \mathbf{A} : \mathbf{B}^{skew} = \mathbf{A}^{skew} : \mathbf{B}^{skew}$$

Q.E.D.

c)

$$\begin{aligned}
\epsilon_{ijk} \mathsf{T}_{jk} &= \epsilon_{ij1} \mathsf{T}_{j1} + \epsilon_{ij2} \mathsf{T}_{j2} + \epsilon_{ij3} \mathsf{T}_{j3} = 0_i \\
&= \epsilon_{i11} \mathsf{T}_{11} + \epsilon_{i21} \mathsf{T}_{21} + \epsilon_{i31} \mathsf{T}_{31} + \epsilon_{i12} \mathsf{T}_{12} + \epsilon_{i22} \mathsf{T}_{22} + \epsilon_{i32} \mathsf{T}_{32} + \epsilon_{i13} \mathsf{T}_{13} + \epsilon_{i23} \mathsf{T}_{23} + \epsilon_{i33} \mathsf{T}_{33} \\
&= \epsilon_{i21} \mathsf{T}_{21} + \epsilon_{i31} \mathsf{T}_{31} + \epsilon_{i12} \mathsf{T}_{12} + \epsilon_{i32} \mathsf{T}_{32} + \epsilon_{i13} \mathsf{T}_{13} + \epsilon_{i23} \mathsf{T}_{23} = 0_i
\end{aligned}$$

Then, the vector components are:

$$i = 1 \quad \Rightarrow \quad \epsilon_{1jk} \mathsf{T}_{jk} = \epsilon_{132} \mathsf{T}_{32} + \epsilon_{123} \mathsf{T}_{23} = -\mathsf{T}_{32} + \mathsf{T}_{23} = 0 \Rightarrow \mathsf{T}_{32} = \mathsf{T}_{23}$$

$$i = 2 \quad \Rightarrow \quad \epsilon_{2jk} \mathsf{T}_{jk} = \epsilon_{231} \mathsf{T}_{31} + \epsilon_{213} \mathsf{T}_{13} = \mathsf{T}_{31} - \mathsf{T}_{13} = 0 \Rightarrow \mathsf{T}_{31} = \mathsf{T}_{13}$$

$$i = 3 \quad \Rightarrow \quad \epsilon_{3jk} \mathsf{T}_{jk} = \epsilon_{321} \mathsf{T}_{21} + \epsilon_{312} \mathsf{T}_{12} = -\mathsf{T}_{21} + \mathsf{T}_{12} = 0 \Rightarrow \mathsf{T}_{21} = \mathsf{T}_{12}$$

with that we have shown that: if $\epsilon_{ijk} \mathsf{T}_{jk} = 0_i$ holds this implies that $\mathbf{T}$ is symmetric, i.e. $\mathbf{T} = \mathbf{T}^T$.

Problem 1.32

Given a second-order tensor $\mathbf{A}$ in which the components of the symmetric part is known in the Cartesian system:

$$\mathsf{A}_{ij}^{sym} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Obtain $\hat{\mathbf{N}} \cdot \mathbf{A} \cdot \hat{\mathbf{N}}$, where the unit vector components are $\hat{\mathsf{N}}_i = [1 \quad 0 \quad 0]$.

Solution:

In **Problem 1.31** it was shown that $\hat{\mathbf{N}} \cdot \mathbf{A} \cdot \hat{\mathbf{N}} = \hat{\mathbf{N}} \cdot \mathbf{A}^{sym} \cdot \hat{\mathbf{N}}$ with that we can obtain:

$$\hat{\mathbf{N}} \cdot \mathbf{A} \cdot \hat{\mathbf{N}} = \hat{\mathbf{N}} \cdot \mathbf{A}^{sym} \cdot \hat{\mathbf{N}} = \mathsf{N}_i \mathsf{A}_{ij}^{sym} \mathsf{N}_j = [1 \quad 0 \quad 0] \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 4$$

Problem 1.33

Let $\mathbf{W}$ be an antisymmetric tensor. a) Show that $\mathbf{W} \cdot \mathbf{W}$ is a symmetric second-order tensor. b) Also show that $(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) : \mathbf{1} = 0$.

Solution:

a) If we show that $(\mathbf{W} \cdot \mathbf{W})^{skew} = \mathbf{0}$ holds, then we prove that $\mathbf{W} \cdot \mathbf{W}$ is symmetric.

$$(\mathbf{W} \cdot \mathbf{W})^{skew} = \frac{1}{2}\left[(\mathbf{W} \cdot \mathbf{W}) - (\mathbf{W} \cdot \mathbf{W})^T\right] = \frac{1}{2}\left[(\mathbf{W} \cdot \mathbf{W}) - \mathbf{W}^T \cdot \mathbf{W}^T\right] = \frac{1}{2}\left[(\mathbf{W} \cdot \mathbf{W}) - \mathbf{W} \cdot \mathbf{W}\right] = \mathbf{0}$$

where we have applied the antisymmetric tensor property $\mathbf{W} = -\mathbf{W}^T$.

Alternative solutions a) The tensor $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$, so, if we can show that $\mathbf{W} \cdot \mathbf{W} = (\mathbf{W} \cdot \mathbf{W})^T$, the tensor $(\mathbf{W} \cdot \mathbf{W})$ is symmetric. Taking into account the definition of antisymmetric tensor, $\mathbf{W} = -\mathbf{W}^T$, we can obtain:

$$\mathbf{W} \cdot \mathbf{W} = (-\mathbf{W}^T) \cdot (-\mathbf{W}^T) = \mathbf{W}^T \cdot \mathbf{W}^T = (\mathbf{W} \cdot \mathbf{W})^T$$

We can also check the symmetry by means of the tensor components:

$$(\mathbf{W} \cdot \mathbf{W})_{ij} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\mathsf{W}_{12}^2 - \mathsf{W}_{13}^2 & -\mathsf{W}_{13}\mathsf{W}_{23} & \mathsf{W}_{12}\mathsf{W}_{23} \\ -\mathsf{W}_{13}\mathsf{W}_{23} & -\mathsf{W}_{12}^2 - \mathsf{W}_{23}^2 & -\mathsf{W}_{12}\mathsf{W}_{13} \\ \mathsf{W}_{12}\mathsf{W}_{23} & -\mathsf{W}_{12}\mathsf{W}_{13} & -\mathsf{W}_{13}^2 - \mathsf{W}_{23}^2 \end{bmatrix}$$

b) Let us try to write the term $(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) : \mathbf{1}$ (scalar) in indicial notation. Note that the tensor $(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W})$ is a second-order tensor, since:

$$\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W} = \mathsf{W}_{ji}\hat{\mathbf{e}}_i \otimes \overbrace{\hat{\mathbf{e}}_j \cdot \mathsf{W}_{lk}\hat{\mathbf{e}}_l}^{\delta_{jl}} \otimes \overbrace{\hat{\mathbf{e}}_k \cdot \mathsf{W}_{pq}\hat{\mathbf{e}}_p}^{\delta_{kp}} \otimes \hat{\mathbf{e}}_q = \mathsf{W}_{ji}\mathsf{W}_{lk}\mathsf{W}_{pq}\delta_{jl}\delta_{kp}\hat{\mathbf{e}}_i \otimes \mathbf{e}_q$$
$$= \mathsf{W}_{ji}\mathsf{W}_{jk}\mathsf{W}_{kq}\hat{\mathbf{e}}_i \otimes \mathbf{e}_q$$

and

$$(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) : \mathbf{1} = (\mathsf{W}_{ji}\mathsf{W}_{jk}\mathsf{W}_{kq}\overbrace{\hat{\mathbf{e}}_i \otimes \underbrace{\hat{\mathbf{e}}_q) : (\delta_{rs}\hat{\mathbf{e}}_r}_{\delta_{qs}}}^{\delta_{ir}} \otimes \hat{\mathbf{e}}_s) = \mathsf{W}_{ji}\mathsf{W}_{jk}\mathsf{W}_{kq}\delta_{rs}\delta_{ir}\delta_{qs} = \mathsf{W}_{ji}\mathsf{W}_{jk}\mathsf{W}_{ki}$$

Note that the following is true:

$$(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) : \mathbf{1} = (\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W})_{iq}\delta_{iq} = (\mathsf{W}_{ji}\mathsf{W}_{jk}\mathsf{W}_{kq})\delta_{iq} = \mathsf{W}_{ji}(\mathsf{W}_{jk}\mathsf{W}_{ki}) = \mathbf{W} : (\mathbf{W} \cdot \mathbf{W}) = 0$$

In **Problem 1.43** we will show that $\mathbf{T} : \mathbf{1} = \mathrm{Tr}(\mathbf{T})$, where $\mathbf{T}$ is a second-order tensor, so,

$$(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) : \mathbf{1} = \mathrm{Tr}(\mathbf{W}^T \cdot \mathbf{W} \cdot \mathbf{W}) = \mathrm{Tr}[\mathbf{W}^T \cdot (\mathbf{W} \cdot \mathbf{W})] = (\mathbf{W}^T)^T : (\mathbf{W} \cdot \mathbf{W}) = \mathbf{W} : (\mathbf{W} \cdot \mathbf{W})$$

where we have applied the trace property $\mathrm{Tr}(\boldsymbol{A} \cdot \boldsymbol{B}^T) = \mathrm{Tr}(\boldsymbol{A}^T \cdot \boldsymbol{B}) = \boldsymbol{A} : \boldsymbol{B}$. Note also that

$$\Rightarrow \underbrace{\mathbf{W}}_{\text{Antisymmetric}} : \underbrace{(\mathbf{W} \cdot \mathbf{W})}_{\text{Symmetric}} = 0$$

As we have seen in **Problem 1.30**, the double scalar product between a symmetric tensor $(\mathbf{W} \cdot \mathbf{W})$ and an antisymmetric tensor ($\mathbf{W}$) is zero.

Problem 1.34

Let $\mathbf{B}$ be a second-order tensor such that $\mathsf{B}_{pq} = \epsilon_{pqs} a_s$ with $a_i = \frac{1}{2}\epsilon_{ijk}\mathsf{B}_{jk}$. Prove that $\mathbf{B}$ is an antisymmetric tensor.

Solution:

$$\mathsf{B}_{pq} = \epsilon_{pqs} a_s = \epsilon_{pqs}\left(\frac{1}{2}\epsilon_{sjk}\mathsf{B}_{jk}\right) = \frac{1}{2}\epsilon_{pqs}\epsilon_{sjk}\mathsf{B}_{jk} = \frac{1}{2}\epsilon_{pqs}\epsilon_{jks}\mathsf{B}_{jk}$$

Taking into account the relationship $\epsilon_{pqs}\epsilon_{jks} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$, the above equation can be rewritten as follows:

$$\mathsf{B}_{pq} = \frac{1}{2}\epsilon_{pqs}\epsilon_{jks}\mathsf{B}_{jk} = \frac{1}{2}(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\mathsf{B}_{jk} = \frac{1}{2}(\delta_{pj}\delta_{qk}\mathsf{B}_{jk} - \delta_{pk}\delta_{qj}\mathsf{B}_{jk}) = \frac{1}{2}(\mathsf{B}_{pq} - \mathsf{B}_{qp}) = \mathsf{B}^{skew}_{pq}$$

Alternative solution:

Taking into account that $\mathsf{B}_{qp} = \epsilon_{qps} a_s$ and $\epsilon_{pqs} = -\epsilon_{qps}$, we can conclude that:

$$\mathsf{B}_{pq} = \epsilon_{pqs} a_s = -\epsilon_{qps} a_s = -\mathsf{B}_{qp} \quad \Rightarrow \quad \mathbf{B} = -\mathbf{B}^T \text{ (antisymmetric)}$$

Problem 1.35

Show that the tensor $\mathbf{A}^{skew}\cdot\mathbf{A}^{sym}+\mathbf{A}^{sym}\cdot\mathbf{A}^{skew}$ is an antisymmetric tensor.

Solution: Denoting by $\mathbf{B}=\mathbf{A}^{skew}\cdot\mathbf{A}^{sym}+\mathbf{A}^{sym}\cdot\mathbf{A}^{skew}$, and by taking into account that $\mathbf{A}^{skew}=-(\mathbf{A}^{skew})^T$, $\mathbf{A}^{sym}=(\mathbf{A}^{sym})^T$, we can conclude that:

$$\begin{aligned}\mathbf{B}&=\mathbf{A}^{skew}\cdot\mathbf{A}^{sym}+\mathbf{A}^{sym}\cdot\mathbf{A}^{skew}=\mathbf{A}^{skew}\cdot\mathbf{A}^{sym}-\mathbf{A}^{sym}\cdot(\mathbf{A}^{skew})^T=\mathbf{A}^{skew}\cdot\mathbf{A}^{sym}-(\mathbf{A}^{skew}\cdot\mathbf{A}^{sym})^T\\&=2(\mathbf{A}^{skew}\cdot\mathbf{A}^{sym})^{skew}\end{aligned}$$

Problem 1.36

Let $\mathbf{T}$ be an arbitrary second-order tensor, and $\vec{\mathbf{n}}$ be a vector. Check if the relationship $\vec{\mathbf{n}}\cdot\mathbf{T}=\mathbf{T}\cdot\vec{\mathbf{n}}$ is valid.

Solution:

$$\begin{aligned}\vec{\mathbf{n}}\cdot\mathbf{T}&=\mathrm{n}_i\hat{\mathbf{e}}_i\cdot\mathrm{T}_{kl}(\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l)\\&=\mathrm{n}_i\mathrm{T}_{kl}\delta_{ik}\hat{\mathbf{e}}_l\\&=\mathrm{n}_k\mathrm{T}_{kl}\hat{\mathbf{e}}_l\\&=(\mathrm{n}_1\mathrm{T}_{1l}+\mathrm{n}_2\mathrm{T}_{2l}+\mathrm{n}_3\mathrm{T}_{3l})\hat{\mathbf{e}}_l\end{aligned}\quad\text{and}\quad\begin{aligned}\mathbf{T}\cdot\vec{\mathbf{n}}&=\mathrm{T}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)\cdot\mathrm{n}_i\hat{\mathbf{e}}_i\\&=\mathrm{n}_i\mathrm{T}_{lk}\delta_{ki}\hat{\mathbf{e}}_l\\&=\mathrm{n}_k\mathrm{T}_{lk}\hat{\mathbf{e}}_l\\&=(\mathrm{n}_1\mathrm{T}_{l1}+\mathrm{n}_2\mathrm{T}_{l2}+\mathrm{n}_3\mathrm{T}_{l3})\hat{\mathbf{e}}_l\end{aligned}$$

With the above we can prove that $\mathrm{n}_k\mathrm{T}_{kl}\neq\mathrm{n}_k\mathrm{T}_{lk}$, then:

$$\vec{\mathbf{n}}\cdot\mathbf{T}\neq\mathbf{T}\cdot\vec{\mathbf{n}}$$

If $\mathbf{T}$ is a symmetric tensor, it follows that the relationship $\vec{\mathbf{n}}\cdot\mathbf{T}^{sym}=\mathbf{T}^{sym}\cdot\vec{\mathbf{n}}$ holds.

Problem 1.37

Obtain the axial vector $\vec{w}$ associated with the antisymmetric second-order tensor $(\vec{x}\otimes\vec{a})^{skew}$.

Solution: Let $\vec{z}$ be an arbitrary vector, it then holds that:

$$(\vec{x}\otimes\vec{a})^{skew}\cdot\vec{z}=\vec{w}\wedge\vec{z}$$

where $\vec{w}$ is the axial vector associated with $(\vec{x}\otimes\vec{a})^{skew}$. Using the definition of an antisymmetric tensor:

$$(\vec{x}\otimes\vec{a})^{skew}=\frac{1}{2}\left[(\vec{x}\otimes\vec{a})-(\vec{x}\otimes\vec{a})^T\right]=\frac{1}{2}\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]$$

and by replacing it with $(\vec{x}\otimes\vec{a})^{skew}\cdot\vec{z}=\vec{w}\wedge\vec{z}$, we obtain:

$$\frac{1}{2}\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{w}\wedge\vec{z}\qquad\Rightarrow\qquad\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=2\vec{w}\wedge\vec{z}$$

By using the equation $\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{z}\wedge(\vec{x}\wedge\vec{a})$, (see **Problem 1.17**), the above equation becomes:

$$\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{z}\wedge(\vec{x}\wedge\vec{a})=(\vec{a}\wedge\vec{x})\wedge\vec{z}=2\vec{w}\wedge\vec{z}$$

with the above we can conclude that:

$$\vec{w}=\frac{1}{2}(\vec{a}\wedge\vec{x})\quad\textit{is the axial vector associated with}\quad(\vec{x}\otimes\vec{a})^{skew}$$

Problem 1.38

Let us consider two symmetric tensors $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$, and their axial vectors represented respectively by $\vec{w}^{(1)}$ and $\vec{w}^{(2)}$. Show that:

$$\boxed{\begin{aligned}&\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}=(\vec{w}^{(2)}\otimes\vec{w}^{(1)})-(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\mathbf{1}\\&\mathrm{Tr}[\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}]\;=-2(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\end{aligned}}$$

Solution: By means of the antisymmetric tensor properties, we can obtain that:

$$\begin{aligned}\mathbf{W}^{(1)}\cdot\vec{\mathbf{a}}&=\vec{w}^{(1)}\wedge\vec{\mathbf{a}}\\\vec{\mathbf{a}}\cdot\mathbf{W}^{(1)^T}&=-\vec{\mathbf{a}}\wedge\vec{w}^{(1)}\\-\vec{\mathbf{a}}\cdot\mathbf{W}^{(1)}&=-\vec{\mathbf{a}}\wedge\vec{w}^{(1)}\\\vec{\mathbf{a}}\cdot\mathbf{W}^{(1)}&=\vec{\mathbf{a}}\wedge\vec{w}^{(1)}\end{aligned}\qquad\text{and}\qquad\mathbf{W}^{(2)}\cdot\vec{\mathbf{a}}=\vec{w}^{(2)}\wedge\vec{\mathbf{a}}$$

Then, by applying the dot product $(\vec{\mathbf{a}}\cdot\mathbf{W}^{(1)})\cdot(\mathbf{W}^{(2)}\cdot\vec{\mathbf{a}})$ we obtain:

$$(\vec{\mathbf{a}}\cdot\mathbf{W}^{(1)})\cdot(\mathbf{W}^{(2)}\cdot\vec{\mathbf{a}})=(\vec{\mathbf{a}}\wedge\vec{w}^{(1)})\cdot(\vec{w}^{(2)}\wedge\vec{\mathbf{a}})$$

We will continue the development in indicial notation:

$$\begin{aligned}(\mathrm{a}_i\mathrm{W}^{(1)}_{ij})(\mathrm{W}^{(1)}_{jk}\mathrm{a}_k)&=(\epsilon_{ijk}\mathrm{a}_j w^{(1)}_k)(\epsilon_{ipq}w^{(2)}_p\mathrm{a}_q)\\\mathrm{a}_i(\mathrm{W}^{(1)}_{ij}\mathrm{W}^{(1)}_{jk})\mathrm{a}_k&=\mathrm{a}_j(\epsilon_{ijk}\epsilon_{ipq}w^{(1)}_k w^{(2)}_p)\mathrm{a}_q=\mathrm{a}_j\left[(\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp})w^{(1)}_k w^{(2)}_p\right]\mathrm{a}_q\\&=\mathrm{a}_j\left[\delta_{jp}\delta_{kq}w^{(1)}_k w^{(2)}_p-\delta_{jq}\delta_{kp}w^{(1)}_k w^{(2)}_p\right]\mathrm{a}_q\\&=\mathrm{a}_j\left[w^{(1)}_q w^{(2)}_j-\delta_{jq}w^{(1)}_k w^{(2)}_k\right]\mathrm{a}_q\end{aligned}$$

In tensorial notation the above equation becomes:

$$\vec{\mathbf{a}}\cdot\left[\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}\right]\cdot\vec{\mathbf{a}}=\vec{\mathbf{a}}\cdot\left[(\vec{w}^{(2)}\otimes\vec{w}^{(1)})-(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\mathbf{1}\right]\cdot\vec{\mathbf{a}}$$

With that we can conclude that $\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}=(\vec{w}^{(2)}\otimes\vec{w}^{(1)})-(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\mathbf{1}$.

b)

$$\begin{aligned}\mathrm{Tr}\left[\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}\right]&=\mathrm{Tr}\left[(\vec{w}^{(2)}\otimes\vec{w}^{(1)})-(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\mathbf{1}\right]=\mathrm{Tr}\left[(\vec{w}^{(2)}\otimes\vec{w}^{(1)})\right]-\mathrm{Tr}\left[(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\mathbf{1}\right]\\&=(\vec{w}^{(2)}\cdot\vec{w}^{(1)})-(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\underbrace{\mathrm{Tr}[\mathbf{1}]}_{=3}=-2(\vec{w}^{(1)}\cdot\vec{w}^{(2)})\end{aligned}$$

Alternative solution

In this alternative solution we use the tensor components in which it fulfills:

$$\mathrm{W}^{(1)}_{ij}=\begin{bmatrix}0&\mathrm{W}^{(1)}_{12}&\mathrm{W}^{(1)}_{13}\\-\mathrm{W}^{(1)}_{12}&0&\mathrm{W}^{(1)}_{23}\\-\mathrm{W}^{(1)}_{12}&-\mathrm{W}^{(1)}_{12}&0\end{bmatrix}=\begin{bmatrix}0&-w^{(1)}_3&w^{(1)}_2\\w^{(1)}_3&0&-w^{(1)}_1\\-w^{(1)}_2&w^{(1)}_1&0\end{bmatrix}$$

$$\mathrm{W}^{(2)}_{ij}=\begin{bmatrix}0&\mathrm{W}^{(2)}_{12}&\mathrm{W}^{(2)}_{13}\\-\mathrm{W}^{(2)}_{12}&0&\mathrm{W}^{(2)}_{23}\\-\mathrm{W}^{(2)}_{12}&-\mathrm{W}^{(2)}_{12}&0\end{bmatrix}=\begin{bmatrix}0&-w^{(2)}_3&w^{(2)}_2\\w^{(2)}_3&0&-w^{(2)}_1\\-w^{(2)}_2&w^{(2)}_1&0\end{bmatrix}$$

With that we can obtain:

$$[\mathbf{W}^{(1)}\cdot\mathbf{W}^{(2)}]_{ij}=\mathsf{W}^{(1)}_{ik}\mathsf{W}^{(2)}_{kj}=\begin{bmatrix}0 & -w_3^{(1)} & w_2^{(1)}\\ w_3^{(1)} & 0 & -w_1^{(1)}\\ -w_2^{(1)} & w_1^{(1)} & 0\end{bmatrix}\begin{bmatrix}0 & -w_3^{(2)} & w_2^{(2)}\\ w_3^{(2)} & 0 & -w_1^{(2)}\\ -w_2^{(2)} & w_1^{(2)} & 0\end{bmatrix}$$

$$\Rightarrow \mathsf{W}^{(1)}_{ik}\mathsf{W}^{(2)}_{kj}=\begin{bmatrix}-w_3^{(1)}w_3^{(2)}-w_2^{(1)}w_2^{(2)} & w_2^{(1)}w_1^{(2)} & w_3^{(1)}w_1^{(2)}\\ w_1^{(1)}w_2^{(2)} & -w_3^{(2)}w_3^{(1)}-w_1^{(1)}w_1^{(2)} & w_2^{(2)}w_3^{(1)}\\ w_3^{(2)}w_1^{(1)} & w_3^{(1)}w_2^{(2)} & -w_2^{(1)}w_2^{(2)}-w_1^{(2)}w_1^{(1)}\end{bmatrix}$$

In the term (11) of the above matrix we sum and subtract the term $w_1^{(2)}w_1^{(1)}$, in the term (22) we sum and subtract the term $w_2^{(2)}w_2^{(1)}$ and in the term (33) we add and subtract the term $w_3^{(2)}w_3^{(1)}$, so,

$$\mathsf{W}^{(1)}_{ik}\mathsf{W}^{(2)}_{kj}=\begin{bmatrix}w_1^{(2)}w_1^{(1)} & w_1^{(2)}w_2^{(1)} & w_1^{(2)}w_3^{(1)}\\ w_2^{(2)}w_1^{(1)} & w_2^{(2)}w_2^{(1)} & w_2^{(2)}w_3^{(1)}\\ w_3^{(2)}w_1^{(1)} & w_3^{(2)}w_2^{(1)} & w_3^{(2)}w_3^{(1)}\end{bmatrix}+$$

$$+\begin{bmatrix}-w_1^{(1)}w_1^{(2)}-w_2^{(1)}w_2^{(2)}-w_3^{(1)}w_3^{(2)} & 0 & 0\\ 0 & -w_1^{(1)}w_1^{(2)}-w_2^{(1)}w_2^{(2)}-w_3^{(1)}w_3^{(2)} & 0\\ 0 & 0 & -w_1^{(1)}w_1^{(2)}-w_2^{(1)}w_2^{(2)}-w_3^{(1)}w_3^{(2)}\end{bmatrix}$$

which is the same as:

$$\mathsf{W}^{(1)}_{ik}\mathsf{W}^{(2)}_{kj}=w_i^{(2)}w_j^{(1)}-(w_1^{(1)}w_1^{(2)}+w_2^{(1)}w_2^{(2)}+w_3^{(1)}w_3^{(2)})\delta_{ij}=w_i^{(2)}w_j^{(1)}-(w_k^{(1)}w_k^{(2)})\delta_{ij}$$

NOTE: The alternative solution by means of components was made only as a check. The reader must give priority to the solution via indicial or tensorial notation, since the solution via components is not always so simple to obtain.

1.5 Cofactor. Adjugate. Inverse. Particular Tensors. Determinant

Problem 1.39

Show that $\mathrm{Tr}(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$.

Solution:

$$\mathrm{Tr}(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\mathrm{Tr}[(\mathsf{a}_i\hat{\mathbf{e}}_i)\otimes(\mathsf{b}_j\hat{\mathbf{e}}_j)]=\mathsf{a}_i\mathsf{b}_j\,\mathrm{Tr}[\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j]\quad=\mathsf{a}_i\mathsf{b}_j(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j)=\mathsf{a}_i\mathsf{b}_j\delta_{ij}=\mathsf{a}_i\mathsf{b}_i=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$$
$$=\mathsf{a}_i\hat{\mathbf{e}}_i\cdot\mathsf{b}_j\hat{\mathbf{e}}_j=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$$

Problem 1.40

Given $T_{ij}=\lambda E_{kk}\delta_{ij}+2\mu E_{ij}$, $W=\frac{1}{2}T_{ij}E_{ij}$, and $P=T_{ij}T_{ij}$, show that:

$$W=\mu\boldsymbol{E}:\boldsymbol{E}+\frac{\lambda}{2}[\mathrm{Tr}(\boldsymbol{E})]^2 \qquad \textit{and} \qquad P=4\mu^2\boldsymbol{E}:\boldsymbol{E}+\lambda(3\lambda+4\mu)[\mathrm{Tr}(\boldsymbol{E})]^2$$

Solution 1: (Indicial notation)

$$W=\frac{1}{2}T_{ij}E_{ij}=\frac{1}{2}\left(\lambda E_{kk}\delta_{ij}+2\mu E_{ij}\right)E_{ij}=\frac{1}{2}\left(\lambda E_{kk}\delta_{ij}E_{ij}+2\mu E_{ij}E_{ij}\right)=\frac{1}{2}\left(\lambda E_{kk}E_{ii}+2\mu E_{ij}E_{ij}\right)$$

since $E_{kk}=E_{ii}=\mathrm{Tr}(\boldsymbol{E})$ and $E_{ij}E_{ij}=\boldsymbol{E}:\boldsymbol{E}$, we can conclude that $W=\mu\boldsymbol{E}:\boldsymbol{E}+\frac{\lambda}{2}[\mathrm{Tr}(\boldsymbol{E})]^2$.

$$\begin{aligned}P=T_{ij}T_{ij}&=\left(\lambda E_{kk}\delta_{ij}+2\mu E_{ij}\right)\left(\lambda E_{qq}\delta_{ij}+2\mu E_{ij}\right)\\&=\lambda E_{kk}\delta_{ij}\lambda E_{qq}\delta_{ij}+\lambda E_{kk}\delta_{ij}2\mu E_{ij}+2\mu E_{ij}\lambda E_{qq}\delta_{ij}+2\mu E_{ij}2\mu E_{ij}\\&=\lambda^2E_{kk}\delta_{ii}E_{qq}+2\mu\lambda E_{kk}E_{ii}+2\mu\lambda E_{ii}E_{qq}+4\mu^2E_{ij}E_{ij}=3\lambda^2E_{kk}E_{qq}+4\mu\lambda E_{kk}E_{ii}+4\mu^2E_{ij}E_{ij}\\&=\lambda(3\lambda+4\mu)E_{kk}E_{qq}+4\mu^2E_{ij}E_{ij}\end{aligned}$$

With that we show that $P=4\mu^2\boldsymbol{E}:\boldsymbol{E}+\lambda(3\lambda+4\mu)[\mathrm{Tr}(\boldsymbol{E})]^2$.

Solution 2: (Tensorial notation)

In tensorial notation we obtain:

$\boldsymbol{T}=\lambda\mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu\boldsymbol{E}$, $W=\frac{1}{2}\boldsymbol{T}:\boldsymbol{E}$, and $P=\boldsymbol{T}:\boldsymbol{T}$

Then

$$\begin{aligned}W&=\frac{1}{2}\boldsymbol{T}:\boldsymbol{E}=\frac{1}{2}\left(\lambda\mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu\boldsymbol{E}\right):\boldsymbol{E}=\frac{1}{2}\left(\lambda\mathrm{Tr}(\boldsymbol{E})\mathbf{1}:\boldsymbol{E}+2\mu\boldsymbol{E}:\boldsymbol{E}\right)=\frac{1}{2}\left(\lambda\mathrm{Tr}(\boldsymbol{E})\mathrm{Tr}(\boldsymbol{E})+2\mu\boldsymbol{E}:\boldsymbol{E}\right)\\&=\frac{\lambda}{2}[\mathrm{Tr}(\boldsymbol{E})]^2+\mu\boldsymbol{E}:\boldsymbol{E}\end{aligned}$$

$$\begin{aligned}P=\boldsymbol{T}:\boldsymbol{T}&=\left(\lambda\mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu\boldsymbol{E}\right):\left(\lambda\mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu\boldsymbol{E}\right)\\&=[\lambda\mathrm{Tr}(\boldsymbol{E})]^2\underbrace{\mathbf{1}:\mathbf{1}}_{=3}+2\mu\lambda\mathrm{Tr}(\boldsymbol{E})\underbrace{\mathbf{1}:\boldsymbol{E}}_{=\mathrm{Tr}(\boldsymbol{E})}+2\mu\lambda\mathrm{Tr}(\boldsymbol{E})\underbrace{\boldsymbol{E}:\mathbf{1}}_{=\mathrm{Tr}(\boldsymbol{E})}+(2\mu)^2\boldsymbol{E}:\boldsymbol{E}\\&=3\lambda^2[\mathrm{Tr}(\boldsymbol{E})]^2+4\mu\lambda[\mathrm{Tr}(\boldsymbol{E})]^2+4\mu^2\boldsymbol{E}:\boldsymbol{E}=\lambda(3\lambda+4\mu)[\mathrm{Tr}(\boldsymbol{E})]^2+4\mu^2\boldsymbol{E}:\boldsymbol{E}\end{aligned}$$

Problem 1.41

Let σ_{ij} be the second-order tensor components which are a function of ε_{ij}, $\sigma_{ij}=\sigma_{ij}(\varepsilon_{ij})$, and is given by:

$$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}\xrightarrow{Tensorial}\boldsymbol{\sigma}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}$$

where λ and μ are scalars. Starting with the above equation, obtain an expression for ε_{ij} in function of σ_{ij}, i.e. $\varepsilon_{ij}=\varepsilon_{ij}(\sigma_{ij})$. Express the result in indicial and tensorial notation.

Solution:

Indicial notation	Tensorial notation
$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$	$\boldsymbol{\sigma}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}$
$\Rightarrow 2\mu\varepsilon_{ij}=\sigma_{ij}-\lambda\varepsilon_{kk}\delta_{ij}$	$\Rightarrow 2\mu\boldsymbol{\varepsilon}=\boldsymbol{\sigma}-\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$
$\Rightarrow \varepsilon_{ij}=\frac{1}{2\mu}\sigma_{ij}-\frac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij}$	$\Rightarrow \boldsymbol{\varepsilon}=\frac{1}{2\mu}\boldsymbol{\sigma}-\frac{\lambda}{2\mu}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$

Next, we need to obtain the following trace ε_{kk}, to do this we obtain the trace of σ_{ij}:

Indicial notation	Tensorial notation
$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij} \qquad (i=j)$	
$\Rightarrow \sigma_{ii}=\lambda\varepsilon_{kk}\delta_{ii}+2\mu\varepsilon_{ii}=\lambda\varepsilon_{kk}3+2\mu\varepsilon_{kk}$	$\boldsymbol{\sigma}:\mathbf{1}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}:\mathbf{1}+2\mu\boldsymbol{\varepsilon}:\mathbf{1}$
$\Rightarrow \sigma_{kk}=(3\lambda+2\mu)\varepsilon_{kk}$	$\mathrm{Tr}(\boldsymbol{\sigma})=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})3+2\mu\mathrm{Tr}(\boldsymbol{\varepsilon})$
$\Rightarrow \varepsilon_{kk}=\dfrac{1}{(3\lambda+2\mu)}\sigma_{kk}$	$\Rightarrow \mathrm{Tr}(\boldsymbol{\varepsilon})=\dfrac{1}{(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})$

Then

Indicial notation	Tensorial notation
$\varepsilon_{ij}=\dfrac{1}{2\mu}\sigma_{ij}-\dfrac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij}$	$\boldsymbol{\varepsilon}=\dfrac{1}{2\mu}\boldsymbol{\sigma}-\dfrac{\lambda}{2\mu}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$
$=\dfrac{1}{2\mu}\sigma_{ij}-\dfrac{\lambda}{2\mu}\dfrac{1}{(3\lambda+2\mu)}\sigma_{kk}\delta_{ij}$	$=\dfrac{1}{2\mu}\boldsymbol{\sigma}-\dfrac{\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$

Problem 1.42

Let $\mathbf{T}$ be a second-order tensor. Show that:

$$(\mathbf{T}^m)^T=(\mathbf{T}^T)^m \quad and \quad \mathrm{Tr}(\mathbf{T}^T)^m=\mathrm{Tr}(\mathbf{T}^m)\,.$$

Solution:

$$(\mathbf{T}^m)^T=(\mathbf{T}\cdot\mathbf{T}\cdots\mathbf{T})^T=\mathbf{T}^T\cdot\mathbf{T}^T\cdots\mathbf{T}^T=(\mathbf{T}^T)^m$$

For the second demonstration we can use the trace property $\mathrm{Tr}(\mathbf{T}^T)=\mathrm{Tr}(\mathbf{T})$, thus:

$$\mathrm{Tr}(\mathbf{T}^T)^m=\mathrm{Tr}(\mathbf{T}^m)^T=\mathrm{Tr}(\mathbf{T}^m)$$

Problem 1.43

Show that $\mathbf{T}:\mathbf{1}=\mathrm{Tr}(\mathbf{T})$, where $\mathbf{T}$ is an arbitrary second-order tensor.

Solution:

$$\mathbf{T}:\mathbf{1}=\mathrm{T}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j:\delta_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l=\mathrm{T}_{ij}\delta_{kl}\delta_{ik}\delta_{jl}=\mathrm{T}_{ij}\delta_{ij}=\mathrm{T}_{ii}=\mathrm{T}_{jj}=\mathrm{Tr}(\mathbf{T})$$

Problem 1.44

Show that if $\boldsymbol{\sigma}$ and $\mathbf{D}$ are second-order tensors, the following relationship is valid:

$$\boldsymbol{\sigma}\cdot\cdot\mathbf{D}=\mathrm{Tr}(\boldsymbol{\sigma}\cdot\mathbf{D})$$

Solution: We start with the following definition:

$$\begin{aligned}
\boldsymbol{\sigma}\cdot\cdot\mathbf{D} &= \sigma_{ij}\mathrm{D}_{ji}\\
&= \sigma_{kj}\mathrm{D}_{jl}\delta_{ik}\delta_{il}=\sigma_{kj}\mathrm{D}_{jl}\delta_{lk}\\
&= \underbrace{\sigma_{kj}\mathrm{D}_{jl}}_{(\boldsymbol{\sigma}\cdot\mathbf{D})_{kl}}\delta_{lk}\\
&= (\boldsymbol{\sigma}\cdot\mathbf{D})_{kl}\delta_{lk}=(\boldsymbol{\sigma}\cdot\mathbf{D})_{kk}=(\boldsymbol{\sigma}\cdot\mathbf{D})_{ll}\\
&= \mathrm{Tr}(\boldsymbol{\sigma}\cdot\mathbf{D})
\end{aligned}$$

An alternative demonstration would be:

$$\boldsymbol{\sigma}\cdot\cdot\mathbf{D}=\sigma_{ij}\mathsf{D}_{ji}=\sigma_{ij}\mathsf{D}_{jk}\delta_{ik}=(\boldsymbol{\sigma}\cdot\mathbf{D}):\mathbf{1}=\mathrm{Tr}(\boldsymbol{\sigma}\cdot\mathbf{D})$$

Problem 1.45

Show that $|\mathbf{A}|\epsilon_{tpq}=\epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$.

Solution:

We start with the following definition:

$$|\mathbf{A}|=\epsilon_{rjk}\mathsf{A}_{r1}\mathsf{A}_{j2}\mathsf{A}_{k3} \quad\Rightarrow\quad |\mathbf{A}|\epsilon_{tpq}=\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{r1}\mathsf{A}_{j2}\mathsf{A}_{k3} \tag{1.23}$$

and also taking into account that the term $\epsilon_{rjk}\epsilon_{tpq}$ can be replaced by:

$$\epsilon_{rjk}\epsilon_{tpq}=\begin{vmatrix}\delta_{rt} & \delta_{rp} & \delta_{rq}\\ \delta_{jt} & \delta_{jp} & \delta_{jq}\\ \delta_{kt} & \delta_{kp} & \delta_{kq}\end{vmatrix}=\delta_{rt}\delta_{jp}\delta_{kq}+\delta_{rp}\delta_{jq}\delta_{kt}+\delta_{rq}\delta_{jt}\delta_{kp}-\delta_{rq}\delta_{jp}\delta_{kt}-\delta_{jq}\delta_{kp}\delta_{rt}-\delta_{kq}\delta_{jt}\delta_{rp} \tag{1.24}$$

Then, by substituting (1.24) into (1.23) we can obtain:

$$\begin{aligned}|\mathbf{A}|\epsilon_{tpq}&=\mathsf{A}_{t1}\mathsf{A}_{p2}\mathsf{A}_{q3}+\mathsf{A}_{p1}\mathsf{A}_{q2}\mathsf{A}_{t3}+\mathsf{A}_{q1}\mathsf{A}_{t2}\mathsf{A}_{p3}-\mathsf{A}_{q1}\mathsf{A}_{p2}\mathsf{A}_{t3}-\mathsf{A}_{t1}\mathsf{A}_{q2}\mathsf{A}_{p3}-\mathsf{A}_{p1}\mathsf{A}_{t2}\mathsf{A}_{q3}\\ &=\mathsf{A}_{t1}(\epsilon_{1jk}\mathsf{A}_{pj}\mathsf{A}_{qk})+\mathsf{A}_{t2}(\epsilon_{2jk}\mathsf{A}_{pj}\mathsf{A}_{qk})+\mathsf{A}_{t3}(\epsilon_{3jk}\mathsf{A}_{pj}\mathsf{A}_{qk})=\epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}=\epsilon_{rjk}\mathsf{A}_{tr}\mathsf{A}_{pj}\mathsf{A}_{qk}\end{aligned}$$

NOTE: Let us consider that $\mathbf{C}=\mathbf{A}\cdot\mathbf{B}$ ($\mathsf{C}_{ij}=\mathsf{A}_{ik}\mathsf{B}_{kj}$), then we can obtain

$$\begin{aligned}|\mathbf{C}|=|\mathbf{A}\cdot\mathbf{B}|&=\epsilon_{rjk}\mathsf{C}_{r1}\mathsf{C}_{j2}\mathsf{C}_{k3}=\epsilon_{rjk}[\mathbf{A}\cdot\mathbf{B}]_{r1}[\mathbf{A}\cdot\mathbf{B}]_{j2}[\mathbf{A}\cdot\mathbf{B}]_{k3}=\epsilon_{rjk}(\mathsf{A}_{rt}\mathsf{B}_{t1})(\mathsf{A}_{jp}\mathsf{B}_{p2})(\mathsf{A}_{kq}\mathsf{B}_{q3})\\ &=\epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}\mathsf{B}_{t1}\mathsf{B}_{p2}\mathsf{B}_{q3}=|\mathbf{A}|\epsilon_{tpq}\mathsf{B}_{t1}\mathsf{B}_{p2}\mathsf{B}_{q3}=|\mathbf{A}||\mathbf{B}|\end{aligned}$$

So, we have shown that $|\mathbf{A}\cdot\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$.

Alternative solution:

Considering that $\epsilon_{tpq}=\begin{vmatrix}\delta_{1t} & \delta_{1p} & \delta_{1q}\\ \delta_{2t} & \delta_{2p} & \delta_{2q}\\ \delta_{3t} & \delta_{3p} & \delta_{3q}\end{vmatrix}=\begin{vmatrix}\delta_{1t} & \delta_{2t} & \delta_{3t}\\ \delta_{1p} & \delta_{2p} & \delta_{3p}\\ \delta_{1q} & \delta_{2q} & \delta_{3q}\end{vmatrix}$ and that $|\mathbf{A}\cdot\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$ we can obtain:

$$|\mathbf{A}|\epsilon_{tpq}=\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{r1}\mathsf{A}_{j2}\mathsf{A}_{k3}=\begin{vmatrix}\delta_{1t} & \delta_{2t} & \delta_{3t}\\ \delta_{1p} & \delta_{2p} & \delta_{3p}\\ \delta_{1q} & \delta_{2q} & \delta_{3q}\end{vmatrix}\begin{vmatrix}\mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13}\\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23}\\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33}\end{vmatrix}=\left|\begin{bmatrix}\delta_{1t} & \delta_{2t} & \delta_{3t}\\ \delta_{1p} & \delta_{2p} & \delta_{3p}\\ \delta_{1q} & \delta_{2q} & \delta_{3q}\end{bmatrix}\begin{bmatrix}\mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13}\\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23}\\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33}\end{bmatrix}\right|$$

Note that $\delta_{1t}\mathsf{A}_{11}+\delta_{2t}\mathsf{A}_{21}+\delta_{3t}\mathsf{A}_{31}=\delta_{st}\mathsf{A}_{s1}=\mathsf{A}_{t1}$, with that we can obtain:

$$|\mathbf{A}|\epsilon_{tpq}=\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{r1}\mathsf{A}_{j2}\mathsf{A}_{k3}=\left|\begin{bmatrix}\mathsf{A}_{t1} & \mathsf{A}_{t2} & \mathsf{A}_{t3}\\ \mathsf{A}_{p1} & \mathsf{A}_{p2} & \mathsf{A}_{p3}\\ \mathsf{A}_{q1} & \mathsf{A}_{q2} & \mathsf{A}_{q3}\end{bmatrix}\right|=\epsilon_{rjk}\mathsf{A}_{tr}\mathsf{A}_{pj}\mathsf{A}_{qk}$$

Problem 1.46

Show that $\boxed{|\mathbf{A}|=\frac{1}{6}\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}}$.

Solution:

Starting with the definition $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$, (see **Problem 1.45**), and by multiplying both sides of the equation by ϵ_{tpq}, we obtain:

$$|\mathbf{A}|\epsilon_{tpq}\epsilon_{tpq} = \epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq} \tag{1.25}$$

Note that $\epsilon_{tpq}\epsilon_{tpq} = \delta_{tt}\delta_{pp} - \delta_{tp}\delta_{tp} = \delta_{tt}\delta_{pp} - \delta_{tt} = 6$. Then, the relationship (1.25) becomes:

$$|\mathbf{A}| = \frac{1}{6}\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$$

Problem 1.47

Show the following property:

$$\boxed{(\mathbf{B}\cdot\vec{\mathbf{a}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - (\mathbf{B}\cdot\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}}) + (\mathbf{B}\cdot\vec{\mathbf{c}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathrm{Tr}(\mathbf{B})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]} \tag{1.26}$$

Solution:

Expressing in Voigt notation the left side of the above equation we obtain:

$$\begin{aligned}
&\epsilon_{ijk}(\mathbf{B}\cdot\vec{\mathbf{a}})_i \mathsf{b}_j \mathsf{c}_k - \epsilon_{ijk}(\mathbf{B}\cdot\vec{\mathbf{b}})_i \mathsf{a}_j \mathsf{c}_k + \epsilon_{ijk}(\mathbf{B}\cdot\vec{\mathbf{c}})_i \mathsf{a}_j \mathsf{b}_k = \\
&\Rightarrow = \epsilon_{ijk}[(\mathsf{B}_{i1}\mathsf{a}_1 + \mathsf{B}_{i2}\mathsf{a}_2 + \mathsf{B}_{i3}\mathsf{a}_3)\mathsf{b}_j\mathsf{c}_k - (\mathsf{B}_{i1}\mathsf{b}_1 + \mathsf{B}_{i2}\mathsf{b}_2 + \mathsf{B}_{i3}\mathsf{b}_3)\mathsf{a}_j\mathsf{c}_k + \\
&\qquad + (\mathsf{B}_{i1}\mathsf{c}_1 + \mathsf{B}_{i2}\mathsf{c}_2 + \mathsf{B}_{i3}\mathsf{c}_3)\mathsf{a}_j\mathsf{b}_k] \\
&\Rightarrow = \epsilon_{ijk}[(\mathsf{B}_{i1}\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k + \mathsf{B}_{i2}\mathsf{a}_2\mathsf{b}_j\mathsf{c}_k + \mathsf{B}_{i3}\mathsf{a}_3\mathsf{b}_j\mathsf{c}_k) - (\mathsf{B}_{i1}\mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{B}_{i2}\mathsf{b}_2\mathsf{a}_j\mathsf{c}_k + \mathsf{B}_{i3}\mathsf{b}_3\mathsf{a}_j\mathsf{c}_k) + \\
&\qquad + (\mathsf{B}_{i1}\mathsf{c}_1\mathsf{a}_j\mathsf{b}_k + \mathsf{B}_{i2}\mathsf{c}_2\mathsf{a}_j\mathsf{b}_k + \mathsf{B}_{i3}\mathsf{c}_3\mathsf{a}_j\mathsf{b}_k)] \\
&\Rightarrow = \epsilon_{ijk}[\mathsf{B}_{i1}(\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_1\mathsf{a}_j\mathsf{b}_k) + \mathsf{B}_{i2}(\mathsf{a}_2\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_2\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_2\mathsf{a}_j\mathsf{b}_k) + \\
&\qquad + \mathsf{B}_{i3}(\mathsf{a}_3\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_3\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_3\mathsf{a}_j\mathsf{b}_k)]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow = &(\epsilon_{1jk}\mathsf{B}_{11} + \epsilon_{2jk}\mathsf{B}_{21} + \epsilon_{3jk}\mathsf{B}_{31})(\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_1\mathsf{a}_j\mathsf{b}_k) + \\
&(\epsilon_{1jk}\mathsf{B}_{12} + \epsilon_{2jk}\mathsf{B}_{22} + \epsilon_{3jk}\mathsf{B}_{32})(\mathsf{a}_2\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_2\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_2\mathsf{a}_j\mathsf{b}_k) + \\
&(\epsilon_{1jk}\mathsf{B}_{13} + \epsilon_{2jk}\mathsf{B}_{23} + \epsilon_{3jk}\mathsf{B}_{33})(\mathsf{a}_3\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_3\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_3\mathsf{a}_j\mathsf{b}_k)
\end{aligned} \tag{1.27}$$

Note that:

$$\epsilon_{1jk}(\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_1\mathsf{a}_j\mathsf{b}_k) = \begin{vmatrix} \mathsf{a}_1 & \mathsf{a}_2 & \mathsf{a}_3 \\ \mathsf{b}_1 & \mathsf{b}_2 & \mathsf{b}_3 \\ \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \end{vmatrix} = \epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k$$

$$\epsilon_{2jk}(\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_1\mathsf{a}_j\mathsf{b}_k) = \epsilon_{3jk}(\mathsf{a}_1\mathsf{b}_j\mathsf{c}_k - \mathsf{b}_1\mathsf{a}_j\mathsf{c}_k + \mathsf{c}_1\mathsf{a}_j\mathsf{b}_k) = 0$$

whereby the equation in (1.27) becomes:

$$\mathsf{B}_{11}\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k + \mathsf{B}_{22}\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k + \mathsf{B}_{33}\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k = (\mathsf{B}_{11} + \mathsf{B}_{22} + \mathsf{B}_{33})\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k = \mathrm{Tr}(\mathbf{B})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]$$

Q.E.D.

Note also that:

$$(\mathbf{B}^T\cdot\vec{\mathbf{a}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - (\mathbf{B}^T\cdot\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}}) + (\mathbf{B}^T\cdot\vec{\mathbf{c}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathrm{Tr}(\mathbf{B})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]$$

since $\mathrm{Tr}(\mathbf{B}^T) = \mathrm{Tr}(\mathbf{B}) \equiv I_{\mathbf{B}}$. It is also valid the following:

$$\boxed{\begin{aligned}&(\mathbf{B}\cdot\vec{\mathbf{a}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})+\vec{\mathbf{a}}\cdot((\mathbf{B}\cdot\vec{\mathbf{b}})\wedge\vec{\mathbf{c}})+\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge(\mathbf{B}\cdot\vec{\mathbf{c}}))=\mathrm{Tr}(\mathbf{B})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]\\ \Rightarrow\ &[(\mathbf{B}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},\vec{\mathbf{c}}]+[\vec{\mathbf{a}},(\mathbf{B}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}]+[\vec{\mathbf{a}},\vec{\mathbf{b}},(\mathbf{B}\cdot\vec{\mathbf{c}})]=I_{\mathbf{B}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]\end{aligned}} \tag{1.28}$$

Problem 1.48

Show the following property:

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})] \ = \det(\mathbf{A})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]} \tag{1.29}$$

where $\mathbf{A}$ is a non-singular second order tensor, and $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are linearly independent vectors.

Solution:

$\mathbf{A}$ non-singular tensor $\Rightarrow \det(\mathbf{A})\equiv|\mathbf{A}|\neq 0$.

$\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ linearly independent vectors $\Rightarrow \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\neq 0$.

We express the scalar triple product in indicial notation, i.e. $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k$, and by multiply both sides of this equation by the determinant of $\mathbf{A}$ we obtain:

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\ |\mathbf{A}|=\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k|\mathbf{A}|$$

It was proven in **Problem 1.45** that $|\mathbf{A}|\epsilon_{ijk}=\epsilon_{pqr}\mathsf{A}_{pi}\mathsf{A}_{qj}\mathsf{A}_{rk}$, thus:

$$\begin{aligned}\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\ |\mathbf{A}|&=\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k|\mathbf{A}|=\epsilon_{pqr}\mathsf{A}_{pi}\mathsf{A}_{qj}\mathsf{A}_{rk}\mathsf{a}_i\mathsf{b}_j\mathsf{c}_k=\epsilon_{pqr}(\mathsf{A}_{pi}\mathsf{a}_i)(\mathsf{A}_{qj}\mathsf{b}_j)(\mathsf{A}_{rk}\mathsf{c}_k)\\&=\epsilon_{pqr}(\mathbf{A}\cdot\vec{\mathbf{a}})_p(\mathbf{A}\cdot\vec{\mathbf{b}})_q(\mathbf{A}\cdot\vec{\mathbf{c}})_r=(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]\end{aligned}$$

Problem 1.49

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors and α, μ be scalars. Show that:

$$\boxed{\det(\mu\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\mu^3+\mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}} \tag{1.30}$$

Solution: The determinant of $\mathbf{A}$ is given by $|\mathbf{A}|=\epsilon_{ijk}\mathsf{A}_{i1}\mathsf{A}_{j2}\mathsf{A}_{k3}$. If we denote by $\mathsf{A}_{ij}=\mu\delta_{ij}+\alpha\,\mathsf{a}_i\mathsf{b}_j$, thus, $\mathsf{A}_{i1}=\mu\delta_{i1}+\alpha\,\mathsf{a}_i\mathsf{b}_1$, $\mathsf{A}_{j2}=\mu\delta_{j2}+\alpha\mathsf{a}_j\mathsf{b}_2$, $\mathsf{A}_{k3}=\delta_{k3}+\alpha\,\mathsf{a}_k\mathsf{b}_3$, then the equation in (1.30) can be rewritten as:

$$\det(\mu\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\epsilon_{ijk}(\mu\delta_{i1}+\alpha\,\mathsf{a}_i\mathsf{b}_1)(\mu\delta_{j2}+\alpha\,\mathsf{a}_j\mathsf{b}_2)(\mu\delta_{k3}+\alpha\,\mathsf{a}_k\mathsf{b}_3) \tag{1.31}$$

By developing the equation (1.31), we obtain:

$$\begin{aligned}\det(\mu\mathbf{1}+\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\epsilon_{ijk}\big[&\mu^3\delta_{i1}\delta_{j2}\delta_{k3}+\mu^2\alpha\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2}+\mu^2\alpha\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3}+\mu^2\alpha\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3}+\\&+\mu\alpha^2\mathsf{a}_j\mathsf{b}_2\mathsf{a}_k\mathsf{b}_3\delta_{i1}+\mu\alpha^2\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}+\mu\alpha^2\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3}+\alpha^3\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3\big]\end{aligned}$$

Note that: $\mu^3\epsilon_{ijk}\delta_{i1}\delta_{j2}\delta_{k3}=\mu^3\epsilon_{123}=\mu^3$,

$$\mu^2\alpha(\epsilon_{ijk}\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2}+\epsilon_{ijk}\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3}+\epsilon_{ijk}\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3})=$$
$$\mu^2\alpha(\epsilon_{12k}\mathsf{a}_k\mathsf{b}_3+\epsilon_{1j3}\mathsf{a}_j\mathsf{b}_2+\epsilon_{i23}\mathsf{a}_i\mathsf{b}_1)=\mu^2\alpha(\mathsf{a}_3\mathsf{b}_3+\mathsf{a}_2\mathsf{b}_2+\mathsf{a}_1\mathsf{b}_1)=\mu^2\alpha(\mathsf{a}_k\mathsf{b}_k)=\mu^2\alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}=\epsilon_{i2k}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3=\mathsf{a}_1\mathsf{a}_3\mathsf{b}_1\mathsf{b}_3-\mathsf{a}_3\mathsf{a}_1\mathsf{b}_1\mathsf{b}_3=0$$
$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3}=\epsilon_{ij3}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2=\epsilon_{123}\mathsf{a}_1\mathsf{a}_2\mathsf{b}_1\mathsf{b}_2-\epsilon_{213}\mathsf{a}_2\mathsf{a}_1\mathsf{b}_1\mathsf{b}_2=0$$
$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3=0$$

Notice that, there was no need to expand the terms $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}$, $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3}$, and $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3$ to realize that these terms equal zero, since $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}=(\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})_j\mathsf{b}_1\mathsf{b}_3\delta_{j2}=0$, similarly for other terms.

Taking into account the above considerations we can prove that:

$$\det(\mu\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\mu^3+\mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$$

For the particular case when $\mu=1$ the above equation becomes:

$$\boxed{\det(\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=1+\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}$$

Then, it is simple to prove that $\det(\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=0$, since

$$\det(\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\alpha^3\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3=\alpha^3\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})\right]=0$$

NOTE: We can extrapolate the equation in (1.30) in such a way that:

$$\boxed{\det(\mu\mathbb{I}^{sym}+\alpha\,\mathbf{A}\otimes\mathbf{B})=\mu^3+\mu^2\alpha\,\mathbf{A}:\mathbf{B}}\tag{1.32}$$

where $\mathbb{I}^{sym}$ is the symmetric fourth-order unit tensor, $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors. Note that $\det(\mathbb{I}^{sym})=(1)^3+(1)^2(0)(\mathbf{0}:\mathbf{0})=1$ and $\det(\mathbf{1}\otimes\mathbf{1})=(0)^3+(0)^2(1)(\mathbf{1}:\mathbf{1})=0$.

Problem 1.50

Let $\mathbf{A}$ be an arbitrary second-order tensor. Show that there is a nonzero vector $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$ so that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$ if and only if $\det(\mathbf{A})=0$, Chadwick (1976).

Solution: Firstly, we show that, if $\det(\mathbf{A})\equiv|\mathbf{A}|=0\Rightarrow\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$. Secondly, we show that, if $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}\Rightarrow\det(\mathbf{A})\equiv|\mathbf{A}|=0$.

We assume that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, and we choose an arbitrary basis $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ (linearly independent), then:

$$\vec{\mathbf{f}}\cdot\left(\vec{\mathbf{g}}\wedge\vec{\mathbf{h}}\right)|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})\right]\,,\quad\text{(see \textbf{Problem 1.48})}$$

Due to the fact that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, the implication is that:

$$(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})\right]=0$$

Thus, we can conclude that the vectors $(\mathbf{A}\cdot\vec{\mathbf{f}})$, $(\mathbf{A}\cdot\vec{\mathbf{g}})$, $(\mathbf{A}\cdot\vec{\mathbf{h}})$, are linearly dependent. This implies that there are nonzero scalars α, β, γ so that:

$$\alpha(\mathbf{A}\cdot\vec{\mathbf{f}})+\beta(\mathbf{A}\cdot\vec{\mathbf{g}})+\gamma(\mathbf{A}\cdot\vec{\mathbf{h}})=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot\left(\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}}\right)=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$$

where $\vec{\mathbf{n}}=\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}}\neq\vec{\mathbf{0}}$ since $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ is linearly independent.

Now we choose two vectors $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, which are linearly independent to $\vec{\mathbf{n}}$. Once more, we apply definition:

$$\vec{\mathbf{k}}\cdot(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}})|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{k}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{m}})\wedge(\mathbf{A}\cdot\vec{\mathbf{n}})\right]$$

Considering that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$, and $\vec{\mathbf{k}}\cdot(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}})\neq 0$ owing to the fact that $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, $\vec{\mathbf{n}}$ are linearly independent, we can conclude that:

$$\underbrace{\vec{\mathbf{k}}\cdot(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}})}_{\neq 0}|\mathbf{A}|=0 \quad\Rightarrow\quad |\mathbf{A}|=0$$

Problem 1.51

Let $\boldsymbol{F}$ be an arbitrary second-order tensor. Show that the resulting tensors $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ are *symmetric tensors* and *semi-positive definite tensors*. Also check in what condition are $\boldsymbol{C}$ and $\boldsymbol{b}$ *positive definite tensors*.

Solution: Symmetry:

$$\boldsymbol{C}^T=(\boldsymbol{F}^T\cdot\boldsymbol{F})^T=\boldsymbol{F}^T\cdot(\boldsymbol{F}^T)^T=\boldsymbol{F}^T\cdot\boldsymbol{F}=\boldsymbol{C}$$
$$\boldsymbol{b}^T=(\boldsymbol{F}\cdot\boldsymbol{F}^T)^T=(\boldsymbol{F}^T)^T\cdot\boldsymbol{F}^T=\boldsymbol{F}\cdot\boldsymbol{F}^T=\boldsymbol{b}$$

Thus, we have shown that $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ are symmetric tensors.

To prove that the tensors $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ are semi-positive definite tensors, we start with the definition of a semi-positive definite tensor, i.e., a tensor $\mathbf{A}$ is semi-positive definite if $\hat{\mathbf{x}}\cdot\mathbf{A}\cdot\hat{\mathbf{x}}\geq 0$ holds, for all $\hat{\mathbf{x}}\neq\vec{\mathbf{0}}$. Thus:

$$\begin{aligned}\hat{\mathbf{x}}\cdot(\boldsymbol{F}^T\cdot\boldsymbol{F})\cdot\hat{\mathbf{x}}&=\boldsymbol{F}\cdot\hat{\mathbf{x}}\cdot\boldsymbol{F}\cdot\hat{\mathbf{x}}\\&=(\boldsymbol{F}\cdot\hat{\mathbf{x}})\cdot(\boldsymbol{F}\cdot\hat{\mathbf{x}})\\&=\left\|\boldsymbol{F}\cdot\hat{\mathbf{x}}\right\|^2\geq 0\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}\hat{\mathbf{x}}\cdot(\boldsymbol{F}\cdot\boldsymbol{F}^T)\cdot\hat{\mathbf{x}}&=\hat{\mathbf{x}}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\hat{\mathbf{x}}\\&=(\boldsymbol{F}^T\cdot\hat{\mathbf{x}})\cdot(\boldsymbol{F}^T\cdot\hat{\mathbf{x}})\\&=\left\|\boldsymbol{F}^T\cdot\hat{\mathbf{x}}\right\|^2\geq 0\end{aligned}$$

Or in indicial notation:

$$\begin{aligned}\mathrm{x}_i C_{ij}\mathrm{x}_j&=\mathrm{x}_i(F_{ki}F_{kj})\mathrm{x}_j\\&=(F_{ki}\mathrm{x}_i)(F_{kj}\mathrm{x}_j)\\&=\left\|F_{ki}\mathrm{x}_i\right\|^2\geq 0\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}\mathrm{x}_i b_{ij}\mathrm{x}_j&=\mathrm{x}_i(F_{ik}F_{jk})\mathrm{x}_j\\&=(F_{ik}\mathrm{x}_i)(F_{jk}\mathrm{x}_j)\\&=\left\|F_{ik}\mathrm{x}_i\right\|^2\geq 0\end{aligned}$$

Thus, we proved that $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ are semi-positive definite tensors. Note that $\hat{\mathbf{x}}\cdot\boldsymbol{C}\cdot\hat{\mathbf{x}}=\left\|\boldsymbol{F}\cdot\hat{\mathbf{x}}\right\|^2$ equals zero, when $\hat{\mathbf{x}}\neq\vec{\mathbf{0}}$, if $\boldsymbol{F}\cdot\hat{\mathbf{x}}=\vec{\mathbf{0}}$. Furthermore, by definition $\boldsymbol{F}\cdot\hat{\mathbf{x}}=\vec{\mathbf{0}}$ with $\hat{\mathbf{x}}\neq\vec{\mathbf{0}}$ if and only if $\det(\boldsymbol{F})=0$, (see **Problem 1.50**). Then, the tensors $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ are positive definite if and only if $\det(\boldsymbol{F})\neq 0$.

Problem 1.52

Let $d\vec{X}^{(1)}$, $d\vec{X}^{(2)}$, $d\vec{X}^{(3)}$, $d\vec{x}^{(1)}$, $d\vec{x}^{(2)}$, $d\vec{x}^{(3)}$ be vectors, and they are related to each other as follows $d\vec{x}^{(1)} = \boldsymbol{F} \cdot d\vec{X}^{(1)}$, $d\vec{x}^{(2)} = \boldsymbol{F} \cdot d\vec{X}^{(2)}$, $d\vec{x}^{(3)} = \boldsymbol{F} \cdot d\vec{X}^{(3)}$, where $\boldsymbol{F}$ is a non-singular second-order tensor and $\exists \boldsymbol{F}^{-1}$. a.1) Considering $dV = d\vec{x}^{(1)} \cdot \left(d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}\right) \neq 0$ and $dV_0 = d\vec{X}^{(1)} \cdot (d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}) \neq 0$, obtain a relationship between the scalars dV and dV_0 in terms of $\boldsymbol{F}$. a.2) Obtain the relationship between $\vec{\boldsymbol{c}} = d\vec{X}^{(2)} \wedge d\vec{X}^{(3)} \neq \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{c}}^* = d\vec{x}^{(2)} \wedge d\vec{x}^{(3)} \neq \vec{\boldsymbol{0}}$.

Solution

a.1) Taking into account the problem statement it fulfills that:

$$dV = d\vec{x}^{(1)} \cdot (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = (\boldsymbol{F} \cdot d\vec{X}^{(1)}) \cdot \left[(\boldsymbol{F} \cdot d\vec{X}^{(2)}) \wedge (\boldsymbol{F} \cdot d\vec{X}^{(3)})\right]$$

In **Problem 1.48** it was proven that $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})\ |\mathbf{A}| = (\mathbf{A} \cdot \vec{\mathbf{a}}) \cdot \left[(\mathbf{A} \cdot \vec{\mathbf{b}}) \wedge (\mathbf{A} \cdot \vec{\mathbf{c}})\right]$, so

$$\begin{aligned} &\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})\ |\mathbf{A}| = (\mathbf{A} \cdot \vec{\mathbf{a}}) \cdot \left[(\mathbf{A} \cdot \vec{\mathbf{b}}) \wedge (\mathbf{A} \cdot \vec{\mathbf{c}})\right] \\ &\Rightarrow d\vec{X}^{(1)} \cdot (d\vec{X}^{(2)} \wedge d\vec{X}^{(3)})\ |\boldsymbol{F}| = (\boldsymbol{F} \cdot d\vec{X}^{(1)}) \cdot \left[(\boldsymbol{F} \cdot d\vec{X}^{(2)}) \wedge (\boldsymbol{F} \cdot d\vec{X}^{(3)})\right] \end{aligned}$$

With that we conclude that:

$$dV = d\vec{x}^{(1)} \cdot (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = (\boldsymbol{F} \cdot d\vec{X}^{(1)}) \cdot \left[(\boldsymbol{F} \cdot d\vec{X}^{(2)}) \wedge (\boldsymbol{F} \cdot d\vec{X}^{(3)})\right] = |\boldsymbol{F}| \left[d\vec{X}^{(1)} \cdot (d\vec{X}^{(2)} \wedge d\vec{X}^{(3)})\right]$$

thus

$$dV = |\boldsymbol{F}|\, dV_0 \tag{1.33}$$

a.2) Since exist the inverse of $\boldsymbol{F}$, i.e. $\exists \boldsymbol{F}^{-1}$, we can obtain that

$$d\vec{x}^{(1)} = \boldsymbol{F} \cdot d\vec{X}^{(1)} \quad \Rightarrow \quad \boldsymbol{F}^{-1} \cdot d\vec{x}^{(1)} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F} \cdot d\vec{X}^{(1)} = \mathbf{1} \cdot d\vec{X}^{(1)} = d\vec{X}^{(1)}$$

And by taking into account the equation (1.33) we can obtain:

$$\begin{aligned} &dV = |\boldsymbol{F}|\, dV_0 \quad \Rightarrow \quad d\vec{x}^{(1)} \cdot (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = |\boldsymbol{F}|\, d\vec{X}^{(1)} \cdot [d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}] \\ &\Rightarrow d\vec{x}^{(1)} \cdot (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = |\boldsymbol{F}|\, (\boldsymbol{F}^{-1} \cdot d\vec{x}^{(1)}) \cdot [d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}] \\ &\Rightarrow d\vec{x}^{(1)} \cdot (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = d\vec{x}^{(1)} \cdot \left(|\boldsymbol{F}|\, \boldsymbol{F}^{-T} \cdot [d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}]\right) \\ &\Rightarrow (d\vec{x}^{(2)} \wedge d\vec{x}^{(3)}) = |\boldsymbol{F}|\, \boldsymbol{F}^{-T} \cdot [d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}] \quad \Rightarrow \quad \vec{\boldsymbol{c}}^* = |\boldsymbol{F}|\, \boldsymbol{F}^{-T} \cdot \vec{\boldsymbol{c}} \end{aligned}$$

NOTE 1: Note that $\vec{\boldsymbol{c}}^* \neq \boldsymbol{F} \cdot \vec{\boldsymbol{c}}$. We can rewrite the above equation as follows

$$d\vec{x}^{(2)} \wedge d\vec{x}^{(3)} = |\boldsymbol{F}|\, \boldsymbol{F}^{-T} \cdot \left[d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}\right] \quad \Rightarrow \quad (\boldsymbol{F} \cdot d\vec{X}^{(2)}) \wedge (\boldsymbol{F} \cdot d\vec{X}^{(3)}) = |\boldsymbol{F}|\, \boldsymbol{F}^{-T} \cdot \left[d\vec{X}^{(2)} \wedge d\vec{X}^{(3)}\right]$$

The tensor $|\boldsymbol{F}|\, \boldsymbol{F}^{-T}$ is known as the cofactor of $\boldsymbol{F}$, i.e. $\mathrm{cof}(\boldsymbol{F}) = |\boldsymbol{F}|\, \boldsymbol{F}^{-T}$ with this we define the inverse of a tensor:

$$\begin{aligned} &\mathrm{cof}(\boldsymbol{F}) = |\boldsymbol{F}|\, \boldsymbol{F}^{-T} \quad \Rightarrow \quad [|\boldsymbol{F}|\, \boldsymbol{F}^{-T}]^T = [\mathrm{cof}(\boldsymbol{F})]^T \quad \Rightarrow \quad |\boldsymbol{F}|\, \boldsymbol{F}^{-1} = [\mathrm{cof}(\boldsymbol{F})]^T \\ &\Rightarrow \boldsymbol{F}^{-1} = \frac{1}{|\boldsymbol{F}|}[\mathrm{cof}(\boldsymbol{F})]^T = \frac{1}{|\boldsymbol{F}|}[\mathrm{adj}(\boldsymbol{F})] \end{aligned}$$

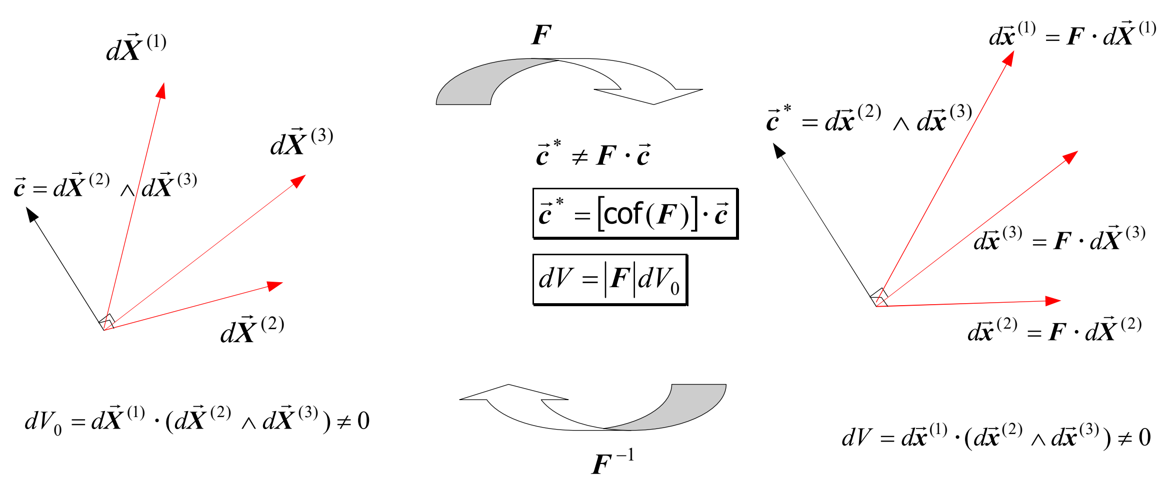


Figure 1.9

NOTE 2: Let us suppose now that $\boldsymbol{F} = \boldsymbol{A} \cdot \boldsymbol{B}$, and let us consider three vectors $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \wedge \vec{c}) \neq 0$, and $\vec{\boldsymbol{a}}^* = \boldsymbol{B} \cdot \vec{\boldsymbol{a}}$, $\vec{\boldsymbol{b}}^* = \boldsymbol{B} \cdot \vec{\boldsymbol{b}}$, $\vec{c}^* = \boldsymbol{B} \cdot \vec{c}$, thus by apply the previous definitions we can state:

$$
\begin{aligned}
|\boldsymbol{F}|\, \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \wedge \vec{c}) = (\boldsymbol{F} \cdot \vec{\boldsymbol{a}}) \cdot \left[(\boldsymbol{F} \cdot \vec{\boldsymbol{b}}) \wedge (\boldsymbol{F} \cdot \vec{c}) \right] \quad &= (\boldsymbol{A} \cdot \boldsymbol{B} \cdot \vec{\boldsymbol{a}}) \cdot \left[(\boldsymbol{A} \cdot \boldsymbol{B} \cdot \vec{\boldsymbol{b}}) \wedge (\boldsymbol{A} \cdot \boldsymbol{B} \cdot \vec{c}) \right] \\
&= (\boldsymbol{A} \cdot \vec{\boldsymbol{a}}^*) \cdot \left[(\boldsymbol{A} \cdot \vec{\boldsymbol{b}}^*) \wedge (\boldsymbol{A} \cdot \vec{c}^*) \right] \\
&= |\boldsymbol{A}|\, \vec{\boldsymbol{a}}^* \cdot (\vec{\boldsymbol{b}}^* \wedge \vec{c}^*) = |\boldsymbol{A}|\, (\boldsymbol{B} \cdot \vec{\boldsymbol{a}}) \cdot \left[(\boldsymbol{B} \cdot \vec{\boldsymbol{b}}) \wedge (\boldsymbol{B} \cdot \vec{c}) \right] \\
&= |\boldsymbol{A}||\boldsymbol{B}|\, \vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \wedge \vec{c})
\end{aligned}
$$

With that we can conclude that: if $\boldsymbol{F} = \boldsymbol{A} \cdot \boldsymbol{B}$ then $|\boldsymbol{F}| = |\boldsymbol{A} \cdot \boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|$.

Problem 1.53

Let $\mathbf{A}$ and $\mathbf{B}$ be orthogonal tensors, show that the tensor $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ is also an orthogonal tensor.

Solution: By definition, a tensor is orthogonal if $\mathbf{C}^{-1} = \mathbf{C}^T$ holds:

$$\mathbf{C}^{-1} = (\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = \mathbf{B}^T \cdot \mathbf{A}^T = (\mathbf{A} \cdot \mathbf{B})^T = \mathbf{C}^T$$

Q.E.D.

Problem 1.54

Show that $\text{adj}(\mathbf{A} \cdot \mathbf{B}) = \text{adj}(\mathbf{B}) \cdot \text{adj}(\mathbf{A})$ and $\text{cof}(\mathbf{A} \cdot \mathbf{B}) = [\text{cof}(\mathbf{A})] \cdot [\text{cof}(\mathbf{B})]$.

Solution:

Based on the definition of the inverse of a tensor we can say that:

$$
\begin{aligned}
&\mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = \frac{[\text{adj}(\mathbf{B})]}{|\mathbf{B}|} \cdot \frac{[\text{adj}(\mathbf{A})]}{|\mathbf{A}|} \\
&\Rightarrow |\mathbf{A}||\mathbf{B}| \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = [\text{adj}(\mathbf{B})] \cdot [\text{adj}(\mathbf{A})] = [\text{cof}(\mathbf{B})]^T \cdot [\text{cof}(\mathbf{A})]^T \\
&\Rightarrow |\mathbf{A}||\mathbf{B}| (\mathbf{A} \cdot \mathbf{B})^{-1} = [\text{adj}(\mathbf{B})] \cdot [\text{adj}(\mathbf{A})] = \left([\text{cof}(\mathbf{A})] \cdot [\text{cof}(\mathbf{B})] \right)^T \\
&\Rightarrow |\mathbf{A}||\mathbf{B}| \frac{[\text{adj}(\mathbf{A} \cdot \mathbf{B})]}{|\mathbf{A} \cdot \mathbf{B}|} = [\text{adj}(\mathbf{B})] \cdot [\text{adj}(\mathbf{A})] = ([\text{cof}(\mathbf{A})] \cdot [\text{cof}(\mathbf{B})])^T \\
&\Rightarrow \text{adj}(\mathbf{A} \cdot \mathbf{B}) = [\text{adj}(\mathbf{B})] \cdot [\text{adj}(\mathbf{A})] = ([\text{cof}(\mathbf{A})] \cdot [\text{cof}(\mathbf{B})])^T
\end{aligned}
\tag{1.34}
$$

Q.E.D.

where we have used the property $|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$. Also taking into account the definition of adjugate and cofactor we can conclude that:

$$\text{adj}(\mathbf{A}\cdot\mathbf{B}) = ([\text{cof}(\mathbf{A}\cdot\mathbf{B})])^T = ([\text{cof}(\mathbf{A})]\cdot[\text{cof}(\mathbf{B})])^T \quad \Rightarrow \quad [\text{cof}(\mathbf{A}\cdot\mathbf{B})] = [\text{cof}(\mathbf{A})]\cdot[\text{cof}(\mathbf{B})] \tag{1.35}$$

Problem 1.55

Show that:

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{b}}) = [\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})} \tag{1.36}$$

Solution:

Starting from the equation $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$, (see **Problem 1.45**), and by multiply both sides by $\mathsf{a}_t\mathsf{b}_p$, we obtain:

$$|\mathbf{A}|\epsilon_{tpq}\mathsf{a}_t\mathsf{b}_p = \epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}\mathsf{a}_t\mathsf{b}_p = \epsilon_{rjk}(\mathsf{A}_{rt}\mathsf{a}_t)(\mathsf{A}_{jp}\mathsf{b}_p)\mathsf{A}_{kq}$$

Multiplying both sides by A^{-1}_{qs} we obtain:

$$|\mathbf{A}|\epsilon_{tpq}\mathsf{a}_t\mathsf{b}_p\mathsf{A}^{-1}_{qs} = \epsilon_{rjk}(\mathsf{A}_{rt}\mathsf{a}_t)(\mathsf{A}_{jp}\mathsf{b}_p)\mathsf{A}_{kq}\mathsf{A}^{-1}_{qs} = \epsilon_{rjk}(\mathsf{A}_{rt}\mathsf{a}_t)(\mathsf{A}_{jp}\mathsf{b}_p)\delta_{ks} = \epsilon_{rjs}(\mathsf{A}_{rt}\mathsf{a}_t)(\mathsf{A}_{jp}\mathsf{b}_p)$$

Note that $\mathsf{A}^{-1}_{qs} = \dfrac{[\text{cof}(\mathbf{A})]_{sq}}{|\mathbf{A}|}$ holds, whereby the above equation becomes:

$$|\mathbf{A}|\epsilon_{tpq}\mathsf{a}_t\mathsf{b}_p\mathsf{A}^{-1}_{qs} = |\mathbf{A}|\epsilon_{tpq}\mathsf{a}_t\mathsf{b}_p\frac{[\text{cof}(\mathbf{A})]_{sq}}{|\mathbf{A}|} = [\text{cof}(\mathbf{A})]_{sq}\epsilon_{tpq}\mathsf{a}_t\mathsf{b}_p = \epsilon_{rjs}(\mathsf{A}_{rt}\mathsf{a}_t)(\mathsf{A}_{jp}\mathsf{b}_p)$$

$$\Rightarrow [\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\mathbf{A}\cdot\vec{\mathbf{a}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{b}})$$

Problem 1.56

Show that:

$$\boxed{\vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[\vec{\mathbf{b}} \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge \vec{\mathbf{c}}\right] = \text{Tr}([\text{cof}(\mathbf{A})])\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})\right]} \tag{1.37}$$

Solution:

In **Problem 1.55** it was demonstrated that $[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\mathbf{A}\cdot\vec{\mathbf{a}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{b}})$, thus the following relationships hold:

$$\vec{\mathbf{a}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = \vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right]$$

$$-\vec{\mathbf{b}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{c}}) = -\vec{\mathbf{b}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{a}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right] = (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[\vec{\mathbf{b}} \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right]$$

$$\vec{\mathbf{c}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \vec{\mathbf{c}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{a}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{b}})\right] = (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge \vec{\mathbf{c}}\right]$$

Summing the three above equations we obtain:

$$\begin{aligned}&\vec{\mathbf{a}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) - \vec{\mathbf{b}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{c}}) + \vec{\mathbf{c}}\cdot[\text{cof}(\mathbf{A})]\cdot(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \\ &\qquad = \vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[\vec{\mathbf{b}} \wedge (\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}}) \wedge \vec{\mathbf{c}}\right]\end{aligned}$$

According to **Problem 1.47** the following is true:

$$\left([\mathrm{cof}(\mathbf{A})]^T \cdot \vec{\mathbf{a}}\right)\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - \left([\mathrm{cof}(\mathbf{A})]^T \cdot \vec{\mathbf{b}}\right)\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}}) + \left([\mathrm{cof}(\mathbf{A})]^T \cdot \vec{\mathbf{c}}\right)\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathrm{Tr}([\mathrm{cof}(\mathbf{A})])[\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})] = I\!I_{\mathbf{A}}[\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})]$$

where $I\!I_{\mathbf{A}} = \mathrm{Tr}[\mathrm{cof}(\mathbf{A})]$ is the second principal invariant of $\mathbf{A}$, thus:

$$\vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[\vec{\mathbf{b}}\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge\vec{\mathbf{c}}\right] = I\!I_{\mathbf{A}}\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]$$

NOTE 1: We can summarize that:

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) + \vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge\vec{\mathbf{c}}\right] + \vec{\mathbf{a}}\cdot\left[\vec{\mathbf{b}}\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] = I_{\mathbf{A}}\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]} \text{ (see **Problem 1.47**)} \quad (1.38)$$

$$\boxed{\vec{\mathbf{a}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[\vec{\mathbf{b}}\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] + (\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge\vec{\mathbf{c}}\right] = I\!I_{\mathbf{A}}\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]} \quad (1.39)$$

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] = I\!I\!I_{\mathbf{A}}\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]} \text{ (see **Problem 1.48**)} \quad (1.40)$$

where $I_{\mathbf{A}} = \mathrm{Tr}(\mathbf{A})$, $I\!I_{\mathbf{A}} = \mathrm{Tr}([\mathrm{cof}(\mathbf{A})])$, $I\!I\!I_{\mathbf{A}} = \det(\mathbf{A})$. Using the notation $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) \equiv [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$, the above equations can also be written as follows:

$$\boxed{[(\mathbf{A}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + [\vec{\mathbf{a}},(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] + [\vec{\mathbf{a}},\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] = I_{\mathbf{A}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\boxed{[\vec{\mathbf{a}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{a}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] = I\!I_{\mathbf{A}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\boxed{[(\mathbf{A}\cdot\vec{\mathbf{a}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] = I\!I\!I_{\mathbf{A}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

NOTE 2: If we consider three linearly independent vectors $[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})] \equiv [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] \neq 0$, and three vectors such as:

$$\begin{cases} \vec{\mathbf{f}} = \alpha_1\vec{\mathbf{a}} + \alpha_2\vec{\mathbf{b}} + \alpha_3\vec{\mathbf{c}} \\ \vec{\mathbf{g}} = \beta_1\vec{\mathbf{a}} + \beta_2\vec{\mathbf{b}} + \beta_3\vec{\mathbf{c}} \\ \vec{\mathbf{h}} = \gamma_1\vec{\mathbf{a}} + \gamma_2\vec{\mathbf{b}} + \gamma_3\vec{\mathbf{c}} \end{cases} \quad \Rightarrow \quad \begin{Bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{g}} \\ \vec{\mathbf{h}} \end{Bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{b}} \\ \vec{\mathbf{c}} \end{Bmatrix} \quad (1.41)$$

And according to Cramer's rule, (see **Problem 1.16**), the following relationships are true:

$$\alpha_1 = \frac{\vec{\mathbf{f}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}{\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})} \equiv \frac{[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \alpha_2 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{f}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \alpha_3 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{f}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\beta_1 = \frac{[\vec{\mathbf{g}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \beta_2 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{g}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \beta_3 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{g}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\gamma_1 = \frac{[\vec{\mathbf{h}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \gamma_2 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{h}},\vec{\mathbf{c}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]} \quad ; \quad \gamma_3 = \frac{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{h}}]}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

By performing the triple scalar product $[\vec{\mathbf{f}}\cdot(\vec{\mathbf{g}}\wedge\vec{\mathbf{h}})] \equiv [\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}]$, we can obtain:

$$[\vec{\mathbf{f}}\cdot(\vec{\mathbf{g}}\wedge\vec{\mathbf{h}})] = \left|\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\right| [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$$

$$= \left| \frac{1}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}\begin{bmatrix}[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{f}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{f}}]\\ [\vec{\mathbf{g}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{g}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{g}}]\\ [\vec{\mathbf{h}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{h}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{h}}]\end{bmatrix}\right| [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] = \left|\boldsymbol{P}\right| [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$$

where

$$\boldsymbol{P} = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix} = \frac{1}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}\begin{bmatrix}[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{f}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{f}}]\\ [\vec{\mathbf{g}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{g}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{g}}]\\ [\vec{\mathbf{h}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{h}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{h}}]\end{bmatrix} \tag{1.42}$$

For the case when $\vec{\mathbf{f}} = \mathbf{A}\cdot\vec{\mathbf{a}}$, $\vec{\mathbf{g}} = \mathbf{A}\cdot\vec{\mathbf{b}}$, $\vec{\mathbf{h}} = \mathbf{A}\cdot\vec{\mathbf{c}}$, the principal invariants of $\boldsymbol{P}$ are:

$$I_{\boldsymbol{P}} = \mathrm{Tr}(\boldsymbol{P}) = \frac{1}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}\left([\mathbf{A}\cdot\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] + [\vec{\mathbf{a}},\mathbf{A}\cdot\vec{\mathbf{b}},\vec{\mathbf{c}}] + [\vec{\mathbf{a}},\vec{\mathbf{b}},\mathbf{A}\cdot\vec{\mathbf{c}}]\right) = I_{\mathbf{A}}$$

$$I\!I_{\boldsymbol{P}} = \frac{1}{([\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}])^2}\left(\begin{vmatrix}[\vec{\mathbf{a}},\mathbf{A}\cdot\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\mathbf{A}\cdot\vec{\mathbf{b}}]\\ [\vec{\mathbf{a}},\mathbf{A}\cdot\vec{\mathbf{c}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\mathbf{A}\cdot\vec{\mathbf{c}}]\end{vmatrix} + \begin{vmatrix}[\mathbf{A}\cdot\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\mathbf{A}\cdot\vec{\mathbf{a}}]\\ [\mathbf{A}\cdot\vec{\mathbf{c}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\mathbf{A}\cdot\vec{\mathbf{c}}]\end{vmatrix} + \begin{vmatrix}[\mathbf{A}\cdot\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\mathbf{A}\cdot\vec{\mathbf{a}},\vec{\mathbf{c}}]\\ [\mathbf{A}\cdot\vec{\mathbf{b}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\mathbf{A}\cdot\vec{\mathbf{b}},\vec{\mathbf{c}}]\end{vmatrix}\right)$$
$$= I\!I_{\mathbf{A}}$$

$$I\!I\!I_{\boldsymbol{P}} = I\!I\!I_{\mathbf{A}} = \det(\mathbf{A})$$

NOTE 3: Let us consider the Cartesian system where

$$\begin{cases}\vec{\mathbf{a}} = \mathrm{a}_1\hat{\mathbf{e}}_1 + \mathrm{a}_2\hat{\mathbf{e}}_2 + \mathrm{a}_3\hat{\mathbf{e}}_3\\ \vec{\mathbf{b}} = \mathrm{b}_1\hat{\mathbf{e}}_1 + \mathrm{b}_2\hat{\mathbf{e}}_2 + \mathrm{b}_3\hat{\mathbf{e}}_3\\ \vec{\mathbf{c}} = \mathrm{c}_1\hat{\mathbf{e}}_1 + \mathrm{c}_2\hat{\mathbf{e}}_2 + \mathrm{c}_3\hat{\mathbf{e}}_3\end{cases} \quad\Rightarrow\quad \begin{Bmatrix}\vec{\mathbf{a}}\\ \vec{\mathbf{b}}\\ \vec{\mathbf{c}}\end{Bmatrix} = \begin{bmatrix}\mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3\\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3\\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{Bmatrix}$$

Also let us consider that $\vec{\mathbf{f}} = \hat{\mathbf{e}}_1$, $\vec{\mathbf{g}} = \hat{\mathbf{e}}_2$, $\vec{\mathbf{h}} = \hat{\mathbf{e}}_3$, so, taking into account the above equation and the equation in (1.41) we can conclude that:

$$\begin{Bmatrix}\vec{\mathbf{f}}\\ \vec{\mathbf{g}}\\ \vec{\mathbf{h}}\end{Bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\begin{Bmatrix}\vec{\mathbf{a}}\\ \vec{\mathbf{b}}\\ \vec{\mathbf{c}}\end{Bmatrix} \quad\Rightarrow\quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{Bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\begin{bmatrix}\mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3\\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3\\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{Bmatrix}$$

thus

$$\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}\begin{bmatrix}\mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3\\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3\\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix} = \begin{bmatrix}\mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3\\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3\\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3\end{bmatrix}^{-1}$$

With that we can obtain the inverse of a tensor. Let us consider the tensor $\mathbf{A}$ where the components are:

$$\mathrm{A}_{ij} = \begin{bmatrix}\mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13}\\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23}\\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33}\end{bmatrix} = \begin{bmatrix}\mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3\\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3\\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3\end{bmatrix} \quad\Rightarrow\quad |\mathbf{A}| = [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$$

Then, the inverse $\boldsymbol{P} = \mathbf{A}^{-1}$, (see equation (1.42)), becomes:

$$
\mathbf{A}^{-1} = \frac{1}{[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}\begin{bmatrix} [\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{f}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{f}}] \\ [\vec{\mathbf{g}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{g}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{g}}] \\ [\vec{\mathbf{h}},\vec{\mathbf{b}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{h}},\vec{\mathbf{c}}] & [\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{h}}] \end{bmatrix}
$$

$$
= \frac{1}{|\mathbf{A}|}\begin{bmatrix}
\begin{vmatrix} 1 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{vmatrix} \\
\begin{vmatrix} 0 & 1 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 1 & 0 \end{vmatrix} \\
\begin{vmatrix} 0 & 0 & 1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix}
\end{bmatrix}
= \frac{1}{|\mathbf{A}|}\begin{bmatrix}
\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\
-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\
\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{bmatrix}
$$

Taking into account that $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[\mathrm{cof}(\mathbf{A})]^T = \frac{1}{|\mathbf{A}|}[\mathrm{adj}(\mathbf{A})]$, we can conclude that:

$$
[\mathrm{cof}(\mathbf{A})]_{ij} = \begin{bmatrix}
\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\
-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\
\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{bmatrix}^T
= \begin{bmatrix}
\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
-\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\
\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{bmatrix}
$$

Note that the coefficient of the above matrix, $[\mathrm{cof}(\mathbf{A})]_{ij}$, can be obtained by solving the determinant of the resulting matrix by removing the i^{th} row and the j^{th} column, which result we multiply by $(-1)^{i+j}$, for example:

$$
[\mathrm{cof}(\mathbf{A})]_{12} = (-1)^{1+2}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}
$$

Problem 1.57

Given the scalars I_C, II_C, III_C in terms of the scalars I_E, II_E, III_E:

$$
\begin{cases} I_C = 2I_E + 3 \\ II_C = 4I_E + 4\,II_E + 3 \\ III_C = 2I_E + 4\,II_E + 8\,III_E + 1 \end{cases} \tag{1.43}
$$

Obtain the reverse form of the above equations, i.e. obtain $I_{\boldsymbol{E}}$, $II_{\boldsymbol{E}}$, $III_{\boldsymbol{E}}$ in terms of $I_{\boldsymbol{C}}$, $II_{\boldsymbol{C}}$, $III_{\boldsymbol{C}}$.

Solution:

The equations in (1.43) can be restructured as follows:

$$\begin{Bmatrix} I_{\boldsymbol{C}} \\ II_{\boldsymbol{C}} \\ III_{\boldsymbol{C}} \end{Bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix} \begin{Bmatrix} I_{\boldsymbol{E}} \\ II_{\boldsymbol{E}} \\ III_{\boldsymbol{E}} \end{Bmatrix} + \begin{Bmatrix} 3 \\ 3 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix} \begin{Bmatrix} I_{\boldsymbol{E}} \\ II_{\boldsymbol{E}} \\ III_{\boldsymbol{E}} \end{Bmatrix} = \begin{Bmatrix} I_{\boldsymbol{C}} \\ II_{\boldsymbol{C}} \\ III_{\boldsymbol{C}} \end{Bmatrix} - \begin{Bmatrix} 3 \\ 3 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix} \begin{Bmatrix} I_{\boldsymbol{E}} \\ II_{\boldsymbol{E}} \\ III_{\boldsymbol{E}} \end{Bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix}^{-1} \left(\begin{Bmatrix} I_{\boldsymbol{C}} \\ II_{\boldsymbol{C}} \\ III_{\boldsymbol{C}} \end{Bmatrix} - \begin{Bmatrix} 3 \\ 3 \\ 1 \end{Bmatrix} \right)$$

$$\begin{Bmatrix} I_{\boldsymbol{E}} \\ II_{\boldsymbol{E}} \\ III_{\boldsymbol{E}} \end{Bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix}^{-1} \begin{Bmatrix} I_{\boldsymbol{C}} - 3 \\ II_{\boldsymbol{C}} - 3 \\ III_{\boldsymbol{C}} - 1 \end{Bmatrix}$$

where

$$\mathcal{A}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix}^{-1} = \frac{1}{|\mathcal{A}|}[\mathrm{cof}(\mathcal{A})]^T = \frac{1}{64} \begin{bmatrix} \begin{vmatrix} 4 & 0 \\ 4 & 8 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 2 & 8 \end{vmatrix} & \begin{vmatrix} 4 & 4 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 4 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 2 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 4 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} \end{bmatrix}$$

with that the scalars $I_{\boldsymbol{E}}$, $II_{\boldsymbol{E}}$, $III_{\boldsymbol{E}}$ can be obtained as follows:

$$\begin{Bmatrix} I_{\boldsymbol{E}} \\ II_{\boldsymbol{E}} \\ III_{\boldsymbol{E}} \end{Bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 4 & 8 \end{bmatrix}^{-1} \begin{Bmatrix} I_{\boldsymbol{C}} - 3 \\ II_{\boldsymbol{C}} - 3 \\ III_{\boldsymbol{C}} - 1 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} \end{bmatrix} \begin{Bmatrix} I_{\boldsymbol{C}} - 3 \\ II_{\boldsymbol{C}} - 3 \\ III_{\boldsymbol{C}} - 1 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(I_{\boldsymbol{C}} - 3) \\ \frac{1}{4}(-2I_{\boldsymbol{C}} + II_{\boldsymbol{C}} + 3) \\ \frac{1}{8}(I_{\boldsymbol{C}} - II_{\boldsymbol{C}} + III_{\boldsymbol{C}} - 1) \end{Bmatrix}$$

1.6 Additive Decomposition of Tensors

Problem 1.58

Find a fourth-order tensor $\mathbb{P}$ so that $\mathbb{P} : \mathbf{A} = \mathbf{A}^{dev}$, where $\mathbf{A}$ is a second-order tensor.

Solution: Taking into account the additive decomposition into spherical and deviatoric parts, we obtain:

$$\mathbf{A} = \mathbf{A}^{sph} + \mathbf{A}^{dev} = \frac{\mathrm{Tr}(\mathbf{A})}{3}\mathbf{1} + \mathbf{A}^{dev} \qquad \Rightarrow \qquad \mathbf{A}^{dev} = \mathbf{A} - \frac{\mathrm{Tr}(\mathbf{A})}{3}\mathbf{1}$$

By definition the fourth-order tensors are:

$$\mathbb{I} = \mathbf{1}\overline{\otimes}\mathbf{1} = \delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \mathbb{I}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.44}$$

$$\bar{\mathbb{I}} = \mathbf{1} \underline{\otimes} \mathbf{1} = \delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \bar{\mathbb{I}}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.45}$$

$$\bar{\bar{\mathbb{I}}} = \mathbf{1} \otimes \mathbf{1} = \delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \bar{\bar{\mathbb{I}}}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.46}$$

where it holds that:

$$\begin{aligned}
\mathbb{I}:\mathbf{A} &= \left(\delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right):\left(\mathsf{A}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q\right) \\
&= \delta_{ik}\delta_{j\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \delta_{ik}\delta_{j\ell}\mathsf{A}_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \mathsf{A}_{ij}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \mathbf{A}
\end{aligned} \tag{1.47}$$

$$\begin{aligned}
\bar{\bar{\mathbb{I}}}:\mathbf{A} &= \left(\delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right):\left(\mathsf{A}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q\right) \\
&= \delta_{ij}\delta_{k\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \delta_{ij}\delta_{k\ell}\mathsf{A}_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \mathsf{A}_{kk}\delta_{ij}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \\
&= \mathrm{Tr}(\mathbf{A})\mathbf{1}
\end{aligned} \tag{1.48}$$

Referring to the definition of fourth-order unit tensors seen in (1.47), and (1.48), where the relations $\bar{\bar{\mathbb{I}}}:\mathbf{A} = \mathrm{Tr}(\mathbf{A})\mathbf{1}$ and $\mathbb{I}:\mathbf{A} = \mathbf{A}$ hold, we can now state:

$$\mathbf{A}^{dev} = \mathbf{A} - \frac{\mathrm{Tr}(\mathbf{A})}{3}\mathbf{1} = \mathbb{I}:\mathbf{A} - \frac{1}{3}\bar{\bar{\mathbb{I}}}:\mathbf{A} = \left(\mathbb{I} - \frac{1}{3}\bar{\bar{\mathbb{I}}}\right):\mathbf{A} = \left(\mathbb{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right):\mathbf{A}$$

Therefore, we can conclude that:

$$\boxed{\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}}$$

The tensor $\mathbb{P}$ is known as a *fourth-order projection tensor*, Holzapfel(2000).

1.7 Transformation Law for Tensor Components. Invariants.

Problem 1.59

Under the base transformation $\hat{\mathbf{e}}'_i = a_{ij}\hat{\mathbf{e}}_j$ and by considering that the second-order tensor components in this new base are given by:

$$\mathsf{T}'_{ij} = a_{ik}a_{jl}\mathsf{T}_{kl}$$

Show that:

a) $\mathsf{T}'_{ii} = \mathsf{T}_{kk} = \mathrm{Tr}(\mathbf{T})$; b) $\mathsf{T}'_{ij}\mathsf{T}'_{ji} = \mathsf{T}_{kl}\mathsf{T}_{lk}$; c) $\det(\mathbf{T}') = \det(\mathbf{T})$

Solution:

Note that the transformation matrix $a_{ij} = \mathcal{A}$ is an orthogonal matrix, i.e. $\mathcal{A}^{-1} = \mathcal{A}^T$, so the relationship $\mathcal{A}^T\mathcal{A} = \mathbf{1}$ ($a_{ki}a_{kj} = \delta_{ij}$) hold.

a) $\mathsf{T}'_{ij} = a_{ik}a_{jl}\mathsf{T}_{kl} \xrightarrow{i=j} \mathsf{T}'_{ii} = a_{ik}a_{il}\mathsf{T}_{kl} = \delta_{kl}\mathsf{T}_{kl} = \mathsf{T}_{kk} = \mathsf{T}_{ll}$

b) $\mathsf{T}'_{ij}\mathsf{T}'_{ji} = (a_{ik}a_{jl}\mathsf{T}_{kl})(a_{jp}a_{iq}\mathsf{T}_{pq}) = \underbrace{a_{ik}a_{iq}}_{=\delta_{kq}}\underbrace{a_{jl}a_{jp}}_{=\delta_{lp}}\mathsf{T}_{kl}\mathsf{T}_{pq} = \delta_{kq}\delta_{lp}\mathsf{T}_{kl}\mathsf{T}_{pq} = \mathsf{T}_{qp}\mathsf{T}_{pq} = \mathsf{T}_{kl}\mathsf{T}_{lk}$

with that we show that $\mathrm{Tr}(\mathbf{T}^2) = \mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = \mathsf{T}_{ij}\mathsf{T}_{ji}$

c) $\det(\mathsf{T}'_{ij}) = \det(a_{ik}a_{jl}\mathsf{T}_{kl}) = \underbrace{\det(a_{ik})}_{=1}\underbrace{\det(a_{jl})}_{=1}\det(\mathsf{T}_{kl}) = \det(\mathsf{T}_{kl})$

we have just shown that $\mathsf{T}_{kk} = \mathrm{Tr}(\mathbf{T})$, $\mathsf{T}_{kl}\mathsf{T}_{lk} = \mathrm{Tr}(\mathbf{T}\cdot\mathbf{T})$ and $\det(\mathbf{T})$ are invariants.

Problem 1.60

Let $\mathbf{T}$ be a symmetric second-order tensor and $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ be scalars, where:

$$\boxed{I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = \mathsf{T}_{ii} \quad ; \quad II_{\mathbf{T}} = \frac{1}{2}\left\{{I_{\mathbf{T}}}^2 - \mathrm{Tr}(\mathbf{T}^2)\right\} \quad ; \quad III_{\mathbf{T}} = \det(\mathbf{T})}$$

Show that $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ are invariant with a change of basis.

Solution:

a) Taking into account the transformation law for the second-order tensor components $\mathsf{T}'_{ij} = a_{ik}a_{jl}\mathsf{T}_{kl}$ or in matrix form $\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T$. Then, T'_{ii} is:

$$\mathsf{T}'_{ii} = a_{ik}a_{il}\mathsf{T}_{kl} = \delta_{kl}\mathsf{T}_{kl} = \mathsf{T}_{kk} = I_{\mathbf{T}}$$

Hence we have proved that $I_{\mathbf{T}}$ is independent of the adopted system.

b) To prove that $II_{\mathbf{T}}$ is an invariant, one only needs to show that $\mathrm{Tr}(\mathbf{T}^2)$ is one also, since ${I_{\mathbf{T}}}^2$ is already an invariant.

$$\begin{aligned}
\mathrm{Tr}(\mathbf{T}'^2) = \mathrm{Tr}(\mathbf{T}'\cdot\mathbf{T}') = \mathbf{T}':\mathbf{T}' = \mathsf{T}'_{ij}\mathsf{T}'_{ij} &= (a_{ik}a_{jl}\mathsf{T}_{kl})(a_{ip}a_{jq}\mathsf{T}_{pq}) \\
&= \underbrace{a_{ik}a_{ip}}_{\delta_{kp}}\underbrace{a_{jl}a_{jq}}_{\delta_{lq}}\mathsf{T}_{kl}\mathsf{T}_{pq} \\
&= \mathsf{T}_{pl}\mathsf{T}_{pl} \\
&= \mathbf{T}:\mathbf{T} = \mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = \mathrm{Tr}(\mathbf{T}^2)
\end{aligned}$$

c) Matrix form: $\det(\mathbf{T}') = \det(\mathcal{T}') = \det(\mathcal{A}\,\mathcal{T}\,\mathcal{A}^T) = \underbrace{\det(\mathcal{A})}_{=1}\det(\mathcal{T})\underbrace{\det(\mathcal{A}^T)}_{=1} = \det(\mathcal{T})$

Problem 1.61

Show that the following relations are invariants:

$$C_1^2 + C_2^2 + C_3^2 \quad ; \quad C_1^3 + C_2^3 + C_3^3 \quad ; \quad C_1^4 + C_2^4 + C_3^4$$

where C_1, C_2, C_3 are the eigenvalues of the second-order tensor $\boldsymbol{C}$.

Solution:

Recall that in the principal space of $\boldsymbol{C}$ the following is true:

$$C'_{ij} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} I_{\boldsymbol{C}} = C_1 + C_2 + C_3 \\ II_{\boldsymbol{C}} = C_2C_3 + C_1C_3 + C_1C_2 \\ III_{\boldsymbol{C}} = C_1C_2C_3 \end{cases}$$

Any combination of invariants is also an invariant, so, on this basis, we can try to express the above expressions in terms of their principal invariants.

$$I_{\boldsymbol{C}}^2 = (C_1 + C_2 + C_3)^2 = C_1^2 + C_2^2 + C_3^2 + 2\underbrace{(C_1C_2 + C_1C_3 + C_2C_3)}_{II_{\boldsymbol{C}}} \Rightarrow C_1^2 + C_2^2 + C_3^2 = I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}}$$

So, we have proved that $C_1^2 + C_2^2 + C_3^2$ is an invariant. Similarly, we can obtain the other relationships, so, we summarize:

$$\boxed{\begin{aligned} &C_1 + C_2 + C_3 = I_C \\ &C_1^2 + C_2^2 + C_3^2 = I_C^2 - 2\,II_C \\ &C_1^3 + C_2^3 + C_3^3 = I_C^3 - 3\,II_C I_C + 3\,III_C \\ &C_1^4 + C_2^4 + C_3^4 = I_C^4 - 4\,II_C I_C^2 + 4\,III_C I_C + 2\,II_C^2 \\ &C_1^5 + C_2^5 + C_3^5 = I_C^5 - 5\,II_C I_C^3 + 5\,III_C I_C^2 + 5\,II_C^2 I_C - 5\,III_C\,II_C \end{aligned}}$$

The following is also true:

$$C_1^{n+1} + C_2^{n+1} + C_3^{n+1} = \left(C_1^n + C_2^n + C_3^n\right) I_C - C_1\left(C_2^{n-1} + C_3^{n-1}\right) - C_2\left(C_1^{n-1} + C_3^{n-1}\right) - C_3\left(C_1^{n-1} + C_2^{n-1}\right)$$

Problem 1.62

Show that, if a symmetric second-order tensor $\mathbf{T}$ has three different real eigenvalues $(\lambda_1 \neq \lambda_2 \neq \lambda_3)$, the principal space of $\mathbf{T}$ is formed by an orthonormal basis.

Solution:

Consider a symmetric second-order tensor $\mathbf{T}$. By the definition of eigenvalues, given in $\mathbf{T}\cdot\hat{\mathbf{n}}^{(a)} = \lambda_a \hat{\mathbf{n}}^{(a)}$, if λ_1, λ_2, λ_3 are the eigenvalues of $\mathbf{T}$, then it follows that:

$$\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_1\hat{\mathbf{n}}^{(1)} \quad ; \quad \mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)} \quad ; \quad \mathbf{T}\cdot\hat{\mathbf{n}}^{(3)} = \lambda_3\hat{\mathbf{n}}^{(3)} \tag{1.49}$$

Applying the dot product between $\hat{\mathbf{n}}^{(2)}$ and $\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_1\hat{\mathbf{n}}^{(1)}$, and the dot product between $\hat{\mathbf{n}}^{(1)}$ and $\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)}$ we obtain:

$$\hat{\mathbf{n}}^{(2)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_1\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} \quad ; \quad \hat{\mathbf{n}}^{(1)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)} \tag{1.50}$$

Since $\mathbf{T}$ is symmetric, it holds that $\hat{\mathbf{n}}^{(2)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(1)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)}$, so:

$$\lambda_1\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_2\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} \tag{1.51}$$

$$\Rightarrow (\lambda_1 - \lambda_2)\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)} = 0 \tag{1.52}$$

To satisfy the equation (1.52), with $\lambda_1 \neq \lambda_2 \neq 0$, the following must be true:

$$\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)} = 0 \tag{1.53}$$

Similarly, it is possible to show that $\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(3)} = 0$ and $\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(3)} = 0$ and then we can conclude that the eigenvectors are mutually orthogonal, and constitute an orthogonal basis, where the transformation matrix between systems is:

$$\mathcal{A} = \begin{bmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{n}_1^{(1)} & \hat{n}_2^{(1)} & \hat{n}_3^{(1)} \\ \hat{n}_1^{(2)} & \hat{n}_2^{(2)} & \hat{n}_3^{(2)} \\ \hat{n}_1^{(3)} & \hat{n}_2^{(3)} & \hat{n}_3^{(3)} \end{bmatrix} \tag{1.54}$$

NOTE: If the tensor is not symmetric the principal space not necessarily is orthogonal.

Problem 1.63

Obtain the components of $\mathbf{T}'$, given by the transformation:

$$\mathbf{T}' = \mathbf{A}\cdot\mathbf{T}\cdot\mathbf{A}^T$$

where the components of $\mathbf{T}$ and $\mathbf{A}$ are shown, respectively, as T_{ij} and a_{ij}. Afterwards, given that a_{ij} are the components of the transformation matrix, represent graphically the components of the tensors $\mathbf{T}$ and $\mathbf{T}'$ on both systems.

Solution: The expression $\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T$ in symbolic notation is given by:

$$\begin{aligned} \mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) &= a_{rs}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_s) \cdot \mathsf{T}_{pq}(\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) \cdot a_{kl}(\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k) = a_{rs}\mathsf{T}_{pq}a_{kl}\delta_{sp}\delta_{ql}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) \\ &= a_{rp}\mathsf{T}_{pq}a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) \end{aligned}$$

To obtain the components of $\mathbf{T}'$ one only need make the double scalar product with the basis $(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$, the result of which is:

$$\begin{aligned} \mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) &= a_{rp}\mathsf{T}_{pq}a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ \mathsf{T}'_{ab}\delta_{ai}\delta_{bj} &= a_{rp}\mathsf{T}_{pq}a_{kq}\delta_{ri}\delta_{kj} \quad \Rightarrow \quad \mathsf{T}'_{ij} = a_{ip}\mathsf{T}_{pq}a_{jq} \end{aligned}$$

The above equation is shown in matrix notation as:

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \xrightarrow{inverse} \mathcal{T} = \mathcal{A}^{-1}\,\mathcal{T}'\,\mathcal{A}^{-T}$$

Since $\mathcal{A}$ is an orthogonal matrix, it holds that $\mathcal{A}^T = \mathcal{A}^{-1}$. Thus, $\mathcal{T} = \mathcal{A}^T\,\mathcal{T}'\,\mathcal{A}$. The graphical representation of the tensor components in both systems can be seen in Figure 1.10.

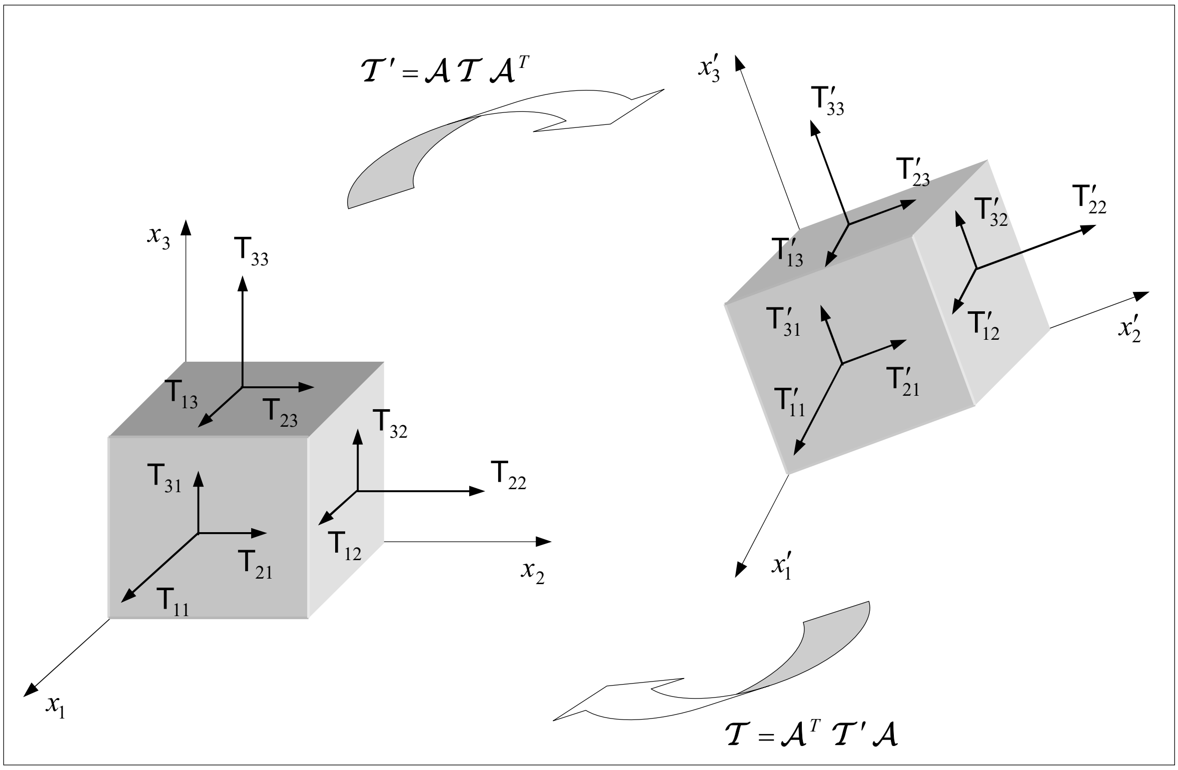


Figure 1.10: Transformation law for the second-order tensor components.

Problem 1.64

Let $\mathbf{T}$ be a second-order tensor whose components in the Cartesian system (x_1, x_2, x_3) are given by:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given that the transformation matrix between two systems, (x_1, x_2, x_3)- (x'_1, x'_2, x'_3), is:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Obtain the tensor components T_{ij} in the new coordinate system (x'_1, x'_2, x'_3).

Solution: The transformation law for second-order tensor components is $\mathsf{T}'_{ij} = a_{ik} a_{jl} \mathsf{T}_{kl}$.

To enable the previous calculation to be carried out in matrix form we use:

$$\mathsf{T}'_{ij} = [a_{ik}]\,[\mathsf{T}_{kl}]\,[a_{lj}]^T$$

Thus

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T$$

$$\mathcal{T}' = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

On carrying out the operation of the previous matrices we now have:

$$\mathcal{T}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Problem 1.65

Find the transformation matrix between the systems: x, y, z and x''', y''', z'''. These systems are represented in Figure 1.11.

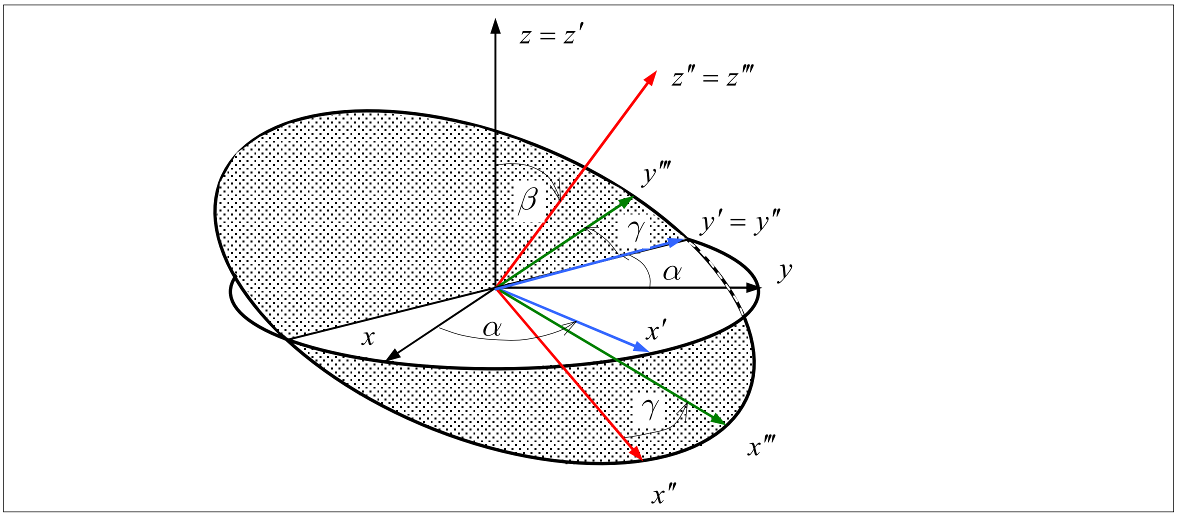


Figure 1.11: Rotation.

Solution: Note that: if we have an initial space and successive transformations until the final space, the transformation law from the initial space to the final space is formed by the product of the transformations in the opposite direction. That is, we place in the final space and we follow opposite direction of the arrows until the initial space, (see Figure 1.12).

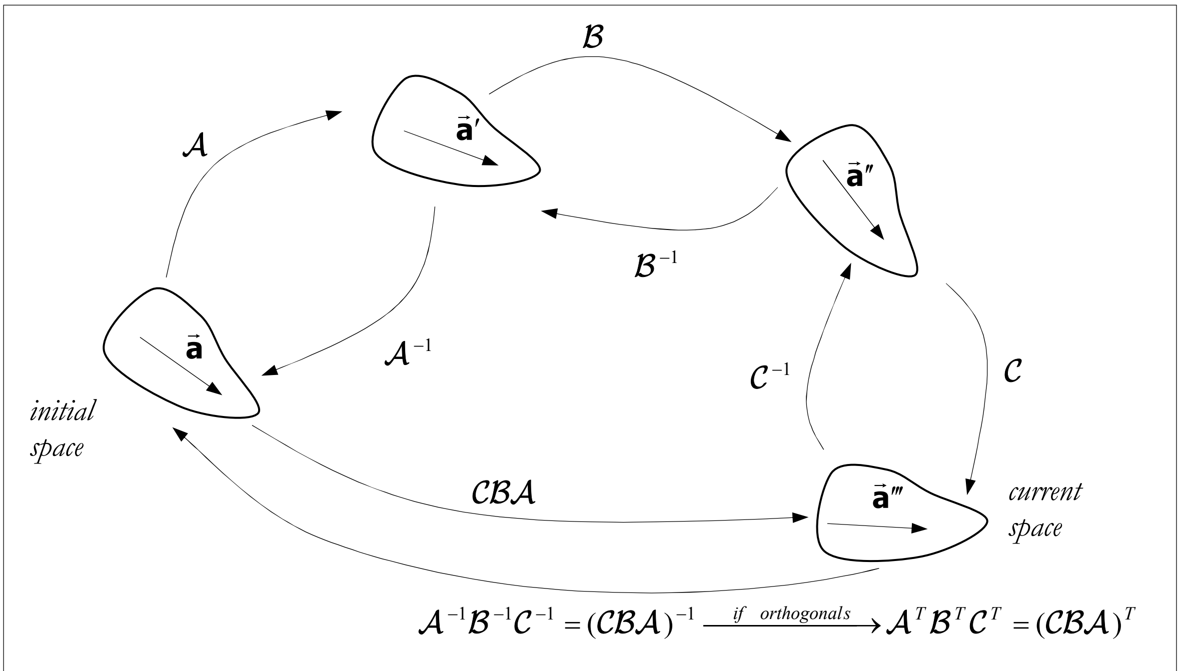


Figure 1.12

The coordinate system x''', y''', z''' can be obtained by different combinations of rotations as follows:

- Rotation along the z-axis, (see Figure 1.13).

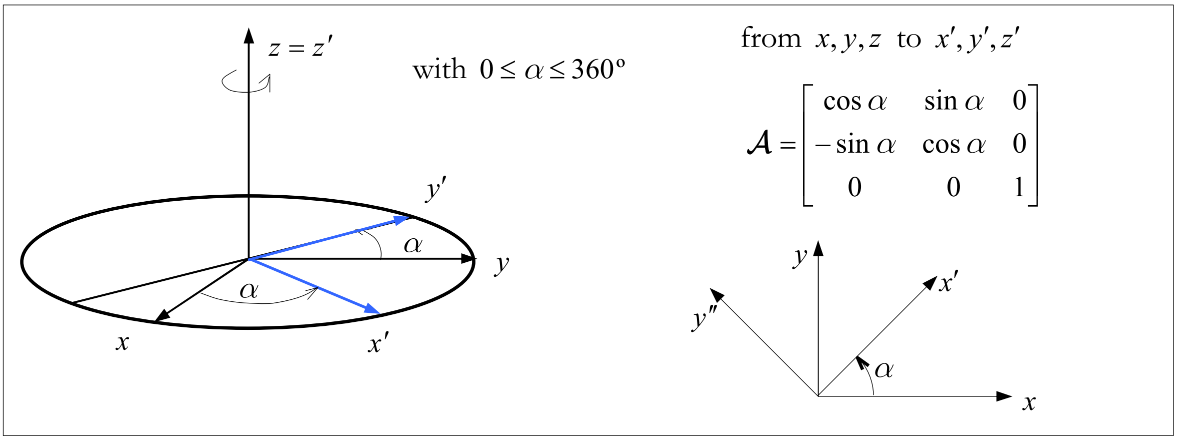


Figure 1.13

- Rotation along the y'-axis, (see Figure 1.14).

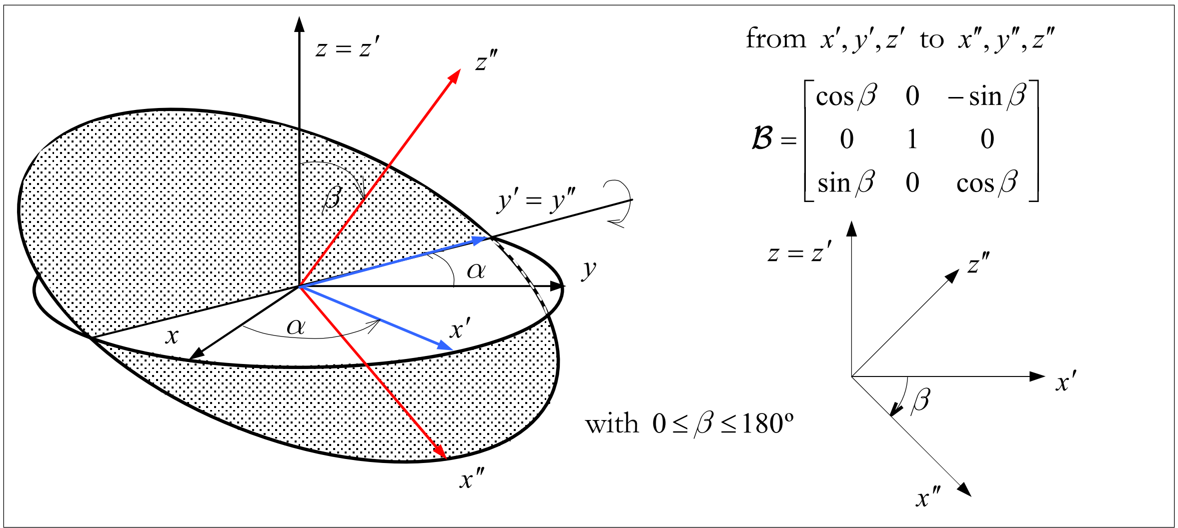


Figure 1.14

- Rotation along the z''-axis, (see Figure 1.15)

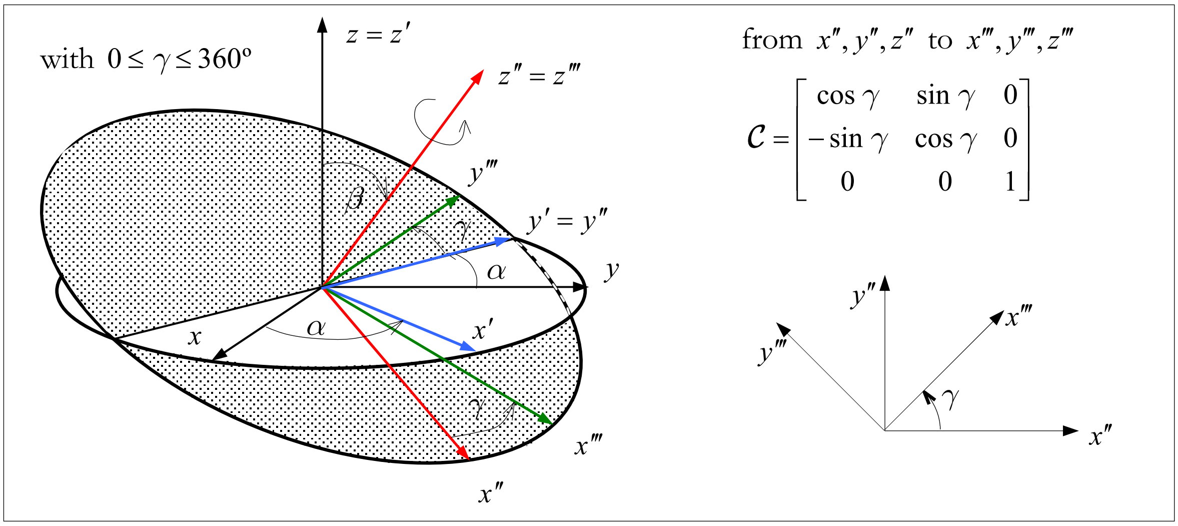


Figure 1.15

The transformation matrix from (x, y, z) to (x''', y''', z''') is given by

$$\mathcal{D} = \mathcal{C}\mathcal{B}\mathcal{A}$$

After multiplying the matrices, we obtain:

$$\mathcal{D} = \begin{bmatrix} (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma) & (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma) & -\sin\beta\cos\gamma \\ (-\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma) & (-\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma) & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix}$$

The angles α, β, γ are known as *Euler angles* and were introduced by Leonhard Euler to describe the orientation of a rigid body motion, which is discussed in Chapter 4.

Problem 1.66

If a_{ij} represent the components of the base transformation matrix, show that the following equations are true:

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \end{cases} \quad or \quad \begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \end{cases}$$

Solution:

We start from the principle that the basis transformation matrix is an orthogonal matrix, i.e. $a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij}$. Then:

$$a_{ik}a_{jk} = a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = \delta_{ij} \quad \Rightarrow \quad \begin{cases} (i=1, j=1) & a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ (i=2, j=2) & a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ (i=3, j=3) & a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \\ (i=1, j=2) & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ (i=2, j=3) & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \\ (i=1, j=3) & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \end{cases}$$

Alternative solution:

$$\mathcal{A}\mathcal{A}^T = \mathbf{1} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the matrix multiplication we obtain:

$$\begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{21}^2 + a_{22}^2 + a_{23}^2 & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 1.67

a) Obtain the transformation matrix, a_{ij}, from the system $O\vec{X}$ to the system $o\vec{x}$, (see Figure 1.16). The plane defined by the triangle $1-2-3$ is lying on the plane $x_2 - x_3$. b) Obtain the triangle area in terms of the node coordinates.

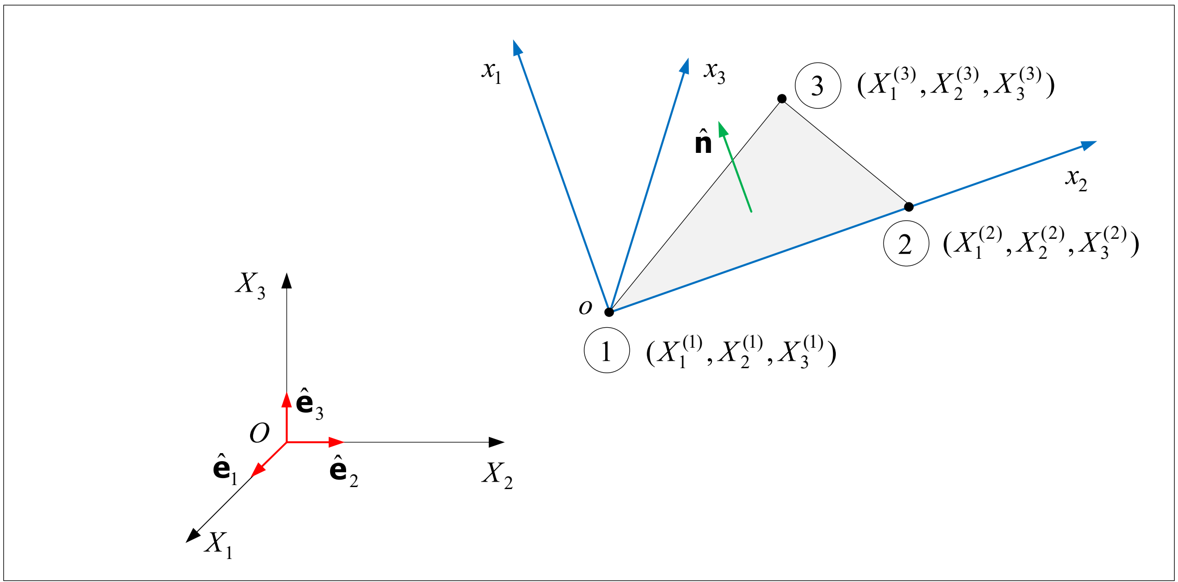


Figure 1.16

Solution:

a) The unit vector associated with the direction $\overrightarrow{\mathbf{12}}$ is given by:

$$\hat{\mathbf{12}} = \frac{(\overrightarrow{\mathbf{12}})_1}{\left\|\overrightarrow{\mathbf{12}}\right\|}\hat{\mathbf{e}}_1 + \frac{(\overrightarrow{\mathbf{12}})_2}{\left\|\overrightarrow{\mathbf{12}}\right\|}\hat{\mathbf{e}}_2 + \frac{(\overrightarrow{\mathbf{12}})_3}{\left\|\overrightarrow{\mathbf{12}}\right\|}\hat{\mathbf{e}}_3 = a_{21}\hat{\mathbf{e}}_1 + a_{22}\hat{\mathbf{e}}_2 + a_{23}\hat{\mathbf{e}}_3$$

where

$$\overrightarrow{\mathbf{12}} = (X_1^{(2)} - X_1^{(1)})\hat{\mathbf{e}}_1 + (X_2^{(2)} - X_2^{(1)})\hat{\mathbf{e}}_2 + (X_3^{(2)} - X_3^{(1)})\hat{\mathbf{e}}_3$$

$$\left\|\overrightarrow{\mathbf{12}}\right\| = \sqrt{(X_1^{(2)} - X_1^{(1)})^2 + (X_2^{(2)} - X_2^{(1)})^2 + (X_3^{(2)} - X_3^{(1)})^2}$$

The unit vector $\hat{\mathbf{n}} // \overrightarrow{ox_1}$ can be obtained by $\hat{\mathbf{n}} = \hat{\mathbf{12}} \wedge \hat{\mathbf{13}}$. And the unit vector $\hat{\mathbf{13}}$ is given by:

$$\hat{\mathbf{13}} = \frac{(\overrightarrow{\mathbf{13}})_1}{\left\|\overrightarrow{\mathbf{13}}\right\|}\hat{\mathbf{e}}_1 + \frac{(\overrightarrow{\mathbf{13}})_2}{\left\|\overrightarrow{\mathbf{13}}\right\|}\hat{\mathbf{e}}_2 + \frac{(\overrightarrow{\mathbf{13}})_3}{\left\|\overrightarrow{\mathbf{13}}\right\|}\hat{\mathbf{e}}_3 = s_1\hat{\mathbf{e}}_1 + s_2\hat{\mathbf{e}}_2 + s_3\hat{\mathbf{e}}_3$$

where

$$\overrightarrow{\mathbf{13}} = (X_1^{(3)} - X_1^{(1)})\hat{\mathbf{e}}_1 + (X_2^{(3)} - X_2^{(1)})\hat{\mathbf{e}}_2 + (X_3^{(3)} - X_3^{(1)})\hat{\mathbf{e}}_3$$

$$\left\|\overrightarrow{\mathbf{13}}\right\| = \sqrt{(X_1^{(3)} - X_1^{(1)})^2 + (X_2^{(3)} - X_2^{(1)})^2 + (X_3^{(3)} - X_3^{(1)})^2}$$

Then, we can calculate:

$$\hat{\mathbf{n}} = \stackrel{\wedge}{\mathbf{12}} \wedge \stackrel{\wedge}{\mathbf{13}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_{21} & a_{22} & a_{23} \\ s_1 & s_2 & s_3 \end{vmatrix} = (a_{22}s_3 - a_{23}s_2)\hat{\mathbf{e}}_1 + (a_{23}s_1 - a_{21}s_3)\hat{\mathbf{e}}_2 + (a_{21}s_2 - a_{22}s_1)\hat{\mathbf{e}}_3$$

$$\Rightarrow \hat{\mathbf{n}} = a_{11}\hat{\mathbf{e}}_1 + a_{12}\hat{\mathbf{e}}_2 + a_{13}\hat{\mathbf{e}}_3$$

The unit vector associated with the direction $\overrightarrow{ox_3}$ can be calculated as follows:

$$\stackrel{\wedge}{ox_3} = \hat{\mathbf{n}} \wedge \stackrel{\wedge}{\mathbf{12}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = (a_{12}a_{23} - a_{13}a_{22})\hat{\mathbf{e}}_1 + (a_{13}a_{21} - a_{11}a_{23})\hat{\mathbf{e}}_2 + (a_{11}a_{22} - a_{12}a_{21})\hat{\mathbf{e}}_3$$

$$\Rightarrow \stackrel{\wedge}{ox_3} = a_{31}\hat{\mathbf{e}}_1 + a_{32}\hat{\mathbf{e}}_2 + a_{33}\hat{\mathbf{e}}_3$$

Then, the transformation matrix from the system $O\vec{X}$ to the system $o\vec{x}$ is given by

$$\mathcal{A} = a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

b) The area vector, formed by $\overrightarrow{\mathbf{12}}$ and $\overrightarrow{\mathbf{13}}$, can be obtained by the means of the cross product as follows:

$$\vec{A} = (\overrightarrow{\mathbf{12}} \wedge \overrightarrow{\mathbf{13}}) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ (\overrightarrow{\mathbf{12}})_1 & (\overrightarrow{\mathbf{12}})_2 & (\overrightarrow{\mathbf{12}})_3 \\ (\overrightarrow{\mathbf{13}})_1 & (\overrightarrow{\mathbf{13}})_2 & (\overrightarrow{\mathbf{13}})_3 \end{vmatrix} = \begin{vmatrix} (\overrightarrow{\mathbf{12}})_2 & (\overrightarrow{\mathbf{12}})_3 \\ (\overrightarrow{\mathbf{13}})_2 & (\overrightarrow{\mathbf{13}})_3 \end{vmatrix} \hat{\mathbf{e}}_1 - \begin{vmatrix} (\overrightarrow{\mathbf{12}})_1 & (\overrightarrow{\mathbf{12}})_3 \\ (\overrightarrow{\mathbf{13}})_1 & (\overrightarrow{\mathbf{13}})_3 \end{vmatrix} \hat{\mathbf{e}}_2 + \begin{vmatrix} (\overrightarrow{\mathbf{12}})_1 & (\overrightarrow{\mathbf{12}})_2 \\ (\overrightarrow{\mathbf{13}})_1 & (\overrightarrow{\mathbf{13}})_2 \end{vmatrix} \hat{\mathbf{e}}_3$$

$$= A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

And the triangle area is defined by

$$A_T = \frac{1}{2}\left\|\vec{A}\right\| = \frac{1}{2}\left\|(\overrightarrow{\mathbf{12}} \wedge \overrightarrow{\mathbf{13}})\right\| = \frac{1}{2}\sqrt{A_1^2 + A_2^2 + A_3^2}$$

$$= \frac{1}{2}\sqrt{\left(\begin{vmatrix} (\overrightarrow{\mathbf{12}})_2 & (\overrightarrow{\mathbf{12}})_3 \\ (\overrightarrow{\mathbf{13}})_2 & (\overrightarrow{\mathbf{13}})_3 \end{vmatrix}\right)^2 + \left(-\begin{vmatrix} (\overrightarrow{\mathbf{12}})_1 & (\overrightarrow{\mathbf{12}})_3 \\ (\overrightarrow{\mathbf{13}})_1 & (\overrightarrow{\mathbf{13}})_3 \end{vmatrix}\right)^2 + \left(\begin{vmatrix} (\overrightarrow{\mathbf{12}})_1 & (\overrightarrow{\mathbf{12}})_2 \\ (\overrightarrow{\mathbf{13}})_1 & (\overrightarrow{\mathbf{13}})_2 \end{vmatrix}\right)^2}$$

$$= \frac{1}{2}\sqrt{[(\overrightarrow{\mathbf{12}})_2(\overrightarrow{\mathbf{13}})_3 - (\overrightarrow{\mathbf{12}})_3(\overrightarrow{\mathbf{13}})_2]^2 + [(\overrightarrow{\mathbf{12}})_1(\overrightarrow{\mathbf{13}})_3 - (\overrightarrow{\mathbf{12}})_3(\overrightarrow{\mathbf{13}})_1]^2 + [(\overrightarrow{\mathbf{12}})_1(\overrightarrow{\mathbf{13}})_2 - (\overrightarrow{\mathbf{12}})_2(\overrightarrow{\mathbf{13}})_1]^2}$$

where the components of the vectors $\overrightarrow{\mathbf{12}}$ and $\overrightarrow{\mathbf{13}}$ are

$$\overrightarrow{\mathbf{12}} = (X_1^{(2)} - X_1^{(1)})\hat{\mathbf{e}}_1 + (X_2^{(2)} - X_2^{(1)})\hat{\mathbf{e}}_2 + (X_3^{(2)} - X_3^{(1)})\hat{\mathbf{e}}_3$$

$$\overrightarrow{\mathbf{13}} = (X_1^{(3)} - X_1^{(1)})\hat{\mathbf{e}}_1 + (X_2^{(3)} - X_2^{(1)})\hat{\mathbf{e}}_2 + (X_3^{(3)} - X_3^{(1)})\hat{\mathbf{e}}_3$$

1.8 Eigenvalues, Eigenvectors, Orthogonal Transformation

Problem 1.68

Let $\mathbf{Q}$ be a proper orthogonal tensor, and $\mathbf{E}$ be an arbitrary second-order tensor. Show that the eigenvalues of $\mathbf{E}$ do not change with the following orthogonal transformation:

$$\mathbf{E}^* = \mathbf{Q}\cdot\mathbf{E}\cdot\mathbf{Q}^T$$

Solution:

We will take into account that the eigenvalues of $\mathbf{E}$ are obtained by solving the determinant $|\mathbf{E}-\lambda\mathbf{1}|=0$, and that the spherical tensor is the same for any transformation $\mathbf{1}=\mathbf{Q}\cdot\lambda\mathbf{1}\cdot\mathbf{Q}^T=\mathbf{1}^*$, then:

$$\begin{aligned}
0 &= \det(\mathbf{E}^*-\lambda\mathbf{1}^*)\\
&= \det(\mathbf{Q}\cdot\mathbf{E}\cdot\mathbf{Q}^T-\mathbf{Q}\cdot\lambda\mathbf{1}\cdot\mathbf{Q}^T)\\
&= \det[\mathbf{Q}\cdot(\mathbf{E}-\lambda\mathbf{1})\cdot\mathbf{Q}^T]\\
&= \underbrace{\det(\mathbf{Q})}_{1}\det(\mathbf{E}-\lambda\mathbf{1})\underbrace{\det(\mathbf{Q}^T)}_{1}\\
&= \det(\mathbf{E}-\lambda\mathbf{1})
\end{aligned}$$

$$\begin{aligned}
0 &= \det(\mathsf{E}^*_{ij}-\lambda\delta^*_{ij})\\
&= \det(\mathsf{Q}_{ik}\mathsf{E}_{kp}\mathsf{Q}_{jp}-\lambda\mathsf{Q}_{ik}\mathsf{Q}_{jp}\delta_{kp})\\
&= \det[\mathsf{Q}_{ik}(\mathsf{E}_{kp}-\lambda\delta_{kp})\mathsf{Q}_{jp}]\\
&= \det(\mathsf{Q}_{ik})\det(\mathsf{E}_{kp}-\lambda\delta_{kp})\det(\mathsf{Q}_{jp})\\
&= \det(\mathsf{E}_{kp}-\lambda\delta_{kp})
\end{aligned}$$

Thus, we have proved that $\mathbf{E}$ and $\mathbf{E}^*$ have the same eigenvalues.

Problem 1.69

Let $\mathbf{A}$ be a second-order tensor and $\mathbf{Q}$ be an orthogonal tensor. If the orthogonal transformation law to $\mathbf{A}$ is given by $\mathbf{A}^*=\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T$, show that $\mathbf{A}^{2^*}=\mathbf{Q}\cdot\mathbf{A}^2\cdot\mathbf{Q}^T$.

Solution:

$$\begin{aligned}
\mathbf{A}^{2^*} &= \mathbf{A}^*\cdot\mathbf{A}^*\\
&= (\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T)\cdot(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T)\\
&= \mathbf{Q}\cdot\mathbf{A}\cdot\underbrace{\mathbf{Q}^T\cdot\mathbf{Q}}_{=\mathbf{1}}\cdot\mathbf{A}\cdot\mathbf{Q}^T\\
&= \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{Q}^T\\
&= \mathbf{Q}\cdot\mathbf{A}^2\cdot\mathbf{Q}^T
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A}^{2^*})_{ij} &= (\mathbf{A}^*\cdot\mathbf{A}^*)_{ij} = \mathsf{A}^*_{ik}\mathsf{A}^*_{kj}\\
&= (\mathsf{Q}_{ip}\mathsf{A}_{pr}\mathsf{Q}_{kr})(\mathsf{Q}_{ks}\mathsf{A}_{st}\mathsf{Q}_{jt})\\
&= \mathsf{Q}_{ip}\mathsf{A}_{pr}\underbrace{\mathsf{Q}_{kr}\mathsf{Q}_{ks}}_{=\delta_{rs}}\mathsf{A}_{st}\mathsf{Q}_{jt}\\
&= \mathsf{Q}_{ip}\mathsf{A}_{pr}\delta_{rs}\mathsf{A}_{st}\mathsf{Q}_{jt} = \mathsf{Q}_{ip}\mathsf{A}_{ps}\mathsf{A}_{st}\mathsf{Q}_{jt}\\
&= \mathsf{Q}_{ip}(\mathbf{A}\cdot\mathbf{A})_{pt}\mathsf{Q}_{jt}\\
&= (\mathbf{Q}\cdot\mathbf{A}^2\cdot\mathbf{Q}^T)_{ij}
\end{aligned}$$

Problem 1.70

Given the tensor components:

$$\mathsf{T}_{ij}=\begin{bmatrix}5&3&3\\2&6&3\\2&2&4\end{bmatrix}$$

a) Obtain the principal invariants of $\mathbf{T}$, i.e. obtain $I_{\mathbf{T}}$, $II_{\mathbf{T}}$ and $III_{\mathbf{T}}$;

b) Obtain the characteristic polynomial associated with $\mathbf{T}$;

c) If λ_1, λ_2 and λ_3 are the eigenvalues of $\mathbf{T}$ and $\lambda_1 = 10$. Obtain λ_2 and $\lambda_3 > 2$.

Solution:

a) The principal invariants of $\mathbf{T}$ are:

$I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = 5 + 6 + 4 = 15$

$$I\!I_{\mathbf{T}} = \begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} = 56$$

$I\!I\!I_{\mathbf{T}} = \det(\mathbf{T}) = 60$

b) The characteristic polynomial can be obtained by solving the determinant:

$$\begin{vmatrix} 5-\lambda & 3 & 3 \\ 2 & 6-\lambda & 3 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda I\!I_{\mathbf{T}} - I\!I\!I_{\mathbf{T}} = 0$$

thus:

$$\lambda^3 - 15\lambda^2 + 56\lambda - 60 = 0$$

c) In the principal space the following is true:

$$\mathsf{T}'_{ij} = \begin{bmatrix} \lambda_1 = 10 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 > 2 \end{bmatrix}$$

where the principal invariants are

$I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = \lambda_1 + \lambda_2 + \lambda_3 = 15 \quad \Rightarrow \quad \lambda_2 + \lambda_3 = 5$

$I\!I\!I_{\mathbf{T}} = \det(\mathbf{T}) = \lambda_1\lambda_2\lambda_3 = 60 \quad \Rightarrow \quad \lambda_2\lambda_3 = 6$

By combining these two equations we can obtain:

$$\left.\begin{aligned} \lambda_2\lambda_3 &= 6 \\ \lambda_2 + \lambda_3 &= 5 \end{aligned}\right\} \Rightarrow (5-\lambda_3)\lambda_3 = 6 \Rightarrow \lambda_3^2 - 5\lambda_3 + 6 = 0 \Rightarrow \begin{cases} \lambda_3^{(1)} = 3 \\ \lambda_3^{(2)} = 2 \end{cases}$$

We discard the solution $\lambda_3^{(2)} = 2$, thus $\lambda_3 = 3$. Then:

$$\mathsf{T}'_{ij} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In this space we can check that $\begin{cases} I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = 10 + 2 + 3 = 15 \\ I\!I_{\mathbf{T}} = 2\times 3 + 10\times 3 + 10\times 2 = 56 \\ I\!I\!I_{\mathbf{T}} = \det(\mathbf{T}) = 10\times 2\times 3 = 60 \end{cases}$

Problem 1.71

Find the principal values and directions of the second-order tensor $\mathbf{T}$, where the Cartesian components of $\mathbf{T}$ are:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: We need to find nontrivial solutions for $(\mathsf{T}_{ij} - \lambda \delta_{ij})\hat{\mathsf{n}}_j = 0_i$, which are constrained by $\hat{\mathsf{n}}_j \hat{\mathsf{n}}_j = 1$ (unit vector). As we have seen, the nontrivial solution requires that:

$$\left| \mathsf{T}_{ij} - \lambda \delta_{ij} \right| = 0$$

Explicitly, the above equation is:

$$\begin{vmatrix} \mathsf{T}_{11} - \lambda & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22} - \lambda & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33} - \lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Developing the above determinant, we can obtain the cubic equation:

$$(1-\lambda)[(3-\lambda)^2 - 1] = 0 \quad \Rightarrow \quad \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

We could have obtained the characteristic equation directly in terms of invariants:

$$I_{\mathbf{T}} = \mathrm{Tr}(\mathsf{T}_{ij}) = \mathsf{T}_{ii} = \mathsf{T}_{11} + \mathsf{T}_{22} + \mathsf{T}_{33} = 7$$

$$II_{\mathbf{T}} = \frac{1}{2}\left(\mathsf{T}_{ii}\mathsf{T}_{jj} - \mathsf{T}_{ij}\mathsf{T}_{ij}\right) = \begin{vmatrix} \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{32} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{13} \\ \mathsf{T}_{31} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{vmatrix} = 14$$

$$III_{\mathbf{T}} = \left|\mathsf{T}_{ij}\right| = \epsilon_{ijk}\mathsf{T}_{i1}\mathsf{T}_{j2}\mathsf{T}_{k3} = 8$$

Then, the characteristic equation becomes:

$$\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0 \quad \rightarrow \quad \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

On solving the cubic equation we obtain three real roots, namely:

$$\lambda_1 = 1; \quad \lambda_2 = 2; \quad \lambda_3 = 4$$

We can also verify that:

$$I_{\mathbf{T}} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 4 = 7 \quad \checkmark$$
$$II_{\mathbf{T}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\times 2 + 2\times 4 + 4\times 1 = 14 \quad \checkmark$$
$$III_{\mathbf{T}} = \lambda_1\lambda_2\lambda_3 = 8 \quad \checkmark$$

Thus, we can see that the invariants are the same as those evaluated previously.

<u>Principal directions:</u>

Each eigenvalue, λ_i, is associated with a corresponding eigenvector, $\hat{\mathbf{n}}^{(i)}$. We can use the equation $(\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = \mathbf{0}_i$ to obtain the principal directions.

- $\lambda_1 = 1$

$$\begin{bmatrix} 3-\lambda_1 & -1 & 0 \\ -1 & 3-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 3-1 & -1 & 0 \\ -1 & 3-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These become the following system of equations:

$$\begin{cases} 2n_1 - n_2 = 0 \\ -n_1 + 2n_2 = 0 \end{cases} \Rightarrow n_1 = n_2 = 0, \qquad 0n_3 = 0$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1$$

Then we can conclude that: $\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$.

This solution could have been directly determined by the specific features of the $\mathcal{T}$ matrix. As the terms $T_{13} = T_{23} = T_{31} = T_{32} = 0$ imply that $T_{33} = 1$ is already a principal value, then, consequently, the original direction is a principal direction.

$\lambda_2 = 2$

$$\begin{bmatrix} 3-\lambda_2 & -1 & 0 \\ -1 & 3-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \\ 0 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} n_1 - n_2 = 0 \Rightarrow n_1 = n_2 \\ -n_1 + n_2 = 0 \\ -n_3 = 0 \end{cases}$$

The first two equations are linearly dependent, after which we need an additional equation:

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_1^2 = 1 \Rightarrow n_1 = \pm\sqrt{\frac{1}{2}}$$

Thus:

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \left[\pm\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0\right]$$

$\lambda_3 = 4$

$$\begin{bmatrix} 3-\lambda_3 & -1 & 0 \\ -1 & 3-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \\ 0 & 0 & 1-4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -n_1 - n_2 = 0 \\ -n_1 - n_2 = 0 \end{cases} \Rightarrow n_1 = -n_2, \qquad -3n_3 = 0$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_2^2 = 1 \Rightarrow n_2 = \pm\sqrt{\frac{1}{2}}$$

Then:

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \left[\mp\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0\right]$$

Afterwards, we summarize the eigenvalues and eigenvectors of $\mathbf{T}$:

$$\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = \begin{bmatrix} 0 & 0 & \pm 1 \end{bmatrix}$$

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \begin{bmatrix} \pm\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \begin{bmatrix} \mp\sqrt{\frac{1}{2}} & \pm\sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$

NOTE 1: The tensor components of this problem are the same as those used in **Problem 1.64**. Additionally, we can verify that the eigenvectors make up the transformation matrix, $\mathcal{A}$, between the original system, (x_1, x_2, x_3), and the principal space, (x_1', x_2', x_3'), (see **Problem 1.64**).

Problem 1.72

Let $\mathbf{Q}$ be a proper orthogonal tensor a) show that $\mathbf{Q}$ has one real eigenvalue and equals to 1. b) Also show that $\mathbf{Q}$ can be represented by means of the angle θ as follows:

$$\mathbf{Q} = \hat{\mathbf{p}} \otimes \hat{\mathbf{p}} + \cos\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{q}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) - \sin\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{r}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{q}})$$

where $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$, are unit vectors which form an orthonormal basis, where $\hat{\mathbf{p}}$ is the direction associated with the eigenvalue $\lambda = 1$, i.e. $\hat{\mathbf{p}}$ is an eigenvector of $\mathbf{Q}$. c) Obtain the principal invariants of $\mathbf{Q}$ in function of the angle θ. d) Given a vector position $\vec{x}$, obtain the new vector originated by the orthogonal transformation $\mathbf{Q} \cdot \vec{x}$ in the space formed by $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$.

Solution:

a) Taking into account the definition of the orthogonal tensor we can state that:

$$\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{1} \quad \Rightarrow \quad \mathbf{Q}^T \cdot \mathbf{Q} - \mathbf{Q}^T = \mathbf{1} - \mathbf{Q}^T \quad \Rightarrow \quad \mathbf{Q}^T \cdot (\mathbf{Q} - \mathbf{1}) = -(\mathbf{Q}^T - \mathbf{1})$$
$$\Rightarrow \mathbf{Q}^T \cdot (\mathbf{Q} - \mathbf{1}) = -(\mathbf{Q} - \mathbf{1})^T$$

Then we obtain the determinant of the two previous tensors:

$$\det[\mathbf{Q}^T \cdot (\mathbf{Q} - \mathbf{1})] = \det[-(\mathbf{Q} - \mathbf{1})^T] = (-1)^3 \det[(\mathbf{Q} - \mathbf{1})^T]$$
$$\Rightarrow \underbrace{\det[\mathbf{Q}^T]}_{=\det\mathbf{Q}=1} \det[(\mathbf{Q} - \mathbf{1})] = -\det[(\mathbf{Q} - \mathbf{1})^T] = -\det[\mathbf{Q} - \mathbf{1}] \quad \Rightarrow \quad \det[\mathbf{Q} - \mathbf{1}] = -\det[\mathbf{Q} - \mathbf{1}]$$

where we have used the following determinant properties: $\det[\alpha\mathbf{A}] = \alpha^3 \det[\mathbf{A}]$, $\det[\mathbf{A}^T] = \det[\mathbf{A}]$, $\det[\mathbf{A} \cdot \mathbf{B}] = \det[\mathbf{A}]\det[\mathbf{B}]$. The unique scalar which satisfies the expression above is zero, then:

$$\det[\mathbf{Q} - \mathbf{1}] = 0$$

Taking into account the definition of eigenvalue, $\det[\mathbf{Q} - \lambda\mathbf{1}] = 0$, we can conclude that when $\lambda = 1$ it fulfills $\det[\mathbf{Q} - \mathbf{1}] = 0$, then $\lambda = 1$ is eigenvalue of $\mathbf{Q}$. Hence, there is a direction (eigenvector) satisfying that $\mathbf{Q} \cdot \hat{\mathbf{e}}_1^* = \lambda \hat{\mathbf{e}}_1^* = \hat{\mathbf{e}}_1^*$.

b) We consider that the vectors $\hat{\mathbf{p}} \equiv \hat{\mathbf{e}}_1^*$, $\hat{\mathbf{q}} \equiv \hat{\mathbf{e}}_2^*$, $\hat{\mathbf{r}} \equiv \hat{\mathbf{e}}_3^*$ form an orthonormal basis.

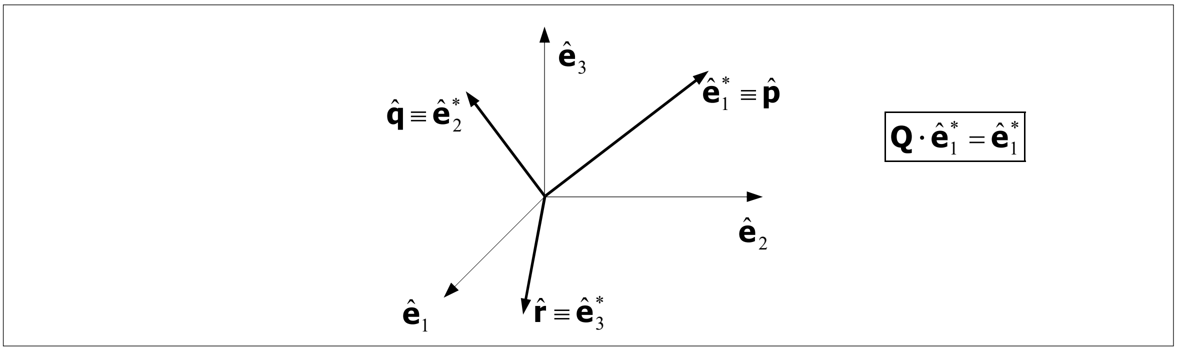


Figure 1.17

The symbolic representation of the tensor $\mathbf{Q}$ in the basis $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$ is given by:

$$\begin{aligned}\mathbf{Q} &= \mathrm{Q}_{ij}^*\hat{\mathbf{e}}_i^* \otimes \hat{\mathbf{e}}_j^* \\ &= \mathrm{Q}_{11}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{12}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{13}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_3^* + \mathrm{Q}_{21}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{22}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{23}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^* + \\ &\qquad \mathrm{Q}_{31}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{32}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{33}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_3^*\end{aligned} \tag{1.55}$$

Taking into account that $\hat{\mathbf{e}}_1^*$ is eigenvector of $\mathbf{Q}$ and is associated with the eigenvalue $\lambda = 1$, it holds that $\mathbf{Q}\cdot\hat{\mathbf{e}}_1^* = \lambda\hat{\mathbf{e}}_1^* = \hat{\mathbf{e}}_1^*$. In addition, making the projection of $\mathbf{Q}$, given by (1.55), according to direction $\hat{\mathbf{e}}_1^*$, we obtain:

$$\begin{aligned}\mathbf{Q}\cdot\hat{\mathbf{e}}_1^* &= \hat{\mathbf{e}}_1^* \\ \mathbf{Q}\cdot\hat{\mathbf{e}}_1^* &= [\ \mathrm{Q}_{11}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{12}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{13}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_3^* + \mathrm{Q}_{21}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{22}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{23}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^* + \\ &\qquad \mathrm{Q}_{31}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{32}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{33}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_3^*\ ]\cdot\hat{\mathbf{e}}_1^* \\ &= \mathrm{Q}_{11}^*\hat{\mathbf{e}}_1^* + \mathrm{Q}_{21}^*\hat{\mathbf{e}}_2^* + \mathrm{Q}_{31}^*\hat{\mathbf{e}}_3^*\end{aligned}$$

with that we conclude that $\mathrm{Q}_{11}^* = 1$, $\mathrm{Q}_{21}^* = 0$, $\mathrm{Q}_{31}^* = 0$.

Remember that two coaxial tensors have the same principal directions (eigenvectors). A tensor and its inverse are coaxial tensors, then if $\mathbf{Q}^{-1} = \mathbf{Q}^T$, this implies that $\mathbf{Q}^T$ and $\mathbf{Q}$ are coaxial tensors, and $\hat{\mathbf{e}}_1^*$ is also principal direction of $\mathbf{Q}^T$, then it fulfills that:

$$\begin{aligned}\mathbf{Q}^T\cdot\hat{\mathbf{e}}_1^* &= \hat{\mathbf{e}}_1^* \\ \mathbf{Q}^T\cdot\hat{\mathbf{e}}_1^* &= [\ \mathrm{Q}_{11}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{21}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{31}^*\hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_3^* + \mathrm{Q}_{12}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{22}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{32}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^* + \\ &\qquad \mathrm{Q}_{13}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{23}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{33}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_3^*\ ]\cdot\hat{\mathbf{e}}_1^* \\ &= \mathrm{Q}_{11}^*\hat{\mathbf{e}}_1^* + \mathrm{Q}_{12}^*\hat{\mathbf{e}}_2^* + \mathrm{Q}_{13}^*\hat{\mathbf{e}}_3^*\end{aligned}$$

with that we conclude that $\mathrm{Q}_{11}^* = 1$, $\mathrm{Q}_{12}^* = 0$, $\mathrm{Q}_{13}^* = 0$. Then, the equation (1.55) becomes:

$$\mathbf{Q} = \hat{\mathbf{e}}_1^* \otimes \hat{\mathbf{e}}_1^* + \mathrm{Q}_{22}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{23}^*\hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^* + \mathrm{Q}_{32}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* + \mathrm{Q}_{33}^*\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_3^* \tag{1.56}$$

In matrix form, the components of $\mathbf{Q}$ in the basis $\hat{\mathbf{e}}_i^*$, (see Figure 1.18), are given by:

$$\mathrm{Q}_{ij}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Q}_{22}^* & \mathrm{Q}_{23}^* \\ 0 & \mathrm{Q}_{32}^* & \mathrm{Q}_{33}^* \end{bmatrix}$$

$$
\mathsf{Q}^*_{ij}=\begin{bmatrix}1 & 0 & 0\\ 0 & \mathsf{Q}^*_{22} & \mathsf{Q}^*_{23}\\ 0 & \mathsf{Q}^*_{32} & \mathsf{Q}^*_{33}\end{bmatrix}
$$

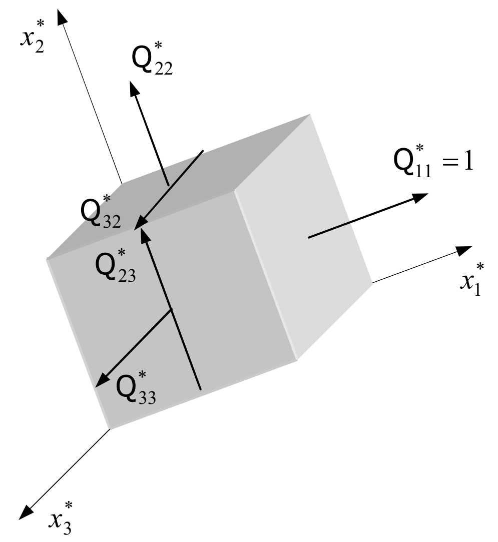


Figure 1.18

Once again we apply the orthogonality condition $\mathbf{Q}^T\cdot\mathbf{Q}=\mathbf{Q}\cdot\mathbf{Q}^T=\mathbf{1}$ that in the space $\hat{\mathbf{e}}^*_i$ can be represented by means of components as follows:

$$
\mathsf{Q}^*_{ki}\mathsf{Q}^*_{kj}=\delta_{ij}\quad\Rightarrow\quad\begin{bmatrix}1 & 0 & 0\\ 0 & \mathsf{Q}^*_{22} & \mathsf{Q}^*_{32}\\ 0 & \mathsf{Q}^*_{23} & \mathsf{Q}^*_{33}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \mathsf{Q}^*_{22} & \mathsf{Q}^*_{23}\\ 0 & \mathsf{Q}^*_{32} & \mathsf{Q}^*_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}
$$
$$
\Rightarrow\begin{bmatrix}1 & 0 & 0\\ 0 & [(\mathsf{Q}^*_{22})^2+(\mathsf{Q}^*_{32})^2] & [\mathsf{Q}^*_{22}\mathsf{Q}^*_{23}+\mathsf{Q}^*_{32}\mathsf{Q}^*_{33}]\\ 0 & [\mathsf{Q}^*_{22}\mathsf{Q}^*_{23}+\mathsf{Q}^*_{32}\mathsf{Q}^*_{33}] & [(\mathsf{Q}^*_{33})^2+(\mathsf{Q}^*_{23})^2]\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\tag{1.57}
$$

The determinant of a proper orthogonal tensor is $\det(\mathbf{Q})=+1$, thus

$$
\begin{vmatrix}1 & 0 & 0\\ 0 & \mathsf{Q}^*_{22} & \mathsf{Q}^*_{23}\\ 0 & \mathsf{Q}^*_{32} & \mathsf{Q}^*_{33}\end{vmatrix}=1\quad\Rightarrow\quad\mathsf{Q}^*_{22}\mathsf{Q}^*_{33}-\mathsf{Q}^*_{23}\mathsf{Q}^*_{32}=1\tag{1.58}
$$

Taking into account (1.57) and (1.58) we obtain the following set of equations:

$$
\begin{cases}(\mathsf{Q}^*_{22})^2+(\mathsf{Q}^*_{32})^2=1\\ \mathsf{Q}^*_{22}\mathsf{Q}^*_{23}+\mathsf{Q}^*_{32}\mathsf{Q}^*_{33}=0\\ (\mathsf{Q}^*_{33})^2+(\mathsf{Q}^*_{23})^2=1\\ \mathsf{Q}^*_{22}\mathsf{Q}^*_{33}-\mathsf{Q}^*_{23}\mathsf{Q}^*_{32}=1\end{cases}\qquad\begin{cases}\cos^2\theta+\sin^2\theta=1\\ \cos\theta(-\sin\theta)+\sin\theta\cos\theta=0\\ \cos^2\theta+\sin^2\theta=1\\ \cos\theta\cos\theta-(-\sin\theta)(\sin\theta)=1\end{cases}
$$

whereupon we have demonstrated the existence of an angle that meets the above conditions:

$$
\mathsf{Q}^*_{ij}=\begin{bmatrix}1 & 0 & 0\\ 0 & \mathsf{Q}^*_{22} & \mathsf{Q}^*_{32}\\ 0 & \mathsf{Q}^*_{23} & \mathsf{Q}^*_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\tag{1.59}
$$

Returning to the equation (1.56), and taking into account (1.59), we conclude that:

$$
\begin{aligned}\mathbf{Q}&=\hat{\mathbf{e}}^*_1\otimes\hat{\mathbf{e}}^*_1+(\cos\theta)\,\hat{\mathbf{e}}^*_2\otimes\hat{\mathbf{e}}^*_2+(-\sin\theta)\hat{\mathbf{e}}^*_2\otimes\hat{\mathbf{e}}^*_3+(\sin\theta)\hat{\mathbf{e}}^*_3\otimes\hat{\mathbf{e}}^*_2+(\cos\theta)\,\hat{\mathbf{e}}^*_3\otimes\hat{\mathbf{e}}^*_3\\ &=\hat{\mathbf{e}}^*_1\otimes\hat{\mathbf{e}}^*_1+\cos\theta\left[\hat{\mathbf{e}}^*_2\otimes\hat{\mathbf{e}}^*_2+\hat{\mathbf{e}}^*_3\otimes\hat{\mathbf{e}}^*_3\right]-\sin\theta\left[\hat{\mathbf{e}}^*_2\otimes\hat{\mathbf{e}}^*_3-\hat{\mathbf{e}}^*_3\otimes\hat{\mathbf{e}}^*_2\right]\end{aligned}
$$

Considering that $\hat{\mathbf{p}} \equiv \hat{\mathbf{e}}_1^*$, $\hat{\mathbf{q}} \equiv \hat{\mathbf{e}}_2^*$, $\hat{\mathbf{r}} \equiv \hat{\mathbf{e}}_3^*$, we show that:

$$\mathbf{Q} = \hat{\mathbf{p}} \otimes \hat{\mathbf{p}} + \cos\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{q}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) - \sin\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{r}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{q}})$$

It is interesting to note that the additive decomposition of $\mathbf{Q}$ in an antisymmetric and a symmetric part, in the space $\hat{\mathbf{e}}_i^*$, is:

$$\underbrace{\mathsf{Q}_{ij}^{*\,sym} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix}}_{[\hat{\mathbf{p}}\otimes\hat{\mathbf{p}}+\cos\theta(\hat{\mathbf{q}}\otimes\hat{\mathbf{q}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})]_{ij}} \quad ; \quad \underbrace{\mathsf{Q}_{ij}^{*\,skew} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{bmatrix}}_{[-\sin\theta(\hat{\mathbf{q}}\otimes\hat{\mathbf{r}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{q}})]_{ij}}$$

Note that the format of $\mathsf{Q}_{ij}^{*\,skew}$ has the same format as the antisymmetric tensor ($\mathbf{W}$) in the space defined by the axial vector:

$$\mathsf{W}_{ij}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

where ω is the magnitude of the axial vector.

c) By means of (1.59) it is easy to show that $I_{\mathbf{Q}} = II_{\mathbf{Q}} = 1 + 2\cos\theta$, $III_{\mathbf{Q}} = 1$.

d) We represent the vector $\vec{\boldsymbol{x}}$ by means its components and the basis $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$, as follows: $\vec{\boldsymbol{x}} = p\hat{\mathbf{p}} + q\hat{\mathbf{q}} + r\hat{\mathbf{r}}$.

Then, it fulfills that: $\vec{\boldsymbol{x}} \cdot \hat{\mathbf{p}} = (p\hat{\mathbf{p}} + q\hat{\mathbf{q}} + r\hat{\mathbf{r}}) \cdot \hat{\mathbf{p}} = p \quad ; \quad \vec{\boldsymbol{x}} \cdot \hat{\mathbf{q}} = q \quad ; \quad \vec{\boldsymbol{x}} \cdot \hat{\mathbf{r}} = r$

Thus, (see Figure 1.19), it holds that:

$$\begin{aligned} \tilde{\vec{\boldsymbol{x}}} = \mathbf{Q} \cdot \vec{\boldsymbol{x}} &= \left[\hat{\mathbf{p}} \otimes \hat{\mathbf{p}} + \cos\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{q}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) - \sin\theta(\hat{\mathbf{q}} \otimes \hat{\mathbf{r}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{q}})\right] \cdot \left[p\hat{\mathbf{p}} + q\hat{\mathbf{q}} + r\hat{\mathbf{r}}\right] \\ &= p\hat{\mathbf{p}} + (q\cos\theta - r\sin\theta)\hat{\mathbf{q}} + (r\cos\theta + q\sin\theta)\hat{\mathbf{r}} \end{aligned}$$

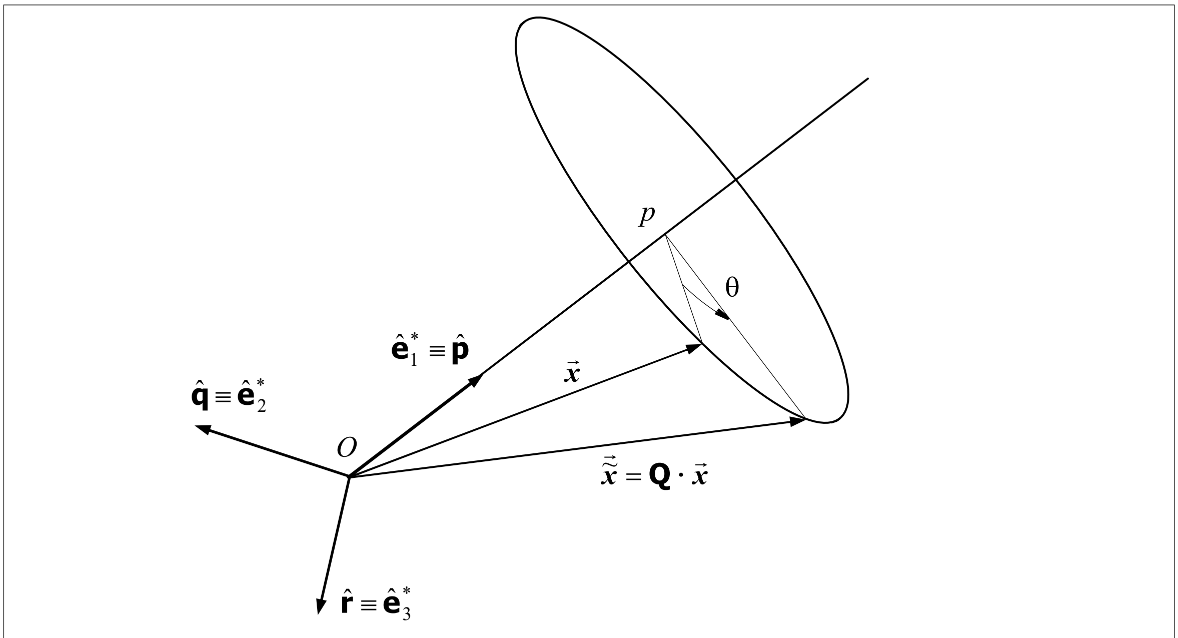


Figure 1.19

Problem 1.73

Let us consider the tensorial transformations $\vec{\mathbf{p}}' = \mathbf{U} \cdot \vec{\mathbf{p}}$ and $\vec{\mathbf{p}}'' = \mathbf{R} \cdot \vec{\mathbf{p}}'$, where $\mathbf{R}$ is an orthogonal tensor and $\mathbf{U}$ is a second-order tensor with $\mathbf{U} \cdot \mathbf{U}^{-1} = \mathbf{1}$, i.e. $\exists\ \mathbf{U}^{-1}$. Obtain the transformation law between $\vec{\mathbf{p}}$ and $\vec{\mathbf{p}}''$, (see Figure 1.20).

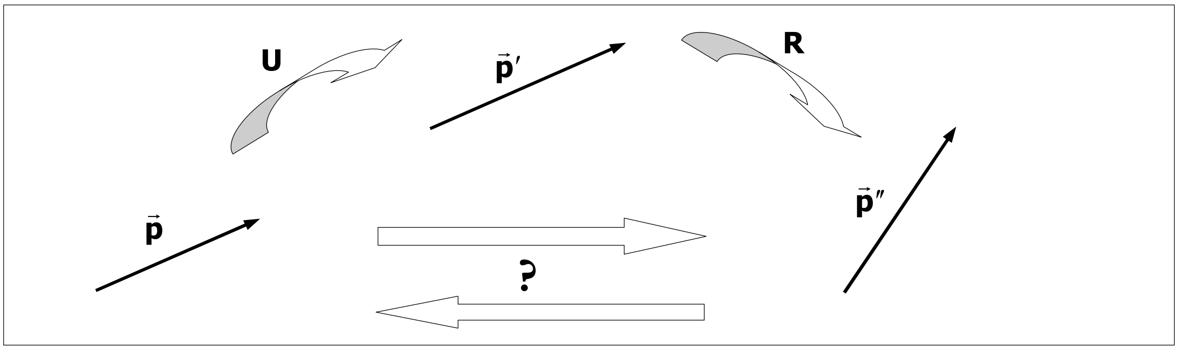


Figure 1.20

Solution:

Taking into account that $\mathbf{R}^{-1} = \mathbf{R}^T$ (orthogonal tensor), we can guarantee that the inverse of $\mathbf{R}$ exists, and considering that $\vec{\mathbf{p}}'' = \mathbf{R} \cdot \vec{\mathbf{p}}'$ we can obtain:

$$\vec{\mathbf{p}}'' = \mathbf{R} \cdot \vec{\mathbf{p}}' \quad \Rightarrow \quad \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{R}^{-1} \cdot \mathbf{R} \cdot \vec{\mathbf{p}}' \quad \Rightarrow \quad \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{1} \cdot \vec{\mathbf{p}}' = \vec{\mathbf{p}}'$$

Substituting $\vec{\mathbf{p}}' = \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}''$ into $\vec{\mathbf{p}}' = \mathbf{U} \cdot \vec{\mathbf{p}}$ we obtain:

$$\begin{array}{l|l}
\vec{\mathbf{p}}' = \mathbf{U} \cdot \vec{\mathbf{p}} & \vec{\mathbf{p}}' = \mathbf{U} \cdot \vec{\mathbf{p}} \\
\Rightarrow \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{U} \cdot \vec{\mathbf{p}} & \Rightarrow \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{U} \cdot \vec{\mathbf{p}} \\
\Rightarrow \mathbf{R} \cdot \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{R} \cdot \mathbf{U} \cdot \vec{\mathbf{p}} & \Rightarrow \mathbf{U}^{-1} \cdot \mathbf{R}^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{U}^{-1} \cdot \mathbf{U} \cdot \vec{\mathbf{p}} \\
\Rightarrow \mathbf{1} \cdot \vec{\mathbf{p}}'' = \mathbf{R} \cdot \mathbf{U} \cdot \vec{\mathbf{p}} & \Rightarrow (\mathbf{R} \cdot \mathbf{U})^{-1} \cdot \vec{\mathbf{p}}'' = \mathbf{1} \cdot \vec{\mathbf{p}} = \vec{\mathbf{p}} \\
\Rightarrow \vec{\mathbf{p}}'' = (\mathbf{R} \cdot \mathbf{U}) \cdot \vec{\mathbf{p}} & \Rightarrow \vec{\mathbf{p}} = (\mathbf{R} \cdot \mathbf{U})^{-1} \cdot \vec{\mathbf{p}}''
\end{array} \tag{1.60}$$

Or in indicial notation:

$$\begin{array}{l|l}
\mathrm{p}'_i = \mathrm{U}_{ij}\mathrm{p}_j & \mathrm{p}'_i = \mathrm{U}_{ij}\mathrm{p}_j \\
\Rightarrow \mathrm{R}^{-1}_{ij}\mathrm{p}''_j = \mathrm{U}_{ij}\mathrm{p}_j & \Rightarrow \mathrm{R}^{-1}_{ij}\mathrm{p}''_j = \mathrm{U}_{ij}\mathrm{p}_j \\
\Rightarrow \mathrm{R}_{ki}\mathrm{R}^{-1}_{ij}\mathrm{p}''_j = \mathrm{R}_{ki}\mathrm{U}_{ij}\mathrm{p}_j & \Rightarrow \mathrm{U}^{-1}_{ki}\mathrm{R}^{-1}_{ij}\mathrm{p}''_j = \mathrm{U}^{-1}_{ki}\mathrm{U}_{ij}\mathrm{p}_j \\
\Rightarrow \delta_{kj}\mathrm{p}''_j = \mathrm{R}_{ki}\mathrm{U}_{ij}\mathrm{p}_j & \Rightarrow (\mathrm{R}_{ki}\mathrm{U}_{ij})^{-1}\mathrm{p}''_j = \delta_{kj}\mathrm{p}_j = \mathrm{p}_k \\
\Rightarrow \mathrm{p}''_k = (\mathrm{R}_{ki}\mathrm{U}_{ij})\mathrm{p}_j & \Rightarrow \mathrm{p}_k = (\mathrm{R}_{ki}\mathrm{U}_{ij})^{-1}\mathrm{p}''_j
\end{array} \tag{1.61}$$

And the graphical representation is presented in Figure 1.21.

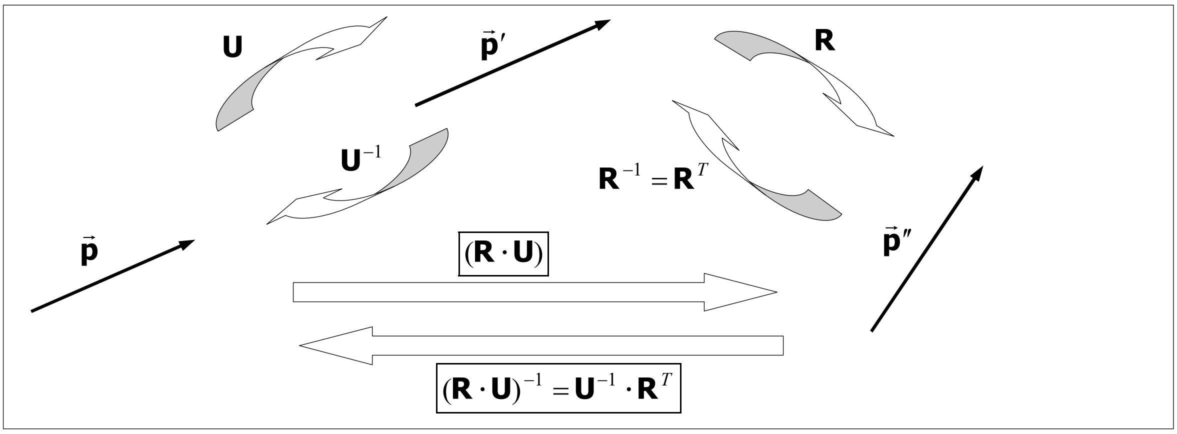


Figure 1.21

1.9 Spectral Representation of Tensors

Problem 1.74

Let $\boldsymbol{\omega}$ be an antisymmetric second-order tensor and $\mathbf{V}$ be a positive definite symmetric tensor whose spectral representation is given by:

$$\mathbf{V} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$$

Show that the antisymmetric tensor $\boldsymbol{\omega}$ can be represented by:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab} \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Also show the following is true:

$$\boldsymbol{\omega}\cdot\mathbf{V} - \mathbf{V}\cdot\boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab} (\lambda_b - \lambda_a) \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Solution:

It is true that

$$\boldsymbol{\omega}\cdot\mathbf{1} = \boldsymbol{\omega}\cdot\left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}\right) = \sum_{a=1}^{3} \boldsymbol{\omega}\cdot\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \left(\vec{\boldsymbol{w}} \wedge \hat{\mathbf{n}}^{(a)}\right) \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a,b=1}^{3} w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(a)}\right) \otimes \hat{\mathbf{n}}^{(a)}$$

where we have applied the antisymmetric tensor property $\boldsymbol{\omega}\cdot\hat{\mathbf{n}} = \vec{\boldsymbol{w}} \wedge \hat{\mathbf{n}}$, where $\vec{\boldsymbol{w}}$ is the axial vector associated with $\boldsymbol{\omega}$. Expanding the above equation, we obtain:

$$\begin{aligned}
\boldsymbol{\omega} &= w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(1)}\right) \otimes \hat{\mathbf{n}}^{(1)} + w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(2)}\right) \otimes \hat{\mathbf{n}}^{(2)} + w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(3)}\right) \otimes \hat{\mathbf{n}}^{(3)} = \\
&= w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(1)}\right) \otimes \hat{\mathbf{n}}^{(1)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(1)}\right) \otimes \hat{\mathbf{n}}^{(1)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)}\right) \otimes \hat{\mathbf{n}}^{(1)} + \\
&\quad + w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)}\right) \otimes \hat{\mathbf{n}}^{(2)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(2)}\right) \otimes \hat{\mathbf{n}}^{(2)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(2)}\right) \otimes \hat{\mathbf{n}}^{(2)} + \\
&\quad + w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(3)}\right) \otimes \hat{\mathbf{n}}^{(3)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(3)}\right) \otimes \hat{\mathbf{n}}^{(3)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(3)}\right) \otimes \hat{\mathbf{n}}^{(3)}
\end{aligned}$$

By simplifying the above equation we can obtain:

$$\boldsymbol{\omega} = -w_2\, \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(1)} + w_3\, \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(1)} + w_1\, \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(2)} - w_3\, \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(2)} + -w_1\, \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(3)} + w_2\, \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(3)}$$

Taking into account that $w_1 = -\omega_{23} = \omega_{32}$, $w_2 = \omega_{13} = -\omega_{31}$, $w_3 = -\omega_{12} = \omega_{21}$, the above equation becomes:

$$\begin{aligned}\boldsymbol{\omega} = \omega_{31}\, \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_{21}\, \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_{32}\, \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(2)} + \omega_{12}\, \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(2)} + \\ + \omega_{23}\, \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(3)} + \omega_{13}\, \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(3)}\end{aligned}$$

which is the same as:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

The terms $\boldsymbol{\omega} \cdot \mathbf{V}$ and $\mathbf{V} \cdot \boldsymbol{\omega}$ can be expressed as follows:

$$\begin{aligned}\boldsymbol{\omega} \cdot \mathbf{V} &= \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \right) \cdot \left(\sum_{b=1}^{3} \lambda_b\, \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} \right) \\ &= \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \cdot \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\end{aligned}$$

and

$$\mathbf{V} \cdot \boldsymbol{\omega} = \left(\sum_{a=1}^{3} \lambda_a\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \right) = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Then,

$$\boldsymbol{\omega} \cdot \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\omega} = \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \right) - \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \right) = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab} (\lambda_b - \lambda_a)\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Similarly, it is possible to show that:

$$\boldsymbol{\omega} \cdot \mathbf{V}^2 - \mathbf{V}^2 \cdot \boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab} (\lambda_b^2 - \lambda_a^2)\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Problem 1.75

Let $\boldsymbol{C}$ be a positive definite tensor, whose Cartesian components are given by:

$$C_{ij} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Obtain the following tensors: a) $\boldsymbol{C}^2$; b) $\mathbf{U} = \sqrt{\boldsymbol{C}}$. c) Check if the tensors $\boldsymbol{C}$ and $\mathbf{U}$ are coaxial.

Solution:

Note that the tensors $\boldsymbol{C}^2$ and $\mathbf{U} = \sqrt{\boldsymbol{C}}$ are coaxial with the tensor $\boldsymbol{C}$. By means of the spectral representation of $\boldsymbol{C}$:

$$\boldsymbol{C} = \sum_{a=1}^{3} \gamma_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$$

where γ_a are the eigenvalues of $\boldsymbol{C}$, and $\hat{\mathbf{N}}^{(a)}$ are the eigenvectors of $\boldsymbol{C}$, we can obtain:

$$\boldsymbol{C}^2 = \sum_{a=1}^{3} \gamma_a^2 \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \mathbf{U} = \sqrt{\boldsymbol{C}} = \sum_{a=1}^{3} \sqrt{\gamma_a} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \tag{1.62}$$

Calculation of the eigenvalues and eigenvectors of the tensor $\boldsymbol{C}$.

Due to the structure of the $\boldsymbol{C}$ tensor components we already know one eigenvalue $\gamma_2 = 4$ which is associated with the principal direction $\hat{\mathrm{N}}_i^{(2)} = [0 \quad \pm 1 \quad 0]$. To calculate the remaining eigenvalues is sufficient to solve the following characteristic determinant:

$$\begin{vmatrix} 2-\gamma & 1 \\ 1 & 2-\gamma \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\gamma)^2 = 1 \quad \Rightarrow \quad (2-\gamma) = \pm 1 \Rightarrow \begin{cases} \gamma_1 = 2-1 = 1 \\ \gamma_3 = 2+1 = 3 \end{cases}$$

Associated with the eigenvalue $\gamma_1 = 1$ we have the following eigenvector:

$$\begin{bmatrix} 2-\gamma_1 & 1 \\ 1 & 2-\gamma_1 \end{bmatrix} \begin{bmatrix} \hat{\mathrm{N}}_1^{(1)} \\ \hat{\mathrm{N}}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathrm{N}}_1^{(1)} \\ \hat{\mathrm{N}}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathrm{N}}_1^{(1)} = -\hat{\mathrm{N}}_3^{(1)}$$

with the restriction $\hat{\mathrm{N}}_i^{(1)} \hat{\mathrm{N}}_i^{(1)} = 1$, thus

$$\hat{\mathrm{N}}_1^{(1)} \hat{\mathrm{N}}_1^{(1)} + \hat{\mathrm{N}}_2^{(1)} \hat{\mathrm{N}}_2^{(1)} + \hat{\mathrm{N}}_3^{(1)} \hat{\mathrm{N}}_3^{(1)} = 1$$

$$\Rightarrow \hat{\mathrm{N}}_1^{(1)} \hat{\mathrm{N}}_1^{(1)} + \hat{\mathrm{N}}_1^{(1)} \hat{\mathrm{N}}_1^{(1)} = 1 \quad \Rightarrow \quad \hat{\mathrm{N}}_1^{(1)} = \pm \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \hat{\mathrm{N}}_3^{(1)} = -\hat{\mathrm{N}}_1^{(1)} = \mp \frac{1}{\sqrt{2}}$$

Associated with the eigenvalue $\gamma_3 = 3$ we have the following eigenvector:

$$\begin{bmatrix} 2-\gamma_3 & 1 \\ 1 & 2-\gamma_3 \end{bmatrix} \begin{bmatrix} \hat{\mathrm{N}}_1^{(3)} \\ \hat{\mathrm{N}}_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{\mathrm{N}}_1^{(3)} \\ \hat{\mathrm{N}}_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathrm{N}}_1^{(3)} = \hat{\mathrm{N}}_3^{(3)}$$

with the restriction $\hat{\mathrm{N}}_i^{(3)} \hat{\mathrm{N}}_i^{(3)} = 1$, thus

$$\hat{\mathrm{N}}_1^{(3)} \hat{\mathrm{N}}_1^{(3)} + \hat{\mathrm{N}}_2^{(3)} \hat{\mathrm{N}}_2^{(3)} + \hat{\mathrm{N}}_3^{(3)} \hat{\mathrm{N}}_3^{(3)} = 1$$

$$\Rightarrow \hat{\mathrm{N}}_1^{(3)} \hat{\mathrm{N}}_1^{(3)} + \hat{\mathrm{N}}_1^{(3)} \hat{\mathrm{N}}_1^{(3)} = 1 \quad \Rightarrow \quad \hat{\mathrm{N}}_1^{(3)} = \pm \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \hat{\mathrm{N}}_3^{(3)} = \hat{\mathrm{N}}_1^{(3)} = \pm \frac{1}{\sqrt{2}}$$

Summarizing we have:

$$\left.\begin{array}{ll} \gamma_1 = 1 \Rightarrow & \hat{\mathrm{N}}_i^{(1)} = \left[\pm \frac{1}{\sqrt{2}} \quad 0 \quad \mp \frac{1}{\sqrt{2}} \right] \\ \gamma_2 = 4 \Rightarrow & \hat{\mathrm{N}}_i^{(2)} = [0 \quad \pm 1 \quad 0] \\ \gamma_3 = 3 \Rightarrow & \hat{\mathrm{N}}_i^{(3)} = \left[\pm \frac{1}{\sqrt{2}} \quad 0 \quad \pm \frac{1}{\sqrt{2}} \right] \end{array} \right\} \xrightarrow{\text{Transformation Matrix}} \mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then it holds that:

$$\mathcal{C}' = \mathcal{A}\, \mathcal{C}\, \mathcal{A}^T \Rightarrow \mathcal{C} = \mathcal{A}^T\, \mathcal{C}'\, \mathcal{A}$$

In the principal space we have:

$$
C'_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} C'^{\,2}_{ij} = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 4^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\ \mathsf{U}'_{ij} = \sqrt{C'_{ij}} = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \end{cases}
$$

Be aware that the above operation can only be done in the principal space, (see Figure 1.22). Note also that the tensor $\boldsymbol{C}$ is a positive definite tensor, so, its eigenvalues are positive. In the original space we have the following components:

$$
C^2_{ij} = \mathcal{A}^T\, \mathcal{C}'^2\, \mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 16 & 0 \\ 4 & 0 & 5 \end{bmatrix} \tag{1.63}
$$

Note that this result could have been obtained easily by means of the operation $\boldsymbol{C}^2 = \boldsymbol{C} \cdot \boldsymbol{C}$, which in components becomes:

$$
C^2_{ij} = C_{ik} C_{kj} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 16 & 0 \\ 4 & 0 & 5 \end{bmatrix}
$$

Similarly, we can obtain the components of $\mathbf{U}$ in the original Cartesian system:

$$
\mathsf{U}_{ij} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & 0 & \frac{\sqrt{3}-1}{2} \\ 0 & 2 & 0 \\ \frac{\sqrt{3}-1}{2} & 0 & \frac{\sqrt{3}+1}{2} \end{bmatrix}
$$

c) The tensors $\boldsymbol{C}$ and $\mathbf{U}$ are coaxial, since they have the same eigenvectors, (see equation (1.62)). Note also that the eigenvalues of $\mathbf{U}$ were obtained in the principal space of $\boldsymbol{C}$. We can also verify that $\boldsymbol{C}$ and $\mathbf{U}$ are coaxial by means of $\boldsymbol{C} \cdot \mathbf{U} = \mathbf{U} \cdot \boldsymbol{C}$, i.e.:

$$
C_{ik} \mathsf{U}_{kj} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}+1}{2} & 0 & \frac{\sqrt{3}-1}{2} \\ 0 & 2 & 0 \\ \frac{\sqrt{3}-1}{2} & 0 & \frac{\sqrt{3}+1}{2} \end{bmatrix} = \begin{bmatrix} 3.098 & 0 & 2.098 \\ 0 & 8 & 0 \\ 2.098 & 0 & 3.098 \end{bmatrix}
$$

$$
\mathsf{U}_{ik} C_{kj} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & 0 & \frac{\sqrt{3}-1}{2} \\ 0 & 2 & 0 \\ \frac{\sqrt{3}-1}{2} & 0 & \frac{\sqrt{3}+1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3.098 & 0 & 2.098 \\ 0 & 8 & 0 \\ 2.098 & 0 & 3.098 \end{bmatrix}
$$

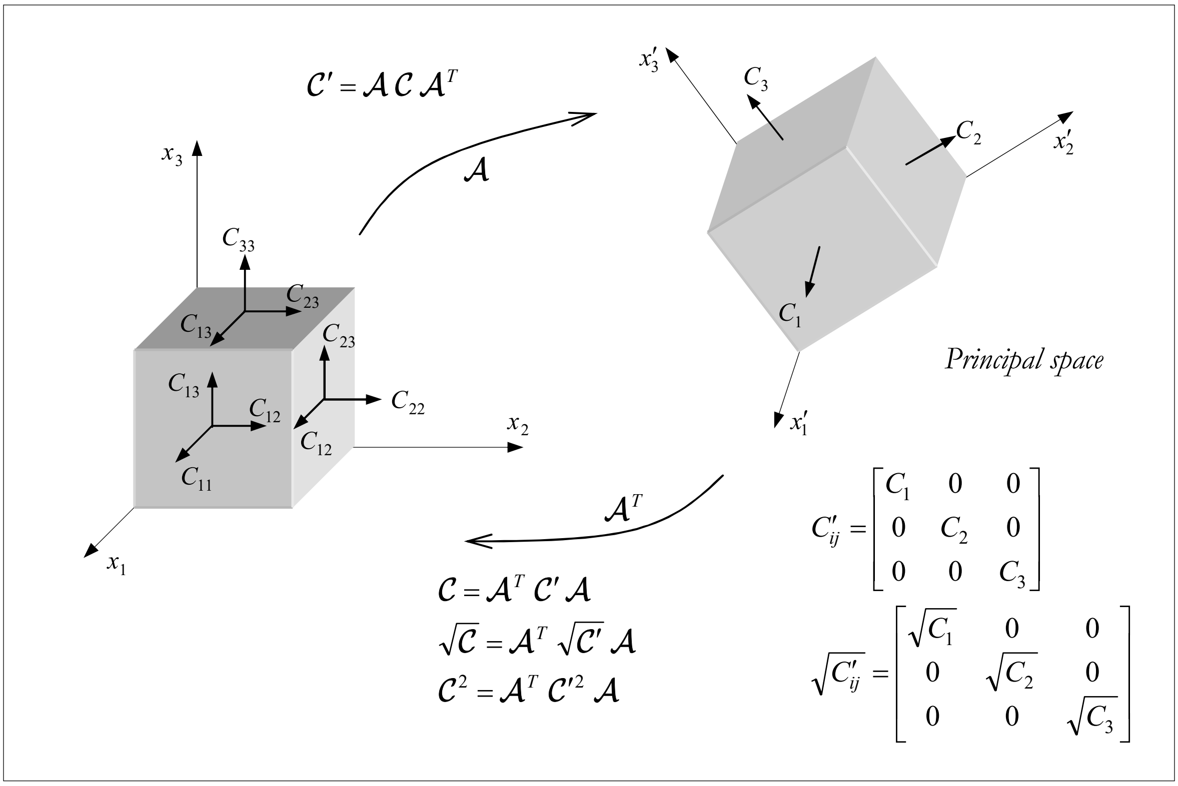


Figure 1.22

Problem 1.76

Let $\boldsymbol{C}$ be a symmetric second-order tensor and $\mathbf{R}$ a proper orthogonal tensor. The components of these tensors, in the Cartesian system, are given by:

$$C_{ij} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad ; \quad \mathsf{R}_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

a) Obtain the following tensors: a.1) $\boldsymbol{C}^8$; a2) $\mathbf{U} = \sqrt{\boldsymbol{C}}$.

b) Obtain the principal invariants of $\boldsymbol{C}$.

c) Taking into account that the tensors $\boldsymbol{b}$ and $\boldsymbol{C}$ are related to each other by the following proper orthogonal transformation $\boldsymbol{C} = \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R}$, obtain the third principal invariant of $\boldsymbol{b}$.

Solution:

a) *Answer:* $\boldsymbol{C}^8 = \begin{bmatrix} 3281 & 0 & 3280 \\ 0 & 65536 & 0 \\ 3280 & 0 & 3281 \end{bmatrix}$, (see **Problem 1.75**).

b) The principal invariants of $\boldsymbol{C}$:

$$I_{\boldsymbol{C}} = \mathrm{Tr}(C_{ij}) = C_{ii} = C_{11} + C_{22} + C_{33} = 8$$

$$II_{\boldsymbol{C}} = \frac{1}{2}\left(C_{ii}C_{jj} - C_{ij}C_{ij}\right) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 19\,;$$

$III_{\boldsymbol{C}} = |\boldsymbol{C}| = \epsilon_{ijk} C_{i1} C_{j2} C_{k3} = 12$.

c) Taking into account the determinant property, the third principal invariant of $\boldsymbol{b}$ can be expressed as follows:

$$|\boldsymbol{C}| \equiv \det(\boldsymbol{C}) = \det(\mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R}) = \underbrace{\det(\mathbf{R}^T)}_{=+1} \det(\boldsymbol{b}) \underbrace{\det(\mathbf{R})}_{=+1} = \det(\boldsymbol{b}) = III_{\boldsymbol{b}} = 12$$

NOTE: If $\mathbf{R}$ is an orthogonal tensor ($\mathbf{R}^{-1} = \mathbf{R}^T$) and the equation $\boldsymbol{C} = \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R}$ holds, this implies that $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues $\gamma_a^{(\boldsymbol{C})} = \gamma_a^{(\boldsymbol{b})} = \gamma_a$, (see **Problem 1.68**), so, they have also the same characteristic equation:

$$\lambda^3 - \lambda^2 I_{\boldsymbol{C}} + \lambda II_{\boldsymbol{C}} - III_{\boldsymbol{C}} = \lambda^3 - \lambda^2 I_{\boldsymbol{b}} + \lambda II_{\boldsymbol{b}} - III_{\boldsymbol{b}} = 0 \quad \Rightarrow \quad \begin{cases} I_{\boldsymbol{C}} = I_{\boldsymbol{b}} \\ II_{\boldsymbol{C}} = II_{\boldsymbol{b}} \\ III_{\boldsymbol{C}} = III_{\boldsymbol{b}} \end{cases}$$

Note also that

$$\boldsymbol{C} = \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R} \quad \Rightarrow \quad \mathbf{R} \cdot \boldsymbol{C} \cdot \mathbf{R}^T = \underbrace{\mathbf{R} \cdot \mathbf{R}^T}_{=\mathbf{1}} \cdot \boldsymbol{b} \cdot \underbrace{\mathbf{R} \cdot \mathbf{R}^T}_{=\mathbf{1}} = \boldsymbol{b}$$

And if we start from the spectral representation $\boldsymbol{C} = \sum_{a=1}^{3} \gamma_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ and by considering $\boldsymbol{b} = \mathbf{R} \cdot \boldsymbol{C} \cdot \mathbf{R}^T$ we can obtain:

$$\begin{aligned}
\boldsymbol{b} &= \mathbf{R} \cdot \boldsymbol{C} \cdot \mathbf{R}^T \\
\boldsymbol{b} &= \mathbf{R} \cdot \left(\sum_{a=1}^{3} \gamma_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right) \cdot \mathbf{R}^T = \sum_{a=1}^{3} \gamma_a \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \cdot \mathbf{R}^T = \sum_{a=1}^{3} \gamma_a \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \\
&= \sum_{a=1}^{3} \gamma_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}
\end{aligned}$$

where $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$ are the eigenvectors of $\boldsymbol{b}$.

Problem 1.77

Let $\mathbf{S}$ be a symmetric second-order tensor with $\det(\mathbf{S}) \neq 0$. Considering that $\mathbf{S}$ has two equal eigenvalues, i.e. $\mathsf{S}_2 = \mathsf{S}_3$ and $\mathsf{S}_1 \neq \mathsf{S}_2$, show that $\mathbf{S}$ can be represented by:

$$\mathbf{S} = \mathsf{S}_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \mathsf{S}_2 (\mathbf{1} - \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)})$$

where $\hat{\mathbf{n}}^{(1)}$ is the eigenvector of $\mathbf{S}$ associated with the eigenvalue S_1, $\mathbf{1}$ is the second-order unit tensor.

Solution: We start from the spectral representation of $\mathbf{S}$:

$$\begin{aligned}
\mathbf{S} &= \sum_{a=1}^{3} \mathsf{S}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \mathsf{S}_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \mathsf{S}_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \mathsf{S}_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \\
&= \mathsf{S}_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \mathsf{S}_2 (\hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)})
\end{aligned} \tag{1.64}$$

Remember that $\mathbf{1}$ is a spherical tensor, whereby any direction is a principal direction. Based on this principle, we adopt the principal space of $\mathbf{S}$ to make the spectral representation of $\mathbf{1}$, i.e.:

$$\mathbf{1}=\sum_{a=1}^{3}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}=\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)}+\hat{\mathbf{n}}^{(2)}\otimes\hat{\mathbf{n}}^{(2)}+\hat{\mathbf{n}}^{(3)}\otimes\hat{\mathbf{n}}^{(3)}$$
$$\Rightarrow\hat{\mathbf{n}}^{(2)}\otimes\hat{\mathbf{n}}^{(2)}+\hat{\mathbf{n}}^{(3)}\otimes\hat{\mathbf{n}}^{(3)}=\mathbf{1}-\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)} \tag{1.65}$$

By substituting the above equation into (1.64) we obtain:

$$\mathbf{S}=\mathsf{S}_1\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)}+\mathsf{S}_2(\hat{\mathbf{n}}^{(2)}\otimes\hat{\mathbf{n}}^{(2)}+\hat{\mathbf{n}}^{(3)}\otimes\hat{\mathbf{n}}^{(3)})=\mathsf{S}_1\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)}+\mathsf{S}_2(\mathbf{1}-\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)})$$

Q.E.D.

1.10 Cayley-Hamilton Theorem

Problem 1.78

Let $\mathbf{T}$ be an arbitrary second-order tensor, show the Cayley-Hamilton theorem, which states that any tensor satisfies its own characteristic equation.

Solution:

We start from the characteristic equation of the tensor: $\lambda^3-\lambda^2 I_{\mathbf{T}}+\lambda II_{\mathbf{T}}-III_{\mathbf{T}}=0$, which fulfils for each eigenvalue $\lambda_1,\lambda_2,\lambda_3$, then:

$$\lambda_1^3-\lambda_1^2 I_{\mathbf{T}}+\lambda_1 II_{\mathbf{T}}-III_{\mathbf{T}}=0$$
$$\lambda_2^3-\lambda_2^2 I_{\mathbf{T}}+\lambda_2 II_{\mathbf{T}}-III_{\mathbf{T}}=0$$
$$\lambda_3^3-\lambda_3^2 I_{\mathbf{T}}+\lambda_3 II_{\mathbf{T}}-III_{\mathbf{T}}=0$$

Restructuring the above equations in matrix form we obtain:

$$\begin{bmatrix}\lambda_1^3&0&0\\0&\lambda_2^3&0\\0&0&\lambda_3^3\end{bmatrix}-\begin{bmatrix}\lambda_1^2&0&0\\0&\lambda_2^2&0\\0&0&\lambda_3^2\end{bmatrix}I_{\mathbf{T}}+\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}II_{\mathbf{T}}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}III_{\mathbf{T}}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{1.66}$$
$$\mathsf{T}'^3_{ij}-\mathsf{T}'^2_{ij}I_{\mathbf{T}}+\mathsf{T}'_{ij}\;II_{\mathbf{T}}-III_{\mathbf{T}}\delta_{ij}=0_{ij}$$

Note that in the principal space of $\mathbf{T}$ the following relationships are true:

$$\mathsf{T}'_{ij}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$$
$$\mathsf{T}'^2_{ij}=\mathsf{T}'_{ik}\mathsf{T}'_{kj}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}\lambda_1^2&0&0\\0&\lambda_2^2&0\\0&0&\lambda_3^2\end{bmatrix}$$
$$\mathsf{T}'^3_{ij}=\mathsf{T}'_{ik}\mathsf{T}'_{kp}\mathsf{T}'_{pj}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=\begin{bmatrix}\lambda_1^3&0&0\\0&\lambda_2^3&0\\0&0&\lambda_3^3\end{bmatrix}$$

The component transformation law between spaces for a second-order tensor is $\mathsf{T}'_{ij}=F_{ik}\mathsf{T}_{kp}F^{-1}_{pj}$, where F_{ij} is the transformation matrix from the original space (T_{ij}) to the principal space (T'_{ij}). Note also that the relationships $\mathsf{T}'^2_{ij}=F_{ik}\mathsf{T}^2_{kp}F^{-1}_{pj}$ and $\mathsf{T}'^3_{ij}=F_{ik}\mathsf{T}^3_{kp}F^{-1}_{pj}$ hold, (see **Problem 1.69**). With that the equation in (1.66) can be rewritten as follows:

$$
\begin{aligned}
&\mathsf{T}'^3_{ij} - \mathsf{T}'^2_{ij} I_{\mathbf{T}} + \mathsf{T}'_{ij}\, II_{\mathbf{T}} - III_{\mathbf{T}}\delta_{ij} = 0_{ij}\\
&\Rightarrow F_{ik}\mathsf{T}^3_{kp}F^{-1}_{pj} - F_{ik}\mathsf{T}^2_{kp}F^{-1}_{pj} I_{\mathbf{T}} + F_{ik}\mathsf{T}_{kp}F^{-1}_{pj}\, II_{\mathbf{T}} - III_{\mathbf{T}}F_{ik}\delta_{kp}F^{-1}_{pj} = 0_{ij}\\
&\Rightarrow F_{ik}\left(\mathsf{T}^3_{kp} - \mathsf{T}^2_{kp} I_{\mathbf{T}} + \mathsf{T}_{kp}\, II_{\mathbf{T}} - III_{\mathbf{T}}\delta_{kp}\right)F^{-1}_{pj} = 0_{ij}\\
&\Rightarrow F^{-1}_{si}F_{ik}\left(\mathsf{T}^3_{kp} - \mathsf{T}^2_{kp} I_{\mathbf{T}} + \mathsf{T}_{kp}\, II_{\mathbf{T}} - III_{\mathbf{T}}\delta_{kp}\right)F^{-1}_{pj}F_{jt} = F^{-1}_{si}0_{ij}F_{jt} = 0_{st}\\
&\Rightarrow \delta_{sk}\left(\mathsf{T}^3_{kp} - \mathsf{T}^2_{kp} I_{\mathbf{T}} + \mathsf{T}_{kp}\, II_{\mathbf{T}} - III_{\mathbf{T}}\delta_{kp}\right)\delta_{pt} = F^{-1}_{si}0_{ij}F_{jt} = 0_{st}\\
&\Rightarrow \mathsf{T}^3_{st} - \mathsf{T}^2_{st} I_{\mathbf{T}} + \mathsf{T}_{st}\, II_{\mathbf{T}} - III_{\mathbf{T}}\delta_{st} = 0_{st}\\
&\Rightarrow \mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T}\, II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}
\end{aligned}
$$

Q.E.D.

Alternative solution:

In **Problem 1.56** (NOTE 1) we have summarized that:

$$\boxed{[(\mathbf{A}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + [\vec{\mathbf{a}},(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] + [\vec{\mathbf{a}},\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] = I_{\mathbf{A}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\boxed{[\vec{\mathbf{a}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{a}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] = II_{\mathbf{A}}\,[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

$$\boxed{[(\mathbf{A}\cdot\vec{\mathbf{a}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] = III_{\mathbf{A}}\,[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]}$$

where $[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] \equiv \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) \neq 0$ holds with $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, $\vec{\mathbf{c}} \neq \vec{\mathbf{0}}$. Now if we consider that the vector $\vec{\mathbf{a}}$ is given by $\vec{\mathbf{a}} = \mathbf{A}\cdot\vec{\mathbf{f}}$ we can obtain:

$$
\begin{aligned}
&[(\mathbf{A}\cdot\vec{\mathbf{a}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + [\vec{\mathbf{a}},(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] + [\vec{\mathbf{a}},\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] = I_{\mathbf{A}}[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]\\
&\Rightarrow [(\mathbf{A}\cdot\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] = I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}]\\
&\Rightarrow [(\mathbf{A}^2\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] = I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}]\\
&\Rightarrow [(\mathbf{A}^2\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] - I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] = -[(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] - [(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})]
\end{aligned}
\tag{1.67}
$$

According to the definition of $II_{\mathbf{A}}$ it is also true that:

$$
\begin{aligned}
&[\vec{\mathbf{f}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] + [(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] = II_{\mathbf{A}}\,[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}]\\
&\Rightarrow [\vec{\mathbf{f}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] - II_{\mathbf{A}}\,[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}] = -[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})] - [(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}]
\end{aligned}
$$

Taking into account the above equation into the equation (1.67) we can obtain:

$$
\begin{aligned}
&[(\mathbf{A}^2\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] - I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] = -[(\mathbf{A}\cdot\vec{\mathbf{f}}),(\mathbf{A}\cdot\vec{\mathbf{b}}),\vec{\mathbf{c}}] - [(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},(\mathbf{A}\cdot\vec{\mathbf{c}})]\\
&\Rightarrow [(\mathbf{A}^2\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] - I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] = [\vec{\mathbf{f}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] - II_{\mathbf{A}}\,[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}]\\
&\Rightarrow [(\mathbf{A}^2\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] - I_{\mathbf{A}}[(\mathbf{A}\cdot\vec{\mathbf{f}}),\vec{\mathbf{b}},\vec{\mathbf{c}}] + II_{\mathbf{A}}\,[\vec{\mathbf{f}},\vec{\mathbf{b}},\vec{\mathbf{c}}] - [\vec{\mathbf{f}},(\mathbf{A}\cdot\vec{\mathbf{b}}),(\mathbf{A}\cdot\vec{\mathbf{c}})] = 0\\
&\Rightarrow (\mathbf{A}^2\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - I_{\mathbf{A}}(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) + II_{\mathbf{A}}\,\vec{\mathbf{f}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - \vec{\mathbf{f}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] = 0
\end{aligned}
$$

In **Problem 1.55** we have shown that $(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}}) = [\mathrm{cof}(\mathbf{A})]\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})$ holds, then the above equation becomes

$$
\begin{aligned}
&(\mathbf{A}^2\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - I_{\mathbf{A}}(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) + II_{\mathbf{A}}\,\vec{\mathbf{f}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - \vec{\mathbf{f}}\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right] = 0\\
&\Rightarrow (\mathbf{A}^2\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - I_{\mathbf{A}}(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) + II_{\mathbf{A}}\,\vec{\mathbf{f}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) - \vec{\mathbf{f}}\cdot[\mathrm{cof}(\mathbf{A})]\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = 0\\
&\Rightarrow \left\{(\mathbf{A}^2\cdot\vec{\mathbf{f}}) - I_{\mathbf{A}}(\mathbf{A}\cdot\vec{\mathbf{f}}) + II_{\mathbf{A}}\,\vec{\mathbf{f}} - \vec{\mathbf{f}}\cdot[\mathrm{cof}(\mathbf{A})]\right\}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = 0
\end{aligned}
$$

Note that the vectors $(\mathbf{A}^2 \cdot \vec{\mathbf{f}})$, $(\mathbf{A} \cdot \vec{\mathbf{f}})$, $\vec{\mathbf{f}} \neq \vec{\mathbf{0}}$, $(\vec{\mathbf{f}} \cdot [\mathrm{cof}(\mathbf{A})])$ are not orthogonal to $(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) \neq \vec{\mathbf{0}}$, so, we can conclude that

$$\begin{aligned}
&\Rightarrow (\mathbf{A}^2 \cdot \vec{\mathbf{f}}) - I_{\mathbf{A}} (\mathbf{A} \cdot \vec{\mathbf{f}}) + I\!I_{\mathbf{A}}\, \vec{\mathbf{f}} - \vec{\mathbf{f}} \cdot [\mathrm{cof}(\mathbf{A})] = \vec{\mathbf{0}} \\
&\Rightarrow \mathbf{A}^2 \cdot \vec{\mathbf{f}} - I_{\mathbf{A}} \mathbf{A} \cdot \vec{\mathbf{f}} + I\!I_{\mathbf{A}}\, \mathbf{1} \cdot \vec{\mathbf{f}} - [\mathrm{cof}(\mathbf{A})]^T \cdot \vec{\mathbf{f}} = \vec{\mathbf{0}} \\
&\Rightarrow \left\{ \mathbf{A}^2 - I_{\mathbf{A}} \mathbf{A} + I\!I_{\mathbf{A}}\, \mathbf{1} - [\mathrm{cof}(\mathbf{A})]^T \right\} \cdot \vec{\mathbf{f}} = \vec{\mathbf{0}} \\
&\Rightarrow \mathbf{A}^2 - I_{\mathbf{A}} \mathbf{A} + I\!I_{\mathbf{A}}\, \mathbf{1} - [\mathrm{cof}(\mathbf{A})]^T = \vec{\mathbf{0}}
\end{aligned}$$

Using the definition $|\mathbf{A}|\mathbf{A}^{-1} = [\mathrm{cof}(\mathbf{A})]^T$, the above equation becomes

$$\begin{aligned}
&\mathbf{A}^2 - I_{\mathbf{A}} \mathbf{A} + I\!I_{\mathbf{A}}\, \mathbf{1} - [\mathrm{cof}(\mathbf{A})]^T = \vec{\mathbf{0}} \\
&\Rightarrow \mathbf{A}^2 - I_{\mathbf{A}} \mathbf{A} + I\!I_{\mathbf{A}}\, \mathbf{1} - |\mathbf{A}|\mathbf{A}^{-1} = \vec{\mathbf{0}} \\
&\Rightarrow \mathbf{A}^2 \cdot \mathbf{A} - I_{\mathbf{A}} \mathbf{A} \cdot \mathbf{A} + I\!I_{\mathbf{A}}\, \mathbf{1} \cdot \mathbf{A} - |\mathbf{A}|\mathbf{A}^{-1} \cdot \mathbf{A} = \vec{\mathbf{0}} \cdot \mathbf{A} \\
&\Rightarrow \mathbf{A}^3 - I_{\mathbf{A}} \mathbf{A}^2 + I\!I_{\mathbf{A}}\, \mathbf{A} - |\mathbf{A}|\mathbf{1} = \vec{\mathbf{0}}
\end{aligned}$$

Q.E.D.

Problem 1.79

Based on the Cayley-Hamilton theorem, find the inverse of a tensor $\mathbf{T}$ in terms of tensor power.

Solution: The Cayley-Hamilton theorem states that:

$$\mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T} I\!I_{\mathbf{T}} - I\!I\!I_{\mathbf{T}} \mathbf{1} = \mathbf{0}$$

Carrying out the dot product between the previous equation and the tensor $\mathbf{T}^{-1}$, we obtain:

$$\mathbf{T}^3 \cdot \mathbf{T}^{-1} - \mathbf{T}^2 \cdot \mathbf{T}^{-1} I_{\mathbf{T}} + \mathbf{T} \cdot \mathbf{T}^{-1} I\!I_{\mathbf{T}} - I\!I\!I_{\mathbf{T}} \mathbf{1} \cdot \mathbf{T}^{-1} = \mathbf{0} \cdot \mathbf{T}^{-1}$$

$$\mathbf{T}^2 - \mathbf{T} I_{\mathbf{T}} + \mathbf{1} I\!I_{\mathbf{T}} - I\!I\!I_{\mathbf{T}} \mathbf{T}^{-1} = \mathbf{0} \quad \Rightarrow \quad \mathbf{T}^{-1} = \frac{1}{I\!I\!I_{\mathbf{T}}} \left(\mathbf{T}^2 - I_{\mathbf{T}} \mathbf{T} + I\!I_{\mathbf{T}} \mathbf{1} \right)$$

Problem 1.80

Check the Cayley-Hamilton theorem by using a second-order tensor whose Cartesian components are given by:

$$\mathcal{T} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

The Cayley-Hamilton theorem states that:

$$\mathcal{T}^3 - \mathcal{T}^2 I_{\mathbf{T}} + \mathcal{T} I\!I_{\mathbf{T}} - I\!I\!I_{\mathbf{T}} \mathbf{1} = \mathbf{0}$$

where $I_{\mathbf{T}} = 5 + 2 + 1 = 8$, $I\!I_{\mathbf{T}} = 10 + 2 + 5 = 17$, $I\!I\!I_{\mathbf{T}} = 10$, and

$$\mathcal{T}^3 = \begin{bmatrix} 5^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \mathcal{T}^2 = \begin{bmatrix} 5^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying the Cayley-Hamilton theorem, we can verify that the following is true:

$$\begin{bmatrix}125&0&0\\0&8&0\\0&0&1\end{bmatrix}-8\begin{bmatrix}25&0&0\\0&4&0\\0&0&1\end{bmatrix}+17\begin{bmatrix}5&0&0\\0&2&0\\0&0&1\end{bmatrix}-10\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Problem 1.81

Given the matrix $\mathcal{P}$ which is represented by its components $P_{ij}\,(i,j=1,2,3,4)$. a) Obtain the inverse of $\mathcal{P}$, b) the invariants, y c) the characteristic equation. Consider that:

$$\mathcal{P}=\begin{bmatrix}1&2&3&1\\2&2&1&2\\4&1&5&3\\3&1&2&4\end{bmatrix}\quad\text{and}\quad \boldsymbol{1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Solution:

By applying the Cayley-Hamilton theorem we can obtain:

$$\begin{aligned}
&\mathcal{P}^4+\mathcal{P}^3 I_1+\mathcal{P}^2 I_2+\mathcal{P}\;I_3+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{P}\left(\mathcal{P}^3+\mathcal{P}^2 I_1+\mathcal{P}\;I_2+\boldsymbol{1}\;I_3\right)+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{P}\left(\mathcal{P}\;\left(\mathcal{P}^2+\mathcal{P}\;I_1+\boldsymbol{1}\;I_2\right)+\boldsymbol{1}\;I_3\right)+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{P}\left(\mathcal{P}\;\left(\underbrace{\mathcal{P}\;\left(\mathcal{P}\;+\boldsymbol{1}\;I_1\right)}_{\mathcal{C}_1}+\boldsymbol{1}\;I_2\right)+\boldsymbol{1}\;I_3\right)+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{P}\left(\underbrace{\mathcal{P}\;\left(\mathcal{C}_1+\boldsymbol{1}\;I_2\right)}_{\mathcal{C}_2}+\boldsymbol{1}\;I_3\right)+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{P}\left(\mathcal{C}_2+\boldsymbol{1}\;I_3\right)+I_4\boldsymbol{1}=\boldsymbol{0}\\
&\Rightarrow \mathcal{C}_3+I_4\boldsymbol{1}=\boldsymbol{0}
\end{aligned}$$

where we have denoted by:

$$\begin{aligned}
&\mathcal{C}_0=\mathcal{P}\\
&\mathcal{C}_1=\mathcal{P}\;\left(\mathcal{C}_0\;+\boldsymbol{1}\;I_1\right)\\
&\mathcal{C}_2=\mathcal{P}\;\left(\mathcal{C}_1\;+\boldsymbol{1}\;I_2\right)\\
&\mathcal{C}_3=\mathcal{P}\;\left(\mathcal{C}_2\;+\boldsymbol{1}\;I_3\right)
\end{aligned}$$

We can obtain the trace of $\mathcal{C}_3+I_4\boldsymbol{1}=\boldsymbol{0}$ as follows:

$$\begin{aligned}
&\mathrm{Tr}(\mathcal{C}_3+I_4\boldsymbol{1})=\mathrm{Tr}(\boldsymbol{0})\\
&\Rightarrow \mathrm{Tr}(\mathcal{C}_3)+\mathrm{Tr}(I_4\boldsymbol{1})=\mathrm{Tr}(\mathcal{C}_3)+I_4\mathrm{Tr}(\boldsymbol{1})=\mathrm{Tr}(\mathcal{C}_3)+4I_4=0 \quad\Rightarrow\quad I_4=\frac{-\mathrm{Tr}(\mathcal{C}_3)}{4}
\end{aligned}$$

Similarly, we can define that:

$$I_3=\frac{-\mathrm{Tr}(\mathcal{C}_2)}{3}\quad;\quad I_2=\frac{-\mathrm{Tr}(\mathcal{C}_1)}{2}\quad;\quad I_1=\frac{-\mathrm{Tr}(\mathcal{C}_0)}{1}$$

With that we can obtain:

$$I_1 = \frac{-\mathrm{Tr}(\mathcal{C}_0)}{1} = -(1+2+5+4) = -12$$

thus we evaluate the matrix $\mathcal{C}_1 = \mathcal{P}\left(\mathcal{C}_0 + \mathbf{1}\, I_1\right)$:

$$\mathcal{C}_1 = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} - 12 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 8 & -14 & -14 & 6 \\ -8 & -13 & 5 & -7 \\ -13 & 6 & -16 & -3 \\ -11 & 2 & 4 & -21 \end{bmatrix}$$

$$I_2 = \frac{-\mathrm{Tr}(\mathcal{C}_1)}{2} = \frac{-(8-13-16-21)}{2} = \frac{-(-42)}{2} = 21$$

In turn we can obtain $\mathcal{C}_2 = \mathcal{P}\left(\mathcal{C}_1 + \mathbf{1}\, I_2\right)$

$$\mathcal{C}_2 = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \left(\begin{bmatrix} 8 & -14 & -14 & 6 \\ -8 & -13 & 5 & -7 \\ -13 & 6 & -16 & -3 \\ -11 & 2 & 4 & -21 \end{bmatrix} + 21 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -37 & 22 & 15 & -17 \\ 7 & -2 & -5 & -5 \\ 10 & -12 & -14 & 2 \\ 9 & -14 & -11 & 5 \end{bmatrix}$$

$$I_3 = \frac{-\mathrm{Tr}(\mathcal{C}_2)}{3} = \frac{-(-37-2-14+5)}{3} = \frac{-(-48)}{3} = 16$$

In turn we can obtain $\mathcal{C}_3 = \mathcal{P}\left(\mathcal{C}_2 + \mathbf{1}\, I_3\right)$

$$\mathcal{C}_3 = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \left(\begin{bmatrix} -37 & 22 & 15 & -17 \\ 7 & -2 & -5 & -5 \\ 10 & -12 & -14 & 2 \\ 9 & -14 & -11 & 5 \end{bmatrix} + 16 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 32 & 0 & 0 & 0 \\ 0 & 32 & 0 & 0 \\ 0 & 0 & 32 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$I_4 = \frac{-\mathrm{Tr}(\mathcal{C}_3)}{4} = \frac{-4(32)}{4} = -32 = \det(\mathcal{P})$$

Then, the characteristic equation becomes:

$$\mathcal{P}^4 + \mathcal{P}^3 I_1 + \mathcal{P}^2 I_2 + \mathcal{P}\, I_3 + I_4 \mathbf{1} = \mathbf{0} \quad \Rightarrow \quad \mathcal{P}^4 - 12\mathcal{P}^3 + 21\mathcal{P}^2 + 16\mathcal{P} - 32\mathbf{1} = \mathbf{0}$$

The characteristic equation coefficients could have been obtained by evaluates the determinant:

$$\det(\mathcal{P} - \lambda\mathbf{1}) \equiv \left|\mathcal{P} - \lambda\mathbf{1}\right| = \begin{vmatrix} 1-\lambda & 2 & 3 & 1 \\ 2 & 2-\lambda & 1 & 2 \\ 4 & 1 & 5-\lambda & 3 \\ 3 & 1 & 2 & 4-\lambda \end{vmatrix} = 0$$

c) The inverse can be obtained by starting from:

$$\mathcal{P}\left(\mathcal{C}_2 + \mathbf{1}\, I_3\right) + I_4 \mathbf{1} = \mathbf{0}$$

$$\Rightarrow \mathcal{P}^{-1}\mathcal{P}\left(\mathcal{C}_2 + \mathbf{1}\, I_3\right) + I_4 \mathcal{P}^{-1}\mathbf{1} = \mathbf{0} \quad \Rightarrow \quad \Rightarrow \left(\mathcal{C}_2 + \mathbf{1}\, I_3\right) + I_4 \mathcal{P}^{-1} = \mathbf{0}$$

$$\Rightarrow \mathcal{P}^{-1} = -\frac{1}{I_4}\left(\mathcal{C}_2 + \mathbf{1}\, I_3\right) \quad \Leftrightarrow \quad \mathcal{P}^{-1} = \frac{1}{\det(\mathcal{P})}\mathrm{adj}[\mathcal{P}] \quad \therefore \quad \mathrm{adj}[\mathcal{P}] = -\left(\mathcal{C}_2 + \mathbf{1}\, I_3\right)$$

thus:

$$\mathcal{P}^{-1} = -\frac{1}{(-32)}\left(\begin{bmatrix} -37 & 22 & 15 & -17 \\ 7 & -2 & -5 & -5 \\ 10 & -12 & -14 & 2 \\ 9 & -14 & -11 & 5 \end{bmatrix} + 16\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{32}\begin{bmatrix} -21 & 22 & 15 & -17 \\ 7 & 14 & -5 & -5 \\ 10 & -12 & 2 & 2 \\ 9 & -14 & -11 & 21 \end{bmatrix}$$

NOTE 1: The procedure performed previously, in the literature, is called *Faddeev-Leverrier method.*

Note that the inverse can also be obtained by using the same procedure as the one used in the equation (1.42), i.e.:

$$\left(\frac{-1}{32}\right)\begin{bmatrix}
\begin{vmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 0 & 0 & 0 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}\end{vmatrix} \\
\begin{vmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix}\end{vmatrix} \\
\begin{vmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}\end{vmatrix} \\
\begin{vmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}\end{vmatrix} & \begin{vmatrix}\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{vmatrix}
\end{bmatrix} = \left(\frac{-1}{32}\right)\begin{pmatrix} 21 & -22 & -15 & 17 \\ -7 & -14 & 5 & 5 \\ -10 & 12 & -2 & -2 \\ -9 & 14 & 11 & -21 \end{pmatrix}$$

NOTE 2: We can also obtain the characteristic coefficients by means of the following procedure. Considering $\mathcal{P}^4 - \mathcal{P}^3 \bar{I}_1 + \mathcal{P}^2 \bar{I}_2 - \mathcal{P}\ \bar{I}_3 + \bar{I}_4 \boldsymbol{1} = \boldsymbol{0}$

The last coefficient is $\bar{I}_4 = \det(\mathcal{P}) = -32$.

The coefficient $\bar{I}_3$ is obtained by the sum of the determinants of the resulting matrices by eliminating one row and one column associated with the main diagonal, i.e.

$$\bar{I}_3 = \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 3 & 1 \\ 4 & 5 & 3 \\ 3 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 4 & 2 & 5 \end{bmatrix} \right\| = -16$$

The coefficient $\bar{I}_2$ is obtained by the sum of the determinants of the resulting matrices by eliminating two rows and two columns associated with the main diagonal, i.e.

$$\bar{I}_2 = \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| +$$

$$+ \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| +$$

$$+ \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \right\| = 21$$

The coefficient $\bar{I}_1$ is obtained by the sum of the determinants of the resulting matrices by eliminating three rows and three columns associated with the main diagonal, i.e.

$$\bar{I}_1 = \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix} \right\|$$

$$= \left\| [4] \right\| + \left\| [1] \right\| + \left\| [2] \right\| + |5| = 12 = \mathrm{Tr}(\mathcal{P})$$

Problem 1.82

Let $\mathbf{A}$ be a second-order tensor, show that:

a) $I\!I_{\mathbf{A}} = \frac{1}{2}\left\{(I_{\mathbf{A}})^2 - \mathrm{Tr}(\mathbf{A}^2)\right\}$

b) $\det(\mathbf{A}) = \frac{1}{6}\left\{[\mathrm{Tr}(\mathbf{A})]^3 + 2\mathrm{Tr}(\mathbf{A}^3) - 3\mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{A}^2)\right\}$

Solution:

a) It was shown in **Problem 1.79** that $I\!I\!I_{\mathbf{A}}\mathbf{A}^{-1} = (\mathbf{A}^2 - \mathbf{A}I_{\mathbf{A}} + \mathbf{1}I\!I_{\mathbf{A}})$, then, by applying the double scalar product with the second-order unit tensor we obtain:

$$I\!I\!I_{\mathbf{A}}\mathbf{A}^{-1}:\mathbf{1} = \left(\mathbf{A}^2 - \mathbf{A}I_{\mathbf{A}} + \mathbf{1}I\!I_{\mathbf{A}}\right):\mathbf{1} = \mathbf{A}^2:\mathbf{1} - \mathbf{A}:\mathbf{1}I_{\mathbf{A}} + \mathbf{1}:\mathbf{1}I\!I_{\mathbf{A}}$$
$$I\!I\!I_{\mathbf{A}}\mathrm{Tr}(\mathbf{A}^{-1}) = \mathrm{Tr}(\mathbf{A}^2) - \mathrm{Tr}(\mathbf{A})I_{\mathbf{A}} + \mathrm{Tr}(\mathbf{1})I\!I_{\mathbf{A}} = \mathrm{Tr}(\mathbf{A}^2) - (I_{\mathbf{A}})^2 + 3I\!I_{\mathbf{A}}$$

Taking into account the inverse of a tensor $\mathbf{A}^{-1} = \dfrac{[\mathrm{cof}(\mathbf{A})]^T}{I\!I\!I_{\mathbf{A}}}$, we can conclude that:

$$I\!I\!I_{\mathbf{A}}\mathrm{Tr}(\mathbf{A}^{-1}) = \mathrm{Tr}(I\!I\!I_{\mathbf{A}}\mathbf{A}^{-1}) = \mathrm{Tr}\left(I\!I\!I_{\mathbf{A}}\frac{[\mathrm{cof}(\mathbf{A})]^T}{I\!I\!I_{\mathbf{A}}}\right) = \mathrm{Tr}\left([\mathrm{cof}(\mathbf{A})]^T\right) = \mathrm{Tr}\left([\mathrm{cof}(\mathbf{A})]\right) = I\!I_{\mathbf{A}}$$

With that, we can obtain:

$$I\!I\!I_{\mathbf{A}}\mathrm{Tr}(\mathbf{A}^{-1}) = I\!I_{\mathbf{A}} = \mathrm{Tr}(\mathbf{A}^2) - (I_{\mathbf{A}})^2 + 3I\!I_{\mathbf{A}} \quad \Rightarrow \quad I\!I_{\mathbf{A}} - 3I\!I_{\mathbf{A}} = \mathrm{Tr}(\mathbf{A}^2) - (I_{\mathbf{A}})^2$$
$$\Rightarrow I\!I_{\mathbf{A}} = \frac{1}{2}\left\{(I_{\mathbf{A}})^2 - \mathrm{Tr}(\mathbf{A}^2)\right\}$$

b) We start from the Cayley-Hamilton theorem, which states that any tensor satisfies its own characteristic equation, i.e.:

$$\mathbf{A}^3 - \mathbf{A}^2 I_{\mathbf{A}} + \mathbf{A}I\!I_{\mathbf{A}} - I\!I\!I_{\mathbf{A}}\mathbf{1} = \mathbf{0} \tag{1.68}$$

where $I_{\mathbf{A}} = [\mathrm{Tr}(\mathbf{A})]$, $I\!I_{\mathbf{A}} = \frac{1}{2}\left\{[\mathrm{Tr}(\mathbf{A})]^2 - \mathrm{Tr}(\mathbf{A}^2)\right\}$ and $I\!I\!I_{\mathbf{A}} = \det(\mathbf{A})$ are the principal invariants of $\mathbf{A}$. Applying the double scalar product between the second-order unit tensor ($\mathbf{1}$) and the equation in (1.68) we obtain:

$$\mathbf{A}^3:\mathbf{1} - \mathbf{A}^2:\mathbf{1}I_{\mathbf{A}} + \mathbf{A}:\mathbf{1}I\!I_{\mathbf{A}} - I\!I\!I_{\mathbf{A}}\mathbf{1}:\mathbf{1} = \mathbf{0}:\mathbf{1}$$
$$\mathrm{Tr}(\mathbf{A}^3) - \mathrm{Tr}(\mathbf{A}^2)I_{\mathbf{A}} + \mathrm{Tr}(\mathbf{A})I\!I_{\mathbf{A}} - I\!I\!I_{\mathbf{A}}[\mathrm{Tr}(\mathbf{1})] = [\mathrm{Tr}(\mathbf{0})]$$
$$\mathrm{Tr}(\mathbf{A}^3) - \mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{A})\frac{1}{2}\left\{[\mathrm{Tr}(\mathbf{A})]^2 - \mathrm{Tr}(\mathbf{A}^2)\right\} - I\!I\!I_{\mathbf{A}}3 = 0$$
$$\mathrm{Tr}(\mathbf{A}^3) - \mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A}) + \frac{1}{2}[\mathrm{Tr}(\mathbf{A})]^3 - \frac{1}{2}\mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{A}^2) - I\!I\!I_{\mathbf{A}}3 = 0$$
$$\frac{1}{2}\left\{2\mathrm{Tr}(\mathbf{A}^3) - 3\mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A}) + [\mathrm{Tr}(\mathbf{A})]^3\right\} - I\!I\!I_{\mathbf{A}}3 = 0$$

with which we obtain:

$$I\!I\!I_{\mathbf{A}} = \det(\mathbf{A}) = \frac{1}{6}\left\{[\mathrm{Tr}(\mathbf{A})]^3 + 2\mathrm{Tr}(\mathbf{A}^3) - 3\mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A})\right\}$$

or in indicial notation:

$$I\!I\!I_{\mathbf{A}} = \det(\mathbf{A}) = \frac{1}{6}\left\{\mathsf{A}_{ii}\mathsf{A}_{jj}\mathsf{A}_{kk} + 2\mathsf{A}_{ij}\mathsf{A}_{jk}\mathsf{A}_{ki} - 3\mathsf{A}_{ij}\mathsf{A}_{ji}\mathsf{A}_{kk}\right\}$$

NOTE: It is interesting to note that the principal invariants of $\mathbf{A}$ are formed by the three fundamental invariants of a second-order tensor, namely $\mathrm{Tr}(\mathbf{A})$, $\mathrm{Tr}(\mathbf{A}^2)$, $\mathrm{Tr}(\mathbf{A}^3)$, i.e.:

$$\boxed{\begin{aligned} I_{\mathbf{A}} &= \mathrm{Tr}(\mathbf{A}) \\ II_{\mathbf{A}} &= \frac{1}{2}\left\{[\mathrm{Tr}(\mathbf{A})]^2 - \mathrm{Tr}(\mathbf{A}^2)\right\} \\ III_{\mathbf{A}} &= \det(\mathbf{A}) = \frac{1}{6}\left\{[\mathrm{Tr}(\mathbf{A})]^3 + 2\mathrm{Tr}(\mathbf{A}^3) - 3\mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A})\right\} \end{aligned}}$$

Problem 1.83

Show that $II_{\mathbf{T}} = III_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^{-1})$, where $II_{\mathbf{T}} = \frac{1}{2}\left\{[\mathrm{Tr}(\mathbf{T})]^2 - \mathrm{Tr}(\mathbf{T}^2)\right\}$ is the second principal invariant of $\mathbf{T}$, and $III_{\mathbf{T}}$ is the third principal invariant (the determinant of $\mathbf{T}$).

Solution:

It was shown in **Problem 1.79** that $\mathbf{T}^{-1} = \frac{1}{III_{\mathbf{T}}}(\mathbf{T}^2 - \mathbf{T}I_{\mathbf{T}} + \mathbf{1}II_{\mathbf{T}})$, then, by applying the double scalar product with the second-order unit tensor we obtain:

$$\mathbf{T}^{-1}:\mathbf{1} = \frac{1}{III_{\mathbf{T}}}\left(\mathbf{T}^2 - \mathbf{T}I_{\mathbf{T}} + \mathbf{1}II_{\mathbf{T}}\right):\mathbf{1} = \frac{1}{III_{\mathbf{T}}}\left(\mathbf{T}^2:\mathbf{1} - \mathbf{T}:\mathbf{1}I_{\mathbf{T}} + \mathbf{1}:\mathbf{1}II_{\mathbf{T}}\right)$$

$$\mathrm{Tr}(\mathbf{T}^{-1}) = \frac{1}{III_{\mathbf{T}}}\left(\mathrm{Tr}(\mathbf{T}^2) - \mathrm{Tr}(\mathbf{T})I_{\mathbf{T}} + \mathrm{Tr}(\mathbf{1})II_{\mathbf{T}}\right)$$

$$\Rightarrow\ III_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^{-1}) = \underbrace{\mathrm{Tr}(\mathbf{T}^2) - I_{\mathbf{T}}^2}_{=-2II_{\mathbf{T}}} + 3II_{\mathbf{T}} \quad \Rightarrow \quad III_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^{-1}) = II_{\mathbf{T}}$$

Problem 1.84

Show that:

$$\boxed{(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\alpha}\mathbf{1} - \frac{\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}}$$

where $\vec{\mathbf{c}}$ and $\vec{\mathbf{b}}$ are vectors, $\mathbf{1}$ is the second-order unit tensor, and α and β are scalars.

Solution:

Let us consider that $\mathbf{T} = (\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$, and the inverse of a tensor obtained in **Problem 1.79**:

$$\mathbf{T}^{-1} = \frac{1}{III_{\mathbf{T}}}\left(\mathbf{T}^2 - \mathbf{T}I_{\mathbf{T}} + \mathbf{1}II_{\mathbf{T}}\right) \tag{1.69}$$

Next, we obtain $\mathbf{T}^2$:

$$\begin{aligned} \mathbf{T}^2 &= \mathbf{T}\cdot\mathbf{T} = (\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) \\ &= \alpha^2\mathbf{1}\cdot\mathbf{1} + \alpha\beta\mathbf{1}\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\mathbf{1} + \beta^2(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) \end{aligned}$$

where it fulfills that $(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = (\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$, (see **Problem 1.20**). Then, the above equation can be rewritten as follows:

$$\mathbf{T}^2 = \alpha^2\mathbf{1} + 2\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$$

and its trace is given by:

$$\begin{aligned}\mathrm{Tr}(\mathbf{T}^2) &= \mathrm{Tr}[\alpha^2\mathbf{1} + 2\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})] = \alpha^2\mathrm{Tr}(\mathbf{1}) + 2\alpha\beta\mathrm{Tr}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathrm{Tr}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\\ &= 3\alpha^2 + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) = 3\alpha^2 + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})^2\end{aligned}$$

Next, we calculate the principal invariants of $\mathbf{T}$

$$I_{\mathbf{T}} = \mathrm{Tr}(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = \alpha\mathrm{Tr}(\mathbf{1}) + \beta\mathrm{Tr}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = 3\alpha + \beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})$$

$$(I_{\mathbf{T}})^2 = [3\alpha + \beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})]^2 = 9\alpha^2 + 6\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})^2$$

$$\begin{aligned}I\!I_{\mathbf{T}} &= \frac{1}{2}\left\{I_{\mathbf{T}}^{\,2} - \mathrm{Tr}(\mathbf{T}^2)\right\} = \frac{1}{2}\left\{9\alpha^2 + 6\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})^2 - [3\alpha^2 + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})^2]\right\}\\ &= 3\alpha^2 + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\end{aligned}$$

$I\!I\!I_{\mathbf{T}} = \det(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = \alpha^3 + \alpha^2\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}$, (see **Problem 1.49**).

Then, the equation in (1.69) becomes:

$$\begin{aligned}I\!I\!I_{\mathbf{T}}\mathbf{T}^{-1} &= \mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + I\!I_{\mathbf{T}}\mathbf{1}\\ &= \alpha^2\mathbf{1} + 2\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\\ &\quad - \left[3\alpha + \beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\right](\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \left[3\alpha^2 + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\right]\mathbf{1}\\ &= \alpha^2\mathbf{1} + 2\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) - 3\alpha^2\mathbf{1} - 3\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) - \alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1}\\ &\quad - \beta^2(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) + 3\alpha^2\mathbf{1} + 2\alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1}\\ &= \mathbf{1}\alpha^2 + \alpha\beta(\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1} - \alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = (\alpha^2 + \alpha\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1} - \alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\\ &= \frac{1}{\alpha}(\alpha^3 + \alpha^2\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1} - \alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = [\mathrm{adj}(\mathbf{T})] = [\mathrm{cof}(\mathbf{T})]^T\end{aligned} \tag{1.70}$$

Taking into account that $\mathbf{T} = (\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$, $I\!I\!I_{\mathbf{T}} = \alpha^3 + \alpha^2\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}}$, the above equation becomes:

$$\mathbf{T}^{-1} = \frac{1}{\alpha}\frac{I\!I\!I_{\mathbf{T}}}{I\!I\!I_{\mathbf{T}}}\mathbf{1} - \frac{\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})}{I\!I\!I_{\mathbf{T}}} = \frac{1}{\alpha}\mathbf{1} - \frac{\alpha\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})}{(\alpha^3 + \alpha^2\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})} \tag{1.71}$$

or:

$$\boxed{(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\alpha}\mathbf{1} - \frac{\beta}{\alpha(\alpha + \beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})} \quad \textit{Tensorial notation} \tag{1.72}$$

$$\boxed{(\alpha\,\delta_{ij} + \beta\,\mathrm{c}_i\mathrm{b}_j)^{-1} = \frac{1}{\alpha}\delta_{ij} - \frac{\beta}{\alpha(\alpha + \beta\,\mathrm{c}_k\mathrm{b}_k)}(\mathrm{c}_i\mathrm{b}_j)} \quad \textit{Indicial notation} \tag{1.73}$$

$$\boxed{\left[\alpha[\boldsymbol{I}] + \beta\left[\{\boldsymbol{c}\}\{\boldsymbol{b}\}^T\right]\right]^{-1} = \frac{1}{\alpha}[\boldsymbol{I}] - \frac{\beta}{\alpha(\alpha + \beta\,\{\boldsymbol{c}\}^T\{\boldsymbol{b}\})}\left[\{\boldsymbol{c}\}\{\boldsymbol{b}\}^T\right]} \quad \textit{Matrix notation} \tag{1.74}$$

NOTE 1: The above equation is also valid for matrices of *n*-dimensions.

In the particular case when $\alpha = 1$, $\beta = 1$, we can obtain:

$$\boxed{(\mathbf{1} + \vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1} = \mathbf{1} - \frac{(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})}{1 + \vec{\mathbf{c}}\cdot\vec{\mathbf{b}}}} \tag{1.75}$$

NOTE 2: The equation in (1.72) or in (1.70) can be rewritten as follows:

$$\mathbf{T}^{-1} = (\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\alpha}\mathbf{1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$$

$$= \frac{1}{(\alpha^3+\alpha^2\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}\left[(\alpha^2+\alpha\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1} - \beta\alpha(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\right] = \frac{1}{\det(\mathbf{T})}\left[\mathrm{adj}(\mathbf{T})\right]$$

with that we can conclude that:

$$\mathrm{adj}(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}) = (\alpha^2 + \alpha\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})\mathbf{1} - \beta\alpha(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$$

NOTE 3: We can extend the equation in (1.72) such that:

$$\boxed{(\alpha\mathbb{I}^{sym} + \beta\,\mathbf{A}\otimes\mathbf{B})^{-1} = \frac{1}{\alpha}\mathbb{I}^{sym} - \frac{\beta}{\alpha(\alpha+\beta\,\mathbf{A}:\mathbf{B})}(\mathbf{A}\otimes\mathbf{B})}$$

where we now have that $\mathbb{I}^{sym}$ is the symmetric fourth-order unit tensor, $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors, and α and β are scalars. With that it is easy to show that $(\mathbb{I}^{sym})^{-1} = \mathbb{I}^{sym}$.

Problem 1.85

Taking into account that $(\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\alpha}\mathbf{1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})$, (see **Problem 1.84**), show that:

$$\boxed{(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})}\left[(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})\right]} \tag{1.76}$$

where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors, $\mathbf{A}$ is a second-order tensor, with $\det(\mathbf{A}) \neq 0$ ($\exists\mathbf{A}^{-1}$), and α and β are scalars.

Solution:

Note that the expression $(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$ can be rewritten as follows:

$$(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = (\alpha\mathbf{A}\cdot\mathbf{1} + \beta\,\mathbf{1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = (\alpha\mathbf{A}\cdot\mathbf{1} + \beta\,(\mathbf{A}\cdot\mathbf{A}^{-1})\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mathbf{A}\cdot(\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$$

Using the inverse property such as $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$, we can obtain:

$$(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1} = \left[\mathbf{A}\cdot(\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\right]^{-1} = (\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1}\cdot\mathbf{A}^{-1}$$

Note that the result of the algebraic operation $\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}$ is a vector in which we denote $\vec{\mathbf{c}} = \mathbf{A}^{-1}\cdot\vec{\mathbf{a}}$, with that we rewrite the above equation as follows:

$$(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1} = (\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1}\cdot\mathbf{A}^{-1} = (\alpha\mathbf{1} + \beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1}\cdot\mathbf{A}^{-1}$$

$$= \left[\frac{1}{\alpha}\mathbf{1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\right]\cdot\mathbf{A}^{-1} = \frac{1}{\alpha}\mathbf{1}\cdot\mathbf{A}^{-1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\mathbf{A}^{-1}$$

$$= \frac{1}{\alpha}\mathbf{A}^{-1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}\vec{\mathbf{c}}\otimes\vec{\mathbf{b}}\cdot\mathbf{A}^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})}(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})$$

$$= \frac{1}{\alpha}\mathbf{A}^{-1} - \frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})}(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})$$

The above equation in indicial notation becomes:

$$(\alpha \mathsf{A}_{ij} + \beta\, \mathsf{a}_i \mathsf{b}_j)^{-1} = \frac{1}{\alpha}\mathsf{A}_{ij}^{-1} - \frac{\beta}{\alpha(\alpha + \beta\, \mathsf{b}_p \mathsf{A}_{pq}^{-1} \mathsf{a}_q)} (\mathsf{A}_{ik}^{-1}\mathsf{a}_k)(\mathsf{b}_s \mathsf{A}_{sj}^{-1})$$

The reader should be aware here with the algebraic operation $(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\cdot\vec{\mathbf{b}} \neq \underbrace{\mathbf{A}^{-1}\cdot(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}_{\text{Invalid Expression}}$, the latter has no consistency, since we cannot have a scalar product (contraction) with the scalar $(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$. We can check this fact by means of indicial notation $\vec{\mathbf{c}}\cdot\vec{\mathbf{b}} = \mathsf{c}_i \mathsf{b}_i = (\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})_i \mathsf{b}_i = \mathsf{A}_{ik}^{-1}\mathsf{a}_k \mathsf{b}_i$, then, the possible expressions tensorial notation are $(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\cdot\vec{\mathbf{b}} = \underbrace{\mathsf{b}_i \mathsf{A}_{ik}^{-1}\mathsf{a}_k}_{\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}} = \underbrace{\mathsf{a}_k \mathsf{A}_{ik}^{-1}\mathsf{b}_i}_{\vec{\mathbf{a}}\cdot\mathbf{A}^{-T}\cdot\vec{\mathbf{b}}} = \underbrace{\mathsf{A}_{ik}^{-1}\mathsf{b}_i \mathsf{a}_k}_{\mathbf{A}^{-1}:(\vec{\mathbf{b}}\otimes\vec{\mathbf{a}})} = \underbrace{\mathsf{A}_{ik}^{-1}\mathsf{a}_k \mathsf{b}_i}_{\mathbf{A}^{-T}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})} = \underbrace{\mathsf{A}_{ik}^{-1}\mathsf{a}_k \mathsf{b}_i}_{\mathbf{A}^{-1}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^T}$.

NOTE 1: For the particular case when $\alpha = 1$, $\beta = 1$, we fall back on the *Sherman-Morrison formula:*

$$\boxed{(\mathbf{A} + \vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1} = \mathbf{A}^{-1} - \frac{(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})}{1 + \vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}}} \qquad (1.77)$$

Sherman-Morrison formula (tensorial notation)

The above equation in matrix notation becomes

$$\boxed{\left[[A] + [\{a\}\{b\}^T]\right]^{-1} = [A]^{-1} - \frac{\left[\left\{[A]^{-1}\{a\}\right\}\left\{\{b\}^T [A]^{-1}\right\}^T\right]}{1 + \{b\}^T [A]^{-1}\{a\}}} \qquad (1.78)$$

Sherman-Morrison formula (matrix notation)

NOTE 2: Note that if $(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mathbf{A}\cdot(\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$, the determinant is defined as follows:

$$\begin{aligned}\det(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) &= \det\left[\mathbf{A}\cdot(\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\right] = \det[\mathbf{A}]\det\left[(\alpha\mathbf{1} + \beta\,\mathbf{A}^{-1}\cdot\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\right]\\ &= \det[\mathbf{A}](\alpha^3 + \alpha^2\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\end{aligned}$$

with that, the equation in (1.76) can be rewritten as follows:

$$\boxed{(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1} = \frac{1}{\gamma}\left\{|\mathbf{A}|(\alpha^2 + \alpha\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\mathbf{A}^{-1} - |\mathbf{A}|\alpha\beta\left[(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})\right]\right\}}$$

with $\gamma = \det(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = |\mathbf{A}|(\alpha^3 + \alpha^2\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})$. (1.79)

with that we conclude that:

$$\operatorname{adj}(\alpha\mathbf{A} + \beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \left\{|\mathbf{A}|(\alpha^2 + \alpha\beta\,\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\mathbf{A}^{-1} - |\mathbf{A}|\alpha\beta\left[(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})\right]\right\}$$

NOTE 3: We can extend the equation in (1.76) such that:

$$\boxed{(\alpha\mathbb{D} + \beta\,\mathbf{A}\otimes\mathbf{B})^{-1} = \frac{1}{\alpha}\mathbb{D}^{-1} - \frac{\beta}{\alpha(\alpha + \beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A})}\left[(\mathbb{D}^{-1}:\mathbf{A})\otimes(\mathbf{B}:\mathbb{D}^{-1})\right]} \qquad (1.80)$$

where we now have that $\mathbb{D}$ is a fourth-order tensor, $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors, and α and β are scalars.

Note that:

$$\boxed{(\alpha\mathbb{D}+\beta\,\mathbf{A}\otimes\mathbf{B})^{-1}=\frac{1}{\gamma}\left\{|\mathbb{D}|(\alpha^2+\alpha\beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A})\mathbb{D}^{-1}-|\mathbb{D}|\alpha\beta\left[(\mathbb{D}^{-1}:\mathbf{A})\otimes(\mathbf{B}:\mathbb{D}^{-1})\right]\right\}}$$

with $\gamma=\det(\alpha\mathbb{D}+\beta\,\mathbf{A}\otimes\mathbf{B})=|\mathbb{D}|(\alpha^3+\alpha^2\beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A})$. (1.81)

where we can conclude that:

$$\det(\alpha\mathbb{D}+\beta\,\mathbf{A}\otimes\mathbf{B})=\det(\mathbb{D})(\alpha^3+\alpha^2\beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A}) \tag{1.82}$$

$$\operatorname{adj}(\alpha\mathbb{D}+\beta\,\mathbf{A}\otimes\mathbf{B})=\left\{|\mathbb{D}|(\alpha^2+\alpha\beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A})\mathbb{D}^{-1}-\alpha\beta|\mathbb{D}|\left[(\mathbb{D}^{-1}:\mathbf{A})\otimes(\mathbf{B}:\mathbb{D}^{-1})\right]\right\} \tag{1.83}$$

Problem 1.86

a) Let $\mathbf{C}=(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$ be a second-order tensor. Show that:

$$\boxed{\left|\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}\right|=\alpha^3+\alpha^2\gamma(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})+\alpha\beta\gamma\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\right]} \tag{1.84}$$

where $\left|\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}\right|\equiv\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$ represents the determinant of the tensor $\mathbf{C}$. b) For the particular case when $\alpha=1$, $\vec{\mathbf{d}}=\vec{\mathbf{a}}$, $\vec{\mathbf{c}}=\vec{\mathbf{b}}$, show that:

$$\det(\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{b}}\otimes\vec{\mathbf{a}})=1+(\beta+\gamma)(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\beta\gamma\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2 \tag{1.85}$$

Solution:

We define an auxiliary tensor $\mathbf{D}=\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}$ and in turn we have $\mathbf{C}=(\mathbf{D}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$. According to **Problem 1.85**, (see equation (1.76)), it holds that:

$$\det(\mathbf{D}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})=|\mathbf{D}|(1+\gamma\,\vec{\mathbf{d}}\cdot\mathbf{D}^{-1}\cdot\vec{\mathbf{c}})\text{, where:}$$

$\det(\mathbf{D})\equiv|\mathbf{D}|=\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})=\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$ and

$$(\mathbf{D})^{-1}=(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1}=\frac{1}{\alpha}\mathbf{1}-\frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$$

With that, we can obtain that:

$$\begin{aligned}
\det(\mathbf{D}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})&=|\mathbf{D}|(1+\gamma\,\vec{\mathbf{d}}\cdot\mathbf{D}^{-1}\cdot\vec{\mathbf{c}})\\
&=\left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[(1+\gamma\,\vec{\mathbf{d}}\cdot\left(\frac{1}{\alpha}\mathbf{1}-\frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\right)\cdot\vec{\mathbf{c}}\right]\\
&=\left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[1+\gamma\left(\frac{1}{\alpha}\vec{\mathbf{d}}\cdot\mathbf{1}\cdot\vec{\mathbf{c}}-\frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\vec{\mathbf{d}}\cdot(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{c}}\right)\right]\\
&=\left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[1+\gamma\left(\frac{1}{\alpha}\vec{\mathbf{d}}\cdot\vec{\mathbf{c}}-\frac{\beta}{\alpha(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{d}}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\right)\right]\\
&=\left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[1+\frac{\gamma}{\alpha}(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})-\frac{\alpha\beta\gamma}{\alpha^2(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\right]
\end{aligned}$$

Note that $(\vec{\mathbf{d}}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}) = \underbrace{(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})}_{scalar}\otimes\underbrace{(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})}_{scalar} \equiv (\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})$.

$$
\begin{aligned}
\det(\mathbf{D}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}) &= \left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[1+\frac{\gamma}{\alpha}(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})\right]-\left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[\frac{\alpha\beta\gamma}{\alpha^2(\alpha+\beta\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\right]\\
&= \left[\alpha^3+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]\left[1+\frac{\gamma}{\alpha}(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})\right]-\alpha\beta\gamma(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\\
&= \left[\alpha^3+\alpha^2\gamma(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})+\alpha\beta\gamma(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})\right]-\alpha\beta\gamma(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})
\end{aligned}
$$

Then:

$$
\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}) = \alpha^3+\alpha^2\gamma(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})+\alpha\beta\gamma[(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})]
$$

with that we can show the equation in (1.84).

For the particular case when $\vec{\mathbf{d}}=\vec{\mathbf{a}}$, $\vec{\mathbf{c}}=\vec{\mathbf{b}}$, we have:

$$
\begin{aligned}
\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}) &= \alpha^3+\alpha^2\gamma(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\alpha\beta\gamma[(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{c}}\cdot\vec{\mathbf{d}})]\\
\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{b}}\otimes\vec{\mathbf{a}}) &= \alpha^3+\alpha^2\gamma(\vec{\mathbf{b}}\cdot\vec{\mathbf{a}})+\alpha^2\beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\alpha\beta\gamma[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{a}})]\\
&= \alpha^3+\alpha^2(\beta+\gamma)(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\alpha\beta\gamma[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})]
\end{aligned}
$$

In **Problem 1.1** we have shown that $\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2 = \left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2$ holds, thus:

$$
\det(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\vec{\mathbf{b}}\otimes\vec{\mathbf{a}}) = \alpha^3+\alpha^2(\beta+\gamma)(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\alpha\beta\gamma\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2
$$

For the particular case when $\alpha=1$ we can obtain:

$$
\det(\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\vec{\mathbf{b}}\otimes\vec{\mathbf{a}}) = 1+(\beta+\gamma)(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})-\beta\gamma\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2
$$

Problem 1.87

a) Obtain the inverse of the tensor $\mathbf{C}=(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$.

b.1) Given the second-order tensor $\mathbf{D}=\boldsymbol{B}+\dfrac{\vec{\boldsymbol{p}}\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-\dfrac{(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}$ where $\boldsymbol{B}=\boldsymbol{B}^T$ and $\exists\boldsymbol{B}^{-1}$, show that:

$$
\boxed{\mathbf{D}^{-1}=\boldsymbol{B}^{-1}+\frac{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}+\vec{\boldsymbol{p}}\cdot\boldsymbol{B}^{-1}\cdot\vec{\boldsymbol{p}})}{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}})^2}[\vec{q}\otimes\vec{q}]-\frac{2}{(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}})}[\vec{q}\otimes(\boldsymbol{B}^{-1}\cdot\vec{\boldsymbol{p}})]^{sym}} \tag{1.86}
$$

b.2) If $\boldsymbol{B}$ is a positive definite tensor, obtain the conditions under which $\mathbf{D}$ is a non-singular tensor.

Solution:

Denoting by $\mathbf{A}=(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$ we can obtain $\mathbf{C}=(\mathbf{A}+\gamma\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})$, and by taking into account

$$
(\alpha\mathbf{A}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1}=\frac{1}{\alpha}\mathbf{A}^{-1}-\frac{\beta}{\alpha(\alpha+\beta\vec{\mathbf{b}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})}\left[(\mathbf{A}^{-1}\cdot\vec{\mathbf{a}})\otimes(\vec{\mathbf{b}}\cdot\mathbf{A}^{-1})\right] \tag{1.87}
$$

which was obtained in **Problem 1.85**, (see equation (1.76)), thus

$$
(\mathbf{A}+\gamma\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})^{-1}=\mathbf{A}^{-1}-\frac{\gamma}{(1+\gamma\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{c}})}\left[(\mathbf{A}^{-1}\cdot\vec{\mathbf{c}})\otimes(\vec{\mathbf{d}}\cdot\mathbf{A}^{-1})\right] \tag{1.88}
$$

It was shown in **Problem 1.84** that:

$$(\alpha\mathbf{1}+\beta\,\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})^{-1}=\frac{1}{\alpha}\mathbf{1}-\frac{\beta(\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{c}}\cdot\vec{\mathbf{b}})} \tag{1.89}$$

With that we can obtain:

$$\mathbf{A}^{-1}=(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})^{-1}=\frac{1}{\alpha}\mathbf{1}-\frac{\beta(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}$$

Furthermore, we have

$$\mathbf{A}^{-1}\cdot\vec{\mathbf{c}}=\left(\frac{1}{\alpha}\mathbf{1}-\frac{\beta(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\right)\cdot\vec{\mathbf{c}}=\frac{1}{\alpha}\mathbf{1}\cdot\vec{\mathbf{c}}-\frac{\beta(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\cdot\vec{\mathbf{c}}=\frac{1}{\alpha}\vec{\mathbf{c}}-\frac{\beta(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\vec{\mathbf{a}}$$

$$\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}=\vec{\mathbf{d}}\cdot\left(\frac{1}{\alpha}\mathbf{1}-\frac{\beta(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\right)=\frac{1}{\alpha}\vec{\mathbf{d}}\cdot\mathbf{1}-\vec{\mathbf{d}}\cdot\frac{\beta(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}=\frac{1}{\alpha}\vec{\mathbf{d}}-\frac{\beta(\vec{\mathbf{d}}\cdot\vec{\mathbf{a}})}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}\vec{\mathbf{b}}$$

With that we conclude that

$$(\alpha\mathbf{1}+\beta\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\gamma\,\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})^{-1}=\theta_{(1)}\mathbf{1}+\theta_{(2)}(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})+\theta_{(3)}[\theta_{(1)}\vec{\mathbf{c}}+\theta_{(2)}(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})\vec{\mathbf{a}}]\otimes[\theta_{(1)}\vec{\mathbf{d}}+\theta_{(2)}(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})\vec{\mathbf{b}}] \tag{1.90}$$

where

$$\theta_{(1)}=\frac{1}{\alpha}$$

$$\theta_{(2)}=\frac{-\beta}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}$$

$$\theta_{(3)}=\frac{-\gamma}{(1+\gamma\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{c}})}$$

$$\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{c}}=\frac{1}{\alpha}(\vec{\mathbf{d}}\cdot\vec{\mathbf{c}})-\frac{\beta}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{d}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})$$

NOTE: The equation in (1.90) is also valid for matrices of *n*-dimensions.

b.1) We can rewrite the tensor $\mathbf{D}$ as follows:

$$\mathbf{D}=\boldsymbol{B}\cdot\mathbf{1}+\mathbf{1}\cdot\frac{\vec{\boldsymbol{p}}\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-\mathbf{1}\cdot\frac{(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}=\boldsymbol{B}\cdot\mathbf{1}+(\boldsymbol{B}\cdot\boldsymbol{B}^{-1})\cdot\frac{\vec{\boldsymbol{p}}\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-(\boldsymbol{B}\cdot\boldsymbol{B}^{-1})\cdot\frac{(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}$$

$$=\boldsymbol{B}\cdot\left[\mathbf{1}+\boldsymbol{B}^{-1}\cdot\frac{\vec{\boldsymbol{p}}\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-\boldsymbol{B}^{-1}\cdot\frac{(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}\right]=\boldsymbol{B}\cdot\left[\mathbf{1}+\frac{(\boldsymbol{B}^{-1}\cdot\vec{\boldsymbol{p}})\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-\frac{(\boldsymbol{B}^{-1}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}\right]$$

$$=\boldsymbol{B}\cdot\left[\mathbf{1}+\frac{(\boldsymbol{B}^{-1}\cdot\vec{\boldsymbol{p}})\otimes\vec{\boldsymbol{p}}}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}-\frac{\vec{\boldsymbol{q}}\otimes(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}\right]$$

and by denoting by

$$\vec{\boldsymbol{a}}=(\boldsymbol{B}^{-1}\cdot\vec{\boldsymbol{p}})\quad;\quad\vec{\boldsymbol{b}}=\vec{\boldsymbol{p}}\quad;\quad\vec{\boldsymbol{c}}=\vec{\boldsymbol{q}}\quad;\quad\vec{\boldsymbol{d}}=(\boldsymbol{B}\cdot\vec{\boldsymbol{q}})\quad;\quad\beta=\frac{1}{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{q}}}\quad;\quad\gamma=\frac{-1}{\vec{\boldsymbol{q}}\cdot\boldsymbol{B}\cdot\vec{\boldsymbol{q}}}$$

Then, we can rewrite $\mathbf{D}$ as follows

$$\mathbf{D} = \boldsymbol{B}\cdot\left[\mathbf{1}+\beta\vec{a}\otimes\vec{b}+\gamma\vec{c}\otimes\vec{d}\right] = \boldsymbol{B}\cdot\boldsymbol{C} \qquad \Rightarrow \qquad \mathbf{D}^{-1} = (\boldsymbol{B}\cdot\boldsymbol{C})^{-1} = \boldsymbol{C}^{-1}\cdot\boldsymbol{B}^{-1}$$

where $\boldsymbol{C} = [\mathbf{1}+\beta\vec{a}\otimes\vec{b}+\gamma\vec{c}\otimes\vec{d}]$. The inverse of $\boldsymbol{C}$ can be obtained via subsection (a) with $\alpha = 1$. Moreover, we have:

$\theta_{(1)} = 1$,

$$\theta_{(2)} = \frac{-\beta}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})} = \frac{-\beta}{(1+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})} = \frac{-1}{\vec{p}\cdot\vec{q}}\frac{1}{(1+\frac{1}{\vec{p}\cdot\vec{q}}(\boldsymbol{B}^{-1}\cdot\vec{p})\cdot\vec{p})} = \frac{-1}{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}$$

$$\begin{aligned}
\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{c}} &= \frac{1}{\alpha}(\vec{\mathbf{d}}\cdot\vec{\mathbf{c}}) - \frac{\beta}{\alpha(\alpha+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})}(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}) \\
&= ((\boldsymbol{B}\cdot\vec{q})\cdot\vec{q}) + \frac{-1}{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}((\boldsymbol{B}^{-1}\cdot\vec{p})\cdot(\boldsymbol{B}\cdot\vec{q}))(\vec{p}\cdot\vec{q}) \\
&= \vec{q}\cdot\boldsymbol{B}\cdot\vec{q} + \frac{-(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}
\end{aligned}$$

$$\begin{aligned}
\theta_{(3)} &= \frac{-\gamma}{(1+\gamma\vec{\mathbf{d}}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{c}})} = \frac{1}{\vec{q}\cdot\boldsymbol{B}\cdot\vec{q}}\frac{1}{\left(1+\frac{-1}{\vec{q}\cdot\boldsymbol{B}\cdot\vec{q}}\vec{q}\cdot\boldsymbol{B}\cdot\vec{q}+\frac{-(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}\right)} \\
&= \frac{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}
\end{aligned}$$

$$\theta_{(2)}\theta_{(3)} = \frac{-1}{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}\frac{(\vec{p}\cdot\vec{q}+\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})} = \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}$$

$$\begin{aligned}
\theta_{(2)}\theta_{(3)}(\vec{a}\cdot\vec{d}) &= \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}((\boldsymbol{B}^{-1}\cdot\vec{p})\cdot(\boldsymbol{B}\cdot\vec{q})) \\
&= \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q}) = \frac{-1}{(\vec{p}\cdot\vec{q})}
\end{aligned}$$

$$\theta_{(2)}\theta_{(3)}(\vec{b}\cdot\vec{c}) = \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}(\vec{p}\cdot\vec{q}) = \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})}$$

$$\theta_{(2)}\theta_{(3)}(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d}) = \frac{-1}{(\vec{p}\cdot(\boldsymbol{B}^{-T}\cdot\boldsymbol{B})\cdot\vec{q})(\vec{p}\cdot\vec{q})}((\boldsymbol{B}^{-1}\cdot\vec{p})\cdot(\boldsymbol{B}\cdot\vec{q}))(\vec{p}\cdot\vec{q}) = -1$$

The equation in (1.90) becomes:

$$\begin{aligned}
\boldsymbol{C}^{-1} &= \mathbf{1}+\theta_{(2)}(\vec{a}\otimes\vec{b})+\theta_{(3)}\left[\vec{c}+\theta_{(2)}(\vec{b}\cdot\vec{c})\vec{a}\right]\otimes\left[\vec{d}+\theta_{(2)}(\vec{a}\cdot\vec{d})\vec{b}\right] \\
\boldsymbol{C}^{-1} &= \mathbf{1}+\theta_{(2)}\left[\vec{a}\otimes\vec{b}\right]+\theta_{(3)}\left[\vec{c}\otimes\vec{d}\right]+\theta_{(3)}\theta_{(2)}(\vec{a}\cdot\vec{d})\left[\vec{c}\otimes\vec{b}\right]+\theta_{(3)}\theta_{(2)}(\vec{b}\cdot\vec{c})\left[\vec{a}\otimes\vec{d}\right] \\
&\quad +\theta_{(3)}\theta_{(2)}^2(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\left[\vec{a}\otimes\vec{b}\right] \\
\boldsymbol{C}^{-1} &= \mathbf{1}+\left\{\theta_{(2)}+\theta_{(3)}\theta_{(2)}^2(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\right\}\left[\vec{a}\otimes\vec{b}\right]+\theta_{(3)}\left[\vec{c}\otimes\vec{d}\right]+\theta_{(3)}\theta_{(2)}(\vec{a}\cdot\vec{d})\left[\vec{c}\otimes\vec{b}\right] \\
&\quad +\theta_{(3)}\theta_{(2)}(\vec{b}\cdot\vec{c})\left[\vec{a}\otimes\vec{d}\right]
\end{aligned}$$

Note that: $\left\{\theta_{(2)}+\theta_{(3)}\theta_{(2)}^2(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\right\} = \theta_{(2)}\left\{1+\theta_{(3)}\theta_{(2)}(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\right\} = \theta_{(2)}\{1-1\} = 0$, thus

$$\boldsymbol{C}^{-1} = \mathbf{1}+\theta_{(3)}\left[\vec{c}\otimes\vec{d}\right]+\theta_{(3)}\theta_{(2)}(\vec{a}\cdot\vec{d})\left[\vec{c}\otimes\vec{b}\right]+\theta_{(3)}\theta_{(2)}(\vec{b}\cdot\vec{c})\left[\vec{a}\otimes\vec{d}\right]$$

$$\boldsymbol{C}^{-1} = \mathbf{1} + \theta_{(3)}\left[\vec{c} \otimes \vec{d}\right] + \theta_{(3)}\theta_{(2)}(\vec{a} \cdot \vec{d})\left[\vec{c} \otimes \vec{b}\right] + \theta_{(3)}\theta_{(2)}(\vec{b} \cdot \vec{c})\left[\vec{a} \otimes \vec{d}\right]$$
$$= \mathbf{1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})\,(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes (\boldsymbol{B} \cdot \vec{q})\right] + \frac{-1}{(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes \vec{p}\right]$$
$$+ \frac{-1}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})}\left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes (\boldsymbol{B} \cdot \vec{q})\right]$$

With that, we can obtain:

$$\mathbf{D}^{-1} = \boldsymbol{C}^{-1} \cdot \boldsymbol{B}^{-1}$$
$$= \left\{\mathbf{1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})\,(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes (\boldsymbol{B} \cdot \vec{q})\right] + \frac{-1}{(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes \vec{p}\right]\right.$$
$$\left. + \frac{-1}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})}\left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes (\boldsymbol{B} \cdot \vec{q})\right]\right\} \cdot \boldsymbol{B}^{-1}$$
$$= \boldsymbol{B}^{-1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})\,(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes (\boldsymbol{B} \cdot \vec{q})\right] \cdot \boldsymbol{B}^{-1} + \frac{-1}{(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes \vec{p}\right] \cdot \boldsymbol{B}^{-1}$$
$$+ \frac{-1}{(\vec{p} \cdot (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q})}\left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes (\boldsymbol{B} \cdot \vec{q})\right] \cdot \boldsymbol{B}^{-1}$$

Note that:

$$\left\{\left[\vec{q} \otimes (\boldsymbol{B} \cdot \vec{q})\right] \cdot \boldsymbol{B}^{-1}\right\}_{ij} = \left[\vec{q} \otimes (\boldsymbol{B} \cdot \vec{q})\right]_{ik} B^{-1}_{kj} = \left[q_i (\boldsymbol{B} \cdot \vec{q})_k\right] B^{-1}_{kj} = \left[q_i (B_{kp} q_p)\right] B^{-1}_{kj} = q_i (B_{kp} B^{-1}_{kj} q_p)$$
$$= \left[\vec{q} \otimes (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q}\right]_{ij}$$

$$\left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes (\boldsymbol{B} \cdot \vec{q})\right] \cdot \boldsymbol{B}^{-1} = (\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes (\boldsymbol{B}^{-T} \cdot \boldsymbol{B}) \cdot \vec{q}$$

Now, if we consider the symmetry of $\boldsymbol{B}$, i.e. $\boldsymbol{B} = \boldsymbol{B}^T$, we obtain:

$$\mathbf{D}^{-1} = \boldsymbol{B}^{-1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot \vec{q})\,(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes \vec{q}\right] + \frac{-1}{(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes (\vec{p} \cdot \boldsymbol{B}^{-1})\right] + \frac{-1}{(\vec{p} \cdot \vec{q})}\left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes \vec{q}\right]$$
$$= \boldsymbol{B}^{-1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot \vec{q})\,(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes \vec{q}\right] + \frac{-1}{(\vec{p} \cdot \vec{q})}\left\{\left[\vec{q} \otimes (\vec{p} \cdot \boldsymbol{B}^{-1})\right] + \left[(\boldsymbol{B}^{-1} \cdot \vec{p}) \otimes \vec{q}\right]\right\}$$
$$= \boldsymbol{B}^{-1} + \frac{(\vec{p} \cdot \vec{q} + \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p})}{(\vec{p} \cdot \vec{q})^2}\left[\vec{q} \otimes \vec{q}\right] - \frac{2}{(\vec{p} \cdot \vec{q})}\left[\vec{q} \otimes (\boldsymbol{B}^{-1} \cdot \vec{p})\right]^{sym}$$

Note that, due to the symmetry of $\boldsymbol{B}$, it holds that $\vec{p} \cdot \boldsymbol{B}^{-1} = \boldsymbol{B}^{-1} \cdot \vec{p} = \vec{s}$, and $\boldsymbol{B}^{-T} \cdot \boldsymbol{B} = \mathbf{1}$.

b.2) A tensor is non-singular if $\det(\mathbf{D}) \neq 0$. By using the equation obtained previously we get:

$$\mathbf{D} = \boldsymbol{B} \cdot \left[\mathbf{1} + \beta \vec{a} \otimes \vec{b} + \gamma \vec{c} \otimes \vec{d}\right]$$
$$\Rightarrow \quad \det(\mathbf{D}) = \det\left(\boldsymbol{B} \cdot \left[\mathbf{1} + \beta \vec{a} \otimes \vec{b} + \gamma \vec{c} \otimes \vec{d}\right]\right) = \det(\boldsymbol{B}) \det\left[\mathbf{1} + \beta \vec{a} \otimes \vec{b} + \gamma \vec{c} \otimes \vec{d}\right]$$

Note that $\det(\boldsymbol{B}) > 0$, since $\boldsymbol{B}$ is a positive definite tensor. Then, the condition under which $\mathbf{D}$ is a non-singular tensor is $\det[\mathbf{1} + \beta \vec{a} \otimes \vec{b} + \gamma \vec{c} \otimes \vec{d}] \neq 0$. By using the determinant expression obtained in **Problem 1.86** we can obtain:

$$\det(\alpha \mathbf{1} + \beta\, \vec{a} \otimes \vec{b} + \gamma\, \vec{c} \otimes \vec{d}) = \alpha^3 + \alpha^2 \gamma (\vec{c} \cdot \vec{d}) - \alpha\beta\gamma\left[(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d})\right]$$

where $\alpha = 1$, $\vec{a} \cdot \vec{b} = (\boldsymbol{B}^{-1} \cdot \vec{p}) \cdot \vec{p} = \vec{p} \cdot \boldsymbol{B}^{-1} \cdot \vec{p}$, $\vec{a} \cdot \vec{d} = (\boldsymbol{B}^{-1} \cdot \vec{p}) \cdot (\boldsymbol{B} \cdot \vec{q}) = \vec{p} \cdot \vec{q}$, $\vec{b} \cdot \vec{c} = \vec{p} \cdot \vec{q}$

$\vec{c} \cdot \vec{d} = \vec{q} \cdot (\boldsymbol{B} \cdot \vec{q}) = \vec{q} \cdot \boldsymbol{B} \cdot \vec{q}$, $\gamma(\vec{c} \cdot \vec{d}) = \dfrac{-1}{\vec{q} \cdot \boldsymbol{B} \cdot \vec{q}}\vec{q} \cdot \boldsymbol{B} \cdot \vec{q} = -1$,

$$\beta\gamma\left[(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})-(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})\right]=\frac{1}{\vec{p}\cdot\vec{q}}\frac{-1}{\vec{q}\cdot\boldsymbol{B}\cdot\vec{q}}\left[(\vec{p}\cdot\vec{q})(\vec{p}\cdot\vec{q})-(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})\right]$$

Thus:

$$\det\left[\mathbf{1}+\beta\vec{a}\otimes\vec{b}+\gamma\vec{c}\otimes\vec{d}\right]=\frac{1}{(\vec{p}\cdot\vec{q})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})}\left[(\vec{p}\cdot\vec{q})(\vec{p}\cdot\vec{q})-(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})\right]\neq 0$$

Then, the conditions are: $\vec{p}\neq\vec{0}$, $\vec{q}\neq\vec{0}$, $(\vec{p}\cdot\vec{q})\neq 0$, i.e. $\vec{p}$ and $\vec{q}$ can not be orthogonal vectors. Another condition that must be met is:

$$\underbrace{(\vec{p}\cdot\vec{q})(\vec{p}\cdot\vec{q})}_{>0}-\underbrace{(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})}_{>0}\neq 0$$

Note that by the fact that $\boldsymbol{B}$ is positive definite tensor, the scalar $(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})>0$ is always positive for any vector $\vec{q}\neq\vec{0}$. The same apply to $(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})>0$, since, if the tensor is positive definite so is its inverse. Note also that $\mathbf{D}$ is a positive definite tensor if $(\vec{p}\cdot\vec{q})^2>(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})$ and $(\vec{p}\cdot\vec{q})>0$. These two conditions can be replaced by $(\vec{p}\cdot\vec{q})>\sqrt{(\vec{p}\cdot\boldsymbol{B}^{-1}\cdot\vec{p})(\vec{q}\cdot\boldsymbol{B}\cdot\vec{q})}$.

Problem 1.88

Let $\mathbf{A}=\mathbf{A}(\tau)$ and τ be a second-order tensor and a scalar respectively, show that:

$$\boxed{\frac{d|\mathbf{A}|}{d\tau}=|\mathbf{A}|\,\mathrm{Tr}\left(\frac{d\mathbf{A}}{d\tau}\cdot\mathbf{A}^{-1}\right)} \tag{1.91}$$

Solution:

In **Problem 1.82** and in **Problem 1.79**, we have demonstrated, respectively, that:

$$I\!I\!I_{\mathbf{A}}=\det(\mathbf{A})=|\mathbf{A}|=\frac{1}{6}\left\{[\mathrm{Tr}(\mathbf{A})]^3+2\mathrm{Tr}(\mathbf{A}^3)-3\mathrm{Tr}(\mathbf{A}^2)\mathrm{Tr}(\mathbf{A})\right\} \tag{1.92}$$

$$I\!I\!I_{\mathbf{A}}\mathbf{A}^{-1}=\mathbf{A}^2-\mathbf{A}I_{\mathbf{A}}+I\!I_{\mathbf{A}}\mathbf{1} \tag{1.93}$$

where $I_{\mathbf{A}}=\mathrm{Tr}(\mathbf{A})$, $I\!I_{\mathbf{A}}=\frac{1}{2}\left\{[\mathrm{Tr}(\mathbf{A})]^2-\mathrm{Tr}(\mathbf{A}^2)\right\}$.

Note also that the following derivatives are true:

$$\frac{d[I_{\mathbf{A}}]}{d\tau}=\frac{d[\mathrm{Tr}(\mathbf{A})]}{d\tau}=\frac{d[\mathsf{A}_{kk}]}{d\tau}=\frac{d[\mathsf{A}_{ik}\delta_{ik}]}{d\tau}=\frac{d[\mathsf{A}_{ik}]}{d\tau}\delta_{ik}=\frac{d\mathbf{A}}{d\tau}:\mathbf{1}=\mathrm{Tr}\left[\frac{d\mathbf{A}}{d\tau}\right]$$

$$\frac{d[\mathrm{Tr}(\mathbf{A}^2)]}{d\tau}=\mathrm{Tr}\left[\frac{d(\mathbf{A}^2)}{d\tau}\right]=\mathrm{Tr}\left[2\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right]=2\mathrm{Tr}\left[\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right]$$

$$\frac{d[\mathrm{Tr}(\mathbf{A}^3)]}{d\tau}=3\mathrm{Tr}\left[\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau}\right]$$

Taking the derivative of (1.92) with respect to τ we can obtain:

$$\begin{aligned}
\frac{d(III_{\mathbf{A}})}{d\tau} &= \frac{1}{6}\frac{d}{d\tau}\left\{[\operatorname{Tr}(\mathbf{A})]^3 + 2\operatorname{Tr}(\mathbf{A}^3) - 3\operatorname{Tr}(\mathbf{A}^2)\operatorname{Tr}(\mathbf{A})\right\} \\
&= \frac{1}{6}\left\{3[\operatorname{Tr}(\mathbf{A})]^2 \frac{d[\operatorname{Tr}(\mathbf{A})]}{dt} + 2\frac{d[\operatorname{Tr}(\mathbf{A}^3)]}{d\tau} - 3\frac{d[\operatorname{Tr}(\mathbf{A}^2)]}{d\tau}\operatorname{Tr}(\mathbf{A}) - 3\operatorname{Tr}(\mathbf{A}^2)\frac{d[\operatorname{Tr}(\mathbf{A})]}{d\tau}\right\} \\
&= \frac{1}{6}\left\{3[\operatorname{Tr}(\mathbf{A})]^2 \operatorname{Tr}\left[\frac{d\mathbf{A}}{d\tau}\right] + 6\operatorname{Tr}\left[\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau}\right] - 6\operatorname{Tr}\left[\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right]\operatorname{Tr}(\mathbf{A}) - 3\operatorname{Tr}(\mathbf{A}^2)\operatorname{Tr}\left[\frac{d\mathbf{A}}{d\tau}\right]\right\} \\
&= \operatorname{Tr}\left[\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau}\right] - \operatorname{Tr}\left[\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right]\operatorname{Tr}(\mathbf{A}) + \frac{1}{2}\left\{[\operatorname{Tr}(\mathbf{A})]^2 - \operatorname{Tr}(\mathbf{A}^2)\right\}\operatorname{Tr}\left[\frac{d\mathbf{A}}{d\tau}\right]
\end{aligned}$$

or

$$\frac{d(III_{\mathbf{A}})}{d\tau} = \operatorname{Tr}\left[\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau}\right] - \operatorname{Tr}\left[\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right]I_{\mathbf{A}} + II_{\mathbf{A}}\operatorname{Tr}\left[\frac{d\mathbf{A}}{d\tau}\right] \tag{1.94}$$

Taking the scalar product of the equation in (1.93) with $\frac{d\mathbf{A}}{d\tau}$, we can obtain:

$$III_{\mathbf{A}}\mathbf{A}^{-1}\cdot\frac{d\mathbf{A}}{d\tau} = \left(\mathbf{A}^2 - \mathbf{A}I_{\mathbf{A}} + II_{\mathbf{A}}\mathbf{1}\right)\cdot\frac{d\mathbf{A}}{d\tau} = \mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau} - \mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}I_{\mathbf{A}} + II_{\mathbf{A}}\frac{d\mathbf{A}}{d\tau}$$

and the trace of the above equation is given by:

$$\begin{aligned}
\operatorname{Tr}\left(\mathbf{A}^{-1}\cdot\frac{d\mathbf{A}}{d\tau}\right)III_{\mathbf{A}} &= \operatorname{Tr}\left(\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau} - \mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}I_{\mathbf{A}} + II_{\mathbf{A}}\frac{d\mathbf{A}}{d\tau}\right) \\
&= \operatorname{Tr}\left(\mathbf{A}^2\cdot\frac{d\mathbf{A}}{d\tau}\right) - \operatorname{Tr}\left(\mathbf{A}\cdot\frac{d\mathbf{A}}{d\tau}\right)I_{\mathbf{A}} + \operatorname{Tr}\left(\frac{d\mathbf{A}}{d\tau}\right)II_{\mathbf{A}}
\end{aligned} \tag{1.95}$$

Comparing equations (1.94) and (1.95), we can conclude that:

$$\frac{d(III_{\mathbf{A}})}{d\tau} = III_{\mathbf{A}}\operatorname{Tr}\left(\mathbf{A}^{-1}\cdot\frac{d\mathbf{A}}{d\tau}\right) = III_{\mathbf{A}}\operatorname{Tr}\left(\frac{d\mathbf{A}}{d\tau}\cdot\mathbf{A}^{-1}\right)$$

1.11 Isotropic and Anisotropic Tensors

Problem 1.89

Let $\mathbb{C}$ be a fourth-order tensor, whose components are

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \tag{1.96}$$

where δ_{ij} are the second-order unit tensor components, and λ, μ and γ are scalar.

a) What kind of symmetry has the tensor $\mathbb{C}$? b) What conditions must be met to guarantee the symmetry of $\mathbb{C}$?

Solution:

The tensor has major symmetry whether $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$ holds. Taking into account the equation in (1.96), we can conclude that the tensor has major symmetry since

$$\mathbb{C}_{klij} = \lambda\delta_{kl}\delta_{ij} + \mu\delta_{ki}\delta_{lj} + \gamma\delta_{kj}\delta_{li} = \mathbb{C}_{ijkl}$$

We check now if the tensor has minor symmetry, e.g. $\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$

$$\mathbb{C}_{ijlk} = \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl} \neq \mathbb{C}_{ijkl}$$

We can easily verify this by the fact that when $i=2$, $j=1$, $k=1$, $l=2$, we have

$\mathbb{C}_{ijkl} = \mathbb{C}_{2112} = \lambda\delta_{21}\delta_{12} + \mu\delta_{21}\delta_{12} + \gamma\delta_{22}\delta_{11} = \gamma$

$\mathbb{C}_{ijlk} = \mathbb{C}_{2121} = \lambda\delta_{21}\delta_{21} + \mu\delta_{22}\delta_{11} + \gamma\delta_{21}\delta_{12} = \mu$

Then, the tensor $\mathbb{C}$ has minor symmetry if and only if $\mu = \gamma$, with that we can obtain:

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Note that $\delta_{ij}\delta_{kl}$ has major and minor symmetry, while the tensors $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$ are not symmetric. Note also that $(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = 2\mathbb{I}^{sym}_{ijkl}$.

Problem 1.90

Let $\mathbb{C}$ be a fourth-order tensor, whose components are given by:

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.97}$$

where λ, μ are constant real numbers. Show that $\mathbb{C}$ is isotropic.

Solution:

Applying the transformation law for the fourth-order tensor components:

$$\mathbb{C}'_{ijkl} = a_{im}a_{jn}a_{kp}a_{lq}\mathbb{C}_{mnpq} \tag{1.98}$$

and by replacing the relation $\mathbb{C}_{mnpq} = \lambda\delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})$ into the above equation, we obtain:

$$\begin{aligned}
\mathbb{C}'_{ijkl} &= a_{im}a_{jn}a_{kp}a_{lq}\left[\lambda\delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})\right] \\
&= \lambda a_{im}a_{jn}a_{kp}a_{lq}\delta_{mn}\delta_{pq} + \mu(a_{im}a_{jn}a_{kp}a_{lq}\delta_{mp}\delta_{nq} + a_{im}a_{jn}a_{kp}a_{lq}\delta_{mq}\delta_{np}) \\
&= \lambda a_{in}a_{jn}a_{kq}a_{lq} + \mu(a_{ip}a_{jq}a_{kp}a_{lq} + a_{iq}a_{jn}a_{kn}a_{lq}) \\
&= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \mathbb{C}_{ijkl}
\end{aligned} \tag{1.99}$$

which proves that $\mathbb{C}$ is an isotropic tensor, i.e. the $\mathbb{C}$-components do not change for any transformation basis.

Problem 1.91

Let $\mathbb{C}$ be a symmetric isotropic fourth-order tensor which is represented by its components as follows:

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \quad \text{(indicial notation)}$$

$$\mathbb{C} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I} \quad \text{(tensorial notation)}$$

where λ and μ are scalars, $\mathbf{1}$ is the second-order unit tensor, $\mathbf{I}$ is the symmetric fourth-order unit tensor, i.e. $\mathbf{I} \equiv \mathbb{I}^{sym}$.

a) Given a symmetric second-order tensor $\boldsymbol{\varepsilon}$, obtain an expression for $\boldsymbol{\sigma}$ knowing that $\boldsymbol{\sigma} = \mathbb{C}:\boldsymbol{\varepsilon}$. Express the result in indicial and tensorial notation.
b) Show that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have the same eigenvectors, i.e. the same principal directions.

c) If $\gamma_{\boldsymbol{\sigma}}$ are the eigenvalues (principal values) of $\boldsymbol{\sigma}$, obtain the eigenvalues of $\boldsymbol{\varepsilon}$ in terms of $\gamma_{\boldsymbol{\sigma}}$.

Solution:

a)

Tensorial notation	Indicial notation
$\begin{aligned}\boldsymbol{\sigma} &= \mathbb{C}:\boldsymbol{\varepsilon} \\ &= (\lambda \mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}):\boldsymbol{\varepsilon} \\ &= \lambda\mathbf{1}\otimes\underbrace{\mathbf{1}:\boldsymbol{\varepsilon}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + 2\mu\underbrace{\mathbf{I}:\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}^{sym}=\boldsymbol{\varepsilon}} \\ &= \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\end{aligned}$	$\begin{aligned}\sigma_{ij} &= \mathbb{C}_{ijkl}\varepsilon_{kl} \\ &= \left[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\varepsilon_{kl} \\ &= \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl}\varepsilon_{kl} + \delta_{il}\delta_{jk}\varepsilon_{kl}) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + \mu(\varepsilon_{ij} + \varepsilon_{ji}) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu(\varepsilon_{ij}^{sym}) \\ &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\end{aligned}$

where we have considered the symmetry of the tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$.

b) and c) Starting from the definition of eigenvalues ($\gamma_{\boldsymbol{\sigma}}$)-eigenvectors ($\hat{\mathbf{n}}$) of the tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}}$$

and by substituting the value of $\boldsymbol{\sigma}$ obtained previously we can obtain:

$$\begin{aligned}&(\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\,\boldsymbol{\varepsilon})\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \\ &\Rightarrow \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\cdot\hat{\mathbf{n}} + 2\mu\,\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \qquad \Rightarrow \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}} + 2\mu\,\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \\ &\Rightarrow 2\mu\,\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} - \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}} \qquad \Rightarrow 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = (\gamma_{\boldsymbol{\sigma}} - \lambda\mathrm{Tr}(\boldsymbol{\varepsilon}))\hat{\mathbf{n}} \\ &\Rightarrow \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \left(\frac{\gamma_{\boldsymbol{\sigma}} - \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\right)\hat{\mathbf{n}} \\ &\Rightarrow \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\varepsilon}}\hat{\mathbf{n}}\end{aligned}$$

Note that the last equation is the definition of eigenvalue-eigenvector of $\boldsymbol{\varepsilon}$. With that we conclude that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have the same eigenvectors (they are coaxial). And the eigenvalues of $\boldsymbol{\varepsilon}$ can be obtained as follows:

$$\gamma_{\boldsymbol{\varepsilon}} = \frac{\gamma_{\boldsymbol{\sigma}} - \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}$$

If we denote by $\gamma_{\boldsymbol{\varepsilon}}^{(1)} = \varepsilon_1$, $\gamma_{\boldsymbol{\varepsilon}}^{(2)} = \varepsilon_2$, $\gamma_{\boldsymbol{\varepsilon}}^{(3)} = \varepsilon_3$ and $\gamma_{\boldsymbol{\sigma}}^{(1)} = \sigma_1$, $\gamma_{\boldsymbol{\sigma}}^{(2)} = \sigma_2$, $\gamma_{\boldsymbol{\sigma}}^{(3)} = \sigma_3$. The explicit form of the above relationship is given by:

$$\begin{bmatrix}\varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3\end{bmatrix} = \frac{1}{2\mu}\begin{bmatrix}\sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3\end{bmatrix} - \frac{\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$$

where it is also true that $\begin{bmatrix}\sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3\end{bmatrix} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} + 2\mu\begin{bmatrix}\varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3\end{bmatrix}$

Problem 1.92

a) Obtain the inverse of the fourth-order tensor $\mathbb{C} = 2\mu\mathbf{I} + \lambda\mathbf{1}\otimes\mathbf{1}$ where $\mathbf{I} \equiv \mathbb{I}^{sym}$ is the symmetric fourth-order unit tensor, $\mathbf{1}$ is the second-order unit tensor, and $\mu > 0$ and λ are scalars. b) Obtain the determinant of $\mathbb{C}$. In addition, if we consider that $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \dfrac{E}{2(1+\nu)}$, find the possible values for E and ν in order to guarantee that the tensor $\mathbb{C}$ is positive definite. c) Obtain also the reciprocal of the equation $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$ in function of $\mu > 0$, λ, where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are symmetric second-order tensors.

Solution:

a) We use the equation obtained in (1.80):

$$(\alpha\mathbb{D} + \beta\,\mathbf{A}\otimes\mathbf{B})^{-1} = \frac{1}{\alpha}\mathbb{D}^{-1} - \frac{\beta}{\alpha(\alpha + \beta\,\mathbf{B}:\mathbb{D}^{-1}:\mathbf{A})}\left[(\mathbb{D}^{-1}:\mathbf{A})\otimes(\mathbf{B}:\mathbb{D}^{-1})\right]$$

By denoting by $\mathbb{D} = \mathbf{I}$, $\mathbf{A} = \mathbf{B} = \mathbf{1}$, $\alpha = 2\mu$, $\beta = \lambda$, the above equation can be rewritten as follows:

$$\mathbb{C}^{-1} = (2\mu\mathbf{I} + \lambda\mathbf{1}\otimes\mathbf{1})^{-1} = \frac{1}{2\mu}\mathbf{I}^{-1} - \frac{\lambda}{2\mu(2\mu + \lambda\,\mathbf{1}:\mathbf{I}^{-1}:\mathbf{1})}\left[(\mathbf{I}^{-1}:\mathbf{1})\otimes(\mathbf{1}:\mathbf{I}^{-1})\right]$$

Remember that it holds that $\mathbf{I}^{-1} = \mathbf{I}$ and $(\mathbf{I}^{-1}:\mathbf{1}) = \mathbf{I}:\mathbf{1} = \mathbf{1}$. Then we can obtain the scalar value of $\mathbf{1}:\mathbf{I}^{-1}:\mathbf{1} = \mathbf{1}:\mathbf{I}:\mathbf{1} = \mathbf{1}:\mathbf{1} = \mathrm{Tr}(\mathbf{1}) = 3$. We also express in indicial notation:

$$\begin{aligned}\mathbf{1}:\mathbf{I}^{-1}:\mathbf{1} = \mathbf{1}:\mathbf{I}:\mathbf{1} = \delta_{ij}\mathbb{I}^{sym}_{ijkl}\delta_{kl} &= \delta_{ij}\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{kl} = \frac{1}{2}(\delta_{ij}\delta_{ik}\delta_{jl}\delta_{kl} + \delta_{ij}\delta_{il}\delta_{jk}\delta_{kl})\\ &= \frac{1}{2}(\delta_{jj} + \delta_{jj}) = 3\end{aligned}$$

Resulting that:

$$\mathbb{C}^{-1} = (2\mu\mathbf{I} + \lambda\mathbf{1}\otimes\mathbf{1})^{-1} = \frac{1}{2\mu}\mathbf{I} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}(\mathbf{1}\otimes\mathbf{1})$$

Let us check whether $\mathbb{C}:\mathbb{C}^{-1} = \mathbb{I}^{sym} \equiv \mathbf{I}$ holds or not:

$$\mathbb{C}:\mathbb{C}^{-1} = (2\mu\mathbf{I} + \lambda\mathbf{1}\otimes\mathbf{1}):\left(\frac{1}{2\mu}\mathbf{I} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}(\mathbf{1}\otimes\mathbf{1})\right)$$

$$\mathbb{C}:\mathbb{C}^{-1} = (\frac{2\mu}{2\mu}\mathbf{I}:\mathbf{I} - \frac{2\mu\lambda}{2\mu(2\mu + 3\lambda)}\mathbf{I}:(\mathbf{1}\otimes\mathbf{1}) + \frac{\lambda}{2\mu}(\mathbf{1}\otimes\mathbf{1}):\mathbf{I} - \frac{\lambda^2}{2\mu(2\mu + 3\lambda)}(\mathbf{1}\otimes\mathbf{1}):(\mathbf{1}\otimes\mathbf{1})$$

According to **Problem 1.27** it fulfills that $\mathbf{I}:\mathbf{I} = \mathbf{I}$, $\mathbf{I}:(\mathbf{1}\otimes\mathbf{1}) = (\mathbf{1}\otimes\mathbf{1}):\mathbf{I} = \mathbf{1}\otimes\mathbf{1}$, and $(\mathbf{1}\otimes\mathbf{1}):(\mathbf{1}\otimes\mathbf{1}) = 3(\mathbf{1}\otimes\mathbf{1})$. With that we can obtain:

$$\mathbb{C}:\mathbb{C}^{-1} = \mathbf{I} + \underbrace{\left(\frac{-2\mu\lambda}{2\mu(2\mu + 3\lambda)} + \frac{\lambda}{2\mu} - \frac{3\lambda^2}{2\mu(2\mu + 3\lambda)}\right)}_{=0}(\mathbf{1}\otimes\mathbf{1}) = \mathbf{I}$$

b) We can use directly the equation (1.32), (see **Problem 1.49**):

$$\det(\alpha\mathbb{I}^{sym} + \beta\mathbf{A}\otimes\mathbf{B}) = \alpha^3 + \alpha^2\beta\,\mathbf{A}:\mathbf{B}$$

and by denoting by $\alpha = 2\mu$, $\beta = \lambda$, $\mathbf{A} = \mathbf{B} = \mathbf{1}$ we can conclude that:

$$\det(2\mu\mathbf{I}+\lambda\mathbf{1}\otimes\mathbf{1})=(2\mu)^3+(2\mu)^2\lambda\,\mathbf{1}:\mathbf{1}=(2\mu)^3+(2\mu)^2\lambda\,3=(2\mu)^2(2\mu+3\lambda)$$

The tensor $\mathbb{C}$ is definite positive if the eigenvalues are positive numbers, i.e.:

$$\mu>0\Rightarrow\mu=\frac{E}{2(1+\nu)}>0$$

$$2\mu+3\lambda>0\Rightarrow 2\frac{E}{2(1+\nu)}+3\frac{E\nu}{(1+\nu)(1-2\nu)}=\frac{E}{(1-2\nu)}>0$$

Denoting by $y_1=(1+\nu)\neq 0$, $y_2=(1-2\nu)\neq 0$, we can conclude that:

$$\mu=\frac{E}{2(1+\nu)}=\frac{E}{2y_1}>0 \quad\Rightarrow\quad \begin{cases}\begin{cases}E>0\\ y_1>0\end{cases}\\ \begin{cases}E<0\\ y_1<0\end{cases}\end{cases} \quad ; \quad 2\mu+3\lambda=\frac{E}{(1-2\nu)}=\frac{E}{y_2}>0 \quad\Rightarrow\quad \begin{cases}\begin{cases}E>0\\ y_2>0\end{cases}\\ \begin{cases}E<0\\ y_2<0\end{cases}\end{cases}$$

The above conditions must fulfill simultaneously. Then, by means of Figure 1.23 we can conclude that $E>0$ and $-1<\nu<0.5$.

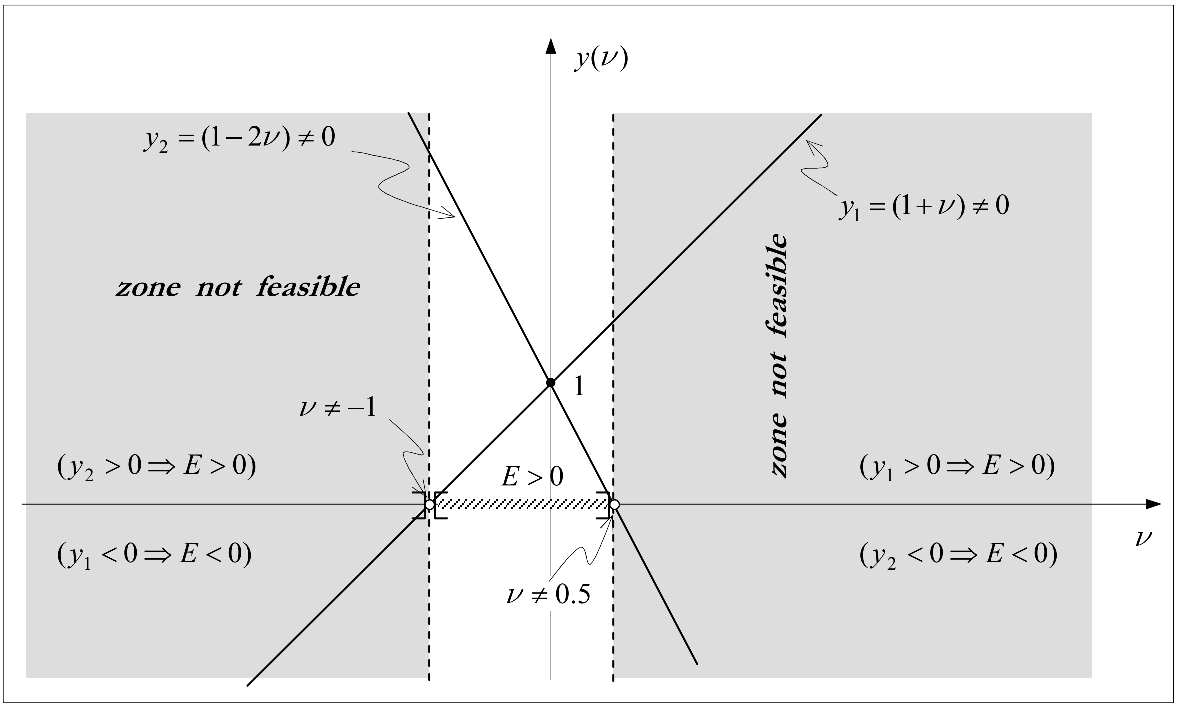


Figure 1.23

c)

$$\boldsymbol{\sigma}=\mathbb{C}:\boldsymbol{\varepsilon} \quad\Rightarrow\quad \mathbb{C}^{-1}:\boldsymbol{\sigma}=\mathbb{C}^{-1}:\mathbb{C}:\boldsymbol{\varepsilon} \quad\Rightarrow\quad \mathbb{C}^{-1}:\boldsymbol{\sigma}=\mathbb{I}^{sym}:\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{sym}=\boldsymbol{\varepsilon}$$

$$\Rightarrow\boldsymbol{\varepsilon}=\mathbb{C}^{-1}:\boldsymbol{\sigma}$$

$$\Rightarrow\boldsymbol{\varepsilon}=\left[\frac{1}{2\mu}\mathbf{I}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathbf{1}\otimes\mathbf{1}\right]:\boldsymbol{\sigma}=\frac{1}{2\mu}\mathbf{I}:\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathbf{1}\otimes\mathbf{1}:\boldsymbol{\sigma}$$

$$\Rightarrow\boldsymbol{\varepsilon}=\frac{1}{2\mu}\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

The transformation between the two hyperspaces can be appreciated in Figure 1.24. It is also worth reviewing **Problem 1.41**.

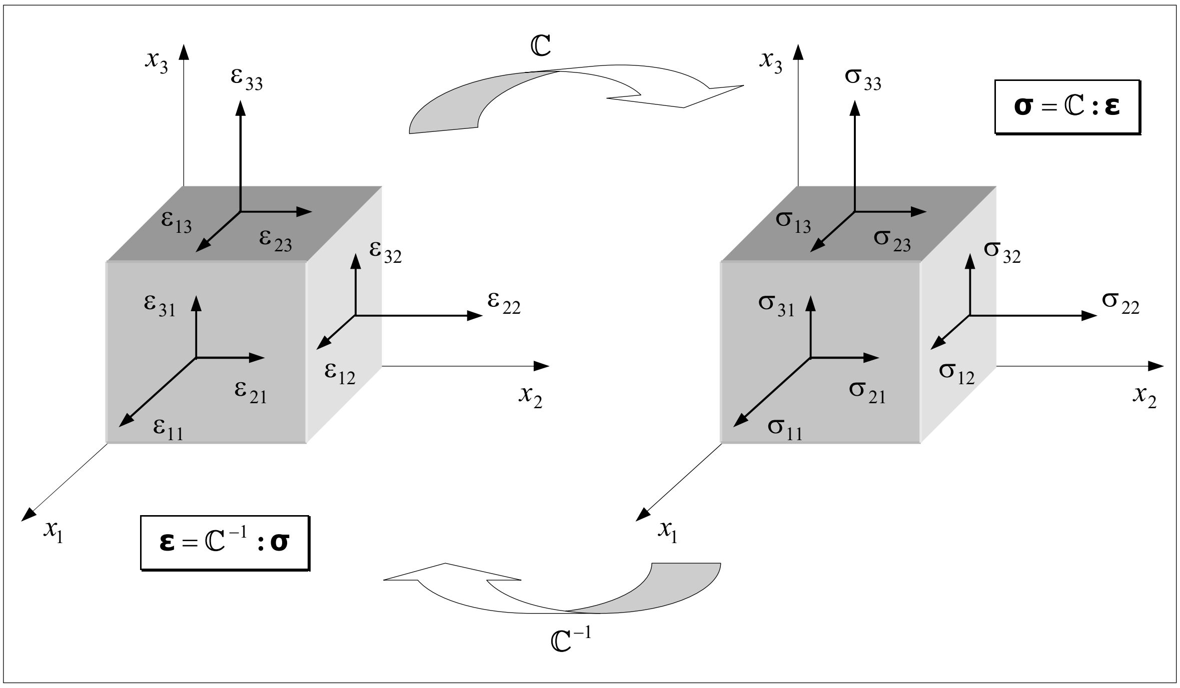


Figure 1.24

Problem 1.93

Let $\boldsymbol{Q}^e(\hat{\mathbf{N}})$ be a second-order tensor, which is known as the elastic acoustic tensor, and is defined as follows:

$$\boldsymbol{Q}^e(\hat{\mathbf{N}}) = \hat{\mathbf{N}} \cdot \mathbb{C}^e \cdot \hat{\mathbf{N}}$$

where $\hat{\mathbf{N}}$ is the unit vector and $\mathbb{C}^e$ is the isotropic symmetric fourth-order tensor and given by $\mathbb{C}^e = \lambda(\mathbf{1} \otimes \mathbf{1}) + 2\mu\mathbf{I}$, whose components are: $\mathbb{C}^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Obtain the components of the elastic acoustic tensor.

Solution:

Using symbolic notation we can obtain:

$$\begin{aligned}
\boldsymbol{Q}^e(\hat{\mathbf{N}}) &= \hat{\mathbf{N}} \cdot \mathbb{C}^e \cdot \hat{\mathbf{N}} = \left(\hat{\mathrm{N}}_i \hat{\mathbf{e}}_i\right) \cdot \left(\mathbb{C}^e_{pqrs} \hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_s\right) \cdot \left(\hat{\mathrm{N}}_j \hat{\mathbf{e}}_j\right) \\
&= \hat{\mathrm{N}}_i \mathbb{C}^e_{pqrs} \hat{\mathrm{N}}_j \delta_{ip}\delta_{sj} \left(\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_r\right) = \hat{\mathrm{N}}_p \mathbb{C}^e_{pqrs} \hat{\mathrm{N}}_s \left(\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_r\right)
\end{aligned}$$

Then, the components of $\boldsymbol{Q}^e(\hat{\mathbf{N}})$ are:

$$\begin{aligned}
Q^e{}_{qr} &= \hat{\mathrm{N}}_p \mathbb{C}^e_{pqrs} \hat{\mathrm{N}}_s = \hat{\mathrm{N}}_p \left[\lambda\delta_{pq}\delta_{rs} + \mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})\right] \hat{\mathrm{N}}_s \\
&= \lambda\delta_{pq}\delta_{rs}\hat{\mathrm{N}}_p\hat{\mathrm{N}}_s + \mu(\hat{\mathrm{N}}_p\delta_{pr}\delta_{qs}\hat{\mathrm{N}}_s + \hat{\mathrm{N}}_p\delta_{ps}\delta_{qr}\hat{\mathrm{N}}_s) = \lambda\hat{\mathrm{N}}_q\hat{\mathrm{N}}_r + \mu(\hat{\mathrm{N}}_r\hat{\mathrm{N}}_q + \hat{\mathrm{N}}_s\delta_{qr}\hat{\mathrm{N}}_s)
\end{aligned}$$

Note that $\hat{\mathbf{N}}$ is the unit vector, then $\hat{\mathrm{N}}_s\hat{\mathrm{N}}_s = 1$ holds. With that we can obtain:

$$Q^e{}_{qr} = \mu\delta_{qr} + (\lambda+\mu)\hat{\mathrm{N}}_q\hat{\mathrm{N}}_r \xrightarrow{\text{tensorial}} \boldsymbol{Q}^e(\hat{\mathbf{N}}) = \mu\mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}$$

Problem 1.94

Let $\boldsymbol{Q}$ be a symmetric second-order tensor and given by:

$$\boldsymbol{Q}(\hat{\mathbf{N}}) = \mu\mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}$$

where λ and μ are scalars, and $\hat{\mathbf{N}}$ is the unit vector.

a) Obtain the eigenvalues of $\boldsymbol{Q}(\hat{\mathbf{N}})$ and determine the restrictions on λ and μ in order to guarantee the existence of $\boldsymbol{Q}^{-1}(\hat{\mathbf{N}})$, i.e. $\exists \boldsymbol{Q}^{-1}$.

b) Taking into account that $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \dfrac{E}{2(1+\nu)}$, determine the possible values of (E,ν) with which $\boldsymbol{Q}(\hat{\mathbf{N}})$ is a positive definite tensor.

c) Obtain the inverse of $\boldsymbol{Q}(\hat{\mathbf{N}})$.

Solution:

a) It was shown in **Problem 1.49** that, given the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ it holds that:

$$\det(\beta \mathbf{1} + \alpha \vec{\mathbf{a}} \otimes \vec{\mathbf{b}}) = \beta^3 + \beta^2 \alpha \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$$

The eigenvalues can be determined by means of the characteristic determinant $\det(\boldsymbol{Q} - \gamma \mathbf{1}) = 0$, where γ_i are the eigenvalues of $\boldsymbol{Q}$. Then:

$$\det[\mu \mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}} \otimes \hat{\mathbf{N}} - \gamma \mathbf{1}] = 0 \quad \Rightarrow \quad \det[(\mu-\gamma)\mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}} \otimes \hat{\mathbf{N}}] = 0$$

Denoting by $\beta = (\mu - \gamma)$ and $\alpha = (\lambda + \mu)$ we conclude that:

$$\det[(\mu-\gamma)\mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}} \otimes \hat{\mathbf{N}}] = 0 \quad \Rightarrow \quad (\mu-\gamma)^3 + (\mu-\gamma)^2(\lambda+\mu)\underbrace{\hat{\mathbf{N}} \cdot \hat{\mathbf{N}}}_{=1} = 0$$

$$(\mu-\gamma)^2\left[(\mu-\gamma) + (\lambda+\mu)\right] = 0 \quad \Rightarrow \quad (\mu-\gamma)^2\left[(\lambda+2\mu) - \gamma\right] = 0$$

The above characteristic equation has the following solutions:

$$(\mu-\gamma)^2\left[(\lambda+2\mu)-\gamma\right] = 0 \quad \xrightarrow{solution} \quad \begin{cases} (\mu-\gamma)^2 = 0 \quad \Rightarrow \quad \begin{cases} \gamma_1 = \mu \\ \gamma_2 = \mu \end{cases} \\ [(\lambda+2\mu)-\gamma] = 0 \quad \Rightarrow \quad \gamma_3 = (\lambda+2\mu) \end{cases}$$

In the principal space of $\boldsymbol{Q}$, the components of $\boldsymbol{Q}$ are:

$$Q'_{ij} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+2\mu) \end{bmatrix}$$

The inverse of $\boldsymbol{Q}$ exits if the determinant of $\boldsymbol{Q}$ is non-zero:

$$|\boldsymbol{Q}| = \mu^2(\lambda+2\mu) \neq 0 \quad \Rightarrow \quad \begin{cases} \mu \neq 0 \\ \lambda + 2\mu \neq 0 \quad \Rightarrow \quad \lambda \neq -2\mu \end{cases}$$

b) A tensor is definite positive if its eigenvalues are greater than zero, then:

$$\begin{cases} \mu = \dfrac{E}{2(1+\nu)} > 0 \\ \lambda + 2\mu = \dfrac{E\nu}{(1+\nu)(1-2\nu)} + 2\dfrac{E}{2(1+\nu)} = \dfrac{E(1-\nu)}{(-2\nu^2 - \nu + 1)} > 0 \end{cases}$$

We check that $\begin{cases} (1+\nu) \neq 0 \quad \Rightarrow \quad \nu \neq -1 \\ (-2\nu^2 - \nu + 1) \neq 0 \quad \Rightarrow \quad \begin{cases} \nu \neq -1 \\ \nu \neq 0.5 \end{cases} \end{cases}$

Denoting by $y_1=(1+\nu)\neq 0$, $y_2=(1-\nu)\neq 0$, $y_3=(-2\nu^2-\nu+1)\neq 0$, (see Figure 1.25), we can rewrite the restrictions as follows:

$$\begin{cases} \mu=\dfrac{E}{2y_1}>0 \quad \Rightarrow \begin{cases}\begin{cases}E>0\\ y_1>0\end{cases}\\ \begin{cases}E<0\\ y_1<0\end{cases}\end{cases} \\ \lambda+2\mu=\dfrac{Ey_2}{y_3}>0 \quad \Rightarrow \begin{cases} E>0\Rightarrow\begin{cases}y_2,y_3>0\\ y_2,y_3<0\end{cases}\\ E<0\Rightarrow\begin{cases}y_2>0,y_3<0\\ y_2<0,y_3>0\end{cases}\end{cases}\end{cases}$$

with which we obtain:

$$\begin{cases} E>0 \quad \Rightarrow \quad \nu\subset\left]-1;0.5\right[\cup\left]1;\infty\right[\\ E<0 \quad \Rightarrow \quad \nu\subset\left]-\infty;-1\right[\end{cases}$$

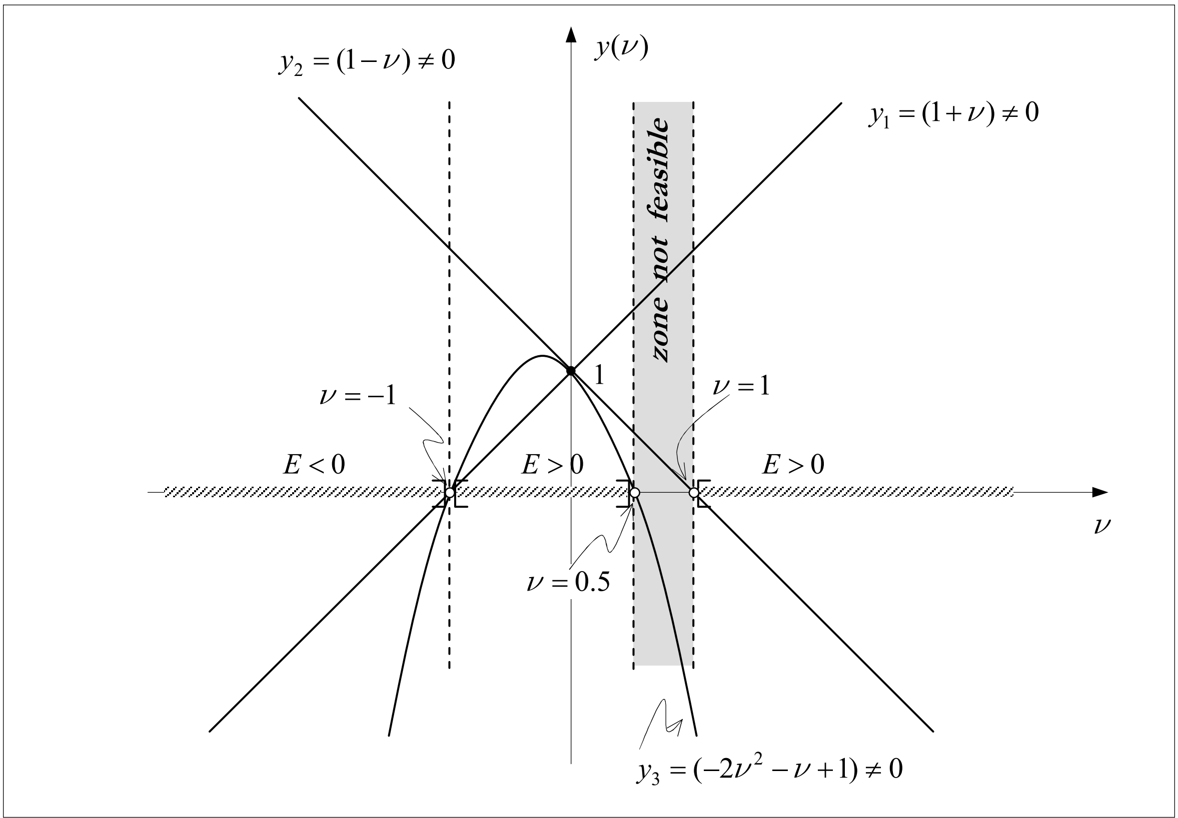


Figure 1.25

c) The inverse of the $\boldsymbol{Q}(\hat{\mathbf{N}})$-components in the principal space of $\boldsymbol{Q}(\hat{\mathbf{N}})$ are given by:

$$Q'_{ij}=\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & (\lambda+2\mu)\end{bmatrix} \xrightarrow{inverse} Q'^{-1}_{ij}=\begin{bmatrix}\dfrac{1}{\mu} & 0 & 0\\ 0 & \dfrac{1}{\mu} & 0\\ 0 & 0 & \dfrac{1}{(\lambda+2\mu)}\end{bmatrix} \quad \therefore \quad \left|\boldsymbol{Q}^{-1}\right|=\frac{1}{\mu^2(\lambda+2\mu)}$$

Then, the eigenvalues of $\boldsymbol{Q}(\hat{\mathbf{N}})^{-1}$ are $Q_1'^{-1} = Q_2'^{-1} = \dfrac{1}{\mu}$, $Q_3'^{-1} = \dfrac{1}{(\lambda + 2\mu)}$. Recall that a tensor and its inverse share the same principal space, i.e. they are coaxial tensors. Moreover, we can express the spectral representation of $\boldsymbol{Q}(\hat{\mathbf{N}})^{-1}$ as follows:

$$
\begin{aligned}
\boldsymbol{Q}^{-1} &= \sum_{a=1}^{3} Q_a^{-1} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = Q_1^{-1} \hat{\mathbf{N}}^{(1)} \otimes \hat{\mathbf{N}}^{(1)} + Q_2^{-1} \hat{\mathbf{N}}^{(2)} \otimes \hat{\mathbf{N}}^{(2)} + Q_3^{-1} \hat{\mathbf{N}}^{(3)} \otimes \hat{\mathbf{N}}^{(3)} \\
&= Q_1^{-1} (\hat{\mathbf{N}}^{(1)} \otimes \hat{\mathbf{N}}^{(1)} + \hat{\mathbf{N}}^{(2)} \otimes \hat{\mathbf{N}}^{(2)}) + Q_3^{-1} \hat{\mathbf{N}}^{(3)} \otimes \hat{\mathbf{N}}^{(3)} = Q_1^{-1} (\mathbf{1} - \hat{\mathbf{N}}^{(3)} \otimes \hat{\mathbf{N}}^{(3)}) + Q_3^{-1} \hat{\mathbf{N}}^{(3)} \otimes \hat{\mathbf{N}}^{(3)} \\
&= Q_1^{-1} (\mathbf{1} - \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}) + Q_3^{-1} \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}
\end{aligned}
$$

where we have considered that $\hat{\mathbf{N}}^{(3)} = \hat{\mathbf{N}}$. It is interesting to see **Problem 1.77**. Then:

$$
\begin{aligned}
\boldsymbol{Q}^{-1} &= Q_1^{-1} (\mathbf{1} - \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}) + Q_3^{-1} \hat{\mathbf{N}} \otimes \hat{\mathbf{N}} = \frac{1}{\mu} (\mathbf{1} - \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}) + \frac{1}{(\lambda + 2\mu)} \hat{\mathbf{N}} \otimes \hat{\mathbf{N}} \\
&= \frac{1}{\mu} \mathbf{1} - \frac{1}{\mu} \hat{\mathbf{N}} \otimes \hat{\mathbf{N}} + \frac{1}{(\lambda + 2\mu)} \hat{\mathbf{N}} \otimes \hat{\mathbf{N}} = \frac{1}{\mu} \mathbf{1} - \left(\frac{1}{\mu} - \frac{1}{(\lambda + 2\mu)} \right) \hat{\mathbf{N}} \otimes \hat{\mathbf{N}} \\
&= \frac{1}{\mu} \mathbf{1} - \left(\frac{\lambda + \mu}{\mu (\lambda + 2\mu)} \right) \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}
\end{aligned}
$$

Note that $\boldsymbol{Q}^{-1} = \left(\hat{\mathbf{N}} \cdot \mathbb{C}^e \cdot \hat{\mathbf{N}} \right)^{-1} \neq \hat{\mathbf{N}} \cdot \mathbb{C}^{e^{-1}} \cdot \hat{\mathbf{N}}$, where $\mathbb{C}^{e^{-1}} = \dfrac{1}{2\mu} \mathbf{I} - \dfrac{\lambda}{2\mu (2\mu + 3\lambda)} (\mathbf{1} \otimes \mathbf{1})$. We evaluate the tensor $\boldsymbol{Qinv} = \hat{\mathbf{N}} \cdot \mathbb{C}^{e^{-1}} \cdot \hat{\mathbf{N}}$:

$$
\begin{aligned}
&\Rightarrow (\boldsymbol{Qinv})_{jk} = \hat{\mathrm{N}}_i \mathbb{C}^{-1}_{ijkl} \hat{\mathrm{N}}_l = \hat{\mathrm{N}}_i \left(\frac{1}{2\mu} \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{\lambda}{2\mu (2\mu + 3\lambda)} \delta_{ij} \delta_{kl} \right) \hat{\mathrm{N}}_l \\
&\Rightarrow (\boldsymbol{Qinv})_{jk} = \frac{1}{2\mu} \frac{1}{2} (\hat{\mathrm{N}}_i \delta_{ik} \delta_{jl} \hat{\mathrm{N}}_l + \hat{\mathrm{N}}_i \delta_{il} \delta_{jk} \hat{\mathrm{N}}_l) - \frac{\lambda}{2\mu (2\mu + 3\lambda)} \hat{\mathrm{N}}_i \delta_{ij} \delta_{kl} \hat{\mathrm{N}}_l \\
&\Rightarrow (\boldsymbol{Qinv})_{jk} = \frac{1}{4\mu} (\hat{\mathrm{N}}_k \hat{\mathrm{N}}_j + \hat{\mathrm{N}}_l \hat{\mathrm{N}}_l \delta_{jk}) - \frac{\lambda}{2\mu (2\mu + 3\lambda)} \hat{\mathrm{N}}_j \hat{\mathrm{N}}_k = \frac{1}{4\mu} \delta_{jk} + \left(\frac{1}{4\mu} - \frac{\lambda}{2\mu (2\mu + 3\lambda)} \right) \hat{\mathrm{N}}_j \hat{\mathrm{N}}_k \\
&\Rightarrow (\boldsymbol{Qinv})_{jk} = \frac{1}{4\mu} \delta_{jk} + \left(\frac{2\mu + \lambda}{4\mu (2\mu + 3\lambda)} \right) \hat{\mathrm{N}}_j \hat{\mathrm{N}}_k
\end{aligned}
$$

Thus:

$$
\boldsymbol{Qinv} = \frac{1}{4\mu} \mathbf{1} + \left(\frac{2\mu + \lambda}{4\mu (2\mu + 3\lambda)} \right) \hat{\mathbf{N}} \otimes \hat{\mathbf{N}}
$$

Note that $\mu \neq 0$ and $(2\mu + 3\lambda) \neq 0$, and moreover, these conditions are the same as those to guarantee that $\exists \mathbb{C}^{-1}$, (see **Problem 1.92**).

1.12 Polar Decomposition

Problem 1.95

Let us consider that $\boldsymbol{F}$ has inverse ($\det(\boldsymbol{F}) \neq 0$) and that can be multiplicatively decomposed as:

$$\boldsymbol{F} = \mathbf{Q} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{Q}$$

If $\mathbf{U}$ has the eigenvalues λ_a associated with the eigenvectors $\hat{\mathbf{N}}^{(a)}$, and $\mathbf{V}$ has the eigenvalues μ_a associated with the eigenvectors $\hat{\mathbf{n}}^{(a)}$, show that:

$$\mu_a = \lambda_a$$

Obtain also the relationship between the eigenvectors $\hat{\mathbf{N}}^{(a)}$ and $\hat{\mathbf{n}}^{(a)}$.

Solution:

By using the definition of $\boldsymbol{F}$ we can obtain the following relationship:

$$\mathbf{Q}^T \cdot \boldsymbol{F} = \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{U} = \mathbf{Q}^T \cdot \mathbf{V} \cdot \mathbf{Q} \quad \Rightarrow \quad \mathbf{Q}^T \cdot \boldsymbol{F} = \mathbf{U} = \mathbf{Q}^T \cdot \mathbf{V} \cdot \mathbf{Q}$$

and by considering the definition of eigenvalue-eigenvector of $\mathbf{U}$ we can obtain:

$$\begin{aligned} \mathbf{U} \cdot \hat{\mathbf{N}}^{(a)} &= \lambda_a \hat{\mathbf{N}}^{(a)} \\ \mathbf{Q}^T \cdot \mathbf{V} \cdot \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)} &= \lambda_a \hat{\mathbf{N}}^{(a)} \qquad \text{(the index here does not indicate summation)} \\ \underbrace{\mathbf{Q} \cdot \mathbf{Q}^T}_{\mathbf{1}} \cdot \mathbf{V} \cdot \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)} &= \lambda_a \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)} \end{aligned}$$

thus,

$$\begin{aligned} \mathbf{V} \cdot \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)} &= \lambda_a \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)} \\ \mathbf{V} \cdot \hat{\mathbf{n}}^{(a)} &= \lambda_a \hat{\mathbf{n}}^{(a)} \end{aligned}$$

where we have assumed that $\hat{\mathbf{n}}^{(a)} = \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)}$. Furthermore, by comparing the two definitions of eigenvalue-eigenvector of the tensors $\mathbf{U}$ and $\mathbf{V}$, we can verify that the they have the same eigenvalues but different eigenvectors and they are related to each other by the orthogonal transformation $\hat{\mathbf{n}}^{(a)} = \mathbf{Q} \cdot \hat{\mathbf{N}}^{(a)}$.

1.13 Spherical and Deviatoric Tensors

Problem 1.96

Let $\boldsymbol{\sigma}$ be a symmetric second-order tensor, and $\mathbf{s} \equiv \boldsymbol{\sigma}^{dev}$ be a deviatoric tensor. Prove that $\mathbf{s} : \dfrac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$. Also show that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors.

Solution: First, we make use of the definition of a deviatoric tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma}^{sph} + \mathbf{s} = \frac{I_{\boldsymbol{\sigma}}}{3} \mathbf{1} + \mathbf{s} \quad \Rightarrow \quad \mathbf{s} = \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3} \mathbf{1}.$$

Afterwards we calculate:

$$\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \frac{\partial\left[\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right]}{\partial \boldsymbol{\sigma}} = \frac{\partial[\boldsymbol{\sigma}]}{\partial \boldsymbol{\sigma}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \boldsymbol{\sigma}}\mathbf{1}$$

which in indicial notation is:

$$\frac{\partial \mathrm{s}_{ij}}{\partial \sigma_{kl}} = \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \sigma_{kl}}\delta_{ij} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}$$

Therefore

$$\mathrm{s}_{ij}\frac{\partial \mathrm{s}_{ij}}{\partial \sigma_{kl}} = \mathrm{s}_{ij}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right) = \mathrm{s}_{ij}\delta_{ik}\delta_{jl} - \frac{1}{3}\mathrm{s}_{ij}\delta_{kl}\delta_{ij} = \mathrm{s}_{kl} - \frac{1}{3}\delta_{kl}\underbrace{\mathrm{s}_{ii}}_{=0} = \mathrm{s}_{kl} \quad \Rightarrow \quad \mathbf{s} : \frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$$

To show that two tensors are coaxial, we must prove that $\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev}$:

$$\begin{aligned}\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev} &= \boldsymbol{\sigma} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{sph}) = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{sph} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \\ &= \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \cdot \boldsymbol{\sigma} = \left(\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right) \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma}\end{aligned}$$

Therefore, we have shown that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors. In other words, they have the same principal directions (eigenvectors).

Problem 1.97

Consider that $J = [\det(\boldsymbol{b})]^{\frac{1}{2}} = (I\!I\!I_{\boldsymbol{b}})^{\frac{1}{2}}$, where $\boldsymbol{b}$ is a symmetric second-order tensor, i.e. $\boldsymbol{b} = \boldsymbol{b}^T$. Obtain the partial derivatives of J and $\ln(J)$ with respect to $\boldsymbol{b}$.

Solution:

$$\frac{\partial J}{\partial \boldsymbol{b}} = \frac{\partial\left[(I\!I\!I_{\boldsymbol{b}})^{\frac{1}{2}}\right]}{\partial \boldsymbol{b}} = \frac{1}{2}(I\!I\!I_{\boldsymbol{b}})^{\frac{-1}{2}}\frac{\partial I\!I\!I_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2}(I\!I\!I_{\boldsymbol{b}})^{\frac{-1}{2}} I\!I\!I_{\boldsymbol{b}}\boldsymbol{b}^{-T} = \frac{1}{2}(I\!I\!I_{\boldsymbol{b}})^{\frac{1}{2}}\boldsymbol{b}^{-1} = \frac{1}{2}J\,\boldsymbol{b}^{-1}$$

$$\Rightarrow \frac{\partial[\mathrm{Ln}(J)]}{\partial \boldsymbol{b}} = \frac{\partial\left[\mathrm{Ln}\left(I\!I\!I_{\boldsymbol{b}}^{\frac{1}{2}}\right)\right]}{\partial \boldsymbol{b}} = \frac{1}{2\,I\!I\!I_{\boldsymbol{b}}}\frac{\partial I\!I\!I_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2}\boldsymbol{b}^{-1}$$

1.14 Voigt Notation

Problem 1.98

a) Write the equation $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$ in Voigt notation, where $\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ is the isotropic symmetric fourth-order tensor, and the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are structured according to Voigt notation as follows:

$$\{\boldsymbol{\sigma}\} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \quad ; \quad \{\boldsymbol{\varepsilon}\} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix}$$

b) Write the equation $\boldsymbol{\varepsilon} = \mathbb{C}^{-1} : \boldsymbol{\sigma}$ in Voigt notation, where the tensor $\mathbb{C}^{-1}$, (see **Problem 1.92**), is given by.

$$\mathbb{C}^{-1} = \frac{1}{2\mu}\mathbf{I} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\mathbf{1} \otimes \mathbf{1}$$

Solution:

We write the equation $\boldsymbol{\sigma} = (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \boldsymbol{\varepsilon}$ in indicial notation:

$$\sigma_{ij} = \left[\lambda \delta_{ij}\delta_{kl} + 2\mu \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right] \varepsilon_{kl} = \left[\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right] \varepsilon_{kl}$$

The second-order unit tensor in Voigt notation is:

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{Voigt} \{\delta\} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, the term $(\mathbf{1} \otimes \mathbf{1})_{ij} = \delta_{ij}\delta_{kl}$ in Voigt notation becomes:

$$\bar{\bar{\mathbb{I}}}_{ijkl} = \delta_{ij}\delta_{kl} \xrightarrow{Voigt} [\bar{\bar{\mathcal{I}}}] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \{\delta\}\{\delta\}^T$$

The symmetric fourth-order unit tensor $\mathrm{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ in Voigt notation is:

$$\mathrm{I}_{ijkl} \xrightarrow{Voigt} [\mathcal{I}] = \begin{bmatrix} \mathrm{I}_{1111} & \mathrm{I}_{1122} & \mathrm{I}_{1133} & \mathrm{I}_{1112} & \mathrm{I}_{1123} & \mathrm{I}_{1113} \\ \mathrm{I}_{2211} & \mathrm{I}_{2222} & \mathrm{I}_{2233} & \mathrm{I}_{2212} & \mathrm{I}_{2223} & \mathrm{I}_{2213} \\ \mathrm{I}_{3311} & \mathrm{I}_{3322} & \mathrm{I}_{3333} & \mathrm{I}_{3312} & \mathrm{I}_{3323} & \mathrm{I}_{3313} \\ \mathrm{I}_{1211} & \mathrm{I}_{1222} & \mathrm{I}_{1233} & \mathrm{I}_{1212} & \mathrm{I}_{1223} & \mathrm{I}_{1213} \\ \mathrm{I}_{2311} & \mathrm{I}_{2322} & \mathrm{I}_{2333} & \mathrm{I}_{2312} & \mathrm{I}_{2323} & \mathrm{I}_{2313} \\ \mathrm{I}_{1311} & \mathrm{I}_{1322} & \mathrm{I}_{1333} & \mathrm{I}_{1312} & \mathrm{I}_{1323} & \mathrm{I}_{1313} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

With these, we can conclude that $\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ in Voigt notation becomes:

$$[\mathcal{C}]=\lambda\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}+2\mu\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&\frac{1}{2}&0&0\\0&0&0&0&\frac{1}{2}&0\\0&0&0&0&0&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu&\lambda&\lambda&0&0&0\\\lambda&\lambda+2\mu&\lambda&0&0&0\\\lambda&\lambda&\lambda+2\mu&0&0&0\\0&0&0&\mu&0&0\\0&0&0&0&\mu&0\\0&0&0&0&0&\mu\end{bmatrix}$$

thus

$$\boldsymbol{\sigma}=(\lambda\mathbf{1}\otimes\mathbf{1}+2\mu\mathbf{I}):\boldsymbol{\varepsilon}\xrightarrow{Voigt}\underbrace{\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{13}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu&\lambda&\lambda&0&0&0\\\lambda&\lambda+2\mu&\lambda&0&0&0\\\lambda&\lambda&\lambda+2\mu&0&0&0\\0&0&0&\mu&0&0\\0&0&0&0&\mu&0\\0&0&0&0&0&\mu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\2\varepsilon_{12}\\2\varepsilon_{23}\\2\varepsilon_{13}\end{bmatrix}}_{\{\boldsymbol{\sigma}\}=[\mathcal{C}]\{\boldsymbol{\varepsilon}\}}$$

b)

$$\boldsymbol{\varepsilon}=\mathbb{C}^{-1}:\boldsymbol{\sigma}$$

$$\Rightarrow\boldsymbol{\varepsilon}=\left[\frac{1}{2\mu}\mathbf{I}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathbf{1}\otimes\mathbf{1}\right]:\boldsymbol{\sigma}=\frac{1}{2\mu}\mathbf{I}:\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathbf{1}\otimes\mathbf{1}:\boldsymbol{\sigma}$$

$$\Rightarrow\boldsymbol{\varepsilon}=\frac{1}{2\mu}\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

$$\Rightarrow\varepsilon_{ij}=\frac{1}{2\mu}\sigma_{ij}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{kk}\delta_{ij}$$

Note that:

$$\varepsilon_{11}=\frac{1}{2\mu}\sigma_{11}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{11}=\left(\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\right)\sigma_{11}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{22}+\sigma_{33})$$

$$\varepsilon_{22}=\frac{1}{2\mu}\sigma_{22}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{22}=\left(\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\right)\sigma_{22}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{33})$$

$$\varepsilon_{33}=\frac{1}{2\mu}\sigma_{33}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{33}=\left(\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\right)\sigma_{33}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{22})$$

$$\varepsilon_{12}=\frac{1}{2\mu}\sigma_{12}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{kk})\underbrace{\delta_{12}}_{=0}=\frac{1}{2\mu}\sigma_{12}\quad\Rightarrow\quad2\varepsilon_{12}=\frac{1}{\mu}\sigma_{12}$$

$$2\varepsilon_{23}=\frac{1}{\mu}\sigma_{23}$$

$$2\varepsilon_{13}=\frac{1}{\mu}\sigma_{13}$$

Restructuring the above in Voigt notation we can obtain:

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}
$$

$$\{\boldsymbol{\varepsilon}\} = [\mathcal{C}]^{-1}\{\boldsymbol{\sigma}\}$$

Problem 1.99

Let $\mathbf{T}(\vec{x},t)$ be a symmetric second-order tensor, which is expressed in terms of the position $(\vec{x})$ and time (t). Also, bear in mind that the tensor components, along direction x_3, are equal to zero, i.e. $\mathsf{T}_{13} = \mathsf{T}_{23} = \mathsf{T}_{33} = 0$.

NOTE: We define $\mathbf{T}(\vec{x},t)$ as a field tensor, i.e. the value of $\mathbf{T}$ depends on position and time. If the tensor is independent of any one direction at all points $(\vec{x})$, e.g. if $\mathbf{T}(\vec{x},t)$ is independent of the x_3-direction, (see Figure 1.26), the problem becomes a two-dimensional problem (plane state) so that the problem is greatly simplified.

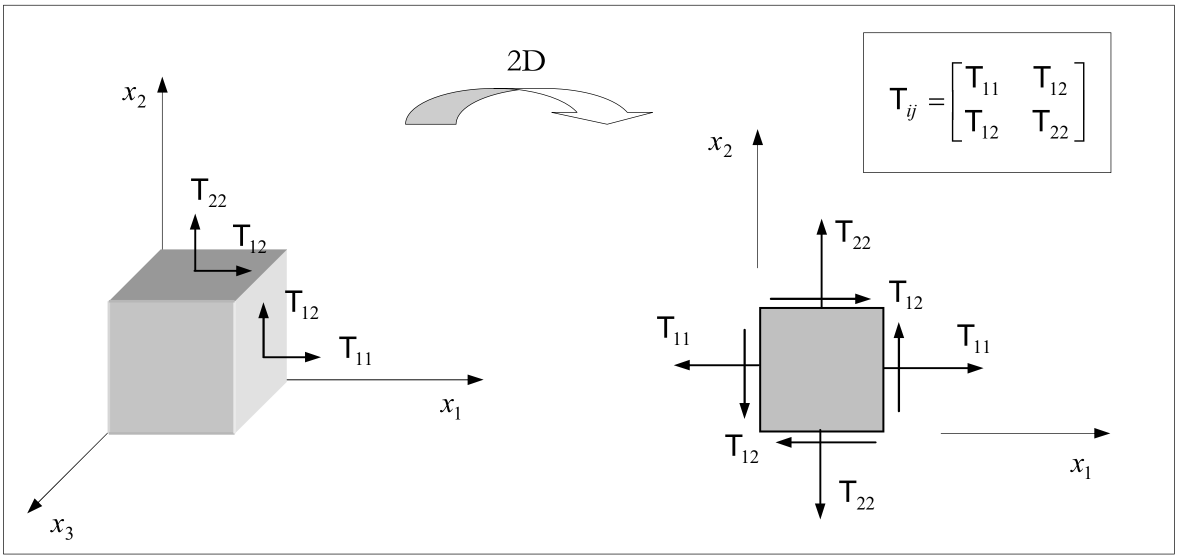


Figure 1.26: A two-dimensional problem (2D).

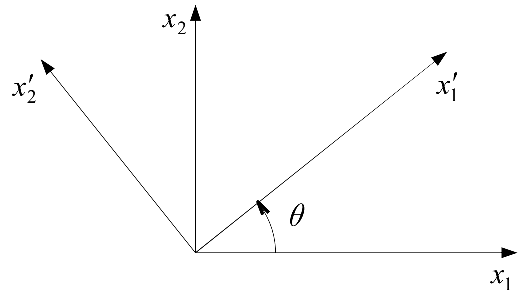
a) Obtain T'_{11}, T'_{22}, T'_{12} in the new reference system $(x'_1 - x'_2)$ defined in Figure 1.27.

b) Obtain the value of θ so that θ corresponds to the principal direction of $\mathbf{T}$, and also find an equation for the principal values of $\mathbf{T}$.

c) Evaluate the values of T'_{ij}, $(i,j=1,2)$, when $\mathsf{T}_{11}=1$, $\mathsf{T}_{22}=2$, $\mathsf{T}_{12}=-4$ and $\theta = 45^\circ$. Also, obtain the principal values and principal directions.

d) Draw a graph that shows the relationship between θ and components T'_{11}, T'_{22} and T'_{12}, and in which the angle varies from 0° to 360°.

Use the Voigt Notation, and express the results in terms of 2θ.



$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 1.27: A two-dimensional problem (2D).

Solution:

a) Here we can apply the transformation law in Voigt notation $\{\mathcal{T}'\} = [\mathcal{M}]\ \{\mathcal{T}\}$, where

$$\{\mathcal{T}'\} = \begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{33} \\ \mathsf{T}'_{12} \\ \mathsf{T}'_{23} \\ \mathsf{T}'_{13} \end{bmatrix} \quad ; \quad \{\mathcal{T}\} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix}$$

$$[\mathcal{M}] = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & a_{13}{}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\ a_{21}{}^2 & a_{22}{}^2 & a_{23}{}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\ a_{31}{}^2 & a_{32}{}^2 & a_{33}{}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\ a_{21}a_{11} & a_{22}a_{12} & a_{13}a_{23} & (a_{11}a_{22}+a_{12}a_{21}) & (a_{13}a_{22}+a_{12}a_{23}) & (a_{13}a_{21}+a_{11}a_{23}) \\ a_{31}a_{21} & a_{32}a_{22} & a_{33}a_{23} & (a_{31}a_{22}+a_{32}a_{21}) & (a_{33}a_{22}+a_{32}a_{23}) & (a_{33}a_{21}+a_{31}a_{23}) \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12}+a_{32}a_{11}) & (a_{33}a_{12}+a_{32}a_{13}) & (a_{33}a_{11}+a_{31}a_{13}) \end{bmatrix}$$

For the particular case shown in Figure 1.27, the transformation matrix $[\mathcal{M}]$, after eliminate the role and column associated with the x_3-direction, becomes:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & 2a_{11}a_{12} \\ a_{21}{}^2 & a_{22}{}^2 & 2a_{21}a_{22} \\ a_{21}a_{11} & a_{22}a_{12} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.100}$$

The transformation matrix, a_{ij}, in the plane, can be evaluated in terms of a single parameter, θ:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.101}$$

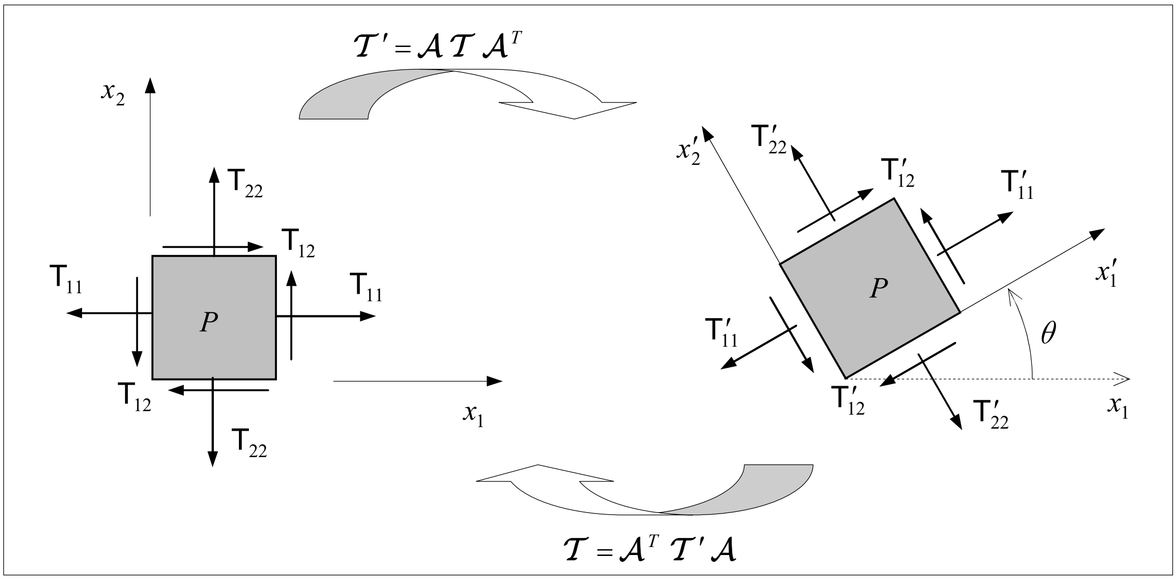


Figure 1.28: Transformation law for (2D) tensor components.

By substituting the matrix components a_{ij} given in (1.101) into (1.100) we obtain:

$$\begin{bmatrix} \mathsf{T}_{11}' \\ \mathsf{T}_{22}' \\ \mathsf{T}_{12}' \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.102}$$

Making use of the following trigonometric identities, $2\cos\theta\sin\theta = \sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, the equation (1.102) becomes:

$$\begin{bmatrix} \mathsf{T}_{11}' \\ \mathsf{T}_{22}' \\ \mathsf{T}_{12}' \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1+\cos 2\theta}{2}\right) & \left(\dfrac{1-\cos 2\theta}{2}\right) & \sin 2\theta \\ \left(\dfrac{1-\cos 2\theta}{2}\right) & \left(\dfrac{1+\cos 2\theta}{2}\right) & -\sin 2\theta \\ \left(-\dfrac{\sin 2\theta}{2}\right) & \left(\dfrac{\sin 2\theta}{2}\right) & \cos 2\theta \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix}$$

Explicitly, the above components are given by:

$$\begin{cases} \mathsf{T}_{11}' = \left(\dfrac{1+\cos 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{1-\cos 2\theta}{2}\right)\mathsf{T}_{22} + \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}_{22}' = \left(\dfrac{1-\cos 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{1+\cos 2\theta}{2}\right)\mathsf{T}_{22} - \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}_{12}' = \left(-\dfrac{\sin 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{\sin 2\theta}{2}\right)\mathsf{T}_{22} + \mathsf{T}_{12}\cos 2\theta \end{cases}$$

Rearranging the previous equation we can obtain:

$$\boxed{\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2}\right) + \left(\dfrac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)\cos 2\theta + \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{22} = \left(\dfrac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2}\right) - \left(\dfrac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)\cos 2\theta - \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{12} = -\left(\dfrac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)\sin 2\theta + \mathsf{T}_{12}\cos 2\theta \end{cases}} \tag{1.103}$$

b) Recalling that the principal directions are characterized by the lack of any tangential components, i.e. $\mathsf{T}_{ij} = 0$ if $i \neq j$, in order to find the principal directions for the plane case, we let $\mathsf{T}'_{12} = 0$, hence:

$$\mathsf{T}'_{12} = -\left(\frac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)\sin 2\theta + \mathsf{T}_{12}\cos 2\theta = 0 \Rightarrow \left(\frac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)\sin 2\theta = \mathsf{T}_{12}\cos 2\theta$$

$$\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\mathsf{T}_{12}}{\mathsf{T}_{11} - \mathsf{T}_{22}} \Rightarrow \tan(2\theta) = \frac{2\mathsf{T}_{12}}{\mathsf{T}_{11} - \mathsf{T}_{22}}$$

Then, the angle corresponding to the principal direction is:

$$\boxed{\theta = \frac{1}{2}\arctan\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11} - \mathsf{T}_{22}}\right)} \tag{1.104}$$

To find the principal values (eigenvalues) we must solve the following characteristic equation:

$$\begin{vmatrix} \mathsf{T}_{11} - \mathsf{T} & \mathsf{T}_{12} \\ \mathsf{T}_{12} & \mathsf{T}_{22} - \mathsf{T} \end{vmatrix} = 0 \quad \Rightarrow \quad \mathsf{T}^2 - \mathsf{T}(\mathsf{T}_{11} + \mathsf{T}_{22}) + \left(\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2\right) = 0$$

And by evaluating the quadratic equation we obtain:

$$\begin{aligned} \mathsf{T}_{(1,2)} &= \frac{-[-(\mathsf{T}_{11} + \mathsf{T}_{22})] \pm \sqrt{[-(\mathsf{T}_{11} + \mathsf{T}_{22})]^2 - 4(1)\left(\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2\right)}}{2(1)} \\ &= \frac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2} \pm \sqrt{\frac{[(\mathsf{T}_{11} + \mathsf{T}_{22})]^2 - 4\left(\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2\right)}{4}} \end{aligned}$$

By rearranging the above equation we obtain the principal values for the two-dimensional case as:

$$\boxed{\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2}} \tag{1.105}$$

c) We directly apply equation (1.103) to evaluate the values of the components T'_{ij}, $(i, j = 1,2)$, where $\mathsf{T}_{11} = 1$, $\mathsf{T}_{22} = 2$, $\mathsf{T}_{12} = -4$ and $\theta = 45^{\mathrm{o}}$, i.e.:

$$\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{1+2}{2}\right) + \left(\dfrac{1-2}{2}\right)\cos 90^{\circ} - 4\sin 90^{\circ} = -2.5 \\ \mathsf{T}'_{22} = \left(\dfrac{1+2}{2}\right) - \left(\dfrac{1-2}{2}\right)\cos 90^{\circ} + 4\sin 90^{\circ} = 5.5 \\ \mathsf{T}'_{12} = -\left(\dfrac{1-2}{2}\right)\sin 90^{\circ} - 4\cos 90^{\circ} = 0.5 \end{cases}$$

And the angle corresponding to the principal direction is:

$$\theta = \frac{1}{2}\arctan\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}\right) = \frac{2\times(-4)}{1-2} \quad \Rightarrow \quad (\theta = 41.4375^{\circ})$$

The principal values of $\mathbf{T}(\vec{x},t)$ can be evaluated as follows:

$$\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2} \quad \Rightarrow \quad \begin{cases} \mathsf{T}_1 = 5.5311 \\ \mathsf{T}_2 = -2.5311 \end{cases}$$

d) By referring to equation in (1.103) and by varying θ from 0° to 360°, we can obtain different values of T'_{11}, T'_{22}, T'_{12}, which are illustrated in the following graph:

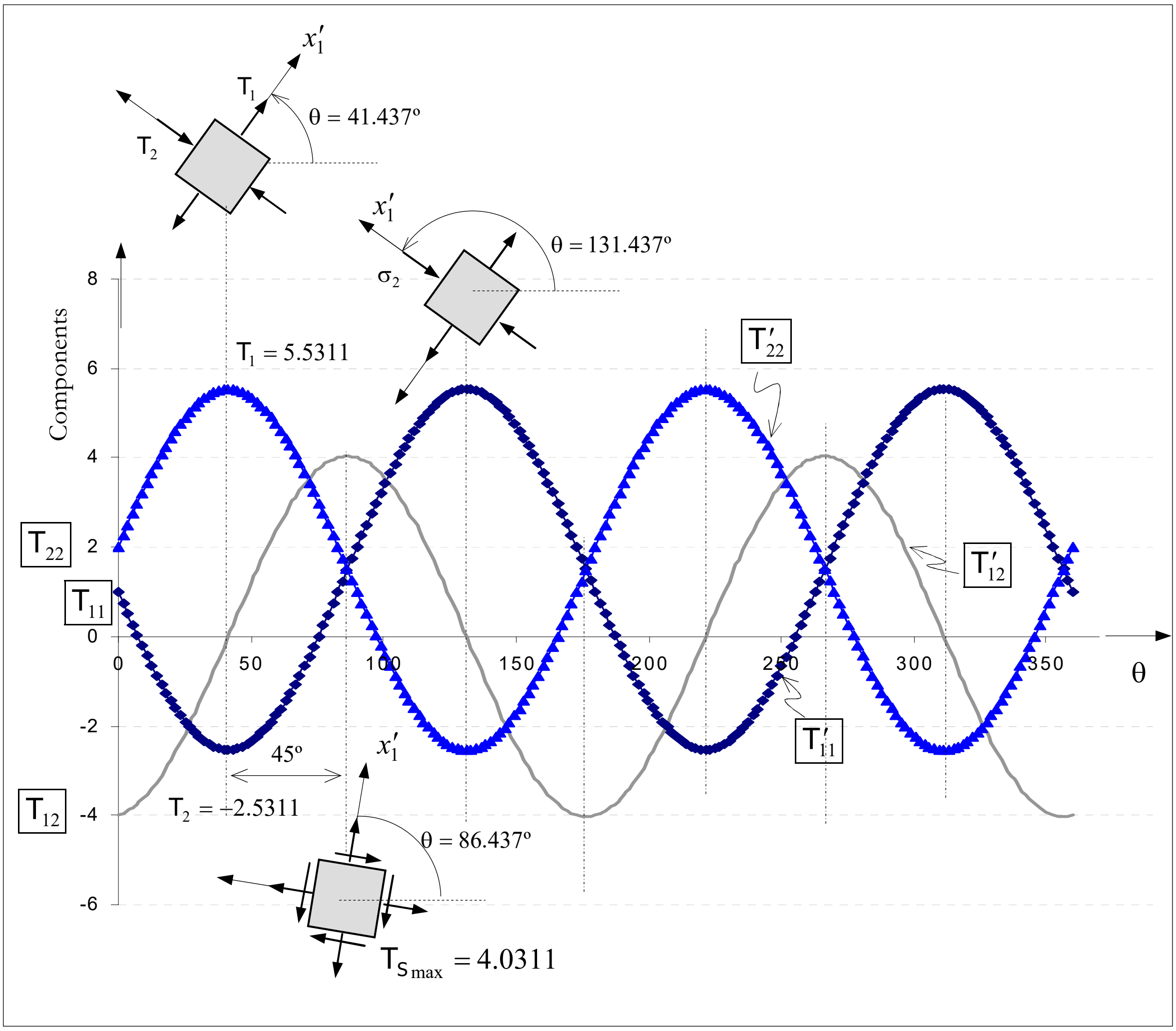


Figure 1.29

Problem 1.100

Obtain the principal values (eigenvalues) and the principal directions (eigenvectors) of the symmetric part of $\mathbf{T}$, whose components in the Cartesian system are given by:

$$\mathsf{T}_{ij} = \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \qquad (i, j = 1,2)$$

Solution:

The symmetric part of the tensor is given by:

$$\mathsf{T}_{ij}^{sym} = \frac{1}{2}\left(\mathsf{T}_{ij} + \mathsf{T}_{ji}\right) = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

The principal values:

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - 9\lambda + 16 = 0$$

The solution of the quadratic equation is given by:

$$\lambda_{(1,2)} = \frac{-9 \pm \sqrt{(-9)^2 - 4\times(1)\times(16)}}{2\times 1} \Rightarrow \begin{cases} \lambda_1 \equiv \mathsf{T}_1 = 6.5615 \\ \lambda_2 \equiv \mathsf{T}_2 = 2.4385 \end{cases}$$

We can draw the Mohr circle (2D) of the tensor $\mathbf{T}^{sym}$:

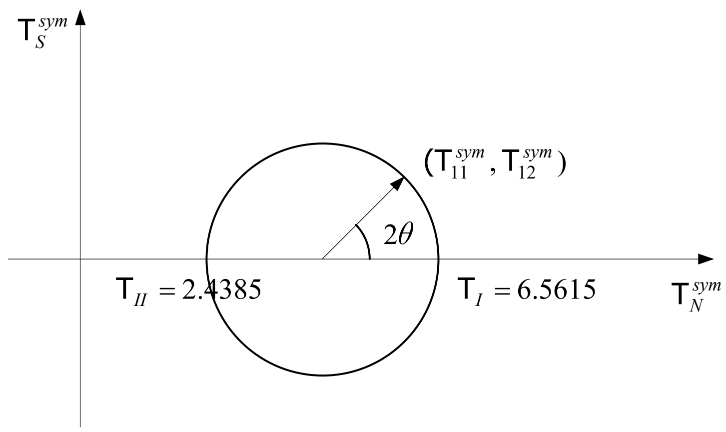
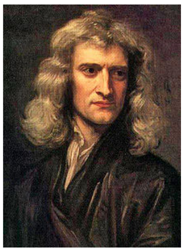


Figure 1.30

For the plane case, the principal direction can be obtained by means of the equation:

$$\tan(2\theta) = \frac{2\mathsf{T}_{12}^{sym}}{\mathsf{T}_{11}^{sym} - \mathsf{T}_{22}^{sym}} = \frac{2\times 2}{5-4} = 4 \quad \Rightarrow \quad \theta = 37.982^{\circ}$$

1.15 Tensor Fields



Sir Isaac **Newton**

(1642 - 1727)

Gottfried Wilhelm **Leibniz**

(1646 - 1716)

Source: **Wikipedia**

Problem 1.101

Find the gradient of the function $f(x_1, x_2) = \cos(x_1) + \exp^{x_1 x_2}$ at the point $(x_1 = 0, x_2 = 1)$.

Solution: By definition, the gradient of a scalar function is given by:

$$\nabla_{\vec{x}} f = \frac{\partial f}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{\partial f}{\partial x_2} \hat{\mathbf{e}}_2$$

where: $\dfrac{\partial f}{\partial x_1} = -\sin(x_1) + x_2 \exp^{x_1 x_2}$; $\dfrac{\partial f}{\partial x_2} = x_1 \exp^{x_1 x_2}$

$$\nabla_{\vec{x}} f(x_1, x_2) = [-\sin(x_1) + x_2 \exp^{x_1 x_2}]\hat{\mathbf{e}}_1 + [x_1 \exp^{x_1 x_2}]\hat{\mathbf{e}}_2 \Rightarrow \nabla_{\vec{x}} f(0,1) = [1]\hat{\mathbf{e}}_1 + [0]\hat{\mathbf{e}}_2 = 2\hat{\mathbf{e}}_1$$

Problem 1.102

Let $\vec{v}$ and φ be, respectively, vector and scalar, and twice continuously differentiable, by using indicial notation, show that:

a) $\nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = 0$

b) $\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \varphi) = \nabla_{\vec{x}}{}^2 \varphi$

c) $\nabla_{\vec{x}}(\phi\mu) = \mu(\nabla_{\vec{x}}\phi) + \phi(\nabla_{\vec{x}}\mu)$

d) $\nabla_{\vec{x}} \cdot (\phi\vec{v}) = (\nabla_{\vec{x}}\phi) \cdot \vec{v} + \phi(\nabla_{\vec{x}} \cdot \vec{v})$

e) $\nabla_{\vec{x}} \cdot (\mathbf{A} \cdot \mathbf{B}) = (\nabla_{\vec{x}}\mathbf{A}) : \mathbf{B} + \mathbf{A} \cdot (\nabla_{\vec{x}} \cdot \mathbf{B})$ ($\mathbf{A}$ and $\mathbf{B}$ are second-order tensors)

Solution:

a) Considering that

$$\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i \quad \text{and} \quad \nabla_{\vec{x}} \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_l} \cdot \hat{\mathbf{e}}_l \tag{1.106}$$

then

$$\nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \frac{\partial}{\partial x_l}(\epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i) \cdot \hat{\mathbf{e}}_l = \frac{\partial}{\partial x_l}(\epsilon_{ijk} v_{k,j} \delta_{il}) = \frac{\partial}{\partial x_l}(\epsilon_{ljk} v_{k,j}) = \epsilon_{ljk} v_{k,jl} \tag{1.107}$$

Note that ϵ_{ljk} is an antisymmetric tensor in lj and $v_{k,jl}$ is a symmetric tensor in lj, thus:

$$\epsilon_{ljk} v_{k,jl} = 0 \tag{1.108}$$

b)

$$\begin{aligned}\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \varphi) &= \frac{\partial}{\partial x_j} (\varphi_{,i} \hat{\mathbf{e}}_i) \cdot \hat{\mathbf{e}}_j = \frac{\partial}{\partial x_j} (\varphi_{,i} \delta_{ij}) = \frac{\partial \varphi_{,j}}{\partial x_j} = \varphi_{,jj} \\ &= \frac{\partial}{\partial x_j} \left(\frac{\partial \varphi}{\partial x_j} \right) = \frac{\partial^2 \varphi}{\partial x_j^2} = \nabla_{\vec{x}}{}^2 \varphi\end{aligned} \tag{1.109}$$

c)

$$[\nabla_{\vec{x}} (\phi\mu)]_i = (\phi\mu)_{,i} = \phi_{,i}\mu + \phi\mu_{,i} = \mu[\nabla_{\vec{x}}\phi]_i + \phi[\nabla_{\vec{x}}\mu]_i \tag{1.110}$$

d) The result of $\nabla_{\vec{x}} \cdot (\phi\vec{v})$ is a scalar which can be expressed as follows:

$$\begin{aligned}\nabla_{\vec{x}} \cdot (\phi\vec{v}) = \quad (\phi v_i)_{,i} \quad &= \phi_{,i} v_i + \phi v_{i,i} \\ &= (\nabla_{\vec{x}}\phi) \cdot \vec{v} + \phi(\nabla_{\vec{x}} \cdot \vec{v})\end{aligned}$$

e) Considering that $(\mathbf{A} \cdot \mathbf{B})_{ij} = \mathsf{A}_{ik}\mathsf{B}_{kj}$, $[\nabla_{\vec{x}} \cdot (\mathbf{A} \cdot \mathbf{B})]_i = (\mathbf{A} \cdot \mathbf{B})_{ij,j} = (\mathsf{A}_{ik}\mathsf{B}_{kj})_{,j}$, thus

$$(\mathsf{A}_{ik}\mathsf{B}_{kj})_{,j} = \mathsf{A}_{ik,j}\mathsf{B}_{kj} + \mathsf{A}_{ik}\mathsf{B}_{kj,j} = [(\nabla_{\vec{x}}\mathbf{A}) : \mathbf{B}]_i + [\mathbf{A} \cdot (\nabla_{\vec{x}} \cdot \mathbf{B})]_i$$

Problem 1.103

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be vectors. Show that the following identity $\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$ holds.

Solution:

Observing that $\vec{\mathbf{a}} = \mathsf{a}_j \hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathsf{b}_k \hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$, we can express $\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}})$ as follows:

$$\frac{\partial(\mathsf{a}_j \hat{\mathbf{e}}_j + \mathsf{b}_k \hat{\mathbf{e}}_k)}{\partial x_i} \cdot \hat{\mathbf{e}}_i = \frac{\partial \mathsf{a}_j}{\partial x_i} \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i + \frac{\partial \mathsf{b}_k}{\partial x_i} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i = \frac{\partial \mathsf{a}_i}{\partial x_i} + \frac{\partial \mathsf{b}_i}{\partial x_i} = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$$

Alternative solution: Working directly with indicial notation we obtain:

$$\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = (\mathsf{a}_i + \mathsf{b}_i)_{,i} = \mathsf{a}_{i,i} + \mathsf{b}_{i,i} = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$$

Problem 1.104

Find the components of $(\nabla_{\vec{x}} \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}$.

Solution: Bearing in mind that $\vec{\mathbf{a}} = \mathsf{a}_j \hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathsf{b}_k \hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$ $(i = 1,2,3)$, the following is true:

$$(\nabla_{\vec{x}} \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} = \left(\frac{\partial(\mathsf{a}_j \hat{\mathbf{e}}_j)}{\partial x_i} \otimes \hat{\mathbf{e}}_i \right) \cdot (\mathsf{b}_k \hat{\mathbf{e}}_k) = \left(\frac{\partial \mathsf{a}_j}{\partial x_i} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \right) \cdot (\mathsf{b}_k \hat{\mathbf{e}}_k) = \mathsf{b}_k \delta_{ik} \frac{\partial \mathsf{a}_j}{\partial x_i} \hat{\mathbf{e}}_j = \mathsf{b}_k \frac{\partial \mathsf{a}_j}{\partial x_k} \hat{\mathbf{e}}_j$$

Expanding the dummy index k, we obtain:

$$\mathsf{b}_k \frac{\partial \mathsf{a}_j}{\partial x_k} = \mathsf{b}_1 \frac{\partial \mathsf{a}_j}{\partial x_1} + \mathsf{b}_2 \frac{\partial \mathsf{a}_j}{\partial x_2} + \mathsf{b}_3 \frac{\partial \mathsf{a}_j}{\partial x_3}$$

Thus,

$$j=1 \quad \Rightarrow \quad \mathsf{b}_1 \frac{\partial \mathsf{a}_1}{\partial x_1} + \mathsf{b}_2 \frac{\partial \mathsf{a}_1}{\partial x_2} + \mathsf{b}_3 \frac{\partial \mathsf{a}_1}{\partial x_3}$$

$$j=2 \quad \Rightarrow \quad \mathsf{b}_1 \frac{\partial \mathsf{a}_2}{\partial x_1} + \mathsf{b}_2 \frac{\partial \mathsf{a}_2}{\partial x_2} + \mathsf{b}_3 \frac{\partial \mathsf{a}_2}{\partial x_3}$$

$$j=3 \quad \Rightarrow \quad \mathsf{b}_1 \frac{\partial \mathsf{a}_3}{\partial x_1} + \mathsf{b}_2 \frac{\partial \mathsf{a}_3}{\partial x_2} + \mathsf{b}_3 \frac{\partial \mathsf{a}_3}{\partial x_3}$$

Problem 1.105

Prove that the following relationship is valid:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T}\right) = \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T$$

where $\vec{\mathbf{q}}(\vec{x},t)$ is an arbitrary vector field, and $T(\vec{x},t)$ is a scalar field.

Solution:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T}\right) = \frac{\partial}{\partial x_i}\left(\frac{\mathsf{q}_i}{T}\right) \equiv \left(\frac{\mathsf{q}_i}{T}\right)_{,i} = \frac{1}{T}\mathsf{q}_{i,i} - \frac{1}{T^2}\mathsf{q}_i T_{,i} = \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \quad \text{(scalar)}$$

Problem 1.106

Show that:

a) $\boxed{\mathrm{rot}(\lambda\vec{\mathbf{a}}) \equiv \vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}}) = \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})}$ (1.111)

b) $\boxed{\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}} - (\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}} + (\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} - (\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}}$ (1.112)

c) $\boxed{\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}}) - \nabla^2_{\vec{x}}\vec{\mathbf{a}}}$ (1.113)

d) $\boxed{\nabla_{\vec{x}} \cdot (\psi \nabla_{\vec{x}}\phi) = \psi\nabla^2_{\vec{x}}\phi + (\nabla_{\vec{x}}\psi) \cdot (\nabla_{\vec{x}}\phi)}$ (1.114)

Solution:

a) The result of the algebraic operation $\vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}})$ is a vector, whose components are given by:

$$\begin{aligned}
[\vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}})]_i &= \epsilon_{ijk}(\lambda \mathsf{a}_k)_{,j} \\
&= \epsilon_{ijk}(\lambda_{,j}\mathsf{a}_k + \lambda \mathsf{a}_{k,j}) \\
&= \epsilon_{ijk}\lambda \mathsf{a}_{k,j} + \epsilon_{ijk}\lambda_{,j}\mathsf{a}_k \\
&= \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})_i + \epsilon_{ijk}(\nabla_{\vec{x}}\lambda)_j \mathsf{a}_k \\
&= \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})_i + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})_i
\end{aligned} \tag{1.115}$$

with that we check the identity: $\mathrm{rot}(\lambda\vec{\mathbf{a}}) = \vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}}) = \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})$.

The components of the vector product $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ are given by $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})_k = \epsilon_{kij}\mathsf{a}_i\mathsf{b}_j$. Then:

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]_l = \epsilon_{lpk}(\epsilon_{kij}\mathsf{a}_i\mathsf{b}_j)_{,p} = \epsilon_{kij}\epsilon_{lpk}(\mathsf{a}_{i,p}\mathsf{b}_j + \mathsf{a}_i\mathsf{b}_{j,p}) \tag{1.116}$$

b) Considering that $\epsilon_{kij} = \epsilon_{ijk}$, the result of $\epsilon_{ijk}\epsilon_{lpk} = \delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl}$ and by substituting into the above equation we can obtain:

$$
\begin{aligned}
[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]_l &= \epsilon_{kij}\epsilon_{lpk}(\mathsf{a}_{i,p}\mathsf{b}_j + \mathsf{a}_i\mathsf{b}_{j,p}) \\
&= (\delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl})(\mathsf{a}_{i,p}\mathsf{b}_j + \mathsf{a}_i\mathsf{b}_{j,p}) \\
&= \delta_{il}\delta_{jp}\mathsf{a}_{i,p}\mathsf{b}_j - \delta_{ip}\delta_{jl}\mathsf{a}_{i,p}\mathsf{b}_j + \delta_{il}\delta_{jp}\mathsf{a}_i\mathsf{b}_{j,p} - \delta_{ip}\delta_{jl}\mathsf{a}_i\mathsf{b}_{j,p} \\
&= \mathsf{a}_{l,p}\mathsf{b}_p - \mathsf{a}_{p,p}\mathsf{b}_l + \mathsf{a}_l\mathsf{b}_{p,p} - \mathsf{a}_p\mathsf{b}_{l,p}
\end{aligned}
\tag{1.117}
$$

Note that $[(\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}]_l = \mathsf{a}_{l,p}\mathsf{b}_p$, $[(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}}]_l = \mathsf{a}_{p,p}\mathsf{b}_l$, $[(\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}}]_l = \mathsf{a}_l\mathsf{b}_{p,p}$, $[(\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}]_l = \mathsf{a}_p\mathsf{b}_{l,p}$.

c) The components of $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})$ are given by $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})_i = \underbrace{\epsilon_{ijk}\mathsf{a}_{k,j}}_{\mathsf{c}_i}$. Then:

$$
\begin{aligned}
[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})]_q &= \epsilon_{qli}\mathsf{c}_{i,l} \\
&= \epsilon_{qli}(\epsilon_{ijk}\mathsf{a}_{k,j})_{,l} \\
&= \epsilon_{qli}\epsilon_{ijk}\mathsf{a}_{k,jl}
\end{aligned}
\tag{1.118}
$$

Considering that $\epsilon_{qli}\epsilon_{ijk} = \epsilon_{qli}\epsilon_{jki} = \delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj}$, the above equation becomes:

$$
[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})]_q = \epsilon_{qli}\epsilon_{ijk}\mathsf{a}_{k,jl} = (\delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj})\mathsf{a}_{k,jl} = \delta_{qj}\delta_{lk}\mathsf{a}_{k,jl} - \delta_{qk}\delta_{lj}\mathsf{a}_{k,jl} = \mathsf{a}_{k,kq} - \mathsf{a}_{q,ll}
$$

Note that $[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})]_q = \mathsf{a}_{k,kq}$ and $[\nabla^2_{\vec{x}}\vec{\mathbf{a}}]_q = \mathsf{a}_{q,ll}$.

d)

$$
\nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) = (\phi\psi_{,i})_{,i} = \phi\psi_{,ii} + \phi_{,i}\psi_{,i} = \phi\nabla^2_{\vec{x}}\psi + (\nabla_{\vec{x}}\phi) \cdot (\nabla_{\vec{x}}\psi)
\tag{1.119}
$$

where ϕ and ψ are scalar functions.

Another interesting identity originating from the above equation is:

$$
\begin{aligned}
&(1)\quad \nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) = \phi\nabla^2_{\vec{x}}\psi + (\nabla_{\vec{x}}\phi) \cdot (\nabla_{\vec{x}}\psi) \\
&(2)\quad \nabla_{\vec{x}} \cdot (\psi\nabla_{\vec{x}}\phi) = \psi\nabla^2_{\vec{x}}\phi + (\nabla_{\vec{x}}\psi) \cdot (\nabla_{\vec{x}}\phi)\,(1)
\end{aligned}
\tag{1.120}
$$

Subtracting the two previous identities, $(1)-(2)$, we can obtain:

$$
\begin{aligned}
&\nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) - \nabla_{\vec{x}} \cdot (\psi\nabla_{\vec{x}}\phi) = \phi\nabla^2_{\vec{x}}\psi - \psi\nabla^2_{\vec{x}}\phi \\
&\Rightarrow \nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi - \psi\nabla_{\vec{x}}\phi) = \phi\nabla^2_{\vec{x}}\psi - \psi\nabla^2_{\vec{x}}\phi
\end{aligned}
\tag{1.121}
$$

Problem 1.107

Let ϕ be a scalar field which is independent of x_1. Show that the following relationship is true $(\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 = -\nabla^2_{\vec{x}}\phi$, where the vector field $\vec{\tau}$ is given by $\tau_1 = 0$, $\tau_2 = \dfrac{\partial\phi}{\partial x_3}$, and $\tau_3 = \dfrac{-\partial\phi}{\partial x_2}$.

Solution:

According to the problem statement we have that $\phi = \phi(x_2, x_3)$. Then, the Laplacian of ϕ becomes:

$$\nabla_{\vec{x}}^2\phi = \nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\phi) = \phi_{,ii} = \phi_{,11} + \phi_{,22} + \phi_{,33} = \underbrace{\frac{\partial^2\phi}{\partial x_1^2}}_{=0} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$

Next we calculate the components of the vector $(\nabla_{\vec{x}} \wedge \vec{\tau})$:

$$\nabla_{\vec{x}} \wedge \vec{\tau} = \frac{\partial}{\partial x_j}\overbrace{\hat{\mathbf{e}}_j \wedge \tau_k\hat{\mathbf{e}}_k}^{\epsilon_{ijk}\hat{\mathbf{e}}_i} = \frac{\partial \tau_k}{\partial x_j}\epsilon_{ijk}\hat{\mathbf{e}}_i = \epsilon_{ijk}\tau_{k,j}\hat{\mathbf{e}}_i$$

thus, the components are:

$$(\nabla_{\vec{x}} \wedge \vec{\tau})_i = \epsilon_{ijk}\tau_{k,j} = \epsilon_{i12}\tau_{2,1} + \epsilon_{i13}\tau_{3,1} + \epsilon_{i21}\tau_{1,2} + \epsilon_{i23}\tau_{3,2} + \epsilon_{i31}\tau_{1,3} + \epsilon_{i32}\tau_{2,3}$$

$$(\nabla_{\vec{x}} \wedge \vec{\tau})_i \quad = \quad \begin{cases} (i=1) \Rightarrow \epsilon_{123}\tau_{3,2} + \epsilon_{132}\tau_{2,3} = \tau_{3,2} - \tau_{2,3} \\ (i=1) \Rightarrow \epsilon_{213}\tau_{3,1} + \epsilon_{231}\tau_{1,3} = \tau_{1,3} - \tau_{3,1} \\ (i=1) \Rightarrow \epsilon_{312}\tau_{2,1} + \epsilon_{321}\tau_{1,2} = \tau_{2,1} - \tau_{1,2} \end{cases}$$

$$(\nabla_{\vec{x}} \wedge \vec{\tau})_i = \begin{bmatrix} \tau_{3,2} - \tau_{2,3} \\ \tau_{1,3} - \tau_{3,1} \\ \tau_{2,1} - \tau_{1,2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \tau_3}{\partial x_2} - \dfrac{\partial \tau_2}{\partial x_3} \\ \dfrac{\partial \tau_1}{\partial x_3} - \dfrac{\partial \tau_3}{\partial x_1} \\ \dfrac{\partial \tau_2}{\partial x_1} - \dfrac{\partial \tau_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \tau_3}{\partial x_2}\left(\dfrac{-\partial\phi}{\partial x_2}\right) - \dfrac{\partial \tau_2}{\partial x_3}\left(\dfrac{\partial\phi}{\partial x_3}\right) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{\partial^2\phi}{\partial x_2^2} + \dfrac{\partial^2\phi}{\partial x_3^2}\right) \\ 0 \\ 0 \end{bmatrix}$$

Then, $(\nabla_{\vec{x}} \wedge \vec{\tau})\cdot\hat{\mathbf{e}}_1 = -\left(\dfrac{\partial^2\phi}{\partial x_2^2} + \dfrac{\partial^2\phi}{\partial x_3^2}\right)$. With that we show $(\nabla_{\vec{x}} \wedge \vec{\tau})\cdot\hat{\mathbf{e}}_1 = -\nabla_{\vec{x}}^2\phi$.

Problem 1.108

Let ϕ be a scalar field, and $\vec{\mathbf{u}}$ be a vector field. a) Show that $\nabla_{\vec{x}}\cdot(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = 0$ and $\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}\phi) = \vec{\mathbf{0}}$.

b) Show that $\vec{\nabla}_{\vec{x}} \wedge [(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{v}\,] = (\nabla_{\vec{x}}\cdot\vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + [\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})]\cdot\vec{v} - (\nabla_{\vec{x}}\vec{v})\cdot(\vec{\nabla}_{\vec{x}} \wedge \vec{v})$;

c) Referring $\vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \wedge \vec{v}$, show that $\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^2\vec{v}) = \nabla_{\vec{x}}^2(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \nabla_{\vec{x}}^2\vec{\boldsymbol{\omega}}$.

Solution:

Regarding that: $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \epsilon_{ijk}v_{k,j}\hat{\mathbf{e}}_i$

$$\nabla_{\vec{x}}\cdot(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \frac{\partial}{\partial x_l}(\epsilon_{ijk}v_{k,j}\hat{\mathbf{e}}_i)\cdot\hat{\mathbf{e}}_l = \epsilon_{ijk}\frac{\partial}{\partial x_l}(v_{k,j})\,\delta_{il} = \epsilon_{ijk}\frac{\partial}{\partial x_i}(v_{k,j}) = \epsilon_{ijk}v_{k,ji}$$

The second derivative of $\vec{v}$ is symmetrical with ij, i.e. $v_{k,ji} = v_{k,ij}$, while ϵ_{ijk} is antisymmetric with ij, i.e., $\epsilon_{ijk} = -\epsilon_{jik}$, thus:

$$\epsilon_{ijk}v_{k,ji} = \epsilon_{ij1}v_{1,ji} + \epsilon_{ij2}v_{2,ji} + \epsilon_{ij3}v_{3,ji} = 0$$

Note that $\epsilon_{ij1}v_{1,ji} = 0$ since the double scalar product between a symmetric and an antisymmetric tensor is zero.

Likewise, we can show that:

$$\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}\phi) = \epsilon_{ijk}\phi_{,kj}\,\hat{\mathbf{e}}_i = 0_i\hat{\mathbf{e}}_i = \vec{\mathbf{0}}$$

b) Denoting by $\vec{\boldsymbol{\omega}} = \vec{\nabla}_{\vec{x}} \wedge \vec{v}$ we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge [(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{v}\,] = \vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v})$$

Observing the equation in (1.112), it holds that:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) = (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} - (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}})\vec{v} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}}$$

Note that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = 0$. Then, we can draw the conclusion that:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) &= (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}} \\ &= (\nabla_{\vec{x}} \cdot \vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + [\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})] \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

c) Observing the equation in (1.113) we obtain:

$$\nabla^2_{\vec{x}}\vec{v} = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}$$

Applying the curl to the above equation we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge (\nabla^2_{\vec{x}}\vec{v}) = \underbrace{\vec{\nabla}_{\vec{x}} \wedge [\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})]}_{=\vec{\mathbf{0}}} - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$$

Referring once again to the equation in (1.113) to express the term $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\nabla^2_{\vec{x}}\vec{v}) &= -\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}) = -\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}}) + \nabla^2_{\vec{x}}\vec{\boldsymbol{\omega}} = -\nabla_{\vec{x}}\underbrace{[\nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})]}_{=0} + \nabla^2_{\vec{x}}\vec{\boldsymbol{\omega}} \\ &= \nabla^2_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

Problem 1.109

Show that:

a) $$\boxed{\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}}) \equiv \mathrm{rot}(\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \mathrm{rot}(\vec{\mathbf{b}})} \tag{1.122}$$

Solution:

The expression $\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ is a scalar which can be expressed as follows:

$$\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)_{,i} = \underbrace{\epsilon_{ijk}\mathrm{a}_{j,i}}_{(\vec{\nabla}\wedge\vec{\mathbf{a}})_k}\mathrm{b}_k + \underbrace{\epsilon_{ijk}\mathrm{b}_{k,i}}_{(\vec{\nabla}\wedge\vec{\mathbf{b}})_j}\mathrm{a}_j = (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}})$$

Problem 1.110

a) Let $\mathbf{T}$ be an arbitrary second-order tensor, obtain the symbolic notation in Cartesian basis for: a.1) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})$, a.2) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})^T$, a.3) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}^T)$, and a.4) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}^T)^T$. a.5) Considering that $\vec{\mathbf{c}}$ is a constant vector, show that:

$$\boxed{\vec{\nabla}_{\vec{x}} \wedge (\mathbf{T} \cdot \vec{\mathbf{c}}) = (\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}) \cdot \vec{\mathbf{c}} = \vec{\mathbf{c}} \cdot [\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}]^T}$$

b) Obtain the symbolic notation of $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \overline{\boldsymbol{\varepsilon}})^T$.

c) Consider the second-order tensor $\boldsymbol{F} = \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\boldsymbol{x}}} + \mathbf{1}$, prove that c.1) $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F})^T = \mathbf{0}$ and $\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F}^T = \mathbf{0}$; c.2) Obtain the explicit components of $\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F}$.

Solution:

a.1) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}) = \frac{\partial}{\partial x_p} \hat{\mathbf{e}}_p \wedge \mathsf{T}_{qj}(\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_j) = \frac{\partial \mathsf{T}_{qj}}{\partial x_p} \hat{\mathbf{e}}_p \wedge \hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_j = \mathsf{T}_{qj,p} \epsilon_{ipq} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \epsilon_{ipq} \mathsf{T}_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$

a.2) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})^T = \epsilon_{ipq} \mathsf{T}_{qj,p} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i = \epsilon_{jpq} \mathsf{T}_{qi,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$

a.3) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}^T) = \frac{\partial}{\partial x_p} \hat{\mathbf{e}}_p \wedge \mathsf{T}_{jq}(\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_j) = \frac{\partial \mathsf{T}_{jq}}{\partial x_p} \hat{\mathbf{e}}_p \wedge \hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_j = \epsilon_{ipq} \mathsf{T}_{jq,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$

a.4) $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}^T)^T = \epsilon_{jpq} \mathsf{T}_{iq,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$

where we have considered the definition $\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k = \epsilon_{ijk} \hat{\mathbf{e}}_i$.

a.5) Let us considere that $\vec{\mathbf{a}} = \mathbf{T} \cdot \vec{\mathbf{c}} = (\mathsf{T}_{qj} \mathsf{c}_j) \hat{\mathbf{e}}_q = \mathsf{a}_q \hat{\mathbf{e}}_q$, thus:

$$
\begin{aligned}
&\vec{\nabla}_{\vec{x}} \wedge (\mathbf{T} \cdot \vec{\mathbf{c}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}} = \frac{\partial}{\partial x_p} \hat{\mathbf{e}}_p \wedge \mathsf{a}_q \hat{\mathbf{e}}_q = \frac{\partial \mathsf{a}_q}{\partial x_p} \epsilon_{ipq} \hat{\mathbf{e}}_i = \epsilon_{ipq} \frac{\partial \mathsf{a}_q}{\partial x_p} \hat{\mathbf{e}}_i = \epsilon_{ipq} \mathsf{a}_{q,p} \hat{\mathbf{e}}_i \\
&\Rightarrow \epsilon_{ipq} \mathsf{a}_{q,p} \hat{\mathbf{e}}_i = \epsilon_{ipq} (\mathsf{T}_{qj} \mathsf{c}_j)_{,p} \hat{\mathbf{e}}_i = \epsilon_{ipq} \mathsf{T}_{qj,p} \mathsf{c}_j \hat{\mathbf{e}}_i + \epsilon_{ipq} \mathsf{T}_{qj} \underbrace{\mathsf{c}_{j,p}}_{=0_{jp}} \hat{\mathbf{e}}_i = \epsilon_{ipq} \mathsf{T}_{qj,p} \mathsf{c}_j \hat{\mathbf{e}}_i
\end{aligned}
$$

where we have considered that $\vec{\mathbf{c}}$ is constant, i.e. $\mathsf{c}_{j,p} = \frac{\partial \mathsf{c}_j}{\partial x_p} = 0_{jp}$.

Note that $\epsilon_{ipq} \mathsf{T}_{qj,p}$ are the components of $(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})_{ij}$, (see (a.1)), thus

$$
\vec{\nabla}_{\vec{x}} \wedge (\mathbf{T} \cdot \vec{\mathbf{c}}) = \epsilon_{ipq} \mathsf{T}_{qj,p} \mathsf{c}_j \hat{\mathbf{e}}_i = (\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})_{ij} \mathsf{c}_j \hat{\mathbf{e}}_i = [(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}) \cdot \vec{\mathbf{c}}]_i \hat{\mathbf{e}}_i = [\vec{\mathbf{c}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \mathbf{T})^T]_i \hat{\mathbf{e}}_i
$$

$$
(\vec{\nabla}_{\vec{x}} \wedge \mathbf{T}) \cdot \vec{\mathbf{c}} = \epsilon_{ipq} \mathsf{T}_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot \mathsf{c}_k \hat{\mathbf{e}}_k = \epsilon_{ipq} \mathsf{T}_{qj,p} \mathsf{c}_k \hat{\mathbf{e}}_i \delta_{jk} = \epsilon_{ipq} \mathsf{T}_{qj,p} \mathsf{c}_j \hat{\mathbf{e}}_i
$$

b) We have already shown that $(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) = \epsilon_{ipq} \bar{\varepsilon}_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, thus

$$
\begin{aligned}
\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T &= \frac{\partial}{\partial x_s} \hat{\mathbf{e}}_s \wedge (\epsilon_{ipq} \bar{\varepsilon}_{qj,p} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) = \epsilon_{ipq} \frac{\partial \bar{\varepsilon}_{qj,p}}{\partial x_s} \hat{\mathbf{e}}_s \wedge (\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) = \epsilon_{ipq} \epsilon_{tsj} \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i \\
&= (-\epsilon_{iqp})(-\epsilon_{tjs}) \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i = \epsilon_{iqp} \epsilon_{tjs} \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i = \epsilon_{qpi} \epsilon_{jst} \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i
\end{aligned}
$$

Note that:

$$
\begin{aligned}
\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) &= \frac{\partial}{\partial x_s} \hat{\mathbf{e}}_s \wedge (\epsilon_{ipq} \bar{\varepsilon}_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \epsilon_{ipq} \frac{\partial \bar{\varepsilon}_{qj,p}}{\partial x_s} \hat{\mathbf{e}}_s \wedge (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \epsilon_{ipq} \bar{\varepsilon}_{qj,ps} \epsilon_{tsi} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_j \\
&= \epsilon_{its} \epsilon_{ipq} \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_j = (\delta_{tp} \delta_{sq} - \delta_{tq} \delta_{sp}) \bar{\varepsilon}_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_j \\
&= (\delta_{tp} \delta_{sq} \bar{\varepsilon}_{qj,ps} - \delta_{tq} \delta_{sp} \bar{\varepsilon}_{qj,ps}) \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_j = (\bar{\varepsilon}_{sj,ts} - \bar{\varepsilon}_{tj,ss}) \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_j
\end{aligned}
$$

c.1) Note that $\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F} = \vec{\nabla}_{\vec{x}} \wedge \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\boldsymbol{x}}} + \mathbf{1} \right) = \vec{\nabla}_{\vec{x}} \wedge \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\boldsymbol{x}}} \right) + \vec{\nabla}_{\vec{x}} \wedge (\mathbf{1}) = \vec{\nabla}_{\vec{x}} \wedge \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\boldsymbol{x}}} \right) = \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}$, where we have denoted by $\boldsymbol{J} = \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\boldsymbol{x}}}$. Taking into account $\bar{\varepsilon}_{qj} = J_{qj} = \frac{\partial \mathsf{u}_q}{\partial x_j} = \mathsf{u}_{q,j}$ into $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T$ we can obtain:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J})^T = \epsilon_{iqp}\epsilon_{tjs} J_{qj,ps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i = \epsilon_{iqp}\epsilon_{tjs} \mathsf{u}_{q,jps} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i$$

Note that $\mathsf{u}_{q,jps} = \mathsf{u}_{q,pjs} = \mathsf{u}_{q,psj}$, i.e. it is symmetric in js, and the tensor $\epsilon_{tjs} = -\epsilon_{tsj}$ is antisymmetric in js, so $\epsilon_{tjs}\mathsf{u}_{q,jps} = 0_{tqp}$, and $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F})^T = 0_{ti} \hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i = \mathbf{0}$.

Alternative solution:

Taking into account that

$$\epsilon_{iqp}\epsilon_{tjs} = \begin{vmatrix} \delta_{it} & \delta_{ij} & \delta_{is} \\ \delta_{qt} & \delta_{qj} & \delta_{qs} \\ \delta_{pt} & \delta_{pj} & \delta_{ps} \end{vmatrix} = \delta_{it}\delta_{qj}\delta_{ps} + \delta_{ij}\delta_{qs}\delta_{pt} + \delta_{is}\delta_{pj}\delta_{qt} - \delta_{is}\delta_{qj}\delta_{pt} - \delta_{qs}\delta_{pj}\delta_{it} - \delta_{ps}\delta_{qt}\delta_{ij}$$

then

$$\begin{aligned}
\epsilon_{iqp}\epsilon_{tjs} F_{qj,ps} &= (\delta_{it}\delta_{qj}\delta_{ps} + \delta_{ij}\delta_{qs}\delta_{pt} + \delta_{is}\delta_{pj}\delta_{qt} - \delta_{is}\delta_{qj}\delta_{pt} - \delta_{qs}\delta_{pj}\delta_{it} - \delta_{ps}\delta_{qt}\delta_{ij})\, \mathsf{u}_{q,jps} \\
&= \delta_{it}\delta_{qj}\delta_{ps}\mathsf{u}_{q,jps} + \delta_{ij}\delta_{qs}\delta_{pt}\mathsf{u}_{q,jps} + \delta_{is}\delta_{pj}\delta_{qt}\mathsf{u}_{q,jps} - \delta_{is}\delta_{qj}\delta_{pt}\mathsf{u}_{q,jps} - \delta_{qs}\delta_{pj}\delta_{it}\mathsf{u}_{q,jps} - \delta_{ps}\delta_{qt}\delta_{ij}\mathsf{u}_{q,jps} \\
&= \delta_{it}\mathsf{u}_{j,jss} + \mathsf{u}_{s,its} + \mathsf{u}_{t,ppi} - \mathsf{u}_{j,jti} - \delta_{it}\mathsf{u}_{s,pps} - \mathsf{u}_{t,ipp} = 0_{ti}
\end{aligned}$$

Note that $\delta_{it}\mathsf{u}_{j,jss} = \delta_{it}\mathsf{u}_{p,pss} = \delta_{it}\mathsf{u}_{p,ssp} = \delta_{it}\mathsf{u}_{s,pps}$, $\mathsf{u}_{s,its} = \mathsf{u}_{j,itj} = \mathsf{u}_{j,jti}$, $\mathsf{u}_{t,ppi} = \mathsf{u}_{t,ipp}$.

We express $\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}^T$ in indicial notation:

$$\begin{aligned}
\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}^T &= \frac{\partial}{\partial x_p} \hat{\mathbf{e}}_p \wedge J_{qj}(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_q) = \frac{\partial J_{qj}}{\partial x_p} \hat{\mathbf{e}}_p \wedge \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_q = J_{qj,p}\epsilon_{ipj} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_q \\
&= \epsilon_{ipj} J_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_q = \epsilon_{ipj}\mathsf{u}_{q,jp} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_q = 0_{ip} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_q
\end{aligned}$$

Note that $\mathsf{u}_{q,jp} = \mathsf{u}_{q,pj}$ is symmetric in jp meanwhile $\epsilon_{ipj} = -\epsilon_{ijp}$ is antisymmetric in jp.

c.2) We express $\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}$ in indicial notation, (see item (a.1)):

$$\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J} = \epsilon_{ipq} J_{qj,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \epsilon_{ipq}\mathsf{u}_{q,jp} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

Expanding the term $\epsilon_{ipq}\mathsf{u}_{q,jp}$ we can obtain:

$$\begin{array}{cccccc}
\epsilon_{ipq}\mathsf{u}_{q,jp} = & \underbrace{\epsilon_{ip1}\mathsf{u}_{1,jp}} & + & \underbrace{\epsilon_{ip2}\mathsf{u}_{2,jp}} & + & \underbrace{\epsilon_{ip3}\mathsf{u}_{3,jp}} \\
& \epsilon_{i11}\mathsf{u}_{1,j1} & + & \epsilon_{i12}\mathsf{u}_{2,j1} & + & \epsilon_{i13}\mathsf{u}_{3,j1} \\
& + & & + & & + \\
& \epsilon_{i21}\mathsf{u}_{1,j2} & + & \epsilon_{i22}\mathsf{u}_{2,j2} & + & \epsilon_{i23}\mathsf{u}_{3,j2} \\
& + & & + & & + \\
& \epsilon_{i31}\mathsf{u}_{1,j3} & + & \epsilon_{i32}\mathsf{u}_{2,j3} & + & \epsilon_{ip3}\mathsf{u}_{3,j3}
\end{array}$$

thus,

$$(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J})_{ij} = \begin{bmatrix} \mathsf{u}_{3,12} - \mathsf{u}_{2,13} & \mathsf{u}_{3,22} - \mathsf{u}_{2,23} & \mathsf{u}_{3,32} - \mathsf{u}_{2,33} \\ \mathsf{u}_{1,13} - \mathsf{u}_{3,11} & \mathsf{u}_{1,23} - \mathsf{u}_{3,21} & \mathsf{u}_{1,33} - \mathsf{u}_{3,31} \\ \mathsf{u}_{2,11} - \mathsf{u}_{1,12} & \mathsf{u}_{2,21} - \mathsf{u}_{1,22} & \mathsf{u}_{2,31} - \mathsf{u}_{1,32} \end{bmatrix} = \begin{bmatrix} J_{31,2} - J_{21,3} & J_{32,2} - J_{22,3} & J_{33,2} - J_{23,3} \\ J_{11,3} - J_{31,1} & J_{12,3} - J_{32,1} & J_{13,3} - J_{33,1} \\ J_{21,1} - J_{11,2} & J_{22,1} - J_{12,2} & J_{23,1} - J_{13,2} \end{bmatrix}$$

Note that

$$(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}^T)_{ij} = \begin{bmatrix} J_{13,2}-J_{12,3} & J_{23,2}-J_{22,3} & J_{33,2}-J_{32,3} \\ J_{11,3}-J_{13,1} & J_{21,3}-J_{23,1} & J_{31,3}-J_{33,1} \\ J_{12,1}-J_{11,2} & J_{22,1}-J_{21,2} & J_{32,1}-J_{31,2} \end{bmatrix} = \begin{bmatrix} \mathrm{u}_{1,32}-\mathrm{u}_{1,23} & \mathrm{u}_{2,32}-\mathrm{u}_{2,23} & \mathrm{u}_{3,32}-\mathrm{u}_{3,23} \\ \mathrm{u}_{1,13}-\mathrm{u}_{1,31} & \mathrm{u}_{2,13}-\mathrm{u}_{2,31} & \mathrm{u}_{3,13}-\mathrm{u}_{3,31} \\ \mathrm{u}_{1,21}-\mathrm{u}_{1,12} & \mathrm{u}_{2,21}-\mathrm{u}_{2,12} & \mathrm{u}_{3,21}-\mathrm{u}_{3,12} \end{bmatrix}$$
$$= 0_{ij}$$

Note that, if

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{J}+\boldsymbol{J}^T) \quad \Rightarrow \quad \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \frac{1}{2}\underbrace{\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J})^T}_{=\boldsymbol{0}} + \frac{1}{2}\vec{\nabla}_{\vec{x}} \wedge (\underbrace{\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{J}^T}_{=\boldsymbol{0}})^T = \boldsymbol{0}$$

where $\boldsymbol{J} = \dfrac{\partial \vec{\mathbf{u}}}{\partial \vec{x}}$.

Problem 1.111

Let $\vec{\mathbf{a}}$ and $\vec{v}$ be vectors, show that

$$(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{\mathbf{a}} = [\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T] \cdot \vec{\mathbf{a}}$$

Solution:

If we consider $(\vec{\nabla}_{\vec{x}} \wedge \vec{v})_i = \epsilon_{ijk} v_{k,j}$, then $[(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{\mathbf{a}}]_s = \epsilon_{sip}\epsilon_{ijk} v_{k,j}\mathrm{a}_p$. Note also that the relationship $\epsilon_{sip}\epsilon_{ijk} = \epsilon_{psi}\epsilon_{jki} = \delta_{pj}\delta_{sk} - \delta_{pk}\delta_{sj}$ holds, then

$$\begin{aligned}[(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{\mathbf{a}}]_s &= \epsilon_{sip}\epsilon_{ijk} v_{k,j}\mathrm{a}_p = (\delta_{pj}\delta_{sk} - \delta_{pk}\delta_{sj}) v_{k,j}\mathrm{a}_p = (\delta_{pj}\delta_{sk} v_{k,j} - \delta_{pk}\delta_{sj} v_{k,j})\mathrm{a}_p \\ &= (v_{s,p} - v_{p,s})\mathrm{a}_p = \left\{ [\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T] \cdot \vec{\mathbf{a}} \right\}_s\end{aligned}$$

Alternative solution:

If we denote by $\boldsymbol{\ell} = \nabla_{\vec{x}}\vec{v}$, then $[\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T] = 2(\nabla_{\vec{x}}\vec{v})^{skew} = 2\boldsymbol{\ell}^{skew}$. Note that the axial vector associated with the antisymmetric tensor $(\nabla_{\vec{x}}\vec{v})^{skew} = (\vec{v} \otimes \vec{\nabla}_{\vec{x}})^{skew}$ is the vector $\vec{\varphi} = \frac{1}{2}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})$, (see **Problem 1.37**). If we recall the property of an antisymmetric tensor $(\nabla_{\vec{x}}\vec{v})^{skew} \cdot \vec{\mathbf{a}} = \vec{\varphi} \wedge \vec{\mathbf{a}}$, we can conclude that

$$\begin{aligned}&(\nabla_{\vec{x}}\vec{v})^{skew} \cdot \vec{\mathbf{a}} = \vec{\varphi} \wedge \vec{\mathbf{a}} \quad \Rightarrow \quad \frac{1}{2}[\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T] \cdot \vec{\mathbf{a}} = \frac{1}{2}(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{\mathbf{a}} \\ &\Rightarrow [\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T] \cdot \vec{\mathbf{a}} = (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \wedge \vec{\mathbf{a}}\end{aligned}$$

Problem 1.112

Let $\vec{\mathbf{u}} = \vec{\mathbf{u}}(\vec{x})$ be a vector field. By means of components of $\vec{\mathbf{u}}$, a) show that $\nabla^2_{\vec{x}}\vec{\mathbf{u}} = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{u}})$ when $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) = \vec{\mathbf{0}}$, b) show that $\nabla^2_{\vec{x}}\vec{\mathbf{u}} = -\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}})$ when $\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{u}}) = \vec{\mathbf{0}}$.

Solution:

We have proven in **Problem 1.106** that the following is true:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}}) - \nabla^2_{\vec{x}}\vec{\mathbf{a}} \quad \xrightarrow{indicial} \quad \epsilon_{ilq}\epsilon_{qjk}\mathrm{a}_{k,jl} = \mathrm{a}_{j,ji} - \mathrm{a}_{i,jj}$$

Then, we can obtain

$$\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}}\vec{\mathbf{u}}) \equiv \nabla^2_{\vec{x}}\vec{\mathbf{u}} = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{u}}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) \quad \xrightarrow{indicial} \quad \mathrm{u}_{i,jj} = \mathrm{u}_{j,ji} - \epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jl}$$

Then, it is easy to verify that:

a) $$\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \vec{\mathbf{u}}) \equiv \nabla^2_{\vec{x}} \vec{\mathbf{u}} = \nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{u}}) - \underbrace{\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}})}_{=\vec{\mathbf{0}}} \quad \Rightarrow \quad \nabla^2_{\vec{x}} \vec{\mathbf{u}} = \nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{u}})$$

Components:

$$\mathrm{u}_{i,jj} = \mathrm{u}_{j,ji} \quad \Rightarrow \quad \mathrm{u}_{i,11} + \mathrm{u}_{i,22} + \mathrm{u}_{i,33} = \mathrm{u}_{1,1i} + \mathrm{u}_{2,2i} + \mathrm{u}_{3,3i}$$

$$\Rightarrow \begin{cases} \mathrm{u}_{1,11} + \mathrm{u}_{1,22} + \mathrm{u}_{1,33} = \mathrm{u}_{1,11} + \mathrm{u}_{2,21} + \mathrm{u}_{3,31} \\ \mathrm{u}_{2,11} + \mathrm{u}_{2,22} + \mathrm{u}_{2,33} = \mathrm{u}_{1,12} + \mathrm{u}_{2,22} + \mathrm{u}_{3,32} \\ \mathrm{u}_{3,11} + \mathrm{u}_{3,22} + \mathrm{u}_{3,33} = \mathrm{u}_{1,13} + \mathrm{u}_{2,23} + \mathrm{u}_{3,33} \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{u}_{1,22} + \mathrm{u}_{1,33} = \mathrm{u}_{2,21} + \mathrm{u}_{3,31} \\ \mathrm{u}_{2,11} + \mathrm{u}_{2,33} = \mathrm{u}_{1,12} + \mathrm{u}_{3,32} \\ \mathrm{u}_{3,11} + \mathrm{u}_{3,22} = \mathrm{u}_{1,13} + \mathrm{u}_{2,23} \end{cases} \tag{1.123}$$

Note that in the Cartesian basis we have:

$$\vec{\mathbf{u}} = \mathrm{u}_i \hat{\mathbf{e}}_i = \mathrm{u}_1 \hat{\mathbf{e}}_1 + \mathrm{u}_2 \hat{\mathbf{e}}_2 + \mathrm{u}_3 \hat{\mathbf{e}}_3$$

$$(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) \equiv \mathrm{rot}(\vec{\mathbf{u}}) = (\mathrm{rot}(\vec{\mathbf{u}}))_i \hat{\mathbf{e}}_i = \underbrace{\left(\frac{\partial \mathrm{u}_3}{\partial x_2} - \frac{\partial \mathrm{u}_2}{\partial x_3} \right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_1} \hat{\mathbf{e}}_1 + \underbrace{\left(\frac{\partial \mathrm{u}_1}{\partial x_3} - \frac{\partial \mathrm{u}_3}{\partial x_1} \right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_2} \hat{\mathbf{e}}_2 + \underbrace{\left(\frac{\partial \mathrm{u}_2}{\partial x_1} - \frac{\partial \mathrm{u}_1}{\partial x_2} \right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_3} \hat{\mathbf{e}}_3$$

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) = \left(\frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_2} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_3} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_1} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_2} \right) \hat{\mathbf{e}}_3$$

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}})]_i = \begin{Bmatrix} \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_2} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_3} \\ \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_3} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_1} \\ \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_1} - \frac{\partial (\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x_2}\left(\frac{\partial \mathrm{u}_2}{\partial x_1} - \frac{\partial \mathrm{u}_1}{\partial x_2} \right) - \frac{\partial}{\partial x_3}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} - \frac{\partial \mathrm{u}_3}{\partial x_1} \right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial \mathrm{u}_3}{\partial x_2} - \frac{\partial \mathrm{u}_2}{\partial x_3} \right) - \frac{\partial}{\partial x_1}\left(\frac{\partial \mathrm{u}_2}{\partial x_1} - \frac{\partial \mathrm{u}_1}{\partial x_2} \right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} - \frac{\partial \mathrm{u}_3}{\partial x_1} \right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \mathrm{u}_3}{\partial x_2} - \frac{\partial \mathrm{u}_2}{\partial x_3} \right) \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathrm{u}_{2,12} - \mathrm{u}_{1,22} - \mathrm{u}_{1,33} + \mathrm{u}_{3,13} \\ \mathrm{u}_{3,23} - \mathrm{u}_{2,33} - \mathrm{u}_{2,11} + \mathrm{u}_{1,21} \\ \mathrm{u}_{1,31} - \mathrm{u}_{3,11} - \mathrm{u}_{3,22} + \mathrm{u}_{2,32} \end{Bmatrix}$$

If we are considering that $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) = \vec{\mathbf{0}}$ then:

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}})]_i = \begin{Bmatrix} \mathrm{u}_{2,12} - \mathrm{u}_{1,22} - \mathrm{u}_{1,33} + \mathrm{u}_{3,13} \\ \mathrm{u}_{3,23} - \mathrm{u}_{2,33} - \mathrm{u}_{2,11} + \mathrm{u}_{1,21} \\ \mathrm{u}_{1,31} - \mathrm{u}_{3,11} - \mathrm{u}_{3,22} + \mathrm{u}_{2,32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} \mathrm{u}_{2,12} + \mathrm{u}_{3,13} = \mathrm{u}_{1,22} + \mathrm{u}_{1,33} \\ \mathrm{u}_{3,23} + \mathrm{u}_{1,21} = \mathrm{u}_{2,33} + \mathrm{u}_{2,11} \\ \mathrm{u}_{1,31} + \mathrm{u}_{2,32} = \mathrm{u}_{3,11} + \mathrm{u}_{3,22} \end{cases}$$

which are the same conditions as those presented in equation (1.123).

b) $$\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \vec{\mathbf{u}}) \equiv \nabla^2_{\vec{x}} \vec{\mathbf{u}} = \underbrace{\nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{u}})}_{=\vec{\mathbf{0}}} - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}}) \quad \Rightarrow \quad \nabla^2_{\vec{x}} \vec{\mathbf{u}} = -\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{u}})$$

Components

$$\mathrm{u}_{i,jj} = -\epsilon_{ilq} \epsilon_{qjk} \mathrm{u}_{k,jl} \quad \Rightarrow \quad \begin{cases} \mathrm{u}_{1,11} + \mathrm{u}_{1,22} + \mathrm{u}_{1,33} = -(\mathrm{u}_{2,12} - \mathrm{u}_{1,22} - \mathrm{u}_{1,33} + \mathrm{u}_{3,13}) \\ \mathrm{u}_{2,11} + \mathrm{u}_{2,22} + \mathrm{u}_{2,33} = -(\mathrm{u}_{3,23} - \mathrm{u}_{2,33} - \mathrm{u}_{2,11} + \mathrm{u}_{1,21}) \\ \mathrm{u}_{3,11} + \mathrm{u}_{3,22} + \mathrm{u}_{3,33} = -(\mathrm{u}_{1,31} - \mathrm{u}_{3,11} - \mathrm{u}_{3,22} + \mathrm{u}_{2,32}) \end{cases} \tag{1.124}$$

And if we consider $\nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{u}}) = \vec{\mathbf{0}}$ we can obtain:

$$[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{\mathbf{u}})]_i = \mathrm{u}_{1,1i}+\mathrm{u}_{2,2i}+\mathrm{u}_{3,3i}=0_i \quad\Rightarrow\quad \begin{cases}\mathrm{u}_{1,11}+\mathrm{u}_{2,21}+\mathrm{u}_{3,31}=0\\ \mathrm{u}_{1,12}+\mathrm{u}_{2,22}+\mathrm{u}_{3,32}=0\\ \mathrm{u}_{1,13}+\mathrm{u}_{2,23}+\mathrm{u}_{3,33}=0\end{cases} \quad\Rightarrow\quad \begin{cases}\mathrm{u}_{3,31}+\mathrm{u}_{2,21}=-\mathrm{u}_{1,11}\\ \mathrm{u}_{1,12}+\mathrm{u}_{3,32}=-\mathrm{u}_{2,22}\\ \mathrm{u}_{1,13}+\mathrm{u}_{2,23}=-\mathrm{u}_{3,33}\end{cases}$$

If we replace the above equations into (1.124) we prove that the equality holds.

Problem 1.113

Let $\boldsymbol{\sigma}$ be a second-order tensor field, and $\vec{\mathbf{a}}$ be a vector field. Show the identities:

a) $$\boxed{\nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}\wedge\boldsymbol{\sigma}) = \boldsymbol{\epsilon}:[(\nabla_{\vec{x}}\vec{\mathbf{a}})\cdot\boldsymbol{\sigma}^T]+\vec{\mathbf{a}}\wedge(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})} \qquad (1.125)$$

b) $$\boxed{\nabla_{\vec{x}}\cdot(\boldsymbol{\sigma}\wedge\vec{\mathbf{a}}) = \vec{\mathbf{a}}\cdot[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\sigma}^T]-\boldsymbol{\sigma}\cdot[\vec{\nabla}_{\vec{x}}\wedge\vec{\mathbf{a}}]} \qquad (1.126)$$

where $\boldsymbol{\epsilon}$ is the Levi-Civita tensor (third-order tensor).

Solution:

a)

$$\vec{\mathbf{a}}\wedge\boldsymbol{\sigma} = \mathrm{a}_i\hat{\mathbf{e}}_i\wedge\sigma_{jk}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k = \sigma_{jk}\mathrm{a}_i\epsilon_{pij}\hat{\mathbf{e}}_p\otimes\hat{\mathbf{e}}_k$$

$$\Rightarrow \nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}\wedge\boldsymbol{\sigma}) = \frac{\partial}{\partial x_q}(\sigma_{jk}\mathrm{a}_i\epsilon_{pij}\hat{\mathbf{e}}_p\otimes\hat{\mathbf{e}}_k)\cdot\hat{\mathbf{e}}_q = \frac{\partial}{\partial x_q}(\sigma_{jk}\mathrm{a}_i\epsilon_{pij}\hat{\mathbf{e}}_p)\delta_{kq} = \frac{\partial}{\partial x_k}(\sigma_{jk}\mathrm{a}_i\epsilon_{pij}\hat{\mathbf{e}}_p)$$

$$\Rightarrow \nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}\wedge\boldsymbol{\sigma}) = (\sigma_{jk}\mathrm{a}_i\epsilon_{pij})_{,k}\hat{\mathbf{e}}_p = (\epsilon_{pij}\sigma_{jk,k}\mathrm{a}_i+\epsilon_{pij}\sigma_{jk}\mathrm{a}_{i,k})\hat{\mathbf{e}}_p$$

Note that $\epsilon_{pij}\sigma_{jk,k}\mathrm{a}_i = \epsilon_{pij}(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})_j\mathrm{a}_i = \epsilon_{pij}(\vec{\mathbf{a}})_i(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})_j = [\vec{\mathbf{a}}\wedge(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})]_p$

and $\epsilon_{pij}\sigma_{jk}\mathrm{a}_{i,k} = \epsilon_{pij}\sigma_{jk}(\nabla_{\vec{x}}\vec{\mathbf{a}})_{ik} = \epsilon_{pij}(\nabla_{\vec{x}}\vec{\mathbf{a}})_{ik}\sigma_{jk} = \epsilon_{pij}[(\nabla_{\vec{x}}\vec{\mathbf{a}})\cdot\boldsymbol{\sigma}^T]_{ij} = \{\boldsymbol{\epsilon}:[(\nabla_{\vec{x}}\vec{\mathbf{a}})\cdot\boldsymbol{\sigma}^T]\}_p$

with that we show the equation in (1.125).

Note that when $\vec{\mathbf{a}}=\vec{x}$ the equation (1.125) becomes:

$$\nabla_{\vec{x}}\cdot(\vec{x}\wedge\boldsymbol{\sigma}) = \boldsymbol{\epsilon}:[(\nabla_{\vec{x}}\vec{x})\cdot\boldsymbol{\sigma}^T]+\vec{x}\wedge(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}) = \boldsymbol{\epsilon}:[\mathbf{1}\cdot\boldsymbol{\sigma}^T]+\vec{x}\wedge(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}) = \boldsymbol{\epsilon}:\boldsymbol{\sigma}^T+\vec{x}\wedge(\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})$$

b)

$$\boldsymbol{\sigma}\wedge\vec{\mathbf{a}} = \sigma_{jk}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\wedge\mathrm{a}_i\hat{\mathbf{e}}_i = \sigma_{jk}\mathrm{a}_i\epsilon_{pki}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_p$$

$$\Rightarrow \nabla_{\vec{x}}\cdot(\boldsymbol{\sigma}\wedge\vec{\mathbf{a}}) = \frac{\partial}{\partial x_q}(\sigma_{jk}\mathrm{a}_i\epsilon_{pki}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_p)\cdot\hat{\mathbf{e}}_q = \frac{\partial}{\partial x_q}(\sigma_{jk}\mathrm{a}_i\epsilon_{pki}\hat{\mathbf{e}}_j)\delta_{pq} = \frac{\partial}{\partial x_p}(\sigma_{jk}\mathrm{a}_i\epsilon_{pki}\hat{\mathbf{e}}_j)$$

$$\Rightarrow \nabla_{\vec{x}}\cdot(\boldsymbol{\sigma}\wedge\vec{\mathbf{a}}) = (\sigma_{jk}\mathrm{a}_i\epsilon_{pki})_{,p}\hat{\mathbf{e}}_j = (\epsilon_{pki}\sigma_{jk,p}\mathrm{a}_i+\epsilon_{pki}\sigma_{jk}\mathrm{a}_{i,p})\hat{\mathbf{e}}_j$$

Note that

$$\epsilon_{pki}\sigma_{jk,p}\mathrm{a}_i = \epsilon_{ipk}\sigma_{jk,p}\mathrm{a}_i = [\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\sigma}^T]_{ij}\mathrm{a}_i = \{\vec{\mathbf{a}}\cdot[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\sigma}^T]\}_j$$

$$\epsilon_{pki}\sigma_{jk}\mathrm{a}_{i,p} = \epsilon_{kip}\mathrm{a}_{i,p}\sigma_{jk} = -\epsilon_{kpi}\mathrm{a}_{i,p}\sigma_{jk} = -(\vec{\nabla}_{\vec{x}}\wedge\vec{\mathbf{a}})_k\sigma_{jk} = -\{\boldsymbol{\sigma}\cdot[\vec{\nabla}_{\vec{x}}\wedge\vec{\mathbf{a}}]\}_j$$

Problem 1.114

Consider that $\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}+\vec{\mathbf{p}}=\vec{\mathbf{q}}$, where $\boldsymbol{\sigma}$ is a second-order tensor field, and $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are vector fields. The equation $\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}+\vec{\mathbf{p}}=\vec{\mathbf{q}}$ fulfills at any point of the volume V which is delimitated by surface S. Show that, if the following equation:

$$\int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_S \vec{x} \wedge \vec{\mathbf{t}}^* dS = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

is also valid, then $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ holds. Consider that $\vec{\mathbf{t}}^* = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the outward pointing unit normal to surface S.

Solution:

$$\int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_S \vec{x} \wedge \vec{\mathbf{t}}^* dS = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

$$\Rightarrow \int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_S \vec{x} \wedge (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) dS = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

Note that $(\vec{x} \wedge \vec{\mathbf{t}}^*)_i = \epsilon_{ijk} x_j \mathrm{t}_k^* = \epsilon_{ijk} x_j (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})_k = \epsilon_{ijk} x_j \sigma_{kp} \hat{\mathrm{n}}_p = (\vec{x} \wedge \boldsymbol{\sigma})_{ip} \hat{\mathrm{n}}_p = (\vec{x} \wedge \boldsymbol{\sigma}) \cdot \hat{\mathbf{n}}$, with that we can obtain:

$$\Rightarrow \int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_S (\vec{x} \wedge \boldsymbol{\sigma}) \cdot \hat{\mathbf{n}} dS = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

By applying the divergence theorem to the surface integral we can obtain:

$$\Rightarrow \int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_V \nabla_{\vec{x}} \cdot (\vec{x} \wedge \boldsymbol{\sigma}) dV = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

It was proven in **Problem 1.113** that $\nabla_{\vec{x}} \cdot (\vec{x} \wedge \boldsymbol{\sigma}) = \boldsymbol{\epsilon} : \boldsymbol{\sigma}^T + \vec{x} \wedge (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma})$, and by replacing it into the above equation we can obtain:

$$\Rightarrow \int_V \vec{x} \wedge \vec{\mathbf{p}} dV + \int_V [\boldsymbol{\epsilon} : \boldsymbol{\sigma}^T + \vec{x} \wedge (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma})] dV = \int_V \vec{x} \wedge \vec{\mathbf{q}} dV$$

$$\Rightarrow \int_V [\vec{x} \wedge \vec{\mathbf{p}} + \boldsymbol{\epsilon} : \boldsymbol{\sigma}^T + \vec{x} \wedge (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) - \vec{x} \wedge \vec{\mathbf{q}}] dV = \vec{\mathbf{0}}$$

$$\Rightarrow \int_V \{\vec{x} \wedge \underbrace{[(\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) + \vec{\mathbf{p}} - \vec{\mathbf{q}}]}_{=\vec{\mathbf{0}}} + \boldsymbol{\epsilon} : \boldsymbol{\sigma}^T\} dV = \vec{\mathbf{0}}$$

$$\Rightarrow \int_V \boldsymbol{\epsilon} : \boldsymbol{\sigma}^T dV = \vec{\mathbf{0}}$$

If $(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^T)$ if valid for the whole volume it is also valid locally, so $\boldsymbol{\epsilon} : \boldsymbol{\sigma}^T = \vec{\mathbf{0}}$. Note that $(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^T)_i = \epsilon_{ijk} \sigma_{kj} = -\epsilon_{ikj} \sigma_{kj} = 0_i$, i.e. the tensor $\boldsymbol{\epsilon}$ is antisymmetric in kj, since the double scalar product between a symmetric and an antisymmetric tensor is zero, then we prove that $\boldsymbol{\sigma}$ is symmetric, i.e. $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$. We can also prove that by means of components:

$$\begin{cases} \epsilon_{1jk}\sigma_{kj} = 0 & \Rightarrow & \epsilon_{123}\sigma_{32} + \epsilon_{132}\sigma_{23} = 0 & \Rightarrow & \sigma_{32} - \sigma_{23} = 0 & \Rightarrow & \sigma_{32} = \sigma_{23} \\ \epsilon_{2jk}\sigma_{kj} = 0 & \Rightarrow & \epsilon_{213}\sigma_{31} + \epsilon_{231}\sigma_{13} = 0 & \Rightarrow & -\sigma_{31} + \sigma_{13} = 0 & \Rightarrow & \sigma_{31} = \sigma_{13} \\ \epsilon_{3jk}\sigma_{kj} = 0 & \Rightarrow & \epsilon_{312}\sigma_{21} + \epsilon_{321}\sigma_{12} = 0 & \Rightarrow & \sigma_{21} - \sigma_{12} = 0 & \Rightarrow & \sigma_{21} = \sigma_{12} \end{cases}$$

Problem 1.115

a) Show that

$$\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge \vec{x}\} = \{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge \vec{x} - (\boldsymbol{\epsilon} \cdot : (\nabla_{\vec{x}} \boldsymbol{\varepsilon}))\mathbf{1} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]$$

where $\boldsymbol{\varepsilon}$ is a second-order tensor, $\vec{x}$ is the vector position, $\boldsymbol{\epsilon}$ is the Levi-Civita tensor (third-order tensor), and $\mathbf{1}$ is the second-order unit tensor. b) Simplify the above equation by considering that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$ is symmetric second-order tensor, i.e. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$.

Solution:

$$\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge \vec{x}\} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \wedge \left\{\left[\frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j \wedge \varepsilon_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q\right]^T \wedge x_k\hat{\mathbf{e}}_k\right\}$$

$$= \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \wedge \left\{\left[\frac{\partial \varepsilon_{pq}}{\partial x_j}\epsilon_{tjp}\hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_q\right]^T \wedge x_k\hat{\mathbf{e}}_k\right\}$$

Applying the transpose property we can obtain:

$$\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge \vec{x}\} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \wedge \left\{\left[\frac{\partial \varepsilon_{pq}}{\partial x_j}\epsilon_{tjp}\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_t\right] \wedge x_k\hat{\mathbf{e}}_k\right\} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \wedge \left\{\frac{\partial \varepsilon_{pq}}{\partial x_j}x_k\epsilon_{tjp}\epsilon_{stk}\hat{\mathbf{e}}_q \otimes \hat{\mathbf{e}}_s\right\}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \varepsilon_{pq}}{\partial x_j}x_k\epsilon_{tjp}\epsilon_{stk}\right)\epsilon_{niq}\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_s = \epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\frac{\partial}{\partial x_i}\left(\frac{\partial \varepsilon_{pq}}{\partial x_j}x_k\right)\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_s$$

$$= \epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\left(\frac{\partial}{\partial x_i}\left(\frac{\partial \varepsilon_{pq}}{\partial x_j}\right)x_k + \frac{\partial x_k}{\partial x_i}\frac{\partial \varepsilon_{pq}}{\partial x_j}\right)\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_s$$

$$= \epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\left(\frac{\partial^2 \varepsilon_{pq}}{\partial x_i \partial x_j}x_k + \delta_{ki}\frac{\partial \varepsilon_{pq}}{\partial x_j}\right)\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_s$$

$$= \left(\epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\frac{\partial^2 \varepsilon_{pq}}{\partial x_i \partial x_j}x_k + \epsilon_{nkq}\epsilon_{tjp}\epsilon_{stk}\frac{\partial \varepsilon_{pq}}{\partial x_j}\right)\hat{\mathbf{e}}_n \otimes \hat{\mathbf{e}}_s$$

Note that the term $\epsilon_{nkq}\epsilon_{tjp}\epsilon_{stk}\frac{\partial \varepsilon_{pq}}{\partial x_j} = \epsilon_{nkq}\epsilon_{tjp}\epsilon_{stk}\varepsilon_{pq,j} = -\epsilon_{nqk}\epsilon_{tjp}\epsilon_{stk}\varepsilon_{pq,j}$ can be expressed as follows:

$$\begin{aligned}
\epsilon_{nkq}\epsilon_{tjp}\epsilon_{stk}\varepsilon_{pq,j} &= -\epsilon_{nqk}\epsilon_{tjp}\epsilon_{stk}\varepsilon_{pq,j} = -(\delta_{ns}\delta_{qt} - \delta_{nt}\delta_{qs})\epsilon_{tjp}\varepsilon_{pq,j} = -\delta_{ns}\delta_{qt}\epsilon_{tjp}\varepsilon_{pq,j} + \delta_{nt}\delta_{qs}\epsilon_{tjp}\varepsilon_{pq,j} \\
&= -\delta_{ns}\epsilon_{tjp}\varepsilon_{pt,j} + \epsilon_{njp}\varepsilon_{ps,j} = -\epsilon_{ptj}\varepsilon_{pt,j}\delta_{ns} + \epsilon_{njp}\varepsilon_{ps,j} \\
&= -\epsilon_{ptj}\varepsilon_{pt,j}\delta_{ns} + \epsilon_{njp}\varepsilon_{ps,j} = -(\epsilon_{ptj}\varepsilon_{pt})_{,j}\delta_{ns} + \epsilon_{njp}\varepsilon_{ps,j} \\
&= -(\boldsymbol{\epsilon}\cdot\!:(\nabla_{\vec{x}}\boldsymbol{\varepsilon}))\delta_{ns} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]_{ns} = -(\nabla_{\vec{x}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\varepsilon}))\delta_{ns} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]_{ns} \\
&= \{-(\boldsymbol{\epsilon}\cdot\!:(\nabla_{\vec{x}}\boldsymbol{\varepsilon}))\mathbf{1} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]\}_{ns} = \{-(\nabla_{\vec{x}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\varepsilon}))\mathbf{1} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]\}_{ns}
\end{aligned}$$

and

$$\epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\frac{\partial^2 \varepsilon_{pq}}{\partial x_i \partial x_j}x_k = \epsilon_{niq}\epsilon_{tjp}\epsilon_{stk}\varepsilon_{pq,ij}x_k = \left(\{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge \vec{x}\right)_{ns}$$

b) If $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$ we can show that $\boldsymbol{\epsilon}\cdot\!:(\nabla_{\vec{x}}\boldsymbol{\varepsilon}) = \nabla_{\vec{x}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\varepsilon}) = 0$, then

$$\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge \vec{x}\} = \{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge \vec{x} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]$$

Problem 1.116

Let $\vec{v}$ be a vector field in function of $\vec{x}$, i.e. $\vec{v} = \vec{v}(\vec{x})$, whose components are given by:

$$\begin{cases} v_1 = x_1 - 5x_2 + 2x_3 \\ v_2 = 5x_1 + x_2 - 3x_3 \\ v_3 = -2x_1 + 3x_2 + x_3 \end{cases}$$

a) Obtain the gradient of $\vec{\boldsymbol{v}}$; b) Obtain $(\nabla_{\vec{x}}\vec{\boldsymbol{v}}):\mathbf{1}$; c) Apply the additive decomposition of the tensor $\nabla_{\vec{x}}\vec{\boldsymbol{v}}$ into a symmetric and antisymmetric parts; d) Obtain the axial vector associated with the antisymmetric tensor $(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}$.

Solution: a)

$$\nabla_{\vec{x}}\vec{\boldsymbol{v}} = \frac{\partial \vec{\boldsymbol{v}}}{\partial \vec{\boldsymbol{x}}} \xrightarrow{components} (\nabla_{\vec{x}}\vec{\boldsymbol{v}})_{ij} = v_{i,j} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 \\ 5 & 1 & -3 \\ -2 & 3 & 1 \end{bmatrix}$$

b) $(\nabla_{\vec{x}}\vec{\boldsymbol{v}}):\mathbf{1} = \mathrm{Tr}(\nabla_{\vec{x}}\vec{\boldsymbol{v}}) = 1+1+1 = 3$

c) $\nabla_{\vec{x}}\vec{\boldsymbol{v}} = (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{sym} + (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew} = \underbrace{\frac{1}{2}\left[(\nabla_{\vec{x}}\vec{\boldsymbol{v}}) + (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^T\right]}_{=(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{sym}} + \underbrace{\frac{1}{2}\left[(\nabla_{\vec{x}}\vec{\boldsymbol{v}}) - (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^T\right]}_{=(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}}$

Then, the components of $(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{sym}$ and $(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}$ are given, respectively, by:

$$[(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{sym}]_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad [(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}]_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right] = \begin{bmatrix} 0 & -5 & 2 \\ 5 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

d) Remember that

$$(\mathbf{W})_{ij} \equiv [(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}]_{ij} \equiv v_{i,j}^{skew} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ \mathsf{W}_{21} & 0 & \mathsf{W}_{23} \\ \mathsf{W}_{31} & \mathsf{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{1.127}$$

where w_1, w_2, w_3 are the components of the axial vector $\vec{\boldsymbol{w}}$ associated with the antisymmetric tensor $\mathbf{W} \equiv (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}$, then, to the proposed problem we have:

$$\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 2 \\ 5 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} w_1 = 3 \\ w_2 = 2 \\ w_3 = 5 \end{cases}$$

The axial vector, in the Cartesian basis, is $\vec{\boldsymbol{w}} = 3\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3$.

Alternative solution d) In **Problem 1.37** where we have shown that $\frac{1}{2}(\vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{x}})$ is the axial vector associated with the antisymmetric tensor $(\vec{\boldsymbol{x}} \otimes \vec{\boldsymbol{a}})^{skew}$. Then, the axial vector associated with the antisymmetric tensor $(\nabla_{\vec{x}} \vec{\boldsymbol{v}})^{skew} = [(\vec{\boldsymbol{v}}) \otimes (\vec{\nabla}_{\vec{x}})]^{skew}$ is the vector $\vec{\boldsymbol{w}} = \frac{1}{2}(\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}})$, thus

$$\vec{\boldsymbol{w}} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \frac{1}{2}\left[\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_1 - \left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\hat{\mathbf{e}}_3\right]$$

$$= \frac{1}{2}[(3-(-3))\hat{\mathbf{e}}_1 - ((-2)-(2))\hat{\mathbf{e}}_2 + (5-(-5))\hat{\mathbf{e}}_3] = 3\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3$$

Problem 1.117

Let $\boldsymbol{\ell} = \nabla_{\vec{x}} \vec{\boldsymbol{v}}$ be a second-order tensor. Considering that $\mathbf{D} = (\nabla_{\vec{x}} \vec{\boldsymbol{v}})^{sym}$ and $\mathbf{W} = (\nabla_{\vec{x}} \vec{\boldsymbol{v}})^{skew}$, show that $\mathbf{W} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{W} = 2(\mathbf{D} \cdot \mathbf{W})^{skew} = [(\nabla_{\vec{x}} \vec{\boldsymbol{v}}) \cdot (\nabla_{\vec{x}} \vec{\boldsymbol{v}})]^{skew} = (\boldsymbol{\ell} \cdot \boldsymbol{\ell})^{skew}$.

Solution:

In **Problem 1.35** we have shown that: given an arbitrary second-order tensor $\boldsymbol{\ell}$ it fulfills that

$$\boldsymbol{\ell}^{skew} \cdot \boldsymbol{\ell}^{sym} + \boldsymbol{\ell}^{sym} \cdot \boldsymbol{\ell}^{skew} = 2(\boldsymbol{\ell}^{skew} \cdot \boldsymbol{\ell}^{sym})^{skew}$$

Then, $\mathbf{W} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{W} = 2(\mathbf{D} \cdot \mathbf{W})^{skew}$ holds. Taking into account the definition of symmetry and antisymmetry, i.e. $\mathbf{D} = \frac{1}{2}\left[\boldsymbol{\ell} + \boldsymbol{\ell}^T\right]$, $\mathbf{W} = \frac{1}{2}\left[\boldsymbol{\ell} - \boldsymbol{\ell}^T\right]$, we can conclude that:

$$\mathbf{W} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{W} = 2(\mathbf{D} \cdot \mathbf{W})^{skew} = \frac{2}{4}\left[(\boldsymbol{\ell} + \boldsymbol{\ell}^T) \cdot (\boldsymbol{\ell} - \boldsymbol{\ell}^T)\right]^{skew} = \frac{1}{2}\left[\boldsymbol{\ell} \cdot \boldsymbol{\ell} + \boldsymbol{\ell} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell} - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell}^T\right]^{skew}$$

$$= \underbrace{\frac{1}{2}\left[\boldsymbol{\ell} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell}\right]^{skew}}_{=\mathbf{0}} + \frac{1}{2}\left[\boldsymbol{\ell} \cdot \boldsymbol{\ell} - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell}^T\right]^{skew}$$

$$= \frac{1}{2}\left[\boldsymbol{\ell} \cdot \boldsymbol{\ell} - (\boldsymbol{\ell} \cdot \boldsymbol{\ell})^T\right]^{skew} = \frac{1}{2}\left[2(\boldsymbol{\ell} \cdot \boldsymbol{\ell})^{skew}\right]^{skew} = (\boldsymbol{\ell} \cdot \boldsymbol{\ell})^{skew} = (\nabla_{\vec{x}} \vec{\boldsymbol{v}} \cdot \nabla_{\vec{x}} \vec{\boldsymbol{v}})^{skew}$$

Obs.: Note that the resulting tensor $\boldsymbol{\ell} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell}$ is a symmetric one, since:

$$(\boldsymbol{\ell} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell})^T = \boldsymbol{\ell} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^T \cdot \boldsymbol{\ell}$$

Problem 1.118

Consider the scalar $J = |\boldsymbol{F}| \equiv \det(\boldsymbol{F})$ and an arbitrary second-order tensor given by $\boldsymbol{\ell} = \nabla_{\vec{x}} \vec{\boldsymbol{v}} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$, where $\dot{\boldsymbol{F}} \equiv \frac{d\boldsymbol{F}}{dt}$ represents the time derivative of $\boldsymbol{F}$. Show that the following is true:

$$\boxed{\frac{d(J)}{dt} \equiv \dot{J} = J(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})} \qquad (1.128)$$

Solution: In **Problem 1.88** we have shown that $\frac{d|\mathbf{A}|}{d\tau} = |\mathbf{A}| \operatorname{Tr}\left(\frac{d\mathbf{A}}{d\tau} \cdot \mathbf{A}^{-1}\right)$ holds, where $\mathbf{A} = \mathbf{A}(\tau)$ is an arbitrary second-order tensor and τ a scalar. Making $\mathbf{A} = \boldsymbol{F}$ and $\tau = t$, we can obtain:

$$\frac{d|\boldsymbol{F}|}{dt}=\frac{dJ}{dt}=|\boldsymbol{F}|\,\mathrm{Tr}\left(\frac{d\boldsymbol{F}}{dt}\cdot\boldsymbol{F}^{-1}\right)=J\,\mathrm{Tr}(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})=J\,\mathrm{Tr}(\boldsymbol{\ell})=J\,\mathrm{Tr}(\boldsymbol{\ell}^{sym})=J\,\mathrm{Tr}(\nabla_{\vec{x}}\vec{\boldsymbol{v}})=J(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})$$

Alternative solution:

In **Problem 1.45** we have shown that given a second-order tensor $\boldsymbol{F}$ the relationship $|\boldsymbol{F}|\epsilon_{tpq}=\epsilon_{rjk}F_{rt}F_{jp}F_{kq}$ holds, and if we take the time derivative of it we can obtain:

$$\frac{D|\boldsymbol{F}|}{Dt}\epsilon_{tpq}=\frac{D}{Dt}(\epsilon_{rjk}F_{rt}F_{jp}F_{kq})=\epsilon_{rjk}\dot{F}_{rt}F_{jp}F_{kq}+\epsilon_{rjk}F_{rt}\dot{F}_{jp}F_{kq}+\epsilon_{rjk}F_{rt}F_{jp}\dot{F}_{kq} \tag{1.129}$$

According to the problem statement we have $\boldsymbol{\ell}=\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} \quad\Rightarrow\quad \dot{\boldsymbol{F}}=\boldsymbol{\ell}\cdot\boldsymbol{F}$, with that the following relations $\dot{F}_{rt}=\ell_{rs}F_{st}$, $\dot{F}_{jp}=\ell_{js}F_{sp}$ and $\dot{F}_{kq}=\ell_{ks}F_{sq}$ hold, and the equation in (1.129) can be rewritten as follows:

$$\begin{aligned}\frac{D|\boldsymbol{F}|}{Dt}\epsilon_{tpq}&=\epsilon_{rjk}\dot{F}_{rt}F_{jp}F_{kq}+\epsilon_{rjk}F_{rt}\dot{F}_{jp}F_{kq}+\epsilon_{rjk}F_{rt}F_{jp}\dot{F}_{kq}\\&=\epsilon_{rjk}\ell_{rs}F_{st}F_{jp}F_{kq}+\epsilon_{rjk}F_{rt}\ell_{js}F_{sp}F_{kq}+\epsilon_{rjk}F_{rt}F_{jp}\ell_{ks}F_{sq}\end{aligned}$$

We multiply both sides of the above equation by $\mathrm{u}_t\mathrm{v}_p\mathrm{w}_q$ we can obtain:

$$\begin{aligned}\frac{D|\boldsymbol{F}|}{Dt}\epsilon_{tpq}\mathrm{u}_t\mathrm{v}_p\mathrm{w}_q&=\epsilon_{rjk}\ell_{rs}F_{st}F_{jp}F_{kq}\mathrm{u}_t\mathrm{v}_p\mathrm{w}_q+\epsilon_{rjk}F_{rt}\ell_{js}F_{sp}F_{kq}\mathrm{u}_t\mathrm{v}_p\mathrm{w}_q+\epsilon_{rjk}F_{rt}F_{jp}\ell_{ks}F_{sq}\mathrm{u}_t\mathrm{v}_p\mathrm{w}_q\\&=\epsilon_{rjk}(\ell_{rs}F_{st}\mathrm{u}_t)(F_{jp}\mathrm{v}_p)(F_{kq}\mathrm{w}_q)+\epsilon_{rjk}(F_{rt}\mathrm{u}_t)(\ell_{js}F_{sp}\mathrm{v}_p)(F_{kq}\mathrm{w}_q)\\&\qquad+\epsilon_{rjk}(F_{rt}\mathrm{u}_t)(F_{jp}\mathrm{v}_p)(\ell_{ks}F_{sq}\mathrm{w}_q)\\&=\epsilon_{rjk}(\ell_{rs}\mathrm{a}_s)(\mathrm{b}_j)(\mathrm{c}_k)+\epsilon_{rjk}(\mathrm{a}_r)(\ell_{js}\mathrm{b}_s)(\mathrm{c}_k)+\epsilon_{rjk}(\mathrm{a}_r)(\mathrm{b}_j)(\ell_{ks}\mathrm{c}_s)\end{aligned}$$

where we have denoted by $\mathrm{a}_s=F_{st}\mathrm{u}_t$, $\mathrm{b}_j=F_{jp}\mathrm{v}_p$, $\mathrm{c}_s=F_{sq}\mathrm{w}_q$. The above equation in tensorial notation becomes:

$$\frac{D|\boldsymbol{F}|}{Dt}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\wedge\vec{\mathbf{w}})=(\boldsymbol{\ell}\cdot\vec{\mathbf{a}})\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})+\vec{\mathbf{a}}\cdot[(\boldsymbol{\ell}\cdot\vec{\mathbf{b}})\wedge\vec{\mathbf{c}}]+\vec{\mathbf{a}}\cdot[\vec{\mathbf{b}}\wedge(\boldsymbol{\ell}\cdot\vec{\mathbf{c}})]=\mathrm{Tr}(\boldsymbol{\ell})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]$$

where we have used the property of trace, (see **Problem 1.47**). The above equation can also be written as follows:

$$\frac{D|\boldsymbol{F}|}{Dt}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\wedge\vec{\mathbf{w}})=\mathrm{Tr}(\boldsymbol{\ell})[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})]=\mathrm{Tr}(\boldsymbol{\ell})\{(\boldsymbol{F}\cdot\vec{\mathbf{u}})\cdot[(\boldsymbol{F}\cdot\vec{\mathbf{v}})\wedge(\boldsymbol{F}\cdot\vec{\mathbf{w}})]\}=\mathrm{Tr}(\boldsymbol{\ell})|\boldsymbol{F}|\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\wedge\vec{\mathbf{w}})$$

where we have used the property of determinant, (see **Problem 1.48**), with that we conclude that $\dfrac{D|\boldsymbol{F}|}{Dt}=\mathrm{Tr}(\boldsymbol{\ell})|\boldsymbol{F}|$.

Problem 1.119

Let us consider a vector field represented by the unit vector field $\hat{\mathbf{b}}(\vec{x})$, (see Figure 1.31). Obtain the second-order projection tensor $\mathbf{P}$ such that $\vec{\mathbf{p}}=\mathbf{P}\cdot\vec{\mathbf{u}}$ holds, where $\vec{\mathbf{u}}$ is an arbitrary vector and $\vec{\mathbf{p}}$ is orthogonal to the field defined by $\hat{\mathbf{b}}(\vec{x})$.

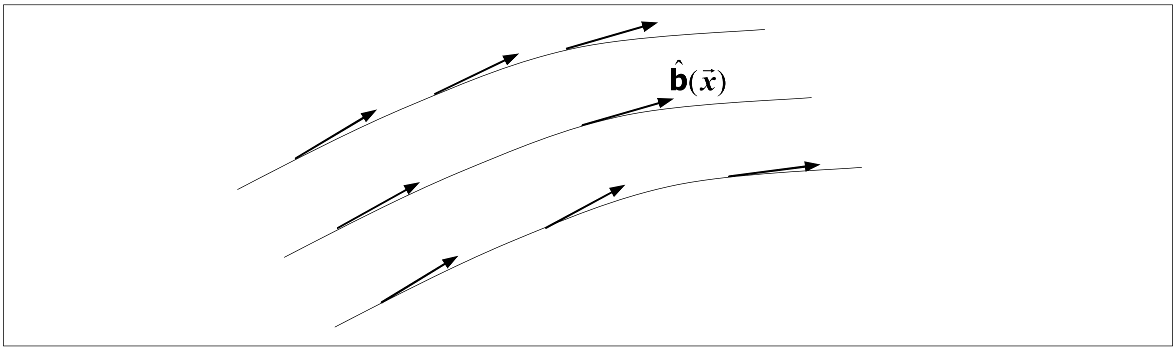


Figure 1.31: Vector field.

Solution:

The proposed problem is represented in Figure 1.32.

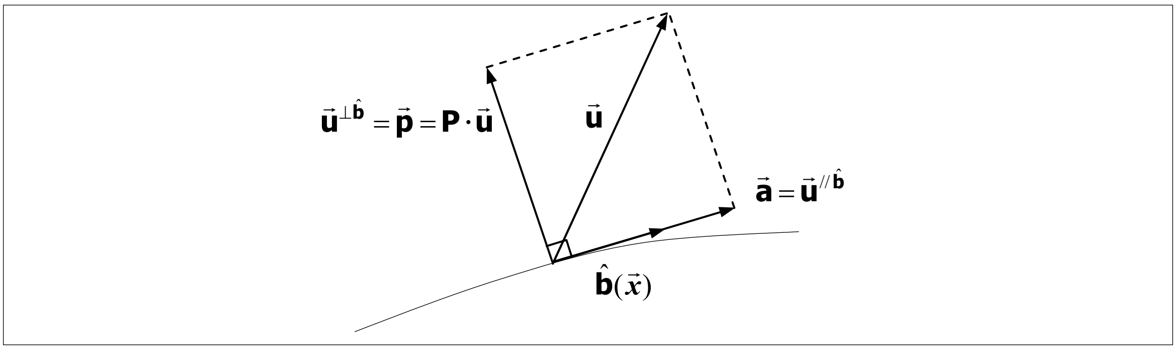


Figure 1.32

And, by considering the vector summation we obtain $\vec{\mathbf{u}} = \vec{\mathbf{a}} + \vec{\mathbf{p}}$. In addition, the vector $\vec{\mathbf{a}}$ can be obtained by means of the projection of $\vec{\mathbf{u}}$ onto the direction $\hat{\mathbf{b}}$: $\vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|\hat{\mathbf{b}} = (\vec{\mathbf{u}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$, note also that $\vec{\mathbf{a}} = (\vec{\mathbf{u}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \vec{\mathbf{u}}\cdot(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})$. With that we can obtain:

$$
\begin{aligned}
\vec{\mathbf{p}} &= \vec{\mathbf{u}} - \vec{\mathbf{a}} \\
&= \vec{\mathbf{u}} - (\vec{\mathbf{u}}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \vec{\mathbf{u}} - (\vec{\mathbf{u}}\cdot\hat{\mathbf{b}})\otimes\hat{\mathbf{b}} \\
&= \mathbf{1}\cdot\vec{\mathbf{u}} - (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\cdot\vec{\mathbf{u}} \\
&= [\mathbf{1} - (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})]\cdot\vec{\mathbf{u}} \\
&= \mathbf{P}\cdot\vec{\mathbf{u}}
\end{aligned}
\qquad
\begin{aligned}
\mathrm{p}_i &= \mathrm{u}_i - \mathrm{a}_i \\
&= \mathrm{u}_i - (\mathrm{u}_k\hat{\mathrm{b}}_k)\hat{\mathrm{b}}_i \\
&= \mathrm{u}_k\delta_{ik} - \mathrm{u}_k\hat{\mathrm{b}}_k\hat{\mathrm{b}}_i \\
&= (\delta_{ik} - \hat{\mathrm{b}}_k\hat{\mathrm{b}}_i)\mathrm{u}_k \\
&= \mathrm{P}_{ik}\mathrm{u}_k
\end{aligned}
$$

Thus, we conclude that the projection second-order tensor is given by:

$$\mathbf{P} = \mathbf{1} - \hat{\mathbf{b}}\otimes\hat{\mathbf{b}}$$

The same result could have been obtained by means of vector product, (see Figure 1.33).

Taking into account that $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}) = \left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1} - \vec{\mathbf{a}}\otimes\vec{\mathbf{a}}\right]\cdot\vec{\mathbf{b}}$, (see **Problem 1.17**), we can obtain $\hat{\mathbf{b}}\wedge(\vec{\mathbf{u}}\wedge\hat{\mathbf{b}}) = \left[(\hat{\mathbf{b}}\cdot\hat{\mathbf{b}})\mathbf{1} - \hat{\mathbf{b}}\otimes\hat{\mathbf{b}}\right]\cdot\vec{\mathbf{u}} = \left[\mathbf{1} - \hat{\mathbf{b}}\otimes\hat{\mathbf{b}}\right]\cdot\vec{\mathbf{u}} = \vec{\mathbf{p}}$.

Then we can present a vector as follows:

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}^{//\hat{\mathbf{b}}} + \vec{\mathbf{u}}^{\perp\hat{\mathbf{b}}} = (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\cdot\vec{\mathbf{u}} + [\mathbf{1} - (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})]\cdot\vec{\mathbf{u}}$$

where $\vec{\mathbf{u}}^{//\hat{\mathbf{b}}} = (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\cdot\vec{\mathbf{u}}$ is the vector parallel to $\hat{\mathbf{b}}$-direction and $\vec{\mathbf{u}}^{\perp\hat{\mathbf{b}}} = [\mathbf{1} - (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})]\cdot\vec{\mathbf{u}}$ is the perpendicular one.

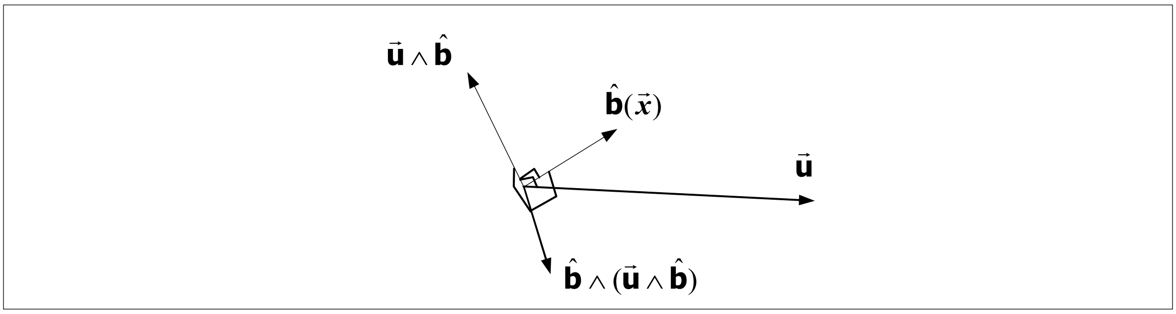


Figure 1.33

Problem 1.120

Given a vector field $\vec{v}(\vec{x})$, show that the following relationship holds:

$$\boxed{(\nabla_{\vec{x}}\vec{v})\cdot\vec{v} = \frac{1}{2}\nabla_{\vec{x}}(v^2) - \vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})}$$

where $v = \|\vec{v}\|$ is the module of $\vec{v}$, so $v^2 = \vec{v}\cdot\vec{v}$.

Solution:

Note that $\frac{1}{2}[\nabla_{\vec{x}}(v^2)]_i = \frac{1}{2}[\nabla_{\vec{x}}(\vec{v}\cdot\vec{v})]_i = \frac{1}{2}(v_k v_k)_{,i} = \frac{1}{2}(v_{k,i}v_k + v_k v_{k,i}) = v_k v_{k,i} = (\vec{v}\cdot\nabla_{\vec{x}}\vec{v})_i$.

At one point of the vector field $\vec{v}$, we consider a plane normal to $\vec{v}$ and recalling that the projection of a second-order tensor onto a direction ($\vec{v}$) is a vector which does not necessary have the same direction as ($\vec{v}$), with that we represent the following vectors $(\nabla_{\vec{x}}\vec{v})\cdot\vec{v}$ and $\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})$:

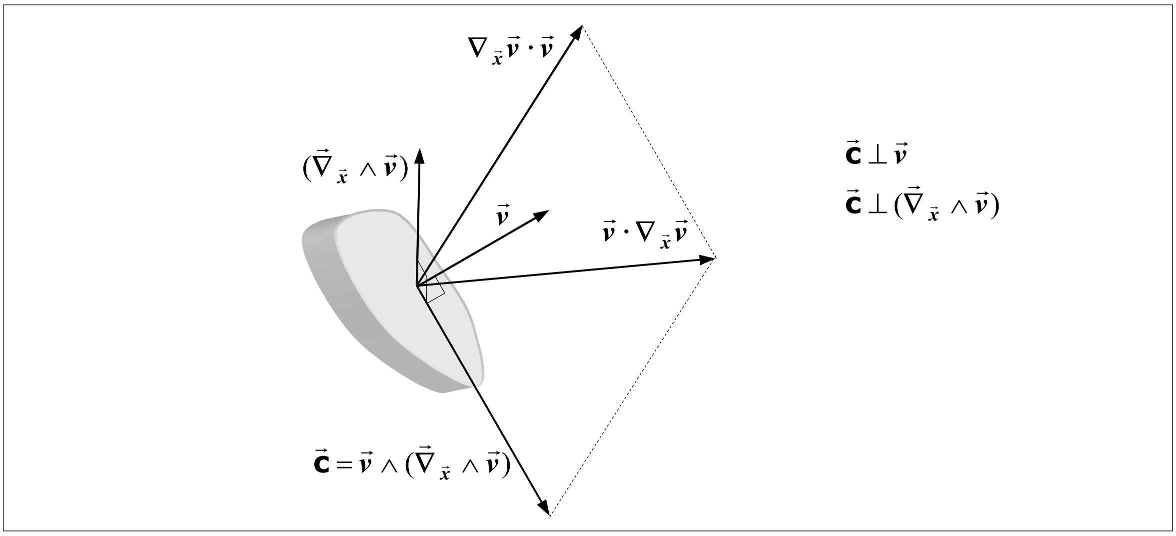


Figure 1.34

Note that, by means of summation of vectors, (see Figure 1.34), we can obtain:

$$(\nabla_{\vec{x}}\vec{v})\cdot\vec{v} + \vec{\mathbf{c}} = \vec{v}\cdot(\nabla_{\vec{x}}\vec{v}) \quad\Rightarrow\quad \vec{\mathbf{c}} = \vec{v}\cdot(\nabla_{\vec{x}}\vec{v}) - (\nabla_{\vec{x}}\vec{v})\cdot\vec{v} \quad\Rightarrow\quad \vec{\mathbf{c}} = \vec{v}\cdot(\nabla_{\vec{x}}\vec{v}) - \vec{v}\cdot(\nabla_{\vec{x}}\vec{v})^T$$

$$\Rightarrow \vec{\mathbf{c}} = \vec{v}\cdot((\nabla_{\vec{x}}\vec{v}) - (\nabla_{\vec{x}}\vec{v})^T) = \vec{v}\cdot 2(\nabla_{\vec{x}}\vec{v})^{skew}$$

If we consider that $\vec{w}$ is the axial vector associated with the antisymmetric tensor $(\nabla_{\vec{x}}\vec{v})^{skew}$, it fulfills that: $(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}=\vec{w}\wedge\vec{v}\Rightarrow\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})^{skew}=\vec{v}\wedge\vec{w}$. In addition, the relationship $\mathrm{rot}(\vec{v})\equiv\vec{\nabla}_{\vec{x}}\wedge\vec{v}=2\vec{w}$ holds. Then,

$$\vec{\mathbf{c}}=\vec{v}\cdot 2(\nabla_{\vec{x}}\vec{v})^{skew}=\vec{v}\wedge 2\vec{w}=\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v}) \tag{1.130}$$

with that we conclude that:

$$(\nabla_{\vec{x}}\vec{v})\cdot\vec{v}+\vec{\mathbf{c}}=\vec{v}\cdot(\nabla_{\vec{x}}\vec{v}) \quad\Rightarrow\quad (\nabla_{\vec{x}}\vec{v})\cdot\vec{v}=\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})-\vec{\mathbf{c}}$$
$$\Rightarrow(\nabla_{\vec{x}}\vec{v})\cdot\vec{v}=\frac{1}{2}\nabla_{\vec{x}}(v^2)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})$$

It is interesting to note that: when $(\nabla_{\vec{x}}\vec{v})$ is a symmetric tensor, i.e. $(\nabla_{\vec{x}}\vec{v})=(\nabla_{\vec{x}}\vec{v})^{sym}$, the following is fulfilled $(\nabla_{\vec{x}}\vec{v})^{skew}=\mathbf{0}$, $\vec{\mathbf{c}}=\vec{\mathbf{0}}$, $(\vec{\nabla}_{\vec{x}}\wedge\vec{v})=\vec{\mathbf{0}}$, and $(\nabla_{\vec{x}}\vec{v})\cdot\vec{v}=\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})$ has the same direction as $\vec{v}$, (see Figure 1.35 (a)).

When $(\nabla_{\vec{x}}\vec{v})=(\nabla_{\vec{x}}\vec{v})^{skew}$ we have that $\vec{\mathbf{c}}=\vec{v}\cdot 2(\nabla_{\vec{x}}\vec{v})^{skew}=2\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})$, (see equation (1.130)). With that, $\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})=-(\nabla_{\vec{x}}\vec{v})\cdot\vec{v}$ holds, and the vector $\vec{v}$ is perpendicular to the vector $(\vec{\nabla}_{\vec{x}}\wedge\vec{v})$, (see Figure 1.35(b)).

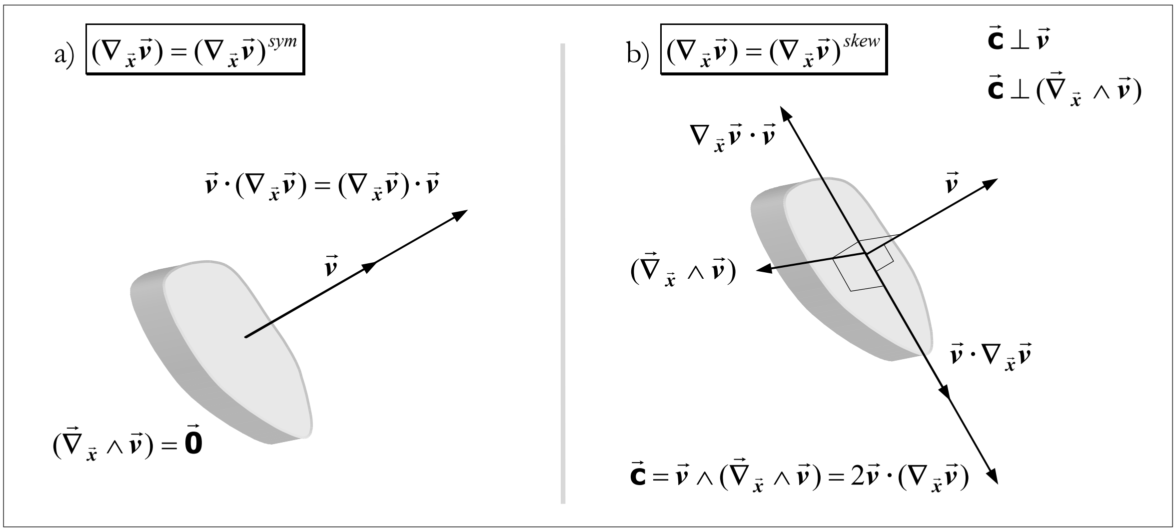


Figure 1.35

Alternative solution:

$$\begin{aligned}
\nabla_{\vec{x}}\vec{v}\cdot\vec{v}&=((\nabla_{\vec{x}}\vec{v})^{sym}+(\nabla_{\vec{x}}\vec{v})^{skew})\cdot\vec{v}=(\nabla_{\vec{x}}\vec{v})^{sym}\cdot\vec{v}+(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}\\
&=(\nabla_{\vec{x}}\vec{v})^{sym}\cdot\vec{v}+(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}+((\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}-(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v})\\
&=((\nabla_{\vec{x}}\vec{v})^{sym}\cdot\vec{v}-(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v})+2(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}\\
&=\frac{1}{2}\left[(\nabla_{\vec{x}}\vec{v}\ +(\nabla_{\vec{x}}\vec{v})^T)-(\nabla_{\vec{x}}\vec{v}\ -(\nabla_{\vec{x}}\vec{v})^T)\right]\cdot\vec{v}+2(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}\\
&=\frac{1}{2}(2(\nabla_{\vec{x}}\vec{v})^T)\cdot\vec{v}+2(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}=\vec{v}\cdot(\nabla_{\vec{x}}\vec{v})+2(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}\\
&=\frac{1}{2}\nabla_{\vec{x}}(v^2)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})
\end{aligned}$$

Remember that $(\nabla_{\vec{x}}\vec{v}^{\,skew})^T=-(\nabla_{\vec{x}}\vec{v})^{skew}$, thus $2(\nabla_{\vec{x}}\vec{v})^{skew}\cdot\vec{v}=-\vec{v}\cdot 2(\nabla_{\vec{x}}\vec{v})^{skew}=-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})$

Problem 1.121

Let $\vec{\mathbf{u}}(\vec{x})$ be a stationary vector field. a) Obtain the components of the differential $d\vec{\mathbf{u}}$. b) Now, consider that $\vec{\mathbf{u}}(\vec{x})$ represents a displacement field, and is independent of x_3. With these conditions, graphically illustrate the displacement field in the differential area element $dx_1 dx_2$.

Solution: According to the differential and gradient definitions, the following equations are true $d\vec{\mathbf{u}} \equiv \vec{\mathbf{u}}(\vec{x}+d\vec{x}) - \vec{\mathbf{u}}(\vec{x})$ and $d\vec{\mathbf{u}} = (\nabla_{\vec{x}}\vec{\mathbf{u}}) \cdot d\vec{x}$, (see Figure 1.36).

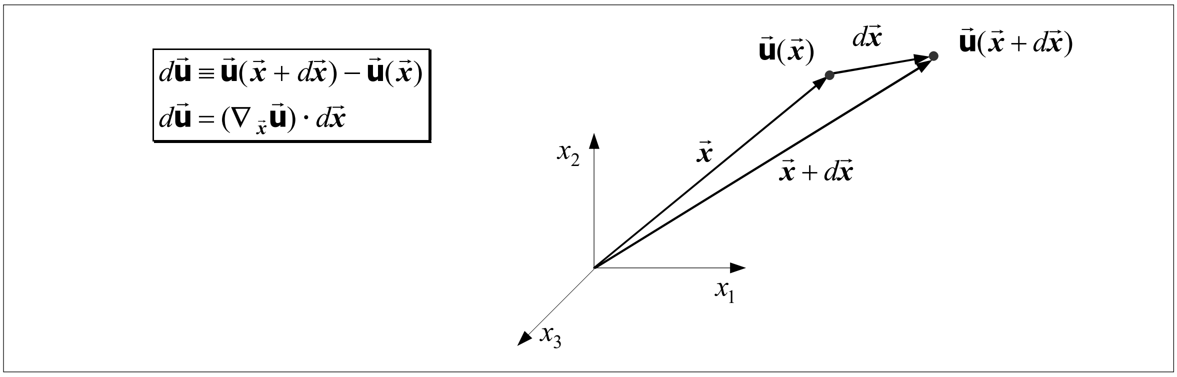


Figure 1.36

Thus, the components are defined as:

$$
d\mathrm{u}_i = \frac{\partial \mathrm{u}_i}{\partial x_j} dx_j \Rightarrow \begin{bmatrix} d\mathrm{u}_1 \\ d\mathrm{u}_2 \\ d\mathrm{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \Rightarrow \begin{cases} d\mathrm{u}_1 = \frac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_1}{\partial x_2} dx_2 + \frac{\partial \mathrm{u}_1}{\partial x_3} dx_3 \\ d\mathrm{u}_2 = \frac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_2}{\partial x_2} dx_2 + \frac{\partial \mathrm{u}_2}{\partial x_3} dx_3 \\ d\mathrm{u}_3 = \frac{\partial \mathrm{u}_3}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_3}{\partial x_2} dx_2 + \frac{\partial \mathrm{u}_3}{\partial x_3} dx_3 \end{cases}
$$

with

$$
\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_1(x_1, x_2, x_3) \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_2(x_1, x_2, x_3) \\ d\mathrm{u}_3 = \mathrm{u}_3(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_3(x_1, x_2, x_3) \end{cases}
$$

As the field is independent of x_3, the displacement field in the differential area element is defined as:

$$
\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_1(x_1, x_2) = \frac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_2(x_1, x_2) = \frac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}
$$

or:

$$
\begin{cases} \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_1(x_1, x_2) + \frac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_2(x_1, x_2) + \frac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \frac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}
$$

Note that the above equation is equivalent to the Taylor series expansion taking into account only up to linear terms. The representation of the displacement field in the differential area element is shown in Figure 1.37.

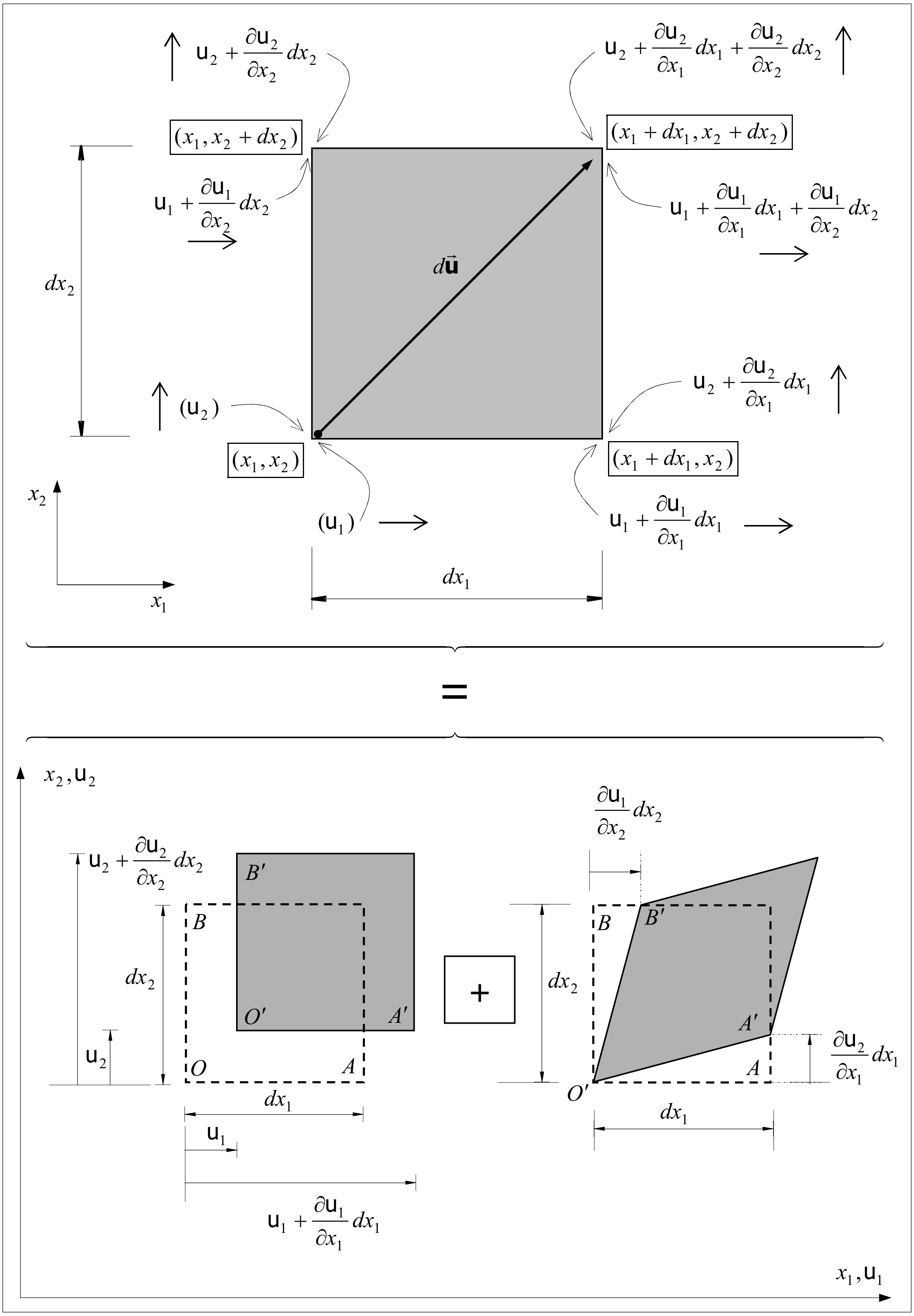


Figure 1.37: Displacement field in the differential area element.

Problem 1.122

Given a second-order tensor field $\mathbf{T}(\vec{x})$. Show that: if there is no source of the field $\mathbf{T}(\vec{x})$ it fulfills that the divergence of $\mathbf{T}(\vec{x})$ is equal to zero, i.e. $\nabla_{\vec{x}} \cdot \mathbf{T} = \vec{\mathbf{0}}$. For the demonstration, consider the tensor field in a differential volume element $dV = dx_1 dx_2 dx_3$ in the Cartesian system.

Solution:

Let us set the tensor field $\mathbf{T}(\vec{x})$ in the differential volume element. For this purpose, we start from the definition of the differential of $\mathbf{T}(\vec{x})$ which is defined by means of gradient as follows:

$$\boxed{\left.\begin{array}{l} d\mathbf{T} \equiv \mathbf{T}(\vec{x}+d\vec{x}) - \mathbf{T}(\vec{x}) \\ d\mathbf{T} = (\nabla_{\vec{x}}\mathbf{T})\cdot d\vec{x} \end{array}\right\} \Rightarrow \mathbf{T}(\vec{x}+d\vec{x}) - \mathbf{T}(\vec{x}) = (\nabla_{\vec{x}}\mathbf{T})\cdot d\vec{x} \Rightarrow \mathbf{T}(\vec{x}+d\vec{x}) = \mathbf{T}(\vec{x}) + (\nabla_{\vec{x}}\mathbf{T})\cdot d\vec{x}}$$

The above equation in indicial notation becomes:

$$\begin{aligned} \mathsf{T}_{ij}(\vec{x}+d\vec{x}) &= \mathsf{T}_{ij}(\vec{x}) + \mathsf{T}_{ij,k}\,dx_k \\ &= \mathsf{T}_{ij}(\vec{x}) + \mathsf{T}_{ij,1}\,dx_1 + \mathsf{T}_{ij,2}\,dx_2 + \mathsf{T}_{ij,3}\,dx_3 \\ &= \mathsf{T}_{ij}(\vec{x}) + \frac{\partial \mathsf{T}_{ij}}{\partial x_1}dx_1 + \frac{\partial \mathsf{T}_{ij}}{\partial x_2}dx_2 + \frac{\partial \mathsf{T}_{ij}}{\partial x_3}dx_3 \end{aligned}$$

The representation of the field components $\mathsf{T}_{ij}(\vec{x}+d\vec{x})$ can be appreciated in Figure 1.38. Note that on the face normal to $x_1 + dx_1$ act the components $\mathsf{T}_{i1}(\vec{x}) + \frac{\partial \mathsf{T}_{i1}}{\partial x_1}dx_1$, since according our convention, the first index indicate the direction in which points out and the second index indicates the normal plane.

Once established the tensor field componets $\mathsf{T}_{ij}(\vec{x}+d\vec{x})$ in the differential volume element, we apply the total balance of the tensor field components $\mathsf{T}_{ij}(\vec{x}+d\vec{x})$ according to the directions x_1, x_2, x_3.

Total balance of $\mathsf{T}_{ij}(\vec{x}+d\vec{x})$ in dV according to x_1-direction is equal to zero (there is no source):

$$\begin{aligned} &\left(\mathsf{T}_{11} + \frac{\partial \mathsf{T}_{11}}{\partial x_1}dx_1\right)dx_2dx_3 + \left(\mathsf{T}_{13} + \frac{\partial \mathsf{T}_{13}}{\partial x_3}dx_3\right)dx_1dx_2 + \left(\mathsf{T}_{12} + \frac{\partial \mathsf{T}_{12}}{\partial x_2}dx_2\right)dx_1dx_3 - \mathsf{T}_{11}dx_2dx_3 \\ &\qquad - \mathsf{T}_{13}dx_1dx_2 - \mathsf{T}_{12}dx_1dx_3 = 0 \end{aligned}$$

By simplifying the above equation we can obtain:

$$\frac{\partial \mathsf{T}_{11}}{\partial x_1}dx_1dx_2dx_3 + \frac{\partial \mathsf{T}_{13}}{\partial x_3}dx_3dx_1dx_2 + \frac{\partial \mathsf{T}_{12}}{\partial x_2}dx_2dx_1dx_3 = 0$$

$$\Rightarrow \frac{\partial \mathsf{T}_{11}}{\partial x_1} + \frac{\partial \mathsf{T}_{12}}{\partial x_2} + \frac{\partial \mathsf{T}_{13}}{\partial x_3} = 0$$

Similarly, according to the directions x_2 and x_3 we will obtain, respectively:

$$\frac{\partial \mathsf{T}_{21}}{\partial x_1} + \frac{\partial \mathsf{T}_{22}}{\partial x_2} + \frac{\partial \mathsf{T}_{23}}{\partial x_3} = 0 \qquad \text{and} \qquad \frac{\partial \mathsf{T}_{31}}{\partial x_1} + \frac{\partial \mathsf{T}_{32}}{\partial x_2} + \frac{\partial \mathsf{T}_{33}}{\partial x_3} = 0$$

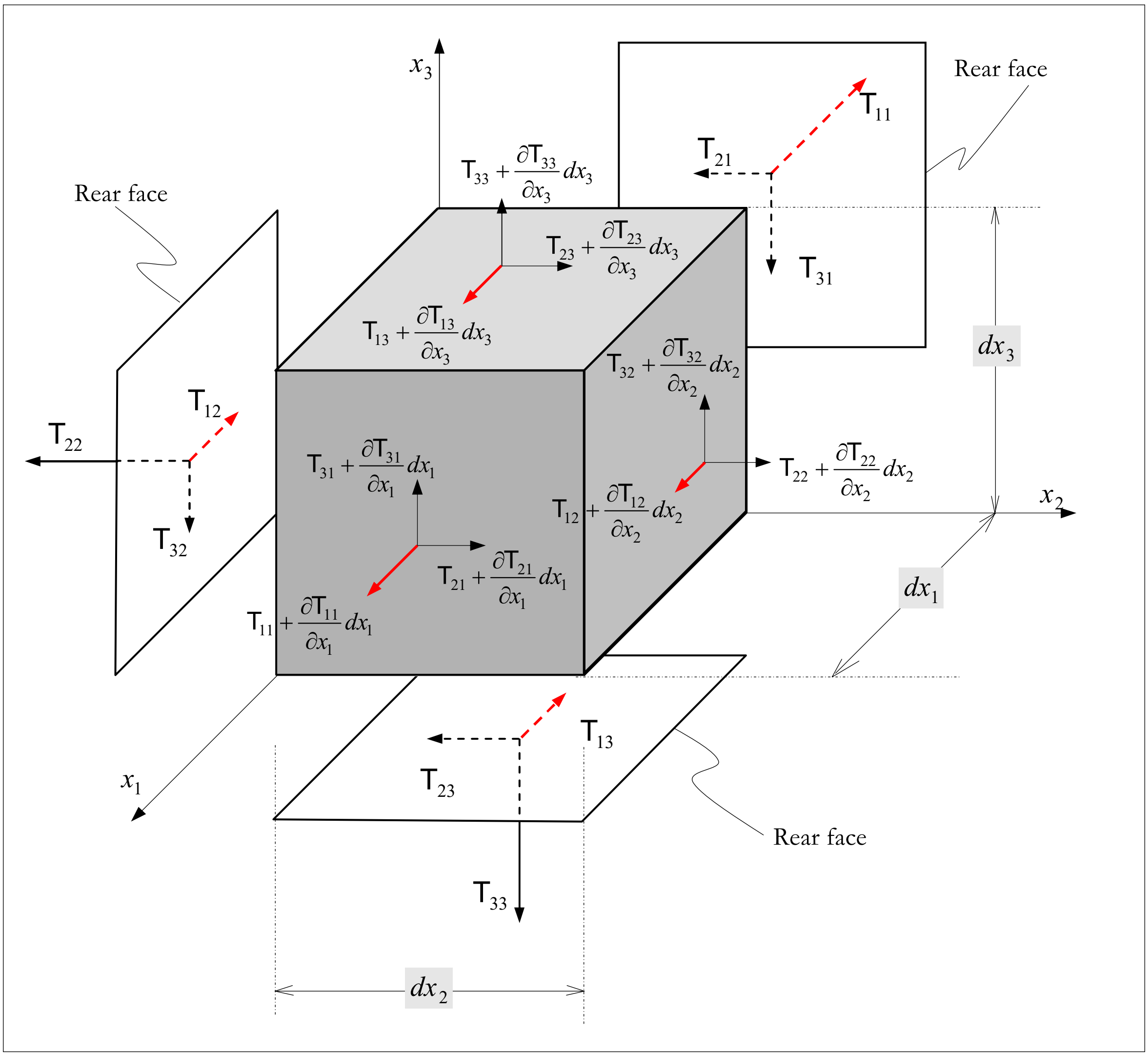


Figure 1.38: Tensor field components in the differential volume element.

Then, we have the following set of equations that must be met simultaneously:

$$\begin{cases}\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=0\\ \frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=0\\ \frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0\end{cases} \Rightarrow \begin{cases}T_{11,1}+T_{12,2}+T_{13,3}=0\\ T_{21,1}+T_{22,2}+T_{23,3}=0\\ T_{31,1}+T_{32,2}+T_{33,3}=0\end{cases} \Rightarrow \begin{cases}T_{1j,j}=0\\ T_{2j,j}=0\\ T_{3j,j}=0\end{cases} \Rightarrow T_{ij,j}=0_i$$

Thus, we have shown that in the absence of source, the divergence is zero:

$$T_{ij,j}=0_i \quad \Leftrightarrow \quad (\nabla_{\vec{x}}\cdot\mathbf{T})_i=0_i \quad \xrightarrow{tensorial} \quad \nabla_{\vec{x}}\cdot\mathbf{T}=\vec{\mathbf{0}}$$

NOTE 1: If we have a tensor field, the tensor order of the source is a minor order of the tensor, e.g. the source of a vector field is represented by a scalar field.

NOTE 2: If the divergence of a tensor field is positive we have a source, on the contrary if the divergence is negative we have a sink.

Problem 1.123

Show that:

$$[(\nabla_{\vec{x}}\mathbf{T})\cdot\vec{u}]\cdot\vec{a} = [\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]\cdot\vec{u} \tag{1.131}$$

where $\mathbf{T}=\mathbf{T}(\vec{x})$ is a second-order tensor field, $\vec{u}=\vec{u}(\vec{x})$ is a vector field, and $\vec{a}$ an arbitrary vector (independent of $\vec{x}$).

Solution: Note that the term $[(\nabla_{\vec{x}}\mathbf{T})\cdot\vec{u}]\cdot\vec{a}$ is a vector, which in indicial notation becomes:

$$\{[(\nabla_{\vec{x}}\mathbf{T})\cdot\vec{u}]\cdot\vec{a}\}_i = [(\nabla_{\vec{x}}\mathbf{T})\cdot\vec{u}]_{ik}(\vec{a})_k = [(\nabla_{\vec{x}}\mathbf{T})_{ikp}u_p]a_k = [\mathsf{T}_{ik,p}u_p]a_k = \mathsf{T}_{ik,p}u_p a_k \tag{1.132}$$

Now we express the term $[\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]\cdot\vec{u}$ in indicial notation:

$$\begin{aligned}
&(\mathbf{T}\cdot\vec{a})_i = \mathsf{T}_{ik}a_k \quad \xrightarrow{gradient} \quad [\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij} = (\mathbf{T}\cdot\vec{a})_{i,j} = (\mathsf{T}_{ik}a_k)_{,j} \\
&\Rightarrow [\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij} = (\mathsf{T}_{ik}a_k)_{,j} = \mathsf{T}_{ik,j}a_k + \mathsf{T}_{ik}\underbrace{a_{k,j}}_{=0_{k,j}} = \mathsf{T}_{ik,j}a_k \\
&or \Rightarrow [\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij} = [\vec{a}\cdot(\nabla_{\vec{x}}\mathbf{T}^T)]_{ij} + [\mathbf{T}\cdot\underbrace{(\nabla_{\vec{x}}\vec{a})}_{=\mathbf{0}}]_{ij} = [\vec{a}\cdot(\nabla_{\vec{x}}\mathbf{T}^T)]_{ij} = \mathsf{T}_{ik,j}a_k
\end{aligned} \tag{1.133}$$

where we have considered that $\vec{a}$ is independent of $(\vec{x})$. If we apply the scalar product between the above equation and $\vec{u}$ we obtain:

$$\{[\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]\cdot\vec{u}\}_i = \{[\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij}u_j\} = \mathsf{T}_{ik,j}a_k u_j = \mathsf{T}_{ik,p}u_p a_k \tag{1.134}$$

If we compare (1.132) with (1.134) we show (1.131).

Not that, if $\vec{a}=\vec{a}(\vec{x})$ depends on $\vec{x}$ and according to the equation in (1.133) we can conclude that $[\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij} = (\mathsf{T}_{ik}a_k)_{,j} = \mathsf{T}_{ik,j}a_k + \mathsf{T}_{ik}a_{k,j} \quad \Rightarrow \quad [\nabla_{\vec{x}}(\mathbf{T}\cdot\vec{a})]_{ij} = [\vec{a}\cdot(\nabla_{\vec{x}}\mathbf{T}^T)]_{ij} + [\mathbf{T}\cdot(\nabla_{\vec{x}}\vec{a})]_{ij}$.

Problem 1.124

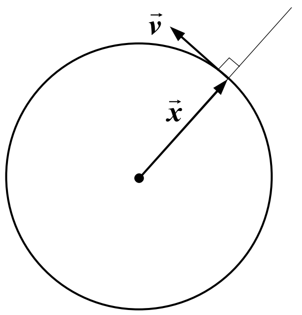
Show that if the magnitude of a vector, $\vec{\boldsymbol{\omega}}=\vec{\boldsymbol{\omega}}(t)$, is constant with time, this implies that $\vec{\boldsymbol{\omega}}$ is orthogonal to $\dfrac{d\vec{\boldsymbol{\omega}}}{dt}$ at any time t.

Solution:

We start from the definition of the magnitude of a vector, where $\|\vec{\boldsymbol{\omega}}\|^2=\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{\omega}}$ holds, thus:

$$\frac{d(\|\vec{\boldsymbol{\omega}}\|^2)}{dt} = \frac{d(\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{\omega}})}{dt} = \frac{d(\vec{\boldsymbol{\omega}})}{dt}\cdot\vec{\boldsymbol{\omega}} + \vec{\boldsymbol{\omega}}\cdot\frac{d(\vec{\boldsymbol{\omega}})}{dt} = 2\vec{\boldsymbol{\omega}}\cdot\frac{d(\vec{\boldsymbol{\omega}})}{dt} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\omega}}\perp\frac{d\vec{\boldsymbol{\omega}}}{dt}$$

NOTE: A particular case of this problem is the circular motion, (see Figure 1.39).



$$\left.\begin{aligned}&\|\vec{x}\| = \text{constant}\\ &\vec{v} = \frac{d\vec{x}}{dt}\end{aligned}\right\} \Rightarrow \vec{x}\perp\frac{d\vec{x}}{dt}$$

Figure 1.39: Circular motion.

1.15.1 Theorems Involving Integrals

Problem 1.125

Check the divergence theorem (Gauss theorem) for the vector field $\vec{\boldsymbol{F}}$ whose Cartesian components are given by $F_i = x_i + (x_3^2 - x_3)\delta_{i3}$. Consider the boundary defined by the cylinder $x_1^2 + x_2^2 \le 1$, $0 \le x_3 \le 1$.

Solution:

The divergence theorem states that:

$$\int_V \nabla_{\vec{x}} \cdot \vec{\boldsymbol{F}}\, dV = \int_S \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} dS$$

where $\hat{\boldsymbol{n}}$ is the normal to the surface and points outwards.

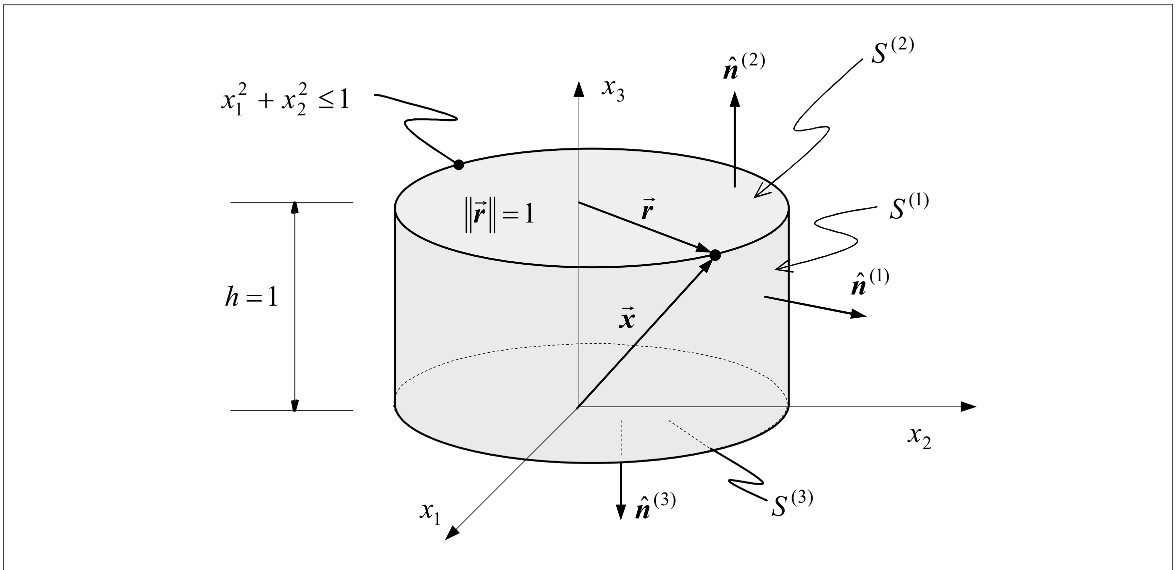


Figure 1.40

Calculation of $\int_V \nabla_{\vec{x}} \cdot \vec{\boldsymbol{F}}\, dV$:

$$\nabla_{\vec{x}} \cdot \vec{\boldsymbol{F}} = F_{i,i} = \left[x_i + (x_3^2 - x_3)\delta_{i3}\right]_{,i} = x_{i,i} + (x_3^2 - x_3)_{,i}\delta_{i3} = \delta_{ii} + (x_3^2 - x_3)_{,3} = 3 + (2x_3 - 1) = 2x_3 + 2$$

Thus

$$\int_V \nabla_{\vec{x}} \cdot \vec{\boldsymbol{F}}\, dV = \int_V (2x_3 + 2)\, dV = \int_A \int_{x_3=0}^{x_3=1} (2x_3 + 2) dx_3 dA = 3\int_A dA = 3(\pi r^2) = 3\pi$$

where A is the area defined by the circle $x_1^2 + x_2^2 \le 1$.

Calculation of $\int_S \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} dS$

We decompose the boundary in three areas, namely: $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, (see Figure 1.40), then

$$\int_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = \int_{S^{(1)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(1)}dS^{(1)} + \int_{S^{(2)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(2)}dS^{(2)} + \int_{S^{(3)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(3)}dS^{(3)}$$

The components of $\vec{F}$ are: $F_1 = x_1 + (x_3^2 - x_3)\delta_{13} = x_1$, $F_2 = x_2$, $F_3 = x_3 + (x_3^2 - x_3)\delta_{33} = x_3^2$. The representation of $\vec{F}$ in the Cartesian basis is given by: $\vec{F} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3^2\hat{\mathbf{e}}_3$. The normal for each surface are defined as follows:

$$\hat{\boldsymbol{n}}^{(1)} // \vec{r} \quad \Rightarrow \quad \hat{\boldsymbol{n}}^{(1)} = \frac{1}{\sqrt{x_1^2 + x_2^2}}(x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2)\,;\;\; \hat{\boldsymbol{n}}^{(2)} = \hat{\mathbf{e}}_3\,;\;\; \hat{\boldsymbol{n}}^{(3)} = -\hat{\mathbf{e}}_3$$

On the surface $S^{(1)}$ it holds that:

$$\begin{aligned}\int_{S^{(1)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(1)}dS^{(1)} &= \int_{S^{(1)}} (x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3^2\hat{\mathbf{e}}_3)\cdot\frac{1}{\sqrt{x_1^2 + x_2^2}}(x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2)dS^{(1)} \\ &= \int_{S^{(1)}} \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2}}dS^{(1)} = \int_{S^{(1)}} 1dS^{(1)} = 2\pi rh = 2\pi\end{aligned}$$

where we have considered the cylinder area ($2\pi rh = 2\pi$).

On the surface $S^{(2)}$ it holds that $x_3 = 1$:

$$\int_{S^{(2)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(2)}dS^{(2)} = \int_{S^{(2)}} (x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + 1\hat{\mathbf{e}}_3)\cdot(\hat{\mathbf{e}}_3)dS^{(2)} = \int_{S^{(2)}} 1dS^{(2)} = \pi r^2 = \pi$$

where we have considered the circle area ($\pi r^2 = \pi$).

On the surface $S^{(3)}$, it holds that $x_3 = 0$:

$$\int_{S^{(3)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(3)}dS^{(3)} = \int_{S^{(3)}} (x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3)\cdot(-\hat{\mathbf{e}}_3)dS^{(3)} = \int_{S^{(3)}} 0dS^{(3)} = 0$$

with that: $\int_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = \int_{S^{(1)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(1)}dS^{(1)} + \int_{S^{(2)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(2)}dS^{(2)} + \int_{S^{(3)}} \vec{F}\cdot\hat{\boldsymbol{n}}^{(3)}dS^{(3)} = 3\pi$

With that we check the divergence theorem: $\int_V \nabla_{\vec{x}}\cdot\vec{F}\,dV = \int_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = 3\pi$.

Problem 1.126

Let Ω be a domain bounded by Γ as shown in Figure 1.41. Further consider that $\mathbf{m}$ is a second-order tensor field and ω is a scalar field. Show that the following relationship holds:

$$\int_\Omega [\mathbf{m} : (\nabla_{\vec{x}}(\nabla_{\vec{x}}\omega))]d\Omega = \int_\Gamma [(\nabla_{\vec{x}}\omega)\cdot\mathbf{m}]\cdot\hat{\mathbf{n}}d\Gamma - \int_\Omega [(\nabla_{\vec{x}}\cdot\mathbf{m})\cdot\nabla_{\vec{x}}\omega]d\Omega$$

$$\int_\Omega [\mathrm{m}_{ij}\omega_{,ij}]\,d\Omega = \int_\Gamma (\omega_{,i}\mathrm{m}_{ij})\hat{\mathrm{n}}_j\,d\Gamma - \int_\Omega [\mathrm{m}_{ij,j}\omega_{,i}]\,d\Omega$$

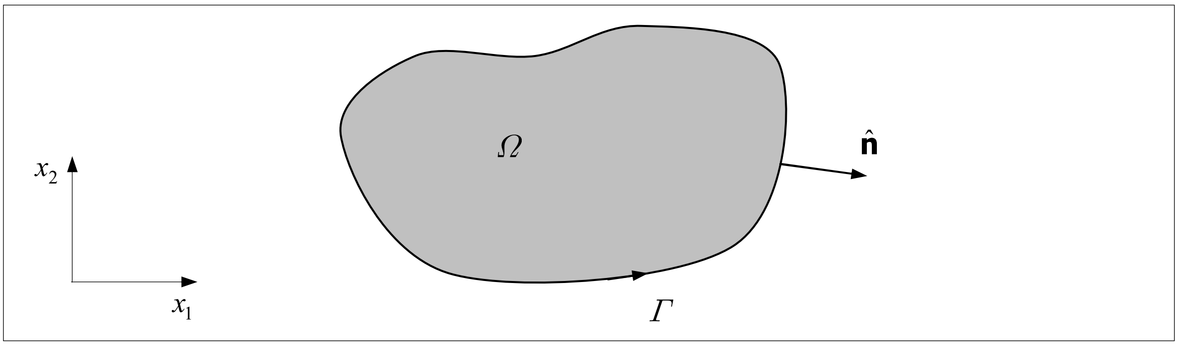


Figure 1.41

Solution:

We could directly apply the definition of integration by parts to demonstrate the above relationship. But, here we will start with the definition of the divergence theorem. That is, given a tensor field $\vec{\mathbf{v}}$, it is true that:

$$\int_{\Omega} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} \; d\Omega = \int_{\Gamma} \vec{\mathbf{v}} \cdot \hat{\mathbf{n}} \, d\Gamma \xrightarrow{indicial} \int_{\Omega} \mathsf{v}_{j,j} \; d\Omega = \int_{\Gamma} \mathsf{v}_j \hat{\mathsf{n}}_j \; d\Gamma$$

Observing that the tensor $\vec{\mathbf{v}}$ can be represented by the result of the algebraic operation $\vec{\mathbf{v}} = \nabla_{\vec{x}} \omega \cdot \mathbf{m}$ and the equivalent in indicial notation is $\mathsf{v}_j = \omega_{,i} \mathsf{m}_{ij}$, and by substituting it in the above equation we obtain:

$$\int_{\Omega} \mathsf{v}_{j,j} \, d\Omega = \int_{\Gamma} \mathsf{v}_j \hat{\mathsf{n}}_j \; d\Gamma \quad \Rightarrow \int_{\Omega} [\omega_{,i} \mathsf{m}_{ij}]_{,j} \; d\Omega = \int_{\Gamma} \omega_{,i} \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma$$

$$\Rightarrow \int_{\Omega} [\omega_{,ij} \mathsf{m}_{ij} + \omega_{,i} \mathsf{m}_{ij,j}] \, d\Omega = \int_{\Gamma} \omega_{,i} \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma$$

$$\Rightarrow \int_{\Omega} [\omega_{,ij} \mathsf{m}_{ij}] \, d\Omega = \int_{\Gamma} \omega_{,i} \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma - \int_{\Omega} [\omega_{,i} \mathsf{m}_{ij,j}] \, d\Omega$$

The above equation in tensorial notation becomes:

$$\int_{\Omega} [\mathbf{m} : (\nabla_{\vec{x}} (\nabla_{\vec{x}} \omega))] d\Omega = \int_{\Gamma} [(\nabla_{\vec{x}} \omega) \cdot \mathbf{m}] \cdot \hat{\mathbf{n}} d\Gamma - \int_{\Omega} [\nabla_{\vec{x}} \omega \cdot (\nabla_{\vec{x}} \cdot \mathbf{m})] d\Omega$$

NOTE: Consider now the domain defined by the volume V, which is bounded by the surface S with the outward unit normal to the surface $\hat{\mathbf{n}}$. If $\vec{N}$ is a vector field and T is a scalar field, it is also true that:

$$\int_V N_i T_{,ij} \, dV = \int_S N_i T_{,i} \hat{\mathsf{n}}_j dS - \int_V N_{i,j} T_{,i} \, dV$$

$$\Rightarrow \int_V \vec{N} \cdot (\nabla_{\vec{x}} (\nabla_{\vec{x}} T)) dV = \int_S (\nabla_{\vec{x}} T \cdot \vec{N}) \otimes \hat{\mathbf{n}} dS - \int_V \nabla_{\vec{x}} T \cdot \nabla_{\vec{x}} \vec{N} \; dV$$

where we have directly applied the definition of integration by parts.

Problem 1.127

Let $\vec{\mathbf{b}}$ be a vector field, which is defined as $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$. Show that:

$$\int_S \lambda \mathsf{b}_i \hat{\mathsf{n}}_i \, dS = \int_V \lambda,_i \mathsf{b}_i \, dV$$

where $\lambda = \lambda(\vec{x})$ represents a scalar field.

Solution 1: The Cartesian components of $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$ are represented by $\mathsf{b}_i = \epsilon_{ijk} \mathsf{v}_{k,j}$ and by substituting them in the above surface integral we obtain:

$$\int_S \lambda \mathsf{b}_i \hat{\mathsf{n}}_i \, dS = \int_S \lambda \epsilon_{ijk} \mathsf{v}_{k,j} \hat{\mathsf{n}}_i \, dS$$

Applying the divergence theorem we obtain:

$$\begin{aligned}
\int_S \lambda \mathsf{b}_i \hat{\mathsf{n}}_i \, dS = \int_S \lambda \epsilon_{ijk} \mathsf{v}_{k,j} \hat{\mathsf{n}}_i \, dS &= \int_V (\epsilon_{ijk} \lambda \mathsf{v}_{k,j}),_i \, dV \\
&= \int_V (\epsilon_{ijk} \lambda,_i \mathsf{v}_{k,j} + \epsilon_{ijk} \lambda \mathsf{v}_{k,ji}) \, dV \\
&= \int_V (\lambda,_i \underbrace{\epsilon_{ijk} \mathsf{v}_{k,j}}_{\mathsf{b}_i} + \lambda \underbrace{\epsilon_{ijk} \mathsf{v}_{k,ji}}_{0}) \, dV = \int_V \lambda,_i \mathsf{b}_i \, dV
\end{aligned}$$

Solution 2:

$$\int_S \lambda \mathsf{b}_i \hat{\mathsf{n}}_i \, dS = \int_V (\lambda \mathsf{b}_i),_i \, dV = \int_V (\lambda,_i \mathsf{b}_i + \lambda \mathsf{b}_i,_i) \, dV$$

note that $\mathsf{b}_i = \epsilon_{ijk} \mathsf{v}_{k,j} \Rightarrow \mathsf{b}_{i,i} = \epsilon_{ijk} \mathsf{v}_{k,ji} = \epsilon_{ijk} \mathsf{v}_{k,ij} = 0$

$$\int_S \lambda \mathsf{b}_i \hat{\mathsf{n}}_i \, dS = \int_V \lambda,_i \mathsf{b}_i \, dV = \int_V \lambda,_i \epsilon_{ijk} \mathsf{v}_{k,j} \, dV$$

Problem 1.128

Let V be a volume domain which is delimited by surface S. a) Show that:

$$\int_S (\vec{x} \otimes \hat{\mathbf{n}} + \hat{\mathbf{n}} \otimes \vec{x}) \, dS = 2V \mathbf{1} \tag{1.135}$$

where $\hat{\mathbf{n}}$ is the outward unit vector to surface S. b) Show also that:

$$\int_V (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x} \, dV = \int_S (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \otimes \vec{x} \, dS - \int_V \boldsymbol{\sigma} \, dV \qquad \int_V \sigma_{ik,k} x_j \, dV = \int_S \sigma_{ik} \hat{\mathsf{n}}_k x_j \, dS - \int_V \sigma_{ij} \, dV$$

and

$$\int_V \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \, dV = \int_S \vec{x} \otimes (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \, dS - \int_V \boldsymbol{\sigma}^T \, dV \qquad \int_V x_i \sigma_{jk,k} \, dV = \int_S x_i \sigma_{jk} \hat{\mathsf{n}}_k \, dS - \int_V \sigma_{ji} \, dV$$

where $\boldsymbol{\sigma}$ is an arbitrary second-order tensor field.

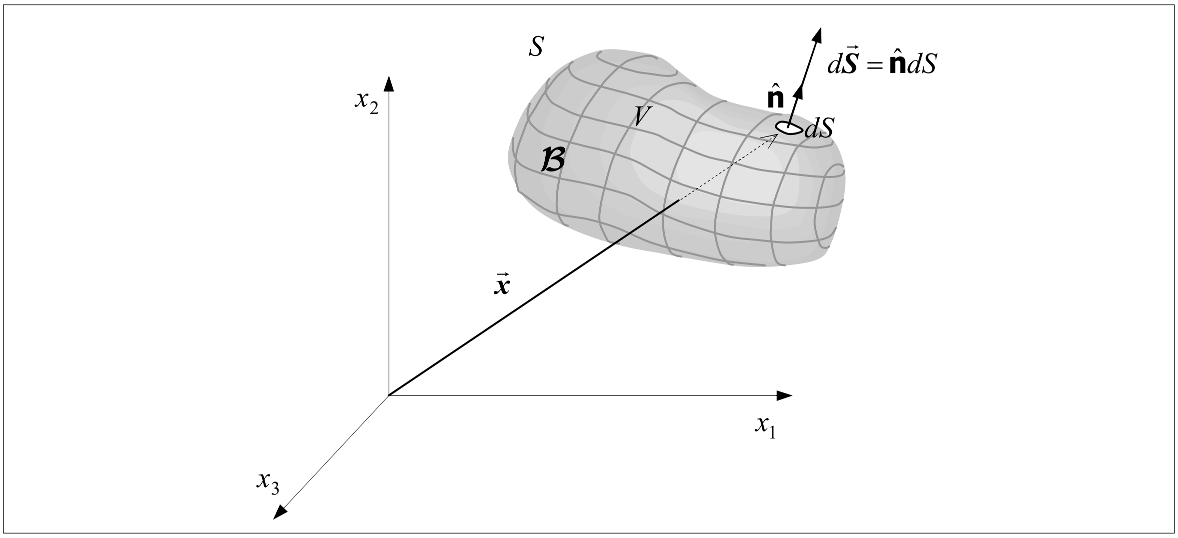


Figure 1.42

Solution:

a) Considering only the first term of the integrand in (1.135), we can obtain:

$$\int_S (\vec{x} \otimes \hat{\mathbf{n}})\, dS = \int_S (\vec{x} \otimes \mathbf{1} \cdot \hat{\mathbf{n}})\, dS = \int_S (\vec{x} \otimes \mathbf{1}) \cdot \hat{\mathbf{n}}\, dS$$

By applying the divergence theorem we can obtain:

$$\int_S (\vec{x} \otimes \hat{\mathbf{n}})\, dS = \int_S (\vec{x} \otimes \mathbf{1}) \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla_{\vec{x}} \cdot (\vec{x} \otimes \mathbf{1})\, dV$$

We will continue the development in indicial notation:

$$\int_S x_i \hat{\mathrm{n}}_j \, dS = \int_S x_i \delta_{jk} \hat{\mathrm{n}}_k \, dS = \int_V (\delta_{jk} x_i)_{,k} \, dV = \int_V (\delta_{jk,k} x_i + \delta_{jk} x_{i,k})\, dV$$

Taking into account that $\delta_{jk,k} = 0_j$, $x_{i,k} = \delta_{ik}$, we can conclude that:

$$\int_S x_i \hat{\mathrm{n}}_j \, dS = \int_V \delta_{ji} \, dV = \delta_{ji} \int_V dV = \delta_{ji} V \qquad \Big| \qquad \int_S (\vec{x} \otimes \hat{\mathbf{n}})\, dS = V \mathbf{1}^T = V\mathbf{1} \tag{1.136}$$

Similarly, we can conclude that $\int_S (\hat{\mathbf{n}} \otimes \vec{x})\, dS = V\mathbf{1}$. With that the following is true:

$$\int_S (\vec{x} \otimes \hat{\mathbf{n}} + \hat{\mathbf{n}} \otimes \vec{x})\, dS = 2V\mathbf{1}$$

b) Note that the following is true

$$(x_j \sigma_{ik})_{,k} = \underbrace{x_{j,k}}_{=\delta_{jk}} \sigma_{ik} + x_j \sigma_{ik,k} \quad \Rightarrow \quad x_j \sigma_{ik,k} = (x_j \sigma_{ik})_{,k} - \sigma_{ij}$$

$$\Rightarrow \quad (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x} = \nabla_{\vec{x}} \cdot (\boldsymbol{\sigma} \otimes \vec{x}) - \boldsymbol{\sigma}$$

with that we can obtain:

$$\int_V (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x}\, dV = \int_V \nabla_{\vec{x}} \cdot (\boldsymbol{\sigma} \otimes \vec{x})\, dV - \int_V \boldsymbol{\sigma}\, dV$$

$$\int_V (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x}\, dV = \int_S (\boldsymbol{\sigma} \otimes \vec{x}) \cdot \hat{\mathbf{n}}\, dS - \int_V \boldsymbol{\sigma}\, dV$$

$$= \int_S (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \otimes \vec{x}\, dS - \int_V \boldsymbol{\sigma}\, dV$$

$$\int_V x_j \sigma_{ik,k}\, dV = \int_V (x_j \sigma_{ik})_{,k}\, dV - \int_V \sigma_{ij}\, dV$$

$$\int_V x_j \sigma_{ik,k}\, dV = \int_S x_j \sigma_{ik} \hat{n}_k\, dS - \int_V \sigma_{ij}\, dV$$

$$= \int_S (\sigma_{ik} \hat{n}_k) x_j\, dS - \int_V \sigma_{ij}\, dV$$

where we have applied the divergence theorem to the first integral on the right side of equation.

Taking into account that

$$\left[(\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x}\right]^T = \left[\nabla_{\vec{x}} \cdot (\boldsymbol{\sigma} \otimes \vec{x}) - \boldsymbol{\sigma}\right]^T$$
$$\Rightarrow \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) = \left[\nabla_{\vec{x}} \cdot (\boldsymbol{\sigma} \otimes \vec{x})\right]^T - \boldsymbol{\sigma}^T$$

$$x_i \sigma_{jk,k} = (x_i \sigma_{jk})_{,k} - \sigma_{ji}$$

we can obtain:

$$\int_V \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma})\, dV = \int_V \left[\nabla_{\vec{x}} \cdot (\boldsymbol{\sigma} \otimes \vec{x})\right]^T\, dV - \int_V \boldsymbol{\sigma}^T\, dV$$

$$\int_V \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma})\, dV = \int_S (\vec{x} \otimes \boldsymbol{\sigma}) \cdot \hat{\mathbf{n}}\, dS - \int_V \boldsymbol{\sigma}^T\, dV$$

$$= \int_S \vec{x} \otimes (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\, dS - \int_V \boldsymbol{\sigma}^T\, dV$$

$$\int_V x_i \sigma_{jk,k}\, dV = \int_V (x_i \sigma_{jk})_{,k}\, dV - \int_V \sigma_{ji}\, dV$$

$$\int_V x_i \sigma_{jk,k}\, dV = \int_S (x_i \sigma_{jk}) \hat{n}_k\, dS - \int_V \sigma_{ji}\, dV$$

$$= \int_S x_i (\sigma_{jk} \hat{n}_k)\, dS - \int_V \sigma_{ji}\, dV$$

NOTE: If we obtain the trace of the equation (1.136) we can also obtain:

$$\int_S x_i \hat{n}_i\, dS = \delta_{ji} \delta_{ji} V = \delta_{ii} V \qquad \int_S (\vec{x} \otimes \hat{\mathbf{n}}) : \mathbf{1}\, dS = \int_S (\vec{x} \cdot \hat{\mathbf{n}})\, dS = V \mathbf{1} : \mathbf{1} \tag{1.137}$$

If we are dealing with a three dimensional case (3D) the trace $\delta_{ii} = 3$, and if we are dealing with two dimensional case (2D) we have that $\delta_{ii} = 2$. With that we can conclude that:

$$\int_S x_i \hat{n}_i\, dS = 3V \qquad \int_S (\vec{x} \cdot \hat{\mathbf{n}})\, dS = 3V \qquad \textit{(3D case)}$$

and

$$\int_\Gamma x_i \hat{n}_i\, d\Gamma = 2A \qquad \int_\Gamma (\vec{x} \cdot \hat{\mathbf{n}})\, d\Gamma = 2A \qquad \textit{(2D case)}$$

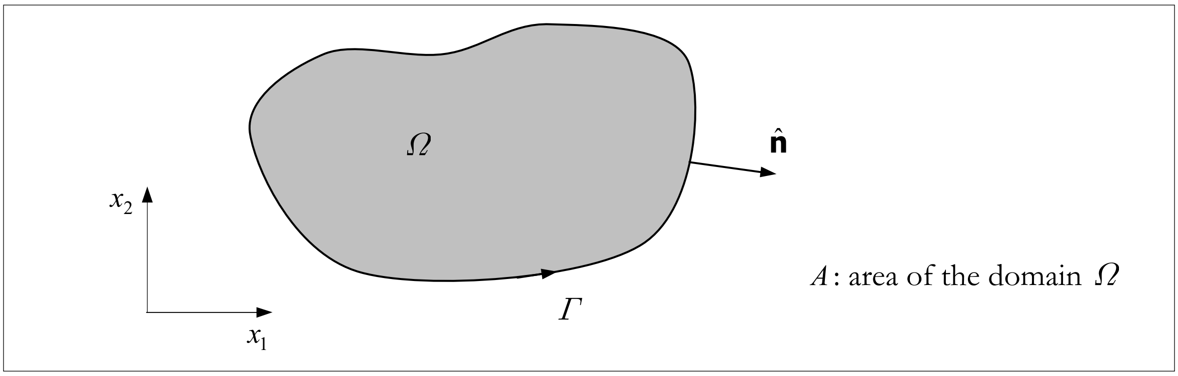


Figure 1.43: Two dimensional case – 2D.

Problem 1.129

Let ϕ be a scalar field which is given by $\phi = -\frac{GM}{\|\vec{a}\|}$, where G and M are scalars and constants, and $\|\vec{a}\|$ is the magnitude of the vector $\vec{a} \neq \vec{0}$. a) Obtain the gradient of ϕ. b) Obtain the gradient of ϕ for the particular case when $\vec{a} = \vec{x}$ and draw the field $\nabla_{\vec{x}}\phi$ in the Cartesian space.

Solution:

$$(\nabla_{\vec{x}}\phi)_i \equiv \left(\frac{\partial\phi}{\partial\vec{x}}\right)_{,i} \equiv \phi_{,i} = \left(\frac{-GM}{\|\vec{a}\|}\right)_{,i} = -GM\left(\frac{-1}{\|\vec{a}\|^2}\right)(\|\vec{a}\|)_{,i} \tag{1.138}$$

The term $(\|\vec{a}\|)_{,i}$ can be expressed as follows:

$$(\|\vec{a}\|)_{,i} = \left[(\vec{a}\cdot\vec{a})^{\frac{1}{2}}\right]_{,i} = \frac{1}{2}(\vec{a}\cdot\vec{a})^{\frac{-1}{2}}(\vec{a}\cdot\vec{a})_{,i} = \frac{1}{2}(\vec{a}\cdot\vec{a})^{\frac{-1}{2}}(a_k a_k)_{,i}$$
$$= \frac{1}{2}(\vec{a}\cdot\vec{a})^{\frac{-1}{2}}(a_{k,i}a_k + a_k a_{k,i}) = (\vec{a}\cdot\vec{a})^{\frac{-1}{2}}(a_{k,i}a_k) = \frac{1}{\|\vec{a}\|}(a_{k,i}a_k)$$

or in indicial notation:

$$\boxed{\nabla_{\vec{x}}(\|\vec{a}\|) = \frac{1}{\|\vec{a}\|}(\vec{a}\cdot\nabla_{\vec{x}}\vec{a})} \tag{1.139}$$

Then, the equation (1.138) becomes:

$$(\nabla_{\vec{x}}\phi)_i \equiv \left(\frac{\partial\phi}{\partial\vec{x}}\right)_{,i} \equiv \phi_{,i} = -GM\left(\frac{-1}{\|\vec{a}\|^2}\right)(\|\vec{a}\|)_{,i} = GM\left(\frac{1}{\|\vec{a}\|^2}\right)\frac{1}{\|\vec{a}\|}(a_{k,i}a_k) = \frac{GM}{\|\vec{a}\|^3}(a_{k,i}a_k) = \frac{GM}{\|\vec{a}\|^3}(\vec{a}\cdot\nabla_{\vec{x}}\vec{a})_i$$

Moreover, considering that the unit vector according to the direction $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$, we can obtain:

$$(\nabla_{\vec{x}}\phi)_i = \frac{GM}{\|\vec{a}\|^3}(\vec{a}\cdot\nabla_{\vec{x}}\vec{a})_i = \frac{GM}{\|\vec{a}\|^2}(\hat{a}\cdot\nabla_{\vec{x}}\vec{a})_i \tag{1.140}$$

b) For the particular case when $\vec{a} = \vec{x}$ we have:

$$(\|\vec{x}\|)_{,i} = \frac{1}{\|\vec{x}\|}(x_{k,i}x_k) = \frac{1}{\|\vec{x}\|}(\delta_{ki}x_k) = \frac{1}{\|\vec{x}\|}(x_i) \quad \text{where } r = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

or in tensorial notation:

$$\boxed{\nabla_{\vec{x}}(\|\vec{x}\|) = \frac{1}{\|\vec{x}\|}(\vec{x}\cdot\nabla_{\vec{x}}\vec{x}) = \frac{1}{\|\vec{x}\|}(\vec{x}\cdot\mathbf{1}) = \frac{1}{\|\vec{x}\|}(\vec{x}) = \hat{x}}$$

whereupon

$$(\nabla_{\vec{x}}\phi)_i \equiv \left(\frac{\partial\phi}{\partial\vec{x}}\right)_i \equiv \phi_{,i} = \left(\frac{-GM}{\|\vec{x}\|}\right)_{,i} = -GM\left(\frac{-1}{\|\vec{x}\|^2}\right)(\|\vec{x}\|)_{,i} = \frac{GM}{\|\vec{x}\|^3}(\vec{x})_i \tag{1.141}$$

or in tensorial notation:

$$\boxed{\nabla_{\vec{x}}\phi = \nabla_{\vec{x}}\left(\frac{-GM}{\|\vec{x}\|}\right) = \frac{GM}{\|\vec{x}\|^3}\vec{x} = \frac{GM}{\|\vec{x}\|^2}\hat{x}} \tag{1.142}$$

Note that the vector field $\nabla_{\vec{x}}\phi$ is radial, i.e. it is normal to the spheres defined by $\vec{x}$ and decreases with $\|\vec{x}\|^2 = r^2$, (see Figure 1.44). The equation (1.142) can also be written as follows:

$$\boxed{\nabla\phi = \nabla\left(\frac{-GM}{r}\right) = \frac{GM}{r^2}\hat{r} = \frac{\partial}{\partial r}\left(\frac{-GM}{r}\right)\hat{r} = \frac{\partial\phi(r)}{\partial r}\hat{r} = \phi'(r)\hat{r}} \tag{1.143}$$

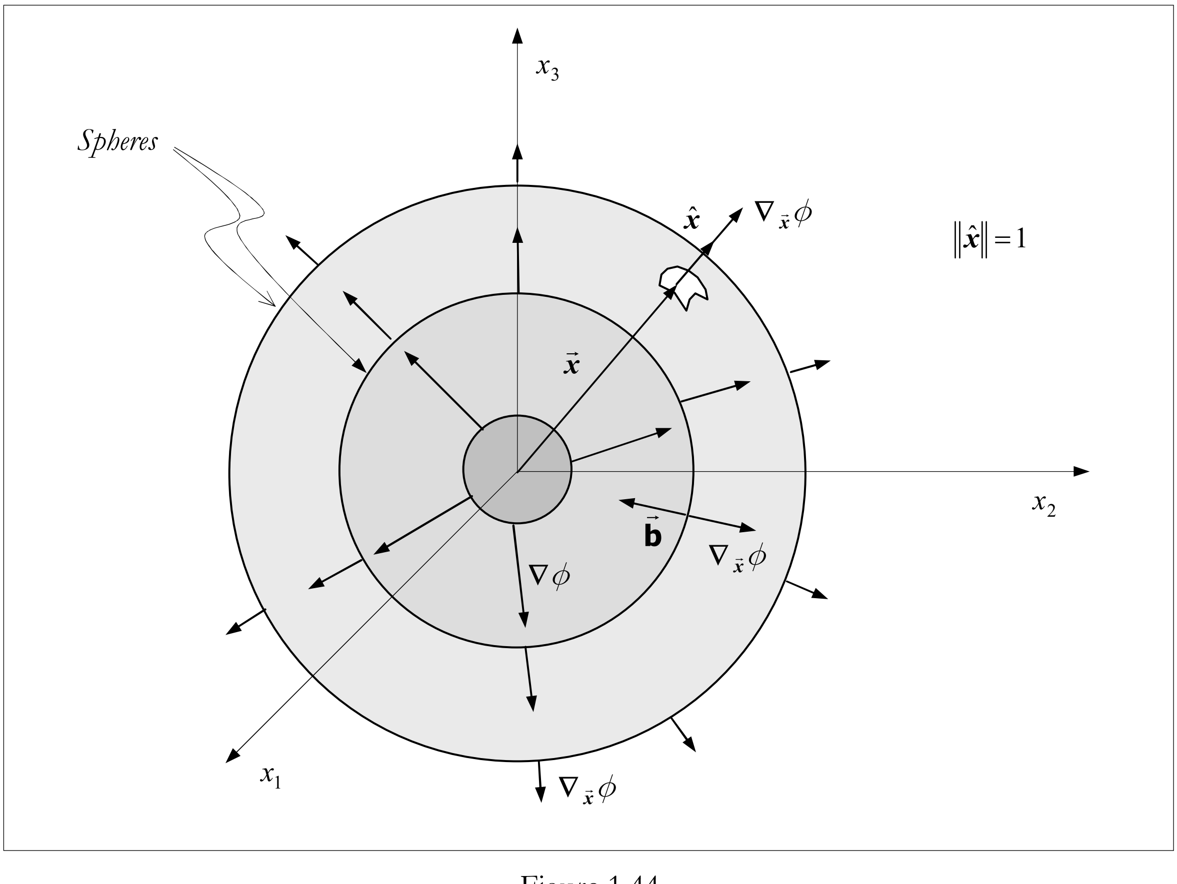


Figure 1.44

NOTE: The function $\phi = -\dfrac{GM}{\lVert \vec{x} \rVert}$ represents the gravitational potential which has the following property $\vec{\mathbf{b}} = -\nabla_{\vec{x}}\phi$, (see Figure 1.44), where $G = 6.67384 \times 10^{-11} \dfrac{m^3}{kg\, s^2}$ is the gravitational constant, M is the total mass of the planet. We check the units:

$$[\phi] = \left[-\frac{GM}{\lVert \vec{x} \rVert} \right] = \frac{m^3}{kg\, s^2}\frac{kg}{m} = \frac{kg\, m}{s^2}\frac{m}{kg} = \frac{N\, m}{kg} = \frac{J}{kg} \text{(Unit of energy per unit mass)} \atop \text{(specific energy)}$$

$$[\vec{\mathbf{b}}] = [-\nabla_{\vec{x}}\phi] = \left[\frac{\partial \phi}{\partial \vec{x}} \right] = \frac{J}{m\, kg} = \frac{N\cancel{m}}{\cancel{m}\, kg} = \frac{kgm}{s^2 kg} = \frac{m}{s^2} \text{(Unit of force per unit mass)} \atop \text{(unit of acceleration)}$$

It is interesting to check that $\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge [-\nabla_{\vec{x}}\phi] = \vec{\mathbf{0}}$, (see **Problem 1.108**).

We can obtain $\vec{\mathbf{b}}$ on the Earth surface by means of

$$\vec{\mathbf{b}} = -\nabla_{\vec{x}}\phi = -\frac{GM}{\lVert \vec{x} \rVert^2}\hat{\boldsymbol{x}}$$

where the total mass of Earth is $M \approx 5.98 \times 10^{24}\, kg$ and the approximate radius is $R \approx 6.37 \times 10^6\, m$, with that we obtain

$$\vec{\mathbf{b}} = -\frac{GM}{\lVert \vec{x} \rVert^2}\hat{\boldsymbol{x}} = -\frac{GM}{R^2}\hat{\boldsymbol{x}} \approx -9.82\hat{\boldsymbol{x}}$$

and its module is denoted by $g = \lVert \vec{\mathbf{b}} \rVert \approx 9.82 \dfrac{m}{s^2}$.

Adopting that the system $\vec{x}'$ has its origin at the center of mass of the body M, and invoking the Newton's second law ($\vec{F} = m\vec{a}$), we can obtain the force that act in a body (m) due to the gravitational field $\vec{\mathbf{b}} = -\nabla_{\vec{x}}\phi$:

$$\vec{F} = m\vec{a} = m\vec{\mathbf{b}} = -\frac{GMm}{\lVert \vec{x}' \rVert^2}\hat{\boldsymbol{x}}' \tag{1.144}$$

We can express the above equation in a generic system ($\vec{x}$), (see Figure 1.45).

Then, for the system $\vec{x}$ the force is given by:

$$\boxed{\vec{F}^{(mM)} = -\frac{GMm}{\lVert \vec{x}^{(m)} - \vec{x}^{(M)} \rVert^2} \frac{(\vec{x}^{(m)} - \vec{x}^{(M)})}{\lVert \vec{x}^{(m)} - \vec{x}^{(M)} \rVert}} \quad \textit{Newton's law of "universal" gravitation} \tag{1.145}$$

where we have adopted the nomenclature $\vec{F}^{(mM)}$ to indicate the force in m due to the influence of M. Note also that in M we have the same force in direction and magnitude, but of opposite sense to $\vec{F}^{(mM)}$.

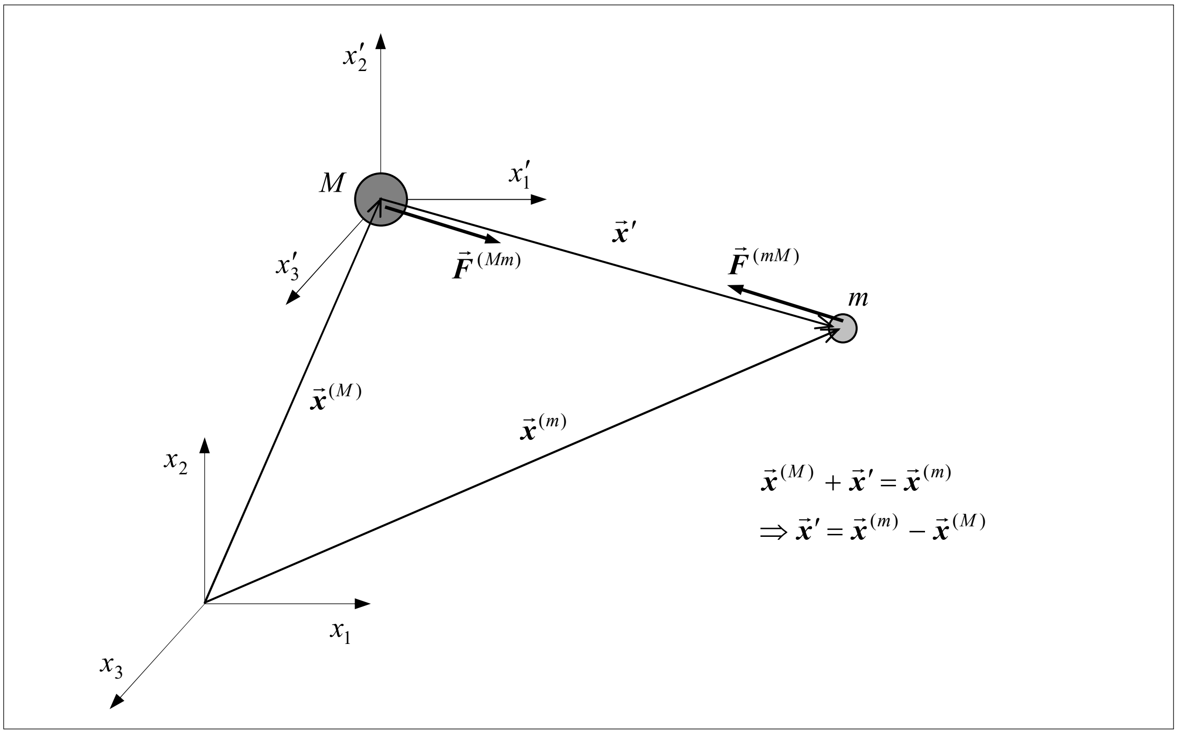


Figure 1.45

Problem 1.130

Consider that $\overline{\phi} = \frac{1}{r}$ where $r = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$:

a) Show that:

$$\nabla_{\vec{x}} \cdot [\nabla_{\vec{x}} \overline{\phi}(\vec{x} - \vec{\mathbf{0}})] \equiv \nabla^2 \overline{\phi} \equiv \frac{\partial^2 \overline{\phi}}{\partial x_1^2} + \frac{\partial^2 \overline{\phi}}{\partial x_2^2} + \frac{\partial^2 \overline{\phi}}{\partial x_3^2} = 0 \quad \Big| \; \textit{Laplace equation} \tag{1.146}$$

for $r \neq 0$. We use the nomenclature $[\nabla_{\vec{x}} \overline{\phi}(\vec{x} - \vec{\mathbf{0}})]$ to indicate that the origin ($\vec{x} = \vec{\mathbf{0}}$) is not included.

b) Given a closed surface S containing the origin, show that:

$$\int_S (\nabla_{\vec{x}} \overline{\phi}) \cdot \hat{\boldsymbol{n}} dS = -4\pi \tag{1.147}$$

where $\hat{\boldsymbol{n}}$ is the outward unit vector to surface.

Solution:

It was obtained in **Problem 1.129** that

$$\boxed{\nabla_{\vec{x}} \overline{\phi} = \nabla_{\vec{x}} \left(\frac{-GM}{\|\vec{x}\|} \right) = \frac{GM}{\|\vec{x}\|^3} \vec{x} = \frac{GM}{\|\vec{x}\|^2} \hat{x}} \tag{1.148}$$

Denoting by $GM = -1$ we obtain:

$$\boxed{\nabla_{\vec{x}} \overline{\phi} = \nabla_{\vec{x}} \left(\frac{1}{\|\vec{x}\|} \right) = \frac{-1}{\|\vec{x}\|^3} \vec{x} = \frac{-1}{\|\vec{x}\|^2} \hat{x}} \tag{1.149}$$

or in indicial notation:

$$(\nabla_{\vec{x}}\overline{\phi})_i = \left(\frac{-1}{\|\vec{x}\|^3}\vec{x}\right)_i = \frac{-1}{\|\vec{x}\|^3}x_i \tag{1.150}$$

By calculating the divergence of the previous relationship we can obtain:

$$\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\overline{\phi}) = \overline{\phi}_{,ii} = \left(\frac{-x_i}{\|\vec{x}\|^3}\right)_{,i} = -\frac{x_{i,i}}{\|\vec{x}\|^3} - x_i\left(\frac{1}{\|\vec{x}\|^3}\right)_{,i} = -\frac{x_{i,i}}{\|\vec{x}\|^3} - x_i\left[\frac{-3}{\|\vec{x}\|^4}(\|\vec{x}\|)_{,i}\right] \tag{1.151}$$

In **Problem 1.129** it was shown that $\nabla_{\vec{x}}(\|\vec{x}\|) = \frac{1}{\|\vec{x}\|}(\vec{x})$, in addition, note that $x_{i,i} = \delta_{ii} = 3$, with that we can obtain:

$$\begin{aligned}\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\overline{\phi}) &= -\frac{3}{\|\vec{x}\|^3} - x_i\left[\frac{-3}{\|\vec{x}\|^4}(\|\vec{x}\|)_{,i}\right] = -\frac{3}{\|\vec{x}\|^3} - x_i\left[\frac{-3}{\|\vec{x}\|^4}\frac{x_i}{\|\vec{x}\|}\right] = -\frac{3}{\|\vec{x}\|^3} + \frac{3x_ix_i}{\|\vec{x}\|^5} \\ &= -\frac{3}{\|\vec{x}\|^3} + \frac{3\|\vec{x}\|^2}{\|\vec{x}\|^5} = 0\end{aligned} \tag{1.152}$$

c) We adopt an arbitrary sphere of radius r, whose surface area is $4\pi r^2$. Then:

$$\int_S (\nabla_{\vec{x}}\overline{\phi})\cdot\hat{\boldsymbol{n}}dS = \int_S\left(\frac{-1}{\|\vec{x}\|^2}\hat{\boldsymbol{x}}\right)\cdot\hat{\boldsymbol{n}}dS = \frac{-1}{\|\vec{x}\|^2}\int_S\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{n}}dS = \frac{-1}{\|\vec{x}\|^2}\int_S dS = \frac{-1}{r^2}\times(Area) = \frac{-1}{r^2}\times(4\pi r^2) = -4\pi \tag{1.153}$$

Note that $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{n}} = 1$ since for the sphere it holds that $\hat{\boldsymbol{x}} // \hat{\boldsymbol{n}}$.

It is interesting to note that by means of the divergence theorem it fulfills that:

$$\int_V \nabla_{\vec{x}}\cdot[\nabla_{\vec{x}}\overline{\phi}]dV = \int_S (\nabla_{\vec{x}}\overline{\phi})\cdot\hat{\boldsymbol{n}}dS \qquad \Big| \qquad \int_V \overline{\phi}_{,ii}dV = \int_S \overline{\phi}_{,i}n_i dS \tag{1.154}$$

We have shown that $\nabla_{\vec{x}}\cdot[\nabla_{\vec{x}}\overline{\phi}(\vec{x}-\vec{\mathbf{0}})] = 0$, but that only apply to $\vec{x} \neq \vec{\mathbf{0}}$ (the origin is not included). That is, taking into account the result in (1.153), the result in (1.154) has consistency if at the point $\vec{x} = \vec{\mathbf{0}}$ there is a sink and equal to (-4π). With that, it is very intuitive to conclude that any closed surface that does not contain the origin the following holds $\int_S (\nabla_{\vec{x}}\overline{\phi})\cdot\hat{\boldsymbol{n}}dS = 0$, (see Parker (2003)).

Problem 1.131

a) Show that:

$$\int_S (\nabla\phi)\cdot\hat{\boldsymbol{n}}dS = 4\pi GM(r) \tag{1.155}$$

where $\phi = \left(\frac{-GM}{r}\right)$ is the gravitational potential, and $M(r)$ is the total mass contained into the sphere whose radius is r, and S-surface represents the sphere boundary.

b) Consider a sphere of radius $r = a$ which represents a planet. Obtain the total mass of the planet in function of the mass density $\rho = \rho(r)$.

c) Obtain the gravitational potential for $r < a$ and $r \ge a$. In this section, consider that the mass density is uniform in the planet $\rho = \rho_0$

Solution:

a) In **Problem 1.130** we have shown that:

$$\int_S (\nabla \overline{\phi}) \cdot \hat{\boldsymbol{n}} dS = \int_S \left[\nabla \left(\frac{1}{r} \right) \right] \cdot \hat{\boldsymbol{n}} dS = -4\pi \tag{1.156}$$

By multiply both sides of the equation by $GM(r)$ we can obtain:

$$\begin{aligned} & -GM(r) \int_S \left[\nabla \left(\frac{1}{r} \right) \right] \cdot \hat{\boldsymbol{n}} dS = 4\pi GM(r) \quad \Rightarrow \quad \int_S \left[\nabla \left(\frac{-GM(r)}{r} \right) \right] \cdot \hat{\boldsymbol{n}} dS = 4\pi GM(r) \\ & \Rightarrow \int_S [\nabla \phi] \cdot \hat{\boldsymbol{n}} dS = 4\pi GM(r) \end{aligned} \tag{1.157}$$

b)

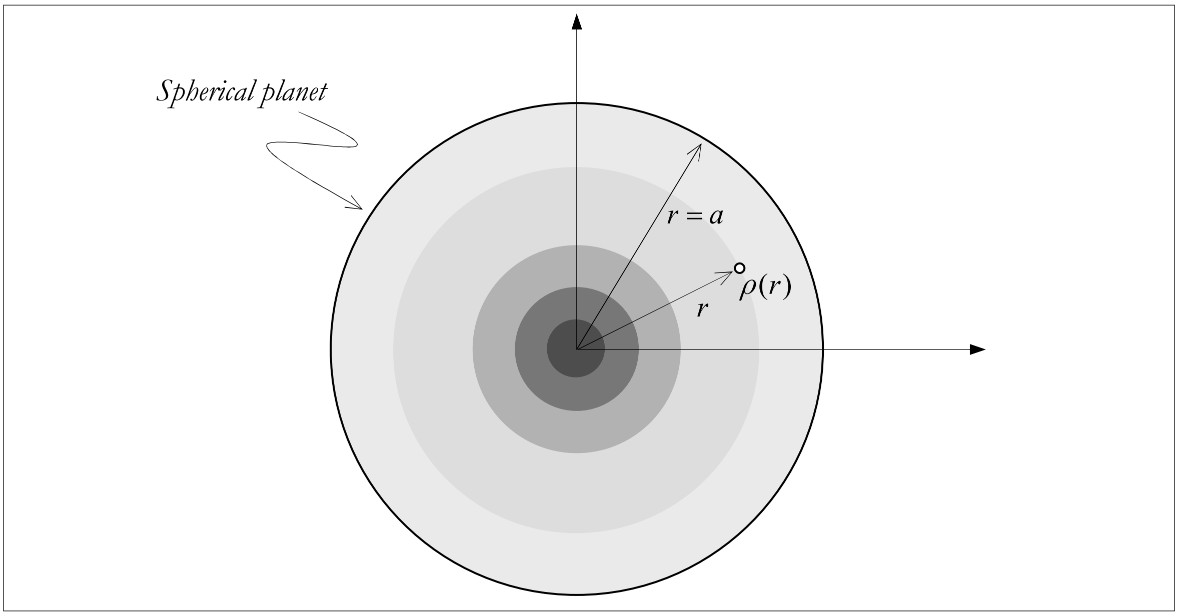


Figure 1.46

The total mass is obtained as follows:

$$M = \int_V \rho(r) dV \tag{1.158}$$

Note that $V_{sphere} = \frac{4}{3} \pi r^3 \Rightarrow dV = \frac{4}{3} \pi 3 r^2 dr = 4\pi r^2 dr$. Then:

$$M = \int_V \rho(r) dV = \int_{r=0}^{r=a} \rho(r) 4\pi r^2 dr \tag{1.159}$$

c) Remember that in **Problem 1.129**, (see equation (1.143)), we have obtained that

$$\nabla \phi = -\vec{\mathbf{b}} = \nabla \left(\frac{-GM}{r} \right) = \frac{GM}{r^2} \hat{\boldsymbol{r}} = \frac{\partial}{\partial r} \left(\frac{-GM}{r} \right) \hat{\boldsymbol{r}} = \frac{\partial \phi(r)}{\partial r} \hat{\boldsymbol{r}} = \phi'(r) \hat{\boldsymbol{r}} \tag{1.160}$$

By using the equation in (1.157) we can obtain:

$$
\begin{aligned}
&\int_S [\nabla \phi] \cdot \hat{\boldsymbol{n}} dS = 4\pi G M(r) \\
&\Rightarrow \int_S -\vec{\mathbf{b}} \cdot \hat{\boldsymbol{n}} dS = \int_S \phi'(r) \underbrace{\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}}_{=1} dS = \phi'(r) \int_S dS = \phi'(r)(4\pi r^2) = 4\pi G M(r) \\
&\Rightarrow \phi'(r) r^2 = G M(r) \qquad \Rightarrow \qquad \phi'(r) = \frac{G M(r)}{r^2}
\end{aligned}
\tag{1.161}
$$

where $M(r) = V\rho_0 = \frac{4}{3}\pi r^3 \rho_0$. Then:

$$
\phi'(r) = \frac{G M(r)}{r^2} = \frac{4G\pi\rho_0}{3} r \quad \Rightarrow \quad \frac{d\phi(r)}{dr} = \frac{4G\pi\rho_0}{3} r \quad \Rightarrow \quad \Rightarrow d\phi(r) = \frac{4G\pi\rho_0}{3} r dr \tag{1.162}
$$

By integrating the above equation we can obtain:

$$
\int d\phi = \int \frac{4G\pi\rho_0}{3} r dr \quad \Rightarrow \quad \phi(r) = \frac{4G\pi\rho_0}{3} \frac{r^2}{2} + C \quad \Rightarrow \quad \phi^{(1)}(r) = \frac{2G\pi\rho_0}{3} r^2 + C \tag{1.163}
$$

where we have denoted $\phi^{(1)}(r) = \phi(r)$ for $r < a$. For values $r \geq a$ the gravitational potential is given by

$$
\phi = \frac{-GM}{r} = \frac{-4G\pi a^3 \rho_0}{3r} = \phi^{(2)} \qquad for \qquad r \geq a \tag{1.164}
$$

where M is the total mass of the planet whose value is $M = V\rho_0 = \frac{4}{3}\pi a^3 \rho_0$. Note that the potential ϕ has to be continuous in $r = a$, (see Parker (2003)), thus:

$$
\begin{aligned}
&\phi^{(1)}(r = a) = \phi^{(2)}(r = a) \qquad \Rightarrow \qquad \frac{2G\pi\rho_0}{3} a^2 + C = \frac{-4G\pi a^3 \rho_0}{3a} \\
&\Rightarrow C = \frac{-2G\pi a^3 \rho_0}{a} = \frac{-2G\pi a^3 \rho_0}{a} \frac{4}{3} \frac{3}{4} = \frac{-2GM}{a} \frac{3}{4} = \frac{-3MG}{2a}
\end{aligned}
\tag{1.165}
$$

With that the equation (1.163) becomes

$$
\phi^{(1)}(r) = \frac{2G\pi\rho_0}{3} r^2 + C = \frac{2G\pi\rho_0}{3} r^2 - \frac{3MG}{2a} = \frac{MG}{2a^3} r^2 - \frac{3MG}{2a} = \frac{MG}{2a^2}\left(\frac{r^2}{2a^2} - \frac{3}{2}\right) \tag{1.166}
$$

We summarize the gravitational potential, (see Figure 1.47 and Figure 1.48), as follows:

$$
\begin{cases}
\phi(r) = \dfrac{MG}{2a^2}\left(\dfrac{r^2}{2a^2} - \dfrac{3}{2}\right) \qquad for \qquad r < a \\[2ex]
\phi(r) = \dfrac{MG}{r} \qquad for \qquad r \geq a
\end{cases}
\tag{1.167}
$$

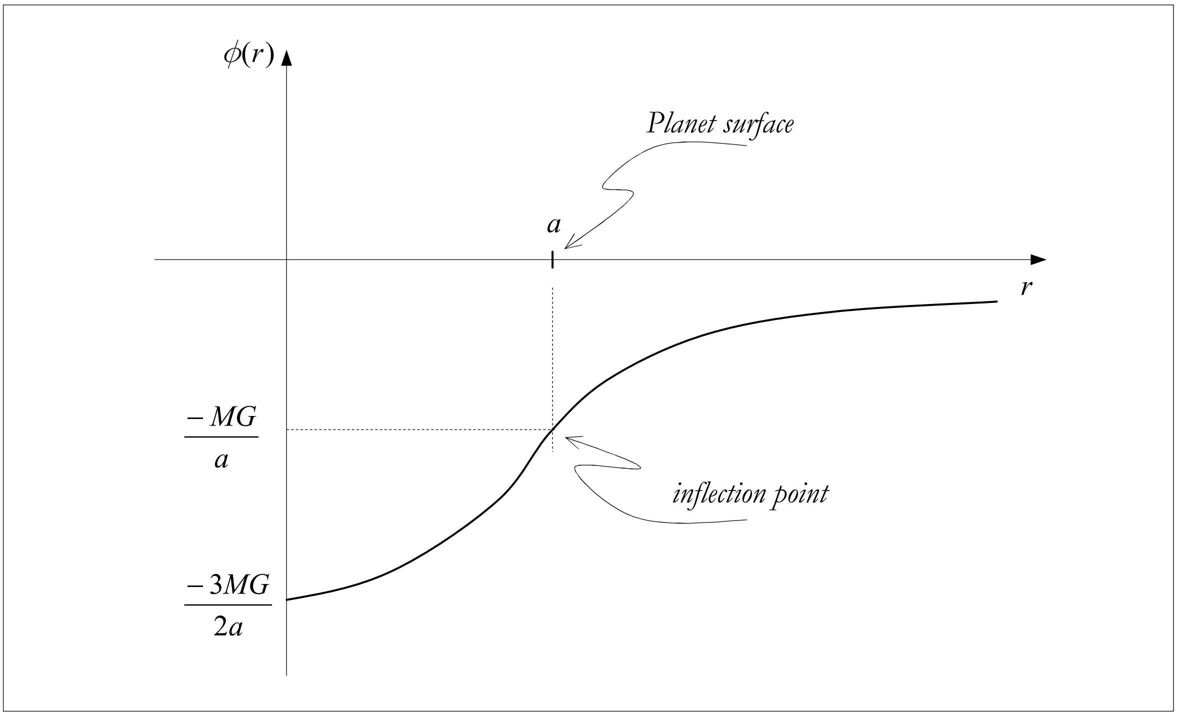


Figure 1.47: Gravitational potential *vs.* radius.

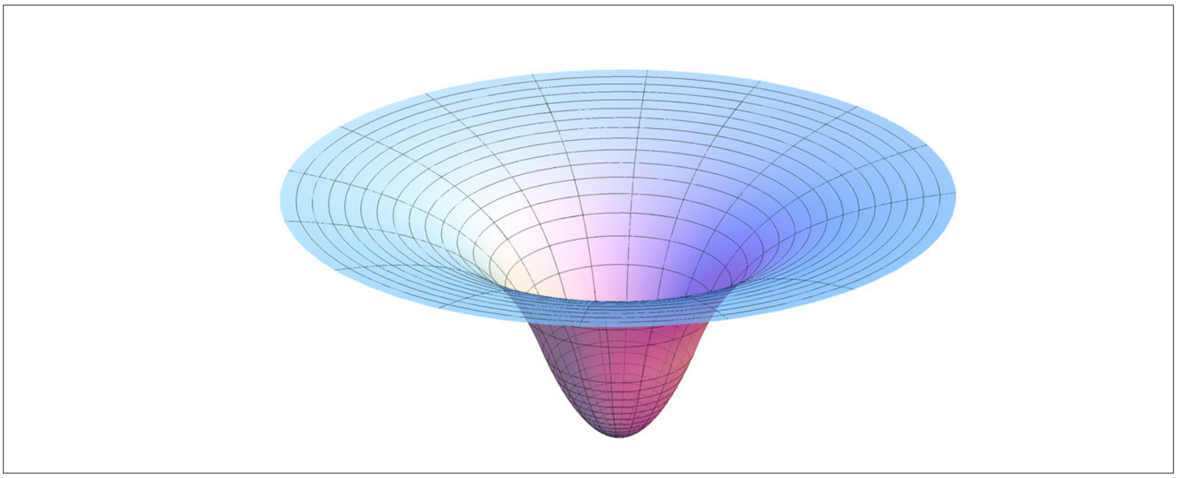


Figure 1.48: Gravitational potential (Ref. Wikipedia: "Gravitational potential").

Problem 1.132

a) Show that the orbit of a planet takes place on a plane. b) Prove the Kepler's laws of planetary motion:

b.1) First Law: The orbit of a planet is described by an ellipse, with the Sun at one of the foci of the ellipse;

b.2) Second Law: The vector position from the Sun to the planet describes an area at a constant rate;

b.3) Third Law: If T (orbital period) represents the time required for the planet to perform a full elliptical orbit, whose major axis of the ellipse is $2a$, the relationship $T^2 = \kappa a^3$ holds, where κ is a constant.

Reminder: Expressions related to the ellipse, (see Figure 1.49):

Equation of the ellipse: $\|\vec{\boldsymbol{x}}\| = r = \dfrac{p}{1 + e\cos\theta}$

Eccentricity: $e = \sqrt{\dfrac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2}}$; $0 < e < 1$, where $\mathrm{a}^2 = \dfrac{p^2}{(1-e^2)^2}$ holds.

Area enclosed by an ellipse: $A = \pi\mathrm{ab}$

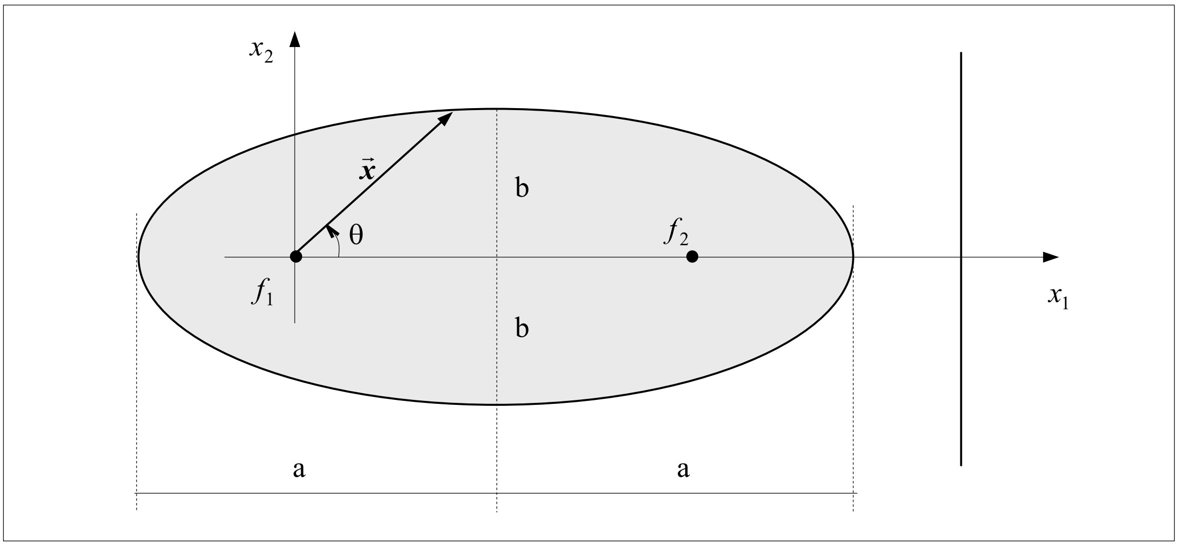


Figure 1.49

Solution:

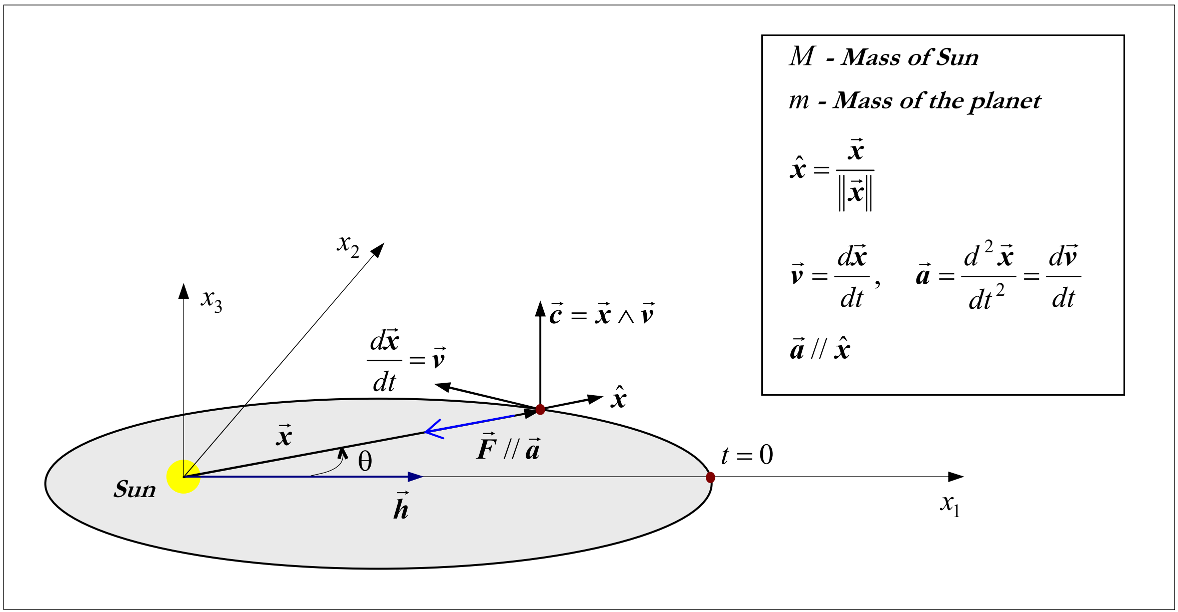


Figure 1.50: Orbit of the planet.

a) To show that the orbit takes place on a plane, we must prove that the vector ($\vec{\boldsymbol{c}}$) normal to the plane which is defined by the vectors $\hat{\boldsymbol{x}}$ and $\vec{\boldsymbol{v}}$ does not change with time, i.e. $\vec{\boldsymbol{c}}$ is constant, (see Figure 1.50).

We recall the equation (1.144) obtained in **Problem 1.132**:

$$\vec{F} = m\vec{a} = m\vec{\mathbf{b}} = -\frac{GMm}{\|\vec{x}\|^2}\hat{x} \quad ; \quad \vec{a} = -\frac{GM}{\|\vec{x}\|^2}\hat{x} \tag{1.168}$$

Next, we obtain the rate of change of $\vec{c} = \vec{x} \wedge \vec{v}$:

$$\frac{d\vec{c}}{dt} = \frac{d}{dt}(\vec{x} \wedge \vec{v}) = \frac{d}{dt}(\vec{x}) \wedge \vec{v} + \vec{x} \wedge \frac{d}{dt}(\vec{v}) = \underbrace{\vec{v} \wedge \vec{v}}_{=\vec{0}} + \underbrace{\vec{x} \wedge \vec{a}}_{=\vec{0}} = \vec{0}$$

Thus we have shown that the vector $\vec{c} = \vec{x} \wedge \vec{v}$ does not change with time, which implies that the orbit takes place on a plane.

b.1) First Law

Since the planet's orbit is performed on a plane, we take $x_1 - x_2$ as the plane of the orbit, then the vector $\vec{c}$ has the same direction as x_3, (see Figure 1.50).

We express $\vec{c}$ in term of $\hat{x}$:

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{d}{dt}(\|\vec{x}\|\hat{x}) = \frac{d(\|\vec{x}\|)}{dt}\hat{x} + \|\vec{x}\|\frac{d\hat{x}}{dt}$$

and

$$\vec{c} = \vec{x} \wedge \vec{v} = (\|\vec{x}\|\hat{x}) \wedge \left(\frac{d(\|\vec{x}\|)}{dt}\hat{x} + \|\vec{x}\|\frac{d\hat{x}}{dt}\right) = \|\vec{x}\|\frac{d(\|\vec{x}\|)}{dt}\underbrace{\hat{x} \wedge \hat{x}}_{=\vec{0}} + \|\vec{x}\|^2 \hat{x} \wedge \frac{d\hat{x}}{dt} = \|\vec{x}\|^2 \hat{x} \wedge \frac{d\hat{x}}{dt}$$

Taking into account that $\vec{a} = -\frac{GM}{\|\vec{x}\|^2}\hat{x}$, we calculate the vector $\vec{a} \wedge \vec{c}$ which has the same direction as $\vec{v}$, i.e. $(\vec{a} \wedge \vec{c}) // \vec{v}$:

$$\begin{aligned}\vec{a} \wedge \vec{c} &= \left(-\frac{GM}{\|\vec{x}\|^2}\hat{x}\right) \wedge \left(\|\vec{x}\|^2 \hat{x} \wedge \frac{d\hat{x}}{dt}\right) = -GM \ \hat{x} \wedge \left(\hat{x} \wedge \frac{d\hat{x}}{dt}\right) = -GM\left[(\hat{x} \cdot \frac{d\hat{x}}{dt})\hat{x} - (\hat{x} \cdot \hat{x})\frac{d\hat{x}}{dt}\right] \\ &= GM\frac{d\hat{x}}{dt}\end{aligned}$$

where we have used the property $\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$, (see **Problem 1.17**). Note also that it holds that $\hat{x} \cdot \frac{d\hat{x}}{dt} = 0$ since $\hat{x} \perp \frac{d\hat{x}}{dt}$, and $\hat{x} \cdot \hat{x} = \|\hat{x}\|^2 = 1$. Considering that GM is a constant, the following is true:

$$\vec{a} \wedge \vec{c} = GM\frac{d\hat{x}}{dt} = \frac{d(GM \ \hat{x})}{dt}$$

Since the vector $\vec{c}$ does not change with time, the following is true:

$$\vec{a} \wedge \vec{c} = \frac{d\vec{v}}{dt} \wedge \vec{c} = \frac{d(\vec{v} \wedge \vec{c})}{dt}$$

Thus

$$\frac{d(\vec{v} \wedge \vec{c})}{dt} = \frac{d(GM \ \hat{x})}{dt}$$

Integrating over time the above equation we can obtain:

$$\vec{v} \wedge \vec{c} = GM \ \hat{x} + \vec{h}$$

where $\vec{\boldsymbol{h}}$ is constant vector of integration and is independent of time. Note that $\vec{\boldsymbol{h}}$ is located on the plane $x_1 - x_2$, since $(\vec{\boldsymbol{v}} \wedge \vec{\boldsymbol{c}})$ and $\hat{\boldsymbol{x}}$ are also on the plane $x_1 - x_2$, (see Figure 1.50).

We calculate:

$$\vec{\boldsymbol{h}} \cdot \hat{\boldsymbol{x}} = \left\|\vec{\boldsymbol{h}}\right\| \left\|\hat{\boldsymbol{x}}\right\| \cos\theta = h\cos\theta$$

where we have denoted by $h = \left\|\vec{\boldsymbol{h}}\right\|$. Then:

$$\begin{aligned} c^2 &= \left\|\vec{\boldsymbol{c}}\right\|^2 = \vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{c}} = (\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{c}} = (\vec{\boldsymbol{v}} \wedge \vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{x}} \\ &= \left(GM\ \hat{\boldsymbol{x}} + \vec{\boldsymbol{h}}\right) \cdot (\left\|\vec{\boldsymbol{x}}\right\| \hat{\boldsymbol{x}}) = \left\|\vec{\boldsymbol{x}}\right\| GM\ \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} + \left\|\vec{\boldsymbol{x}}\right\| \vec{\boldsymbol{h}} \cdot \hat{\boldsymbol{x}} = \left\|\vec{\boldsymbol{x}}\right\| GM + \left\|\vec{\boldsymbol{x}}\right\| h\cos\theta \\ &= \left\|\vec{\boldsymbol{x}}\right\| \left(GM + h\cos\theta\right) = r\left(GM + h\cos\theta\right) \end{aligned}$$

where we have considered that $r = \left\|\vec{\boldsymbol{x}}\right\|$. Then, we can obtain the following equation of the ellipse:

$$\Rightarrow r = \frac{c^2}{\left(GM + h\cos\theta\right)} = \frac{\dfrac{c^2}{GM}}{\dfrac{\left(GM + h\cos\theta\right)}{GM}} = \frac{p}{1 + e\cos\theta}$$

where we have considered that:

$$p = \frac{c^2}{GM} \qquad and \qquad e = \frac{h}{GM} \tag{1.169}$$

b.2) Second Law

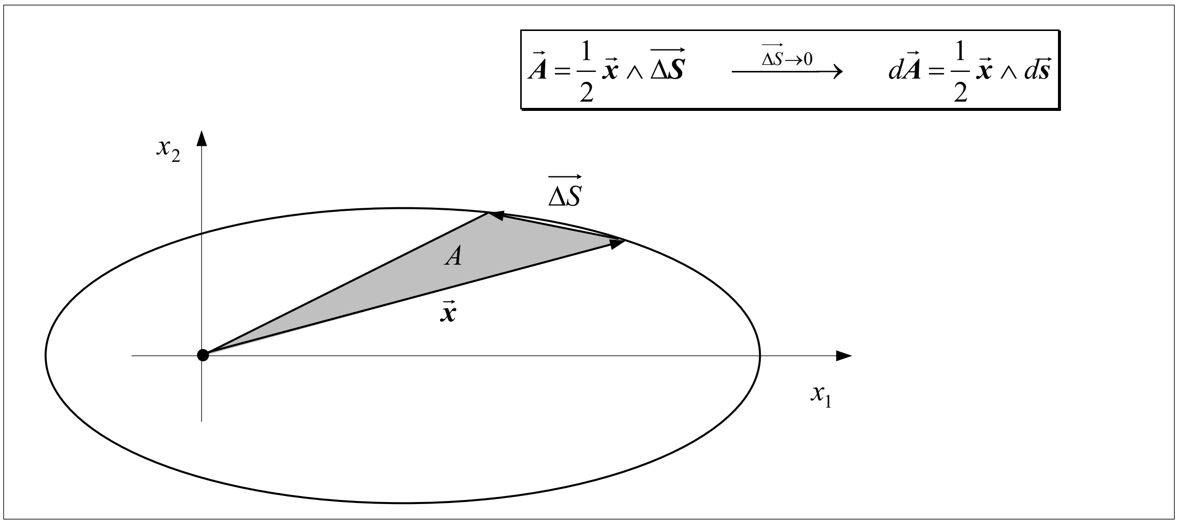


Figure 1.51

The rate of change of $d\vec{\boldsymbol{A}}$ becomes:

$$\begin{aligned} \frac{D(d\vec{\boldsymbol{A}})}{Dt} &= \frac{1}{2}\frac{D(\vec{\boldsymbol{x}} \wedge d\vec{\boldsymbol{s}})}{Dt} = \frac{1}{2}\frac{D(\vec{\boldsymbol{x}})}{Dt} \wedge d\vec{\boldsymbol{s}} + \frac{1}{2}\vec{\boldsymbol{x}} \wedge \frac{D(d\vec{\boldsymbol{s}})}{Dt} \\ &= \frac{1}{2}\underbrace{\frac{D(\vec{\boldsymbol{x}})}{Dt} \wedge d\vec{\boldsymbol{s}}}_{=\vec{\boldsymbol{0}}} + \frac{1}{2}\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{v}} = \frac{1}{2}\vec{\boldsymbol{c}} \quad \text{(constant)} \end{aligned}$$

and its magnitude:

$$\left\| \frac{D(d\vec{A})}{Dt} \right\| = \frac{D(dA)}{Dt} = \frac{1}{2}\|\vec{c}\| = \frac{1}{2}c$$

NOTE: As a consequence of second law it follows that if the areas of two sectors are equal, the time required to perform their paths are equal, that is, according to Figure 1.52 as the areas of the sectors OCD and EFO are equal the times to perform $C \to D$ and $E \to F$ are equal. As result, when the planet is closer to the Sun its velocity is greater than when it is far.

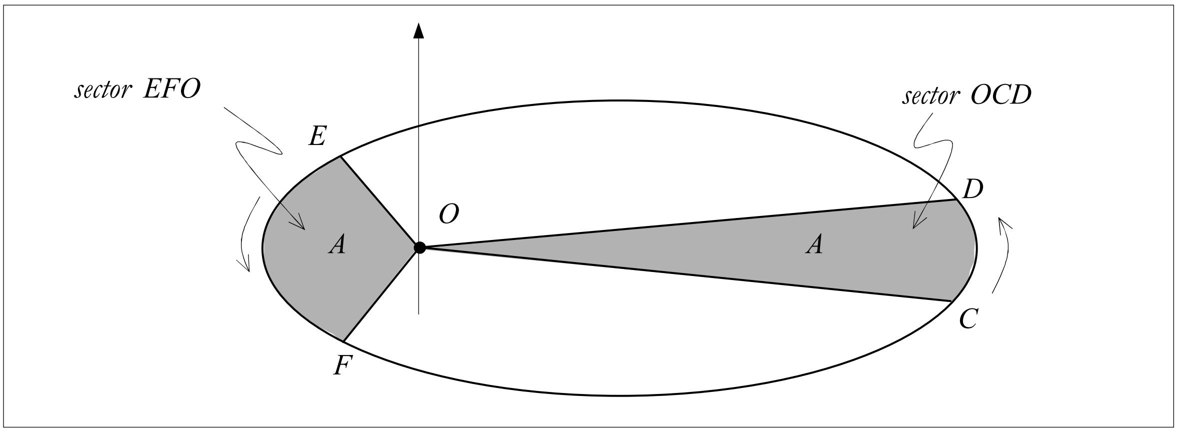


Figure 1.52: Orbit of the planet.

b.3) Third Law

If T is the total time for a complete orbit (orbital period), we can obtain:

$$A = \int_0^T \frac{D(dA)}{Dt}\, dt = \int_0^T \frac{1}{2} c\, dt = \frac{1}{2} cT$$

Taking into account the area enclosed by the ellipse: $A = \pi \mathrm{ab}$, we conclude that $\frac{1}{2}cT = \pi \mathrm{ab}$, thus:

$$T = \frac{2\pi \mathrm{ab}}{c} \quad \Rightarrow \quad T^2 = \frac{4\pi^2 \mathrm{a}^2 \mathrm{b}^2}{c^2} \tag{1.170}$$

Considering the equation of the eccentricity, we can obtain:

$$e = \sqrt{\frac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2}} \quad \Rightarrow \quad \mathrm{b}^2 = \mathrm{a}^2 - \mathrm{a}^2 e^2 \quad \Rightarrow \quad \mathrm{b}^2 = \mathrm{a}^2(1 - e^2)$$

and taking into account $\mathrm{a}^2 = \frac{p^2}{(1-e^2)^2} \Rightarrow \mathrm{a} = \frac{p}{(1-e^2)} \Rightarrow (1-e^2)\mathrm{a} = p$ into the above equation, we can obtain:

$$\mathrm{b}^2 = \mathrm{a}^2(1-e^2) \quad \Rightarrow \quad \mathrm{b}^2 = \mathrm{a}p \quad \Rightarrow \quad p = \frac{\mathrm{b}^2}{\mathrm{a}}$$

Whereby the equation (1.170) can be rewritten as follows:

$$T^2 = \frac{4\pi^2 \mathrm{a}^2 \mathrm{b}^2}{c^2} = \frac{4\pi^2 \mathrm{a}^2 \mathrm{ab}^2}{c^2 \mathrm{a}} = \frac{4\pi^2 \mathrm{a}^3 p}{c^2} = \frac{4\pi^2}{GM}\mathrm{a}^3 = \kappa\, \mathrm{a}^3 \tag{1.171}$$

where we have considered that $\frac{p}{c^2} = \frac{1}{GM}$, (see equation (1.169)).

COMPLEMENTARY NOTE 1

Geometrical Properties of Curves

Let us consider the curve defined by the function $y = y(x)$, (see Figure 1.53), we denote by:

First derivative: $\frac{dy(x)}{dx} \equiv y' \equiv y_{,x}$ (tangent of the curve at a point)

Second derivative: $\frac{d^2 y(x)}{dx^2} \equiv y'' \equiv y_{,xx}$.

Infinitesimal arc length $d\mathscr{s}$:

According to Figure 1.53 we can obtain:

$$\Delta\mathscr{s} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(\Delta x^2 + \Delta y^2)\frac{\Delta x^2}{\Delta x^2}} = \sqrt{\frac{(\Delta x^2 + \Delta y^2)}{\Delta x^2}}\ \Delta x = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Then, we define the *differential arc-length element* as follows:

$$d\mathscr{s} = \lim_{\Delta x \to 0}\left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \equiv \sqrt{1 + (y')^2}\ dx = \left[1 + (y')^2\right]^{\frac{1}{2}}\ dx$$

$$\Rightarrow \frac{d\mathscr{s}}{dx} = \left[1 + (y')^2\right]^{\frac{1}{2}}$$

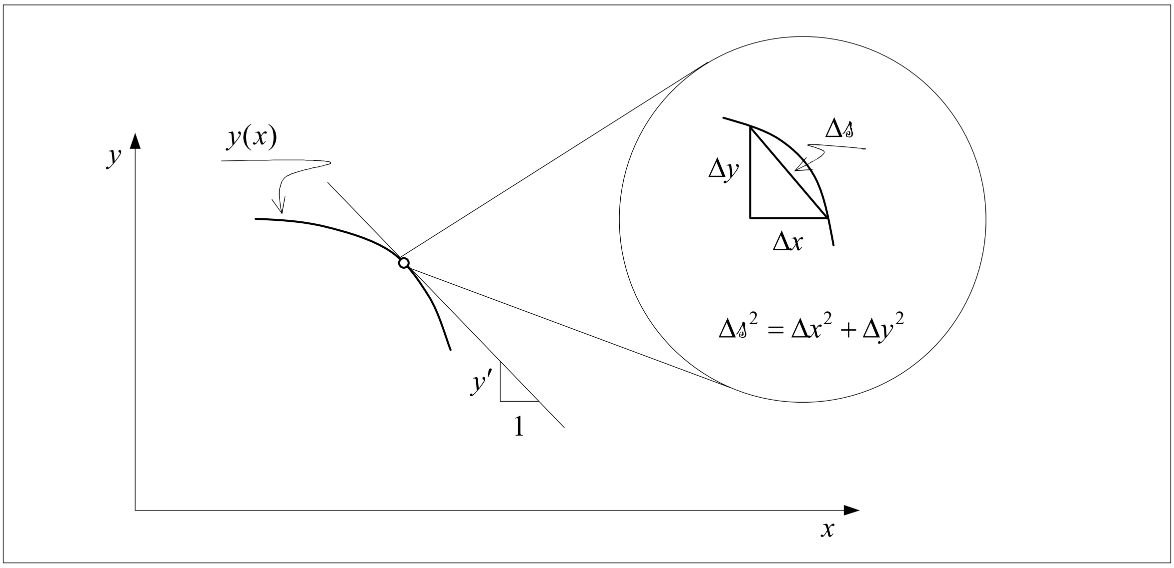


Figure 1.53

Curvature

Curvature measures how quickly the direction of $\hat{\mathbf{s}}$ changes with respect to a change in arc length $\mathcal{s}$, where $\hat{\mathbf{s}}$ is the unit vector according to the (y') -direction, (see Figure 1.57). So, we define the vector curvature as follows:

$$\vec{\boldsymbol{\kappa}} = \frac{d\hat{\mathbf{s}}}{d\mathcal{s}} \xrightarrow{\text{curvature}} \kappa = \left\| \vec{\boldsymbol{\kappa}} \right\| = \left\| \frac{d\hat{\mathbf{s}}}{d\mathcal{s}} \right\|$$

where $\kappa(x)$ is the curvature of the curve at point x.

Let us consider that there is an angle ψ such as $\tan(\psi) = \frac{dy}{dx} \equiv y'$ and if we differentiate with respect to x we obtain:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\tan(\psi)\right] = \frac{d\left[\tan(\psi)\right]}{d\psi}\frac{d\psi}{dx} = \sec^2(\psi)\frac{d\psi}{dx} = \left[1+\tan^2(\psi)\right]\frac{d\psi}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} \equiv y'' = \left[1+\tan^2(\psi)\right]\frac{d\psi}{dx} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d\psi}{dx} = \left[1+(y')^2\right]\frac{d\psi}{dx}$$

$$\Rightarrow \frac{d\psi}{dx} = \frac{y''}{[1+(y')^2]}$$

The curvature can be obtained as follows:

$$\kappa = \frac{d\psi}{d\mathcal{s}} = \frac{d\psi}{dx}\frac{dx}{d\mathcal{s}} = \frac{y''}{[1+(y')^2]}\frac{1}{[1+(y')^2]^{\frac{1}{2}}} = \frac{y''}{[1+(y')^2]^{\frac{3}{2}}}$$

where $\frac{dx}{d\mathcal{s}} = \frac{1}{[1+(y')^2]^{\frac{1}{2}}}$ holds.

Note that the curvature of a circumference is constant, (see Figure 1.54):

$$\begin{aligned} &\kappa = \frac{d\psi}{d\mathcal{s}} = \quad \Rightarrow \quad \kappa d\mathcal{s} = d\psi \xrightarrow{\text{integrating}} \int \kappa d\mathcal{s} = \int d\psi \\ &\Rightarrow \kappa \int d\mathcal{s} = \int d\psi \quad \Rightarrow \quad \Rightarrow \kappa 2\pi r = 2\pi \quad \Rightarrow \quad \kappa = \frac{2\pi}{2\pi r} = \frac{1}{r} \end{aligned} \tag{1.172}$$

where $(2\pi r)$ is the length of the circumference of radius r.

If we consider Figure 1.54 we can conclude that the curvature of the circumference of radius $r^{(1)}$ is greater than the circumference of radius $r^{(2)}$:

$$r^{(1)} < r^{(2)} \quad \Rightarrow \quad \frac{1}{r^{(1)}} > \frac{1}{r^{(2)}} \quad \Rightarrow \quad \kappa^{(1)} > \kappa^{(2)}$$

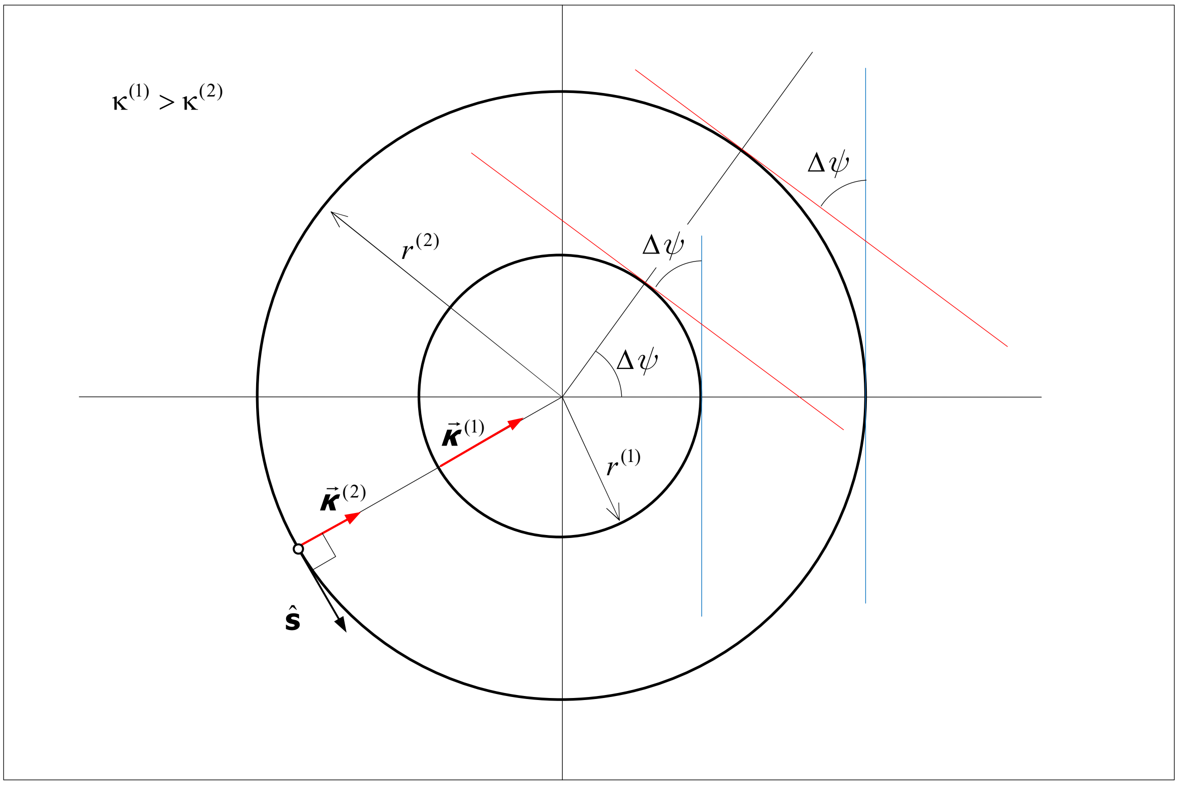


Figure 1.54

By considering the curvature, (see equation (1.172)), we can obtain:

$$\int \kappa d\mathfrak{s} = \int_A^B d\psi = \psi_B - \psi_A \equiv \Delta\psi_{B\_A}$$

Then, the area defined in Figure 1.55 can be obtained by $Area = \int \kappa d\mathfrak{s} = \psi_B - \psi_A \equiv \Delta\psi_{B\_A}$.

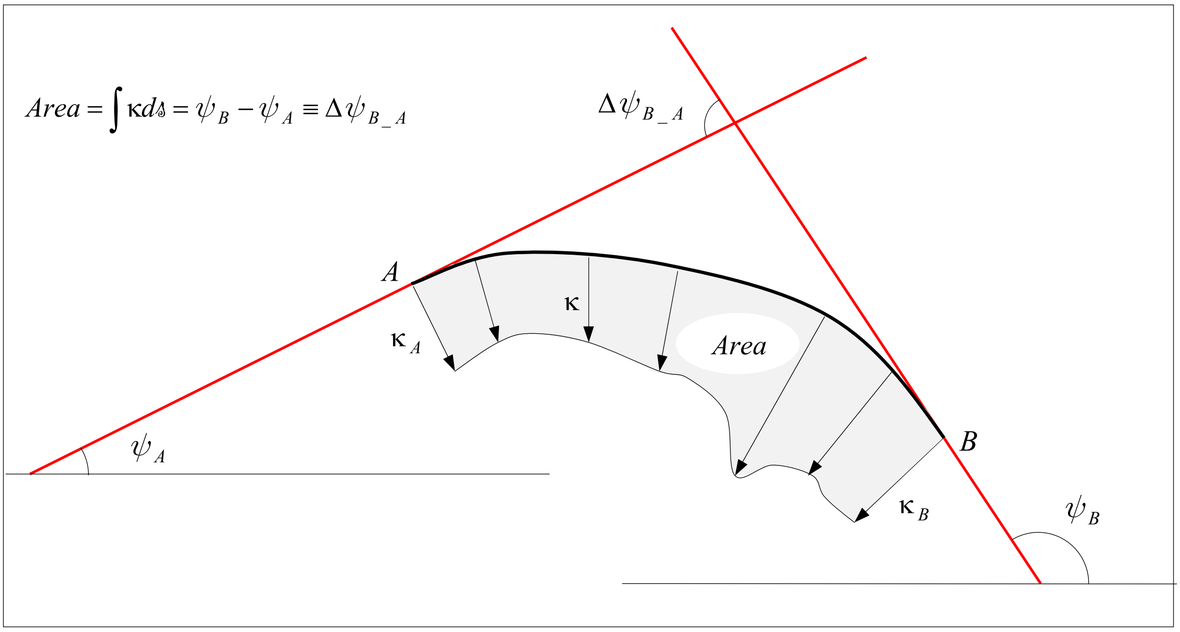


Figure 1.55

For example, let us consider a circumference of radius r, and the variation of angle from A to B can be obtained as follows $Area = \psi_B - \psi_A \equiv \Delta\psi_{B\_A}$, (see Figure 1.56):

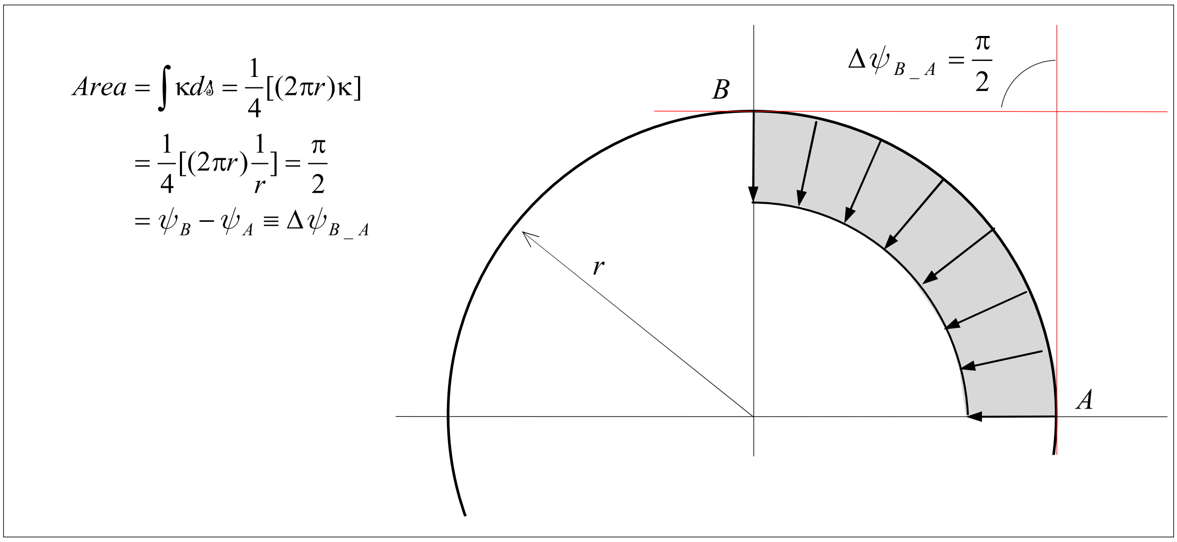


Figure 1.56

The *Curvature Vector* is defined as follows

$$\vec{\boldsymbol{\kappa}} = \frac{d\hat{\mathbf{s}}}{d\mathscr{s}}$$

where $\hat{\mathbf{s}}$ is the unit vector (tangent to the curve), and if we use the unit vector properties we can conclude that:

$$\left\|\hat{\mathbf{s}}\right\| = 1 \qquad \left\|\hat{\mathbf{s}}\right\|^2 = \hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = 1 \quad \Rightarrow \quad \frac{d(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}})}{d\mathscr{s}} = \frac{d(1)}{d\mathscr{s}} = 0$$

$$\Rightarrow \frac{d(\hat{\mathbf{s}})}{d\mathscr{s}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{s}} \cdot \frac{d(\hat{\mathbf{s}})}{d\mathscr{s}} = 2\frac{d\hat{\mathbf{s}}}{d\mathscr{s}} \cdot \hat{\mathbf{s}} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\kappa}} \cdot \hat{\mathbf{s}} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\kappa}} \perp \hat{\mathbf{s}}$$

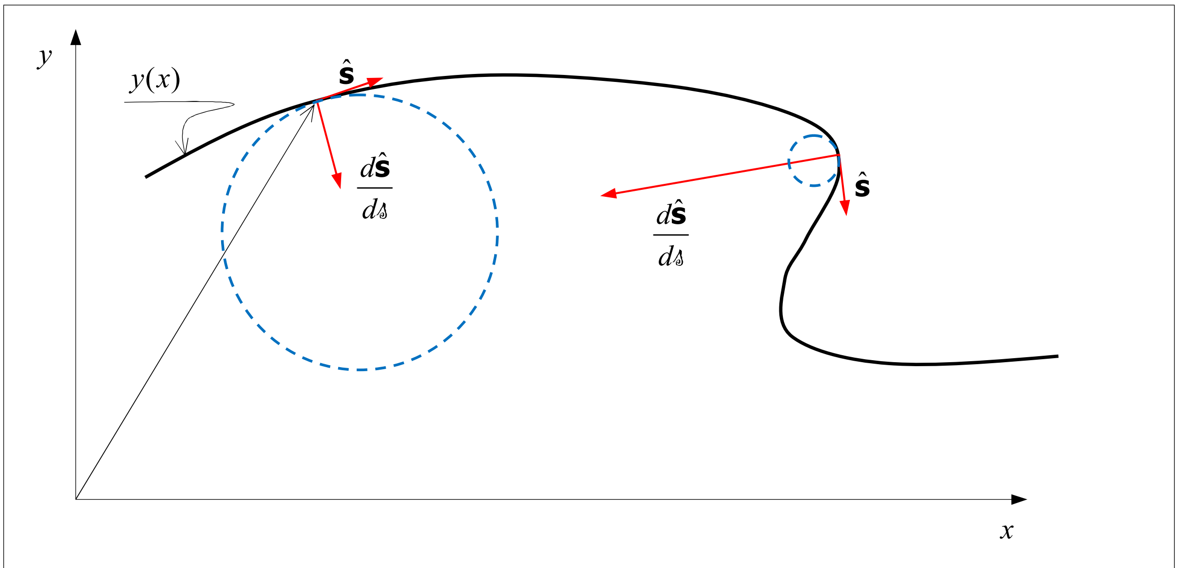


Figure 1.57

Geometrical relations among coordinate increments

The transformation matrix from the system $\vec{x}$ to the system $\hat{\mathbf{n}}$-$\hat{\mathbf{s}}$ (which is denoted by $\vec{\bar{x}}$), (see Figure 1.58), is given by:

$$a_{ij} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 \\ -\hat{n}_2 & \hat{n}_1 \end{bmatrix}$$

Then, it fulfils that:

$$d\bar{x}_i = a_{ij} dx_j \quad \Rightarrow \quad dx_i = a_{ji} d\bar{x}_j$$

$$\begin{Bmatrix} dx_1 \\ dx_2 \end{Bmatrix} = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 \\ -\hat{n}_2 & \hat{n}_1 \end{bmatrix}^T \begin{Bmatrix} 0 \\ ds \end{Bmatrix} = \begin{bmatrix} \hat{n}_1 & -\hat{n}_2 \\ \hat{n}_2 & \hat{n}_1 \end{bmatrix} \begin{Bmatrix} 0 \\ ds \end{Bmatrix} = \begin{Bmatrix} -\hat{n}_2 ds \\ \hat{n}_1 ds \end{Bmatrix}$$

With that we can obtain:

$$\begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{dx_2}{ds} \\ -\dfrac{dx_1}{ds} \end{Bmatrix} \quad \Rightarrow \quad \hat{n}_i = \epsilon_{3ij} \frac{dx_j}{ds}$$

where ϵ_{kij} is the permutation symbol.

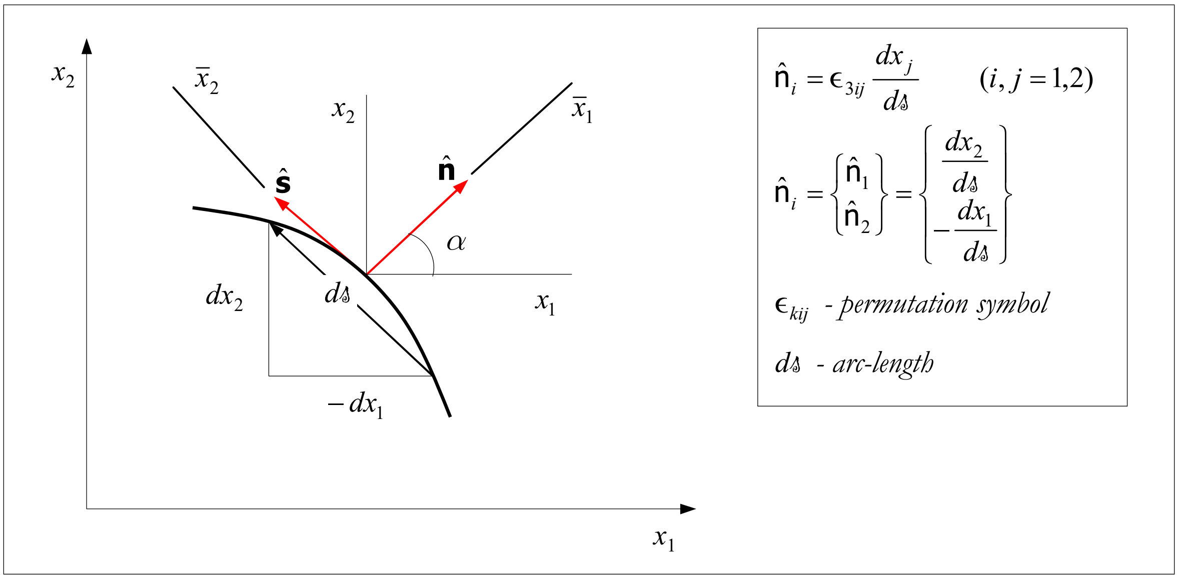


Figure 1.58

COMPLEMENTARY NOTE 2

Geometrical Center (Centroid – C.Geo.)

Let us consider $V=\int_V dV$ the volume delimited by the surface S, and also let us consider the systems $\vec{x}$ and $\vec{x}'$, (see Figure 1.59). By means of vector summation we can obtain:

$$\vec{x}=\vec{\bar{x}}+\vec{x}'$$

By integrate over the volume we can obtain:

$$\int_V \vec{x}dV=\int_V(\vec{\bar{x}}+\vec{x}')dV=\int_V\vec{\bar{x}}dV+\int_V\vec{x}'dV$$

The volume centroid is the point $(\vec{\bar{x}}^{(V)})$ where the following equation fulfils:

$$\int_V \vec{x}'dV=\vec{0} \qquad \Rightarrow \qquad \vec{\bar{x}}=\vec{\bar{x}}^{(V)}$$

Then, the centroid can be calculated as follows:

$$\int_V\vec{x}dV=\int_V\vec{\bar{x}}^{(V)}dV+\underbrace{\int_V\vec{x}'dV}_{=\vec{0}}=\int_V\vec{\bar{x}}^{(V)}dV=\vec{\bar{x}}^{(V)}\int_V dV \quad \Rightarrow \quad \boxed{\vec{\bar{x}}^{(V)}=\frac{\int_V\vec{x}dV}{\int_V dV}=\frac{\int_V\vec{x}dV}{V}}$$

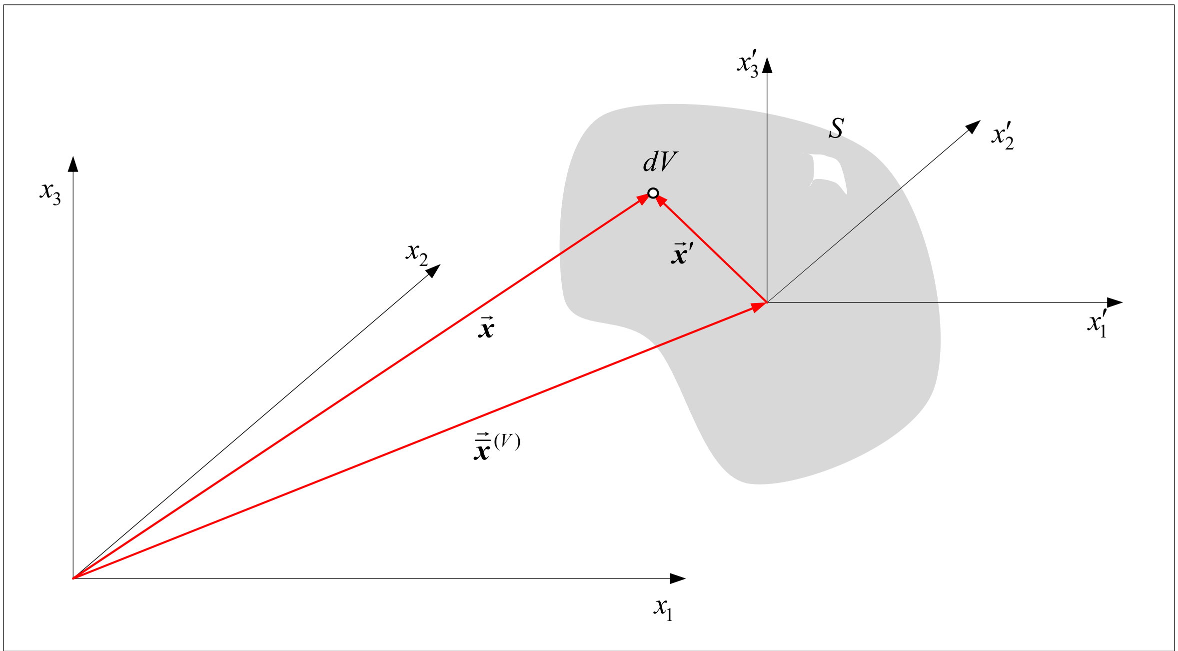


Figure 1.59

The components of the volume centroid $(\bar{x}_1^{(V)},\bar{x}_2^{(V)},\bar{x}_3^{(V)})$ can be obtained as follows:

$$\bar{x}_1^{(V)}=\frac{\int_V x_1 dV}{\int_V dV} \quad ; \quad \bar{x}_2^{(V)}=\frac{\int_V x_2 dV}{\int_V dV} \quad ; \quad \bar{x}_3^{(V)}=\frac{\int_V x_3 dV}{\int_V dV} \quad \textit{Volume Centroid} \tag{1.173}$$

where

$\int_V x_1 dV$ is the first moment of the volume about the x_1 axis;

$\int_V x_2 dV$ is the first moment of the volume about the x_2 axis;

$\int_V x_3 dV$ is the first moment of the volume about the x_3 axis.

Note that, if the volume is given by $V = V^{(A)} + V^{(B)}$ then the following is true:

$$\bar{\vec{\boldsymbol{x}}}^{(V)} = \frac{\int_V \vec{\boldsymbol{x}} dV}{\int_V dV} = \frac{\int_V \vec{\boldsymbol{x}} dV}{V} \qquad \Rightarrow \qquad V\,\bar{\vec{\boldsymbol{x}}}^{(V)} = \int_V \vec{\boldsymbol{x}} dV$$

$$\Rightarrow V\,\bar{\vec{\boldsymbol{x}}}^{(V)} = \int_V \vec{\boldsymbol{x}} dV = \int_{V^{(A)}+V^{(B)}} \vec{\boldsymbol{x}} dV = \int_{V^{(A)}} \vec{\boldsymbol{x}} dV^{(A)} + \int_{V^{(B)}} \vec{\boldsymbol{x}} dV^{(B)} = V^{(A)} \bar{\vec{\boldsymbol{x}}}^{(V^A)} + V^{(B)} \bar{\vec{\boldsymbol{x}}}^{(V^B)}$$

where we have used the definitions:

$$\bar{\vec{\boldsymbol{x}}}^{(V^A)} = \frac{\int_{V^{(A)}} \vec{\boldsymbol{x}} dV^{(A)}}{\int_{V^{(A)}} dV^{(A)}} \quad ; \quad \bar{\vec{\boldsymbol{x}}}^{(V^B)} = \frac{\int_{V^{(B)}} \vec{\boldsymbol{x}} dV^{(B)}}{\int_{V^{(B)}} dV^{(B)}}$$

In a same fashion to the derivation of the equations in (1.173), we can also define the area centroid as follows:

$$\bar{x}_1^{(A)} = \frac{\int_A x_1 dA}{\int_A dA} \quad ; \quad \bar{x}_2^{(A)} = \frac{\int_A x_2 dA}{\int_A dA} \quad ; \quad \bar{x}_3^{(A)} = \frac{\int_A x_3 dA}{\int_A dA} \quad \textit{Area centroid} \tag{1.174}$$

and the line centroid:

$$\bar{x}_1^{(L)} = \frac{\int_L x_1 dL}{\int_L dL} \quad ; \quad \bar{x}_2^{(L)} = \frac{\int_L x_2 dL}{\int_L dL} \quad ; \quad \bar{x}_3^{(L)} = \frac{\int_L x_3 dL}{\int_L dL} \quad \textit{Line centroid} \tag{1.175}$$

Center of the Scalar Field of a Domain

Let us consider the scalar field ($\phi = \phi(\vec{\boldsymbol{x}})$), by means of vector summation we can write:

$$\phi\vec{\boldsymbol{x}} = \phi\bar{\vec{\boldsymbol{x}}}^{(V-\phi)} + \phi\vec{\boldsymbol{x}}'' \xrightarrow{\text{Integrating}} \int_V \phi\vec{\boldsymbol{x}} dV = \int_V \phi\bar{\vec{\boldsymbol{x}}}^{(V-\phi)} dV + \int_V \phi\vec{\boldsymbol{x}}'' dV$$

The center of the scalar field $\phi = \phi(\vec{\boldsymbol{x}})$ delimited by the domain V is defined by:

$$\int_V \phi\vec{\boldsymbol{x}}'' dV = \vec{\boldsymbol{0}}$$

With that we can conclude that:

$$\int_V \phi \vec{\mathbf{x}} dV = \int_V \phi \vec{\bar{\mathbf{x}}}^{(V\_\phi)} dV \quad \Rightarrow \quad \vec{\bar{\mathbf{x}}}^{(V\_\phi)} = \frac{\int_V \phi \vec{\mathbf{x}} dV}{\int_V \phi dV} = \frac{\int_V \phi \vec{\mathbf{x}} dV}{V^{(\phi)}}$$

whose components are:

$$\bar{x}_1^{(V\_\phi)} = \frac{\int_V \phi x_1 dV}{\int_V \phi dV} \quad ; \quad \bar{x}_2^{(V\_\phi)} = \frac{\int_V \phi x_2 dV}{\int_V \phi dV} \quad ; \quad \bar{x}_3^{(V\_\phi)} = \frac{\int_V \phi x_3 dV}{\int_V \phi dV}$$

Note that, if the scalar field is uniform inside the volume, the center of the scalar field and the geometrical center are the same:

$$\vec{\bar{\mathbf{x}}}^{(V\_\phi)} = \frac{\int_V \phi \vec{\mathbf{x}} dV}{\int_V \phi dV} = \frac{\phi \int_V \vec{\mathbf{x}} dV}{\phi \int_V dV} = \frac{\int_V \vec{\mathbf{x}} dV}{\int_V dV} = \vec{\bar{\mathbf{x}}}^{(V)}$$

Similarly we can define the center of the scalar field into an Area:

$$\vec{\bar{\mathbf{x}}}^{(A\_\phi)} = \frac{\int_A \phi \vec{\mathbf{x}} dA}{\int_A \phi dA}$$

and the center of a scalar field of the curve

$$\vec{\bar{\mathbf{x}}}^{(L\_\phi)} = \frac{\int_L \phi \vec{\mathbf{x}} dL}{\int_L \phi dL}$$

If the scalar field ϕ represents the mass density (ρ) the center of the scalar is denoted by *Center of Mass (C.M.).*

$$\vec{\bar{\mathbf{x}}}^{(V\_\rho)} = \frac{\int_V \rho \, \vec{\mathbf{x}} dV}{\int_V \rho \, dV} \quad \textit{Center of mass (defined by a volume)} \qquad (1.176)$$

Center of Vector Field delimited by a Domain

Consider that the body of volume V is subjected to the vector field $\vec{\mathbf{b}} = \vec{\mathbf{b}}(\vec{x})$, and by a scalar field $\phi(\vec{x})$, (see Figure 1.60).

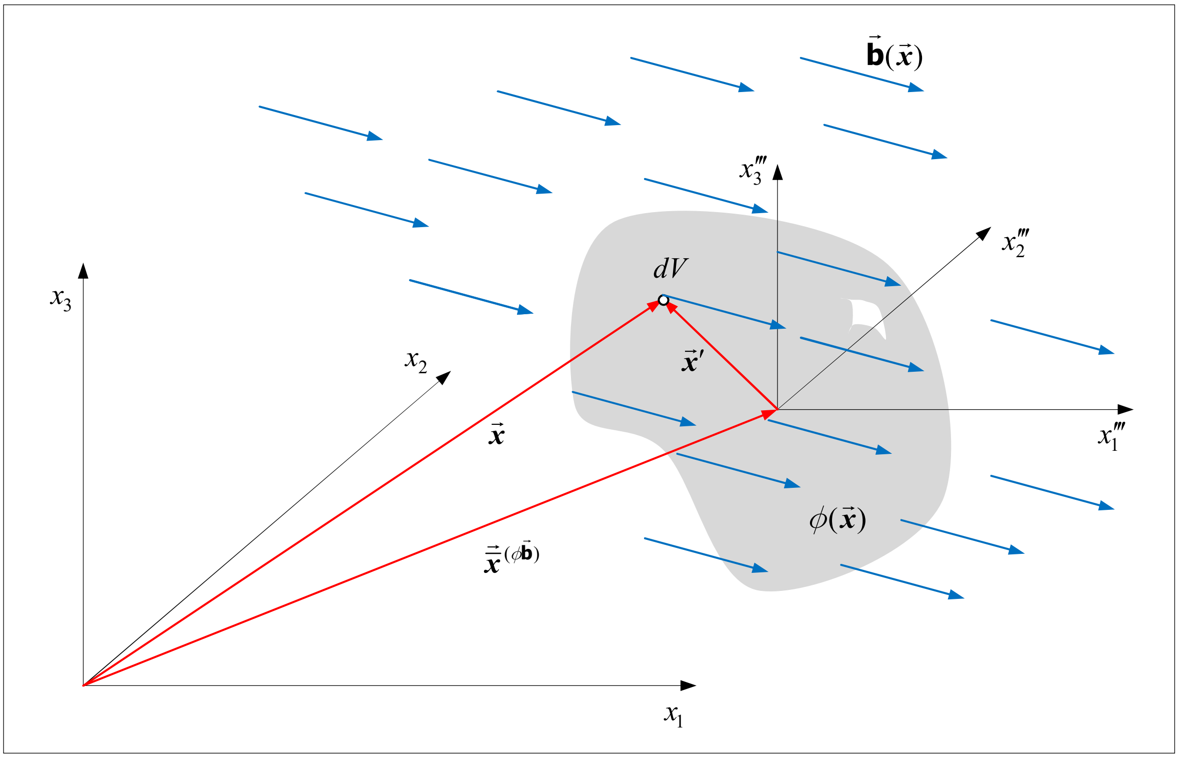


Figure 1.60

By considering the systems $\vec{x}$ and $\vec{x}'''$, we can obtain:

$$\vec{x} = \vec{\bar{x}}^{(\phi\vec{\mathbf{b}})} + \vec{x}''' \quad \Rightarrow \quad \phi\vec{x}\cdot\vec{\mathbf{b}} = \phi\vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\vec{\mathbf{b}} + \phi\vec{x}'''\cdot\vec{\mathbf{b}}$$
$$\Rightarrow \int_V \phi\,\vec{x}\cdot\vec{\mathbf{b}}dV = \int_V \phi\vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\vec{\mathbf{b}}dV + \int_V \phi\vec{x}'''\cdot\vec{\mathbf{b}}dV$$

The center of the vector field delimited by the volume V is defined by:

$$\int_V \phi\vec{x}'''\cdot\vec{\mathbf{b}}dV = 0 \tag{1.177}$$

Then

$$\int_V \phi\vec{x}\cdot\vec{\mathbf{b}}dV = \int_V \phi\vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\vec{\mathbf{b}}dV = \vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\left(\int_V \phi\vec{\mathbf{b}}dV\right) = \vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\vec{F} \quad \Rightarrow \quad \vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}\cdot\vec{F} = \int_V \phi\vec{x}\cdot\vec{\mathbf{b}}dV$$

where we have denoted by

$$\vec{F} = \int_V \phi\vec{\mathbf{b}}dV \tag{1.178}$$

Note that $\vec{F}$ is the resultant force and is located at the point $\vec{\bar{x}}^{(\phi\vec{\mathbf{b}})}$.

If the arbitrary field $\vec{\mathbf{b}}$ is uniform inside of volume V we can obtain:

$$\int_V \phi \vec{x} \cdot \vec{\mathbf{b}} dV = \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} \cdot \left(\int_V \phi \vec{\mathbf{b}} dV \right) \quad \Rightarrow \quad \left(\int_V \phi \vec{x} dV \right) \cdot \vec{\mathbf{b}} = \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} \cdot \vec{\mathbf{b}} \left(\int_V \phi dV \right)$$

$$\Rightarrow \left(\int_V \phi dV \right) \vec{\bar{x}}^{(V\_\phi)} \cdot \vec{\mathbf{b}} = \left(\int_V \phi dV \right) \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} \cdot \vec{\mathbf{b}} \quad \Rightarrow \quad \left(\int_V \phi dV \right) \left(\vec{\bar{x}}^{(V\_\phi)} - \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} \right) \cdot \vec{\mathbf{b}} = 0$$

$$\Rightarrow \vec{\bar{x}}^{(V\_\phi)} - \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} = \vec{\boldsymbol{0}}$$

$$\Rightarrow \vec{\bar{x}}^{(V\_\phi)} = \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})}$$

and if in addition the scalar field ϕ is uniform we can obtain:

$$\vec{\bar{x}}^{(V\_\phi)} = \vec{\bar{x}}^{(\phi \vec{\mathbf{b}})} = \vec{\bar{x}}^{(V)}$$

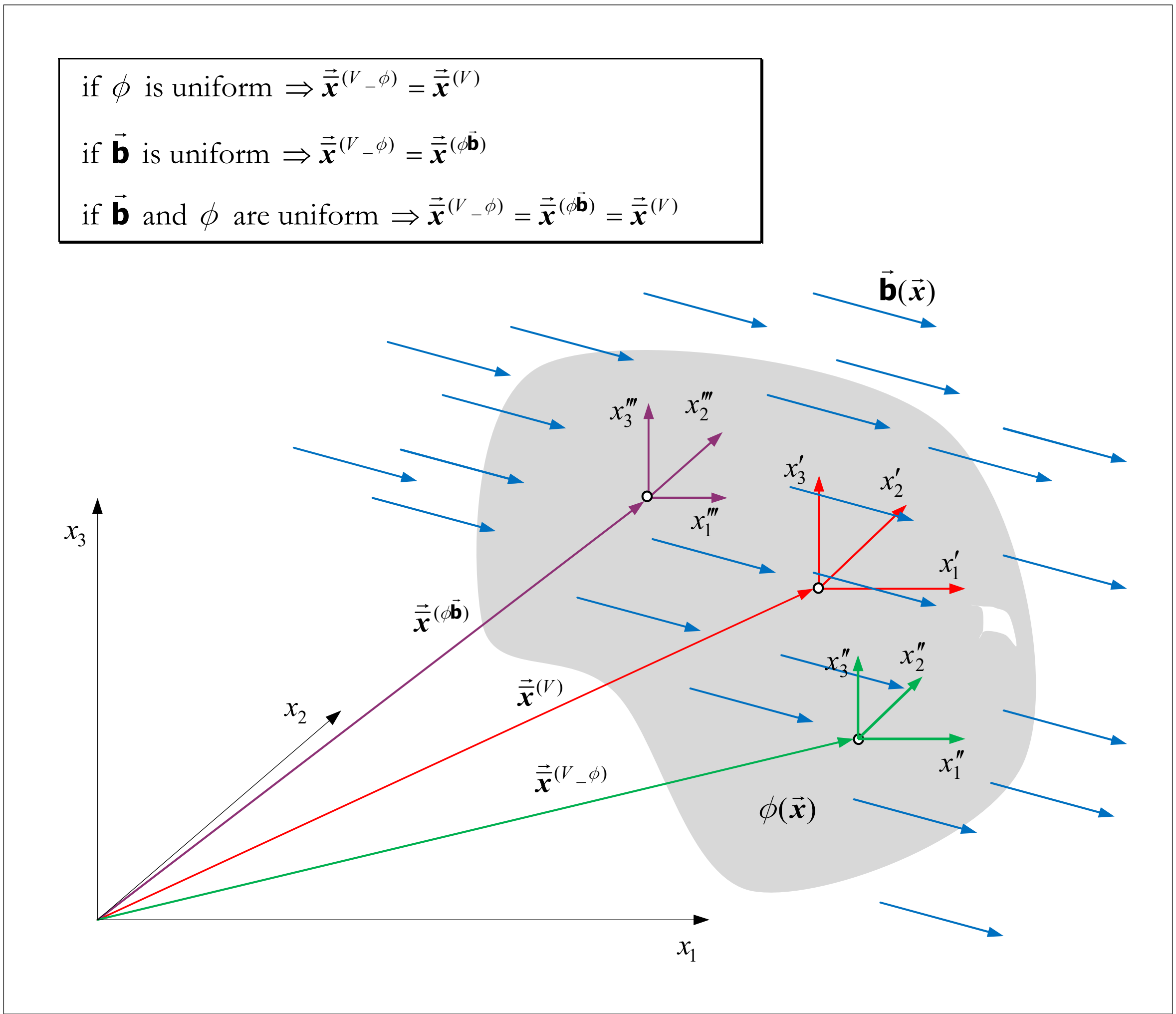


Figure 1.61

If the scalar field $\phi = \rho$ is the mass density, and $\vec{\mathbf{b}}$ represents the gravitational field on the proximity of the Earth surface, the equation (1.178) becomes:

$$F_i = \int_V \rho \mathrm{b}_i dV = \begin{Bmatrix} 0 \\ 0 \\ \int_V -\rho g dV \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -mg \end{Bmatrix}$$

where $m = \int_V \rho dV$ stands for the total mass of the body.

Center of Null Rotation

Consider that:

$$\vec{x} = \vec{\vec{x}}^{(\vec{\tau})} + \vec{x}' \quad \Rightarrow \quad \vec{x} \wedge \vec{\mathbf{b}} = \vec{\vec{x}}^{(\vec{\tau})} \wedge \vec{\mathbf{b}} + \vec{x}' \wedge \vec{\mathbf{b}}$$
$$\Rightarrow \int_V \phi\vec{x} \wedge \vec{\mathbf{b}} dV = \int_V \phi\vec{\vec{x}}^{(\vec{\tau})} \wedge \vec{\mathbf{b}} dV + \int_V \phi\vec{x}' \wedge \vec{\mathbf{b}} dV$$

The center of the vector field ($\phi\vec{\mathbf{b}}$) where the rotation is null is defined by:

$$\int_V \phi\vec{x}' \wedge \vec{\mathbf{b}} dV = \vec{\mathbf{0}} \tag{1.179}$$

with that we can obtain:

$$\int_V \phi\vec{x} \wedge \vec{\mathbf{b}} dV = \int_V \phi\vec{\vec{x}}^{(\vec{\tau})} \wedge \vec{\mathbf{b}} dV = \vec{\vec{x}}^{(\vec{\tau})} \wedge \underbrace{\left(\int_V \phi\vec{\mathbf{b}} dV \right)}_{\vec{F}} \quad \Rightarrow \quad \vec{\vec{x}}^{(\vec{\tau})} \wedge \vec{F} = \vec{\tau}$$

where $\vec{\tau}$ is the torque that the field $\vec{\mathbf{b}}$ produces into the body and is defined by:

$$\vec{\tau} = \int_V \phi\vec{x} \wedge \vec{\mathbf{b}} dV \tag{1.180}$$

If the scalar field represents the mass density ($\phi = \rho$), and $\vec{\mathbf{b}}$ represents the gravitational field, $\vec{\vec{x}}^{(\vec{\tau})}$ is denoted by *Center of Gravity (G)*. Note also that $\vec{\vec{x}}^{(\vec{\tau})} = \vec{\vec{x}}^{(\phi\vec{\mathbf{b}})}$.

Next we will obtain the torque of the vector field $(\phi\vec{\mathbf{b}} \wedge \vec{x})$:

$$\vec{x} = \vec{\vec{\vec{x}}} + \vec{x}' \quad \Rightarrow \quad \vec{x} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) = \vec{\vec{\vec{x}}} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) + \vec{x}' \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x})$$

By integrating over the volume the above equation we can obtain:

$$\int_V \vec{x} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV = \int_V \vec{\vec{\vec{x}}} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV + \int_V \vec{x}' \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV \tag{1.181}$$

The center of the vector field $(\phi\vec{\mathbf{b}} \wedge \vec{x})$ of null rotation is defined by:

$$\int_V \vec{x}' \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV = \vec{\mathbf{0}} \tag{1.182}$$

with that the equation in (1.181) becomes:

$$\int_V \vec{x} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV = \int_V \vec{\vec{\vec{x}}} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV = \vec{\vec{\vec{x}}} \wedge \left(\int_V (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV \right) = \vec{\vec{\vec{x}}} \wedge \left(\vec{\vec{x}}^{(\vec{\tau})} \wedge \vec{F} \right) = \vec{\vec{\vec{x}}} \wedge \vec{\tau} \tag{1.183}$$

where we have used the equation (1.180). In **Problem 1.17** we have shown that the equation $\vec{x} \wedge (\vec{\mathbf{b}} \wedge \vec{x}) = [(\vec{x} \cdot \vec{x})\mathbf{1} - \vec{x} \otimes \vec{x}] \cdot \vec{\mathbf{b}}$ holds, so that the above equation can be rewritten as follows:

$$\int_V \vec{x} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x})dV = \overline{\overline{\overline{x}}} \wedge \vec{\tau} \quad \Rightarrow \quad \int_V \{[(\vec{x}\cdot\vec{x})\mathbf{1} - \vec{x}\otimes\vec{x}]\cdot\phi\vec{\mathbf{b}}\}dV = \overline{\overline{\overline{x}}} \wedge \vec{\tau}$$

$$\Rightarrow \int_V \{\mathbf{j}_O \cdot \phi\vec{\mathbf{b}}\}dV = \overline{\overline{\overline{x}}} \wedge \vec{\tau}$$

where we have introduced the second-order pseudo-tensor:

$$\mathbf{j}_O = (\vec{x}\cdot\vec{x})\mathbf{1} - \vec{x}\otimes\vec{x} \quad ; \quad \mathrm{j}_{Oij} = x_k x_k \delta_{ij} - x_i x_j$$

whose components are:

$$\begin{aligned}
\mathrm{j}_{Oij} = x_k x_k \delta_{ij} - x_i x_j &= (x_1^2 + x_2^2 + x_3^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3x_3 \end{bmatrix} \\
&= \begin{bmatrix} (x_2^2 + x_3^2) & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & (x_1^2 + x_3^2) & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & (x_1^2 + x_2^2) \end{bmatrix}
\end{aligned} \tag{1.184}$$

If $\phi = \rho$ is the mass density, and $\vec{\mathbf{b}}$ is uniform vector field, we can obtain:

$$\int_V \{\mathbf{j}_O \cdot \phi\vec{\mathbf{b}}\}dV = \overline{\overline{\overline{x}}} \wedge \vec{\tau} \quad \Rightarrow \quad \left(\int_V \rho\, \mathbf{j}_O dV\right)\cdot\vec{\mathbf{b}} = \overline{\overline{\overline{x}}} \wedge \vec{\tau} \quad \Rightarrow \quad \Rightarrow \mathbf{I}_O \cdot \vec{\mathbf{b}} = \overline{\overline{\overline{x}}} \wedge \vec{\tau}$$

where $\mathbf{I}_O$ is the inertia tensor of mass density and is defined by:

$$\mathbf{I}_O^{(\rho)} = \int_V \rho\, \mathbf{j}_O dV \quad ; \quad \mathrm{I}_{O\ ij}^{(\rho)} = \begin{bmatrix} \int_V \rho(x_2^2 + x_3^2)dV & -\int_V \rho x_1x_2 dV & -\int_V \rho x_1x_3 dV \\ -\int_V \rho x_1x_2 dV & \int_V \rho(x_1^2 + x_3^2)dV & -\int_V \rho x_2x_3 dV \\ -\int_V \rho x_1x_3 dV & -\int_V \rho x_2x_3 dV & \int_V \rho(x_1^2 + x_2^2)dV \end{bmatrix} \tag{1.185}$$

whose SI-unit is $[\mathbf{I}_O^{(\rho)}] = \dfrac{kg}{m^3} m^2 m^3 = kg\, m^2$.

Similarly, we define the inertia tensor of area:

$$\mathrm{I}_{O\ ij}^{(A)} = \begin{bmatrix} \int_A (x_2^2 + x_3^2)dA & -\int_A x_1x_2 dA & -\int_A x_1x_3 dA \\ -\int_A x_1x_2 dA & \int_A (x_1^2 + x_3^2)dA & -\int_A x_2x_3 dA \\ -\int_A x_1x_3 dA & -\int_A x_2x_3 dA & \int_A (x_1^2 + x_2^2)dA \end{bmatrix}$$

NOTE: A list of inertia tensor for several solids can be found in Wikipedia
http://en.wikipedia.org/wiki/List_of_moments_of_inertia

Note that if we consider the torque $(\phi\vec{\mathbf{b}}\wedge\vec{x})$ and vector $\vec{x}=\vec{\bar{\bar{x}}}+\vec{x}'$ we can obtain:

$$
\begin{aligned}
\int_V \vec{x}\wedge(\phi\vec{\mathbf{b}}\wedge\vec{x})dV &= \int_V \{[(\vec{x}\cdot\vec{x})\mathbf{1}-\vec{x}\otimes\vec{x}]\cdot\phi\vec{\mathbf{b}}\}dV \\
&= \int_V \{\ \{[(\vec{\bar{\bar{x}}}+\vec{x}')\cdot(\vec{\bar{\bar{x}}}+\vec{x}')]\mathbf{1}-[(\vec{\bar{\bar{x}}}+\vec{x}')\otimes(\vec{\bar{\bar{x}}}+\vec{x}')]\}\cdot\phi\vec{\mathbf{b}}\}dV \\
&= \int_V \{\ \{[(\vec{\bar{\bar{x}}}\cdot\vec{\bar{\bar{x}}})+(\vec{\bar{\bar{x}}}\cdot\vec{x}')+(\vec{x}'\cdot\vec{\bar{\bar{x}}})+(\vec{x}'\cdot\vec{x}')]\mathbf{1} \\
&\qquad -[(\vec{\bar{\bar{x}}}\otimes\vec{\bar{\bar{x}}})+(\vec{\bar{\bar{x}}}\otimes\vec{x}')+(\vec{x}'\otimes\vec{\bar{\bar{x}}})+(\vec{x}'\otimes\vec{x}')\}\cdot\phi\vec{\mathbf{b}}\}dV \\
&= \int_V \{[(\vec{x}'\cdot\vec{x}')\mathbf{1}-\vec{x}'\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\}dV+\int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{\bar{\bar{x}}})\mathbf{1}-\vec{\bar{\bar{x}}}\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\}dV+ \\
&\qquad +\int_V \{[(\vec{x}'\cdot\vec{\bar{\bar{x}}})\mathbf{1}-\vec{\bar{\bar{x}}}\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\}dV+\int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{x}')\mathbf{1}-\vec{x}'\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\}dV
\end{aligned}
\tag{1.186}
$$

Note that, if the field $\vec{\mathbf{b}}$ is uniform then we can obtain:

$$
\begin{aligned}
\int_V \{[(\vec{x}'\cdot\vec{\bar{\bar{x}}})\mathbf{1}-\vec{\bar{\bar{x}}}\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\}dV &= \left(\int_V \phi[(\vec{x}'\cdot\vec{\bar{\bar{x}}})\mathbf{1}-\vec{\bar{\bar{x}}}\otimes\vec{x}']\,dV\right)\cdot\vec{\mathbf{b}} \\
&= \left(\int_V \phi(\vec{x}'\cdot\vec{\bar{\bar{x}}})\mathbf{1}\,dV\right)\cdot\vec{\mathbf{b}}+\left(\int_V \phi\vec{\bar{\bar{x}}}\otimes\vec{x}'\,dV\right)\cdot\vec{\mathbf{b}} \\
&= \left[\left(\int_V \phi\vec{x}'\,dV\right)\cdot\vec{\bar{\bar{x}}}\right]\mathbf{1}\cdot\vec{\mathbf{b}}+\left[\vec{\bar{\bar{x}}}\otimes\left(\int_V \phi\vec{x}'\,dV\right)\right]\cdot\vec{\mathbf{b}}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{x}')\mathbf{1}-\vec{x}'\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\}dV &= \left(\int_V \phi[(\vec{\bar{\bar{x}}}\cdot\vec{x}')\mathbf{1}-\vec{x}'\otimes\vec{\bar{\bar{x}}}]\,dV\right)\cdot\vec{\mathbf{b}} \\
&= \left(\int_V \phi(\vec{\bar{\bar{x}}}\cdot\vec{x}')\mathbf{1}\,dV\right)\cdot\vec{\mathbf{b}}+\left(\int_V \phi\vec{x}'\otimes\vec{\bar{\bar{x}}}\,dV\right)\cdot\vec{\mathbf{b}} \\
&= \left[\vec{\bar{\bar{x}}}\cdot\left(\int_V \phi\vec{x}'\,dV\right)\right]\mathbf{1}\cdot\vec{\mathbf{b}}+\left[\left(\int_V \phi\vec{x}'\,dV\right)\otimes\vec{\bar{\bar{x}}}\right]\cdot\vec{\mathbf{b}}
\end{aligned}
$$

Note that we have considered that $\vec{\mathbf{b}}$ is uniform, hence $\vec{\bar{\bar{x}}}^{(V\_\phi)}=\vec{\bar{\bar{x}}}^{(\phi\vec{\mathbf{b}})}=\vec{\bar{\bar{x}}}$, and the equation $\int_V \phi\vec{x}'\,dV=\vec{\mathbf{0}}$ holds. With that the equation in (1.186) becomes:

$$
\begin{aligned}
\int_V \vec{x}\wedge(\phi\vec{\mathbf{b}}\wedge\vec{x})dV &= \int_V \{[(\vec{x}\cdot\vec{x})\mathbf{1}-\vec{x}\otimes\vec{x}]\cdot\phi\vec{\mathbf{b}}\}dV \\
&= \int_V \{[(\vec{x}'\cdot\vec{x}')\mathbf{1}-\vec{x}'\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\}dV+\int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{\bar{\bar{x}}})\mathbf{1}-\vec{\bar{\bar{x}}}\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\}dV
\end{aligned}
\tag{1.187}
$$

If we consider that:

$$
\begin{aligned}
&\int_V \vec{x} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}) dV = \int_V \{[(\vec{x}\cdot\vec{x})\mathbf{1} - \vec{x}\otimes\vec{x}]\cdot\phi\vec{\mathbf{b}}\} dV = \int_V \phi\{\mathbf{j}_O \cdot \vec{\mathbf{b}}\} dV \\
&\int_V \vec{\bar{\bar{x}}} \wedge (\phi\vec{\mathbf{b}} \wedge \vec{\bar{\bar{x}}}) dV = \int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{\bar{\bar{x}}})\mathbf{1} - \vec{\bar{\bar{x}}}\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\} dV = \int_V \phi\{\bar{\mathbf{j}}_O \cdot \vec{\mathbf{b}}\} dV = \bar{\mathbf{j}}_O \cdot \int_V \{\phi\vec{\mathbf{b}}\} dV \\
&\int_V \vec{x}' \wedge (\phi\vec{\mathbf{b}} \wedge \vec{x}') dV = \int_V \{[(\vec{x}'\cdot\vec{x}')\mathbf{1} - \vec{x}'\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\} dV = \int_V \phi\{\mathbf{j}'_O \cdot \vec{\mathbf{b}}\} dV
\end{aligned}
\tag{1.188}
$$

The equation in (1.187) becomes:

$$
\begin{aligned}
&\int_V \{[(\vec{x}\cdot\vec{x})\mathbf{1} - \vec{x}\otimes\vec{x}]\cdot\phi\vec{\mathbf{b}}\} dV = \int_V \{[(\vec{x}'\cdot\vec{x}')\mathbf{1} - \vec{x}'\otimes\vec{x}']\cdot\phi\vec{\mathbf{b}}\} dV + \int_V \{[(\vec{\bar{\bar{x}}}\cdot\vec{\bar{\bar{x}}})\mathbf{1} - \vec{\bar{\bar{x}}}\otimes\vec{\bar{\bar{x}}}]\cdot\phi\vec{\mathbf{b}}\} dV \\
&\Rightarrow \int_V \phi\{\mathbf{j}_O \cdot \vec{\mathbf{b}}\} dV = \int_V \phi\{\mathbf{j}'_O \cdot \vec{\mathbf{b}}\} dV + \int_V \phi\{\bar{\mathbf{j}}_O \cdot \vec{\mathbf{b}}\} dV \\
&\Rightarrow \int_V \phi\{\mathbf{j}_O \cdot \vec{\mathbf{b}}\} dV - \int_V \phi\{\mathbf{j}'_O \cdot \vec{\mathbf{b}}\} dV - \int_V \phi\{\bar{\mathbf{j}}_O \cdot \vec{\mathbf{b}}\} dV = \vec{\mathbf{0}} \\
&\Rightarrow \left(\int_V \phi[\mathbf{j}_O - \mathbf{j}'_O - \bar{\mathbf{j}}_O] \; dV \right) \cdot \vec{\mathbf{b}} = \vec{\mathbf{0}}
\end{aligned}
\tag{1.189}
$$

Note that the above equation must be true for any uniform vector field $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, thus:

$$
\begin{aligned}
&\int_V \phi[\mathbf{j}_O - \mathbf{j}'_O - \bar{\mathbf{j}}_O] \; dV = \mathbf{0} \\
&\Rightarrow \int_V \phi\, \mathbf{j}_O \; dV = \int_V \phi\, \mathbf{j}'_O \; dV + \int_V \phi\, \bar{\mathbf{j}}_O \; dV
\end{aligned}
\tag{1.190}
$$

where the components, (see equation (1.184)), of $\mathbf{j}_O$, $\bar{\mathbf{j}}_O$, and $\mathbf{j}'_O$ are given by:

$$
(\mathbf{j}_O)_{ij} = \begin{bmatrix} (x_2^2 + x_3^2) & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & (x_1^2 + x_3^2) & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & (x_1^2 + x_2^2) \end{bmatrix}, \quad (\mathbf{j}'_O)_{ij} = \begin{bmatrix} (x_2'^2 + x_3'^2) & -x'_1 x'_2 & -x'_1 x'_3 \\ -x'_1 x'_2 & (x_1'^2 + x_3'^2) & -x'_2 x'_3 \\ -x'_1 x'_3 & -x'_2 x'_3 & (x_1'^2 + x_2'^2) \end{bmatrix},
$$

$$
(\bar{\mathbf{j}}_O)_{ij} = \begin{bmatrix} (\bar{\bar{x}}_2^2 + \bar{\bar{x}}_3^2) & -\bar{\bar{x}}_1 \bar{\bar{x}}_2 & -\bar{\bar{x}}_1 \bar{\bar{x}}_3 \\ -\bar{\bar{x}}_1 \bar{\bar{x}}_2 & (\bar{\bar{x}}_1^2 + \bar{\bar{x}}_3^2) & -\bar{\bar{x}}_2 \bar{\bar{x}}_3 \\ -\bar{\bar{x}}_1 \bar{\bar{x}}_3 & -\bar{\bar{x}}_2 \bar{\bar{x}}_3 & (\bar{\bar{x}}_1^2 + \bar{\bar{x}}_2^2) \end{bmatrix}
$$

Let us assume that the given systems $(x_1 - x_2 - x_3)$ are related by the transformation law $x_i^* = \boldsymbol{\mathcal{A}}_{ij} x_j$, where $\boldsymbol{\mathcal{A}}_{ij}$ is the orthogonal matrix, then it follows that $x_i = \boldsymbol{\mathcal{A}}_{ji} x_j^*$. Thus being able to express I_{Oij}, (see equation (1.185)), as follows:

$$
\begin{aligned}
\mathrm{I}_{Oij} &= \int_V \rho \left[x_k x_k \delta_{ij} - x_i x_j \right] dV = \int_V \rho \left[(x_k^* x_k^*) \boldsymbol{\mathcal{A}}_{ip} \delta_{pq} \boldsymbol{\mathcal{A}}_{jq} - \boldsymbol{\mathcal{A}}_{ip} x_p^* \boldsymbol{\mathcal{A}}_{jq} x_q^* \right] dV \\
&= \int_V \boldsymbol{\mathcal{A}}_{ip} \left\{ \rho \left[(x_k^* x_k^*) \delta_{pq} - x_p^* x_q^* \right] \right\} \boldsymbol{\mathcal{A}}_{jq} \; dV = \boldsymbol{\mathcal{A}}_{ip} \left\{ \int_V \rho \left[(x_k^* x_k^*) \delta_{pq} - x_p^* x_q^* \right] dV \right\} \boldsymbol{\mathcal{A}}_{jq} \\
&= \boldsymbol{\mathcal{A}}_{ip} \mathrm{I}_{Oij}^* \boldsymbol{\mathcal{A}}_{jq}
\end{aligned}
$$

Note that $x_k x_k = \boldsymbol{\mathcal{A}}_{ks} x_s^* \boldsymbol{\mathcal{A}}_{kt} x_t^* = x_s^* x_t^* \boldsymbol{\mathcal{A}}_{ks} \boldsymbol{\mathcal{A}}_{kt} = x_s^* x_t^* \delta_{st} = x_s^* x_s^* = x_t^* x_t^* = x_k^* x_k^*$.

Abusing a little bit of notation, we also use tensorial notation, but keep in mind that we are working with tensor components, and we are not doing an orthogonal transformation.

$$\begin{aligned}\mathbf{I}_O &= \int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV = \int_V \rho\,[(\vec{x}^*\cdot\vec{x}^*)\boldsymbol{\mathcal{A}}^T\cdot\mathbf{1}\cdot\boldsymbol{\mathcal{A}} - (\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*\otimes\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*)]\,dV\\
&= \int_V \rho\,[(\vec{x}^*\cdot\vec{x}^*)\boldsymbol{\mathcal{A}}^T\cdot\mathbf{1}\cdot\boldsymbol{\mathcal{A}} - (\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*\otimes\vec{x}^*\cdot\boldsymbol{\mathcal{A}})]\,dV\\
&= \int_V \boldsymbol{\mathcal{A}}^T\cdot\left\{\rho\,[(\vec{x}^*\cdot\vec{x}^*)\mathbf{1} - (\vec{x}^*\otimes\vec{x}^*)]\right\}\cdot\boldsymbol{\mathcal{A}}\,dV\\
&= \boldsymbol{\mathcal{A}}^T\cdot\left\{\int_V \rho\,[(\vec{x}^*\cdot\vec{x}^*)\mathbf{1} - (\vec{x}^*\otimes\vec{x}^*)]\,dV\right\}\cdot\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}^T\cdot\mathbf{I}_O^*\cdot\boldsymbol{\mathcal{A}}\end{aligned}$$

$$\boxed{\begin{aligned}\mathbf{I}_O &= \boldsymbol{\mathcal{A}}^T\cdot\mathbf{I}_O^*\cdot\boldsymbol{\mathcal{A}}\\ \mathrm{I}_{O\,ij} &= \mathcal{A}_{ip}\mathrm{I}^*_{O\,ij}\mathcal{A}_{jq}\end{aligned}}\quad \textit{Inertia tensor components after a base change (rotation)} \qquad (1.191)$$

Then, it is also true $\mathbf{I}_O^* = \boldsymbol{\mathcal{A}}\cdot\mathbf{I}_O\cdot\boldsymbol{\mathcal{A}}^T$, which are the inertia tensor components in the system $x_1^*x_2^*x_3^*$. Note that the equation (1.191) is the same component transformation law for a second-order tensor, where $\boldsymbol{\mathcal{A}}$ is the transformation matrix from the $x_1x_2x_3$-system to $x_1^*x_2^*x_3^*$-system.

We can also define the relationship between the Inertia Tensor of Area in the same fashion as the one defined in (1.190), by considering ϕ to be constant i.e.:

$$\begin{aligned}\phi\int_A \mathbf{j}_O\;dA = \phi\int_A \mathbf{j}'_O\,dA + \phi\int_A \bar{\mathbf{j}}_O\,dA \quad &\Rightarrow \quad \int_A \mathbf{j}_O\;dA = \int_A \mathbf{j}'_O\,dA + \int_A \bar{\mathbf{j}}_O\,dA\\
&\Rightarrow \mathbf{I}_O = \mathbf{I}'_{O\vec{x}'} + \bar{\mathbf{I}}_O\end{aligned} \qquad (1.192)$$

where $\mathbf{I}_{O\vec{x}}$ is the inertia tensor of area for the system $O\vec{x}$, $\mathbf{I}'_{O\vec{x}'}$ is calculated by considering the system at the Area Centroid $O\vec{x}'$ and $\bar{\mathbf{I}}_{O\vec{x}}$ is the relation between the two systems:

$$\mathbf{I}_{O\vec{x}} = \int_A [(\vec{x}\cdot\vec{x})\mathbf{1} - \vec{x}\otimes\vec{x}]dA \quad ; \quad \mathbf{I}'_{O\vec{x}'} = \int_A [(\vec{x}'\cdot\vec{x}')\mathbf{1} - \vec{x}'\otimes\vec{x}']dA$$

$$\bar{\mathbf{I}}_{O\vec{x}} = \int_A [(\bar{\vec{x}}\cdot\bar{\vec{x}})\mathbf{1} - \bar{\vec{x}}\otimes\bar{\vec{x}}]dA = [(\bar{\vec{x}}\cdot\bar{\vec{x}})\mathbf{1} - \bar{\vec{x}}\otimes\bar{\vec{x}}]\int_A dA = A[(\bar{\vec{x}}\cdot\bar{\vec{x}})\mathbf{1} - \bar{\vec{x}}\otimes\bar{\vec{x}}]$$

Then, the equation (1.192) in indicial notation can be written as follows

$$\begin{aligned}&\mathbf{I}_O = \mathbf{I}'_{O\vec{x}'} + \bar{\mathbf{I}}_O = \mathbf{I}'_{O\vec{x}'} + A[(\bar{\vec{x}}\cdot\bar{\vec{x}})\mathbf{1} - \bar{\vec{x}}\otimes\bar{\vec{x}}] \xrightarrow{\textit{indicial}} \mathrm{I}_{Oij} = \mathrm{I}'_{O\vec{x}'ij} + \bar{\mathrm{I}}_{Oij}\\
&\Rightarrow \mathrm{I}_{Oij} = \mathrm{I}'_{O\vec{x}'ij} + A[(\bar{x}_k\bar{x}_k)\delta_{ij} - \bar{x}_i\bar{x}_j]\\
&\Rightarrow \mathrm{I}_{Oij} = \mathrm{I}'_{O\vec{x}'ij} + A\begin{bmatrix}(\bar{x}_2^2+\bar{x}_3^2) & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3\\ -\bar{x}_1\bar{x}_2 & (\bar{x}_1^2+\bar{x}_3^2) & -\bar{x}_2\bar{x}_3\\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & (\bar{x}_1^2+\bar{x}_2^2)\end{bmatrix}\end{aligned} \qquad (1.193)$$

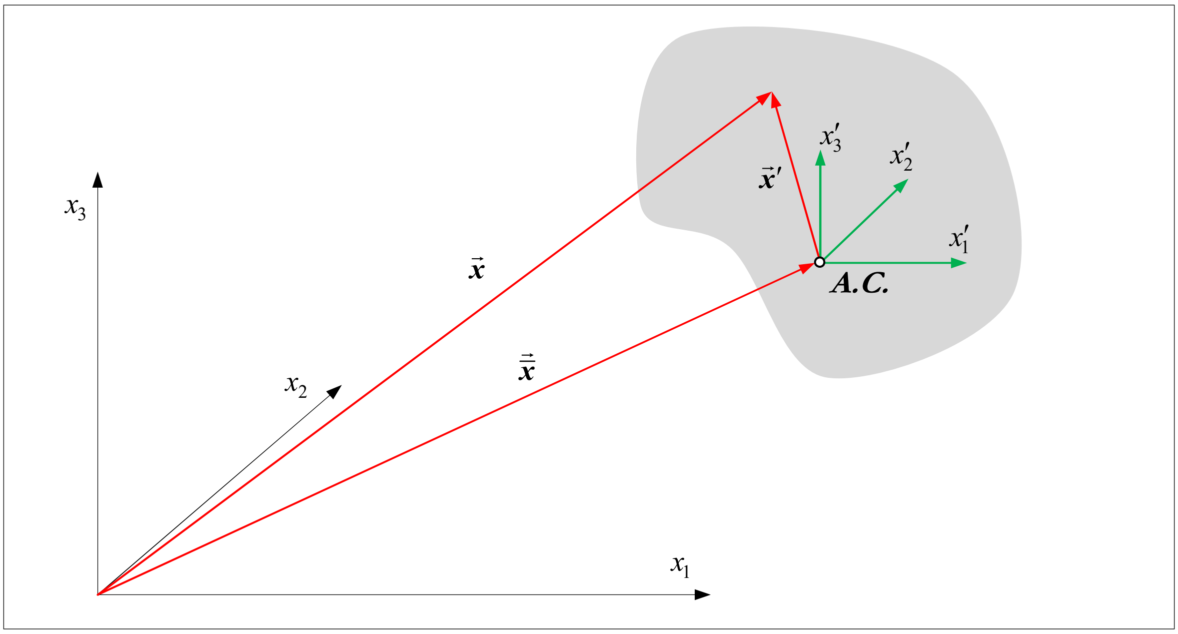


Figure 1.62

Example: Let us calculate the Inertia Tensor of Area in the system $O\vec{X}$ (plane X_2-X_3) for the triangle described in Figure 1.63.

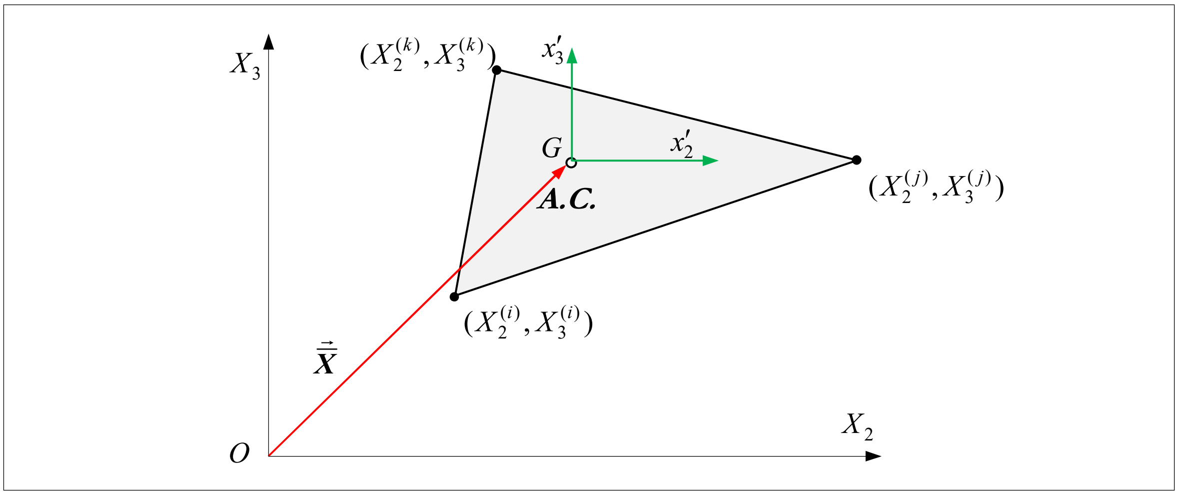


Figure 1.63

If we know the inertia tensor of area ($\mathbf{I}'_{G\vec{x}'}$) for the triangle related to the system $G\vec{x}'$ we can calculate the inertia tensor for the system $O\vec{X}$ as follows:

$$
\begin{aligned}
\mathrm{I}_{O\vec{X}\,ij} &= \mathrm{I}'_{G\vec{x}'ij} + A\begin{bmatrix} (\bar{X}_2^2+\bar{X}_3^2) & -\bar{X}_1\bar{X}_2 & -\bar{X}_1\bar{X}_3 \\ -\bar{X}_1\bar{X}_2 & (\bar{X}_1^2+\bar{X}_3^2) & -\bar{X}_2\bar{X}_3 \\ -\bar{X}_1\bar{X}_3 & -\bar{X}_2\bar{X}_3 & (\bar{X}_1^2+\bar{X}_2^2) \end{bmatrix} \\
&= \mathrm{I}'_{G\vec{x}'ij} + A\begin{bmatrix} (\bar{X}_2^2+\bar{X}_3^2) & 0 & 0 \\ 0 & \bar{X}_3^2 & -\bar{X}_2\bar{X}_3 \\ 0 & -\bar{X}_2\bar{X}_3 & \bar{X}_2^2 \end{bmatrix}
\end{aligned}
\tag{1.194}
$$

where we have considered that $\overline{X}_1 = 0$.

In order to calculate the inertia tensor of area $\mathbf{I}'_{G\vec{x}'}$ we will define three systems, $O\vec{X}'$, $o\vec{x}$ and $G\vec{x}''$, as indicated in Figure 1.64.

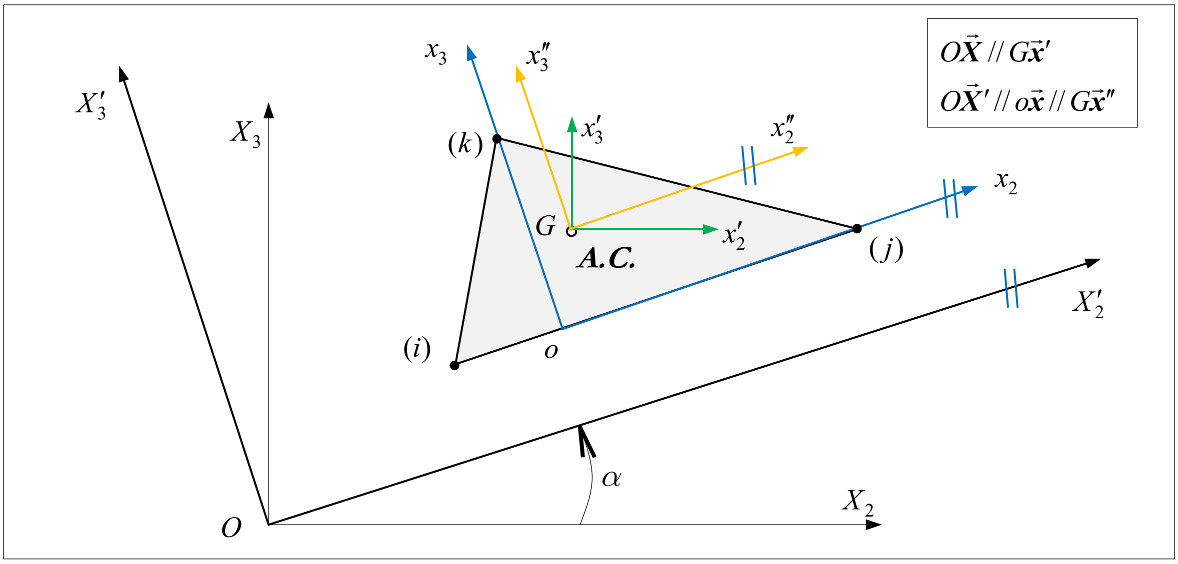


Figure 1.64

We define some parameters by considering the system $o\vec{x}$ as described in Figure 1.65.

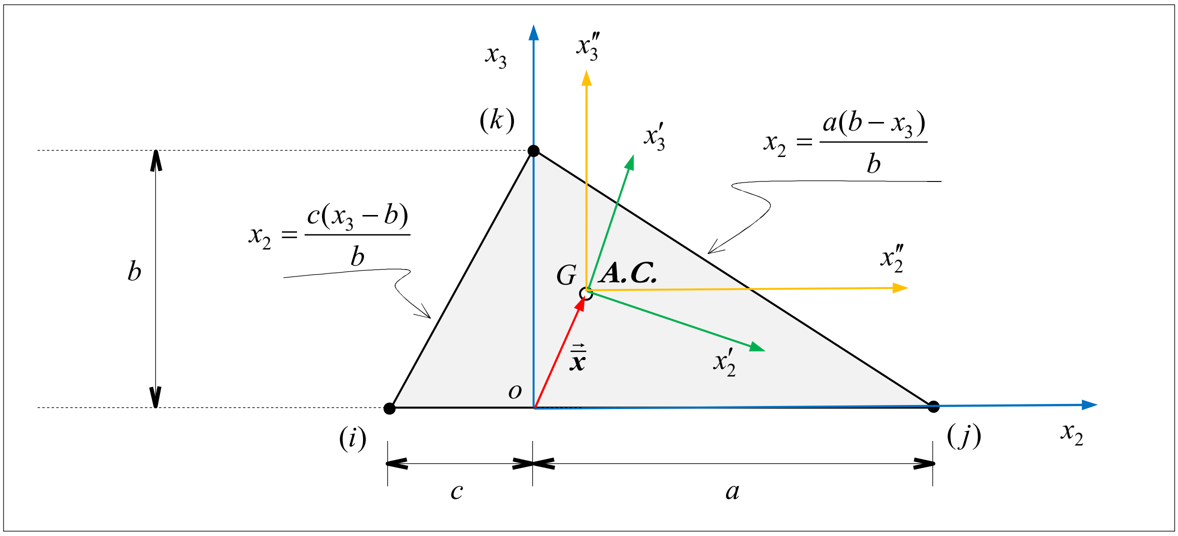


Figure 1.65

By definition the Area Centroid, (see equation (1.174)), is given by:

$$\overline{x}_1 = \frac{\int_A x_1 dA}{\int_A dA} \quad ; \quad \overline{x}_2 = \frac{\int_A x_2 dA}{\int_A dA} \quad ; \quad \overline{x}_3 = \frac{\int_A x_3 dA}{\int_A dA} \quad \textit{Area centroid} \qquad (1.195)$$

Then we can calculate:

Area:

$$A=\int_A dA=\int_{x_3=0}^{x_3=b}\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} dx_2\right)dx_3=\frac{b}{2}(a+c) \tag{1.196}$$

The first moment of area:

$$\int_A x_2 dA=\int_{x_3=0}^{x_3=b}\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} x_2 dx_2\right)dx_3=\frac{b}{6}(a^2-c^2) \quad ; \quad \int_A x_3 dA=\int_{x_3=0}^{x_3=b} x_3\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} dx_2\right)dx_3=\frac{b^2}{6}(a+c) \tag{1.197}$$

Then;

$$\bar{x}_1=0 \quad ; \quad \bar{x}_2=\frac{\frac{b}{6}(a^2-c^2)}{\frac{b}{2}(a+c)}=\frac{(a-c)}{3} \quad ; \quad \bar{x}_3=\frac{\frac{b^2}{6}(a+c)}{\frac{b}{2}(a+c)}=\frac{b}{3}$$

As expected, since the Area centroid for the triangle is:

$$\bar{x}_2=\frac{x_2^{(i)}+x_2^{(j)}+x_2^{(k)}}{3}=\frac{(a-c)}{3} \quad ; \quad \bar{x}_3=\frac{x_3^{(i)}+x_3^{(j)}+x_3^{(k)}}{3}=\frac{b}{3}$$

The inertia tensor in the system $o\vec{x}$

$$(\mathbf{I}_{o\vec{x}})_{ij}=\begin{bmatrix}\mathrm{I}_{O\vec{x}11} & \mathrm{I}_{O\vec{x}12} & \mathrm{I}_{O\vec{x}13}\\ \mathrm{I}_{O\vec{x}12} & \mathrm{I}_{O\vec{x}22} & \mathrm{I}_{O\vec{x}23}\\ \mathrm{I}_{O\vec{x}13} & \mathrm{I}_{O\vec{x}23} & \mathrm{I}_{O\vec{x}33}\end{bmatrix}=\begin{bmatrix}\int_A (x_2^2+x_3^2)dA & 0 & 0\\ 0 & \int_A (x_3^2)dA & -\int_A (x_2x_3)dA\\ 0 & -\int_A (x_2x_3)dA & \int_A (x_2^2)dA\end{bmatrix}$$

where

$$\mathrm{I}_{o\vec{x}22}=\int_A (x_3^2)dA=\int_{x_3=0}^{x_3=b} x_3^2\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} dx_2\right)dx_3=\frac{b^3}{12}(a+c)$$

$$\mathrm{I}_{o\vec{x}33}=\int_A (x_2^2)dA=\int_{x_3=0}^{x_3=b}\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} x_2^2 dx_2\right)dx_3=\frac{b}{12}(a^3+c^3)$$

$$\mathrm{I}_{o\vec{x}23}=-\int_A (x_2x_3)dA=-\int_{x_3=0}^{x_3=b} x_3\left(\int_{x_2=\frac{c(x_3-b)}{b}}^{x_2=\frac{a(b-x_3)}{b}} x_2 dx_2\right)dx_3=-\frac{b^2}{24}(a^2-c^2)=\frac{b^2}{24}(c^2-a^2)$$

Then

$$
(\mathbf{I}_{o\vec{x}})_{ij}=\begin{bmatrix}\mathrm{I}_{o\vec{x}11} & \mathrm{I}_{o\vec{x}12} & \mathrm{I}_{o\vec{x}13}\\ \mathrm{I}_{o\vec{x}12} & \mathrm{I}_{o\vec{x}22} & \mathrm{I}_{o\vec{x}23}\\ \mathrm{I}_{o\vec{x}13} & \mathrm{I}_{o\vec{x}23} & \mathrm{I}_{o\vec{x}33}\end{bmatrix}=\begin{bmatrix}\mathrm{I}_{o\vec{x}22}+\mathrm{I}_{o\vec{x}33} & 0 & 0\\ 0 & \dfrac{b^3}{12}(a+c) & \dfrac{b^2}{24}(c^2-a^2)\\ 0 & \dfrac{b^2}{24}(c^2-a^2) & \dfrac{b}{12}(a^3+c^3)\end{bmatrix}
$$

Calculation of the inertia tensor of area in the system $G\vec{x}''$.

We can use the equation

$$
\mathrm{I}_{o\vec{x}ij}=\mathrm{I}''_{G\vec{x}''ij}+A\begin{bmatrix}(\bar{x}_2^2+\bar{x}_3^2) & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3\\ -\bar{x}_1\bar{x}_2 & (\bar{x}_1^2+\bar{x}_3^2) & -\bar{x}_2\bar{x}_3\\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & (\bar{x}_1^2+\bar{x}_2^2)\end{bmatrix}
$$

$$
\Rightarrow \mathrm{I}''_{G\vec{x}''ij}=\mathrm{I}_{o\vec{x}ij}-A\begin{bmatrix}(\bar{x}_2^2+\bar{x}_3^2) & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3\\ -\bar{x}_1\bar{x}_2 & (\bar{x}_1^2+\bar{x}_3^2) & -\bar{x}_2\bar{x}_3\\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & (\bar{x}_1^2+\bar{x}_2^2)\end{bmatrix}
$$

where $A=\dfrac{b}{2}(a+c)$, $\bar{x}_1=0$, $\bar{x}_2=\dfrac{(a-c)}{3}$ and $\bar{x}_3=\dfrac{b}{3}$. Then, the above equation becomes:

$$
\mathrm{I}''_{G\vec{x}''ij}=\begin{bmatrix}\mathrm{I}_{o\vec{x}22}+\mathrm{I}_{o\vec{x}33} & 0 & 0\\ 0 & \dfrac{b^3}{12}(a+c) & \dfrac{b^2}{24}(c^2-a^2)\\ 0 & \dfrac{b^2}{24}(c^2-a^2) & \dfrac{b}{12}(a^3+c^3)\end{bmatrix}-A\begin{bmatrix}\left(\dfrac{a-c}{3}\right)^2+\left(\dfrac{b}{3}\right)^2 & 0 & 0\\ 0 & \left(\dfrac{b}{3}\right)^2 & -\dfrac{(a-c)}{3}\dfrac{b}{3}\\ 0 & -\dfrac{(a-c)}{3}\dfrac{b}{3} & \left(\dfrac{a-c}{3}\right)^2\end{bmatrix}
$$

After simplification we can obtain:

$$
\mathrm{I}''_{G\vec{x}''ij}=\begin{bmatrix}\dfrac{b(a+c)}{36}(a^2+b^2+c^2+ac) & 0 & 0\\ 0 & \dfrac{b^3}{36}(a+c) & \dfrac{b^2}{72}(a^2-c^2)\\ 0 & \dfrac{b^2}{72}(a^2-c^2) & \dfrac{b(a+c)}{36}(a^2+c^2+ac)\end{bmatrix} \tag{1.198}
$$

Calculation of the inertia tensor of area in the system $G\vec{x}'$.

The transformation matrix from the system $O\vec{X}//G\vec{x}'$ to $O\vec{X}'//o\vec{x}//G\vec{x}''$, (see Figure 1.64), is given by:

$$
\boldsymbol{\mathcal{A}}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & d & s\\ 0 & -s & d\end{bmatrix}
$$

Then the transformation matrix from $O\vec{X}'//o\vec{x}//G\vec{x}''$ to $O\vec{X}//G\vec{x}'$ is $\boldsymbol{\mathcal{A}}^T$. And we use the transformation defined in (1.191) in order to calculate:

$$
\mathbf{I}'_{G\vec{x}'}=\boldsymbol{\mathcal{A}}^T\cdot\mathbf{I}''_{G\vec{x}''}\cdot\boldsymbol{\mathcal{A}}
$$

The above equation after matrix multiplications becomes:

$$\mathrm{I}'_{G\vec{x}'ij} = \begin{bmatrix} \mathrm{I}''_{G\vec{x}''11} & 0 & 0 \\ 0 & d^2\mathrm{I}''_{G\vec{x}''22} - 2sd\,\mathrm{I}''_{G\vec{x}''23} + s^2\mathrm{I}''_{G\vec{x}''33} & \mathrm{I}''_{G\vec{x}''23}(d^2 - s^2) + sd(\mathrm{I}''_{G\vec{x}''22} - \mathrm{I}''_{G\vec{x}''33}) \\ 0 & \mathrm{I}''_{G\vec{x}''23}(d^2 - s^2) + sd(\mathrm{I}''_{G\vec{x}''22} - \mathrm{I}''_{G\vec{x}''33}) & s^2\mathrm{I}''_{G\vec{x}''22} + 2sd\,\mathrm{I}''_{G\vec{x}''23} + d^2\mathrm{I}''_{G\vec{x}''33} \end{bmatrix} \tag{1.199}$$

With that we can use the equation in (1.194) in order to calculate $\mathbf{I}_{O\vec{X}}$:

$$\mathrm{I}_{O\vec{X}ij} = \mathrm{I}'_{G\vec{x}'ij} + A\,\overline{\mathrm{I}}'_{G\vec{x}'ij}$$

$$= \begin{bmatrix} \mathrm{I}'_{G\vec{x}'11} & 0 & 0 \\ 0 & \mathrm{I}'_{G\vec{x}'22} & \mathrm{I}'_{G\vec{x}'23} \\ 0 & \mathrm{I}'_{G\vec{x}'23} & \mathrm{I}'_{G\vec{x}'33} \end{bmatrix} + A \begin{bmatrix} (\overline{X}_2^2 + \overline{X}_3^2) & 0 & 0 \\ 0 & \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ 0 & -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix}$$

$$\mathrm{I}_{O\vec{X}ij} = \begin{bmatrix} \mathrm{I}'_{G\vec{x}'11} + A(\overline{X}_2^2 + \overline{X}_2^2) & 0 & 0 \\ 0 & \mathrm{I}'_{G\vec{x}'22} + A\overline{X}_3^2 & \mathrm{I}'_{G\vec{x}'23} - A\overline{X}_2\overline{X}_3 \\ 0 & \mathrm{I}'_{G\vec{x}'23} - A\overline{X}_2\overline{X}_3 & \mathrm{I}'_{G\vec{x}'33} + A\overline{X}_2^2 \end{bmatrix} \tag{1.200}$$

NOTE: The node connectivity $i - j - k$ must be oriented according to the counterclockwise direction, (see Figure 1.66).

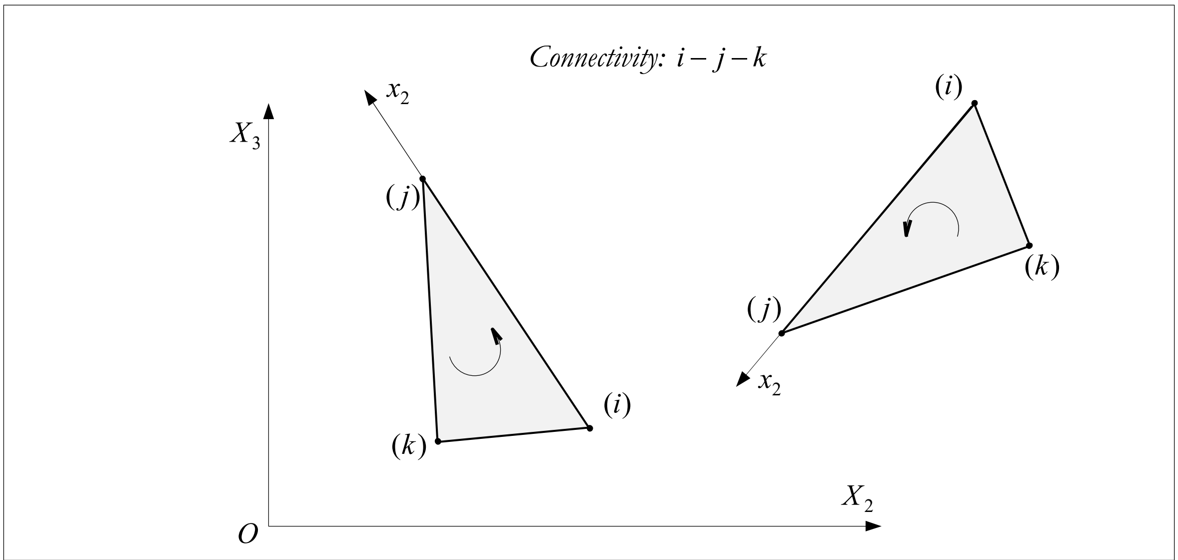


Figure 1.66

Procedure

Given the node coordinates: $i(X_2^{(i)}, X_3^{(i)}) - j(X_2^{(j)}, X_3^{(j)}) - k(X_2^{(k)}, X_3^{(k)})$

Calculate the transformation matrix:

$$L = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$$

$$d = \cos\alpha = \frac{X_2^{(j)} - X_2^{(i)}}{L}$$

$$s = \sin\alpha = \frac{X_3^{(j)} - X_3^{(i)}}{L}$$

Node coordinates in the system $O\vec{X}'$

$$(\text{Node } i) \Rightarrow \begin{Bmatrix} X_2'^{(i)} \\ X_3'^{(i)} \end{Bmatrix} = \begin{bmatrix} d & s \\ -s & d \end{bmatrix} \begin{Bmatrix} X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} X_2^{(i)} d + s X_3^{(i)} \\ X_3^{(i)} d - s X_2^{(i)} \end{Bmatrix}$$

$$(\text{Node } j) \Rightarrow \begin{Bmatrix} X_2'^{(j)} \\ X_3'^{(j)} \end{Bmatrix} = \begin{bmatrix} d & s \\ -s & d \end{bmatrix} \begin{Bmatrix} X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = \begin{Bmatrix} X_2^{(j)} d + s X_3^{(j)} \\ X_3^{(j)} d - s X_2^{(j)} \end{Bmatrix}$$

$$(\text{Node } k) \Rightarrow \begin{Bmatrix} X_2'^{(k)} \\ X_3'^{(k)} \end{Bmatrix} = \begin{bmatrix} d & s \\ -s & d \end{bmatrix} \begin{Bmatrix} X_2^{(k)} \\ X_3^{(k)} \end{Bmatrix} = \begin{Bmatrix} X_2^{(k)} d + s X_3^{(k)} \\ X_3^{(k)} d - s X_2^{(k)} \end{Bmatrix}$$

Then

$a = X_2'^{(j)} - X_2'^{(k)}$, $c = X_2'^{(k)} - X_2'^{(i)}$, $b = X_3'^{(k)} - X_3'^{(i)}$.

Area: $A = \dfrac{(a+c)b}{2}$

Calculate the area centroid related to the system $o\vec{x}$:

$$\bar{x}_2 = \frac{(a-c)}{3} \quad ; \quad \bar{x}_3 = \frac{b}{3}$$

Calculation of the area centroid ($\vec{\bar{X}}$) related to the system $O\vec{X}$:

Coordinate of o in the system $O\vec{X}'$ is $(X_2'^{(k)}; X_3'^{(i)})$

Coordinate of G in the system $O\vec{X}'$ is $(X_2'^{(k)} + \bar{x}_2; X_3'^{(i)} + \bar{x}_3)$

Coordinate of G in the system $O\vec{X}$ is:

$$\begin{Bmatrix} \bar{X}_2 \\ \bar{X}_3 \end{Bmatrix} = \begin{Bmatrix} X_2^{(G)} \\ X_3^{(G)} \end{Bmatrix} = \begin{bmatrix} d & -s \\ s & d \end{bmatrix} \begin{Bmatrix} X_2'^{(k)} + \bar{x}_2 \\ X_3'^{(i)} + \bar{x}_3 \end{Bmatrix} = \begin{Bmatrix} (X_2'^{(k)} + \bar{x}_2)d - s(X_3'^{(i)} + \bar{x}_3) \\ (X_2'^{(k)} + \bar{x}_2)s + d(X_3'^{(i)} + \bar{x}_3) \end{Bmatrix}$$

Calculate $\mathbf{I}''_{G\vec{x}''}$, (see equation (1.198));

Calculate $\mathbf{I}'_{G\vec{x}'}$, (see equation (1.199));

Calculate $\mathbf{I}_{O\vec{X}}$, (see equation (1.200)).

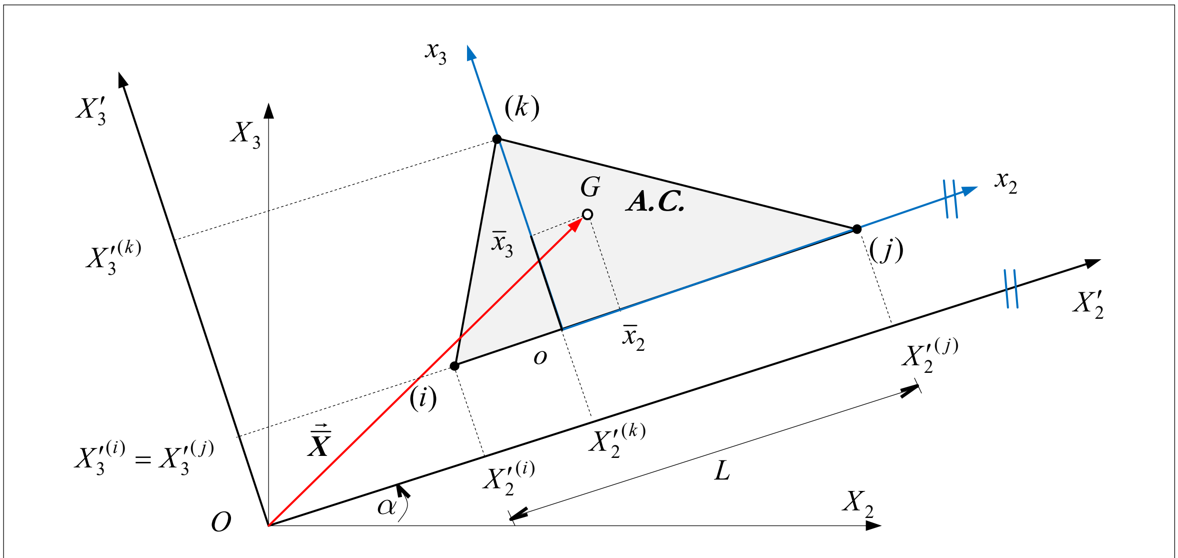


Figure 1.67

If we have n triangles the following is true:

$$\mathbf{I}_{O\vec{x}} = \sum_{e=1}^{n} \mathbf{I}_{O\vec{x}}^{(e)}$$

and

$$\overline{X}_2 = \frac{\sum_{e=1}^{n} \overline{X}_2^{(e)} A^{(e)}}{\sum_{e=1}^{n} A^{(e)}} \quad ; \quad \overline{X}_3 = \frac{\sum_{e=1}^{n} \overline{X}_3^{(e)} A^{(e)}}{\sum_{e=1}^{n} A^{(e)}}$$

Then, we can obtain the inertia tensor of area for any geometric shape in two dimensions, (see Figure 1.68).

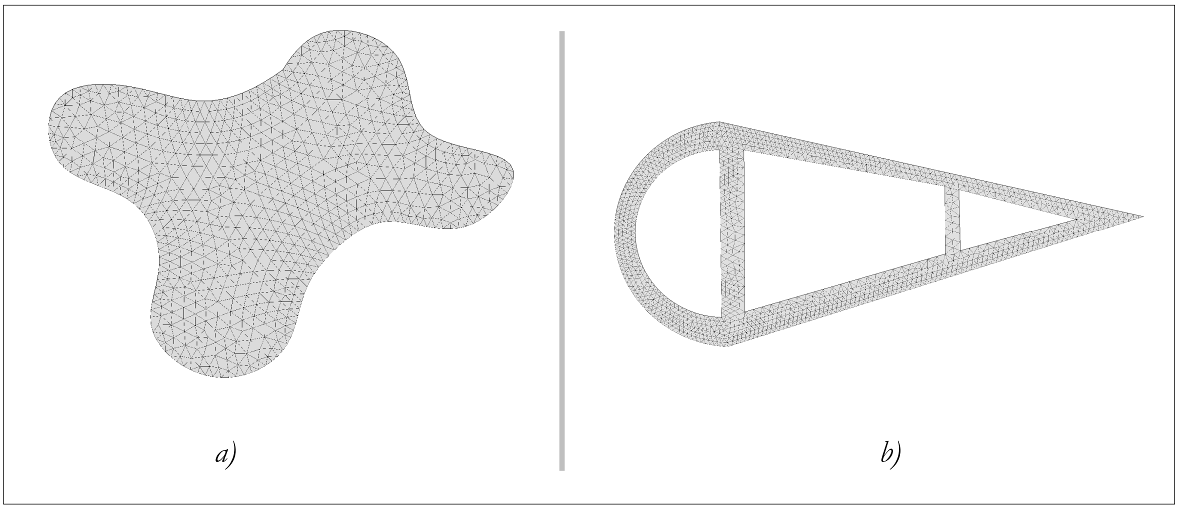


Figure 1.68

NOTE 1: Let us suppose now that the system $o\vec{x}$ is located as shown in Figure 1.69.

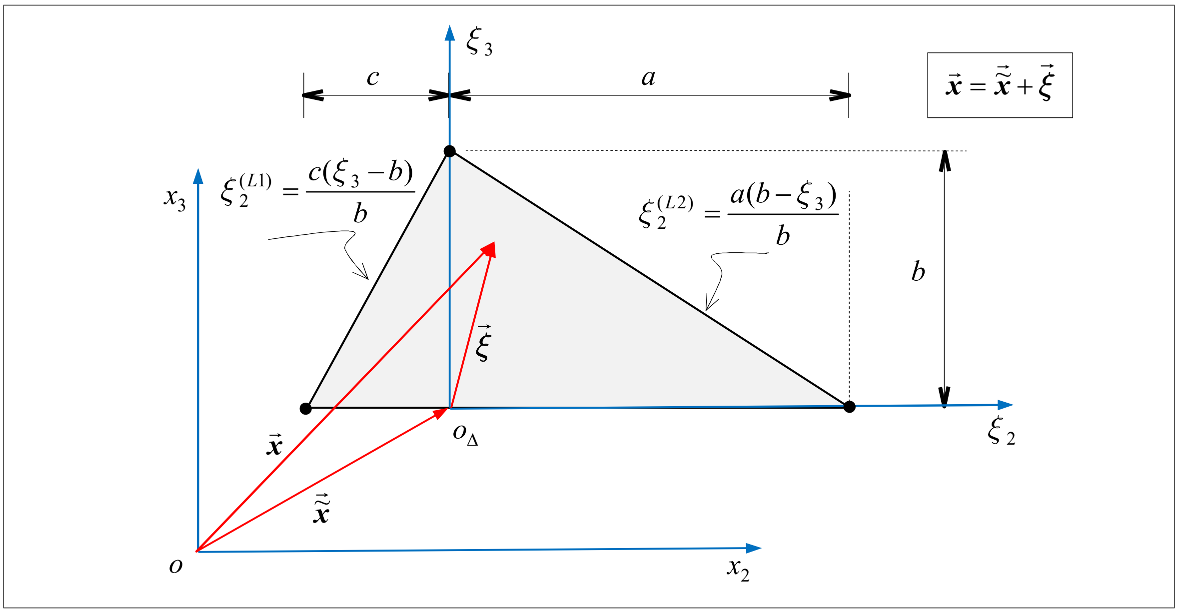


Figure 1.69

According to Figure 1.69 we can state that

$$\vec{\boldsymbol{x}} = \vec{\tilde{\boldsymbol{x}}} + \vec{\boldsymbol{\xi}} \quad \Rightarrow \quad x_i = \tilde{x}_i + \xi_i \quad \Rightarrow \quad \xi_i = x_i - \tilde{x}_i \quad \Rightarrow \quad \begin{cases} \xi_2 = x_2 - \tilde{x}_2 \\ \xi_3 = x_3 - \tilde{x}_3 \end{cases}$$

Then

$$\xi_2^{(L1)} = \frac{c(\xi_3 - b)}{b} \quad \Rightarrow \quad x_2^{(L1)} - \tilde{x}_2 = \frac{c(x_3 - \tilde{x}_3 - b)}{b} \quad \Rightarrow \quad x_2^{(L1)} = \frac{c(x_3 - \tilde{x}_3 - b)}{b} + \tilde{x}_2$$

$$\xi_2^{(L2)} = \frac{a(b - \xi_3)}{b} \quad \Rightarrow \quad x_2^{(L2)} - \tilde{x}_2 = \frac{a(b - (x_3 - \tilde{x}_3))}{b} \quad \Rightarrow \quad x_2^{(L2)} = \frac{a(b - x_3 + \tilde{x}_3)}{b} + \tilde{x}_2$$

Area:

$$A = \int_A dA = \int_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} \left(\int_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} dx_2 \right) dx_3 = \frac{b}{2}(a+c) \tag{1.201}$$

The first moment of area:

about x_2:

$$\int_A x_2 dA = \int_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} \left(\int_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} x_2 dx_2 \right) dx_3 = \frac{b}{6}(a+c)(a-c+3\tilde{x}_2) \tag{1.202}$$

about x_3

$$\int_A x_3 dA = \int_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} x_3 \left(\int_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} dx_2 \right) dx_3 = \frac{b}{6}(a+c)(b+3\tilde{x}_3) \tag{1.203}$$

Then;

$$\bar{x}_1 = 0 \quad ; \quad \bar{x}_2 = \frac{\frac{b}{6}(a+c)(a-c+3\tilde{x}_2)}{\frac{b}{2}(a+c)} = \frac{(a-c)}{3} + \tilde{x}_2 \quad ; \quad \bar{x}_3 = \frac{\frac{b}{6}(a+c)(b+3\tilde{x}_3)}{\frac{b}{2}(a+c)} = \frac{b}{3} + \tilde{x}_3$$

The inertia tensor in the system $o\vec{x}$

$$(\mathbf{I}_{o\vec{x}})_{ij} = \begin{bmatrix} \mathrm{I}_{O\vec{x}11} & \mathrm{I}_{O\vec{x}12} & \mathrm{I}_{O\vec{x}13} \\ \mathrm{I}_{O\vec{x}12} & \mathrm{I}_{O\vec{x}22} & \mathrm{I}_{O\vec{x}23} \\ \mathrm{I}_{O\vec{x}13} & \mathrm{I}_{O\vec{x}23} & \mathrm{I}_{O\vec{x}33} \end{bmatrix} = \begin{bmatrix} \int_A (x_2^2 + x_3^2) dA & 0 & 0 \\ 0 & \int_A (x_3^2) dA & -\int_A (x_2 x_3) dA \\ 0 & -\int_A (x_2 x_3) dA & \int_A (x_2^2) dA \end{bmatrix}$$

where

$$\mathrm{I}_{o\vec{x}22} = \int_A (x_3^2) dA = \int_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} x_3^2 \left(\int_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} dx_2 \right) dx_3 = \frac{b}{12}(a+c)(b^2 + 4b\tilde{x}_3 + 6\tilde{x}_3^2)$$

$$\mathrm{I}_{o\tilde{x}33} = \int_A (x_2^2)\,dA = \int\limits_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} \left(\int\limits_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} x_2^2\,dx_2 \right) dx_3 = \frac{b}{12}(a+c)(a^2+4a\tilde{x}_2-ac+6\tilde{x}_2^2-4c\tilde{x}_2+c^2)$$

$$\mathrm{I}_{o\tilde{x}23} = -\int_A (x_2 x_3)\,dA = -\int\limits_{x_3=\tilde{x}_3}^{x_3=b+\tilde{x}_3} x_3 \left(\int\limits_{x_2=\frac{c(x_3-\tilde{x}_3-b)}{b}+\tilde{x}_2}^{x_2=\frac{a(b-x_3+\tilde{x}_3)}{b}+\tilde{x}_2} x_2\,dx_2 \right) dx_3 = \frac{-b}{24}(a+c)(ab+4a\tilde{x}_3+12\tilde{x}_2\tilde{x}_3+4b\tilde{x}_2-cb-4c\tilde{x}_3)$$

Note that when the system is located at the area centroid we have $\tilde{x}_2 = \frac{-(a-c)}{3}$ and $\tilde{x}_2 = \frac{-b}{3}$, and by substituting into the above equations we will obtain the same expressions as those for $\mathrm{I}''_{G\vec{x}''ij}$ given by the equations in (1.198).

Problem 1.133

Obtain the inertia tensor of mass density related to the system $o\vec{x}$ for the tetrahedron described in Figure 1.70. The tetrahedron base plane (formed by the triangle $1-2-3$) is lying on the plane x_2-x_3. Consider that the mass density is constant.

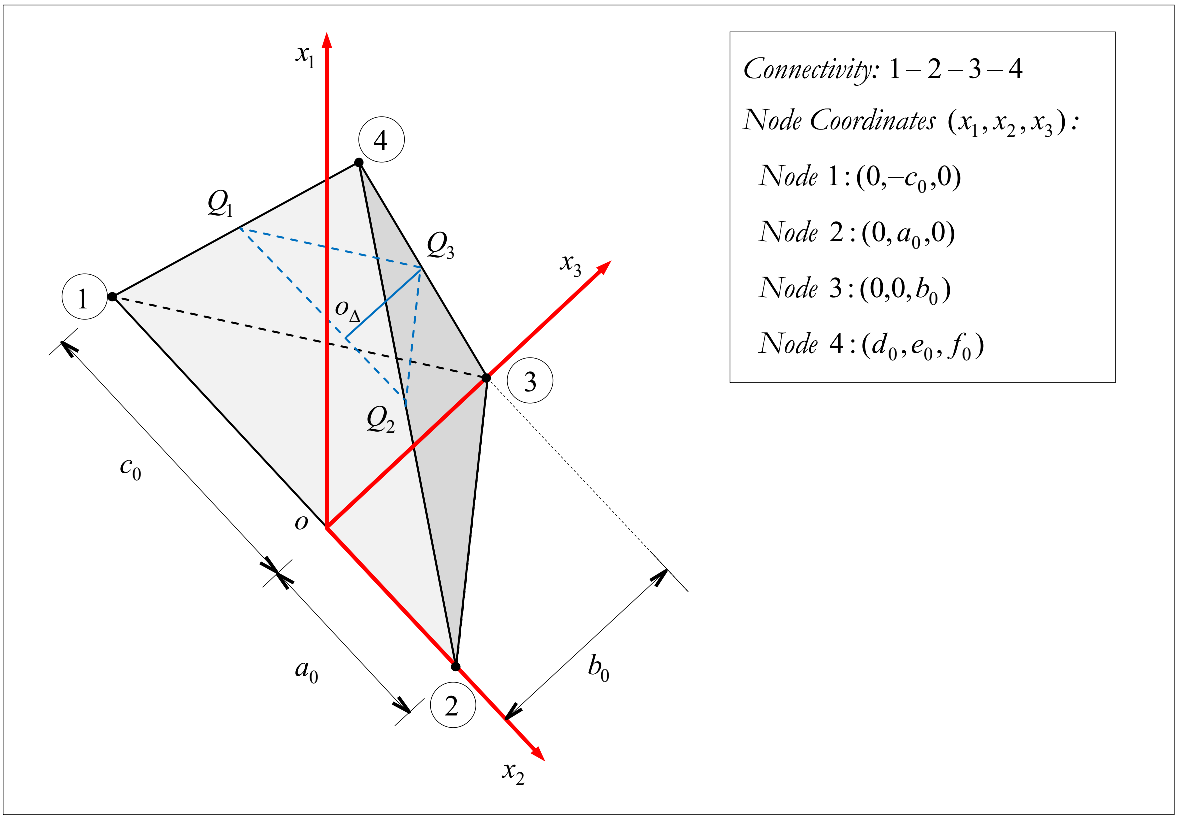


Figure 1.70

Solution:

By definition the inertia tensor of mass density is given by the equation in (1.185), i.e.:

$$\mathrm{I}^{(\rho)}_{o\vec{x}\,ij} = \begin{bmatrix} \mathrm{I}_{o\vec{x}11} & \mathrm{I}_{o\vec{x}12} & \mathrm{I}_{o\vec{x}13} \\ \mathrm{I}_{o\vec{x}12} & \mathrm{I}_{o\vec{x}22} & \mathrm{I}_{o\vec{x}23} \\ \mathrm{I}_{o\vec{x}13} & \mathrm{I}_{o\vec{x}23} & \mathrm{I}_{o\vec{x}33} \end{bmatrix} = \begin{bmatrix} \int_V \rho(x_2^2 + x_3^2)dV & -\int_V \rho x_1 x_2 dV & -\int_V \rho x_1 x_3 dV \\ -\int_V \rho x_1 x_2 dV & \int_V \rho(x_1^2 + x_3^2)dV & -\int_V \rho x_2 x_3 dV \\ -\int_V \rho x_1 x_3 dV & -\int_V \rho x_2 x_3 dV & \int_V \rho(x_1^2 + x_2^2)dV \end{bmatrix} \tag{1.204}$$

From now on we will adopt $\mathbf{I}^{(\rho)}_{o\vec{x}} \equiv \mathbf{I}_{o\vec{x}}$.

Note that the volume integral can be expressed as follows:

$$\int_{x_1} \left(\int_A f(\vec{\mathbf{x}}) dA \right) dx_1$$

where the area is defined on the plane $x_2 - x_3$. For the tetrahedron defined in Figure 1.70 the area on the plane $x_2 - x_3$ is defined by the arbitrary triangle formed by the points $Q_1 - Q_2 - Q_3$ which is parallel to the base plane defined by the nodes $1-2-3$. In order to calculate the integral related to the area we can use the same equations derived by considering Figure 1.69, but now the parameters $a(x_1)$, $b(x_1)$, $c(x_1)$, $\tilde{x}_2(x_1)$ and $\tilde{x}_3(x_1)$ will depend on x_1. Now, in order to define these parameters we have to define the points Q_1, Q_2, Q_3 and o_Δ, (see Figure 1.70 and Figure 1.71).

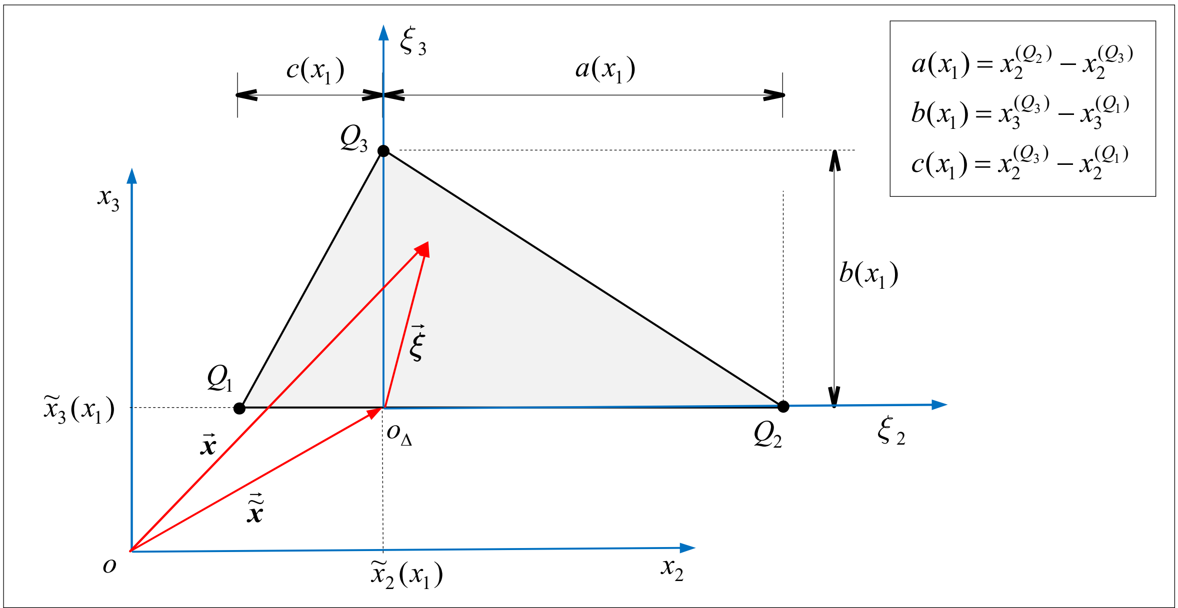


Figure 1.71

According to Figure 1.70 we can define the following coordinates:

$$\text{Point } Q_1 : \begin{cases} x_2^{(Q_1)} = -c_0 + \dfrac{(e_0 + c_0)}{d_0} x_1 \\ x_3^{(Q_1)} = \dfrac{f_0}{d_0} x_1 \end{cases}$$

$$\text{Point } Q_2 : \begin{cases} x_2^{(Q_2)} = a_0 + \dfrac{(e_0 - a_0)}{d_0} x_1 \\ x_3^{(Q_2)} = \dfrac{f_0}{d_0} x_1 \end{cases}$$

$$\text{Point } Q_3 : \begin{cases} x_2^{(Q_3)} = \dfrac{e_0}{d_0} x_1 \\ x_3^{(Q_3)} = b_0 + \dfrac{(f_0 - b_0)}{d_0} x_1 \end{cases}$$

$$\text{Point } o_\Delta : \begin{cases} x_2^{(o_\Delta)} \equiv \tilde{x}_2 = x_2^{(Q_3)} = \dfrac{e_0}{d_0} x_1 \\ x_3^{(o_\Delta)} \equiv \tilde{x}_3 = x_3^{(Q_1)} = x_3^{(Q_2)} = \dfrac{f_0}{d_0} x_1 \end{cases}$$

Once these points are defined we can obtain:

$$a(x_1) = x_2^{(Q_2)} - x_2^{(Q_3)} = \left(a_0 + \frac{(e_0 - a_0)}{d_0} x_1 \right) - \left(\frac{e_0}{d_0} x_1 \right) = a_0 - \frac{a_0}{d_0} x_1$$

$$b(x_1) = x_3^{(Q_3)} - x_3^{(Q_1)} = \left(b_0 + \frac{(f_0 - b_0)}{d_0} x_1 \right) - \left(\frac{f_0}{d_0} x_1 \right) = b_0 - \frac{b_0}{d_0} x_1$$

$$c(x_1) = x_2^{(Q_3)} - x_2^{(Q_1)} = \left(\frac{e_0}{d_0} x_1 \right) - \left(-c_0 + \frac{(e_0 + c_0)}{d_0} x_1 \right) = c_0 - \frac{c_0}{d_0} x_1$$

$$\tilde{x}_2(x_1) = \frac{e_0}{d_0} x_1$$

$$\tilde{x}_3(x_1) = \frac{f_0}{d_0} x_1$$

Now we can easily obtain the volume integrals:

Volume: If we take into account the equation for the area defined in equation (1.201), $A = \frac{b}{2}(a + c)$, we can obtain:

$$\begin{aligned} V &= \int_{x_1=0}^{x_1=d_0} \left(\int_A dA \right) dx_1 = \int_{x_1=0}^{x_1=d_0} [A(x_1)] dx_1 = \int_{x_1=0}^{x_1=d_0} \left(\frac{b(x_1)}{2} [a(x_1) + c(x_1)] \right) dx_1 \\ &= \int_{x_1=0}^{x_1=d_0} \left(\frac{\left(b_0 - \frac{b_0}{d_0} x_1 \right)}{2} \left[\left(a_0 - \frac{a_0}{d_0} x_1 \right) + \left(c_0 - \frac{c_0}{d_0} x_1 \right) \right] \right) dx_1 = \frac{1}{6} b_0 d_0 (a_0 + c_0) \end{aligned} \tag{1.205}$$

Note that the Volume Centroid ($\vec{\bar{x}}_1^{(V)}$) and the Centroid of the mass density field ($\vec{\bar{x}}_1^{(V\_\rho)}$) are the same, since the mass density is a constant filed.

First Moment of Volume

About x_1

$$\int_V x_1 dV = \int_{x_1=0}^{x_1=d_0} x_1\left(\int_A dA\right)dx_1 = \int_{x_1=0}^{x_1=d_0} x_1 A(x_1)dx_1 = \int_{x_1=0}^{x_1=d_0} x_1\left(\frac{\left(b_0-\dfrac{b_0}{d_0}x_1\right)}{2}\left[\left(a_0-\frac{a_0}{d_0}x_1\right)+\left(c_0-\frac{c_0}{d_0}x_1\right)\right]\right)dx_1$$

$$=\frac{1}{24}b_0 d_0^2(a_0+c_0)$$

About x_2, (see equation (1.202)):

$$\int_V x_2 dV = \int_{x_1=0}^{x_1=d_0}\left(\int_A x_2 dA\right)dx_1 = \int_{x_1=0}^{x_1=d_0}\left(\frac{b(x_1)}{6}[a(x_1)+c(x_1)][a(x_1)-c(x_1)+3\tilde{x}_2(x_1)]\right)dx_1$$

$$=\frac{1}{24}b_0 d_0(a_0+c_0)(a_0-c_0+e_0)$$

About x_3, (see equation (1.203)):

$$\int_V x_3 dV = \int_{x_1=0}^{x_1=d_0}\left(\int_A x_3 dA\right)dx_1 = \int_{x_1=0}^{x_1=d_0}\left(\frac{b(x_1)}{6}[a(x_1)+c(x_1)][b(x_1)+3\tilde{x}_3(x_1)]\right)dx_1$$

$$=\frac{1}{24}b_0 d_0(a_0+c_0)(b_0+f_0)$$

The Volume Centroid related to the system $o\vec{x}$

By definition the Volume Centroid, (see equation (1.173)), is given by:

$$\bar{x}_1^{(V)} = \frac{\int_V x_1 dV}{\int_V dV} \quad ; \quad \bar{x}_2^{(V)} = \frac{\int_V x_2 dV}{\int_V dV} \quad ; \quad \bar{x}_3^{(V)} = \frac{\int_V x_3 dV}{\int_V dV} \quad \textit{Volume Centroid} \tag{1.206}$$

Then

$$\begin{aligned}
\bar{x}_1^{(V)} &= \frac{\int_V x_1 dV}{\int_V dV} = \frac{\frac{1}{24}b_0 d_0^2(a_0+c_0)}{\frac{1}{6}b_0 d_0(a_0+c_0)} = \frac{d_0}{4}\\
\bar{x}_2^{(V)} &= \frac{\int_V x_2 dV}{\int_V dV} = \frac{\frac{1}{24}b_0 d_0(a_0+c_0)(a_0-c_0+e_0)}{\frac{1}{6}b_0 d_0(a_0+c_0)} = \frac{(a_0-c_0+e_0)}{4}\\
\bar{x}_3^{(V)} &= \frac{\int_V x_3 dV}{\int_V dV} = \frac{\frac{1}{24}b_0 d_0(a_0+c_0)(b_0+f_0)}{\frac{1}{6}b_0 d_0(a_0+c_0)} = \frac{(b_0+f_0)}{4}
\end{aligned} \tag{1.207}$$

As expected, since the tetrahedron volume centroid is given by:

$$\overline{x}_1 = \frac{x_1^{(1)} + x_1^{(2)} + x_1^{(3)} + x_1^{(4)}}{4} \quad ; \quad \overline{x}_2 = \frac{x_2^{(1)} + x_2^{(2)} + x_2^{(3)} + x_2^{(4)}}{4} \quad ; \quad \overline{x}_3 = \frac{x_3^{(1)} + x_3^{(2)} + x_3^{(3)} + x_3^{(4)}}{4}$$

By considering the area integrals given in **Problem 1.132**-NOTE 1 and by considering $a(x_1) = a_0 - \frac{a_0}{d_0} x_1$, $b(x_1) = b_0 - \frac{b_0}{d_0} x_1$, $c(x_1) = c_0 - \frac{c_0}{d_0} x_1$, $\tilde{x}_2(x_1) = \frac{e_0}{d_0} x_1$ and $\tilde{x}_3(x_1) = \frac{f_0}{d_0} x_1$, the components of the inertia tensor of mass density, (see equation in (1.204)), can be obtained as follows:

$$\mathrm{I}_{o\vec{x}11} = \int_V \rho(x_2^2 + x_3^2) dV = \rho\left[\int_{x_1=0}^{x_1=d_0}\left(\int_A (x_2^2) dA\right) dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\int_A (x_3^2) dA\right) dx_1\right]$$

$$= \rho\left\{\int_{x_1=0}^{x_1=d_0}\left[\left(\frac{b}{12}(a+c)(a^2 + 4a\tilde{x}_2 - ac + 6\tilde{x}_2^2 - 4c\tilde{x}_2 + c^2)\right) + \left(\frac{b}{12}(a+c)(b^2 + 4b\tilde{x}_3 + 6\tilde{x}_3^2)\right)\right] dx_1\right\}$$

$$= \frac{\rho}{60} b_0 d_0 (a_0 + c_0)(b_0 f_0 + b_0^2 + a_0^2 + c_0^2 + a_0 e_0 - e_0 c_0 + e_0^2 + f_0^2 - a_0 c_0)$$

$$\mathrm{I}_{o\vec{x}12} = -\int_V \rho x_1 x_2 dV = -\rho \int_{x_1=0}^{x_1=d_0} x_1\left(\int_A (x_2) dA\right) dx_1 = -\rho \int_{x_1=0}^{x_1=d_0} x_1\left(\frac{b}{6}(a+c)(a - c + 3\tilde{x}_2)\right) dx_1$$

$$= \frac{-\rho}{120} b_0 d_0^2 (a_0 + c_0)(a_0 + 2e_0 - c_0)$$

$$\mathrm{I}_{o\vec{x}13} = -\int_V \rho x_1 x_3 dV = -\rho \int_{x_1=0}^{x_1=d_0} x_1\left(\int_A (x_3) dA\right) dx_1 = -\rho \int_{x_1=0}^{x_1=d_0} x_1\left(\frac{b}{6}(a+c)(b + 3\tilde{x}_3)\right) dx_1$$

$$= \frac{-\rho}{120} b_0 d_0^2 (a_0 + c_0)(b_0 + 2 f_0)$$

$$\mathrm{I}_{o\vec{x}22} = \int_V \rho(x_1^2 + x_3^2) dV = \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2\left(\int_A dA\right) dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\int_A x_3^2 dA\right) dx_1\right] = \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2 A dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\int_A x_3^2 dA\right) dx_1\right]$$

$$= \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2\left(\frac{b}{2}[a+c]\right) dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\frac{b}{12}(a+c)(b^2 + 4b\tilde{x}_3 + 6\tilde{x}_3^2)\right) dx_1\right]$$

$$= \frac{\rho}{60} b_0 d_0 (a_0 + c_0)(b_0^2 + b_0 f_0 + f_0^2 + d_0^2)$$

$$\mathrm{I}_{o\vec{x}23} = -\int_V \rho x_2 x_3 dV = \rho \int_{x_1=0}^{x_1=d_0}\left(\int_A (-x_2 x_3) dA\right) dx_1$$

$$= \rho \int_{x_1=0}^{x_1=d_0}\left(\frac{-b}{24}(a+c)(ab + 4a\tilde{x}_3 + 12\tilde{x}_2\tilde{x}_3 + 4b\tilde{x}_2 - cb - 4c\tilde{x}_3)\right) dx_1$$

$$= \frac{-\rho}{120} b_0 d_0 (a_0 + c_0)(b_0 a_0 + f_0 a_0 + 2 f_0 e_0 - b_0 c_0 + e_0 b_0 - c_0 f_0)$$

$$\mathrm{I}_{o\vec{x}33} = \int_V \rho(x_1^2 + x_2^2)dV = \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2\left(\int_A dA\right)dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\int_A x_2^2 dA\right)dx_1\right] = \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2 A dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\int_A x_2^2 dA\right)dx_1\right]$$

$$= \rho\left[\int_{x_1=0}^{x_1=d_0} x_1^2\left(\frac{b}{2}[a+c]\right)dx_1 + \int_{x_1=0}^{x_1=d_0}\left(\frac{b}{12}(a+c)(a^2 + 4a\tilde{x}_2 - ac + 6\tilde{x}_2^2 - 4c\tilde{x}_2 + c^2)\right)dx_1\right]$$

$$= \frac{\rho}{60} b_0 d_0 (a_0 + c_0)(c_0^2 - c_0 e_0 - c_0 a_0 + d_0^2 + e_0^2 + a_0 e_0 + a_0^2)$$

If we consider that m stands for the total mass of the tetrahedron, and by the fact that the mass density field is constant the following holds:

$$m = \rho V \qquad \Rightarrow \qquad \rho = \frac{m}{V} = \frac{m}{\frac{1}{6} b_0 d_0 (a_0 + c_0)}$$

Then, the inertial tensor of mass density can be written in terms of total mass as follows:

$$\mathrm{I}_{o\vec{x}11} = \frac{m}{10}(b_0 f_0 + b_0^2 + a_0^2 + c_0^2 + a_0 e_0 - c_0 e_0 + e_0^2 + f_0^2 - c_0 a_0)$$

$$\mathrm{I}_{o\vec{x}12} = \frac{-m}{20} d_0 (a_0 + 2e_0 - c_0)$$

$$\mathrm{I}_{o\vec{x}13} = \frac{-m}{20} d_0 (b_0 + 2 f_0)$$

$$\mathrm{I}_{o\vec{x}22} = \frac{m}{10}(b_0^2 + b_0 f_0 + f_0^2 + d_0^2)$$

$$\mathrm{I}_{o\vec{x}23} = \frac{-m}{20}(b_0 a_0 + f_0 a_0 + 2 f_0 e_0 - b_0 c_0 + e_0 b_0 - c_0 f_0)$$

$$\mathrm{I}_{o\vec{x}33} = \frac{m}{10}(c_0^2 - c_0 e_0 - c_0 a_0 + d_0^2 + e_0^2 + a_0 e_0 + a_0^2)$$

Calculation of the inertia tensor of mass density in the system $G\vec{x}''$.

Now if we want to calculate the inertia tensor in the system located at the volume centroid $G\vec{x}''$, (see Figure 1.72), we can use the definition:

$$\mathrm{I}_{o\vec{x}ij} = \mathrm{I}''_{G\vec{x}''ij} + m\begin{bmatrix} (\bar{x}_2^2 + \bar{x}_3^2) & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & (\bar{x}_1^2 + \bar{x}_3^2) & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & (\bar{x}_1^2 + \bar{x}_2^2) \end{bmatrix}$$

$$\Rightarrow \mathrm{I}''_{G\vec{x}''ij} = \mathrm{I}_{o\vec{x}ij} - m\begin{bmatrix} (\bar{x}_2^2 + \bar{x}_3^2) & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & (\bar{x}_1^2 + \bar{x}_3^2) & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & (\bar{x}_1^2 + \bar{x}_2^2) \end{bmatrix}$$

and by considering that $\bar{x}_1 = \frac{d_0}{4}$, $\bar{x}_2 = \frac{(a_0 - c_0 + e_0)}{4}$ and $\bar{x}_3 = \frac{(b_0 + f_0)}{4}$, we can obtain:

$$
\mathrm{I}''_{G\vec{x}''ij} = \begin{bmatrix} \mathrm{I}''_{G\vec{x}''11} & \mathrm{I}''_{G\vec{x}''12} & \mathrm{I}''_{G\vec{x}''13} \\ \mathrm{I}''_{G\vec{x}''12} & \mathrm{I}''_{G\vec{x}''22} & \mathrm{I}''_{G\vec{x}''23} \\ \mathrm{I}''_{G\vec{x}''13} & \mathrm{I}''_{G\vec{x}''23} & \mathrm{I}''_{G\vec{x}''33} \end{bmatrix} \tag{1.208}
$$

where

$$
\mathrm{I}''_{G\vec{x}''11} = \frac{m}{80}(-2b_0 f_0 + 3b_0^2 + 3a_0^2 + 3c_0^2 - 2a_0 e_0 + 2c_0 e_0 + 3e_0^2 + 3f_0^2 + 2c_0 a_0)
$$

$$
\mathrm{I}''_{G\vec{x}''12} = \frac{-m}{80} d_0(-a_0 + 3e_0 + c_0)
$$

$$
\mathrm{I}''_{G\vec{x}''13} = \frac{-m}{80} d_0(-b_0 + 3f_0)
$$

$$
\mathrm{I}''_{G\vec{x}''22} = \frac{m}{80}(3b_0^2 - 2b_0 f_0 + 3f_0^2 + 3d_0^2)
$$

$$
\mathrm{I}''_{G\vec{x}''23} = \frac{-m}{80}(-b_0 a_0 - f_0 a_0 + 3f_0 e_0 + b_0 c_0 - e_0 b_0 + c_0 f_0)
$$

$$
\mathrm{I}''_{G\vec{x}''33} = \frac{m}{80}(3c_0^2 + 2c_0 e_0 + 2c_0 a_0 + 3d_0^2 + 3e_0^2 - 2a_0 e_0 + 3a_0^2)
$$

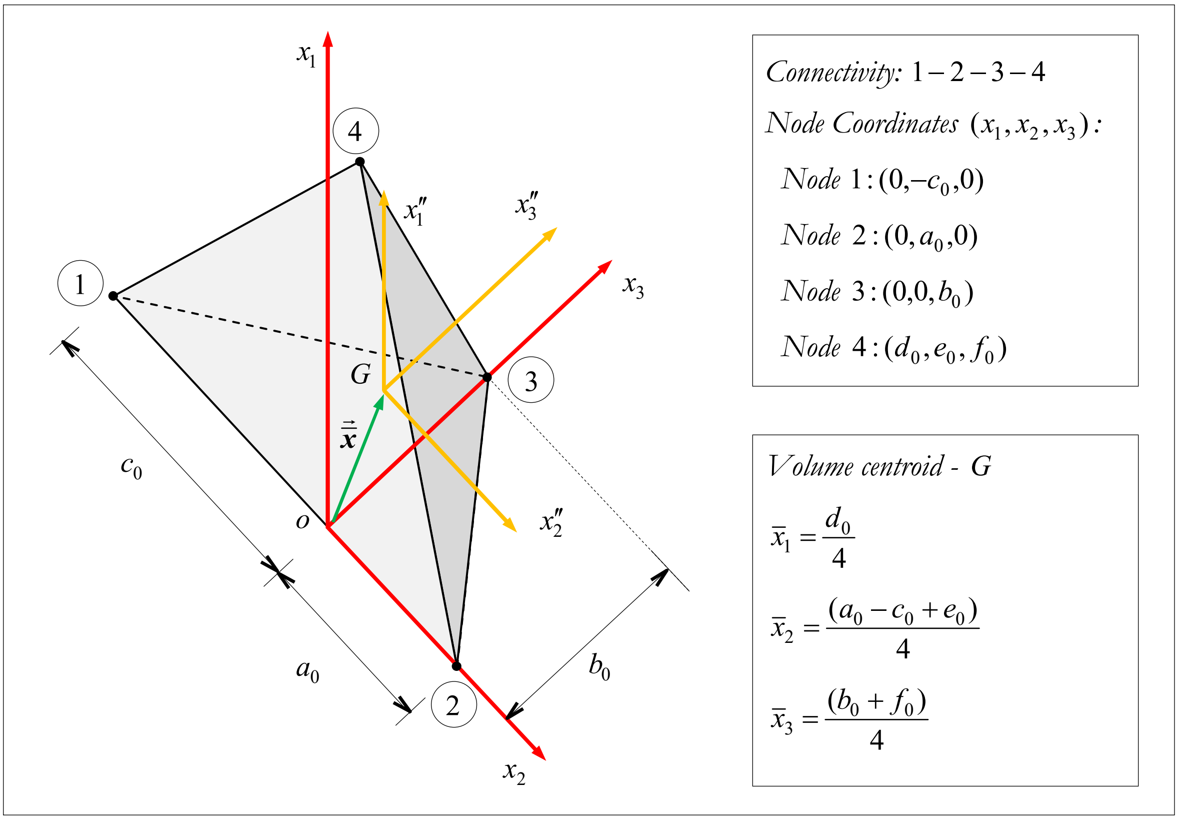


Figure 1.72

2 Continuum Kinematics

2.1 Description of Motion, Material Time Derivative, Lagrangian and Eulerian Variables

Problem 2.1

A continuum is defined by a square with sides b, subjected to rigid body motion which is defined by rotating the continuum counterclockwise by an angle of 30° to the origin. Find the equations of motion. Also obtain the new position of particle D.

Hint: Consider the systems $\vec{x}$ and $\vec{X}$ to be superimposed.

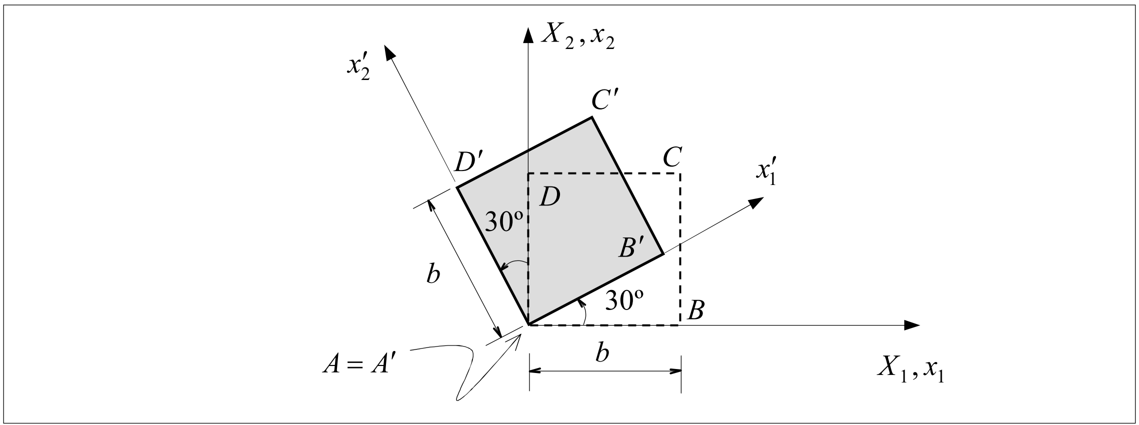


Figure 2.1

Solution:

We apply the rigid body motion equations $\vec{x} = \vec{\mathbf{c}} + \mathbf{Q} \cdot \vec{X} = \mathbf{Q} \cdot \vec{X}$, to $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. The components of $\mathbf{Q}$ are the same as the components of the transformation matrix from the $\vec{x}'$-system to the $\vec{x}$-system, i.e.:

$$\mathsf{Q}_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathcal{A}^T$$

where $\mathcal{A}$ is the transformation matrix form the $\vec{x}$-system to the $\vec{x}'$-system. So, the continuum particles are governed by the equations of motion:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos 30^{o} & -\sin 30^{o} & 0 \\ \sin 30^{o} & \cos 30^{o} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

A particle which initially was at point D $(X_1 = 0, X_2 = b, X_3 = 0)$ moves into the following position:

$$\begin{Bmatrix} x_1^D \\ x_2^D \\ x_3^D \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ b \\ 0 \end{Bmatrix} = \begin{Bmatrix} -b\sin 30^\circ \\ b\cos 30^\circ \\ 0 \end{Bmatrix}$$

Problem 2.2

Consider the following equations of motion:

$$\begin{cases} x_1 = \exp^t X_1 - \exp^{-t} X_2 \\ x_2 = \exp^t X_1 + \exp^{-t} X_2 \qquad (t > 0) \\ x_3 = X_3 \end{cases} \tag{2.1}$$

Find velocity, acceleration in material and spatial descriptions.

Solution:

Material velocity (Lagrangian):

$$\vec{V}(\vec{X},t) = \frac{D\vec{x}(\vec{X},t)}{Dt} \xrightarrow{components} \begin{cases} V_1 = \exp^t X_1 + \exp^{-t} X_2 \\ V_2 = \exp^t X_1 - \exp^{-t} X_2 \\ V_3 = 0 \end{cases} \tag{2.2}$$

Acceleration:

$$A_1 = \exp^t X_1 - \exp^{-t} X_2 \quad ; \quad A_2 = \exp^t X_1 + \exp^{-t} X_2 \quad ; \quad A_3 = 0 \tag{2.3}$$

To find the velocity and acceleration components in the spatial description we substitute the equations of motion, i.e.:

Eulerian velocity (spatial description)

$$v_1 = x_2 \quad ; \quad v_2 = x_1 \quad ; \quad v_3 = 0 \tag{2.4}$$

Eulerian acceleration (spatial description)

$$a_1 = x_1 = v_2 \quad ; \quad a_2 = x_2 = v_1 \quad ; \quad a_3 = 0 \tag{2.5}$$

Problem 2.3

The velocity field of a fluid is given by:

$$\vec{v} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 \tag{2.6}$$

and the temperature field is $T(\vec{x},t) = 3x_2 + x_3 t$. Find the rate of change of temperature.

Solution:

The rate of change of any property is given by the material time derivative, and when we are dealing with Eulerian variables the material derivative is given by:

$$\frac{DT}{Dt} = \frac{\partial T(\vec{x},t)}{\partial t} + \frac{\partial T(\vec{x},t)}{\partial x_j} v_j = \frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial x_1} v_1 + \frac{\partial T}{\partial x_2} v_2 + \frac{\partial T}{\partial x_3} v_3 \right) \tag{2.7}$$

$$\frac{DT}{Dt} = x_3 + (0 \times x_1 + 3 \times x_2 + t x_3) = x_3 + (3x_2 + t x_3) \tag{2.8}$$

Problem 2.4

Given the following equations motion:

$$x_i = X_i + 0.2tX_2\delta_{1i} \tag{2.9}$$

and the temperature field (steady):

$$T(\vec{x}) = 2x_1 + x_2^2 \tag{2.10}$$

a) Find the temperature field in material description;

b) Find the rate of change of temperature for one particle that in the reference configuration was at the position $(0,1,0)$.

Solution:

According to the equations of motion we have:

$$\begin{aligned} x_1 &= X_1 + 0.2tX_2\delta_{11} = X_1 + 0.2tX_2 \\ x_2 &= X_2 + 0.2tX_2\delta_{12} = X_2 \\ x_3 &= X_3 + 0.2tX_2\delta_{13} = X_3 \end{aligned}$$

And by means of the above equations it is possible to express the temperature in material description:

$$T(\vec{x}(\vec{X},t)) = 2x_1(\vec{X},t) + [x_2(\vec{X},t)]^2 = 2(X_1 + 0.2tX_2) + (X_2)^2 = 2X_1 + (X_2 + 0.4t)X_2 = T(\vec{X},t)$$

b) The material time derivative of temperature is given by:

$$\frac{DT(\vec{X},t)}{Dt} \equiv \dot{T}(\vec{X},t) = 0.4X_2$$

For the particle $(X_1 = 0; X_2 = 1; X_3 = 0)$ we have:

$$\dot{T}((X_1 = 0; X_2 = 1; X_3 = 0), t) = 0.4X_2 = 0.4$$

Note that, although the Eulerian temperature $(T(\vec{x}))$ is independent of time, the Lagrangian temperature $T(\vec{X},t)$ depends on time, in other words, the temperature at a point does not change meanwhile the particle temperature changes.

Problem 2.5

Find the velocity field $\vec{V}(\vec{X},t)$ in the material description and the acceleration field $\vec{A}(\vec{X},t)$ of the particle at time t in function of the rate of change of displacement $\vec{U}(\vec{X},t)$.

Solution:

$$\vec{V}(\vec{X},t) = \frac{D}{Dt}\vec{U}(\vec{X},t) = \dot{\vec{U}} \tag{2.11}$$

$$\vec{A}(\vec{X},t) = \frac{D}{Dt}\vec{V}(\vec{X},t) = \dot{\vec{V}} = \frac{D^2}{Dt^2}\vec{U}(\vec{X},t) = \ddot{\vec{U}} \tag{2.12}$$

Problem 2.6

Consider the following equations of motion in the Lagrangian description:

$$\begin{cases} x_1(\vec{X},t) = X_2t^2 + X_1 \\ x_2(\vec{X},t) = X_3t + X_2 \\ x_3(\vec{X},t) = X_3 \end{cases} \xrightarrow{\text{Matrix form}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & t^2 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{2.13}$$

Is the motion above possible? If so, find the displacement, velocity and acceleration fields in Lagrangian and Eulerian descriptions. Consider a particle P that at time $t=0$ was at the point defined by the triple equation $X_1=2, X_2=1, X_3=3$. Find the velocity of P at time $t=1s$ and $t=2s$.

Solution:

Motion is possible if $J \neq 0$, thus

$$J=\left|\frac{\partial x_i}{\partial X_j}\right|=\begin{vmatrix}\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}\end{vmatrix}=\begin{vmatrix}1 & t^2 & 0\\ 0 & 1 & t\\ 0 & 0 & 1\end{vmatrix}=1\neq 0$$

The displacement vector field is given by the definition $\vec{\mathbf{u}}=\vec{x}-\vec{X}$. Using the equations of motion (2.78) we obtain:

$$\begin{cases}\mathsf{u}_1(\vec{X},t)=x_1(\vec{X},t)-X_1=[X_2t^2+X_1]-X_1=X_2t^2\\ \mathsf{u}_2(\vec{X},t)=x_2(\vec{X},t)-X_2=[X_3t+X_2]-X_2=X_3t\\ \mathsf{u}_3(\vec{X},t)=x_3(\vec{X},t)-X_3=[X_3]-X_3=0\end{cases} \tag{2.14}$$

which are the components of the displacement vector in the Lagrangian description. Here, velocity and acceleration can be evaluated as follows:

$$\begin{cases}V_1\equiv v_1(\vec{X},t)=\dfrac{d\mathsf{u}_1(\vec{X},t)}{dt}=\dfrac{d}{dt}\left(X_2t^2\right)=2X_2t\\ V_2\equiv v_2(\vec{X},t)=\dfrac{d\mathsf{u}_2(\vec{X},t)}{dt}=\dfrac{d}{dt}\left(X_3t\right)=X_3\\ V_3\equiv v_3(\vec{X},t)=\dfrac{d\mathsf{u}_3(\vec{X},t)}{dt}=\dfrac{d}{dt}\left(X_2t\right)=0\end{cases} \quad ; \quad \begin{cases}A_1\equiv a_1(\vec{X},t)=\dfrac{dV_1}{dt}=2X_2\\ A_2\equiv a_2(\vec{X},t)=\dfrac{dV_2}{dt}=0\\ A_3\equiv a_3(\vec{X},t)=\dfrac{dV_3}{dt}=0\end{cases} \tag{2.15}$$

The inverse form of (2.13) provides us the inverse equations of motion (Eulerian description):

$$\begin{Bmatrix}X_1\\X_2\\X_3\end{Bmatrix}=\begin{bmatrix}1 & -t^2 & t^3\\ 0 & 1 & -t\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}\Rightarrow\begin{cases}X_1(\vec{x},t)=x_1-t^2x_2+t^3x_3\\ X_2(\vec{x},t)=x_2-tx_3\\ X_3(\vec{x},t)=x_3\end{cases} \tag{2.16}$$

Then, the displacement, velocity and acceleration fields in Eulerian description can be evaluated by substituting equation (2.16) into the equations (2.14) and (2.15), i.e.:

$$\begin{cases}\mathsf{u}_1\left(\vec{X}(\vec{x},t),t\right)=X_2(\vec{x},t)t^2=(x_2-tx_3)t^2=\mathsf{u}_1(\vec{x},t)\\ \mathsf{u}_2\left(\vec{X}(\vec{x},t),t\right)=X_3(\vec{x},t)t=x_3t=\mathsf{u}_2(\vec{x},t)\\ \mathsf{u}_3\left(\vec{X}(\vec{x},t),t\right)=\mathsf{u}_3(\vec{x},t)=0\end{cases} \tag{2.17}$$

$$\begin{cases}V_1\left(\vec{X}(\vec{x},t),t\right)=2X_2(\vec{x},t)t=2(x_2-tx_3)t=v_1(\vec{x},t)\\ V_2\left(\vec{X}(\vec{x},t),t\right)=X_3(\vec{x},t)=x_3=v_2(\vec{x},t)\\ V_3\left(\vec{X}(\vec{x},t),t\right)=v_3(\vec{x},t)=0\end{cases} \tag{2.18}$$

$$\begin{cases} A_1\left(\vec{X}(\vec{x},t),t\right) = 2X_2(\vec{x},t) = 2(x_2 - tx_3) = a_1(\vec{x},t) \\ A_2\left(\vec{X}(\vec{x},t),t\right) = a_2(\vec{x},t) = 0 \\ A_3\left(\vec{X}(\vec{x},t),t\right) = a_3(\vec{x},t) = 0 \end{cases} \tag{2.19}$$

Taking into account the Lagrangian description of velocity given in (2.15), the velocity of particle $P\,(X_1 = 2, X_2 = 1, X_3 = 3)$ at time $t = 1s$ is given by:

$$v_1(\vec{X},t) = 2X_2 t = 2m/s \quad ; \quad v_2(\vec{X},t) = X_3 = 3m/s \quad ; \quad v_3(\vec{X},t) = 0$$

We can also observe that at time $t = 1s$ the particle P occupies the position:

$$x_1 = X_2 t^2 + X_1 = 3 \quad ; \quad x_2 = X_3 t + X_2 = 4 \quad ; \quad x_3 = X_3 = 3$$

So, the velocity of the particle P, (see Figure 2.2), can also be evaluated by (2.18) as:

$$\begin{cases} v_1(\vec{x},t) = 2(x_2 - tx_3)t = 2(4 + 1\times 3)\times 1 = 2m/s \\ v_2(\vec{x},t) = x_3 = 3m/s \\ v_3(\vec{x},t) = 0 \end{cases}$$

Note that, the velocities obtained via the Lagrangian or Eulerian description are the same, since velocity is an intrinsic property of the particle.

We can also provide the velocity of the particle P at time $t = 2s$:

$$\begin{cases} V_1 \equiv v_1(\vec{X},t) = 2X_2 t = 2\times 2\times 1 = 4m/s \\ V_2 \equiv v_2(\vec{X},t) = X_3 = 3m/s \\ V_3 \equiv v_3(\vec{X},t) = 0 \end{cases}$$

At time $t = 2s$ the new position of P is:

$$x_1(\vec{X},t) = X_2 t^2 + X_1 = 6 \quad ; \quad x_2(\vec{X},t) = X_3 t + X_2 = 7 \quad ; \quad x_3(\vec{X},t) = X_3 = 3$$

As we can verify the Lagrangian description of motion $\vec{x}(\vec{X},t)$ describes the trajectory of P.

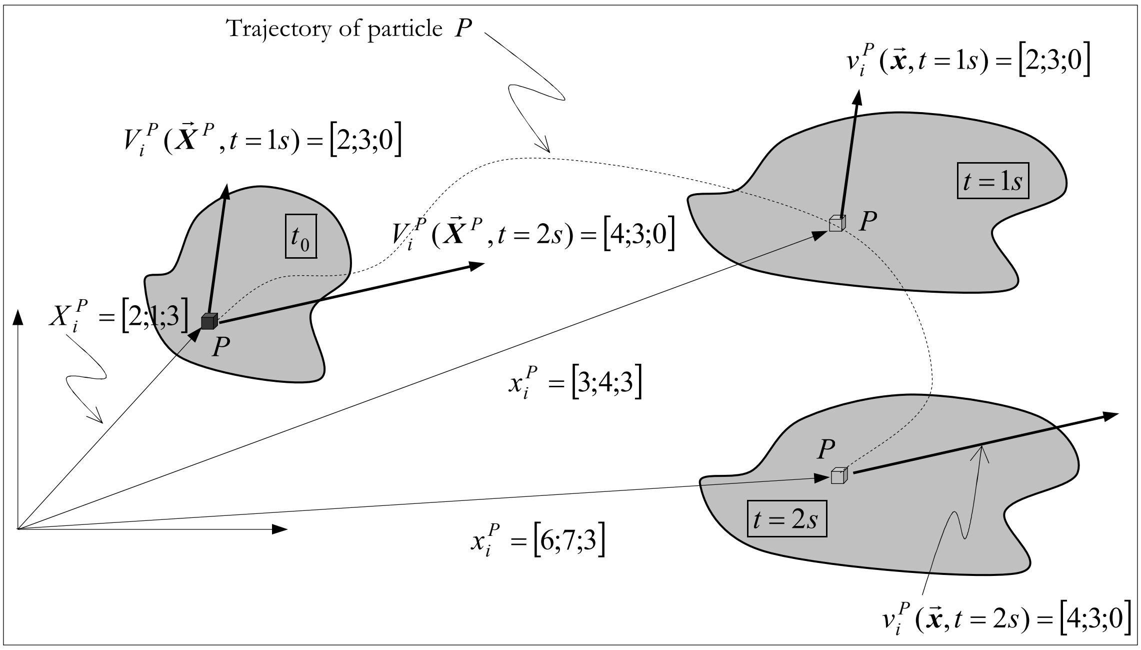


Figure 2.2

NOTE: Note that, the Eulerian velocity cannot be obtained by means of $\frac{D\vec{X}(\vec{x},t)}{Dt} = \vec{0} \neq \vec{v}(\vec{x},t)$. We can verify this by means of the proposed problem:

$$\frac{DX_i(\vec{x},t)}{Dt} = \frac{\partial X_i(\vec{x},t)}{\partial t} + \left[\frac{\partial X_i(\vec{x},t)}{\partial x_1} v_1(\vec{x},t) + \frac{\partial X_i(\vec{x},t)}{\partial x_2} v_2(\vec{x},t) + \frac{\partial X_i(\vec{x},t)}{\partial x_3} v_3(\vec{x},t)\right]$$

thus:

$$\begin{aligned}
\frac{DX_1(\vec{x},t)}{Dt} &= \frac{\partial X_1(\vec{x},t)}{\partial t} + \left[\frac{\partial X_1(\vec{x},t)}{\partial x_1} v_1(\vec{x},t) + \frac{\partial X_1(\vec{x},t)}{\partial x_2} v_2(\vec{x},t) + \frac{\partial X_1(\vec{x},t)}{\partial x_3} v_3(\vec{x},t)\right] \\
&= \left(-2tx_2 + 3t^2 x_3\right) + \left[1 \times 2(x_2 - tx_3)t - t^2 \times x_3 + t^3 \times 0\right] = 0 \\
\frac{DX_2(\vec{x},t)}{Dt} &= \frac{\partial X_2(\vec{x},t)}{\partial t} + \left[\frac{\partial X_2(\vec{x},t)}{\partial x_1} v_1(\vec{x},t) + \frac{\partial X_2(\vec{x},t)}{\partial x_2} v_2(\vec{x},t) + \frac{\partial X_2(\vec{x},t)}{\partial x_3} v_3(\vec{x},t)\right] \\
&= \left(-x_3\right) + \left[0 \times 2(x_2 - tx_3)t + 1 \times x_3 - t \times 0\right] = 0 \\
\frac{DX_3(\vec{x},t)}{Dt} &= \frac{\partial X_3(\vec{x},t)}{\partial t} + \left[\frac{\partial X_3(\vec{x},t)}{\partial x_1} v_1(\vec{x},t) + \frac{\partial X_3(\vec{x},t)}{\partial x_2} v_2(\vec{x},t) + \frac{\partial X_3(\vec{x},t)}{\partial x_3} v_3(\vec{x},t)\right] \\
&= \left(0\right) + \left[0 \times 2(x_2 - tx_3)t + 0 \times x_3 + 1 \times 0\right] = 0
\end{aligned}$$

Remember that $\vec{\mathbf{u}} = \vec{x} - \vec{X}$, then:

$$\vec{v}(\vec{X},t) = \frac{D\vec{x}(\vec{X},t)}{Dt} = \frac{D}{Dt}\left(\vec{\mathbf{u}}(\vec{X},t) - \vec{X}(\vec{x},t)\right) = \frac{D\vec{\mathbf{u}}(\vec{X},t)}{Dt} \equiv \dot{\vec{\mathbf{u}}}(\vec{X},t)$$

Also, it fulfills that:

$$\vec{v}(\vec{x},t) = \dot{\vec{\mathbf{u}}}(\vec{x},t) \equiv \frac{D\vec{\mathbf{u}}(\vec{x},t)}{Dt} = \frac{\partial \vec{\mathbf{u}}(\vec{x},t)}{\partial t} + \frac{\partial \vec{\mathbf{u}}(\vec{x},t)}{\partial \vec{x}} \cdot \vec{v}(\vec{x},t)$$

Problem 2.7

The velocity field of the continuum, in Eulerian description, is given by:

$$v_1 = \frac{x_1}{1+t} \quad ; \quad v_2 = \frac{2x_2}{1+t} \quad ; \quad v_3 = \frac{3x_3}{1+t} \tag{2.20}$$

a) Obtain the relationship between material and spatial coordinates $x_i = x_i(\vec{X},t)$;

b) Obtain the acceleration components by means of the spatial motion description.

c) Obtain the acceleration components by means of the Lagrangian motion.

Solution:

a) Considering that $v_i = \frac{dx_i(\vec{X},t)}{dt}$ we can obtain:

$$v_1 = \frac{dx_1}{dt} = \frac{x_1}{1+t} \Rightarrow \frac{dx_1}{x_1} = \frac{dt}{1+t} \tag{2.21}$$

$$\begin{aligned}
&\int \frac{1}{x_1} dx_1 = \int \frac{1}{1+t} dt \Rightarrow \mathrm{Ln} x_1 = \mathrm{Ln}(1+t) + \mathrm{Ln}(C_1) \Rightarrow \\
&\Rightarrow x_1 = C_1(1+t)
\end{aligned} \tag{2.22}$$

The initial condition is $t=0 \Rightarrow x_1 = X_1$, with that the constant of integration is obtained: $C_1 = X_1$, then

$$x_1 = X_1(1+t) \tag{2.23}$$

$$v_2 = \frac{dx_2}{dt} = \frac{2x_2}{1+t} \Rightarrow \frac{dx_2}{x_2} = \frac{2dt}{1+t} \tag{2.24}$$

$$\int \frac{1}{x_2} dx_2 = \int \frac{2}{1+t} dt \Rightarrow \mathrm{Ln} x_2 = 2\mathrm{Ln}(1+t) + \mathrm{Ln} C_2 \qquad \Rightarrow \qquad x_2 = C_2(1+t)^2 \tag{2.25}$$

for $t=0 \Rightarrow x_2 = X_2 \Rightarrow C_2 = X_2$

$$x_2 = X_2(1+t)^2 \tag{2.26}$$

$$v_3 = \frac{dx_3}{dt} = \frac{3x_3}{1+t} \Rightarrow \frac{dx_3}{x_3} = \frac{3dt}{1+t} \tag{2.27}$$

$$\int \frac{1}{x_3} dx_3 = \int \frac{3}{1+t} dt \Rightarrow \mathrm{Ln} x_3 = 3\mathrm{Ln}(1+t) + \mathrm{Ln} C_3 \qquad \Rightarrow \qquad x_3 = C_3(1+t)^3 \tag{2.28}$$

and $t=0 \Rightarrow x_3 = X_3 \Rightarrow C_3 = X_3$

$$x_3 = X_3(1+t)^3 \tag{2.29}$$

Then, the equations of motion are:

$$x_1 = X_1(1+t) \quad ; \quad x_2 = X_2(1+t)^2 \quad ; \quad x_3 = X_3(1+t)^3 \tag{2.30}$$

b) By applying the material time derivative to the Eulerian velocity $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)$ we can obtain the acceleration as follows:

$$\frac{D\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{Dt} = \vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t) = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} + \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \tag{2.31}$$

which is indicial notation is represented by

$$a_i = \frac{\partial v_i}{\partial t} + (v_{i,k})v_k = \frac{\partial v_i}{\partial t} + (v_{i,1}v_1 + v_{i,2}v_2 + v_{i,3}v_3) \tag{2.32}$$

thus,

$$\begin{aligned}
a_1 &= -\frac{x_1}{(1+t)^2} + \left(\frac{x_1}{1+t}\frac{1}{1+t} + 0 + 0 \right) = 0 \\
a_2 &= -\frac{2x_2}{(1+t)^2} + \left(0 + \frac{2x_2}{1+t}\frac{2}{1+t} + 0 \right) = \frac{2x_2}{(1+t)^2} \\
a_3 &= -\frac{3x_3}{(1+t)^2} + \left(0 + 0 + \frac{3x_3}{1+t}\frac{3}{1+t} \right) = \frac{6x_3}{(1+t)^2}
\end{aligned} \tag{2.33}$$

c) The Lagrangian velocity components are obtained by substituting the equations of motion given by (2.30), i.e:

$$V_1 = X_1 \quad ; \quad V_2 = 2X_2(1+t) \quad ; \quad V_3 = 3X_3(1+t)^2 \tag{2.34}$$

In the same fashion we can obtain the Lagrangian acceleration components:

$$a_1 = \frac{dV_1}{dt} = 0 \quad ; \quad a_2 = \frac{dV_2}{dt} = 2X_2 \quad ; \quad a_3 = \frac{dV_3}{dt} = 6X_3(1+t) \tag{2.35}$$

Problem 2.8

Consider the equations of motion:

$$x_1 = X_1 \quad ; \quad x_2 = X_2 + AX_3 \quad ; \quad x_3 = X_3 + AX_2 \tag{2.36}$$

where A is constant. Find the displacement vector field components in the material and spatial descriptions.

Solution:

Lagrangian displacement vector:

$$\vec{\mathbf{u}} = \vec{x}(\vec{X},t) - \vec{X} \xrightarrow{\text{components}} \begin{cases} \mathsf{u}_1 = x_1 - X_1 = 0 \\ \mathsf{u}_2 = x_2 - X_2 = (X_2 + AX_3) - X_2 = AX_3 \\ \mathsf{u}_3 = x_3 - X_3 = (X_3 + AX_2) - X_3 = AX_2 \end{cases} \tag{2.37}$$

The equations of motion (2.36) in matrix form become:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}}_{=[\mathcal{A}]} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{where } [\mathcal{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} \tag{2.38}$$

The determinant and the inver of are represented, respectively, by:

$$\det[\mathcal{A}] \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{vmatrix} = 1 - A^2 \qquad [\mathcal{A}]^{-1} = \frac{1}{1-A^2} \begin{bmatrix} 1-A^2 & 0 & 0 \\ 0 & 1 & -A \\ 0 & -A & 1 \end{bmatrix} \tag{2.39}$$

thus the inverse of motion can be obtained as

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{1-A^2} \begin{bmatrix} 1-A^2 & 0 & 0 \\ 0 & 1 & -A \\ 0 & -A & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} X_1 = x_1 \\ X_2 = \dfrac{1}{1-A^2}(x_2 - Ax_3) \\ X_3 = \dfrac{1}{1-A^2}(x_3 - Ax_2) \end{cases} \tag{2.40}$$

The Eulerian displacement vector components (spatial description) become:

$$\begin{cases} \mathsf{u}_1 = x_1 - X_1 = 0 \\ \mathsf{u}_2 = x_2 - X_2 = x_2 - \dfrac{1}{1-A^2}(x_2 - Ax_3) = \dfrac{A(x_3 - Ax_2)}{1-A^2} \\ \mathsf{u}_1 = x_3 - X_3 = x_3 - \dfrac{1}{1-A^2}(x_3 - Ax_2) = \dfrac{A(x_2 - Ax_3)}{1-A^2} \end{cases} \tag{2.41}$$

Problem 2.9

Consider the equations of motion:

$$x_1 = X_1 \quad ; \quad x_2 = X_2 + X_3 t \quad ; \quad x_3 = X_3 + X_3 t \tag{2.42}$$

Obtain the velocity of the particles that are passing through the point $(0,1,2)$ at time $t_1 = 0s$ and $t_2 = 1s$.

Solution:

The velocity field is given by:

$$\vec{V}(\vec{X},t)=\frac{D\vec{x}(\vec{X},t)}{Dt} \tag{2.43}$$

And by considering the equations of motion (2.42), $\vec{x}(\vec{X},t)$, we can obtain the the Lagrangian velocity components:

$$V_1=0 \quad ; \quad V_2=X_3 \quad ; \quad V_3=X_3 \tag{2.44}$$

Note that, in order to obtain the velocity of a particle we have to identify the particle, the is identified by its material coordinate, i.e. the coordinate of the particle at time $t=0s$.

At time $t=0s$ we have $\vec{x}=\vec{X}$, then the particle in question is $(X_1=0, X_2=1, X_3=2)$, thus its velocity is:

$$V_1=0 \quad ; \quad V_2=2 \quad ; \quad V_3=2 \tag{2.45}$$

The material coordinates for the particle that is passing through the point $(x_1=0, x_2=1, x_3=2)$ at time $t=1s$, can be obtained as follows:

$$\left.\begin{array}{l} x_1=0=X_1 \\ x_2=1=X_2+X_3 \\ x_3=2=X_3+X_3 \end{array}\right\} \Rightarrow (X_1=0;\, X_2=0;\, X_3=1) \tag{2.46}$$

And by means of Lagrangian velocity (2.44) we can obtain:

$$V_1=0 \quad ; \quad V_2=1 \quad ; \quad V_3=1 \tag{2.47}$$

Problem 2.10

By adopting the Cartesian system the particle motion is defined as follows:

$$\begin{aligned} x_1(\vec{X},t)&=X_1\sin\left(\frac{ct}{X_1^2+X_2^2}\right)+X_2\cos\left(\frac{ct}{X_1^2+X_2^2}\right) \\ x_2(\vec{X},t)&=-X_1\cos\left(\frac{ct}{X_1^2+X_2^2}\right)+X_2\sin\left(\frac{ct}{X_1^2+X_2^2}\right) \\ x_3(\vec{X},t)&=X_3 \end{aligned} \tag{2.48}$$

where c is a constant. Obtain the velocity components in spatial and material descriptions.

Solution:

The velocity components in the material (Lagrangian) description are:

$$\begin{aligned} V_1(\vec{X},t)&=\frac{Dx_1(\vec{X},t)}{Dt}=\frac{c}{X_1^2+X_2^2}\left[X_1\cos\left(\frac{ct}{X_1^2+X_2^2}\right)-X_2\sin\left(\frac{ct}{X_1^2+X_2^2}\right)\right] \\ V_2(\vec{X},t)&=\frac{Dx_2(\vec{X},t)}{Dt}=\frac{c}{X_1^2+X_2^2}\left[X_1\sin\left(\frac{ct}{X_1^2+X_2^2}\right)+X_2\cos\left(\frac{ct}{X_1^2+X_2^2}\right)\right] \\ V_3(\vec{X},t)&=\frac{Dx_3(\vec{X},t)}{Dt}=0 \end{aligned} \tag{2.49}$$

Taking into account the equation (2.48), we can note that the following relationship holds:

$$x_1^2 + x_2^2 = X_1^2 + X_2^2 \tag{2.50}$$

And by considering the above equation into the equation (2.48), the velocity components in the spatial (Eulerian) description are:

$$v_1(\vec{\boldsymbol{x}},t) = \frac{-c\,x_2}{x_1^2 + x_2^2} \quad ; \quad v_2(\vec{\boldsymbol{x}},t) = \frac{c\,x_1}{x_1^2 + x_2^2} \quad ; \quad v_3(\vec{\boldsymbol{x}},t) = 0 \tag{2.51}$$

The inverse equations of motion, $\vec{\boldsymbol{X}} = \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t)$, are:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \sin\left(\dfrac{c\,t}{x_1^2 + x_2^2}\right) & -\cos\left(\dfrac{c\,t}{x_1^2 + x_2^2}\right) & 0 \\ \cos\left(\dfrac{c\,t}{x_1^2 + x_2^2}\right) & \sin\left(\dfrac{c\,t}{x_1^2 + x_2^2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.52}$$

Problem 2.11

The Eulerian velocity field components are:

$$v_1 = x_1 \quad ; \quad v_2 = \frac{x_2}{2t+3} \quad ; \quad v_3 = 0 \tag{2.53}$$

Find the parametric equations of the trajectory of the particle which was at (X_1, X_2, X_3) in the reference configuration.

Solution:

Remember that the trajectory of a particle is given by the equation $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$. Then, to find the path line (trajectory) we must solve the system:

$$\frac{dx_1}{dt} = x_1 \quad ; \quad \frac{dx_2}{dt} = \frac{x_2}{2t+3} \quad ; \quad \frac{dx_3}{dt} = 0 \tag{2.54}$$

with the initial conditions

$$\begin{cases} x_1(t=0) = X_1 \\ x_2(t=0) = X_2 \\ x_3(t=0) = X_3 \end{cases} \tag{2.55}$$

$$\begin{aligned} &\int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_0^t dt \quad \Rightarrow \quad \mathrm{Ln}\left(\frac{x_1}{X_1}\right) = t \quad \Rightarrow \quad x_1 = X_1 \mathrm{exp}^t \\ &\int_{X_2}^{x_2} \frac{dx_2}{x_2} = \int_0^t \frac{dt}{2t+3} \quad \Rightarrow \quad \mathrm{Ln}\left(\frac{x_2}{X_2}\right) = \mathrm{Ln}\left(\sqrt{2t+3}\right) - \mathrm{Ln}\left(\sqrt{3}\right) \quad \Rightarrow \quad x_2 = X_2\sqrt{\frac{2}{3}t+1} \\ &x_3 = X_3 \end{aligned} \tag{2.56}$$

Then, the equations of motion are given by:

$$x_1 = X_1 \mathrm{exp}^t \quad ; \quad x_2 = X_2\sqrt{\frac{2}{3}t+1} \quad ; \quad x_3 = X_3 \tag{2.57}$$

Problem 2.12

Consider the following equations of motion:

$$x_1 = X_1 \quad ; \quad x_2 = 2\,t\,X_3 + X_2 \quad ; \quad x_3 = X_3 \tag{2.58}$$

and a physical quantity represented by the scalar field $q(\vec{\boldsymbol{x}},t)$ in the Eulerian description:

$$q(\vec{\boldsymbol{x}},t) = 2x_1 + x_2 - x_3 + 1 \tag{2.59}$$

a) Obtain the Lagrangian description of the physical quantity;

b) Obtain the Lagrangian and Eulerian velocities;

c) Obtain the rate of change of the physical quantity.

d) Obtain the local rate of change of q at the spatial point $(1,3,2)$.

Solution:

a) The Lagrangian description can be obtained by substituting the equations of motion $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$ into the Eulerian variable, i.e. $q(\vec{\boldsymbol{x}},t) = q(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t) = Q(\vec{\boldsymbol{X}},t)$. Then, by substituting the equations of motion (2.58) into the equation of the variable $q(\vec{\boldsymbol{x}},t)$ given by (2.59) we can obtain:

$$Q(\vec{\boldsymbol{X}},t) = 2X_1 + X_2 + (2t-1)X_3 + 1 \tag{2.60}$$

b) The velocity vector is defined by

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{X}},t) = \frac{D\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)}{Dt} \tag{2.61}$$

And by considering the equations of motion (2.58) we can obtain the Lagrangian velocity:

$$V_1 = 0 \quad ; \quad V_2 = 2X_3 \quad ; \quad V_3 = 0 \tag{2.62}$$

The inverse of the equations of motion is:

$$\begin{cases} x_1 = X_1 \\ x_2 = 2\,t\,X_3 + X_2 \\ x_3 = X_3 \end{cases} \xrightarrow{inverse} \begin{cases} X_1 = x_1 \\ X_2 = x_2 - 2\,t\,x_3 \\ X_3 = x_3 \end{cases}$$

Then, the Eulerian velocity components are given by:

$$v_1 = 0 \quad ; \quad v_2 = 2x_3 \quad ; \quad v_3 = 0 \tag{2.63}$$

c) The rate of change of the variable is obtained by applying the material time derivative. If we are dealing with Lagrangian variable the material time derivative is given by:

$$\dot{Q} = \frac{D}{Dt} Q(\vec{\boldsymbol{X}},t) = 2X_3 \tag{2.64}$$

and if we are dealing with Eulerian variables the material time derivative is given by

$$\dot{q} = \underbrace{\frac{\partial q(\vec{\boldsymbol{x}},t)}{\partial t}}_{=0(steady)} + (\nabla_{\vec{x}} q) \cdot \vec{\boldsymbol{v}} \tag{2.65}$$

$$\dot{q} = 0 + q_{,i}\, v_i \quad = 0 + \left[\frac{\partial q}{\partial x_1} v_1 + \frac{\partial q}{\partial x_2} v_2 + \frac{\partial q}{\partial x_3} v_3 \right] = \left[(2)(0) + (1)(2x_3) + (-1)(0) \right] \; = 2x_3 \tag{2.66}$$

We could have obtained the same result by starting from $\dot{Q} = 2X_3$ in which we substitute $X_3 = x_3$, thus

$$\dot{q}(\vec{x},t) = \dot{Q}(\vec{X}(\vec{x},t),t) \quad \Rightarrow \quad \dot{q}(\vec{x},t) = 2x_3 \tag{2.67}$$

d) Note that the physical quantity field is stationary, i.e. $q = q(\vec{x})$, then the local rate of change is $\frac{\partial q(\vec{x})}{\partial t} = 0$ at any spatial point.

Problem 2.13

Given the Lagrangian displacement field:

$$\mathsf{u}_1 = ktX_2 \quad ; \quad \mathsf{u}_2 = 0 \quad ; \quad \mathsf{u}_3 = 0$$

and the Eulerian temperature field $T(\vec{x},t) = (x_1 + x_2)\,t$.

a) Find the rate of change of temperature for a particle that at time $t = 1s$ is passing through the point $(1,1,1)$.

Solution:

We can apply the definition $\dot{T}(\vec{x},t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial t}$ or $\dot{T}(\vec{X},t) = \frac{DT(\vec{X},t)}{Dt}$ in order to obtain the material time derivative.

By means of the equation $\mathsf{u}_i = x_i - X_i$ we can obtain the equations of motion:

$$\begin{aligned} \mathsf{u}_1 = x_1 - X_1 \quad &\Rightarrow \quad x_1 = \mathsf{u}_1 + X_1 \quad \Rightarrow \quad x_1 = X_1 + ktX_2 \\ \mathsf{u}_2 = x_2 - X_2 \quad &\Rightarrow \quad x_2 = \mathsf{u}_2 + X_2 \quad \Rightarrow \quad x_2 = X_2 \\ \mathsf{u}_3 = x_3 - X_3 \quad &\Rightarrow \quad x_3 = \mathsf{u}_3 + X_3 \quad \Rightarrow \quad x_3 = X_3 \end{aligned}$$

The Lagrangian temperature field (material description) can be obtained as follows:

$$T(\vec{x}(\vec{X},t),t) = (x_1 + x_2)t = ((X_1 + ktX_2) + (X_2))t = X_1 t + kX_2 t^2 + X_2 t = T(\vec{X},t)$$

Then, the material time derivative becomes:

$$\dot{T}(\vec{X},t) = \frac{DT(\vec{X},t)}{Dt} = \frac{D}{Dt}[X_1 t + kX_2 t^2 + X_2 t] = X_1 + 2kX_2 t + X_2 \tag{2.68}$$

If we want to find the rate of change of temperature of the particle which is passing through the point $x_1 = 1, x_2 = 1, x_3 = 1$ at $t = 1s$, we have two possibilities, namely: 1) Finding the position of said particle in the reference configuration and replacing it in the above equation. 2) The other possibility is by means of the equation of the rate of change of temperature in the spatial (Eulerian) description. To do this, we will need to establish the inverse of the equations of motion, i.e.: $\vec{X} = \vec{X}(\vec{x},t)$:

$$\begin{cases} x_1 = X_1 + ktX_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases} \quad \Rightarrow \quad \begin{cases} X_1 = x_1 - ktx_2 \\ X_2 = x_2 \\ X_3 = x_3 \end{cases}$$

And by substituting the above equations into the equation (2.68) we can obtain:

$$\dot{T}(\vec{X}(\vec{x},t),t) = X_1 + 2kX_2 t + X_2 = (x_1 - ktx_2) + 2kt(x_2) + (x_2) = \dot{T}(\vec{x},t)$$

by simplifying the above equation we can obtain $\dot{T}(\vec{x},t) = x_1 + ktx_2 + x_2$. Then:

$$\dot{T}(x_1 = 1, x_2 = 1, x_3 = 1, t = 1) = (1-k) + 2k + 1 = k + 2$$

Alternative solution:

$$\dot{T}(\vec{\boldsymbol{x}}, t) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \vec{\boldsymbol{x}}} \cdot \frac{\partial \vec{\boldsymbol{x}}}{\partial t} = (x_1 + x_2) + \left(\frac{\partial T}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial T}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial T}{\partial x_3}\frac{\partial x_3}{\partial t} \right)$$
$$= (x_1 + x_2) + (tkX_2 + t(0) + (0)(0)) = x_1 + x_2 + tkX_2$$

Note that $x_2 = X_2$, then:

$$\dot{T}(\vec{\boldsymbol{x}}, t) = x_1 + x_2 + tkx_2$$

Problem 2.14

Let us consider the following equations of motion:

$$x_1 = X_1 \quad ; \quad x_2 = X_2 + \frac{t}{2}X_3 \quad ; \quad x_3 = X_3 + \frac{t}{2}X_2 \tag{2.69}$$

a) Is this motion possible? Justify your answer;

b) Obtain the velocity components in the Lagrangian and Eulerian descriptions;

c) Obtain the path line (trajectory equation).

Solution:

a) Obtaining the Jacobian determinant:

$$J = |\boldsymbol{F}| = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{t}{2} \\ 0 & \frac{t}{2} & 1 \end{vmatrix} = 1 - \frac{t^2}{4} \tag{2.70}$$

The motion is possible if $J = |\boldsymbol{F}| > 0$:

$$J = 1 - \frac{t^2}{4} > 0 \Rightarrow t < 2\,s \tag{2.71}$$

b) The Lagrangian velocity components are obtained as follows:

$$\begin{cases} V_1 = \dfrac{Dx_1(\vec{\boldsymbol{X}}, t)}{Dt} = 0 \\ V_2 = \dfrac{Dx_1(\vec{\boldsymbol{X}}, t)}{Dt} = \dfrac{D}{Dt}\left(X_2 + \dfrac{t}{2}X_3 \right) = \dfrac{X_3}{2} \\ V_3 = \dfrac{Dx_1(\vec{\boldsymbol{X}}, t)}{Dt} = \dfrac{D}{Dt}\left(X_3 + \dfrac{t}{2}X_2 \right) = \dfrac{X_2}{2} \end{cases} \tag{2.72}$$

The inverse of the equations of motion is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{t}{2} \\ 0 & \frac{t}{2} & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \xrightarrow{inverse} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{J} \begin{bmatrix} J & 0 & 0 \\ 0 & 1 & -\frac{t}{2} \\ 0 & -\frac{t}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.73}$$

By substituting the values of X_i given by the above equations into the Lagrangian velocity (2.72) we can obtain the velocity in the spatial description:

$$v_1 = 0 \quad ; \quad v_2 = \frac{x_3 - \frac{t}{2}x_2}{2 - \frac{t^2}{2}} = \frac{2x_3 - tx_2}{4 - t^2} \quad ; \quad v_3 = \frac{x_2 - \frac{t}{2}x_3}{2 - \frac{t^2}{2}} = \frac{2x_2 - tx_3}{4 - t^2} \tag{2.74}$$

c) The trajectory can be obtained by eliminating t of the equations of motion (2.69):

$$\begin{cases} x_1 = X_1 \\ (x_3 - X_3)X_3 = (x_2 - X_2)X_2 \quad \Rightarrow \quad x_3 = \dfrac{X_2}{X_3}x_2 - \dfrac{X_2^2}{X_3} + X_3 \end{cases} \tag{2.75}$$

Problem 2.15

The Eulerian velocity field for a steady fluid is given by:

$$\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) = U\frac{b^2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}\hat{\mathbf{e}}_1 + 2U\frac{b^2 x_1 x_2}{(x_1^2 + x_2^2)^2}\hat{\mathbf{e}}_2 + V\hat{\mathbf{e}}_3 \tag{2.76}$$

where U and V are constants.

Show that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0$ and find the Eulerian acceleration field.

Solution:

$$\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = v_{i,i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = -2Ub^2\frac{x_1(x_1^2 - 3x_2^2)}{(x_1^2 + x_2^2)^3} + 2Ub^2\frac{x_1(x_1^2 - 3x_2^2)}{(x_1^2 + x_2^2)^3} = 0$$

The Eulerian acceleration field:

$$\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}) = \underbrace{\frac{\partial \vec{\boldsymbol{v}}}{\partial t}}_{=\vec{\mathbf{0}}} + (\nabla_{\vec{x}}\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}} = (\nabla_{\vec{x}}\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}}$$

The components of the spatial velocity gradient are given by:

$$(\nabla_{\vec{x}}\vec{\boldsymbol{v}})_{ij} \equiv \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \frac{2Ub^2}{(x_1^2 + x_2^2)^3}\begin{bmatrix} x_1(3x_2^2 - x_1^2) & -x_2(3x_1^2 - x_2^2) & 0 \\ -x_2(3x_1^2 - x_2^2) & -x_1(3x_2^2 - x_1^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The acceleration components are given by $a_i = (\nabla_{\vec{x}}\vec{\boldsymbol{v}})_{ij}(\vec{\boldsymbol{v}})_j$:

$$a_i = \frac{\partial v_i}{\partial x_j}v_j = \frac{2Ub^2}{(x_1^2 + x_2^2)^3}\begin{bmatrix} x_1(3x_2^2 - x_1^2) & -x_2(3x_1^2 - x_2^2) & 0 \\ -x_2(3x_1^2 - x_2^2) & -x_1(3x_2^2 - x_1^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U\frac{b^2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} \\ 2U\frac{b^2 x_1 x_2}{(x_1^2 + x_2^2)^2} \\ V \end{bmatrix} = \begin{bmatrix} \frac{-2x_1U^2b^4}{(x_1^2 + x_2^2)^3} \\ \frac{-2x_2U^2b^4}{(x_1^2 + x_2^2)^3} \\ 0 \end{bmatrix}$$

Problem 2.16

Calculate the material time derivative for the property ϕ when said property is described as follows:

- Material description: $\phi(\vec{X},t) = X_1 t^2$;
- Spatial description: $\phi(\vec{x},t) = \dfrac{x_1 t^2}{(1+t)}$.

Consider that the equations of motion by $x_1 = x_1(X_1)$, i.e. it is independent of X_2 and X_3.

Solution:

a) Material time derivative of $\phi(\vec{X},t) = X_1 t^2$:

$$\frac{D}{Dt}\phi(\vec{X},t) \equiv \dot{\phi}(\vec{X},t) = 2X_1 t$$

b) Material time derivative of $\phi(\vec{x},t) = \dfrac{x_1 t^2}{(1+t)}$:

$$\begin{aligned}
\frac{D}{Dt}\phi(\vec{x},t) &= \frac{\partial\phi(\vec{x},t)}{\partial t} + (\nabla_{\vec{x}}\phi)\cdot\vec{v} = \frac{\partial\phi(\vec{x},t)}{\partial t} + \frac{\partial\phi(\vec{x},t)}{\partial x_i}v_i \\
&= \frac{\partial\phi(\vec{x},t)}{\partial t} + \left[\frac{\partial\phi(\vec{x},t)}{\partial x_1}v_1 + \frac{\partial\phi(\vec{x},t)}{\partial x_2}v_2 + \frac{\partial\phi(\vec{x},t)}{\partial x_3}v_3\right] \\
&= \frac{\partial}{\partial t}\left(\frac{x_1 t^2}{(1+t)}\right) + \left[\frac{\partial\phi(\vec{x},t)}{\partial x_1}v_1 + 0 + 0\right]
\end{aligned} \tag{2.77}$$

We need to know the velocity component v_1. We start from the principle that a property is intrinsic to the particle, then:

$$\phi(\vec{X},t) = X_1 t^2 \quad\Rightarrow\quad \phi(\vec{X}(\vec{x},t),t) = \phi(\vec{x},t) = \frac{x_1 t^2}{(1+t)} \quad\Rightarrow\quad X_1 = \frac{x_1}{(1+t)}$$

The velocity becomes:

$$\vec{v}(\vec{X},t) = \frac{D}{Dt}\left(X_1 t^2\right) = 2X_1 t\,\hat{\mathbf{e}}_1 \quad\Rightarrow\quad \vec{v}(\vec{x},t) = 2\frac{x_1}{(1+t)}t\,\hat{\mathbf{e}}_1$$

Then, the material time derivative (2.77) becomes:

$$\begin{aligned}
\frac{D}{Dt}\phi(\vec{x},t) &= \frac{\partial}{\partial t}\left(\frac{x_1 t^2}{(1+t)}\right) + \left[\frac{\partial\phi(\vec{x},t)}{\partial x_1}v_1\right] = \left(\frac{2x_1 t}{(1+t)} - \frac{x_1 t^2}{(1+t)^2}\right) + \left[\frac{t^2}{(1+t)}X_1\right] \\
&= \left(\frac{2x_1 t}{(1+t)} - \frac{x_1 t^2}{(1+t)^2}\right) + \left[\frac{t^2}{(1+t)}2\frac{x_1}{(1+t)}t\right] = \frac{2x_1 t}{(1+t)}
\end{aligned}$$

We could also have obtained the same result by starting from $\dfrac{D}{Dt}\phi(\vec{X},t) \equiv \dot{\phi}(\vec{X},t) = 2X_1 t$ and by substituting $X_1 = \dfrac{x_1}{(1+t)}$, i.e.:

$$\frac{D}{Dt}\phi(\vec{X},t) \equiv \dot{\phi}(\vec{X},t) = 2X_1 t \quad\Rightarrow\quad \frac{D}{Dt}\phi(\vec{X}(\vec{x},t),t) = \dot{\phi}(\vec{X}(\vec{x},t),t) = \dot{\phi}(\vec{x},t) = 2\frac{x_1}{(1+t)}t$$

Problem 2.17

Consider the following equations of motion in the Lagrangian description:

$$\begin{cases} x_1 = X_1 t^2 + 2X_2 t + X_1 \\ x_2 = 2X_1 t^2 + X_2 t + X_2 \\ x_3 = \frac{1}{2} X_3 t + X_3 \end{cases} \xrightarrow[\text{form}]{\text{Matrix}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} t^2+1 & 2t & 0 \\ 2t^2 & t+1 & 0 \\ 0 & 0 & \frac{1}{2}t+1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{2.78}$$

Find the components of the displacement vector in Lagrangian and Eulerian descriptions.

Solution:

By definition the displacement vector is obtained by $\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}$, then by substituting the equations of motion (2.78) we can obtain:

$$\begin{cases} \mathsf{u}_1(\vec{\boldsymbol{X}},t) = x_1(\vec{\boldsymbol{X}},t) - X_1 = (X_1 t^2 + 2X_2 t + X_1) - X_1 = X_1 t^2 + 2X_2 t \\ \mathsf{u}_2(\vec{\boldsymbol{X}},t) = x_2(\vec{\boldsymbol{X}},t) - X_2 = (2X_1 t^2 + X_2 t + X_2) - X_2 = 2X_1 t^2 + X_2 t \\ \mathsf{u}_3(\vec{\boldsymbol{X}},t) = x_3(\vec{\boldsymbol{X}},t) - X_3 = (\frac{1}{2} X_3 t + X_3) - X_3 = \frac{1}{2} X_3 t \end{cases}$$

which are the displacement components in the Lagrangian description (material).

To obtain the Eulerian displacement we will need to obtain the inverse equations of motion (2.78), which is:

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{1}{3t^3 - 1 - t - t^2} \begin{bmatrix} -(1+t) & 2t & 0 \\ 2t^2 & -(1+t^2) & 0 \\ 0 & 0 & \dfrac{3t^3 - 1 - t - t^2}{\frac{1}{2}(t+2)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \Rightarrow \begin{cases} X_1 = \dfrac{2tx_2 - x_1(1+t)}{3t^3 - 1 - t - t^2} \\ X_2 = \dfrac{2x_1 t^2 - x_2(1+t^2)}{3t^3 - 1 - t - t^2} \\ X_3 = \dfrac{2x_3}{(t+2)} \end{cases}$$

We can also use the definition $\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}$, but now we replace the material coordinate to obtain the displacement vector components in the Eulerian description:

$$\begin{cases} \mathsf{u}_1(\vec{\boldsymbol{x}},t) = x_1 - X_1(\vec{\boldsymbol{x}},t) = x_1 - \dfrac{2tx_2 - x_1(1+t)}{3t^3 - 1 - t - t^2} \\ \mathsf{u}_2(\vec{\boldsymbol{x}},t) = x_2 - X_2(\vec{\boldsymbol{x}},t) = x_2 - \dfrac{2x_1 t^2 - x_2(1+t^2)}{3t^3 - 1 - t - t^2} \\ \mathsf{u}_3(\vec{\boldsymbol{x}},t) = x_3 - X_3(\vec{\boldsymbol{x}},t) = x_3 - \dfrac{2x_3}{(t+2)} \end{cases}$$

Problem 2.18

The following equations describe the motion of a body, (see Figure 2.3):

$$\begin{cases} x_1 = X_1 + 0.2X_2 t \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

At time $t=0$, the cube of side 1 has one vertex at the origin of the system which is indicated by point *O*, (see Figure 2.3). Obtain the configuration of the body at time $t=2s$.

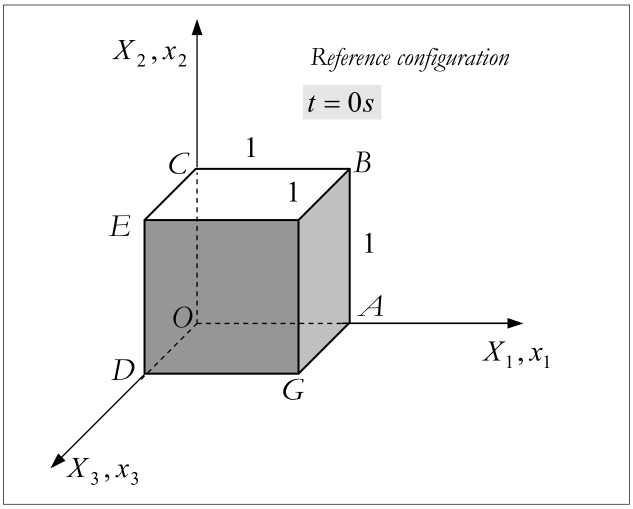


Figure 2.3: Reference configuration $t=0$.

Solution:

To obtain the current configuration of the body at time $t=2s$, we will analyze the particle motion. The particle which occupies position O (origin) at $t=0$ has material coordinate:

$$X_1 = 0 \quad ; \quad X_2 = 0 \quad ; \quad X_3 = 0$$

and by substituting the above coordinates into the equations of motion we can obtain:

$$x_i(X_1 = 0, X_2 = 0, X_3 = 0, t) \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Then, we can conclude that the particle O does not change its position during motion.

The particles lying on the $\overline{OA}$ line, in the initial configuration, have the reference coordinate $(X_1, X_2 = 0, X_3 = 0)$. In spatial coordinates:

$$\begin{cases} x_1 = X_1 + 0.2X_2 t = X_1 \\ x_2 = X_2 = 0 \\ x_3 = X_3 = 0 \end{cases}$$

That is, all particles lying on the $\overline{OA}$ line do not move during motion. Similarly, we can verify that the line $(X_1, X_2 = 0, X_3 = 1)$ in the reference configuration $(X_1, X_2 = 0, X_3 = 1)$ does not move:

$$x_1 = X_1 + 0.2 \times 0 \times 2 = X_1 \quad ; \quad x_2 = X_2 = 0 \quad ; \quad x_3 = X_3 = 0$$

The particles lying on the $\overline{CB}$ line, $(X_1, X_2 = 1, X_3 = 0)$, at time $t=2s$ will move according to:

$$x_1 = X_1 + 0.2 \times 1 \times 2 = X_1 + 0.4 \quad ; \quad x_2 = X_2 = 1 \quad ; \quad x_3 = X_3 = 0$$

Then, all particles lying on the $\overline{CB}$ line will move 0.4 according to x_1-direction.

The particles belonging to line $\overline{OC}$ at $t=0$, will move to positions:

$$\begin{cases} x_1 = X_1 + 0.2X_2 t = 0 + 0.2 \times 2 \times X_2 = 0.4X_2 \\ x_2 = X_2 \\ x_3 = X_3 = 0 \end{cases}$$

Following the same procedure for the remaining particles, we can obtain the final configuration of the body at time $t = 2s$, (see Figure 2.4).

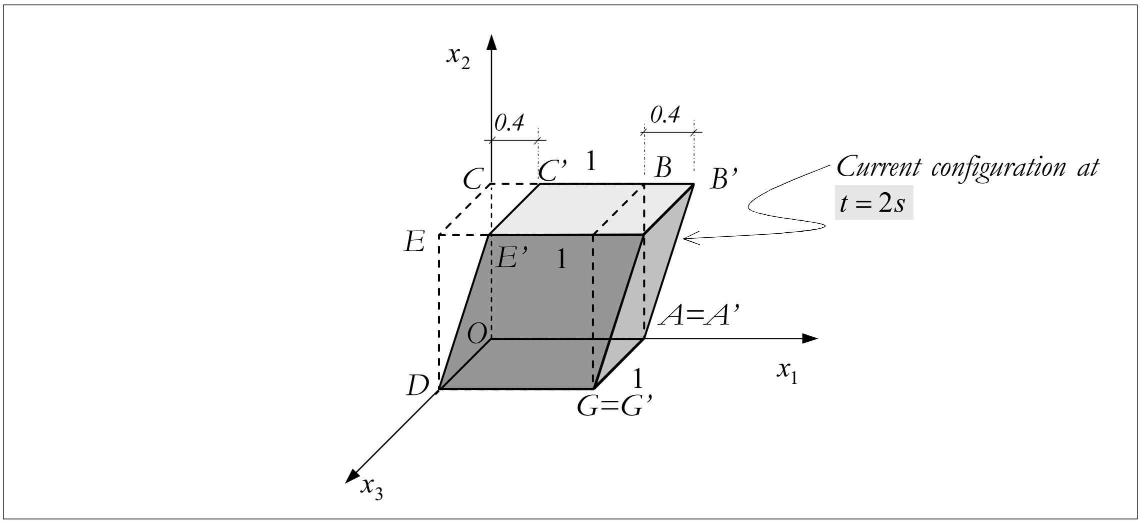


Figure 2.4: Body configuration at time t (deformed configuration).

Problem 2.19

Consider the equations of motion:

$$\begin{cases} x_1 = X_1 + t^2 X_2 \\ x_2 = t^2 X_1 + X_2 \\ x_3 = X_3 \end{cases} \xrightarrow{\text{Matrix form}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & t^2 & 0 \\ t^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

a) Obtain the trajectory of particle Q which originally at time t_0 was at $X_i = (1,2,1)$;

b) By considering the current configuration at $t = 0.5s$, obtain the velocity and acceleration components of the particle P that was originally at $X_i = \left(\frac{16}{15}; \frac{-4}{15}; 1\right)$;

c) Obtain the equations of motion in the Eulerian description;

d) Obtain the velocity and acceleration components of one particle that at time ($t = 0.5s$) is passing through the point $x_i = (1,0,1)$.

Obs.: Consider the International System of Units (SI-Units).

Solution:

1)Using the equations of motion and by substituting the material coordinates of the point $X_i = (1,2,1)$, we can obtain:

$$x_1 = 1 + 2t^2 \quad ; \quad x_2 = 2 + t^2 \quad ; \quad x_3 = 1$$

The above equations represent the motion of the particle. To obtain the trajectory, we eliminate the time of the equations of motion, i.e.:

$$x_1 - 2x_2 = -3 \quad ; \quad x_3 = 1$$

which indicates that the particle moves in a straight line defined by $(x_1 - 2x_2 = -3)$ on the plane $x_3 = 1$, (see Figure 2.5).

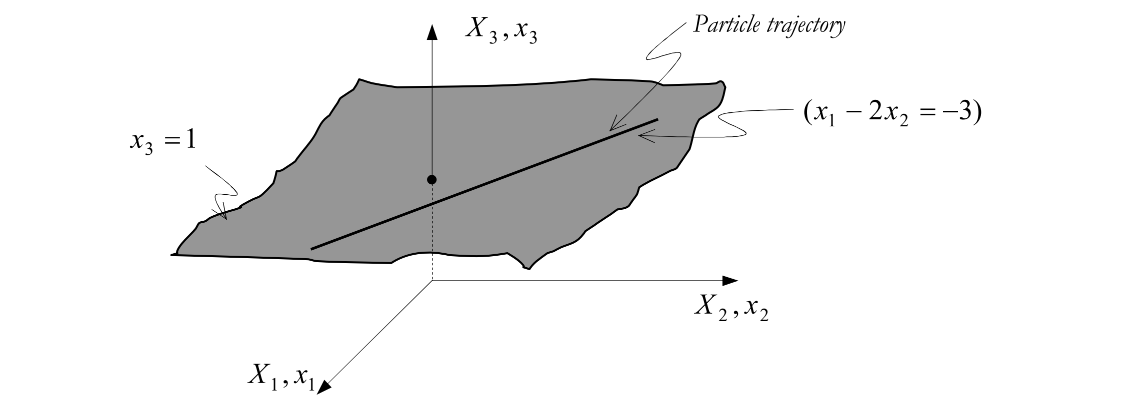


Figure 2.5

2) The velocity and acceleration components of the particle P are given by:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{X}},t) = \frac{D\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)}{Dt} \xrightarrow{components} \begin{cases} V_1 = 2tX_2 \\ V_2 = 2tX_1 \\ V_3 = 0 \end{cases}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}},t) = \frac{D\vec{\boldsymbol{v}}(\vec{\boldsymbol{X}},t)}{Dt} \xrightarrow{components} \begin{cases} A_1 = 2X_2 \\ A_2 = 2X_1 \\ A_3 = 0 \end{cases}$$

Then, the particle which was originally located at the point $X_i = \left(\frac{16}{15}; \frac{-4}{15}; 1\right)$ will achieve a new configuration at $t = 0.5s$. In this configuration, velocity and acceleration for the particle are respectively:

$$\begin{cases} V_1 = 2 \times 0.5 \times \frac{-4}{15} = \frac{-4}{15}\, m/s \\ V_2 = 2 \times 0.5 \times \left(\frac{16}{15}\right) = \frac{16}{15}\, m/s \\ V_3 = 0 \end{cases} \quad and \quad \begin{cases} A_1 = 2 \times \frac{-4}{15} = \frac{-8}{15}\, m/s^2 \\ A_2 = 2 \times \left(\frac{16}{15}\right) = \frac{32}{15}\, m/s^2 \\ A_3 = 0 \end{cases}$$

3) The inverse of the equations of motion can be obtained as follows:

$$\begin{cases} x_1 = X_1 + t^2 X_2 \Rightarrow \ X_1 = x_1 - t^2 X_2 \\ x_2 = t^2 X_1 + X_2 \Rightarrow \ X_2 = x_2 - t^2 X_1 \\ x_3 = X_3 \Rightarrow X_3 = x_3 \end{cases} \quad \Rightarrow \quad \begin{cases} X_1 = \dfrac{x_1 - t^2 x_2}{1 - t^4} \\ X_2 = \dfrac{x_2 - t^2 x_1}{1 - t^4} \\ X_3 = x_3 \end{cases} \tag{2.79}$$

4) The velocity and acceleration of the particle that at time ($t = 0.5s$) is passing through the point $x_i = (1,0,1)$ can be obtained by means of velocity and acceleration in Eulerian description:

Velocity:

$$\begin{cases} V_1 = 2tX_2 \\ V_2 = 2X_1 t \\ V_3 = 0 \end{cases} \xrightarrow[X_1, X_2]{substituting} \begin{cases} v_1 = 2t\dfrac{x_2 - t^2 x_1}{1 - t^4} \\ v_2 = 2t\dfrac{x_1 - t^2 x_2}{1 - t^4} \\ v_3 = 0 \end{cases} \xrightarrow[x(1,0,1)]{t=0.5s} \begin{cases} v_1 = \dfrac{-4}{15} m/s \\ v_2 = \dfrac{16}{15} m/s \\ v_3 = 0 \end{cases}$$

Acceleration:

$$\begin{cases} A_1 = 2X_2 \\ A_2 = 2X_1 \\ A_3 = 0 \end{cases} \xrightarrow[X_1, X_2]{substituting} \begin{cases} a_1 = 2\dfrac{x_2 - t^2 x_1}{1 - t^4} \\ a_2 = 2\dfrac{x_1 - t^2 x_2}{1 - t^4} \\ a_3 = 0 \end{cases} \xrightarrow[x(1,0,1)]{t=0.5s} \begin{cases} a_1 = -\dfrac{8}{15} m/s^2 \\ a_2 = \dfrac{32}{15} m/s^2 \\ a_3 = 0 \end{cases}$$

We can obtain the initial position X_i of the particle by using the inverse of the equations of motion which is represented by the equations in (2.79), in which we consider $x_i(1,0,1)$:

$$\begin{cases} X_1 = \dfrac{x_1 - t^2 x_2}{1 - t^4} = \dfrac{1 - (0.5^2)(0)}{1 - (0.5)^4} = \dfrac{16}{15} \\ X_2 = \dfrac{x_2 - t^2 x_1}{1 - t^4} = \dfrac{0 - (0.5^2)(1)}{1 - (0.5)^4} = -\dfrac{4}{15} \\ X_3 = x_3 = 1 \end{cases}$$

We can verify that it is the same particle P referred to in paragraph 2. It is logical that we have obtained the same velocity and acceleration using either the material or spatial description, since the velocity and acceleration are intrinsic properties of the particle.

Problem 2.20

The acceleration vector field is described by:

$$\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t) = \frac{D\vec{\boldsymbol{v}}}{Dt} = \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + (\nabla_{\vec{x}} \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}}$$

Show that acceleration can also be written as:

$$\frac{D\vec{\boldsymbol{v}}}{Dt} = \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + \nabla_{\vec{x}}\left(\frac{v^2}{2}\right) - \vec{\boldsymbol{v}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}}) \equiv \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + \nabla_{\vec{x}}\left(\frac{v^2}{2}\right) - \vec{\boldsymbol{v}} \wedge \mathrm{rot}\, \vec{\boldsymbol{v}}$$

Solution:

To show the above relationship one need only demonstrate that:

$$(\nabla_{\vec{x}} \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}} = \nabla_{\vec{x}}\left(\frac{v^2}{2}\right) - \vec{\boldsymbol{v}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}})$$

Expressing the terms on the right of the equation in symbolic notation we can obtain:

$$\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{\boldsymbol{v}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{\boldsymbol{v}})=\frac{1}{2}\left[\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(v_j v_j)\right]-(v_i\hat{\mathbf{e}}_i)\wedge\left[\frac{\partial}{\partial x_r}\hat{\mathbf{e}}_r\wedge(v_s\hat{\mathbf{e}}_s)\right]$$

Using the definition of the permutation symbol, (see Chapter 1), we can express the vector product as:

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{\boldsymbol{v}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{\boldsymbol{v}}) &=\frac{1}{2}\left[\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(v_j v_j)\right]-(v_i\hat{\mathbf{e}}_i)\wedge\epsilon_{rst}\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_t\\ &=\frac{1}{2}\left[\hat{\mathbf{e}}_i 2v_j\frac{\partial v_j}{\partial x_i}\right]-\epsilon_{rst}\epsilon_{itk}v_i\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_k\end{aligned}$$

where we have used the equation $\hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_t=\epsilon_{itk}\hat{\mathbf{e}}_k$. In Chapter 1 we also proved that $\epsilon_{rst}\epsilon_{itk}=\epsilon_{rst}\epsilon_{kit}=\delta_{rk}\delta_{si}-\delta_{ri}\delta_{sk}$, then:

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{\boldsymbol{v}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{\boldsymbol{v}}) &=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-\left(\delta_{rk}\delta_{si}-\delta_{ri}\delta_{sk}\right)v_i\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_k\\ &=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-\left(\delta_{rk}\delta_{si}v_i\frac{\partial v_s}{\partial x_r}-\delta_{ri}\delta_{sk}v_i\frac{\partial v_s}{\partial x_r}\right)\hat{\mathbf{e}}_k\\ &=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-\left(v_s\frac{\partial v_s}{\partial x_k}-v_i\frac{\partial v_k}{\partial x_i}\right)\hat{\mathbf{e}}_k\end{aligned}$$

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{\boldsymbol{v}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{\boldsymbol{v}}) &=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_k}\hat{\mathbf{e}}_k+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\ &=\delta_{sj}v_s\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_k}\delta_{ik}\hat{\mathbf{e}}_i+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\ &=v_s\frac{\partial v_s}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_i}\hat{\mathbf{e}}_i+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\ &=\frac{\partial v_k\hat{\mathbf{e}}_k}{\partial x_i}v_i=\frac{\partial(\vec{\boldsymbol{v}})}{\partial x_i}v_i=\frac{\partial(\vec{\boldsymbol{v}})}{\partial x_i}\delta_{ij}v_j=\frac{\partial(\vec{\boldsymbol{v}})}{\partial x_i}(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j)v_j\\ &=\left(\frac{\partial(\vec{\boldsymbol{v}})}{\partial x_i}\hat{\mathbf{e}}_i\right)\cdot\hat{\mathbf{e}}_j v_j=(\nabla_{\vec{x}}\vec{\boldsymbol{v}})\cdot\vec{\boldsymbol{v}}\end{aligned}$$

NOTE: We have already discussed this problem in Chapter 1, (see **Problem 1.120**).

Problem 2.21

Consider the equations of motion $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$ and the temperature field $T(\vec{\boldsymbol{x}},t)$ given respectively by:

$$\begin{cases}x_1=X_1(1+t)\\ x_2=X_2(1+t)\\ x_3=X_3\end{cases}\quad\text{and}\quad T(\vec{\boldsymbol{x}})=x_1^2+x_2^2$$

Find the rate of change of temperature for the particle P at time $t=1s$ given that particle P was at point $(X_1=3, X_2=1, X_3=0)$ at time $t=0$.

Solution 1:

In this first solution we first obtain the material time derivative of the Lagrangian temperature, so, we have to obtain the temperature in Lagrangian description $T(\vec{\boldsymbol{X}},t)$ (Lagrangian temperature):

$$T(\vec{\boldsymbol{x}}) = x_1^2 + x_2^2$$
$$\downarrow$$
$$\textit{By substituting the equations of motion}$$
$$\downarrow$$
$$T(\vec{\boldsymbol{X}},t) = X_1^2(1+t)^2 + X_2^2(1+t)^2$$

The material time derivative of the Lagrangian temperature is given by:

$$\dot{T}(\vec{\boldsymbol{X}},t) \equiv \frac{DT}{Dt} = \frac{dT(\vec{\boldsymbol{X}},t)}{dt} = 2X_1^2(1+t) + 2X_2^2(1+t)$$

By substituting $t = 1s$, $(X_1 = 3, X_2 = 1, X_3 = 0)$, into the above equation we obtain:

$$\Rightarrow \dot{T}(\vec{\boldsymbol{X}},t) = 2X_1^2(1+t) + 2X_2^2(1+t) = 2(3)^2(1+1) + 2(1)^2(1+1) = 40\frac{K}{s}$$

Solution 2:

In this alternative solution we directly use the definition of material time derivative of the Eulerian variable, i.e. $\dot{T}(\vec{\boldsymbol{x}},t) = \dfrac{DT}{Dt} = \dfrac{\partial T(\vec{\boldsymbol{x}})}{\partial t} + \dfrac{\partial T(\vec{\boldsymbol{x}})}{\partial x_k} v_k(\vec{\boldsymbol{x}},t)$.

From the equations of motion we obtain:

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \xrightarrow{\textit{velocity}} \begin{cases} v_1(\vec{\boldsymbol{X}},t) = X_1 \\ v_2(\vec{\boldsymbol{X}},t) = X_2 \\ v_3(\vec{\boldsymbol{X}},t) = 0 \end{cases}$$

The equations of motion in Eulerian description are given by:

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \xrightarrow{\textit{inverse of motion}} \begin{cases} X_1 = \dfrac{x_1}{(1+t)} \\ X_2 = \dfrac{x_2}{(1+t)} \\ X_3 = x_3 \end{cases}$$

So, it is possible to obtain the Eulerian velocity as follows:

$$\begin{cases} V_1(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = X_1(\vec{\boldsymbol{x}},t) = \dfrac{x_1}{(1+t)} = v_1(\vec{\boldsymbol{x}},t) \\ V_2 = (\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = X_2(\vec{\boldsymbol{x}},t) = \dfrac{x_2}{(1+t)} = v_2(\vec{\boldsymbol{x}},t) \\ V_3 = v_3(\vec{\boldsymbol{x}},t) = 0 \end{cases}$$

Afterwards, the material time derivative of the Eulerian temperature, $T(\vec{\boldsymbol{x}},t)$, is given by:

$$\Rightarrow \frac{DT(\vec{\boldsymbol{x}},t)}{Dt} \equiv \dot{T}(\vec{\boldsymbol{x}},t) = \underbrace{\frac{\partial T(\vec{\boldsymbol{x}})}{\partial t}}_{=0\,(\text{Stationary field})} + \left[\frac{\partial T}{\partial x_1}v_1 + \frac{\partial T}{\partial x_2}v_2 + \frac{\partial T}{\partial x_3}v_3\right]$$

$$\Rightarrow \dot{T}(\vec{\boldsymbol{x}},t) = 2x_1\frac{x_1}{1+t} + 2x_2\frac{x_2}{1+t} + 0 \quad \Rightarrow \quad \dot{T}(\vec{\boldsymbol{x}},t) = \frac{2x_1^2}{1+t} + \frac{2x_2^2}{1+t} = \frac{2}{1+t}(x_1^2 + x_2^2)$$

The position of particle P at time $t = 1s$ is evaluated as follows:

$$\begin{cases} x_1 = X_1(1+t) = 3(1+1) = 6 \\ x_2 = X_2(1+t) = 1(1+1) = 2 \\ x_3 = X_3 = 0 \end{cases}$$

Then, by substituting the spatial coordinates in the expression of the material time derivative of temperature we obtain:

$$\dot{T}(\vec{x},t) = \dot{T}(x_1 = 6, x_2 = 2, x_3 = 0, t = 1) = \frac{2}{1+t}(x_1^2 + x_2^2) = \frac{2}{1+1}(6^2 + 2^2) = 40$$

Alternatively, the expression $\dot{T}(\vec{x},t)$ could also have been obtained as:

$$\dot{T}(\vec{X},t) = 2X_1^2(1+t) + 2X_2^2(1+t)$$

$$\dot{T}(\vec{X}(\vec{x},t),t) = 2[X_1(\vec{x},t)]^2(1+t) + 2[X_2(\vec{x},t)]^2(1+t) = 2\left[\frac{x_1}{(1+t)}\right]^2(1+t) + 2\left[\frac{x_2}{(1+t)}\right]^2(1+t)$$

$$= \frac{2}{(1+t)}(x_1^2 + x_2^2) = \dot{T}(\vec{x},t)$$

Problem 2.22

Consider the motion:

$$x_i = X_i(1+t) \qquad (t > 0)$$

Obtain the velocity field in the spatial description.

Solution:

The velocity is obtained by means of the material time derivative of the equations of motion:

$$V_i = \dot{x}_i = \frac{d}{dt}[X_i(1+t)] = X_i \tag{2.80}$$

To find the velocity in the spatial description we will need to obtain the inverse of the equations of motion which is

$$x_i = X_i(1+t) \Rightarrow X_i = \frac{x_i}{(1+t)}$$

and by substituting into the equation (2.80) we obtain the Eulerian velocity:

$$v_i = X_i(\vec{x},t) = \frac{x_i}{1+t}$$

Problem 2.23

The equations of motion and the temperature field $T(\vec{x})$ are given respectively by:

$$x_i = X_i(1+t) \qquad (i = 1,2) \qquad ; \qquad T(\vec{x}) = 2(x_1^2 + x_2^2)$$

Find the rate of change of temperature at time $t = 1s$ for one particle that was at position $(1,1)$ in the reference configuration.

Note that the temperature field is a steady field, i.e. $T = T(\vec{x})$.

Solution 1:

We can obtain the equation for temperature in the material description:

$$\begin{cases} T(\vec{\boldsymbol{x}}) = 2(x_1^2 + x_2^2) \\ \quad\downarrow \\ \text{by substituting the equations of motion} \\ \quad\downarrow \\ T(\vec{\boldsymbol{X}},t) = 2\left[X_1^2(1+t)^2 + X_2^2(1+t)^2\right] \end{cases}$$

Then, the material time derivative can be obtained as follows:

$$\Rightarrow \dot{T}(\vec{\boldsymbol{X}},t) = \frac{DT}{Dt} = \frac{dT(\vec{\boldsymbol{X}},t)}{dt} = 2\left[2X_1^2(1+t) + 2X_2^2(1+t)\right]$$

By substituting $t = 1s$ and the material coordinates $(X_1 = 1; X_2 = 1)$ into the above equation we can obtain:

$$\Rightarrow \dot{T}(X_1 = 1; X_2 = 1; t = 1) = 16\frac{K}{s}$$

Solution 2:

In this alternative solution we directly use the definition of the material time derivative of Eulerian property:

$$T(\vec{\boldsymbol{x}}) = 2(x_1^2 + x_1^2) \quad ; \quad x_i = (1+t)X_i \qquad (i = 1,2)$$

$$\Rightarrow \dot{T}(\vec{\boldsymbol{x}},t) = \frac{DT}{Dt} = \frac{\partial T(\vec{\boldsymbol{x}})}{\partial t} + \frac{\partial T(\vec{\boldsymbol{x}})}{\partial x_k}\frac{\partial x_k}{\partial t} \qquad (i = 1,2)$$

Note that $T(\vec{\boldsymbol{x}})$ is not a function of time, so $\frac{\partial T(\vec{\boldsymbol{x}})}{\partial t} = 0$:

$$\Rightarrow \dot{T}(\vec{\boldsymbol{x}},t) = \frac{\partial T(\vec{\boldsymbol{x}})}{\partial x_k}\frac{\partial x_k}{\partial t} = \frac{\partial T}{\partial x_1}\underbrace{\frac{\partial x_1}{\partial t}}_{V_1 = X_1} + \frac{\partial T}{\partial x_2}\underbrace{\frac{\partial x_2}{\partial t}}_{V_2 = X_2}$$

$$\Rightarrow \dot{T}(\vec{\boldsymbol{x}},t) = 4x_1\frac{x_1}{1+t} + 4x_2\frac{x_2}{1+t} \quad \Rightarrow \quad \dot{T}(\vec{\boldsymbol{x}},t) = \frac{4x_1^2}{1+t} + \frac{4x_2^2}{1+t}$$

The particle that at reference configuration was at position $(1,1)$, at time $t = 1s$ will be at position $x_i = (1+t)X_i = 2X_i$, i.e. $(x_1 = 2; x_2 = 2)$:

$$\dot{T}(x_1 = 2; x_2 = 2; t = 1) = \frac{4(2)^2}{1+1} + \frac{4(2)^2}{1+1} = 16\frac{K}{s}$$

Problem 2.24

Consider the equations of motion:

$$\begin{cases} x_1 = X_1 \exp^t + X_3(\exp^t - 1) \\ x_2 = X_2 + X_3(\exp^t - \exp^{-t}) \\ x_3 = X_3 \end{cases}$$

Obtain the velocity and acceleration components in Lagrangian and Eulerian descriptions.

Solution:

First we obtain the inverse of the equations of motion:

$$\begin{cases} x_1 = X_1 \exp^t + X_3(\exp^t - 1) \\ x_2 = X_2 + X_3(\exp^t - \exp^{-t}) \\ x_3 = X_3 \Rightarrow \quad X_3 = x_3 \end{cases} \Rightarrow \quad \begin{cases} x_1 - X_1 \exp^t = x_3(\exp^t - 1) \\ x_2 - X_2 = x_3(\exp^t - \exp^{-t}) \\ x_3 = X_3 \Rightarrow \quad X_3 = x_3 \end{cases}$$

thus:

$$\begin{cases} X_1 = x_1 \exp^{-t} - \exp^{-t}(\exp^t - 1) \\ X_2 = x_2 - x_3(\exp^{2t} - 1)\exp^{-t} \\ X_3 = x_3 \end{cases} \tag{2.81}$$

or

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \exp^t & 0 & (\exp^t - 1) \\ 0 & 1 & (\exp^t - \exp^{-t}) \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \xrightarrow{inverse} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \exp^{-t} & 0 & -\exp^{-t}(\exp^t - 1) \\ 0 & 1 & -(\exp^{2t} - 1)\exp^{-t} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

a) The velocity components in the material description are given by:

$$V_i = \frac{D}{Dt} x_j(\vec{\boldsymbol{X}}, t) \longrightarrow \begin{cases} V_1 = X_1 \exp^t + X_3 \exp^t \\ V_2 = X_3 \exp^t + X_3 \exp^{-t} = X_3(\exp^t + \exp^{-t}) \\ V_3 = 0 \end{cases} \tag{2.82}$$

b) The acceleration components in the material description are given by:

$$A_i(\vec{\boldsymbol{X}}, t) = \frac{DV_i(\vec{\boldsymbol{X}}, t)}{Dt} \longrightarrow \begin{cases} A_1 = X_1 \exp^t + X_3 \exp^t \\ A_2 = X_3(\exp^t - \exp^{-t}) \\ A_3 = 0 \end{cases} \tag{2.83}$$

To obtain the velocity and acceleration in the spatial description it is sufficient to replace the values of X_1, X_2, X_3, given by the equation (2.81), into the equations (2.82) and (2.83), thus, we can obtain:

$$\underbrace{\begin{cases} v_1 = x_1 + x_3 \\ v_2 = x_3(\exp^t + \exp^{-t}) \\ v_3 = 0 \end{cases}}_{\text{Velocity in the spatial description}} \quad ; \quad \underbrace{\begin{cases} a_1 = x_1 + x_3 \\ a_2 = x_3(\exp^t - \exp^{-t}) \\ a_3 = 0 \end{cases}}_{\text{Acceleration in the spatial description}}$$

Problem 2.25

The motion of the continuum, $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$, is given by the following equations:

$$\begin{cases} x_1 = \frac{1}{2}(X_1 + X_2)\exp^t + \frac{1}{2}(X_1 - X_2)\exp^{-t} \\ x_2 = \frac{1}{2}(X_1 + X_2)\exp^t - \frac{1}{2}(X_1 - X_2)\exp^{-t} \\ x_3 = X_3 \end{cases}$$

$$0 \leq t \leq \text{constant}$$

Express the velocity components in the material and spatial descriptions.

Solution:

The velocity components using material description are:

$$\begin{cases} V_1 = \dfrac{Dx_1(\vec{\boldsymbol{X}},t)}{Dt} = \dfrac{1}{2}(X_1 + X_2)\exp^{t} - \dfrac{1}{2}(X_1 - X_2)\exp^{-t} \\ V_2 = \dfrac{Dx_2(\vec{\boldsymbol{X}},t)}{Dt} = \dfrac{1}{2}(X_1 + X_2)\exp^{t} + \dfrac{1}{2}(X_1 - X_2)\exp^{-t} \\ V_3 = 0 \end{cases} \tag{2.84}$$

To express the velocity components in the spatial description we will need the inverse of the equations of motion, i.e. we will need to find $\vec{\boldsymbol{X}} = \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t)$:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} (\exp^{t} + \exp^{-t}) & (\exp^{t} - \exp^{-t}) & 0 \\ (\exp^{t} - \exp^{-t}) & (\exp^{t} + \exp^{-t}) & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$\xrightarrow{inverse} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} (\exp^{2t} + 1)\exp^{-t} & -(\exp^{2t} - 1)\exp^{-t} & 0 \\ -(\exp^{2t} - 1)\exp^{-t} & (\exp^{2t} + 1)\exp^{-t} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Then, to obtain the Eulerian velocity we substitute the above equations into the Lagrangian velocity (2.84), with which we can obtain:

$$v_1 = x_2 \quad ; \quad v_2 = x_1 \quad ; \quad v_3 = 0$$

Problem 2.26

Given the motion:

$$x_i = (X_1 + ktX_2)\delta_{i1} + X_2\delta_{i2} + X_3\delta_{i3} \qquad (i = 1,2,3)$$

and the temperature field $T(\vec{\boldsymbol{x}}) = x_1 + x_2$.

Obtain the rate of change of T of a particle that in the current configuration is located at the point $(1,1,1)$.

Solution:

Considering the equations of motion:

$$x_1 = X_1 + ktX_2 \quad ; \quad x_2 = X_2 \quad ; \quad x_3 = X_3$$

and by substituting the values of x_i into the temperature field $T(\vec{\boldsymbol{x}},t)$, we can obtain the temperature field in the material description $T(\vec{\boldsymbol{X}},t)$:

$$T(\vec{\boldsymbol{x}}) = x_1 + x_2 \quad \Rightarrow \quad T(\vec{\boldsymbol{X}},t) = X_1 + ktX_2 + X_2$$

The material time derivative is given by:

$$\dot{T}(\vec{\boldsymbol{X}},t) = \frac{DT(\vec{\boldsymbol{X}},t)}{Dt} = \frac{D(X_1 + ktX_2 + X_2)}{Dt} = kX_2 = k\,x_2 \xrightarrow{(1,1,1)} \dot{T} = k$$

Alternative solution:

The material time derivative for a property expressed in the spatial description is given by:

$$\dot{T}(x_1,x_2,x_3,t)=\frac{DT(\vec{\boldsymbol{x}},t)}{Dt}=\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_k}\frac{\partial x_k(\vec{\boldsymbol{X}},t)}{\partial t}$$

Considering $T(\vec{\boldsymbol{x}})=x_1+x_2$, we can obtain:

$$\dot{T}(x_1,x_2,x_3,t)=\underbrace{\frac{\partial T}{\partial t}}_{=0}+\left(\frac{\partial T}{\partial x_1}\frac{\partial x_1}{\partial t}+\frac{\partial T}{\partial x_2}\underbrace{\frac{\partial x_2}{\partial t}}_{=0}+\underbrace{\frac{\partial T}{\partial x_3}}_{=0}\underbrace{\frac{\partial x_3}{\partial t}}_{=0}\right)$$

$$\Rightarrow \dot{T}(x_1,x_2,x_3,t)=kX_2$$

we obtain the inverse equations of motion:

$$\begin{cases} x_1 = X_1+ktX_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases} \xrightarrow{inverse} \begin{cases} X_1 = x_1-ktx_2 \\ X_2 = x_2 \\ X_3 = x_3 \end{cases}$$

With that the equation $\dot{T}(\vec{\boldsymbol{X}})=kX_2$ can be expressed as follows:

$$\Rightarrow \dot{T}(x_1,x_2,x_3,t)=kX_2=kx_2$$

For the particle in the current configuration at the position $(1,1,1)$ we have:

$$\dot{T}(x_1=1,x_2=1,x_3=1,t)=k$$

Problem 2.27

Given a steady velocity field: it asks readers to give their opinion on whether particle velocities are constant or not. If not, in which situation is met. Justify the answer.

Solution:

A field $\phi(\vec{\boldsymbol{x}},t)$ is said to be steady if the local rate of change does not vary over time, so:

$$\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial t}=\boldsymbol{0} \quad \Rightarrow \quad \phi=\phi(\vec{\boldsymbol{x}}) \quad \textit{Steady state (stationary) field} \tag{2.85}$$

For example, let us consider a stationary (steady state) velocity field as shown in Figure 2.6. Then, as we can verify, the field representation for any time, e.g. t_1 and t_2, does not change. However, that does not mean that the velocities of the particles do not change over time. In light of Figure 2.6, we can now focus our attention on the fixed spatial point $\vec{\boldsymbol{x}}^*$. At time t_1 the particle Q is passing through point $\vec{\boldsymbol{x}}^*$ with velocity $\vec{\boldsymbol{v}}^*$. Let us also consider another particle P, which is passing through another point with velocity $\vec{\boldsymbol{v}}^P(t_1)\neq\vec{\boldsymbol{v}}^*$. At time t_2 the particle P is now passing through the point $\vec{\boldsymbol{x}}^*$. It follows that if we are dealing with a steady state velocity field, then the velocity of particle P at $\vec{\boldsymbol{x}}^*$ must be $\vec{\boldsymbol{v}}^*$, i.e. $\vec{\boldsymbol{v}}^P(t_2)=\vec{\boldsymbol{v}}^*$. We can easily contrast this with the material time derivative of velocity, which is always associated with the same particle, i.e.:

$$\frac{D\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{Dt}\equiv\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t)=\underbrace{\frac{\partial\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t}}_{=\vec{\boldsymbol{0}}\text{(Stationay)}}+(\nabla_{\vec{x}}\vec{\boldsymbol{v}})\cdot\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})=(\nabla_{\vec{x}}\vec{\boldsymbol{v}})\cdot\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})=\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}) \tag{2.86}$$

The rate of change of velocity (acceleration) will be zero if the velocity field is stationary $\left(\frac{\partial\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t}=\vec{\boldsymbol{0}}\right)$ and homogeneous ($\nabla_{\vec{x}}\vec{\boldsymbol{v}}=\boldsymbol{0}$).

We can also verify that, although spatial velocity is independent of time, that does not mean material velocity is also, since:

$$\vec{v}(\vec{x}) = \vec{v}(\vec{x}(\vec{X},t)) = \vec{v}(\vec{X},t) \tag{2.87}$$

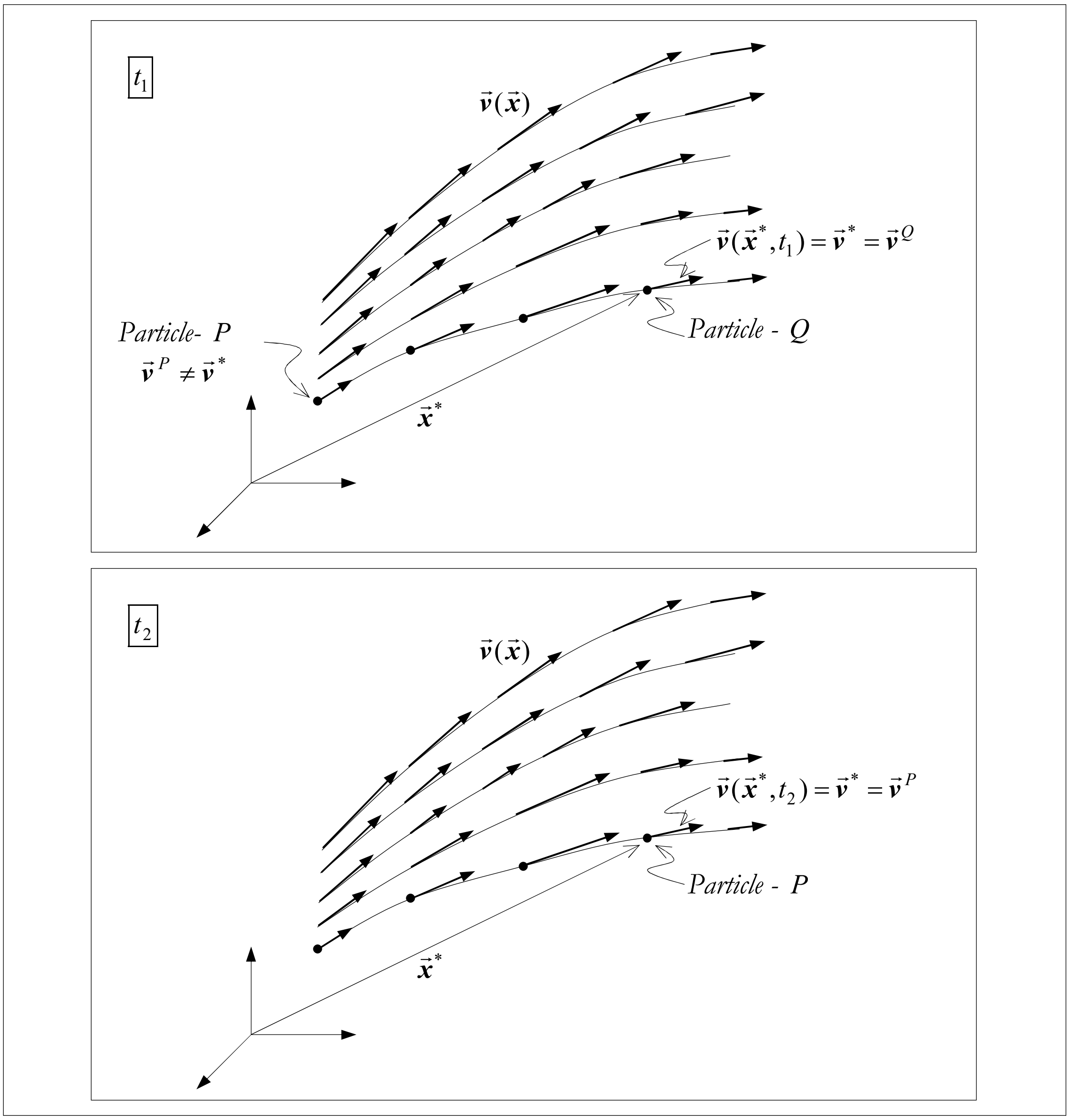


Figure 2.6: Steady velocity field.

2.2 Deformation Gradient, Deformation/strain Tensors, Homogeneous Deformation

Problem 2.28

A rod, which can be considered as a one-dimensional solid, undergoes a uniform stretching which is given by $\lambda = \exp^{at}$ where $a = constant$.

a) Obtain the equations of motion $\vec{x} = \vec{x}(\vec{X},t)$;

b) Obtain the rate-of-deformation tensor components, i.e. $\mathbf{D}$ -components.

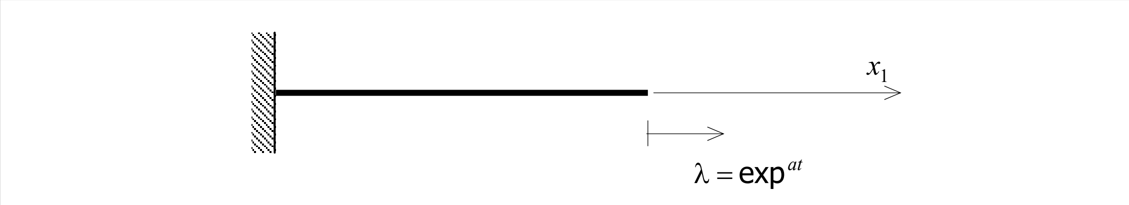


Figure 2.7

Solution:

Using the $1D$ approaching we have:

$$\lambda = \frac{ds}{dS} = \frac{dx}{dX} = \exp^{at} \Rightarrow dx = \exp^{at} dX \tag{2.88}$$

$$\int dx = \int \exp^{at} dX \xrightarrow{\text{Integrating}} x_1 = \exp^{at} X_1 + C \tag{2.89}$$

at $t = 0 \Rightarrow x = X$, thus

$$x = \exp^{0} X_1 + C \Rightarrow X = X + C \Rightarrow C = 0 \tag{2.90}$$

with that we can obtain the equations of motion:

$$x_1 = \exp^{at} X_1 \quad ; \quad x_2 = X_2 \quad ; \quad x_3 = X_3 \tag{2.91}$$

The velocity field components become:

$$v_1 = \frac{dx_1}{dt} = a\, X_1 \exp^{at} = a\, x_1 \quad ; \quad v_2 = 0 \quad ; \quad v_3 = 0 \tag{2.92}$$

And the rate-of-deformation tensor components can be obtained as follows:

$$\mathsf{D}_{ij} = \frac{1}{2}\left(\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_j} + \frac{\partial v_j(\vec{\boldsymbol{x}},t)}{\partial x_i} \right) \quad \Rightarrow \quad \mathsf{D}_{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.93}$$

Problem 2.29

Consider the equations of motion:

$$x_1 = X_1 + 2X_3 \quad ; \quad x_2 = X_2 - 2X_3 \quad ; \quad x_3 = X_3 - 2X_1 + 2X_2$$

Obtain the Green-Lagrange strain tensor components, i.e. $\boldsymbol{E}$-components.

Solution 1:

The displacement field components are given by

$$\begin{cases} \mathsf{u}_1 = x_1 - X_1 = 2X_3 \\ \mathsf{u}_2 = x_2 - X_2 = -2X_3 \\ \mathsf{u}_3 = x_3 - X_3 = -2X_1 + 2X_2 \end{cases}$$

The Green-Lagrange strain tensor can be expressed in function of Lagrangian displacement as follows:

$$
\begin{aligned}
E_{ij} &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i} + \frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) \\
&= \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) = \left(\frac{\partial \mathrm{u}_i}{\partial X_j}\right)^{sym} + \frac{1}{2}\left(\frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right)
\end{aligned} \tag{2.94}
$$

where the material (Lagrangian) displacement gradient is given by:

$$
\frac{\partial \mathrm{u}_i}{\partial X_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial X_1} & \frac{\partial \mathrm{u}_1}{\partial X_2} & \frac{\partial \mathrm{u}_1}{\partial X_3} \\ \frac{\partial \mathrm{u}_2}{\partial X_1} & \frac{\partial \mathrm{u}_2}{\partial X_2} & \frac{\partial \mathrm{u}_2}{\partial X_3} \\ \frac{\partial \mathrm{u}_3}{\partial X_1} & \frac{\partial \mathrm{u}_3}{\partial X_2} & \frac{\partial \mathrm{u}_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}
$$

Note that, for this case, the displacement gradient is an antisymmetric tensor. That is, the symmetric part is the null tensor. Then, the equation in (2.94) becomes:

$$
E_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) = \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}
$$

Solution 2:

We can directly apply the definition $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1})$, $E_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right)$, where:

$$
F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}
$$

Thus

$$
E_{ij} = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}
$$

Problem 2.30

Consider a homogeneous transformation defined by the following equations:

$$
x_1 = X_1 + 2X_2 + X_3 \quad ; \quad x_2 = 2X_2 \quad ; \quad x_3 = X_1 + 2X_3 \tag{2.95}
$$

Show that, for a homogeneous transformation, vectors whose are parallel in the reference configuration remain parallel after deformation.

For the demonstration consider two vectors defined by the vector position of two particles A and B in the reference configuration:

$$\vec{X}^A = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \quad ; \quad \vec{X}^B = 2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \tag{2.96}$$

Solution:

The vector connecting the two particles in the reference configuration is given by:

$$\vec{\mathbf{V}} = \vec{B} - \vec{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \tag{2.97}$$

and the deformation gradient is:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{2.98}$$

We can obtain the vector position of the particle in the current configuration by means of:

$$d\vec{x} = \boldsymbol{F} \cdot d\vec{X} \quad \Rightarrow \text{Homogeneous transformation} \Rightarrow \quad \vec{x} = \boldsymbol{F} \cdot \vec{X} \tag{2.99}$$

thus,

$$x_i^A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad ; \quad x_i^B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix} \tag{2.100}$$

and the vector that connect these two points is:

$$\vec{\mathbf{v}} = \vec{x}^B - \vec{x}^A = 4\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3 \tag{2.101}$$

then any vector parallel to $\vec{\mathbf{V}}$, for example the vector $2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3$, after transformation becomes: $8\hat{\mathbf{e}}_1 + 4\hat{\mathbf{e}}_2 + 6\hat{\mathbf{e}}_3$, which is parallel to $\vec{\mathbf{v}}$. Note that, since we are dealing with homogeneous deformation the equation $\vec{\mathbf{v}} = \boldsymbol{F} \cdot \vec{\mathbf{V}}$ is valid, i.e.:

$$\mathrm{v}_i = F_{ij}\mathrm{V}_i \quad \Rightarrow \quad \mathrm{v}_i = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Problem 2.31

Consider a pure shear deformation represented by homogenous deformation:

$$\vec{x} = \vec{X} + ktX_2\hat{\mathbf{e}}_1 \tag{2.102}$$

where $\hat{\mathbf{e}}_i$ is the Cartesian basis, and the components of the above equation are:

$$x_1 = X_1 + ktX_2 \quad ; \quad x_2 = X_2 \quad ; \quad x_3 = X_3 \tag{2.103}$$

Obtain the new geometry (deformed configuration) for the body (rectangle) described in Figure 2.8.

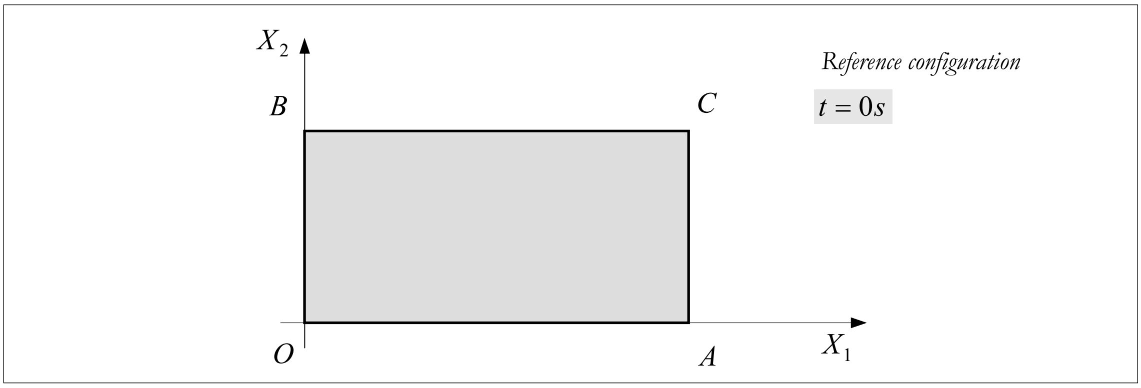


Figure 2.8

Solution:

The deformation gradient components are:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & kt & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.104}$$

Note that this is a case of homogenous deformation, i.e. $\vec{\boldsymbol{x}} = \boldsymbol{F} \cdot \vec{\boldsymbol{X}} + \vec{\mathbf{c}}$ with $\vec{\mathbf{c}} = \vec{\mathbf{0}}$.

The Jacobian determinant:

$$J = |\boldsymbol{F}| = 1 \tag{2.105}$$

Since $J = 1$ there is no dilatancy (variation of volume).

The particles lying on the BC-line, coordinates $(X_1, X_2, 0)$, in the current configuration will become:

$$x_1^{(BC)} = X_1 + k\,t\,X_2 \quad ; \quad x_2^{(BC)} = X_2 \quad ; \quad x_3^{(BC)} = 0 \tag{2.106}$$

The particles lying on the OA-line, coordinates $(X_1, 0, 0)$, in the current configuration assume the position:

$$x_1^{(OA)} = X_1 \quad ; \quad x_2^{(OA)} = 0 \quad ; \quad x_3^{(OA)} = 0 \tag{2.107}$$

then, the OA-line does not change its position during motion, (see Figure 2.9).

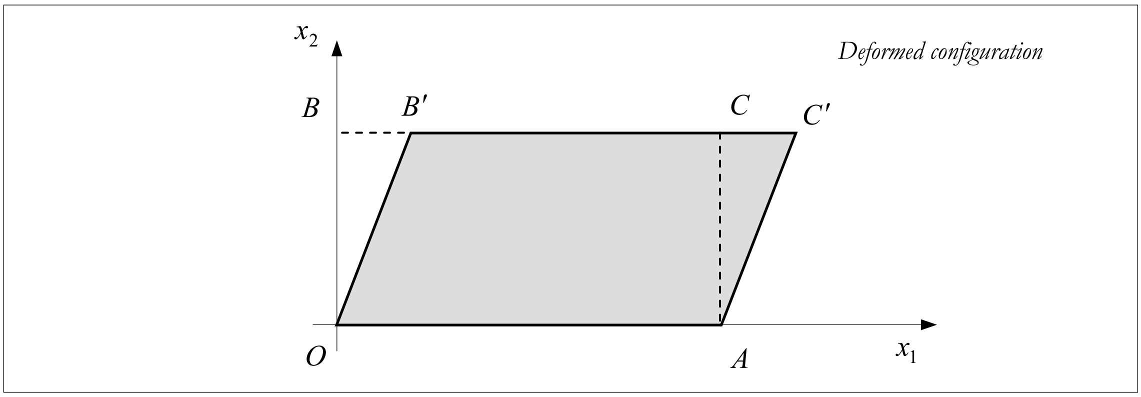


Figure 2.9

Problem 2.32

Consider the equations of motion:

$$x_1 = X_1 + \frac{\sqrt{2}}{2} X_2 \quad ; \quad x_2 = \frac{\sqrt{2}}{2} X_1 + X_2 \quad ; \quad x_3 = X_3 \tag{2.108}$$

a) Show that this deformation is characterized by a homogeneous transformation;

b) Obtain the displacement field components in material and spatial descriptions;

c) Consider the particles located according to the equation:

$$X_1^2 + X_2^2 = 2 \quad ; \quad X_3 = 0$$

Obtain the new configuration of these particles in the current configuration;

d) Obtain the right Cauchy-Green deformation tensor components ($\boldsymbol{C}$) and the Green-Lagrange strain tensor ($\boldsymbol{E}$).

e) Obtain the principal values of $\boldsymbol{C}$ and $\boldsymbol{E}$.

Solution:

a) The equation of a homogeneous deformation is described by $x_i = F_{ij} X_j$, where

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.109}$$

Note that $\boldsymbol{F}$ is independent of $\vec{\boldsymbol{x}}$, so, $\boldsymbol{F}$ is a homogeneous transformation, and the equation $x_i = F_{ij} X_j$ is in accordance with (2.108):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \Leftrightarrow \quad x_i = F_{ij} X_j \tag{2.110}$$

And the inverse of (2.110) is represented by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} X_1 = 2x_1 - \sqrt{2}\, x_2 \\ X_2 = -\sqrt{2}\, x_1 + 2\, x_2 \\ X_3 = x_3 \end{cases} \tag{2.111}$$

b) The Lagrangian displacement field is given by:

$$\vec{\mathbf{u}}(\vec{X},t)=\vec{x}(\vec{X},t)-\vec{X}\Rightarrow\begin{cases}\mathrm{u}_1(\vec{X},t)=x_1(\vec{X},t)-X_1=X_1+\dfrac{\sqrt{2}}{2}X_2-X_1=\dfrac{\sqrt{2}}{2}X_2\\ \mathrm{u}_2(\vec{X},t)=x_2(\vec{X},t)-X_2=\dfrac{\sqrt{2}}{2}X_1+X_2-X_2=\dfrac{\sqrt{2}}{2}X_1\\ \mathrm{u}_3(\vec{X},t)=x_3(\vec{X},t)-X_3=0\end{cases} \tag{2.112}$$

which, in spatial coordinates, becomes:

$$\begin{aligned}&\mathrm{u}_1(\vec{x},t)=x_1-X_1(\vec{x},t)=x_1-(2x_1-\sqrt{2}\,x_2)=-x_1+\sqrt{2}\,x_2\\&\mathrm{u}_2(\vec{x},t)=x_2-X_2(\vec{x},t)=x_2-(-\sqrt{2}\,x_1+2x_2)=\sqrt{2}\,x_1-x_2\\&\mathrm{u}_3(\vec{x},t)=x_3-X_3(\vec{x},t)=x_3-x_3=0\end{aligned} \tag{2.113}$$

c) The particles describing the circle $X_1^2+X_2^2=2$ in the reference configuration, in the current configuration becomes:

$$\left(2x_1-\sqrt{2}\,x_2\right)^2+\left(-\sqrt{2}\,x_1+2\,x_2\right)^2=2 \tag{2.114}$$

which is the same as:

$$3x_1^2+3x_2^2-4\sqrt{2}\,x_1x_2=1 \text{ (an ellipse equation)} \tag{2.115}$$

The curve made up by the same particle during motion is called *material curve*. The material curve for this example is described in Figure 2.10.

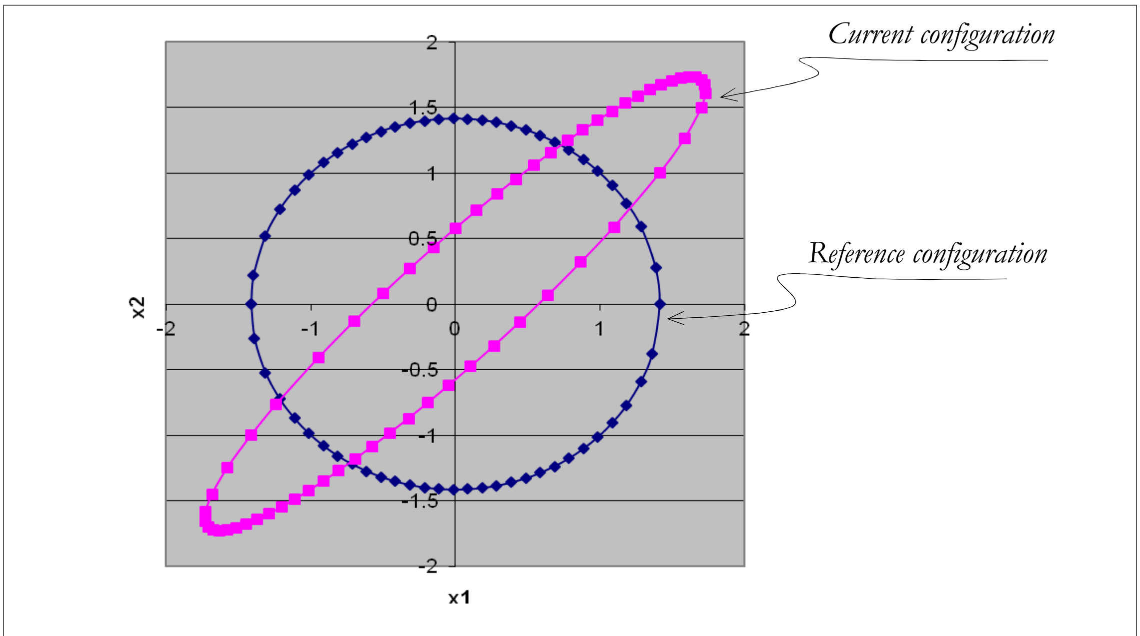


Figure 2.10: Material curves.

d) The right Cauchy-Green deformation tensor and the Green-Lagrange strain tensor are given, respectively, by:

$$\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F} \quad ; \quad \boldsymbol{E}=\frac{1}{2}(\boldsymbol{C}-\mathbf{1}) \tag{2.116}$$

Then, the $\boldsymbol{C}$-components are:

$$C_{ij}=\begin{bmatrix}1 & \frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^T\begin{bmatrix}1 & \frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{3}{2} & \sqrt{2} & 0\\ \sqrt{2} & \frac{3}{2} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{2.117}$$

And the eigenvalues of (2.117) can be evaluated by means of $|\boldsymbol{C}-\lambda\mathbf{1}|=0$, in which the result is:

$$C_1=\frac{3}{2}+\sqrt{2}\approx 2.914 \quad ; \quad C_2=\frac{3}{2}-\sqrt{2}\approx 0.086 \quad ; \quad C_3=1 \tag{2.118}$$

The $\boldsymbol{E}$-components are:

$$E_{ij}=\frac{1}{2}\left(C_{ij}-\delta_{ij}\right)=\frac{1}{2}\left(\begin{bmatrix}\frac{3}{2} & \sqrt{2} & 0\\ \sqrt{2} & \frac{3}{2} & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=\frac{1}{4}\begin{bmatrix}1 & 2\sqrt{2} & 0\\ 2\sqrt{2} & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{2.119}$$

The eigenvalues of $\boldsymbol{E}$ can be obtained by means of $|\boldsymbol{E}-\lambda\mathbf{1}|=0$. Since $E_{33}=E_3=0$ is already an eigenvalue, then in order to obtain the remaining eigenvalues we just need to solve:

$$\begin{vmatrix}\frac{1}{4}-\lambda & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{1}{4}-\lambda\end{vmatrix}=0 \quad \Rightarrow \quad \lambda^2-\frac{\lambda}{2}-\frac{7}{16}=0 \Rightarrow \begin{cases}\lambda_1=\dfrac{1+2\sqrt{2}}{4}\\ \lambda_2=\dfrac{1-2\sqrt{2}}{4}\end{cases} \tag{2.120}$$

Then, the three eigenvalues of $\boldsymbol{E}$ are:

$$E_1=\frac{1+2\sqrt{2}}{4}\approx 0.957 \quad ; \quad E_2=\frac{1-2\sqrt{2}}{4}\approx -0.457 \quad ; \quad E_3=0 \tag{2.121}$$

Problem 2.33

Let us consider the following equations of motion:

$$x_1=X_1+\frac{1}{2}X_2 \quad ; \quad x_2=\frac{1}{2}X_1+X_2 \quad ; \quad x_3=X_3 \tag{2.122}$$

a) Obtain the displacement field ($\vec{\mathbf{u}}$) in the Lagrangian and Eulerian descriptions;

b) Determine the material curve in the current configuration for a material circle defined in the reference configuration as:

$$X_1^2+X_2^2=2 \quad ; \quad X_3=0$$

c) Obtain the components of the right Cauchy-Green deformation tensor and the Green-Lagrange strain tensor;

d) Obtain the principal stretches.

Solution:

The deformation gradient is given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad ; \quad J = |\boldsymbol{F}| = 0.75$$

Note that $\boldsymbol{F}$ is independent of $\vec{\boldsymbol{x}}$, so, $\boldsymbol{F}$ is a homogeneous transformation. And by comparing this with the equations of motion in (2.122) we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \Leftrightarrow \quad x_i = F_{ij} X_j$$

So, we can verify that the proposed example is a case of homogeneous deformation in which $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. The inverse form of the above equation is given by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} X_1 = \frac{4}{3}x_1 - \frac{2}{3}x_2 \\ X_2 = -\frac{2}{3}x_1 + \frac{4}{3}x_2 \\ X_3 = x_3 \end{cases} \tag{2.123}$$

The displacement field is defined by $\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}$, after which the components of the Lagrangian displacement become:

$$\mathsf{u}_i(\vec{\boldsymbol{X}},t) = x_i(\vec{\boldsymbol{X}},t) - X_i \Rightarrow \begin{cases} \mathsf{u}_1(\vec{\boldsymbol{X}},t) = x_1(\vec{\boldsymbol{X}},t) - X_1 = X_1 + \frac{1}{2}X_2 - X_1 = \frac{1}{2}X_2 \\ \mathsf{u}_2(\vec{\boldsymbol{X}},t) = x_2(\vec{\boldsymbol{X}},t) - X_2 = \frac{1}{2}X_1 + X_2 - X_2 = \frac{1}{2}X_1 \\ \mathsf{u}_3(\vec{\boldsymbol{X}},t) = x_3(\vec{\boldsymbol{X}},t) - X_3 = 0 \end{cases} \tag{2.124}$$

The components of the Eulerian displacement can be obtained by substituting the Eulerian description of motion (2.123) into (2.124), the result of which is:

$$\begin{cases} \mathsf{u}_1(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = \frac{1}{2}X_2(\vec{\boldsymbol{x}},t) = \frac{1}{2}\left[-\frac{2}{3}x_1 + \frac{4}{3}x_2\right] = \mathsf{u}_1(\vec{\boldsymbol{x}},t) \\ \mathsf{u}_2(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = \frac{1}{2}X_1(\vec{\boldsymbol{x}},t) = \frac{1}{2}\left[-\frac{2}{3}x_1 + \frac{4}{3}x_2\right] = \mathsf{u}_2(\vec{\boldsymbol{x}},t) \\ \mathsf{u}_3(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = \mathsf{u}_3(\vec{\boldsymbol{x}},t) = 0 \end{cases} \tag{2.125}$$

The particles belonging to the circle $X_1^2 + X_2^2 = 2$ in the reference configuration will form a new curve in the current configuration which is defined by:

$$X_1^2 + X_2^2 = 2 \Rightarrow \left[\frac{4}{3}x_1 - \frac{2}{3}x_2\right]^2 + \left[-\frac{2}{3}x_1 + \frac{4}{3}x_2\right]^2 = 2 \Rightarrow 20x_1^2 - 32x_1x_2 + 20x_2^2 = 18$$

which is an ellipse equation (Figure 2.10 shows the material curve in different configurations).

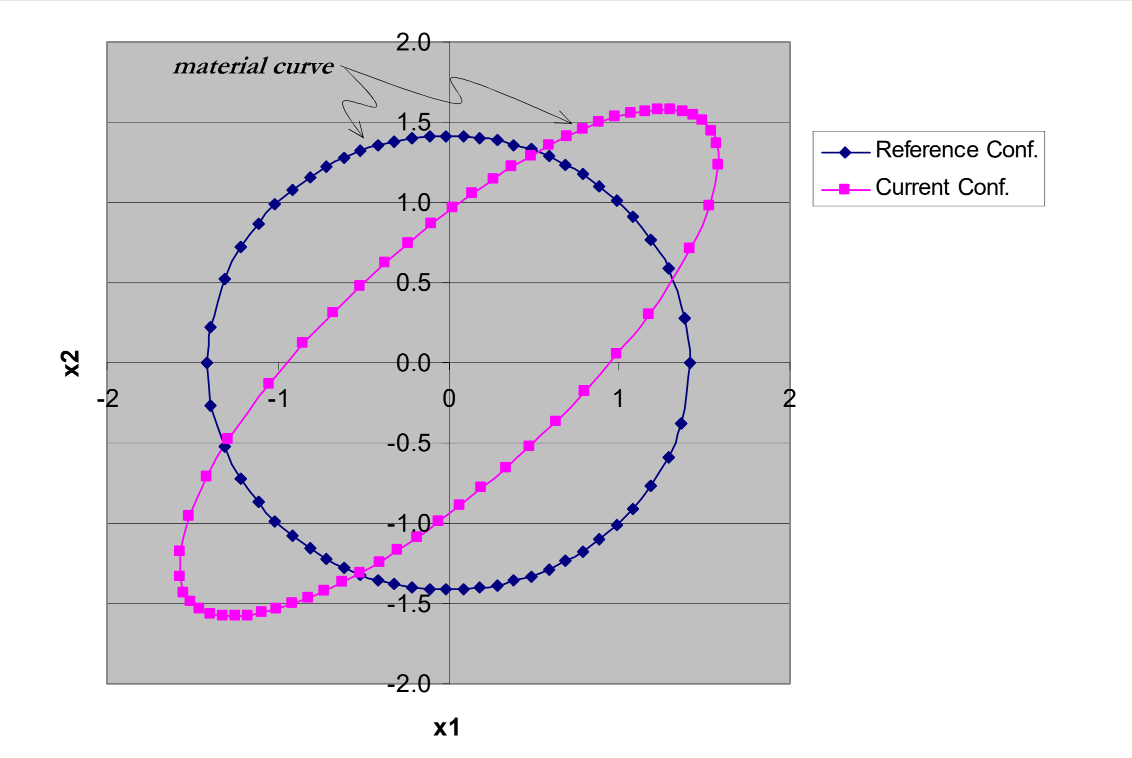


Figure 2.11: Material curve.

The components of $\boldsymbol{C}$ and $\boldsymbol{E}$ can be obtained by using the definitions $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$:

$$C_{ij} = F_{ki}F_{kj} \quad \Rightarrow \quad C_{ij} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1.25 & 1 & 0 \\ 1 & 1.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{ij} = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right) \quad \Rightarrow \quad E_{ij} = \frac{1}{2}\left(\begin{bmatrix} 1.25 & 1 & 0 \\ 1 & 1.25 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0.125 & 0.5 & 0 \\ 0.5 & 0.125 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the principal space of $\boldsymbol{C}$ its components are given by:

$$C'_{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad \Rightarrow \quad \sqrt{C'_{ij}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where λ_i show the principal stretches. Therefore, to calculate these we need to obtain the $\boldsymbol{C}$ eigenvalues:

$$\begin{vmatrix} 1.25 - C & 1 \\ 1 & 1.25 - C \end{vmatrix} = 0 \Rightarrow C^2 - 2.5C + 0.5625 = 0 \Rightarrow \begin{cases} C_1 = 2.25 \\ C_2 = 0.25 \end{cases}$$

$$C'_{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 2.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2.34

Show that

$$\nabla_{\vec{X}} \cdot [(\det \boldsymbol{F})\boldsymbol{F}^{-T}] = \vec{\mathbf{0}} \tag{2.126}$$

Hint: The Nanson's formula $d\vec{\boldsymbol{a}} = J\,\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}$, or $d\vec{\boldsymbol{a}} = da\,\hat{\mathbf{n}} = J\,\boldsymbol{F}^{-T} \cdot \hat{\mathbf{N}}\,dA$.

Solution:

Considering the Nanson's formula in indicial notation $da\,\hat{\mathsf{n}}_i = J\,F_{ki}^{-1}\hat{\mathsf{N}}_k\,dA$, with $J = \det(\boldsymbol{F})$ we can apply the surface integral in order to obtain:

$$\int_S \hat{\mathsf{n}}_i da = \int_{S_0} J\,F_{ki}^{-1}\hat{\mathsf{N}}_k dA \tag{2.127}$$

Note that, if we consider the function f, the following is true:

$$\int_S \hat{\mathsf{n}}_i f\,da = \int_V f_{,i}\,dV = \int_V \frac{\partial f}{\partial x_i} dV$$

and denoting by $f = 1$, we can obtain:

$$\int_S \hat{\mathsf{n}}_i\,da = 0_i$$

Returning to equation (2.127), and applying the divergence theorem to the integral on the right of the equation we obtain:

$$\begin{gathered} \int_S \hat{\mathsf{n}}_i da = 0_i = \int_{S_0} J\,F_{ki}^{-1}\hat{\mathsf{N}}_k dA = \int_{V_0} (J\,F_{ki}^{-1})_{,k}\,dV_0 = \int_{V_0} \frac{\partial (J\,F_{ki}^{-1})}{\partial X_k} dV_0 = 0_i \\ \int_{V_0} \nabla_{\vec{X}} \cdot [(\det \boldsymbol{F})\boldsymbol{F}^{-T}]\,dV_0 = \vec{\mathbf{0}} \end{gathered} \tag{2.128}$$

Then, if the above volume integral is valid for the whole volume we can guarantee that is also valid locally, i.e.:

$$\nabla_{\vec{X}} \cdot [(\det \boldsymbol{F})\boldsymbol{F}^{-T}] = \vec{\mathbf{0}} \tag{2.129}$$

Problem 2.35

Show that $\dot{\boldsymbol{E}} = [\boldsymbol{F}^{-T} \cdot \nabla_{\vec{X}} \dot{\vec{\mathbf{u}}}(\vec{\boldsymbol{X}},t)]^{sym}$ and b) $\mathbf{D} = [\nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{\boldsymbol{x}},t)]^{sym}$, where $\boldsymbol{E}$ is the Green-Lagrange strain tensor and $\mathbf{D}$ is the rate-of-deformation tensor.

Solution:

$$\dot{\boldsymbol{E}} \equiv \frac{D}{Dt}\boldsymbol{E} = \frac{D}{Dt}\left[\frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}\right)\right] = \frac{1}{2}(\dot{\boldsymbol{F}}^T \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}) = \frac{1}{2}\left[(\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}})^T + (\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}})\right] = [\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}]^{sym}$$

Note that:

$$\dot{F}_{ij} = \frac{D}{Dt}\left(\frac{\partial x_i(X,t)}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\frac{Dx_i(X,t)}{Dt} = \frac{\partial}{\partial X_j}\left[\dot{\mathrm{u}}_i(X,t)\right] = \frac{\partial \dot{\mathrm{u}}_i(X,t)}{\partial X_j} = \left(\nabla_{\vec{X}}\dot{\vec{\mathbf{u}}}(\vec{X},t)\right)_{ij}$$

with that we demonstrate that:

$$\dot{\boldsymbol{E}} \equiv \frac{D}{Dt}\boldsymbol{E} = [\boldsymbol{F}^T \cdot \dot{\boldsymbol{F}}]^{sym} = [\boldsymbol{F}^T \cdot \nabla_{\vec{X}}\dot{\vec{\mathbf{u}}}(\vec{X},t)]^{sym}$$

b)

$$\mathbf{D} = \boldsymbol{\ell}^{sym} = \frac{1}{2}[\boldsymbol{\ell} + \boldsymbol{\ell}^T] = \frac{1}{2}[\nabla_{\vec{x}}\vec{\boldsymbol{v}} + (\nabla_{\vec{x}}\vec{\boldsymbol{v}})^T] = [\nabla_{\vec{x}}\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)]^{sym} = [\nabla_{\vec{x}}\dot{\vec{\mathbf{u}}}(\vec{\boldsymbol{x}},t)]^{sym}$$

where we have considered $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \dot{\vec{\mathbf{u}}}(\vec{\boldsymbol{x}},t)$.

Problem 2.36

Obtain the relationship $\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$ starting from the definition $(ds)^2 - (dS)^2 = d\vec{\boldsymbol{X}} \cdot 2\boldsymbol{E} \cdot d\vec{\boldsymbol{X}}$. Get also the relationship between $\frac{D}{Dt}\left[(ds)^2\right]$ and $\mathbf{D}$.

If we are dealing with a rigid body motion, find the condition in order to guarantee the rigid body motion.

Solution:

Taking the material time derivative of $(ds)^2 - (dS)^2 = d\vec{\boldsymbol{X}} \cdot 2\boldsymbol{E} \cdot d\vec{\boldsymbol{X}}$ we obtain:

$$\begin{aligned}
\frac{D}{Dt}[(ds)^2 - (dS)^2] &= \frac{D}{Dt}[(ds)^2] &&= \frac{D}{Dt}[d\vec{\boldsymbol{X}} \cdot 2\boldsymbol{E} \cdot d\vec{\boldsymbol{X}}] \\
&= \frac{D}{Dt}[d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}] &&= 2\underbrace{d\dot{\vec{\boldsymbol{X}}}}_{=0} \cdot \boldsymbol{E} \cdot d\vec{\boldsymbol{X}} + 2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}} + 2d\vec{\boldsymbol{X}} \cdot \boldsymbol{E} \cdot \underbrace{d\dot{\vec{\boldsymbol{X}}}}_{=0} \\
&= 2d\vec{\boldsymbol{x}} \cdot \frac{D}{Dt}[d\vec{\boldsymbol{x}}] &&= 2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}}
\end{aligned}$$

The term $\frac{D}{Dt}[d\vec{\boldsymbol{x}}]$ can be expressed as follows:

$$\left\{\begin{aligned}
\frac{D}{Dt}[d\vec{\boldsymbol{x}}] &= \frac{D}{Dt}[\boldsymbol{F} \cdot d\vec{\boldsymbol{X}}] \\
&= \dot{\boldsymbol{F}} \cdot d\vec{\boldsymbol{X}} \\
&= \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}
\end{aligned}\right. \xrightarrow{Indicial} \left\{\begin{aligned}
\frac{D}{Dt}[dx_k] &= \frac{D}{Dt}\left[\frac{\partial x_k}{\partial X_i}dX_i\right] \\
&= \frac{D}{Dt}\left(\frac{\partial x_k}{\partial X_i}\right)dX_i = \frac{D}{DX_i}\left(\frac{\partial x_k}{\partial t}\right)dX_i \\
&= \frac{\partial v_k}{\partial X_i}dX_i
\end{aligned}\right.$$

with that we conclude that:

$$\begin{aligned}
2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}} &= 2d\vec{\boldsymbol{x}} \cdot \frac{D}{Dt}[d\vec{\boldsymbol{x}}] \\
&= 2d\vec{\boldsymbol{x}} \cdot \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} \\
&= 2(\boldsymbol{F} \cdot d\vec{\boldsymbol{X}}) \cdot \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} \\
&= 2d\vec{\boldsymbol{X}} \cdot \boldsymbol{F}^T \cdot \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}
\end{aligned}$$

We can apply the additive decomposition of the spatial velocity gradient ($\boldsymbol{\ell}$) into a symmetric ($\mathbf{D}$) and an antisymmetric ($\mathbf{W}$) part:

$$
\begin{aligned}
2d\vec{X}\cdot\dot{\boldsymbol{E}}\cdot d\vec{X} &= 2d\vec{X}\cdot\boldsymbol{F}^T\cdot\boldsymbol{\ell}\cdot\boldsymbol{F}\cdot d\vec{X}\\
&= 2d\vec{X}\cdot\boldsymbol{F}^T\cdot(\mathbf{D}+\mathbf{W})\cdot\boldsymbol{F}\cdot d\vec{X}\\
&= 2d\vec{X}\cdot\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}\cdot d\vec{X}+2d\vec{X}\cdot\boldsymbol{F}^T\cdot\mathbf{W}\cdot\boldsymbol{F}\cdot d\vec{X}\\
&= 2d\vec{X}\cdot\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}\cdot d\vec{X}
\end{aligned}
$$

Note that $d\vec{X}\cdot\boldsymbol{F}^T\cdot\mathbf{W}\cdot\boldsymbol{F}\cdot d\vec{X}=d\vec{\boldsymbol{x}}\cdot\mathbf{W}\cdot d\vec{\boldsymbol{x}}=\mathbf{W}:(d\vec{\boldsymbol{x}}\otimes d\vec{\boldsymbol{x}})=0$, since $\mathbf{W}$ is an antisymmetric tensor and $(d\vec{\boldsymbol{x}}\otimes d\vec{\boldsymbol{x}})$ is a symmetric tensor. Then, we conclude that:

$$
\dot{\boldsymbol{E}}=\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}
$$

With that it is possible to relate $\frac{D}{Dt}[(ds)^2]$ and $\mathbf{D}$ as follows:

$$
\frac{D}{Dt}[(ds)^2]=2d\vec{X}\cdot\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}\cdot d\vec{X}=2d\vec{\boldsymbol{x}}\cdot\mathbf{D}\cdot d\vec{\boldsymbol{x}}
$$

During the rigid body motion the distances between particles do not change during motion, i.e. $\frac{D}{Dt}[(ds)^2]=0$, and according to the above equation we can conclude that the rigid body motion is guaranteed by $\mathbf{D}=\mathbf{0}$,since $d\vec{\boldsymbol{x}}\neq\vec{\mathbf{0}}$.

Problem 2.37

Consider the velocity field:

$$
v_1=-5x_2+2x_3 \quad ; \quad v_2=5x_1-3x_3 \quad ; \quad v_3=-2x_1+3x_2
$$

Show that this motion corresponds to a rigid body motion.

Solution:

At first we obtain the spatial velocity gradient $(\boldsymbol{\ell})$, whose components are given by:

$$
\ell_{ij}=\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_j}=\begin{bmatrix}\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3}\\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3}\\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3}\end{bmatrix}=\begin{bmatrix}0 & -5 & 2\\ 5 & 0 & -3\\ -2 & 3 & 0\end{bmatrix} \tag{2.130}
$$

Remember that $(\boldsymbol{\ell})$ can be decomposed into a symmetric ($\mathbf{D}$) and antisymmetric ($\mathbf{W}$) part, i.e. $\boldsymbol{\ell}=\mathbf{D}+\mathbf{W}=\mathbf{W}$. Since $\mathbf{D}=\mathbf{0}$, there is no strain during motion, i.e. a rigid body motion.

Problem 2.38

Let us consider the following velocity field:

$$
v_1=-3x_2+1x_3 \quad ; \quad v_2=3x_1-5x_3 \quad ; \quad v_3=-1x_1+5x_2
$$

Show that this motion corresponds to rigid body motion.

Solution: First we obtain the components of the spatial velocity gradient $(\boldsymbol{\ell})$:

$$\ell_{ij} = \frac{\partial v_i(\vec{x},t)}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = \ell_{ij}^{skew}$$

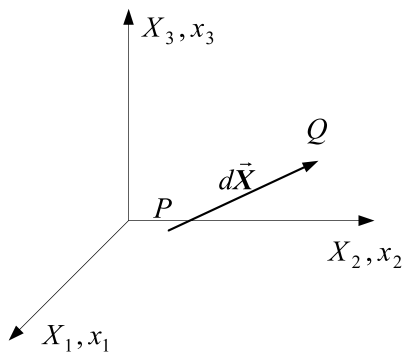
Taking into account that $\boldsymbol{\ell}$ can be decomposed into a symmetric ($\boldsymbol{\ell}^{sym} \equiv \mathbf{D}$) and an antisymmetric ($\boldsymbol{\ell}^{skew} \equiv \mathbf{W}$) part, i.e. $\boldsymbol{\ell} = \mathbf{D} + \mathbf{W}$, we can thus conclude that $\mathbf{D} = \mathbf{0}$, which is a characteristic of rigid body motion.

Problem 2.39

The displacement field components are given by:

$$\mathrm{u}_1 = 3X_1^2 + X_2 \quad ; \quad \mathrm{u}_2 = 2X_2^2 + X_3 \quad ; \quad \mathrm{u}_3 = 4X_3^2 + X_1$$

Obtain the vector $d\vec{x}$ (current configuration) correspondent to the vector in the reference configuration represented by $d\vec{X}$ at the point $P(1,1,1)$, (see Figure 2.12).



$$dX_k = \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$

Figure 2.12

Solution:

To determine the vector $d\vec{x}$ we need to obtain the deformation gradient $\boldsymbol{F}$, which can be obtained by using the relationship:

$$F_{ij} = \delta_{ij} + \frac{\partial \mathrm{u}_i}{\partial X_j} \quad \Rightarrow \quad F_{ij} = \begin{bmatrix} 1+6X_1 & 1 & 0 \\ 0 & 1+4X_2 & 1 \\ 1 & 0 & 1+8X_3 \end{bmatrix}$$

And the deformation gradient components evaluated at the point $P(1,1,1)$ are:

$$F_{ij}\big|_P = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix}$$

Then, the vector components dx_i are given by:

$$dx_i = F_{ij} dX_j$$

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} 7dX_1 + dX_2 \\ 5dX_2 + dX_3 \\ dX_1 + 9dX_3 \end{bmatrix}$$

Problem 2.40

Consider a continuum in which the displacement field is described by the following equations:

$$\mathrm{u}_1 = 2X_1^2 + X_1X_2 \quad ; \quad \mathrm{u}_2 = X_2^2 \quad ; \quad \mathrm{u}_3 = 0$$

By definition, a *material curve* is always formed by the same particles. Let $\overline{OP}$ and $\overline{OT}$ be material lines in the reference configuration, where $O(X_1 = 0, X_2 = 0, X_3 = 0)$, $P(X_1 = 1, X_2 = 1, X_3 = 0)$ and $T(X_1 = 1, X_2 = 0, X_3 = 0)$. Find the material curves in the current configuration. Also find the deformation gradient.

Solution:

a) The equations of motion can be obtained by means of the displacement field, i.e.:

$$\mathrm{u}_i = x_i - X_i \quad \Rightarrow \quad \begin{cases} x_1 = \mathrm{u}_1 + X_1 \\ x_2 = \mathrm{u}_2 + X_2 \\ x_3 = \mathrm{u}_3 + X_3 \end{cases} \xrightarrow[\text{the values of } \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3]{\text{substituting}} \begin{cases} x_1 = X_1 + 2X_1^2 + X_1X_2 \\ x_2 = X_2 + X_2^2 \\ x_3 = X_3 \end{cases}$$

Then, to obtain the material curve, one need only substitute the material coordinates with the particles belonging to the line $\overline{OP}$ in the equations of motion, (see Figure 2.13). Notice that the material curve $\overline{OP}$ in the current configuration is no longer a straight line, but the line $\overline{OT}$ is still a straight line in the current configuration, (see Figure 2.14).

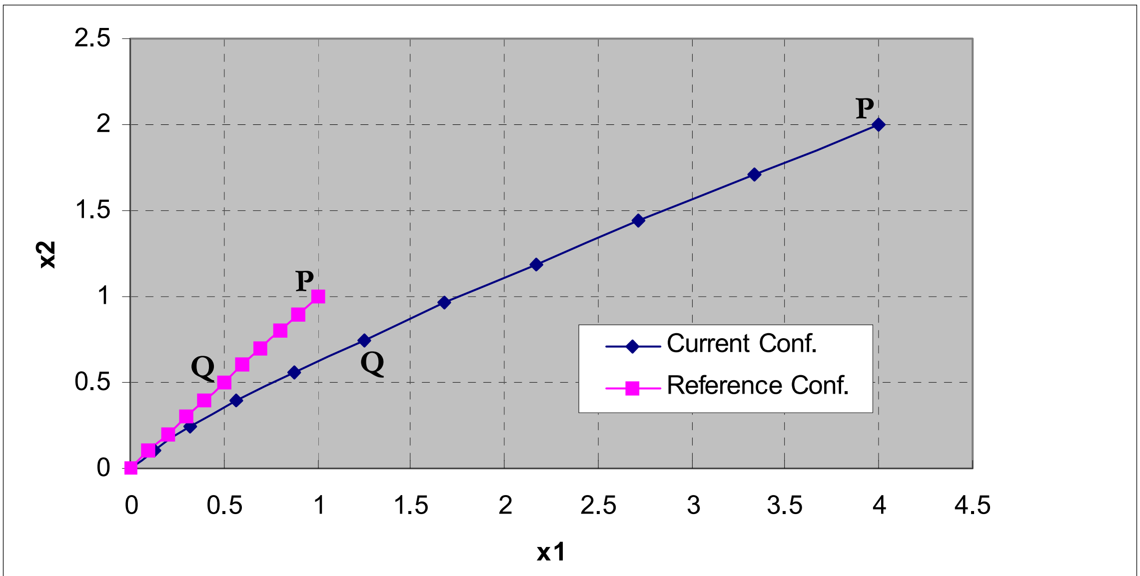


Figure 2.13: Deformation of the material curve $\overline{OP}$.

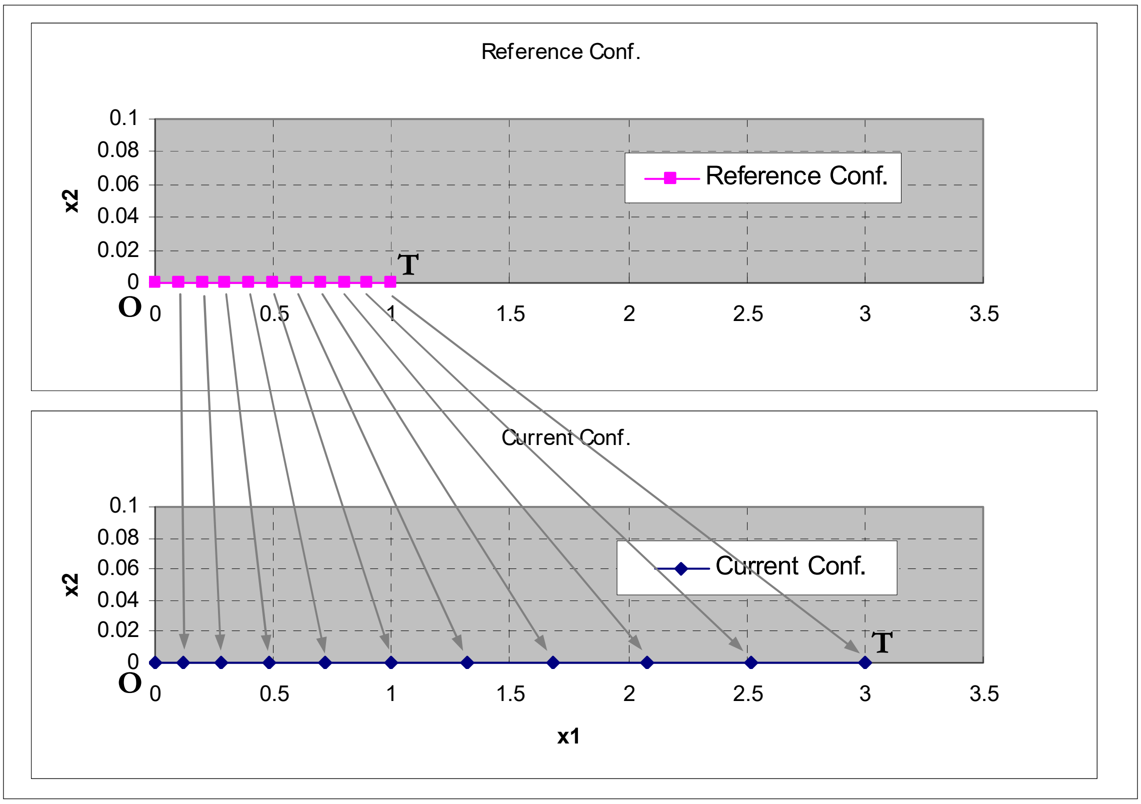


Figure 2.14: Deformation of the material curve $\overline{OT}$.

The components of the deformation gradient can be obtained as follows:

$$F_{jk}=\begin{bmatrix}\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}\end{bmatrix}=\begin{bmatrix}(1+4X_1+X_2) & X_1 & 0\\ 0 & 1+2X_2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Note that we are not dealing with homogeneous deformation, since $\boldsymbol{F}$ depends on $\vec{\boldsymbol{X}}$.

Problem 2.41

Starting from the definition $\frac{D}{Dt}[\det(\boldsymbol{F})]=\frac{DF_{ij}}{Dt}\operatorname{cof}(F_{ij})$, show that the equation $\dot{J}=J\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}$, is valid.

Solution: Considering that $F_{ij}=\frac{\partial x_i}{\partial X_j}$, the material time derivative of $|\boldsymbol{F}|\equiv\det(\boldsymbol{F})$ is given by:

$$\frac{D}{Dt}[\det(\boldsymbol{F})]=\frac{D}{Dt}\left(\frac{\partial x_i(\boldsymbol{X},t)}{\partial X_j}\right)\operatorname{cof}(F_{ij})=\frac{D}{\partial X_j}\left(\frac{\partial x_i(\boldsymbol{X},t)}{Dt}\right)\operatorname{cof}(F_{ij})=\frac{D}{\partial X_j}(v_i)\operatorname{cof}(F_{ij})$$

and considering that $v_i(\vec{\boldsymbol{x}}(\boldsymbol{X},t),t)$, we can state that:

$$\frac{D}{Dt}[\det(\boldsymbol{F})] = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}\operatorname{cof}(F_{ij})$$

By referring to the definition of the cofactor: $[\operatorname{cof}(F_{ij})]^T = (F_{ij})^{-1}\det(\boldsymbol{F})$, we can also state the following is valid:

$$\frac{D}{Dt}[\det(\boldsymbol{F})] = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}(F_{ij})^{-T}\det(\boldsymbol{F}) = \frac{\partial v_i}{\partial x_k}F_{kj}(F_{ij})^{-1}\det(\boldsymbol{F}) = \frac{\partial v_i}{\partial x_k}\delta_{ki}\det(\boldsymbol{F}) = \frac{\partial v_i}{\partial x_i}\det(\boldsymbol{F})$$
$$= J v_{i,i}$$

An alternative solution is presented in **Problem 1.118** in Chapter 1.

Problem 2.42

Let $d\vec{\boldsymbol{x}}$ be a differential line element in the current configuration. Find the material time derivative of $d\vec{\boldsymbol{x}}$.

Solution:

$$\frac{D}{Dt}d\vec{\boldsymbol{x}} = \frac{D}{Dt}(\boldsymbol{F}\cdot d\vec{\boldsymbol{X}}) = \frac{D}{Dt}(\boldsymbol{F})\cdot d\vec{\boldsymbol{X}} + \boldsymbol{F}\cdot\underbrace{\frac{D}{Dt}(d\vec{\boldsymbol{X}})}_{\boldsymbol{0}} = \boldsymbol{\ell}\cdot\underbrace{\boldsymbol{F}\cdot d\vec{\boldsymbol{X}}}_{d\vec{x}} = \boldsymbol{\ell}\cdot d\vec{\boldsymbol{x}} \equiv (\nabla_{\vec{x}}\vec{\boldsymbol{v}})\cdot d\vec{\boldsymbol{x}}$$

And, whose components are represented by:

$$\left(\frac{D}{Dt}d\vec{\boldsymbol{x}}\right)_i = v_{i,k}\,dx_k = \frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_k}dx_k$$

Problem 2.43

Let us consider the equations of motion:

$$x_1 = X_1 + 4X_1X_2 \quad ; \quad x_2 = X_2 + X_2^2 \quad ; \quad x_3 = X_3 + X_3^2$$

Find the Green-Lagrange strain tensor ($\boldsymbol{E}$).

Solution: Referring to the $\boldsymbol{E}$ equation:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\cdot\boldsymbol{F} - \mathbf{1}) \quad ; \quad E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \tag{2.131}$$

where the components of $\boldsymbol{F}$ are derived as:

$$F_{kj} = \frac{\partial x_k}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} (1+4X_2) & 4X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}$$

And,

$$F_{ki}F_{kj} = \begin{bmatrix} (1+4X_2) & 0 & 0 \\ 4X_1 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}\begin{bmatrix} (1+4X_2) & 4X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}$$
$$= \begin{bmatrix} (1+4X_2)^2 & (1+4X_2)4X_1 & 0 \\ (1+4X_2)4X_1 & (4X_1)^2 + (1+2X_2)^2 & 0 \\ 0 & 0 & (1+2X_3)^2 \end{bmatrix}$$

Then, by substituting the above into the equation in (2.131) we can obtain:

$$E_{ij} = \frac{1}{2}\begin{bmatrix} (1+4X_2)^2 - 1 & (1+4X_2)4X_1 & 0 \\ (1+4X_2)4X_1 & (4X_1)^2 + (1+2X_2)^2 - 1 & 0 \\ 0 & 0 & (1+2X_3)^2 - 1 \end{bmatrix}$$

Problem 2.44

Obtain the principal invariants of $\boldsymbol{E}$ in terms of the principal invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$.

Solution:

The principal invariants of $\boldsymbol{E}$ are given by:

$$I_{\boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E}) \quad ; \quad II_{\boldsymbol{E}} = \frac{1}{2}\left[I_{\boldsymbol{E}}^2 - \mathrm{Tr}(\boldsymbol{E}^2)\right] \quad ; \quad III_{\boldsymbol{E}} = \det(\boldsymbol{E})$$

Considering $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$, the principal invariants can also be expressed as follows:

The First Invariant:

$$I_{\boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E}) = \mathrm{Tr}\left[\frac{1}{2}(\boldsymbol{C} - \mathbf{1})\right] = \frac{1}{2}\mathrm{Tr}(\boldsymbol{C} - \mathbf{1}) = \frac{1}{2}\left[\mathrm{Tr}(\boldsymbol{C}) - \mathrm{Tr}(\mathbf{1})\right] = \frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)$$

The Second Invariant:

$$II_{\boldsymbol{E}} = \frac{1}{2}\left[I_{\boldsymbol{E}}^2 - \mathrm{Tr}(\boldsymbol{E}^2)\right]$$

where

$$I_{\boldsymbol{E}}^2 = \left[\frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)\right]^2 = \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 6I_{\boldsymbol{C}} + 9\right)$$

$$\mathrm{Tr}(\boldsymbol{E}^2) = \mathrm{Tr}\left[\frac{1}{2}(\boldsymbol{C} - \mathbf{1})\right]^2 = \frac{1}{4}\mathrm{Tr}\left[(\boldsymbol{C} - \mathbf{1})^2\right] = \frac{1}{4}\mathrm{Tr}\left(\boldsymbol{C}^2 - 2\boldsymbol{C} + \mathbf{1}\right) = \frac{1}{4}\left[\mathrm{Tr}(\boldsymbol{C}^2) - 2\mathrm{Tr}(\boldsymbol{C}) + \mathrm{Tr}(\mathbf{1})\right]$$
$$= \frac{1}{4}\left[\mathrm{Tr}(\boldsymbol{C}^2) - 2I_{\boldsymbol{C}} + 3\right]$$

The term $\mathrm{Tr}(\boldsymbol{C}^2)$ can be obtained as follows:

$$\boldsymbol{C} \cdot \boldsymbol{C} = \boldsymbol{C}^2 \quad \Rightarrow \quad C_{ij}'^2 = \begin{bmatrix} C_1^2 & 0 & 0 \\ 0 & C_2^2 & 0 \\ 0 & 0 & C_3^2 \end{bmatrix} \Rightarrow \mathrm{Tr}(\boldsymbol{C}^2) = C_1^2 + C_2^2 + C_3^2$$

It is also true that:

$$I_{\boldsymbol{C}}^2 = \left(C_1 + C_2 + C_3\right)^2 = C_1^2 + C_2^2 + C_3^2 + 2\underbrace{\left(C_1 C_2 + C_1 C_3 + C_2 C_3\right)}_{II_{\boldsymbol{C}}}$$

$$\Rightarrow C_1^2 + C_2^2 + C_3^2 = I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}}$$

Therefore we have:

$$\mathrm{Tr}(\boldsymbol{E}^2) = \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}} - 2I_{\boldsymbol{C}} + 3\right)$$

Whereupon, the second invariant can also be expressed as:

$$I\!I_{\boldsymbol{E}} = \frac{1}{2}\left[\frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 6I_{\boldsymbol{C}} + 9\right) - \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 2\,I\!I_{\boldsymbol{C}} - 2I_{\boldsymbol{C}} + 3\right)\right] = \frac{1}{4}\left(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3\right)$$

The Third Invariant:

$$I\!I\!I_{\boldsymbol{E}} = \det(\boldsymbol{E}) = \det\left[\frac{1}{2}(\boldsymbol{C} - \mathbf{1})\right] = \left(\frac{1}{2}\right)^3 \det[(\boldsymbol{C} - \mathbf{1})]$$

The term $\det[(\boldsymbol{C} - \mathbf{1})]$ can also be expressed as:

$$\det(\boldsymbol{C} - \mathbf{1}) = \begin{vmatrix} C_1 - 1 & 0 & 0 \\ 0 & C_2 - 1 & 0 \\ 0 & 0 & C_3 - 1 \end{vmatrix} = (C_1 - 1)(C_2 - 1)(C_3 - 1)$$
$$= C_1 C_2 C_3 - C_1 C_2 - C_1 C_3 - C_2 C_3 + C_1 + C_2 + C_3 - 1 = I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1$$

Then:

$$I\!I\!I_{\boldsymbol{E}} = \frac{1}{8}\left(I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1\right)$$

In short we have:

$$\boxed{\begin{aligned} I_{\boldsymbol{E}} &= \frac{1}{2}(I_{\boldsymbol{C}} - 3) \\ I\!I_{\boldsymbol{E}} &= \frac{1}{4}(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3) \\ I\!I\!I_{\boldsymbol{E}} &= \frac{1}{8}(I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1) \end{aligned}} \quad ; \quad \boxed{\begin{aligned} I_{\boldsymbol{C}} &= 2I_{\boldsymbol{E}} + 3 \\ I\!I_{\boldsymbol{C}} &= 4\,I\!I_{\boldsymbol{E}} + 4I_{\boldsymbol{E}} + 3 \\ I\!I\!I_{\boldsymbol{C}} &= 8\,I\!I\!I_{\boldsymbol{E}} + 4\,I\!I_{\boldsymbol{E}} + 2I_{\boldsymbol{E}} + 1 \end{aligned}}$$

Problem 2.45

Let $\Psi = \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}})$ be a scalar-valued tensor function, where $I_{\boldsymbol{C}}$, $I\!I_{\boldsymbol{C}}$, $I\!I\!I_{\boldsymbol{C}}$ are the principal invariants of the right Cauchy-Green deformation tensor $\boldsymbol{C}$. Obtain the derivative of Ψ with respect to $\boldsymbol{C}$ and with respect to $\boldsymbol{b}$. Check whether the following equation is valid $\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \Psi_{,\boldsymbol{b}} \cdot \boldsymbol{b}$ or not.

Solution:

Using the chain rule of derivative we can obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}})}{\partial \boldsymbol{C}} = \frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} \tag{2.132}$$

Considering the partial derivative of the invariants:

$\dfrac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}$, $\dfrac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}^T = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}$, $\dfrac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-T} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1}$, we can obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\mathbf{1} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\left(I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}\right) + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\,I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1}$$

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\boldsymbol{C} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\,I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1}} \tag{2.133}$$

It is also true that:

$$\boxed{\Psi_{,\boldsymbol{b}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} I_{\boldsymbol{b}} \right) \mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} \boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}} \boldsymbol{b}^{-1}} \tag{2.134}$$

We apply the dot product of the above equation with $\boldsymbol{F}$ on the left and with $\boldsymbol{F}^T$ on the right, i.e.:

$$\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}} \right) \boldsymbol{F} \cdot \mathbf{1} \cdot \boldsymbol{F}^T - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} \boldsymbol{F} \cdot \boldsymbol{C} \cdot \boldsymbol{F}^T + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}} \boldsymbol{F} \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{F}^T \tag{2.135}$$

And by considering the following relationships:

$$\Rightarrow \boldsymbol{F} \cdot \mathbf{1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{b}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} \quad \Rightarrow \quad \boldsymbol{F} \cdot \boldsymbol{C} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{b} \cdot \boldsymbol{b} = \boldsymbol{b}^2$$

And considering the relationship $\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{b}^{-1} \cdot \boldsymbol{F}$ we conclude that:

$$\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{b}^{-1} \cdot \boldsymbol{F} \Rightarrow \boldsymbol{F} \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{b}^{-1} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{b}^{-1} \cdot \boldsymbol{b}$$

Then, the equation in (2.135) can be rewritten as follows:

$$\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}} \right) \boldsymbol{b} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} \boldsymbol{b}^2 + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}} \boldsymbol{b}^{-1} \cdot \boldsymbol{b}$$

$$\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}} \right) \mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} \boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}} \boldsymbol{b}^{-1} \right] \cdot \boldsymbol{b}$$

It is also valid that:

$$\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} I_{\boldsymbol{b}} \right) \mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} \boldsymbol{B} + \frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}} \boldsymbol{b}^{-1} \right] \cdot \boldsymbol{b}$$

$$\boxed{\boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \Psi_{,\boldsymbol{b}} \cdot \boldsymbol{b}}$$

Taking into account the equation (2.134) we can conclude that the equation $\Psi_{,\boldsymbol{b}} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \Psi_{,\boldsymbol{b}}$ is valid, indicating that the tensors $\Psi_{,\boldsymbol{b}}$ and $\boldsymbol{b}$ are coaxial.

Problem 2.46

Show that the Green-Lagrange strain tensor ($\boldsymbol{E}$) and the right Cauchy-Green deformation tensor ($\boldsymbol{C}$) are coaxial tensors.

Solution:

Two tensors are coaxial if they have the same principal directions. Coaxiality can also be demonstrated if the relation $\boldsymbol{C} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot \boldsymbol{C}$ holds.

Starting with the definition $\boldsymbol{C} = \mathbf{1} + 2\boldsymbol{E}$, we can conclude that:

$$\boldsymbol{C} \cdot \boldsymbol{E} = (\mathbf{1} + 2\boldsymbol{E}) \cdot \boldsymbol{E} = \mathbf{1} \cdot \boldsymbol{E} + 2\boldsymbol{E} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot (\mathbf{1} + 2\boldsymbol{E}) = \boldsymbol{E} \cdot \boldsymbol{C}$$

Thus, we can prove that $\boldsymbol{E}$ and $\boldsymbol{C}$ are coaxial tensors.

Problem 2.47

Obtain the material time derivative of the Jacobian determinant ($\dot{J}$) in terms of ($\dot{\boldsymbol{E}}$), ($\dot{\boldsymbol{C}}$), ($\dot{\boldsymbol{F}}$).

Solution:

We starting from the relationship $\dot{J} = J\,\mathrm{Tr}(\mathbf{D})$, where $\mathbf{D}$ is the rate-of-deformation tensor which is related to $\dot{\boldsymbol{E}}$ by means of the relationship $\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}$, then:

$$\dot{J} = J\,\mathrm{Tr}(\mathbf{D}) = J\,\mathrm{Tr}\left(\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) : \mathbf{1}$$

In indicial notation we have:

$$\dot{J} = J\,F^{-1}_{ki}\dot{E}_{kp}F^{-1}_{pj}\delta_{ij} = J\,F^{-1}_{ki}F^{-1}_{pi}\dot{E}_{kp} = J\,(\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}) : \dot{\boldsymbol{E}} = J\,\boldsymbol{C}^{-1} : \dot{\boldsymbol{E}} = \frac{J}{2}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}}$$

The $\dot{J}$ can still be expressed in terms of $\dot{\boldsymbol{F}}$. To this end let us consider the following equation $\dot{E}_{kp} = \frac{1}{2}\left(\dot{F}_{sk}F_{sp} + F_{sk}\dot{F}_{sp}\right)$. Then, $\dot{J}$ can also be expressed by:

$$\begin{aligned}
\dot{J} &= J\,F^{-1}_{ki}F^{-1}_{pi}\dot{E}_{kp} = J\,F^{-1}_{ki}F^{-1}_{pi}\frac{1}{2}\left(\dot{F}_{sk}F_{sp} + F_{sk}\dot{F}_{sp}\right) = \frac{J}{2}\left(F^{-1}_{ki}F^{-1}_{pi}\dot{F}_{sk}F_{sp} + F^{-1}_{ki}F^{-1}_{pi}F_{sk}\dot{F}_{sp}\right)\\
&= \frac{J}{2}\left(\delta_{si}F^{-1}_{ki}\dot{F}_{sk} + \delta_{si}F^{-1}_{pi}\dot{F}_{sp}\right) = \frac{J}{2}\left(F^{-1}_{ks}\dot{F}_{sk} + F^{-1}_{ps}\dot{F}_{sp}\right) = JF^{-1}_{ts}\dot{F}_{st} = J\dot{F}_{st}F^{-1}_{ts}\\
&= J\boldsymbol{F}^{-T} : \dot{\boldsymbol{F}} = J\dot{\boldsymbol{F}} : \boldsymbol{F}^{-T}
\end{aligned}$$

In short, there are various different ways to express the material time derivative of the Jacobian determinant:

$$\boxed{\begin{aligned}
\dot{J} = J\,\mathrm{Tr}(\mathbf{D}) \quad &= J\,\boldsymbol{C}^{-1} : \dot{\boldsymbol{E}} &&= \frac{J}{2}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}} &&= J\dot{\boldsymbol{F}} : \boldsymbol{F}^{-T}\\
&= J\,\mathrm{Tr}(\boldsymbol{C}^{-1}\cdot\dot{\boldsymbol{E}}) &&= \frac{J}{2}\mathrm{Tr}(\boldsymbol{C}^{-1}\cdot\dot{\boldsymbol{C}}) &&= J\,\mathrm{Tr}(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})
\end{aligned}}$$

where we have used the trace property: $\mathbf{A} : \mathbf{B} = \mathrm{Tr}(\mathbf{A}\cdot\mathbf{B}^T) = \mathrm{Tr}(\mathbf{A}^T\cdot\mathbf{B})$ in which $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-order tensors.

Problem 2.48

The displacement field components are given by:

$$\mathrm{u}_1 = 0.1X_2^2 \quad ; \quad \mathrm{u}_2 = 0 \quad ; \quad \mathrm{u}_3 = 0$$

a) Is this motion possible? Justify;

b) Obtain the right Cauchy-Green deformation tensor;

c) Consider two vectors at the point $P(1,1,0)$ in the reference configuration, namely: $\vec{\mathbf{b}} = 0.01\hat{\mathbf{e}}_1$ and $\vec{\mathbf{c}} = 0.015\hat{\mathbf{e}}_2$. Find the correspondent vectors in current configuration;

d) Obtain the stretches of the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, at the point $P(1,1,0)$;

e) Find the angle variation defined by the two vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

Solution:

a) A motion is possible if the Jacobian determinant is positive. The deformation gradient components can be obtained as follows:

$$F_{ij} = \delta_{ij} + \frac{\partial \mathrm{u}_i}{\partial X_j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.2X_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant is $\left|F_{ij}\right| = J = 1 > 0$. Then, the motion is possible.

b) The right Cauchy-Green deformation tensor is defined as $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, then the components are given by:

$$C_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0.2X_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0.2X_2 & 0.2^2 X_2^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) The vector $\vec{\mathbf{b}} = 0.01\hat{\mathbf{e}}_1$ at the point $P(1,1,0)$ deforms according to:

$$\vec{\mathbf{b}}' = \boldsymbol{F}\big|_P \cdot \vec{\mathbf{b}} \qquad \Rightarrow \qquad \begin{bmatrix} \mathrm{b}'_1 \\ \mathrm{b}'_2 \\ \mathrm{b}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0.2\times 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}$$

and the vector $\vec{\mathbf{c}} = 0.015\hat{\mathbf{e}}_2$ in the current configuration becomes:

$$\begin{bmatrix} \mathrm{c}'_1 \\ \mathrm{c}'_2 \\ \mathrm{c}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0.2\times 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.015 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.003 \\ 0.015 \\ 0 \end{bmatrix}$$

d) The stretch can be obtained as follows:

$$\lambda_{\vec{\mathbf{b}}} = \frac{\| \vec{\mathbf{b}}' \|}{\| \vec{\mathbf{b}} \|} = \frac{\sqrt{0.01^2}}{0.01} = 1$$

and the stretch of $\vec{\mathbf{c}}$ is given as:

$$\lambda_{\vec{\mathbf{c}}} = \frac{\| \vec{\mathbf{c}}' \|}{\| \vec{\mathbf{c}} \|} = \frac{\sqrt{0.003^2 + 0.015^2}}{0.015} = 1.0198 \approx 1.02$$

Alternative solution: Taking into account that $\lambda_{\hat{\boldsymbol{M}}} = \sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}}$ and by evaluating $\boldsymbol{C}$ at the point P we obtain:

$$C_{ij}(X_1 = 1, X_2 = 1, X_3 = 0) = \left. \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0.2X_2 & 0.2^2 X_2^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|_P = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1.04 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, by applying $\lambda_{\hat{\mathbf{b}}} = \sqrt{\hat{\mathbf{b}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{b}}}$ and $\lambda_{\hat{\mathbf{c}}} = \sqrt{\hat{\mathbf{c}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{c}}}$ we can obtain:

$$\lambda_{\hat{\mathbf{b}}}^2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1.04 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \qquad \Rightarrow \qquad \lambda_{\hat{\mathbf{b}}} = 1$$

$$\lambda_{\hat{\mathbf{c}}}^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1.04 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1.04 \qquad \Rightarrow \qquad \lambda_{\hat{\mathbf{c}}} = 1.0198$$

e) In the current configuration the angle between the vectors $\vec{\mathbf{b}}'$ and $\vec{\mathbf{c}}'$ can be obtained according to the relationship:

$$\cos\theta = \frac{\vec{\mathbf{b}}' \cdot \vec{\mathbf{c}}'}{\left\|\vec{\mathbf{b}}'\right\| \left\|\vec{\mathbf{c}}'\right\|}$$

$$\cos\theta = \frac{(0.01\hat{\mathbf{e}}_1 + 0\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3)\cdot(0.003\hat{\mathbf{e}}_1 + 0.015\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3)}{\sqrt{0.01^2}\sqrt{0.003^2 + 0.015^2}} = \frac{0.00003}{0.01\sqrt{0.000234}} = 0.196116135$$

$$\theta = \arccos(0.196116135) \approx 78.69^{\mathrm{o}}$$

In the reference configuration the angle between these two vectors is 90^{o}, then the angle variation is:

$$\Delta\theta = 90^{\mathrm{o}} - 78.69^{\mathrm{o}} = 11.3^{\mathrm{o}}$$

Alternative solution: Given two directions in the reference configuration represented by their unit vectors $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$, the angle formed by these unit vectors in the current configuration (after motion) is given by:

$$\cos\theta = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}}\sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}} = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}}\lambda_{\hat{N}}}$$

Denoting by $\hat{\boldsymbol{M}} = \hat{\mathbf{b}}$ and $\hat{\boldsymbol{N}} = \hat{\mathbf{c}}$ it fulfills that:

$$\hat{\mathbf{b}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{c}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1.04 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0.2$$

Then,

$$\cos\theta = \frac{\hat{\mathbf{b}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{c}}}{\sqrt{\hat{\mathbf{b}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{b}}}\sqrt{\hat{\mathbf{c}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{c}}}} = \frac{\hat{\mathbf{b}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{c}}}{\lambda_{\hat{\mathbf{b}}}\lambda_{\hat{\mathbf{c}}}} = \frac{0.2}{\sqrt{1}\sqrt{1.04}} = 0.196116135$$

Problem 2.49

Let $\phi(\vec{\boldsymbol{X}},t)$ be a scalar field in Lagrangian (material) description. Find the relationship between the material gradient of $\phi(\vec{\boldsymbol{X}},t)$, i.e. $\nabla_{\vec{X}}\phi(\vec{\boldsymbol{X}},t)$, and the spatial gradient of $\phi(\vec{\boldsymbol{x}},t)$, i.e. $\nabla_{\vec{x}}\phi(\vec{\boldsymbol{x}},t)$.

Solution:

Remember that a Lagrangian variable $\phi(\vec{\boldsymbol{X}},t)$ can be expressed in the Eulerian (current) configuration by means of the equations of motion, i.e.:

$$\phi(\vec{\boldsymbol{X}},t) = \phi(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = \phi(\vec{\boldsymbol{x}},t) .$$

Then, from the scalar gradient definition we obtain:

$$\nabla_{\vec{X}}\phi(\vec{\boldsymbol{X}},t) = \frac{\partial\phi(\vec{\boldsymbol{X}},t)}{\partial\vec{\boldsymbol{X}}} = \frac{\partial\phi(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t)}{\partial\vec{\boldsymbol{x}}} \cdot \frac{\partial\vec{\boldsymbol{x}}}{\partial\vec{\boldsymbol{X}}} = \frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial\vec{\boldsymbol{x}}} \cdot \boldsymbol{F} = \nabla_{\vec{x}}\phi(\vec{\boldsymbol{x}},t) \cdot \boldsymbol{F}$$

In addition we have the inverse form:

$$\nabla_{\vec{x}}\phi(\vec{\boldsymbol{x}},t) = \frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial\vec{\boldsymbol{x}}} = \frac{\partial\phi(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t)}{\partial\vec{\boldsymbol{X}}} \cdot \frac{\partial\vec{\boldsymbol{X}}}{\partial\vec{\boldsymbol{x}}} = \frac{\partial\phi(\vec{\boldsymbol{X}},t)}{\partial\vec{\boldsymbol{X}}} \cdot \boldsymbol{F}^{-1} = \nabla_{\vec{X}}\phi(\vec{\boldsymbol{X}},t) \cdot \boldsymbol{F}^{-1}$$

Problem 2.50

Given the following Eulerian velocity field components:

$$v_1 = 0 \quad ; \quad v_2 = 0 \quad ; \quad v_3 = f(x_1, x_2)x_3$$

a) Find the particle trajectory;
b) Obtain the mass density (ρ), knowing that at $t = 0$ we have $\rho = f(x_1, x_2)$.

Solution:

$$\frac{dx_1}{dt} = v_1 = 0 \Rightarrow x_1(t) = C_1 \text{ at } t = 0 \Rightarrow x_1 = X_1 \Rightarrow x_1(t=0) = C_1 = X_1;$$

$$\frac{dx_2}{dt} = v_2 = 0 \Rightarrow x_2(t) = C_2 \text{ at } t = 0 \Rightarrow x_2 = X_2 \Rightarrow x_2(t=0) = C_2 = X_2$$

$$\frac{dx_3}{dt} = v_3 = f(x_1, x_2)x_3 = f(C_1, C_2)x_3 \Rightarrow \int \frac{dx_3}{x_3} = \int f(C_1, C_2)dt \Rightarrow \mathsf{Ln}(x_3) = f(C_1, C_2)t + k$$

By denoting by $k = \mathsf{Ln}(C_3)$ the above equation becomes:

$$\mathsf{Ln}(x_3) - \mathsf{Ln}(C_3) = f(X_1, X_2)t$$

$$\Rightarrow \mathsf{Ln}\left(\frac{x_3}{C_3}\right) = f(X_1, X_2)t \Rightarrow \frac{x_3}{C_3} = \mathsf{exp}^{f(X_1, X_2)t} \Rightarrow x_3 = C_3 \mathsf{exp}^{f(X_1, X_2)t}$$

at $t = 0 \Rightarrow x_3 = X_3 \Rightarrow x_3(t=0) = C_3 = X_3$

Summarizing:

$$x_1 = X_1 \quad ; \quad x_2 = X_2 \quad ; \quad x_3 = X_3 \mathsf{exp}^{f(X_1, X_2)t} \tag{2.136}$$

The mass density:

$$\rho = \frac{\rho_0}{|\boldsymbol{F}|} \quad \text{with} \quad F_{ij} = \frac{\partial x_i}{\partial X_j}$$

$$J = |\boldsymbol{F}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & \mathsf{exp}^{f(X_1, X_2)t} \end{vmatrix} = \mathsf{exp}^{f(X_1, X_2)t}$$

As we can see, the values marked by (?) are not necessary in order to obtain the above determinant, then:

$$\rho = \frac{\rho_0}{|\boldsymbol{F}|} = \frac{f(X_1, X_2)}{\mathsf{exp}^{f(X_1, X_2)t}}$$

Note that according to the problem statement, $t = 0$, $\rho = f(x_1, x_2)$, and according to the equations in (2.136) we can conclude that $\rho_0 = f(X_1, X_2)$.

Problem 2.51

Obtain the equation for mass density in terms of the third invariant of the right Cauchy-Green deformation tensor, i.e. $\rho_0 = \rho_0(\mathit{III}_{\boldsymbol{C}})$.

Solution:

Starting by the definition:

$$\rho_0(\vec{X}) = \rho(\vec{x}, t) J$$

and considering that the third invariant $I\!I\!I_C = \det(C) = \det(F^T \cdot F) = J^2$, we obtain $J = \sqrt{I\!I\!I_C}$, then:

$$\rho_0 = \rho \sqrt{I\!I\!I_C} \tag{2.137}$$

Problem 2.52

Consider the displacement field of a continuous medium by:

$$\mathrm{u}_1 = (a_1 - 1) X_1 \quad ; \quad \mathrm{u}_2 = (a_2 - 1) X_2 + a_1 \alpha X_1 \quad ; \quad \mathrm{u}_3 = (a_3 - 1) X_3$$

where α is a constant. Determine a_1, a_2 and a_3 knowing that the solid is incompressible, that a segment parallel to the X_3-axis does not stretch and that any element area defined in the plane $X_1 - X_3$ remains unchanged.

Solution:

Based on the definition of the displacement field, i.e. $\vec{\mathbf{u}} = \vec{x} - \vec{X}$, we can obtain:

$$\begin{aligned}
\mathrm{u}_1 &= x_1 - X_1 = (a_1 - 1) X_1 \quad &\Rightarrow& \quad x_1 = a_1 X_1 \\
\mathrm{u}_2 &= x_2 - X_2 = (a_2 - 1) X_2 + a_1 \alpha X_1 \quad &\Rightarrow& \quad x_2 = a_2 X_2 + a_1 \alpha X_1 \\
\mathrm{u}_3 &= x_3 - X_3 = (a_3 - 1) X_3 \quad &\Rightarrow& \quad x_3 = a_3 X_3
\end{aligned}$$

Then, the equations of motion are:

$$\begin{cases} x_1 = a_1 X_1 \\ x_2 = a_2 X_2 + a_1 \alpha X_1 \\ x_3 = a_3 X_3 \end{cases} \quad \Rightarrow \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_1 \alpha & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \text{ (homogeneous deformation)}$$

which is possible to establish that $|F| = a_1 a_2 a_3 > 0$.

By means of the incompressibility condition $dV = |F| dV_0 \Rightarrow |F| \equiv J = 1$, the following relationship is true:

$$a_1 a_2 a_3 = 1$$

By the fact that a segment parallel to the X_3-axis, e.g. $\hat{\mathrm{M}}_i = [0 \quad 0 \quad 1]$, does not stretch that implies that the stretching according to this direction is unitary, i.e. $\lambda_{\hat{\mathbf{M}}} = 1$, thus

$$\lambda_{\hat{\mathbf{M}}} = \sqrt{1 + 2 \hat{\mathbf{M}} \cdot E \cdot \hat{\mathbf{M}}} = \sqrt{1 + 2 E_{33}} = 1 \quad \Rightarrow \quad E_{33} = 0$$

The components of the Green-Lagrange strain tensor, $E = \frac{1}{2}(F^T \cdot F - \mathbf{1})$, can be obtained as follows:

$$E_{ij} = \frac{1}{2} \left\{ \begin{bmatrix} a_1 & a_1 \alpha & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ a_1 \alpha & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} a_1^2 + a_1^2 \alpha^2 - 1 & a_1 a_2 \alpha & 0 \\ a_1 a_2 \alpha & a_2^2 - 1 & 0 \\ 0 & 0 & a_3^2 - 1 \end{bmatrix}$$

thus:

$$E_{33} = a_3^2 - 1 = 0 \quad \Rightarrow \quad a_3 = \pm 1$$

Any element area on the plane $X_1 - X_3$ does not change

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_1\alpha & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

By defining two vectors on the plane $X_1 - X_3$, i.e. $\hat{\mathsf{N}}_i^{(1)} = [1 \quad 0 \quad 0]$ and $\hat{\mathsf{N}}_i^{(3)} = [0 \quad 0 \quad 1]$ we can obtain:

$$\mathbf{n}_i^{(1)} = \begin{bmatrix} a_1 & 0 & 0 \\ a_1\alpha & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_1\alpha \\ 0 \end{Bmatrix} \quad ; \quad \mathbf{n}_i^{(3)} = \begin{bmatrix} a_1 & 0 & 0 \\ a_1\alpha & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ a_3 \end{Bmatrix}$$

Then, the area in the current configuration is obtained as follows:

$$\vec{\mathbf{n}}^{(1)} \wedge \vec{\mathbf{n}}^{(3)} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_1\alpha & 0 \\ 0 & 0 & a_3 \end{vmatrix} = a_1\alpha\hat{\mathbf{e}}_1 - a_1 a_3 \hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$$

and its module does not change $\left\| \mathbf{N}^{(1)} \wedge \mathbf{N}^{(3)} \right\| = \left\| \vec{\mathbf{n}}^{(1)} \wedge \vec{\mathbf{n}}^{(3)} \right\| = 1$:

$$\left\| \vec{\mathbf{n}}^{(1)} \wedge \vec{\mathbf{n}}^{(3)} \right\| = 1 = \sqrt{(a_1\alpha)^2 + (-a_1 a_3)^2} \Rightarrow a_1^2 a_3^2 \alpha^2 + a_1^2 a_3^2 = 1$$

We have previously obtained that $a_3^2 = 1$, with that we can obtain:

$$a_1^2 a_3^2 \alpha^2 + a_1^2 a_3^2 = 1 \quad \Rightarrow \quad a_1^2 \alpha^2 + a_1^2 = 1 \Rightarrow a_1^2 = \frac{1}{(1+\alpha^2)} \Rightarrow a_1 = \pm \frac{1}{\sqrt{(1+\alpha^2)}}$$

with that we can conclude that:

$$a_1 = \frac{1}{\sqrt{(1+\alpha^2)}} \quad ; \quad a_2 = \sqrt{(1+\alpha^2)} \quad ; \quad a_3 = 1$$

Problem 2.53

Consider the solid shown in Figure 2.15 which is subjected to a homogenous deformation.

a) Obtain the general expression for the material displacement field $\vec{\mathbf{u}}(\vec{X},t)$ in function of the material displacement gradient tensor $\boldsymbol{J}$.

b) Obtain $\vec{\mathbf{u}}(\vec{X},t)$ knowing that also holds the following boundary conditions:

$$\mathsf{u}_2(\vec{X},t) = \mathsf{u}_3(\vec{X},t) = 0 \qquad \forall X_1, X_2, X_3$$
$$\mathsf{u}_1(X_1 = 0, X_2, X_3, t) = 0$$
$$\mathsf{u}_1(X_1 = L, X_2, X_3, t) = \Delta_L$$

c) Justify the possible values (positive and negative) that can take Δ_L.

d) Calculate the material and spatial strain tensors and the infinitesimal strain tensor.

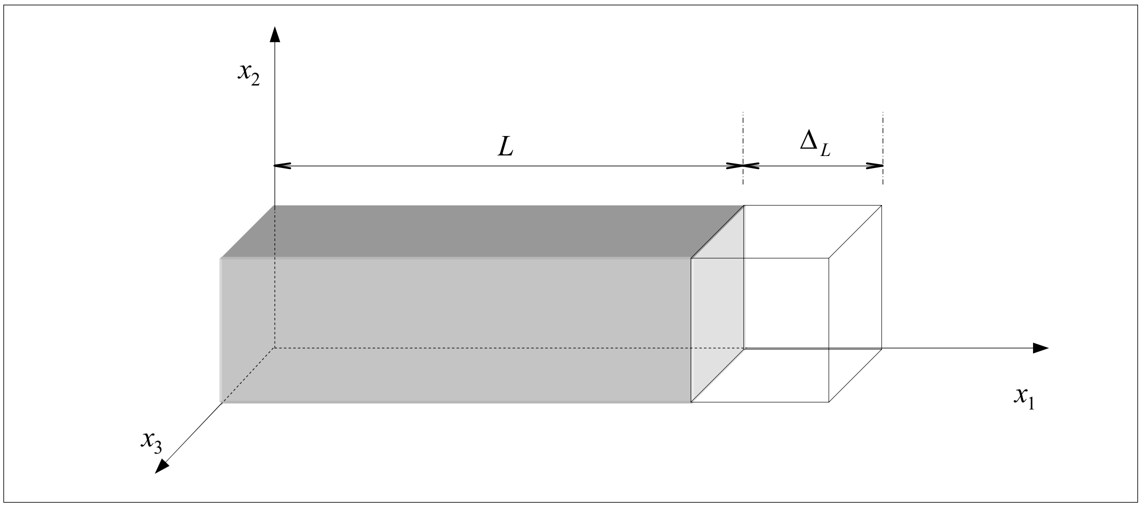


Figure 2.15

Solution:

A homogeneous deformation is characterized by $\boldsymbol{F}(\vec{X},t)=\boldsymbol{F}(t)$. In addition, we know that:

$$\boldsymbol{F}(\vec{X},t)=\mathbf{1}+\boldsymbol{J}(\vec{X},t)\xrightarrow{\text{Homogeneous deformation}}\boldsymbol{F}(t)=\mathbf{1}+\boldsymbol{J}(t)$$

where $\boldsymbol{J}$ is the material displacement gradient tensor. Note that the homogenous deformation is independent of the vector position, with that we can obtain:

$$\boldsymbol{J}(t)=\frac{\partial\vec{\mathbf{u}}(\vec{X},t)}{\partial\vec{X}} \quad\Rightarrow\quad \int\boldsymbol{J}(t)\cdot d\vec{X}=\int d\vec{\mathbf{u}}(\vec{X},t) \quad\Rightarrow\quad \vec{\mathbf{u}}(\vec{X},t)=\boldsymbol{J}(t)\cdot\vec{X}+\vec{\mathbf{c}}(t)$$

where $\vec{\mathbf{c}}(t)$ is the constant of integration. And the components of $\vec{\mathbf{u}}(\vec{X},t)$ are:

$$\begin{Bmatrix}\mathrm{u}_1\\\mathrm{u}_2\\\mathrm{u}_3\end{Bmatrix}=\begin{Bmatrix}J_{11}X_1+J_{12}X_2+J_{13}X_3\\J_{21}X_1+J_{22}X_2+J_{23}X_3\\J_{31}X_1+J_{32}X_2+J_{33}X_3\end{Bmatrix}+\begin{Bmatrix}\mathrm{c}_1\\\mathrm{c}_2\\\mathrm{c}_3\end{Bmatrix}$$

b) From the conditions in paragraph b) we can conclude that:

condition 1) $\mathrm{u}_2(\vec{X},t)=\mathrm{u}_3(\vec{X},t)=0 \qquad \forall X_1,X_2,X_3$:

$$\begin{Bmatrix}\mathrm{u}_1\\\mathrm{u}_2=0\\\mathrm{u}_3=0\end{Bmatrix}=\begin{Bmatrix}J_{11}X_1+J_{12}X_2+J_{13}X_3\\J_{21}X_1+J_{22}X_2+J_{23}X_3\\J_{31}X_1+J_{32}X_2+J_{33}X_3\end{Bmatrix}+\begin{Bmatrix}\mathrm{c}_1\\\mathrm{c}_2\\\mathrm{c}_3\end{Bmatrix}\Rightarrow\begin{cases}J_{21}=0;J_{22}=0;J_{23}=0,\mathrm{c}_2=0\\J_{31}=0;J_{32}=0;J_{33}=0,\mathrm{c}_3=0\end{cases}$$

condition 2) $\mathrm{u}_1(X_1=0,X_2,X_3,t)=0$:

$$\begin{Bmatrix}\mathrm{u}_1=0\\\mathrm{u}_2\\\mathrm{u}_3\end{Bmatrix}=\begin{Bmatrix}J_{11}X_1+J_{12}X_2+J_{13}X_3\\0\\0\end{Bmatrix}+\begin{Bmatrix}\mathrm{c}_1\\0\\0\end{Bmatrix}\Rightarrow\left\{J_{12}=0;J_{13}=0,\mathrm{c}_1=0\right.$$

condition 3) $\mathrm{u}_1(X_1=L,X_2,X_3,t)=\Delta_L$

$$\begin{Bmatrix}\mathrm{u}_1\\\mathrm{u}_2\\\mathrm{u}_3=\Delta_L\end{Bmatrix}=\begin{Bmatrix}J_{11}L\\0\\0\end{Bmatrix}+\begin{Bmatrix}0\\0\\0\end{Bmatrix}\Rightarrow\left\{J_{11}=\frac{\Delta_L}{L}\right.$$

Hence, the components of the displacement gradient are:

$$J_{ij} = \begin{bmatrix} \frac{\Delta_L}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad \begin{Bmatrix} \mathsf{c}_1 \\ \mathsf{c}_2 \\ \mathsf{c}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

with that the displacement field components are:

$$\vec{\mathbf{u}}(\vec{X},t) = \boldsymbol{J}(t)\cdot\vec{X} + \vec{\mathbf{c}}(t) \xrightarrow{components} \mathsf{u}_i(\vec{X},t) = \begin{bmatrix} \frac{\Delta_L}{L}X_1 \\ 0 \\ 0 \end{bmatrix}$$

c) The motion is possible and has physical meaning if $|\boldsymbol{F}| > 0$:

$$\boldsymbol{F}(t) = \mathbf{1} + \boldsymbol{J}(t) \xrightarrow{components} F_{ij} = \begin{bmatrix} 1+\frac{\Delta_L}{L} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |\boldsymbol{F}| = 1 + \frac{\Delta_L}{L} > 0 \Rightarrow \Delta_L > -L$$

The material strain tensor (the Green-Lagrange strain tensor):

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\cdot\boldsymbol{F} - \mathbf{1}\right) \xrightarrow{components} E_{ij} = \begin{bmatrix} \frac{\Delta_L}{L} + \frac{1}{2}\left(\frac{\Delta_L}{L}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\frac{m^2}{m^2}\right]$$

The spatial strain tensor (the Almansi strain tensor):

$$\boldsymbol{e} = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right) \xrightarrow{components} e_{ij} = \frac{\left[\frac{\Delta_L}{L} + \frac{1}{2}\left(\frac{\Delta_L}{L}\right)^2\right]}{\left(1+\frac{\Delta_L}{L}\right)^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\frac{m^2}{m^2}\right]$$

The infinitesimal strain tensor is defined by $\boldsymbol{\varepsilon} = (\nabla_{\vec{x}}\vec{\mathbf{u}})^{sym} = \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}^T)$, and its components are:

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\Delta_L}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\frac{m}{m}\right] \text{ (dimensionless)}$$

Note that, when $\left(\frac{\Delta_L}{L}\right) << 1$ is very small the term $\left(\frac{\Delta_L}{L}\right)^2 \approx 0$ can be discarded, and in this scenario we have $\boldsymbol{E} \approx \boldsymbol{e} \approx \boldsymbol{\varepsilon}$, i.e. we are dealing with the small deformation regime.

Problem 2.54

The tetrahedron shown in Figure 2.16 undergoes homogeneous deformation ($\boldsymbol{F} = const.$) with the following consequences:

1. The points O, A and B do not move;
2. The solid volume becomes p times the initial volume;
3. The length of the segment $\overline{AC}$ becomes $\frac{p}{\sqrt{2}}$ times the initial length;
4. The angle AOC becomes 45°.

a) Justify why we cannot use the infinitesimal deformation theory;
b) Obtain the deformation gradient, and the possible values for p and the displacement field in material and spatial descriptions;
c) Draw the deformed solid.

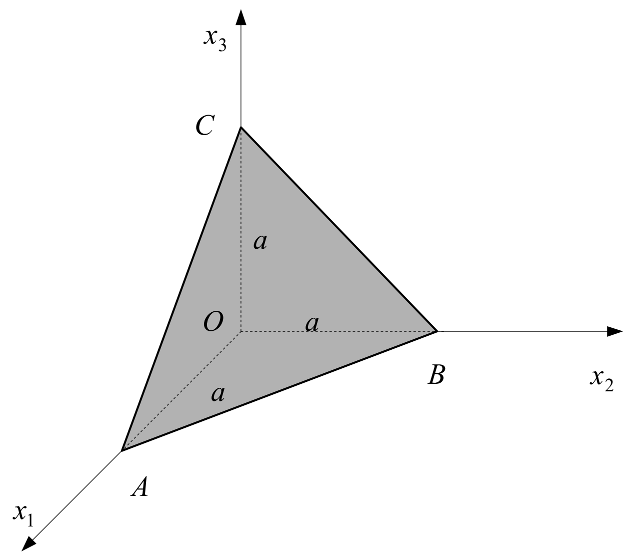


Figure 2.16

Solution:

a) The angle $AOC = 90^{\circ}$ becomes 45°, so we are not dealing with a small deformation, since for the small deformation case the condition $\Delta\phi << 1$ must fulfill, and in this problem we have $\Delta\phi << \frac{\pi}{4} \approx 0.7854$;

b) We have a case of homogeneous deformation. Then, the equations of motion are given by:

$$\vec{\boldsymbol{x}} = \boldsymbol{F}(t) \cdot \vec{\boldsymbol{X}} + \vec{\mathbf{c}}(t) \quad \Rightarrow \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{Bmatrix} \mathsf{c}_1 \\ \mathsf{c}_2 \\ \mathsf{c}_3 \end{Bmatrix}$$

The point $O(X_1 = 0, X_2 = 0, X_3 = 0)$ *does not move:*

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \mathsf{c}_1 \\ \mathsf{c}_2 \\ \mathsf{c}_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \mathsf{c}_1 \\ \mathsf{c}_2 \\ \mathsf{c}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The point $A(X_1 = a, X_2 = 0, X_3 = 0)$ *does not move:*

$$\begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} aF_{11} \\ aF_{21} \\ aF_{31} \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} F_{11} = 1 \\ F_{21} = 0 \\ F_{31} = 0 \end{cases}$$

The point $B(X_1 = 0, X_2 = a, X_3 = 0)$ *does not move:*

$$\begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & F_{32} & F_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{Bmatrix} aF_{12} \\ aF_{22} \\ aF_{32} \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} F_{12} = 0 \\ F_{22} = 1 \\ F_{32} = 0 \end{cases}$$

Gathering the above information, we have:

$$F_{ij} = \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} \quad ; \quad |\boldsymbol{F}| = F_{33} > 0$$

The volume of the solid becomes $"p"$ *times the initial volume.* The relationship between the initial (reference) volume and the current (final) volume is given by:

$$dV = |\boldsymbol{F}| dV_0 \quad \Rightarrow \quad \int dV = \int |\boldsymbol{F}| dV_0 \quad \Rightarrow \quad V_{final} = |\boldsymbol{F}| V_{initial} = F_{33} V_{initial}$$

where we have considered the homogeneous deformation case. With this, we can conclude that $F_{33} = p$

(The length of segment $\overline{AC}$ *becomes* $\frac{p}{\sqrt{2}}$ *times the initial length).* As we are dealing with a homogeneous deformation, a line in the reference configuration will remain a line in the current configuration.

The point $C(X_1 = 0, X_2 = 0, X_3 = a)$ moves to:

$$\begin{Bmatrix} x_1^C \\ x_2^C \\ x_3^C \end{Bmatrix} = \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & p \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} x_1^C \\ x_2^C \\ x_3^C \end{Bmatrix} = \begin{Bmatrix} aF_{13} \\ aF_{23} \\ ap \end{Bmatrix}$$

The length of segment $\overline{AC}$ in the reference configuration is $L_{\overline{AC}} = a\sqrt{2}$. The vector that connect the points $A' \equiv A(x_1 = a, x_2 = 0, x_3 = 0)$ and $C'(x_1 = aF_{13}, x_2 = aF_{23}, x_3 = ap)$ in the current configuration is given by:

$$\overrightarrow{AC} = (aF_{13} - a)\hat{\mathbf{e}}_1 + (aF_{23})\hat{\mathbf{e}}_2 + (ap)\hat{\mathbf{e}}_3$$

and its magnitude is:

$$\left\| \overrightarrow{AC} \right\| = \ell_{\overline{AC}} = \sqrt{(a(F_{13} - 1))^2 + (aF_{23})^2 + (ap)^2} = a\sqrt{(F_{13} - 1)^2 + (F_{23})^2 + (p)^2}$$

Using the information provided by the problem $\ell_{\overline{AC}} = \frac{p}{\sqrt{2}} L_{\overline{AC}}$, we can obtain:

$$\ell_{\overline{AC}} = \frac{p}{\sqrt{2}} L_{\overline{AC}}$$

$$a\sqrt{(F_{13} - 1)^2 + (F_{23})^2 + (p)^2} = \frac{p}{\sqrt{2}} a\sqrt{2}$$

$$\sqrt{(F_{13} - 1)^2 + (F_{23})^2 + (p)^2} = p$$

thus

$$(F_{13}-1)^2+(F_{23})^2+p^2=p^2 \quad\Rightarrow\quad (F_{13}-1)^2+(F_{23})^2=0 \quad\Rightarrow\quad \begin{cases} F_{13}=1 \\ F_{23}=0 \end{cases}$$

Then, the deformation gradient components are:

$$F_{ij}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}$$

The angle AOC *changes to* 45°.

$$dX_i^{(1)}=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad\Rightarrow\quad dx_i^{(1)}=F_{ij}dX_j^{(1)} \quad\Rightarrow\quad \begin{Bmatrix} dx_1^{(1)} \\ dx_2^{(1)} \\ dx_3^{(1)} \end{Bmatrix}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$dX_i^{(2)}=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad\Rightarrow\quad dx_i^{(2)}=F_{ij}dX_j^{(2)} \quad\Rightarrow\quad \begin{Bmatrix} dx_1^{(2)} \\ dx_2^{(2)} \\ dx_3^{(2)} \end{Bmatrix}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ p \end{Bmatrix}$$

$$\cos(AOC')=\cos(45^{\circ})=\frac{d\vec{\mathbf{x}}^{(1)}\cdot d\vec{\mathbf{x}}^{(2)}}{\left\| d\vec{\mathbf{x}}^{(1)}\right\|\left\| d\vec{\mathbf{x}}^{(2)}\right\|}=\frac{\sqrt{2}}{2}$$

where $\left\| d\vec{\mathbf{x}}^{(1)}\right\|=1$, $\left\| d\vec{\mathbf{x}}^{(2)}\right\|=\sqrt{1+p^2}$, $d\vec{\mathbf{x}}^{(1)}\cdot d\vec{\mathbf{x}}^{(2)}=1$. Then:

$$\frac{1}{\sqrt{1+p^2}}=\frac{\sqrt{2}}{2} \quad\Rightarrow\quad p=\pm 1$$

As the Jacobian determinant must be greater than zero $|\boldsymbol{F}|=p>0$, this implies that $p=1$:

$$F_{ij}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The equations of motion become:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} X_1+X_3 \\ X_2 \\ X_3 \end{Bmatrix}$$

The material displacement field becomes:

$$\vec{\mathbf{u}}(\vec{X},t)=\vec{x}(\vec{X},t)-\vec{X} \xrightarrow{\text{components}} \begin{Bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \mathsf{u}_3 \end{Bmatrix}=\begin{Bmatrix} X_1+X_3 \\ X_2 \\ X_3 \end{Bmatrix}-\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}=\begin{Bmatrix} X_3 \\ 0 \\ 0 \end{Bmatrix}$$

The spatial displacement field becomes:

$$\begin{Bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \mathsf{u}_3 \end{Bmatrix}=\begin{Bmatrix} x_3 \\ 0 \\ 0 \end{Bmatrix}$$

c) The deformed body is described in Figure 2.17.

$$\begin{Bmatrix} x_1^C \\ x_2^C \\ x_3^C \end{Bmatrix} = \begin{Bmatrix} aF_{13} \\ aF_{23} \\ ap \end{Bmatrix} = \begin{Bmatrix} a \\ 0 \\ a \end{Bmatrix}$$

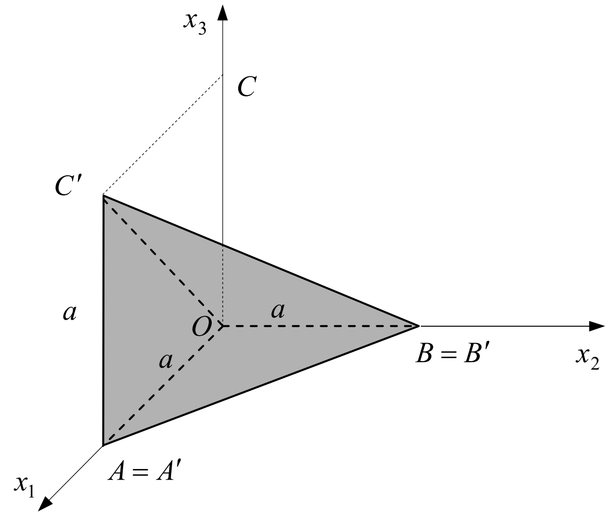


Figure 2.17

Problem 2.55

A rigid body motion is characterized by the following equation:

$$\vec{x} = \vec{\mathbf{c}}(t) + \mathbf{Q}(t) \cdot \vec{X} \tag{2.138}$$

Find the velocity and the acceleration fields as a function of $\vec{\omega}$, where $\vec{\omega}$ is the axial vector associated with the antisymmetric tensor ($\Omega = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T$).

Solution:

The material time derivative of $\vec{x} = \vec{\mathbf{c}}(t) + \mathbf{Q}(t) \cdot \vec{X}$ is given by

$$\vec{v} = \frac{D}{Dt}\vec{x} \equiv \dot{\vec{x}} = \dot{\vec{\mathbf{c}}} + \dot{\mathbf{Q}} \cdot \vec{X}$$

Let us consider that $\Omega = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \Rightarrow \dot{\mathbf{Q}} = \Omega \cdot \mathbf{Q}$. The above equation can also be expressed as:

$$\vec{v} = \dot{\vec{\mathbf{c}}} + \Omega \cdot \mathbf{Q} \cdot \vec{X} \qquad \Rightarrow \qquad \vec{v} = \dot{\vec{\mathbf{c}}} + \Omega \cdot (\vec{x} - \vec{\mathbf{c}})$$

If Ω is an antisymmetric tensor, it holds that $\Omega \cdot \vec{\mathbf{a}} = \vec{\omega} \wedge \vec{\mathbf{a}}$, where $\vec{\omega}$ (*angular velocity vector*) is the axial vector associated with the antisymmetric tensor Ω. Then, the associated velocity can be expressed as:

$$\boxed{\vec{v} = \dot{\vec{\mathbf{c}}} + \Omega \cdot (\vec{x} - \vec{\mathbf{c}}) = \dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})} \tag{2.139}$$

Note that $\mathbf{Q}(t)$ is only dependent on time, hence the axial vector (angular velocity) associated with Ω is also time-dependent, i.e. $\vec{\omega} = \vec{\omega}(t)$.

Then, its acceleration is given by:

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{x}} = \ddot{\vec{\mathbf{c}}} + \ddot{\mathbf{Q}} \cdot \vec{X}$$

By referring to $\ddot{\mathbf{Q}} = \dot{\Omega} \cdot \mathbf{Q} + \Omega \cdot \dot{\mathbf{Q}}$, the above equation can also be expressed as:

$$\vec{a} = \ddot{\vec{\mathbf{c}}} + (\dot{\Omega} \cdot \mathbf{Q} + \Omega \cdot \dot{\mathbf{Q}}) \cdot \vec{X} = \ddot{\vec{\mathbf{c}}} + \dot{\Omega} \cdot \mathbf{Q} \cdot \vec{X} + \Omega \cdot \dot{\mathbf{Q}} \cdot \vec{X}$$
$$= \ddot{\vec{\mathbf{c}}} + \dot{\Omega} \cdot \mathbf{Q} \cdot \vec{X} + \Omega \cdot \Omega \cdot \mathbf{Q} \cdot \vec{X} = \ddot{\vec{\mathbf{c}}} + \dot{\Omega} \cdot (\vec{x} - \vec{\mathbf{c}}) + \Omega \cdot \Omega \cdot (\vec{x} - \vec{\mathbf{c}})$$

we can state that:

$$\boxed{\vec{a} = \ddot{\vec{\mathbf{c}}} + \dot{\vec{\omega}} \wedge (\vec{x} - \vec{\mathbf{c}}) + \vec{\omega} \wedge [\vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})]} \qquad (2.140)$$

where $\vec{\alpha} \equiv \dot{\vec{\omega}}$ shows the *angular acceleration*.

For a rigid body motion when $\vec{\mathbf{c}} = \vec{\mathbf{0}}$, the velocity becomes $\vec{v} = \vec{\omega} \wedge \vec{x}$ whose components are $v_i = \epsilon_{ipq} \omega_p x_q$, and the rate-of-deformation tensor $\mathbf{D}$ becomes:

$$\mathsf{D}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \frac{1}{2}\left(\frac{\partial(\epsilon_{ipq}\omega_p x_q)}{\partial x_j} + \frac{\partial(\epsilon_{jpq}\omega_p x_q)}{\partial x_i} \right) = \frac{1}{2}\left(\epsilon_{ipq}\omega_p \frac{\partial x_q}{\partial x_j} + \epsilon_{jpq}\omega_p \frac{\partial x_q}{\partial x_i} \right)$$
$$= \frac{1}{2}\left(\epsilon_{ipq}\omega_p \delta_{qj} + \epsilon_{jpq}\omega_p \delta_{qi} \right) = \frac{1}{2}\left(\epsilon_{ipj}\omega_p + \epsilon_{jpi}\omega_p \right) = \frac{1}{2}\left(\epsilon_{ipj}\omega_p - \epsilon_{ipj}\omega_p \right) = 0_{ij}$$

So, once again we have proved that $\mathbf{D} = \mathbf{0}$ for a rigid body motion, (see **Problem 2.36**).

Problem 2.56

Given a coordinate system $\vec{x}$ which is fixed in space, and the mobile system $\vec{x}^*$ characterized only by rotation, (see Figure 2.18). Show that the rate of change of a vector $\vec{b}$ can be represented by:

$$\boxed{\left(\frac{D\vec{b}}{Dt} \right)_{fixed} = \left(\frac{D\vec{b}}{Dt} \right)_{mobile} + \vec{\varphi} \wedge \vec{b} = \left(\frac{D\vec{b}}{Dt} \right)_{mobile} + \Omega^T \cdot \vec{b}} \qquad (2.141)$$

where $\left(\frac{D\vec{b}}{Dt} \right)_{fijo}$ represents the rate of change of $\vec{b}$ with respect to the fixed system $\vec{x}$, $\left(\frac{D\vec{b}}{Dt} \right)_{móvil}$ represents the rate of change of $\vec{b}$ with respect to the mobile system which its angular velocity is $\vec{\varphi}$.

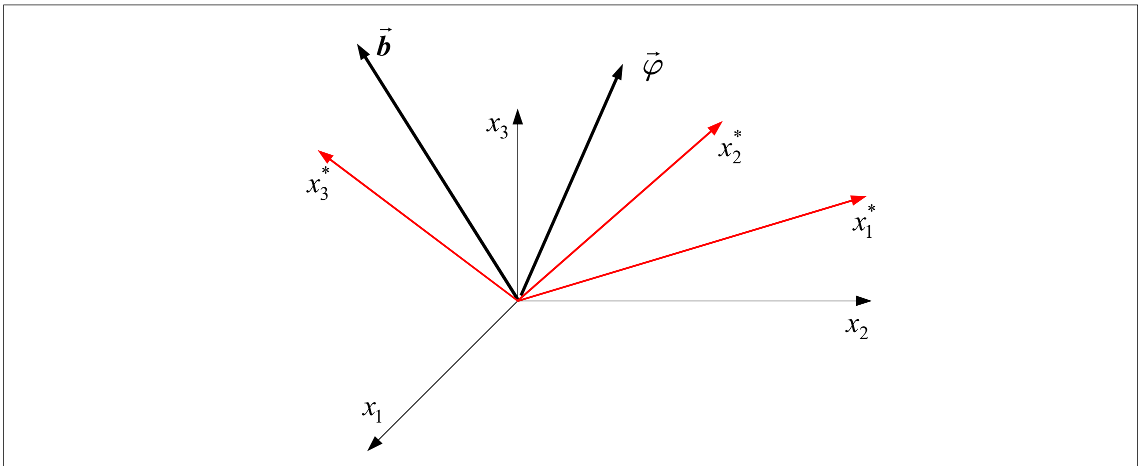


Figure 2.18

Solution:

By means of component transformation law the following relationships are true:

$$\vec{b}^* = \boldsymbol{\mathcal{A}} \cdot \vec{b} \quad \Leftrightarrow \quad \vec{b} = \boldsymbol{\mathcal{A}}^T \cdot \vec{b}^* \quad \text{(components)}$$

where $\boldsymbol{\mathcal{A}}$ is the matrix transformation from the system $\vec{x}$ to the system $\vec{x}^*$.

The rate of change of the vector $\vec{b} = \boldsymbol{\mathcal{A}}^T \cdot \vec{b}^*$ can be evaluated as follows:

$$\frac{D}{Dt}\vec{b} \equiv \dot{\vec{b}} = \frac{D}{Dt}\left[\boldsymbol{\mathcal{A}}^T \cdot \vec{b}^*\right] = \dot{\boldsymbol{\mathcal{A}}}^T \cdot \vec{b}^* + \boldsymbol{\mathcal{A}}^T \cdot \dot{\vec{b}}^* \tag{2.142}$$

Making an analogy with rate of change of the orthogonal tensor, (see Chaves(2013) – Chapter 2), we can state that $\Omega = \dot{\boldsymbol{\mathcal{A}}} \cdot \boldsymbol{\mathcal{A}}^T \Rightarrow \dot{\boldsymbol{\mathcal{A}}}^T = \boldsymbol{\mathcal{A}}^T \cdot \Omega^T$, where Ω^T is an antisymmetric tensor and represents the rate of change of rotate of the system $\vec{x}^*$ with respect to the fixed system $\vec{x}$. Then, the equation in (2.142) can be rewritten as follows:

$$\frac{D}{Dt}\vec{b} \equiv \dot{\vec{b}} = \dot{\boldsymbol{\mathcal{A}}}^T \cdot \vec{b}^* + \boldsymbol{\mathcal{A}}^T \cdot \dot{\vec{b}} = \boldsymbol{\mathcal{A}}^T \cdot \Omega^T \cdot \vec{b}^* + \boldsymbol{\mathcal{A}}^T \cdot \dot{\vec{b}}^* = \boldsymbol{\mathcal{A}}^T \cdot \left[\Omega^T \cdot \vec{b}^* + \dot{\vec{b}}^*\right] \tag{2.143}$$

Recall the antisymmetric tensor property $\Omega^T \cdot \vec{b}^* = \vec{\varphi} \wedge \vec{b}^*$, where $\vec{\varphi}$ is the axial vector associated with the antisymmetric tensor Ω^T, i.e. $\vec{\varphi} = \vec{\varphi}(t)$ is the angular velocity of the mobile system $\vec{x}^*$. Then, the equation in (2.143) can also be rewritten as:

$$\dot{\vec{b}} = \boldsymbol{\mathcal{A}}^T \cdot \left[\Omega^T \cdot \vec{b}^* + \dot{\vec{b}}^*\right] = \boldsymbol{\mathcal{A}}^T \cdot \left[\vec{\varphi}^* \wedge \vec{b}^* + \dot{\vec{b}}^*\right] \quad \text{(components)} \tag{2.144}$$

Note that the term $\boldsymbol{\mathcal{A}} \cdot \dot{\vec{b}}$ represents the components of $\dot{\vec{b}}$ in the system $\vec{x}^*$, and also note that $\boldsymbol{\mathcal{A}} \cdot \dot{\vec{b}} \neq \dot{\vec{b}}^*$, thus:

$$\left[\boldsymbol{\mathcal{A}} \cdot \dot{\vec{b}}\right]^* = \dot{\vec{b}}^* + \vec{\varphi}^* \wedge \vec{b}^* \quad \text{(components)} \tag{2.145}$$

which in tensorial notation becomes:

$$\left(\frac{D\vec{b}}{Dt}\right)_{fijo} = \left(\frac{D\vec{b}}{Dt}\right)_{móvil} + \vec{\varphi} \wedge \vec{b} \quad \text{(tensorial notation)} \tag{2.146}$$

Problem 2.57

a) A continuum is rotating as a rigid body with a constant angular velocity $\vec{\omega} = \omega_3 \hat{\mathbf{e}}_3$, (see Figure 2.19):

a.1) Obtain the velocity components in the spatial and material descriptions;

a.2) Obtain the acceleration in the spatial (Eulerian) description;

a.3) When $\omega_3 = 3rad/s$, obtain the vector position, velocity and acceleration at time $t = 2.5s$ of the particle that in the reference configuration was at $(1,1,0)$.

b) Taking into account **Problem 1.129** where we have obtained the body force vector (per unit mass) $\vec{\mathbf{b}} = -\frac{GM}{\|\vec{x}\|}\hat{x}$ where $g = \|\vec{\mathbf{b}}\|$ is the acceleration of gravity caused by gravitational field. Now, if we consider the Earth as a sphere that rotates around its axis with angular velocity $\vec{\omega} = \omega_3 \hat{\mathbf{e}}_3$, obtaining the acceleration of gravity (g_ϕ) at sea level in terms of the latitude ϕ.

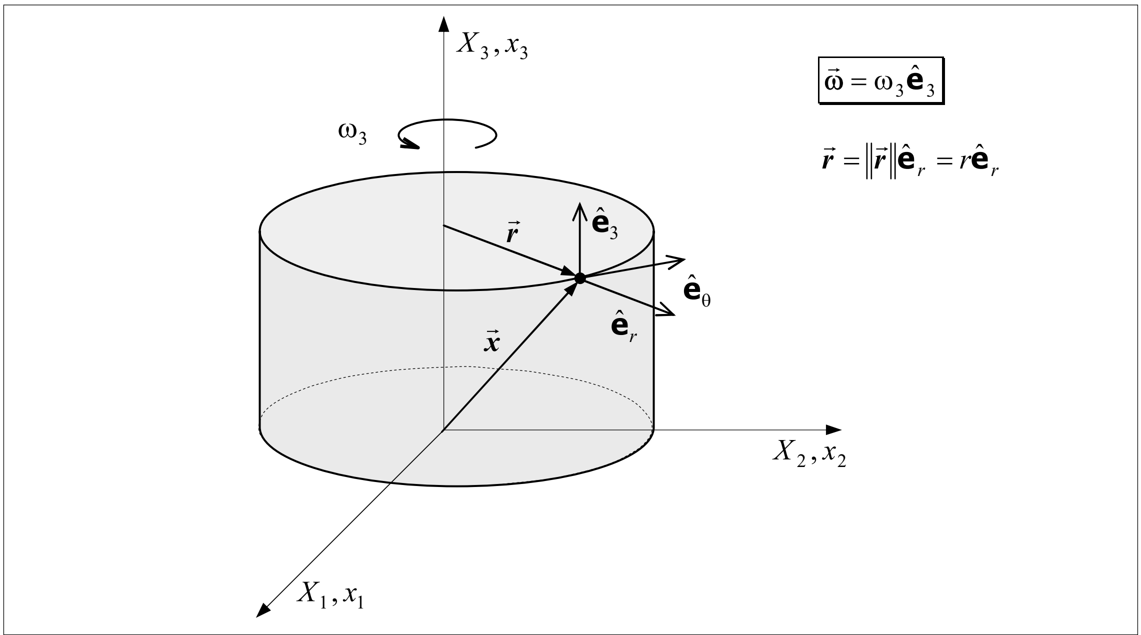


Figure 2.19

Solution:

a.1) By means of **Problem 2.55** we can conclude that $\vec{v}(\vec{x},t) = \vec{\omega} \wedge \vec{x}$, or in indicial notation:

$$
\begin{aligned}
v_i &= \epsilon_{ijk}\omega_j x_k = \epsilon_{i1k}\underbrace{\omega_1}_{=0} x_k + \epsilon_{i2k}\underbrace{\omega_2}_{=0} x_k + \epsilon_{i3k}\omega_3 x_k = \epsilon_{i3k}\omega_3 x_k \\
&= \epsilon_{i31}\omega_3 x_1 + \epsilon_{i32}\omega_3 x_2 + \underbrace{\epsilon_{i33}}_{=0_i}\omega_3 x_3 = \epsilon_{i31}\omega_3 x_1 + \epsilon_{i32}\omega_3 x_2
\end{aligned}
$$

Then:

$$
v_1 = \epsilon_{132}\omega_3 x_2 = -\omega_3 x_2 \quad ; \quad v_2 = \epsilon_{231}\omega_3 x_1 = \omega_3 x_1 \quad ; \quad v_3 = 0 \tag{2.147}
$$

Note that the field $\vec{v}(\vec{x},t)$ is stationary, i.e. $\vec{v} = \vec{v}(\vec{x})$.

For a rigid body motion when $\vec{\mathbf{c}}(t) = \vec{\mathbf{0}}$, the equations of motion are governed by:

$$
\vec{x} = \mathbf{Q}(t) \cdot \vec{X}
$$

where the orthogonal matrix components are given by the transformation matrix from the system $\vec{x}'$ to $\vec{x}$, thus:

$$
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} \cos\theta(t) X_1 - \sin\theta(t) X_2 \\ \sin\theta(t) X_1 + \cos\theta(t) X_2 \\ X_3 \end{Bmatrix}
$$

Considering that $\omega = \dfrac{d\theta(t)}{dt}$ and by integrating we can obtain:

$$
\int d\theta(t) = \int \omega dt \quad \Rightarrow \quad \theta(t) = \omega t
$$

Then, we can rewrite the equations of motion as follows:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_1\cos(\omega t) - X_2\sin(\omega t) \\ X_1\sin(\omega t) + X_2\cos(\omega t) \\ X_3 \end{Bmatrix} \tag{2.148}$$

To obtain the expression for velocity in the material (Lagrangian) description, we replace the equations of motion (2.148) into the equations (2.147):

$$\begin{cases} v_1(\vec{X},t) = -\omega_3(X_1\sin(\omega t) + X_2\cos(\omega t)) \\ v_2(\vec{X},t) = \omega_3(X_1\cos(\omega t) - X_2\sin(\omega t)) \\ v_3(\vec{X},t) = 0 \end{cases} \tag{2.149}$$

a.2) The Eulerian acceleration can be obtained by means of the definition of material time derivative of $\vec{v}(\vec{x},t)$, i.e.:

$$\vec{a}(\vec{x},t) = \underbrace{\frac{\partial \vec{v}(\vec{x},t)}{\partial t}}_{\vec{0}} + \frac{\partial \vec{v}(\vec{x},t)}{\partial \vec{x}} \cdot \frac{\partial \vec{x}(\vec{X},t)}{\partial t} = (\nabla_{\vec{x}}\vec{v}) \cdot \vec{v}(\vec{x},t)$$

where the spatial velocity gradient components are given by:

$$\left(\frac{\partial \vec{v}(\vec{x},t)}{\partial \vec{x}}\right)_{ij} = (\nabla_{\vec{x}}\vec{v})_{ij} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (antisymmetric)}$$

With that, we check that we are dealing with a rigid body motion. Then, the Eulerian acceleration components are given by:

$$a_i(\vec{x},t) = \left[(\nabla_{\vec{x}}\vec{v}) \cdot \vec{v}(\vec{x},t)\right]_i = \begin{bmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\omega_3 x_2 \\ \omega_3 x_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\omega_3^2 x_1 \\ -\omega_3^2 x_2 \\ 0 \end{Bmatrix}$$

We can express the acceleration $\vec{a}(\vec{x},t) = -\omega_3^2 x_1 \hat{\mathbf{e}}_1 - \omega_3^2 x_2 \hat{\mathbf{e}}_2$ in the cylindrical coordinate, (see Figure 2.19). Note that:

$x_1 = r\cos\theta$, $x_1 = r\cos\theta$, $\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_r \sin\theta + \hat{\mathbf{e}}_\theta \cos\theta$. Then, the acceleration in the cylindrical coordinate system becomes:

$$\begin{aligned} \vec{a} &= -\omega_3^2 x_1 \hat{\mathbf{e}}_1 - \omega_3^2 x_2 \hat{\mathbf{e}}_2 \\ &= -\omega_3^2 (r\cos\theta)(\hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta) - \omega_3^2 (r\sin\theta)(\hat{\mathbf{e}}_r \sin\theta + \hat{\mathbf{e}}_\theta \cos\theta) \\ &= -\omega_3^2 r(\cos^2\theta + \sin^2\theta)\hat{\mathbf{e}}_r = -\omega_3^2 r\hat{\mathbf{e}}_r = -\omega_3^2 \vec{r} \end{aligned}$$

The latter is known as the centripetal acceleration.

a.3) The particle at position $(1,1,0)$ in the reference configuration describes a circular path of radius $r = \sqrt{2}$ on the $x_1 - x_2$-plane, (see Figure 2.20).

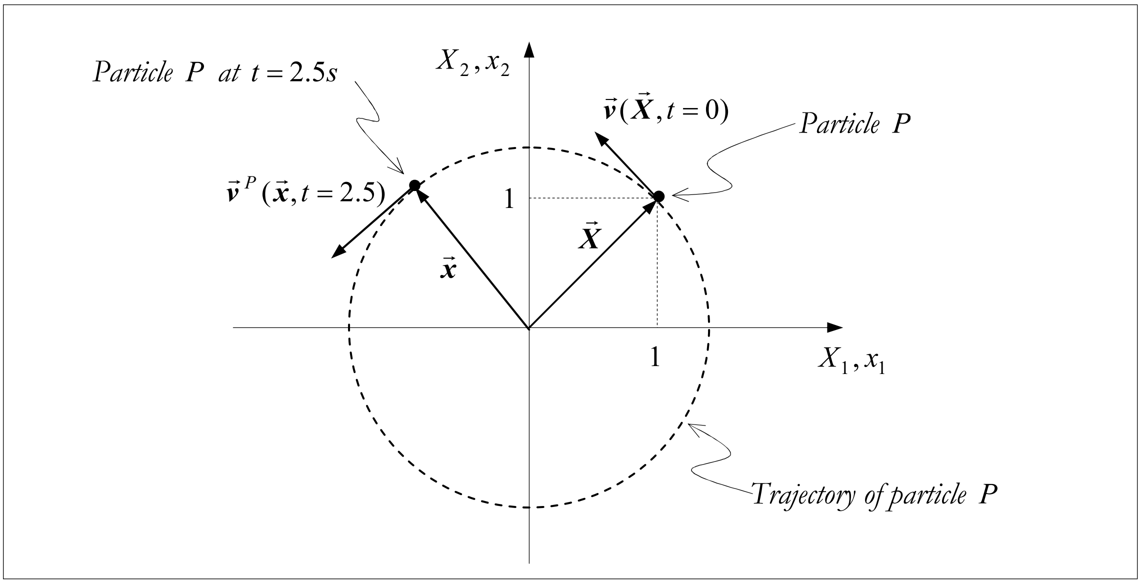


Figure 2.20

In the reference configuration $(t=0)$ it fulfills that $\vec{X}=\vec{x}$. For the particle P we have:

$$\begin{cases} v_1^P(\vec{x}, t=0) = -\omega_3 x_2 = -\omega_3 X_2 = -(3)(1) = -3 \\ v_2^P(\vec{x}, t=0) = \omega_3 x_1 = \omega_3 X_1 = (3)(1) = 3 \\ v_3^P = 0 \end{cases}$$

$$a_i^P(\vec{x}, t=0) = \begin{Bmatrix} -\omega_3^2 X_1 \\ -\omega_3^2 X_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -9 \\ -9 \\ 0 \end{Bmatrix}$$

At time $t=2.5s$ the position, velocity, and acceleration of the particle P are given by:

$$\begin{Bmatrix} x_1^P \\ x_2^P \\ x_3^P \end{Bmatrix} = \begin{Bmatrix} X_1 \cos(\omega t) - X_2 \sin(\omega t) \\ X_1 \sin(\omega t) + X_2 \cos(\omega t) \\ X_3 \end{Bmatrix} = \begin{Bmatrix} \cos(3\times 2.5) - \sin(3\times 2.5) \\ \sin(3\times 2.5) + \cos(3\times 2.5) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.59136 \\ 1.28464 \\ 0 \end{Bmatrix}$$

$$\begin{cases} v_1^P(\vec{x}, t=2.5) = -\omega_3 x_2 = -(3)(1.28464) = -3.85391 \\ v_2^P(\vec{x}, t=2.5) = \omega_3 x_1 = (3)(-0.59136) = -1.77409 \\ v_3^P = 0 \end{cases}$$

$$a_i^P(\vec{x}, t=2.5) = \begin{Bmatrix} -\omega_3^2 x_1 \\ -\omega_3^2 x_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5.322 \\ -11.562 \\ 0 \end{Bmatrix}$$

b) For a particle located on the surface of the Earth, due to rotation, this particle will feel as being projected outward according to r-direction, (see Figure 2.21). Keep in mind that the real force is the Centripetal due to the centripetal acceleration. For convenience, we adopt a fictitious force, centrifugal force, which would be the cause of this apparent outward projection. Associated with this force we have the centrifugal acceleration ($\vec{a}_{ctfu}$) which is equal but opposite to the centripetal acceleration ($\vec{a}_{ctpe}$).

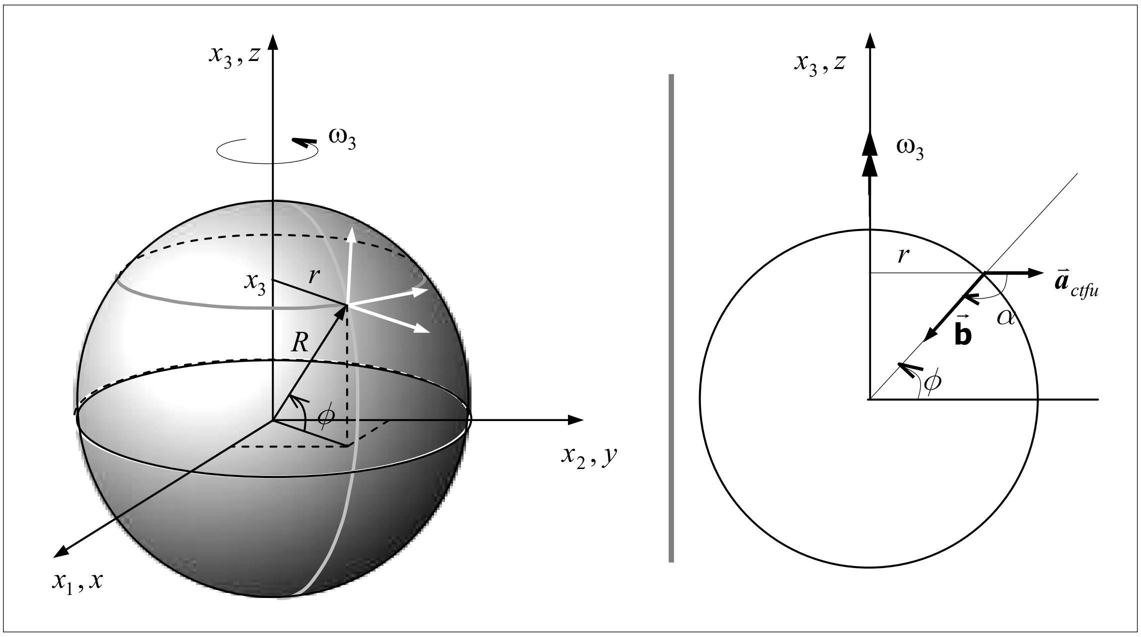


Figure 2.21

Acceleration of gravity for defined latitude ϕ is given by:

$$g_\phi = \left\| \vec{a}_{ctfu} + \vec{\mathbf{b}} \right\|$$

Remember that given two vectors it holds that $\left\| \vec{a}_{ctfu} + \vec{\mathbf{b}} \right\| = \sqrt{\left\| \vec{a}_{ctfu} \right\|^2 + 2\left\| \vec{a}_{ctfu} \right\| \left\| \vec{\mathbf{b}} \right\| \cos\alpha + \left\| \vec{\mathbf{b}} \right\|^2}$, (see **Problem 1.02**). For this particular case, we have $\left\| \vec{\mathbf{b}} \right\| = g$, $\left\| \vec{a}_{ctpe} \right\| = \left\| \vec{a}_{ctfu} \right\| = \left\| -\omega_3^2 \vec{r} \right\| = \omega_3^2 r$. Also check that $r = R\cos\phi$ and $\cos\alpha = \cos(\pi - \phi) = -\cos\phi$. With that, we can obtain:

$$g_\phi = \left\| \vec{a}_{ctfu} + \vec{\mathbf{b}} \right\| = \sqrt{\left\| \vec{a}_{ctpe} \right\|^2 - 2\left\| \vec{a}_{ctpe} \right\| \left\| \vec{\mathbf{b}} \right\| \cos\phi + \left\| \vec{\mathbf{b}} \right\|^2} = \sqrt{(\omega_3^2 r)^2 - 2(\omega_3^2 r)\, g\cos\phi + g^2}$$
$$= \sqrt{(\omega_3^2 R\cos\phi)^2 - 2(\omega_3^2 R\cos\phi)\, g\cos\phi + g^2}$$

thus

$$g_\phi = \sqrt{g^2 - 2g\omega_3^2 R\cos^2\phi + \omega_3^4 R^2\cos^2\phi}$$

Note that at the poles ($\phi = 90^\circ$) we have $g_\phi^{Pol} = g$, and in the line of Ecuador it holds that $g_\phi^{Ecu} = \sqrt{g^2 - 2g\omega_3^2 R + \omega_3^4 R^2} = \sqrt{(g - \omega_3^2 R)^2} = g - \omega_3^2 R$.

Problem 2.58

Consider a rod subjected to successive displacements as shown in Figure 2.22. Show that the engineering strain (also known as the Cauchy strain or the infinitesimal strain) is not additive to successive increments of strain, i.e. $\varepsilon^{(1)} + \varepsilon^{(2)} \neq \varepsilon$.

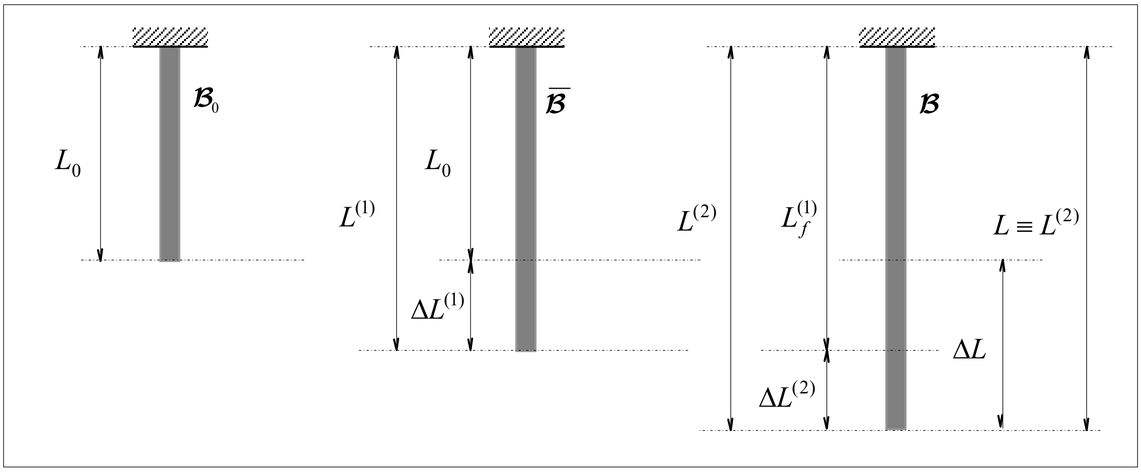


Figure 2.22

Solution:

The engineering strain was obtained as:

$$\varepsilon_C = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} = \lambda - 1$$

Then, the total strain experienced by the body, i.e. from the $\mathcal{B}_0$-configuration to the $\mathcal{B}$-configuration is:

$$\varepsilon_C = \frac{L^{(2)} - L_0}{L_0} = \frac{L^{(2)}}{L_0} - 1$$

In the $\overline{\mathcal{B}}$ -configuration the engineering strain is:

$$\varepsilon_C^{(1)} = \frac{L^{(1)} - L_0}{L_0} = \frac{L^{(1)}}{L_0} - 1$$

In the $\mathcal{B}$ -configuration considering only the displacement increment $\mathbf{u}^{(2)}$, we obtain:

$$\varepsilon_C^{(2)} = \frac{L^{(2)} - L^{(1)}}{L^{(1)}} = \frac{L^{(2)}}{L^{(1)}} - 1$$

thus

$$\varepsilon_C^{(1)} + \varepsilon_C^{(2)} = \left(\frac{L^{(1)}}{L_0} - 1 + \frac{L^{(2)}}{L^{(1)}} - 1 \right) \neq \left(\frac{L^{(2)}}{L_0} - 1 \right) = \varepsilon_C$$

An essential requirement for any strain is that it can be possible to characterize the real displacement. For this case the final length is:

$$\left.\begin{aligned}\int_0^{L_0}\varepsilon_C^{(1)}dx &= \int_0^{L_0}\left(\frac{L^{(1)}}{L_0}-1\right)dx = L^{(1)}-L_0 = \Delta L^{(1)}\\ \int_0^{L_1}\varepsilon_C^{(2)}dx &= \int_0^{L_1}\left(\frac{L^{(2)}}{L^{(1)}}-1\right)dx = L^{(2)}-L^{(1)} = \Delta L^{(2)}\end{aligned}\right\}\Rightarrow \Delta L^{(1)}+\Delta L^{(2)} = \Delta L$$

$$\int_0^{L_0}\varepsilon_C\,dx = \int_0^{L_0}\left(\frac{L}{L_0}-1\right)dx = L-L_0 = \Delta L$$

The Green-Lagrange strain tensor

Note that the Green-Lagrange strain tensor in the $\mathcal{B}$-configuration is given by:

$$\varepsilon_G = \frac{L^2-L_0^2}{2L_0^2} = \frac{1}{2}\left(\lambda^2-1\right)$$

We could have obtained the same expression by using the relationship $\boldsymbol{E} = \boldsymbol{E}^{(1)} + \boldsymbol{F}^{(1)^T}\cdot\boldsymbol{E}^{(2)}\cdot\boldsymbol{F}^{(1)}$, where for the uniaxial case we have $\boldsymbol{E}\to\varepsilon_{\boldsymbol{G}}$, $\boldsymbol{E}^{(1)}\to\varepsilon_G^{(1)}$, $\boldsymbol{E}^{(2)}\to\varepsilon_G^{(2)}$, $\boldsymbol{F}^{(1)}\to\lambda^{(1)} = \dfrac{L^{(1)}}{L_0}$. Then:

$$\boldsymbol{E} = \boldsymbol{E}^{(1)} + \boldsymbol{F}^{(1)^T}\cdot\boldsymbol{E}^{(2)}\cdot\boldsymbol{F}^{(1)}$$

$$\varepsilon_G = \varepsilon_G^{(1)} + \lambda^{(1)}\varepsilon_G^{(2)}\lambda^{(1)} = \frac{1}{2}\left[\left(\frac{L^{(1)}}{L_0}\right)^2-1\right] + \left(\frac{L^{(1)}}{L_0}\right)\left\{\frac{1}{2}\left[\left(\frac{L^{(2)}}{L^{(1)}}\right)^2-1\right]\right\}\left(\frac{L^{(1)}}{L_0}\right)$$

$$= \frac{\left(L^{(2)^2}-L_0^2\right)}{2L_0^2} = \frac{(L^2-L_0^2)}{2L_0^2}$$

2.3 Polar Decomposition of the Deformation Gradient

Problem 2.59

Let us consider the Cartesian components of the deformation gradient:

$$F_{ij} = \begin{bmatrix} 5 & 3 & 3\\ 2 & 6 & 3\\ 2 & 2 & 4\end{bmatrix}$$

obtain the tensors $\mathbf{U}$ (right stretch tensor), $\mathbf{V}$ (left stretch tensor), and $\mathbf{R}$ (rotation tensor).

Solution:

Before obtaining the tensors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{R}$, we analyze the deformation gradient $\boldsymbol{F}$.

The motion is possible if the determinant of $\boldsymbol{F}$ is greater than zero, $\det(\boldsymbol{F}) = 60 > 0$. The eigenvalues and eigenvectors of $\boldsymbol{F}$ are given by:

$F'_{11} = 10$ associated with eigenvector $\hat{m}_i^{(1)} = \left[0.6396021491;\quad 0.6396021491;\quad 0.4264014327\right]$

$F'_{22} = 3$ associated with $\hat{m}_i^{(2)} = \left[-0.5570860145;\quad 0.7427813527;\quad -0.3713906764\right]$

$F'_{33} = 2$ associated with $\hat{m}_i^{(3)} = [-0.4082482905; \quad -0.4082482905; \quad 0.8164965809]$

It is easy to check that the basis formed by these eigenvectors does not form an orthogonal basis, i.e. $\hat{m}_i^{(1)}\hat{m}_i^{(2)} \neq 0$, $\hat{m}_i^{(1)}\hat{m}_i^{(3)} \neq 0$, $\hat{m}_i^{(2)}\hat{m}_i^{(3)} \neq 0$. We can also verify that if $\mathcal{D}$ is the matrix containing the eigenvectors of $\boldsymbol{F}$:

$$\mathcal{D} = \begin{bmatrix} \hat{m}_i^{(1)} \\ \hat{m}_i^{(2)} \\ \hat{m}_i^{(3)} \end{bmatrix} = \begin{bmatrix} 0.6396021491; & 0.6396021491; & 0.4264014327 \\ -0.5570860145; & 0.7427813527; & -0.3713906764 \\ -0.4082482905; & -0.4082482905; & 0.8164965809 \end{bmatrix}$$

we find that $\det(\mathcal{D}) = 0.905 \neq 1$, and $\mathcal{D}^{-1} \neq \mathcal{D}^T$. However, it holds that:

$$\mathcal{D}^{-1}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\mathcal{D} = \begin{bmatrix} 5 & 2 & 2 \\ 3 & 6 & 2 \\ 3 & 3 & 4 \end{bmatrix} = (\boldsymbol{F}^T)_{ij} \qquad and \qquad \mathcal{D}\begin{bmatrix} 5 & 2 & 2 \\ 3 & 6 & 2 \\ 3 & 3 & 4 \end{bmatrix}\mathcal{D}^{-1} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The right Cauchy-Green deformation tensor components, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, are given by:

$$C_{ij} = F_{ki}F_{kj} = \begin{bmatrix} 33 & 31 & 29 \\ 31 & 49 & 35 \\ 29 & 35 & 34 \end{bmatrix}$$

Then the eigenvalues and eigenvectors of $\boldsymbol{C}$ are given by:

$C'_{11} = 9.274739 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(1)} = [0.6861511933; \quad -0.7023576528; \quad 0.1894472683]$

$C'_{22} = 3.770098 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(2)} = [0.5105143234; \quad 0.2793856273; \quad -0.8132215099]$

$C'_{33} = 102.955163 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(3)} = [-0.518239; \quad -0.65470405; \quad -0.550264423]$

These eigenvectors constitute an orthogonal basis, so, it holds that $\mathcal{A}_{\boldsymbol{C}}^{-1} = \mathcal{A}_{\boldsymbol{C}}^T$, and $\det(\mathcal{A}_{\boldsymbol{C}}) = -1$ (improper orthogonal tensor):

$$\mathcal{A}_{\boldsymbol{C}} = \begin{bmatrix} \hat{\mathsf{N}}_i^{(1)} \\ \hat{\mathsf{N}}_i^{(2)} \\ \hat{\mathsf{N}}_i^{(3)} \end{bmatrix} = \begin{bmatrix} 0.6861511933 & -0.7023576528 & 0.1894472683 \\ 0.5105143234 & 0.2793856273 & -0.8132215099 \\ -0.518239 & -0.65470405 & -0.550264423 \end{bmatrix}$$

Furthermore, it holds that:

$$\mathcal{A}_{\boldsymbol{C}}^T\begin{bmatrix} C'_{11} & 0 & 0 \\ 0 & C'_{22} & 0 \\ 0 & 0 & C'_{33} \end{bmatrix}\mathcal{A}_{\boldsymbol{C}} = \begin{bmatrix} 33 & 31 & 29 \\ 31 & 49 & 35 \\ 29 & 35 & 34 \end{bmatrix} = C_{ij} \quad ; \quad \mathcal{A}_{\boldsymbol{C}}\begin{bmatrix} 33 & 31 & 29 \\ 31 & 49 & 35 \\ 29 & 35 & 34 \end{bmatrix}\mathcal{A}_{\boldsymbol{C}}^T = \begin{bmatrix} C'_{11} & 0 & 0 \\ 0 & C'_{22} & 0 \\ 0 & 0 & C'_{33} \end{bmatrix}$$

In the $\boldsymbol{C}$ principal space we obtain the components of the right stretch tensor, $\mathbf{U}$, as:

$$\mathcal{U}' = \mathsf{U}'_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \sqrt{C'_{11}} & 0 & 0 \\ 0 & \sqrt{C'_{22}} & 0 \\ 0 & 0 & \sqrt{C'_{33}} \end{bmatrix} = \begin{bmatrix} 3.0454455 & 0 & 0 \\ 0 & 1.9416741 & 0 \\ 0 & 0 & 10.1466824 \end{bmatrix}$$

and its inverse:

$$\mathcal{U}'^{-1} = \mathsf{U}'^{-1}_{ij} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3.0454455} & 0 & 0 \\ 0 & \frac{1}{1.9416741} & 0 \\ 0 & 0 & \frac{1}{10.1466824} \end{bmatrix}$$

We can evaluate the components of the tensor $\mathbf{U}$ in the original space by means of the transformation law:

$$\mathcal{A}_C^T \mathcal{U}' \mathcal{A}_C = \begin{bmatrix} 4.66496626 & 2.25196988 & 2.48328843 \\ 2.25196988 & 6.00314487 & 2.80907159 \\ 2.48328843 & 2.80907159 & 4.46569091 \end{bmatrix} = \mathsf{U}_{ij}$$

and

$$\mathcal{A}_C^T \mathcal{U}'^{-1} \mathcal{A}_C = \begin{bmatrix} 0.31528844 & -0.05134777 & -0.14302659 \\ 2.25196988 & 0.24442627 & -0.12519889 \\ -0.14302659 & -0.12519889 & 0.38221833 \end{bmatrix} = \mathsf{U}^{-1}_{ij}$$

Then, the rotation tensor of the polar decomposition is given by the equation $\mathbf{R} = \boldsymbol{F} \cdot \mathbf{U}^{-1}$, which is a proper orthogonal tensor, i.e. $\det(\mathbf{R}) = 1$.

$$\mathsf{R}_{ij} = F_{ik} \mathsf{U}^{-1}_{kj} = \begin{bmatrix} 0.9933191 & 0.10094326 & 0.05592536 \\ -0.10658955 & 0.98826538 & 0.10940847 \\ -0.04422505 & -0.11463858 & 0.9924224 \end{bmatrix}$$

The left Cauchy-Green deformation tensor components, $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$, are given by:

$$b_{ij} = F_{ik} F_{jk} = \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix}$$

Next, the eigenvalues and eigenvectors of $\boldsymbol{b}$ are given by:

$$b'_{11} = 9.274739 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(1)} = [0.6212637156 \quad -0.7465251613 \quad 0.238183919]$$

$$b'_{22} = 3.770098 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(2)} = [0.4898263742 \quad 0.1327190337 \quad -0.8616587383]$$

$$b'_{33} = 102.95516 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(3)} = [-0.611638389 \quad -0.6519860747 \quad -0.448121233]$$

Note that, the tensors $\boldsymbol{b}$ and $\boldsymbol{C}$ have the same eigenvalues but different eigenvectors. If the eigenvectors of $\boldsymbol{b}$ constitute an orthogonal basis then it holds that $\mathcal{A}_{\boldsymbol{b}}^{-1} = \mathcal{A}_{\boldsymbol{b}}^T$, and $\det(\mathcal{A}_{\boldsymbol{b}}) = -1$:

$$\mathcal{A}_{\boldsymbol{b}} = \begin{bmatrix} \hat{\mathsf{n}}_i^{(1)} \\ \hat{\mathsf{n}}_i^{(2)} \\ \hat{\mathsf{n}}_i^{(3)} \end{bmatrix} = \begin{bmatrix} 0.6212637156 & -0.7465251613 & 0.238183919 \\ 0.4898263742 & 0.1327190337 & -0.8616587383 \\ -0.611638389 & -0.6519860747 & -0.448121233 \end{bmatrix}$$

and, it also holds that:

$$\mathcal{A}_{\boldsymbol{b}}^T \begin{bmatrix} b'_{11} & 0 & 0 \\ 0 & b'_{22} & 0 \\ 0 & 0 & b'_{33} \end{bmatrix} \mathcal{A}_{\boldsymbol{b}} = \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix} = b_{ij} \quad ; \quad \mathcal{A}_{\boldsymbol{b}} \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix} \mathcal{A}_{\boldsymbol{b}}^T = \begin{bmatrix} b'_{11} & 0 & 0 \\ 0 & b'_{22} & 0 \\ 0 & 0 & b'_{33} \end{bmatrix}$$

Since $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues, it follows that $\mathsf{U}'_{ij} = \mathsf{V}'_{ij}$, i.e. they have the same components in their respectively principal space. Additionally, it holds that $\mathsf{U}'^{-1}_{ij} = \mathsf{V}'^{-1}_{ij}$.

The components of the tensor $\mathbf{V}$ in the original space can be evaluated by:

$$\mathcal{A}_b^T \mathcal{V}' \mathcal{A}_b = \mathcal{A}_b^T \mathcal{U}' \mathcal{A}_b = \begin{bmatrix} 5.3720129 & 2.76007379 & 2.41222612 \\ 2.76007379 & 6.04463857 & 2.20098553 \\ 2.41222612 & 2.20098553 & 3.6519622 \end{bmatrix} = \mathsf{V}_{ij}$$

and

$$\mathcal{A}_b^T \mathcal{V}'^{-1} \mathcal{A}_b = \mathcal{A}_b^T \mathcal{U}'^{-1} \mathcal{A}_b = \begin{bmatrix} 0.28717424 & -0.07950684 & -0.14176921 \\ -0.07950684 & 0.23396031 & -0.08848799 \\ -0.14176921 & -0.08848799 & 0.42079849 \end{bmatrix} = \mathsf{V}^{-1}_{ij}$$

The polar decomposition rotation tensor obtained previously has to be the same as the one obtained by $\mathbf{R} = \mathbf{V}^{-1} \cdot \boldsymbol{F}$.

We could also have obtained the tensors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{R}$, by means of their spectral representation. That is, if we know the principal stretches, λ_i, and the eigenvectors of $\boldsymbol{C}$ ($\hat{\mathbf{N}}^{(i)}$), and the eigenvectors of $\boldsymbol{b}$ ($\hat{\mathbf{n}}^{(i)}$), it is easy to show that:

$$\mathsf{U}_{ij} = \left(\sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \lambda_1 \, \hat{\mathsf{N}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \lambda_2 \, \hat{\mathsf{N}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \lambda_3 \, \hat{\mathsf{N}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$\mathsf{V}_{ij} = \left(\sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right)_{ij} = \lambda_1 \, \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{n}}_j^{(1)} + \lambda_2 \, \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{n}}_j^{(2)} + \lambda_3 \, \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{n}}_j^{(3)}$$

$$\mathsf{R}_{ij} = \left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$F_{ij} = \left(\sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \lambda_1 \, \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \lambda_2 \, \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \lambda_3 \, \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$\begin{aligned} \boldsymbol{F} &= \sum_{a=1}^{3} \lambda_a \, \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} \\ &= \mathbf{R} \cdot \left(\sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right) = \left(\sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \end{aligned}$$

As we can verify, the representations of the tensors $\mathbf{R}$ and $\boldsymbol{F}$ are not the spectral representations in the strict sense of the word, i.e., λ_i are not eigenvalues of $\boldsymbol{F}$, and neither $\hat{\mathbf{n}}^{(i)}$ nor $\hat{\mathbf{N}}^{(i)}$ are eigenvectors of $\boldsymbol{F}$.

Problem 2.60

The deformation gradient at one point of the body is given by:

$$\boldsymbol{F} = 0.2 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 - 0.1 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + 0.3 \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + 0.4 \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + 0.1 \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$$

where $\hat{\mathbf{e}}_i \quad (i = 1,2,3)$ represents the Cartesian basis.

a) Obtain the deformation tensors $\boldsymbol{b}$ and $\boldsymbol{C}$;
b) Obtain the eigenvalues and eigenvectors of $\boldsymbol{b}$ and $\boldsymbol{C}$;

c) Write the "spectral representation" of $\boldsymbol{F}$ in function of the eigenvalues of $\boldsymbol{C}$ (C_a) and check if $\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ holds, where λ_a are the principal stretches, $\hat{\mathbf{n}}$ are the eigenvectors of $\boldsymbol{b}$, and $\hat{\mathbf{N}}$ are the eigenvectors of $\boldsymbol{C}$;

d) Obtain the spectral representation and components of: ($\mathbf{R}$) spin tensor of the polar decomposition; the stretch tensors $\mathbf{U}$ and $\mathbf{V}$;

Solution

The deformation gradient components can be represented as follows:

$$\boldsymbol{F} = F_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = 0.2\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 - 0.1\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + 0.3\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + 0.4\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + 0.1\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3$$

$$F_{ij} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

a) The left Cauchy-Green deformation tensor ($\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$) components are given by:

$$b_{ij} = F_{ik} F_{jk} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}^T = \begin{bmatrix} 0.05 & 0.02 & 0 \\ 0.02 & 0.25 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \qquad (2.150)$$

The right Cauchy-Green deformation tensor ($\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$) components are given by:

$$C_{ij} = F_{ki} F_{kj} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}^T \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.13 & 0.1 & 0 \\ 0.1 & 0.17 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \qquad (2.151)$$

b) The eigenvalues and eigenvectors of $\boldsymbol{b}$ and $\boldsymbol{C}$ are obtained as follows;

$$\boldsymbol{C} \cdot \hat{\mathbf{N}} = C_{(a)} \hat{\mathbf{N}}^{(a)} \quad \Rightarrow \quad \left| \boldsymbol{C} - C\mathbf{1} \right| = 0$$

where the index (a) does not indicate summation. Note that we already know one eigenvalue of $\boldsymbol{C}$, i.e. $C_{(3)} = 0.01$, (see $\boldsymbol{C}$-components in (2.151)). Then, the characteristic determinant becomes:

$$\begin{vmatrix} 0.13 - C & 0.1 \\ 0.1 & 0.17 - C \end{vmatrix} = 0 \quad \Rightarrow \quad (0.13 - C)(0.17 - C) - 0.01 = 0$$

The solution of the quadratic equation is:

$$C_{(1)} = 0.25198 \quad ; \quad C_{(2)} = 0.04802$$

Then:

$$Cc_{(1)} = 0.25198 \quad \Rightarrow \quad \hat{\mathsf{N}}_i^{(1)} = \begin{bmatrix} 0.633399 \\ 0.77334 \\ 0 \end{bmatrix} \quad ; \quad C_{(2)} = 0.04802 \quad \Rightarrow \quad \hat{\mathsf{N}}_i^{(2)} = \begin{bmatrix} -0.77334 \\ 0.63399 \\ 0 \end{bmatrix}$$

$$C_{(3)} = 0.01 \quad \Rightarrow \quad \hat{\mathsf{N}}_i^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors of the tensor $\boldsymbol{b}$:

$$\boldsymbol{b} \cdot \hat{\mathbf{n}} = b_{(a)} \hat{\mathbf{n}}^{(a)}$$

where the index (a) does not indicate summation. Then

$$b_{(1)} = 0.25198 \quad \Rightarrow \quad \hat{\mathrm{n}}_i^{(1)} = \begin{bmatrix} 0.098538 \\ 0.995133 \\ 0 \end{bmatrix} \quad ; \quad b_{(2)} = 0.04802 \quad \Rightarrow \quad \hat{\mathrm{n}}_i^{(2)} = \begin{bmatrix} -0.995133 \\ 0.098538 \\ 0 \end{bmatrix}$$

$$b_{(3)} = 0.01 \quad \Rightarrow \quad \hat{\mathrm{n}}_i^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

As expected, the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues, i.e.

$$C'_{ij} = \begin{bmatrix} 0.252 & 0 & 0 \\ 0 & 0.048 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \quad ; \quad b'_{ij} = \begin{bmatrix} 0.252 & 0 & 0 \\ 0 & 0.048 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

but they have different eigenvectors. In addition, the spectral representations of the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are given respectively by:

$$\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \boldsymbol{b} = \sum_{a=1}^{3} \lambda_a^2 \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$$

where $\lambda_a > 0$ are the principal stretches. Considering that $\lambda_a^2 = C_a$ are the eigenvalues of $\boldsymbol{C}$ and of $\boldsymbol{b}$, the principal stretches are:

$$\lambda_{(1)} = \sqrt{0.25198} \approx 0.501976 \quad ; \quad \lambda_{(2)} = \sqrt{0.04802} \approx 0.219134 \quad ; \quad \lambda_{(3)} = \sqrt{0.01} = 0.1$$

c) To check if $\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ holds we calculate the components of $\left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij}$ with the results obtained previously, i.e.:

$$\left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \lambda_1 \hat{\mathrm{n}}_i^{(1)} \otimes \hat{\mathrm{N}}_j^{(1)} + \lambda_2 \hat{\mathrm{n}}_i^{(2)} \otimes \hat{\mathrm{N}}_j^{(2)} + \lambda_3 \hat{\mathrm{n}}_i^{(3)} \otimes \hat{\mathrm{N}}_j^{(3)}$$

$$= 0.50197 \begin{bmatrix} 0.76958 & -0.6309 & 0 \\ -0.0762 & 0.06247 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.219134 \begin{bmatrix} 0.06247 & 0.0762 & 0 \\ 0.6309 & 0.76958 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ 0.1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} = F_{ij}$$

With that the equation $\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ holds.

d) The orthogonal tensor is given by

$$\mathbf{R} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \qquad \textit{components} \qquad (\mathbf{R})_{ij} = \begin{bmatrix} 0.832 & -0.554 & 0 \\ 0.554 & 0.832 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which can be verified with:

$$\mathsf{R}_{ij} = \left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \hat{\mathsf{n}}_i^{(1)} \otimes \hat{\mathsf{N}}_j^{(1)} + \hat{\mathsf{n}}_i^{(2)} \otimes \hat{\mathsf{N}}_j^{(2)} + \hat{\mathsf{n}}_i^{(3)} \otimes \hat{\mathsf{N}}_j^{(3)}$$

$$\mathsf{R}_{ij} = \begin{bmatrix} 0.76958 & -0.6309 & 0 \\ -0.0762 & 0.06247 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.06247 & 0.0762 & 0 \\ 0.6309 & 0.76958 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.832 & -0.5547 & 0 \\ 0.5547 & 0.832 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The right stretch tensor:

$$\mathbf{U} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \qquad \textit{components} \qquad (\mathbf{U})_{ij} \approx \begin{bmatrix} 0.333 & 0.139 & 0 \\ 0.139 & 0.388 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

The left stretch tensor:

$$\mathbf{V} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \qquad \textit{components} \qquad (\mathbf{V})_{ij} \approx \begin{bmatrix} 0.222 & 0.028 & 0 \\ 0.028 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Problem 2.61

Consider the following equations of motion:

$$x_1 = X_1 \quad ; \quad x_2 = X_2 - \alpha X_3 \quad ; \quad x_3 = X_3 + \alpha X_2$$

a) Obtain the deformation gradient, the right Cauchy-Green deformation tensor, the left Cauchy-Green deformation tensor, the Green-Lagrange strain tensor and the Almansi strain tensor. Check whether this case represents a homogeneous deformation.

b) Obtain the right stretch tensor, the spin tensor of polar decomposition and the principal space of the left Cauchy-Green deformation tensor of the polar decomposition.

c) Obtain the final length of an initial length element equal to 2 which is in the X_3-direction, and the angular distortion of an initial angle 30° which is in the plane $X_1 - X_2$.

d) Obtain the strain tensor by considering the small deformation regime.

Solution:

a) The deformation gradient ($\boldsymbol{F}$) components are:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix}$$

In general we have $d\vec{x} = \boldsymbol{F} \cdot d\vec{X}$, and if we are dealing with a homogeneous deformation (a particular case of motion) the relationship $\vec{x} = \boldsymbol{F} \cdot \vec{X} + \vec{\mathbf{c}}(t)$ holds, a fact that can be checked by means of the equations of motion in matrix form with $\vec{\mathbf{c}}(t) = \vec{\mathbf{0}}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

The right Cauchy-Green deformation tensor, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, components are:

$$C_{ij} = F_{ki} F_{kj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 1+\alpha^2 \end{bmatrix}$$

The left Cauchy-Green deformation tensor, $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$, components are:

$$b_{ij} = F_{ik} F_{jk} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 1+\alpha^2 \end{bmatrix}$$

The Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$, and the Almansi strain tensor, $\boldsymbol{e} = \frac{1}{2}(\mathbf{1} - \boldsymbol{b}^{-1})$, are defined by their components as follows:

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 1+\alpha^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

$$e_{ij} = \frac{1}{2}(\delta_{ij} - b^{-1}_{ij}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1+\alpha^2} & 0 \\ 0 & 0 & \frac{1}{1+\alpha^2} \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\alpha^2}{1+\alpha^2} & 0 \\ 0 & 0 & \frac{\alpha^2}{1+\alpha^2} \end{bmatrix}$$

We can check the results by the relationship $\boldsymbol{E} = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}$:

$$E_{ij} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\alpha^2}{1+\alpha^2} & 0 \\ 0 & 0 & \frac{\alpha^2}{1+\alpha^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

b) According to the format of the Cartesian components of $\boldsymbol{C}$, we can verify that the original space is already the principal space of $\boldsymbol{C}$, i.e. the principal directions are $\hat{\mathrm{N}}_i^{(1)} = [1 \quad 0 \quad 0]$, $\hat{\mathrm{N}}_i^{(1)} = [0 \quad 1 \quad 0]$, $\hat{\mathrm{N}}_i^{(1)} = [0 \quad 0 \quad 1]$. By definition, the right stretch tensor is given by $\mathbf{U} = \sqrt{\boldsymbol{C}}$, and its components are:

$$\mathsf{U}_{ij} = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1+\alpha^2} & 0 \\ 0 & 0 & \sqrt{1+\alpha^2} \end{bmatrix} \xrightarrow{inverse} \mathsf{U}^{-1}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{1+\alpha^2}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{1+\alpha^2}} \end{bmatrix}$$

By means of the right polar decomposition, $\boldsymbol{F} = \mathbf{R}\cdot\mathbf{U} \Rightarrow \mathbf{R} = \boldsymbol{F}\cdot\mathbf{U}^{-1}$, we can obtain:

$$\mathsf{R}_{ij} = F_{ik}\mathsf{U}^{-1}_{kj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{1+\alpha^2}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{1+\alpha^2}} \end{bmatrix} = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} \sqrt{1+\alpha^2} & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix}$$

Note that by means of the format of the Cartesian components of $\boldsymbol{b}$ indicate that the principal directions are $[1 \quad 0 \quad 0]$, $[0 \quad 1 \quad 0]$, $[0 \quad 0 \quad 1]$, but this is not the principal directions of $\boldsymbol{b}$ related to the polar decomposition. Note that there are two equal eigenvalues related to the directions $[0 \quad 1 \quad 0]$, $[0 \quad 0 \quad 1]$, then any direction in the plane $x_2 - x_3$ is a principal direction, (see Figure 2.23).

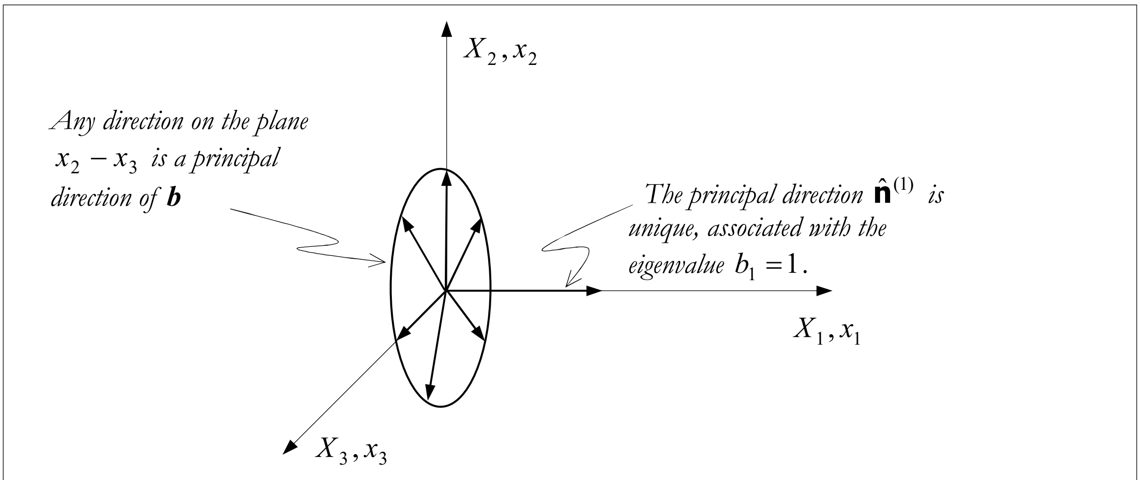


Figure 2.23: Principal space of $\boldsymbol{b}$.

Remember that the polar decomposition is unique, i.e. there is one principal base $\boldsymbol{b}$ for the polar decomposition associated with $\hat{\mathbf{N}}^{(a)}$. By means of the relation $\hat{\mathbf{n}}^{(a)} = \mathbf{R}\cdot\hat{\mathbf{N}}^{(a)}$ we can obtain the principal base of $\boldsymbol{b}$ for the polar decomposition:

$$\hat{\mathsf{n}}^{(2)}_i = (\mathbf{R}\cdot\hat{\mathbf{N}}^{(2)})_i = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} \sqrt{1+\alpha^2} & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix}$$

$$\hat{\mathsf{n}}^{(3)}_i = (\mathbf{R}\cdot\hat{\mathbf{N}}^{(3)})_i = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} \sqrt{1+\alpha^2} & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} 0 \\ -\alpha \\ 1 \end{bmatrix}$$

In addition, we can check that the relation $\mathbf{R} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ holds:

$$
\mathsf{R}_{ij} = \hat{\mathsf{n}}_i^{(1)}\hat{\mathsf{N}}_j^{(1)} + \hat{\mathsf{n}}_i^{(2)}\hat{\mathsf{N}}_j^{(2)} + \hat{\mathsf{n}}_i^{(3)}\hat{\mathsf{N}}_j^{(3)}
$$

$$
= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} 0 \\ -\alpha \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathsf{R}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 0 \end{bmatrix} + \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} \sqrt{1+\alpha^2} & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix}
$$

c) By means of the stretch definition according to the $\hat{\boldsymbol{M}}$-direction, i.e. $\lambda_{\hat{\boldsymbol{M}}} = \dfrac{\|d\vec{\mathbf{x}}\|}{\|d\vec{\mathbf{X}}\|} = \dfrac{ds}{dS}$, and considering that the stretch is not dependent on line integral (homogeneous deformation), it holds that:

$$
L_{final} = \int ds = \int \lambda_{\hat{\boldsymbol{M}}} dS = \lambda_{\hat{\boldsymbol{M}}} \int dS = \lambda_{\hat{\boldsymbol{M}}} L_{initial}
$$

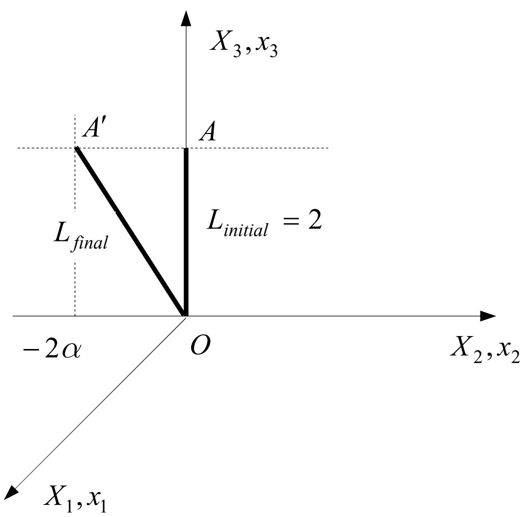
The stretch according to X_3-direction is given by:

$$
\lambda_{X_3} = \sqrt{C_{33}} = \sqrt{1+2E_{33}} = \sqrt{1+\alpha^2}
$$

Then:

$$
L_{final} = \lambda_{\hat{\boldsymbol{M}}} \int_0^2 dX_2 = \sqrt{1+\alpha^2}\,(L_{initial}) = 2\sqrt{1+\alpha^2}
$$

As we are dealing with a homogeneous deformation, a line in the reference configuration remains a line in the current configuration, (see Figure 2.24).



$$
\begin{bmatrix} x_1^A \\ x_2^A \\ x_3^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix}\begin{bmatrix} X_1^A \\ X_2^A \\ X_3^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\alpha \\ 2 \end{bmatrix}
$$

Figure 2.24

According to Figure 2.24 we can check that:

$$L^2_{initial} = 2^2 + (-2\alpha)^2 = 4(1+\alpha^2) \quad \Rightarrow \quad L_{initial} = 2\sqrt{1+\alpha^2}$$

To obtain the angle in the current configuration formed by two unit vectors, we can use the equation:

$$\cos\theta = \frac{\cos\Theta + 2\hat{\boldsymbol{M}}\cdot\boldsymbol{E}\cdot\hat{\boldsymbol{N}}}{\lambda_{\hat{\boldsymbol{M}}}\lambda_{\hat{\boldsymbol{N}}}} \tag{2.152}$$

where Θ is the angle between the unit vectors $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$ in the reference configuration, and θ is the angle between the two new unit vectors in the current configuration.

Considering that the Green-Lagrange strain tensor is independent of $\vec{\boldsymbol{X}}$, we adopt two unit vectors forming an angle $\Theta = 30^\circ$ in the plane $X_1 - X_2$, e.g. $\hat{N}_i = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and $\hat{M}_i = \begin{bmatrix}\cos 30^\circ & \sin 30^\circ & 0\end{bmatrix}$. With these data we have:

$$\hat{\boldsymbol{M}}\cdot\boldsymbol{E}\cdot\hat{\boldsymbol{N}} = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & \alpha^2 & 0\\ 0 & 0 & \alpha^2\end{bmatrix}\begin{bmatrix}\cos 30^\circ\\ \sin 30^\circ\\ 0\end{bmatrix} = 0$$

The stretches:

$$\lambda^2_{\hat{\boldsymbol{M}}} = \hat{\boldsymbol{M}}\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{M}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1+\alpha^2 & 0\\ 0 & 0 & 1+\alpha^2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} = 1 \quad \Rightarrow \quad \lambda_{\hat{\boldsymbol{M}}} = 1$$

and

$$\lambda^2_{\hat{\boldsymbol{N}}} = \hat{\boldsymbol{N}}\cdot\boldsymbol{C}\cdot\hat{\boldsymbol{N}} = \begin{bmatrix}\cos 30^\circ & \sin 30^\circ & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1+\alpha^2 & 0\\ 0 & 0 & 1+\alpha^2\end{bmatrix}\begin{bmatrix}\cos 30^\circ\\ \sin 30^\circ\\ 0\end{bmatrix}$$

$$= \cos^2 30^\circ + (1+\alpha^2)\sin^2 30^\circ = 1 + \alpha^2 \sin^2 30^\circ$$

Then, $\lambda_{\hat{\boldsymbol{N}}} = \sqrt{1+\alpha^2\sin^2 30^\circ}$. Then, we can obtain:

$$\cos\theta = \frac{\cos\Theta + 2\hat{\boldsymbol{M}}\cdot\boldsymbol{E}\cdot\hat{\boldsymbol{N}}}{\lambda_{\hat{\boldsymbol{M}}}\lambda_{\hat{\boldsymbol{N}}}} = \frac{\cos 30^\circ}{\sqrt{1+\alpha^2\sin^2 30^\circ}}$$

As we are dealing with a homogeneous deformation, we adopt two lines in the reference configuration and we obtain the angle formed by these lines in the current configuration. For example, adopting the lines $\overrightarrow{OB} = \begin{bmatrix}\cos 30^\circ & 0 & 0\end{bmatrix}$ and $\overrightarrow{OC} = \begin{bmatrix}\cos 30^\circ & \sin 30^\circ & 0\end{bmatrix}$. And according to the equations of motion, the point O does not move. Then, we obtain the new position of the points B and C, (see Figure 2.25):

$$\begin{bmatrix}x_1^B\\ x_2^B\\ x_3^B\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\alpha\\ 0 & \alpha & 1\end{bmatrix}\begin{bmatrix}X_1^B\\ X_2^B\\ X_3^B\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\alpha\\ 0 & \alpha & 1\end{bmatrix}\begin{bmatrix}\cos 30^\circ\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\cos 30^\circ\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x_1^C\\ x_2^C\\ x_3^C\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\alpha\\ 0 & \alpha & 1\end{bmatrix}\begin{bmatrix}X_1^C\\ X_2^C\\ X_3^C\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\alpha\\ 0 & \alpha & 1\end{bmatrix}\begin{bmatrix}\cos 30^\circ\\ \sin 30^\circ\\ 0\end{bmatrix} = \begin{bmatrix}\cos 30^\circ\\ \sin 30^\circ\\ \alpha\sin 30^\circ\end{bmatrix}$$

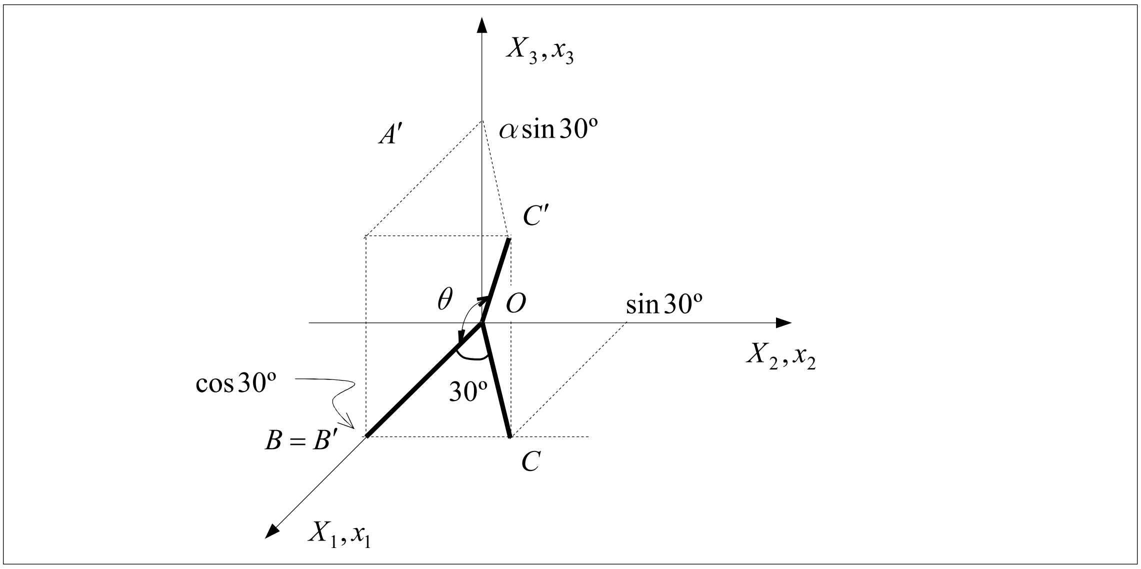


Figure 2.25

Then, the angle formed by the new unit vectors $\overrightarrow{O'B'}$ and $\overrightarrow{O'C'}$ is:

$$\overrightarrow{O'B'} \cdot \overrightarrow{O'C'} = \left\| O'B' \right\| \left\| O'C' \right\| \cos\theta$$

$$\cos^2 30^\circ = \sqrt{\cos^2 30^\circ} \sqrt{\cos^2 30^\circ + \sin^2 30^\circ + \alpha^2 \sin^2 30^\circ} \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\cos 30^\circ}{\sqrt{1+\alpha^2 \sin^2 30^\circ}}$$

d)

$$\varepsilon_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2.62

For a given motion (shear deformation):

$$x_1 = X_1 + kX_2 \quad ; \quad x_2 = X_2 \quad ; \quad x_3 = X_3$$

where k is a constant. Obtain the tensors: $\boldsymbol{F}$ (deformation gradient), $\boldsymbol{C}$ (the right Cauchy-Green deformation tensor), $\boldsymbol{b}$ (the left Cauchy-Green deformation tensor), $\boldsymbol{E}$ (the Green-Lagrange strain tensor), $\mathbf{U}$ (the right stretch tensor), $\mathbf{V}$ (the left stretch tensor) and $\mathbf{R}$ (the spin tensor of the polar decomposition).

Solution:

The deformation gradient components:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The right Cauchy-Green deformation tensor, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, components are:

$$C_{ij} = F_{ki} F_{kj} = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The left Cauchy-Green deformation tensor, $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$, components are:

$$b_{ij} = F_{ik} F_{jk} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1})$, components are:

$$E_{ij} = \frac{1}{2}\left(\begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that there is only deformation on the $x_1 - x_2$-plane.

Considering the polar decomposition $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$, we can obtain:

$$\boldsymbol{C} = (\mathbf{V} \cdot \mathbf{R})^T \cdot (\mathbf{V} \cdot \mathbf{R}) = \mathbf{R}^T \cdot \mathbf{V}^T \cdot \mathbf{V} \cdot \mathbf{R} = \mathbf{R}^T \cdot \mathbf{V} \cdot \mathbf{V} \cdot \mathbf{R} = \mathbf{R}^T \cdot \mathbf{V}^2 \cdot \mathbf{R} = \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R}$$

For simplicity, we will work on the $x_1 - x_2$-plane, with that we represent the rotation tensor components as follows:

$$\mathsf{R}_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad (i, j = 1,2)$$

where $\cos^2\theta + \sin^2\theta = c^2 + s^2 = 1$ holds. The relationship $\boldsymbol{C} = \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R}$ becomes:

$$\begin{bmatrix} 1 & k \\ k & 1+k^2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} 1+k^2 & k \\ k & 1 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$
$$= \begin{bmatrix} (c^2 + c^2k^2 + 2sck + s^2) & (-sck^2 - s^2k + c^2k) \\ (-sck^2 - s^2k + c^2k) & (c^2 + s^2k^2 - 2sck + s^2) \end{bmatrix}$$

From the relationship $(c^2 + c^2k^2 + 2sck + s^2) = 1 \Rightarrow (c^2k^2 + 2sck + 1) = 1$ we can obtain $s = \frac{-k}{2} c$. Then, starting from the relation $(-sck^2 - s^2k + c^2k) = k$ and by considering that $s = \frac{-k}{2} c$, we can obtain:

$$c = \frac{1}{\sqrt{\frac{k^2}{4} + 1}} = \frac{2}{\sqrt{k^2 + 4}} \quad ; \quad s = \frac{\frac{-k}{2}}{\sqrt{\frac{k^2}{4} + 1}} = \frac{-k}{\sqrt{k^2 + 4}}$$

thus:

$$\mathsf{R}_{ij}=\begin{bmatrix}\dfrac{2}{\sqrt{k^2+4}} & \dfrac{k}{\sqrt{k^2+4}} & 0\\ \dfrac{-k}{\sqrt{k^2+4}} & \dfrac{2}{\sqrt{k^2+4}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

From the polar decomposition $\boldsymbol{F}=\mathbf{R}\cdot\mathbf{U}=\mathbf{V}\cdot\mathbf{R}$, we obtain $\mathbf{U}=\mathbf{R}^T\cdot\boldsymbol{F}$ and $\mathbf{V}=\boldsymbol{F}\cdot\mathbf{R}^T$, whose components are:

$$\mathsf{U}_{ij}=\mathsf{R}_{ki}F_{kj}=\begin{bmatrix}\dfrac{2}{\sqrt{k^2+4}} & \dfrac{-k}{\sqrt{k^2+4}} & 0\\ \dfrac{k}{\sqrt{k^2+4}} & \dfrac{2}{\sqrt{k^2+4}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & k & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\dfrac{2}{\sqrt{k^2+4}} & \dfrac{k}{\sqrt{k^2+4}} & 0\\ \dfrac{k}{\sqrt{k^2+4}} & \dfrac{2+k^2}{\sqrt{k^2+4}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\mathsf{V}_{ij}=F_{ik}\mathsf{R}_{jk}=\begin{bmatrix}1 & k & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dfrac{2}{\sqrt{k^2+4}} & \dfrac{-k}{\sqrt{k^2+4}} & 0\\ \dfrac{k}{\sqrt{k^2+4}} & \dfrac{2}{\sqrt{k^2+4}} & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\dfrac{2+k^2}{\sqrt{k^2+4}} & \dfrac{k}{\sqrt{k^2+4}} & 0\\ \dfrac{k}{\sqrt{k^2+4}} & \dfrac{2}{\sqrt{k^2+4}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

Problem 2.63

A deformable parallelepiped of dimensions $2\times 2\times 1$ is in the reference configuration as indicated in Figure 2.26. This body is subjected to motion:

$$\vec{x}(\vec{X},t)=-\exp^{X_2 t}\hat{\mathbf{e}}_1+tX_1^2\hat{\mathbf{e}}_2+X_3\hat{\mathbf{e}}_3 \tag{2.153}$$

where (X_1,X_2,X_3) are the material coordinates, and t stands for time.

a) Obtain the deformation gradient $\boldsymbol{F}$.
b) Obtain the right Cauchy-Green deformation tensor $\boldsymbol{C}$, and the principal stretches.
c) Obtain the right stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{R}$. Check if $\mathbf{R}$ is a proper orthogonal tensor.
d) Find the volume of the deformed parallelepiped at time $t=1s$.

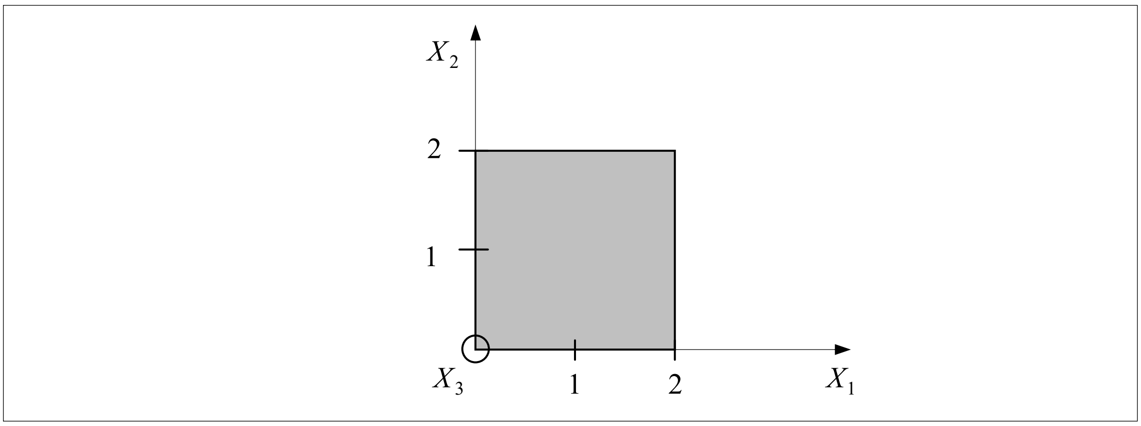


Figure 2.26

Solution:

a) According to the equation (2.153), the vector position components are $x_1 = -\exp^{X_2 t}$, $x_2 = tX_1^2$, $x_3 = X_3$, then the deformation gradient ($\boldsymbol{F}$) components are given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 0 & -t\exp^{X_2 t} & 0 \\ 2tX_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) The right Cauchy-Green deformation tensor are defined by $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, whose components are $C_{ij} = F_{ki} F_{kj}$:

$$C_{ij} = \begin{bmatrix} 0 & 2tX_1 & 0 \\ -t\exp^{X_2 t} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -t\exp^{X_2 t} & 0 \\ 2tX_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4t^2 X_1^2 & 0 & 0 \\ 0 & t^2 \exp^{2X_2 t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this space is the principal space (principal directions) of $\boldsymbol{C}$. Considering that λ_i are the principal stretches, the following relationship is fulfilled:

$$\boldsymbol{C} = \mathbf{U}^2 = \sum_{a=1}^{3} \lambda_a^2 \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad \Rightarrow \quad \mathbf{U} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$$

As we are working in the principal space of $\boldsymbol{C}$, we can obtain the principal stretches as follows:

$$\lambda_1 = +\sqrt{4t^2 X_1^2} \quad ; \quad \lambda_2 = +\sqrt{t^2 \exp^{2X_2 t}} \quad ; \quad \lambda_3 = +\sqrt{1}$$

which are positive numbers, since $\mathbf{U} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ is a positive definite tensor by definition, thus:

$$\lambda_1 = 2tX_1 \quad ; \quad \lambda_2 = t\exp^{X_2 t} \quad ; \quad \lambda_3 = 1$$

c)

$$\mathsf{U}_{ij} = \begin{bmatrix} 2tX_1 & 0 & 0 \\ 0 & t\exp^{X_2 t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathsf{U}_{ij}^{-1} = \begin{bmatrix} \frac{1}{2tX_1} & 0 & 0 \\ 0 & \frac{1}{t\exp^{X_2 t}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

According to the polar decomposition, $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} \Rightarrow \mathbf{R} = \boldsymbol{F} \cdot \mathbf{U}^{-1}$, we can obtain the rotation tensor ($\mathbf{R}$) components as follows:

$$\mathsf{R}_{ij} = \begin{bmatrix} 0 & -t\exp^{X_2 t} & 0 \\ 2tX_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2tX_1} & 0 & 0 \\ 0 & \frac{1}{t\exp^{X_2 t}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the orthogonality condition $\mathbf{R}\cdot\mathbf{R}^{-1}=\mathbf{R}\cdot\mathbf{R}^{T}=\mathbf{1}$ holds:

$$\mathsf{R}_{ik}\mathsf{R}_{jk}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

and the proper condition $\det(\mathbf{R})=+1$.

d) To calculate the final volume we use the relationship $dV=JdV_0$, where $J=|\boldsymbol{F}|$ is the Jacobian determinant and is given by:

$$J=\begin{vmatrix}0&-t\exp^{X_2t}&0\\2tX_1&0&0\\0&0&1\end{vmatrix}=2t^2X_1\exp^{X_2t}$$

At time $t=1s$ we have $J=2X_1\exp^{X_2}$. Then, the volume at time $t=1s$ is given by:

$$\int dV=\int_{V_0}JdV_0=\int_{X_1=0}^{2}\int_{X_2=0}^{2}\int_{X_3=0}^{1}(2X_1\exp^{X_2})dX_3dX_2dX_1=4(\exp^2-1)\approx 25.556$$

NOTE: We cannot use the equation $V=JV_0$ because we are not dealing with homogeneous deformation case.

Problem 2.64

A body is subjected to motion:

$$x_1=X_1\quad;\quad x_2=X_2+kX_3\quad;\quad x_3=X_3+kX_2$$

where k is a constant.

a) Obtain the deformation gradient ($\boldsymbol{F}$); the right Cauchy-Green deformation tensor ($\boldsymbol{C}$); the Green-Lagrange strain tensor ($\boldsymbol{E}$).

b) Calculate the displacement field, the magnitude $(dx)^2$ of sides $\overline{OA}$ and $\overline{OB}$, and diagonal $\overline{OC}$ after deformation of Figure 2.27.

c) Consider now a square as indicated in Figure 2.28:

c.1) Obtain the stretches according to directions $\overline{OC}$ and $\overline{BA}$; c.2) Obtain the angle θ_{23} in the current configuration in function of k.

c.3) Apply the polar decomposition of the tensor $\boldsymbol{F}$ in order to obtain $\mathbf{U}$ and $\mathbf{R}$.

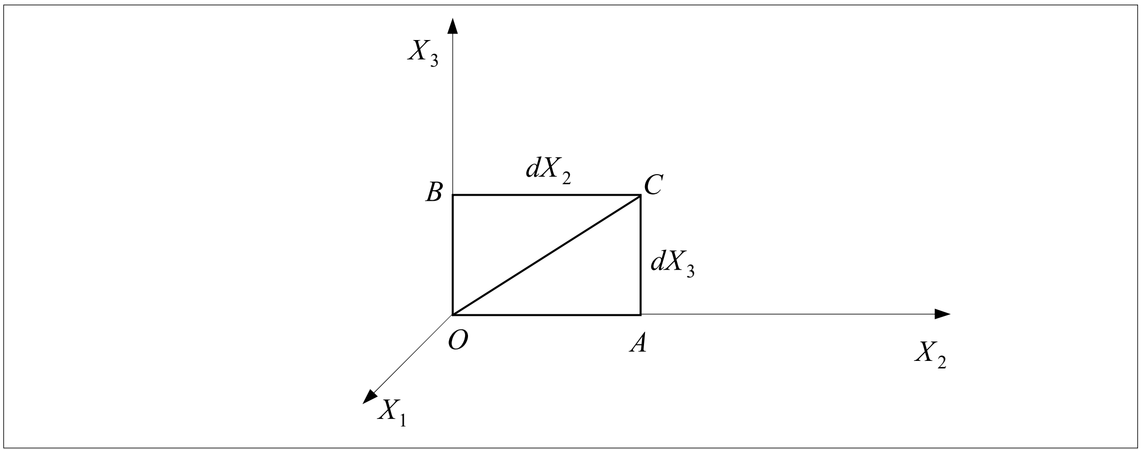


Figure 2.27

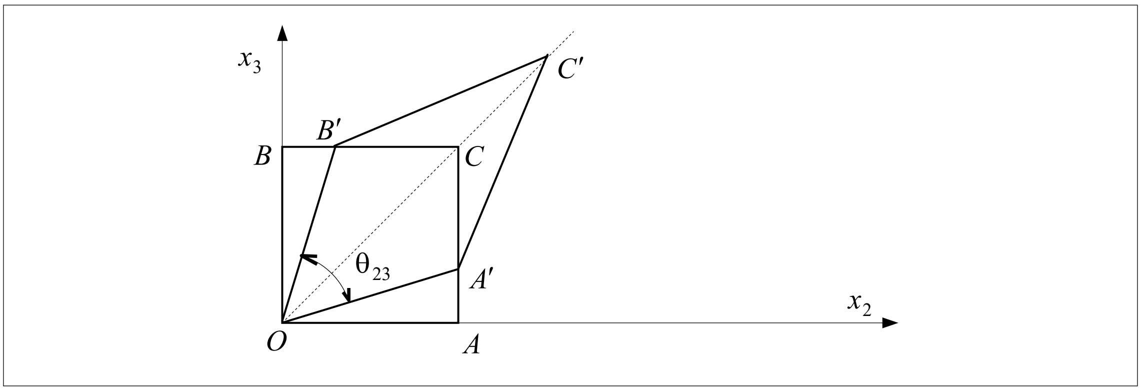


Figure 2.28

Solution:

a) The deformation gradient components are:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k & 1 \end{bmatrix}$$

The right Cauchy-Green deformation tensor is given by $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, thus:

$$C_{ij} = F_{ki} F_{kj} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix}$$

The Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1})$, components are:

$$E_{ij} = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k^2 & 2k \\ 0 & 2k & k^2 \end{bmatrix}$$

b.1) The displacement field, $\vec{\mathbf{u}} = \vec{x} - \vec{X}$, components are:

$$\mathrm{u}_1 = x_1 - X_1 = 0 \quad ; \quad \mathrm{u}_2 = x_2 - X_2 = kX_3 \quad ; \quad \mathrm{u}_3 = x_3 - X_3 = kX_2$$

b.2) Calculation of $\|d\vec{\mathbf{x}}\|^2 = (ds)^2$:

$$(ds)^2 = \|d\vec{\mathbf{x}}\|^2 = d\vec{\mathbf{x}} \cdot d\vec{\mathbf{x}} = \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} = d\vec{\boldsymbol{X}} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} = d\vec{\boldsymbol{X}} \cdot \boldsymbol{C} \cdot d\vec{\boldsymbol{X}}$$

thus:

$$(ds)^2 = \begin{bmatrix} dX_1 & dX_2 & dX_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$
$$= (dX_1)^2 + (dX_2)^2(1+k^2) + (dX_3)^2(1+k^2) + 4k(dX_2)(dX_3)$$

Then, for the diagonal $\overline{OC}$ we have $\begin{bmatrix} 0 & dX_2 & dX_3 \end{bmatrix}$, with that we can obtain:

$$(dx)^2 = (dX_2)^2(1+k^2) + (dX_3)^2(1+k^2) + 4k(dX_2)(dX_3)$$

For the side $\overline{OA}$ we have $\begin{bmatrix} 0 & dX_2 & 0 \end{bmatrix}$, with that we can obtain:

$$(dx)^2 = (dX_2)^2(1+k^2)$$

For the side $\overline{OB}$ we have $\begin{bmatrix} 0 & 0 & dX_3 \end{bmatrix}$, with that we can obtain:

$$(dx)^2 = (dX_3)^2(1+k^2)$$

c) The stretch according to the $\hat{\mathbf{N}}$-direction (reference configuration) is given by the equation $(\lambda_{\hat{\mathbf{N}}})^2 = \hat{\mathbf{N}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{N}}$.

c.1) The stretch according to the $\overline{OC}$-direction: $\hat{\mathrm{N}}_i = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, is:

$$(\lambda_{\overline{OC}})^2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = (1+k)^2$$

The stretch according to the $\overline{BA}$-direction: $\hat{\mathrm{N}}_i = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, with that we can obtain:

$$(\lambda_{\overline{BA}})^2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = (1-k)^2$$

c.2) The variation of the angle can be calculated by means of the equation:

$$\cos\theta = \frac{\hat{\mathbf{M}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{N}}}{\sqrt{\hat{\mathbf{M}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{M}}}\sqrt{\hat{\mathbf{N}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{N}}}} = \frac{\hat{\mathbf{M}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{N}}}{\lambda_{\hat{\mathbf{M}}}\lambda_{\hat{\mathbf{N}}}}$$

where the unit vector according to the $\overline{OB}$-direction is $\hat{\mathsf{M}}_i = [0 \quad 0 \quad 1]$, and according to the $\overline{OA}$-direction is $\hat{\mathsf{N}}_i = [0 \quad 1 \quad 0]$. With that we can obtain:

$$(\lambda_{\overline{OB}})^2 = [0 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1+k^2$$

$$(\lambda_{\overline{OA}})^2 = [0 \quad 1 \quad 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1+k^2$$

$$\hat{\mathsf{M}}_i C_{ij} \hat{\mathsf{N}}_j = [0 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2k$$

Then:

$$\cos\theta_{23} = \frac{\hat{\mathbf{M}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{N}}}{\lambda_{\hat{\mathbf{M}}}\lambda_{\hat{\mathbf{N}}}} = \frac{2k}{1+k^2}$$

c.3) The polar decomposition of $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$, where:

$$\boldsymbol{C} = \mathbf{U}^2 = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \qquad \Rightarrow \qquad \mathbf{U} = \sqrt{\boldsymbol{C}} = \sum_{a=1}^{3} \sqrt{\lambda_a} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$$

Calculation of the eigenvalues of $\boldsymbol{C}$. Note that according to the format of $\boldsymbol{C}$-components, there is only deformation according to $x_2 - x_3$-plane. In addition, we know one eigenvalue $\overline{\lambda}_1 = 1$ associated with the direction $\mathsf{N}_i^{(1)} = [1 \quad 0 \quad 0]$. By means of the characteristic determinant we can obtain:

$$\begin{vmatrix} (1+k^2) - \overline{\lambda} & 2k \\ 2k & (1+k^2) - \overline{\lambda} \end{vmatrix} = 0 \qquad \Rightarrow \qquad \overline{\lambda}^2 - 2(1+k^2)\overline{\lambda} + (1 - 2k^2 + k^4) = 0$$

$$\Rightarrow \overline{\lambda}^2 - 2(1+k^2)\overline{\lambda} + (1-k^2)^2 = 0$$

The roots are: $\overline{\lambda}_2 = 1 + k^2 + 2k = (1+k)^2$; $\overline{\lambda}_3 = 1 + k^2 - 2k = (1-k)^2$

Then, in the principal space of $\boldsymbol{C}$ we have:

$$C'_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+k)^2 & 0 \\ 0 & 0 & (1-k)^2 \end{bmatrix}$$

The principal directions are $\overline{\lambda}_2 \Rightarrow \mathsf{N}_i^{(2)} = \left[0 \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$, $\overline{\lambda}_3 \Rightarrow \mathsf{N}_i^{(3)} = \left[0 \quad \frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$. Then, the transformation matrix between the original space and the principal space is:

$$a_{ij} = \mathcal{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

That is, the following must be true:

$$\mathcal{C}' = \mathcal{A}\,\mathcal{C}\,\mathcal{A}^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+k)^2 & 0 \\ 0 & 0 & (1-k)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & 2k \\ 0 & 2k & 1+k^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

Then, in the principal space of $\boldsymbol{C}$, we have:

$$C'_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+k)^2 & 0 \\ 0 & 0 & (1-k)^2 \end{bmatrix} \quad \Rightarrow \quad \mathsf{U}_{ij} = \begin{bmatrix} +\sqrt{1} & 0 & 0 \\ 0 & +\sqrt{(1+k)^2} & 0 \\ 0 & 0 & +\sqrt{(1-k)^2} \end{bmatrix}$$

$$\Rightarrow \mathsf{U}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+k) & 0 \\ 0 & 0 & (1-k) \end{bmatrix}$$

The inverse tensor in the principal space can be obtained as follows:

$$\mathsf{U}'^{-1}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(1+k)} & 0 \\ 0 & 0 & \frac{1}{(1-k)} \end{bmatrix}$$

The components of $\mathbf{U}$ in the original space are given by:

$$\mathcal{U}'^{-1} = \mathcal{A}^T\,\mathcal{U}^{-1}\,\mathcal{A}$$

$$\mathsf{U}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(1+k)} & 0 \\ 0 & 0 & \frac{1}{(1-k)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(1-k^2)} & \frac{-k}{(1-k^2)} \\ 0 & \frac{-k}{(1-k^2)} & \frac{1}{(1-k^2)} \end{bmatrix}$$

From the polar decomposition we have $\boldsymbol{F} = \mathbf{R}\cdot\mathbf{U} \Rightarrow \mathbf{R} = \boldsymbol{F}\cdot\mathbf{U}^{-1}$, thus

$$\mathsf{R}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(1-k^2)} & \frac{-k}{(1-k^2)} \\ 0 & \frac{-k}{(1-k^2)} & \frac{1}{(1-k^2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2.65

Given the following equations of motion:

$$x_1 = \lambda_1 X_1 \quad ; \quad x_2 = -\lambda_3 X_3 \quad ; \quad x_3 = \lambda_2 X_2$$

a) Obtain the final volume to a unit cube;

b) Obtain the deformed area to a unit square defined in the $X_1 - X_2$-plane, and draw the deformed area;

c) Apply the polar decomposition and obtain the tensors $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$

Solution:

a)

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad \Rightarrow \quad F_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \text{(homogenous deformation)}$$

The determinant of $\boldsymbol{F}$ is given by $|\boldsymbol{F}| \equiv J = \lambda_1 \lambda_2 \lambda_3$, and the deformed volume:

$$dV = |\boldsymbol{F}| dV_0 \xrightarrow{\text{integrating}} V_{final} = |\boldsymbol{F}| V_{initial} = \lambda_1 \lambda_2 \lambda_3$$

b) Applying the Nanson's formula and by considering the particular case (homogeneous deformation):

$$d\vec{\boldsymbol{a}} = J\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}} \xrightarrow{\text{integrating}} \vec{\boldsymbol{a}}_{final} = J\boldsymbol{F}^{-T} \cdot \vec{\boldsymbol{A}}_{initial}$$

where

$$\vec{\boldsymbol{A}}_{initial} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\mathbf{e}}_3 \, ; \quad F_{ij}^{-1} = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \begin{bmatrix} \lambda_2 \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_1 \lambda_3 \\ 0 & -\lambda_1 \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_2} \\ 0 & \frac{-1}{\lambda_3} & 0 \end{bmatrix}$$

With that the deformed area vector is:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \lambda_1 \lambda_2 \lambda_3 \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & \frac{-1}{\lambda_3} \\ 0 & \frac{1}{\lambda_2} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\lambda_1 \lambda_2 \\ 0 \end{Bmatrix}$$

and its magnitude is:

$$\left\| \vec{\boldsymbol{a}}_{final} \right\| = \sqrt{(-\lambda_1 \lambda_2)^2} = \lambda_1 \lambda_2$$

where the points $A(1,0,0)$, $B(0,1,0)$ and $C(1,1,0)$, (see Figure 2.29), move according to the equations of motion:

$$\begin{Bmatrix} x_1^A \\ x_2^A \\ x_3^A \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 \\ 0 \\ 0 \end{Bmatrix} \quad ; \quad \begin{Bmatrix} x_1^B \\ x_2^B \\ x_3^B \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \lambda_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1^C \\ x_2^C \\ x_3^C \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 \\ 0 \\ \lambda_2 \end{Bmatrix}$$

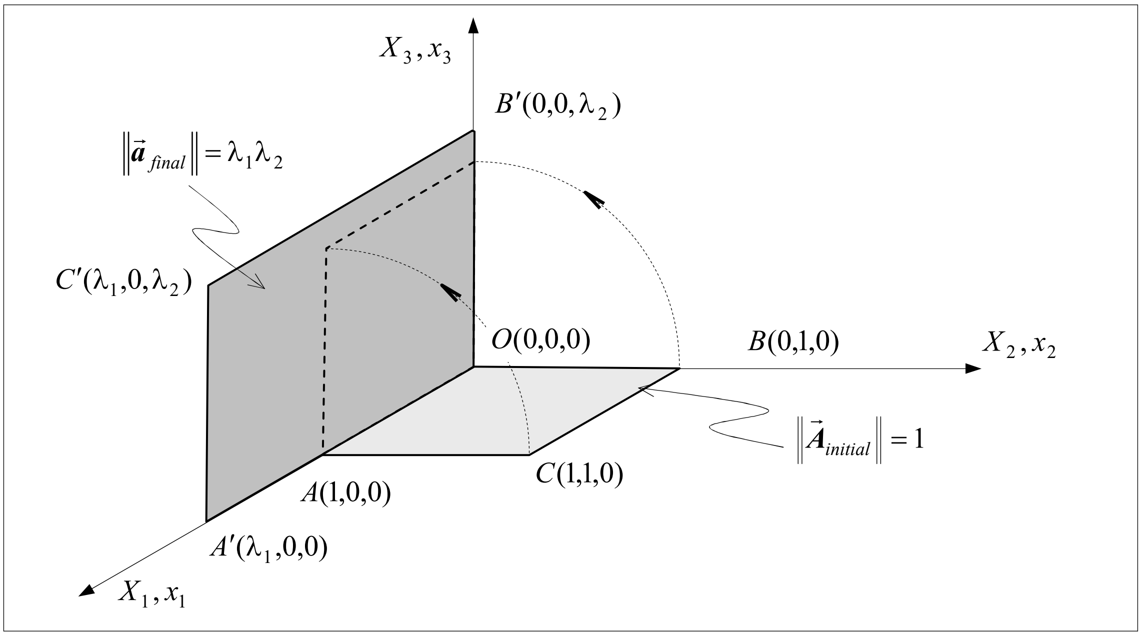


Figure 2.29

c) According to the polar decomposition definition $\boldsymbol{F} = \mathbf{R}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R}$ where $\mathbf{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T\cdot\boldsymbol{F}}$ and $\mathbf{V} = \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{F}\cdot\boldsymbol{F}^T}$ we obtain:

$$C_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad \Rightarrow \quad \mathsf{U}_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$b_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_3^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix} \quad \Rightarrow \quad \mathsf{V}_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

Note that the original space coincides with the principal space of $\boldsymbol{C}$. Note also that $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues but different principal directions. To obtain the spin tensor of the polar decomposition we apply $\mathbf{R} = \boldsymbol{F}\cdot\mathbf{U}^{-1} = \mathbf{V}^{-1}\cdot\boldsymbol{F}$:

$$\mathsf{R}_{ij} = F_{ik}\mathsf{U}_{kj}^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2.66

Consider the equations of motion:

$$x_1 = \sqrt{3}X_1 \quad ; \quad x_2 = 2X_2 \quad ; \quad x_3 = \sqrt{3}X_3 - X_2$$

Obtain the material ellipsoid associated with the material sphere defined in the reference configuration by $X_1^2 + X_2^2 + X_3^2 = 1$, (see Figure 2.30). Check that the ellipsoid in the principal space of the left stretch tensor $\mathbf{V}$ has the format $\frac{x_1'^2}{\lambda_1^2} + \frac{x_2'^2}{\lambda_2^2} + \frac{x_3'^2}{\lambda_3^2} = 1$, where λ_1, λ_2, λ_3 are the principal stretches.

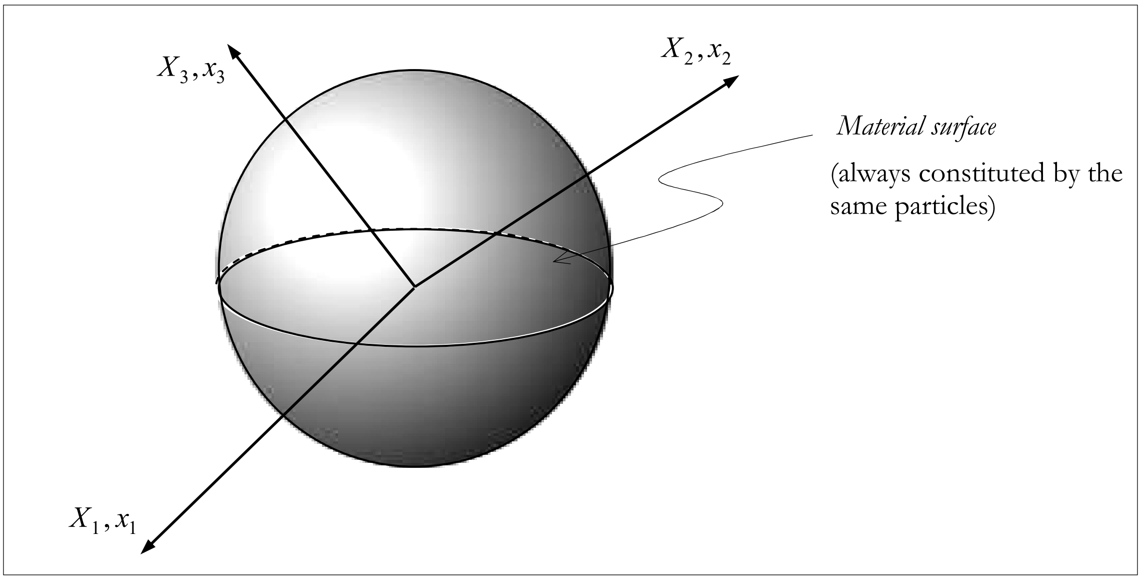


Figure 2.30: Material sphere.

Solution:

The equations of motion and its inverse, in matrix form, are given by:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & \sqrt{3} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \xrightarrow{\text{inverse}} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

The equations of motion in the spatial description are given by:

$$X_1 = \frac{\sqrt{3}}{3}x_1 \quad ; \quad X_2 = \frac{x_2}{2} \quad ; \quad X_3 = \frac{\sqrt{3}}{6}x_2 + \frac{\sqrt{3}}{3}x_3$$

By substituting the above equations into the sphere equation we can obtain:

$$X_1^2 + X_2^2 + X_3^2 = 1 \quad \Rightarrow \quad \left(\frac{\sqrt{3}}{3}x_1\right)^2 + \left(\frac{x_2}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}x_2 + \frac{\sqrt{3}}{3}x_3\right)^2 = 1$$

By simplifying the above equation we can obtain:

$$x_1^2 + x_2^2 + x_3^2 + x_2 x_3 = 3$$

which is the equation of an ellipsoid. We now represent the ellipsoid equation in the principal space of the left stretch tensor $\mathbf{V}$. Recall that the tensor $\mathbf{V}$ and $\boldsymbol{b}$ are coaxial, i.e. they have the same principal directions), and is also true that:

$$\mathbf{V} = \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T}$$

The components of $\boldsymbol{b}$ are

$$b_{ij} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & \sqrt{3} \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & -\sqrt{3} \\ 0 & -\sqrt{3} & 3 \end{bmatrix}$$

Note that we know already one eigenvalue $b_1 = 3$ associated with the eigenvector $\hat{\mathsf{n}}_i^{(1)} = [1 \quad 0 \quad 0]$. Then, the other principal directions are in the plane $x_2 - x_3$, with that we obtain

$$b_2 = 6 \xrightarrow{\text{eigenvector}} \hat{\mathsf{n}}_i^{(2)} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix} \; ; \; b_3 = 2 \xrightarrow{\text{eigenvector}} \hat{\mathsf{n}}_i^{(3)} = \begin{bmatrix} 0 & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$$

thus:

$$b'_{ij} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{Transformation matrix}} a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathsf{V}'_{ij} = \begin{bmatrix} \lambda_1 = \sqrt{3} & 0 & 0 \\ 0 & \lambda_2 = \sqrt{6} & 0 \\ 0 & 0 & \lambda_3 = \sqrt{2} \end{bmatrix}$$

Then, applying the transformation law from x_1, x_2, x_3-system to the x'_1, x'_2, x'_3-system we obtain:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = x'_1 \\ x_2 = \frac{-\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \\ x_3 = \frac{\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \end{cases}$$

with that, the equation of the ellipsoid in the principal space of $\mathbf{V}$, (see Figure 2.31), is represented by:

$$x_1^2 + x_2^2 + x_3^2 + x_2 x_3 = 3$$

$$(x'_1)^2 + \left(-\frac{\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \right)^2 + \left(\frac{\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \right)^2 + \left(-\frac{\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \right) \left(\frac{\sqrt{2}}{2} x'_2 + \frac{\sqrt{2}}{2} x'_3 \right) = 3$$

Simplifying the above equation we can obtain:

$$\frac{x_1'^2}{3}+\frac{x_2'^2}{6}+\frac{x_3'^2}{2}=\frac{x_1'^2}{(\sqrt{3})^2}+\frac{x_2'^2}{(\sqrt{6})^2}+\frac{x_3'^2}{(\sqrt{2})^2}=\frac{x_1'^2}{\lambda_1^2}+\frac{x_2'^2}{\lambda_2^2}+\frac{x_3'^2}{\lambda_3^2}=1$$

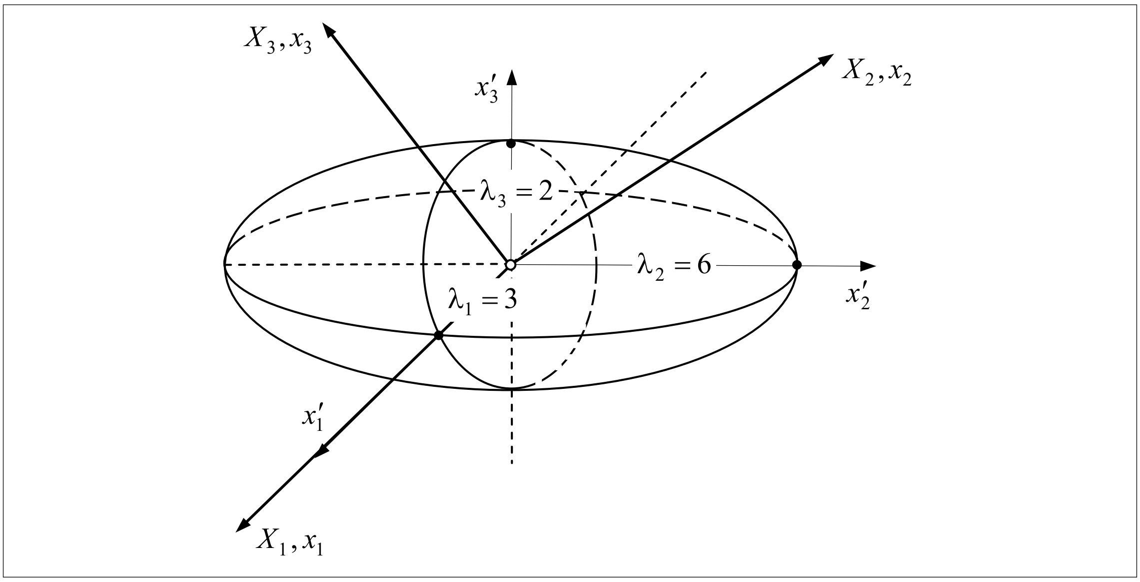


Figure 2.31: The material ellipsoid (deformed configuration).

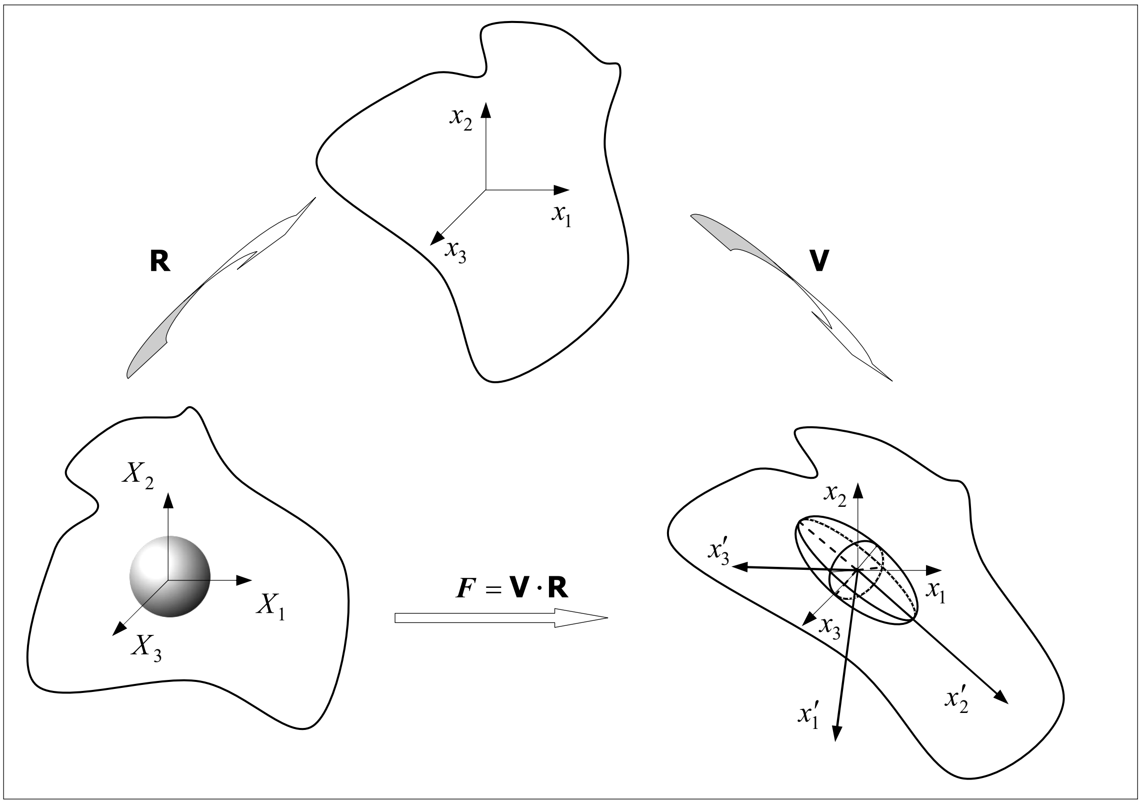


Figure 2.32: The left polar decomposition.

Problem 2.67

A square of side b turns counterclockwise of 30°. After turning the square is deformed such that the base maintains its initial length and the height is doubled, (see Figure 2.33). Calculate the deformation gradient, the right Cauchy-Green deformation tensor, and the Green-Lagrange strain tensor.

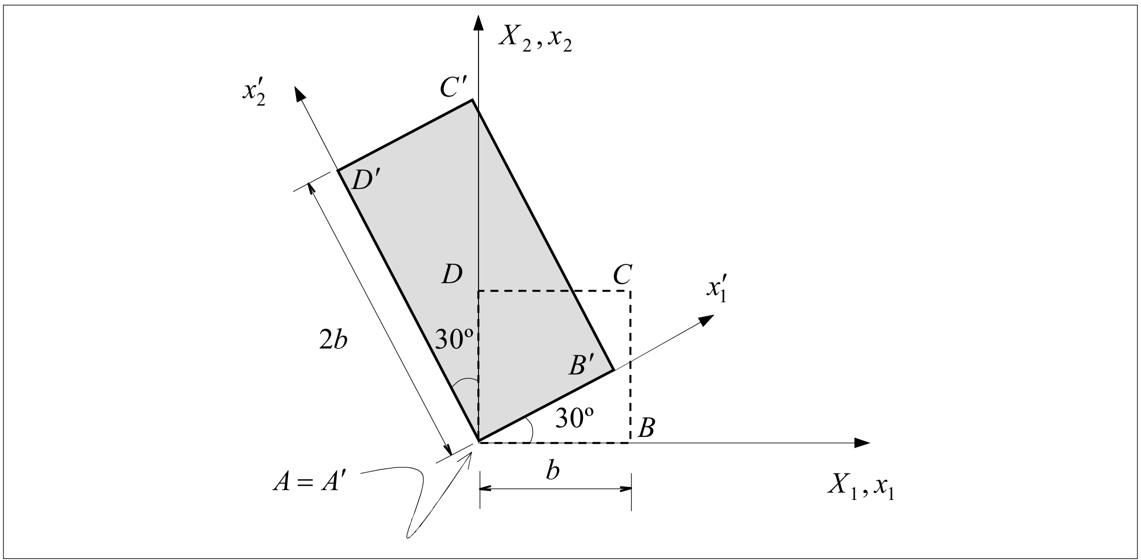


Figure 2.33: Body under rotation/deformation.

Solution:

Note that we can apply the decomposition of motion: first we apply a pure deformation and then a rotation is applied, (see Figure 2.34). The motion is governed by the right stretch tensor of the polar decomposition:

$$\mathsf{U}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where we have applied the definition of stretch. Note that they are principal values. We then apply a rotation, where the components of $\mathbf{R}$ are the same as the transformation matrix from the $\vec{x}'$-system to the $\vec{x}$-system:

$$\mathsf{R}_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

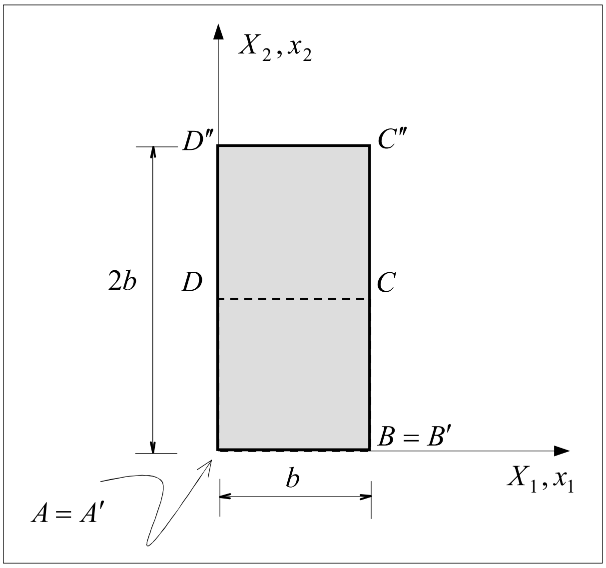


Figure 2.34

Then, by applying the left polar decomposition, $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$, we can obtain:

$$F_{ij} = \mathsf{R}_{ik}\mathsf{U}_{kj} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -2\sin\theta & 0 \\ \sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the proposed problem, we have:

$$F_{ij} = \begin{bmatrix} \cos 30^\circ & -2\sin 30^\circ & 0 \\ \sin 30^\circ & 2\cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As we are dealing with a homogenous deformation the equation $\vec{x} = \boldsymbol{F} \cdot \vec{X} + \vec{\mathbf{c}}$ holds, for this case with $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. For example, for a particle at point D in the reference configuration moves to the point:

$$\begin{Bmatrix} x_1^D \\ x_2^D \\ x_3^D \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & -2\sin 30^\circ & 0 \\ \sin 30^\circ & 2\cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1^D \\ X_2^D \\ X_3^D \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & -2\sin 30^\circ & 0 \\ \sin 30^\circ & 2\cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ b \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2b\sin 30^\circ \\ 2b\cos 30^\circ \\ 0 \end{Bmatrix}$$

a fact that can be easily checked by means of Figure 2.33.

By means of definition of the right Cauchy-Green deformation tensor, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, we can obtain the Cartesian components:

$$C_{ij} = F_{ki}F_{kj} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -2\sin\theta & 0 \\ \sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} + \mathbf{1})$, and its components are:

$$E_{ij} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the original space coincides with the principal space. We could also have obtained the components of $\boldsymbol{C}$ and $\boldsymbol{E}$ by means of its spectral representations: $\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, $\boldsymbol{E} = \sum_{a=1}^{3} \frac{1}{2}(\lambda_a^2 - 1)\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, where λ_a are the principal stretches.

2.4 Infinitesimal Deformation Regime

Problem 2.68

Given the equations of motion

$$x_1 = X_1 + 4X_1X_2t \quad ; \quad x_2 = X_2 + X_2^2 t \quad ; \quad x_3 = X_3 + X_3^2 t \tag{2.154}$$

a) Obtain the velocity field;

b) Obtain the infinitesimal strain tensor field;

c) At time $t = 1\,s$, obtain the infinitesimal strain tensor.

Solution:

a) Velocity field:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{X}},t)=\frac{d\vec{\boldsymbol{x}}}{dt}\Rightarrow\begin{cases}V_1=4X_1X_2\\V_2=X_2^2\\V_3=X_3^2\end{cases}\tag{2.155}$$

b) Acceleration field:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}},t)=\frac{d\vec{\boldsymbol{V}}}{dt}\Rightarrow\begin{cases}A_1=0\\A_2=0\\A_3=0\end{cases}\tag{2.156}$$

c) Displacement field:

$$\begin{cases}\mathsf{u}_1(\vec{\boldsymbol{X}},t)=x_1(\vec{\boldsymbol{X}},t)-X_1=X_1+4X_1X_2-X_1=4X_1X_2\\\mathsf{u}_2(\vec{\boldsymbol{X}},t)=x_2(\vec{\boldsymbol{X}},t)-X_2=X_2+X_2^2-X_2=X_2^2\\\mathsf{u}_3(\vec{\boldsymbol{X}},t)=x_3(\vec{\boldsymbol{X}},t)-X_3=X_3+X_3^2-X_3=X_3^2\end{cases}\tag{2.157}$$

Then, the infinitesimal strain tensor components are given by $\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial\mathsf{u}_i}{\partial x_j}+\frac{\partial\mathsf{u}_j}{\partial x_i}\right)$, and the displacement gradient can be obtained as follows:

$$\frac{\partial\mathsf{u}_i(\vec{\boldsymbol{X}},t)}{\partial X_j}=\begin{bmatrix}\frac{\partial\mathsf{u}_1}{\partial X_1}&\frac{\partial\mathsf{u}_1}{\partial X_2}&\frac{\partial\mathsf{u}_1}{\partial X_3}\\\frac{\partial\mathsf{u}_2}{\partial X_1}&\frac{\partial\mathsf{u}_2}{\partial X_2}&\frac{\partial\mathsf{u}_2}{\partial X_3}\\\frac{\partial\mathsf{u}_3}{\partial X_1}&\frac{\partial\mathsf{u}_3}{\partial X_2}&\frac{\partial\mathsf{u}_3}{\partial X_3}\end{bmatrix}=\begin{bmatrix}4X_2&4X_1&0\\0&2X_2&0\\0&0&2X_3\end{bmatrix}\tag{2.158}$$

thus:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial\mathsf{u}_i}{\partial x_j}+\frac{\partial\mathsf{u}_j}{\partial x_i}\right)=\begin{bmatrix}4X_2&2X_1&0\\2X_1&2X_2&0\\0&0&2X_3\end{bmatrix}\tag{2.159}$$

which is independent of time.

Problem 2.69

Consider the infinitesimal strain tensor:

$$\varepsilon_{ij}=\begin{bmatrix}0&0&0\\0&\mu\frac{X_2X_3}{\ell^2}&-\mu\frac{X_3^2}{\ell^2}\\0&-\mu\frac{X_3^2}{\ell^2}&-\mu\frac{X_2X_3}{\ell^2}\end{bmatrix}\tag{2.160}$$

and the infinitesimal spin tensor:

$$\omega_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{2\ell^2}\left(X_2^2 - X_3^2\right) \\ 0 & -\frac{\mu}{2\ell^2}\left(X_2^2 - X_3^2\right) & 0 \end{bmatrix} \tag{2.161}$$

Obtain the displacement field components.

Solution:

The displacement gradient is related to the infinitesimal strain tensor and the infinitesimal spin tensor as follows:

$$\mathrm{u}_{i,j} = \varepsilon_{ij} + \omega_{ij} \tag{2.162}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(\mathrm{u}_{i,j} + \mathrm{u}_{i,j}) \quad ; \quad \omega_{ij} = \frac{1}{2}(\mathrm{u}_{i,j} - \mathrm{u}_{i,j}) \tag{2.163}$$

thus:

$$\mathrm{u}_{i,j} = \frac{\mu}{2\ell^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2X_2X_3 & X_2^2 - 3X_3^2 \\ 0 & -(X_2^2 + X_3^2) & -2X_2X_3 \end{bmatrix} \tag{2.164}$$

$$\frac{\partial \mathrm{u}_1}{\partial x_1} = 0 \longrightarrow \mathrm{u}_1 = 0 \tag{2.165}$$

$$\begin{aligned} &\frac{\partial \mathrm{u}_2}{\partial x_2} = \frac{\mu}{2\ell^2}\left(2X_2X_3\right) \\ &\Rightarrow \int \partial \mathrm{u}_2 = \int \frac{\mu}{2\ell^2}\left(2X_2X_3\right)\partial x_2 \Rightarrow \mathrm{u}_2 = \frac{\mu}{2\ell^2}\left[X_2^2X_3 + C_1(X_3)\right] \end{aligned} \tag{2.166}$$

$$\begin{aligned} &\frac{\partial \mathrm{u}_3}{\partial x_3} = -\frac{\mu}{2\ell^2}\left(2X_2X_3\right) \\ &\Rightarrow \int \partial \mathrm{u}_3 = \int -\frac{\mu}{2\ell^2}\left(2X_2X_3\right)\partial x_3 \Rightarrow \mathrm{u}_3 = -\frac{\mu}{2\ell^2}\left[X_3^2X_2 + C_2(X_2)\right] \end{aligned} \tag{2.167}$$

To determine the constant $C_1(X_3)$ from the result (2.166) we take the derivative of u_3 with respect to X_3:

$$\begin{aligned} &\frac{\partial \mathrm{u}_2}{\partial X_3} = \frac{\mu}{2\ell^2}\left[X_2^2 + \frac{\partial C_1(X_3)}{\partial X_3}\right] = \frac{\mu}{2\ell^2}\left[X_2^2 - 3X_3^2\right] \Rightarrow \frac{\partial C_1(X_3)}{\partial X_3} = -3X_3^2 \\ &\Rightarrow C_1(X_3) = -X_3^3 \end{aligned} \tag{2.168}$$

In the same fashion we find the constant $C_2(X_2)$:

$$\begin{aligned} &\frac{\partial \mathrm{u}_3}{\partial X_2} = -\frac{\mu}{2\ell^2}\left[X_3^2 + \frac{\partial C_2(X_2)}{\partial X_2}\right] = -\frac{\mu}{2\ell^2}\left[X_2^2 + X_3^2\right] \Rightarrow \frac{\partial C_2(X_2)}{\partial X_2} = X_2^2 \\ &\Rightarrow C_2(X_2) = \frac{X_2^3}{3} \end{aligned} \tag{2.169}$$

Then, the displacement field is given by:

$$\mathrm{u}_1 = 0 \quad ; \quad \mathrm{u}_2 = \frac{\mu}{2\ell^2}\left[X_2^2 X_3 - X_3^3\right] \quad ; \quad \mathrm{u}_3 = -\frac{\mu}{2\ell^2}\left[X_3^2 X_2 + \frac{X_2^3}{3}\right] \tag{2.170}$$

Problem 2.70

Show that, if we are dealing with the small deformation regime, the rate of change of the infinitesimal strain tensor ($\dot{\boldsymbol{\varepsilon}}$) is equal to the rate-of-deformation tensor ($\mathbf{D}$).

Solution:

Consider the relationship between the rate of change of the Green-Lagrange deformation tensor ($\dot{\boldsymbol{E}}$) and the rate-of-deformation tensor ($\mathbf{D}$):

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F} \tag{2.171}$$

For the case of small deformation $\boldsymbol{F} \approx \mathbf{1}$ holds, in addition it fulfills that $\dot{\boldsymbol{E}} \approx \dot{\boldsymbol{e}} \approx \dot{\boldsymbol{\varepsilon}}$ then:

$$\dot{\boldsymbol{E}} = \dot{\boldsymbol{\varepsilon}} = \mathbf{D} \tag{2.172}$$

Problem 2.71

Given the equations of motion

$$x_1 = X_1 \quad ; \quad x_2 = X_2 + X_1(\exp^{-2t} - 1) \quad ; \quad x_3 = X_3 + X_1(\exp^{-3t} - 1) \tag{2.173}$$

Obtain the rate-of-deformation ($\mathbf{D}$) and compare with the rate of change of the infinitesimal strain tensor ($\dot{\boldsymbol{\varepsilon}}$).

Solution:

By definition, the rate-of-deformation tensor ($\mathbf{D}$) is the symmetric part of the spatial velocity gradient ($\boldsymbol{\ell} = \nabla_{\boldsymbol{x}} \vec{\boldsymbol{v}}$):

$$\mathbf{D} = \frac{1}{2}(\boldsymbol{\ell} + \boldsymbol{\ell}^T) \tag{2.174}$$

And by definition, the infinitesimal strain tensor is equal to the symmetric part of the displacement gradient:

$$\boldsymbol{\varepsilon}(\vec{x}, t) = \nabla^{sym} \vec{\mathbf{u}} \equiv (\nabla \vec{\mathbf{u}})^{sym} \Rightarrow \dot{\boldsymbol{\varepsilon}} \equiv \frac{D\boldsymbol{\varepsilon}}{Dt} \tag{2.175}$$

The displacement field is given by $\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}$. Considering the equations of motion, the displacement field components become:

$$\begin{cases} \mathrm{u}_1(\vec{\boldsymbol{X}}, t) = x_1(\vec{\boldsymbol{X}}, t) - X_1 = X_1 - X_1 = 0 \\ \mathrm{u}_2(\vec{\boldsymbol{X}}, t) = x_2(\vec{\boldsymbol{X}}, t) - X_2 = X_2 + X_1(\exp^{-2t} - 1) - X_2 = X_1(\exp^{-2t} - 1) \\ \mathrm{u}_3(\vec{\boldsymbol{X}}, t) = x_3(\vec{\boldsymbol{X}}, t) - X_3 = X_3 + X_1(\exp^{-3t} - 1) - X_3 = X_1(\exp^{-3t} - 1) \end{cases}$$

The velocity field is given by $\left(\vec{\boldsymbol{v}}(\vec{\boldsymbol{X}}, t) = \dfrac{D\vec{\mathbf{u}}(\vec{\boldsymbol{X}}, t)}{Dt}\right)$. Then, the velocity field components, in material coordinates, are:

$$v_1(\vec{\boldsymbol{X}}, t) = 0 \quad ; \quad v_2(\vec{\boldsymbol{X}}, t) = X_1(-2\exp^{-2t}) \quad ; \quad v_3(\vec{\boldsymbol{X}}, t) = X_1(-3\exp^{-3t}) \tag{2.176}$$

Given the inverse of the equations of motion:

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + X_1(\exp^{-2t} - 1) \\ x_3 = X_3 + X_1(\exp^{-3t} - 1) \end{cases} \xrightarrow{inverse} \begin{cases} X_1 = x_1 \\ X_2 = x_2 - x_1(\exp^{-2t} - 1) \\ X_3 = x_3 - x_1(\exp^{-3t} - 1) \end{cases} \tag{2.177}$$

we can obtain the velocity field in spatial coordinates:

$$v_1(\vec{x},t) = 0 \quad ; \quad v_2(\vec{x},t) = -2x_1\exp^{-2t} \quad ; \quad v_3(\vec{x},t) = -3x_1\exp^{-3t} \tag{2.178}$$

The spatial velocity gradient $(\boldsymbol{\ell})$ components are given by:

$$(\boldsymbol{\ell})_{ij} = (\nabla_{x}\vec{v})_{ij} = \frac{\partial v_i(\vec{x},t)}{\partial x_j} = \begin{bmatrix} 0 & 0 & 0 \\ -2\exp^{-2t} & 0 & 0 \\ -3\exp^{-3t} & 0 & 0 \end{bmatrix} \tag{2.179}$$

and

$$(\mathbf{D})_{ij} = \frac{1}{2}(\ell_{ij} + \ell_{ji}) = \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & 0 \\ -2\exp^{-2t} & 0 & 0 \\ -3\exp^{-3t} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2\exp^{-2t} & 0 & 0 \\ -3\exp^{-3t} & 0 & 0 \end{bmatrix}^T \right)$$
$$= \begin{bmatrix} 0 & -\exp^{-2t} & -\frac{3}{2}\exp^{-3t} \\ -\exp^{-2t} & 0 & 0 \\ -\frac{3}{2}\exp^{-3t} & 0 & 0 \end{bmatrix} \tag{2.180}$$

We can also obtain the spin tensor $\mathbf{W} = \boldsymbol{\ell}^{skew}$ components as follows

$$\mathrm{W}_{ij} = \frac{1}{2}(\ell_{ij} - \ell_{ji}) = \begin{bmatrix} 0 & \exp^{-2t} & \frac{3}{2}\exp^{-3t} \\ -\exp^{-2t} & 0 & 0 \\ -\frac{3}{2}\exp^{-3t} & 0 & 0 \end{bmatrix} \tag{2.181}$$

The infinitesimal strain tensor $(\boldsymbol{\varepsilon})$

Starting from the displacement field:

$$\begin{cases} \mathrm{u}_1(\vec{x},t) = 0 \\ \mathrm{u}_2(\vec{x},t) = x_1(\exp^{-2t} - 1) \\ \mathrm{u}_3(\vec{x},t) = x_1(\exp^{-3t} - 1) \end{cases} \tag{2.182}$$

the displacement gradient components can be obtained as follows:

$$(\nabla\vec{\mathbf{u}})_{ij} = \frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} 0 & 0 & 0 \\ (\exp^{-2t} - 1) & 0 & 0 \\ (\exp^{-3t} - 1) & 0 & 0 \end{bmatrix} \tag{2.183}$$

We can decompose $(\nabla\vec{\mathbf{u}})$ into a symmetric and an antisymmetric part:

$$(\nabla\vec{\mathbf{u}})_{ij} = (\nabla^{sym}\vec{\mathbf{u}})_{ij} + (\nabla^{skew}\vec{\mathbf{u}})_{ij} = (\boldsymbol{\varepsilon})_{ij} + (\boldsymbol{\omega})_{ij} \tag{2.184}$$

The symmetric part:

$$
\begin{aligned}
(\nabla^{sym}\vec{\mathbf{u}})_{ij} &= \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & 0 \\ (\exp^{-2t}-1) & 0 & 0 \\ (\exp^{-3t}-1) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ (\exp^{-2t}-1) & 0 & 0 \\ (\exp^{-3t}-1) & 0 & 0 \end{bmatrix}^T\right) \\
&= \frac{1}{2}\begin{bmatrix} 0 & \exp^{-2t}-1 & \exp^{-3t}-1 \\ \exp^{-2t}-1 & 0 & 0 \\ \exp^{-3t}-1 & 0 & 0 \end{bmatrix} = \varepsilon_{ij}
\end{aligned}
\tag{2.185}
$$

We can also provide the infinitesimal spin tensor:

$$
(\boldsymbol{\omega})_{ij} = \frac{1}{2}\begin{bmatrix} 0 & -(\exp^{-2t}-1) & -(\exp^{-3t}-1) \\ (\exp^{-2t}-1) & 0 & 0 \\ (\exp^{-3t}-1) & 0 & 0 \end{bmatrix}
\tag{2.186}
$$

Then, the rate of change of $\boldsymbol{\varepsilon}$ is:

$$
\begin{aligned}
(\dot{\boldsymbol{\varepsilon}})_{ij} &= \frac{D}{Dt}(\boldsymbol{\varepsilon})_{ij} = \frac{D}{Dt}\left(\frac{1}{2}\begin{bmatrix} 0 & \exp^{-2t}-1 & \exp^{-3t}-1 \\ \exp^{-2t}-1 & 0 & 0 \\ \exp^{-3t}-1 & 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & -\exp^{-2t} & -\frac{3}{2}\exp^{-3t} \\ -\exp^{-2t} & 0 & 0 \\ -\frac{3}{2}\exp^{-3t} & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{2.187}
$$

with that we can conclude that:

$$
\mathbf{D} = \dot{\boldsymbol{\varepsilon}}
\tag{2.188}
$$

Problem 2.72

Consider a material body in a small deformation regime, which is subjected to the following displacement field:

$$
\mathrm{u}_1 = (-2x_1 + 7x_2)\times 10^{-3} \quad ; \quad \mathrm{u}_2 = (-10x_2 - x_1)\times 10^{-3} \quad ; \quad \mathrm{u}_3 = x_3 \times 10^{-3}
$$

a) Find the infinitesimal spin and strain tensor;

b) Find the principal invariants of the infinitesimal strain tensor, as well as the correspondent characteristic equation;

c) Draw the Mohr's circle in strain, and obtain the maximum shear strain;

d) Find the dilatation and the deviatoric infinitesimal strain tensor.

Solution

a) For the displacement gradient we obtain:

$$
(\nabla\vec{\mathbf{u}})_{ij}=\frac{\partial \mathrm{u}_i}{\partial x_j}=\begin{bmatrix}\dfrac{\partial \mathrm{u}_1}{\partial x_1} & \dfrac{\partial \mathrm{u}_1}{\partial x_2} & \dfrac{\partial \mathrm{u}_1}{\partial x_3}\\ \dfrac{\partial \mathrm{u}_2}{\partial x_1} & \dfrac{\partial \mathrm{u}_2}{\partial x_2} & \dfrac{\partial \mathrm{u}_2}{\partial x_3}\\ \dfrac{\partial \mathrm{u}_3}{\partial x_1} & \dfrac{\partial \mathrm{u}_3}{\partial x_2} & \dfrac{\partial \mathrm{u}_3}{\partial x_3}\end{bmatrix}=\begin{bmatrix}-2 & 7 & 0\\ -1 & -10 & 0\\ 0 & 0 & 1\end{bmatrix}\times 10^{-3} \qquad \left[\frac{m}{m}\right]
$$

In the International System of Units the displacement gradient is dimensionless, i.e. $[\nabla\vec{\mathbf{u}}]=\left[\dfrac{\partial\vec{\mathbf{u}}}{\partial\vec{x}}\right]=\dfrac{m}{m}$.

As for the infinitesimal spin tensor we obtain:

$$
\omega_{ij}=(\nabla^{skew}\vec{\mathbf{u}})_{ij}=\frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j}-\frac{\partial \mathrm{u}_j}{\partial x_i}\right)=\begin{bmatrix}0 & 4 & 0\\ -4 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\times 10^{-3}
$$

Then for the infinitesimal strain tensor we have:

$$
\varepsilon_{ij}=(\nabla^{sym}\vec{\mathbf{u}})_{ij}=\frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j}+\frac{\partial \mathrm{u}_j}{\partial x_i}\right)=\begin{bmatrix}-2 & 3 & 0\\ 3 & -10 & 0\\ 0 & 0 & 1\end{bmatrix}\times 10^{-3}
$$

b) The principal invariants are defined as $I_{\boldsymbol{\varepsilon}}=\mathrm{Tr}(\boldsymbol{\varepsilon})$, $II_{\boldsymbol{\varepsilon}}=\frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2-\mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\}$, $III_{\boldsymbol{\varepsilon}}=\det(\boldsymbol{\varepsilon})$, (see Chapter 1). Then, it follows that:

$$
I_{\boldsymbol{\varepsilon}}=\mathrm{Tr}(\boldsymbol{\varepsilon})=(-2-10+1)\times 10^{-3}=-11\times 10^{-3}
$$

$$
II_{\boldsymbol{\varepsilon}}=\frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2-\mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\}=\left(\begin{vmatrix} & & \\ & -10 & 0\\ & 0 & 1\end{vmatrix}+\begin{vmatrix}-2 & & 0\\ & & \\ 0 & & 1\end{vmatrix}+\begin{vmatrix}-2 & 3 & \\ 3 & -10 & \\ & & \end{vmatrix}\right)\times 10^{-6}=-1\times 10^{-6}
$$

$$
III_{\boldsymbol{\varepsilon}}=\det(\boldsymbol{\varepsilon})=11\times 10^{-9}
$$

Then, the characteristic determinant is:

$$
\begin{vmatrix}-2\times 10^{-3}-\varepsilon & 3\times 10^{-3} & 0\\ 3\times 10^{-3} & -10\times 10^{-3}-\varepsilon & 0\\ 0 & 0 & 1\times 10^{-3}-\varepsilon\end{vmatrix}=0
$$

whilst the characteristic equation is:

$$
\varepsilon^3-I_{\boldsymbol{\varepsilon}}\varepsilon^2+II_{\boldsymbol{\varepsilon}}\varepsilon-III_{\boldsymbol{\varepsilon}}=0 \quad\Rightarrow\quad \varepsilon^3+11\times 10^{-3}\varepsilon^2+\varepsilon\times 11\times 10^{-6}-11\times 10^{-9}=0
$$

c) To draw the Mohr's circle for strain, (see Chaves (2013) - Appendix A), we need to evaluate the eigenvalues of $\boldsymbol{\varepsilon}$. But, if we take a look at the components of $\boldsymbol{\varepsilon}$ we can verify that $\varepsilon=1$ is already an eigenvalue associated with the direction $\hat{\mathrm{n}}_i=[0 \quad 0 \quad \pm 1]$. So, to obtain the remaining eigenvalues one only need solve the following system:

$$
\begin{vmatrix}-2\times 10^{-3}-\varepsilon & 3\times 10^{-3}\\ 3\times 10^{-3} & -10\times 10^{-3}-\varepsilon\end{vmatrix}=0 \quad\Rightarrow\quad \varepsilon^2+12\times 10^{-3}\varepsilon+11\times 10^{-6}=0\Rightarrow\begin{cases}\varepsilon_1=-1.0\times 10^{-3}\\ \varepsilon_2=-11.0\times 10^{-3}\end{cases}
$$

Then by restructuring the eigenvalues such that $\varepsilon_I>\varepsilon_{II}>\varepsilon_{III}$, we obtain:

$$\varepsilon_I = 1.0\times10^{-3} \quad ; \quad \varepsilon_{II} = -1.0\times10^{-3} \quad ; \quad \varepsilon_{III} = -11.0\times10^{-3}$$

Then the maximum shear (tangential) strain is evaluated as follows:

$$\varepsilon_{S\max} = \frac{\varepsilon_I - \varepsilon_{III}}{2} = 6\times10^{-3}$$

Finally, the Mohr's circle for strain can be depicted as:

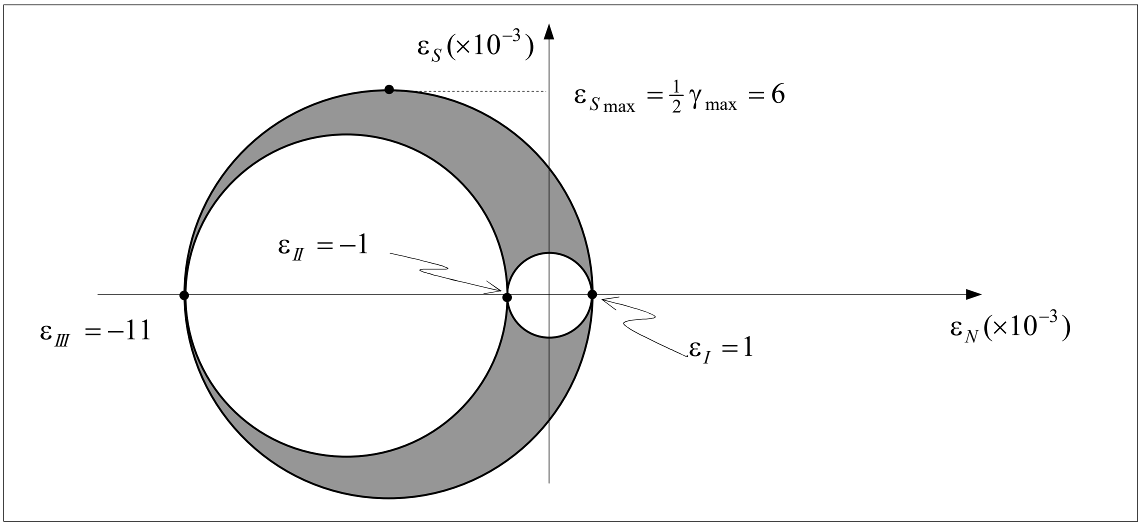


Figure 2.35: Mohr's circle in strain.

d) The variation of volume (dilatation) - ε_V, for small deformation regime, is given by:

$$\varepsilon_V = I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = -12\times10^{-3}$$

The additive decomposition of $\boldsymbol{\varepsilon}$ into a spherical and a deviatoric part is denoted by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}$, where the spherical part is given by:

$$\varepsilon_{ij}^{sph} = \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\delta_{ij} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\times10^{-3}$$

And, the deviatoric part is given by:

$$\varepsilon_{ij}^{dev} = \varepsilon_{ij} - \varepsilon_{ij}^{sph} = \left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \right)\times10^{-3} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix}\times10^{-3}$$

Problem 2.73

At one point of the continuum, the displacement gradient is represented by its components as follows:

$$(\nabla\vec{\mathbf{u}})_{ij} = \begin{bmatrix} 4 & -1 & -4 \\ 1 & -4 & 2 \\ 4 & 0 & 6 \end{bmatrix}\times10^{-3} \tag{2.189}$$

Obtain:

a) the infinitesimal strain and spin tensors;

b) the components of the spherical and deviatoric parts of the infinitesimal strain tensor;

c) the principal invariants of $\boldsymbol{\varepsilon}$: $I_{\boldsymbol{\varepsilon}}, II_{\boldsymbol{\varepsilon}}, III_{\boldsymbol{\varepsilon}}$;

d) the eigenvalues and eigenvectors of the rate-of-deformation tensor.

Solution:

a) The infinitesimal strain tensor ($\boldsymbol{\varepsilon}$) is the symmetric part of the displacement gradient:

$$\boldsymbol{\varepsilon} = \nabla^{sym}\vec{\mathbf{u}} = \frac{1}{2}\left[(\nabla\vec{\mathbf{u}}) + (\nabla\vec{\mathbf{u}})^T\right] \tag{2.190}$$

Then:

$$\varepsilon_{ij} = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & -4 \\ 1 & -4 & 2 \\ 4 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 4 \\ -1 & -4 & 0 \\ -4 & 2 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 & 0 & 0 \\ 0 & -8 & 2 \\ 0 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \qquad [\times 10^{-3}]$$

The infinitesimal spin tensor $\boldsymbol{\omega} = \nabla^{skew}\vec{\mathbf{u}}$

$$\omega_{ij} = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & -4 \\ 1 & -4 & 2 \\ 4 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 4 \\ -1 & -4 & 0 \\ -4 & 2 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -2 & -8 \\ 2 & 0 & 2 \\ 8 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} \qquad [\times 10^{-3}]$$

b) The tensor can be additively decomposed into a spherical and deviatoric part:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev} \tag{2.191}$$

where the spherical part is given by:

$$\boldsymbol{\varepsilon}^{sph} = \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} = \frac{6}{3}\mathbf{1} = 2\mathbf{1} \quad\Rightarrow\quad \varepsilon_{ij}^{sph} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad [\times 10^{-3}] \tag{2.192}$$

The deviatoric part is given by:

$$\varepsilon_{ij}^{dev} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad [\times 10^{-3}] \tag{2.193}$$

c) The principal invariants of $\boldsymbol{\varepsilon}$ are:

$$\begin{aligned} I_{\boldsymbol{\varepsilon}} &= \mathrm{Tr}(\boldsymbol{\varepsilon}) = 6 \qquad [\times 10^{-3}] \\ II_{\boldsymbol{\varepsilon}} &= \begin{vmatrix} -4 & 1 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & -4 \end{vmatrix} = -17 \qquad [\times 10^{-3}]^2 \\ III_{\boldsymbol{\varepsilon}} &= 4 \times (-4) \times 6 - 4 = -100 \qquad [\times 10^{-3}]^3 \end{aligned} \tag{2.194}$$

d) The infinitesimal strain tensor components:

$$\varepsilon_{ij} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \qquad [\times 10^{-3}] \tag{2.195}$$

Note that $\varepsilon_1 = 4 \times 10^{-3}$ is one eigenvalue associated with the eigenvector $[\pm 1,0,0]$. To obtain the remaining eigenvalues, we need to solve the characteristic determinant:

$$\begin{vmatrix} -4-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-4-\lambda)(6-\lambda) - 1 = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda - 25 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4\times 1\times(-25)}}{2\times 1} = \frac{2 \pm \sqrt{4 + 4\times 25}}{2} = 1 \pm \sqrt{26} \tag{2.196}$$

$$\Rightarrow \begin{cases} \lambda_1 = 6.0990 \\ \lambda_2 = -4.099 \end{cases}$$

thus:

$$\varepsilon_1 = 4\times 10^{-3}; \quad \varepsilon_2 = 6.0990\times 10^{-3}; \quad \varepsilon_3 = -4.099\times 10^{-3} \tag{2.197}$$

Restructuring we obtain:

$$\varepsilon_I = 6.0990\times 10^{-3}; \qquad \varepsilon_{II} = 4\times 10^{-3}; \qquad \varepsilon_{III} = -4.099\times 10^{-3} \tag{2.198}$$

Problem 2.74

Obtain the infinitesimal strain tensor and the infinitesimal spin tensor for the following displacement field:

$$\mathrm{u}_1 = x_1^2 \quad ; \quad \mathrm{u}_2 = x_1 x_2 \quad ; \quad \mathrm{u}_3 = 0$$

Solution:

In the small deformation regime, the infinitesimal strain tensor is given by:

$$E_{ij}^L \approx e_{ij}^L \approx \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right)$$

We need to obtain the displacement gradient components:

$$\frac{\partial \mathrm{u}_j}{\partial x_k} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with that we can obtain:

$$E_{ij}^L \approx e_{ij}^L \approx \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2x_1 & x_2 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 & \frac{x_2}{2} & 0 \\ \frac{x_2}{2} & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The infinitesimal spin tensor:

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} - \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2x_1 & x_2 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \frac{-x_2}{2} & 0 \\ \frac{x_2}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2.75

Figure 2.36 shows the transformation experienced by the square $ABCD$ of unit side.

a) State the equations of motion;

b) Is the theory valid for small deformation? justify the answer;

c) Is the finite deformation valid? Justify.

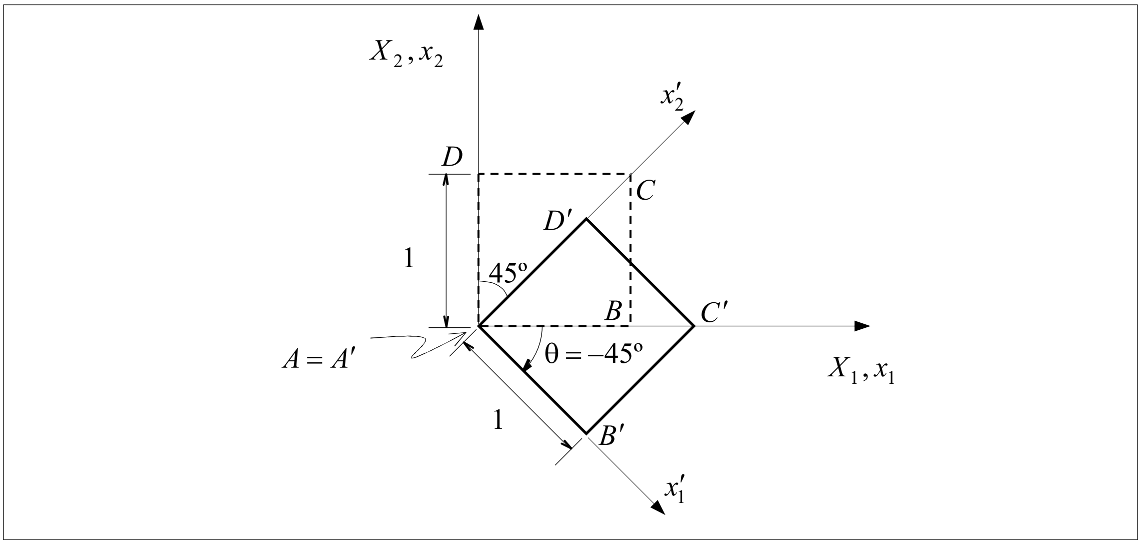


Figure 2.36: Body subjected to rotation.

Solution:

The transformation law between systems $x \Rightarrow x'$ is given by:

$$\begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \xrightarrow{\theta=-45º} \begin{Bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{2.199}$$

NOTE: Remember that by definition of the transformation matrix a_{ij} from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{x}}'$ is given by:

$$a_{ij} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) & \cos(x'_1, x_3) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) & \cos(x'_2, x_3) \\ \cos(x'_3, x_1) & \cos(x'_3, x_2) & \cos(x'_3, x_3) \end{bmatrix} = \begin{bmatrix} \cos(315º) & \cos(225º) & \cos(90º) \\ \cos(405º) & \cos(345º) & \cos(90º) \\ \cos(90º) & \cos(90º) & \cos(0º) \end{bmatrix}$$

Considering the spatial and material coordinates are superimposed, the equations of motion are defined by the inverse of the equation in (2.199):

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = \frac{\sqrt{2}}{2} X_1 + \frac{\sqrt{2}}{2} X_2 \\ x_2 = -\frac{\sqrt{2}}{2} X_1 + \frac{\sqrt{2}}{2} X_2 \end{cases}$$

For example, the point C in the reference configuration has the material coordinates $X_1^C = 1$, $X_2^C = 1$. After the motion we have $x_1^C = \frac{\sqrt{2}}{2}(1) + \frac{\sqrt{2}}{2}(1) = \sqrt{2}$, $x_2^C = -\frac{\sqrt{2}}{2}(1) + \frac{\sqrt{2}}{2}(1) = 0$

Displacement field:

$$\begin{cases} \mathrm{u}_1 = x_1 - X_1 = \frac{\sqrt{2}}{2}X_1 - \frac{\sqrt{2}}{2}X_2 - X_1 = X_1\left(\frac{\sqrt{2}}{2} - 1\right) - \frac{\sqrt{2}}{2}X_2 \\ \mathrm{u}_2 = x_2 - X_2 = \frac{\sqrt{2}}{2}X_1 + \frac{\sqrt{2}}{2}X_2 - X_2 = \frac{\sqrt{2}}{2}X_1 + X_2\left(\frac{\sqrt{2}}{2} - 1\right) \end{cases}$$

Displacement material gradient:

$$\frac{\partial \mathrm{u}_i(\vec{\boldsymbol{X}})}{\partial X_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial X_1} & \frac{\partial \mathrm{u}_1}{\partial X_2} & \frac{\partial \mathrm{u}_1}{\partial X_3} \\ \frac{\partial \mathrm{u}_2}{\partial X_1} & \frac{\partial \mathrm{u}_2}{\partial X_2} & \frac{\partial \mathrm{u}_2}{\partial X_3} \\ \frac{\partial \mathrm{u}_3}{\partial X_1} & \frac{\partial \mathrm{u}_3}{\partial X_2} & \frac{\partial \mathrm{u}_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} - 1 & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The infinitesimal strain tensor is given by $\boldsymbol{\varepsilon} = \nabla^{sym}\vec{\mathbf{u}} = \frac{1}{2}\left[(\nabla\vec{\mathbf{u}}) + (\nabla\vec{\mathbf{u}})^T\right]$, thus:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} - 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0_{ij}$$

Note that, for a rigid body motion the strain tensors must be equal to zero, i.e. $\boldsymbol{\varepsilon} = \mathbf{0}$ (the infinitesimal strain tensor), $\boldsymbol{E} = \mathbf{0}$ (the Green-Lagrange strain tensor), $\boldsymbol{e} = \mathbf{0}$ (the Almansi strain tensor). Calculating the Green-Lagrange strain tensor components we have:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i} + \frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 Stress

3.1 Force, Stress Tensor, Stress vector

Problem 3.1

Ignoring the curvature of the Earth's surface, the gravitational field can be assumed to be uniform as shown in Figure 3.1, where g is the acceleration caused by gravity (the gravity of the Earth). Find the resultant force acting on the body $\mathcal{B}$.

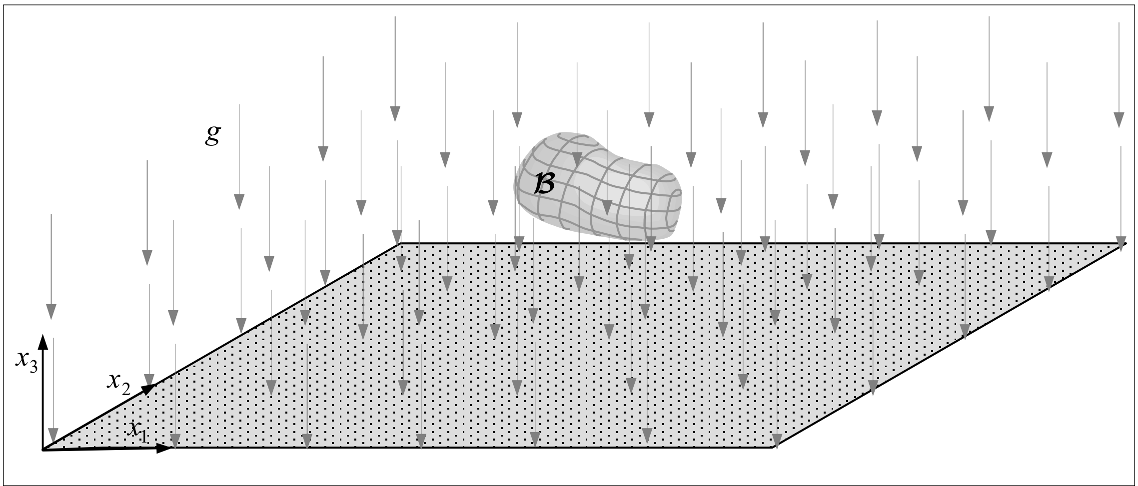


Figure 3.1: Gravitational field.

Solution:

All bodies immersed in a force field are subjected to the specific body force $\vec{\mathbf{b}}$, and in the special case presented in Figure 3.1 this is given by:

$$\mathsf{b}_i(\vec{\boldsymbol{x}},t) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \qquad \left[\frac{m}{s^2}\right]$$

Hence, the total force acting on the body can be evaluated as follows:

$$\mathsf{F}_i = \int_V \rho \mathsf{b}_i(\vec{\boldsymbol{x}},t)\, dV = \begin{bmatrix} 0 \\ 0 \\ -\int_V \rho g \, dV \end{bmatrix}$$

We can also verify the F unit: $[\mathsf{F}] = \int_V \left[\frac{kg}{m^3}\right]\left[\frac{m}{s^2}\right] \overbrace{dV}^{[m^3]} = \frac{kg\, m}{s^2} = N(Newton)$.

Problem 3.2

The Cauchy stress tensor components at a point P are given by:

$$\sigma_{ij} = \begin{bmatrix} 8 & -4 & 1 \\ -4 & 3 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix} Pa$$

a) Calculate the traction vector ($\vec{\mathbf{t}}^{(\hat{n})}$) at P which is associated with the plane ABC defined in Figure 3.2.

b) With reference to paragraph a).

Obtain the normal ($\vec{\boldsymbol{\sigma}}_N$) and tangential ($\vec{\boldsymbol{\sigma}}_S$) traction vectors at P, Chaves(2013)-Appendix A.

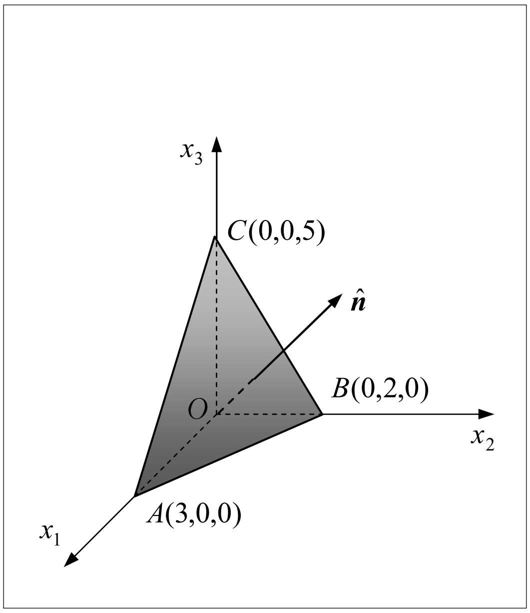


Figure 3.2: Plane ABC.

Solution:

First, we obtain the unit vector which is normal to the plane ABC. To do this we choose two vectors on the plane:

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3 \quad ; \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 0\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3$$

Then, the normal vector associated with the plane ABC is obtained by means of the cross product between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, i.e.:

$$\vec{\boldsymbol{n}} = \overrightarrow{BC} \wedge \overrightarrow{BA} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & -2 & 5 \\ 3 & -2 & 0 \end{vmatrix} = 10\hat{\mathbf{e}}_1 + 15\hat{\mathbf{e}}_2 + 6\hat{\mathbf{e}}_3$$

Additionally, the unit vector codirectional with $\vec{\boldsymbol{n}}$ is given by:

$$\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{n}}}{\|\vec{\boldsymbol{n}}\|} = \frac{10}{19}\hat{\mathbf{e}}_1 + \frac{15}{19}\hat{\mathbf{e}}_2 + \frac{6}{19}\hat{\mathbf{e}}_3$$

Then by using the equation $\vec{\mathbf{t}}^{(\hat{n})} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$, we can obtain the traction components as:

$$\mathrm{t}_i^{(\hat{n})} = \sigma_{ij}\hat{\mathrm{n}}_j \quad \Rightarrow \quad \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 8 & -4 & 1 \\ -4 & 3 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 15 \\ 6 \end{bmatrix} Pa \quad \Rightarrow \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 26 \\ 8 \\ 29.5 \end{bmatrix} Pa$$

b) The traction vector $\vec{\mathbf{t}}^{(\hat{n})}$ associated with the normal $\hat{\boldsymbol{n}}$ can be broken down into a normal ($\vec{\boldsymbol{\sigma}}_N$) and a tangential ($\vec{\boldsymbol{\sigma}}_S$) vector as shown in Figure 3.3. Then,

$$\vec{\mathbf{t}}^{(\hat{n})} = \vec{\boldsymbol{\sigma}}_N + \vec{\boldsymbol{\sigma}}_S \qquad \text{or} \qquad \vec{\mathbf{t}}^{(\hat{n})} = \sigma_N \hat{\boldsymbol{n}} + \sigma_S \hat{\boldsymbol{s}}$$

where σ_N and σ_S are the magnitudes of $\vec{\boldsymbol{\sigma}}_N$ and $\vec{\boldsymbol{\sigma}}_S$, respectively.

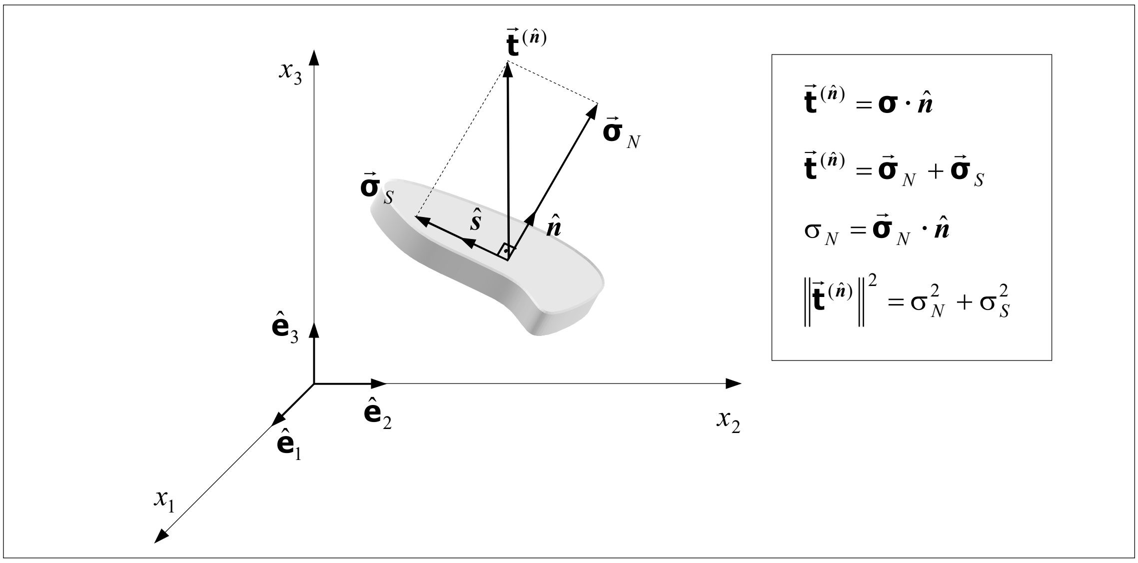


Figure 3.3: Normal and tangential stress vectors.

The normal component, σ_N, can be evaluated as follows:

$$\sigma_N = \vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} \cdot \hat{\boldsymbol{n}} = (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \boldsymbol{\sigma} : (\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}) = \mathrm{t}_i^{(\hat{\boldsymbol{n}})} \hat{n}_i = (\sigma_{ij} \hat{n}_j) \hat{n}_i = \hat{n}_i \sigma_{ij} \hat{n}_j = \sigma_{ij} (\hat{n}_i \hat{n}_j)$$

Thus:

$$\sigma_N = \mathrm{t}_i \hat{n}_i \quad \Rightarrow \quad \sigma_N = \frac{1}{19^2} \begin{bmatrix} 26 & 8 & 29.5 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \\ 6 \end{bmatrix} \approx 1.54\ Pa$$

Then the tangential component, σ_S, can be obtained by means of the Pythagorean Theorem, i.e.:

$$\left\| \vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} \right\|^2 = \sigma_N^2 + \sigma_S^2 \quad \Rightarrow \quad \sigma_S^2 = \mathrm{t}_i^{(\hat{\boldsymbol{n}})} \mathrm{t}_i^{(\hat{\boldsymbol{n}})} - \sigma_N^2$$

where

$$\mathrm{t}_i^{(\hat{\boldsymbol{n}})} \mathrm{t}_i^{(\hat{\boldsymbol{n}})} = \frac{1}{19^2} \begin{bmatrix} 26 & 8 & 29.5 \end{bmatrix} \begin{bmatrix} 26 \\ 8 \\ 29.5 \end{bmatrix} \approx 4.46$$

Thus,

$$\sigma_S = \sqrt{\mathrm{t}_i^{(\hat{\boldsymbol{n}})} \mathrm{t}_i^{(\hat{\boldsymbol{n}})} - \sigma_N^2} = \sqrt{4.46 - 2.3716} \approx 2.0884\ Pa$$

Problem 3.3

The stress state at a point in the continuum is represented by the Cartesian components of the Cauchy stress tensor as:

$$\sigma_{ij} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

a) Obtain the components of $\boldsymbol{\sigma}$ in a new system x_1', x_2', x_3', where the transformation matrix is given by Figure 3.4.

b) Obtain the principal invariants of $\boldsymbol{\sigma}$;

c) Obtain the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$. Also verify if the eigenvectors form a basis transformation between the original and the principal space;

d) Illustrate the Cauchy stress tensor graphically by using the Mohr's circle in stress, (see Appendix A in Chaves (2013));

e) Obtain the spherical ($\boldsymbol{\sigma}^{sph}$) and the deviatoric ($\boldsymbol{\sigma}^{dev}$) part of $\boldsymbol{\sigma}$. Also, find the principal invariants of $\boldsymbol{\sigma}^{dev}$;

f) Obtain the octahedral normal (σ_N^{oct}) and tangential (σ_S^{oct}) components of $\boldsymbol{\sigma}$.

$$a_{ij} = \mathcal{A} = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

where

$$a_{11} = \cos\alpha_1$$
$$a_{12} = \cos\beta_1$$
$$a_{13} = \cos\gamma_1$$
$$\vdots$$

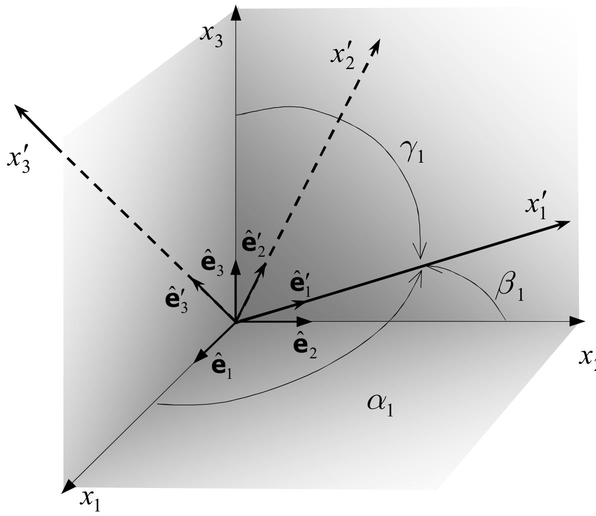


Figure 3.4

Solution:

a) The transformation law for the components of a second-order tensor is given by:

$$\sigma'_{ij} = a_{ik}a_{jl}\sigma_{kl} \xrightarrow{\text{Matrix form}} \sigma' = \mathcal{A}\,\sigma\,\mathcal{A}^T$$

Thus,

$$\sigma'_{ij} = \frac{1}{5^2}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0.6 & 0 \\ 0.6 & 2 & 0.8 \\ 0 & 0.8 & 2 \end{bmatrix}$$

These new components σ'_{ij} can be appreciated in Figure 3.5.

b) The principal invariants of the Cauchy stress tensor can be calculated as follows:

$$I_{\boldsymbol{\sigma}} = \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$II_{\boldsymbol{\sigma}} = \frac{1}{2}\left[(\mathrm{Tr}\boldsymbol{\sigma})^2 - \mathrm{Tr}(\boldsymbol{\sigma}^2)\right] = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$
$$= \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{33}\sigma_{22} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$III_{\boldsymbol{\sigma}} = \det(\boldsymbol{\sigma}) = \epsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} = \frac{1}{6}\left(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ii}\sigma_{jk}\sigma_{jk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}\right)$$
$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

By substituting the values of σ_{ij} for those in the proposed problem we can obtain:

$$I_{\boldsymbol{\sigma}} = 6 \quad ; \quad II_{\boldsymbol{\sigma}} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 11 \quad ; \quad III_{\boldsymbol{\sigma}} = 6$$

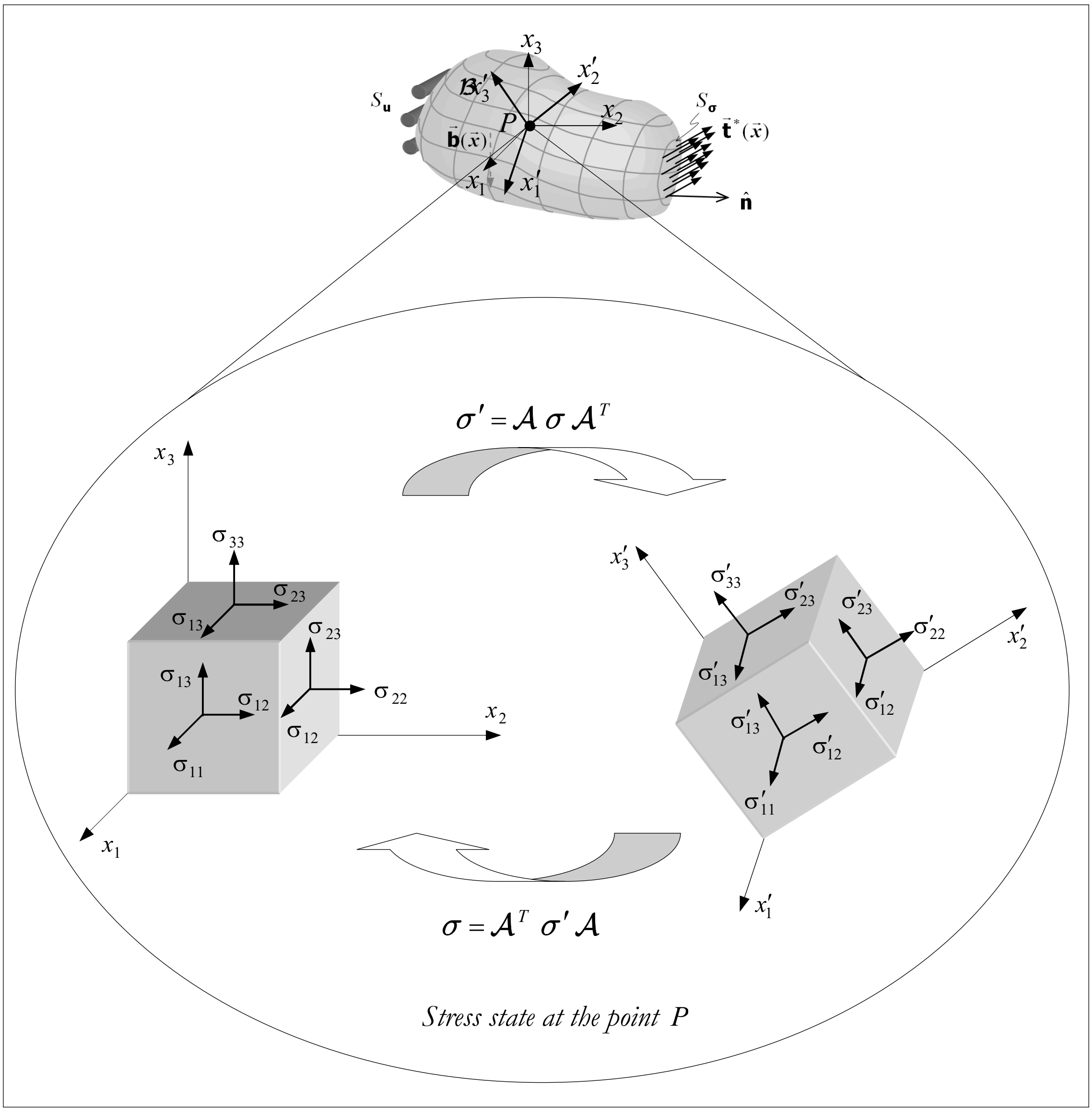


Figure 3.5: Basis transformation.

c) The principal stresses (σ_i) and principal directions ($\hat{\mathbf{n}}^{(i)}$) are obtained by solving the following set of equations:

$$\begin{bmatrix} 2-\sigma & 1 & 0 \\ 1 & 2-\sigma & 0 \\ 0 & 0 & 2-\sigma \end{bmatrix} \begin{bmatrix} \mathrm{n}_1 \\ \mathrm{n}_2 \\ \mathrm{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To obtain the nontrivial solutions of $\hat{\mathbf{n}}^{(i)}$ we have to solve the characteristic determinant, which is a cubic equation for the unknown magnitude σ:

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \quad \Rightarrow \quad \sigma^3 - I_{\boldsymbol{\sigma}}\sigma^2 + II_{\boldsymbol{\sigma}}\sigma - III_{\boldsymbol{\sigma}} = 0$$

However, if we look at the format of the Cauchy stress tensor components, we can notice that we already have one solution since in the x_3-direction the tangential components are equal to zero, then:

$$\sigma_3 = 2 \xrightarrow{\text{Principal direction}} \mathsf{n}_1^{(3)} = \mathsf{n}_2^{(3)} = 0\,,\ \mathsf{n}_3^{(3)} = 1$$

To obtain the other two eigenvalues, one only need solve:

$$\begin{vmatrix} 2-\sigma & 1 \\ 1 & 2-\sigma \end{vmatrix} = (2-\sigma)^2 - 1 = 0 \quad \Rightarrow \quad \begin{cases} \sigma_1 = 1 \\ \sigma_2 = 3 \end{cases}$$

Then we can express the Cauchy stress tensor components in the principal space as:

$$\sigma''_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

Additionally, the principal direction associated with $\sigma_1 = 1$ is calculated as follows:

$$\begin{bmatrix} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} \mathsf{n}_1^{(1)} \\ \mathsf{n}_2^{(1)} \\ \mathsf{n}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \mathsf{n}_1^{(1)} + \mathsf{n}_2^{(1)} = 0 \\ \mathsf{n}_1^{(1)} + \mathsf{n}_2^{(1)} = 0 \end{cases} \Rightarrow \mathsf{n}_1^{(1)} = -\mathsf{n}_2^{(1)}$$

with $\mathsf{n}_3^{(1)} = 0$ and by using the condition ${\mathsf{n}_1^{(1)}}^2 + {\mathsf{n}_2^{(1)}}^2 = 1$ we can obtain:

$$\mathsf{n}_1^{(1)} = -\mathsf{n}_2^{(1)} = \frac{1}{\sqrt{2}} \text{ then } \hat{\mathsf{n}}_i^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Since $\boldsymbol{\sigma}$ is a symmetric tensor, the principal space is formed by an orthogonal basis, so, it is valid that:

$$\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \quad ; \quad \hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \quad ; \quad \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(2)}$$

Thus, the second principal direction can be obtained by the cross product between $\hat{\mathbf{n}}^{(3)}$ and $\hat{\mathbf{n}}^{(1)}$, i.e.:

$$\hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \hat{\mathbf{e}}_1 + \frac{1}{\sqrt{2}} \hat{\mathbf{e}}_2$$

which can also be checked by the following analysis:

The Principal direction associated with $\sigma_2 = 3$:

$$\begin{bmatrix} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \\ 0 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} \mathsf{n}_1^{(2)} \\ \mathsf{n}_2^{(2)} \\ \mathsf{n}_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -\mathsf{n}_1^{(2)} + \mathsf{n}_2^{(2)} = 0 \\ \mathsf{n}_1^{(2)} - \mathsf{n}_2^{(2)} = 0 \end{cases} \Rightarrow \mathsf{n}_1^{(2)} = \mathsf{n}_2^{(2)}$$

With $\mathsf{n}_3^{(3)} = 0$ and using the condition ${\mathsf{n}_1^{(3)}}^2 + {\mathsf{n}_2^{(3)}}^2 = 1$ we can obtain:

$$\mathsf{n}_1^{(2)} = \mathsf{n}_2^{(2)} = \frac{1}{\sqrt{2}} \text{ then } \hat{\mathsf{n}}_i^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

As we have seen in Chapter 1, the eigenvectors of a symmetric tensor form the transformation matrix $\mathcal{D}$, from the original system to the principal space, i.e. $\sigma'' = \mathcal{D}\,\sigma\,\mathcal{D}^T$, thus:

$$\begin{bmatrix} \sigma_1 = 1 & 0 & 0 \\ 0 & \sigma_2 = 3 & 0 \\ 0 & 0 & \sigma_3 = 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) The graphical representation of a second-order tensor can be obtained from the description in Appendix A-Chaves(2013). To do this we have to restructure the eigenvalues of $\boldsymbol{\sigma}$ so that $\sigma_I > \sigma_{II} > \sigma_{III}$, thus:

$$\sigma_I = 3 \quad ; \quad \sigma_{II} = 2 \quad ; \quad \sigma_{III} = 1$$

Then the three circumferences are defined by:

$$\text{Circle 1} \Rightarrow \quad ; \quad (center) C_1 = \frac{1}{2}(\sigma_{II} + \sigma_{III}) = 1.5 \quad ; \quad (radius) R_1 = \frac{1}{2}(\sigma_{II} - \sigma_{III}) = 0.5$$

$$\text{Circle 2} \Rightarrow \quad ; \quad (center) C_2 = \frac{1}{2}(\sigma_I + \sigma_{III}) = 2.0 \quad ; \quad (radius) R_2 = \frac{1}{2}(\sigma_I - \sigma_{III}) = 1.0$$

$$\text{Circle 3} \Rightarrow \quad ; \quad (center) C_3 = \frac{1}{2}(\sigma_I + \sigma_{II}) = 2.5 \quad ; \quad (radius) R_3 = \frac{1}{2}(\sigma_I - \sigma_{II}) = 0.5$$

Then, we can illustrate the Cauchy stress tensor at P by means of Mohr's circle in stress as shown in Figure 3.6.

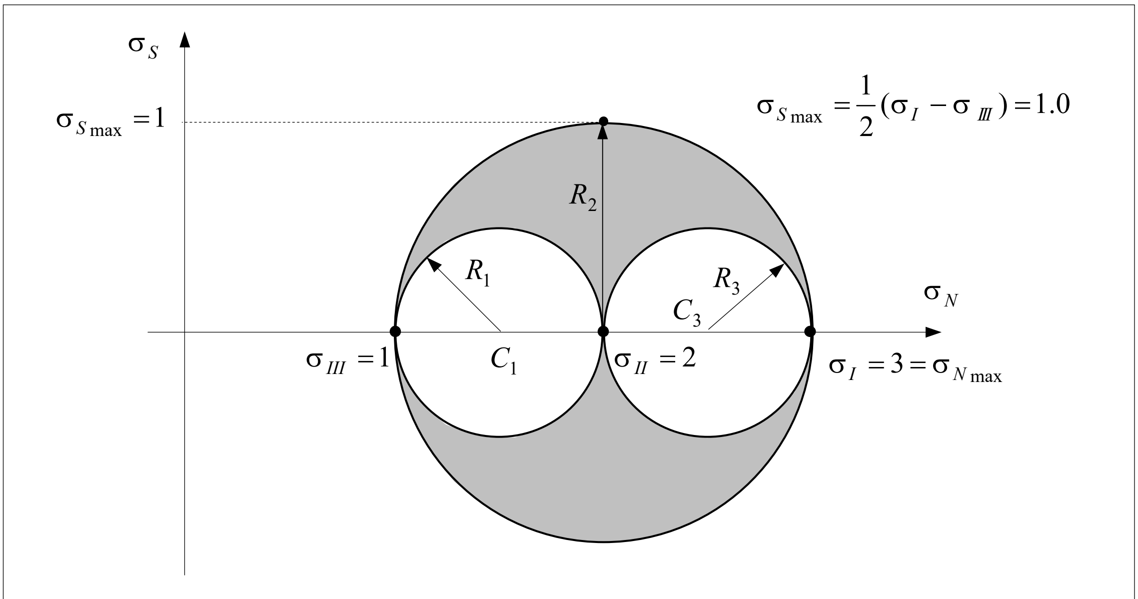


Figure 3.6: Mohr's circle in stress at the point P.

NOTE: Sign convention for stress when using Mohr's circle

When we are using scientific notation, the positive value of σ_{ij} is when it is oriented according to the axis, (see Figure 3.5). As we know the Mohr's circle is the representation of the three-dimensional stress state by means of two-dimensional graph ($\sigma_N \times \sigma_S$), due to this fact we lost some information about stress orientation (sense), so, we need to establish a new sign convention. Here, for Solid Mechanics, we adopt that: the normal stress is positive ($\sigma_N > 0$) when it is dealing with traction (tensile), otherwise, i.e. if we are dealing with compression $\sigma_N < 0$; the tangential (shear) stress is positive as indicated in Figure 3.7 (a). In general, materials have different behavior when they under traction or compression loads, but in the case of tangential stress its magnitude is only what really matters, (see Figure 3.7(c)), for this reason, sometimes the Mohr's circle is drawn only by considering the positive value of σ_S.

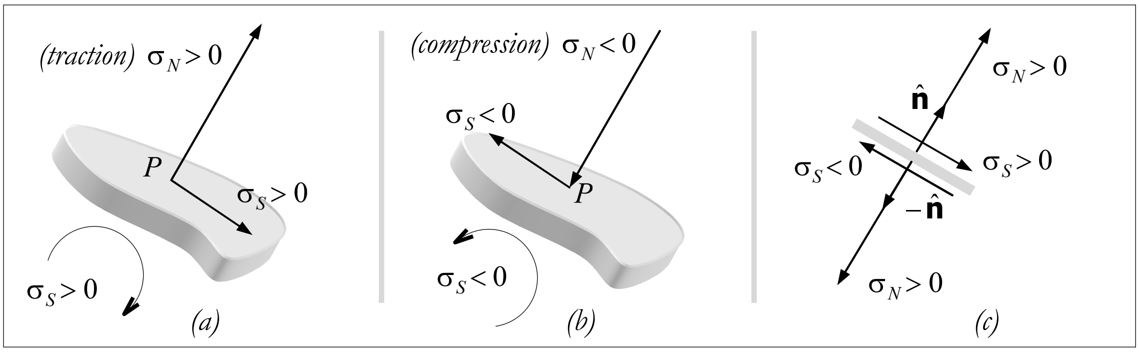


Figure 3.7: Stress sign convention.

e) A second-order tensor can be broken down additively into a spherical and a deviatoric part, i.e.:

Tensorial notation	Indicial notation
$\boxed{\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} \\ &= \sigma_m \mathbf{1} + \boldsymbol{\sigma}^{dev}\end{aligned}}$	$\begin{aligned}\sigma_{ij} &= \sigma_{ij}^{sph} + \sigma_{ij}^{dev} \\ &= \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma_{ij}^{dev} \\ &= \sigma_m \delta_{ij} + \sigma_{ij}^{dev}\end{aligned}$ (3.1)

A schematic representation of these components in the Cartesian basis can be appreciated in Figure 3.8 and the value of the scalar σ_m is evaluated as follows:

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{I_{\boldsymbol{\sigma}}}{3} = \frac{6}{3} = 2$$

Then the spherical part becomes:

$$\sigma_{ij}^{sph} = \sigma_m \delta_{ij} = 2\delta_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

And, the deviatoric part can be evaluated as follows:

$$\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix}$$

Thus,

$$\sigma_{ij}^{dev} = \begin{bmatrix} 2-2 & 1 & 0 \\ 1 & 2-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now let us remember from Chapter 1-Chaves(2013) that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors, i.e., they have the same principal directions, so we can use this information to operate in the principal space of $\boldsymbol{\sigma}$ to obtain the eigenvalues of $\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{sph}$. With that we can obtain:

$$\sigma_{ij}'^{dev} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the invariants of $\boldsymbol{\sigma}^{dev}$ are given by:

$$I_{\boldsymbol{\sigma}^{dev}} = \mathrm{Tr}(\boldsymbol{\sigma}^{dev}) = 0 \quad ; \quad II_{\boldsymbol{\sigma}^{dev}} = -1 \quad ; \quad III_{\boldsymbol{\sigma}^{dev}} = 0$$

Traditionally, in engineering, the invariants of the deviatoric stress tensor are represented by:

$$\boxed{\begin{aligned} \mathrm{J}_1 &= I_{\boldsymbol{\sigma}^{dev}} = 0 \\ \mathrm{J}_2 &= -II_{\boldsymbol{\sigma}^{dev}} = \frac{1}{3}\left(I_{\boldsymbol{\sigma}}^2 - 3\,II_{\boldsymbol{\sigma}}\right) \\ \mathrm{J}_3 &= III_{\boldsymbol{\sigma}^{dev}} = \frac{1}{27}\left(2I_{\boldsymbol{\sigma}}^3 - 9I_{\boldsymbol{\sigma}}\,II_{\boldsymbol{\sigma}} + 27\,III_{\boldsymbol{\sigma}}\right) \end{aligned}}$$

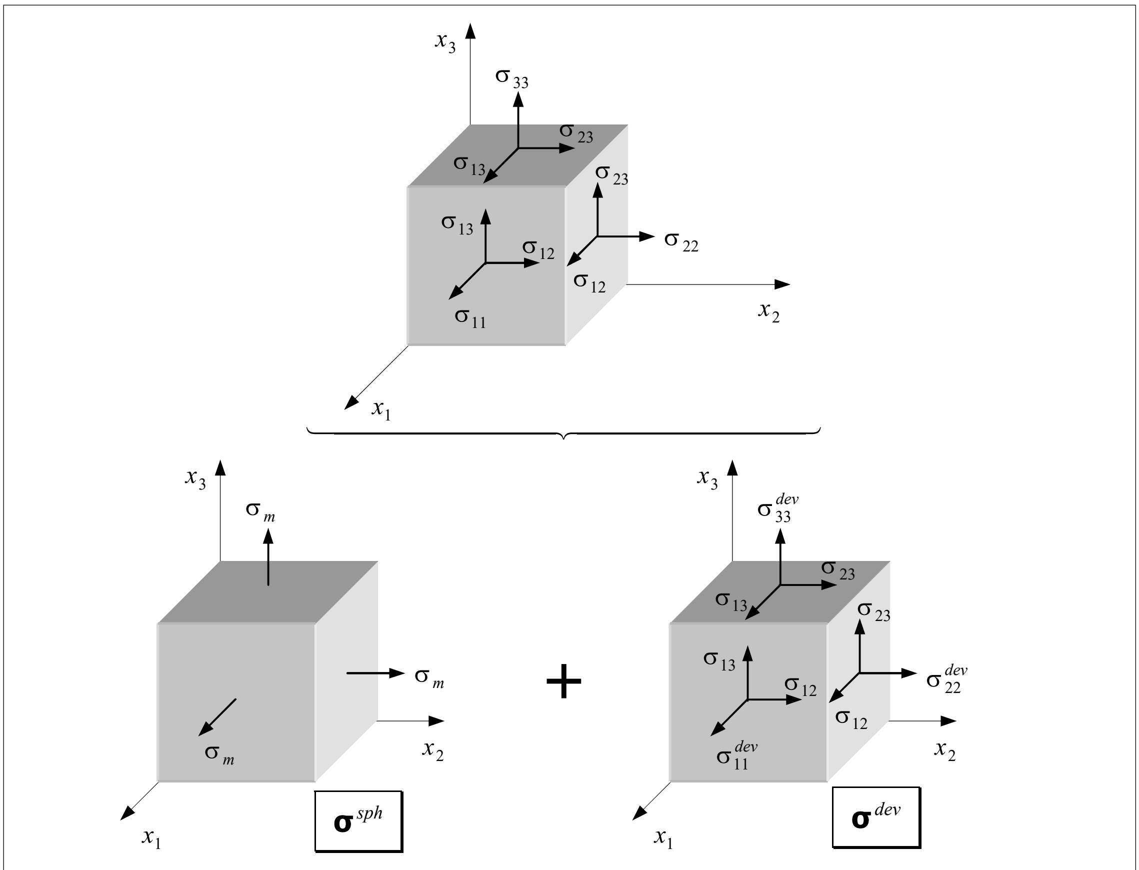


Figure 3.8: The spherical and deviatoric part of $\boldsymbol{\sigma}$.

f) The octahedral normal and tangential components, (see Appendix A in Chaves (2013)), can be expressed as:

$$\boxed{\sigma_N^{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}\sigma_{ii} = \frac{I_{\boldsymbol{\sigma}}}{3} = \sigma_m}$$

$$\boxed{\sigma_S^{oct} \equiv \tau_{oct} = \frac{1}{3}\sqrt{2I_{\boldsymbol{\sigma}}^2 - 6\,II_{\boldsymbol{\sigma}}} = \sqrt{\frac{2}{3}\mathrm{J}_2} = \sqrt{\frac{(\sigma_1^{dev})^2 + (\sigma_2^{dev})^2 + (\sigma_3^{dev})^2}{3}}}$$

Then, by substituting the values of the proposed problem we can obtain:

$$\sigma_N^{oct} = \sigma_m = 6 \quad ; \quad \tau_{oct} = \sqrt{\frac{2}{3}\mathrm{J}_2} = \sqrt{\frac{2}{3}}$$

Problem 3.4

At a point P the Cauchy stress tensor Cartesian components are given by:

$$\sigma_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} MPa \tag{3.2}$$

Find:

a) the traction vector $\vec{\mathbf{t}}$ related to the plane which is normal to the x_1-axis;

b) the traction vector $\vec{\mathbf{t}}$ associated with the plane whose normal is $(1,-1,2)$;

c) the traction vector $\vec{\mathbf{t}}$ associated with the plane parallel to the plane $2x_1 - 2x_2 - x_3 = 0$;

d) the principal stress at the point P;

e) the principal directions of $\boldsymbol{\sigma}$ at the point P.

Solution:

Recall that the traction vector is obtained by means of the equation $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$ which in indicial notation becomes $\mathrm{t}_i^{(\hat{\boldsymbol{n}})} = \sigma_{ij}\hat{\mathrm{n}}_j$.

a) In this case, the unit vector is $\hat{\mathrm{n}}_i = [1,0,0]$. Then, the traction vector is given by:

$$\mathrm{t}_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{\mathrm{n}}_j = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{3.3}$$

b) The unit vector associated with the direction $\mathrm{n}_i = [1,-1,2]$ can be obtained as follows:

$$\hat{\mathrm{n}}_i = \frac{\mathrm{n}_i}{\|\vec{\mathbf{n}}\|} \quad \Rightarrow \quad \hat{\mathrm{n}}_i = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \tag{3.4}$$

where the module of $\vec{\mathbf{n}}$ is $\|\vec{\mathbf{n}}\| = \sqrt{(1)^2 + (-1)^2 + (2)^2} = \sqrt{6}$. Thus,

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 5 \\ 10 \\ -1 \end{bmatrix} \tag{3.5}$$

c) For this case, the vector $n_i = [2,-2,-1]$ is normal to the plane and the unit vector associated with this direction is:

$$\hat{n}_i = \frac{n_i}{\|\vec{\mathbf{n}}\|} \quad \Rightarrow \quad \hat{n}_i = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \quad \textit{where} \quad \|\vec{\mathbf{n}}\| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

Thus,

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j = \frac{1}{3}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 \\ -10 \\ -7 \end{bmatrix} \tag{3.6}$$

d) The principal stresses can be obtained by solving the characteristic determinant:

$$\begin{vmatrix} 1-\sigma & 2 & 3 \\ 2 & 4-\sigma & 6 \\ 3 & 6 & 1-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad \sigma^3 - I_{\boldsymbol{\sigma}}\sigma^2 + II_{\boldsymbol{\sigma}}\sigma - III_{\boldsymbol{\sigma}} = 0 \tag{3.7}$$

where $I_{\boldsymbol{\sigma}} = 6$, $I_{\boldsymbol{\sigma}} = -40$ and $III_{\boldsymbol{\sigma}} = 0$, thus $\sigma^3 - 6\sigma^2 - 40\sigma = 0$. And the solutions are:

$$\sigma_1 = 10\,;\ \sigma_2 = 0\,;\ \sigma_3 = -4 \tag{3.8}$$

e) The principal directions are:

Associated with $\sigma_1 = 10$

$$\begin{cases} -9n_1 + 2n_2 + 3n_3 = 0 \\ 2n_1 - 6n_2 + 6n_3 = 0 \\ 3n_1 + 6n_2 - 9n_3 = 0 \end{cases} \Rightarrow \begin{cases} n_2^{(1)} = 2n_1^{(1)} \\ n_3^{(1)} = \frac{5}{3}n_1^{(1)} \end{cases} \quad \textit{with} \quad \hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1 \quad \Rightarrow \quad n_1^2 = \pm\frac{3}{\sqrt{70}}$$

$$\Rightarrow n_i^{(1)} = \pm\frac{1}{\sqrt{70}}\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$$

In the same fashion we can obtain:

$$\sigma_2 = 0 \xrightarrow{\text{eigenvector}} \hat{n}_i^{(2)} = \pm\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \sigma_2 = -4 \xrightarrow{\text{eigenvector}} \hat{n}_i^{(3)} = \mp\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

The eigenvectors are represented by the following unit vectors:

$$\hat{n}_i^{(1)} = \pm\frac{1}{\sqrt{70}}\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \quad ; \quad \hat{n}_i^{(2)} = \pm\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \hat{n}_i^{(3)} = \mp\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Note also that the eigenvectors formed an orthogonal basis, for example:

$$\hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)} \qquad \Rightarrow \qquad \hat{\mathbf{n}}^{(3)} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{3}{\sqrt{70}} & \frac{6}{\sqrt{70}} & \frac{5}{\sqrt{70}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{vmatrix} = -\frac{1}{\sqrt{14}}\hat{\mathbf{e}}_1 - \frac{2}{\sqrt{14}}\hat{\mathbf{e}}_2 + \frac{3}{\sqrt{14}}\hat{\mathbf{e}}_3$$

Problem 3.5

Show that $\vec{\boldsymbol{\sigma}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\mathbf{1} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$, where $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is the traction vector obtained by projecting the second-order tensor $\boldsymbol{\sigma}$ onto the $\hat{\mathbf{n}}$-direction, and $\vec{\boldsymbol{\sigma}}_S$ is the tangential stress vector associated with the plane, (see Figure 3.9).

Solution 1:

By using vector addition it is true that $\vec{\mathbf{t}}^{(\hat{n})} = \vec{\boldsymbol{\sigma}}_N + \vec{\boldsymbol{\sigma}}_S \Rightarrow \vec{\boldsymbol{\sigma}}_S = \vec{\mathbf{t}}^{(\hat{n})} - \vec{\boldsymbol{\sigma}}_N$. The vector $\vec{\boldsymbol{\sigma}}_N$ can be represented by $\vec{\boldsymbol{\sigma}}_N = \|\vec{\boldsymbol{\sigma}}_N\|\hat{\mathbf{n}}$ and the magnitud $\vec{\boldsymbol{\sigma}}_N = \|\vec{\boldsymbol{\sigma}}_N\|\hat{\mathbf{n}}$ can be obtained by the projection of $\vec{\mathbf{t}}^{(\hat{n})}$ according to $\hat{\mathbf{n}}$-direction, i.e. $\|\vec{\boldsymbol{\sigma}}_N\| = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}}$, so, $\vec{\boldsymbol{\sigma}}_N = \|\vec{\boldsymbol{\sigma}}_N\|\hat{\mathbf{n}} = (\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$. Note that is also true that $\vec{\boldsymbol{\sigma}}_N = (\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$, so

$$\vec{\boldsymbol{\sigma}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \left[\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}}\right] \hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\mathbf{1} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$$

Solution 2:

We can also solve the problem by using the components of the following equation $\vec{\boldsymbol{\sigma}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - [\boldsymbol{\sigma} : (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})]\hat{\mathbf{n}}$, i.e.:

$$\sigma_{S_i} = \mathrm{t}_i^{(\hat{\mathbf{n}})} - [(\hat{\mathrm{n}}_k \hat{\mathrm{n}}_l \sigma_{kl})]\hat{\mathrm{n}}_i = \mathrm{t}_i^{(\hat{\mathbf{n}})} - \hat{\mathrm{n}}_i \hat{\mathrm{n}}_k \mathrm{t}_k^{(\hat{\mathbf{n}})} = \mathrm{t}_k^{(\hat{\mathbf{n}})} \delta_{ik} - \hat{\mathrm{n}}_i \hat{\mathrm{n}}_k \mathrm{t}_k^{(\mathbf{n})} = \mathrm{t}_k^{(\hat{\mathbf{n}})} \left(\delta_{ik} - \hat{\mathrm{n}}_i \hat{\mathrm{n}}_k\right)$$

which in tensorial notation becomes

$$\vec{\boldsymbol{\sigma}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\mathbf{1} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$$

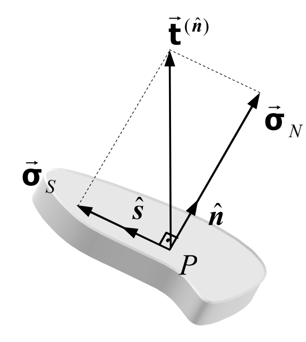


Figure 3.9: Normal and tangential stress vectors.

Problem 3.6

The stress state at one point P of the continuum is schematically represented in Figure 3.10. Obtain the value of the component σ_{22} of the Cauchy stress tensor such that there is at least one plane passing through P in which is free of stress, and obtain the direction of this plane.

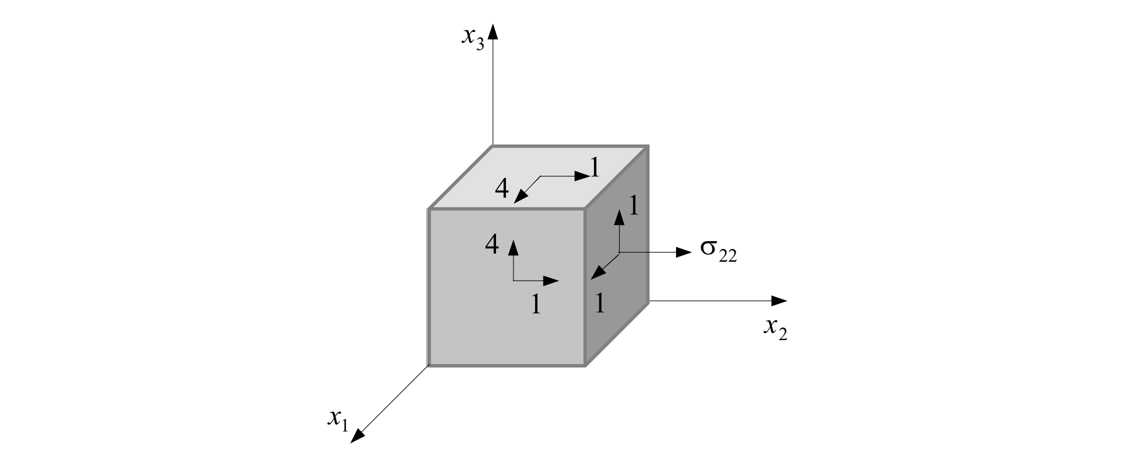


Figure 3.10

Solution:

We seek to find a plane whose direction is $\hat{\mathbf{n}}$ (unit vector) such that $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \vec{\mathbf{0}}$. We can relate the Cauchy stress tensor to the traction vector by means of the equation $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, thus:

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & \sigma_{22} & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with that we can obtain the following set of equations:

$$\begin{cases} n_2 + 4n_3 = 0 \Rightarrow n_3 = -\dfrac{1}{4} n_2 \\ n_1 + \sigma_{22} n_2 + n_3 = 0 \\ 4n_1 + n_2 = 0 \Rightarrow n_1 = -\dfrac{1}{4} n_2 \end{cases} \tag{3.9}$$

By combining the above equations we can obtain:

$$n_1 + \sigma_{22} n_2 + n_3 = 0 \quad \Rightarrow \quad -\frac{1}{4} n_2 + \sigma_{22} n_2 - \frac{1}{4} n_2 = 0 \quad \Rightarrow \quad \left(-\frac{1}{4} + \sigma_{22} - \frac{1}{4} \right) n_2 = 0$$

Then, for $\vec{\mathbf{n}} \neq \vec{\mathbf{0}}$, we have: $\left(-\dfrac{1}{4} + \sigma_{22} - \dfrac{1}{4} \right) = 0 \Rightarrow \sigma_{22} = \dfrac{1}{2}$.

To define the direction of the plane we will start by the restriction $\hat{n}_i \hat{n}_i = 1$, then:

$$\hat{n}_i \hat{n}_i = 1 \quad \Rightarrow \quad \hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1 \quad \Rightarrow \quad \left(-\frac{1}{4} \hat{n}_2 \right)^2 + \hat{n}_2^2 + \left(-\frac{1}{4} \hat{n}_2 \right)^2 = 1$$

$$\Rightarrow \hat{n}_2 = \frac{2\sqrt{2}}{3} \quad ; \quad \hat{n}_1 = \hat{n}_3 = -\frac{\sqrt{2}}{6}$$

where we have used the relationships in (3.9). Thus, the normal vector to the plane in which $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \vec{\mathbf{0}}$ is the unit vector:

$$\hat{n}_i = \frac{\sqrt{2}}{6} \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

3.2 Eigenvalues and Eigenvectors of the Stress Tensor

Problem 3.7

The Cauchy stress field components are presented by:

$$\sigma_{ij} = \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & 4x_1 \\ 2x_2 & 4x_1 & 1 \end{bmatrix} \tag{3.10}$$

where x_i are the Cartesian coordinates.

a) Obtain the traction vector acting at the point $(x_1 = 1, x_2 = 2, x_3 = 3)$ associated with the plane $x_1 + x_2 + x_3 = 6$;

b) Obtain the projection of the traction vector according to the normal and tangential direction to the plane $x_1 + x_2 + x_3 = 6$;

Solution:

a) The unit vector which is normal to the plane $x_1 + x_2 + x_3 = 6$ is:

$$\mathsf{n}_i = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathsf{n}}_i = \frac{\mathsf{n}_i}{\|\vec{\mathbf{n}}\|} = \frac{\mathsf{n}_i}{\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.11}$$

The traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is obtained by the equation $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$ and the components of $\boldsymbol{\sigma}$ at the point are:

$$\sigma_{ij}(x_1 = 1, x_2 = 2, x_3 = 3) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} \tag{3.12}$$

thus,

$$\mathsf{t}_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{\mathsf{n}}_j = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix} \tag{3.13}$$

b) The normal stress associated with this plane is

$$\sigma_N = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 & 5 & 9 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}(5 + 5 + 9) = \frac{19}{3} \tag{3.14}$$

and the tangential stress is

$$\sigma_S^2 = -\sigma_N^2 + \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \tag{3.15}$$

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 & 5 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix} \frac{1}{\sqrt{3}} = \frac{131}{3} \tag{3.16}$$

thus

$$\sigma_S^2 = -\left(\frac{19}{3}\right)^2 + \frac{131}{3} = \sqrt{\frac{32}{9}} \tag{3.17}$$

Problem 3.8

Given a continuum where the stress state is known at one point and is represented by the Cauchy stress tensor Cartesian components:

$$\sigma_{ij} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa \tag{3.18}$$

a) Find the principal stresses (eigenvalues) and the principal directions (eigenvectors).

Solution:

To obtain the principal stresses $\lambda_i = \sigma_i$ and principal directions $\hat{\mathbf{n}}^{(i)}$ we must solve the following set of equations:

$$(\sigma_{ij} - \lambda\delta_{ij})\mathrm{n}_j = 0_i \qquad \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} \mathrm{n}_1 \\ \mathrm{n}_2 \\ \mathrm{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.19}$$

for nontrivial solutions of $\hat{\mathbf{n}}^{(i)}$ the above set of equation has solution if and only if:

$$\left|\sigma_{ij} - \lambda\delta_{ij}\right| = 0$$

Note that, a direction is called principal if there is no tangential stress on the plane normal to such direction, and according to the format of the matrix (3.18) we can note that we have one solution (one principal direction), since the tangential components in the x_3-direction are zero, then:

$$\lambda_3 = 2 \xrightarrow{direction} \mathrm{n}_1^{(1)} = \mathrm{n}_2^{(1)} = 0\,,\ \mathrm{n}_3^{(1)} = \pm 1$$

To obtain the remaining solutions it is sufficient to solve:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda(2-\lambda) = 0$$

We can easily verify that the roots of the above equations are:

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 0$$

Then, we can express the stress tensor components in the principal space as follows:

$$\sigma'_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

Note that we have a unique eigenvalue $\sigma_2 = 0$ associated with the unique direction $\hat{\mathbf{n}}^{(2)}$, and we have two equal eigenvalues $\sigma_1 = \sigma_3 = 2$, so, any direction orthogonal to $\hat{\mathbf{n}}^{(2)}$ is also a principal direction.

b) To obtain the principal direction associated with the solution $\lambda_1 = 2$ we substitute this solution into the equation in (3.19), i.e.:

$$\begin{bmatrix} 1-2 & 1 & 0 \\ 1 & 1-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1^{(1)} \\ \mathsf{n}_2^{(1)} \\ \mathsf{n}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} -\mathsf{n}_1^{(1)} + \mathsf{n}_2^{(1)} = 0 \\ \mathsf{n}_1^{(1)} - \mathsf{n}_2^{(1)} = 0 \end{cases} \quad \Rightarrow \quad \mathsf{n}_1^{(1)} = \mathsf{n}_2^{(1)}$$

and by using the restriction ${\mathsf{n}_1^{(1)}}^2 + {\mathsf{n}_2^{(1)}}^2 = 1$ we can obtain: $\mathsf{n}_1^{(1)} = \mathsf{n}_2^{(1)} = \dfrac{1}{\sqrt{2}}$, then the unit vector is $\hat{\mathsf{n}}_i^{(1)} = \left[\dfrac{1}{\sqrt{2}} \quad \dfrac{1}{\sqrt{2}} \quad 0\right]$.

For the solution $\lambda_2 = 0$, we can obtain:

$$\begin{bmatrix} 1-0 & 1 & 0 \\ 1 & 1-0 & 0 \\ 0 & 0 & 2-0 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1^{(2)} \\ \mathsf{n}_2^{(2)} \\ \mathsf{n}_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \mathsf{n}_1^{(2)} + \mathsf{n}_2^{(2)} = 0 \\ \mathsf{n}_1^{(2)} + \mathsf{n}_2^{(2)} = 0 \\ 2\mathsf{n}_3^{(2)} = 0 \end{cases} \quad \Rightarrow \quad \mathsf{n}_2^{(2)} = -\mathsf{n}_1^{(2)}$$

and by using the restriction ${\mathsf{n}_1^{(2)}}^2 + {\mathsf{n}_2^{(2)}}^2 = 1$, we can obtain: $\mathsf{n}_1^{(2)} = \dfrac{1}{\sqrt{2}}$, $\mathsf{n}_2^{(2)} = \dfrac{-1}{\sqrt{2}}$, then the unit vector is $\hat{\mathsf{n}}_i^{(2)} = \left[\dfrac{1}{\sqrt{2}} \quad \dfrac{-1}{\sqrt{2}} \quad 0\right]$.

As we have seen, the eigenvectors form a matrix transformation ($\mathcal{A}$) between the two systems, i.e. $\boldsymbol{\sigma}' = \mathcal{A}\,\boldsymbol{\sigma}\,\mathcal{A}^T$, thus:

$$\begin{bmatrix} \sigma_1 = 2 & 0 & 0 \\ 0 & \sigma_2 = 0 & 0 \\ 0 & 0 & \sigma_3 = 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

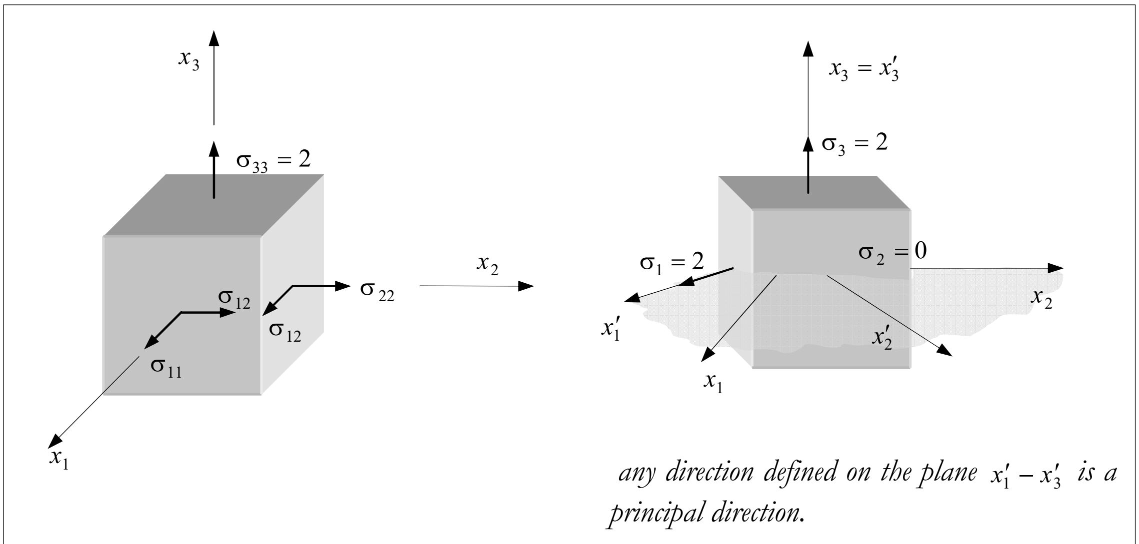


Figure 3.11: Stress state for the **Problem 3.8**.

Problem 3.9

A prismatic dam is subjected to water pressure. The dam has thickness equal to b and height equal to h, (see Figure 3.12). Obtain the restrictions for the Cauchy stress tensor Cartesian components on the faces BC, OB and AC.

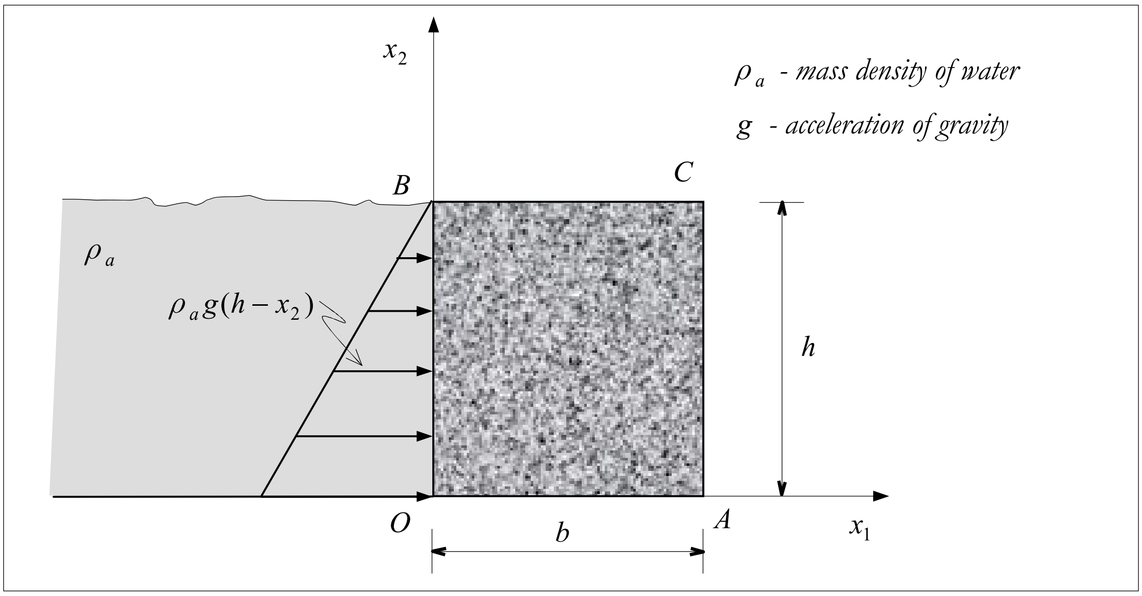


Figure 3.12: Dam under water pressure.

Solution:

The face BC has normal unit vector $\hat{\mathsf{n}}_i^{(BC)} = [0 \quad 1 \quad 0]$. Considering that this face has no traction vector, we can conclude that:

$$\mathsf{t}_i^{(BC)} = \mathsf{0}_i = \sigma_{ij}\hat{\mathsf{n}}_j \quad \Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is the same as $\sigma_{i2} = 0$ and due to the symmetry of $\boldsymbol{\sigma}$ we have $\sigma_{2i} = 0$.

The face OB has as normal unit vector $\hat{\mathsf{n}}_i^{(BC)} = [-1 \quad 0 \quad 0]$. Considering that in this face the traction vector components are $\mathsf{t}_i^{(OB)} = [\rho_a g(h-x_2) \quad 0 \quad 0]$, we can conclude that:

$$\mathsf{t}_i^{(OB)} = \begin{bmatrix} \rho_a g(h-x_2) \\ 0 \\ 0 \end{bmatrix} = \sigma_{ij}\hat{\mathsf{n}}_j \quad \Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_{11} \\ -\sigma_{21} \\ -\sigma_{31} \end{bmatrix} = \begin{bmatrix} \rho_a g(h-x_2) \\ 0 \\ 0 \end{bmatrix}$$

which is the same as $\sigma_{i1} = \rho_a g(h-x_2)\delta_{i1}$.

The face AC has normal unit vector $\hat{\mathsf{n}}_i^{(BC)} = [1 \quad 0 \quad 0]$. Considering that in this face there is no traction vector, we can conclude that:

$$\mathsf{t}_i^{(AC)} = \mathsf{0}_i = \sigma_{ij}\hat{\mathsf{n}}_j \quad \Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is the same as $\sigma_{i1} = 0_i$ and due to the symmetry of $\boldsymbol{\sigma}$ we have $\sigma_{1i} = 0$.

3.3 Maximum shear stress, Mohr circle in stress

Problem 3.10

Obtain the maximum shear stress at a point in which the Cauchy stress Cartesian is represented as indicated in Figure 3.13.

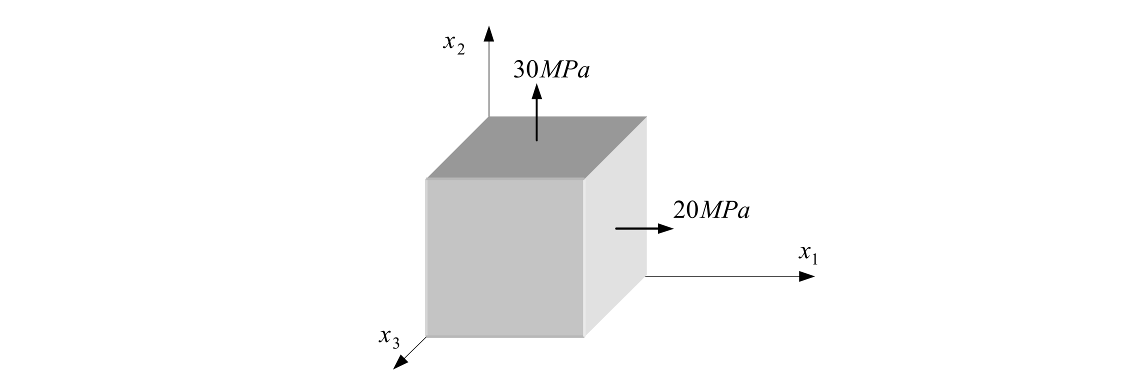


Figure 3.13

Solution:

Note that the coordinate axes x_i are principal directions. We can draw the Mohr's circle by considering the principal stresses: $\sigma_I = 30MPa$, $\sigma_{II} = 20MPa$ and $\sigma_{III} = 0$, (see Figure 3.14).

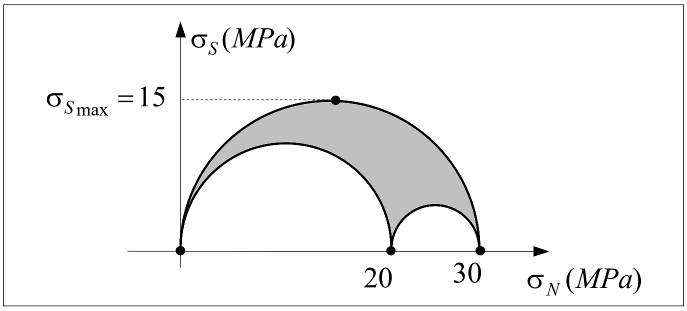


Figure 3.14

The maximum shear stress is calculated as follows:

$$\sigma_{S\max} = \frac{\sigma_I - \sigma_{III}}{2} = \frac{30-0}{2} = 15\ MPa \tag{3.20}$$

Problem 3.11

Consider the Cauchy stress Cartesian components at a point as indicated in Figure 3.15.

a) Draw the Mohr's circle;

b) Obtain the maximum normal stress, and indicate the plane in which occurs;

c) Obtain the maximum shear stress.

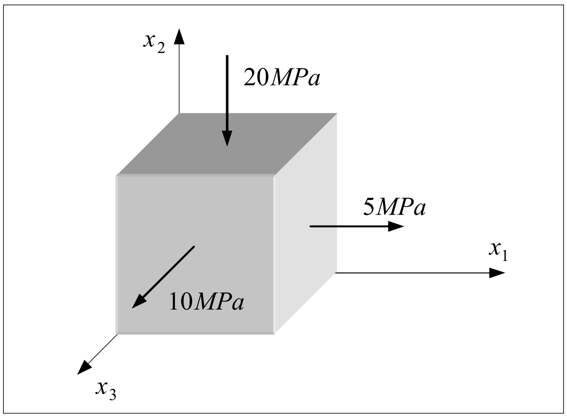


Figure 3.15

Solution:

The stresses represented in Figure 3.15 are in fact the eigenvalues of the stress tensor, since there are no tangential stresses on the planes. By restructuring the eigenvalues such that $\sigma_I > \sigma_{II} > \sigma_{III}$ we have $\sigma_I = 10$, $\sigma_{II} = 5$ and $\sigma_{III} = -20$, then the Mohr's circle in stress can be drawn as indicated in Figure 3.16. The maximum normal stress is $\sigma_{N\max} = \sigma_I = 10$, and the maximum shear stress is defined by the radius of the circumference defined by σ_I and σ_{III}, i.e. $\sigma_{S\max} = \dfrac{\sigma_I - \sigma_{III}}{2} = \dfrac{10-(-20)}{2} = 15MPa$.

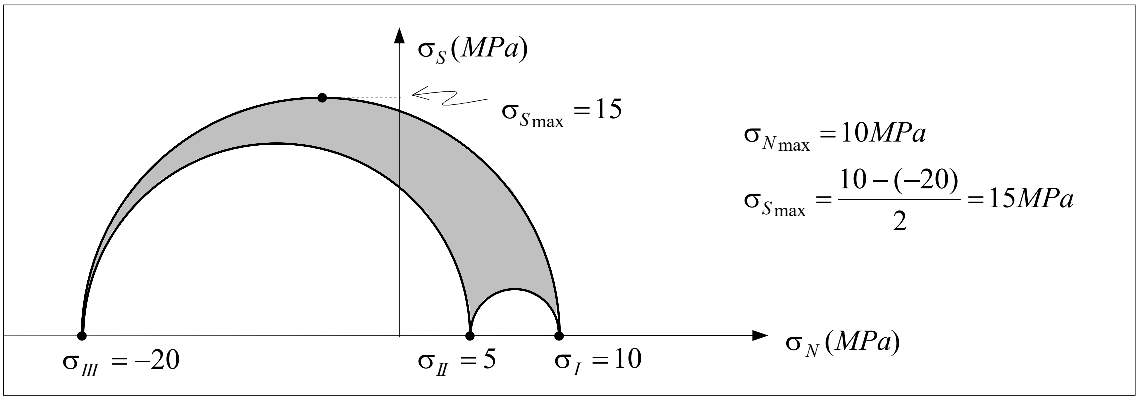


Figure 3.16: Mohr's circle in stress

Problem 3.12

At a point the Cauchy stress tensor components are defined as indicated in Figure 3.17. Determine for which values of σ^* are possible for the following stress cases:

Case a) $(\sigma_N = 4; \sigma_S = 2)$

Case b) $(\sigma_N = 4; \sigma_S = 1)$

Case c) $(\sigma_N = 7; \sigma_S = 0)$

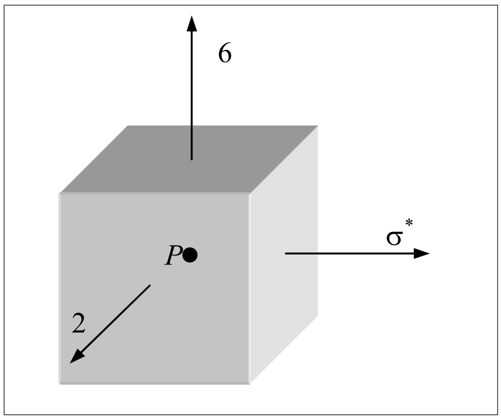


Figure 3.17

Solution

Recall that the pair $(\sigma_N; \sigma_S)$ is only feasible if and only if belongs to the gray zone of the Mohr's circle including the circumferences, (see Figure 3.18). If the pair $(\sigma_N; \sigma_S)$ is located outside the Mohr's circle that means that there is no plane in which the normal stress and tangential stress are defined simultaneously by $(\sigma_N; \sigma_S)$.

According to the problem data, (see Figure 3.17), we can draw the circumference formed by the principal values 2 and 6, (see Figure 3.19). In the smae figure we also draw the three cases.

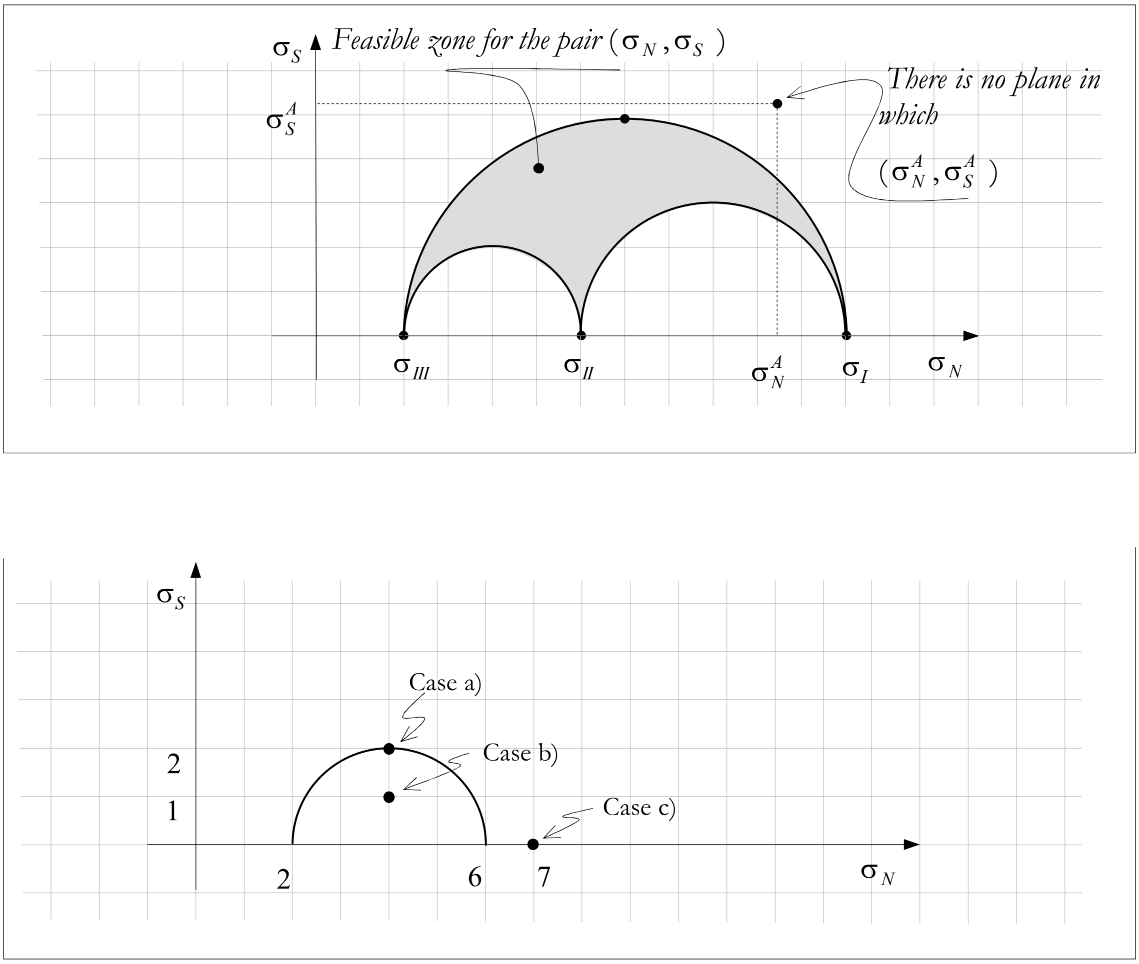


Figure 3.19

Case a): In this case the pair $(\sigma_N = 4; \sigma_S = 2)$ belongs to the circumference formed by the principal stresses 2 and 6, thus σ^* can assume any value, (see Figure 3.20).

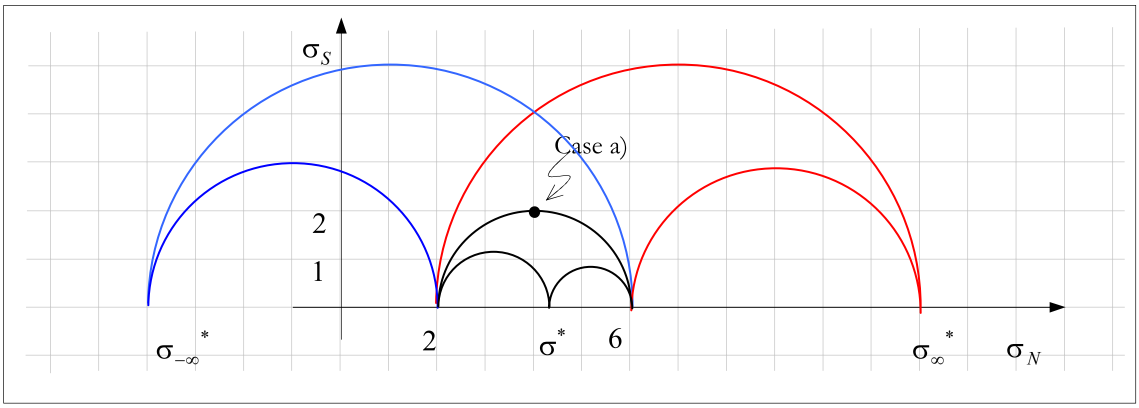


Figure 3.20

Case b) In this case the solution is defined by:

$$\sigma^*_{(2)} \leq \sigma^* \leq \sigma^*_{(1)} \tag{3.21}$$

where $\sigma^*_{(2)}$ and $\sigma^*_{(1)}$ are identified in Figure 3.21.

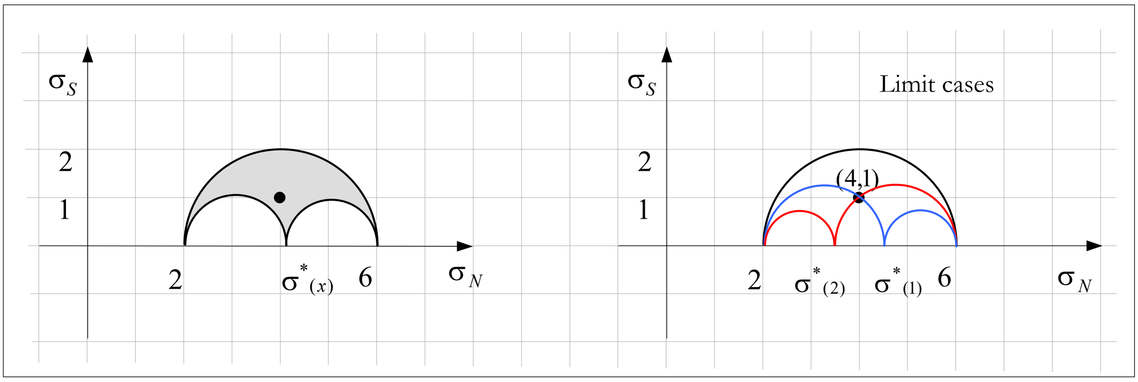


Figure 3.21

Starting from the circumference equation:

$$(x-x_C)^2+(y-y_C)^2=R^2 \tag{3.22}$$

For the case $\sigma^*_{(1)}$, we have: $x=4;\ x_C=\dfrac{(\sigma^*_{(1)}+2)}{2};\quad y=1;\ y_C=0;\ R=\dfrac{(\sigma^*_{(1)}-2)}{2}$

Substituting these values into the circumference equation we can obtain:

$$(x-x_C)^2+(y-y_C)^2=R^2 \quad \Rightarrow \quad \left(4-\frac{(\sigma^*_{(1)}+2)}{2}\right)^2+(1-0)^2=\left(\frac{(\sigma^*_{(1)}-2)}{2}\right)^2 \Rightarrow \sigma^*_{(1)}=4.5$$

For the case $\sigma^*_{(2)}$, we have: $x=4;\ x_C=\dfrac{(6+\sigma^*_{(2)})}{2};\quad y=1;\ y_C=0;\ R=\dfrac{(6-\sigma^*_{(2)})}{2}$

substituting these values into the circumference equation we can obtain:

$$(x-x_C)^2+(y-y_C)^2=R^2 \quad \Rightarrow \quad \left(4-\frac{(6+\sigma^*_{(2)})}{2}\right)^2+(1-0)^2=\left(\frac{(6-\sigma^*_{(2)})}{2}\right)^2 \Rightarrow \sigma^*_{(2)}=3.5$$

thus:

$$3.5\le\sigma^*\le 4.5 \tag{3.23}$$

Case c) In this case the only possible solution is that σ_N is a principal stress, then

$$\sigma^*=7 \tag{3.24}$$

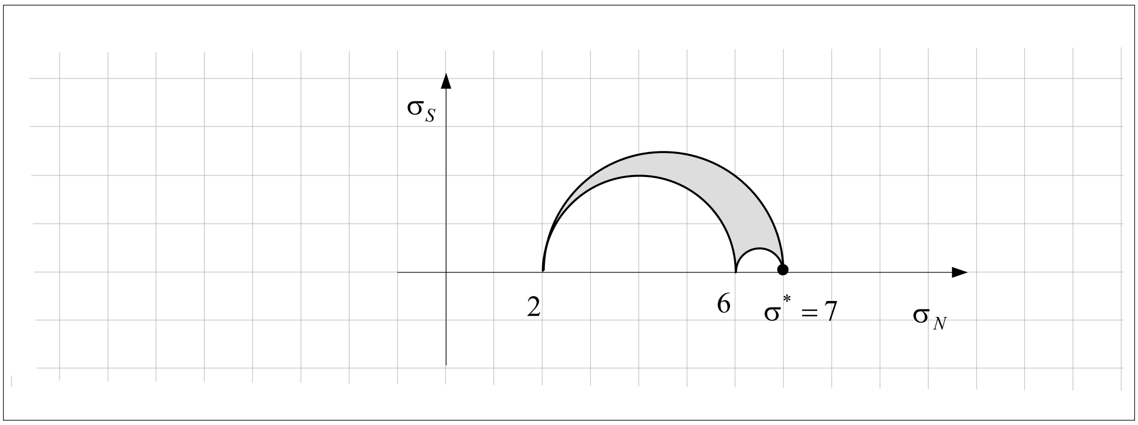


Figure 3.22

Problem 3.13

Obtain the maximum normal and tangential stresses and draw the corresponding Mohr's circle in stress for the following stress state cases:

$$\text{a)}\quad \sigma_{ij} = \begin{bmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{b)}\quad \sigma_{ij} = \begin{bmatrix} -2\tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & -\tau \end{bmatrix} \qquad \text{c)}\quad \sigma_{ij} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \tag{3.25}$$

with $\tau > 0$.

Solution:

Case a) If we check the format of the Cauchy stress tensor components for this case, we can observe that the value $\lambda_{(3)} = 0$ is already an eigenvalue. Then, to obtain the remaining eigenvalues, it is sufficient to solve:

$$\begin{bmatrix} \tau & \tau \\ \tau & \tau \end{bmatrix} \rightarrow \begin{vmatrix} \tau-\lambda & \tau \\ \tau & \tau-\lambda \end{vmatrix} = (\tau-\lambda)^2 - \tau^2 = 0 \Rightarrow \tau - \lambda = \tau \Rightarrow \lambda = 0 \tag{3.26}$$

$$(\tau-\lambda)^2 - \tau^2 = 0 \Rightarrow \tau^2 - 2\lambda\tau + \lambda^2 - \tau^2 = 0 \Rightarrow \lambda(-2\tau + \lambda) = 0 \Rightarrow \begin{cases} \lambda_{(1)} = 0 \\ \lambda_{(2)} = 2\tau \end{cases} \tag{3.27}$$

Then, the Mohr's circle in stress for this case is presented in Figure 3.23.

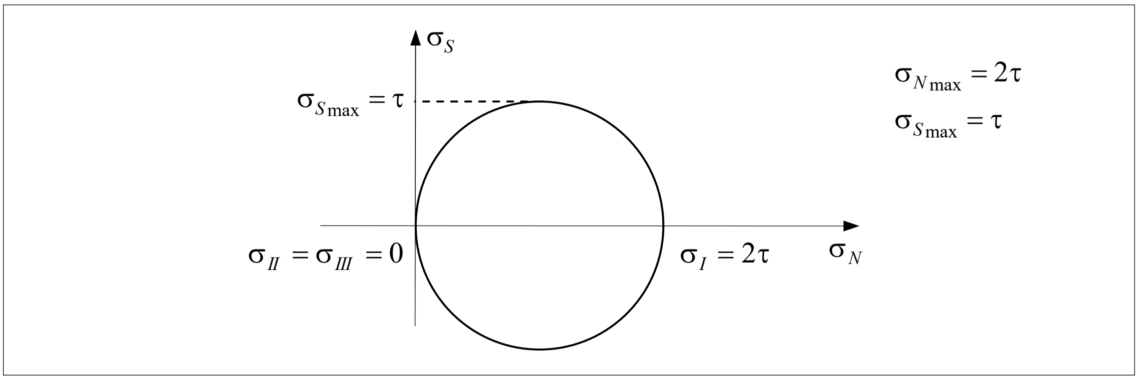


Figure 3.23

b) For the case (b) we have $\sigma_I = \tau$, $\sigma_{II} = -\tau$ and $\sigma_{III} = -2\tau$, and the Mohr's circle is presented in Figure 3.24.

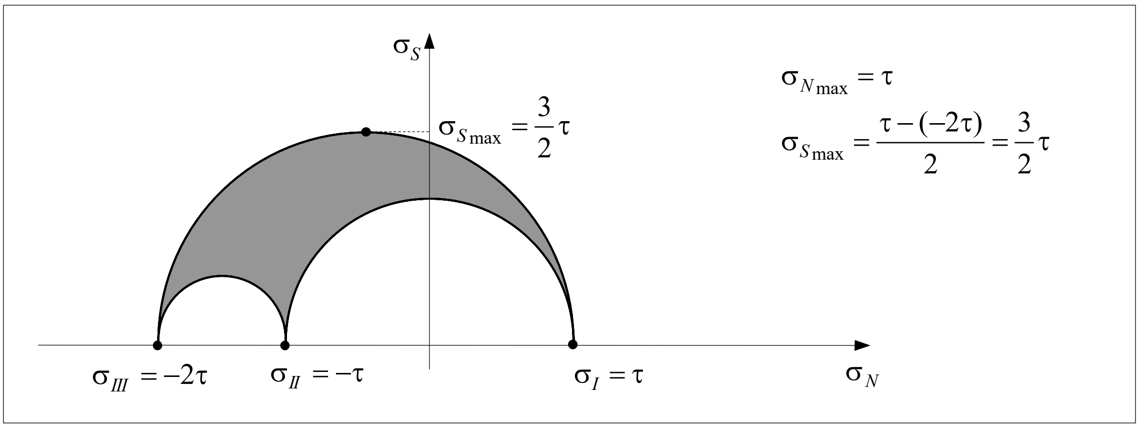


Figure 3.24

c) For the case (c) the eigenvalues can be calculated as follows:

$$\begin{vmatrix} 0-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0-\lambda & 0 \\ \sigma_{13} & 0 & 0-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda^3 + \sigma_{13}^2\lambda + \sigma_{12}^2\lambda = 0 \quad \Rightarrow \quad \lambda(-\lambda^2 + \sigma_{13}^2 + \sigma_{12}^2) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = +\sqrt{\sigma_{13}^2 + \sigma_{12}^2} = \tau \\ \lambda_3 = -\sqrt{\sigma_{13}^2 + \sigma_{12}^2} = -\tau \end{cases}$$

By restructuring the eigenvalues such that $\sigma_I > \sigma_{II} > \sigma_{III}$ we have $\sigma_I = \tau$, $\sigma_{II} = 0$ and $\sigma_{III} = -\tau$, then the Mohr's circle is drawn as indicated in Figure 3.25.

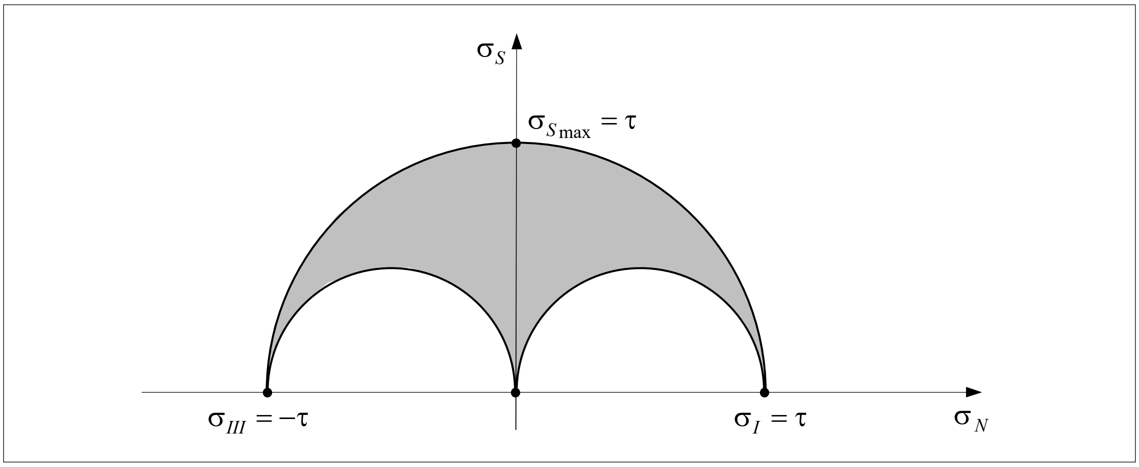


Figure 3.25

Problem 3.14

Make the representation of the Mohr's circle for the following cases:

1) One-dimensional case (traction); 2)One-dimensional case (compression); 3) Two-dimensional case (traction); 4) Three-dimensional case.

Solution:

1) The one-dimensional case (traction) in described in Figure 3.26.

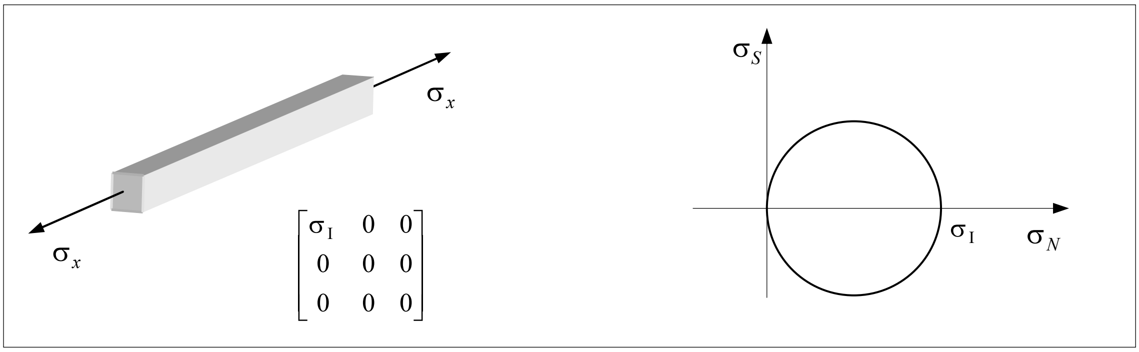


Figure 3.26

2) The one-dimensional compression is described in Figure 3.27.

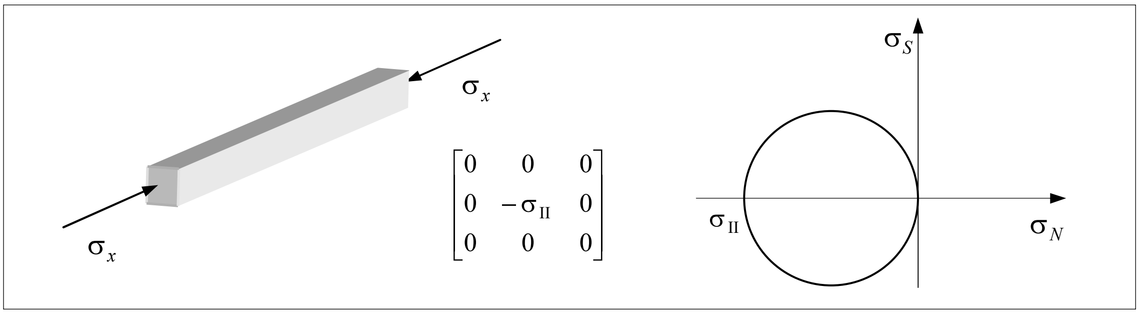


Figure 3.27

3) The two-dimensional case is described in Figure 3.28.

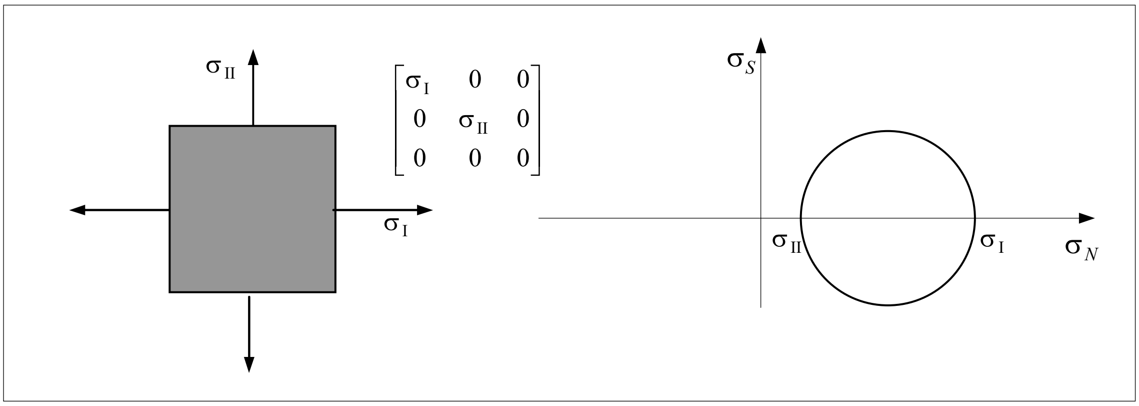


Figure 3.28

4) The three-dimensional case is described in Figure 3.29.

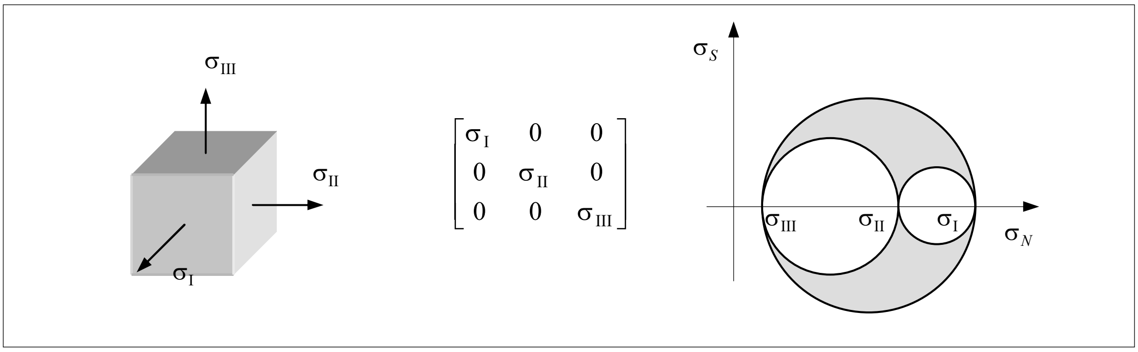


Figure 3.29

3.4 Feature of the stress tensor

Problem 3.15

The Cauchy stress tensor components at the point P are given by:

$$\sigma_{ij} = \begin{bmatrix} 5 & 6 & 7 \\ 6 & 8 & 9 \\ 7 & 9 & 2 \end{bmatrix} GPa \tag{3.28}$$

a) Obtain the mean stress; b) obtain the deviatoric and spherical part of the tensor $\boldsymbol{\sigma}$.

Solution:

a) The mean stress

$$\sigma_m = \frac{\sigma_{kk}}{3} = \frac{5+8+2}{3} = 5 \tag{3.29}$$

b) The spherical part of $\boldsymbol{\sigma}$ is given by

$$\sigma_{ij}^{sph} = \frac{I_\sigma}{3}\delta_{ij} = \sigma_m \delta_{ij} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and the deviatoric part becomes:

$$\sigma_{ij} = \sigma_{ij}^{sph} + \sigma_{ij}^{dev} \quad \Rightarrow \quad \sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} \quad \Rightarrow \quad \sigma_{ij}^{dev} = \begin{bmatrix} 0 & 6 & 7 \\ 6 & 3 & 9 \\ 7 & 9 & -3 \end{bmatrix}$$

Problem 3.16

Consider the Cauchy stress tensor components, in the Cartesian base $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$:

$$\sigma_{ij} = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \tag{3.30}$$

Given the transformation law between the systems x and x':

	x'_1	x'_2	x'_3
x_1	$\frac{3}{5}$	0	$\frac{4}{5}$
x_2	0	1	0
x_3	$-\frac{4}{5}$	0	$\frac{3}{5}$

where the system x' is represented by the basis $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$.

a) Obtain the traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}'_2)}$ associated with the plane whose normal is $\hat{\mathbf{e}}'_2$. Express the result in the Cartesian system $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$ according to the format:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{e}}'_2)} = (\quad)\hat{\mathbf{e}}'_1 + (\quad)\hat{\mathbf{e}}'_2 + (\quad)\hat{\mathbf{e}}'_3 \tag{3.31}$$

b) Obtain the spherical and deviatoric parts of the Cauchy stress tensor.

Solution:

a) Recall that the first row of the transformation matrix is formed by the direction cosines formed between the x'_1-axis and the axes x_1, x_2 and x_3, thus:

$$\boldsymbol{\mathcal{A}} = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix} \tag{3.32}$$

and the transformation law for the second-order tensor components is given by:

$$\boldsymbol{\sigma}' = \boldsymbol{\mathcal{A}}\,\boldsymbol{\sigma}\,\boldsymbol{\mathcal{A}}^T \tag{3.33}$$

thus:

$$\sigma'_{ij} = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 5 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 & 9 & 2 \\ 9 & 5 & 12 \\ 2 & 12 & 31 \end{bmatrix} \tag{3.34}$$

$$\mathrm{t}_i^{(\hat{\mathbf{e}}'_2)} = \frac{1}{5}\begin{bmatrix} 9 \\ 5 \\ 12 \end{bmatrix} \quad \Rightarrow \quad \vec{\mathbf{t}}^{(\hat{\mathbf{e}}'_2)} = \left(\frac{9}{5}\right)\hat{\mathbf{e}}'_1 + (1)\hat{\mathbf{e}}'_2 + \left(\frac{12}{5}\right)\hat{\mathbf{e}}'_3 \tag{3.35}$$

since:

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} t_1^{(\hat{\mathbf{e}}'_1)} & t_1^{(\hat{\mathbf{e}}'_2)} & t_1^{(\hat{\mathbf{e}}'_3)} \\ t_2^{(\hat{\mathbf{e}}'_1)} & t_2^{(\hat{\mathbf{e}}'_2)} & t_2^{(\hat{\mathbf{e}}'_3)} \\ t_3^{(\hat{\mathbf{e}}'_1)} & t_3^{(\hat{\mathbf{e}}'_2)} & t_3^{(\hat{\mathbf{e}}'_3)} \end{bmatrix} \tag{3.36}$$

b) The spherical (σ_{ij}^{sph}) and the deviatoric (σ_{ij}^{dev}) parts are given by

$$\sigma_{ij} = \sigma_{ij}^{sph} + \sigma_{ij}^{dev} = \frac{I_\sigma}{3}\delta_{ij} + \sigma_{ij}^{dev} \tag{3.37}$$

where $I_\sigma = \mathrm{Tr}(\boldsymbol{\sigma}) = 5+1+3 = 9$, then

$$\sigma_{ij}^{sph} = \frac{I_\sigma}{3}\delta_{ij} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{3.38}$$

$$\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} = \begin{bmatrix} 5-3 & 3 & 2 \\ 3 & 1-3 & 0 \\ 2 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 3 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \tag{3.39}$$

Problem 3.17

Consider the Cauchy stress tensor field Cartesian components:

$$\sigma_{ij} = \begin{bmatrix} 0 & Cx_3 & 0 \\ Cx_3 & 0 & -Cx_1 \\ 0 & -Cx_1 & 0 \end{bmatrix}$$

where C is a constant. Also consider that the body is free of body force.

a) Calculate the traction vector at the point $P(4,–4,7)$ associated with the plane whose normal vector is given by $\vec{\mathbf{n}} = 2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - 1\hat{\mathbf{e}}_3$.

b) Represent the Mohr's circle in stress at the point P.

Solution:

a) The traction vector can be obtained by the equation $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ or in indicial notation $t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j$, where $\hat{\mathbf{n}}$ is the unit vector which is normal to the plane. The vector $\vec{\mathbf{n}}$ has module equal to $\|\vec{\mathbf{n}}\| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3$, thus the unit vector is given by:

$$\hat{n}_i = \frac{n_i}{\|\vec{\mathbf{n}}\|} \quad \Rightarrow \quad \hat{n}_i = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Then, by considering the stress components at the point $P(4,–4,7)$

$$\sigma_{ij}(x_1 = 4; x_2 = -4; x_3 = 7) = \begin{bmatrix} 0 & Cx_3 & 0 \\ Cx_3 & 0 & -Cx_1 \\ 0 & -Cx_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7C & 0 \\ 7C & 0 & -4C \\ 0 & -4C & 0 \end{bmatrix} \tag{3.40}$$

we can obtain:

$$\vec{t}_i^{(\hat{n})} = \sigma_{ij}\hat{n}_j = \begin{bmatrix} 0 & 7C & 0 \\ 7C & 0 & -4C \\ 0 & -4C & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 14C \\ 18C \\ -8C \end{bmatrix} \tag{3.41}$$

b) Let us consider that

$$\sigma_{ij} = C\begin{bmatrix} 0 & 7 & 0 \\ 7 & 0 & -4 \\ 0 & -4 & 0 \end{bmatrix} = C\overline{\sigma}_{ij} \tag{3.42}$$

The eigenvalues of the tensor $\overline{\sigma}_{ij}$ can be obtained by solving the determinant:

$$\begin{vmatrix} -\lambda & 7 & 0 \\ 7 & -\lambda & -4 \\ 0 & -4 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda^3 + 16\lambda + 49\lambda = 0 \quad \Rightarrow \quad -\lambda^2 + 65 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \sqrt{65} \\ \lambda_3 = -\sqrt{65} \end{cases}$$

With that we can obtain $\sigma_I = C\sqrt{65}$, $\sigma_{II} = 0$ and $\sigma_{III} = -C\sqrt{65}$. Then, the Mohr's circle can be represented as indicated in Figure 3.30.

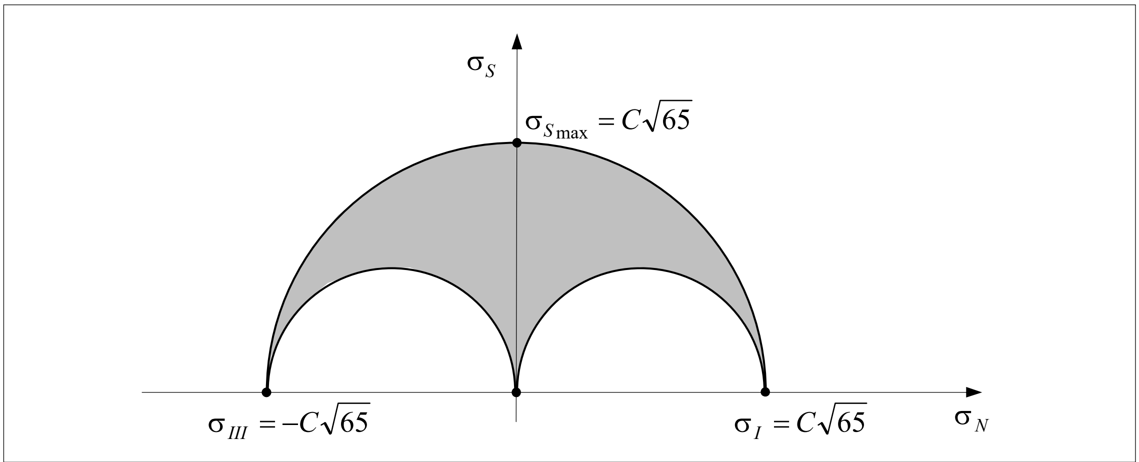


Figure 3.30

Problem 3.18

The stress state at a point of the body is represented by the traction vectors as indicated in Figure 3.31.

a) Obtain the deviatoric part of the stress tensor;

b) Obtain the principal stresses (σ_I, σ_{II}, σ_{III}) and the principal directions;

c) Draw the Mohr's circle in stress;

d) Obtain the maximum shear stress at the point;

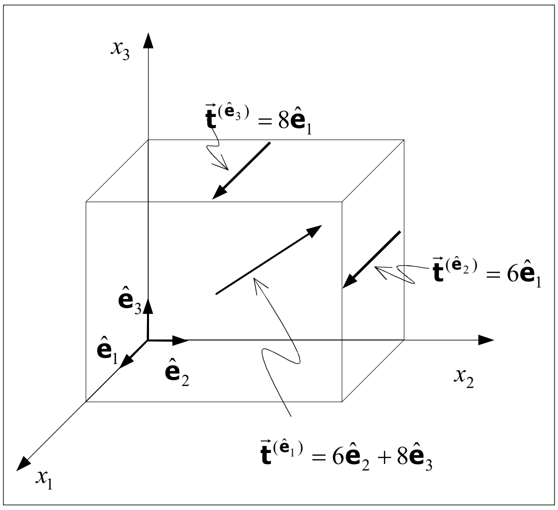


Figure 3.31

e) Find the traction vector associated with the plane whose normal vector is $\hat{\mathbf{n}} = 0.75\hat{\mathbf{e}}_1 + 0.25\hat{\mathbf{e}}_2 - \frac{\sqrt{6}}{4}\hat{\mathbf{e}}_3$;

f) Obtain the normal and tangential stress vector associated with the plane described in paragraph (e).

Solution:

According to Figure 3.31 we can obtain the Cauchy stress tensor components as follows:

$$\sigma_{ij} = \begin{bmatrix} 0 & 6 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix}$$

a)

$$\sigma_{ij} = \sigma_{ij}^{sph} + \sigma_{ij}^{dev}$$

The spherical part is $\sigma_{ij}^{sph} = \frac{I_{\boldsymbol{\sigma}}}{3}\delta_{ij} = 0_{ij}$ since $I_{\boldsymbol{\sigma}} = 0$. Then, the deviatoric part is given by:

$$\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} = \begin{bmatrix} 0 & 6 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix}$$

b) The eigenvalues can be obtained by means of the characteristic determinant:

$$\begin{vmatrix} -\lambda & 6 & 8 \\ 6 & -\lambda & 0 \\ 8 & 0 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda^3 + 100\lambda = 0 \quad \Rightarrow \quad \lambda\left(-\lambda^2 + 100\right) = 0$$

whose solutions are $\lambda_1 = 0$, $\lambda_2 = 10$, $\lambda_3 = -10$, (principal stresses). The principal directions are:

$$\sigma_1 = 0 \xrightarrow{eigenvector} \hat{\mathbf{n}}_i^{(1)} = \begin{bmatrix} 0 & -0.8 & 0.6 \end{bmatrix}$$
$$\sigma_2 = -10 \xrightarrow{eigenvector} \hat{\mathbf{n}}_i^{(2)} = \begin{bmatrix} -0.707 & 0.424 & 0.566 \end{bmatrix}$$
$$\sigma_3 = 10 \xrightarrow{eigenvector} \hat{\mathbf{n}}_i^{(3)} = \begin{bmatrix} 0.707 & 0.424 & 0.566 \end{bmatrix}$$

$$\sigma_I = 10, \ \sigma_{II} = 0, \ \sigma_{III} = -10$$

c) The Mohr's circle in stress can be appreciated in Figure 3.32.

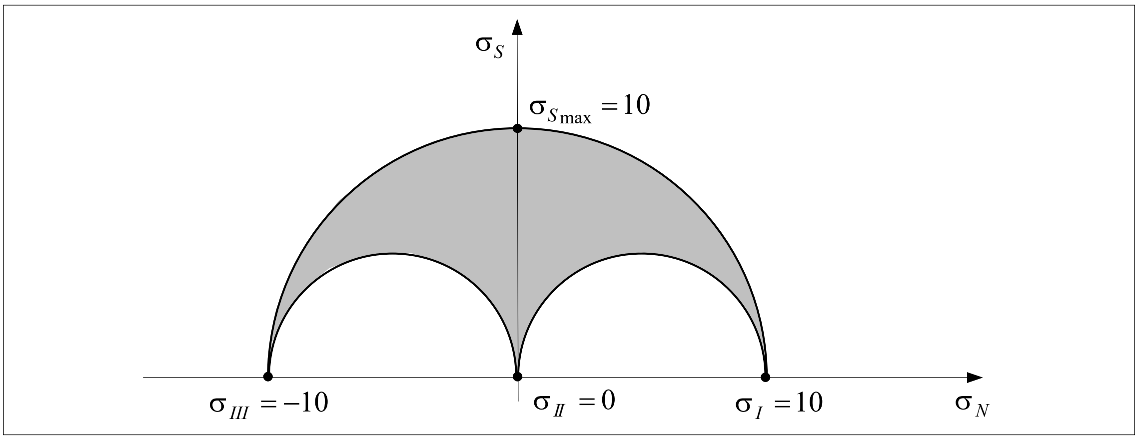


Figure 3.32

d) We can directly obtain the maximum shear stress by means of the Mohr's circle, (Figure 3.32):

$$\tau_{\max} = \frac{\sigma_I - \sigma_{III}}{2} = 10$$

e) Considering $t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j$, we can obtain the traction vector associated with the plane whose normal vector is $\hat{\mathbf{n}} = 0.75\hat{\mathbf{e}}_1 + 0.25\hat{\mathbf{e}}_2 - \frac{\sqrt{6}}{4}\hat{\mathbf{e}}_3$:

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j \quad \Rightarrow \quad \begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \\ -\frac{\sqrt{6}}{4} \end{bmatrix} \approx \begin{bmatrix} -3.39898 \\ 4.5 \\ 6 \end{bmatrix}$$

f) Let us consider the vectors in Figure 3.33.

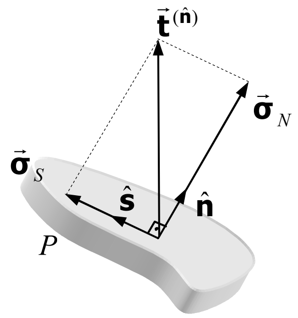


Figure 3.33

The magnitude of $\vec{\boldsymbol{\sigma}}_N$ can be obtained by the projection $\|\vec{\boldsymbol{\sigma}}_N\| = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = t_i^{(\hat{\mathbf{n}})}\hat{n}_i$, thus:

$$\|\vec{\boldsymbol{\sigma}}_N\| = t_i^{(\hat{\mathbf{n}})}\hat{n}_i \approx \begin{bmatrix} -3.39898 & 4.5 & 6 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \\ -\frac{\sqrt{6}}{4} \end{bmatrix} \approx -5.09847$$

The vector $\vec{\boldsymbol{\sigma}}_N$ is given by:

$$\vec{\boldsymbol{\sigma}}_N = \|\vec{\boldsymbol{\sigma}}_N\|\hat{\mathbf{n}} = -3.82385\hat{\mathbf{e}}_1 - 1.27462\hat{\mathbf{e}}_2 + 3.12216\hat{\mathbf{e}}_3$$

In addition, the relationship $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \vec{\boldsymbol{\sigma}}_N + \vec{\boldsymbol{\sigma}}_S$ holds, with that the tangent stress vector is obtained as follows:

$$\begin{aligned} \vec{\boldsymbol{\sigma}}_S &= \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \vec{\boldsymbol{\sigma}}_N \approx (-3.39898 + 3.82385)\hat{\mathbf{e}}_1 + (4.5 + 1.27462)\hat{\mathbf{e}}_2 + (6 - 3.12216)\hat{\mathbf{e}}_3 \\ &\approx (0.42487)\hat{\mathbf{e}}_1 + (5.77462)\hat{\mathbf{e}}_2 + (2.87784)\hat{\mathbf{e}}_3 \end{aligned}$$

and its module as:

$$\|\vec{\boldsymbol{\sigma}}_S\| \approx \sqrt{(0.42487)^2 + (5.77462)^2 + (2.87784)^2} \approx \sqrt{41.808713} = 6.465966$$

NOTE: We could also have used the equation $\|\vec{\boldsymbol{\sigma}}_S\|^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \|\vec{\boldsymbol{\sigma}}_N\|^2$ to obtain the module of $\vec{\boldsymbol{\sigma}}_S$.

Problem 3.19

The Cauchy stress tensor field in the continuum is represented by:

$$\sigma_{ij}(\vec{x}) = \begin{bmatrix} 3x_1 & 5x_2^2 & 0 \\ \sigma_{21} & 3x_2 & 2x_3 \\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix}$$

a) Obtain the body force (per unit volume) to ensure the balance of the continuum.

b) For a particular point ($x_1 = 1, x_2 = 1, x_3 = 0$):

b.1) Draw the Mohr's circle. Obtain the maximum normal and tangential stress component.

b.2) Obtain the traction vector associated with the plane whose normal is $\hat{n}_i = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

b.2.1) Obtain the normal and tangential in this plane.

Solution:

a) Due to the symmetry of the Cauchy stress tensor ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$) we have:

$$\sigma_{ij}(\vec{x}) = \begin{bmatrix} 3x_1 & 5x_2^2 & 0 \\ 5x_2^2 & 3x_2 & 2x_3 \\ 0 & 2x_3 & 0 \end{bmatrix}$$

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}} \xrightarrow{components} \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = -\rho \mathrm{b}_1 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = -\rho \mathrm{b}_1 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = -\rho \mathrm{b}_1 \end{cases} \Rightarrow \begin{cases} 3 + 10x_2 + 0 = -\rho \mathrm{b}_1 \\ 0 + 3 + 2 = -\rho \mathrm{b}_2 \\ 0 + 0 + 0 = -\rho \mathrm{b}_3 \end{cases}$$

with that we can obtain:

$$\rho \mathrm{b}_i = \begin{bmatrix} -10x_2 - 3 \\ -5 \\ 0 \end{bmatrix} \qquad \textit{(Force per unit volume)} \left[\frac{N}{m^3}\right] \qquad (3.43)$$

b) For the particular point ($x_1 = 1, x_2 = 1, x_3 = 0$) we have:

$$\sigma_{ij} = \begin{bmatrix} 3 & 5 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where we can verify that $\sigma_3 = 0$ is one principal value. For the other eigenvalues, it is sufficient to solve:

$$\begin{vmatrix} 3-\sigma & 5 \\ 5 & 3-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad (3-\sigma) = (5)^2 \quad \Rightarrow \quad 3-\sigma = \pm 5 \quad \Rightarrow \quad \begin{cases} \sigma_1 = 8 \\ \sigma_2 = -2 \end{cases}$$

Restructuring the eigenvalues:

$$\sigma_I = 8 \,,\; \sigma_{II} = 0 \,,\; \sigma_{III} = -2$$

b.1) The Mohr's circle in stress is drawn in Figure 3.34.

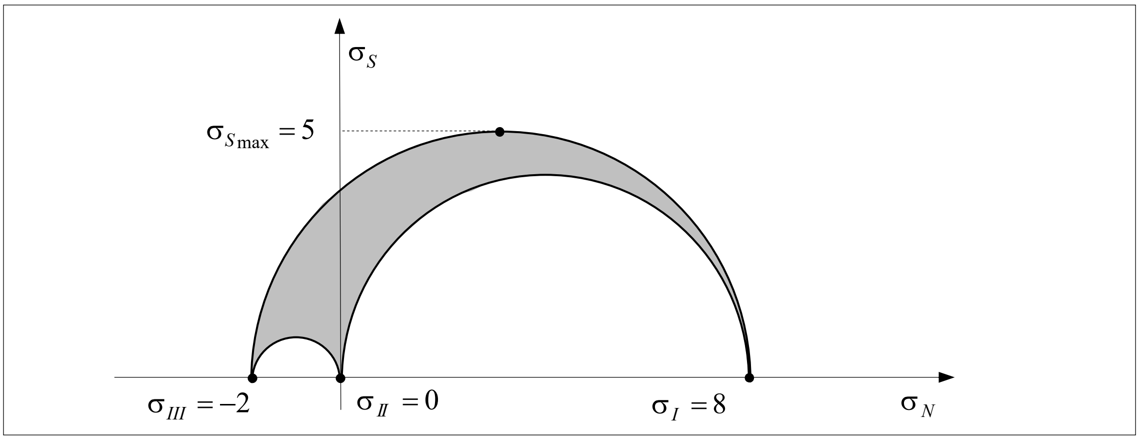


Figure 3.34

By means of the Mohr's circle we can obtain the maximum shear stress $\sigma_{S\max} = 5$ and the maximum normal stress $\sigma_{N\max} = \sigma_I = 10$.

e) Considering that $t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j$, we can obtain the traction vector associated with the plane whose normal vector is $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}\hat{\mathbf{e}}_1 + \frac{1}{\sqrt{3}}\hat{\mathbf{e}}_2 + \frac{1}{\sqrt{3}}\hat{\mathbf{e}}_3$:

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & 5 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 8 \\ 8 \\ 0 \end{bmatrix}$$

b.2) The normal stress component is obtained as follows:

$$\sigma_N = t_i^{(\hat{\mathbf{n}})}\hat{n}_i = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 8 & 8 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{16}{3} \approx 5.333$$

To obtain the tangential component we apply directly $\left\|\vec{\boldsymbol{\sigma}}_S\right\|^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \left\|\vec{\boldsymbol{\sigma}}_N\right\|^2$, where $\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = t_i^{(\hat{\mathbf{n}})} t_i^{(\hat{\mathbf{n}})} = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 8 & 8 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 8 \\ 0 \end{bmatrix} = \frac{128}{3}$. Then:

$$\left\|\vec{\boldsymbol{\sigma}}_S\right\|^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \left\|\vec{\boldsymbol{\sigma}}_N\right\|^2 = \frac{128}{3} - \left(\frac{16}{3}\right)^2 = \frac{128}{9} \quad \Rightarrow \quad \sigma_S = \frac{\sqrt{128}}{3} \approx 3.771$$

Problem 3.20

The stress state at one point of the body is given by means of the spherical and deviatoric part of the Cauchy stress tensor as follows:

$$\sigma_{ij}^{sph} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \sigma_{ij}^{dev} = \begin{bmatrix} 0 & 6 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix}$$

a) Obtain the Cauchy stress tensor components;

b) Find the principal stresses (σ_I, σ_{II}, σ_{III}) and principal directions;

c) Obtain the maximum shear stress;

d) Draw the Mohr's circle in stress for the cases: d.1) the Cauchy stress tensor (σ_{ij}), d.2) the spherical part (σ_{ij}^{sph}) and; d.3) the deviatoric part (σ_{ij}^{dev});

Solution:

a)
$$\sigma_{ij} = \sigma_{ij}^{sph} + \sigma_{ij}^{dev} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 8 \\ 6 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}$$

In **Problem 3.18** we have obtained the principal values of σ_{ij}^{dev} whose values are the same as for the proposed problem. As the tensor and its deviatoric part have the same principal directions, i.e. they are coaxial, we can automatically obtain the principal stresses:

$$\sigma'_{ij} = \sigma'^{sph}_{ij} + \sigma'^{dev}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

The principal directions are the same as those provided in **Problem 3.18**.

d) Mohr's circle in stress can be appreciated in Figure 3.35.

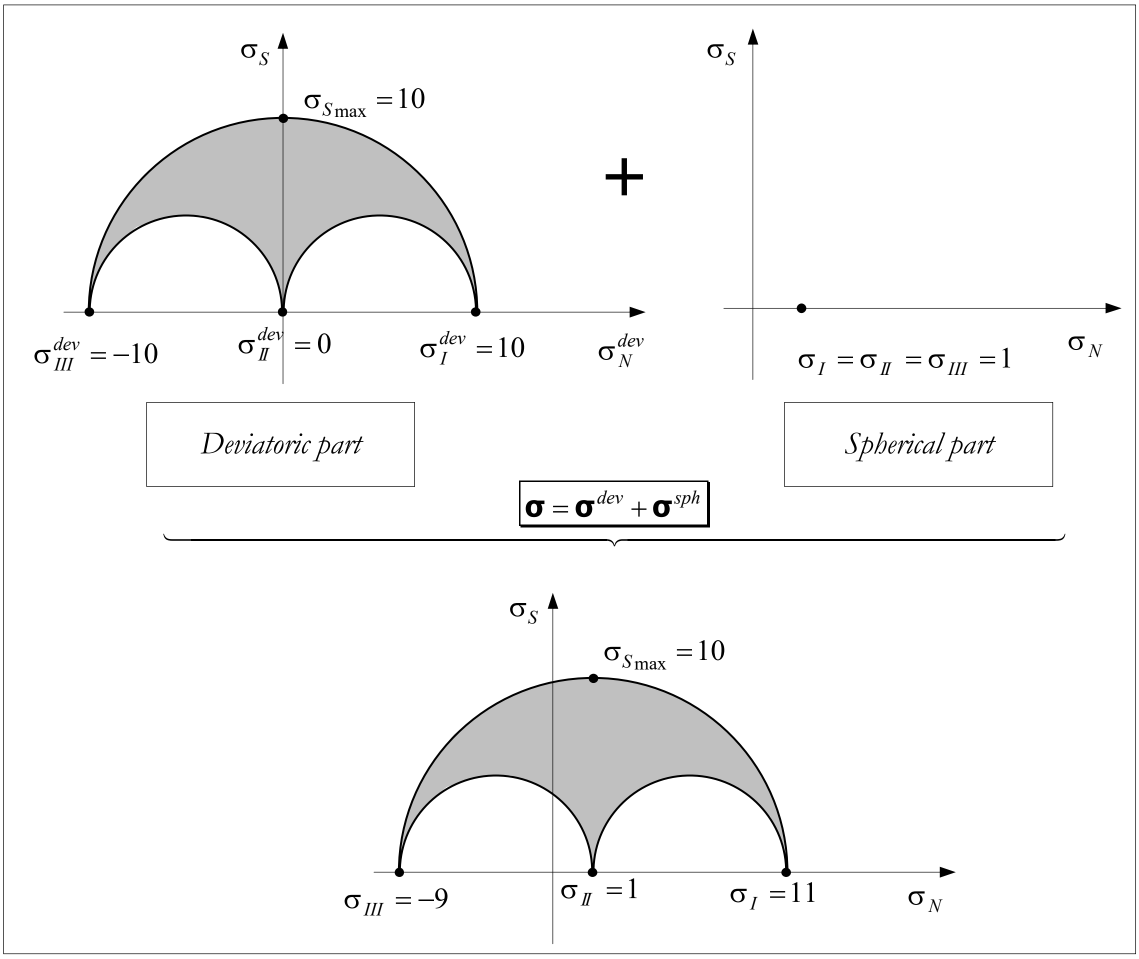


Figure 3.35

Note that the spherical part contribution deviates (translate) the Mohr's circle of the deviatoric part according the σ_N-axis, and does not alter the value of the maximum shear stress.

Problem 3.21

At one point P in the continuum medium, The Cauchy stress tensor $\boldsymbol{\sigma}$ is represented by its Cartesian components as follows:

$$\sigma_{ij} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} MPa ,$$

a) Obtain the principal stresses and principal directions at the point P;

b) Obtain the maximum shear stress;

c) Draw the Mohr's circle for the cases: c.1) the Cauchy stress tensor (σ_{ij}), c.2) the spherical part (σ_{ij}^{sph}) and; c.3) the deviatoric part (σ_{ij}^{dev});

d) i.) Find the traction vector associated with the plane whose normal vector is $\vec{\mathbf{n}} = 1.0\hat{\mathbf{e}}_1 + 1.0\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$;

ii.) Obtain the normal and tangential stress on the plane.

f) Obtain the eigenvalues and eigenvectors of the deviatoric part ($\boldsymbol{\sigma}^{dev}$).

Solution:

a) The eigenvalues are $\sigma_I = 2$, $\sigma_{II} = 2$, $\sigma_{III} = 0$, (see **Problem 3.8**).

b) and c)

$$\sigma'^{dev}_{ij} = \sigma'_{ij} - \sigma'^{sph}_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

d) The traction vector is obtained by $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, we need to normalize the normal vector to the plane, i.e. $\hat{\mathbf{n}} = \dfrac{\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{e}}_1 + \dfrac{1}{\sqrt{2}}\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$. Thus:

$$\begin{bmatrix} t_1^{(\hat{\mathbf{n}})} \\ t_2^{(\hat{\mathbf{n}})} \\ t_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

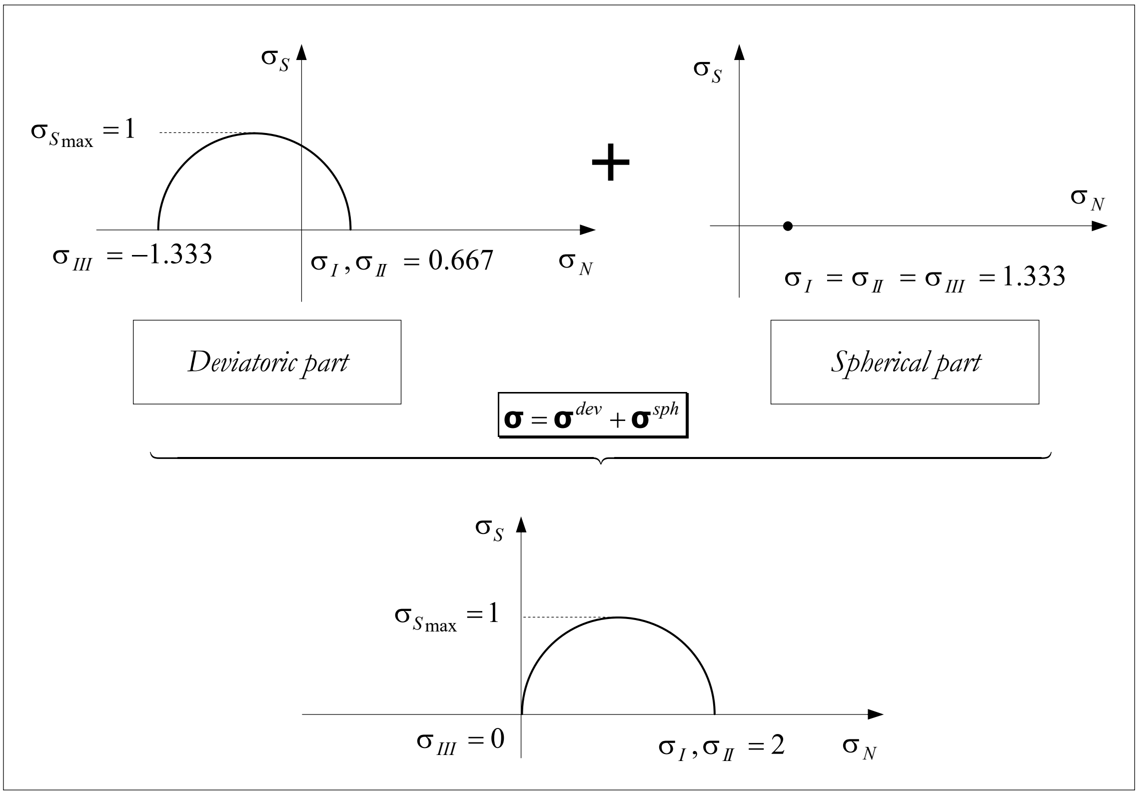


Figure 3.36

Problem 3.22

The Cauchy stress tensor components at one point of the continuum are:

$$\sigma_{ij} = \begin{bmatrix} 29 & 0 & 0 \\ 0 & -26 & 6 \\ 0 & 6 & 9 \end{bmatrix} Pa$$

Decompose the stress tensor in a spherical and a deviatoric part, and obtain the principal stresses and principal directions of the deviatoric part.

Solution:

Consider the additive decomposition of the stress tensor into a spherical and deviatoric part:

$$\sigma_{ij} = \sigma_{ij}^{dev} + \sigma_{ij}^{sph}$$

The deviatoric part is given by

$$\sigma_{ij}^{dev} = \begin{bmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_m \end{bmatrix}$$

where the mean stress is:

$$\sigma_m = \frac{1}{3}\sigma_{ii} = \frac{(29 - 26 + 9)}{3} = 4$$

thus:

$$\sigma_{ij}^{dev} = \begin{bmatrix} 29-4 & 0 & 0 \\ 0 & -26-4 & 6 \\ 0 & 6 & 9-4 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & 6 \\ 0 & 6 & 5 \end{bmatrix} Pa$$

The spherical part components are:

$$\sigma_{ij}^{hyd} \equiv \sigma_{ij}^{sph} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} Pa$$

To verify the above operations, the following relationship must be verified:

$$\sigma_{ij} = \sigma_{ij}^{dev} + \sigma_{ij}^{sph} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & 6 \\ 0 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 0 & 0 \\ 0 & -26 & 6 \\ 0 & 6 & 9 \end{bmatrix} Pa \quad \checkmark$$

To obtain the eigenvalues we solve the characteristic determinant of the deviatoric part:

$$\left|\sigma_{ij}^{dev} - \lambda\delta_{ij}\right| = 0 \longrightarrow \lambda^3 - \lambda \mathrm{J}_2 - \mathrm{J}_3 = 0$$

By solving the above cubic equation we can obtain the following principal values:

$$\begin{cases} \sigma_1^{dev} = 25Pa \\ \sigma_2^{dev} = 6Pa \\ \sigma_3^{dev} = -31Pa \end{cases}$$

Problem 3.23

Consider the Cauchy stress tensor components:

$$\sigma_{ij} = \begin{bmatrix} 12 & 4 & 0 \\ \sigma_{21} & 9 & -2 \\ \sigma_{31} & \sigma_{32} & 3 \end{bmatrix} MPa$$

a) Obtain the spherical and the deviatoric part.

b) Obtain the principal invariants of the deviatoric part.

c) Obtain the normal octahedral stress and the mean stress at this point.

Solution:

Due to the symmetry of the Cauchy stress tensor we can conclude that:

$$\sigma_{ij} = \begin{bmatrix} 12 & 4 & 0 \\ 4 & 9 & -2 \\ 0 & -2 & 3 \end{bmatrix} MPa$$

The mean stress is given by $\sigma_m = \sigma_{oct} = \frac{I_{\boldsymbol{\sigma}}}{3} = \frac{12+9+3}{3} = \frac{24}{3} = 8$.

The spherical and deviatoric parts are:

$$\sigma_{ij}^{sph} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad ; \quad \sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} = \begin{bmatrix} 12 & 4 & 0 \\ 4 & 9 & -2 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 4 & 1 & -2 \\ 0 & -2 & -5 \end{bmatrix}$$

The principal invariants of the deviatoric part are:

$I_{\boldsymbol{\sigma}^{dev}} \equiv \mathsf{J}_1 = 4 + 1 - 5 = 0$, as expected, since the trace of any deviatoric tensor is zero.

$$I\!I_{\boldsymbol{\sigma}^{dev}} = \begin{vmatrix} 1 & -2 \\ -2 & -5 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & -5 \end{vmatrix} + \begin{vmatrix} 4 & 4 \\ 4 & 1 \end{vmatrix} = -41 = -\mathsf{J}_2$$

or by using the definition: $\mathsf{J}_2 = \frac{1}{3}\left(I_{\boldsymbol{\sigma}}^2 - 3 I\!I_{\boldsymbol{\sigma}}\right) = \frac{1}{3}\left(24^2 - 3 \times 151\right) = 41$

$$I\!I\!I_{\boldsymbol{\sigma}^{dev}} \equiv \mathsf{J}_3 = \det(\boldsymbol{\sigma}^{dev}) = 44$$

Problem 3.24

The stress state at one point is represented by the Cauchy stress tensor components:

$$\sigma_{ij} = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$$

where a, b and c are constants. Determine the constants a, b and c such that the traction vector on the octahedral plane is the null vector.

Solution:

The octahedral plane has the following unit vector: $\hat{\mathsf{n}}_i = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, and the traction vector related to this plane is defined by $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, whose components are:

$$\begin{bmatrix} \mathsf{t}_1^{(\hat{\mathbf{n}})} \\ \mathsf{t}_2^{(\hat{\mathbf{n}})} \\ \mathsf{t}_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 + a + b \\ a + 1 + c \\ b + c + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a + b = -1 \\ a + c = -1 \\ b + c = -1 \end{cases}$$

by solving the above set of equations we can obtain $b = \frac{-1}{2}$, $c = \frac{-1}{2}$, $a = \frac{-1}{2}$.

Problem 3.25

At one point P in the continuous medium the Cauchy stress tensor $\boldsymbol{\sigma}$ is represented by its Cartesian components as follows:

$$\sigma_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ \sigma_{21} & 50 & 0 \\ \sigma_{31} & \sigma_{32} & 43 \end{bmatrix} MPa,$$

a) Obtain the principal stresses and principal directions at the point P;
b) Obtain the maximum tangential and normal stress at this point;
c) Draw the Mohr's circle in stress;
d) Obtain the traction vector $\vec{\mathbf{t}}^{(\mathbf{n})}$ on the octahedral plane of the Haigh-Westergaard space. Obtain the normal octahedral stress and the tangential octahedral stress.

Solution:

Considering the symmetry of the Cauchy stress tensor we can conclude that:

$$\sigma_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} MPa$$

Note that the stress $\sigma_{22} = 50$ is already a principal stress and is associated with the principal direction $\hat{n}^{(2)} = [0 \quad \pm 1 \quad 0]$. To find the other principal stresses we must solve the following system:

$$\begin{vmatrix} 57-\sigma & 24 \\ 24 & 43-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad \sigma^2 - 100\sigma + 1875 = 0 \Rightarrow \begin{cases} \sigma_1 = 25 \\ \sigma_3 = 75 \end{cases}$$

Using the definition of eigenvalue-eigenvector, we can obtain the following eigenvectors:

$$\sigma_1 = 50 \quad \Rightarrow \quad \hat{n}^{(1)} = [0 \quad \pm 1.0 \quad 0]$$

$$\sigma_1 = 25 \quad \Rightarrow \quad \hat{n}^{(1)} = [\mp 0.6 \quad 0 \quad \pm 0.8]$$

$$\sigma_3 = 75 \quad \Rightarrow \quad \hat{n}^{(3)} = [\pm 0.8 \quad 0 \quad \pm 0.6]$$

Mohr's circle in stress:

Restructuring the principal stresses such that $\sigma_I > \sigma_{II} > \sigma_{III}$ we have:

$$\sigma_I = 75\,,\ \sigma_{II} = 50\,,\ \sigma_{III} = 25$$

b, c) The Mohr's circle can be appreciated in Figure 3.37 as well as the maximum shear stress.

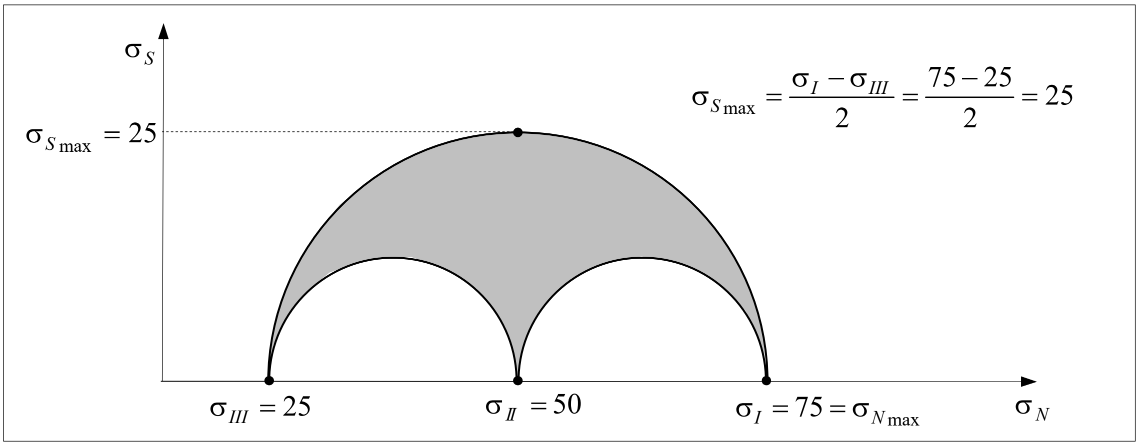


Figure 3.37

d) The Haigh-Westergaard space is formed by principal stress directions, and by definition the traction vector is given by $\vec{\mathbf{t}}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$. The normal vector associated with the octahedral plane is given by $\hat{n}_i = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, then

$$\vec{\mathbf{t}}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \xrightarrow{\text{components}} \begin{Bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{Bmatrix} = \begin{bmatrix} 75 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 25 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 75 \\ 50 \\ 25 \end{bmatrix}$$

and its module is given by:

$$\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 = \frac{1}{3}\left(75^2 + 50^2 + 25^2\right) = \frac{8750}{3} \qquad \Rightarrow \qquad \left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\| = 54.00617$$

The normal octahedral stress is given by $\sigma_{oct} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}}$:

$$\sigma_{oct} = \frac{1}{\sqrt{3}\sqrt{3}}\begin{bmatrix} 75 & 50 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 50MPa$$

We could have applied directly the definition of octahedral normal stress:

$$\sigma_{oct} = \frac{I_\sigma}{3} = \sigma_m = \frac{75 + 50 + 25}{3} = 50MPa$$

The tangential octahedral stress can be obtained by means of the Pythagorean theorem:

$$\tau_{oct} = \sqrt{\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 - \sigma_{oct}^2} = \sqrt{\frac{8750}{3} - 50^2} = 20.4124MPa$$

We could also have applied the definition:

$$\tau_{oct} = \frac{1}{3}\sqrt{2I_\sigma^2 - 6\,II_\sigma} = \frac{1}{3}\sqrt{2 \times 150^2 - 6 \times 6875} = 20.41241MPa$$

where $I_\sigma = 150$, $II_\sigma = 75 \times 50 + 75 \times 25 + 50 \times 25 = 6875$.

e) The spherical part:

$$\sigma_{ij}^{sph} = \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\delta_{ij} = \sigma_m \delta_{ij} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} MPa$$

and the deviatoric part:

$$\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} - \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & -7 \end{bmatrix} MPa$$

f) Considering that the tensor and its deviatoric part are coaxial tensors, we can use the principal space in order to obtain the eigenvalues of the deviatoric part:

$$\sigma_{ij}'^{dev} = \sigma_{ij}' - \sigma_{ij}'^{sph} = \begin{bmatrix} 75 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -25 \end{bmatrix} MPa$$

3.5 Stress state in two-dimensional case (2D)

Problem 3.26

Consider at the point P we know some stresses acting on some planes as indicated in Figure 3.38. By considering the state of plane stress, obtain the state of plane stress at the point σ_{ij}.

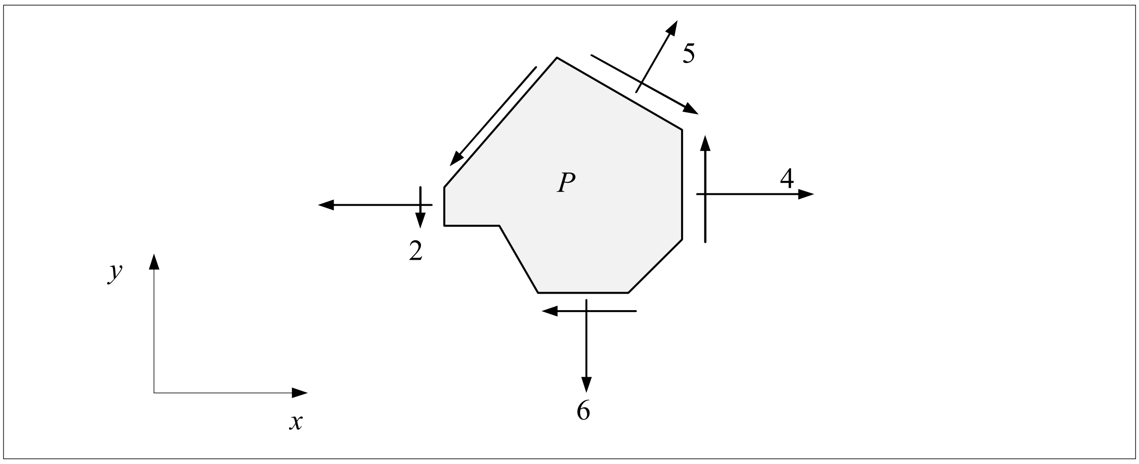


Figure 3.38

Solution:

In the state of plane stress $\sigma_{ij}\,(i,j=1,2)$ we only need two planes to define the stress state at the point:

$$\sigma_{ij}=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix} \tag{3.44}$$

According to Figure 3.38 we can verify that $\sigma_x=4$, $\tau_{xy}=2$ and $\sigma_y=6$, then:

$$\sigma_{ij}=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}=\begin{bmatrix}4 & 2\\ 2 & 6\end{bmatrix} \tag{3.45}$$

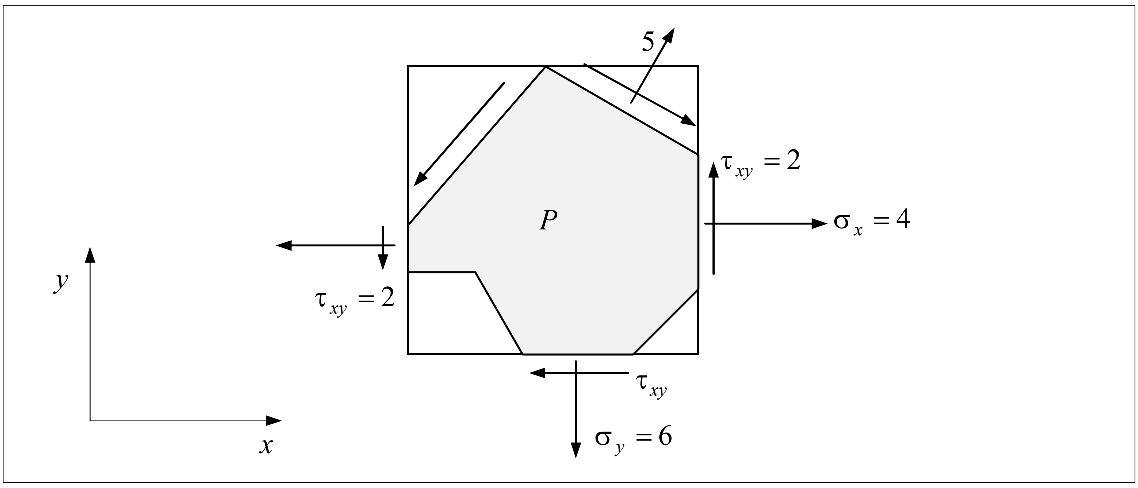


Figure 3.39

Problem 3.27

Consider a composite material, which is made up of matrix and fiber along direction of 45° such as shows in Figure 3.40. This composite material can break if the shear stress along the fiber exceeds the value $3.8\times 10^{6}\,Pa\,(N/m^{2})$.

For the normal stress $\sigma_x = 2.8\times 10^{6}\,Pa$, obtain the maximum value of σ_y for which the material does not break.

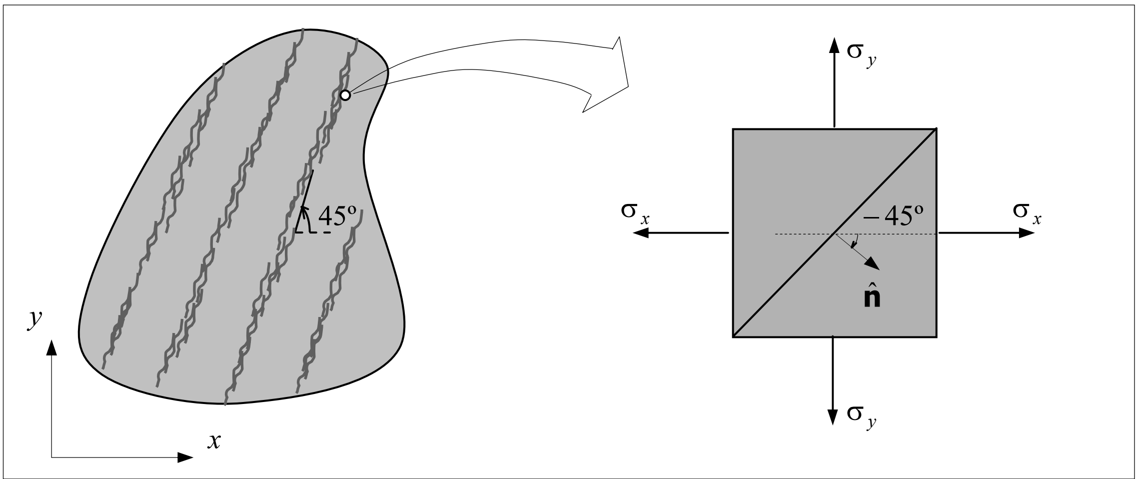


Figure 3.40: Composite material (fiber-matrix).

Solution:

We need to obtain the traction vector on the plane defined by $\theta = -45^{\circ}$, and the tangential components can directly be obtained by means of:

$$\tau'_{xy} \equiv \tau_{(\theta)} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau'_{xy} \equiv \tau_{(\theta=-45^{\circ})} = -\frac{2.8\times 10^{6} - \sigma_y}{2}\sin(-90^{\circ}) = 3.8\times 10^{6}\,Pa$$

$$\boxed{\Rightarrow \sigma_y \approx -4.8\times 10^{6}\,Pa} \text{ (compression)}$$

In **Problem 1.99** was the transformation law for a second-order tensor for 2D case.

Problem 3.28

The stress acting on two planes passing through the point P are shown in Figure 3.41. Obtain the value of the shear stress τ on the plane $a-a$ and the principal stresses at this point.

Solution:

To obtain the stress state at a point in the two dimensional case, we need to determine σ_x, σ_y and τ_{xy}, (see Figure 3.42).

Considering Figure 3.42, we can directly obtain σ_x and τ_{xy} by means of the projection of the traction vector $60Pa$, (see Figure 3.42(b)), i.e.:

$$\sigma_x = 60\cos(30^{\circ}) = 51.962\,Pa$$
$$\tau_{xy} = 60\cos(60^{\circ}) = 30\,Pa$$

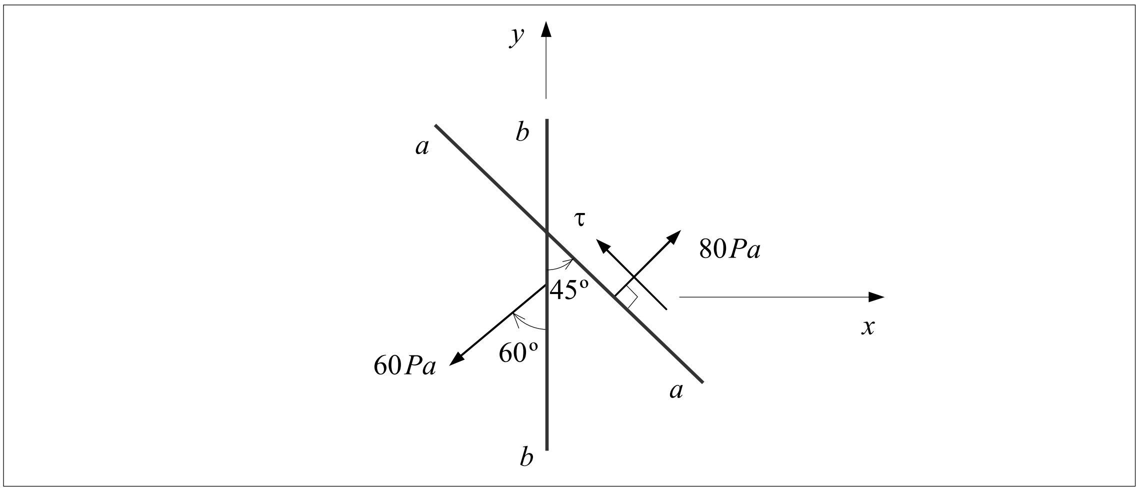


Figure 3.41: Stress state at one point, according to the planes a and b.

In order to obtain σ_y we can employ the following equations:

$$\sigma'_x \equiv \sigma_{(\theta)} \equiv \sigma_N = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} \equiv \sigma_S \equiv \tau_{(\theta)} = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

and by substituting the numerical values we can obtain:

$$\sigma_{(\theta=45^o)} = \frac{51.962 + \sigma_y}{2} + \frac{51.962 - \sigma_y}{2}\cos(90^o) + 30\sin(90^o) = 80Pa$$

$$\tau_{(\theta=45^o)} = \frac{51.962 - \sigma_y}{2}\sin(90^o) - 30\cos(90^o)$$

From the first above equations we can obtain the value of σ_y: $\sigma_y = 48.038Pa$.

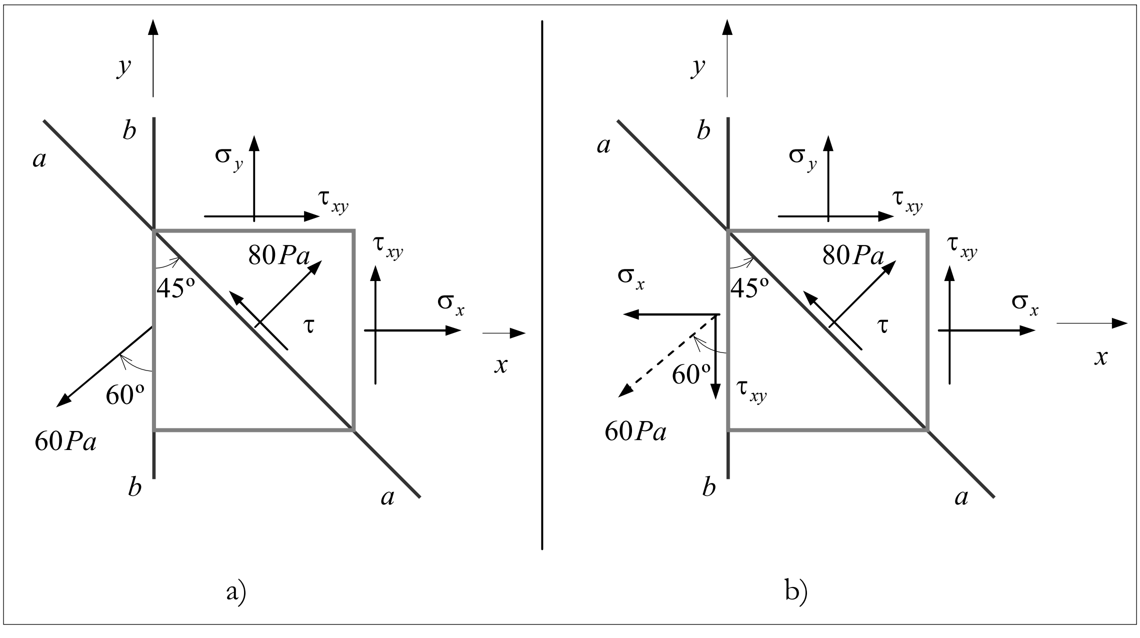


Figure 3.42: Stress state at a point, according to the planes a and b.

Once σ_y is determined, we can obtain the component $\tau_{(\theta=45^\circ)}$:

$$\tau_{(\theta=45^\circ)} = 1.96Pa$$

The principal stresses can be obtained by means of the components σ_x, σ_y, τ_{xy}, such as indicated in the equations:

$$\sigma_{(1,2)} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{(1,2)} = \frac{51.962 + 48.038}{2} \pm \sqrt{\left(\frac{51.962 - 48.038}{2}\right)^2 + 30^2} \Rightarrow \begin{cases} \sigma_1 = 80.1Pa \\ \sigma_2 = 19.9Pa \end{cases}$$

or by means of the characteristic determinant:

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 51.962 - \sigma & 30 \\ 30 & 48.038 - \sigma \end{vmatrix} = 0$$

Problem 3.29

Given a stress state $\sigma_x = 1Pa$, $\tau_{xy} = -4Pa$ and $\sigma_y = 2Pa$. Draw a graph of angle vs. stresses $(\theta - \sigma_x, \sigma_y, \tau_{xy})$, where θ is the rotation angle of the coordinate system, (see Figure 3.43).

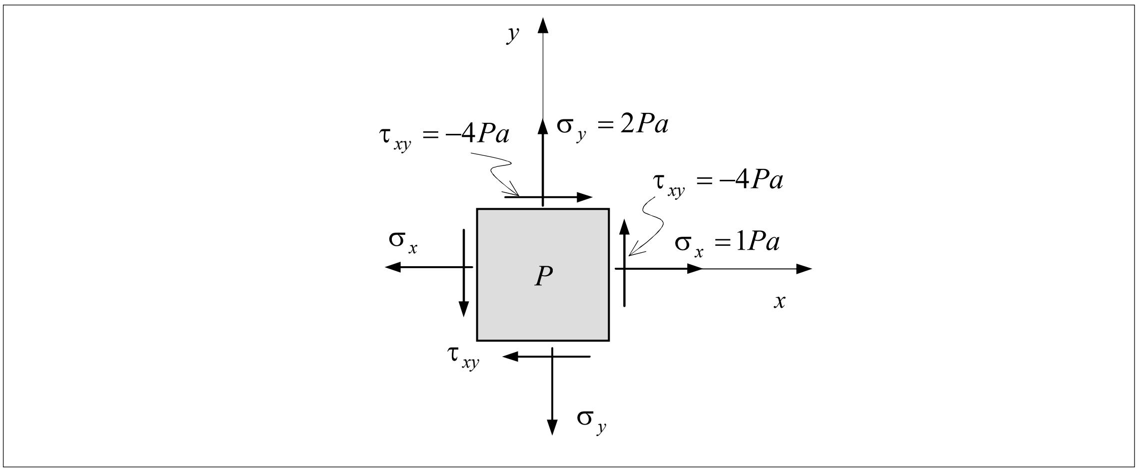


Figure 3.43: Stress state at one point.

Solution:

We can calculate the values σ'_x, σ'_y, τ'_{xy} by using the equations:

$$\boxed{\begin{aligned} \sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau'_{xy} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \\ \sigma'_y &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{aligned}}$$

We can calculate the angle corresponding to the principal direction by means of the equation:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times(-4)}{1-2} = 8 \Rightarrow (\theta = 41.437^\circ)$$

and the principal stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \Rightarrow \quad \begin{cases} \sigma_1 = 5.5311P \\ \sigma_2 = -2.5311Pa \end{cases}$$

Considering the transformation law, we can obtain the values of $\sigma'_x, \sigma'_y, \tau'_{xy}$ for different values of θ. Making θ vary from 0 to 360° we can represent the stresses $\sigma'_x, \sigma'_y, \tau'_{xy}$ in function of the angle, (see Figure 3.44). We can observe that when $\theta = 41.437^\circ$ we have a principal direction, then the tangent stress is zero ($\tau_{xy} = 0$) and the principal stresses are $\sigma_{\text{I}} = 5.5311Pa$ and $\sigma_{\text{II}} = -2.5311Pa$.

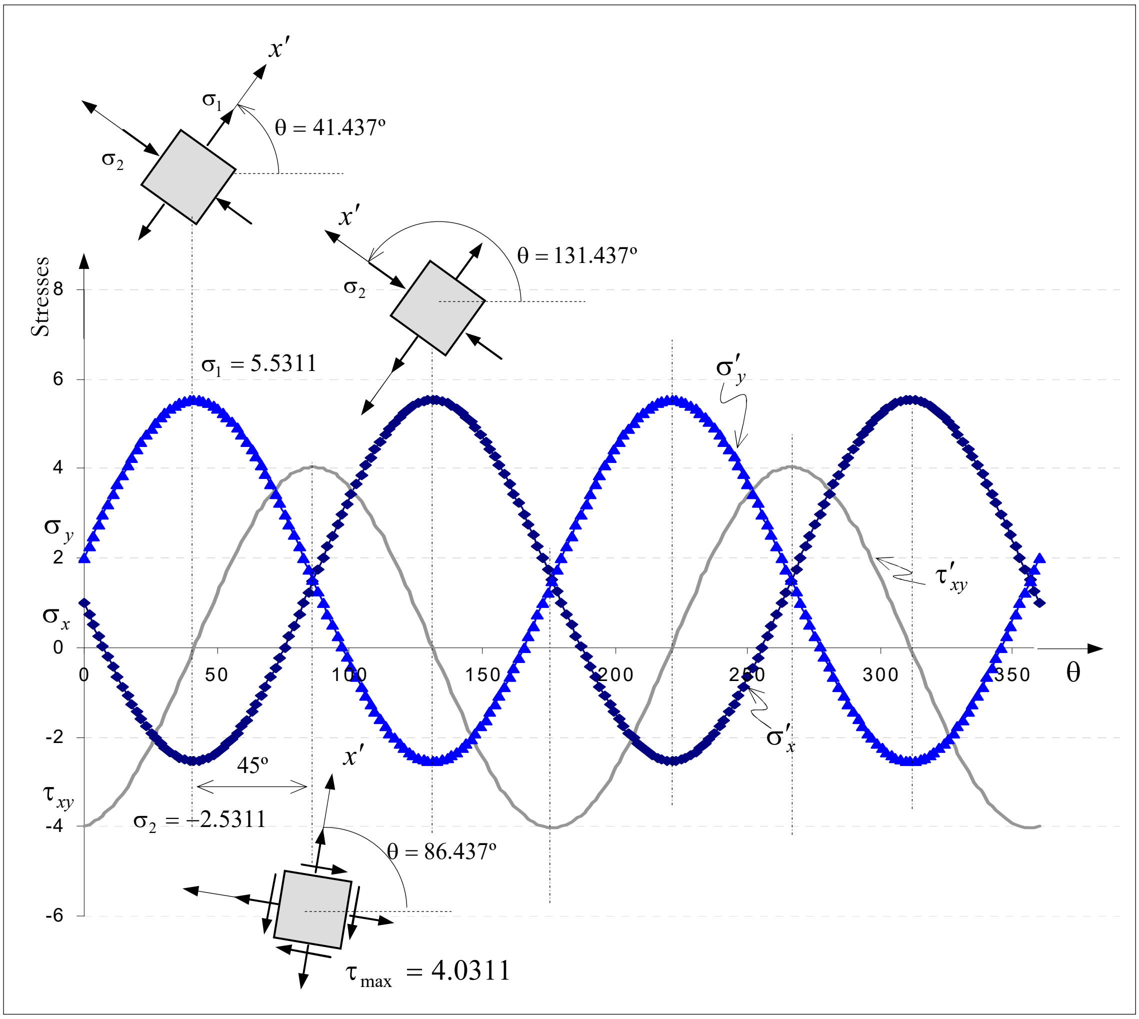


Figure 3.44: Stress components in function of the angle θ.

Problem 3.30

a) Consider the stress field $\sigma_{ij}\,(i,j=1,2)$ in the Cartesian system $x_1-x_2-x_3$, and the following equations:

$$m_{11}=\int_{\frac{-t}{2}}^{\frac{t}{2}}\sigma_{11}x_3dx_3 \quad ; \quad m_{12}=\int_{\frac{-t}{2}}^{\frac{t}{2}}\sigma_{12}x_3dx_3 \quad ; \quad m_{22}=\int_{\frac{-t}{2}}^{\frac{t}{2}}\sigma_{22}x_3dx_3$$

Obtain the component transformation law of $m_{ij}\,(i,j=1,2)$ in the new system $x'_1-x'_2-x'_3$ which is formed by a rotation around the x_3-axis, (see Figure 3.45).

Solution:

Due to the symmetry of $\sigma_{ij}=\sigma_{ji}$, we can conclude that $m_{12}=m_{21}$. The transformation matrix from $x_1-x_2-x_3$ to $x'_1-x'_2-x'_3$ is given as follows:

$$a_{ij}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \xrightarrow{2D} \boldsymbol{\mathcal{A}}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

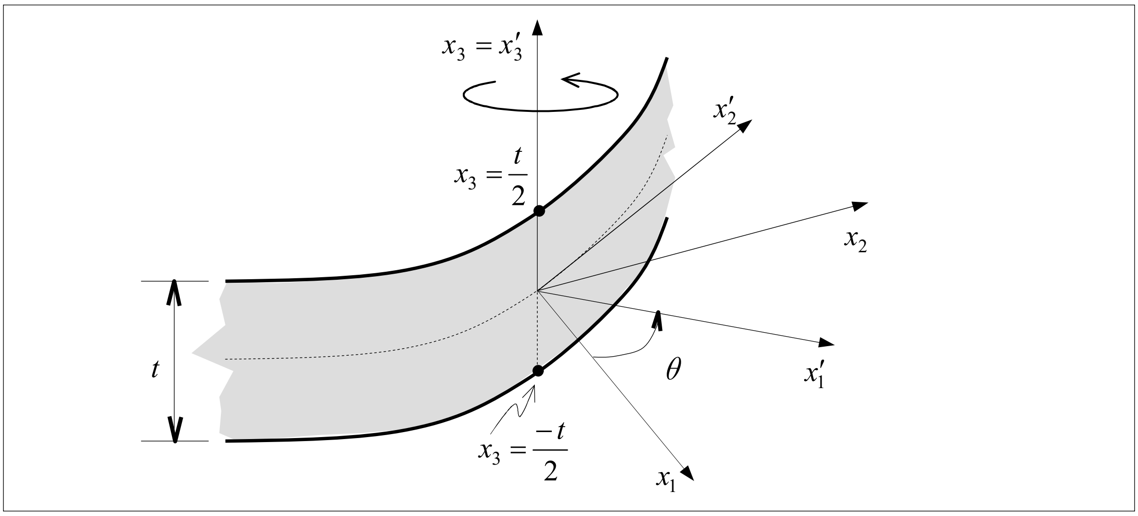


Figure 3.45

By using the Voigt notation, we can obtain:

$$\{\boldsymbol{m}'\}=\begin{Bmatrix}m'_{11}\\ m'_{22}\\ m'_{12}\end{Bmatrix}=\int_{\frac{-t}{2}}^{\frac{t}{2}}\begin{Bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{12}\end{Bmatrix}x'_3dx'_3=\int_{\frac{-t}{2}}^{\frac{t}{2}}\begin{Bmatrix}\sigma'_{11}\\ \sigma'_{22}\\ \sigma'_{12}\end{Bmatrix}x_3dx_3=\int_{\frac{-t}{2}}^{\frac{t}{2}}[\boldsymbol{\mathcal{M}}]\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}x_3dx_3=[\boldsymbol{\mathcal{M}}]\int_{\frac{-t}{2}}^{\frac{t}{2}}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}x_3dx_3$$

with that, we can conclude that:

$$\{\boldsymbol{m}'\}=\begin{Bmatrix}m'_{11}\\ m'_{22}\\ m'_{12}\end{Bmatrix}=[\boldsymbol{\mathcal{M}}]\begin{Bmatrix}m_{11}\\ m_{22}\\ m_{12}\end{Bmatrix}=[\boldsymbol{\mathcal{M}}]\{\boldsymbol{m}\} \tag{3.46}$$

where $[\mathcal{M}]$ is the transformation matrix in Voigt notation for a second-order tensor, (see **Problem 1.99**), and is given by:

$$[\mathcal{M}] = \begin{bmatrix} {a_{11}}^2 & {a_{12}}^2 & 2a_{11}a_{12} \\ {a_{21}}^2 & {a_{22}}^2 & 2a_{21}a_{22} \\ a_{21}a_{11} & a_{22}a_{12} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}$$

Also considering that $[\mathcal{M}]^{-1} = [\mathcal{N}]^T$, we can obtain $\{\boldsymbol{m}\} = [\mathcal{N}]^T\{\boldsymbol{m}'\}$, where

$$[\mathcal{N}] = \begin{bmatrix} {a_{11}}^2 & {a_{12}}^2 & a_{11}a_{12} \\ {a_{21}}^2 & {a_{22}}^2 & a_{21}a_{22} \\ 2a_{21}a_{11} & 2a_{22}a_{12} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}$$

The same result (3.46) could have been obtained by consider m_{ij} as a second-order tensor in two dimensional case (2D), and by means of the transformation law of a second-order tensor we can obtain:

$$\begin{aligned} &m'_{ij} = a_{ik}a_{jl}m_{kl} \quad ; \quad (i,j=1,2) \quad or \quad \boldsymbol{m}' = \boldsymbol{\mathcal{A}}\boldsymbol{m}\boldsymbol{\mathcal{A}}^T \\ &\Rightarrow \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{12} & m'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned} \tag{3.47}$$

3.6 Other measures of stress

Problem 3.31

Prove that the following relationship are valid:

$$\mathbf{P} = J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T} + J\sigma_m\boldsymbol{F}^{-T} \quad ; \quad \mathbf{S} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T} + J\sigma_m\boldsymbol{C}^{-1}$$

where $\mathbf{P}$ and $\mathbf{S}$ are the first and second Piola-Kirchhoff stress tensors, respectively, $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor, $\boldsymbol{F}$ is the deformation gradient, J is the Jacobian determinant, and the scalar σ_m is the mean normal Cauchy stress. Also show that the following relationships are true:

$$\mathbf{P}:\boldsymbol{F} = \mathbf{S}:\boldsymbol{C} = 3J\sigma_m$$

Solution:

Next, we will show the equation $\mathbf{P}:\boldsymbol{F} = \mathbf{S}:\boldsymbol{C}$:

$$\mathbf{P}:\boldsymbol{F} = \mathsf{P}_{ij}F_{ij} = (F_{ik}\mathsf{S}_{kj})F_{ij} = \mathsf{S}_{kj}(F_{ik}F_{ij}) = \mathsf{S}_{kj}(\boldsymbol{F}^T\cdot\boldsymbol{F})_{kj} = \mathsf{S}_{kj}(\boldsymbol{C})_{kj} = \mathbf{S}:\boldsymbol{C}$$

where we have the relationship $\mathbf{P} = \boldsymbol{F}\cdot\mathbf{S}$. By means of the additive decomposition of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} = \sigma_m\mathbf{1} + \boldsymbol{\sigma}^{dev}$, and by considering the definition $\mathbf{P} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}$, we can obtain:

$$\mathbf{P} = J\,(\boldsymbol{\sigma}^{dev} + \sigma_m\mathbf{1})\cdot\boldsymbol{F}^{-T} = J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T} + J\sigma_m\mathbf{1}\cdot\boldsymbol{F}^{-T} = J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T} + J\sigma_m\boldsymbol{F}^{-T}$$

Taking into account the definition $\mathbf{S} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}$, and by breaking down $\boldsymbol{\sigma}$ into $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev}$, we can obtain:

$$
\begin{aligned}
\mathbf{S} &= J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1}\cdot(\boldsymbol{\sigma}^{dev}+\sigma_m\mathbf{1})\cdot\boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}+J\boldsymbol{F}^{-1}\cdot\sigma_m\mathbf{1}\cdot\boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}+J\sigma_m(\boldsymbol{F}^{T}\cdot\boldsymbol{F})^{-1} \\
&= J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}+J\sigma_m\boldsymbol{C}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{S}_{ij} &= JF^{-1}_{ik}\sigma_{kp}F^{-1}_{jp} \\
&= JF^{-1}_{ik}(\sigma^{dev}_{kp}+\sigma_{(m)}\delta_{kp})F^{-1}_{jp} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp}+JF^{-1}_{ik}\sigma_{(m)}\delta_{kp}F^{-1}_{jp} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp}+J\sigma_{(m)}F^{-1}_{ik}F^{-1}_{jk} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp}+J\sigma_{(m)}C^{-1}_{ij}
\end{aligned}
$$

Then by applying the double scalar product between $\mathbf{S}$ and $\boldsymbol{C}$ we can obtain:

$$
\mathbf{S}:\boldsymbol{C}=(J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}+J\sigma_m\boldsymbol{C}^{-1}):\boldsymbol{C}=J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{C}+J\sigma_m\boldsymbol{C}^{-1}:\boldsymbol{C}
$$

where the term $J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{C}$ becomes:

$$
\begin{aligned}
J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{C} \quad &= (J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}):\underbrace{\boldsymbol{C}}_{\boldsymbol{F}^{T}\cdot\boldsymbol{F}} \\
(J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T})_{ij}(\boldsymbol{F}^{T}\cdot\boldsymbol{F})_{ij} \quad &= (F^{-1}_{ip}\sigma^{dev}_{pk}F^{-1}_{jk})(F_{qi}F_{qj}) \\
&= J\,\delta_{qp}\delta_{qk}\sigma^{dev}_{pk} \\
&= J\,\sigma^{dev}_{pk}\delta_{pk}=J\,\sigma^{dev}_{kk} \\
&= J\underbrace{\boldsymbol{\sigma}^{dev}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\sigma}^{dev})=0}=0
\end{aligned}
$$

Thus:

$$
\mathbf{S}:\boldsymbol{C}=J\sigma_m\boldsymbol{C}^{-1}:\boldsymbol{C}=J\sigma_m\mathrm{Tr}(\boldsymbol{C}^{-1}\cdot\boldsymbol{C})=J\sigma_m\mathrm{Tr}(\mathbf{1})=3J\sigma_m
$$

Now, by taking the double scalar product between $\mathbf{P}$ and $\boldsymbol{F}$ we obtain:

$$
\mathbf{P}:\boldsymbol{F}=J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{F}+J\sigma_m\boldsymbol{F}^{-T}:\boldsymbol{F}
$$

Then by analyzing the term $J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{F}$ we can conclude that:

$$
J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{F}=(J\,\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T})_{ij}(\boldsymbol{F})_{ij}=J\,\sigma^{dev}_{ik}\,F^{-1}_{jk}\,F_{ij}=J\sigma^{dev}_{ik}\delta_{ik}=J\underbrace{\boldsymbol{\sigma}^{dev}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\sigma}^{dev})=0}=0
$$

Thus,

$$
\mathbf{P}:\boldsymbol{F}=J\sigma_m\boldsymbol{F}^{-T}:\boldsymbol{F}=J\sigma_m\mathrm{Tr}(\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T})=J\sigma_m\mathrm{Tr}(\mathbf{1})=3J\sigma_m
$$

4 The Fundamental Equations of Continuum Mechanics

Problem 4.1

Show that Reynolds' transport theorem is valid in the following equation:

$$\frac{D}{Dt}\int_V \Phi dV = \frac{D}{Dt}\int_{V_0} \Phi J dV_0 \tag{4.1}$$

where V is the volume in the current configuration, V_0 is the volume in the reference configuration, J is the Jacobian determinant and Φ is a scalar field that describes the physical quantity of a particle per unit volume at time t.

Solution:

$$\frac{D}{Dt}\int_{V_0} \Phi J dV_0 = \int_{V_0}\left(J\frac{D\Phi}{Dt} + \Phi\frac{DJ}{Dt}\right) dV_0 = \int_{V_0}\left(J\frac{D\Phi}{Dt} + J\Phi\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right) dV_0 = \int_V\left(\frac{D\Phi}{Dt} + \Phi\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right) dV \tag{4.2}$$

Problem 4.2

Show that

$$\boxed{\frac{D}{Dt}\int_V \rho\, P_{ij\cdots}(\vec{\boldsymbol{x}},t)\, dV = \int_V \rho\frac{DP_{ij\cdots}(\vec{\boldsymbol{x}},t)}{Dt} dV} \tag{4.3}$$

where $P_{ij\cdots}(\vec{\boldsymbol{x}},t)$ is a continuum property per unit mass, which can be a scalar, a vector or higher order tensor.

Solution:

It was proven in the textbook, (Chaves (2013)), that:

$$\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t) dV = \int_V \left[\frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\frac{\partial v_p}{\partial x_p}\right] dV$$

Then by making $\Phi = \rho\, P_{ij\cdots}$, and by considering it in the above equation we can obtain:

$$\frac{D}{Dt}\int_V \rho\, P_{ij\ldots} dV = \int_V \left[\frac{D}{Dt}(\rho P_{ij\ldots}) + \rho P_{ij\ldots} \frac{\partial v_p}{\partial x_p} \right] dV = \int_V \left[\rho \frac{D}{Dt} P_{ij\ldots} + P_{ij\ldots} \frac{D\rho}{Dt} + \rho P_{ij\ldots} \frac{\partial v_k}{\partial x_k} \right] dV$$

$$= \int_V \left[\rho \frac{D}{Dt} P_{ij\ldots} + P_{ij\ldots} \underbrace{\left(\frac{D\rho}{Dt} + \rho \frac{\partial v_k}{\partial x_k} \right)}_{\substack{=0 \\ mass\ continuity\ equation}} \right] dV$$

Thus, we can conclude that:

$$\frac{D}{Dt}\int_V \rho P_{ij\ldots} dV = \int_V \left[\rho \frac{D P_{ij\ldots}}{Dt} \right] dV$$

Problem 4.3

Prove that the following relationship is valid:

$$\boxed{\rho\, \vec{\boldsymbol{a}} = \frac{\partial}{\partial t}(\rho\, \vec{\boldsymbol{v}}) + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}})} \tag{4.4}$$

Solution:

Based on the Reynolds' transport theorem:

$$\frac{D}{Dt}\int_V \Phi dV = \int_V \frac{\partial \Phi}{\partial t} dV + \int_S \Phi(\vec{\boldsymbol{v}} \cdot \hat{\mathbf{n}}) dS$$

and if we consider that $\Phi = \rho\, \vec{\boldsymbol{v}}$ we can obtain:

$$\frac{D}{Dt}\int_V \rho\, \vec{\boldsymbol{v}}\, dV = \int_V \frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} dV + \int_S \rho\, \vec{\boldsymbol{v}} \otimes (\vec{\boldsymbol{v}} \cdot \hat{\mathbf{n}}) dS$$

The above equation in indicial notation becomes:

$$\frac{D}{Dt}\int_V \rho\, v_i\, dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_S \rho\, v_i (v_k \hat{\mathsf{n}}_k) dS \Rightarrow \int_V \rho \underbrace{\frac{D}{Dt} v_i}_{=a_i} dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_S (\rho\, v_i v_k) \hat{\mathsf{n}}_k\, dS$$

Additionally, by applying the divergence theorem to the surface integral we can obtain:

$$\int_V \rho\, a_i\, dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_V (\rho\, v_i v_k)_{,k}\, dV = \int_V \left[\frac{\partial(\rho\, v_i)}{\partial t} + (\rho\, v_i v_k)_{,k} \right] dV$$

which in tensorial notation is:

$$\int_V \rho\, \vec{\boldsymbol{a}}\, dV = \int_V \left[\frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}}) \right] dV \qquad \Rightarrow \qquad \rho\, \vec{\boldsymbol{a}} = \frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}})$$

Problem 4.4

Let us consider the following velocity field:

$$v_i = \frac{x_i}{1+t} \qquad \text{for } t \geq 0$$

a) Find the mass density field;

b) Show that for this motion the equation $\rho\, x_1\, x_2\, x_3 = \rho_0\, X_1\, X_2\, X_3$ is satisfied.

Solution:

a) By applying the mass continuity equation we can obtain:

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_k}{\partial x_k} = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = -\rho \frac{\partial v_k}{\partial x_k}$$

and by using the given velocity field, we can find that:

$$\frac{\partial v_i}{\partial x_i} = \frac{1}{1+t}\frac{\partial x_i}{\partial x_i} = \frac{\delta_{ii}}{1+t} = \frac{3}{1+t}$$

Thus,

$$\frac{d\rho}{dt} = -\frac{3\rho}{1+t} \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{3dt}{1+t}$$

Then by integrating the both sides of the above equation we can obtain:

$$\int \frac{d\rho}{\rho} = \int -\frac{3dt}{1+t} \quad \Rightarrow \quad \mathrm{Ln}\rho = -3\,\mathrm{Ln}(1+t) + C$$

The constant of integration C is obtained by applying the initial condition at $t=0$, in which $\rho(\vec{\boldsymbol{x}}, t=0) = \rho_0$ holds, thus

$$\mathrm{Ln}\rho_0 = -3\,\mathrm{Ln}(1+0) + C \quad \Rightarrow \quad C = \mathrm{Ln}\rho_0$$

$$\mathrm{Ln}\rho = -3\,\mathrm{Ln}(1+t) + \mathrm{Ln}\rho_0 = \mathrm{Ln}\left(\frac{1}{(1+t)^3}\right) + \mathrm{Ln}\rho_0 = \mathrm{Ln}\left(\frac{\rho_0}{(1+t)^3}\right)$$

Thus, we can conclude that:

$$\rho = \frac{\rho_0}{(1+t)^3}$$

b) By using the velocity definition we can obtain:

$$v_i = \frac{dx_i}{dt} = \frac{x_i}{1+t} \quad \Rightarrow \frac{dx_i}{x_i} = \frac{dt}{1+t}$$

Additionally, by integrating the both sides of the above equation we can obtain:

$$\int \frac{dx_i}{x_i} = \int \frac{dt}{1+t} \quad \Rightarrow \quad \mathrm{Ln}x_i = \mathrm{Ln}(1+t) + K_i \tag{4.5}$$

Then, by applying the initial condition, i.e. at time $t=0 \ \Rightarrow x_i = X_i$, we can obtain:

$$\mathrm{Ln}X_i = \mathrm{Ln}(1+0) + K_i \quad \Rightarrow \quad K_i = \mathrm{Ln}X_i$$

And by substituting the value of K_i into the equation (4.5) we can obtain:

$$\mathrm{Ln}x_i = \mathrm{Ln}(1+t) + \mathrm{Ln}X_i \quad \Rightarrow \quad \mathrm{Ln}(x_i) = \mathrm{Ln}[X_i(1+t)]$$

Hence we can conclude that $x_i = X_i(1+t)$, which gives us $x_1 = X_1(1+t)$, $x_2 = X_2(1+t)$, $x_3 = X_3(1+t)$, and if we consider that $\rho = \dfrac{\rho_0}{(1+t)^3}$, we can obtain:

$$\rho\underbrace{(1+t)}_{=\frac{x_1}{X_1}}\underbrace{(1+t)}_{=\frac{x_2}{X_2}}\underbrace{(1+t)}_{=\frac{x_3}{X_3}} = \rho_0 \quad \Rightarrow \quad \rho\, x_1\, x_2\, x_3 = \rho_0\, X_1\, X_2\, X_3$$

Problem 4.5

The equations of motion of a body are given, in Lagrangian description, by:

$$\begin{cases} x_1 = X_1 + \alpha t X_3 \\ x_2 = X_2 + \alpha t X_3 \\ x_3 = X_3 - \alpha t (X_1 + X_2) \end{cases}$$

where α is a constant scalar. Find the mass density in the current configuration (ρ) in terms of the mass density of the reference configuration (ρ_0), i.e. $\rho = \rho(\rho_0)$.

Solution:

We can apply the equation $\rho_0 = J\rho$, where J is the Jacobian determinant and is given by:

$$J = |F| = \left|\frac{\partial x_i}{\partial X_j}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \alpha t \\ 0 & 1 & \alpha t \\ -\alpha t & -\alpha t & 1 \end{vmatrix} = 1 + 2(\alpha t)^2$$

Thus, we can obtain $\rho = \dfrac{\rho_0}{J} = \dfrac{\rho_0}{1 + 2(\alpha t)^2}$

Problem 4.6

Given the velocity field components:

$$v_1 = a x_1 - b x_2 \quad ; \quad v_2 = b x_1 - a x_2 \quad ; \quad v_3 = c\sqrt{x_1^2 + x_2^2}$$

where a, b and c are constants.
a) Check whether the mass continuity equation is fulfilled or not;
b) Is the motion isochoric?

Solution:

The mass continuity equation:

$$\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0$$

where:

$$\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = v_{i,i} = v_{1,1} + v_{2,2} + v_{3,3} = a - a + 0 = 0$$

The motion is isochoric (incompressible medium), since $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0$ or $\dfrac{D\rho}{Dt} = 0$.

Problem 4.7

Consider a continuous medium and an Eulerian property $\phi(\vec{\boldsymbol{x}}, t)$ assigned by density, i.e. unit of the property per unit volume. Obtain the rate of change of the property by considering the control volume and the control surface.

Solution:

Remember that the rate of change of a property is always associated with the same particles. By means of the material time derivative we can obtain the rate of change of a property when this property is in Eulerian description. Then, the total rate of change of $\phi(\vec{\boldsymbol{x}},t)$ in the volume V which is bounded by the surface S is given by:

$$
\begin{aligned}
\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV = \int_V \frac{D}{Dt}(\phi\, dV) &= \int_V \left[dV \frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\frac{D}{Dt}(dV) \right] \\
&= \int_V \left[dV \frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})\, dV \right] \\
&= \int_V \left[\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) \right] dV
\end{aligned}
\tag{4.6}
$$

We apply the definition of the material time derivative to $\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t)$:

$$
\begin{aligned}
\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV &= \int_V \left[\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) \right] dV \\
&= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t) + \frac{\partial \phi(\vec{\boldsymbol{x}},t)}{\partial \vec{\boldsymbol{x}}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) \right] dV \\
&= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t) \right] dV + \int_V \left[\frac{\partial \phi(\vec{\boldsymbol{x}},t)}{\partial \vec{\boldsymbol{x}}} \cdot \vec{\boldsymbol{v}} + \phi(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) \right] dV \\
&= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t) \right] dV + \int_V \left[\nabla_{\vec{x}} \cdot (\phi\vec{\boldsymbol{v}}) \right] dV
\end{aligned}
\tag{4.7}
$$

We can apply the divergence theorem to the second integral on the right side of the equation to obtain:

$$
\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV = \int_V \underbrace{\frac{\partial \phi(\vec{\boldsymbol{x}},t)}{\partial t}}_{\text{local}} dV + \overbrace{\int_S \underbrace{(\phi\vec{\boldsymbol{v}})}_{\text{flux of } \phi} \cdot \hat{\boldsymbol{n}}\, dS}^{\text{flux of } \phi \text{ through suface S}}
\tag{4.8}
$$

the term $\frac{\partial \phi(\vec{\boldsymbol{x}},t)}{\partial t}$ is local, the volume integral of the right side of the equation is a control volume and the integral surface is a control surface, since the variable $(\phi\vec{\boldsymbol{v}})$ is in Eulerian description. The term $(\phi\vec{\boldsymbol{v}})$ represents the flux of the property ϕ.

When there is no source or sink of the property it is true that $\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV = 0$. And, note also that when the property is the mass density ($\phi = \rho$) the equation in (4.7) becomes the *mass continuity equation.*

$$
\frac{D}{Dt}\int_V \rho(\vec{\boldsymbol{x}},t)dV = \int_V \left[\frac{D}{Dt}\rho(\vec{\boldsymbol{x}},t) + \rho(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) \right] dV = 0 = \int_V \left[\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{x}},t) + \nabla_{\vec{x}} \cdot (\rho\vec{\boldsymbol{v}}) \right] dV = 0
\tag{4.9}
$$

If the above equation is valid for the entire volume then it is valid locally, so

$$
\boxed{\frac{D}{Dt}\rho(\vec{\boldsymbol{x}},t) + \rho(\vec{\boldsymbol{x}},t)(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0} \quad \textit{Mass continuity equation}
\tag{4.10}
$$

or

$$\boxed{\frac{\partial}{\partial t}\rho(\vec{\boldsymbol{x}},t)+\nabla_{\vec{x}}\cdot(\rho\vec{\boldsymbol{v}})=0} \quad \textit{Mass continuity equation} \tag{4.11}$$

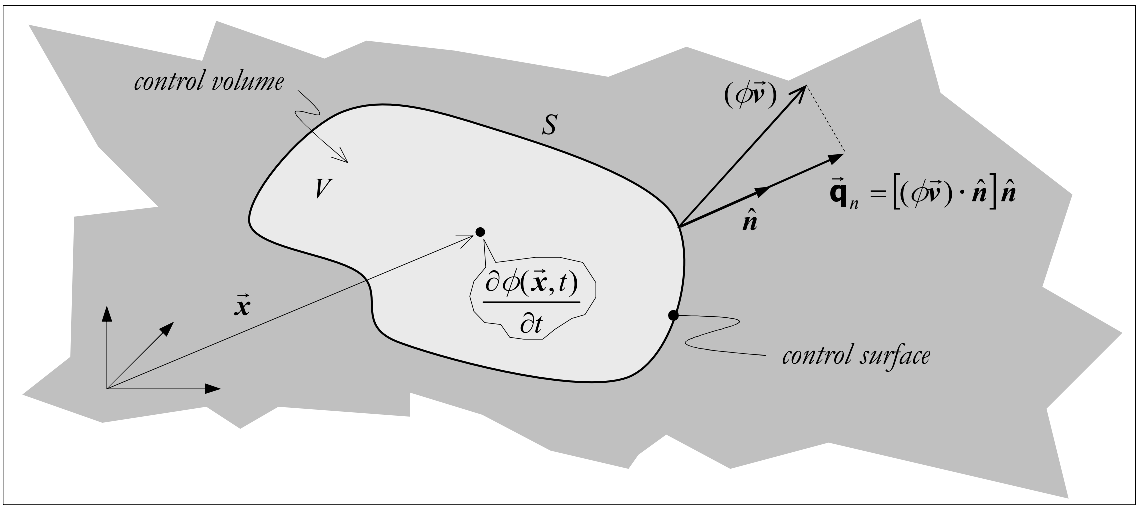


Figure 4.1: Control volume and control surface.

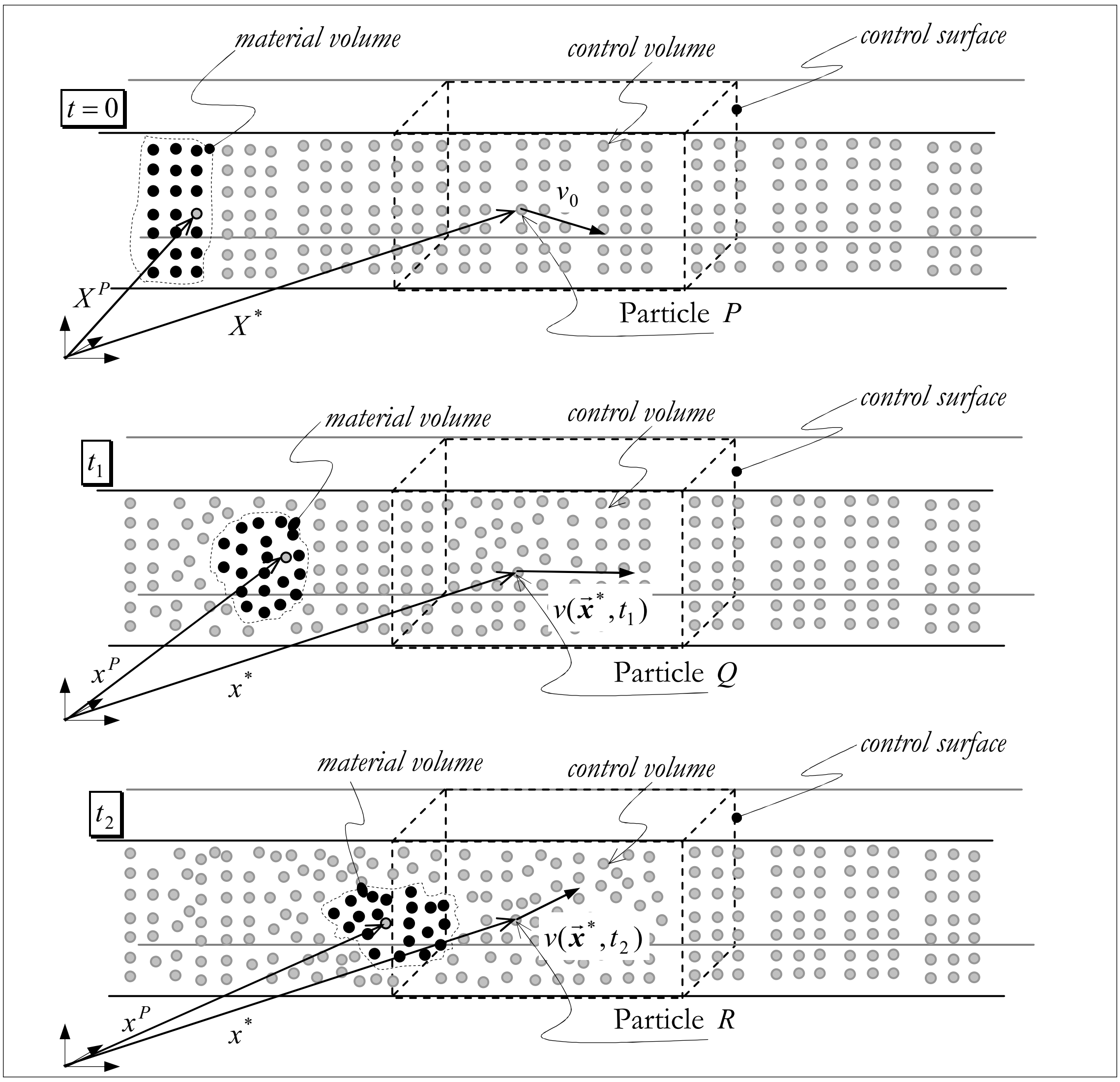


Figure 4.2: Material volume vs. control volume.

Problem 4.8

Show that the following equation is true:

Tensorial notation	Indicial notation	
$\boxed{\rho \frac{\partial \phi}{\partial t} + \rho (\nabla_{\vec{x}} \phi) \cdot \vec{\boldsymbol{v}} = \frac{\partial}{\partial t}(\rho \phi) + \nabla_{\vec{x}} \cdot (\rho \phi \vec{\boldsymbol{v}})}$	$\boxed{\rho \frac{\partial \phi}{\partial t} + \rho \phi_{,i} v_i = \frac{\partial}{\partial t}(\rho \phi) + (\rho \phi v_i)_{,i}}$	(4.12)

where $\phi(\vec{\boldsymbol{x}},t)$ is a scalar field, $\rho(\vec{\boldsymbol{x}},t)$ is the mass density field, and $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)$ is the velocity field.

Solution:

$$\frac{\partial}{\partial t}(\phi\rho) + (\phi\rho v_i)_{,i} = \rho \frac{\partial \phi}{\partial t} + \phi \frac{\partial \rho}{\partial t} + \phi_{,i}(\rho v_i) + \phi(\rho v_i)_{,i} = \rho \frac{\partial \phi}{\partial t} + \phi_{,i}\rho v_i + \phi\left[\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i}\right]$$

$$= \rho \frac{\partial \phi}{\partial t} + \phi_{,i}\rho v_i$$

where we have considered the mass continuity equation $\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0$.

4.1 Equations of Motion. Equilibrium Equations

Problem 4.9

Find the equilibrium equations by means of the differential volume element equilibrium ($dx\,dy\,dz$). For this purpose consider that the Cauchy stress tensor field in the differential volume element varies as indicated in Figure 4.3. Adopt the engineering notation.

Solution:

To obtain the equilibrium equations we will apply the force equilibrium condition in the volume element. First, we evaluate the equilibrium force according to the x-direction:

$$\sum F_x = 0$$

$$\rho \mathsf{b}_x dxdydz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)dydz - \sigma_x dydz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)dxdz$$
$$- \tau_{xy}dxdz + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}dz\right)dxdy - \tau_{xz}dxdy = 0$$

Then, by simplifying the above equation we can obtain:

$$\rho \mathsf{b}_x dxdydz + \frac{\partial \sigma_x}{\partial x}dxdydz + \frac{\partial \tau_{xy}}{\partial y}dxdydz + \frac{\partial \tau_{xz}}{\partial z}dxdydz = 0$$

$$\Rightarrow \rho \mathsf{b}_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

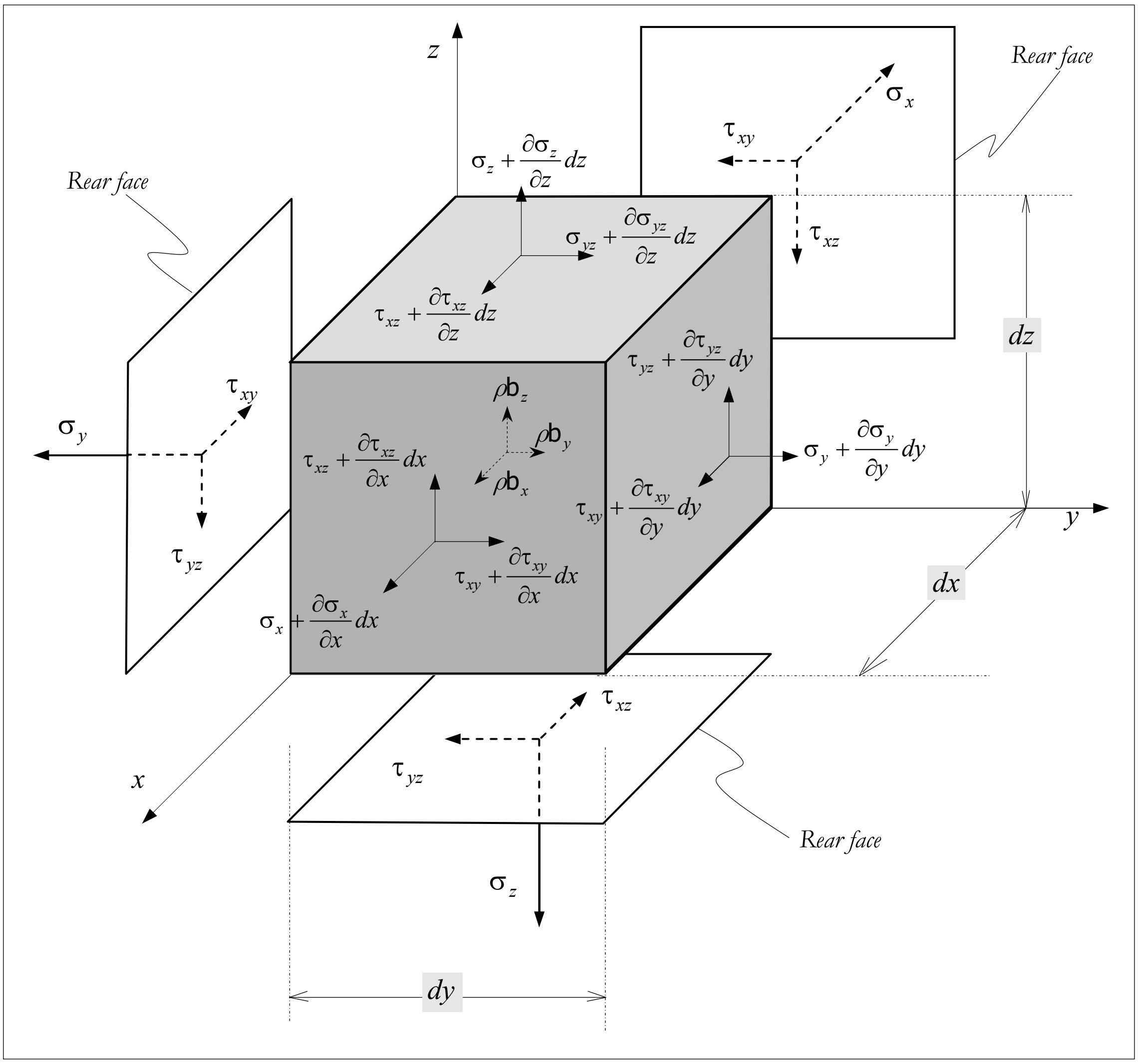


Figure 4.3: The stress field in the differential volume element.

The equilibrium force according to the y-direction, $\sum F_y = 0$, can be expressed as follows

$$\rho \mathrm{b}_y dxdydz + \left(\sigma_{22} + \frac{\partial \sigma_y}{\partial y} dy\right) dxdz - \sigma_y dxdz + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial z} dz\right) dxdy$$
$$- \tau_{yz} dxdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dydz - \tau_{xy} dydz = 0$$

Then by simplifying the above equation we can obtain:

$$\rho \mathrm{b}_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial x_z} = 0$$

Finally, the equilibrium according to the z-direction, $\sum F_z = 0$, is given by:

$$\rho \mathsf{b}_z dxdydz + \left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz \right) dxdy - \sigma_z dxdy + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx \right) dzdy$$
$$- \tau_{xz} dzdy + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy \right) dxdz - \tau_{yz} dxdz = 0$$

and after the simplification is taken place the above equation becomes:

$$\rho \mathsf{b}_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

Then, the equilibrium equations in engineering notation become:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + \rho \mathsf{b}_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial x_z} + \rho \mathsf{b}_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + \rho \mathsf{b}_z = 0 \end{cases}$$

Problem 4.10

Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor field, which is represented by its components in the Cartesian basis as:

$$\sigma_{11} = x_1^2; \qquad \sigma_{22} = x_2^2; \qquad \sigma_{33} = x_1^2 + x_2^2$$
$$\sigma_{12} = \sigma_{21} = 2x_1 x_2; \qquad \sigma_{23} = \sigma_{32} = \sigma_{31} = \sigma_{13} = 0$$

Considering that the body is in equilibrium, find the body forces acting on the continuum.

Solution:

By applying the equilibrium equations, $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$, we can obtain:

$$\sigma_{ij,j} + \rho \mathsf{b}_i = \mathsf{0}_i \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + \rho \mathsf{b}_1 = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + \rho \mathsf{b}_2 = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + \rho \mathsf{b}_3 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} 2x_1 + 2x_1 + \rho \mathsf{b}_1 = 0 \\ 2x_2 + 2x_2 + \rho \mathsf{b}_2 = 0 \\ \rho \mathsf{b}_3 = 0 \end{cases}$$

Thus, to satisfy the equilibrium equations the following condition must be met:

$$\left. \begin{array}{l} 4x_1 = -\rho \mathsf{b}_1 \Rightarrow \rho \mathsf{b}_1 = -4x_1 \\ 4x_2 = -\rho \mathsf{b}_2 \Rightarrow \rho \mathsf{b}_2 = -4x_2 \\ \Rightarrow \rho \mathsf{b}_3 = 0 \end{array} \right\} \quad \Rightarrow \quad \rho \vec{\mathbf{b}} = -4(x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2) \quad \textit{(Body force density)} \quad \left[\frac{N}{m^3} \right]$$

Problem 4.11

Given the velocity field components:

$$v_1 = x_1 x_3 \quad ; \quad v_2 = x_2^2 t \quad ; \quad v_3 = x_2 x_3 t$$

and the Cauchy stress tensor field components:

$$\sigma_{ij} = \alpha \begin{bmatrix} x_2 x_1 & -x_2 x_3 & 0 \\ -x_2 x_3 & x_2^2 & -x_2 \\ 0 & -x_2 & x_3^2 \end{bmatrix}$$

where α is a constant. Obtain the body force (per unit volume) to guarantee the principle of conservation of the linear momentum.

Solution:

From the principle of conservation of linear momentum we obtain the equations of motion:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \vec{a} \qquad \Rightarrow \qquad \rho \vec{\mathbf{b}} = \rho \vec{a} - \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}$$

The acceleration field:

$$\vec{a} = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \frac{\partial \vec{v}(\vec{x},t)}{\partial \vec{x}} \cdot \vec{v}(\vec{x},t) \qquad ; \qquad a_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j$$

where

$$\left(\frac{\partial \vec{v}}{\partial t}\right)_i = \frac{\partial v_i}{\partial t} = \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix} \qquad ; \qquad (\nabla_{\vec{x}} \vec{v})_{ij} \equiv \left(\frac{\partial \vec{v}}{\partial \vec{x}}\right)_{ij} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} x_3 & 0 & x_1 \\ 0 & 2x_2 t & 0 \\ 0 & x_3 t & x_2 t \end{bmatrix}$$

Then

$$a_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j = \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix} + \begin{bmatrix} x_3 & 0 & x_1 \\ 0 & 2x_2 t & 0 \\ 0 & x_3 t & x_2 t \end{bmatrix} \begin{bmatrix} x_1 x_3 \\ x_2^2 t \\ x_2 x_3 t \end{bmatrix} = \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix} + \begin{bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ 2x_2^3 t \\ x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ x_2^2 + 2x_2^3 t \\ x_2 x_3 + x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{bmatrix}$$

The divergence of the Cauchy stress tensor is given by:

$$\sigma_{ij,j} = \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = \alpha(x_2 - x_3) \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = \alpha(2x_2) \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = \alpha(2x_3 - 1) \end{cases} \qquad \Rightarrow \qquad \sigma_{ij,j} = \alpha \begin{bmatrix} x_2 - x_3 \\ 2x_2 \\ 2x_3 - 1 \end{bmatrix}$$

with that the body force density (per unit volume) becomes:

$$\rho \vec{\mathbf{b}} = \rho \vec{a} - \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}$$

$$\rho \mathbf{b}_i = \rho a_i - \sigma_{ij,j} \qquad \Rightarrow \qquad \rho \mathbf{b}_i = \rho \begin{bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ x_2^2 + 2x_2^3 t \\ x_2 x_3 + x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{bmatrix} - \alpha \begin{bmatrix} x_2 - x_3 \\ 2x_2 \\ 2x_3 - 1 \end{bmatrix}$$

Problem 4.12

The Cauchy stress tensor field in the medium in equilibrium is represented by its Cartesian components as follows:

$$\sigma_{ij} = \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & 4x_1 \\ 2x_2 & 4x_1 & 1 \end{bmatrix} \tag{4.13}$$

where x_i are the Cartesian coordinates.

a) By neglecting body forces, is the body in balance?

Solution:

The equilibrium equations:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \underbrace{\rho\vec{\mathbf{b}}}_{=\vec{\mathbf{0}}} = \vec{\mathbf{0}} \quad \xrightarrow{indicial} \quad \sigma_{ij,j} = 0_i \tag{4.14}$$

and by expanding the above equation we can obtain:

$$\sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0_i \quad \Rightarrow \quad \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \end{cases} \tag{4.15}$$

where we have used:

$$\begin{aligned} &\sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial x_1} = 0; \quad \sigma_{12,2} = \frac{\partial \sigma_{12}}{\partial x_2} = 0; \quad \sigma_{13,3} = \frac{\partial \sigma_{13}}{\partial x_3} = 0 \\ &\sigma_{21,2} = \frac{\partial \sigma_{21}}{\partial x_2} = 0; \quad \sigma_{22,2} = \frac{\partial \sigma_{22}}{\partial x_2} = 0; \quad \sigma_{23,3} = \frac{\partial \sigma_{23}}{\partial x_3} = 0 \\ &\sigma_{31,3} = \frac{\partial \sigma_{31}}{\partial x_3} = 0; \quad \sigma_{32,2} = \frac{\partial \sigma_{32}}{\partial x_2} = 0; \quad \sigma_{33,3} = \frac{\partial \sigma_{33}}{\partial x_3} = 0 \end{aligned} \tag{4.16}$$

Problem 4.13

Given a body in equilibrium in which the Cauchy stress tensor field is represented by its components:

$$\begin{aligned} \sigma_{11} &= 6x_1^3 + x_2^2 \quad ; \quad \sigma_{12} = x_3^2 \\ \sigma_{22} &= 12x_1^3 + 60 \quad ; \quad \sigma_{23} = x_2 \\ \sigma_{33} &= 18x_2^3 + 6x_3^3 \quad ; \quad \sigma_{31} = x_1^2 \end{aligned}$$

Obtain the body force density vector (per unit volume) at the point ($x_1 = 2; x_2 = 4; x_3 = 2$).

Solution:

The equilibrium equations are represented by:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \vec{\mathbf{0}} \tag{4.17}$$

and the explicit form:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1}+\dfrac{\partial \sigma_{12}}{\partial x_2}+\dfrac{\partial \sigma_{13}}{\partial x_3}+\rho \mathsf{b}_1=0 \Rightarrow \rho \mathsf{b}_1=-\dfrac{\partial \sigma_{11}}{\partial x_1}-\dfrac{\partial \sigma_{12}}{\partial x_2}-\dfrac{\partial \sigma_{13}}{\partial x_3} \\ \dfrac{\partial \sigma_{21}}{\partial x_1}+\dfrac{\partial \sigma_{22}}{\partial x_2}+\dfrac{\partial \sigma_{23}}{\partial x_3}+\rho \mathsf{b}_2=0 \Rightarrow \rho \mathsf{b}_2=-\dfrac{\partial \sigma_{21}}{\partial x_1}-\dfrac{\partial \sigma_{22}}{\partial x_2}-\dfrac{\partial \sigma_{23}}{\partial x_3} \\ \dfrac{\partial \sigma_{31}}{\partial x_1}+\dfrac{\partial \sigma_{32}}{\partial x_2}+\dfrac{\partial \sigma_{33}}{\partial x_3}+\rho \mathsf{b}_3=0 \Rightarrow \rho \mathsf{b}_3=-\dfrac{\partial \sigma_{31}}{\partial x_1}-\dfrac{\partial \sigma_{32}}{\partial x_2}-\dfrac{\partial \sigma_{33}}{\partial x_3} \end{cases} \tag{4.18}$$

$$\begin{cases} \rho \mathsf{b}_1=-18x_1^2-0-0 \\ \rho \mathsf{b}_2=-0-0-0 \\ \rho \mathsf{b}_3=-2x_1-1-18x_2^2 \end{cases} \Rightarrow \quad \rho \mathsf{b}_i=\begin{bmatrix} -18x_1^2 \\ 0 \\ -2x_1-1-18x_2^2 \end{bmatrix} \tag{4.19}$$

At the point $x_1=2; x_2=4; x_3=2$ the body force density becomes:

$$\rho \mathsf{b}_i=\begin{bmatrix} -72 \\ 0 \\ -77 \end{bmatrix} \qquad \textit{(Force per unit volume)} \left[\frac{N}{m^3}\right] \tag{4.20}$$

Problem 4.14

The Cauchy stress tensor field is represented by its components as follows:

$$\sigma_{ij}=k\begin{bmatrix} x_1^2 x_2 & (a^2-x_2^2)x_1 & 0 \\ (a^2-x_2^2)x_1 & \frac{1}{3}(x_2^3-3a^2 x_2) & 0 \\ 0 & 0 & 2ax_3^2 \end{bmatrix} \tag{4.21}$$

where k and a are constants. Obtain the specific body force field $\vec{\mathbf{b}}$ (per unit mass) in order to achieve equilibrium.

Solution:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1}+\dfrac{\partial \sigma_{12}}{\partial x_2}+\dfrac{\partial \sigma_{13}}{\partial x_3}+\rho \mathsf{b}_1=0 \Rightarrow \rho \mathsf{b}_1=-2x_1x_2k+2x_1x_2k=0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1}+\dfrac{\partial \sigma_{22}}{\partial x_2}+\dfrac{\partial \sigma_{23}}{\partial x_3}+\rho \mathsf{b}_2=0 \Rightarrow \rho \mathsf{b}_2=-k(a^2-x_2^2)-\dfrac{k}{3}(3x_2^2-3a^2)=0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1}+\dfrac{\partial \sigma_{32}}{\partial x_2}+\dfrac{\partial \sigma_{33}}{\partial x_3}+\rho \mathsf{b}_3=0 \Rightarrow \rho \mathsf{b}_3=-4kax_3 \end{cases} \tag{4.22}$$

Then:

$$\mathsf{b}_i=\frac{4kax_3}{\rho}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad \textit{(Force per unit mass)} \left[\frac{N}{kg}\right] \tag{4.23}$$

Problem 4.15

Let us assume that the specific body force is $\vec{\mathbf{b}}=-g\hat{\mathbf{e}}_3$, where g is a constant and consider the Cauchy stress tensor field components:

$$\sigma_{ij} = \alpha \begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & p \end{bmatrix} \tag{4.24}$$

Find p such that satisfies the equilibrium equations. Consider that α is a constant and that the mass density field is homogeneous, i.e. it is independent of the vector position ($\vec{\boldsymbol{x}}$).

Solution:

The equilibrium equations:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}} \xrightarrow{\text{indicial}} \sigma_{ij,j} + \rho \mathrm{b}_i = 0_i \tag{4.25}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + \rho \mathrm{b}_1 = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + \rho \mathrm{b}_2 = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + \rho \mathrm{b}_3 = 0 \end{cases} \Rightarrow \begin{cases} 0+0+0+\rho \mathrm{b}_1 = 0 \Rightarrow \mathrm{b}_1 = 0 \\ 0+0+0+\rho \mathrm{b}_2 = 0 \Rightarrow \mathrm{b}_2 = 0 \\ 0 - \alpha + \dfrac{\partial \sigma_{33}}{\partial x_3} + \rho \mathrm{b}_3 = 0 \end{cases}$$

thus

$$\frac{\partial \sigma_{33}}{\partial x_3} = \frac{\partial (\alpha p)}{\partial x_3} = \alpha \frac{\partial p}{\partial x_3} = \alpha - \rho \mathrm{b}_3 \quad \Rightarrow \quad \frac{\partial p}{\partial x_3} = 1 + \frac{\rho g}{\alpha}$$

$$\Rightarrow dp = \left(1 + \frac{\rho g}{\alpha}\right) dx_3$$

$$p = \int \left(1 + \frac{\rho g}{\alpha}\right) \partial x_3 \quad \Rightarrow \quad p = \left(1 + \frac{\rho g}{\alpha}\right) x_3$$

Problem 4.16

Show that the equilibrium equations are satisfied by considering the following Cauchy stress field Cartesian components:

$$\begin{aligned} &\sigma_{11} = x_2^2 + \nu(x_1^2 - x_2^2) \quad ; \quad \sigma_{12} = -2\nu x_1 x_2 \quad ; \quad \sigma_{23} = \sigma_{13} = 0 \\ &\sigma_{22} = x_1^2 + \nu(x_2^2 - x_1^2) \quad ; \quad \sigma_{33} = \nu(x_1^2 + x_2^2) \end{aligned}$$

Consider that there are no body forces.

Solution:

The equilibrium equations:

$$\sigma_{ij,j} + \underbrace{\rho \mathrm{b}_i}_{=0_i} = 0_i \quad \Rightarrow \quad \sigma_{ij,j} = 0_i \quad \Rightarrow \quad \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0_i$$

$$\begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix} \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0 \end{cases}$$

thus:

$$\Rightarrow \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{31,3} = 2x_1\nu - 2\nu x_1 = 0 \\ \sigma_{12,1} + \sigma_{22,2} + \sigma_{23,3} = -2x_2\nu + 2\nu x_2 = 0 \\ \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0 \end{cases}$$

with that we have shown that the body is in balance.

Problem 4.17

Consider a body in equilibrium in which the Cauchy stress field components are:

$$\sigma_{ij}(\vec{x}) = \begin{bmatrix} x_1 + x_2 & \sigma_{12} & 0 \\ \sigma_{12} & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$$

Find σ_{12}, knowing that σ_{12} is a function of x_1 and x_2, i.e. $\sigma_{12} = \sigma_{12}(x_1, x_2)$. It is also known that the medium is free of body forces and the traction vector associated with the plane $x_1 = 1$ is given by $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = (1 + x_2)\hat{\mathbf{e}}_1 + (5 - x_2)\hat{\mathbf{e}}_2$.

Solution:

As the body is in equilibrium, it must satisfy the equilibrium equations:

$$\sigma_{ij,j} + \underbrace{\rho \mathrm{b}_i}_{=0_i} = 0_i \quad \Rightarrow \quad \sigma_{ij,j} = 0_i \quad \Rightarrow \quad \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0_i$$

thus

$$\Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 1 + \dfrac{\partial \sigma_{12}}{\partial x_2} + 0 = 0 \quad \Rightarrow \quad \dfrac{\partial \sigma_{12}}{\partial x_2} = -1 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = \dfrac{\partial \sigma_{12}}{\partial x_1} - 2 + 0 = 0 \quad \Rightarrow \quad \dfrac{\partial \sigma_{12}}{\partial x_1} = 2 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0 + 0 + 0 = 0 \end{cases}$$

Now considering that for the plane $x_1 = 1$, the stress tensor and the traction vector, when $x_1 = 1$, become respectively:

$$\sigma_{ij}(x_1 = 1, x_2) = \begin{bmatrix} 1 + x_2 & \sigma_{12} & 0 \\ \sigma_{12} & 1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix} \qquad \text{and} \quad \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = (1 + x_2)\hat{\mathbf{e}}_1 + (5 - x_2)\hat{\mathbf{e}}_2$$

Remember that the traction vector can be obtained by the equation $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, then

$$\mathrm{t}^{(\hat{\mathbf{n}})} = \sigma_{ij}(x_1 = 1, x_2)\hat{\mathrm{n}}_j = \begin{bmatrix} 1 + x_2 & \sigma_{12} & 0 \\ \sigma_{12} & 1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + x_2 \\ 5 - x_2 \\ 0 \end{bmatrix} \tag{4.26}$$

$$\mathrm{t}^{(\hat{\mathbf{n}})} = \sigma_{ij}(x_1 = 1, x_2)\,\hat{\mathrm{n}}_j$$

$$\Rightarrow \begin{bmatrix} 1 + x_2 & \sigma_{12}(x_1 = 1, x_2) & 0 \\ \sigma_{12}(x_1 = 1, x_2) & 1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + x_2 \\ \sigma_{12}(x_1 = 1, x_2) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + x_2 \\ 5 - x_2 \\ 0 \end{bmatrix}$$

By means of the equilibrium equations:

$$\frac{\partial\sigma_{12}}{\partial x_1}=2 \quad\Rightarrow\quad \int\partial\sigma_{12}=\int 2\partial x_1 \quad\Rightarrow\quad \sigma_{12}(x_1,x_2)=2x_1+C(x_2)$$

And by using the information given by (4.26) we can obtain the constant of integration:

$$\sigma_{12}(x_1=1,x_2)=5-x_2=2+C(x_2) \quad\Rightarrow\quad C(x_2)=3-x_2$$

thus:

$$\sigma_{12}(x_1,x_2)=2x_1-x_2+3$$

Problem 4.18

The stress state in an continuous medium is given by the Cauchy stress tensor Cartesian components:

$$\sigma_{ij}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{bmatrix}=\begin{bmatrix}0 & Cx_3 & 0\\ Cx_3 & 0 & -Cx_1\\ 0 & -Cx_1 & 0\end{bmatrix}$$

where C is a constant. Consider that the body is free of body force.

a) Show whether the body is in balance or not;

Solution:

a) The continuous medium is in equilibrium if the following equations hold:

$$\nabla\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\vec{\mathbf{0}} \quad ; \quad \sigma_{ij,j}+\rho\,\mathrm{b}_i=\mathrm{0}_i \quad \text{(the equilibrium equations)} \tag{4.27}$$

For the proposed problem we have $\rho\,\mathrm{b}_i=\mathrm{0}_i$, and the vector $\sigma_{ij,j}$ is evaluated as follows:

$$\sigma_{ij,j}=\frac{\partial\sigma_{ij}}{\partial x_j}=\frac{\partial\sigma_{i1}}{\partial x_1}+\frac{\partial\sigma_{i2}}{\partial x_2}+\frac{\partial\sigma_{i3}}{\partial x_3}\Rightarrow\begin{cases}\dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma_{12}}{\partial x_2}+\dfrac{\partial\sigma_{13}}{\partial x_3}\\ \dfrac{\partial\sigma_{21}}{\partial x_1}+\dfrac{\partial\sigma_{22}}{\partial x_2}+\dfrac{\partial\sigma_{23}}{\partial x_3}\\ \dfrac{\partial\sigma_{31}}{\partial x_1}+\dfrac{\partial\sigma_{32}}{\partial x_2}+\dfrac{\partial\sigma_{33}}{\partial x_3}\end{cases}\begin{cases}i=1\Rightarrow 0+0+0=0 & \Rightarrow\quad \sigma_{1j,j}=0\\ i=2\Rightarrow 0+0+0=0 & \Rightarrow\quad \sigma_{2j,j}=0\\ i=3\Rightarrow 0+0+0=0 & \Rightarrow\quad \sigma_{3j,j}=0\end{cases}$$

with that we can conclude that $\sigma_{ij,j}=0_i$ then the body is in equilibrium.

Problem 4.19

Considering the principle of conservation of angular momentum, show that:

$$\int_V \rho[(\vec{x}\otimes(\vec{a}-\vec{\mathbf{b}})-(\vec{a}-\vec{\mathbf{b}})\otimes\vec{x}]\,dV=\int_{S_{\boldsymbol{\sigma}}}[(\vec{x}\otimes\vec{\mathbf{t}}^*-\vec{\mathbf{t}}^*\otimes\vec{x}]\,dS$$

where $\vec{x}$ is the vector position, $\rho(\vec{x},t)$ is the mass density, $\vec{a}(\vec{x},t)$ is the acceleration, $\vec{\mathbf{b}}(\vec{x},t)$ is the specific body force (per unit mass), and $\vec{\mathbf{t}}^*(\vec{x},t)$ is the prescribed traction vector (surface force) on surface $S_{\boldsymbol{\sigma}}$.

Solution:

The principle of conservation of angular momentum states that:

$$\int_{S_{\boldsymbol{\sigma}}} (\vec{x} \wedge \vec{\mathbf{t}}^*) dS + \int_V (\vec{x} \wedge \rho \vec{\mathbf{b}}) dV = \frac{D}{Dt} \int_V (\vec{x} \wedge \rho\, \vec{v}) dV = \int_V (\vec{x} \wedge \rho\, \vec{a}) dV$$

Then, we apply the cross product of the above equation with an arbitrary vector $\vec{z}$, which is independent of $\vec{x}$, and we obtain:

$$\vec{z} \wedge \int_V (\vec{x} \wedge \rho\, \vec{a}) dV = \vec{z} \wedge \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \wedge \vec{\mathbf{t}}^*) dS + \vec{z} \wedge \int_V (\vec{x} \wedge \rho \vec{\mathbf{b}}) dV$$
$$\Rightarrow \int_V \vec{z} \wedge (\vec{x} \wedge \rho\, \vec{a}) dV = \int_{S_{\boldsymbol{\sigma}}} \vec{z} \wedge (\vec{x} \wedge \vec{\mathbf{t}}^*) dS + \int_V \vec{z} \wedge (\vec{x} \wedge \rho \vec{\mathbf{b}}) dV$$

We have shown in Chapter 1 that given three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, the relationship $\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{b}} \otimes \vec{\mathbf{c}} - \vec{\mathbf{c}} \otimes \vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}$ holds, (see **Problem 1.17**). Then, the above equation can be rewritten as follows:

$$\int_V (\vec{x} \otimes \rho\, \vec{a} - \rho\, \vec{a} \otimes \vec{x}) \cdot \vec{z}\, dV = \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* - \vec{\mathbf{t}}^* \otimes \vec{x}) \cdot \vec{z}\, dS + \int_V (\vec{x} \otimes \rho \vec{\mathbf{b}} - \rho \vec{\mathbf{b}} \otimes \vec{x}) \cdot \vec{z}\, dV$$
$$\Rightarrow \int_V \rho\, (\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}) \cdot \vec{z}\, dV - \int_V \rho (\vec{x} \otimes \vec{\mathbf{b}} - \vec{\mathbf{b}} \otimes \vec{x}) \cdot \vec{z}\, dV = \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* - \vec{\mathbf{t}}^* \otimes \vec{x}) \cdot \vec{z}\, dS$$
$$\Rightarrow \int_V \rho \left[\vec{x} \otimes (\vec{a} - \vec{\mathbf{b}}) - (\vec{a} - \vec{\mathbf{b}}) \otimes \vec{x} \right] \cdot \vec{z}\, dV = \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* - \vec{\mathbf{t}}^* \otimes \vec{x}) \cdot \vec{z}\, dS$$
$$\Rightarrow \left\{ \int_V \rho \left[\vec{x} \otimes (\vec{a} - \vec{\mathbf{b}}) - (\vec{a} - \vec{\mathbf{b}}) \otimes \vec{x} \right] dV \right\} \cdot \vec{z} = \left\{ \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* - \vec{\mathbf{t}}^* \otimes \vec{x})\, dS \right\} \cdot \vec{z}$$

with that we can conclude that:

$$\int_V \rho\, [\vec{x} \otimes (\vec{a} - \vec{\mathbf{b}}) - (\vec{a} - \vec{\mathbf{b}}) \otimes \vec{x}]\, dV = \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* - \vec{\mathbf{t}}^* \otimes \vec{x})\, dS$$

Problem 4.20

1) Considering the definition of the *mean stress tensor* ($\overline{\boldsymbol{\sigma}}$):

$$V\, \overline{\boldsymbol{\sigma}} = \int_V \boldsymbol{\sigma}\, dV$$

and based on the principle that the continuum is in static equilibrium, show that:

$$\overline{\boldsymbol{\sigma}} = \frac{1}{2V} \int_V \rho\, [\vec{x} \otimes \vec{\mathbf{b}} + \vec{\mathbf{b}} \otimes \vec{x}]\, dV + \frac{1}{2V} \int_{S_{\boldsymbol{\sigma}}} (\vec{x} \otimes \vec{\mathbf{t}}^* + \vec{\mathbf{t}}^* \otimes \vec{x})\, dS$$

2) Considering that the volume can be decomposed by $V = V^{(1)} - V^{(2)}$, (see Figure 4.4), and by considering that the continuum is subjected to pressure $p^{(1)}$ on surface $S^{(1)}$ and to pressure $p^{(2)}$ on surface $S^{(2)}$, show that:

$$\overline{\boldsymbol{\sigma}} = \frac{-1}{(V^{(1)} - V^{(2)})} (p^{(1)} V^{(1)} - p^{(2)} V^{(2)}) \mathbf{1}$$

Consider that the continuum is free of body forces.

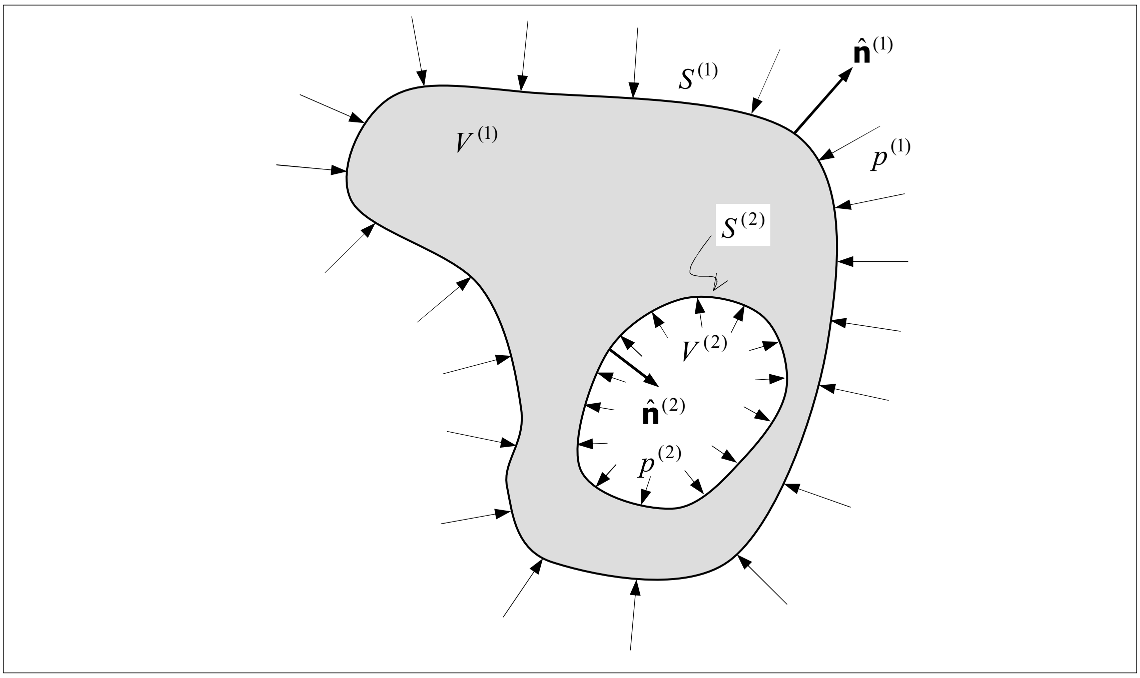


Figure 4.4

Solution:

Taking into account the equilibrium equations $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \vec{a} = \vec{\mathbf{0}}$ (the principle of conservation of linear momentum) for the whole continuum, it must fulfill that:

$$\int_V \vec{x} \otimes [\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}}] \, dV = \vec{\mathbf{0}} \quad \Rightarrow \quad \int_V \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \, dV + \int_V \vec{x} \otimes \rho \vec{\mathbf{b}} \, dV = \vec{\mathbf{0}} \tag{4.28}$$

In Chapter 1, (see **Problem 1.128**), we have shown that the following holds:

$$\int_V (\nabla \cdot \boldsymbol{\sigma}) \otimes \vec{x} \, dV = \int_S (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \otimes \vec{x} \, dS - \int_V \boldsymbol{\sigma} \, dV = \int_S \vec{\mathbf{t}}^* \otimes \vec{x} \, dS - \int_V \boldsymbol{\sigma} \, dV \tag{4.29}$$

$$\int_V \vec{x} \otimes (\nabla \cdot \boldsymbol{\sigma}) \, dV = \int_S \vec{x} \otimes (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \, dS - \int_V \boldsymbol{\sigma}^T \, dV = \int_S \vec{x} \otimes \vec{\mathbf{t}}^* \, dS - \int_V \boldsymbol{\sigma}^T \, dV \tag{4.30}$$

where we have considered the prescribed traction vector $\vec{\mathbf{t}}^* = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$. By substituting (4.30) into the equation (4.28), we can obtain:

$$\begin{aligned} &\int_V \vec{x} \otimes (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \, dV + \int_V \vec{x} \otimes \rho \vec{\mathbf{b}} \, dV = \vec{\mathbf{0}} \Rightarrow \int_S \vec{x} \otimes \vec{\mathbf{t}}^* \, dS - \int_V \boldsymbol{\sigma}^T \, dV + \int_V \vec{x} \otimes \rho \vec{\mathbf{b}} \, dV = \vec{\mathbf{0}} \\ &\Rightarrow \int_V \boldsymbol{\sigma}^T \, dV = \int_S \vec{x} \otimes \vec{\mathbf{t}}^* \, dS + \int_V \vec{x} \otimes \rho \vec{\mathbf{b}} \, dV \end{aligned} \tag{4.31}$$

Then, the following is true:

$$\int_V \boldsymbol{\sigma} \, dV = \int_S \vec{\mathbf{t}}^* \otimes \vec{x} \, dS + \int_V \rho \vec{\mathbf{b}} \otimes \vec{x} \, dV \tag{4.32}$$

Note that the tensors $\vec{x} \otimes \vec{\mathbf{t}}^*$ and $\vec{x} \otimes \rho \vec{\mathbf{b}}$ are not symmetric. This means that the equation in (4.28) does not take in account the principle of conservation of angular momentum, i.e. the symmetry of the Cauchy stress tensor. To guarantee the symmetry of $\boldsymbol{\sigma}$ we do:

$$\begin{aligned}&\int_V \frac{\boldsymbol{\sigma}+\boldsymbol{\sigma}^T}{2}\,dV = \frac{1}{2}\left[\int_S \vec{\mathbf{t}}^* \otimes \vec{x}\,dS + \int_V \rho\vec{\mathbf{b}}\otimes\vec{x}\;dV\right] + \frac{1}{2}\left[\int_S \vec{x}\otimes\vec{\mathbf{t}}^*\,dS + \int_V \vec{x}\otimes\rho\vec{\mathbf{b}}\;dV\right]\\ &\Rightarrow \int_V \boldsymbol{\sigma}^{sym}\,dV = \frac{1}{2}\int_V \rho[\vec{x}\otimes\vec{\mathbf{b}}+\vec{\mathbf{b}}\otimes\vec{x}]\,dV + \frac{1}{2}\int_S [\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS\end{aligned} \tag{4.33}$$

By considering the definition of the mean stress tensor, we can conclude:

$$\begin{aligned}&\int_V \frac{\boldsymbol{\sigma}+\boldsymbol{\sigma}^T}{2}\,dV = \frac{1}{2}\left[\int_S \vec{\mathbf{t}}^* \otimes \vec{x}\,dS + \int_V \rho\vec{\mathbf{b}}\otimes\vec{x}\;dV\right] + \frac{1}{2}\left[\int_S \vec{x}\otimes\vec{\mathbf{t}}^*\,dS + \int_V \vec{x}\otimes\rho\vec{\mathbf{b}}\;dV\right]\\ &\Rightarrow \int_V \boldsymbol{\sigma}^{sym}\,dV = \frac{1}{2}\int_V \rho[\vec{x}\otimes\vec{\mathbf{b}}+\vec{\mathbf{b}}\otimes\vec{x}]\,dV + \frac{1}{2}\int_S [\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS\\ &\Rightarrow V\,\overline{\boldsymbol{\sigma}} = \frac{1}{2}\int_V \rho[\vec{x}\otimes\vec{\mathbf{b}}+\vec{\mathbf{b}}\otimes\vec{x}]\,dV + \frac{1}{2}\int_S [\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS\\ &\Rightarrow \overline{\boldsymbol{\sigma}} = \frac{1}{2V}\int_V \rho[\vec{x}\otimes\vec{\mathbf{b}}+\vec{\mathbf{b}}\otimes\vec{x}]\,dV + \frac{1}{2V}\int_S [\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS\end{aligned} \tag{4.34}$$

In addition, if we consider that the body is free of body force, the above equation becomes:

$$\overline{\boldsymbol{\sigma}} = \frac{1}{2V}\int_S [\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS \tag{4.35}$$

For the particular case described in Figure 4.4 we have $V = V^{(1)} - V^{(2)}$, $S = S^{(1)} + S^{(2)}$, $\vec{\mathbf{t}}^{*(1)} = -p^{(1)}\hat{\mathbf{n}}^{(1)}$ and $\vec{\mathbf{t}}^{*(2)} = -p^{(2)}\hat{\mathbf{n}}^{(2)}$. In this case, the equation (4.35) becomes:

$$\begin{aligned}\overline{\boldsymbol{\sigma}} &= \frac{1}{2(V^{(1)}-V^{(2)})}\left\{\int_{S^{(1)}}[\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS^{(1)} + \int_{S^{(2)}}[\vec{x}\otimes\vec{\mathbf{t}}^* + \vec{\mathbf{t}}^*\otimes\vec{x}]\,dS^{(2)}\right\}\\ &= \frac{1}{2(V^{(1)}-V^{(2)})}\left\{\int_{S^{(1)}} -p^{(1)}[\vec{x}\otimes\hat{\mathbf{n}}^{(1)} + \hat{\mathbf{n}}^{(1)}\otimes\vec{x}]\,dS^{(1)} + \int_{S^{(2)}} -p^{(2)}[\vec{x}\otimes\hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(2)}\otimes\vec{x}]\,dS^{(2)}\right\}\\ &= \frac{-1}{2(V^{(1)}-V^{(2)})}\left\{p^{(1)}\int_{S^{(1)}}[\vec{x}\otimes\hat{\mathbf{n}}^{(1)} + \hat{\mathbf{n}}^{(1)}\otimes\vec{x}]\,dS^{(1)} + p^{(2)}\int_{S^{(2)}}[\vec{x}\otimes\hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(2)}\otimes\vec{x}]\,dS^{(2)}\right\}\end{aligned}$$

We have shown in Chapter 1 that is true $\int_S (\vec{x}\otimes\hat{\mathbf{n}} + \hat{\mathbf{n}}\otimes\vec{x})\,dS = 2V\mathbf{1}$, where $\hat{\mathbf{n}}$ is the outward unit normal to surface S, (see **Problem 1.128**). For this example, $\hat{\mathbf{n}}^{(2)}$ is the inward unit normal to surface $S^{(2)}$, then, we have $\int_{S^{(2)}}[\vec{x}\otimes\hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(2)}\otimes\vec{x}]\,dS^{(2)} = -2V^{(2)}\mathbf{1}$, with that we can obtain:

$$\begin{aligned}\overline{\boldsymbol{\sigma}} &= \frac{-1}{2(V^{(1)}-V^{(2)})}\left\{p^{(1)}\int_{S^{(1)}}[\vec{x}\otimes\hat{\mathbf{n}}^{(1)} + \hat{\mathbf{n}}^{(1)}\otimes\vec{x}]\,dS^{(1)} + p^{(2)}\int_{S^{(2)}}[\vec{x}\otimes\hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(2)}\otimes\vec{x}]\,dS^{(2)}\right\}\\ &= \frac{-1}{2(V^{(1)}-V^{(2)})}\left\{p^{(1)}2V^{(1)}\mathbf{1} - p^{(2)}2V^{(2)}\mathbf{1}\right\} = \frac{-1}{(V^{(1)}-V^{(2)})}\left\{p^{(1)}V^{(1)} - p^{(2)}V^{(2)}\right\}\mathbf{1}\end{aligned}$$

Problem 4.21

Starting from $\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$, show that the energy equation can also be written as follows:

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(u + \frac{1}{2}v^2 \right) &= \nabla_{\vec{x}} \cdot (\vec{v} \cdot \boldsymbol{\sigma}) + \rho \vec{\mathbf{b}} \cdot \vec{v} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \\
\rho \frac{D}{Dt}\left(u + \frac{1}{2}v^2 \right) &= (v_j \sigma_{ji})_{,i} + \rho\, \mathrm{b}_i v_i - \mathrm{q}_{i,i} + \rho r
\end{aligned}
\qquad (4.36)
$$

or

$$
\begin{aligned}
\rho \frac{\partial}{\partial t}\left(u + \frac{1}{2}v^2 \right) + \rho \left[\nabla_{\vec{x}} \left(u + \frac{1}{2}v^2 \right) \right] \cdot \vec{v} &= \nabla_{\vec{x}} \cdot (\vec{v} \cdot \boldsymbol{\sigma}) + \rho \vec{\mathbf{b}} \cdot \vec{v} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \\
\rho \frac{\partial}{\partial t}\left(u + \frac{1}{2}v^2 \right) + \rho \left(u + \frac{1}{2}v^2 \right)_{,i} v_i &= (v_j \sigma_{ji})_{,i} + \rho\, \mathrm{b}_i v_i - \mathrm{q}_{i,i} + \rho r
\end{aligned}
\qquad (4.37)
$$

or

$$
\begin{aligned}
\rho \frac{\partial}{\partial t}\left(u + \frac{1}{2}v^2 \right) + \nabla_{\vec{x}} \cdot \left[\rho \left(u + \frac{1}{2}v^2 \right) \vec{v} \right] &= \nabla_{\vec{x}} \cdot (\vec{v} \cdot \boldsymbol{\sigma}) + \rho \vec{\mathbf{b}} \cdot \vec{v} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \\
\frac{\partial}{\partial t}\left[\rho \left(u + \frac{1}{2}v^2 \right) \right] + \left[\rho \left(u + \frac{1}{2}v^2 \right) v_i \right]_{,i} &= (v_j \sigma_{ji})_{,i} + \rho\, \mathrm{b}_i v_i - \mathrm{q}_{i,i} + \rho r
\end{aligned}
\qquad (4.38)
$$

where ρ is the mass density, u is specific internal energy, v is magnitude of the velocity ($v^2 = \|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$), $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\vec{\mathbf{b}}$ is the specific body force (per unit mass), $\vec{\mathbf{q}}$ is the flux vector, r is the radiant heat constant (also called the heat source).

Solution:

Taking into account the energy equation:

$$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \qquad \Big| \qquad \rho \dot{u} = \sigma_{ij} \mathrm{D}_{ij} - \mathrm{q}_{i,i} + \rho r$$

where $\mathbf{D}$ is the rate-of-deformation tensor which is the symmetric part of the spatial velocity gradient ($\mathbf{D} = (\nabla_{\vec{x}} \vec{v})^{sym} \equiv \boldsymbol{\ell}^{sym}$). Note also that $\boldsymbol{\sigma} : \mathbf{D} = \boldsymbol{\sigma} : \boldsymbol{\ell}^{sym} = \boldsymbol{\sigma} : \boldsymbol{\ell}$ since the double scalar between symmetric ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$) and antisymmetric tensor ($\boldsymbol{\ell}^{skew}$) is zero, i.e. $\boldsymbol{\sigma} : \boldsymbol{\ell}^{skew} = 0$, thus

$$\sigma_{ij} \mathrm{D}_{ij} = \sigma_{ij} (\boldsymbol{\ell})_{ij} = \sigma_{ij} (\nabla_{\vec{x}} \vec{v})_{ij} = \sigma_{ij} v_{i,j}$$

Note also that $(\sigma_{ij} v_i)_{,j} = \sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} \Rightarrow \sigma_{ij} v_{i,j} = (\sigma_{ij} v_i)_{,j} - \sigma_{ij,j} v_i = \sigma_{ij} \mathrm{D}_{ij}$, thus the energy equation becomes:

$$
\begin{aligned}
&\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \\
&\Rightarrow \rho \dot{u} = \nabla_{\vec{x}} \cdot (\vec{v} \cdot \boldsymbol{\sigma}) - (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \cdot \vec{v} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r
\end{aligned}
\qquad \Bigg| \qquad
\begin{aligned}
&\rho \dot{u} = \sigma_{ij} \mathrm{D}_{ij} - \mathrm{q}_{i,i} + \rho r \\
&\Rightarrow \rho \dot{u} = (\sigma_{ij} v_i)_{,j} - \sigma_{ij,j} v_i - \mathrm{q}_{i,i} + \rho r
\end{aligned}
$$

Taking into account the equations of motion we can obtain that:

$$
\begin{aligned}
&\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \ddot{\vec{\mathbf{u}}} = \rho \dot{\vec{v}} \\
&\Rightarrow \nabla_{\vec{x}} \cdot \boldsymbol{\sigma} = \rho \dot{\vec{v}} - \rho \vec{\mathbf{b}} \\
&\Rightarrow (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \cdot \vec{v} = \rho \dot{\vec{v}} \cdot \vec{v} - \rho \vec{\mathbf{b}} \cdot \vec{v}
\end{aligned}
\qquad \Bigg| \qquad
\begin{aligned}
&\sigma_{ij,j} + \rho\, \mathrm{b}_i = \rho\, \ddot{\mathrm{u}}_i = \rho \dot{v}_i \\
&\Rightarrow \sigma_{ij,j} = \rho \dot{v}_i - \rho\, \mathrm{b}_i \\
&\Rightarrow \sigma_{ij,j} v_i = \rho \dot{v}_i v_i - \rho\, \mathrm{b}_i v_i
\end{aligned}
$$

With that the energy equation can also be written as follows:

$$\rho\,\dot{u} = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) - (\nabla_{\vec{x}}\cdot\boldsymbol{\sigma})\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r \qquad \rho\,\dot{u} = \sigma_{ij}\mathsf{D}_{ij} - \mathsf{q}_{i,i} + \rho r$$

$$\rho\,\dot{u} = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) - (\rho\dot{\vec{v}}\cdot\vec{v} - \rho\vec{\mathbf{b}}\cdot\vec{v}) - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r \qquad \rho\,\dot{u} = (\sigma_{ij}v_i)_{,j} - (\rho\dot{v}_i v_i - \rho\,\mathsf{b}_i v_i) - \mathsf{q}_{i,i} + \rho r$$

$$\Rightarrow \rho\,\dot{u} + \rho\dot{\vec{v}}\cdot\vec{v} = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r \qquad \Rightarrow \rho\,\dot{u} + \rho\dot{v}_i v_i = (\sigma_{ij}v_i)_{,j} + \rho\,\mathsf{b}_i v_i - \mathsf{q}_{i,i} + \rho r$$

Note that $\frac{D}{Dt}(\vec{v}\cdot\vec{v}) = (\dot{\vec{v}}\cdot\vec{v}) + (\vec{v}\cdot\dot{\vec{v}}) = 2(\dot{\vec{v}}\cdot\vec{v}) \Rightarrow (\dot{\vec{v}}\cdot\vec{v}) = \frac{1}{2}\frac{D}{Dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\frac{D}{Dt}(v^2)$. Thus, the energy equation becomes

$$\begin{aligned}
&\Rightarrow \rho\,\dot{u} + \rho\dot{\vec{v}}\cdot\vec{v} = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r\\
&\Rightarrow \rho\left(\dot{u} + \frac{1}{2}\frac{D}{Dt}(v^2)\right) = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r\\
&\Rightarrow \rho\frac{D}{Dt}\left(u + \frac{1}{2}v^2\right) = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r
\end{aligned} \tag{4.39}$$

which in indicial notation becomes

$$\begin{aligned}
&\Rightarrow \rho\,\dot{u} + \rho\dot{v}_i v_i = (\sigma_{ij}v_i)_{,j} + \rho\,\mathsf{b}_i v_i - \mathsf{q}_{i,i} + \rho r\\
&\Rightarrow \rho\left(\frac{Du}{Dt} + \frac{1}{2}\frac{D}{Dt}(v^2)\right) = (\sigma_{ij}v_i)_{,j} + \rho\,\mathsf{b}_i v_i - \mathsf{q}_{i,i} + \rho r\\
&\Rightarrow \rho\frac{D}{Dt}\left(u + \frac{1}{2}v^2\right) = (\sigma_{ij}v_i)_{,j} + \rho\,\mathsf{b}_i v_i - \mathsf{q}_{i,i} + \rho r
\end{aligned}$$

with which we show the equation in (4.36). The equation in (4.37) can be easily obtained if we apply the material time derivative $\frac{D\bullet}{Dt} \equiv \dot{\bullet} = \frac{\partial\bullet}{\partial t} + (\nabla_{\vec{x}}\bullet)\cdot\vec{v}$ to the equation in (4.39), i.e.:

$$\begin{aligned}
&\rho\frac{D}{Dt}\left(u + \frac{1}{2}v^2\right) = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r\\
&\Rightarrow \rho\frac{\partial}{\partial t}\left(u + \frac{1}{2}v^2\right) + \rho\left[\nabla_{\vec{x}}\left(u + \frac{1}{2}v^2\right)\right]\cdot\vec{v} = \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) + \rho\vec{\mathbf{b}}\cdot\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}} + \rho r
\end{aligned}$$

In **Problem 4.8** we have shown that $\rho\frac{\partial\phi}{\partial t} + \rho(\nabla_{\vec{x}}\phi)\cdot\vec{v} = \frac{\partial}{\partial t}(\rho\phi) + \nabla_{\vec{x}}\cdot(\rho\phi\vec{v})$ holds, and if we consider that $\phi = \left(u + \frac{1}{2}v^2\right)$ we show the equation in (4.38).

4.2 Some Useful Concepts for the Classical "Mechanics of Materials"

Problem 4.22

Consider a prismatic bar, (see Figure 4.5), in which the Cauchy stress field ($\boldsymbol{\sigma} = \boldsymbol{\sigma}(\vec{x})$) Cartesian components are:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{4.40}$$

Obtain the resultant force and momentum acting on the cross section A. The cross section is characterized by the unit vector $\hat{\mathsf{n}}_i = [1 \quad 0 \quad 0]$, in which it is also subjected to the stress state described in (4.40). Adopt the Cartesian system $O\vec{x}$ described in Figure 4.5.

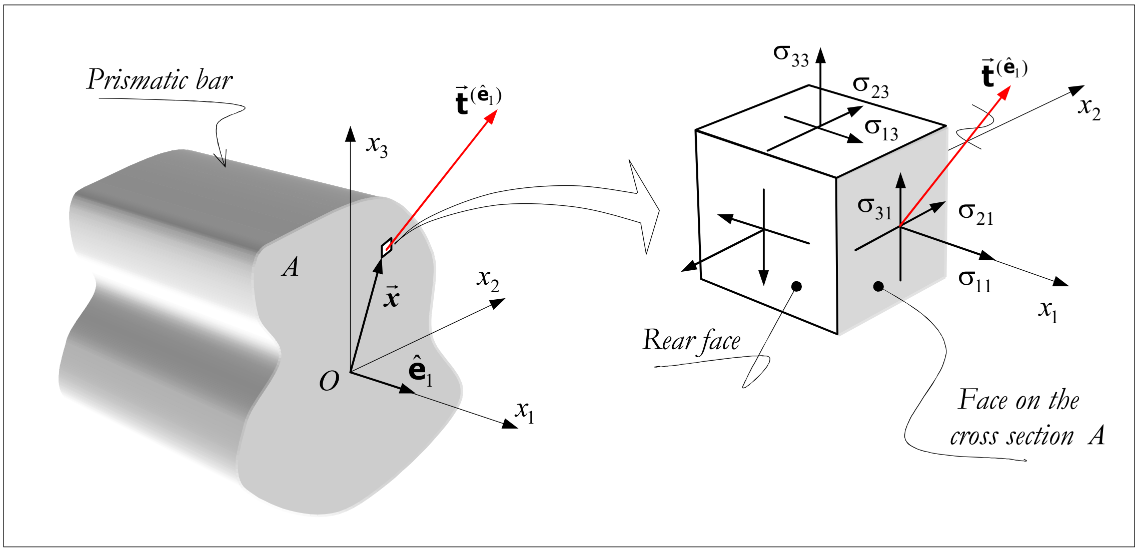


Figure 4.5: Stress field on the cross section A.

Solution:

The traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is related to the Cauchy stress tensor as follows $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$. For this problem we have $\hat{\mathbf{n}} = \hat{\mathbf{e}}_1$, thus:

$$\mathrm{t}_i^{(\hat{\mathbf{e}}_1)} = \sigma_{ij}\hat{\mathrm{n}}_j = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} \\ \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} \\ \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} \end{bmatrix} \tag{4.41}$$

The Resultant Forces

$$\sum \vec{F} = \int_A \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)} dA \quad ; \quad \sum F_{x_i} = \int_A \mathrm{t}_i^{(\hat{\mathbf{e}}_1)} dA$$

Thus,

$$\sum F_{x_i} = \int_A \mathrm{t}_i^{(\hat{\mathbf{e}}_1)} dA \quad \Rightarrow \quad \begin{bmatrix} \sum F_{x_1} \\ \sum F_{x_2} \\ \sum F_{x_3} \end{bmatrix} \equiv \begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} = \begin{bmatrix} \int_A \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} dA \end{bmatrix} = \begin{bmatrix} \int_A \sigma_{11} dA \\ \int_A \sigma_{21} dA \\ \int_A \sigma_{31} dA \end{bmatrix} \tag{4.42}$$

The Resultant Moments

Resultant moment due to $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$ (related to the system $O - x_1 - x_2 - x_3$):

$$\sum \vec{M}_O = \int_A \vec{x} \wedge \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)} dA \quad ; \quad \sum M_i = \int_A \epsilon_{ijk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \tag{4.43}$$

The explicit form of the term $\epsilon_{ijk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)}$ is given by:

$$\begin{aligned} \epsilon_{ijk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} &= \epsilon_{ij1} x_j \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + \epsilon_{ij2} x_j \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} + \epsilon_{ij3} x_j \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} \\ &= \epsilon_{i21} x_2 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + \epsilon_{i31} x_3 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + \epsilon_{i12} x_1 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} + \epsilon_{i32} x_3 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} + \epsilon_{i13} x_1 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} + \epsilon_{i23} x_2 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} \end{aligned}$$

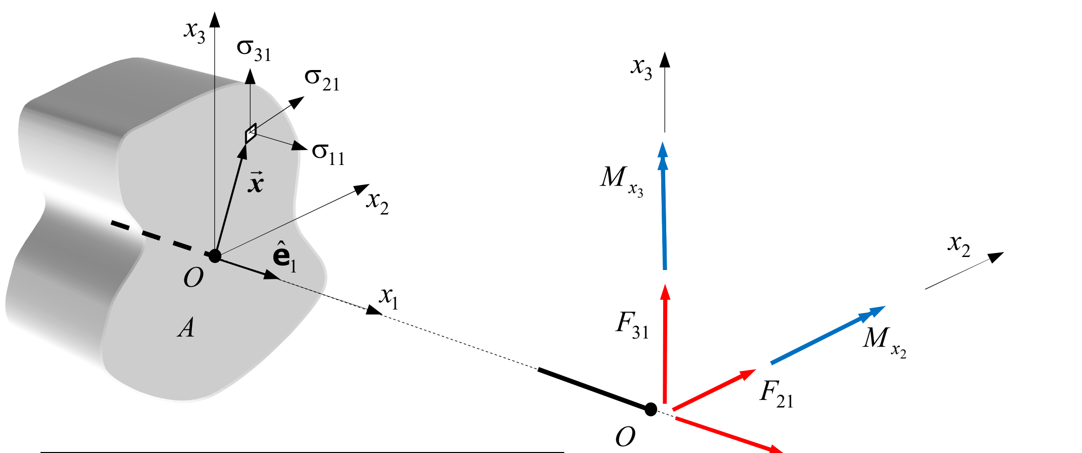
$$
\epsilon_{ijk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} \Rightarrow \begin{cases} (i=1) \Rightarrow \epsilon_{1jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} = \epsilon_{132} x_3 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} + \epsilon_{123} x_2 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} = -x_3 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} + x_2 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} \\ (i=2) \Rightarrow \epsilon_{1jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} = \epsilon_{231} x_3 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + \epsilon_{213} x_1 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} = x_3 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} - x_1 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} \\ (i=3) \Rightarrow \epsilon_{1jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} = \epsilon_{321} x_2 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + \epsilon_{312} x_1 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} = -x_2 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} + x_1 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} \end{cases}
$$

Then, the explicit components of (4.43) are, (see Figure 4.6):

$$
\sum M_i = \int_A \epsilon_{ijk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \quad \Rightarrow \quad \begin{bmatrix} \sum M_1 \\ \sum M_2 \\ \sum M_3 \end{bmatrix} \equiv \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} = \begin{bmatrix} \int_A \epsilon_{1jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \epsilon_{2jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \epsilon_{3jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} = \begin{bmatrix} \int_A \epsilon_{1jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \epsilon_{2jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \\ \int_A \epsilon_{3jk} x_j \mathrm{t}_k^{(\hat{\mathbf{e}}_1)} dA \end{bmatrix} = \begin{bmatrix} \int_A (x_2 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)} - x_3 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)}) dA \\ \int_A (x_3 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)} - x_1 \mathrm{t}_3^{(\hat{\mathbf{e}}_1)}) dA \\ \int_A (x_1 \mathrm{t}_2^{(\hat{\mathbf{e}}_1)} - x_2 \mathrm{t}_1^{(\hat{\mathbf{e}}_1)}) dA \end{bmatrix} = \begin{bmatrix} \int_A (x_2 \sigma_{31} - x_3 \sigma_{21}) dA \\ \int_A (x_3 \sigma_{11} - x_1 \sigma_{31}) dA \\ \int_A (x_1 \sigma_{21} - x_2 \sigma_{11}) dA \end{bmatrix} \tag{4.44}
$$

Note that the equations (4.42) and (4.44) are valid if the section is defined by a plane otherwise these equations are no longer valid.



$$
\begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} = \begin{bmatrix} \int_A \sigma_{11} dA \\ \int_A \sigma_{21} dA \\ \int_A \sigma_{31} dA \end{bmatrix} \quad ; \quad \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} = \begin{bmatrix} \int_A (x_2 \sigma_{31} - x_3 \sigma_{21}) dA \\ \int_A (x_3 \sigma_{11} - x_1 \sigma_{31}) dA \\ \int_A (x_1 \sigma_{21} - x_2 \sigma_{11}) dA \end{bmatrix}
$$

Figure 4.6: Resultant force and moment on cross section A.

If we are using Engineering notation the equations for forces and moments are represented by:

$$\begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} \equiv \begin{bmatrix} N \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \int_A \sigma_{11} dA \\ \int_A \sigma_{21} dA \\ \int_A \sigma_{31} dA \end{bmatrix} \equiv \begin{bmatrix} \int_A \sigma_x dA \\ \int_A \tau_{xy} dA \\ \int_A \tau_{xz} dA \end{bmatrix} \quad ; \quad \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} \equiv \begin{bmatrix} M_T \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \int_A (y\tau_{xz} - z\tau_{xy}) dA \\ \int_A (z\sigma_x - x\tau_{xz}) dA \\ \int_A (x\tau_{xy} - y\sigma_x) dA \end{bmatrix}$$

NOTE 1: Note that, if the body is in equilibrium we have $\sum \vec{F} = \vec{\mathbf{0}}$ and $\sum \vec{M}_O = \vec{\mathbf{0}}$, so, in the cross section at the point O we have to apply forces and moments with the same magnitudes and directions but with opposite senses as those indicated in Figure 4.6, (see Figure 4.7).

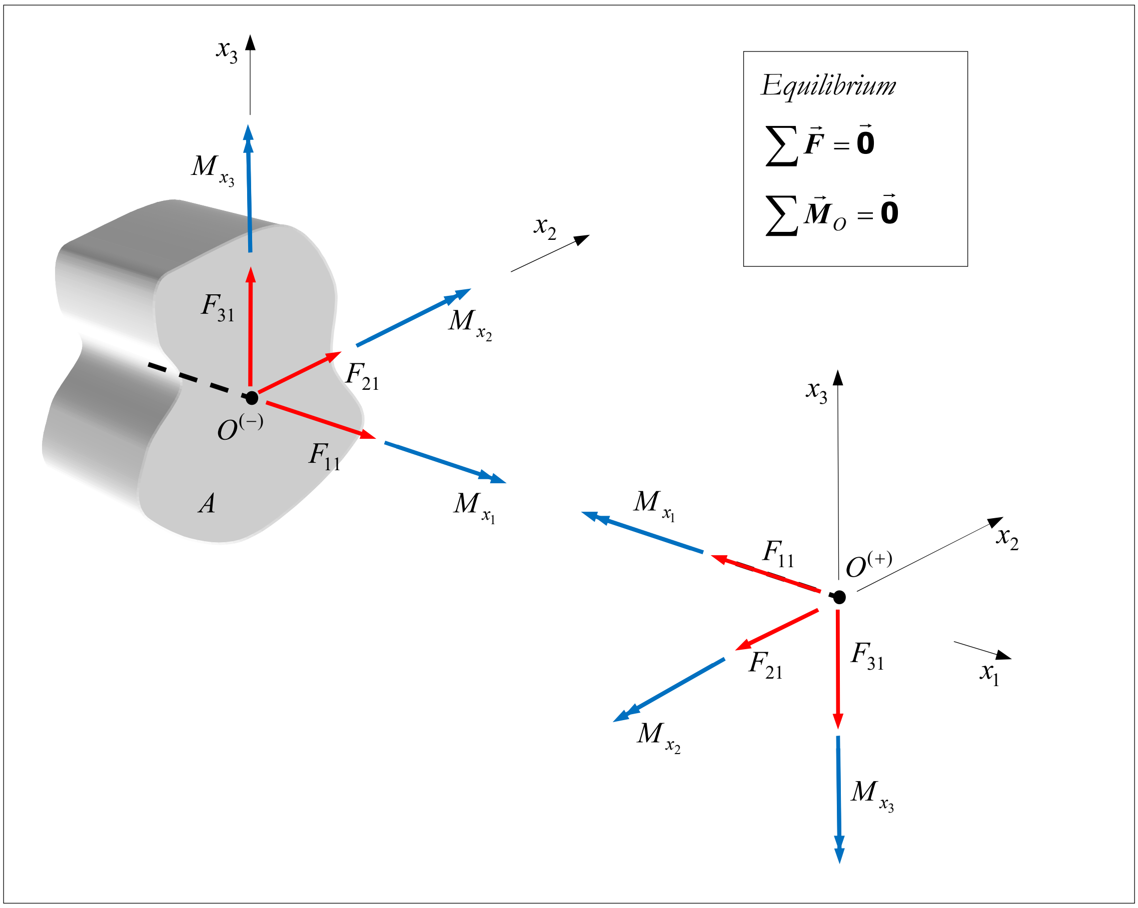


Figure 4.7: Cross section in equilibrium.

NOTE 2: Let us suppose now we have another system defined by $O\vec{x}'$, (see Figure 4.8), where it holds $\vec{x} = \vec{\bar{x}} + \vec{x}'$. Then, the resultant force and resultant moment at the new system are defined by:

The resultant forces

$$\sum F'_{x_i} = \int_A t_i^{(\hat{\mathbf{e}}_1)} dA \quad \Rightarrow \quad \begin{bmatrix} \sum F'_{x_1} \\ \sum F'_{x_2} \\ \sum F'_{x_3} \end{bmatrix} \equiv \begin{bmatrix} F'_{11} \\ F'_{21} \\ F'_{31} \end{bmatrix} = \begin{bmatrix} \int_A \sigma_{11} dA \\ \int_A \sigma_{21} dA \\ \int_A \sigma_{31} dA \end{bmatrix} = \begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix}$$

The resultant moments

$$\begin{bmatrix} M'_{x_1} \\ M'_{x_2} \\ M'_{x_3} \end{bmatrix} = \begin{bmatrix} \int_A (x'_2\sigma_{31} - x'_3\sigma_{21}) dA \\ \int_A (x'_3\sigma_{11} - x'_1\sigma_{31}) dA \\ \int_A (x'_1\sigma_{21} - x'_2\sigma_{11}) dA \end{bmatrix} = \begin{bmatrix} \int_A ((x_2 - \bar{\bar{x}}_2)\sigma_{31} - (x_3 - \bar{\bar{x}}_3)\sigma_{21}) dA \\ \int_A ((x_3 - \bar{\bar{x}}_3)\sigma_{11} - (x_1 - \bar{\bar{x}}_1)\sigma_{31}) dA \\ \int_A ((x_1 - \bar{\bar{x}}_1)\sigma_{21} - (x_2 - \bar{\bar{x}}_2)\sigma_{11}) dA \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} M'_{x_1} \\ M'_{x_2} \\ M'_{x_3} \end{bmatrix} = \begin{bmatrix} \int_A (x_2\sigma_{31} - x_3\sigma_{21}) dA - \bar{\bar{x}}_2 \int_A \sigma_{31} dA + \bar{\bar{x}}_3 \int_A \sigma_{21} dA \\ \int_A (x_3\sigma_{11} - x_1\sigma_{31}) dA - \bar{\bar{x}}_3 \int_A \sigma_{11} dA + \bar{\bar{x}}_1 \int_A \sigma_{31} dA \\ \int_A (x_1\sigma_{21} - x_2\sigma_{11}) dA - \bar{\bar{x}}_1 \int_A \sigma_{21} dA + \bar{\bar{x}}_2 \int_A \sigma_{11} dA \end{bmatrix} = \begin{bmatrix} M_{x_1} - \bar{\bar{x}}_2 F_{31} + \bar{\bar{x}}_3 F_{21} \\ M_{x_2} - \bar{\bar{x}}_3 F_{11} + \bar{\bar{x}}_1 F_{31} \\ M_{x_3} - \bar{\bar{x}}_1 F_{21} + \bar{\bar{x}}_2 F_{11} \end{bmatrix} \tag{4.45}$$

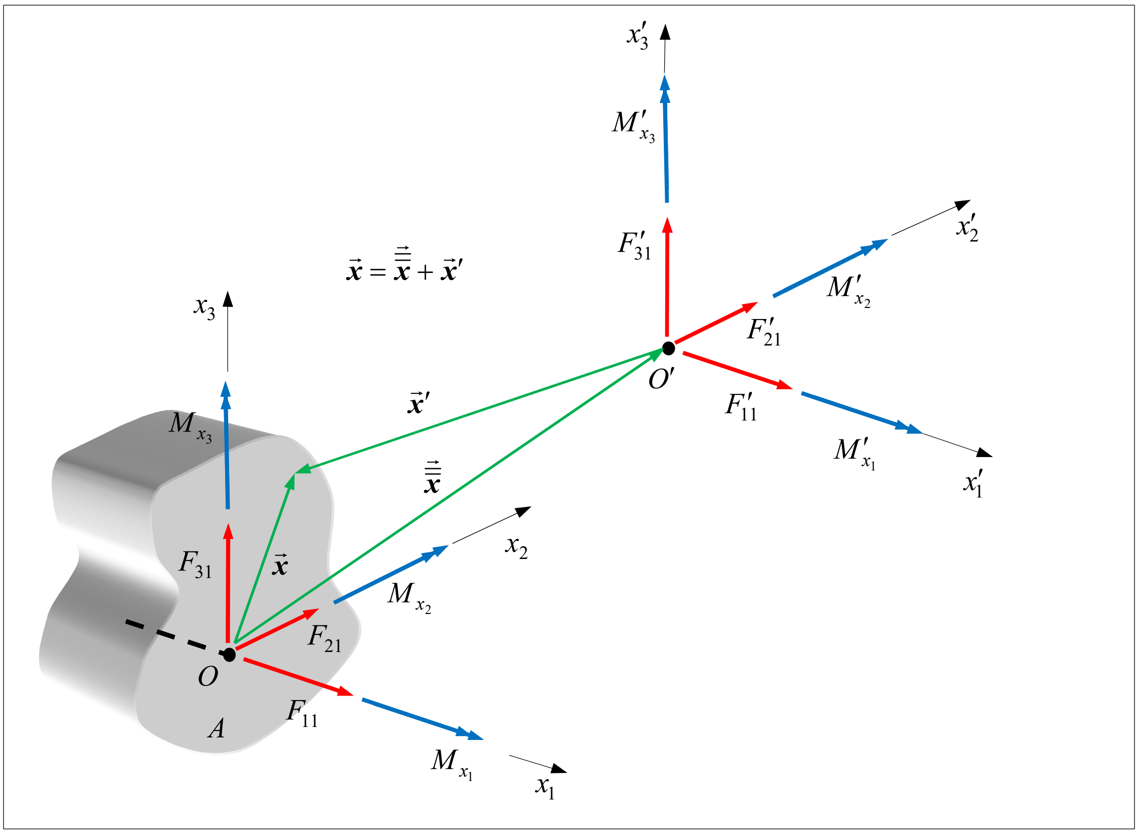


Figure 4.8

Note that, the result given by equation in (4.45) could also have been obtained by:

$$\vec{M}'_{O\vec{x}'} = \vec{M}_{O\vec{x}} + (-\vec{\bar{\bar{x}}}) \wedge \vec{F}_{O\vec{x}} = \vec{M}_{O\vec{x}} - \vec{\bar{\bar{x}}} \wedge \vec{F}_{O\vec{x}}$$

where the above vectors are defined in Figure 4.9. The term $\vec{\bar{\bar{x}}} \wedge \vec{F}_{O\vec{x}}$ can be evaluated as follows:

$$\vec{\bar{\bar{x}}} \wedge \vec{F}_{O\vec{x}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \bar{\bar{x}}_1 & \bar{\bar{x}}_2 & \bar{\bar{x}}_3 \\ F_{11} & F_{21} & F_{31} \end{vmatrix} = (\bar{\bar{x}}_2 F_{31} - \bar{\bar{x}}_3 F_{21})\hat{\mathbf{e}}_1 + (\bar{\bar{x}}_3 F_{11} - \bar{\bar{x}}_1 F_{31})\hat{\mathbf{e}}_2 + (\bar{\bar{x}}_1 F_{21} - \bar{\bar{x}}_2 F_{11})\hat{\mathbf{e}}_3$$

Then, the components of $\vec{M}'_{O\vec{x}'}$ are:

$$\left(\vec{M}'_{O\vec{x}'}\right)_i = \left(\vec{M}_{O\vec{x}} - \vec{\bar{\bar{x}}} \wedge \vec{F}_{O\vec{x}}\right)_i = \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} - \begin{bmatrix} \bar{\bar{x}}_2 F_{31} - \bar{\bar{x}}_3 F_{21} \\ \bar{\bar{x}}_3 F_{11} - \bar{\bar{x}}_1 F_{31} \\ \bar{\bar{x}}_1 F_{21} - \bar{\bar{x}}_2 F_{11} \end{bmatrix}$$

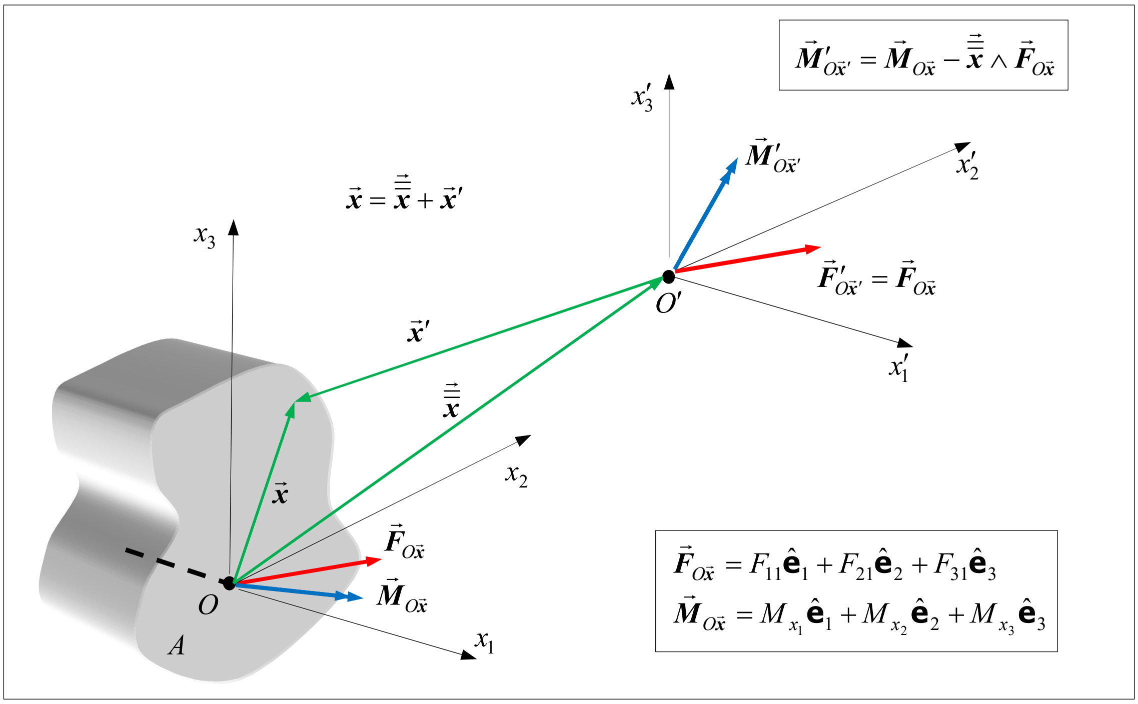


Figure 4.9

NOTE 3: In sight of Figure 4.9 there is a point O' in which $M'_{x_1} = 0$, this point is called *Shear Center (S.C.)* and fulfills that $M'_{x_1} = M_{x_1} - \bar{\bar{x}}_2 F_{31} + \bar{\bar{x}}_3 F_{21} = 0$. Note also that when $F_{21} = 0$ the center can be obtained by $M_{x_1} - \bar{\bar{x}}_2 F_{31} = 0 \Rightarrow \bar{\bar{x}}_2 = x_2^{(S.C.)} = \dfrac{M_{x_1}}{F_{31}}$ and when $F_{31} = 0$ we can obtain $\bar{\bar{x}}_3 = x_3^{(S.C.)} = \dfrac{-M_{x_1}}{F_{21}}$ and the point where these two lines intercept is the Shear Center, (see Figure 4.10). Note also that when F_{21} and F_{31} are applied at the shear center there is no torsion moment, i.e. the beam will only be subjected to flexural moments (M'_{x_2}, M'_{x_3}), and due to this reason the shear center is also called *Flexural Center*.

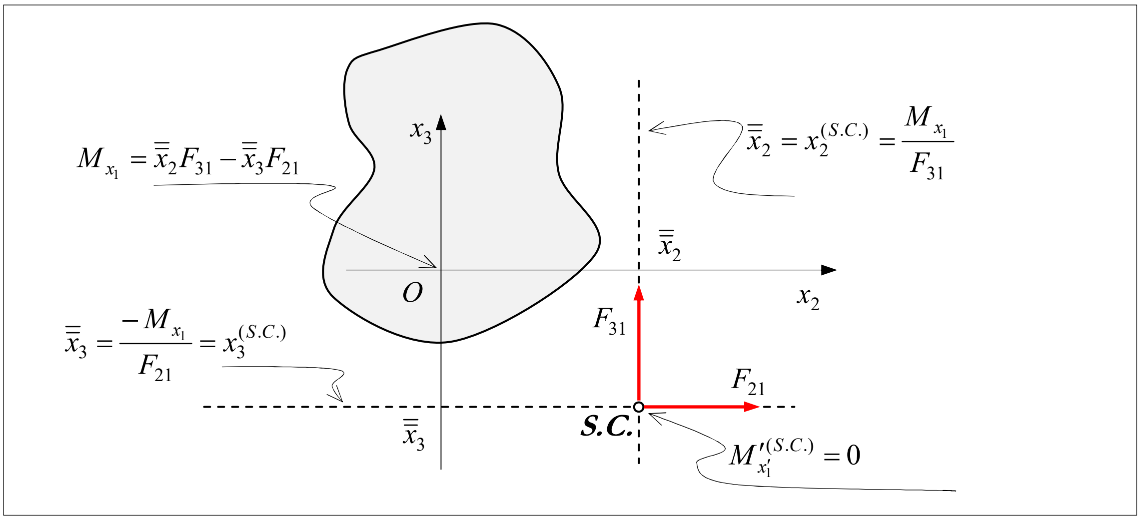


Figure 4.10: Shear center.

NOTE 4: Now instead of considering the stress in the infinitesimal element described in Figure 4.5, let us consider a differential element according to x_1-direction, (see Figure 4.3). Then, on the face $x_1 + dx_1$ we can calculate the resultant forces, (see equation (4.42)), and moments, (see equation (4.44)), by considering the stress distribution given in Figure 4.11.

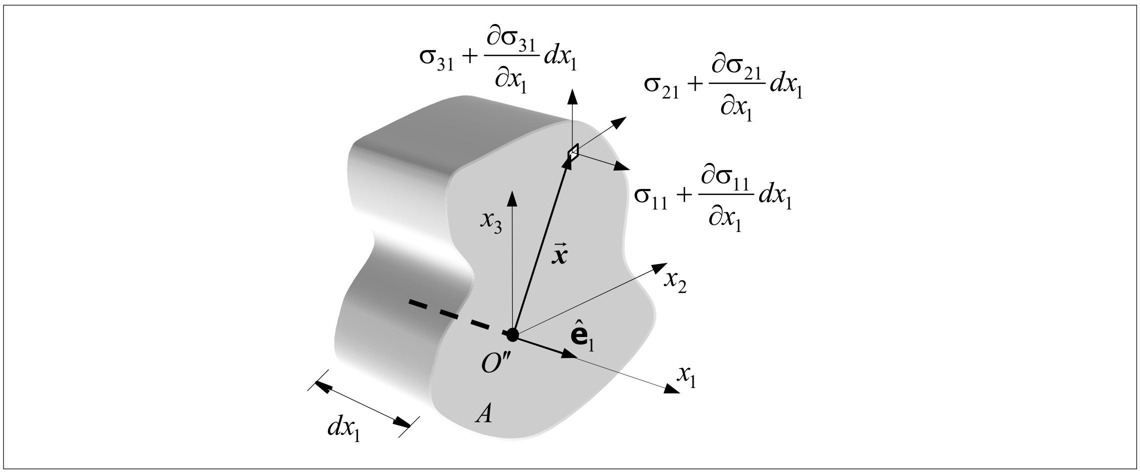


Figure 4.11: Stress distribution in the differential $x_1 + dx_1$.

And according to the Figure 4.11 we can obtain:

The resultant forces by means of the equation (4.42):

$$\begin{cases} F_{11+dx_1} = \int\limits_A \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) dA = \int\limits_A \sigma_{11} dA + \int\limits_A \left(\frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) dA = F_{11} + \frac{\partial}{\partial x_1} \left(\int\limits_A \sigma_{11} dA \right) dx_1 \\ F_{21+dx_1} = \int\limits_A \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} dx_1 \right) dA = \int\limits_A \sigma_{21} dA + \int\limits_A \left(\frac{\partial \sigma_{21}}{\partial x_1} dx_1 \right) dA = F_{21} + \frac{\partial}{\partial x_1} \left(\int\limits_A \sigma_{21} dA \right) dx_1 \\ F_{31+dx_1} = \int\limits_A \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_1} dx_1 \right) dA = \int\limits_A \sigma_{31} dA + \int\limits_A \left(\frac{\partial \sigma_{31}}{\partial x_1} dx_1 \right) dA = F_{31} + \frac{\partial}{\partial x_1} \left(\int\limits_A \sigma_{31} dA \right) dx_1 \end{cases}$$

with which we can conclude that

$$F_{11+dx_1} = F_{11} + \frac{\partial F_{11}}{\partial x_1} dx_1 \quad ; \quad F_{21+dx_1} = F_{21} + \frac{\partial F_{21}}{\partial x_1} dx_1 \quad ; \quad F_{31+dx_1} = F_{31} + \frac{\partial F_{31}}{\partial x_1} dx_1$$

By considering the definition (4.44), the resultant moments at the point O'', (see Figure 4.11), are:

$$\begin{aligned}
M_{x_1+dx_1} &= \int_A \left(x_2 \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_1} dx_1 \right) - x_3 \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} dx_1 \right) \right) dA \\
&= \int_A (x_2\sigma_{31} - x_3\sigma_{21}) dA + \frac{\partial}{\partial x_1} \left(\int_A (x_2\sigma_{31} - x_3\sigma_{21}) dA \right) dx_1 = M_{x_1} + \frac{\partial M_{x_1}}{\partial x_1} dx_1 \\
M_{x_2+dx_1} &= \int_A \left(x_3 \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) - x_1 \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_1} dx_1 \right) \right) dA \\
&= \int_A (x_3\sigma_{11} - x_1\sigma_{31}) dA + \frac{\partial}{\partial x_1} \left(\int_A (x_3\sigma_{11} - x_1\sigma_{31}) dA \right) dx_1 = M_{x_2} + \frac{\partial M_{x_2}}{\partial x_1} dx_1 \\
M_{x_3+dx_1} &= \int_A \left(x_1 \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} dx_1 \right) - x_2 \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) \right) dA \\
&= \int_A (x_1\sigma_{21} - x_2\sigma_{11}) dA + \frac{\partial}{\partial x_1} \left(\int_A (x_1\sigma_{21} - x_2\sigma_{11}) dA \right) dx_1 = M_{x_3} + \frac{\partial M_{x_3}}{\partial x_1} dx_1
\end{aligned}$$

If we take into account the differential element dx_1, (see Figure 4.12), and by means of $\sum \vec{\boldsymbol{M}}_{O''} = \vec{\boldsymbol{0}}$ we can obtain that:

According to x_1-direction:

$$\sum M_{O''x_1} = 0 \quad \Rightarrow \quad M_{x_1+dx_1} - M_{x_1} = M_{x_1} + \frac{\partial M_{x_1}}{\partial x_1} dx_1 - M_{x_1} = 0$$
$$\Rightarrow \boxed{\frac{\partial M_{x_1}}{\partial x_1} = 0 \xrightarrow{\text{Engineering Notation}} \frac{\partial M_T}{\partial x} = 0} \tag{4.46}$$

According to x_2-direction:

$$\sum M_{O''x_2} = 0 \quad \Rightarrow \quad M_{x_2+dx_1} - M_{x_2} - F_{31}dx_1 = M_{x_2} + \frac{\partial M_{x_2}}{\partial x_1} dx_1 - M_{x_2} - F_{31}dx_1 = 0$$
$$\Rightarrow \boxed{\frac{\partial M_{x_2}}{\partial x_1} = F_{31} \xrightarrow{\text{Engineering Notation}} \frac{\partial M_y}{\partial x} = F_z} \tag{4.47}$$

According to x_3-direction:

$$\sum M_{O''x_3} = 0 \quad \Rightarrow \quad M_{x_3+dx_1} - M_{x_3} + F_{21}dx_1 = M_{x_3} + \frac{\partial M_{x_3}}{\partial x_1} dx_1 - M_{x_3} + F_{21}dx_1 = 0$$
$$\Rightarrow \boxed{\frac{\partial M_{x_3}}{\partial x_1} = -F_{21} \xrightarrow{\text{Engineering Notation}} \frac{\partial M_z}{\partial x} = -F_y} \tag{4.48}$$

That is, which cause the shearing force F_{31} is the variation of the moment M_{x_2} along x_1, and which cause the shearing force F_{21} is the variation of the moment M_{x_3} along x_1.

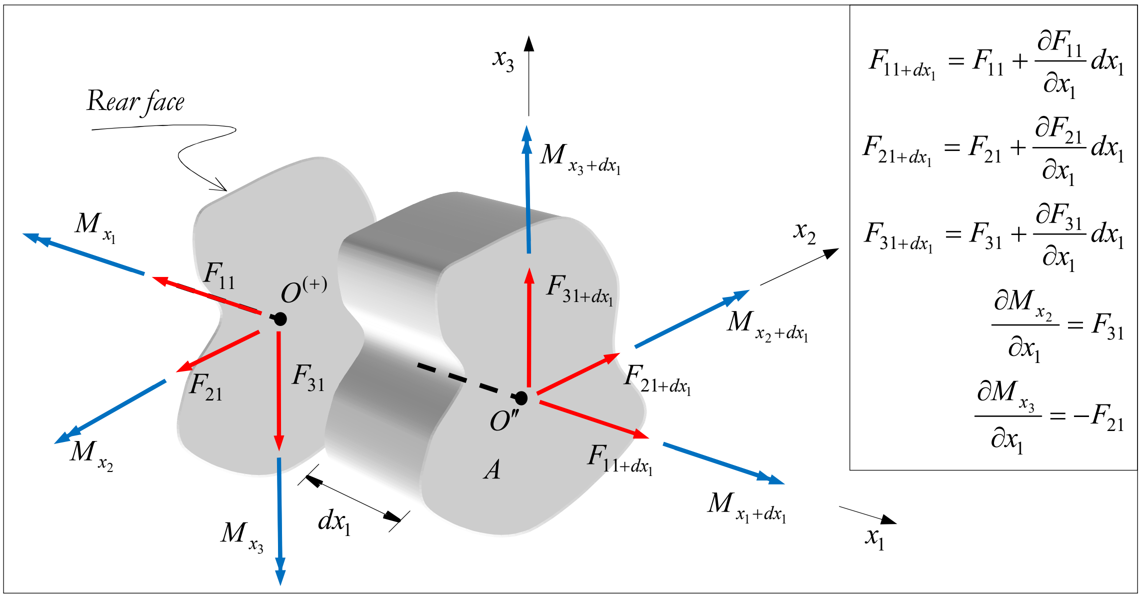


Figure 4.12: Resultant forces and moments in the differential dx_1.

Problem 4.23

Consider the problem established in **Problem 4.22**, (see Figure 4.5). Now let us suppose that the stress state on the cross section A is given by:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.49}$$

Express the normal stress field component $\sigma_{11}(x_2, x_3)$ in terms of resultant force and moment, (see equations (4.42) and (4.44)).

Hypothesis (approximation): Consider that the normal stress field on the cross section varies according to the plane equation.

Solution:

For this particular case, and according to the results established in **Problem 4.22**, we can conclude that:

$$\begin{bmatrix} F_{11} \\ F_{21} \\ F_{31} \end{bmatrix} = \begin{bmatrix} \int_A \sigma_{11} dA \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \end{bmatrix} = \begin{bmatrix} 0 \\ \int_A (x_3 \sigma_{11}) dA \\ -\int_A (x_2 \sigma_{11}) dA \end{bmatrix} \tag{4.50}$$

Since the stress $\sigma_{11}(x_2, x_3)$ varies according to the plane equation, we can adopt:

$$\sigma_{11}(x_2, x_3) = c_1 + c_2 x_2 + c_3 x_3$$

where c_1, c_2 and c_3 are constant to be determined. According to equations in (4.50) we can obtain:

$$\begin{aligned} F_{11} &= \int_A \sigma_{11} dA = \int_A (c_1 + c_2 x_2 + c_3 x_3) dA = c_1 \int_A dA + c_2 \int_A x_2 dA + c_3 \int_A x_3 dA \\ &= c_1 A + c_2 \bar{x}_2 A + c_3 \bar{x}_3 A \end{aligned} \tag{4.51}$$

where we have applied the definition of area centroid, (see Figure 4.13 and the Complementary NOTE 2 at the end of Chapter 1).

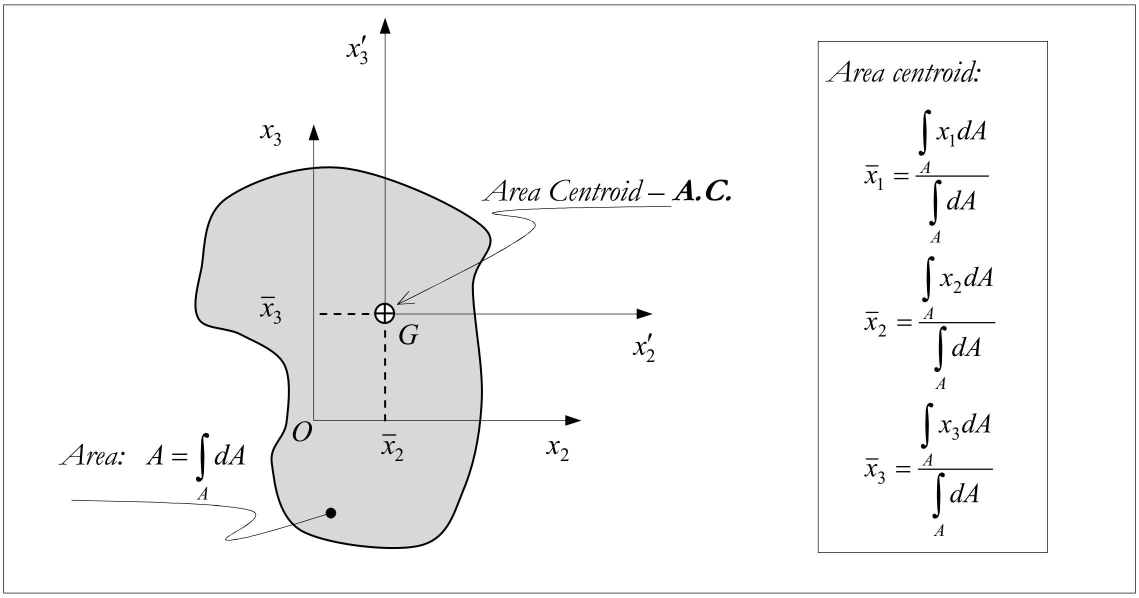


Figure 4.13: Area centroid of cross section A.

From the moment equations (4.50) we can conclude that:

$$\begin{cases} M_{x_2} = \int_A (x_3\sigma_{11})dA = \int_A [x_3(c_1 + c_2x_2 + c_3x_3)]dA = c_1\int_A x_3 dA + c_2\int_A x_3x_2 dA + c_3\int_A x_3^2 dA \\ M_{x_3} = -\int_A (x_2\sigma_{11})dA = -\int_A [x_2(c_1 + c_2x_2 + c_3x_3)]dA = -c_1\int_A x_2 dA - c_2\int_A x_2^2 dA - c_3\int_A x_2x_3 dA \end{cases}$$

Recall that in the Complementary Note at the end of Chapter 1 we have defined some area geometrical properties such as the inertia tensor of area (second-order pseudo-tensor):

$$\mathrm{I}_{O\,ij}^{(A)} = \begin{bmatrix} \mathrm{I}_{11} & \mathrm{I}_{12} & \mathrm{I}_{13} \\ \mathrm{I}_{12} & \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{13} & \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \begin{bmatrix} \int_A (x_2^2 + x_3^2)dA & -\int_A x_1x_2 dA & -\int_A x_1x_3 dA \\ -\int_A x_1x_2 dA & \int_A (x_1^2 + x_3^2)dA & -\int_A x_2x_3 dA \\ -\int_A x_1x_3 dA & -\int_A x_2x_3 dA & \int_A (x_1^2 + x_2^2)dA \end{bmatrix}$$

which for our particular reference system we can obtain:

$$\mathrm{I}_{O\,ij}^{(A)} = \begin{bmatrix} \int_A (x_2^2 + x_3^2)dA & 0 & 0 \\ 0 & \int_A x_3^2 dA & -\int_A x_2x_3 dA \\ 0 & -\int_A x_2x_3 dA & \int_A x_2^2 dA \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{11} & -\mathcal{I}_{12} & -\mathcal{I}_{13} \\ -\mathcal{I}_{12} & \mathrm{I}_{22} & -\mathcal{I}_{23} \\ -\mathcal{I}_{13} & -\mathcal{I}_{23} & \mathrm{I}_{33} \end{bmatrix} \tag{4.52}$$

Then, the moment equations can be rewritten as follows:

$$\begin{cases} M_{x_2} = c_1\int_A x_3 dA + c_2\int_A x_3x_2 dA + c_3\int_A x_3^2 dA = c_1\bar{x}_3 A - c_2\mathrm{I}_{23} + c_3\mathrm{I}_{22} \\ M_{x_3} = -c_1\int_A x_2 dA - c_2\int_A x_2^2 dA - c_3\int_A x_2x_3 dA = -c_1\bar{x}_2 A - c_2\mathrm{I}_{33} + c_3\mathrm{I}_{23} \end{cases} \tag{4.53}$$

Taking into account the equations (4.51) and (4.53) we can obtain the following set of equations:

$$\begin{cases} F_{11} \equiv N = c_1 A + c_2 \bar{x}_2 A + c_3 \bar{x}_3 A \\ M_{x_2} \equiv M_y = c_1 \bar{x}_3 A - c_2 \mathrm{I}_{23} + c_3 \mathrm{I}_{22} \\ M_{x_3} \equiv M_z = -c_1 \bar{x}_2 A - c_2 \mathrm{I}_{33} + c_3 \mathrm{I}_{23} \end{cases} \Rightarrow \frac{1}{A}\begin{Bmatrix} N \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 1 & \bar{x}_2 & \bar{x}_3 \\ \bar{x}_3 & -\frac{\mathrm{I}_{23}}{A} & \frac{\mathrm{I}_{22}}{A} \\ -\bar{x}_2 & -\frac{\mathrm{I}_{33}}{A} & \frac{\mathrm{I}_{23}}{A} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{4.54}$$

$$\Rightarrow \frac{1}{A}\begin{Bmatrix} N \\ M_y \\ M_z \end{Bmatrix} = [\boldsymbol{B}] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [\boldsymbol{B}]^{-1} \frac{1}{A}\begin{Bmatrix} N \\ M_y \\ M_z \end{Bmatrix} \tag{4.55}$$

and $[\boldsymbol{B}]^{-1} = \frac{1}{|\boldsymbol{B}|}[\mathrm{adj}(\boldsymbol{B})]$, where $|\boldsymbol{B}| \equiv \det(\boldsymbol{B}) = \frac{-1}{A^2}(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33} + 2A\bar{x}_2\bar{x}_3\mathrm{I}_{23} + A\bar{x}_3^2\mathrm{I}_{33} + A\bar{x}_2^2\mathrm{I}_{22})$,

$$[\mathrm{adj}(\boldsymbol{B})] = \begin{bmatrix} \frac{-(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}{A^2} & \frac{-(\bar{x}_2\mathrm{I}_{23} + \bar{x}_3\mathrm{I}_{33})}{A} & \frac{(\bar{x}_2\mathrm{I}_{22} + \bar{x}_3\mathrm{I}_{23})}{A} \\ \frac{-(\bar{x}_2\mathrm{I}_{22} + \bar{x}_3\mathrm{I}_{23})}{A} & \frac{(\mathrm{I}_{23} + \bar{x}_2\bar{x}_3 A)}{A} & \frac{(-\mathrm{I}_{22} + \bar{x}_3^2 A)}{A} \\ \frac{-(\bar{x}_2\mathrm{I}_{23} + \bar{x}_3\mathrm{I}_{33})}{A} & \frac{-(-\mathrm{I}_{33} + \bar{x}_2^2 A)}{A} & \frac{-(\mathrm{I}_{23} + \bar{x}_2\bar{x}_3 A)}{A} \end{bmatrix}$$

With that the coefficients c_1, c_2 and c_3 can be determined:

$$\begin{aligned} c_1 &= \frac{-(-N\ \mathrm{I}_{23}^2 + N\ \mathrm{I}_{22}\mathrm{I}_{33} - M_y A\bar{x}_2\mathrm{I}_{23} - M_y A\bar{x}_3\mathrm{I}_{33} + M_z A\bar{x}_2\mathrm{I}_{22} + M_z A\bar{x}_3\mathrm{I}_{23})}{A(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33} + 2A\bar{x}_2\bar{x}_3\mathrm{I}_{23} + A\bar{x}_3^2\mathrm{I}_{33} + A\bar{x}_2^2\mathrm{I}_{22})} \\ c_2 &= \frac{-(-N\ \bar{x}_2\mathrm{I}_{22} - N\ \bar{x}_3\mathrm{I}_{23} + M_y\mathrm{I}_{23} + M_y A\bar{x}_2\bar{x}_3 - M_z\mathrm{I}_{22} + M_z A\bar{x}_3^2)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33} + 2A\bar{x}_2\bar{x}_3\mathrm{I}_{23} + A\bar{x}_3^2\mathrm{I}_{33} + A\bar{x}_2^2\mathrm{I}_{22})} \\ c_3 &= \frac{(N\ \bar{x}_2\mathrm{I}_{23} + N\ \bar{x}_3\mathrm{I}_{33} - M_y\mathrm{I}_{33} + M_y A\bar{x}_2^2 + M_z\mathrm{I}_{23} + M_z A\bar{x}_3\bar{x}_2)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33} + 2A\bar{x}_2\bar{x}_3\mathrm{I}_{23} + A\bar{x}_3^2\mathrm{I}_{33} + A\bar{x}_2^2\mathrm{I}_{22})} \end{aligned} \tag{4.56}$$

Note that the set of equations (4.54) can also be written as follows

$$\begin{Bmatrix} N \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{4.57}$$

And the solution can be obtained by means of Cramer's rule, (see **Problem 1.16**):

$$c_1 = \frac{\begin{vmatrix} N & A\bar{x}_2 & A\bar{x}_3 \\ M_y & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ M_z & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} \quad ; \quad c_2 = \frac{\begin{vmatrix} A & N & A\bar{x}_3 \\ A\bar{x}_3 & M_y & \mathrm{I}_{22} \\ -A\bar{x}_2 & M_z & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} \quad ; \quad c_3 = \frac{\begin{vmatrix} A & A\bar{x}_2 & N \\ A\bar{x}_3 & -\mathrm{I}_{23} & M_y \\ -A\bar{x}_2 & -\mathrm{I}_{33} & M_z \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} \tag{4.58}$$

which must match the equations in (4.56). Once the coefficients $c_i = c_i(N, M_y, M_z, \mathrm{I}_{ij}, \bar{x}_i)$ are obtained, the normal stress component can be defined in terms of resultant forces and moments:

$$\boxed{\sigma_{11}(x_2,x_3)=c_1+c_2x_2+c_3x_3} \quad \textit{(For any system in which the plane } x_2-x_3 \textit{ is lying on the plane defined by the cross section)} \tag{4.59}$$

Note that all variables (N, M_y, M_z, I_{ij}, x_i, $\bar{x}_i$) must be expressed in the adopted system.

Note that when $x_2=\bar{x}_2$ and $x_3=\bar{x}_3$ the equation (4.59) reduces to $\sigma_{11}(\bar{x}_2,\bar{x}_3)=\dfrac{N}{A}$.

If the adopted system is located at the area centroid, we have $\bar{x}_2=0$, $\bar{x}_3=0$. In this situation the coefficients (4.56) become:

$$c_1=\frac{N}{A} \quad ; \quad c_2=\frac{-(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})} \quad ; \quad c_3=\frac{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})} \tag{4.60}$$

and the normal stress component can be evaluated as follows:

$$\boxed{\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3} \quad \textit{(The system is located at the Area Centroid)} \tag{4.61}$$

or

$$\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{(M_z\mathrm{I}_{22}+M_y\mathcal{I}_{23})}{(\mathrm{I}_{22}\mathrm{I}_{33}-\mathrm{I}_{23}^2)}x_2+\frac{(M_y\mathrm{I}_{33}+M_z\mathcal{I}_{23})}{(\mathrm{I}_{22}\mathrm{I}_{33}-\mathrm{I}_{23}^2)}x_3 \tag{4.62}$$

The above two equations can also be written as follows:

$$\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}M_y+\frac{(\mathrm{I}_{22}x_2+\mathrm{I}_{23}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}M_z$$

or (4.63)

$$\sigma_{11}(x_2,x_3)=\frac{N}{A}+\frac{(\mathrm{I}_{33}x_3-\mathcal{I}_{23}x_2)}{(\mathrm{I}_{22}\mathrm{I}_{33}-\mathrm{I}_{23}^2)}M_y-\frac{(\mathrm{I}_{22}x_2-\mathcal{I}_{23}x_3)}{(\mathrm{I}_{22}\mathrm{I}_{33}-\mathrm{I}_{23}^2)}M_z$$

In view of the equation (4.61) note that if the adopted system is at the area centroid, we have $\bar{x}_2=0$, $\bar{x}_3=0$, and if the system is also the principal axes of inertia the product of area inertia is zero, i.e. $\mathrm{I}_{23}=0$. In this situation the coefficients (4.56) become:

$$c_1=\frac{N}{A} \quad ; \quad c_2=\frac{-M_z}{\mathrm{I}_{33}} \quad ; \quad c_3=\frac{M_y}{\mathrm{I}_{22}} \tag{4.64}$$

and the normal stress σ_{11} can be obtained as follows:

$$\boxed{\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{M_z}{\mathrm{I}_{33}}x_2+\frac{M_y}{\mathrm{I}_{22}}x_3} \quad \textit{(The system is located at the Area Centroid and is the principal axes of inertia)} \tag{4.65}$$

NOTE 1: The Neutral Axis (N.A.)

The neutral axis is defined by the absence of normal stress $\sigma_{11}(x_2,x_3)=0$ and the neutral axis equation can be obtained as follows:

By considering an arbitrary system, (see equation (4.59)), i.e.:

$$\begin{aligned}\sigma_{11}(x_2,x_3)=c_1+c_2x_2+c_3x_3=0\\ \Rightarrow \quad c_2x_2+c_3x_3=-c_1\end{aligned} \quad \textit{(Neutral Axis -for any system in which the plane } x_2-x_3 \textit{ is lying on the plane defined by the cross section)} \tag{4.66}$$

where the coefficients c_i are given by the equations in (4.58).

If the adopted system is located at the area centroid, the neutral axis can be defined by means of the equation in (4.61), i.e.:

$$\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3=0$$
$$\Rightarrow -A(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})x_2+A(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})x_3=-N(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33}) \quad \text{(The system is located at the Area Centroid)} \tag{4.67}$$

which represents a straight line defined on the plane (x_2-x_3). When $N\neq 0$, the canonic form of the above equation is represented by:

$$\frac{x_2}{a}+\frac{x_3}{b}=1 \qquad \text{with} \quad a=\frac{N(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}{A(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})} \quad ; \quad b=\frac{N(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}{A(M_y\mathrm{I}_{33}-M_z\mathrm{I}_{23})} \tag{4.68}$$

where a and b are the points in which the Neutral Axis intercepts the axis x_2 and x_3 respectively.

Note also that, when $N=0$ the neutral axis pass through the area centroid. And by means of the equation in (4.61) the neutral line can be established as follows

$$\sigma_{11}(x_2,x_3)=-\frac{(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3=0$$
$$\Rightarrow (-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})x_3-(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})x_2=0 \quad \Rightarrow \quad x_3=\frac{(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}x_2 \tag{4.69}$$

And taking into account that $\mathrm{I}_{23}=-\int_A x_2x_3\,dA=-\mathcal{I}_{23} \quad \therefore \quad \mathcal{I}_{23}=\int_A x_2x_3\,dA$, the equation in (4.69) can be rewritten as follows:

$$(M_y\mathrm{I}_{33}+M_z\mathcal{I}_{23})x_3-(M_y\mathcal{I}_{23}+M_z\mathrm{I}_{22})x_2=0 \tag{4.70}$$

NOTE 2: An interesting relationship is the derivative of σ_{11} with respect to x_1. If we take into account the equation (4.59) we can obtain:

$$\boxed{\frac{\partial\sigma_{11}}{\partial x_1}=\frac{\partial c_1}{\partial x_1}+\frac{\partial c_2}{\partial x_1}x_2+\frac{\partial c_3}{\partial x_1}x_3} \quad \text{(For any system in which the plane } x_2-x_3 \text{ is lying on the plane defined by the cross section)} \tag{4.71}$$

If we consider that the cross section is constant along x_1-direction and N is also constant along x_1-direction or absent we can obtain

$$\frac{\partial c_1}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{\begin{vmatrix} N & A\bar{x}_2 & A\bar{x}_3\\ M_y & -\mathrm{I}_{23} & \mathrm{I}_{22}\\ M_z & -\mathrm{I}_{33} & \mathrm{I}_{23}\end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3\\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22}\\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23}\end{vmatrix}}\right)=\frac{\frac{\partial}{\partial x_1}\begin{vmatrix} N & A\bar{x}_2 & A\bar{x}_3\\ M_y & -\mathrm{I}_{23} & \mathrm{I}_{22}\\ M_z & -\mathrm{I}_{33} & \mathrm{I}_{23}\end{vmatrix}}{\mathcal{X}} \qquad \therefore \quad \mathcal{X}=\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3\\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22}\\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23}\end{vmatrix}$$

$$\Rightarrow \frac{\partial c_1}{\partial x_1} = \frac{1}{\mathcal{X}} \begin{vmatrix} \frac{\partial N}{\partial x_1} & A\bar{x}_2 & A\bar{x}_3 \\ \frac{\partial M_y}{\partial x_1} & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ \frac{\partial M_z}{\partial x_1} & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = \frac{1}{\mathcal{X}} \begin{vmatrix} 0 & A\bar{x}_2 & A\bar{x}_3 \\ F_z & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -F_y & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}$$

where we have applied the equations (4.47) and (4.48). In the same fashion we can obtain

$$\frac{\partial c_2}{\partial x_1} = \frac{\frac{\partial}{\partial x_1} \begin{vmatrix} A & N & A\bar{x}_3 \\ A\bar{x}_3 & M_y & \mathrm{I}_{22} \\ -A\bar{x}_2 & M_z & \mathrm{I}_{23} \end{vmatrix}}{\mathcal{X}} = \frac{1}{\mathcal{X}} \begin{vmatrix} A & \frac{\partial N}{\partial x_1} & A\bar{x}_3 \\ A\bar{x}_3 & \frac{\partial M_y}{\partial x_1} & \mathrm{I}_{22} \\ -A\bar{x}_2 & \frac{\partial M_z}{\partial x_1} & \mathrm{I}_{23} \end{vmatrix} = \frac{1}{\mathcal{X}} \begin{vmatrix} A & 0 & A\bar{x}_3 \\ A\bar{x}_3 & F_z & \mathrm{I}_{22} \\ -A\bar{x}_2 & -F_y & \mathrm{I}_{23} \end{vmatrix}$$

and

$$\frac{\partial c_3}{\partial x_1} = \frac{\frac{\partial}{\partial x_1} \begin{vmatrix} A & A\bar{x}_2 & N \\ A\bar{x}_3 & -\mathrm{I}_{23} & M_y \\ -A\bar{x}_2 & -\mathrm{I}_{33} & M_z \end{vmatrix}}{\mathcal{X}} = \frac{1}{\mathcal{X}} \begin{vmatrix} A & A\bar{x}_2 & \frac{\partial N}{\partial x_1} \\ A\bar{x}_3 & -\mathrm{I}_{23} & \frac{\partial M_y}{\partial x_1} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \frac{\partial M_z}{\partial x_1} \end{vmatrix} = \frac{1}{\mathcal{X}} \begin{vmatrix} A & A\bar{x}_2 & 0 \\ A\bar{x}_3 & -\mathrm{I}_{23} & F_z \\ -A\bar{x}_2 & -\mathrm{I}_{33} & -F_y \end{vmatrix}$$

Then, the equation (4.71) can be written as follows

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} = \frac{1}{\mathcal{X}} \left(\begin{vmatrix} 0 & A\bar{x}_2 & A\bar{x}_3 \\ F_z & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -F_y & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} + \begin{vmatrix} A & 0 & A\bar{x}_3 \\ A\bar{x}_3 & F_z & \mathrm{I}_{22} \\ -A\bar{x}_2 & -F_y & \mathrm{I}_{23} \end{vmatrix} x_2 + \begin{vmatrix} A & A\bar{x}_2 & 0 \\ A\bar{x}_3 & -\mathrm{I}_{23} & F_z \\ -A\bar{x}_2 & -\mathrm{I}_{33} & -F_y \end{vmatrix} x_3 \right)} \quad (4.72)$$

(For any system in which the plane $x_2 - x_3$ is lying on the plane defined by the cross section)

If we are adopting the system at the Area Centroid and if we take the derivative of the equation (4.63) with respect to x_1 we can obtain:

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} = -\frac{(\mathrm{I}_{23}x_2 + \mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\frac{\partial M_y}{\partial x_1} + \frac{(\mathrm{I}_{22}x_2 + \mathrm{I}_{23}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\frac{\partial M_z}{\partial x_1}} \quad \textit{(The system is located at the Area Centroid)} \quad (4.73)$$

or

$$\frac{\partial \sigma_{11}}{\partial x_1} = \frac{(\mathrm{I}_{33}x_3 - \mathcal{I}_{O23}x_2)}{(\mathrm{I}_{22}\mathrm{I}_{33} - \mathrm{I}_{23}^2)}\frac{\partial M_y}{\partial x_1} - \frac{(\mathrm{I}_{22}x_2 - \mathcal{I}_{O23}x_3)}{(\mathrm{I}_{22}\mathrm{I}_{33} - \mathrm{I}_{23}^2)}\frac{\partial M_z}{\partial x_1}$$

and by considering the equations (4.47) and (4.48), the equation (4.73) becomes:

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} = -\frac{(\mathrm{I}_{23}x_2 + \mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}F_z - \frac{(\mathrm{I}_{22}x_2 + \mathrm{I}_{23}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}F_y} \quad \textit{(The system is located at the Area Centroid)} \quad (4.74)$$

or

$$\frac{\partial \sigma_{11}}{\partial x_1} = \frac{(\mathrm{I}_{33}x_3 - \mathcal{I}_{O23}x_2)}{(\mathrm{I}_{22}\mathrm{I}_{33} - \mathrm{I}_{23}^2)}F_z + \frac{(\mathrm{I}_{22}x_2 - \mathcal{I}_{O23}x_3)}{(\mathrm{I}_{22}\mathrm{I}_{33} - \mathrm{I}_{23}^2)}F_y$$

Example: Let us consider the cross section described in Figure 4.14, (Ugural&Fenster (1984)). And the geometrical characteristics for the quadrangular cross section are described in Figure 4.15. The cross section described in Figure 4.14 can be constructed by the two rectangles as described in Figure 4.16.

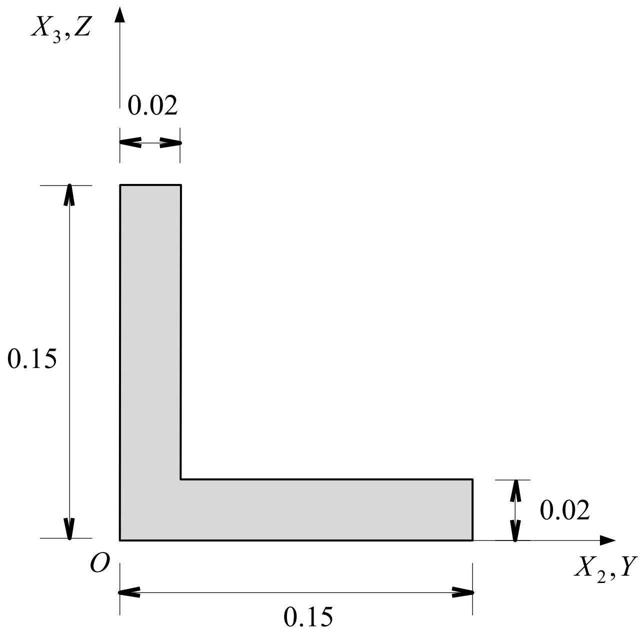
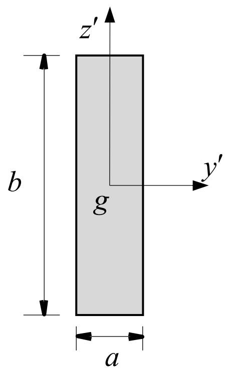


Figure 4.14: Cross section – Dimensions in meter (m).



The inertia tensor of area

$$\bar{\mathrm{I}}_{gij} = \begin{bmatrix} \bar{\mathrm{I}}_{g22} + \bar{\mathrm{I}}_{g33} & 0 & 0 \\ 0 & \dfrac{ab^3}{12} & 0 \\ 0 & 0 & \dfrac{ba^3}{12} \end{bmatrix}$$

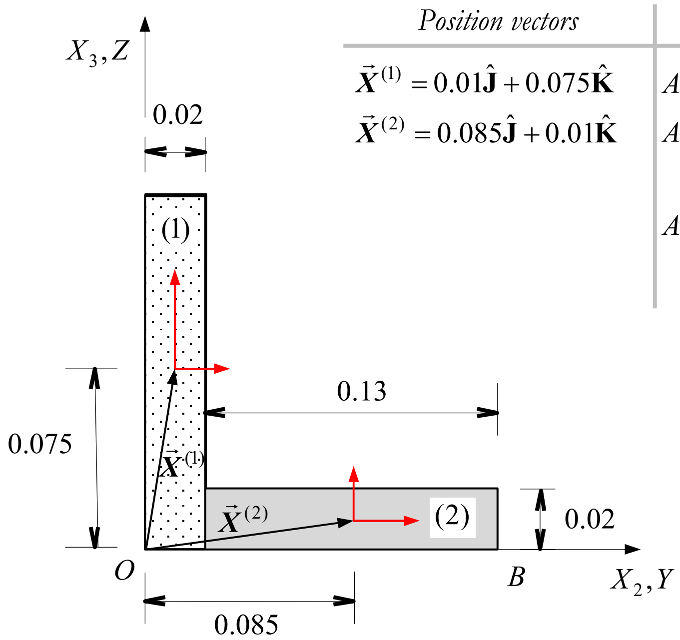
Area: $A = ab$

g *- area centroid*

Figure 4.15: Rectangular cross section.

<u>Area Centroid (***A.C.***) Calculation:</u>

Geometrical decomposition, (see Figure 4.16).



Position vectors	*Areas*
$\vec{X}^{(1)} = 0.01\hat{\mathbf{J}} + 0.075\hat{\mathbf{K}}$	$A^{(1)} = 0.02 \times 0.15 = 0.003m^2$
$\vec{X}^{(2)} = 0.085\hat{\mathbf{J}} + 0.01\hat{\mathbf{K}}$	$A^{(2)} = 0.13 \times 0.02 = 0.0026m^2$
	$A = A^{(1)} + A^{(2)} = 0.0056m^2$

Figure 4.16: Geometric decomposition of the cross section.

By using equations described in Figure 4.13 we can obtain the Area Centroid related to the system X, Y, Z:

$$\overline{X}_2 \equiv \overline{Y} = \frac{\int_A X_2 dA}{\int_A dA} = \frac{\sum_{a=1}^{2} A^{(a)} X_2^{(a)}}{A} = \frac{A^{(1)} X_2^{(1)} + A^{(2)} X_2^{(2)}}{(A^{(1)} + A^{(2)})} = \frac{0.003 \times 0.01 + 0.0026 \times 0.085}{0.0056}$$

$$= 0.04482143m \approx 0.045m$$

$$\overline{X}_3 \equiv \overline{Z} = \frac{\int_A X_3 dA}{\int_A dA} = \frac{\sum_{a=1}^{2} A^{(a)} X_3^{(a)}}{A} = \frac{A^{(1)} X_3^{(1)} + A^{(2)} X_3^{(2)}}{(A^{(1)} + A^{(2)})} = \frac{0.003 \times 0.075 + 0.0026 \times 0.01}{0.0056}$$

$$= 0.04482143m \approx 0.045m$$

Then, the area centroid of the cross section is given by $\overline{Y} = 0.045m; \overline{Z} = 0.045m$, (see Figure 4.17). Note that the cross section has a symmetric axis, so, as expected the point ***A.C.*** lies on the symmetric axis. At the point ***A.C.*** we define a new system x, y, z, (see Figure 4.17).

The position vector $\vec{\boldsymbol{x}}^{(a)} = \vec{\boldsymbol{X}}^{(a)} - \vec{\overline{\boldsymbol{X}}}$, (see Figure 4.17), can be obtained as follows:

$$\vec{\boldsymbol{x}}^{(1)} = \vec{\boldsymbol{X}}^{(1)} - \vec{\overline{\boldsymbol{X}}} = (X_2^{(1)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(1)} - \overline{Z})\hat{\mathbf{k}} = (0.01 - 0.045)\hat{\mathbf{j}} + (0.075 - 0.045)\hat{\mathbf{k}}$$
$$= -0.035\hat{\mathbf{j}} + 0.03\hat{\mathbf{k}}$$

$$\vec{\boldsymbol{x}}^{(2)} = \vec{\boldsymbol{X}}^{(2)} - \vec{\overline{\boldsymbol{X}}} = (X_2^{(2)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(2)} - \overline{Z})\hat{\mathbf{k}} = (0.085 - 0.045)\hat{\mathbf{j}} + (0.01 - 0.045)\hat{\mathbf{k}}$$
$$= 0.04\hat{\mathbf{j}} - 0.035\hat{\mathbf{k}}$$

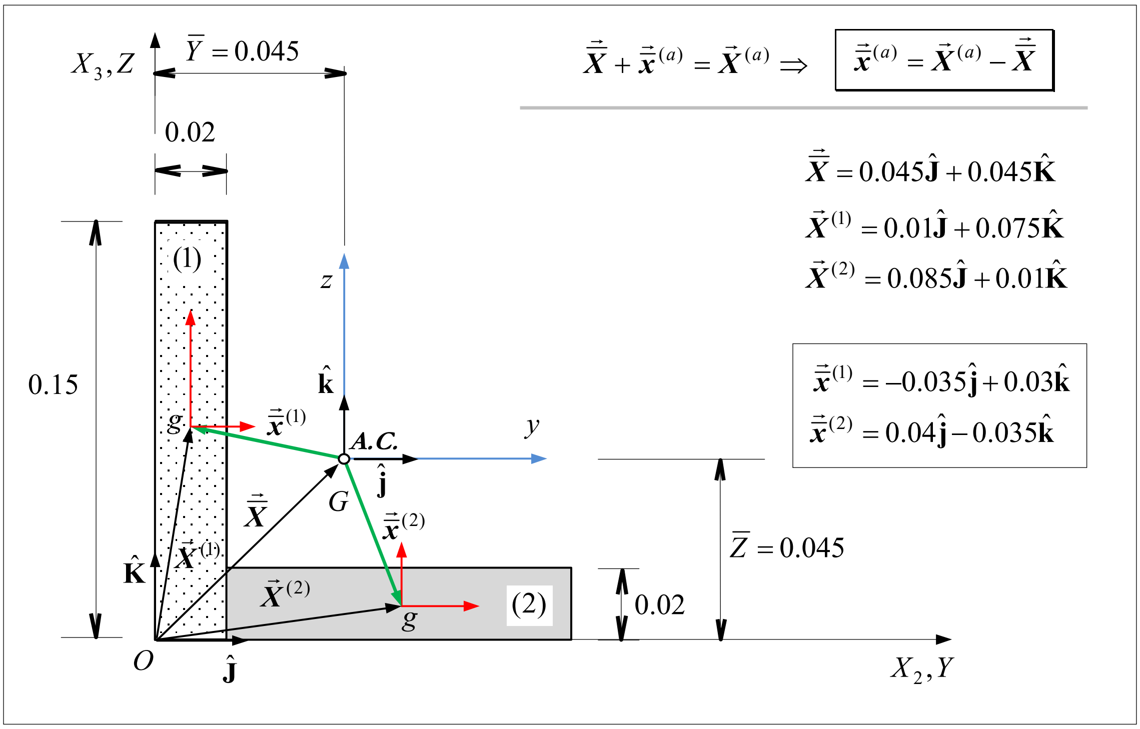


Figure 4.17: Area Centroid - Cross section.

<u>Find the inertia tensor of area for the system located at the Area Centroid (***A.C.***):</u>

We will use the definition of the parallel theorem, (see Chapter 1 COMPLEMENTARY NOTES at the end of the Chapter 1). We use $\mathbf{I}^{(sys)}_{G\vec{x}} = \mathbf{I}^{(1)}_{G\vec{x}} + \mathbf{I}^{(2)}_{G\vec{x}}$, (see **Problem 4.32** -NOTA 1), where $\mathbf{I}_{G\vec{x}} = \bar{\mathbf{I}}_g - A[(\vec{\bar{x}} \otimes \vec{\bar{x}}) - (\vec{\bar{x}} \cdot \vec{\bar{x}})\mathbf{1}]$ (the Steiner's theorem), which in indicial notation becomes:

$$\mathrm{I}_{G\vec{x}ij} = \bar{\mathrm{I}}_{gij} - A[\bar{x}_i\bar{x}_j - (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)\delta_{ij}]$$

or in matrix notation:

$$\begin{aligned}\mathrm{I}_{G\vec{x}ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A\begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\ &= \left[\begin{array}{c:cc} \bar{\mathrm{I}}_{11} & 0 & 0 \\ \hdashline 0 & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ 0 & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{array}\right] + A\left[\begin{array}{c:cc} \bar{x}_2^2 + \bar{x}_3^2 & 0 & 0 \\ \hdashline 0 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ 0 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{array}\right]\end{aligned} \tag{4.75}$$

Note that this problem can be treated as two-dimensional case on the plane defined by $(x_2 - x_3)$.

Rectangle 1 - $\mathbf{I}^{(1)}_{G\vec{x}}$

$$(\bar{\mathbf{I}}^{(1)}_g)_{ij} = \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{0.02 \times 0.15^3}{12} & 0 \\ 0 & \dfrac{0.15 \times 0.02^3}{12} \end{bmatrix} = \begin{bmatrix} 562.5 & 0 \\ 0 & 10.0 \end{bmatrix} \times 10^{-8} m^4$$

Area centroid vector position: $\vec{\bar{x}}^{(1)} = -0.035\hat{\boldsymbol{j}} + 0.03\hat{\boldsymbol{k}}$, $(\bar{x}_1^{(1)} = 0, \bar{x}_2^{(2)} = -0.035, \bar{x}_3^{(2)} = 0.03)$:

$$
\begin{aligned}
(\mathbf{I}_{G\bar{x}}^{(1)})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(1)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 562.5 & 0 \\ 0 & 10.0 \end{bmatrix} \times 10^{-8} + 0.003 \begin{bmatrix} (0.03)^2 & -(-0.035)(0.03) \\ -(-0.035)(0.03) & (-0.035)^2 \end{bmatrix} \\
&= \begin{bmatrix} 832.5 & 315 \\ 315 & 377.5 \end{bmatrix} \times 10^{-8}
\end{aligned}
$$

Rectangle 2 - $\mathbf{I}_{G\bar{x}}^{(2)}$

$$
\begin{aligned}
(\bar{\mathbf{I}}_{g}^{(2)})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{0.13 \times 0.02^3}{12} & 0 \\ 0 & \dfrac{0.02 \times 0.13^3}{12} \end{bmatrix} \\
&= \begin{bmatrix} 8.667 & 0 \\ 0 & 366.167 \end{bmatrix} \times 10^{-8} m^4
\end{aligned}
$$

Area Centroid vector position: $\vec{\bar{x}}^{(2)} = 0.04\hat{\boldsymbol{j}} - 0.035\hat{\boldsymbol{k}}$, $(\bar{x}_1^{(1)} = 0, \bar{x}_2^{(2)} = 0.04, \bar{x}_3^{(2)} = -0.035)$:

By applying the equation (4.75) we can obtain:

$$
\begin{aligned}
(\mathbf{I}_{G\bar{x}}^{(2)})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(2)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 8.667 & 0 \\ 0 & 366.167 \end{bmatrix} \times 10^{-8} + 0.0026 \begin{bmatrix} (-0.035)^2 & -(0.04)(-0.035) \\ -(0.04)(-0.035) & (0.04)^2 \end{bmatrix} \\
&= \begin{bmatrix} 327.16 & 364 \\ 364 & 782.167 \end{bmatrix} \times 10^{-8}
\end{aligned}
$$

Then, we can calculate the inertia tensor of the cross section related to the system located at the Area Centroid:

$$
\begin{aligned}
(\mathbf{I}_{G\bar{x}}^{(sys)})_{ij} &= (\mathbf{I}_{G\bar{x}}^{(1)})_{ij} + (\mathbf{I}_{G\bar{x}}^{(2)})_{ij} = \begin{bmatrix} 832.5 & 315 \\ 315 & 377.5 \end{bmatrix} \times 10^{-8} + \begin{bmatrix} 327.16 & 364 \\ 364 & 782.167 \end{bmatrix} \times 10^{-8} \\
&= \begin{bmatrix} 1159.66 & 679 \\ 679 & 1159.66 \end{bmatrix} \times 10^{-8} m^4
\end{aligned}
\tag{4.76}
$$

<u>Calculation of the normal stress at any point of the cross section:</u>

Let us suppose that on the cross section is acting the moment $M_z = 11000.0 Nm$, $M_y = 0$ and $N = 0$. To obtain the normal stress $\sigma_{11}(x_2, x_3)$ we can apply the equation (4.61), i.e.:

$$
\begin{aligned}
\sigma_{11}(x_2, x_3) &= -\frac{(M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_2 + \frac{(-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_3 \\
&= -\frac{-11000 \times 1159.66 \times 10^{-8}}{((679 \times 10^{-8})^2 - (1159.66 \times 10^{-8})^2)} x_2 + \frac{11000 \times 679 \times 10^{-8}}{((679 \times 10^{-8})^2 - (1159.66 \times 10^{-8})^2)} x_3 \\
&= (-1443.39 \times 10^6 x_2 - 845.129 \times 10^6 x_3) Pa = (-1443.39 x_2 - 845.129 x_3) MPa
\end{aligned}
$$

For the points O, B, C, D, E and F, (see Figure 4.19), and by using the definition $\vec{x}^{(P)} = \vec{X}^{(P)} - \vec{\bar{X}}$, the normal stresses are given by:

Point coordinates (meter-m)					Normal stress
	$X_2[m]$	$X_3[m]$	$x_2 = X_2 - \bar{Y}[m]$	$x_3 = X_3 - \bar{Z}[m]$	$\sigma_{11}[MPa]$
O	0.0	0.0	-0.045	-0.045	102.98
B	0.15	0.0	0.105	-0.045	-113.52
C	0.15	0.02	0.105	-0.025	-130.42
D	0.02	0.02	-0.025	-0.025	57.21
E	0.0	0.15	-0.045	0.105	-23.78
F	0.02	0.15	-0.025	0.105	-52.65

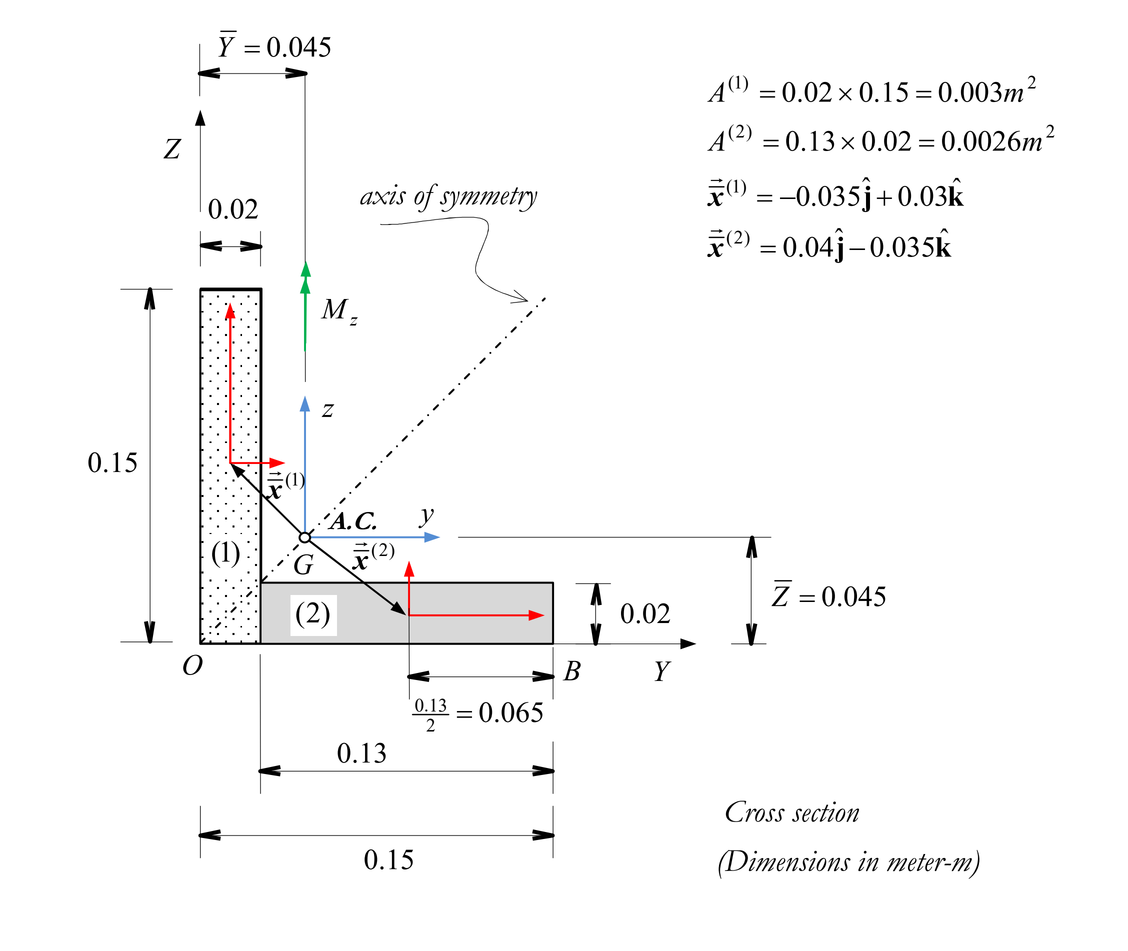


Figure 4.18: Cross section.

The Neutral Axis can be obtained by means of the equation (4.69) with $\sigma_{11}(x_2, x_3) = 0$, i.e.:

$$\sigma_{11}(x_2, x_3) = -\frac{(M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_2 + \frac{(-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_3 = 0$$

$$\Rightarrow (-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23}) x_3 - (M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22}) x_2 = 0$$

$$\Rightarrow (0+11000\times 679\times 10^{-8})x_3 - (0-11000\times 1159.667\times 10^{-8})x_2 = 0$$
$$\Rightarrow 7.469\times 10^{-2}x_3 + 12.756337\times 10^{-2}x_2 = 0$$
$$\Rightarrow x_3 = \frac{-12.756337\times 10^{-2}}{7.469\times 10^{-2}}x_2 = -1.708x_2 \qquad or \qquad z = -1.708y$$
$$\Rightarrow z = \tan(\alpha)y \qquad \therefore \qquad \alpha = \arctan(-1.708) = -59.65^{\circ}$$

<u>Calculation of principal inertia tensor of area:</u>

The inertia tensor of area is a second-order pseudo-tensor and this tensor has same transformation described for a second-order tensor. For example, for component transformation law for a second-order tensor:

$$\boxed{\begin{aligned}\mathbf{I}'_O &= \boldsymbol{\mathcal{A}}\cdot\mathbf{I}_O\cdot\boldsymbol{\mathcal{A}}^T\\ \mathrm{I}'_{Oij} &= \mathcal{A}_{ip}\mathrm{I}_{O\,pj}\mathcal{A}_{qj}\end{aligned}} \quad \textit{Inertia tensor components after a base change (rotation)} \tag{4.77}$$

where $\boldsymbol{\mathcal{A}}$ is the transformation matrix from the $x_1x_2x_3$-system to $x'_1x'_2x'_3$-system.

For two-dimensional problem, (see **Problem 1.99**), we have shown that:

$$\begin{bmatrix}\mathsf{T}'_{11}\\ \mathsf{T}'_{22}\\ \mathsf{T}'_{12}\end{bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{bmatrix}\mathsf{T}_{11}\\ \mathsf{T}_{22}\\ \mathsf{T}_{12}\end{bmatrix} \tag{4.78}$$

and the principal direction is characterized by:

$$\boxed{\theta = \frac{1}{2}\arctan\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}\right)} \tag{4.79}$$

where $\mathsf{T}_{ij}\,(i,j=1,2)$ are the second-order tensor components for 2D problems.

For the problem proposed here

$$(\mathbf{I}^{(sys)}_{G\vec{x}})_{ij} = \begin{bmatrix}1159.66 & 679\\ 679 & 1159.66\end{bmatrix}\times 10^{-8}m^4$$

Then, we can obtain:

$$\theta = \frac{1}{2}\arctan\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}\right) = \frac{1}{2}\arctan\left(\frac{2(679)}{(1159.66)-(1159.66)}\right) = \frac{1}{2}\arctan(\infty) = 45^{\circ} \tag{4.80}$$

And by applying the equation in (4.78) when $\theta = 45^{\circ}$ we can obtain:

$$\begin{aligned}\begin{bmatrix}\mathsf{T}'_{11}\\ \mathsf{T}'_{22}\\ \mathsf{T}'_{12}\end{bmatrix} &= \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{bmatrix}\mathsf{T}_{11}\\ \mathsf{T}_{22}\\ \mathsf{T}_{12}\end{bmatrix}\\ &= \begin{bmatrix}0.5 & 0.5 & 1\\ 0.5 & 0.5 & -1\\ -0.5 & 0.5 & 0\end{bmatrix}\begin{bmatrix}1159.66\\ 1159.66\\ 679\end{bmatrix}\times 10^{-8} = \begin{bmatrix}18.39\\ 4.807\\ 0\end{bmatrix}\times 10^{-6}m^4\end{aligned} \tag{4.81}$$

Then, for the principal system of the inertia tensor $y'-z'$ we have:

$$(\mathbf{I}'^{(P\_sys)}_{G\vec{x}})_{ij} = \begin{bmatrix}18.39 & 0\\ 0 & 4.807\end{bmatrix}\times 10^{-6}m^4$$

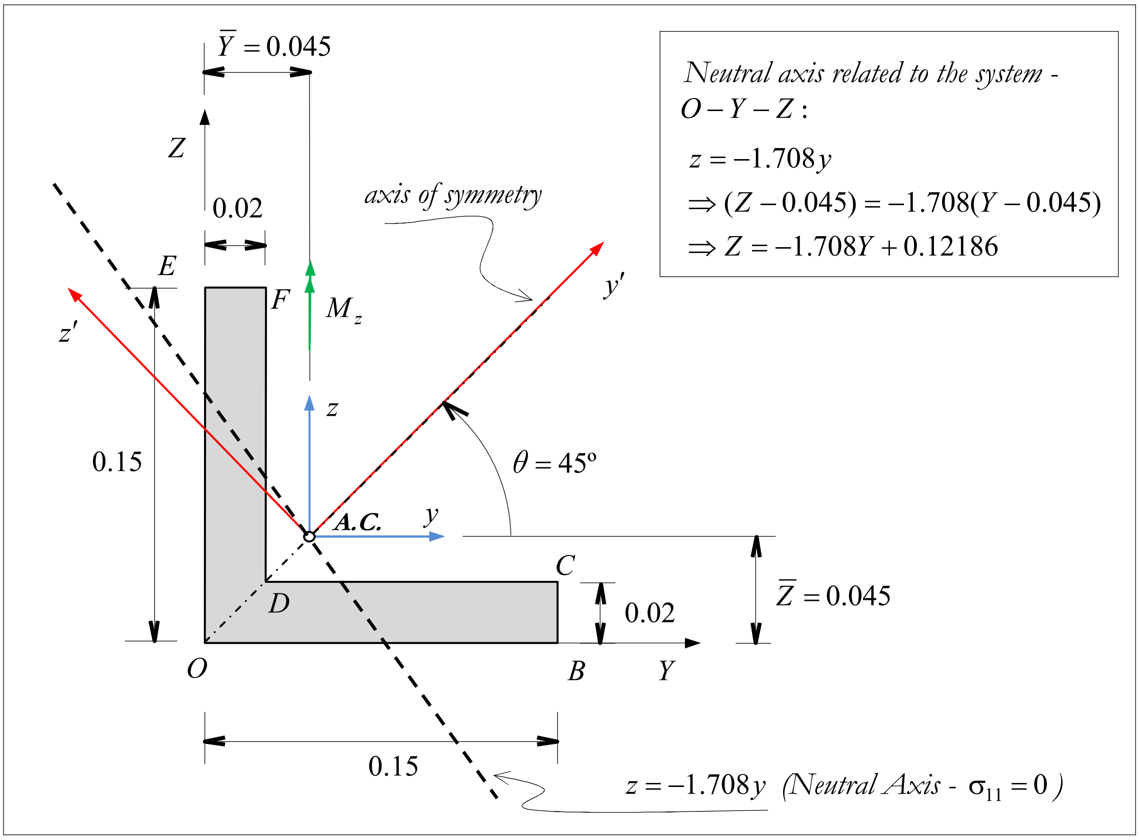


Figure 4.19: Cross section (dimensions in m).

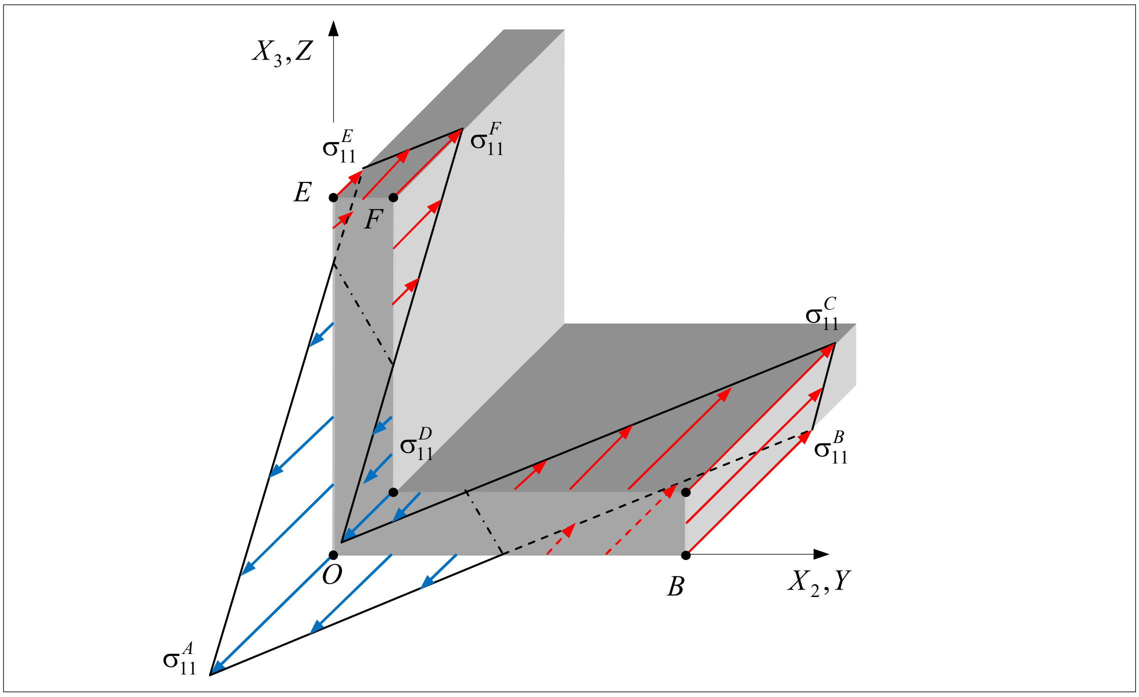


Figure 4.20: Normal stress distribution on the cross section.

NOTE 3: The normal stress components could also have been obtained by adopting the system $X_2 - X_3$, in this case we have to apply the equation in (4.59), $\sigma_{11} = c_1 + c_2 X_2 + c_3 X_3$, and all variables must be expressed in the system $X_2 - X_3$. The inertia tensor for this system can be obtained by considering Figure 4.16 and equation (4.75) in which x_2 and x_3 are now related to the system $X_2 - X_3$:

$$
\mathrm{I}^{(q)}_{O\vec{X}\,ij} = \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A \begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} + ab \begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix}
$$

$$
\Rightarrow \mathrm{I}^{(q)}_{O\vec{X}\,ij} = \frac{ab}{12} \begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} \tag{4.82}
$$

where

Rectangle $q=1$: $a = 0.02$, $b = 0.15$, $\overline{X}_2 = 0.01$, $\overline{X}_3 = 0.075$:

$$
\mathrm{I}^{(1)}_{O\vec{X}\,ij} = \frac{ab}{12} \begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} = \frac{(0.02)(0.15)}{12} \begin{bmatrix} (0.15)^2 + 12(0.075)^2 & -12(0.01)(0.075) \\ -12(0.01)(0.075) & (0.02)^2 + 12(0.01)^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 22.5 & -2.25 \\ -2.25 & 0.4 \end{bmatrix} \times 10^{-6} m^4
$$

Rectangle $q=2$: $a = 0.13$, $b = 0.02$, $\overline{X}_2 = 0.085$, $\overline{X}_3 = 0.01$:

$$
\mathrm{I}^{(2)}_{O\vec{X}\,ij} = \frac{ab}{12} \begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} = \frac{(0.13)(0.02)}{12} \begin{bmatrix} (0.02)^2 + 12(0.01)^2 & -12(0.085)(0.01) \\ -12(0.085)(0.01) & (0.13)^2 + 12(0.085)^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0.3466667 & -2.21 \\ -2.21 & 22.4466667 \end{bmatrix} \times 10^{-6} m^4
$$

Then

$$
\mathrm{I}^{(Sys)}_{O\vec{X}\;ij} = \mathrm{I}^{(1)}_{O\vec{X}\,ij} + \mathrm{I}^{(2)}_{O\vec{X}\,ij} = \left(\begin{bmatrix} 22.5 & -2.25 \\ -2.25 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.3466667 & -2.21 \\ -2.21 & 22.4466667 \end{bmatrix} \right) \times 10^{-6}
$$

$$
\Rightarrow \mathrm{I}^{(Sys)}_{O\vec{X}\;ij} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \begin{bmatrix} 22.8466667 & -4.46 \\ -4.46 & 22.8466667 \end{bmatrix} \times 10^{-6} m^4
$$

And by considering the parameters, $\bar{x}_2 \leftarrow \overline{X}_2 = 0.04482143m$, $\bar{x}_3 \leftarrow \overline{X}_3 = 0.04482143m$, $A = 0.0056m^2$, $N = 0$, $M_y = 0$ and $M_z = 11000Nm$, the coefficients c_i can be obtained as follows:

$$
c_1 = \frac{\begin{vmatrix} N & A\bar{x}_2 & A\bar{x}_3 \\ M_y & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ M_z & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = 102.581 \times 10^6 Pa \quad ; \quad c_2 = \frac{\begin{vmatrix} A & N & A\bar{x}_3 \\ A\bar{x}_3 & M_y & \mathrm{I}_{22} \\ -A\bar{x}_2 & M_z & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = -1443.46 \times 10^6 \frac{Pa}{m}
$$

$$c_3 = \frac{\begin{vmatrix} A & A\bar{x}_2 & N \\ A\bar{x}_3 & -\mathrm{I}_{23} & M_y \\ -A\bar{x}_2 & -\mathrm{I}_{33} & M_z \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = -845.2\times 10^6 \frac{Pa}{m}$$

Thus

$$\sigma_{11}(X_2, X_3) = c_1 + c_2 X_2 + c_3 X_3 = 102.581 - 1443.46 X_2 - 845.2 X_3 \qquad (MPa)$$

For example, for the point O we have

$\sigma_{11}(X_2 = 0, X_3 = 0) = 102.581 - 1443.46 X_2 - 845.2 X_3 = 102.581 (MPa)$

Point B: $\sigma_{11}(X_2 = 0.15, X_3 = 0) = 102.581 - 1443.46 X_2 - 845.2 X_3 = -113.938 (MPa)$

Point E: $\sigma_{11}(X_2 = 0, X_3 = 0.15) = 102.581 - 1443.46 X_2 - 845.2 X_3 = -24.199 (MPa)$

The Neutral axis

If we are adopting the system $X_2 - X_3$, the neutral axis can be obtained by means of the equation (4.66):

$$c_2 X_2 + c_3 X_3 = -c_1 \quad \Rightarrow \quad c_3 X_3 = -c_2 X_2 - c_1 \quad \Rightarrow \quad X_3 = \frac{-c_1}{c_3} - \frac{c_2}{c_3} X_2$$

$$X_3 = \frac{-(102.581\times 10^6)}{(-845.2\times 10^6)} - \frac{(-1443.46\times 10^6)}{(-845.2\times 10^6)} X_2 \quad \Rightarrow \quad X_3 = 0.12137 - 1.7078 X_2$$

which matches the equation presented in Figure 4.19.

The inertia tensor at the Area Centroid, (see equation in (4.75)), can be obtained by means of the Steiner's theorem:

$$\mathbf{I}_{O\vec{X}}^{(Sys)} = \bar{\mathbf{I}}_{G\vec{x}}^{(Sys)} - A[(\bar{\vec{X}} \otimes \bar{\vec{X}}) - (\bar{\vec{X}} \cdot \bar{\vec{X}})\mathbf{1}] \quad \Rightarrow \quad \bar{\mathbf{I}}_{G\vec{x}}^{(Sys)} = \mathbf{I}_{O\vec{X}}^{(Sys)} + A[(\bar{\vec{X}} \otimes \bar{\vec{X}}) - (\bar{\vec{X}} \cdot \bar{\vec{X}})\mathbf{1}]$$

whose components are:

$$(\bar{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \begin{bmatrix} \bar{\mathrm{I}}_{G22} & \bar{\mathrm{I}}_{G23} \\ \bar{\mathrm{I}}_{G23} & \bar{\mathrm{I}}_{G33} \end{bmatrix} = \mathrm{I}_{O\vec{X}\,ij}^{(Sys)} - A \begin{bmatrix} \bar{X}_3^2 & -\bar{X}_2\bar{X}_3 \\ -\bar{X}_2\bar{X}_3 & \bar{X}_2^2 \end{bmatrix} \tag{4.83}$$

where $\bar{\vec{X}}$ is the vector position of the Area Centroid of the cross section related to the system $O\vec{X}$. By substituting the variable values ($\mathbf{I}_{O\vec{X}}^{(Sys)}$, $A = 0.0056 m^2$, $\bar{X}_2 = 0.04482143 m$, $\bar{X}_3 = 0.04482143 m$) we can obtain:

$$(\bar{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \begin{bmatrix} 22.8466667 & -4.46 \\ -4.46 & 22.8466667 \end{bmatrix} \times 10^{-6} - (0.0056) \begin{bmatrix} (0.0448)^2 & -(0.0448)^2 \\ -(0.0448)^2 & (0.0448)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1159.6487 & 679.01793 \\ 679.01793 & 1159.6487 \end{bmatrix} \times 10^{-8} m^4$$

which matches the equation in (4.76).

NOTE 4: Let us consider the rectangle as the one indicated in Figure 4.21. Next we will obtain the Inertia Tensor of Area related to the system $O\vec{X}$.

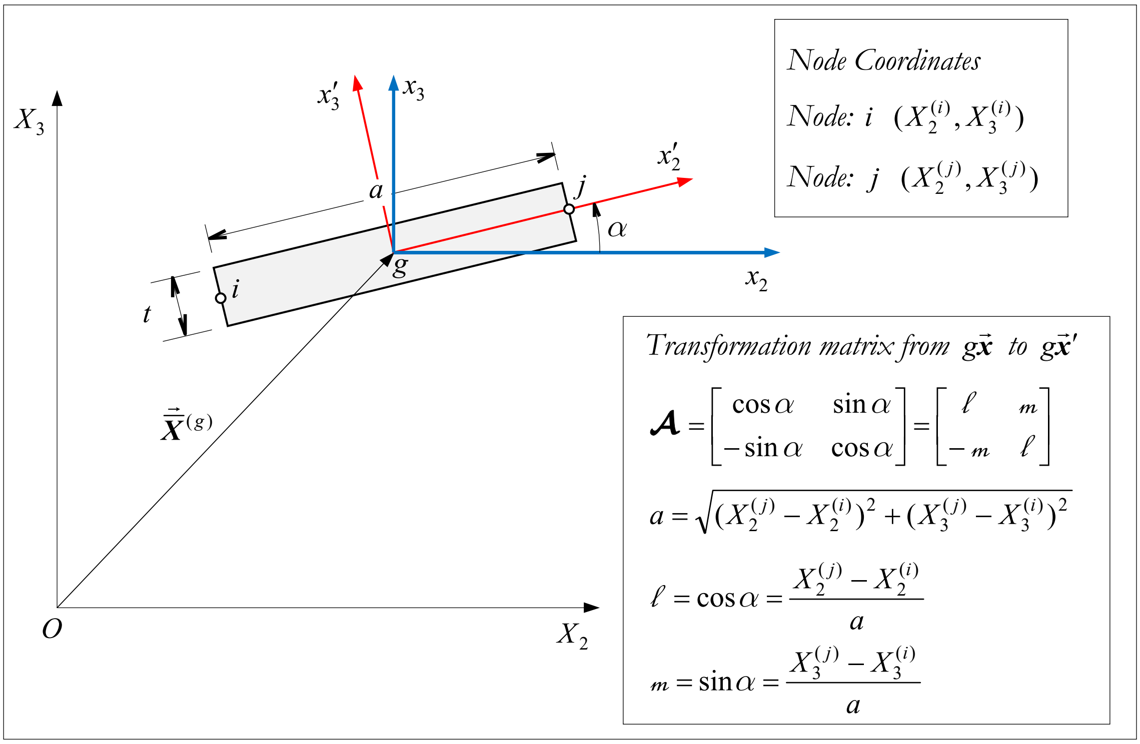


Figure 4.21

The inertia tensor of area related to the system $g\vec{x}'$ is given by

$$\mathrm{I}'_{g\vec{x}'\,ij} = \frac{1}{12}\begin{bmatrix} at^3 & 0 \\ 0 & ta^3 \end{bmatrix}$$

Taking into account the component transformation law for a second-order tensor, (see equation (4.77)), we can obtain the inertia tensor of area in the system $g\vec{x}$, i.e.:

$$\mathrm{I}_{g\vec{x}\,ij} = \mathcal{A}_{ip}\mathrm{I}'_{g\vec{x}'\,ij}\mathcal{A}_{qj} = \frac{1}{12}\begin{bmatrix} \ell & -m \\ m & \ell \end{bmatrix}\begin{bmatrix} at^3 & 0 \\ 0 & ta^3 \end{bmatrix}\begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix}$$

$$\Rightarrow \mathrm{I}_{g\vec{x}\,ij} = \frac{at}{12}\begin{bmatrix} (t^2\ell^2 + a^2 m^2) & \ell m(t^2 - a^2) \\ \ell m(t^2 - a^2) & (t^2 m^2 + a^2 \ell^2) \end{bmatrix}$$

Then, by means of the equation in (4.75) we can obtain:

$$\mathrm{I}_{O\vec{X}\,ij} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \mathrm{I}_{g\vec{x}\,ij} + A\begin{bmatrix} (X_3^{(g)})^2 & -X_2^{(g)}X_3^{(g)} \\ -X_2^{(g)}X_3^{(g)} & (X_2^{(g)})^2 \end{bmatrix}$$

where $A = at$, $X_2^{(g)} = \dfrac{X_2^{(i)} + X_2^{(j)}}{2}$ and $X_3^{(g)} = \dfrac{X_3^{(i)} + X_3^{(j)}}{2}$. The above equation can also be written as follows

$$\boxed{\mathrm{I}_{O\vec{X}\,ij} = \frac{at}{12}\begin{bmatrix} t^2\ell^2 + a^2 m^2 + 12(X_3^{(g)})^2 & \ell m(t^2 - a^2) - 12X_2^{(g)}X_3^{(g)} \\ \ell m(t^2 - a^2) - 12X_2^{(g)}X_3^{(g)} & t^2 m^2 + a^2\ell^2 + 12(X_2^{(g)})^2 \end{bmatrix}} \tag{4.84}$$

Problem 4.24

Consider a cross section described in Figure 4.22, (Buchanan (1988)), in which acts only the normal stress σ_{11}. Knowing that the moments at the Area Centroid are $M_y = -67.5kNm$ and $M_z = -28.13kNm$, obtain the normal stress σ_{11} at the points O, B, C and D.

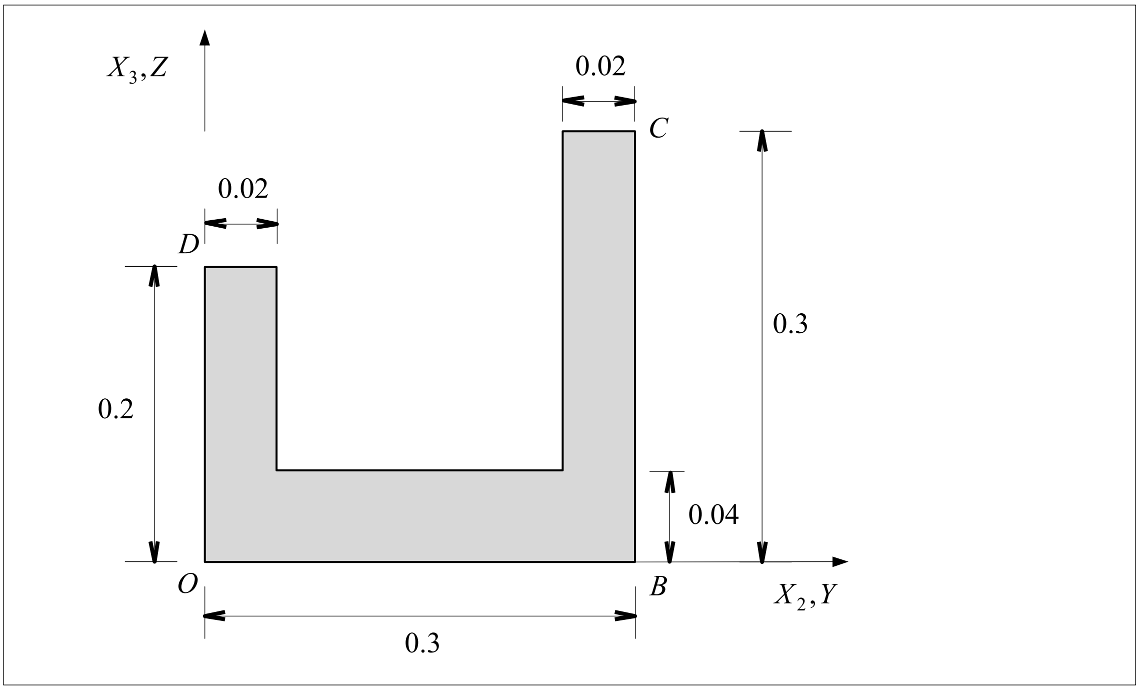
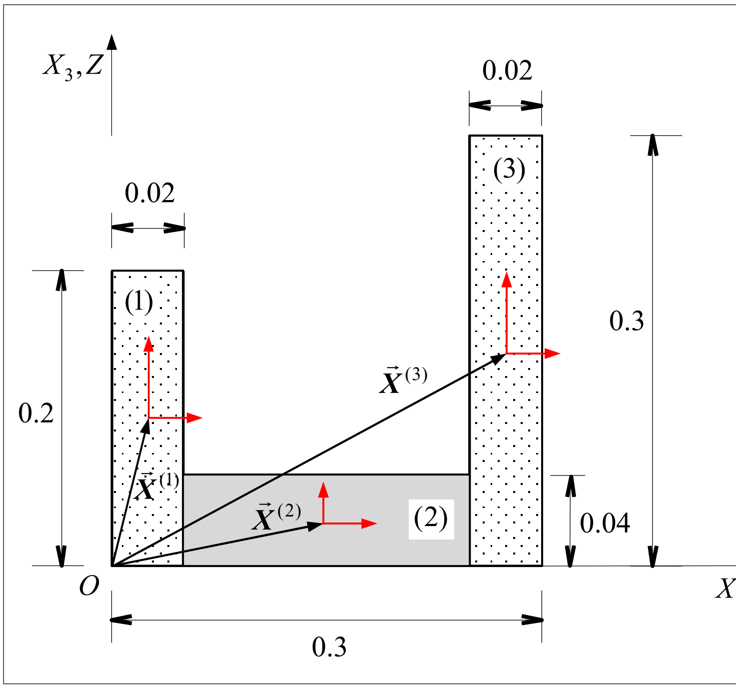


Figure 4.22: Cross section – Dimensions in meter (m).

Solution:

Geometry decomposition:



$$\vec{X}^{(1)} = 0.01\hat{\mathbf{J}} + 0.1\hat{\mathbf{K}}$$
$$\vec{X}^{(2)} = 0.15\hat{\mathbf{J}} + 0.02\hat{\mathbf{K}}$$
$$\vec{X}^{(3)} = 0.29\hat{\mathbf{J}} + 0.15\hat{\mathbf{K}}$$

$$A^{(1)} = 0.02 \times 0.2 = 0.004m^2$$
$$A^{(2)} = 0.26 \times 0.04 = 0.0104m^2$$
$$A^{(3)} = 0.02 \times 0.3 = 0.006m^2$$

$$A = A^{(1)} + A^{(2)} + A^{(3)} = 0.0204m^2$$

Figure 4.23: Geometric decomposition of the cross section.

Calculation of the Area Centroid- G :

$$\overline{X}_2 \equiv \overline{Y} = \frac{\int_A X_2 dA}{\int_A dA} = \frac{\sum_{a=1}^{3} A^{(a)} X_2^{(a)}}{A} = \frac{A^{(1)}X_2^{(1)} + A^{(2)}X_2^{(2)} + A^{(3)}X_2^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})}$$

$$= \frac{0.004 \times 0.01 + 0.0104 \times 0.15 + 0.006 \times 0.29}{0.0204} = 0.163723m$$

$$\overline{X}_3 \equiv \overline{Z} = \frac{\int_A X_3 dA}{\int_A dA} = \frac{\sum_{a=1}^{3} A^{(a)} X_3^{(a)}}{A} = \frac{A^{(1)}X_3^{(1)} + A^{(2)}X_3^{(2)} + A^{(3)}X_3^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})}$$

$$= \frac{0.004 \times 0.1 + 0.0104 \times 0.02 + 0.006 \times 0.15}{0.0204} = 0.073922m$$

Calculation of the position vector - $\vec{\overline{x}}^{(a)}$

The position vector $\vec{\overline{x}}^{(a)} = \vec{X}^{(a)} - \vec{\overline{X}}$ can be obtained as follows:

$$\vec{\overline{x}}^{(1)} = \vec{X}^{(1)} - \vec{\overline{X}} = (X_2^{(1)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(1)} - \overline{Z})\hat{\mathbf{k}} = (0.01 - 0.163723)\hat{\mathbf{j}} + (0.1 - 0.073922)\hat{\mathbf{k}}$$
$$= -0.153723\hat{\mathbf{j}} + 0.026078\hat{\mathbf{k}}$$

$$\vec{\overline{x}}^{(2)} = \vec{X}^{(2)} - \vec{\overline{X}} = (X_2^{(2)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(2)} - \overline{Z})\hat{\mathbf{k}} = (0.15 - 0.163723)\hat{\mathbf{j}} + (0.02 - 0.073922)\hat{\mathbf{k}}$$
$$= -0.013723\hat{\mathbf{j}} - 0.053922\hat{\mathbf{k}}$$

$$\vec{\overline{x}}^{(3)} = \vec{X}^{(3)} - \vec{\overline{X}} = (X_2^{(3)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(3)} - \overline{Z})\hat{\mathbf{k}} = (0.29 - 0.163723)\hat{\mathbf{j}} + (0.15 - 0.073922)\hat{\mathbf{k}}$$
$$= 0.126277\hat{\mathbf{j}} + 0.076078\hat{\mathbf{k}}$$

Calculation of the inertia tensor at the Area Centroid:

Rectangle 1 - $\mathbf{I}_{G\vec{x}}^{(1)}$: $a = 0.02; b = 0.2$

$$\overline{\mathrm{I}}_{g\ ij}^{(1)} = \begin{bmatrix} \overline{\mathrm{I}}_{22} & \overline{\mathrm{I}}_{23} \\ \overline{\mathrm{I}}_{23} & \overline{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{0.02 \times 0.2^3}{12} & 0 \\ 0 & \dfrac{0.2 \times 0.02^3}{12} \end{bmatrix} = \begin{bmatrix} 1.333 & 0 \\ 0 & 0.01333 \end{bmatrix} \times 10^{-5} m^4$$

Area centroid vector position: $\vec{\overline{x}}^{(1)} = -0.153723\hat{\mathbf{j}} + 0.026078\hat{\mathbf{k}}$, $A^{(1)} = 0.004m^2$:

$$(\mathbf{I}_{G\vec{x}}^{(1)})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{22} & \overline{\mathrm{I}}_{23} \\ \overline{\mathrm{I}}_{23} & \overline{\mathrm{I}}_{33} \end{bmatrix} + A^{(1)} \begin{bmatrix} \overline{x}_1^2 + \overline{x}_3^2 & -\overline{x}_2\overline{x}_3 \\ -\overline{x}_2\overline{x}_3 & \overline{x}_1^2 + \overline{x}_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.333 & 0 \\ 0 & 0.01333 \end{bmatrix} \times 10^{-5} + 0.004 \begin{bmatrix} (0.026078)^2 & -(-0.153723)(0.026078) \\ -(-0.153723)(0.026078) & (-0.153723)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.605 & 1.604 \\ 1.604 & 9.466 \end{bmatrix} \times 10^{-5} m^4$$

Rectangle 2 - $\mathbf{I}_{G\vec{x}}^{(2)}$: $a = 0.26; b = 0.04$

$$\overline{\mathrm{I}}_{g\ ij}^{(2)} = \begin{bmatrix} \overline{\mathrm{I}}_{22} & \overline{\mathrm{I}}_{23} \\ \overline{\mathrm{I}}_{23} & \overline{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{0.26 \times 0.04^3}{12} & 0 \\ 0 & \dfrac{0.04 \times 0.26^3}{12} \end{bmatrix} = \begin{bmatrix} 0.13867 & 0 \\ 0 & 5.859 \end{bmatrix} \times 10^{-5} m^4$$

Area centroid vector position: $\bar{\vec{x}}^{(2)} = -0.013723\hat{\mathbf{j}} - 0.053922\hat{\mathbf{k}}$, $A^{(2)} = 0.0104m^2$:

$$
\begin{aligned}
(\mathbf{I}^{(2)}_{G\vec{x}})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(2)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 0.13867 & 0 \\ 0 & 5.859 \end{bmatrix} \times 10^{-5} + 0.0104 \begin{bmatrix} (0.026078)^2 & -(0.013723)(0.053922) \\ -(0.013723)(0.053922) & (-0.013723)^2 \end{bmatrix} \\
&= \begin{bmatrix} 3.163 & -0.76957 \\ -0.76957 & 6.0545 \end{bmatrix} \times 10^{-5} m^4
\end{aligned}
$$

Rectangle 3 - $\mathbf{I}^{(3)}_{G\vec{x}}$: $a = 0.02; b = 0.3$

$$
\bar{\mathrm{I}}^{(3)}_{g\ ij} = \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{0.02 \times 0.3^3}{12} & 0 \\ 0 & \dfrac{0.3 \times 0.02^3}{12} \end{bmatrix} = \begin{bmatrix} 4.5 & 0 \\ 0 & 0.02 \end{bmatrix} \times 10^{-5} m^4
$$

Area centroid vector position: $\bar{\vec{x}}^{(3)} = 0.126277\hat{\mathbf{j}} + 0.076078\hat{\mathbf{k}}$, $A^{(3)} = 0.006m^2$:

$$
\begin{aligned}
(\mathbf{I}^{(3)}_{G\vec{x}})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(3)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 4.5 & 0 \\ 0 & 0.02 \end{bmatrix} \times 10^{-5} + 0.006 \begin{bmatrix} (0.076078)^2 & -(0.126277)(0.076078) \\ -(0.126277)(0.076078) & (0.126277)^2 \end{bmatrix} \\
&= \begin{bmatrix} 7.9727 & -5.764 \\ -5.764 & 9.5875 \end{bmatrix} \times 10^{-5} m^4
\end{aligned}
$$

Then, we can calculate the inertia tensor of the cross section related to the system located at the Area Centroid - G :

$$
\begin{aligned}
(\mathbf{I}^{(sys)}_{G\vec{x}})_{ij} &= \sum_{a=1}^{Nb} (\mathbf{I}^{(a)}_{G\vec{x}})_{ij} = (\mathbf{I}^{(1)}_{G\vec{x}})_{ij} + (\mathbf{I}^{(2)}_{G\vec{x}})_{ij} + (\mathbf{I}^{(3)}_{G\vec{x}})_{ij} \\
&= \left(\begin{bmatrix} 1.605 & 1.604 \\ 1.604 & 9.466 \end{bmatrix} + \begin{bmatrix} 3.163 & -0.76957 \\ -0.76957 & 6.0545 \end{bmatrix} + \begin{bmatrix} 7.9727 & -5.764 \\ -5.764 & 9.5875 \end{bmatrix} \right) \times 10^{-5} \\
&= \begin{bmatrix} 1.2741 & -0.49302 \\ -0.49302 & 2.5108 \end{bmatrix} \times 10^{-4} m^4 = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix}
\end{aligned}
\tag{4.85}
$$

<u>Calculation of the normal stress:</u>

$$
\begin{aligned}
\sigma_{11}(x_2, x_3) &= \frac{(-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23})x_3 - (M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22})x_2}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} \\
\Rightarrow \sigma_{11}(x_2, x_3) &= -\left(\frac{M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22}}{\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33}} \right) x_2 + \left(\frac{-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23}}{\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33}} \right) x_3
\end{aligned}
\tag{4.86}
$$

Taking into account the values for $(\mathbf{I}^{(sys)}_{G\vec{x}})_{ij}$, (see equation (4.85)), and $M_y = -67.5kNm$ and $M_z = -28.13kNm$, the above equation becomes:

$$
\sigma_{11}(x_2, x_3) = 233.83859x_2 - 620.2893x_3 \qquad [MPa] \tag{4.87}
$$

Point coordinates (meter-m)					Normal stress
	$X_2[m]$	$X_3[m]$	$x_2 = X_2 - \overline{Y}[m]$	$x_3 = X_3 - \overline{Z}[m]$	$\sigma_{11}[MPa]$
O	0.0	0.0	-0.16372	-0.07392	7.568242
B	0.3	0.0	0.13628	-0.07392	77.71982
C	0.3	0.3	0.13628	0.22608	-108.366858
D	0.0	0.2	-0.16372	0.12608	-116.489543
E	0.02	0.2	-0.14372	0.12608	-111.81
F	0.02	0.04	-0.14372	-0.03392	-12.57
Q	0.28	0.04	0.116277	-0.03392	48.23
H	0.28	0.3	0.116277	0.22608	-113.04

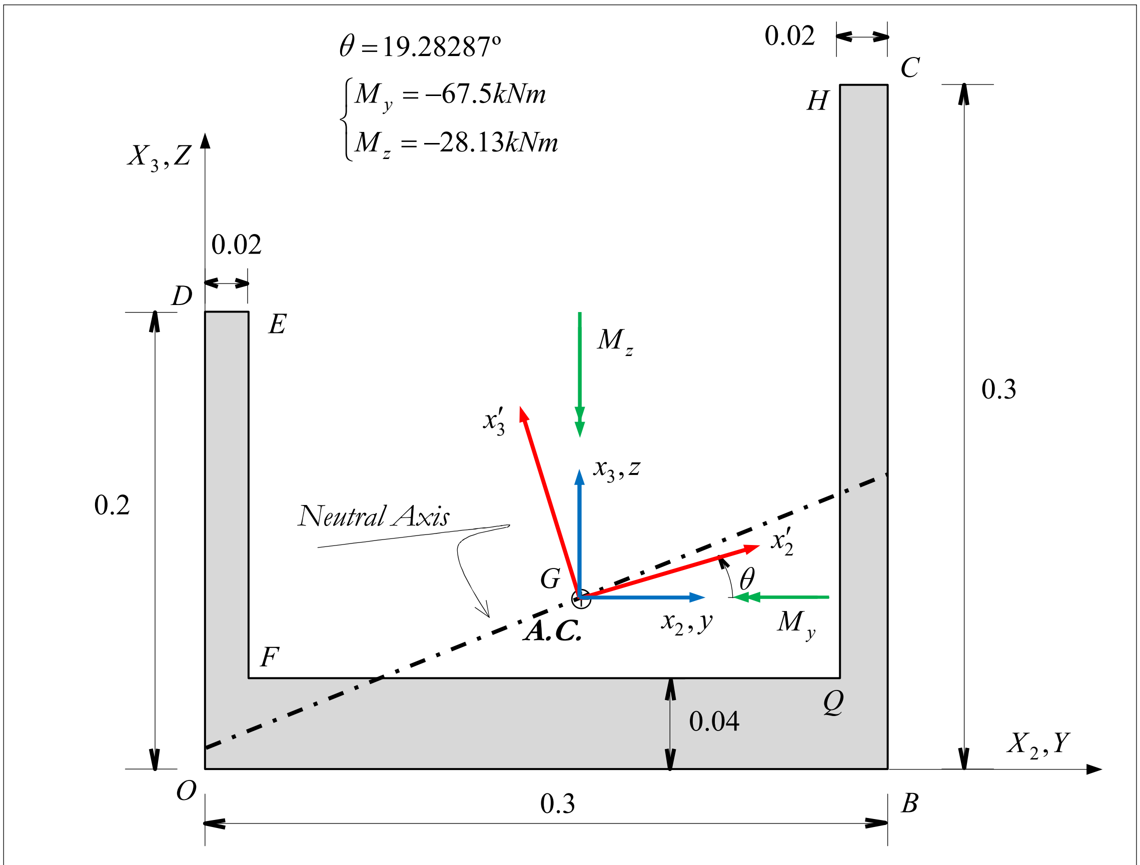


Figure 4.24: Cross section – Dimensions in meter (m).

The Neutral Axis (N.A.) can be obtained as follows:

$$\sigma_{11}(x_2, x_3) = 233.83859 x_2 - 620.2893 x_3 = 0 \quad \Rightarrow \quad x_3 = \frac{233.83859}{620.2893} x_2$$

then, the neutral axis is given by:

$$\begin{aligned} & x_3 = 0.3769833 x_2 \\ & \Rightarrow x_3 = \tan(\alpha) x_2 \qquad \therefore \qquad \alpha = \arctan(0.3769833) = 20.656^{\circ} \end{aligned} \tag{4.88}$$

Calculation of the Inertia Tensor Principal Space:

$$
\begin{aligned}
\theta &= \frac{1}{2}\arctan\left(\frac{2\mathrm{I}_{23}}{\mathrm{I}_{22}-\mathrm{I}_{33}}\right) = \frac{1}{2}\arctan\left(\frac{2(-0.49302)}{(1.2741)-(2.5108)}\right) = \frac{1}{2}\arctan(0.797311) \\
&= 0.3365496\mathit{rad} \\
&\Rightarrow \theta = 0.3365496\frac{180}{\pi} = 19.28287^{\circ}
\end{aligned}
\tag{4.89}
$$

Then the principal values for the inertia tensor are:

$$
\begin{aligned}
\begin{bmatrix} \mathrm{I}'_{22} \\ \mathrm{I}'_{33} \\ \mathrm{I}'_{23} \end{bmatrix} &= \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{bmatrix} \mathrm{I}_{22} \\ \mathrm{I}_{33} \\ \mathrm{I}_{23} \end{bmatrix} \\
&= \begin{bmatrix} 0.89095 & 0.10905 & 0.62341 \\ 0.10905 & 0.89095 & -0.62341 \\ -0.62341 & 0.31171 & 0.78189 \end{bmatrix} \begin{bmatrix} 1.2741 \\ 2.5108 \\ -0.49302 \end{bmatrix} \times 10^{-4} = \begin{bmatrix} 1.10158 \\ 2.68326 \\ 0 \end{bmatrix} \times 10^{-4}\, m^4
\end{aligned}
\tag{4.90}
$$

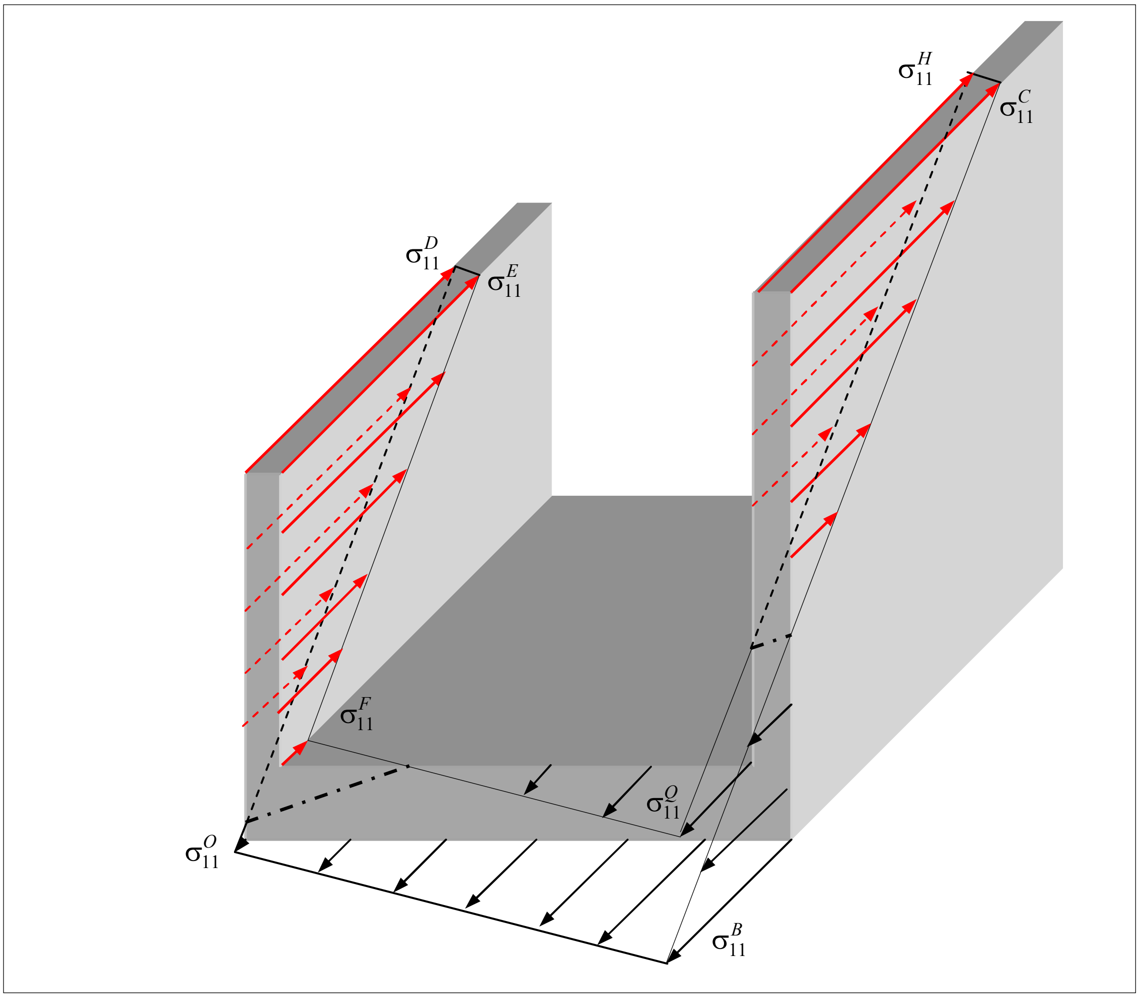


Figure 4.25: Normal stress distribution on the cross section.

Problem 4.25

Consider a cross section described in Figure 4.26, (Cervera&Blanco (2001)), in which acts only the normal stress σ_{11}. Knowing that at the point $p(X_2 = 9.5cm; X_3 = 19.5cm)$ there is a compression force $\vec{\boldsymbol{P}} = -150kN\hat{\mathbf{I}}$, obtain the normal stress σ_{11} at the points B, C, D, E and F.

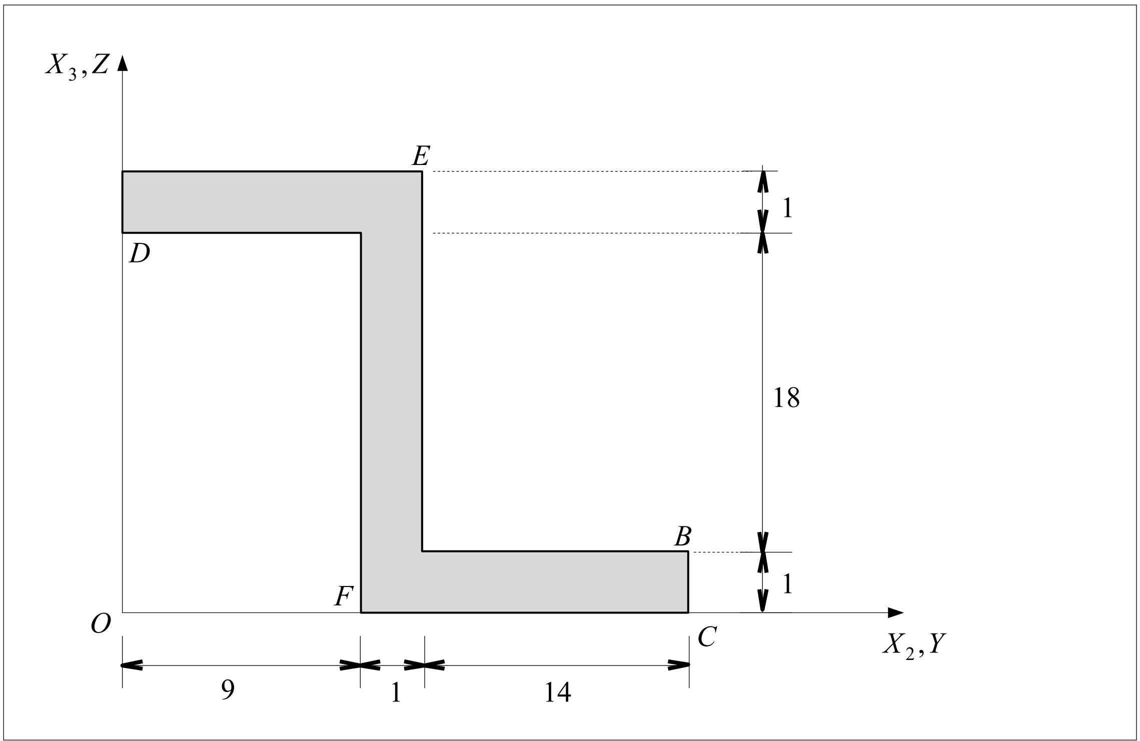


Figure 4.26: Cross section – Dimensions in centimeter (cm).

Solution:

Geometry decomposition:

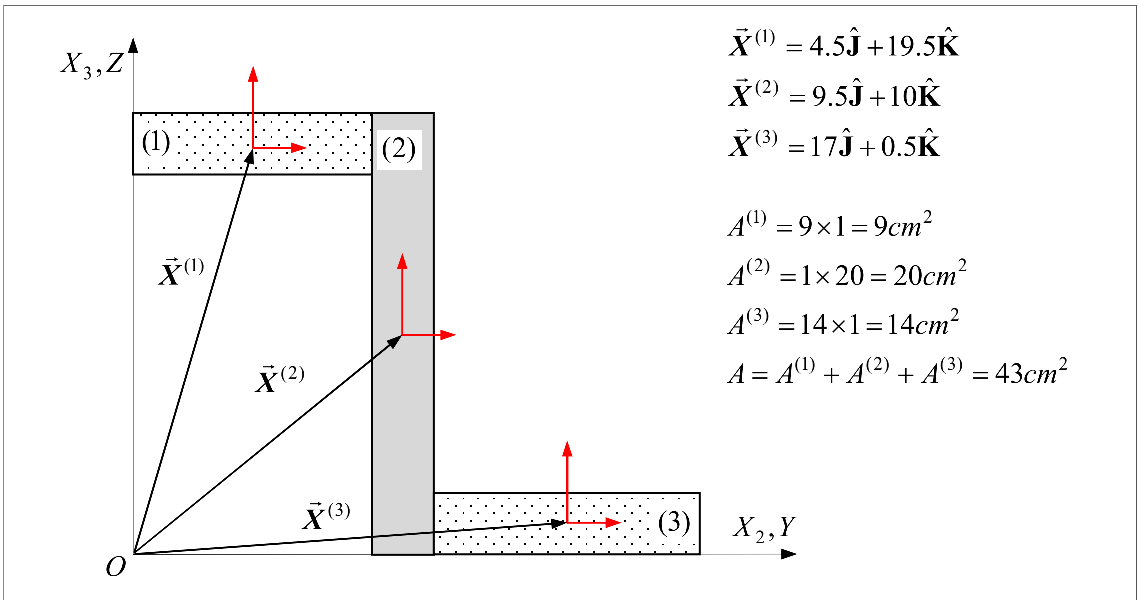


Figure 4.27: Geometric decomposition of the cross section.

Calculation of the Area Centroid - G :

$$\overline{X}_2 \equiv \overline{Y} = \frac{\int_A X_2 dA}{\int_A dA} = \frac{\sum_{a=1}^{3} A^{(a)} X_2^{(a)}}{A} = \frac{A^{(1)} X_2^{(1)} + A^{(2)} X_2^{(2)} + A^{(3)} X_2^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})}$$

$$= \frac{9 \times 4.5 + 20 \times 9.5 + 14 \times 17}{43} = 10.895 cm$$

$$\overline{X}_3 \equiv \overline{Z} = \frac{\int_A X_3 dA}{\int_A dA} = \frac{\sum_{a=1}^{3} A^{(a)} X_3^{(a)}}{A} = \frac{A^{(1)} X_3^{(1)} + A^{(2)} X_3^{(2)} + A^{(3)} X_3^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})}$$

$$= \frac{9 \times 19.5 + 20 \times 10 + 14 \times 0.5}{43} = 8.895 cm$$

Vector position of the Area Centroid: $\vec{\overline{\boldsymbol{X}}} = 10.895 \hat{\mathbf{J}} + 8.895 \hat{\mathbf{K}}$, (see Figure 4.28).

Calculation of the position vector

The position vector $\vec{\overline{\boldsymbol{x}}}^{(a)} = \vec{\boldsymbol{X}}^{(a)} - \vec{\overline{\boldsymbol{X}}}$ can be obtained as follows:

$$\vec{\overline{\boldsymbol{x}}}^{(1)} = \vec{\boldsymbol{X}}^{(1)} - \vec{\overline{\boldsymbol{X}}} = (X_2^{(1)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(1)} - \overline{Z})\hat{\mathbf{k}} = (4.5 - 10.895)\hat{\mathbf{j}} + (19.5 - 8.895)\hat{\mathbf{k}}$$
$$= -6.395\hat{\mathbf{j}} + 10.605\hat{\mathbf{k}}$$

$$\vec{\overline{\boldsymbol{x}}}^{(2)} = \vec{\boldsymbol{X}}^{(2)} - \vec{\overline{\boldsymbol{X}}} = (X_2^{(2)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(2)} - \overline{Z})\hat{\mathbf{k}} = (9.5 - 10.895)\hat{\mathbf{j}} + (10 - 8.895)\hat{\mathbf{k}}$$
$$= -1.395\hat{\mathbf{j}} + 1.105\hat{\mathbf{k}}$$

$$\vec{\overline{\boldsymbol{x}}}^{(3)} = \vec{\boldsymbol{X}}^{(3)} - \vec{\overline{\boldsymbol{X}}} = (X_2^{(3)} - \overline{Y})\hat{\mathbf{j}} + (X_3^{(3)} - \overline{Z})\hat{\mathbf{k}} = (17 - 10.895)\hat{\mathbf{j}} + (0.5 - 8.895)\hat{\mathbf{k}}$$
$$= 6.105\hat{\mathbf{j}} - 8.395\hat{\mathbf{k}}$$

Calculation of the inertia tensor at the Area Centroid:

Rectangle 1 - $\mathbf{I}_{G\vec{x}}^{(1)}$: $a = 9cm; b = 1cm$

$$\overline{\mathrm{I}}_{g\ ij}^{(1)} = \begin{bmatrix} \overline{\mathrm{I}}_{22} & \overline{\mathrm{I}}_{23} \\ \overline{\mathrm{I}}_{23} & \overline{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \frac{ab^3}{12} & 0 \\ 0 & \frac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \frac{9 \times 1^3}{12} & 0 \\ 0 & \frac{1 \times 9^3}{12} \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 60.75 \end{bmatrix} cm^4$$

Area centroid vector position: $\vec{\overline{\boldsymbol{x}}}^{(1)} = -6.395\hat{\mathbf{j}} + 10.605\hat{\mathbf{k}}$, $A^{(1)} = 9cm^2$:

$$(\mathbf{I}_{G\vec{x}}^{(1)})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{22} & \overline{\mathrm{I}}_{23} \\ \overline{\mathrm{I}}_{23} & \overline{\mathrm{I}}_{33} \end{bmatrix} + A^{(1)} \begin{bmatrix} \overline{x}_1^2 + \overline{x}_3^2 & -\overline{x}_2 \overline{x}_3 \\ -\overline{x}_2 \overline{x}_3 & \overline{x}_1^2 + \overline{x}_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0 \\ 0 & 60.75 \end{bmatrix} + 9 \begin{bmatrix} (10.605)^2 & -(-6.395)(10.605) \\ -(-6.395)(10.605) & (-6.395)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.01294 \times 10^3 & 610.37077 \\ 610.37077 & 428.81422 \end{bmatrix} cm^4$$

Rectangle 2 - $\mathbf{I}^{(2)}_{G\vec{x}}$: $a = 1cm; b = 20cm$

$$\bar{\mathrm{I}}^{(2)}_{g\ ij} = \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{1\times 20^3}{12} & 0 \\ 0 & \dfrac{20\times 1^3}{12} \end{bmatrix} = \begin{bmatrix} 666.6667 & 0 \\ 0 & 666.6667 \end{bmatrix} cm^4$$

Area centroid vector position: $\vec{\bar{x}}^{(2)} = -1.395\hat{\mathbf{j}} + 1.105\hat{\mathbf{k}}$, $A^{(2)} = 20cm^2$:

$$\begin{aligned}
(\mathbf{I}^{(2)}_{G\vec{x}})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(2)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 666.6667 & 0 \\ 0 & 666.6667 \end{bmatrix} + 20 \begin{bmatrix} (1.105)^2 & -(-1.395)(1.105) \\ -(-1.395)(1.105) & (-1.395)^2 \end{bmatrix} \\
&= \begin{bmatrix} 691.08717 & 30.8295 \\ 30.8295 & 40.58717 \end{bmatrix} cm^4
\end{aligned}$$

Rectangle 3 - $\mathbf{I}^{(3)}_{G\vec{x}}$: $a = 14cm; b = 1cm$

$$\bar{\mathrm{I}}^{(3)}_{g\ ij} = \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{ab^3}{12} & 0 \\ 0 & \dfrac{ba^3}{12} \end{bmatrix} = \begin{bmatrix} \dfrac{14\times 1^3}{12} & 0 \\ 0 & \dfrac{1\times 14^3}{12} \end{bmatrix} = \begin{bmatrix} 1.166667 & 0 \\ 0 & 228.66667 \end{bmatrix} cm^4$$

Area centroid vector position: $\vec{\bar{x}}^{(3)} = 6.105\hat{\mathbf{j}} - 8.395\hat{\mathbf{k}}$, $A^{(3)} = 14cm^2$:

$$\begin{aligned}
(\mathbf{I}^{(3)}_{G\vec{x}})_{ij} &= \begin{bmatrix} \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + A^{(3)} \begin{bmatrix} \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 1.166667 & 0 \\ 0 & 228.66667 \end{bmatrix} + 14 \begin{bmatrix} (-8.395)^2 & -(6.105)(-8.395) \\ -(6.105)(-8.395) & (6.105)^2 \end{bmatrix} \\
&= \begin{bmatrix} 987.83102 & 717.52065 \\ 717.52065 & 750.46102 \end{bmatrix} cm^4
\end{aligned}$$

Then, we can calculate the inertia tensor of the cross section related to the system located at the Area Centroid - G :

$$\begin{aligned}
(\mathbf{I}^{(sys)}_{G\vec{x}})_{ij} &= (\mathbf{I}^{(1)}_{G\vec{x}})_{ij} + (\mathbf{I}^{(2)}_{G\vec{x}})_{ij} + (\mathbf{I}^{(3)}_{G\vec{x}})_{ij} \\
&= \begin{bmatrix} 1.01294\times 10^3 & 610.37077 \\ 610.37077 & 428.81422 \end{bmatrix} + \begin{bmatrix} 691.08717 & 30.8295 \\ 30.8295 & 40.58717 \end{bmatrix} + \begin{bmatrix} 987.83102 & 717.52065 \\ 717.52065 & 750.46102 \end{bmatrix} \\
&= \begin{bmatrix} 2.69186 & 1.35872 \\ 1.35872 & 1.21986 \end{bmatrix} \times 10^3 cm^4 = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix}
\end{aligned} \tag{4.91}$$

<u>Calculation of the normal stress:</u>

Note that the cross section is under a compression force, so that we must use the equation in (4.61), i.e.:

$$\sigma_{11}(x_2, x_3) = \frac{N}{A} - \frac{(M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_2 + \frac{(-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_3 \tag{4.92}$$

To use the above equations, all the variables must be expressed in the system $G-x_1-x_2-x_3$.

p-point Vector position in the system $X_1-X_2-X_3 \Rightarrow \vec{\boldsymbol{X}}^{(P)} = 9.5\hat{\mathbf{J}} + 19.5\hat{\mathbf{K}}$

Knowing that $\vec{\overline{\boldsymbol{X}}} + \vec{\boldsymbol{x}} = \vec{\boldsymbol{X}} \Rightarrow \vec{\boldsymbol{x}} = \vec{\boldsymbol{X}} - \vec{\overline{\boldsymbol{X}}}$, where $\vec{\overline{\boldsymbol{X}}} = 10.895\hat{\mathbf{J}} + 8.895\hat{\mathbf{K}}$, we can calculate the p-point Vector position in the system $G-x_1-x_2-x_3$ as follows:

$$\vec{\boldsymbol{x}}^{(P)} = \vec{\boldsymbol{X}}^{(P)} - \vec{\overline{\boldsymbol{X}}} \quad \Rightarrow \quad \vec{\boldsymbol{x}}^{(P)} = (9.5-10.895)\hat{\mathbf{j}} + (19.5-8.895)\hat{\mathbf{k}} = -1.395\hat{\mathbf{j}} + 10.605\hat{\mathbf{k}}$$

Then, the moment due to the force $\vec{\boldsymbol{P}} = -150kN\hat{\mathbf{I}} = -150kN\hat{\mathbf{i}}$, at the point G, can be obtained as follows:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{G\vec{x}} &= \vec{\boldsymbol{x}}^{(P)} \wedge \vec{\boldsymbol{P}} = (-1.395\hat{\mathbf{j}} + 10.605\hat{\mathbf{k}}) \wedge (-150\hat{\mathbf{i}}) \\
&= (-1.395)\times(-150) + \underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{=-\hat{\mathbf{k}}}(10.605)\times(-150)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{=\hat{\mathbf{j}}} \\
&= -209.25\hat{\mathbf{k}} - 1590.75\hat{\mathbf{j}} \qquad (kNcm) \\
&= M_z\hat{\mathbf{k}} + M_y\hat{\mathbf{j}}
\end{aligned}$$

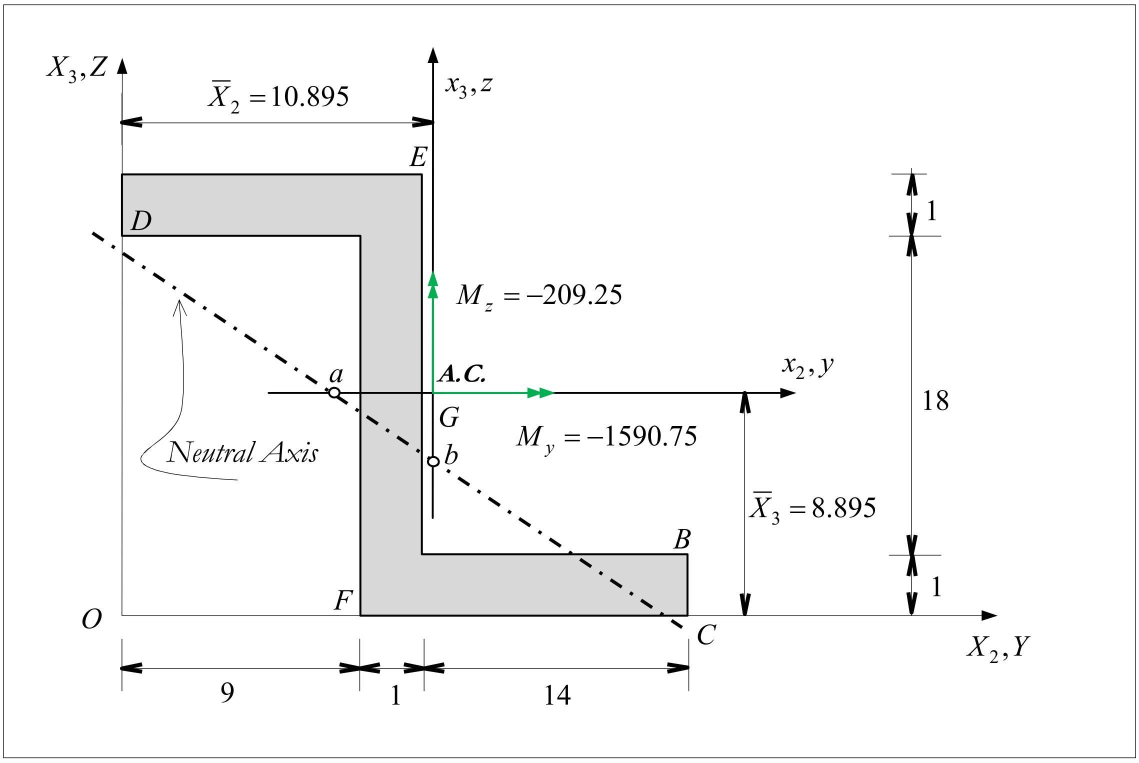


Figure 4.28: Cross section – Dimensions in centimeter (cm).

Taking into account the values for $(\mathbf{I}^{(sys)}_{G\vec{x}})_{ij}$, (see equation (4.91)), and $N = -150kN$, $A = 43cm^2$, $M_y = -1590.75kNcm$ and $M_z = -209.25kNcm$, the equation in (4.92) becomes:

$$\sigma_{11}(x_2, x_3) = -34.88372 - 11.520644x_3 - 11.1167x_2 \qquad [MPa] \qquad (4.93)$$

By means of the above equation and by considering the coordinates of the points we can obtain:

	Point coordinates ($\times 10^{-2} m$)				Normal stress
	X_2	X_3	$x_2 = X_2 - \overline{X}_2$	$x_3 = X_3 - \overline{X}_3$	$\sigma_{11}[MPa]$
B	24	1	13.105	-7.895	-89.6125
C	24	0	13.105	-8.895	-78.092
D	0	19	-0.895	11.105	-30.1834
E	10	20	0.13628	0.22608	-152.8710
F	9	0	-1.895	-8.895	88.6585

The Neutral Axis (N.A.), (see Figure 4.28), can be obtained as follows:

$$\sigma_{11}(x_2, x_3) = -34.88372 - 11.520644 x_3 - 11.1167 x_2 = 0$$
$$\Rightarrow -11.520644 x_3 - 11.1167 x_2 = 34.88372$$
$$\Rightarrow \frac{x_3}{(-3.02793)} + \frac{x_2}{(-3.13796)} = 1.0$$

then, the neutral axis is given by its canonic form as follows:

$$\frac{x_3}{(-3.02793)} + \frac{x_2}{(-3.13796)} = 1.0 \quad \Leftrightarrow \quad \frac{x_3}{b} + \frac{x_2}{a} = 1.0 \tag{4.94}$$

Calculation of the Inertia Tensor Principal Space:

$$\theta = \frac{1}{2}\operatorname{arctan}\left(\frac{2\mathrm{I}_{23}}{\mathrm{I}_{22} - \mathrm{I}_{33}}\right) = \frac{1}{2}\operatorname{arctan}\left(\frac{2(1.35872)}{(2.69186)-(1.21986)}\right) = \frac{1}{2}\operatorname{arctan}(1.8460882)$$
$$= 0.5371793 rad \tag{4.95}$$
$$\Rightarrow \theta = 0.5371793\frac{180}{\pi} = 30.778^{\circ}$$

Then the principal values for the inertia tensor are:

$$\begin{bmatrix} \mathrm{I}'_{22} \\ \mathrm{I}'_{33} \\ \mathrm{I}'_{23} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \mathrm{I}_{22} \\ \mathrm{I}_{33} \\ \mathrm{I}_{23} \end{bmatrix}$$
$$= \begin{bmatrix} 0.73815 & 0.26185 & 0.87928 \\ 0.26185 & 0.73815 & -0.87928 \\ -0.43964 & 0.43964 & 0.4763 \end{bmatrix} \begin{bmatrix} 2.69186 \\ 1.21986 \\ 1.35872 \end{bmatrix} = \begin{bmatrix} 3501.12 \\ 410.6056 \\ 0 \end{bmatrix} cm^4 \tag{4.96}$$

Problem 4.26

A foundation in Engineering is a structural element which serves to transmit the load from the structure to the soil. Consider a *Mat-Slab Foundation* described in Figure 4.29 in which we have six columns. Knowing that soils cannot resist to traction stress, verify whether the design (dimensions) of the foundation, from a structural stability point of view, is appropriated or not.

Hypothesis (approximation): Consider that the mat foundation is infinitely rigid, so that the normal stress distribution in the soil will be a planar distribution.

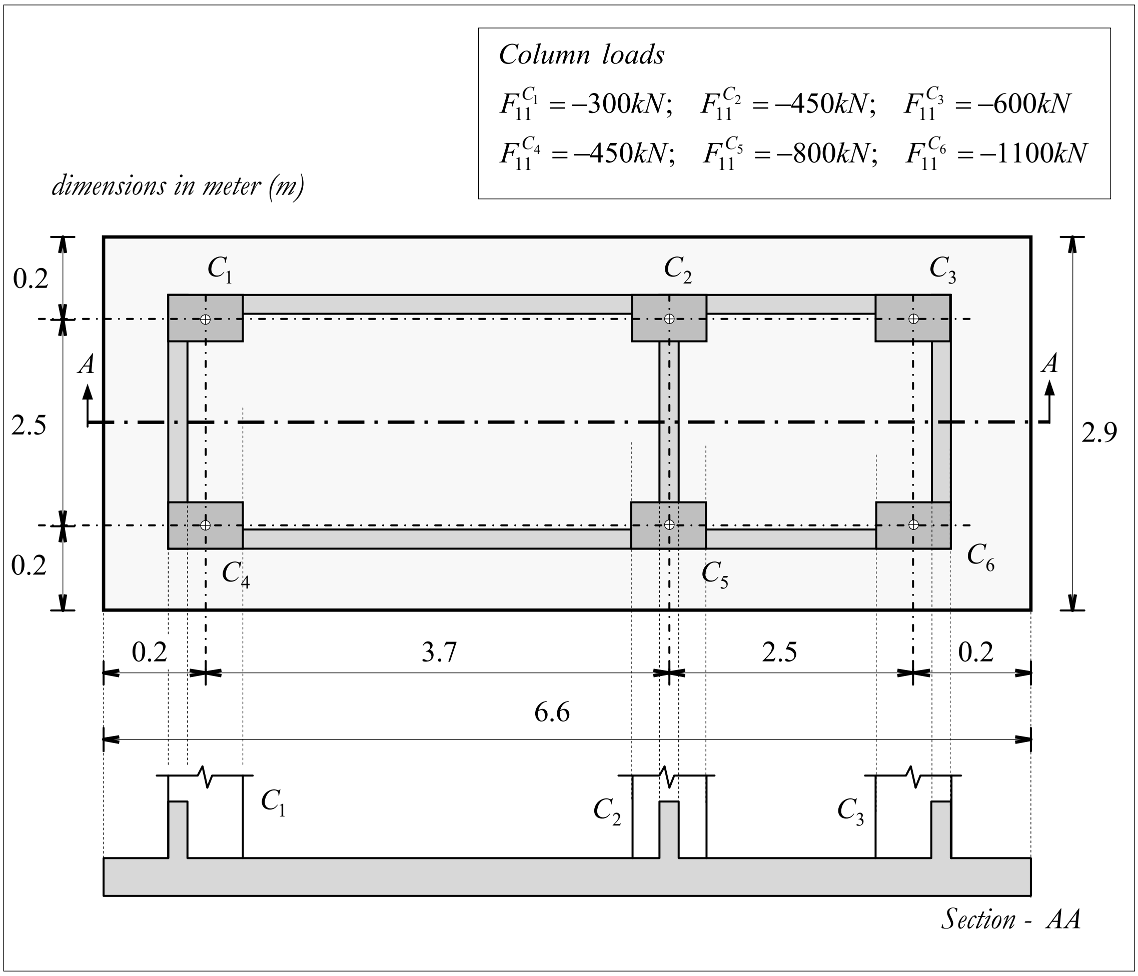


Figure 4.29: Mat foundation.

Solution:

The structural stability will be acceptable if on the ground (mat foundation base) there is only normal stress of compression. And based on the fact that the mat foundation is infinitely rigid we can apply the equation (4.65) if the reference system is located at the area centroid-G:

$$\boxed{\sigma_{11}(x_2, x_3) = \frac{N}{A} - \frac{M_z}{\mathrm{I}_{33}} x_2 + \frac{M_y}{\mathrm{I}_{22}} x_3} \quad \text{(The system is located at the Area Centroid and is the principal axes of inertia)} \tag{4.97}$$

Let us adopt the system located at the area centroid as indicated in Figure 4.30.

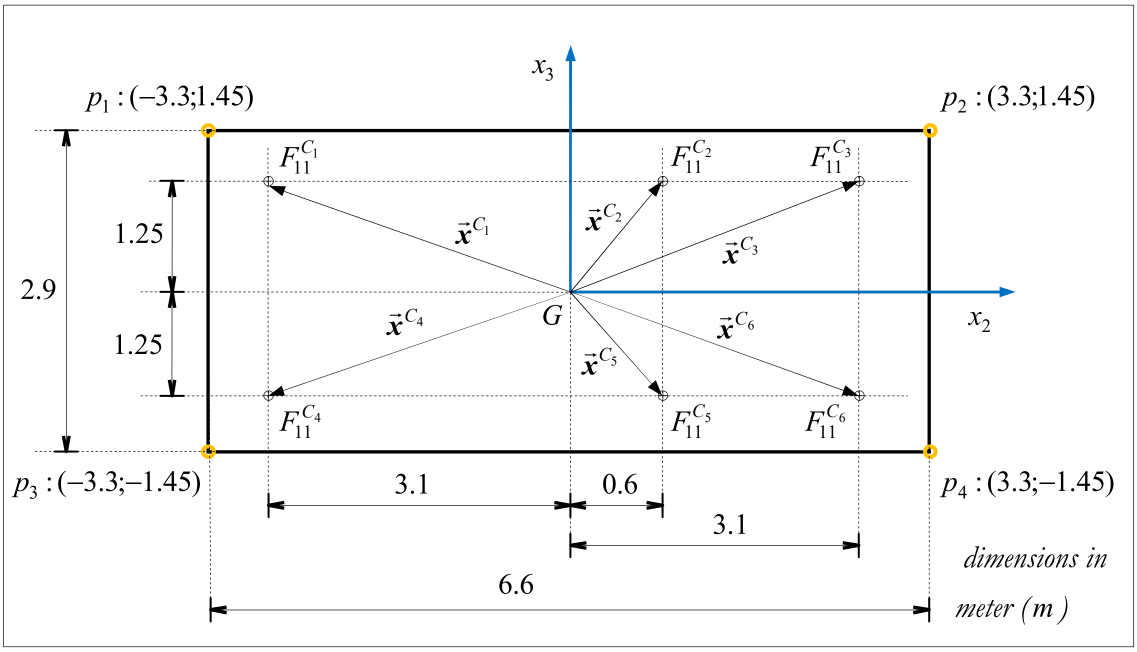


Figure 4.30

Calculation of the total force (F_{11}^{Total}):

$$\vec{F}_{11}^{Total} = \sum_{a=1}^{6} \vec{F}_{11}^{C_a} = \vec{F}_{11}^{C_1} + \vec{F}_{11}^{C_2} + \vec{F}_{11}^{C_3} + \vec{F}_{11}^{C_4} + \vec{F}_{11}^{C_5} + \vec{F}_{11}^{C_6}$$

$$\vec{F}_{11}^{Total} = -300\hat{\mathbf{i}} - 450\hat{\mathbf{i}} - 600\hat{\mathbf{i}} - 450\hat{\mathbf{i}} - 800\hat{\mathbf{i}} - 1100\hat{\mathbf{i}} = -3700\hat{\mathbf{i}}\ kN$$

Calculation of the total moment at the area centroid ($\vec{M}^{Total}$):

$$\vec{M}^{Total} = \sum_{a=1}^{6} (\vec{x}^{C_a} \wedge \vec{F}_{11}^{C_a})$$

$$\vec{M}^{Total} = (\vec{x}^{C_1} \wedge \vec{F}_{11}^{C_1}) + (\vec{x}^{C_2} \wedge \vec{F}_{11}^{C_2}) + (\vec{x}^{C_3} \wedge \vec{F}_{11}^{C_3}) + (\vec{x}^{C_4} \wedge \vec{F}_{11}^{C_4}) + (\vec{x}^{C_5} \wedge \vec{F}_{11}^{C_5}) + (\vec{x}^{C_6} \wedge \vec{F}_{11}^{C_6})$$

where

$$\vec{x}^{C_1} \wedge \vec{F}_{11}^{C_1} = (-3.1\hat{\mathbf{j}} + 1.25\hat{\mathbf{k}}) \wedge (-300\hat{\mathbf{i}}) = (-3.1)(-300)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (1.25)(-300)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = -930\hat{\mathbf{k}} - 375\hat{\mathbf{j}}$$

$$\vec{x}^{C_2} \wedge \vec{F}_{11}^{C_2} = (0.6\hat{\mathbf{j}} + 1.25\hat{\mathbf{k}}) \wedge (-450\hat{\mathbf{i}}) = (0.6)(-450)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (1.25)(-450)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = 270\hat{\mathbf{k}} - 562.5\hat{\mathbf{j}}$$

$$\vec{x}^{C_3} \wedge \vec{F}_{11}^{C_3} = (3.1\hat{\mathbf{j}} + 1.25\hat{\mathbf{k}}) \wedge (-600\hat{\mathbf{i}}) = (3.1)(-600)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (1.25)(-600)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = 1860\hat{\mathbf{k}} - 750\hat{\mathbf{j}}$$

$$\vec{x}^{C_4} \wedge \vec{F}_{11}^{C_4} = (-3.1\hat{\mathbf{j}} - 1.25\hat{\mathbf{k}}) \wedge (-450\hat{\mathbf{i}}) = (-3.1)(-450)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (-1.25)(-450)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = -1395\hat{\mathbf{k}} + 562.5\hat{\mathbf{j}}$$

$$\vec{x}^{C_5} \wedge \vec{F}_{11}^{C_5} = (0.6\hat{\mathbf{j}} - 1.25\hat{\mathbf{k}}) \wedge (-800\hat{\mathbf{i}}) = (0.6)(-800)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (-1.25)(-800)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = 480\hat{\mathbf{k}} + 1000\hat{\mathbf{j}}$$

$$\vec{x}^{C_6} \wedge \vec{F}_{11}^{C_6} = (3.1\hat{\mathbf{j}} - 1.25\hat{\mathbf{k}}) \wedge (-1100\hat{\mathbf{i}}) = (0.6)(-1100)\underbrace{\hat{\mathbf{j}} \wedge \hat{\mathbf{i}}}_{-\hat{\mathbf{k}}} + (-1.25)(-1100)\underbrace{\hat{\mathbf{k}} \wedge \hat{\mathbf{i}}}_{\hat{\mathbf{j}}} = 3410\hat{\mathbf{k}} + 1375\hat{\mathbf{j}}$$

Then,

$$\vec{\boldsymbol{M}}^{Total} = \sum_{a=1}^{6} (\vec{\boldsymbol{x}}^{C_a} \wedge \vec{\boldsymbol{F}}_{11}^{C_a}) = 3695\hat{\mathbf{k}} + 1250\hat{\mathbf{j}} = M_z \hat{\mathbf{k}} + M_y \hat{\mathbf{j}} \quad \Rightarrow \quad \begin{cases} M_y = 1250 kN\, m \\ M_z = 3695 kN\, m \end{cases}$$

Calculation of the normal stress field

For the rectangular cross section, (see Figure 4.15), by considering the system $o\vec{\boldsymbol{x}}$ we have:

$$\mathrm{I}_{22} = \frac{(6.6)(2.9)^3}{12} = 13.414 m^4 \quad ; \quad \mathrm{I}_{33} = \frac{(6.6)^3(2.9)}{12} = 69.478 m^4 \quad ; \quad \mathrm{I}_{23} = 0$$

$$A = (6.6)(2.9) = 19.14 m^2$$

Then,

$$\sigma_{11}(x_2, x_3) = \frac{N}{A} - \frac{M_z}{\mathrm{I}_{33}} x_2 + \frac{M_y}{\mathrm{I}_{22}} x_3 = \frac{-3700}{19.14} - \frac{(3695)}{69.478} x_2 + \frac{(1250)}{13.414} x_3$$

$$\Rightarrow \sigma_{11}(x_2, x_3) = -193.3124 - 53.182 x_2 + 93.187 x_3 \qquad \left[\frac{kN}{m^2} = kPa\right]$$

Point p_1: $\sigma_{11}^{p1}(x_2 = -3.3; x_3 = 1.45) = -193.3124 - 53.182 x_2 + 93.187 x_3 = 117.309$

Point p_2: $\sigma_{11}^{p2}(x_2 = 3.3; x_3 = 1.45) = -193.3124 - 53.182 x_2 + 93.187 x_3 = -233.693$

Point p_3: $\sigma_{11}^{p3}(x_2 = -3.3; x_3 = -1.45) = -193.3124 - 53.182 x_2 + 93.187 x_3 = -152.932$

Point p_4: $\sigma_{11}^{p4}(x_2 = 3.3; x_3 = -1.45) = -193.3124 - 53.182 x_2 + 93.187 x_3 = -503.934$

So, as we can see, the design established for the mat foundation is not the appropriated one, since traction stress appears in the soil, (see Figure 4.31).

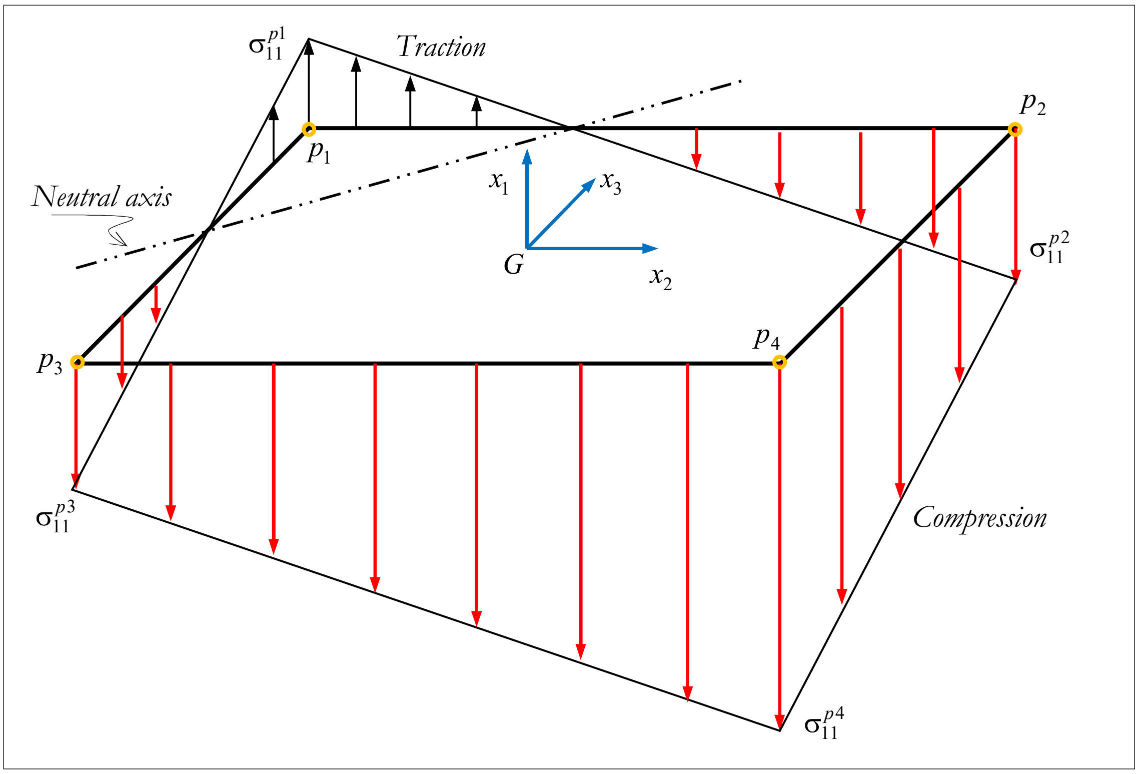


Figure 4.31: Normal stress distribution on the ground.

NOTE: The centroid of the concentrated forces ($\vec{\tilde{x}}$) is defined by $\vec{\tilde{x}} \wedge \vec{F}_{11}^{Total} = \vec{M}^{Total}$, (see Figure 4.32). Then, by considering that $\vec{M}^{Total} = 1250\hat{\mathbf{j}} + 3695\hat{\mathbf{k}}$ and $\vec{F}_{11}^{Total} = -3700\hat{\mathbf{i}}\,kN$ we can obtain:

$$\vec{\tilde{x}} \wedge \vec{F}_{11}^{Total} = \vec{M}^{Total} \quad \Rightarrow \quad \vec{\tilde{x}} \wedge (-3700\hat{\mathbf{i}}) = 1250\hat{\mathbf{j}} + 3695\hat{\mathbf{k}}$$

where

$$(\tilde{x}_1\hat{\mathbf{i}} + \tilde{x}_2\hat{\mathbf{j}} + \tilde{x}_3\hat{\mathbf{k}}) \wedge (-3700\hat{\mathbf{i}}) = -3700\tilde{x}_1\hat{\mathbf{i}} \wedge \hat{\mathbf{i}} - 3700\tilde{x}_2\hat{\mathbf{j}} \wedge \hat{\mathbf{i}} - 3700\tilde{x}_3\hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = 3700\tilde{x}_2\hat{\mathbf{k}} - 3700\tilde{x}_3\hat{\mathbf{j}}$$

so, by applying $\vec{\tilde{x}} \wedge \vec{F}_{11}^{Total} = \vec{M}^{Total}$ we can obtain:

$$3700\tilde{x}_2\hat{\mathbf{k}} - 3700\tilde{x}_3\hat{\mathbf{j}} = 1250\hat{\mathbf{j}} + 3695\hat{\mathbf{k}} \quad \Rightarrow \quad \begin{cases} -3700\tilde{x}_3 = 1250 & \rightarrow \quad \tilde{x}_3 = \dfrac{1250}{-3700} = -0.338 \\ 3700\tilde{x}_2 = 3695 & \rightarrow \quad \tilde{x}_2 = \dfrac{3695}{3700} = 0.999 \end{cases}$$

This position ($\vec{\tilde{x}}$) is called *eccentricity* of $\vec{F}_{11}^{Total}$ which represents the centroid of the forces. The centroid of the forces can also be obtained in the same fashion as the volume/area centroid definition, i.e.:

$$\tilde{x}_2 = \frac{\sum_{a=1}^{6} F_{11}^{C_a} x_2^{C_a}}{\sum_{a=1}^{6} F_{11}^{C_a}} = \frac{F_{11}^{C_1}x_2^{C_1} + F_{11}^{C_2}x_2^{C_2} + F_{11}^{C_3}x_2^{C_3} + F_{11}^{C_4}x_2^{C_4} + F_{11}^{C_5}x_2^{C_5} + F_{11}^{C_6}x_2^{C_6}}{\sum_{a=1}^{6} F_{11}^{C_a}} = \frac{3695}{3700} = 0.999$$

$$\tilde{x}_3 = \frac{-\sum_{a=1}^{6} F_{11}^{C_a} x_3^{C_a}}{\sum_{a=1}^{6} F_{11}^{C_a}} = \frac{-(F_{11}^{C_1}x_3^{C_1} + F_{11}^{C_2}x_3^{C_2} + F_{11}^{C_3}x_3^{C_3} + F_{11}^{C_4}x_3^{C_4} + F_{11}^{C_5}x_3^{C_5} + F_{11}^{C_6}x_3^{C_6})}{\sum_{a=1}^{6} F_{11}^{C_a}} = \frac{-1250}{3700} = -0.338$$

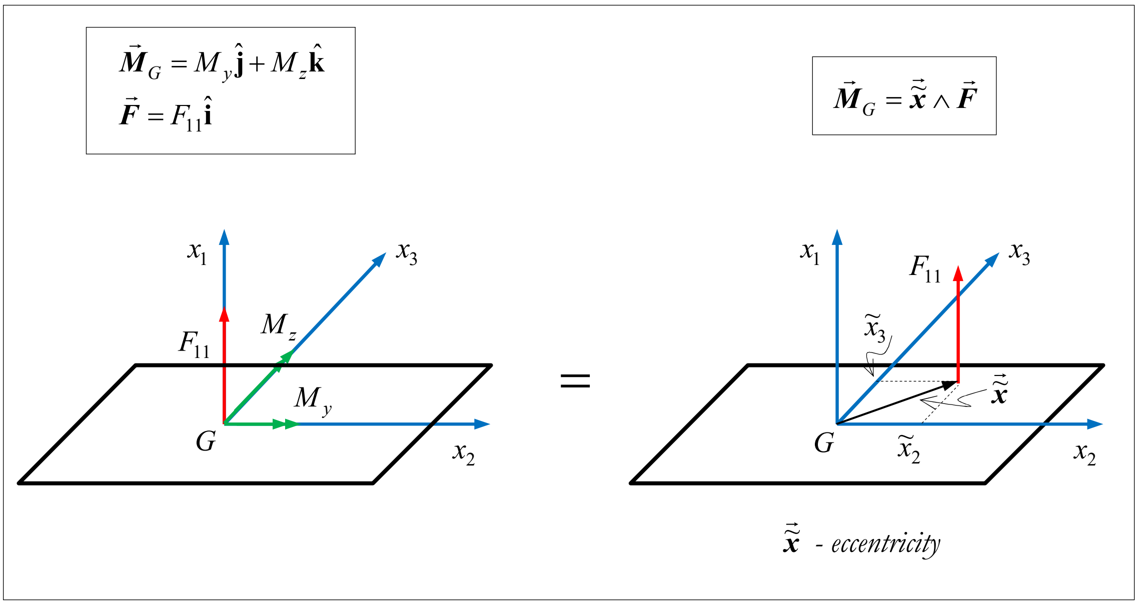


Figure 4.32

Problem 4.27

Consider a prismatic bar with a walled cross section as indicated in Figure 4.33(a). By considering the differential element given by Figure 4.33(b), obtain the equilibrium equations by considering the system (x_1, s).

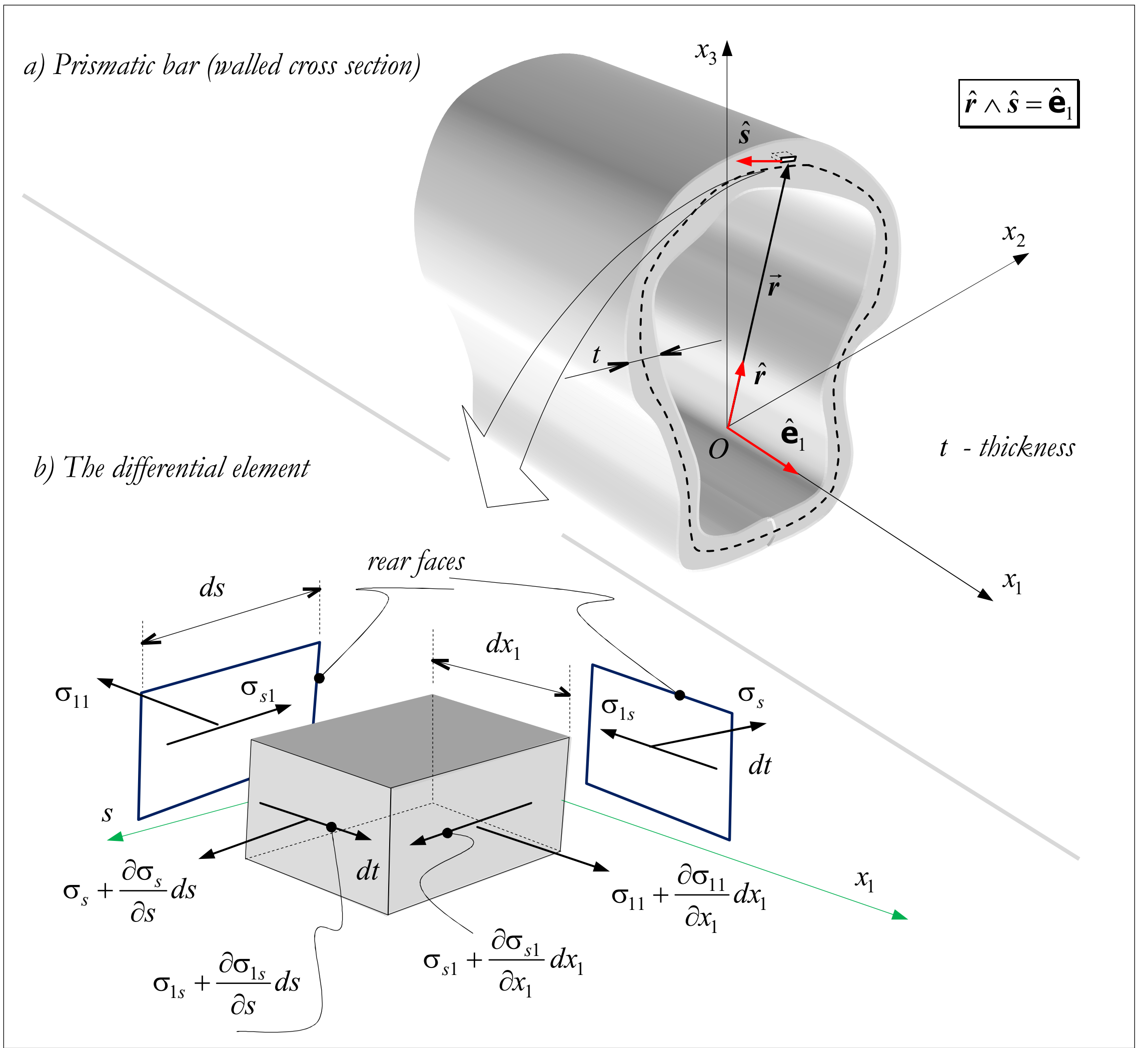


Figure 4.33: Thin-walled member.

Solution:

Equilibrium according to x_1-direction:

$$\sum F_{x_1} = 0 \quad \Rightarrow \quad \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) ds\, dt - \sigma_{11} ds\, dt + \left(\sigma_{1s} + \frac{\partial \sigma_{1s}}{\partial s} ds\right) dx_1\, dt - \sigma_{1s} dx_1\, dt = 0$$

$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1} dx_1 ds\, dt + \frac{\partial \sigma_{1s}}{\partial s} ds dx_1\, dt = 0 \quad \Rightarrow \quad \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{1s}}{\partial s} = 0$$

Equilibrium according to s-direction:

$$\sum F_s = 0 \quad \Rightarrow \quad \left(\sigma_s + \frac{\partial \sigma_s}{\partial s} ds\right) dx_1\, dt - \sigma_s dx_1\, dt + \left(\sigma_{s1} + \frac{\partial \sigma_{s1}}{\partial x_1} dx_1\right) ds\, dt - \sigma_{s1} ds\, dt = 0$$

$$\Rightarrow \frac{\partial \sigma_s}{\partial s} ds dx_1\, dt + \frac{\partial \sigma_{s1}}{\partial x_1} dx_1 ds\, dt = 0 \quad \Rightarrow \quad \frac{\partial \sigma_s}{\partial s} + \frac{\partial \sigma_{s1}}{\partial x_1} = 0$$

The equilibrium equations at a material point on the cross section are:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{1s}}{\partial s} = 0 \\ \dfrac{\partial \sigma_{s}}{\partial s} + \dfrac{\partial \sigma_{s1}}{\partial x_1} = 0 \end{cases}$$

NOTE 1: Thin-Walled Members

Let us consider that the thickness t is very small and the stress distributions in the thickness is given as indicated in Figure 4.34(b), then, we can adopt that $q = \sigma_{s1} t$ and σ_{11} are constant along the thickness, where q is called the *shear flow*. By changing the nomenclature $\sigma_{s1} = \tau$ we rewrite $q = \sigma_{s1} t = \tau t$.

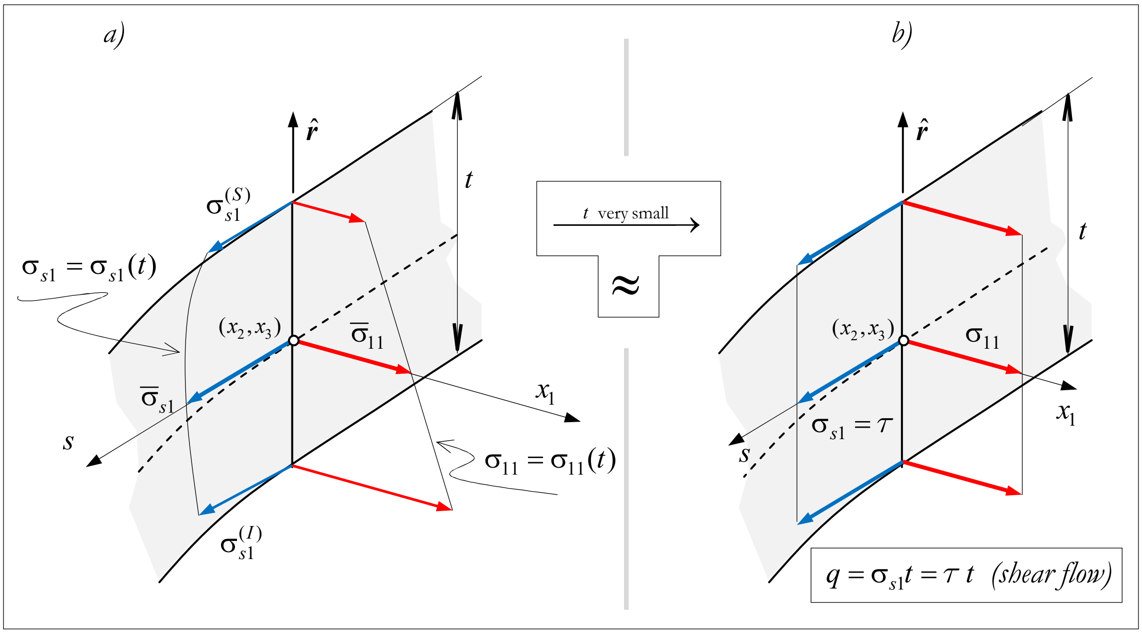


Figure 4.34

Then, the equilibrium equations can be rewritten as follows:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{1s}}{\partial s} = 0 \\ \dfrac{\partial \sigma_{s}}{\partial s} + \dfrac{\partial \sigma_{s1}}{\partial x_1} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \left(\dfrac{q}{t}\right)}{\partial s} = 0 \\ \dfrac{\partial \sigma_{s}}{\partial s} + \dfrac{\partial \left(\dfrac{q}{t}\right)}{\partial x_1} = 0 \end{cases} \Rightarrow \begin{cases} t\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial q}{\partial s} = 0 \\ t\dfrac{\partial \sigma_{s}}{\partial s} + \dfrac{\partial q}{\partial x_1} = 0 \end{cases} \tag{4.98}$$

If we integrate the first equilibrium equation over s-coordinate we can obtain:

$$t\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial q}{\partial s} = 0 \xrightarrow{\text{by integrating over } s} \int_0^s \frac{\partial q}{\partial s} ds = -\int_0^s \left(t\frac{\partial \sigma_{11}}{\partial x_1} \right) ds$$
$$\Rightarrow q(s) - q(0) = -\int_0^s \left(t\frac{\partial \sigma_{11}}{\partial x_1} \right) ds \tag{4.99}$$

The s-coordinate is measured along the cross-section perimeter, (see Figure 4.35). By considering that N is independent of x_1 or $(N=0)$ we can rewrite the equation (4.74) as follows:

$$\frac{\partial \sigma_{11}}{\partial x_1} = -\frac{(\mathrm{I}_{23}x_2 + \mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} F_z - \frac{(\mathrm{I}_{22}x_2 + \mathrm{I}_{23}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} F_y$$
$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1} = -\frac{(\mathrm{I}_{23}F_z + \mathrm{I}_{22}F_y)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_2 - \frac{(\mathrm{I}_{23}F_y + \mathrm{I}_{33}F_z)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_3 \tag{4.100}$$

And by substituting the above equation into the equation in (4.99) we can obtain:

$$q(s) = q(0) - \int_0^s \left(t \frac{\partial \sigma_{11}}{\partial x_1} \right) ds \tag{4.101}$$

$$\boxed{q(s) = q(0) + \left(\frac{\mathrm{I}_{23}F_z + \mathrm{I}_{22}F_y}{\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33}} \right) \int_0^s (t\,x_2) ds + \left(\frac{\mathrm{I}_{23}F_y + \mathrm{I}_{33}F_z}{\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33}} \right) \int_0^s (t\,x_3) ds} \tag{4.102}$$

The system is located at the Area Centroid

The above shear flow is indeterminate because we do not know $q(0)$ when we are dealing with closed cross section. The solution strategy will be discussed in Chapter 6.

If we have an open cross section we can assume that at $s=0 \Rightarrow q(0)=0$, (see Figure 4.35).

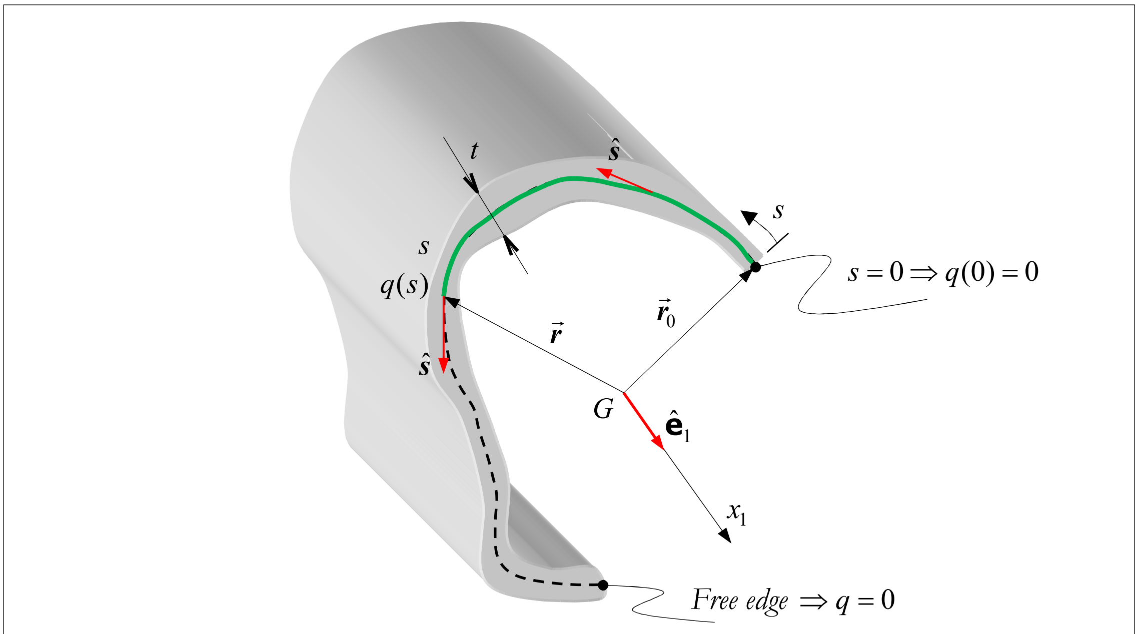


Figure 4.35

NOTE 1.1: The equation in (4.72) can be rewritten in a compact form as follows:

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} = \frac{\mathcal{Y}_0}{\mathcal{X}} + \frac{\mathcal{Y}_2}{\mathcal{X}} x_2 + \frac{\mathcal{Y}_3}{\mathcal{X}} x_3} \tag{4.103}$$

(For any system in which the plane $x_2 - x_3$ is lying on the plane defined by the cross section)

where

$$\mathcal{Y}_0 = \begin{vmatrix} 0 & A\bar{x}_2 & A\bar{x}_3 \\ F_z & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -F_y & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} [Nm^7] \quad ; \quad \mathcal{Y}_2 = \begin{vmatrix} A & 0 & A\bar{x}_3 \\ A\bar{x}_3 & F_z & \mathrm{I}_{22} \\ -A\bar{x}_2 & -F_y & \mathrm{I}_{23} \end{vmatrix} [Nm^6] \tag{4.104}$$

$$\mathcal{Y}_3 = \begin{vmatrix} A & A\bar{x}_2 & 0 \\ A\bar{x}_3 & -\mathrm{I}_{23} & F_z \\ -A\bar{x}_2 & -\mathrm{I}_{33} & -F_y \end{vmatrix} [Nm^6] \quad ; \quad \mathcal{X} = \begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} [m^{10}] \tag{4.105}$$

Then, the equation in (4.101) becomes

$$\boxed{q(s) = q(0) - \int_0^s \left(t\frac{\partial \sigma_{11}}{\partial x_1} \right) ds = q(0) - \frac{\mathcal{Y}_0}{\mathcal{X}} \int_0^s t\, ds - \frac{\mathcal{Y}_2}{\mathcal{X}} \int_0^s (t\, x_2) ds - \frac{\mathcal{Y}_3}{\mathcal{X}} \int_0^s (t\, x_3) ds \qquad \left[\frac{N}{m} \right]} \tag{4.106}$$

(For any system in which the plane $x_2 - x_3$ is lying on the plane defined by the cross section)

NOTE 2: Closed Thin-Walled Cross Section under Torsion only.

If the thickness is very small we can adopt the stress distribution as the one shown in Figure 4.34(b) and if we also consider that $\sigma_{11} = 0$ we can conclude, according to the equilibrium equation (4.98), ($t\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial q}{\partial s} = 0$), that $\frac{\partial q}{\partial s} = 0$, i.e. the shear flow does not vary with s-coordinate, i.e. q is constant. For this scenario we can calculate the moment according to x_1-coordinate as follows:

$$\vec{\boldsymbol{M}}_{Ox_1} = \oint_s \vec{\boldsymbol{r}} \wedge \vec{\boldsymbol{q}} ds = \oint_s q r \hat{\boldsymbol{r}} \wedge \hat{\boldsymbol{s}} ds = q \oint_s r \hat{\mathbf{e}}_1 ds = q \left(\oint_s r ds \right) \hat{\mathbf{e}}_1 = q(2A) \hat{\mathbf{e}}_1 = 2qA \hat{\mathbf{e}}_1 \tag{4.107}$$

where we have applied the equation $\oint_s r ds = \oint_s \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} ds = 2A$, (see NOTE in **Problem 1.128**).

Then, given the moment of torsion $M_{Ox_1} \equiv M_T$ we can calculate the shear flow (q) and the tangential stress (τ) as follows:

$$M_T = 2qA \quad \Rightarrow \quad q = \frac{M_T}{2A} \quad \Rightarrow \quad \tau = \frac{M_T}{2At} \tag{4.108}$$

where $q = \sigma_{s1} t = \tau t$. The above equation could be a good approximation if the thickness (t) is very small. In Chapter 6 we will discuss another approximation to tackle this problem.

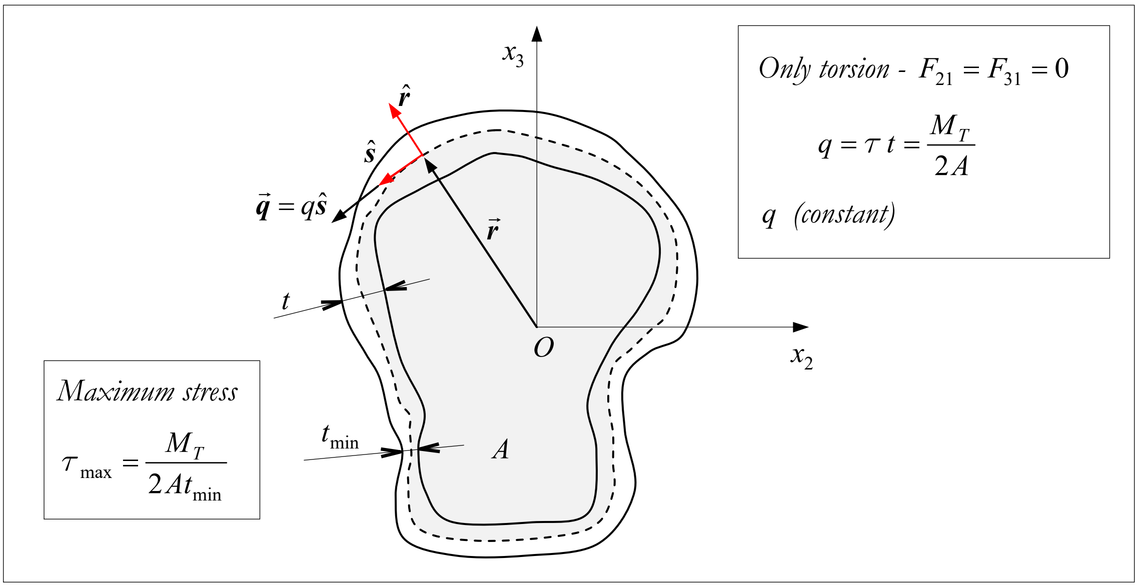


Figure 4.36

Example: Let us consider a circular cross section, (see Figure 4.37), and the applied torsion is $M_T = 5.0\times10^3\,Nmm$. Then, the shear stress can be calculated as follows:

$$\tau = \frac{q}{t} = \frac{\frac{M_T}{2A}}{t} = \frac{M_T}{2At} = \frac{5.0\times10^3}{2\times498.76\times1.2} \approx 4.2\frac{N}{mm^2}$$

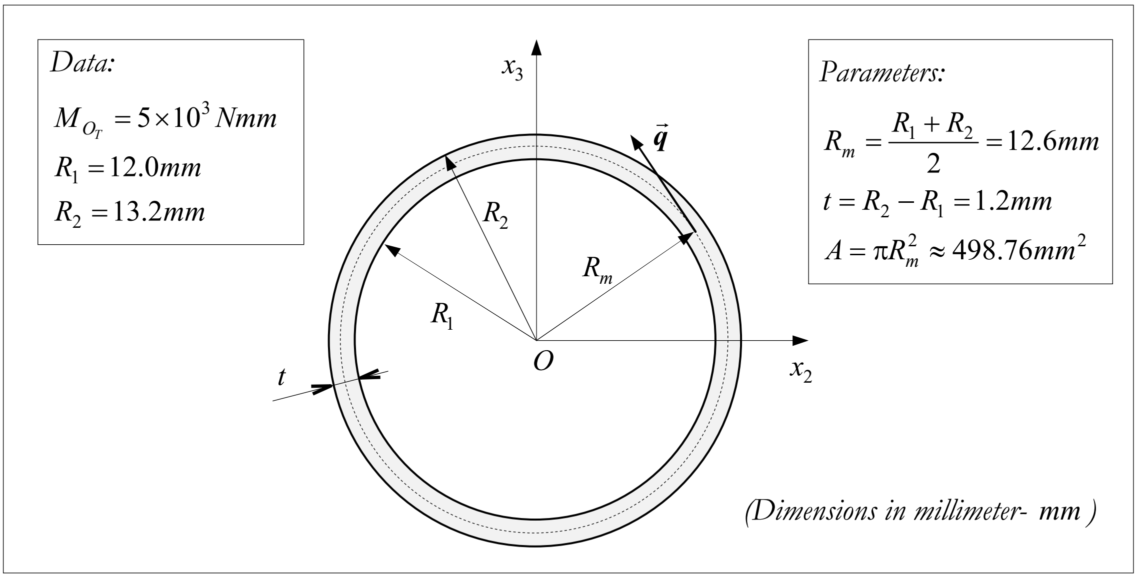


Figure 4.37

Problem 4.28

Consider an arbitrary cross section in which is acting a compression force $F_{11} = -P$, where P is a positive real number, and the adopted system $(o\vec{x})$ is located at the Area Centroid. Consider that when $F_{11} = -P$ is located at the Area Centroid there is no moment, i.e. $M_y = M_z = 0$.

Obtain the loci for the all possible position for $F_{11} = -P$ such as there is no traction on the cross-section, i.e. $\sigma_{11}(x_2, x_3) \geq 0$.

NOTE 1: The region in which the compression force does not produce traction on the cross section is called the *core* or "*kernel*" *of a section*.

Solution:

We can adopt the equation in (4.61), since the adopted system is located at the Area Centroid of the cross section:

$$\sigma_{11}(x_2, x_3) = \frac{N}{A} - \frac{(M_y \mathrm{I}_{23} - M_z \mathrm{I}_{22})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_2 + \frac{(-M_y \mathrm{I}_{33} + M_z \mathrm{I}_{23})}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} x_3 \tag{4.109}$$

Then, for this problem we have $N = F_{11} = -P$, $M_y = -P\tilde{x}_3$, and $M_z = P\tilde{x}_2$, (see Figure 4.38). Note that we are looking for $(\tilde{x}_2, \tilde{x}_3)$ in such a way that in the cross section there is no traction, i.e. $\sigma_{11}(x_2, x_3) \geq 0$, so we can establish that in the boundary of the section the normal stress is zero $\sigma_{11}(x_2, x_3) = 0$, i.e.:

$$\sigma_{11}(x_2,x_3)=\frac{N}{A}-\frac{(M_y\mathrm{I}_{23}-M_z\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(-M_y\mathrm{I}_{33}+M_z\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3=0$$

$$\Rightarrow\frac{(-P)}{A}-\frac{(-P\tilde{x}_3\mathrm{I}_{23}-(P\tilde{x}_2)\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(-(-P\tilde{x}_3)\mathrm{I}_{33}+(P\tilde{x}_2)\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3=0$$

$$\Rightarrow\frac{-1}{A}+\frac{(\tilde{x}_3\mathrm{I}_{23}+\tilde{x}_2\mathrm{I}_{22})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2+\frac{(\tilde{x}_3\mathrm{I}_{33}+\tilde{x}_2\mathrm{I}_{23})}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3=0$$

$$\Rightarrow(\tilde{x}_3\mathrm{I}_{23}+\tilde{x}_2\mathrm{I}_{22})x_2+(\tilde{x}_3\mathrm{I}_{33}+\tilde{x}_2\mathrm{I}_{23})x_3=\frac{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}{A}$$

or

$$\boxed{(\mathrm{I}_{22}x_2+\mathrm{I}_{23}x_3)\tilde{x}_2+(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)\tilde{x}_3=\frac{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}{A}}\quad \textit{The system is located at the Area Centroid} \tag{4.110}$$

Then, for the position (x_2,x_3) in which we assume that $\sigma_{11}(x_2,x_3)=0$ we can find the geometric position $(\tilde{x}_2,\tilde{x}_3)$ for $F_{11}=-P$.

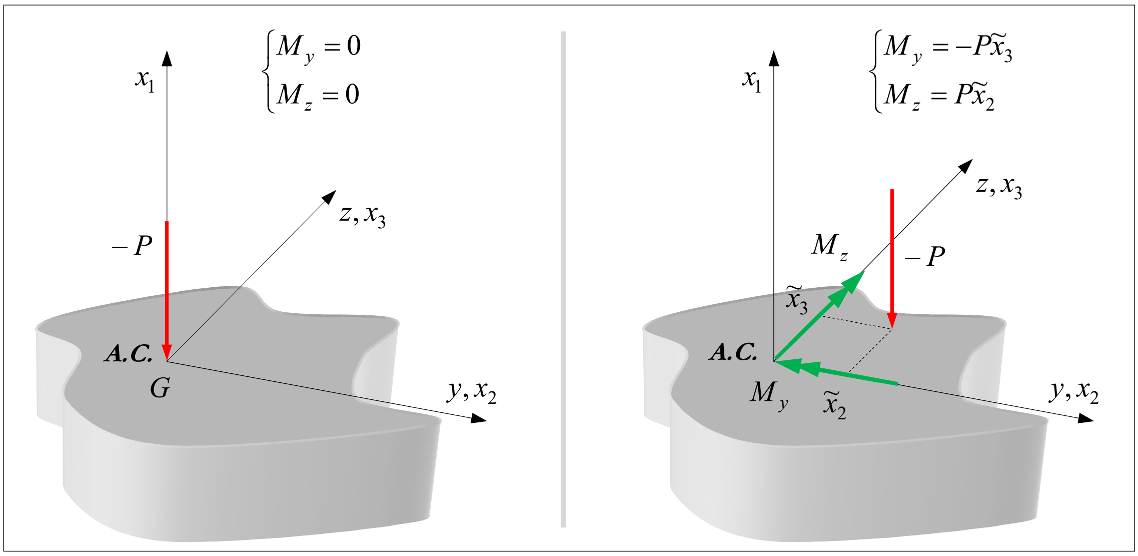


Figure 4.38

For example, if the cross section is a circle of radius r, (see Figure 4.39), and as the system is located at the area centroid the following is true $A=\pi r^2$, $\mathrm{I}_{22}=\mathrm{I}_{33}=\frac{\pi r^4}{4}=\mathrm{I}$, $\mathrm{I}_{23}=0$. The coordinates (x_2,x_3) for the boundary can be represented in terms of the radius: $x_2=r\cos\theta$ and $x_3=r\sin\theta$, therefore

$$(\mathrm{I}_{22}x_2+\mathrm{I}_{23}x_3)\tilde{x}_2+(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)\tilde{x}_3=\frac{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}{A}$$

$$\Rightarrow \mathrm{I}x_2\tilde{x}_2+\mathrm{I}x_3\tilde{x}_3=\frac{-\mathrm{I}^2}{A}\quad\Rightarrow\quad r\cos\theta\tilde{x}_2+r\sin\theta\tilde{x}_3=\frac{-\mathrm{I}}{A}\quad\Rightarrow\quad\cos\theta\tilde{x}_2+\sin\theta\tilde{x}_3=\frac{-\mathrm{I}}{Ar}$$

$$\cos\theta\tilde{x}_2+\sin\theta\tilde{x}_3=\frac{-\mathrm{I}}{Ar}\quad\Rightarrow\quad\cos\theta\tilde{x}_2+\sin\theta\tilde{x}_3=\frac{-\left(\frac{\pi r^4}{4}\right)}{\pi r^2 r}=\frac{-r}{4}$$

$$\Rightarrow \cos\theta(\tilde{r}\cos\theta)+\sin\theta(\tilde{r}\sin\theta)=\frac{-r}{4} \quad \Rightarrow \quad \tilde{r}(\cos^2\theta+\sin^2\theta)=\frac{-r}{4} \quad \Rightarrow \quad \tilde{r}=\frac{-r}{4}$$

Then, if $F_{11}=-P$ assume any position inside the circle of radius $\tilde{r}$ it will not produce traction on the cross section.

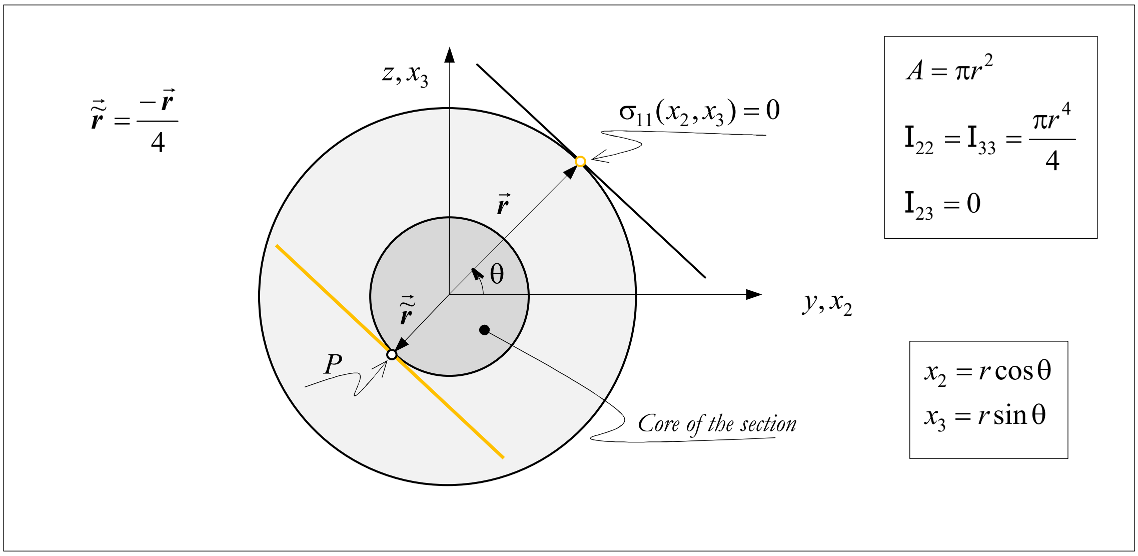


Figure 4.39: Core of the circular cross section.

Let us consider a rectangular cross section, (see Figure 4.40). In this case we have $A=ah$, $I_{22}=\frac{ah^3}{12}$, $I_{33}=\frac{a^3h}{12}$ and $I_{23}=0$. Then, the equation (4.110) becomes:

$$(I_{22}x_2+I_{23}x_3)\tilde{x}_2+(I_{23}x_2+I_{33}x_3)\tilde{x}_3=\frac{(I_{23}^2-I_{22}I_{33})}{A} \quad \Rightarrow \quad I_{22}x_2\tilde{x}_2+I_{33}x_3\tilde{x}_3=\frac{-I_{22}I_{33}}{A}$$

$$\Rightarrow \quad \left(\frac{ah^3}{12}\right)x_2\tilde{x}_2+\left(\frac{a^3h}{12}\right)x_3\tilde{x}_3=\frac{-\left(\frac{ah^3}{12}\right)\left(\frac{a^3h}{12}\right)}{bh} \quad \Rightarrow \quad h^2x_2\tilde{x}_2+a^2x_3\tilde{x}_3=\frac{-h^2a^2}{12}$$

For the point $\left(x_2=\frac{a}{2}, x_3=\frac{h}{2}\right)$ we have:

$$h^2x_2\tilde{x}_2+a^2x_3\tilde{x}_3=\frac{-h^2a^2}{12} \quad \Rightarrow \quad h^2\frac{a}{2}\tilde{x}_2+a^2\frac{h}{2}\tilde{x}_3=\frac{-h^2a^2}{12}$$

$$\Rightarrow h^2\frac{a}{2}\tilde{x}_2+a^2\frac{h}{2}\tilde{x}_3=\frac{-h^2a^2}{12} \quad \Rightarrow \quad \frac{\tilde{x}_2}{\left(\frac{-a}{6}\right)}+\frac{\tilde{x}_3}{\left(\frac{-h}{6}\right)}=1$$

which is a line that intercepts x_2 in $\left(\frac{-a}{6}\right)$ and intercepts x_3 in $\left(\frac{-h}{6}\right)$. If we make the same procedure for other points we can obtain the geometric shape of the core of the rectangular section as the one indicated in Figure 4.40.

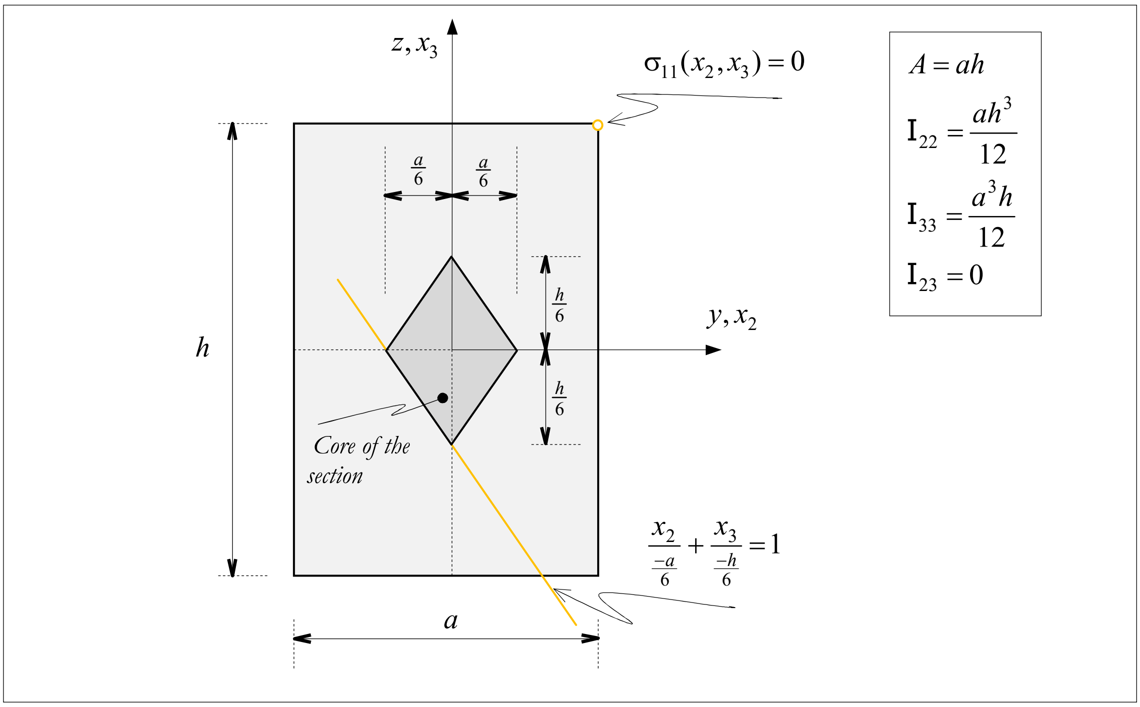


Figure 4.40: Core of the rectangular cross section.

Problem 4.29

Consider the cantilever (beam fixed at one end) under static equilibrium, (see Figure 4.41(a)), in which we have the concentrated force ($F_{31} = P$) applied at the end $x_1 = L$. Obtain the equation for the tangential stress field σ_{31} on the cross section, (see Figure 4.41(b)).

Hypotheses (approximations):

a) Consider that on the cross section, σ_{31} does not vary with x_2, (see Figure 4.41(b)), i.e. $\sigma_{31} = \sigma_{31}(x_3)$, and $\sigma_{2i} = 0_i$.

b) Consider the problem without body forces.

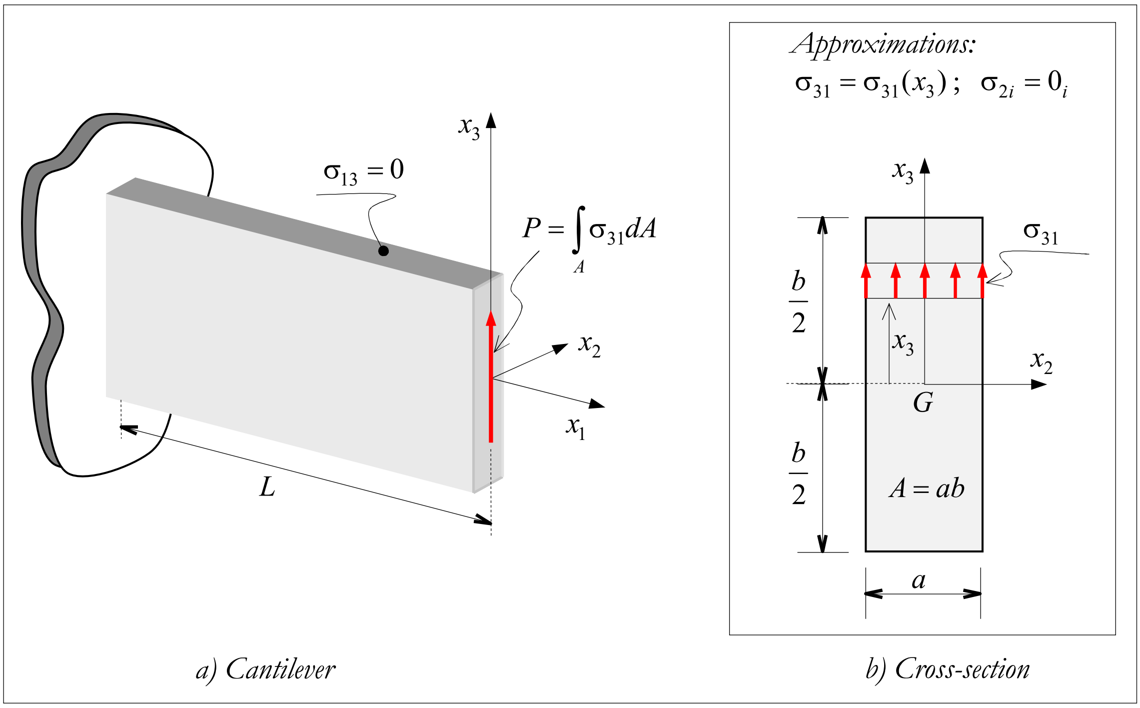


Figure 4.41: Beam fixed at one end (cantilever).

Solution:

The equilibrium equations state that

$$\sigma_{ij,j}+\underbrace{\rho \mathrm{b}_i}_{=0_i}=\underbrace{\rho \ddot{\mathrm{u}}_i}_{=0_i} \quad\Rightarrow\quad \begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma_{12}}{\partial x_2}+\dfrac{\partial\sigma_{13}}{\partial x_3}=0 \\ \dfrac{\partial\sigma_{21}}{\partial x_1}+\dfrac{\partial\sigma_{22}}{\partial x_2}+\dfrac{\partial\sigma_{23}}{\partial x_3}=0 \\ \dfrac{\partial\sigma_{31}}{\partial x_1}+\dfrac{\partial\sigma_{32}}{\partial x_2}+\dfrac{\partial\sigma_{33}}{\partial x_3}=0 \end{cases} \quad\Rightarrow\quad \begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma_{13}}{\partial x_3}=0 \\ 0=0 \\ \dfrac{\partial\sigma_{31}}{\partial x_1}=0 \end{cases}$$

If the system is located at the Area Centroid, the term $\dfrac{\partial\sigma_{11}}{\partial x_1}$ can be expressed by the equation in (4.74):

$$\frac{\partial\sigma_{11}}{\partial x_1}=-\frac{(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}F_z-\frac{(\mathrm{I}_{22}x_2+\mathrm{I}_{23}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}F_y=\frac{-(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}P \tag{4.111}$$

where we have considered $F_z=F_{31}=P$, $F_y=0$. Then, the first equilibrium equation becomes

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{13}}{\partial x_3}=0 \quad\Rightarrow\quad \frac{-(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}P+\frac{\partial\sigma_{13}}{\partial x_3}=0 \quad\Rightarrow\quad \frac{\partial\sigma_{13}}{\partial x_3}=\frac{(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}P \tag{4.112}$$

by integrating over x_3 we can obtain

$$\sigma_{13}=\frac{\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\frac{x_3^2}{2}+\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2x_3+K$$

The constant of integration can be obtained by the condition

$$x_3 = \pm\frac{b}{2} \Rightarrow \sigma_{13} = \sigma_{31} = 0 \quad \Rightarrow \quad \sigma_{13} = \frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\pm\frac{b}{2}\right)^2 + \frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\pm\frac{b}{2}\right)x_2 + K = 0$$

$$\Rightarrow K = -\frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\frac{b}{2}\right)^2 \mp \frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\frac{b}{2}\right)x_2$$

Then, the tangential stress can be expressed as follows

$$\sigma_{13} = \frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}x_3^2 + \frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}x_2 x_3 + K$$

$$\Rightarrow \sigma_{13} = \frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}x_3^2 + \frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}x_2 x_3 - \left(\frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\frac{b}{2}\right)^2 \pm \frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left(\frac{b}{2}\right)x_2\right)$$

$$\Rightarrow \sigma_{13} = \frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left[x_3^2 - \left(\frac{b}{2}\right)^2\right] + \frac{\mathrm{I}_{23}Px_2}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left[x_3 - \left(\pm\frac{b}{2}\right)\right]$$

The adopted system is at the centroid area and is the axis of symmetry, then

$$\mathrm{I}_{22} = \frac{ab^3}{12} \quad ; \quad \mathrm{I}_{33} = \frac{ba^3}{12} \quad ; \quad \mathrm{I}_{23} = 0 \quad ; \quad A = ab$$

Thus, the equation for $\sigma_{13} = \sigma_{31}$, (see Figure 4.42), is given by

$$\sigma_{13} = \frac{\mathrm{I}_{33}P}{2(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left[x_3^2 - \left(\frac{b}{2}\right)^2\right] + \frac{\mathrm{I}_{23}Px_2}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})}\left[x_3 - \left(\pm\frac{b}{2}\right)\right] = \frac{-P}{2\mathrm{I}_{22}}\left[x_3^2 - \left(\frac{b}{2}\right)^2\right]$$

The maximum value of σ_{13} occurs at $x_3 = 0$, which value is

$$\sigma_{13\max} = \sigma_{13}(x_3 = 0) = \frac{P}{2\mathrm{I}_{22}}\left(\frac{b}{2}\right)^2 = \frac{P}{2\left(\frac{ab^3}{12}\right)}\left(\frac{b}{2}\right)^2 = \frac{3P}{2A}$$

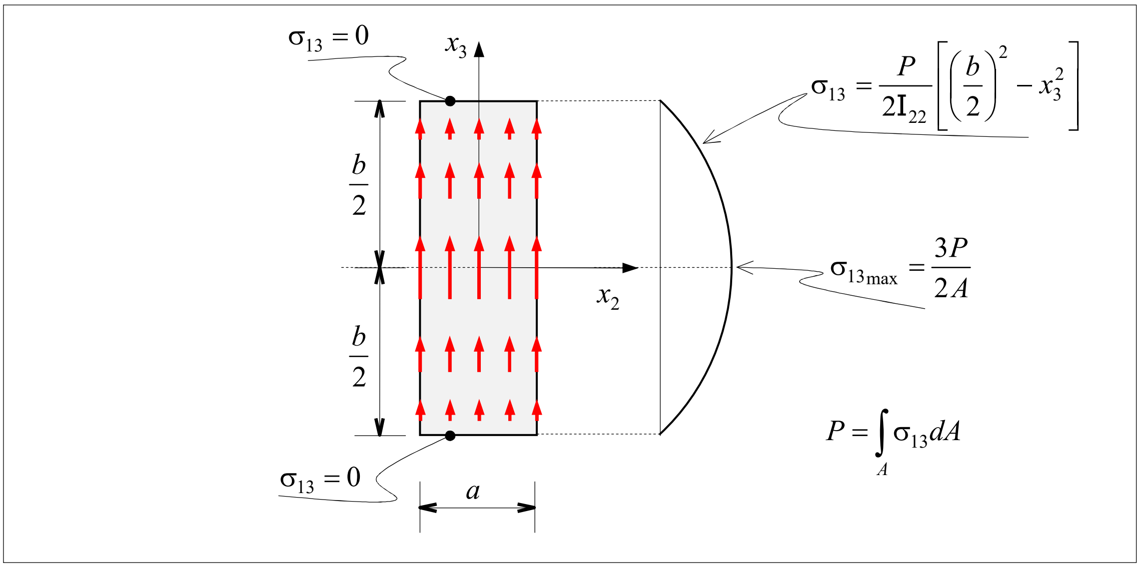


Figure 4.42: Tangential stress distribution on the cross section.

NOTE: We can generalize the previous equations. Let us consider the equation (4.112), i.e.

$$\frac{\partial\sigma_{13}}{\partial x_3}=\frac{(\mathrm{I}_{23}x_2+\mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}P=\frac{\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_3+\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}x_2 \tag{4.113}$$

and consider the approximation made in Figure 4.43.

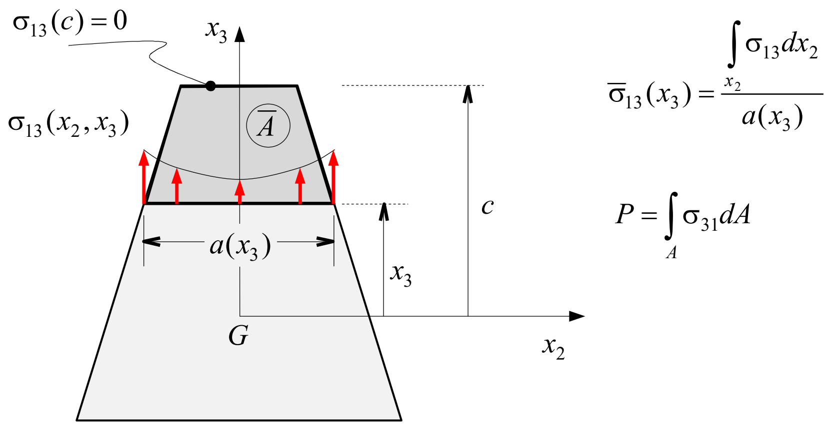


Figure 4.43

And by integrating the equation (4.113) over x_3 from x_3 to c we can obtain

$$\int_{x_3}^{c}\partial\sigma_{13}dx_3=\frac{\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_3\partial x_3+\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_2\partial x_3$$

$$\Rightarrow \underbrace{\sigma_{13}(c)}_{=0}-\sigma_{13}(x_3)=\frac{\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_3\partial x_3+\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_2\partial x_3$$

$$\Rightarrow \sigma_{13}(x_3)=\frac{-\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_3\partial x_3-\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_3}^{c}x_2\partial x_3$$

Now by integrating over x_2 we can obtain

$$\int_{x_2}\sigma_{13}(x_3)dx_2=\frac{-\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_2}\int_{x_3}^{c}x_3dx_3dx_2-\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{x_2}\int_{x_3}^{c}x_2dx_3dx_2$$

$$\Rightarrow \overline{\sigma}_{13}(x_3)a(x_3)=\frac{-\mathrm{I}_{33}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{\overline{A}}x_3dA-\frac{\mathrm{I}_{23}P}{(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{\overline{A}}x_2dA$$

thus

$$\boxed{\overline{\sigma}_{13}(x_3)=\frac{-\mathrm{I}_{33}P}{a(x_3)(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{\overline{A}}x_3dA-\frac{\mathrm{I}_{23}P}{a(x_3)(\mathrm{I}_{23}^2-\mathrm{I}_{22}\mathrm{I}_{33})}\int_{\overline{A}}x_2dA} \tag{4.114}$$

The system is located at the Area Centroid

where $\int_{\overline{A}}x_3dA$ is the first moment of area $\overline{A}$ about the x_2-axis, and $\int_{\overline{A}}x_2dA$ is the first moment of area about the x_3-axis. Note that, if the x_3-axis is an axis of symmetry, then $\int_{\overline{A}}x_2dA=0$ holds, and the above equation reduces to:

$$\boxed{\overline{\sigma}_{13}(x_3) = \frac{P}{a(x_3)\mathrm{I}_{22}} \int_{x_2} \int_{x_3}^{c} x_3 dx_3 dx_2 \equiv \frac{P}{a(x_3)\mathrm{I}_{22}} \int_A x_3 dA} \quad \textit{The system is located at the Area Centroid and is the principal axes of inertia} \tag{4.115}$$

Example: Let us consider a circular cross section, (see Figure 4.44), and the adopted system is at the area centroid and is the principal axis of inertia, then we can apply the equation in (4.115):

$$\overline{\sigma}_{13}(x_3) = \frac{P}{a(x_3)\mathrm{I}_{22}} \int_A x_3 dA = \frac{P}{a(x_3)\mathrm{I}_{22}} \int_{x_3}^{c=R} x_3 a(x_3) dx_3 = \frac{P}{2\sqrt{(R^2 - x_3^2)}\mathrm{I}_{22}} \int_{x_3}^{c=R} 2x_3\sqrt{(R^2 - x_3^2)}$$

$$= \frac{P}{2\mathrm{I}_{22}\sqrt{(R^2 - x_3^2)}} \frac{2}{3}\sqrt{(R^2 - x_3^2)^3} = \frac{P(R^2 - x_3^2)}{3\mathrm{I}_{22}}$$

Note that $\frac{\partial \sigma_{11}}{\partial x_1} = \frac{-(\mathrm{I}_{23}x_2 + \mathrm{I}_{33}x_3)}{(\mathrm{I}_{23}^2 - \mathrm{I}_{22}\mathrm{I}_{33})} P = \frac{Px_3}{\mathrm{I}_{22}}$ and $\frac{\partial \overline{\sigma}_{13}(x_3)}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{P(R^2 - x_3^2)}{3\mathrm{I}_{22}}\right) = \frac{-2Px_3}{3\mathrm{I}_{22}}$, and by replacing it into the equilibrium equation we can obtain:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad \Rightarrow \quad \frac{Px_3}{\mathrm{I}_{22}} + \frac{-2Px_3}{3\mathrm{I}_{22}} = \frac{Px_3}{3\mathrm{I}_{22}} \neq 0$$

which does not satisfy the equilibrium equations.

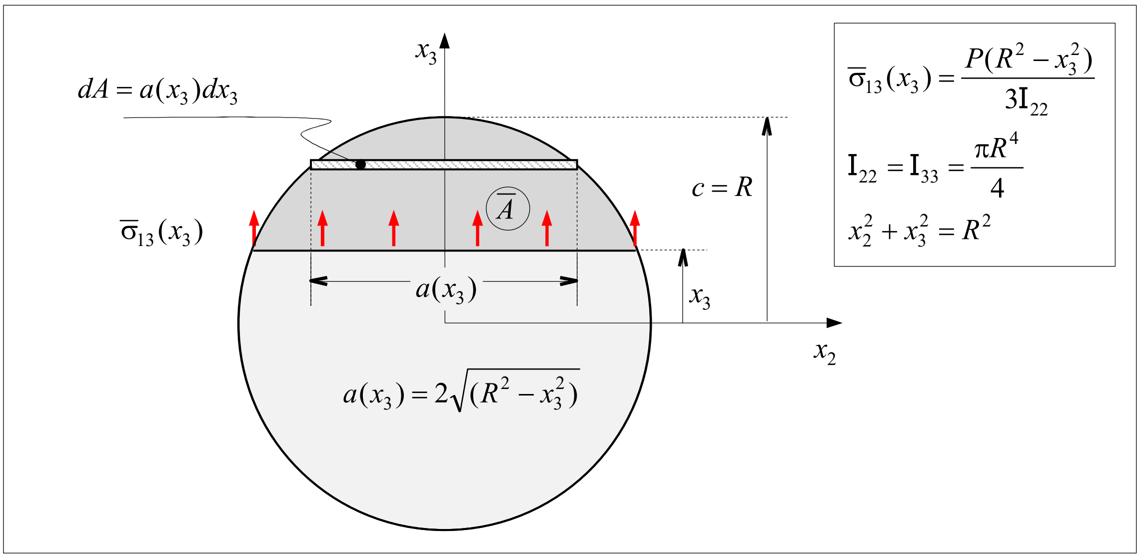


Figure 4.44: Circular cross section.

Problem 4.30

Consider the cantilever (beam fixed at one end) with an I-shaped cross section, (see Figure 4.45). Obtain the tangential stress distribution on the cross section due to the shearing force $F_{31} = F_z = 11000N$.

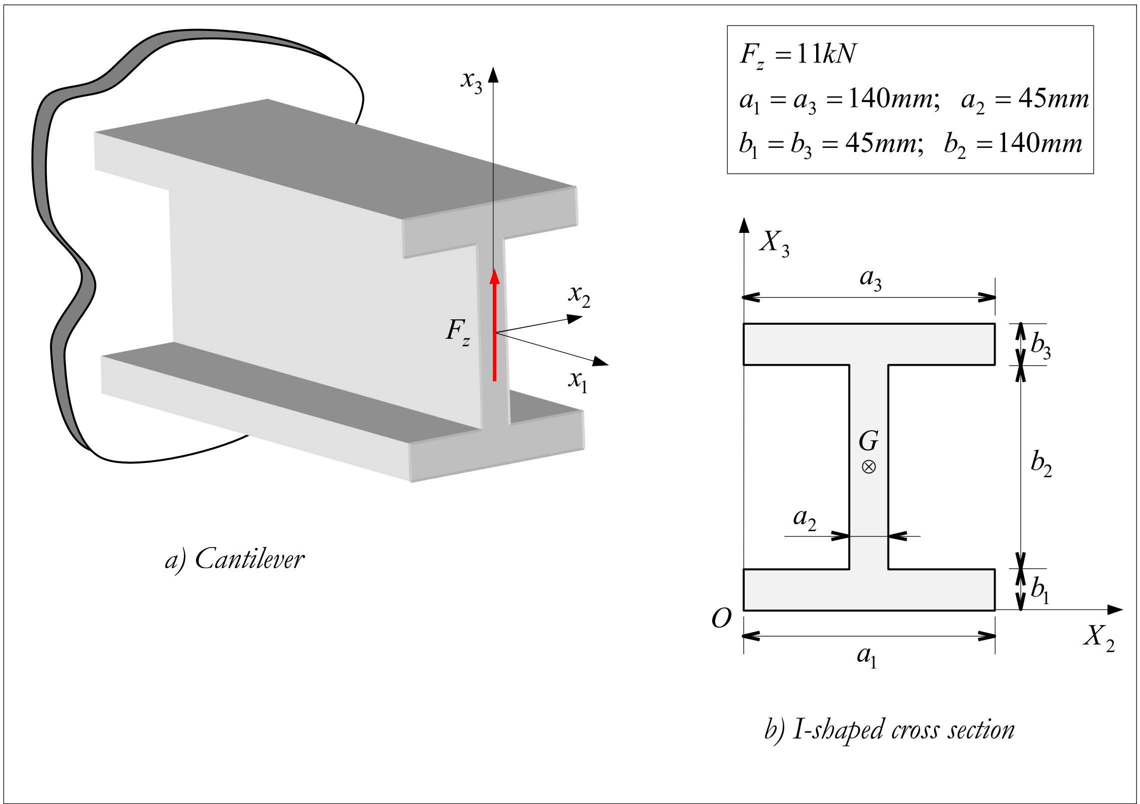


Figure 4.45: I-Shaped cross section.

Solution:

Area Centroid

Due to the geometrical symmetry the Area Centroid-G is located at

$$\overline{X}_2 = \frac{a_1}{2} = 0.07m \quad ; \quad \overline{X}_3 = b_1 + \frac{b_2}{2} = 0.115m$$

And the total area is given by $A = A_1 + A_2 + A_3 = a_1 b_1 + a_2 b_2 + a_3 b_3 = 2a_1 b_1 + a_2 b_2 = 0.0189 m^2$.

The inertia tensor related to the system $X_2 - X_3$**:**

For each rectangle we will apply the equation in (4.82) in which $\bar{x}_2$ and $\bar{x}_3$ are now related to the system $O\vec{\boldsymbol{X}}(X_2 - X_3)$:

$$\mathrm{I}^{(q)}_{O\vec{X}ij} = \frac{ab}{12}\begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} \tag{4.116}$$

Rectangle $q=1$: $a = a_1 = 0.140$, $b = b_1 = 0.045$, $\overline{X}_2 = \frac{a_1}{2} = 0.070$, $\overline{X}_3 = \frac{b_1}{2} = 0.0225$:

$$\mathrm{I}^{(1)}_{O\vec{X}ij} = \frac{ab}{12}\begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} = \begin{bmatrix} 4.2525 & -9.9225 \\ -9.9225 & 41.16 \end{bmatrix} \times 10^{-6} m^4$$

Rectangle $q=2$: $a = a_2 = 0.045$, $b = b_1 = 0.140$, $\overline{X}_2 = \frac{a_1}{2} = 0.070$, $\overline{X}_3 = b_1 + \frac{b_2}{2} = 0.115$:

$$\mathrm{I}^{(2)}_{O\vec{X}ij} = \frac{ab}{12}\begin{bmatrix} b^2 + 12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2 + 12\overline{X}_2^2 \end{bmatrix} = \begin{bmatrix} 93.6075 & -50.715 \\ -50.715 & 31.933125 \end{bmatrix} \times 10^{-6} m^4$$

Rectangle $q=3$: $a=a_1=0.140$, $b=b_1=0.045$, $\overline{X}_2=\frac{a_1}{2}=0.070$, $\overline{X}_3=0.230-\frac{b_3}{2}=0.2075$:

$$\mathrm{I}^{(3)}_{O\bar{X}\,ij}=\frac{ab}{12}\begin{bmatrix} b^2+12\overline{X}_3^2 & -12\overline{X}_2\overline{X}_3 \\ -12\overline{X}_2\overline{X}_3 & a^2+12\overline{X}_2^2 \end{bmatrix}=\begin{bmatrix} 272.3175 & -91.5075 \\ -91.5075 & 41.16 \end{bmatrix}\times 10^{-6}\,m^4$$

Then

$$\mathrm{I}^{(Sys)}_{O\bar{X}\;ij}=\mathrm{I}^{(1)}_{O\bar{X}\,ij}+\mathrm{I}^{(2)}_{O\bar{X}\,ij}+\mathrm{I}^{(3)}_{O\bar{X}\,ij}=\begin{bmatrix} 3.701775 & -1.52145 \\ -1.52145 & 1.142531 \end{bmatrix}\times 10^{-4}\,m^4$$

The inertia tensor related to the system located at the Area Centroid:

The inertia tensor at the Area Centroid, (see equation in (4.75)), can be obtained by means of the Steiner's theorem:

$$\mathbf{I}^{(Sys)}_{O\vec{X}}=\overline{\mathbf{I}}_{G\vec{x}}-A[(\vec{\overline{X}}\otimes\vec{\overline{X}})-(\vec{\overline{X}}\cdot\vec{\overline{X}})\mathbf{1}] \quad\Rightarrow\quad \overline{\mathbf{I}}_{G\vec{x}}=\mathbf{I}_{O\vec{X}}+A[(\vec{\overline{X}}\otimes\vec{\overline{X}})-(\vec{\overline{X}}\cdot\vec{\overline{X}})\mathbf{1}]$$

whose components are:

$$(\overline{\mathbf{I}}_{G\vec{x}})_{ij}=\begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix}=\mathrm{I}^{(Sys)}_{O\vec{X}\;ij}-A\begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix} \tag{4.117}$$

By substituting the variable values we can obtain:

$$\begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix}=\begin{bmatrix} 1.20225 & 0 \\ 0 & 0.2164312 \end{bmatrix}\times 10^{-4}\,m^4$$

Tangential stress on the cross section

Since the axes are axes of symmetry we can apply the equation in (4.115) in order to obtain the tangential stress (shear stress):

$$\overline{\sigma}_{13}(x_3)=\frac{P}{a(x_3)\mathrm{I}_{G22}}\int_{x_2}\int_{x_3}^{c} x_3\,dx_3dx_2 \equiv \frac{P}{a(x_3)\mathrm{I}_{G22}}\int_{A} x_3\,dA \tag{4.118}$$

$$\overline{\sigma}^{(f)}_{13}(x_3=0.115)=0$$

$$\overline{\sigma}^{(+g)}_{13}(x_3=0.070)=\frac{P}{a_3\mathrm{I}_{G22}}\int_{A} x_3\,dA=\frac{P}{a_3\mathrm{I}_{G22}}[A_3\overline{x}_3^{(A_3)}]=\frac{11000}{(0.140)(1.20225)}[(0.0063)(0.0925)]$$

$$\Rightarrow \overline{\sigma}^{(+g)}_{13}(x_3=0.070)=3.808484\times 10^5\,Pa$$

$$\overline{\sigma}^{(-g)}_{13}(x_3=0.070)=\frac{P}{a_2\mathrm{I}_{G22}}\int_{A} x_3\,dA=\frac{P}{a_2\mathrm{I}_{G22}}[A_3\overline{x}_3^{(A_3)}]=\frac{11000}{(0.045)(1.20225)}[(0.0063)(0.0925)]$$

$$\Rightarrow \overline{\sigma}^{(-g)}_{13}(x_3=0.070)=11.848462\times 10^5\,Pa$$

At the neutral axis:

$$\overline{\sigma}^{(h)}_{13}(x_3=0.0)=\frac{P}{a_2\mathrm{I}_{G22}}\int_{A} x_3\,dA=\frac{P}{a_2\mathrm{I}_{G22}}\left[A_3\overline{x}_3^{(A_3)}+\frac{A_2}{2}\frac{b_2}{4}\right]=\frac{11000}{(0.045)(1.20225)}[(0.0063)(0.0925)]$$

$$\Rightarrow \overline{\sigma}^{(h)}_{13}(x_3=0.0)=14.09025\times 10^5\,Pa$$

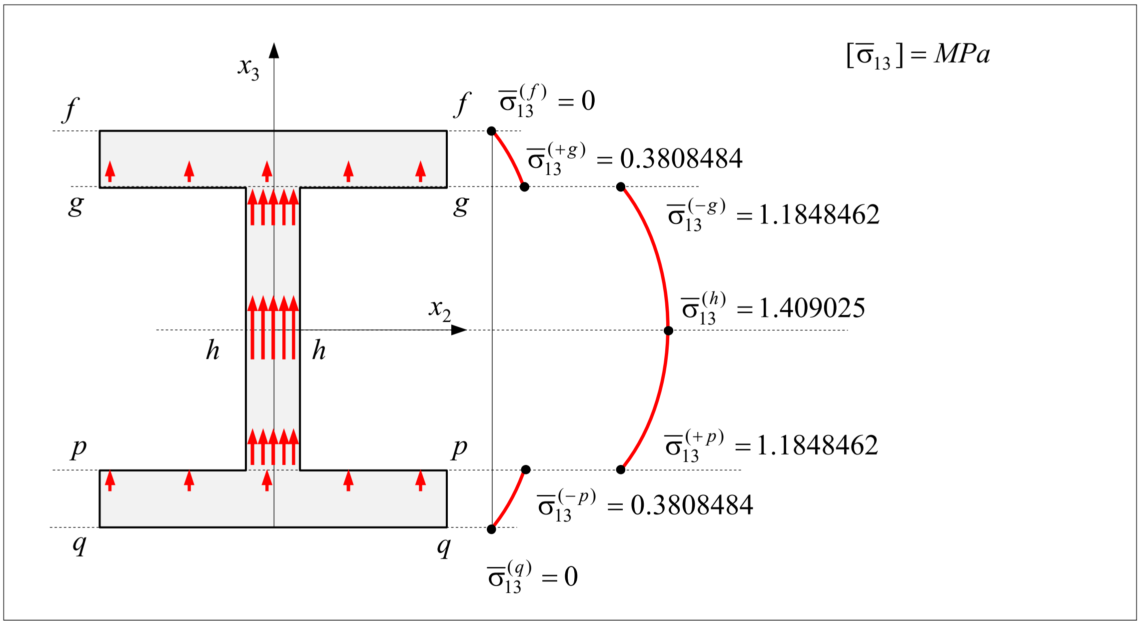


Figure 4.46: Tangential stress distribution on the I-Shaped cross section.

Problem 4.31

Consider the cross section described in Figure 4.47 in which is acting the shearing forces $F_{21} \equiv F_y$ and $F_{31} \equiv F_z$. a) Obtain the shear flux on the flanges. b) Locate the Shear Center (S.C.) of the cross section by adopting the system $O\vec{X}$.

Hypothesis (approximation): Consider that the thickness ($\bar{t}$) is very small when compared with a, (see Figure 4.47).

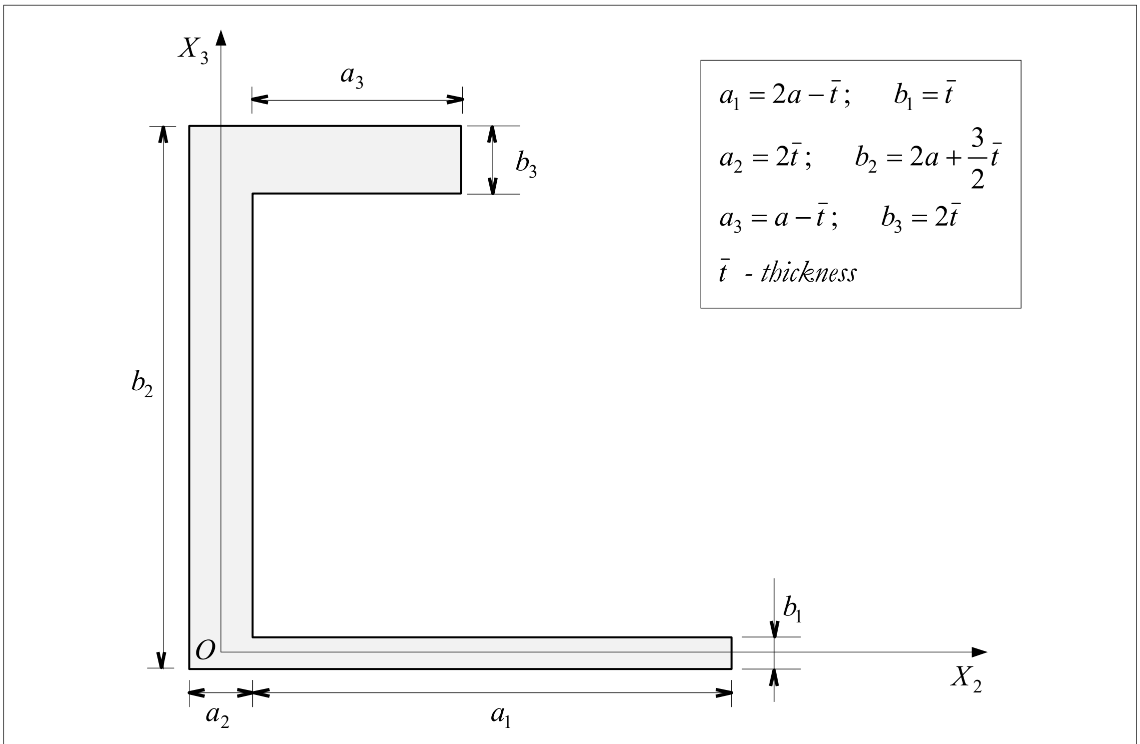


Figure 4.47: Cross section.

Solution:

Since the thickness is very small we can apply the equation in (4.106) in order to obtain the shear flux.

Geometric Properties

For each rectangle we will apply the equation in (4.82) in which $\bar{x}_2$ and $\bar{x}_3$ are now related to the system $O\vec{\boldsymbol{X}}(X_2 - X_3)$:

$$\mathrm{I}^{(q)}_{O\vec{X}\,ij} = \frac{a_q b_q}{12}\begin{bmatrix} b_q^2 + 12\bar{X}_3^2 & -12\bar{X}_2\bar{X}_3 \\ -12\bar{X}_2\bar{X}_3 & a_q^2 + 12\bar{X}_2^2 \end{bmatrix}$$

Rectangle $q=1$: $a_1 = 2a - \bar{t} \approx 2a$, $b_1 = \bar{t}$, $\bar{X}_2^{(1)} = a + \dfrac{\bar{t}}{2} \approx a$, $\bar{X}_3^{(1)} = 0$:

Area: $A_1 = 2a\bar{t}$

$$\mathrm{I}^{(1)}_{O\vec{X}\,ij} = \frac{a_1 b_1}{12}\begin{bmatrix} b_1^2 + 12\bar{X}_3^2 & -12\bar{X}_2\bar{X}_3 \\ -12\bar{X}_2\bar{X}_3 & a_1^2 + 12\bar{X}_2^2 \end{bmatrix} = \frac{2a\bar{t}}{12}\begin{bmatrix} \bar{t}^2 & 0 \\ 0 & 4a^2 + 12a^2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} \bar{t}^3 a & 0 \\ 0 & 16a^3\bar{t} \end{bmatrix}$$

Rectangle $q=2$: $a_2 = 2\bar{t}$, $b_2 = 2a + \dfrac{3}{2}\bar{t} \approx 2a$, $\bar{X}_2^{(2)} = 0$, $\bar{X}_3^{(2)} = a$:

Area: $A_2 = 4a\bar{t}$

$$\mathrm{I}^{(2)}_{O\vec{X}\,ij} = \frac{a_2 b_2}{12}\begin{bmatrix} b_2^2 + 12\bar{X}_3^2 & -12\bar{X}_2\bar{X}_3 \\ -12\bar{X}_2\bar{X}_3 & a_2^2 + 12\bar{X}_2^2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 32a^3\bar{t} & 0 \\ 0 & 8\bar{t}^3 a \end{bmatrix}$$

Rectangle $q=3$: $a_3 = a - \bar{t} \approx a$, $b_3 = 2\bar{t}$, $\bar{X}_2^{(3)} = \dfrac{a+\bar{t}}{2} \approx \dfrac{a}{2}$, $\bar{X}_3^{(3)} = 2a$:

Area: $A_3 = 2a\bar{t}$

$$\mathrm{I}^{(3)}_{O\vec{X}\,ij} = \frac{a_3 b_3}{12}\begin{bmatrix} b_3^2 + 12\bar{X}_3^2 & -12\bar{X}_2\bar{X}_3 \\ -12\bar{X}_2\bar{X}_3 & a_3^2 + 12\bar{X}_2^2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 48a^3\bar{t} & -12a^3\bar{t} \\ -12a^3\bar{t} & 4a^3\bar{t} \end{bmatrix}$$

Then

Total Area: $A = A_1 + A_2 + A_3 = 8a\bar{t}$

The Inertia Tensor or Area:

$$\mathrm{I}^{(Sys)}_{O\vec{X}\;ij} = \mathrm{I}^{(1)}_{O\vec{X}\,ij} + \mathrm{I}^{(2)}_{O\vec{X}\,ij} + \mathrm{I}^{(3)}_{O\vec{X}\,ij} = \frac{1}{6}\begin{bmatrix} 80a^3\bar{t} + 48\bar{t}^3 a & -12a^3\bar{t} \\ -12a^3\bar{t} & 20a^3\bar{t} + 8\bar{t}^3 a \end{bmatrix} \approx \frac{1}{6}\begin{bmatrix} 80a^3\bar{t} & -12a^3\bar{t} \\ -12a^3\bar{t} & 20a^3\bar{t} \end{bmatrix}$$

Calculation of the Area Centroid - G:

$$\bar{X}_2 = \frac{\sum\limits_{q=1}^{3} A^{(q)} X_2^{(q)}}{A} = \frac{A^{(1)}X_2^{(1)} + A^{(2)}X_2^{(2)} + A^{(3)}X_2^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})} \approx \frac{(4a\bar{t})(a) + (2a\bar{t})(0) + (2a\bar{t})\left(\dfrac{a}{2}\right)}{8a\bar{t}} = \frac{3}{8}a$$

$$\bar{X}_3 = \frac{\sum\limits_{q=1}^{3} A^{(q)} X_3^{(q)}}{A} = \frac{A^{(1)}X_3^{(1)} + A^{(2)}X_3^{(2)} + A^{(3)}X_3^{(3)}}{(A^{(1)} + A^{(2)} + A^{(3)})} \approx \frac{(4a\bar{t})(0) + (2a\bar{t})(a) + (2a\bar{t})(2a)}{8a\bar{t}} = a$$

The Shear Flux

Since we are adopting a system which is not at the Area Centroid we have to consider the equation (4.106) in order to obtain the shear flux:

$$q(s) = q(0) - \int_0^s \left(t \frac{\partial \sigma_{11}}{\partial x_1} \right) ds = q(0) - \frac{\mathcal{Y}_0}{\mathcal{X}} \int_0^s t\, ds - \frac{\mathcal{Y}_2}{\mathcal{X}} \int_0^s (t\, X_2) ds - \frac{\mathcal{Y}_3}{\mathcal{X}} \int_0^s (t\, X_3) ds \tag{4.119}$$

where the coefficients $\mathcal{Y}_0$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{X}$ are given by the equations (4.107) and (4.108).

For this problem we have:, $\mathrm{I}^{(Sys)}_{O\bar{x}\ ij} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} \approx \frac{1}{6} \begin{bmatrix} 80a^3\bar{t} & -12a^3\bar{t} \\ -12a^3\bar{t} & 20a^3\bar{t} \end{bmatrix}$, $\bar{x}_2 \leftarrow \bar{X}_2 \approx \frac{3}{8}a$, $\bar{x}_3 \leftarrow \bar{X}_3 \approx a$, $A = 8a\bar{t}$. With that the coefficients become:

$$\frac{\mathcal{Y}_0}{\mathcal{X}} = \frac{\begin{vmatrix} 0 & A\bar{x}_2 & A\bar{x}_3 \\ F_z & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -F_y & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = -F_z \frac{\begin{vmatrix} A\bar{x}_2 & A\bar{x}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}}{\mathcal{X}} - F_y \frac{\begin{vmatrix} A\bar{x}_2 & A\bar{x}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix}}{\mathcal{X}} = \frac{-27}{97a^2\bar{t}} F_y - \frac{93}{388a^2\bar{t}} F_z; \tag{4.120}$$

$$\frac{\mathcal{Y}_2}{\mathcal{X}} = \frac{\begin{vmatrix} A & 0 & A\bar{x}_3 \\ A\bar{x}_3 & F_z & \mathrm{I}_{22} \\ -A\bar{x}_2 & -F_y & \mathrm{I}_{23} \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = F_z \frac{\begin{vmatrix} A & A\bar{x}_3 \\ -A\bar{x}_2 & \mathrm{I}_{23} \end{vmatrix}}{\mathcal{X}} + F_y \frac{\begin{vmatrix} A & A\bar{x}_3 \\ A\bar{x}_3 & \mathrm{I}_{22} \end{vmatrix}}{\mathcal{X}} = \frac{48}{97a^3\bar{t}} F_y + \frac{9}{97a^3\bar{t}} F_z; \tag{4.121}$$

$$\frac{\mathcal{Y}_3}{\mathcal{X}} = \frac{\begin{vmatrix} A & A\bar{x}_2 & 0 \\ A\bar{x}_3 & -\mathrm{I}_{23} & F_z \\ -A\bar{x}_2 & -\mathrm{I}_{33} & -F_y \end{vmatrix}}{\begin{vmatrix} A & A\bar{x}_2 & A\bar{x}_3 \\ A\bar{x}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\bar{x}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}} = -F_z \frac{\begin{vmatrix} A & A\bar{x}_2 \\ -A\bar{x}_2 & -\mathrm{I}_{33} \end{vmatrix}}{\mathcal{X}} - F_y \frac{\begin{vmatrix} A & A\bar{x}_2 \\ A\bar{x}_3 & -\mathrm{I}_{23} \end{vmatrix}}{\mathcal{X}} = \frac{9}{97a^3\bar{t}} F_y + \frac{159}{776a^3\bar{t}} F_z \tag{4.122}$$

Then, the equation in (4.119) becomes

$$\begin{aligned} q(s) = q(0) &- \left(\frac{-27}{97a^2\bar{t}} F_y - \frac{93}{388a^2\bar{t}} F_z \right) \int_0^s t\, ds - \left(\frac{48}{97a^3\bar{t}} F_y + \frac{9}{97a^3\bar{t}} F_z \right) \int_0^s (t\, x_2) ds \\ &- \left(\frac{9}{97a^3\bar{t}} F_y + \frac{159}{776a^3\bar{t}} F_z \right) \int_0^s (t\, x_3) ds \end{aligned} \tag{4.123}$$

We will discretize the path s such as $s = s^{(43)} + s^{(32)} + s^{(21)}$, (see Figure 4.48), where the origin of $s = 0$ is located at point 4.

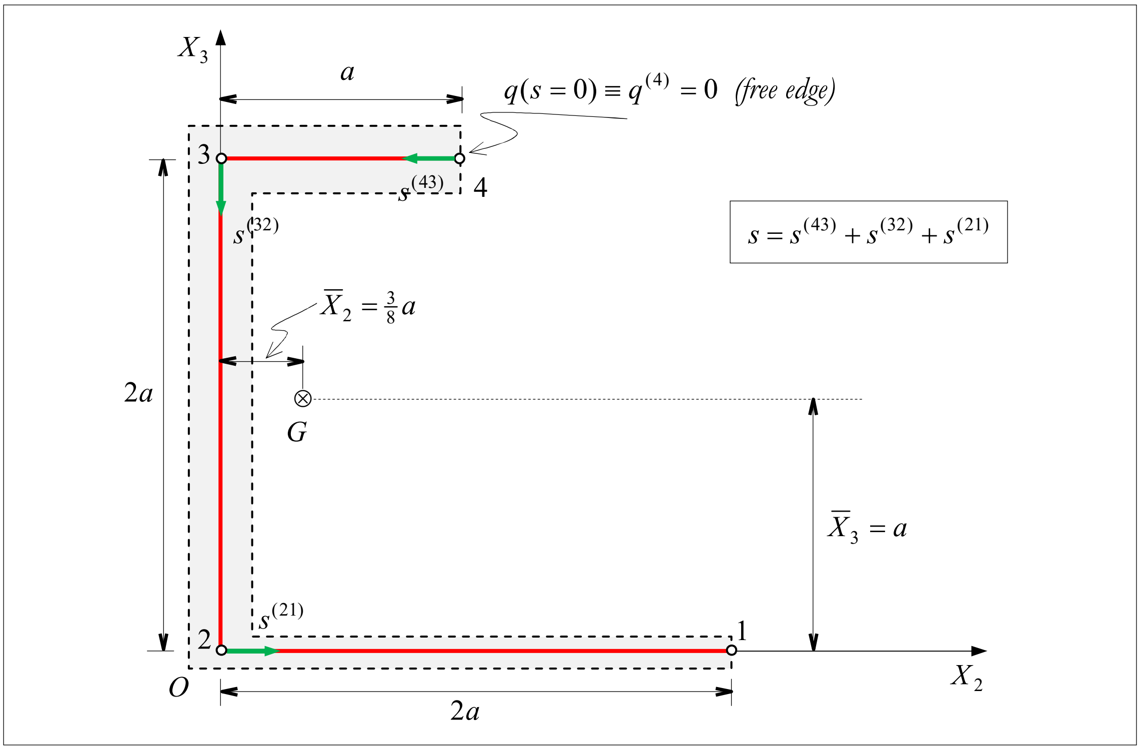


Figure 4.48: Path of s.

Path $4 \to 3$: $X_2 = a - s^{(43)}$, $X_3 = 2a$, $t = 2\bar{t}$, $0 \le s^{(43)} \le a$

Then, the equation (4.123) becomes:

$$q^{(43)} = \underbrace{q^{(4)}}_{=0} - \left(\frac{-27}{97a^2\bar{t}} F_y - \frac{93}{388a^2\bar{t}} F_z \right) \int_0^s (2\bar{t}) ds^{(43)} - \left(\frac{48}{97a^3\bar{t}} F_y + \frac{9}{97a^3\bar{t}} F_z \right) \int_0^s (2\bar{t})\,(a - s^{(43)}) ds^{(43)}$$
$$- \left(\frac{9}{97a^3\bar{t}} F_y + \frac{159}{776a^3\bar{t}} F_z \right) \int_0^s (2\bar{t})(2a) ds^{(43)}$$

After the integrals are solved we can obtain:

$$q^{(43)}(s) = \frac{-3}{97a^3} s(26a - 16s) F_y - \frac{3}{97a^3} s(17a - 3s) F_z$$

And at the point 3, $(s = a)$, we can obtain:

$$q^{(43)}(s=a) = q^{(3)} = \frac{-3}{97a^3} a(26a - 16a) F_y - \frac{3}{97a^3} a(17a - 3a) F_z = \frac{-30}{97a} F_y - \frac{42}{97a} F_z$$

Path $3 \to 2$: $X_2 = 0$, $X_3 = 2a - s^{(32)}$, $t = 2\bar{t}$, $0 \le s^{(32)} \le 2a$

Then, the equation (4.123) becomes:

$$q^{(32)} = q^{(3)} - \left(\frac{-27}{97a^2\bar{t}} F_y - \frac{93}{388a^2\bar{t}} F_z \right) \int_0^s (2\bar{t}) ds^{(32)} - \left(\frac{48}{97a^3\bar{t}} F_y + \frac{9}{97a^3\bar{t}} F_z \right) \int_0^s (2\bar{t})\,(0) ds^{(32)}$$
$$- \left(\frac{9}{97a^3\bar{t}} F_y + \frac{159}{776a^3\bar{t}} F_z \right) \int_0^s (2\bar{t})(2a - s^{(32)}) ds^{(32)}$$

After the integrals are solved we can obtain:

$$q^{(32)}(s) = q^{(3)} + \frac{3}{776a^3}s(48a+24s)F_y - \frac{3}{776a^3}s(88a-53s)F_z$$

$$\Rightarrow q^{(32)}(s) = \left(\frac{-30}{97a}F_y - \frac{42}{97a}F_z\right) + \frac{3}{776a^3}s(48a+24s)F_y - \frac{3}{776a^3}s(88a-53s)F_z$$

And at the point 2, $(s=2a)$, we can obtain:

$$q^{(2)} = \left(\frac{-30}{97a}F_y - \frac{42}{97a}F_z\right) + \frac{3}{776a^3}2a(48a+24(2a))F_y - \frac{3}{776a^3}2a(88a-53(2a))F_z$$

$$\Rightarrow q^{(2)} = \frac{42}{97a}F_y - \frac{57}{194a}F_z$$

Path $2 \to 1$: $X_2 = s^{(21)}$, $X_3 = 0$, $t = \bar{t}$, $0 \le s^{(21)} \le 2a$

Then, the equation (4.123) becomes:

$$q^{(21)} = q^{(2)} - \left(\frac{-27}{97a^2\bar{t}}F_y - \frac{93}{388a^2\bar{t}}F_z\right)\int_0^s (\bar{t})ds^{(21)} - \left(\frac{48}{97a^3\bar{t}}F_y + \frac{9}{97a^3\bar{t}}F_z\right)\int_0^s (\bar{t})\,(s^{(21)})ds^{(21)}$$
$$- \left(\frac{9}{97a^3\bar{t}}F_y + \frac{159}{776a^3\bar{t}}F_z\right)\int_0^s (\bar{t})(0)ds^{(21)}$$

After the integrals are solved we can obtain:

$$q^{(21)}(s) = q^{(2)} + \frac{3}{388a^3}s(36a-32s)F_y + \frac{3}{388a^3}s(31a-6s)F_z$$

$$\Rightarrow q^{(21)}(s) = \left(\frac{42}{97a}F_y - \frac{57}{194a}F_z\right) + \frac{3}{388a^3}s(36a-32s)F_y + \frac{3}{388a^3}s(31a-6s)F_z$$

And at the point 1, $(s=2a)$, we can obtain:

$$q^{(1)} = \left(\frac{42}{97a}F_y - \frac{57}{194a}F_z\right) + \frac{3}{388a^3}2a(36a-32(2a))F_y + \frac{3}{388a^3}2a(31a-6(2a))F_z = 0$$

as expected, $q^{(1)} = 0$ is zero, since we are dealing with a free edge. The Shear Flux can be appreciated in Figure 4.49.

We have defined the *Shear Center (S.C.)* in **Problem 4.22**-NOTE 3. We can calculate the torsion moment at any point, but by simplicity we will adopt the point $2 \equiv O$, since the shear fluxes $q^{(32)}(s)$ and $q^{(21)}(s)$ will not contribute to the torsion moment, and the torsion moment produced by the shear flux $q^{(43)}(s)$ is given by:

$$\vec{\boldsymbol{M}}_O = \int_0^s \vec{\boldsymbol{X}} \wedge \vec{\boldsymbol{q}} ds = \int_0^s (X_3 q^{(43)})\hat{\mathbf{e}}_1 ds = \int_0^s ((2a)q^{(43)})\hat{\mathbf{e}}_1 ds = 2a\underbrace{\int_0^a q^{(43)} ds}_{=f^{(43)}}\hat{\mathbf{e}}_1 = 2af^{(43)}\hat{\mathbf{e}}_1$$

where we have considered

$$\vec{\boldsymbol{X}} \wedge \vec{\boldsymbol{q}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ X_1 & X_2 & X_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & X_2 & X_3 \\ 0 & -q^{(43)} & 0 \end{vmatrix} = (X_3 q_2^{(43)})\hat{\mathbf{e}}_1 = (2aq_2^{(43)})\hat{\mathbf{e}}_1$$

and $F^{(43)}$ is the total force on the flange $4 \to 3$. Then, in order to obtain the torsion moment we have to solve the integral:

$$\vec{\boldsymbol{M}}_O = 2a\int_0^a q^{(43)} ds\hat{\mathbf{e}}_1 = 2a\int_0^a \left(\frac{-3}{97a^3} s(26a-16s)F_y - \frac{3}{97a^3} s(17a-3s)F_z \right) ds\hat{\mathbf{e}}_1$$

$$\Rightarrow \vec{\boldsymbol{M}}_O = \left(\frac{-45a}{97}F_z - \frac{46a}{97}F_y \right)\hat{\mathbf{e}}_1 \qquad \therefore \qquad M_{X_1} = \frac{-45a}{97}F_{31} - \frac{46a}{97}F_{21}$$

And if we compare with the equation in Figure 4.10 we can conclude that

$$X_2^{(S.C.)} = \frac{-45a}{97} \quad ; \quad X_3^{(S.C.)} = \frac{46a}{97}$$

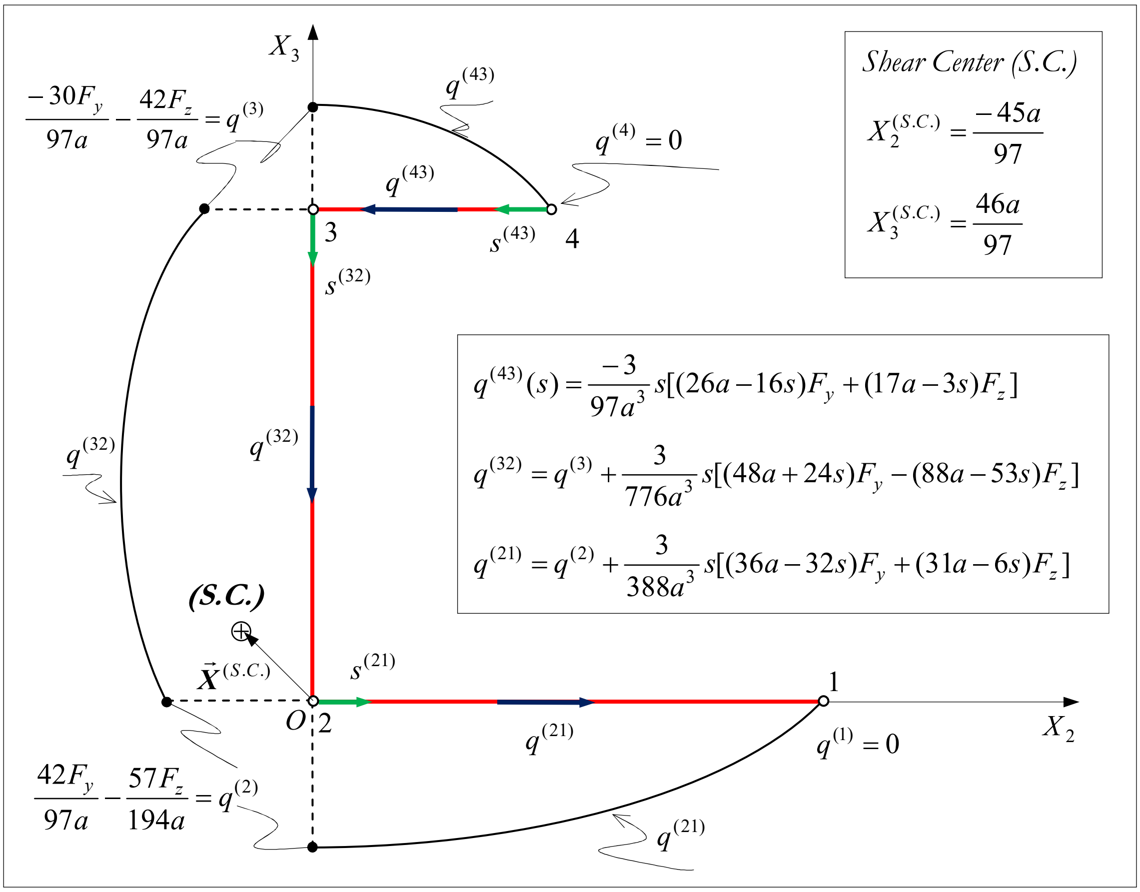


Figure 4.49: Shear Flux on the flanges.

NOTE 1: Let us consider a generic flange element with length a and thickness t which is constant along the flange element, (see Figure 4.50). By considering the systems $O\vec{X}$ and $i\vec{x}$ we can obtain that $\vec{X} = \vec{X}^{(i)} + \vec{x}$ and note that the systems $i\vec{x}$ and $i\vec{x}'$ are related to each other by the transformation matrix $\mathcal{A}$ as follows $\vec{x}' = \mathcal{A}\vec{x}$ and $\vec{x} = \mathcal{A}^T\vec{x}'$, (see Figure 4.21), where the transformation matrix is given by

$$\mathcal{A} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix}$$

Then $\vec{X} = \vec{X}^{(i)} + \vec{x} = \vec{X}^{(i)} + \mathcal{A}^T\vec{x}'$ whose components are:

$$\vec{X} = \vec{X}^{(i)} + \mathcal{A}^T\vec{x}' \xrightarrow{components} \begin{Bmatrix} X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} + \begin{bmatrix} \ell & -m \\ m & \ell \end{bmatrix} \begin{Bmatrix} x_2' = s \\ x_3' = 0 \end{Bmatrix} = \begin{Bmatrix} X_2^{(i)} + \ell s \\ X_3^{(i)} + m s \end{Bmatrix}$$

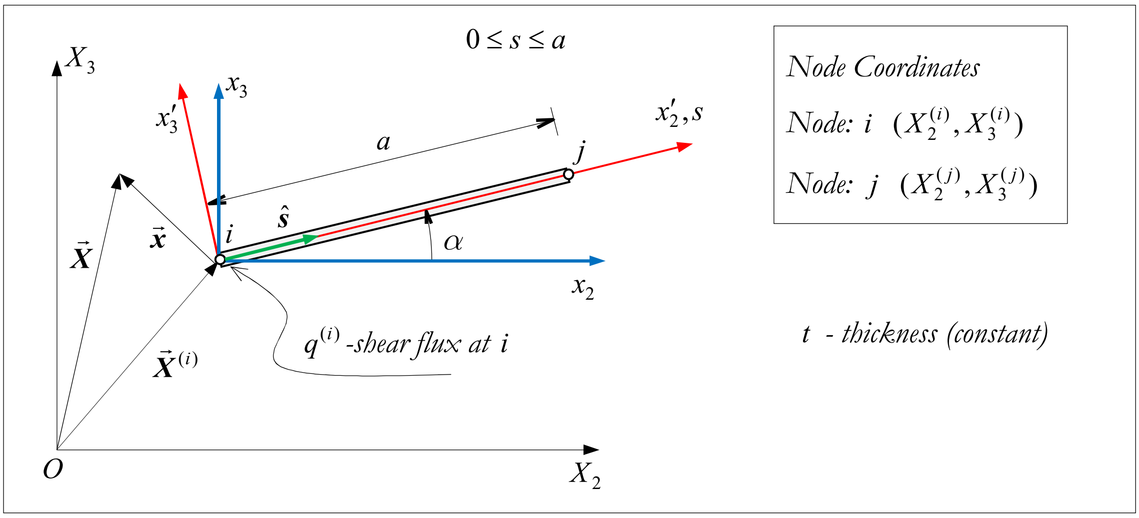


Figure 4.50

By means of the previous considerations we can solve the following integrals:

$$\int_0^s X_2 ds = \int_0^s (X_2^{(i)} + \ell\, s) ds = X_2^{(i)} s + \frac{s^2 \ell}{2} \quad ; \quad \int_0^s X_3 ds = \int_0^s (X_3^{(i)} + m\, s) ds = X_3^{(i)} s + \frac{s^2 m}{2}$$

And the equation in (4.119) can be written as follows

$$\begin{aligned}
& q(s) = q^{(i)} - \frac{\mathcal{Y}_0\, t}{\mathcal{X}} \int_0^s ds - \frac{\mathcal{Y}_2\, t}{\mathcal{X}} \int_0^s X_2 ds - \frac{\mathcal{Y}_3\, t}{\mathcal{X}} \int_0^s X_3 ds \\
& \Rightarrow q(s) = q^{(i)} - \frac{\mathcal{Y}_0\, t}{\mathcal{X}} s - \frac{\mathcal{Y}_2\, t}{\mathcal{X}} \left(X_2^{(i)} s + \frac{s^2 \ell}{2} \right) - \frac{\mathcal{Y}_3\, t}{\mathcal{X}} \left(X_3^{(i)} s + \frac{s^2 m}{2} \right) \\
& \Rightarrow q(s) = q^{(i)} - t \left(\frac{\mathcal{Y}_0}{\mathcal{X}} + \frac{\mathcal{Y}_2 X_2^{(i)}}{\mathcal{X}} + \frac{\mathcal{Y}_3 X_3^{(i)}}{\mathcal{X}} \right) s - t \left(\frac{\mathcal{Y}_2\, \ell}{2\mathcal{X}} + \frac{\mathcal{Y}_3\, m}{2\mathcal{X}} \right) s^2 \\
& \Rightarrow q(s) = q^{(i)} + t\, d_1 s + t\, d_2 s^2
\end{aligned}$$

where

$$d_1 = -\left(\frac{\mathcal{Y}_0}{\mathcal{X}} + \frac{\mathcal{Y}_2 X_2^{(i)}}{\mathcal{X}} + \frac{\mathcal{Y}_3 X_3^{(i)}}{\mathcal{X}} \right) \quad ; \quad d_2 = -\left(\frac{\mathcal{Y}_2\, \ell}{2\mathcal{X}} + \frac{\mathcal{Y}_3\, m}{2\mathcal{X}} \right)$$

The coefficients $\mathcal{Y}_0$, $\mathcal{Y}_2$ and $\mathcal{Y}_3$, (see equations (4.120)-(4.122)), can be rewritten as follows

$$\mathcal{Y}_0 = \begin{vmatrix} 0 & A\bar{X}_2 & A\bar{X}_3 \\ F_z & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -F_y & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = -\begin{vmatrix} A\bar{X}_2 & A\bar{X}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix} F_y - \begin{vmatrix} A\bar{X}_2 & A\bar{X}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} F_z = p_0^{(1)} F_y + p_0^{(2)} F_z \tag{4.124}$$

$$\mathcal{Y}_2 = \begin{vmatrix} A & 0 & A\bar{X}_3 \\ A\bar{X}_3 & F_z & \mathrm{I}_{22} \\ -A\bar{X}_2 & -F_y & \mathrm{I}_{23} \end{vmatrix} = \begin{vmatrix} A & A\bar{X}_3 \\ A\bar{X}_3 & \mathrm{I}_{22} \end{vmatrix} F_y + \begin{vmatrix} A & A\bar{X}_3 \\ -A\bar{X}_2 & \mathrm{I}_{23} \end{vmatrix} F_z = p_2^{(1)} F_y + p_2^{(2)} F_z \tag{4.125}$$

$$\mathcal{Y}_3 = \begin{vmatrix} A & A\bar{X}_2 & 0 \\ A\bar{X}_3 & -\mathrm{I}_{23} & F_z \\ -A\bar{X}_2 & -\mathrm{I}_{33} & -F_y \end{vmatrix} = -\begin{vmatrix} A & A\bar{X}_2 \\ A\bar{X}_3 & -\mathrm{I}_{23} \end{vmatrix} F_y - \begin{vmatrix} A & A\bar{X}_2 \\ -A\bar{X}_2 & -\mathrm{I}_{33} \end{vmatrix} F_z = p_3^{(1)} F_y + p_3^{(2)} F_z \tag{4.126}$$

And the coefficients d_1 and d_2 can be rewritten as follows:

$$d_1 = -\left(\frac{\mathcal{Y}_0}{\mathcal{X}} + \frac{\mathcal{Y}_2 X_2^{(i)}}{\mathcal{X}} + \frac{\mathcal{Y}_3 X_3^{(i)}}{\mathcal{X}}\right)$$

$$\Rightarrow d_1 = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} F_y + p_0^{(2)} F_z + (p_2^{(1)} F_y + p_2^{(2)} F_z) X_2^{(i)} + (p_3^{(1)} F_y + p_3^{(2)} F_z) X_3^{(i)}\right]$$

$$\Rightarrow d_1 = \left\{\frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right]\right\} F_y + \left\{\frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right]\right\} F_z = d_1^{F_y} F_y + d_1^{F_z} F_z$$

$$d_2 = -\left(\frac{\mathcal{Y}_2\,\ell}{2\mathcal{X}} + \frac{\mathcal{Y}_3\,m}{2\mathcal{X}}\right) = \frac{-1}{2\mathcal{X}}\left[(p_2^{(1)} F_y + p_2^{(2)} F_z)\,\ell + (p_3^{(1)} F_y + p_3^{(2)} F_z)\,m\right]$$

$$\Rightarrow d_2 = \left\{\frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right]\right\} F_y + \left\{\frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right]\right\} F_z = d_2^{F_y} F_y + d_2^{F_z} F_z$$

Then, we can represent the shear flux ($q(s) = q^{(i)} + t\,d_1 s + t\,d_2 s^2$) as follows

$$\boxed{q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2) F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2) F_z} \quad \textit{(Shear Flux)}$$

where

$$d_1 = d_1^{F_y} F_y + d_1^{F_z} F_z$$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] \quad ; \quad d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right]$$

$$d_2 = d_2^{F_y} F_y + d_2^{F_z} F_z$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] \quad ; \quad d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right]$$

with $\ell = \dfrac{X_2^{(j)} - X_2^{(i)}}{a}$, $m = \dfrac{X_3^{(j)} - X_3^{(i)}}{a}$ (4.127)

$$p_0^{(1)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix} \quad ; \quad p_0^{(2)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}$$

$$p_2^{(1)} = \begin{vmatrix} A & A\overline{X}_3 \\ A\overline{X}_3 & \mathrm{I}_{22} \end{vmatrix} \quad ; \quad p_2^{(2)} = \begin{vmatrix} A & A\overline{X}_3 \\ -A\overline{X}_2 & \mathrm{I}_{23} \end{vmatrix}$$

$$p_3^{(1)} = -\begin{vmatrix} A & A\overline{X}_2 \\ A\overline{X}_3 & -\mathrm{I}_{23} \end{vmatrix} \quad ; \quad p_3^{(2)} = -\begin{vmatrix} A & A\overline{X}_2 \\ -A\overline{X}_2 & -\mathrm{I}_{33} \end{vmatrix}, \quad \mathcal{X} = \begin{vmatrix} A & A\overline{X}_2 & A\overline{X}_3 \\ A\overline{X}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\overline{X}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}$$

The shear flux at the end of the flange (node j) is given by

$$q_{F_y}^{(j)}(s=a) = q_{F_y}^{(j)} = q_{F_y}^{(i)} + t\,d_1^{F_y} a + t\,d_2^{F_y} a^2 \quad ; \quad q_{F_z}^{(j)}(s=a) = q_{F_z}^{(j)} = q_{F_z}^{(i)} + t\,d_1^{F_z} a + t\,d_2^{F_z} a^2$$

The torsion moment at the point O

The torsion moment due to the shear flux $\vec{\boldsymbol{q}} = q\hat{\boldsymbol{s}}$ can be calculated as follows:

$$\vec{\boldsymbol{M}}_O = \int_0^a \vec{\boldsymbol{X}} \wedge \vec{\boldsymbol{q}} ds$$

where

$$\begin{Bmatrix} \hat{\mathbf{e}}_2' = \hat{s} \\ \hat{\mathbf{e}}_3' \end{Bmatrix} = \begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix} = \begin{Bmatrix} \ell\, \hat{\mathbf{e}}_2 + m\, \hat{\mathbf{e}}_3 \\ -m\, \hat{\mathbf{e}}_2 + \ell\, \hat{\mathbf{e}}_3 \end{Bmatrix}$$

$$\begin{aligned} \vec{\boldsymbol{X}} \wedge \vec{\boldsymbol{q}} &= (X_2 \hat{\mathbf{e}}_2 + X_3 \hat{\mathbf{e}}_3) \wedge (q\hat{s}) = q(X_2 \hat{\mathbf{e}}_2 + X_3 \hat{\mathbf{e}}_3) \wedge (\ell\, \hat{\mathbf{e}}_2 + m\, \hat{\mathbf{e}}_3) \\ &= \ell\, X_2 q \underbrace{\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_2}_{=\vec{\mathbf{0}}} + m\, X_2 q \underbrace{\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_3}_{=\hat{\mathbf{e}}_1} + \ell\, X_3 q \underbrace{\hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_2}_{=-\hat{\mathbf{e}}_1} + m\, X_3 q \underbrace{\hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_3}_{=\vec{\mathbf{0}}} \\ &= (m\, X_2 q - \ell\, X_3 q)\hat{\mathbf{e}}_1 \end{aligned}$$

Then, the magnitude of the torsion moment is given by:

$$M_O = \int_0^a m\, X_2 q ds - \int_0^a \ell\, X_3 q ds$$

$$\Rightarrow M_O = \int_0^a m\, (X_2^{(i)} + \ell\, s)(q^{(i)} + t\, d_1 s + t\, d_2 s^2) ds - \int_0^a \ell\, (X_3^{(i)} + m\, s)(q^{(i)} + t\, d_1 s + t\, d_2 s^2) ds$$

After the integrals are solved we can obtain:

$$\boxed{M_O = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1 + 2a^2 td_2 + 6q^{(i)})} \tag{4.128}$$

Taking into account the equations in (4.127), the torsion moment can be split additively into

$$M_O = M_O^{F_y} F_y + M_O^{F_z} F_z = -(X_3^{(S.C.)}) F_y + (X_2^{(S.C.)}) F_z \tag{4.129}$$

where

$$M_O^{F_y} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) \tag{4.130}$$

and

$$M_O^{F_z} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) \tag{4.131}$$

If we compare the equation (4.129) with the equation in Figure 4.10 we can conclude that

$$X_2^{(S.C.)} = M_O^{F_z} \quad ; \quad X_3^{(S.C.)} = -M_O^{F_y} \quad \textit{(Shear Center)}$$

The total force in the flange

The total force in the flange can be obtained as follows:

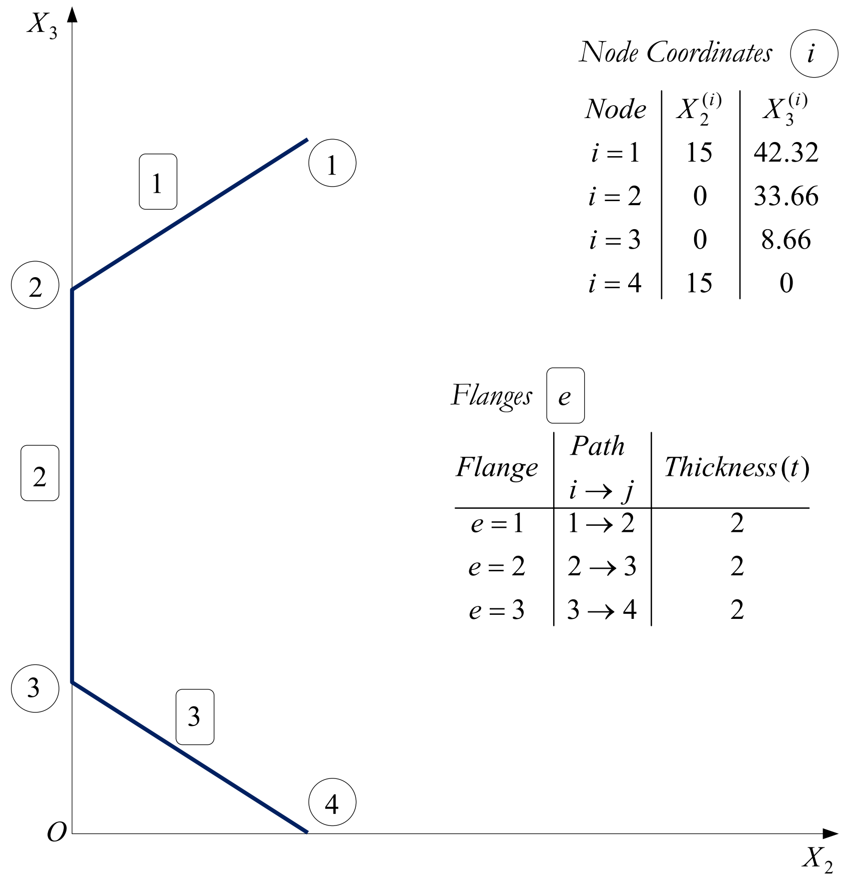
$$f^{(e)} = \int_0^a q(s) ds = F_y \int_0^a (q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2) ds + F_z \int_0^a (q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2) ds$$

$$\Rightarrow f^{(e)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z = f_{F_y}^{(e)} F_y + f_{F_z}^{(e)} F_z$$

Problem 4.32

Consider the cross section described in Figure 4.51 in which is acting the shearing forces $F_{21} \equiv F_y$ and $F_{31} \equiv F_z$. a) Obtain the shear flux on the flanges. b) Locate the Shear Center (S.C.) of the cross section by adopting the system $O\vec{X}$.

Hypothesis (approximation): Consider that the thickness (t) is very small when compared with length flange, (see Figure 4.51).



Node Coordinates (i)

Node	$X_2^{(i)}$	$X_3^{(i)}$
$i=1$	15	42.32
$i=2$	0	33.66
$i=3$	0	8.66
$i=4$	15	0

Flanges $\boxed{e}$

Flange	*Path* $i \to j$	*Thickness* (t)
$e=1$	$1 \to 2$	2
$e=2$	$2 \to 3$	2
$e=3$	$3 \to 4$	2

Figure 4.51: Cross section – Dimensions in centimeter (cm).

Solution:

We will use the information provided in **Problem 4.23**-NOTE 4, in which we have defined

$$\mathbf{I}^{(e)}_{O\vec{X}\,ij} = \frac{at}{12}\begin{bmatrix} t^2\ell^2 + a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & \ell m(t^2 - a^2) - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ \ell m(t^2 - a^2) - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & t^2 m^2 + a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix} \tag{4.132}$$

where $a = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$, $\ell = \dfrac{X_2^{(j)} - X_2^{(i)}}{a}$, $m = \dfrac{X_3^{(j)} - X_3^{(i)}}{a}$,

$\overline{X}_2^{(g)} = \dfrac{X_2^{(i)} + X_2^{(j)}}{2}$, $\overline{X}_3^{(g)} = \dfrac{X_3^{(i)} + X_3^{(j)}}{2}$

And the inertia tensor of area for the compound if given by:

$$\mathbf{I}^{(Sys)}_{O\vec{X}} = \sum_{e=1}^{3} \mathbf{I}^{(e)}_{O\vec{X}}$$

The flange geometric characteristics are described in Table 4.1.

Table 4.1

Flange	$(X_2^{(i)};X_3^{(i)})$	$(X_2^{(j)};X_3^{(j)})$	t	a	*Area* $A^{(e)}$	ℓ	m	$\overline{X}_2^{(g)}$	$\overline{X}_3^{(g)}$
$e=1$	$(15;42.32)$	$(0;33.662)$	2	17.32	34.64	-0.866	-0.5	7.5	37.99
$e=2$	$(0;33.66)$	$(0;8.66)$	2	25	50	0	-1	0	21.16
$e=3$	$(0;8.66)$	$(15;0)$	2	17.32	34.64	0.866	-0.5	7.5	4.33

Area Centroid of the Compound and Total Area

The total area is $A = A^{(1)} + A^{(2)} + A^{(3)} = 34.64 + 50 + 34.64 = 119.28 cm^2$

And the area centroid is given by

$$\overline{X}_2 = \frac{A^{(1)}\overline{X}_2^{(g1)} + A^{(2)}\overline{X}_2^{(g2)} + A^{(3)}\overline{X}_2^{(g3)}}{A^{(1)} + A^{(2)} + A^{(3)}} = 4.35614 cm$$

$$\overline{X}_3 = \frac{A^{(1)}\overline{X}_3^{(g1)} + A^{(2)}\overline{X}_3^{(g2)} + A^{(3)}\overline{X}_3^{(g3)}}{A^{(1)} + A^{(2)} + A^{(3)}} = 21.16 cm$$

The Inertia Tensor of Area

As we are adopting that the thickness is very small the terms related to t^2 in the equation (4.132) can be discarded, so the equation (4.132) becomes

$$\mathrm{I}_{O\vec{X}\,ij}^{(e)} \approx \frac{at}{12}\begin{bmatrix} a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix} \tag{4.133}$$

Flange 1: By substituting the data related to the flange $e=1$, (see Table 4.1), we can obtain

$$\mathrm{I}_{O\vec{X}\,ij}^{(1)} \approx \begin{bmatrix} 5.02103 & -1.02448 \\ -1.02448 & 0.259792 \end{bmatrix} \times 10^4 cm^4$$

Flange 2: By substituting the data related to the flange $e=2$, (see Table 4.1), we can obtain

$$\mathrm{I}_{O\vec{X}\,ij}^{(2)} \approx \begin{bmatrix} 2.49914 & 0 \\ 0 & 0 \end{bmatrix} \times 10^4 cm^4$$

Flange 3: By substituting the data related to the flange $e=3$, (see Table 4.1), we can obtain

$$\mathrm{I}_{O\vec{X}\,ij}^{(3)} \approx \begin{bmatrix} 0.086594919 & -0.0749978 \\ -0.0749978 & 0.259792 \end{bmatrix} \times 10^4 cm^4$$

Then, the inertia tensor for the compound is given by

$$\left(\mathbf{I}_{O\vec{X}}^{(Sys)}\right)_{ij} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \sum_{e=1}^{3} (\mathbf{I}_{O\vec{X}}^{(e)})_{ij} = \mathrm{I}_{O\vec{X}\,ij}^{(1)} + \mathrm{I}_{O\vec{X}\,ij}^{(2)} + \mathrm{I}_{O\vec{X}\,ij}^{(3)} = \begin{bmatrix} 7.60677 & -1.09947 \\ -1.09947 & 5.19585 \end{bmatrix} \times 10^4 cm^4$$

Shear Flux

By means of the equations in (4.127) we can calculate the shear flux in the flanges. Taking into account the geometric properties calculated previously we can calculate the coefficients related to the cross section:

$$p_0^{(1)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix} = -1.17732\times 10^7 \quad ; \quad p_0^{(2)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = -7.40145\times 10^6$$

$$p_2^{(1)} = \begin{vmatrix} A & A\overline{X}_3 \\ A\overline{X}_3 & \mathrm{I}_{22} \end{vmatrix} = 2.70296\times 10^6 \quad ; \quad p_2^{(2)} = \begin{vmatrix} A & A\overline{X}_3 \\ -A\overline{X}_2 & \mathrm{I}_{23} \end{vmatrix} = -41.19111$$

$$p_3^{(1)} = -\begin{vmatrix} A & A\overline{X}_2 \\ A\overline{X}_3 & -\mathrm{I}_{23} \end{vmatrix} = -41.19111 \quad ; \quad p_3^{(2)} = -\begin{vmatrix} A & A\overline{X}_2 \\ -A\overline{X}_2 & -\mathrm{I}_{33} \end{vmatrix} = 3.49793\times 10^5 ,$$

$$\boldsymbol{\mathcal{X}} = \begin{vmatrix} A & A\overline{X}_2 & A\overline{X}_3 \\ A\overline{X}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\overline{X}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = 7.92654\times 10^9$$

Flange 1: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.1)

$q_{F_y}^{(i)} = 0$, $q_{F_z}^{(i)} = 0$ (free edge)

$$d_1^{F_y} = \frac{-1}{\boldsymbol{\mathcal{X}}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = -3.62951\times 10^{-3}$$

$$d_1^{F_z} = \frac{-1}{\boldsymbol{\mathcal{X}}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -9.33723\times 10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\boldsymbol{\mathcal{X}}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 1.47652\times 10^{-4}$$

$$d_2^{F_z} = \frac{-1}{2\boldsymbol{\mathcal{X}}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 1.10301\times 10^{-5}$$

Shear Flux in the Flange 1

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=1)} F_y + q_{F_z}^{(e=1)} F_z$$

The terms $q_{F_y}^{(e=1)}$ and $q_{F_z}^{(e=1)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=1)}(s) = \underbrace{q_{F_y}^{(i)}}_{=0} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = t[(-3.62951\times 10^{-3})s + (1.47652\times 10^{-4})s^2]$$

$$q_{F_z}^{(e=1)}(s) = \underbrace{q_{F_z}^{(i)}}_{=0} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = t[(-9.33723\times 10^{-4})s + (1.10301\times 10^{-5})s^2]$$

at the end $s = 17.32$, (node 2)

$$q_{F_y}^{(e=1)}(s = 17.32) = t[(-3.62951\times 10^{-3})s + (1.47652\times 10^{-4})s^2] = -0.01857t = q_{F_y}^{(j)}$$

$$q_{F_z}^{(e=1)}(s = 17.32) = t[(-9.33723\times 10^{-4})s + (1.10301\times 10^{-5})s^2] = -0.01286t = q_{F_z}^{(j)}$$

Torsion Moment at O due to Flange 1, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = -16.82927$$

and

$$\left(M_O^{F_z}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) = -7.05101$$

Update Variables

$$q_{F_y}^{(i)} \leftarrow q_{F_y}^{(j)} = -0.01857t \quad ; \quad q_{F_z}^{(i)} \leftarrow q_{F_z}^{(j)} = -0.01286t$$

Flange 2: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.1)

$q_{F_y}^{(i)} = -0.01857t$, $q_{F_z}^{(i)} = -0.01286t$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = 1.48547\times 10^{-3}$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -5.5164\times 10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -2.5983\times 10^{-9}$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 2.20647\times 10^{-5}$$

Shear Flux in the Flange 2

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=2)} F_y + q_{F_z}^{(e=2)} F_z$$

The terms $q_{F_y}^{(e=2)}$ and $q_{F_z}^{(e=2)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=2)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = t[(-0.01857) + (1.48547\times 10^{-3})s + (-2.5983\times 10^{-9})s^2]$$

$$q_{F_z}^{(e=2)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = t[(-0.01286) + (-5.5164\times 10^{-4})s + (2.20647\times 10^{-5})s^2]$$

at the end $s = 25$, (node 3)

$$q_{F_y}^{(e=2)}(s=25) = t[(-0.01857) + (1.48547\times 10^{-3})s + (-2.5983\times 10^{-9})s^2] = 0.01857t = q_{F_y}^{(j)}$$

$$q_{F_z}^{(e=2)}(s=25) = t[(-0.01286) + (-5.5164\times 10^{-4})s + (2.20647\times 10^{-5})s^2] = -0.01286t = q_{F_z}^{(j)}$$

Torsion Moment at O due to Flange 2, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=2)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 t d_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

and

$$\left(M_O^{F_z}\right)^{(e=2)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 t d_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Update Variables

$$q_{F_y}^{(i)} \leftarrow q_{F_y}^{(j)} = 0.01857t \quad ; \quad q_{F_z}^{(i)} \leftarrow q_{F_z}^{(j)} = -0.01286t$$

Flange 3: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.1)

$q_{F_y}^{(i)} = 0.01857t$, $q_{F_z}^{(i)} = -0.01286t$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = 1.48534\times 10^{-3}$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = 5.51595\times 10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\boldsymbol{\mathcal{X}}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -1.47655\times 10^{-4}$$

$$d_2^{F_z} = \frac{-1}{2\boldsymbol{\mathcal{X}}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 1.10346\times 10^{-5}$$

Shear Flux in the Flange 3

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=3)}F_y + q_{F_z}^{(e=3)}F_z$$

The terms $q_{F_y}^{(e=3)}$ and $q_{F_z}^{(e=3)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=3)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = t[(0.01857) + (1.48534\times 10^{-3})s + (-1.47655\times 10^{-4})s^2]$$

$$q_{F_z}^{(e=3)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = t[(-0.01286) + (5.51595\times 10^{-4})s + (1.10346\times 10^{-5})s^2]$$

at the end $s = 17.32$, (node 4)

$$q_{F_y}^{(e=3)} = t[(0.01857) + (1.48534\times 10^{-3})s + (-1.47655\times 10^{-4})s^2] = -2.78\times 10^{-6}t = q_{F_y}^{(j)} \approx 0$$

$$q_{F_z}^{(e=3)} = t[(-0.01286) + (5.51595\times 10^{-4})s + (1.10346\times 10^{-5})s^2] = 4.24\times 10^{-11}t = q_{F_z}^{(j)} \approx 0$$

Torsion Moment at O due to Flange 3, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=3)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q_{F_y}^{(i)}) = -4.32891$$

and

$$\left(M_O^{F_z}\right)^{(e=3)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 1.81423$$

Update Variables

$$q_{F_y}^{(i)} \leftarrow q_{F_y}^{(j)} = -2.78\times 10^{-6}t \approx 0 \quad ; \quad q_{F_z}^{(i)} \leftarrow q_{F_z}^{(j)} = 4.24\times 10^{-11}t \approx 0$$

The Total Torsion Moment at O

$$\left(M_O^{F_y}\right)^{(Sys)} = \left(M_O^{F_y}\right)^{(e=1)} + \left(M_O^{F_y}\right)^{(e=2)} + \left(M_O^{F_y}\right)^{(e=3)} = (-16.82927) + (0) + (-4.32891) = -21.1582$$

$$\left(M_O^{F_z}\right)^{(Sys)} = \left(M_O^{F_z}\right)^{(e=1)} + \left(M_O^{F_z}\right)^{(e=2)} + \left(M_O^{F_z}\right)^{(e=3)} = (-7.05101) + (0) + (1.81423) = -5.2367$$

Then

$$M_O = M_O^{F_y}F_y + M_O^{F_z}F_z = -(21.1582)F_y + (-5.2367)F_z = -(X_3^{(S.C.)})F_y + (X_2^{(S.C.)})F_z$$

The Shear Center

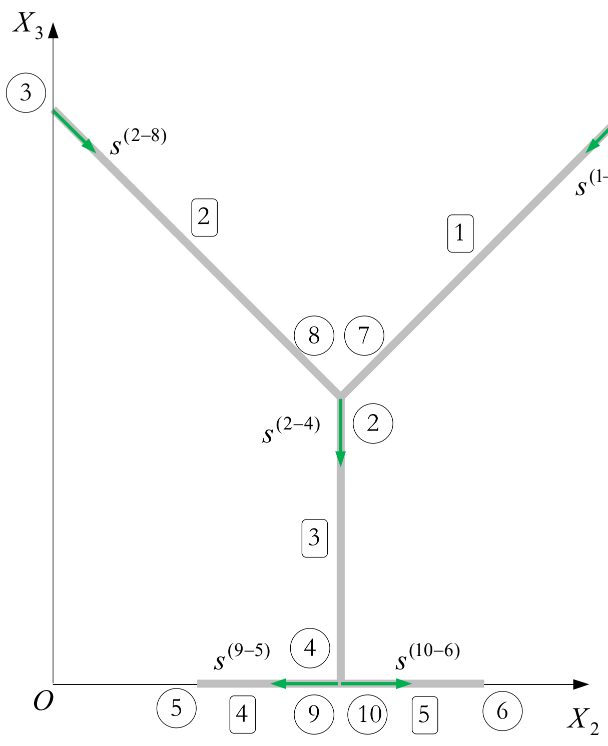
If we compare the above equation with the equation in Figure 4.10 we can conclude that

$$X_2^{(S.C.)} = -5.2367cm \quad ; \quad X_3^{(S.C.)} = 21.1582cm$$

Note that the cross section has one axis of symmetry at $X_3^{(A.C.)} = X_3^{(S.C.)} = 21.1582cm$.

Problem 4.33

Obtain the shear flux in each flange and locate the shear center for the cross section described in Figure 4.52. Note that the nodes 2, 7 and 8 have the same coordinates, and the same for the nodes 4, 9 and 10.



Nodes	*Coordinates* X_2	X_3
1	60	60
2	30	30
3	0	60
4	30	0
5	15	0
6	45	0
7	30	30
8	30	30
9	30	0
10	30	0

Element e	*Connectivity* $i \to j$	*Thickness*
1	$1 \to 7$	$t_1 = 1$
2	$3 \to 8$	$t_2 = 1$
3	$2 \to 4$	$t_3 = 1$
4	$9 \to 5$	$t_4 = 2$
5	$10 \to 6$	$t_5 = 2$

Figure 4.52: Cross section – dimensions in centimeter (cm).

Solution:

The element length can be calculated by means of $a = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$. The flange geometric characteristics are described in Table 4.2.

Table 4.2

Flange	$(X_2^{(i)};X_3^{(i)})$	$(X_2^{(j)};X_3^{(j)})$	t	a	*Area* $A^{(e)}$	ℓ	m	$\overline{X}_2^{(ge)}$	$\overline{X}_3^{(ge)}$
$e=1$	$(60;60)$	$(30;30)$	1	$30\sqrt{2}$	$30\sqrt{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	45	45
$e=2$	$(0;60)$	$(30;30)$	1	$30\sqrt{2}$	$30\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	15	45
$e=3$	$(30;30)$	$(30;0)$	1	30	30	0	-1	30	15
$e=4$	$(30;0)$	$(15;0)$	2	15	30	-1	0	22.5	0
$e=5$	$(30;0)$	$(45;0)$	2	15	30	1	0	37.5	0

Area Centroid of the Compound and Total Area

The total area is $A = A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} + A^{(5)} = 174.8528$

And the area centroid is given by

$$\overline{X}_2 = \frac{A^{(1)}\overline{X}_2^{(g1)} + A^{(2)}\overline{X}_2^{(g2)} + A^{(3)}\overline{X}_2^{(g3)} + A^{(4)}\overline{X}_2^{(g4)} + A^{(5)}\overline{X}_2^{(g5)}}{A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} + A^{(5)}} = 30$$

$$\overline{X}_3 = \frac{A^{(1)}\overline{X}_3^{(g1)} + A^{(2)}\overline{X}_3^{(g2)} + A^{(3)}\overline{X}_3^{(g3)} + A^{(4)}\overline{X}_3^{(g4)} + A^{(5)}\overline{X}_3^{(g5)}}{A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} + A^{(5)}} = 24.411255$$

The Inertia Tensor of Area (for the flange e)

$$\mathrm{I}_{O\vec{X}\,ij}^{(e)} \approx \frac{at}{12}\begin{bmatrix} a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix} \tag{4.134}$$

The inertia tensors for the flanges related to the system $O\vec{X}$ are

$$\mathrm{I}_{O\vec{X}\,ij}^{(1)} \approx \begin{bmatrix} 8.90955 & -8.90955 \\ -8.90955 & 8.90955 \end{bmatrix}\times 10^4 \,;\; \mathrm{I}_{O\vec{X}\,ij}^{(2)} \approx \begin{bmatrix} 8.90955 & -2.54558 \\ -2.54558 & 1.27279 \end{bmatrix}\times 10^4 \,;$$

$$\mathrm{I}_{O\vec{X}\,ij}^{(3)} \approx \begin{bmatrix} 0.9 & -1.35 \\ -1.35 & 2.7 \end{bmatrix}\times 10^4 ; \mathrm{I}_{O\vec{X}\,ij}^{(4)} \approx \begin{bmatrix} 0 & 0 \\ 0 & 1.575 \end{bmatrix}\times 10^4 \text{ and } \mathrm{I}_{O\vec{X}\,ij}^{(4)} \approx \begin{bmatrix} 0 & 0 \\ 0 & 4.275 \end{bmatrix}\times 10^4$$

Then, the inertia tensor for the compound is given by

$$\mathrm{I}_{O\vec{X}\;\;ij}^{(Sys)} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \sum_{e=1}^{5} (\mathbf{I}_{O\vec{X}}^{(e)})_{ij} = \mathrm{I}_{O\vec{X}\,ij}^{(1)} + \mathrm{I}_{O\vec{X}\,ij}^{(2)} + \mathrm{I}_{O\vec{X}\,ij}^{(3)} + \mathrm{I}_{O\vec{X}\,ij}^{(4)} + \mathrm{I}_{O\vec{X}\,ij}^{(5)}$$

$$\Rightarrow \mathrm{I}_{O\vec{X}\;\;ij}^{(Sys)} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \begin{bmatrix} 1.87191 & -1.28051 \\ -1.28051 & 1.87323 \end{bmatrix}\times 10^5$$

Just as exercise, let us calculate the inertia tensor at the Area Centroid, which can be obtained by means of the Steiner's theorem:

$$\mathbf{I}_{O\vec{X}}^{(Sys)} = \overline{\mathbf{I}}_{G\vec{x}}^{(Sys)} - A[(\vec{\overline{X}} \otimes \vec{\overline{X}}) - (\vec{\overline{X}} \cdot \vec{\overline{X}})\mathbf{1}] \quad \Rightarrow \quad \overline{\mathbf{I}}_{G\vec{x}}^{(Sys)} = \mathbf{I}_{O\vec{X}}^{(Sys)} + A[(\vec{\overline{X}} \otimes \vec{\overline{X}}) - (\vec{\overline{X}} \cdot \vec{\overline{X}})\mathbf{1}]$$

whose components are:

$$(\overline{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix} = \mathrm{I}_{O\vec{X}\;\;ij}^{(Sys)} - A\begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix}$$

$$(\overline{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix} = \begin{bmatrix} 8.2994479 & 0 \\ 0 & 2.9955844 \end{bmatrix}\times 10^4$$

Shear Flux

By means of the equations in (4.127) we can calculate the shear flux in the flanges. Taking into account the geometric properties calculated previously we can calculate the coefficients related to the cross section:

$$p_0^{(1)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix} = -4.35355\times 10^8 \quad ; \quad p_0^{(2)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = -1.27863\times 10^8$$

$$p_2^{(1)} = \begin{vmatrix} A & A\overline{X}_3 \\ A\overline{X}_3 & \mathrm{I}_{22} \end{vmatrix} = 1.45118\times 10^7 \quad ; \quad p_2^{(2)} = \begin{vmatrix} A & A\overline{X}_3 \\ -A\overline{X}_2 & \mathrm{I}_{23} \end{vmatrix} = 9.85892\times 10^{-10}$$

$$p_3^{(1)} = -\begin{vmatrix} A & A\overline{X}_2 \\ A\overline{X}_3 & -\mathrm{I}_{23} \end{vmatrix} = 9.85892\times 10^{-10} \quad ; \quad p_3^{(2)} = -\begin{vmatrix} A & A\overline{X}_2 \\ -A\overline{X}_2 & -\mathrm{I}_{33} \end{vmatrix} = 5.23786\times 10^6 ,$$

$$\mathcal{X} = \begin{vmatrix} A & A\overline{X}_2 & A\overline{X}_3 \\ A\overline{X}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\overline{X}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix} = 4.34714\times 10^{11}$$

Flange 1: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.2)

$q_{F_y}^{(i)} = 0$, $q_{F_z}^{(i)} = 0$ (free edge)

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = -1.00147\times 10^{-3} \quad \Rightarrow \quad td_1^{F_y} = -1.00147\times 10^{-3}$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -4.28809\times 10^{-4} \quad \Rightarrow \quad td_1^{F_z} = -4.28809\times 10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 1.18025\times 10^{-5} \quad \Rightarrow \quad td_2^{F_y} = 1.18025\times 10^{-5}$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 4.25996\times 10^{-6} \quad \Rightarrow \quad td_2^{F_z} = 4.25996\times 10^{-6}$$

Shear Flux in the Flange 1

$q_{F_y}^{(i)} = q_{F_y}^{(1)} = 0$, $q_{F_z}^{(i)} = q_{F_z}^{(1)} = 0$

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=1)} F_y + q_{F_z}^{(e=1)} F_z$$

The terms $q_{F_y}^{(e=1)}$ and $q_{F_z}^{(e=1)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=1)}(s) = \underbrace{q_{F_y}^{(1)}}_{=0} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = -1.00147\times 10^{-3} s + 1.18025\times 10^{-5} s^2$$

$$q_{F_z}^{(e=1)}(s) = \underbrace{q_{F_z}^{(i)}}_{=0} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = -4.28809\times 10^{-4} s + 4.25996\times 10^{-6} s^2$$

at the end $s = 30\sqrt{2}$, (node 7)

$$q_{F_y}^{(e=1)}(s = 30\sqrt{2}) = -1.00147\times 10^{-3} s + 1.18025\times 10^{-5} s^2 = -0.0212445 = q_{F_y}^{(7)}$$

$$q_{F_z}^{(e=1)}(s = 30\sqrt{2}) = -4.28809\times 10^{-4} s + 4.25996\times 10^{-6} s^2 = -0.0105249 = q_{F_z}^{(7)}$$

Torsion Moment at O due to Flange 1, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

and

$$\left(M_O^{F_z}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Flange 2: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.2)

$q_{F_y}^{(i)} = q_{F_y}^{(3)} = 0$, $q_{F_z}^{(i)} = q_{F_z}^{(3)} = 0$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = 1.001474\times 10^{-3} \quad \Rightarrow \quad td_1^{F_y} = 1.001474\times 10^{-3}$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -4.2880858\times 10^{-4} \quad \Rightarrow \quad td_1^{F_z} = -4.2880858\times 10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -1.1802485\times 10^{-5} \qquad \Rightarrow \qquad td_2^{F_y} = -1.1802485\times 10^{-5}$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 4.2599628\times 10^{-6} \qquad \Rightarrow \qquad td_2^{F_z} = 4.2599628\times 10^{-6}$$

Shear Flux in the Flange 2

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=2)} F_y + q_{F_z}^{(e=2)} F_z$$

The terms $q_{F_y}^{(e=2)}$ and $q_{F_z}^{(e=2)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=2)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = 1.001474\times 10^{-3} s - 1.1802485\times 10^{-5} s^2$$

$$q_{F_z}^{(e=2)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = -4.2880858\times 10^{-4} s + 4.2599628\times 10^{-6} s^2$$

at the end $s = 30\sqrt{2}$, (node 8)

$$q_{F_y}^{(e=2)}(s = 30\sqrt{2}) = 1.001474\times 10^{-3} s - 1.1802485\times 10^{-5} s^2 = 0.0212445 = q_{F_y}^{(8)}$$

$$q_{F_z}^{(e=2)}(s = 30\sqrt{2}) = -4.2880858\times 10^{-4} s + 4.2599628\times 10^{-6} s^2 = -0.01052487 = q_{F_z}^{(8)}$$

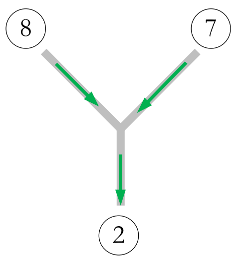
Torsion Moment at O due to Flange 2, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=2)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = -25.49336686$$

and

$$\left(M_O^{F_z}\right)^{(e=2)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) = 11.772767$$

Shear flux compatibility at node 2



$$q^{(7)} + q^{(8)} = q^{(2)}$$

$$\begin{cases} q_{F_y}^{(2)} = -0.0212445 + 0.0212445 = 0 \\ q_{F_z}^{(2)} = -0.0105249 - 0.0105249 = -0.0210497 \end{cases}$$

Flange 3: (Flange data $(X_2^{(i)}, X_3^{(i)}, t, \ell, m)$ in Table 4.2)

$q_{F_y}^{(2)} = 0$, $q_{F_z}^{(2)} = -0.0210497$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = 0 \qquad \Rightarrow \qquad td_1^{F_y} = 0$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -6.73388\times 10^{-5} \qquad \Rightarrow \qquad td_1^{F_z} = -6.73388\times 10^{-5}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 0 \qquad \Rightarrow \qquad td_2^{F_y} = 0$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 6.0245\times 10^{-6} \qquad \Rightarrow \qquad td_2^{F_z} = 6.0245\times 10^{-6}$$

Shear Flux in the Flange 3

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=3)} F_y + q_{F_z}^{(e=3)} F_z$$

The terms $q_{F_y}^{(e=3)}$ and $q_{F_z}^{(e=3)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=3)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = 0$$

$$q_{F_z}^{(e=3)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = q_{F_z}^{(2)} - 6.73388\times10^{-5} s + 6.0245\times10^{-6} s^2$$
$$= -0.0210497 - 6.73388\times10^{-5} s + 6.0245\times10^{-6} s^2$$

at the end $s = 30$, (node 4)

$$q_{F_y}^{(e=3)} = 0 = q_{F_y}^{(4)}$$

$$q_{F_z}^{(e=3)} = -0.0210497 - 6.73388\times10^{-5} s + 6.0245\times10^{-6} s^2 = -0.0176479 = q_{F_z}^{(4)}$$

Torsion Moment at O due to Flange 3, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=3)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

and

$$\left(M_O^{F_z}\right)^{(e=3)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) = -0.717541 - 900 q_{F_z}^{(2)} = 18.227233$$

Flange 4: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(9)},\ q_{F_y}^{(j)} = q_{F_y}^{(5)} = 0$$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = 0 \quad \Rightarrow \quad td_1^{F_y} = 0$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = 2.94131\times10^{-4} \quad \Rightarrow \quad td_1^{F_z} = 5.8826214\times10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 1.66912\times10^{-5} \quad \Rightarrow \quad td_2^{F_y} = 3.3382468\times10^{-5}$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0 \quad \Rightarrow \quad td_2^{F_z} = 0$$

Shear Flux in the Flange 4

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z = q_{F_y}^{(e=3)} F_y + q_{F_z}^{(e=3)} F_z$$

The terms $q_{F_y}^{(e=4)}$ and $q_{F_z}^{(e=4)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=4)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2 = q_{F_y}^{(9)} + 3.3382468\times10^{-5} s^2$$

$$q_{F_z}^{(e=4)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2 = q_{F_z}^{(9)} + 5.8826214\times10^{-4} s$$

at the end $s = 15$, (node 5)

$$q_{F_y}^{(e=4)}(s=15) = q_{F_y}^{(9)} + 3.3382468\times10^{-5} s^2 = q_{F_y}^{(9)} + 7.5110552\times10^{-3} = q_{F_y}^{(5)} = 0$$
$$\Rightarrow q_{F_y}^{(9)} = -7.5110552\times10^{-3}$$

$$q_{F_z}^{(e=4)}(s=15) = q_{F_z}^{(9)} + 5.8826214\times10^{-4}s = q_{F_z}^{(9)} + 0.0088239 = q_{F_z}^{(5)} = 0$$
$$\Rightarrow q_{F_z}^{(9)} = -0.0088239$$

Torsion Moment at O due to Flange 4, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=4)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

and

$$\left(M_O^{F_z}\right)^{(e=4)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Flange 5: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 4.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(10)}\,,\; q_{F_y}^{(j)} = q_{F_y}^{(6)} = 0$$

$$d_1^{F_y} = \frac{-1}{\mathcal{X}}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = 0 \quad\Rightarrow\quad td_1^{F_y} = 0$$

$$d_1^{F_z} = \frac{-1}{\mathcal{X}}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 2.94131\times10^{-4} \quad\Rightarrow\quad td_1^{F_z} = 5.8826214\times10^{-4}$$

$$d_2^{F_y} = \frac{-1}{2\mathcal{X}}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -1.66912\times10^{-5} \quad\Rightarrow\quad td_2^{F_y} = -3.3382468\times10^{-5}$$

$$d_2^{F_z} = \frac{-1}{2\mathcal{X}}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0 \quad\Rightarrow\quad td_2^{F_z} = 0$$

Shear Flux in the Flange 5

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=3)}F_y + q_{F_z}^{(e=3)}F_z$$

The terms $q_{F_y}^{(e=5)}$ and $q_{F_z}^{(e=5)}$ can be evaluated, respectively, as follows:

$$q_{F_y}^{(e=5)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(10)} - 3.3382468\times10^{-5}s^2$$

$$q_{F_z}^{(e=5)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(10)} + 5.8826214\times10^{-4}s$$

at the end $s=15$, (node 6)

$$q_{F_y}^{(e=5)}(s=15) = q_{F_y}^{(10)} - 3.3382468\times10^{-5}s^2 = q_{F_y}^{(10)} - 7.5110552\times10^{-3} = q_{F_y}^{(6)} = 0$$
$$\Rightarrow q_{F_y}^{(10)} = 7.5110552\times10^{-3}$$

$$q_{F_z}^{(e=5)}(s=15) = q_{F_z}^{(10)} + 5.8826214\times10^{-4}s = q_{F_z}^{(10)} + 0.0088239 = q_{F_z}^{(6)} = 0$$
$$\Rightarrow q_{F_z}^{(10)} = -0.0088239$$

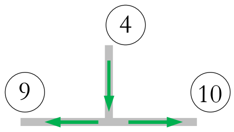
Torsion Moment at O due to Flange 5, (see equations (4.130) and (4.131)):

$$\left(M_O^{F_y}\right)^{(e=5)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

and

$$\left(M_O^{F_z}\right)^{(e=5)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Checking compatibility at the node 4



$$q^{(4)} = q^{(9)} + q^{(10)}$$

$$\begin{cases} q_{F_y}^{(4)} = -7.5110552\times10^{-3} + 7.5110552\times10^{-3} = 0 \\ q_{F_z}^{(4)} = -0.008823 - 0.0088239 = -0.017646 \end{cases}$$

The Total Torsion Moment at O

$$\left(M_O^{F_y}\right)^{(Sys)} = \left(M_O^{F_y}\right)^{(e=1)} + \left(M_O^{F_y}\right)^{(e=2)} + \left(M_O^{F_y}\right)^{(e=3)} + \left(M_O^{F_y}\right)^{(e=4)} + \left(M_O^{F_y}\right)^{(e=5)}$$
$$= (0) + (-25.49336686) + (0) + (0) + (0) = -25.49336686$$

$$\left(M_O^{F_z}\right)^{(Sys)} = \left(M_O^{F_z}\right)^{(e=1)} + \left(M_O^{F_z}\right)^{(e=2)} + \left(M_O^{F_z}\right)^{(e=3)} + \left(M_O^{F_z}\right)^{(e=4)} + \left(M_O^{F_z}\right)^{(e=5)}$$
$$= (0) + (11.772767) + (18.227233) + (0) + (0) = 30$$

Then

$$M_O = M_O^{F_y} F_y + M_O^{F_z} F_z = -(25.49336686)F_y + (30)F_z = -(X_3^{(S.C.)})F_y + (X_2^{(S.C.)})F_z$$

The Shear Center

If we compare the above equation with the equation in Figure 4.10 we can conclude that

$$X_2^{(S.C.)} = 30cm \quad ; \quad X_3^{(S.C.)} = 25.49336686$$

Note that the cross section has one axis of symmetry at $X_2^{(A.C.)} = X_2^{(S.C.)} = 30cm$.

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4.3 Introduction to Flux Problems

Problem 4.34

1) Consider a continuum motion in which the stress power is equal to zero. Also, consider that the heat flux is given by $\vec{\mathbf{q}} = -\mathbf{K}(T) \cdot \nabla_{\vec{x}} T$, which is known as *Fourier's law of thermal conduction*, where $\mathbf{K}(T)$ is a second-order tensor called the *thermal conductivity tensor* (the thermal property of the material), and $c = \dfrac{\partial u(T)}{\partial T}$, where c is the *specific heat capacity* at a constant deformation (the thermal property of the material) and is expressed in units of joule per kelvin, i.e. $[c] = \dfrac{J}{K}$. Taking into account all previous considerations, find the energy equation for this process. Then also provide the unit of $\mathbf{K}(T)$ in the International System of Units (SI).

2) Consider the stress power is equal to zero, and that there is a continuous medium with no internal heat source. Also consider that there is a heterogeneous material where $\mathbf{K} = \mathbf{K}(\vec{x})$ is an arbitrary second-order tensor (not necessarily symmetrical). a) Show that the thermal conductivity tensor is semi-definite positive, b) Check in which scenario the skew part of $\mathbf{K}(\vec{x})$ does not affect the outcome of the heat conduction problem. c) Taking into account that the material is isotropic, in what format is $\mathbf{K}$?

Solution: For this problem we know that the stress power is equal to zero, $\boldsymbol{\sigma} : \mathbf{D} = 0$. It then follows that, the energy equation becomes:

$$\rho \dot{u} = \rho \frac{\partial u}{\partial T} \frac{\partial T}{\partial t} = \underbrace{\boldsymbol{\sigma} : \mathbf{D}}_{=0} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r = -\nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$$

$$\Rightarrow \rho c \frac{\partial T}{\partial t} = -\nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r \qquad \Rightarrow \qquad \rho c \frac{\partial T}{\partial t} = -\nabla_{\vec{x}} \cdot \left[-\mathbf{K}(T) \cdot \nabla_{\vec{x}} T \right] + \rho r$$

or

$$\boxed{\nabla_{\vec{x}} \cdot \left[\mathbf{K}(T) \cdot \nabla_{\vec{x}} T \right] + \rho r = \rho c \frac{\partial T}{\partial t}}$$

The above equation is called the *heat flux equation* which is applied to the thermal conduction problem.

Obs.: If there is no mass transport it fulfills $\dot{T} \equiv \dfrac{DT}{Dt} = \dfrac{\partial T}{\partial t}$.

Then if we take into account the following units: $[\vec{\mathbf{q}}] = \dfrac{J}{m^2 s} = \dfrac{W}{m^2}$, $\left[\nabla_{\vec{x}} T \equiv \dfrac{\partial T}{\partial \vec{\boldsymbol{x}}} \right] = \dfrac{K}{m}$, we can ensure that the units are consistent if the following is met:

$$\begin{array}{ccc} [\vec{\mathbf{q}}] & = & [\mathbf{K}] \cdot [\nabla_{\vec{x}} T] \\ \left[\dfrac{J}{m^2 s} = \dfrac{W}{m^2} \right] & = & \left[\dfrac{J}{s\, m\, K} = \dfrac{W}{m\, K} \right] \left[\dfrac{K}{m} \right] \end{array}$$

thus, we can draw the conclusion that $[\mathbf{K}] = \left[\dfrac{J}{s\, m\, K} = \dfrac{W}{m\, K} \right]$.

NOTE: As we will see later, when the stress power is equal to zero, we can decouple the thermal and mechanical problem. That is, we can study these problems separately. ■

2) a) We start from the heat conductivity inequality:

$$
\begin{aligned}
&-\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T = -(-\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T)\cdot\nabla_{\vec{x}}T \ge 0 \\
&\nabla_{\vec{x}}T\cdot\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T \ge 0
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
&-\mathsf{q}_i T_{,i} = -(-K_{ij}T_{,j})T_{,i} \ge 0 \\
&T_{,i}K_{ij}T_{,j} \ge 0
\end{aligned}
$$

Remember that the arbitrary tensor $\mathbf{A}$ is semi-definite positive if it holds that $\vec{x}\cdot\mathbf{A}\cdot\vec{x}\ge 0$ for all $\vec{x}\neq\vec{\mathbf{0}}$ thereby demonstrating that $\mathbf{K}(\vec{x})$ is a semi-definite positive tensor. Then, as a result the eigenvalues of $\mathbf{K}(\vec{x})$ are all real values greater than or equal to zero, i.e. $\mathsf{K}_1\ge 0$, $\mathsf{K}_2\ge 0$, $\mathsf{K}_3\ge 0$. Also remember that since $\mathbf{K}(\vec{x})$ is not symmetric, the principal space of $\mathbf{K}(\vec{x})$ does not define an orthonormal basis. Moreover, it is noteworthy that: the antisymmetric part of $\mathbf{K}(\vec{x})$ does not affect the heat conduction inequality since:

$$
\begin{aligned}
\nabla_{\vec{x}}T\cdot\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T = \nabla_{\vec{x}}T\cdot[\mathbf{K}^{sym}+\mathbf{K}^{skew}]\cdot\nabla_{\vec{x}}T &= \nabla_{\vec{x}}T\cdot\mathbf{K}^{sym}\cdot\nabla_{\vec{x}}T+\nabla_{\vec{x}}T\cdot\mathbf{K}^{skew}\cdot\nabla_{\vec{x}}T\ge 0 \\
&\nabla_{\vec{x}}T\cdot\mathbf{K}^{sym}\cdot\nabla_{\vec{x}}T+\mathbf{K}^{skew}:(\nabla_{\vec{x}}T\otimes\nabla_{\vec{x}}T)\ge 0
\end{aligned}
$$

Notice that $\mathbf{K}^{skew}:(\nabla_{\vec{x}}T\otimes\nabla_{\vec{x}}T)=0$, since the double scalar product between an antisymmetric tensor ($\mathbf{K}^{skew}$) and a symmetric one $(\nabla_{\vec{x}}T\otimes\nabla_{\vec{x}}T)$ is equal to zero, then:

$$0\le\nabla_{\vec{x}}T\cdot\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T=\nabla_{\vec{x}}T\cdot\mathbf{K}^{sym}\cdot\nabla_{\vec{x}}T\ge 0$$

That is, the above inequality is always true whether $\mathbf{K}(\vec{x})$ is symmetric or not.

b) For the proposed problem the only remaining governing equation is the energy equation: $\rho\dfrac{Du}{Dt}\equiv\rho\dot{u}=\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r=-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}$, where u is the specific internal energy, $\boldsymbol{\sigma}:\mathbf{D}$ is the stress power, and ρr is the internal heat source per unit volume. Then:

$$
\begin{aligned}
\rho\dot{u} &= -\mathsf{q}_{i,i} = -(-\mathsf{K}_{ij}T_{,j})_{,i} = \mathsf{K}_{ij,i}T_{,j}+\mathsf{K}_{ij}T_{,ji} = (\nabla_{\vec{x}}\cdot\mathbf{K}^T)\cdot(\nabla_{\vec{x}}T)+\mathbf{K}:\nabla_{\vec{x}}(\nabla_{\vec{x}}T) \\
&= (\nabla_{\vec{x}}\cdot\mathbf{K}^T)\cdot(\nabla_{\vec{x}}T)+[\mathbf{K}^{sym}+\mathbf{K}^{skew}]:\nabla_{\vec{x}}(\nabla_{\vec{x}}T) \\
&= (\nabla_{\vec{x}}\cdot\mathbf{K}^T)\cdot(\nabla_{\vec{x}}T)+\mathbf{K}^{sym}:\nabla_{\vec{x}}(\nabla_{\vec{x}}T)+\mathbf{K}^{skew}:\nabla_{\vec{x}}(\nabla_{\vec{x}}T) \\
&= (\nabla_{\vec{x}}\cdot\mathbf{K}^T)\cdot(\nabla_{\vec{x}}T)+\mathbf{K}^{sym}:\nabla_{\vec{x}}(\nabla_{\vec{x}}T)
\end{aligned}
$$

where we have considered the symmetry of $[\nabla_{\vec{x}}(\nabla_{\vec{x}}T)]_{ij}=T_{,ij}=T_{,ji}$. If the material is homogeneous the implication is that the $\mathbf{K}$ field does not depend on $(\vec{x})$, so $\mathsf{K}_{ij,i}=\mathbf{0}_j$. In this scenario the heat equation reduces to:

$$\rho\dot{u}=\mathbf{K}^{sym}:\nabla_{\vec{x}}(\nabla_{\vec{x}}T)$$

Therefore, when the material is homogeneous, the antisymmetric part of $\mathbf{K}$ does not affect the outcome.

c) The feature of isotropic materials is that their properties (at one material point) do not change if the coordinate system is changed. It follows then that $\mathbf{K}$ must be an isotropic tensor. An isotropic second-order tensor has the format of a spherical tensor, (see Chapter 1), then the tensor $\mathbf{K}$ must be of the type: $\mathbf{K}=\mathsf{K}\mathbf{1}$, where K is a scalar:

$$\mathsf{K}_{ij}=\mathsf{K}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Problem 4.35

Consider a thermal conduction problem, (see **Problem 4.34**), in a wall with thickness equal to h in which the temperature at the outer face $(x_1 = 0)$ is equal to $38^o C$ and the temperature in the interior face $(x_1 = h)$ is equal to $21^o C$, (see Figure 4.53). Obtain the heat flow for case defined by: stationary problem, the temperature field according to x_2 and x_3-directions is homogeneous, there is no heat source, and the material is isotropic and homogeneous.

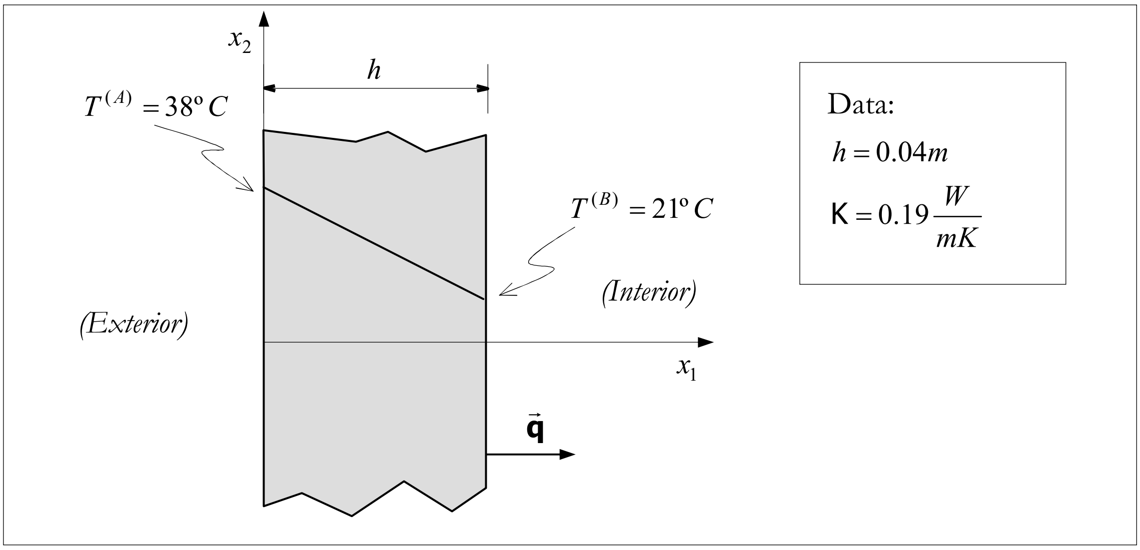


Figure 4.53

Solution:

As we saw in **Problem 4.34** the governing equation for this problem is the equation $\nabla_{\vec{x}} \cdot [\mathbf{K} \cdot \nabla_{\vec{x}} T] + \rho r = \rho c \frac{\partial T}{\partial t}$. If we consider the stationary problem we have $\frac{\partial T}{\partial t} = 0$. If there is no heat source this implies that $r = 0$. With these simplifications the governing equation becomes $\nabla_{\vec{x}} \cdot [\mathbf{K} \cdot \nabla_{\vec{x}} T] = 0$, in addition, if the material is homogenous, the tensor with the thermal properties $\mathbf{K}$ do not vary with $\vec{x}$, then $\nabla_{\vec{x}} \cdot [\mathbf{K} \cdot \nabla_{\vec{x}} T] = \mathbf{K} : \nabla_{\vec{x}} [\nabla_{\vec{x}} T] = 0$, which in indicial notation is $[\mathsf{K}_{ij} T,_j]_{,i} = \underbrace{\mathsf{K}_{ij,i} T,_j}_{=0} + \mathsf{K}_{ij} T,_{ji} = \mathsf{K}_{ij} T,_{ji} = 0$. By expanding this equation we obtain:

$$
\begin{aligned}
&\mathsf{K}_{11} \frac{\partial^2 T}{\partial x_1^2} + \mathsf{K}_{12} \frac{\partial^2 T}{\partial x_2 \partial x_1} + \mathsf{K}_{13} \frac{\partial^2 T}{\partial x_3 \partial x_1} + \mathsf{K}_{21} \frac{\partial^2 T}{\partial x_1 \partial x_2} + \mathsf{K}_{22} \frac{\partial^2 T}{\partial x_2^2} + \mathsf{K}_{23} \frac{\partial^2 T}{\partial x_3 \partial x_2} + \\
&\quad + \mathsf{K}_{31} \frac{\partial^2 T}{\partial x_1 \partial x_3} + \mathsf{K}_{32} \frac{\partial^2 T}{\partial x_2 \partial x_3} + \mathsf{K}_{33} \frac{\partial^2 T}{\partial x_3^2} = 0
\end{aligned}
\tag{4.135}
$$

If the temperature field according to x_2 and x_3-directions is homogenous, this implies that the temperature gradient components according to these directions are equal to zero, i.e. $\frac{\partial T}{\partial x_2} = \frac{\partial T}{\partial x_3} = 0$. For an isotropic material, the thermal conductivity tensor components, (see Chapter 5 of the textbook), are given by:

$$\mathsf{K}_{ij} = \begin{bmatrix} \mathsf{K} & 0 & 0 \\ 0 & \mathsf{K} & 0 \\ 0 & 0 & \mathsf{K} \end{bmatrix}$$

With these considerations the equation (4.135) becomes:

$$\mathsf{K}_{11}\frac{\partial^2 T}{\partial x_1^2} = 0 \xrightarrow{\mathsf{K}_{11}=\mathsf{K}} \Rightarrow \mathsf{K}\frac{\partial^2 T}{\partial x_1^2} = 0 \tag{4.136}$$

By integrating the equation $\mathsf{K}\frac{\partial^2 T}{\partial x_1^2} = 0$ we can obtain:

$$\mathsf{K}\frac{\partial^2 T}{\partial x_1^2} = 0 \xrightarrow{\text{integrating}} \mathsf{K}\frac{\partial T}{\partial x_1} + \mathsf{q}_1 = 0 \quad \Rightarrow \quad \mathsf{q}_1 = -\mathsf{K}\frac{dT}{dx_1}$$

which is the *Fourier's law of thermal conduction*. Note that for this case q_1 is a constant, i.e. it is independent of x_1. By integrating once more we can obtain:

$$\int dT = \int \frac{-\mathsf{q}_1}{\mathsf{K}} dx_1 \quad \Rightarrow \quad T(x_1) = \frac{-\mathsf{q}_1}{\mathsf{K}} x_1 + C$$

Applying the boundary condition, $x_1 = 0 \Rightarrow T = T^{(A)}$, we can obtain the constant of integration $C = T^{(A)}$, and in turn the equation $T(x_1) = \frac{-\mathsf{q}_1}{\mathsf{K}} x_1 + T^{(A)}$. In addition, for $x_1 = h$ we have

$$T(x_1 = h) = T^{(B)} = \frac{-\mathsf{q}_1}{\mathsf{K}} h + T^{(A)} \quad \Rightarrow \quad \mathsf{q}_1 = -\mathsf{K}\frac{(T^{(B)} - T^{(A)})}{h}$$

In this case (one-dimensional case), the temperature gradient can be represented by the slope of the line defined by the temperature, which varies linearly in the wall, (see Figure 4.53).

By replacing the problem data, (see Figure 4.53), we can obtain the heat flux:

$$\mathsf{q}_1 = -\mathsf{K}\frac{(T^{(B)} - T^{(A)})}{h} = -0.19\left(\frac{W}{mK}\right)\frac{(21-38)(K)}{0.04(m)} = 80.75\frac{W}{m^2} = 80.75\frac{J}{m^2 s}$$

Note that the temperature conversion from degrees Celsius to Kelvin is given by $K = {}^{o}C + 273.15$, then the temperature variation (ΔT) either in degrees Celsius or in Kelvin is the same. Note also that the heat flux flows from the higher temperature to the lower temperature region.

NOTE: Let us suppose now that we have two walls with different properties, (see Figure 4.54).

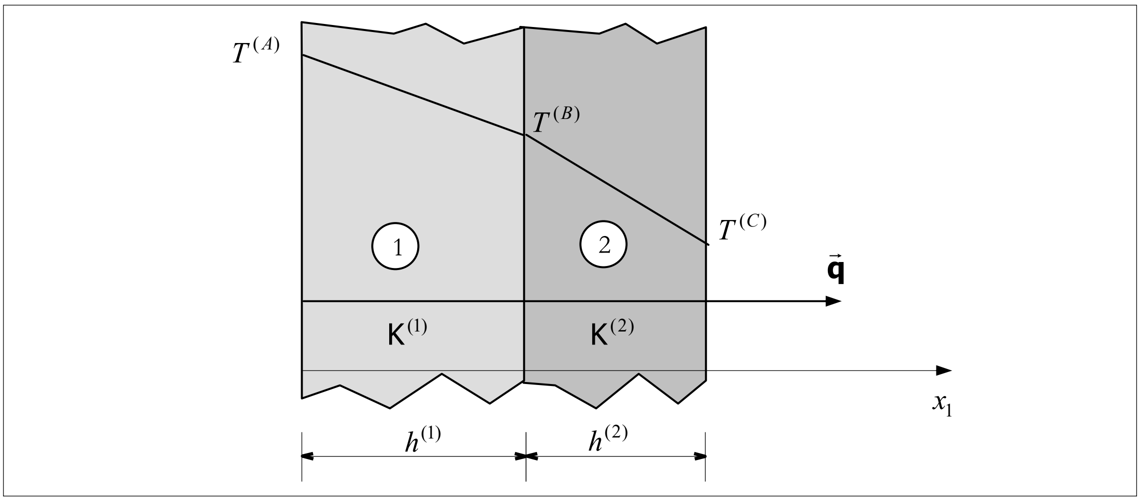


Figure 4.54

Note that the equation $q_1 = -K^{(1)} \dfrac{(T^{(B)} - T^{(A)})}{h^{(1)}}$ is still valid. This also applies to the material 2: $q_1 = -K^{(2)} \dfrac{(T^{(C)} - T^{(B)})}{h^{(2)}}$. To obtain the heat flux we must apply the compatibility in temperature on the face B, i.e.:

$$q_1 = -K^{(1)} \frac{(T^{(B\_1)} - T^{(A)})}{h^{(1)}} \quad \Rightarrow \quad T^{(B\_1)} = T^{(A)} - \frac{q_1 h^{(1)}}{K^{(1)}}$$

$$q_1 = -K^{(2)} \frac{(T^{(C)} - T^{(B\_2)})}{h^{(2)}} \quad \Rightarrow \quad T^{(B\_2)} = T^{(C)} + \frac{q_1 h^{(2)}}{K^{(2)}}$$

$$T^{(B\_1)} = T^{(B\_2)}$$

$$T^{(A)} - \frac{q_1 h^{(1)}}{K^{(1)}} = T^{(C)} + \frac{q_1 h^{(2)}}{K^{(2)}}$$

thus:

$$q_1 = \frac{-(T^{(C)} - T^{(A)})}{\left(\dfrac{h^{(1)}}{K^{(1)}} + \dfrac{h^{(2)}}{K^{(2)}}\right)}$$

Problem 4.36

Next, we assume that at a material point there are two types of material that are represented by a physical quantity per unit volume in such a way that $c = c^f + c^s$, and the following holds $\vec{v} = \vec{v}^f + \vec{v}^s$, (see Figure 4.55). Considering an isothermal process, an incompressible medium, and that the property c^s does not affect the velocity of the material f and that the c^f-field is homogeneous, and there is no source of the material f. Show that:

$$\boxed{Q^s - \nabla_{\vec{x}} \cdot (\vec{v}^f c^s) + \nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} c^s) = \frac{\partial c^s}{\partial t}} \quad \textit{Convection-diffusion equation} \qquad (4.137)$$

where the flux of the property s is given by $\vec{\mathbf{q}}^{(D)} = -\mathbf{D} \cdot \nabla_{\vec{x}} c^s$.

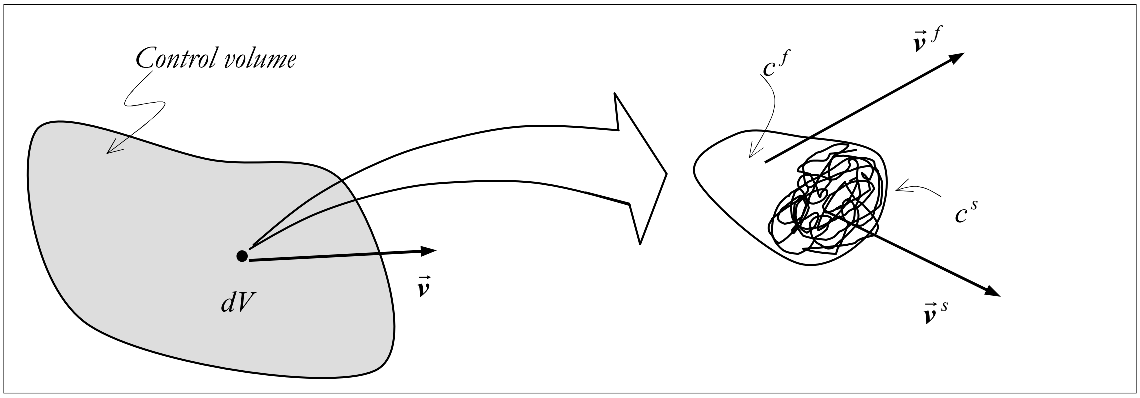


Figure 4.55: Heterogeneous medium.

Solution:

Starting from the continuity equation for this physical quantity, we can obtain:

$$Q = \frac{\partial \Phi}{\partial t} + \nabla_{\vec{x}} \cdot (\Phi \vec{\boldsymbol{v}}) \quad \Rightarrow \quad Q = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{\boldsymbol{x}}} \left[(c^f + c^s)(\vec{\boldsymbol{v}}^f + \vec{\boldsymbol{v}}^s) \right] \tag{4.138}$$

with $Q = Q^s + Q^f$. Thus:

$$\begin{aligned}
& Q^s + Q^f = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{\boldsymbol{x}}} \left[(c^f + c^s)(\vec{\boldsymbol{v}}^f + \vec{\boldsymbol{v}}^s) \right] \\
& \Rightarrow Q^s + Q^f = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{\boldsymbol{x}}} \left[c^f \vec{\boldsymbol{v}}^f + c^f \vec{\boldsymbol{v}}^s + c^s \vec{\boldsymbol{v}}^f + c^s \vec{\boldsymbol{v}}^s \right] \\
& \Rightarrow Q^s + Q^f = \frac{\partial c^f}{\partial t} + \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot \left[c^f \vec{\boldsymbol{v}}^f + c^f \vec{\boldsymbol{v}}^s + c^s \vec{\boldsymbol{v}}^f + c^s \vec{\boldsymbol{v}}^s \right] \\
& \Rightarrow Q^s + Q^f = \left[\frac{\partial c^f}{\partial t} + \nabla_{\vec{x}} \cdot (c^f \vec{\boldsymbol{v}}^f) \right] + \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^f) + \nabla_{\vec{x}} \cdot \left[c^f \vec{\boldsymbol{v}}^s + c^s \vec{\boldsymbol{v}}^s \right]
\end{aligned} \tag{4.139}$$

If we are assuming that there is no (f)-material source, then $\frac{\partial c^f}{\partial t} + \nabla_{\vec{x}} \cdot (c^f \vec{\boldsymbol{v}}^f) = 0$ and $Q^f = 0$ hold, which is the continuity equation of the physical quantity c^f with which the equation in (4.139) becomes:

$$Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^f) + \nabla_{\vec{x}} \cdot \left[c^f \vec{\boldsymbol{v}}^s + c^s \vec{\boldsymbol{v}}^s \right] \tag{4.140}$$

$$\Rightarrow Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^s) + \nabla_{\vec{x}} \cdot (c^f \vec{\boldsymbol{v}}^s) \tag{4.141}$$

$$\Rightarrow Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{\boldsymbol{v}}^s) + (\nabla_{\vec{x}} c^f) \cdot \vec{\boldsymbol{v}}^s + c^f (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}^s) \tag{4.142}$$

If the physical quantity c^f does not change with $\vec{\boldsymbol{x}}$, then the gradient of c^f becomes $\nabla_{\vec{x}} c^f = \vec{\mathbf{0}}$. In addition if we consider the medium (s) to be incompressible we can consider $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}^s = 0$. These simplifications indicate that the material (s) does not affect the velocity field of the material (f). So, if the amount of the material (s) is significant, this approach is no longer valid. Then, with these approximations we can obtain:

$$Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{v}^s) = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}^{(D)} \tag{4.143}$$

Notice that the term $(c^s \vec{v}^s) \equiv \vec{\mathbf{q}}^{(D)}$ represents the flux caused by the (s)-material concentration, the diffusive term. The term $(c^s \vec{v}^f) \equiv \vec{\mathbf{q}}^{(C)}$ is related to mass transport, the convective term. Considering that $\vec{\mathbf{q}}^{(D)} = -\mathbf{D} \cdot \nabla_{\vec{x}} c^s$ the equation (4.143) becomes:

$$\begin{aligned}
& Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}^{(D)} \\
& \Rightarrow Q^s = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (-\mathbf{D} \cdot \nabla_{\vec{x}} c^s) \\
& \Rightarrow Q^s - \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} c^s) = \frac{\partial c^s}{\partial t}
\end{aligned} \tag{4.144}$$

with that we have shown the equation in (4.137).

Problem 4.37

Consider a water reservoir with sediment concentration, (see Figure 4.56). The sediment concentration (concentration density) is given by $c(x_3, t) = C\, t \exp^{(-kx_3 t)}$, *per unit volume*, where C and k are positive constants. a) Obtain the total mass of sediment in the reservoir; b) Obtain the sediment flux knowing that the flux is only a function of x_3 and time t, i.e. $\vec{\mathbf{q}} = \vec{\mathbf{q}}(x_3, t)$.

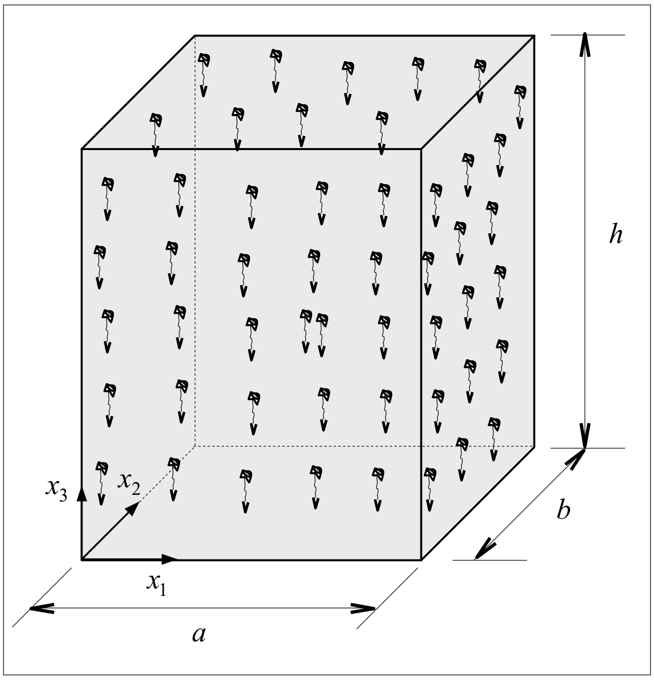


Figure 4.56: Reservoir with sediments.

Solution:

To obtain the total mass we have to solve the integral:

$$\begin{aligned}
M &= \int_V c^s \, dV = \int_0^h \int_0^b \int_0^a C\, t \exp^{(-kx_3 t)} dx_1 dx_2 dx_3 = ab \int_0^h C\, t \exp^{(-kx_3 t)} dx_3 \\
&= ab \left[\frac{-C}{k} \exp^{(-kx_3 t)} \right] \Bigg|_0^h = ab \left[\frac{-C}{k} \exp^{(-kht)} + \frac{C}{k} \right] = \frac{-abC}{k} \left[\exp^{(-kht)} - 1 \right]
\end{aligned}$$

To obtain the flux, we can apply the continuity equation of the concentration:

$$Q = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} \quad \Rightarrow \quad \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} = \mathsf{q}_{i,i} = -\frac{\partial c^s}{\partial t} \tag{4.145}$$

where we have considered that there is no source of the sediment, i.e. $Q = 0$. For this problem, the flux is not dependent on x_2 and x_1. With this condition we have $\mathsf{q}_{1,1} = \mathsf{q}_{2,2} = 0$. Then:

$$\mathsf{q}_{i,i} = \mathsf{q}_{1,1} + \mathsf{q}_{2,2} + \mathsf{q}_{3,3} = \frac{\partial \mathsf{q}_1}{\partial x_1} + \frac{\partial \mathsf{q}_2}{\partial x_2} + \frac{\partial \mathsf{q}_3}{\partial x_3} = -\frac{\partial c^s}{\partial t} \quad \Rightarrow \quad \frac{\partial \mathsf{q}_3}{\partial x_3} = -\frac{\partial c^s}{\partial t} \tag{4.146}$$

where $\frac{\partial c^s}{\partial t} = \frac{\partial}{\partial t}[C\, t \exp^{(-kx_3 t)}] = C \exp^{(-kx_3 t)} - C\, t\, k\, x_3 \exp^{(-kx_3 t)}$ and by substituting into the equation (4.146) we can obtain:

$$\begin{aligned}
&\frac{d\mathsf{q}_3}{dx_3} = -\frac{\partial c^s}{\partial t} = -C \exp^{(-kx_3 t)} + C\, t\, k\, x_3 \exp^{(-kx_3 t)} \\
&\Rightarrow \int d\mathsf{q}_3 = \int [-C \exp^{(-kx_3 t)} + C\, t\, k\, x_3 \exp^{(-kx_3 t)}] dx_3 \\
&\Rightarrow \mathsf{q}_3 = \frac{C}{kt} \exp^{(-kx_3 t)} - \frac{C}{kt} \exp^{(-kx_3 t)} - \frac{C\, k\, x_3\, t}{kt} \exp^{(-kx_3 t)} + K_3 \\
&\Rightarrow \mathsf{q}_3 = -C\, x_3 \exp^{(-kx_3 t)} + \underbrace{K_3}_{=0}
\end{aligned} \tag{4.147}$$

The flux vector in the Cartesian basis is given by $\vec{\mathbf{q}} = -C\, x_3 \exp^{(-kx_3 t)} \hat{\mathbf{e}}_3$.

4.4 Introduction to Rigid Body Motion

Problem 4.38

Starting from the Fundamental Equations of Continuum Mechanics, obtain the governing equations for a rigid solid problem.

Solution:

The fundamental equations of Continuum Mechanics are:

The Fundamental Equations of Continuum Mechanics (Current configuration)		
The Mass Continuity Equation (The principle of conservation of mass)	$\dfrac{D\rho}{Dt}+\rho(\nabla_{\vec{x}}\cdot\vec{v})=0$	(4.148)
The Equations of Motion (The principle of conservation of linear momentum)	$\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\rho\dot{\vec{v}}$	(4.149)
Cauchy Stress Tensor symmetry (The principle of conservation of angular momentum)	$\boldsymbol{\sigma}=\boldsymbol{\sigma}^T$	(4.150)
The Energy Equation (The principle of conservation of energy)	$\rho\dot{u}=\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r$	(4.151)
The Entropy Inequality (The principle of irreversibility)	$\rho\dot{\eta}(\vec{x},t)+\dfrac{1}{T}\boldsymbol{\sigma}:\mathbf{D}-\dfrac{1}{T}\rho\dot{u}-\dfrac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0$	(4.152)

For a rigid body motion there is no mass transportation, so, the principle of conservation of mass plays no rule. The rigid body can be treated as the whole mass is concentrated at one point, so, at the time of establishing the governing equations for rigid body we do not use the local form of the equations (4.149)-(4.150). We will adopt the global formulation of the Principles.

We can start from the definition of the principle of conservation of linear momentum which states that:

$$\sum\vec{\mathbf{F}}=\frac{D}{Dt}\int_V\rho\,\vec{v}\;dV=\dot{\vec{L}}$$

Then we can use the equation of linear momentum $\vec{L}=m\,\vec{\bar{v}}$, (see **Problem 4.39**), to obtain:

$$\sum\vec{\mathbf{F}}=\frac{D}{Dt}\int_V\rho\,\vec{v}\;dV=\dot{\vec{L}}=m\,\dot{\vec{\bar{v}}}=m\,\vec{\bar{a}}$$

Then we have:

$$\boxed{\sum\vec{\mathbf{F}}=m\,\vec{\bar{a}}}$$

Now let us consider the principle of conservation of angular momentum which states:

$$\sum\vec{M}_O=\frac{D}{Dt}\int_V(\vec{x}\wedge\rho\,\vec{v})dV=\frac{D}{Dt}\vec{H}_O\equiv\dot{\vec{H}}_O$$

By which we can obtain:

$$\boxed{\sum \vec{M}_O = \dot{\vec{H}}_O} \quad \text{or} \quad \boxed{\sum \vec{M}_G = \dot{\vec{H}}_G}$$

where the equation of angular momentum $\vec{H}_O$ was obtained in **Problem 4.39**. The set of equations $\sum \vec{\mathbf{F}} = m\vec{a}$ and $\sum \vec{M}_G = \dot{\vec{H}}_G$ inform us that the two systems, described in Figure 4.57, are equivalent.

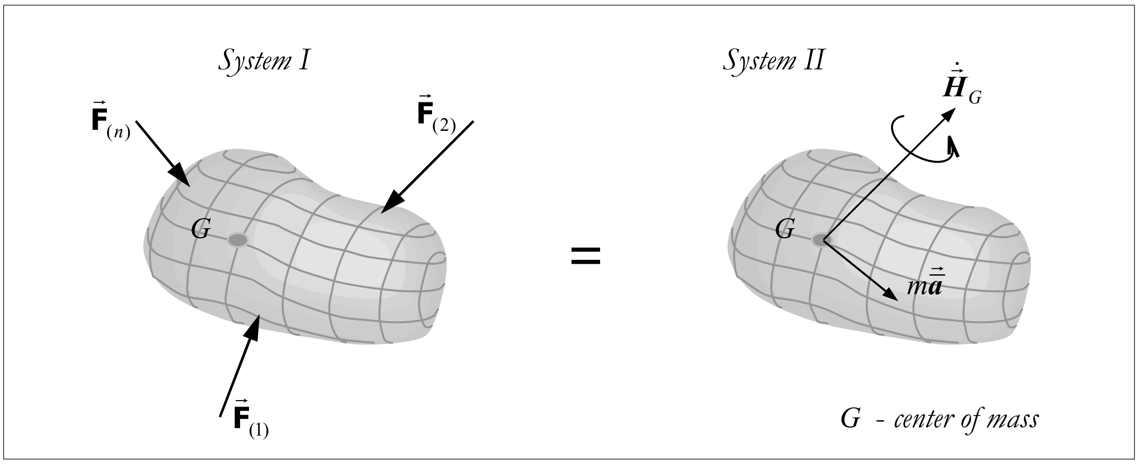


Figure 4.57

Example: Consider the beam with the load and boundary conditions as described in Figure 4.58. Obtain the support reactions V_A, V_B and H_A in order to achieve equilibrium.

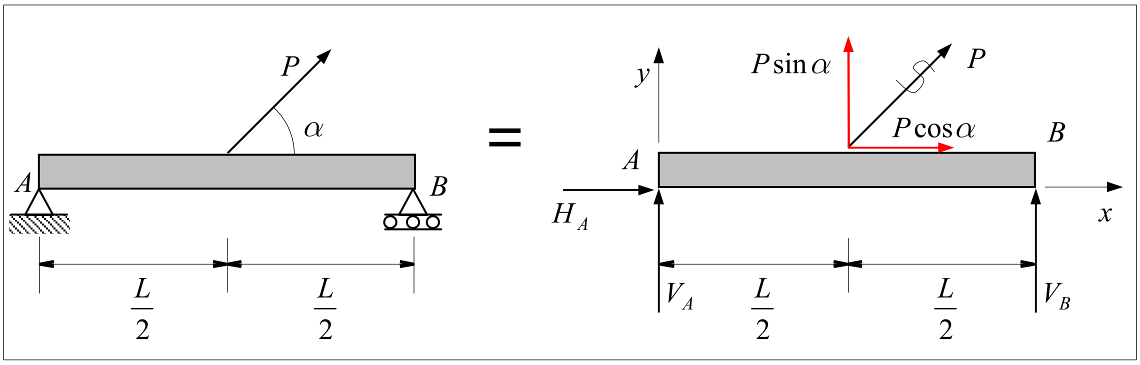


Figure 4.58: Isostatic beam.

Solution:

Although in the beam there is deformation (small deformation regime) and stress, for purposes of support reaction calculation of an isostatic beam we can consider as a rigid body case and the necessary equations, (see equations in (4.153)), are:

$$\sum \vec{\mathbf{F}} = m\vec{a} = \vec{\mathbf{0}} \quad \Rightarrow \begin{cases} \sum \mathrm{F}_x = 0 \\ \sum \mathrm{F}_y = 0 \\ \sum \mathrm{F}_z = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \sum \mathrm{F}_x = H_A + P\cos\alpha = 0 \Rightarrow H_A = -P\cos\alpha \\ \sum \mathrm{F}_y = V_A + V_B + P\sin\alpha = 0 \Rightarrow V_A = -V_B - \sin\alpha \end{cases}$$

$$\sum \vec{\boldsymbol{M}}_A = \dot{\vec{\boldsymbol{H}}}_A = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \begin{cases} \sum M_x = 0 \\ \sum M_y = 0 \\ \sum M_z = 0 \end{cases} \quad \Rightarrow \quad \left\{ \sum M_z = V_B L + P \sin\alpha \frac{L}{2} = 0 \Rightarrow V_B = \frac{-P\sin\alpha}{2} \right.$$

with which we can obtain $V_A = -V_B - P\sin\alpha = \dfrac{-P\sin\alpha}{2}$. Note that we have 3 equations and 3 unknowns (a statically determinate system or isostatic). If we have a system in which there are more unknowns than equations (a statically indeterminate system or hyperstatic), this procedure is no longer valid since the reactions will depend on the beam deformation and this depends on the beam stiffness.

NOTE: If we are dealing with rigid body motion, the governing equations are:

$$\boxed{\sum \vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}} \qquad and \qquad \sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O} \quad \text{Governing equations for rigid body motion} \tag{4.153}$$

The set of equations in (4.153) governs several problems such as: machine components which are in rotation, satellite motion, navigation, etc. In navigation, motion is governed by a device called gyroscope, which are governed by the set of equations in (4.153).

The set of equations in (4.153), in general, are non-linear and the analytical solution is very complex to be obtained. Next, we will try to express the set of equations (4.153) more friendly.

Problem 4.39

Find the linear and angular momentum for the solid described in Figure 4.59 and subjected to rigid body motion.

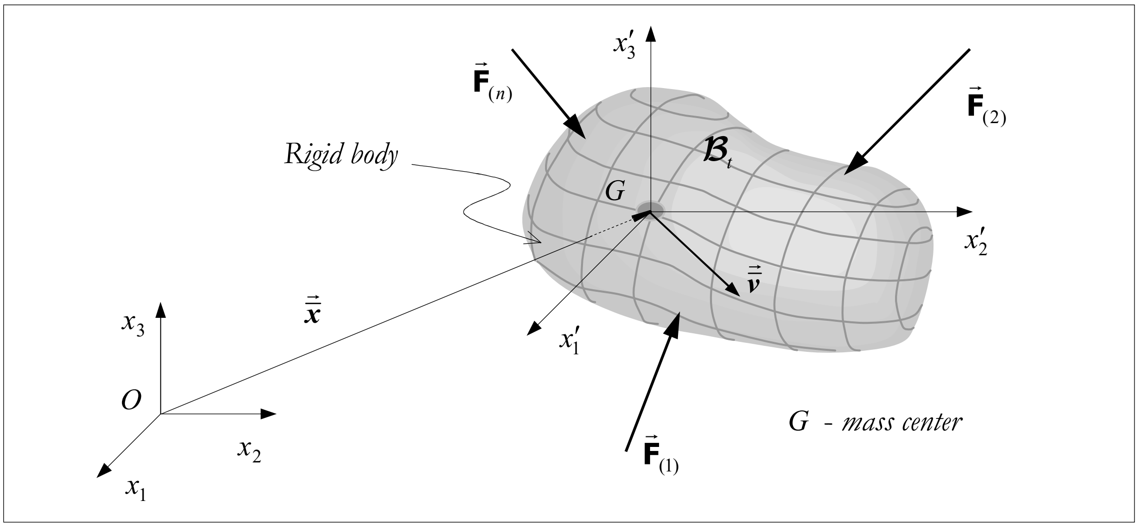


Figure 4.59: Rigid body motion.

Solution:

According to **Problem 2.55** in Chapter 2, we have obtained the velocity for rigid body motion as:

$$\vec{\boldsymbol{v}} = \dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{x}} - \vec{\mathbf{c}})$$

where $\vec{\boldsymbol{\omega}}$ is the axial vector (angular velocity) associated with the antisymmetric tensor $\mathbf{W}$ (the spin tensor).

Linear momentum:

$$\begin{aligned}\vec{\boldsymbol{L}} &= \int_V \rho\, \vec{\boldsymbol{v}}\, dV = \int_V \rho \left(\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{x}} - \vec{\mathbf{c}})\right) dV = \int_V \rho\, \dot{\vec{\mathbf{c}}}\, dV + \int_V \rho\, \vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{x}}\, dV - \int_V \rho\, \vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}}\, dV \\ &= \dot{\vec{\mathbf{c}}} \int_V \rho\, dV + \vec{\boldsymbol{\omega}} \wedge \int_V \rho\, \vec{\boldsymbol{x}}\, dV - \vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}} \int_V \rho\, dV\end{aligned}$$

By definition $\int_V \rho\, \vec{\boldsymbol{x}}\, dV = m\vec{\bar{\boldsymbol{x}}}$ is the first moment of inertia, where m is the total mass, and $\vec{\bar{\boldsymbol{x}}}_k$ is the vector position of the center of mass G. The first moment of inertia is equal to zero if the Cartesian system originates at the center of mass, so, $\int_V \rho\, \vec{\boldsymbol{x}}'\, dV = m\vec{\bar{\boldsymbol{x}}}' = \vec{\mathbf{0}}$.

$$\boxed{\begin{aligned}\vec{\boldsymbol{L}} &= m[\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{\boldsymbol{x}}} - \vec{\mathbf{c}})] \\ &= m\, \vec{\bar{\boldsymbol{v}}}\end{aligned}} \quad \textit{(Linear momentum for rigid body motion)} \tag{4.154}$$

where $\vec{\bar{\boldsymbol{v}}} = \dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{\boldsymbol{x}}} - \vec{\mathbf{c}})$ is the velocity of the center of mass.

Angular momentum:

$$\vec{\boldsymbol{H}}_O = \int_V (\vec{\boldsymbol{x}} \wedge \rho \vec{\boldsymbol{v}})\, dV = \int_V \left[\vec{\boldsymbol{x}} \wedge \rho\left(\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{x}} - \vec{\mathbf{c}})\right)\right] dV$$

Thus

$$\begin{aligned}\vec{H}_O &= \int_V \rho\,\vec{x}\wedge\dot{\vec{c}}\,dV + \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV - \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{c})\,dV \\ &= \left[\int_V \rho\,\vec{x}\,dV\right]\wedge\dot{\vec{c}} + \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV - \left[\int_V \rho\,\vec{x}\,dV\right]\wedge(\vec{\omega}\wedge\vec{c})\end{aligned} \tag{4.155}$$

Next, we discuss the second integral of the previous equation.

It was shown in Chapter 1 that given three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, the relationship $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}$ holds, thus when $\vec{\mathbf{a}}=\vec{\mathbf{c}}$ it holds that $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{a}}$, so, $\int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV = \int_V \rho\left[(\vec{x}\cdot\vec{x})\vec{\omega} - (\vec{x}\cdot\vec{\omega})\vec{x}\right]dV$, with which we can obtain:

$$\begin{aligned}\int_V \rho\left[x_k x_k \omega_i - x_p\omega_p x_i\right]dV &= \int_V \rho\left[x_k x_k \omega_p \delta_{pi} - x_p\omega_p x_i\right]dV = \int_V \rho\left[x_k x_k \delta_{pi} - x_p x_i\right]\omega_p\,dV \\ &= \int_V \rho\left[x_k x_k \delta_{pi} - x_p x_i\right]dV\ \ \omega_p = \mathrm{I}_{Oip}\omega_p\end{aligned}$$

or in tensorial notation:

$$\int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV = \left[\int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV\right]\cdot\vec{\omega} = \mathbf{I}_O\cdot\vec{\omega}$$

where $\mathbf{I}_O = \int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV$ is the inertia tensor with respect to the origin O. As we can observe, $\mathbf{I}_O$ is a second-order pseudo-tensor, since it depends on the reference system, and the components $\mathrm{I}_{Oij} = \int_V \rho\,[x_k x_k \delta_{ij} - x_i x_j]\,dV$ can be explicitly expressed as:

$$\mathrm{I}_{O11} = \int_V \rho\left[(x_1x_1 + x_2x_2 + x_3x_3)\delta_{11} - x_1x_1\right]dV = \int_V \rho\left[x_2^2 + x_3^2\right]dV$$

$$\mathrm{I}_{O22} = \int_V \rho\left[x_1^2 + x_3^2\right]dV \quad ; \quad \mathrm{I}_{O33} = \int_V \rho\left[x_1^2 + x_2^2\right]dV$$

$$\mathrm{I}_{O12} = \int_V \rho\left[(x_1x_1 + x_2x_2 + x_3x_3)\delta_{12} - x_1x_2\right]dV = -\int_V \rho\left[x_1x_2\right]dV = -\mathcal{I}_{O12}$$

$$\mathrm{I}_{O13} = -\int_V \rho\left[x_1x_3\right]dV = -\mathcal{I}_{O13} \quad ; \quad \mathrm{I}_{O23} = -\int_V \rho\left[x_2x_3\right]dV = -\mathcal{I}_{O23}$$

where I_{O11}, I_{O22}, I_{O33}, are *moments of inertia* of the body relative to the reference point O, and $\mathcal{I}_{O12}$, $\mathcal{I}_{O13}$, $\mathcal{I}_{O23}$, are the *products of inertia* of the body relative to the reference point O. Note also that the SI-unit for the inertia tensor is:

$$[\mathbf{I}_O] = \left[\int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV\right] = [\rho][\vec{x}][\vec{x}][dV] = \frac{kg}{m^3}\,m\,m\,m^3 = kg\,m^2$$

The inertia tensor components in matrix form are represented as follows

$$
\mathrm{I}_{Oij} = \begin{bmatrix} \int_V \rho\,[x_2^2 + x_3^2]\,dV & -\int_V \rho\,[x_1x_2]\,dV & -\int_V \rho\,[x_1x_3]\,dV \\ -\int_V \rho\,[x_1x_2]\,dV & \int_V \rho\,[x_1^2 + x_3^2]\,dV & -\int_V \rho\,[x_2x_3]\,dV \\ -\int_V \rho\,[x_1x_3]\,dV & -\int_V \rho\,[x_2x_3]\,dV & \int_V \rho\,[x_1^2 + x_2^2]\,dV \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{O11} & -\mathcal{I}_{O12} & -\mathcal{I}_{O13} \\ -\mathcal{I}_{O12} & \mathrm{I}_{O22} & -\mathcal{I}_{O23} \\ -\mathcal{I}_{O13} & -\mathcal{I}_{O23} & \mathrm{I}_{O33} \end{bmatrix}
\tag{4.156}
$$

Returning to the equation in (4.155) we can state that:

$$
\begin{aligned}
\vec{\boldsymbol{H}}_O &= \left[\int_V \rho\,\vec{\boldsymbol{x}}\,dV\right] \wedge \dot{\vec{\mathbf{c}}} + \int_V \rho\,\vec{\boldsymbol{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{x}})\,dV - \left[\int_V \rho\,\vec{\boldsymbol{x}}\,dV\right] \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}}) \\
&= m\,\vec{\bar{\boldsymbol{x}}} \wedge \dot{\vec{\mathbf{c}}} + \mathbf{I}_O \cdot \vec{\boldsymbol{\omega}} - m\,\vec{\bar{\boldsymbol{x}}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}}) = m\,\vec{\bar{\boldsymbol{x}}} \wedge [\dot{\vec{\mathbf{c}}} - (\vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}})] + \mathbf{I}_O \cdot \vec{\boldsymbol{\omega}}
\end{aligned}
$$

Then by adding and subtracting the term $m\,\vec{\bar{\boldsymbol{x}}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\bar{\boldsymbol{x}}})$ in the above equation we can obtain:

$$
\begin{aligned}
\vec{\boldsymbol{H}}_O &= m\,\vec{\bar{\boldsymbol{x}}} \wedge (\dot{\vec{\mathbf{c}}} - \vec{\boldsymbol{\omega}} \wedge \vec{\mathbf{c}}) + \mathbf{I}_O \cdot \vec{\boldsymbol{\omega}} = m\,\vec{\bar{\boldsymbol{x}}} \wedge [\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{\boldsymbol{x}}} - \vec{\mathbf{c}})] - m\,\vec{\bar{\boldsymbol{x}}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\bar{\boldsymbol{x}}}) + \mathbf{I}_O \cdot \vec{\boldsymbol{\omega}} \\
&= m\,\vec{\bar{\boldsymbol{x}}} \wedge \vec{\bar{\boldsymbol{v}}} - m\left[(\vec{\bar{\boldsymbol{x}}} \cdot \vec{\bar{\boldsymbol{x}}})\,\mathbf{1} - (\vec{\bar{\boldsymbol{x}}} \otimes \vec{\bar{\boldsymbol{x}}})\right] \cdot \vec{\boldsymbol{\omega}} + \mathbf{I}_O \cdot \vec{\boldsymbol{\omega}} = m\,\vec{\bar{\boldsymbol{x}}} \wedge \vec{\bar{\boldsymbol{v}}} + \left\{m\left[(\vec{\bar{\boldsymbol{x}}} \otimes \vec{\bar{\boldsymbol{x}}}) - (\vec{\bar{\boldsymbol{x}}} \cdot \vec{\bar{\boldsymbol{x}}})\,\mathbf{1}\right] + \mathbf{I}_O\right\} \cdot \vec{\boldsymbol{\omega}} \\
&= m\,\vec{\bar{\boldsymbol{x}}} \wedge \vec{\bar{\boldsymbol{v}}} + \bar{\mathbf{I}} \cdot \vec{\boldsymbol{\omega}} \\
&= m\,\vec{\bar{\boldsymbol{x}}} \wedge \vec{\bar{\boldsymbol{v}}} + \vec{\boldsymbol{H}}_G
\end{aligned}
\tag{4.157}
$$

where $\bar{\mathbf{I}} = \mathbf{I}_O + m[(\vec{\bar{\boldsymbol{x}}} \otimes \vec{\bar{\boldsymbol{x}}}) - (\vec{\bar{\boldsymbol{x}}} \cdot \vec{\bar{\boldsymbol{x}}})\,\mathbf{1}]$ is the inertia pseudo-tensor, which is related to the reference system at the center of mass. By means of this equation we can calculate the inertia tensor in any reference system if we know the inertia tensor at the center of mass: $\mathrm{I}_{Oij} = \bar{\mathrm{I}}_{ij} - m[\bar{x}_i\bar{x}_j - (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)\delta_{ij}]$. Explicitly, these components can be expressed as follows:

$$
\boxed{\begin{array}{lcl} \mathrm{I}_{O11} = \bar{\mathrm{I}}_{11} + m(\bar{x}_2^2 + \bar{x}_3^2) & ; & \mathrm{I}_{O12} = \bar{\mathrm{I}}_{12} - m(\bar{x}_1\bar{x}_2) \\ \mathrm{I}_{O22} = \bar{\mathrm{I}}_{22} + m(\bar{x}_1^2 + \bar{x}_3^2) & ; & \mathrm{I}_{O23} = \bar{\mathrm{I}}_{23} - m(\bar{x}_2\bar{x}_3) \\ \mathrm{I}_{O33} = \bar{\mathrm{I}}_{33} + m(\bar{x}_1^2 + \bar{x}_2^2) & ; & \mathrm{I}_{O13} = \bar{\mathrm{I}}_{13} - m(\bar{x}_1\bar{x}_3) \end{array}} \quad \textit{Steiner's theorem}
\tag{4.158}
$$

Note that, the above equations represent the parallel axis theorem (Steiner's theorem) from Classical Mechanics, which in matrix notation is given by:

$$
\mathrm{I}_{Oij} = \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + m \begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \quad \textit{Steiner's theorem}
\tag{4.159}
$$

NOTE 1: If we have two bodies $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ we can conclude that

$$
\begin{aligned}
\vec{\boldsymbol{H}}_O &= \int_{V^{(1)}} (\vec{\boldsymbol{x}} \wedge \rho^{(1)}\vec{\boldsymbol{v}})\,dV^{(1)} + \int_{V^{(2)}} (\vec{\boldsymbol{x}} \wedge \rho^{(2)}\vec{\boldsymbol{v}})\,dV^{(2)} \\
&= m^{(1)}\vec{\bar{\boldsymbol{x}}}^{(1)} \wedge \vec{\bar{\boldsymbol{v}}}^{(1)} + \bar{\mathbf{I}}^{(1)} \cdot \vec{\boldsymbol{\omega}}^{(1)} + m^{(2)}\,\vec{\bar{\boldsymbol{x}}}^{(2)} \wedge \vec{\bar{\boldsymbol{v}}}^{(2)} + \bar{\mathbf{I}}^{(2)} \cdot \vec{\boldsymbol{\omega}}^{(2)} \\
&= m^{(1)}\vec{\bar{\boldsymbol{x}}}^{(1)} \wedge \vec{\bar{\boldsymbol{v}}}^{(1)} + m^{(2)}\,\vec{\bar{\boldsymbol{x}}}^{(2)} \wedge \vec{\bar{\boldsymbol{v}}}^{(2)} + \bar{\mathbf{I}}^{(1)} \cdot \vec{\boldsymbol{\omega}}^{(1)} + \bar{\mathbf{I}}^{(2)} \cdot \vec{\boldsymbol{\omega}}^{(2)} \\
&= m^{(sys)}\vec{\bar{\boldsymbol{x}}}^{(sys)} \wedge \vec{\bar{\boldsymbol{v}}}^{(sys)} + \bar{\mathbf{I}}^{(sys)} \cdot \vec{\boldsymbol{\omega}}^{(sys)}
\end{aligned}
$$

where $\bullet^{(1)}$ and $\bullet^{(2)}$ stand for properties of the bodies $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ respectively. If the two bodies are attached they have the same angular velocity $\vec{\boldsymbol{\omega}}^{(1)} = \vec{\boldsymbol{\omega}}^{(2)} = \vec{\boldsymbol{\omega}}^{(sys)} = \vec{\boldsymbol{\omega}}$, so, we can conclude that:

$$\vec{\boldsymbol{H}}_O = m^{(sys)} \vec{\bar{\boldsymbol{x}}}^{(sys)} \wedge \vec{\bar{\boldsymbol{v}}}^{(sys)} + [\bar{\mathbf{I}}^{(1)} + \bar{\mathbf{I}}^{(2)}] \cdot \vec{\boldsymbol{\omega}}$$

and if the system $O\vec{\boldsymbol{x}}$ is at the center of mass of the system ($\vec{\bar{\boldsymbol{v}}}^{(sys)} = \vec{\boldsymbol{0}}$) we can obtain:

$$\vec{\boldsymbol{H}}_O = \bar{\mathbf{I}}^{(sys)} \cdot \vec{\boldsymbol{\omega}} = [\bar{\mathbf{I}}^{(1)} + \bar{\mathbf{I}}^{(2)}] \cdot \vec{\boldsymbol{\omega}}$$

Problem 4.40

Consider a parallelepiped whose dimensions are $a \times b \times c$, (see Figure 4.60), in which the mass density field, $\rho(\vec{\boldsymbol{x}})$, is homogeneous. Obtain the inertia tensor with respect to system in the center of gravity.

Solution:

We will use the equation in (4.156):

$$\mathrm{I}_{Oij} = \begin{bmatrix} \int_V \rho [x_2^2 + x_3^2] \, dV & -\int_V \rho [x_1 x_2] \, dV & -\int_V \rho [x_1 x_3] \, dV \\ -\int_V \rho [x_1 x_2] \, dV & \int_V \rho [x_1^2 + x_3^2] \, dV & -\int_V \rho [x_2 x_3] \, dV \\ -\int_V \rho [x_1 x_3] \, dV & -\int_V \rho [x_2 x_3] \, dV & \int_V \rho [x_1^2 + x_2^2] \, dV \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{O11} & -\mathcal{I}_{O12} & -\mathcal{I}_{O13} \\ -\mathcal{I}_{O12} & \mathrm{I}_{O22} & -\mathcal{I}_{O23} \\ -\mathcal{I}_{O13} & -\mathcal{I}_{O23} & \mathrm{I}_{O33} \end{bmatrix}$$

Note that, for this problem, the mass density is independent of $\vec{\boldsymbol{x}}$ (homogeneous material), with which it fulfills that:

$$m = \int_V \rho \ dV = \rho \int_V dV = \rho V = \rho abc$$

Then, the moment of inertia I_{O11} becomes:

$$\mathrm{I}_{O11} = \int_V \rho [x_2^2 + x_3^2] \, dV = \rho \int_{\frac{-c}{2}}^{\frac{c}{2}} \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-a}{2}}^{\frac{a}{2}} [x_2^2 + x_3^2] dx_1 dx_2 dx_3 = \rho \frac{abc}{12}(b^2 + c^2) = \frac{m}{12}(b^2 + c^2)$$

In the same fashion we can obtain $\mathrm{I}_{O22} = \dfrac{m}{12}(a^2 + c^2)$ and $\mathrm{I}_{O33} = \dfrac{m}{12}(a^2 + b^2)$.

We leave to the reader show that $\mathcal{I}_{O12} = \mathcal{I}_{O13} = \mathcal{I}_{O23} = 0$. Recall that the inertia tensor give us information about how the mass is distributed according to the adopted system, and note that the mass is equally distributed according to the plane $x_1 x_2$, thus $\int_V \rho [x_1 x_2] \, dV = 0$.

Note also that the adopted axes are principal axes of inertia:

$$\mathrm{I}'_{Oij} = \begin{bmatrix} \frac{m}{12}(b^2 + c^2) & 0 & 0 \\ 0 & \frac{m}{12}(a^2 + c^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2 + b^2) \end{bmatrix} \quad [kg \ m^2]$$

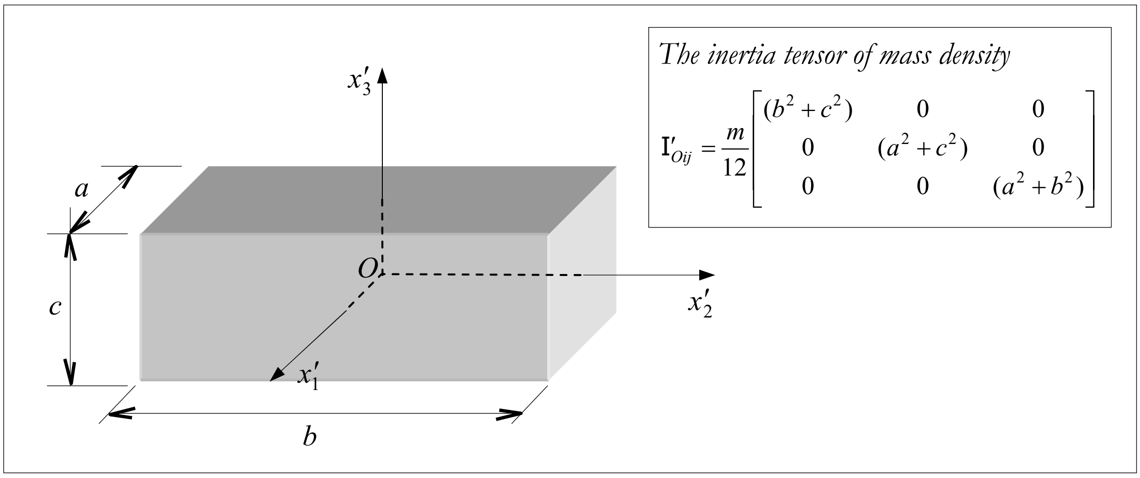


Figure 4.60: Parallelepiped.

Problem 4.41

Consider three thin rods of length a and mass m, (see Figure 4.61). Obtain the inertia tensor of the compound related to the system $O\vec{x}$.

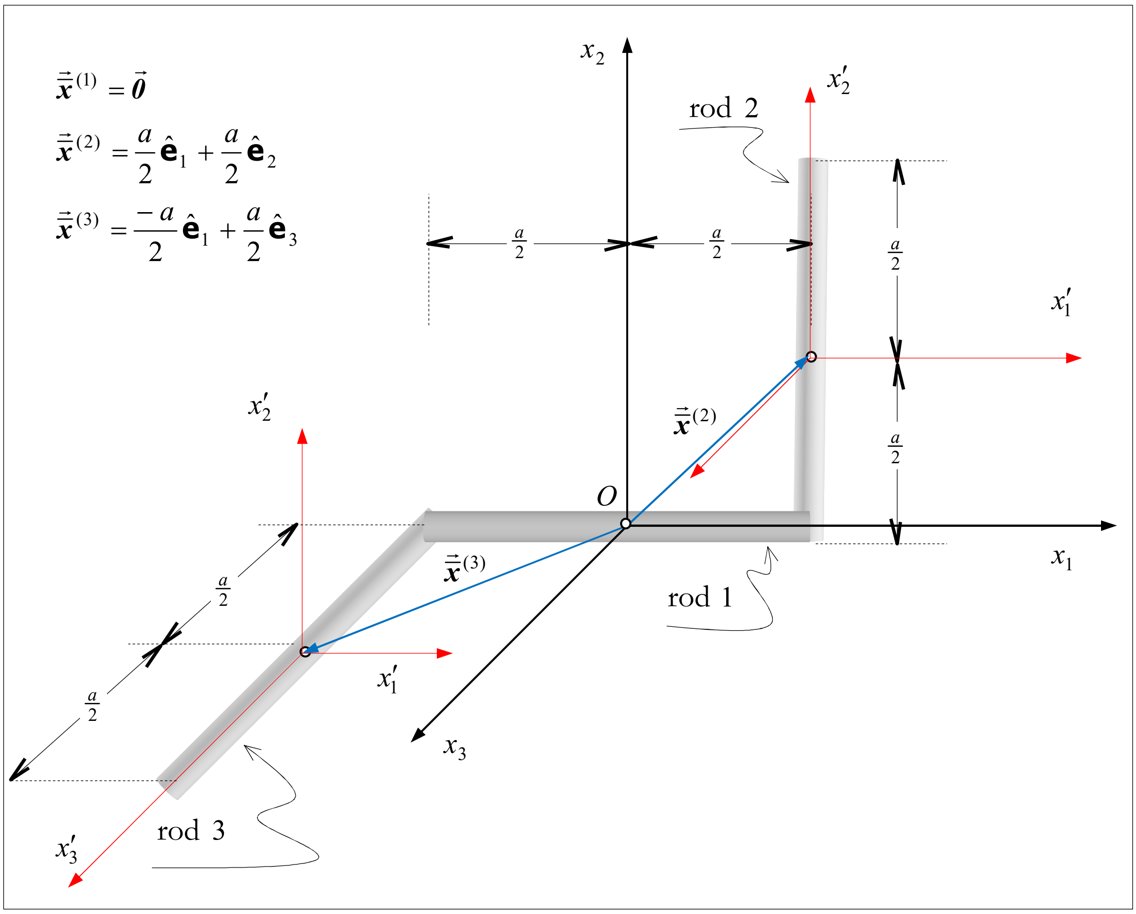
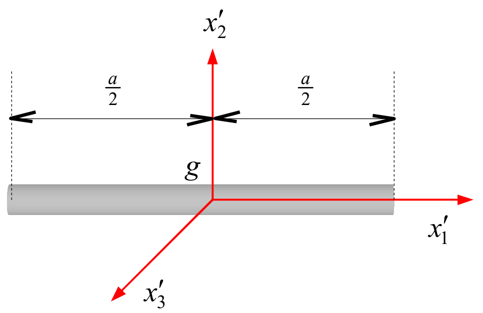


Figure 4.61: System compounded by three rods.

Data: The inertia tensor for the thin rod, in which the all mass is distributed along its axis, is given by Figure 4.62.



$$\bar{\mathrm{I}}_{gij} = \frac{ma^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 4.62: Inertia tensor of the rod related to the principal system.

Solution:

To calculate the inertia tensor of the system we will use the equation $\mathbf{I}^{(sys)}_{O\vec{x}} = \mathbf{I}^{(1)}_{O\vec{x}} + \mathbf{I}^{(2)}_{O\vec{x}} + \mathbf{I}^{(3)}_{O\vec{x}}$ where $\mathrm{I}_{Oij} = \bar{\mathrm{I}}_{ij} - m[\bar{x}_i\bar{x}_j - (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)\delta_{ij}]$ (the Steiner's theorem), (see **Problem 4.39** (NOTA 1)):

$$\mathrm{I}_{O\,ij} = \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + m\begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \qquad (4.160)$$

Rod 1 - $\mathbf{I}^{(1)}_{O\vec{x}}$

Mass center vector position: $(\bar{x}_1^{(1)} = 0, \bar{x}_2^{(1)} = 0, \bar{x}_3^{(1)} = 0)$

$$(\mathbf{I}^{(1)}_{O\vec{x}})_{ij} = \frac{ma^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rod 2 - $\mathbf{I}^{(2)}_{O\vec{x}}$

Mass center vector position: $(\bar{x}_1^{(2)} = \frac{a}{2}, \bar{x}_2^{(2)} = \frac{a}{2}, \bar{x}_3^{(2)} = 0)$

By applying the equation (4.160) we can obtain:

$$(\mathbf{I}^{(2)}_{O\vec{x}})_{ij} = \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + m\begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix}$$

$$= \frac{ma^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + m\begin{bmatrix} \left(\frac{a}{2}\right)^2 + 0^2 & -\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) & 0 \\ -\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) & \left(\frac{a}{2}\right)^2 + 0^2 & 0 \\ 0 & 0 & \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \end{bmatrix} = \frac{ma^2}{12}\begin{bmatrix} 4 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Rod 3 - $\mathbf{I}_{O\bar{x}}^{(3)}$

Mass center vector position: $(\bar{x}_1^{(2)} = \frac{-a}{2}, \bar{x}_2^{(2)} = 0, \bar{x}_3^{(2)} = \frac{a}{2})$

$$(\mathbf{I}_{O\bar{x}}^{(2)})_{ij} = \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + m \begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix}$$

$$= \frac{ma^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + m \begin{bmatrix} (0)^2 + \left(\frac{a}{2}\right)^2 & 0 & -\left(\frac{-a}{2}\right)\left(\frac{a}{2}\right) \\ 0 & \left(\frac{-a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 & 0 \\ -\left(\frac{-a}{2}\right)\left(\frac{a}{2}\right) & 0 & \left(\frac{-a}{2}\right)^2 + 0^2 \end{bmatrix} = \frac{ma^2}{12}\begin{bmatrix} 4 & 0 & 3 \\ 0 & 7 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

Then, we can calculate

$$(\mathbf{I}_{O\bar{x}}^{(sys)})_{ij} = (\mathbf{I}_{O\bar{x}}^{(1)})_{ij} + (\mathbf{I}_{O\bar{x}}^{(2)})_{ij} + (\mathbf{I}_{O\bar{x}}^{(3)})_{ij}$$

$$= \frac{ma^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{ma^2}{12}\begin{bmatrix} 4 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \frac{ma^2}{12}\begin{bmatrix} 4 & 0 & 3 \\ 0 & 7 & 0 \\ 3 & 0 & 3 \end{bmatrix} = \frac{ma^2}{12}\begin{bmatrix} 8 & -3 & 3 \\ -3 & 11 & 0 \\ 3 & 0 & 11 \end{bmatrix}$$

Problem 4.42

Find the kinetic energy related to rigid body motion in terms of the inertia tensor, (see **Problem 4.39** and **Problem 4.38**).

Solution: The rigid body motion velocity can be expressed as $\vec{v} = \dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})$. Then, the kinetic energy becomes:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho(\vec{v}\cdot\vec{v})dV = \frac{1}{2}\int_V \rho\left\{ \left[\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})\right]\cdot\left[\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})\right] \right\} dV$$

Using the following vector sum $\vec{x} = \vec{\bar{x}} + \vec{x}'$, where $\vec{\bar{x}}$ is the mass center vector position, and $\vec{x}'$ is the particle vector position with respect to the system that has its origin in the center of mass, the energy equation becomes:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\left\{ \left[\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge ((\vec{\bar{x}} + \vec{x}') - \vec{\mathbf{c}})\right]\cdot\left[\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge ((\vec{\bar{x}} + \vec{x}') - \vec{\mathbf{c}})\right] \right\} dV$$
$$= \frac{1}{2}\int_V \rho\left\{ \left[\left(\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})\right) + (\vec{\omega} \wedge \vec{x}')\right]\cdot\left[\left(\dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})\right) + (\vec{\omega} \wedge \vec{x}')\right] \right\} dV$$

Note that $\vec{\bar{v}} = \dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})$ is the center of mass velocity, thus:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\left\{ \left[\vec{\bar{v}} + (\vec{\omega} \wedge \vec{x}')\right]\cdot\left[\vec{\bar{v}} + (\vec{\omega} \wedge \vec{x}')\right] \right\} dV$$

or:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot\vec{\bar{v}}\, dV + \frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot(\vec{\omega} \wedge \vec{x}')\, dV + \frac{1}{2}\int_V \rho(\vec{\omega} \wedge \vec{x}')\cdot\vec{\bar{v}}\, dV + \frac{1}{2}\int_V \rho(\vec{\omega} \wedge \vec{x}')\cdot(\vec{\omega} \wedge \vec{x}')\, dV$$

Then by simplifying the above equation we can obtain:

$$\mathcal{K}(t)=\frac{1}{2}\int_V \rho\vec{\bar{\boldsymbol{v}}}\cdot\vec{\bar{\boldsymbol{v}}}\,dV+\int_V \rho\vec{\bar{\boldsymbol{v}}}\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\,dV+\frac{1}{2}\int_V \rho(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\,dV$$

Next, we will discuss separately the terms of the previous equation:

1) $\frac{1}{2}\int_V \rho\vec{\bar{\boldsymbol{v}}}\cdot\vec{\bar{\boldsymbol{v}}}\,dV=\frac{1}{2}\left\|\vec{\bar{\boldsymbol{v}}}\right\|^2\int_V \rho\,dV=\frac{1}{2}m\bar{v}^2$

2) $\int_V \rho\vec{\bar{\boldsymbol{v}}}\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\,dV=\vec{\bar{\boldsymbol{v}}}\cdot\left[\vec{\boldsymbol{\omega}}\wedge\int_V \rho\vec{\boldsymbol{x}}'\,dV\right]=\vec{\bar{\boldsymbol{v}}}\cdot(\vec{\boldsymbol{\omega}}\wedge m\underbrace{\vec{\bar{\boldsymbol{x}}}'}_{=\vec{\boldsymbol{0}}})=0$

Note that, the system $\vec{\boldsymbol{x}}'$ is located at the center of mass (G), hence the center of mass vector position related to the system $\vec{\boldsymbol{x}}'$ is zero.

3)
$$\begin{aligned}\int_V \rho\left[(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\right]dV&=\int_V \rho\,\epsilon_{ijk}\omega_j x'_k\epsilon_{ipq}\omega_p x'_q\,dV=\int_V \rho\,(\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp})\omega_j x'_k\omega_p x'_q\,dV\\&=\int_V \rho\,\omega_j(\delta_{jp}\delta_{kq}x'_k x'_q-\delta_{jq}\delta_{kp}x'_k x'_q)\omega_p\,dV\\&=\int_V \rho\,\omega_j(\delta_{jp}x'_k x'_k-x'_p x'_j)\omega_p\,dV=\omega_j\left(\int_V \rho\,(\delta_{jp}x'_k x'_k-x'_j x'_p)\,dV\right)\omega_p\\&=\omega_j\bar{\mathrm{I}}_{jp}\omega_p\end{aligned}$$

or in tensorial notation as:

$$\int_V \rho\left[(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\right]dV=\vec{\boldsymbol{\omega}}\cdot\left[\int_V \rho\left[(\vec{\boldsymbol{x}}'\cdot\vec{\boldsymbol{x}}')\mathbf{1}-(\vec{\boldsymbol{x}}'\otimes\vec{\boldsymbol{x}}')\right]dV\right]\cdot\vec{\boldsymbol{\omega}}=\vec{\boldsymbol{\omega}}\cdot\bar{\mathbf{I}}\cdot\vec{\boldsymbol{\omega}}$$

where $\bar{\mathbf{I}}$ is the inertia pseudo-tensor related to the system located at the center of mass, (see **Problem 4.39**).

Then if we bear in mind all the above considerations, the kinetic energy equation for rigid body motion becomes:

$$\mathcal{K}(t)=\frac{1}{2}\int_V \rho\vec{\bar{\boldsymbol{v}}}\cdot\vec{\bar{\boldsymbol{v}}}\,dV+\underbrace{\frac{1}{2}\int_V 2\rho\vec{\bar{\boldsymbol{v}}}\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\,dV}_{=0}+\frac{1}{2}\int_V \rho(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\cdot(\vec{\boldsymbol{\omega}}\wedge\vec{\boldsymbol{x}}')\,dV$$

$$\boxed{\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\vec{\boldsymbol{\omega}}\cdot\bar{\mathbf{I}}\cdot\vec{\boldsymbol{\omega}}}\quad \textit{Kinetic energy for rigid body motion}\qquad(4.161)$$

Additionally, if we take into account that:

$$\bar{\mathrm{I}}_{ij}=\begin{bmatrix}\int_V \rho\,[x_2'^2+x_3'^2]\,dV & -\int_V \rho\,[x'_1x'_2]\,dV & -\int_V \rho\,[x'_1x'_3]\,dV\\ -\int_V \rho\,[x'_1x'_2]\,dV & \int_V \rho\,[x_1'^2+x_3'^2]\,dV & -\int_V \rho\,[x'_2x'_3]\,dV\\ -\int_V \rho\,[x'_1x'_3]\,dV & -\int_V \rho\,[x'_2x'_3]\,dV & \int_V \rho\,[x_1'^2+x_2'^2]\,dV\end{bmatrix}=\begin{bmatrix}\bar{\mathcal{I}}_{11} & -\bar{\mathcal{I}}_{12} & -\bar{\mathcal{I}}_{13}\\ -\bar{\mathcal{I}}_{12} & \bar{\mathcal{I}}_{22} & -\bar{\mathcal{I}}_{23}\\ -\bar{\mathcal{I}}_{13} & -\bar{\mathcal{I}}_{23} & \bar{\mathcal{I}}_{33}\end{bmatrix}$$

we can obtain an explicit equation for the kinetic energy as:

$$
\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\omega_k \bar{\mathrm{I}}_{kj}\omega_j=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}\begin{bmatrix}\bar{\mathcal{I}}_{11} & -\bar{\mathcal{I}}_{12} & -\bar{\mathcal{I}}_{13}\\ -\bar{\mathcal{I}}_{12} & \bar{\mathcal{I}}_{22} & -\bar{\mathcal{I}}_{23}\\ -\bar{\mathcal{I}}_{13} & -\bar{\mathcal{I}}_{23} & \bar{\mathcal{I}}_{33}\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}
$$

$$
=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\left[\bar{\mathcal{I}}_{11}\omega_1^2+\bar{\mathcal{I}}_{22}\omega_2^2+\bar{\mathcal{I}}_{33}\omega_3^2-2\bar{\mathcal{I}}_{12}\omega_1\omega_2-2\bar{\mathcal{I}}_{13}\omega_1\omega_3-2\bar{\mathcal{I}}_{23}\omega_2\omega_3\right]
$$

$$
\boxed{\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\left[\bar{\mathcal{I}}_{11}\omega_1^2+\bar{\mathcal{I}}_{22}\omega_2^2+\bar{\mathcal{I}}_{33}\omega_3^2-2\bar{\mathcal{I}}_{12}\omega_1\omega_2-2\bar{\mathcal{I}}_{13}\omega_1\omega_3-2\bar{\mathcal{I}}_{23}\omega_2\omega_3\right]} \qquad (4.162)
$$

Problem 4.43

Consider the inertia pseudo-tensor, $\mathbf{I}_O$, with respect to the system $x_1x_2x_3$, (see Figure 4.63). a) Make the physical interpretation of the inertia tensor. b) Given another orthonormal system, represented by $x_1^*x_2^*x_3^*$. Obtain the inertia tensor components in this new system. c) Show that the inertia tensor is positive definite tensor. For a solid in motion, find in which situation the term $\frac{D\mathbf{I}_O}{Dt}\equiv\dot{\mathbf{I}}_O$ is equal to zero.

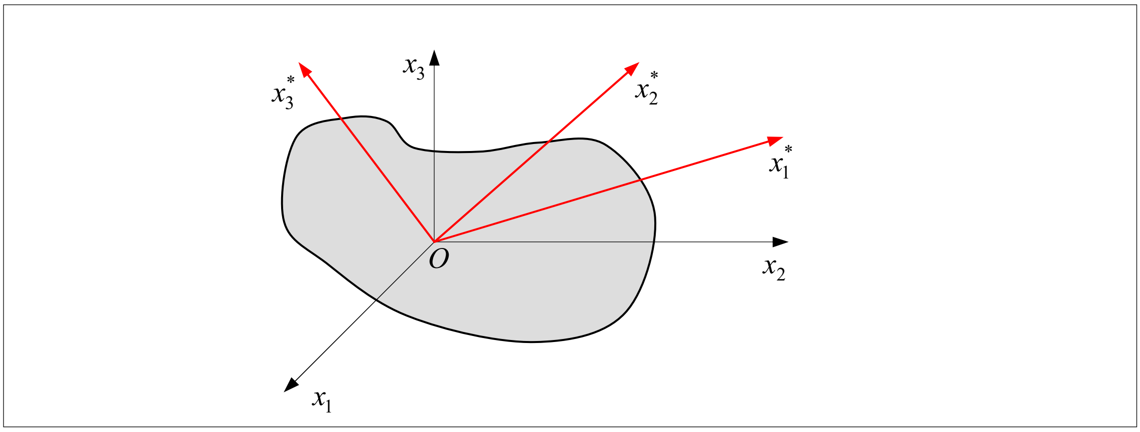


Figure 4.63

Solution:

The inertia pseudo-tensor depends on the adopted coordinate system, and by definition is given by:

$$
\mathbf{I}_O=\int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1}-(\vec{x}\otimes\vec{x})\right]dV \quad ; \quad \mathrm{I}_{Oij}=\int_V \rho\,[x_kx_k\delta_{ij}-x_ix_j]\,dV \qquad [kg\ m^2]
$$

or in components

$$
\mathrm{I}_{ij}=\begin{bmatrix}\int_V \rho\,[x_2^2+x_3^2]\,dV & -\int_V \rho\,[x_1x_2]\,dV & -\int_V \rho\,[x_1x_3]\,dV\\ -\int_V \rho\,[x_1x_2]\,dV & \int_V \rho\,[x_1^2+x_3^2]\,dV & -\int_V \rho\,[x_2x_3]\,dV\\ -\int_V \rho\,[x_1x_3]\,dV & -\int_V \rho\,[x_2x_3]\,dV & \int_V \rho\,[x_1^2+x_2^2]\,dV\end{bmatrix}
$$

a) The inertia tensor gives us the information as the body mass is distributed according to the adopted system.

The term $\int_V \rho[x_1 x_2]\, dV$ indicates how the mass is distributed along the plane $x_1 - x_2$. Then, if the material is homogeneous, i.e. the mass density field is independent of $\vec{\boldsymbol{x}}$, and $x_1 - x_2$ is a plane of symmetry, i.e. the mass is distributed equally with respect to plane $x_1 - x_2$, the term $\int_V \rho[x_1 x_2]\, dV$ is equal to zero. With this, we can conclude that: if the planes $x_1 - x_2$, $x_1 - x_3$, $x_2 - x_3$, are planes of symmetry, the inertia matrix is a diagonal matrix.

Let us consider a student attached to a disc with outstretched arms, each hand holding a weight, (see Figure 4.64(a) – initial system). The disk rotates with angular velocity $\vec{\boldsymbol{\omega}}^{(i)}$ and the inertia tensor according to the system $\vec{\boldsymbol{x}}$ is given by $\mathbf{I}_O^{(i)}$. If we consider a system without energy dissipation, what will it happen when the student moves the arms inwardly as shown in Figure 4.64(b) – final system? As we are dealing with a conservative system, the angular momentum is conserved too, i.e.:

$$\vec{\boldsymbol{H}}_O^{(i)} = \vec{\boldsymbol{H}}_O^{(f)}$$
$$\mathbf{I}_O^{(i)} \cdot \vec{\boldsymbol{\omega}}^{(i)} = \mathbf{I}_O^{(f)} \cdot \vec{\boldsymbol{\omega}}^{(f)}$$

Since for the final system the mass is more concentrated according to the rotation axis than to the initial system the inequality $\mathbf{I}_O^{(f)} < \mathbf{I}_O^{(i)}$ holds and as consequence $\vec{\boldsymbol{\omega}}^{(f)} > \vec{\boldsymbol{\omega}}^{(i)}$.

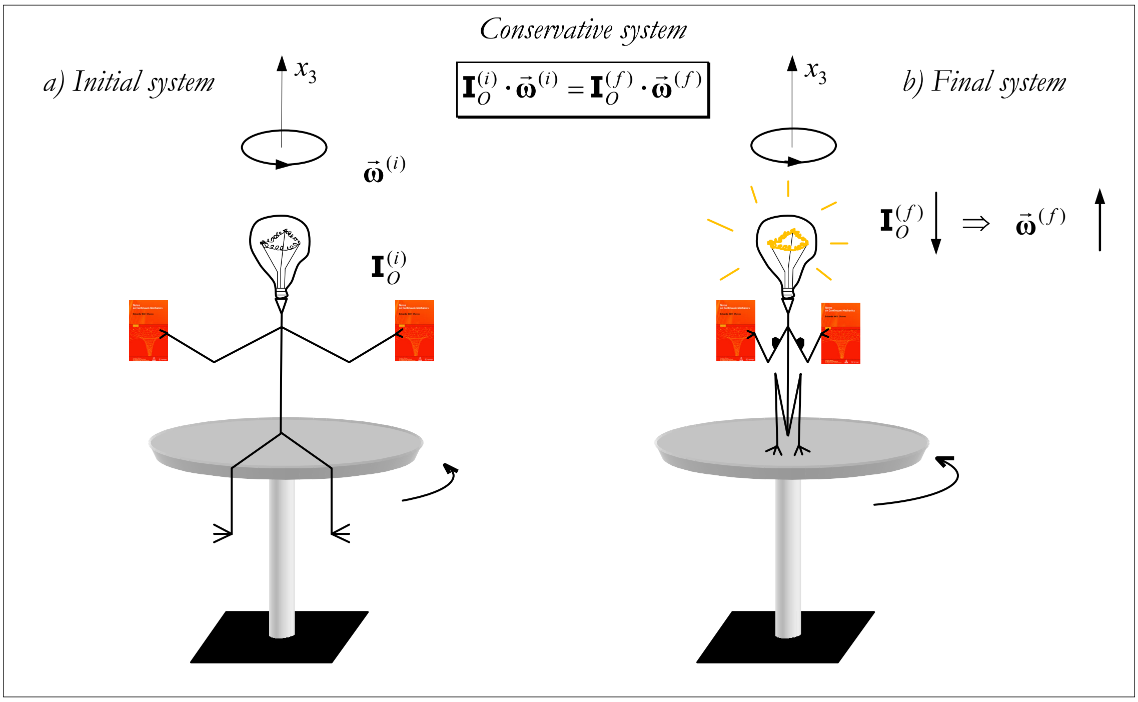


Figure 4.64

b) Let us assume that the given systems, (see Figure 4.63), are related by the transformation law $x_i^* = \mathcal{A}_{ij} x_j$, where $\mathcal{A}_{ij}$ is the orthogonal matrix, then it follows that $x_i = \mathcal{A}_{ji} x_j^*$. Thus, it is possible to express $\mathrm{I}_{O\,ij}$ as follows:

$$
\begin{aligned}
\mathrm{I}_{O\,ij} &= \int_V \rho\left[x_k x_k \delta_{ij} - x_i x_j\right] dV = \int_V \rho\left[(x_k^* x_k^*)\boldsymbol{\mathcal{A}}_{ip}\delta_{pq}\boldsymbol{\mathcal{A}}_{jq} - \boldsymbol{\mathcal{A}}_{ip} x_p^* \boldsymbol{\mathcal{A}}_{jq} x_q^*\right] dV \\
&= \int_V \boldsymbol{\mathcal{A}}_{ip}\left\{\rho\left[(x_k^* x_k^*)\delta_{pq} - x_p^* x_q^*\right]\right\}\boldsymbol{\mathcal{A}}_{jq}\; dV = \boldsymbol{\mathcal{A}}_{ip}\left\{\int_V \rho\left[(x_k^* x_k^*)\delta_{pq} - x_p^* x_q^*\right] dV\right\}\boldsymbol{\mathcal{A}}_{jq} \\
&= \boldsymbol{\mathcal{A}}_{ip}\mathrm{I}^*_{O\,ij}\boldsymbol{\mathcal{A}}_{jq}
\end{aligned}
$$

Note that $x_k x_k = \boldsymbol{\mathcal{A}}_{ks} x_s^* \boldsymbol{\mathcal{A}}_{kt} x_t^* = x_s^* x_t^* \boldsymbol{\mathcal{A}}_{ks}\boldsymbol{\mathcal{A}}_{kt} = x_s^* x_t^* \delta_{st} = x_s^* x_s^* = x_t^* x_t^* = x_k^* x_k^*$.

Abusing a little bit of notation, we also use tensorial notation, but bear in mind that we are working with tensor components, and we are not doing an orthogonal transformation.

$$
\begin{aligned}
\mathbf{I}_O &= \int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right] dV = \int_V \rho\left[(\vec{x}^*\cdot\vec{x}^*)\boldsymbol{\mathcal{A}}^T\cdot\mathbf{1}\cdot\boldsymbol{\mathcal{A}} - (\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*\otimes\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*)\right] dV \\
&= \int_V \rho\left[(\vec{x}^*\cdot\vec{x}^*)\boldsymbol{\mathcal{A}}^T\cdot\mathbf{1}\cdot\boldsymbol{\mathcal{A}} - (\boldsymbol{\mathcal{A}}^T\cdot\vec{x}^*\otimes\vec{x}^*\cdot\boldsymbol{\mathcal{A}})\right] dV \\
&= \int_V \boldsymbol{\mathcal{A}}^T\cdot\left\{\rho\left[(\vec{x}^*\cdot\vec{x}^*)\mathbf{1} - (\vec{x}^*\otimes\vec{x}^*)\right]\right\}\cdot\boldsymbol{\mathcal{A}}\; dV \\
&= \boldsymbol{\mathcal{A}}^T\cdot\left\{\int_V \rho\left[(\vec{x}^*\cdot\vec{x}^*)\mathbf{1} - (\vec{x}^*\otimes\vec{x}^*)\right] dV\right\}\cdot\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}^T\cdot\mathbf{I}_O^*\cdot\boldsymbol{\mathcal{A}}
\end{aligned}
$$

$$
\boxed{\begin{aligned}\mathbf{I}_O &= \boldsymbol{\mathcal{A}}^T\cdot\mathbf{I}_O^*\cdot\boldsymbol{\mathcal{A}} \\ \mathrm{I}_{O\,ij} &= \boldsymbol{\mathcal{A}}_{ip}\mathrm{I}^*_{O\,ij}\boldsymbol{\mathcal{A}}_{jq}\end{aligned}} \quad \textit{Inertia tensor components after a base change (rotation)} \tag{4.163}
$$

Then, it is also true $\mathbf{I}_O^* = \boldsymbol{\mathcal{A}}\cdot\mathbf{I}_O\cdot\boldsymbol{\mathcal{A}}^T$, which are the inertia tensor components in the system $x_1^* x_2^* x_3^*$. Note that the equation (4.163) is the same component transformation law for a second-order tensor, where $\boldsymbol{\mathcal{A}}$ is the transformation matrix from the $x_1 x_2 x_3$-system to $x_1^* x_2^* x_3^*$-system.

c) For a positive definite tensor, by definition, its eigenvalues are greater than zero.

We will start from the kinetic energy obtained in **Problem 4.42**, i.e.:

$$
\mathcal{K}(t) = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left[\bar{\mathcal{I}}_{11}\omega_1^2 + \bar{\mathcal{I}}_{22}\omega_2^2 + \bar{\mathcal{I}}_{33}\omega_3^2 - 2\bar{\mathcal{I}}_{12}\omega_1\omega_2 - 2\bar{\mathcal{I}}_{13}\omega_1\omega_3 - 2\bar{\mathcal{I}}_{23}\omega_2\omega_3\right]
$$

The kinetic energy is a scalar and is always a positive number, and only in two situations the kinetic energy is zero, namely: when there is no mass or when the body is at rest. We adopt a system such that the origin is at the center of mass and the adopted axes are axes of symmetry (inertia principal system) and that the body is rotating around the origin (center of mass). In this situation the kinetic energy becomes:

$$
\mathcal{K}(t) = \frac{1}{2}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}\underbrace{\begin{bmatrix}\mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 \\ 0 & 0 & \mathcal{I}_3\end{bmatrix}}_{\text{Eigenvalues of the Inertia tensor}}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix} = \frac{1}{2}\left[\mathcal{I}_1\omega_1^2 + \mathcal{I}_2\omega_2^2 + \mathcal{I}_3\omega_3^2\right] > 0
$$

In addition, if we have a motion such that $\omega_2 = \omega_3 = 0$, we have $\mathcal{K}(t) = \frac{1}{2}\mathcal{I}_1\omega_1^2$, then, the only way that the kinetic energy is always positive is when $\mathcal{I}_1 > 0$ holds. Similarly, we can conclude that $\mathcal{I}_2 > 0$ and $\mathcal{I}_3 > 0$. Hence, the inertia tensor is a positive definite tensor.

d) As the inertia pseudo-tensor is dependent on the adopted system, for the following situations the inertia tensor related to a solid in motion does not change with time:

1) If the adopted system is attached to the solid.

2) If the solid is rotating along the axis of symmetry, for example, if a cylinder is rotating along the prismatic axis, then during motion the mass distribution is not changing with respect to the adopted system, (see Figure 4.65).

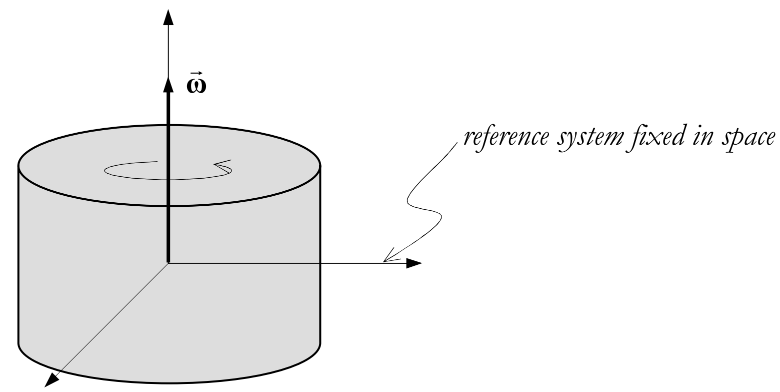


Figure 4.65

Problem 4.44

Consider a homogeneous cylinder of radius r and height $h = 3r$ with total mass equal to m, (see Figure 4.66). Find the inertia tensor for the cylinder related to the system $Ox_1'x_2'x_3'$. The system $Ox_1'x_2'x_3'$ is given by the rotation of the system $Ox_1''x_2''x_3''$ of 45º along the axis x_1''. The systems $Gx_1x_2x_3$ and $Ox_1''x_2''x_3''$ have the same orientation.

Hint: For the reference system $Gx_1x_2x_3$ we know the inertia tensor components and are given by:

$$I_{Gij} = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} = \frac{mr^2}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

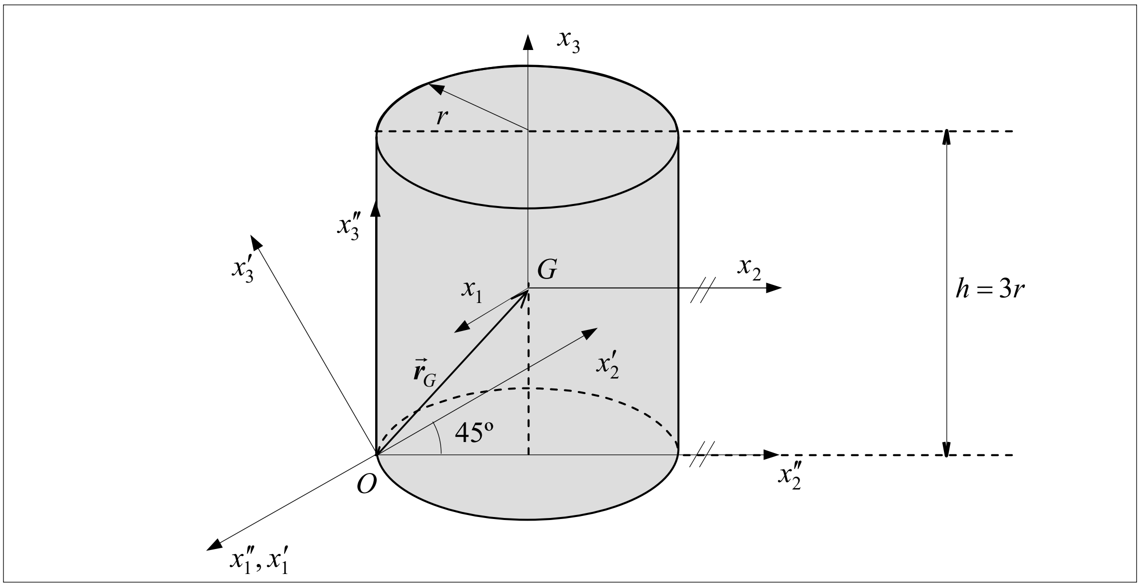


Figure 4.66

Solution:

We can obtain the inertia tensor related to the system $Ox_1''x_2''x_3''$ by means of the Steiner theorem, (see equation (4.158) in **Problem 4.39**). After that, we can obtain the components due to a rotation by means of the equation (4.163), (see **Problem 4.43**).

By means of the equations in (4.159):

$$\mathrm{I}''_{O\,ij} = \begin{bmatrix} \bar{\mathrm{I}}_{11} & \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{13} \\ \bar{\mathrm{I}}_{12} & \bar{\mathrm{I}}_{22} & \bar{\mathrm{I}}_{23} \\ \bar{\mathrm{I}}_{13} & \bar{\mathrm{I}}_{23} & \bar{\mathrm{I}}_{33} \end{bmatrix} + m \begin{bmatrix} \bar{x}_2^2 + \bar{x}_3^2 & -\bar{x}_1\bar{x}_2 & -\bar{x}_1\bar{x}_3 \\ -\bar{x}_1\bar{x}_2 & \bar{x}_1^2 + \bar{x}_3^2 & -\bar{x}_2\bar{x}_3 \\ -\bar{x}_1\bar{x}_3 & -\bar{x}_2\bar{x}_3 & \bar{x}_1^2 + \bar{x}_2^2 \end{bmatrix} \tag{4.164}$$

where $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ are the coordinates of the center of mass with respect to the system $Ox_1''x_2''x_3''$, and by consider the vector $\vec{r}_G = \bar{x}_1\hat{\mathbf{e}}_1'' + \bar{x}_2\hat{\mathbf{e}}_2'' + \bar{x}_3\hat{\mathbf{e}}_3'' = 0\hat{\mathbf{e}}_1'' + r\,\hat{\mathbf{e}}_2'' + \frac{3}{2}r\,\hat{\mathbf{e}}_3''$, we can obtain:

$$\mathrm{I}''_{O\,ij} = \frac{mr^2}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + m \begin{bmatrix} \left[r^2 + (\frac{3}{2}r)^2\right] & 0 & 0 \\ 0 & \left[0^2 + (\frac{3}{2}r)^2\right] & \left[(r)(\frac{3}{2}r)\right] \\ 0 & \left[(r)(\frac{3}{2}r)\right] & \left[0^2 + r^2\right] \end{bmatrix} = \frac{mr^2}{4}\begin{bmatrix} 34 & 0 & 0 \\ 0 & 13 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

Considering the transformation matrix from the $Ox_1''x_2''x_3''$-system to the $Ox_1'x_2'x_3'$-system:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & \sin 45^\circ \\ 0 & -\sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

and by applying the equation (4.163) we can obtain:

$$I'_{O\,ij} = \boldsymbol{\mathcal{A}}^T \mathbf{I}''_O \boldsymbol{\mathcal{A}} = \mathcal{A}_{ip}\mathrm{I}''_{O\,ij}\mathcal{A}_{jq} = \frac{mr^2}{8}\begin{bmatrix} 34 & 0 & 0 \\ 0 & 7 & -7 \\ 0 & -7 & 31 \end{bmatrix}$$

Problem 4.45

Taking into account the angular momentum $\vec{H}_O = m\,\vec{\bar{x}} \wedge \vec{\bar{v}} + \mathbf{I}\cdot\vec{\omega} = m\,\vec{\bar{x}} \wedge \vec{\bar{v}} + \vec{H}_G$, (see **Problem 4.39**),find the rate of change of the angular momentum in such a way that we do not need to calculate at each instant of time the inertia tensor.

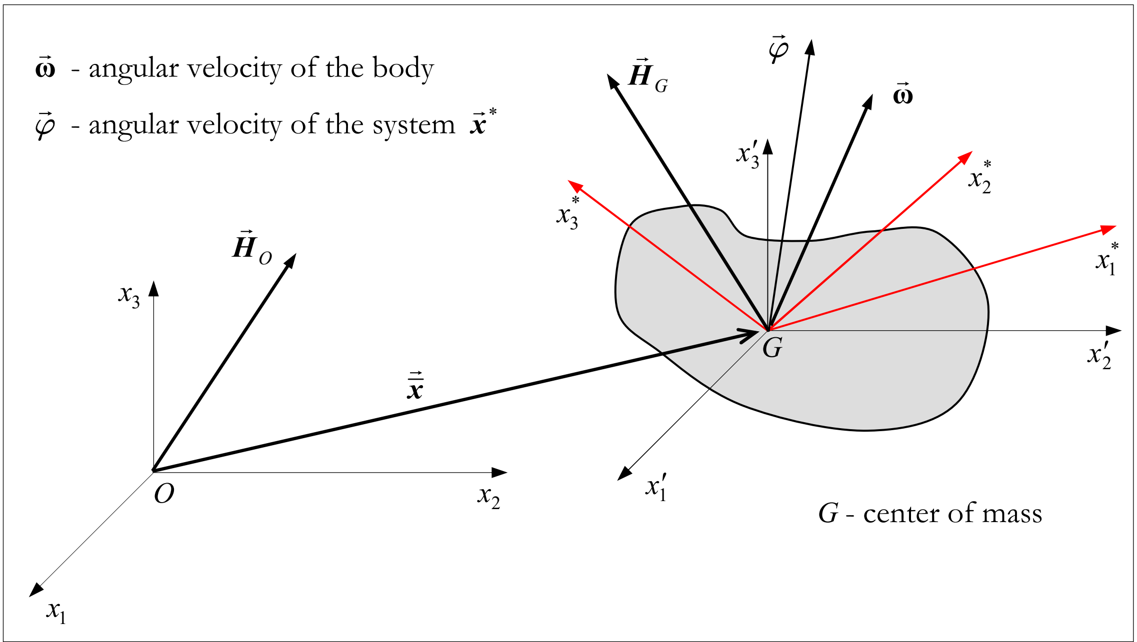


Figure 4.67

Solution: Applying the material time derivative we can obtain:

$$\frac{D(\vec{H}_O)}{Dt} \equiv \dot{\vec{H}}_O = \frac{D}{Dt}\left[m\,\vec{\bar{x}} \wedge \vec{\bar{v}} + \vec{H}_G\right] = \frac{D}{Dt}\left[m\,\vec{\bar{x}} \wedge \vec{\bar{v}}\right] + \frac{D}{Dt}\left[\vec{H}_G\right] = m\frac{D\vec{\bar{x}}}{Dt} \wedge \vec{\bar{v}} + m\,\vec{\bar{x}} \wedge \frac{D\vec{\bar{v}}}{Dt} + \dot{\vec{H}}_G$$

$$= m\,\underbrace{\vec{\bar{v}} \wedge \vec{\bar{v}}}_{=\vec{\mathbf{0}}} + m\,\vec{\bar{x}} \wedge \vec{\bar{a}} + \dot{\vec{H}}_G$$

Thus,

$$\boxed{\frac{D(\vec{H}_O)}{Dt} \equiv \dot{\vec{H}}_O = m\,\vec{\bar{x}} \wedge \vec{\bar{a}} + \dot{\vec{H}}_G} \qquad (4.165)$$

where $\vec{\bar{a}}$ is the acceleration of the center of mass. Next, we will discuss the term $\dot{\vec{H}}_G$. We adopt the mobile system $x_1'x_2'x_3'$ but with fixed orientation in space which is parallel to the fixed system $x_1x_2x_3$, (see Figure 4.67). By expressing the components of $\mathbf{I}$ and $\vec{\omega}$ in the system $x_1'x_2'x_3'$, we can obtain:

$$\vec{H}_G' = \mathbf{I}'\cdot\vec{\omega}' \quad \xrightarrow{\text{rate of change}} \quad \frac{D(\vec{H}_G')}{Dt} \equiv \dot{\vec{H}}_G' = \dot{\mathbf{I}}'\cdot\vec{\omega}' + \mathbf{I}'\cdot\dot{\vec{\omega}}'$$

Note that, as the solid is rotating with respect to the system $\vec{x}'$ the inertia tensor changes, since the mass distribution is changing with respect to the system $\vec{x}'$. Then, at each time step we have to calculate the inertia tensor. This procedure is very laborious. To solve this problem, we adopt a new system $\vec{x}^*$, which has origin at the center of mass, (see Figure 4.67). By means of the component transformation law, the following is true:

$$\text{(components)}\quad\begin{cases}\vec{\boldsymbol{H}}_G^* = \boldsymbol{\mathcal{A}} \cdot \vec{\boldsymbol{H}}'_G & ; \quad \vec{\boldsymbol{H}}'_G = \boldsymbol{\mathcal{A}}^T \cdot \vec{\boldsymbol{H}}_G^* \\ \vec{\boldsymbol{\omega}}^* = \boldsymbol{\mathcal{A}} \cdot \vec{\boldsymbol{\omega}}' & ; \quad \vec{\boldsymbol{\omega}}' = \boldsymbol{\mathcal{A}}^T \cdot \vec{\boldsymbol{\omega}}^* \\ \overline{\mathbf{I}}_O^* = \boldsymbol{\mathcal{A}} \cdot \overline{\mathbf{I}}'_O \cdot \boldsymbol{\mathcal{A}}^T & ; \quad \overline{\mathbf{I}}'_O = \boldsymbol{\mathcal{A}}^T \cdot \overline{\mathbf{I}}_O^* \cdot \boldsymbol{\mathcal{A}}\end{cases}$$

where $\boldsymbol{\mathcal{A}}$ is the transformation matrix from the $\vec{\boldsymbol{x}}'$-system to $\vec{\boldsymbol{x}}^*$-system.

The rate of change of $\vec{\boldsymbol{H}}'_G = \boldsymbol{\mathcal{A}}^T \cdot \vec{\boldsymbol{H}}_G^*$ becomes:

$$\frac{D}{Dt}\vec{\boldsymbol{H}}'_G \equiv \dot{\vec{\boldsymbol{H}}}'_G = \frac{D}{Dt}\left[\boldsymbol{\mathcal{A}}^T \cdot \vec{\boldsymbol{H}}_G^*\right] = \dot{\boldsymbol{\mathcal{A}}}^T \cdot \vec{\boldsymbol{H}}_G^* + \boldsymbol{\mathcal{A}}^T \cdot \dot{\vec{\boldsymbol{H}}}_G^* \tag{4.166}$$

By analogy with the rate of change of the orthogonal tensor, (see Chapter of the textbook), we can conclude that $\Omega = \dot{\boldsymbol{\mathcal{A}}} \cdot \boldsymbol{\mathcal{A}}^T \Rightarrow \dot{\boldsymbol{\mathcal{A}}}^T = \boldsymbol{\mathcal{A}}^T \cdot \Omega^T$, where Ω^T is the antisymmetric tensor and represents the rate of change of rotation of the system $\vec{\boldsymbol{x}}^*$ with respect to the system $\vec{\boldsymbol{x}}'$. Then, we can express (4.166) as follows:

$$\dot{\vec{\boldsymbol{H}}}'_G = \boldsymbol{\mathcal{A}}^T \cdot \Omega^T \cdot \vec{\boldsymbol{H}}_G^* + \boldsymbol{\mathcal{A}}^T \cdot \dot{\vec{\boldsymbol{H}}}_G^* = \boldsymbol{\mathcal{A}}^T \cdot \left[\Omega^T \cdot \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right] \text{ (components)} \tag{4.167}$$

Resorting to the antisymmetric tensor property such that $\Omega^T \cdot \vec{\boldsymbol{H}}_G^* = \vec{\varphi} \wedge \vec{\boldsymbol{H}}_G^*$, where $\vec{\varphi}$ is the axial vector associated with the antisymmetric tensor Ω^T, i.e. $\vec{\varphi} = \vec{\varphi}(t)$ is the angular velocity of the rotating system $\vec{\boldsymbol{x}}^*$. Proving that (4.167) can still be written as follows:

$$\boxed{\dot{\vec{\boldsymbol{H}}}'_G = \boldsymbol{\mathcal{A}}^T \cdot \left[\Omega^T \cdot \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right] = \boldsymbol{\mathcal{A}}^T \cdot \left[\vec{\varphi}^* \wedge \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right]} \text{ (components)} \tag{4.168}$$

where

$$\dot{\vec{\boldsymbol{H}}}_G^* = \frac{D}{Dt}\left[\overline{\mathbf{I}}^* \cdot \vec{\boldsymbol{\omega}}^*\right] = \frac{D\overline{\mathbf{I}}^*}{Dt} \cdot \vec{\boldsymbol{\omega}}^* + \overline{\mathbf{I}}^* \cdot \frac{D\vec{\boldsymbol{\omega}}^*}{Dt}$$

The term $\dfrac{D\overline{\mathbf{I}}^*}{Dt}$ is equal to zero when one of the two possibilities holds:

1) $\dfrac{D\overline{\mathbf{I}}^*}{Dt} = \mathbf{0}$ if the system $\vec{\boldsymbol{x}}^*$ is attached to the solid. In this case, the equation $\vec{\varphi} = \vec{\boldsymbol{\omega}}$ holds, i.e. the mobile system velocity is equal to the angular velocity of the solid.

2) $\dfrac{D\overline{\mathbf{I}}^*}{Dt} = \mathbf{0}$ if the solid rotates around a prismatic axis, (see Figure 4.65 in **Problem 4.43**).

NOTE 1: The equation in (4.168) can be rewritten as follows:

$$\begin{aligned}&\dot{\vec{\boldsymbol{H}}}'_G = \boldsymbol{\mathcal{A}}^T \cdot \left[\vec{\varphi}^* \wedge \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right] \\ &\Rightarrow \quad \boldsymbol{\mathcal{A}} \cdot \dot{\vec{\boldsymbol{H}}}'_G = \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{A}}^T \cdot \left[\vec{\varphi}^* \wedge \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right] = \left[\vec{\varphi}^* \wedge \vec{\boldsymbol{H}}_G^* + \dot{\vec{\boldsymbol{H}}}_G^*\right]\end{aligned} \quad \text{(components)} \tag{4.169}$$

Note that the term $\boldsymbol{\mathcal{A}} \cdot \dot{\vec{\boldsymbol{H}}}'_G$ are the components of $\dot{\vec{\boldsymbol{H}}}'_G$ in the system $\vec{\boldsymbol{x}}^*$, and note also that $\boldsymbol{\mathcal{A}} \cdot \dot{\vec{\boldsymbol{H}}}'_G \neq \dot{\vec{\boldsymbol{H}}}_G^*$, then:

$$\left[\boldsymbol{\mathcal{A}} \cdot \dot{\vec{\boldsymbol{H}}}'_G\right]^* = \dot{\vec{\boldsymbol{H}}}_G^* + \vec{\varphi}^* \wedge \vec{\boldsymbol{H}}_G^* \quad \text{(components)} \tag{4.170}$$

we can also express the above equation in tensorial notation:

$$\boxed{\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_f = \left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_r + \vec{\varphi}\wedge\vec{\boldsymbol{H}}_G} \quad \text{(tensorial notation)} \tag{4.171}$$

where $\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_f$ represents the rate of change of $\vec{\boldsymbol{H}}_G$ with respect to the fixed system, $\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_r$ represents the rate of change of $\vec{\boldsymbol{H}}_G$ with respect to the rotating system with an angular velocity $\vec{\varphi}$.

NOTE 2: The equation in (4.171) is valid for any vector, (see Figure 4.68), i.e. the rate of change of the vector $\vec{\boldsymbol{b}}$ respect to the fixed system $\vec{\boldsymbol{x}}'$ is equal to the rate of change of the vector $\vec{\boldsymbol{b}}$ respect to the rotating system $\vec{\boldsymbol{x}}^*$ plus the vector product between angular velocity of the system ($\vec{\varphi}$ which is associated to the antisimetric tensor Ω^T) and the vector $\vec{\boldsymbol{b}}$:

$$\boxed{\left(\frac{D\vec{\boldsymbol{b}}}{Dt}\right)_{fixed} = \left(\frac{D\vec{\boldsymbol{b}}}{Dt}\right)_{rotating} + \Omega^T\cdot\vec{\boldsymbol{b}} = \left(\frac{D\vec{\boldsymbol{b}}}{Dt}\right)_{rotating} + \vec{\varphi}\wedge\vec{\boldsymbol{b}}} \tag{4.172}$$

Note also that $\left(\frac{D\vec{\varphi}}{Dt}\right)_f = \left(\frac{D\vec{\varphi}}{Dt}\right)_r + \underbrace{\vec{\varphi}\wedge\vec{\varphi}}_{=\vec{\mathbf{0}}} = \left(\frac{D\vec{\varphi}}{Dt}\right)_r$.

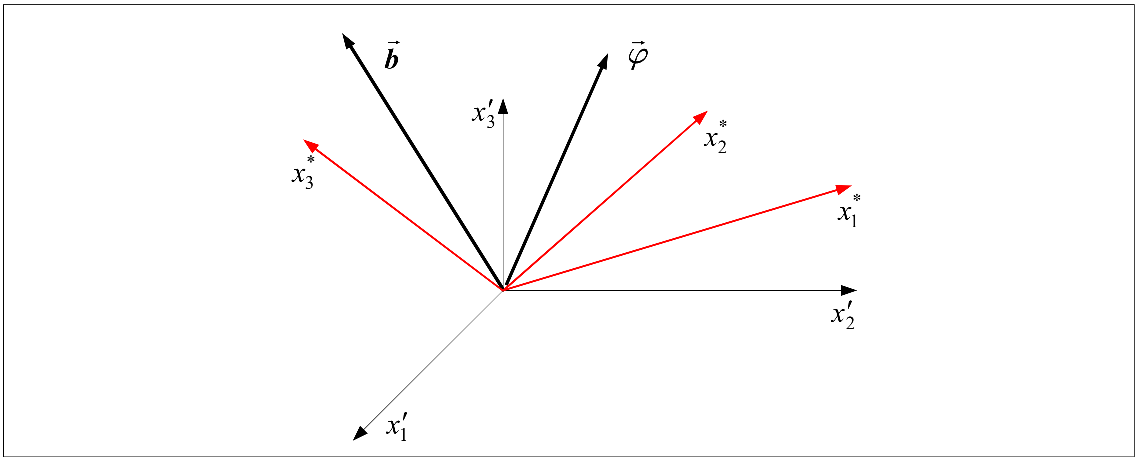


Figure 4.68

NOTE 3: Note that the equation (4.172) is the convective rate, (see Chapter on The Objectivity of Tensors in the textbook), which is defined by $\boxed{\overset{C}{\mathbf{a}} = \dot{\mathbf{a}} + \boldsymbol{\ell}^T\cdot\mathbf{a}}$, where $\boldsymbol{\ell} = \mathbf{D}+\mathbf{W}$, then $\overset{C}{\mathbf{a}} = \dot{\mathbf{a}} + \boldsymbol{\ell}^T\cdot\mathbf{a} = \dot{\mathbf{a}} + (\mathbf{D}+\mathbf{W})^T\cdot\mathbf{a}$. Recall from Chapter 2 (Chaves (2013)) that $\mathbf{W} = \frac{1}{2}\mathbf{R}\cdot\left[\dot{\mathbf{U}}\cdot\mathbf{U}^{-1} - \mathbf{U}^{-1}\cdot\dot{\mathbf{U}}\right]\cdot\mathbf{R}^T + \dot{\mathbf{R}}\cdot\mathbf{R}^T$ holds. And if we are considering rigid solid motion we have $\mathbf{D}=\mathbf{0}$, $\dot{\mathbf{U}}=\mathbf{0}$, and $\mathbf{W}=\Omega=\dot{\mathbf{R}}\cdot\mathbf{R}^T$, with that we obtain $\overset{C}{\mathbf{a}} = \dot{\mathbf{a}} + \Omega^T\cdot\mathbf{a}$.

NOTE 4: Let us expose a simple example to obtain $\mathit{\Omega}^T$. Let us assume that the $\hat{\mathbf{e}}_i$-system is rotating according to the $\hat{\hat{\mathbf{e}}}_i$-system, (see Figure 4.69), and to obtain $\mathit{\Omega}^T$ we procedure as follows. The transformation matrix from $\hat{\hat{\mathbf{e}}}_i$ to $\hat{\mathbf{e}}_i$ is given by:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.173}$$

$$\frac{d(\boldsymbol{\mathcal{A}})}{dt} \equiv \dot{\boldsymbol{\mathcal{A}}} = \begin{bmatrix} \dfrac{d(\cos\theta)}{dt} & \dfrac{d(\sin\theta)}{dt} & 0 \\ -\dfrac{d(\sin\theta)}{dt} & \dfrac{d(\cos\theta)}{dt} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta & 0 \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\theta}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathit{\Omega} = \dot{\boldsymbol{\mathcal{A}}} \cdot \boldsymbol{\mathcal{A}}^T = \dot{\theta}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\theta}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathit{\Omega}^T = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_3 & 0 \end{bmatrix} \quad \Rightarrow \quad \varphi_i = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

where $\vec{\varphi}$ is the axial vector associated with the antisymmetric tensor $\mathit{\Omega}^T$.

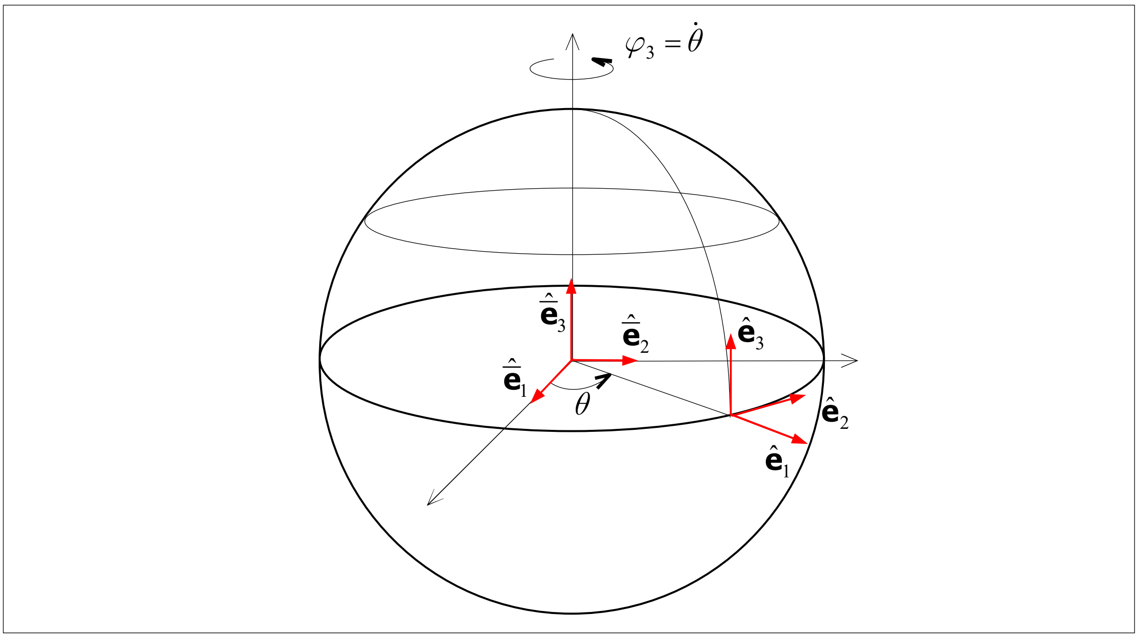


Figure 4.69

NOTE 5: Inertial forces

Let us consider the system $OX_1X_2X_3$, (see Figure 4.70), which is fixed in space. This system is denoted by *inertial reference frame*. To this system the Newton's law is applied, and if there is a falling body it is true that:

$$\vec{F} = m\vec{A}$$

Let us consider also that an observer (attached to the system $ox_1x_2x_3$) is moving (for simplicity's sake we will just consider translation). Since the system $ox_1x_2x_3$ is moving we denote it by *non-inertial reference frame*. By means of vector summation, (see Figure 4.70), we can obtain:

$$\vec{X} = \vec{\mathbf{c}} + \vec{x}$$

The material time derivative of the above equation becomes:

$$\dot{\vec{X}} = \dot{\vec{\mathbf{c}}} + \dot{\vec{x}} \quad \xrightarrow{\frac{D}{Dt}} \quad \ddot{\vec{X}} = \ddot{\vec{\mathbf{c}}} + \ddot{\vec{x}} \quad \Rightarrow \quad \vec{A} = \ddot{\vec{\mathbf{c}}} + \ddot{\vec{x}}$$

and if we multiply by mass (m) we can obtain:

$$m\vec{A} = m\ddot{\vec{\mathbf{c}}} + m\ddot{\vec{x}} \quad \Rightarrow \quad m\ddot{\vec{x}} = m\vec{A} - m\ddot{\vec{\mathbf{c}}} = \vec{F} - m\ddot{\vec{\mathbf{c}}} \quad \Rightarrow \quad m\vec{a} = \vec{F} - m\ddot{\vec{\mathbf{c}}}$$

Note that, for the observer it appears the additional force $(-m\ddot{\vec{\mathbf{c}}})$ to the "Newton's law". This additional force is a fictitious force or pseudo force which is denoted by *inertial force*. In addition, inertial forces appear if the observer's system is rotating, e.g. centripetal force.

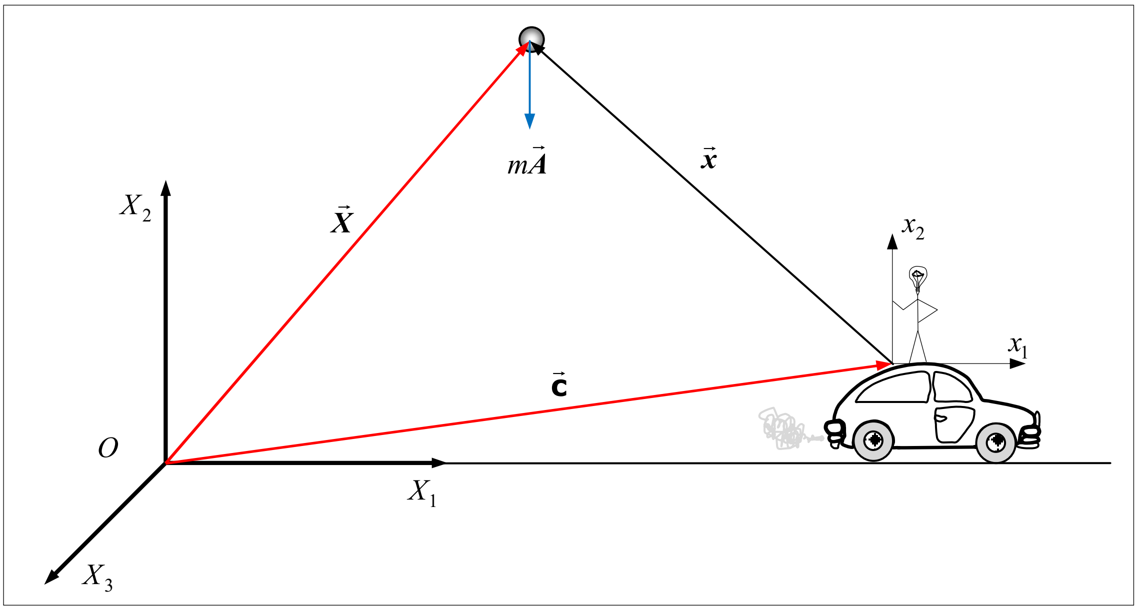


Figure 4.70

Problem 4.46

Show that the acceleration at a fixed system $\vec{a}_f$ can be expressed as:

$$\boxed{\vec{a}_f = \vec{a}_r + 2(\vec{\omega} \wedge \vec{v}_r) + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})} \tag{4.174}$$

where $\vec{a}_r$ and $\vec{v}_r$ are, respectively, the acceleration and the velocity of a particle with respect to an observer that is rotating with the system $\vec{x}^*$, (see Figure 4.68). Consider also that $\dot{\vec{\varphi}} = \vec{\omega}$ is the angular velocity of the system $\vec{x}^*$, which is constant with time.

Solution:

We use directly the equation in (4.172) to obtain the velocity:

$$\left(\frac{D\vec{x}}{Dt}\right)_f = \left(\frac{D\vec{x}}{Dt}\right)_r + \vec{\omega} \wedge \vec{x} \qquad \Rightarrow \qquad \vec{v}_f = \vec{v}_r + \vec{\omega} \wedge \vec{x}$$

We apply the same definition to the above equation in order to obtain the acceleration, i.e.:

$$\begin{aligned}
\left[\frac{D\vec{v}_f}{Dt}\right]_f &= \left[\frac{D[\vec{v}_r + \vec{\omega} \wedge \vec{x}]}{Dt}\right]_f = \left[\frac{D[\vec{v}_r + \vec{\omega} \wedge \vec{x}]}{Dt}\right]_r + \vec{\omega} \wedge [\vec{v}_r + \vec{\omega} \wedge \vec{x}] \\
\vec{a}_f &= \left[\frac{D\vec{v}_r}{Dt}\right]_r + \left[\frac{D[\vec{\omega} \wedge \vec{x}]}{Dt}\right]_r + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) \\
\vec{a}_f &= \left[\frac{D\vec{v}_r}{Dt}\right]_r + \left[\frac{D\vec{\omega}}{Dt}\right]_r \wedge \vec{x} + \vec{\omega} \wedge \left[\frac{D\vec{x}}{Dt}\right]_r + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) \\
\vec{a}_f &= \vec{a}_r + \dot{\vec{\omega}} \wedge \vec{x} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) \\
\vec{a}_f &= \vec{a}_r + \dot{\vec{\omega}} \wedge \vec{x} + 2(\vec{\omega} \wedge \vec{v}_r) + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})
\end{aligned}$$

As we are assuming angular velocity constant $\dot{\vec{\omega}} = \vec{0}$, i.e. the angular acceleration is zero, with that we obtain the equation in (4.174). Then, we can conclude:

$$\boxed{\vec{a}_f = \vec{a}_r + \dot{\vec{\omega}} \wedge \vec{x} + 2(\vec{\omega} \wedge \vec{v}_r) + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})} \tag{4.175}$$

Note that to obtain the above equation we have not used any principle of conservation. The above equation is just relating the acceleration in a fixed system in function of parameters defined in the rotating system.

NOTE 1: Using the identity $\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$, (see **Problem 1.17**), we can conclude that $\vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) = (\vec{\omega} \cdot \vec{x})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{x} = (\vec{\omega} \cdot \vec{x})\vec{\omega} - \|\vec{\omega}\|^2 \vec{x}$. Note that, if $\vec{\omega} = \omega_3 \hat{\mathbf{e}}_3$, (see Figure 4.71), and also if we adopt the system ($\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_3$) and taking into account that $\vec{\omega} \cdot \vec{r} = 0$ we can obtain the following equation $\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}) = (\vec{\omega} \cdot \vec{r})\vec{\omega} - \|\vec{\omega}\|^2 \vec{r} = -\|\vec{\omega}\|^2 \vec{r}$, which is the centripetal acceleration, (see **Problem 2.57**). Earth rotates at a rate $\omega_3 = 2\pi \frac{rad}{day} = \frac{2\pi}{86400} \frac{rad}{s} \approx 0.727 \times 10^{-4} \frac{rad}{s}$. Note that the term $\vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})$ is very small compared with the term $2(\vec{\omega} \wedge \vec{v}_r)$.

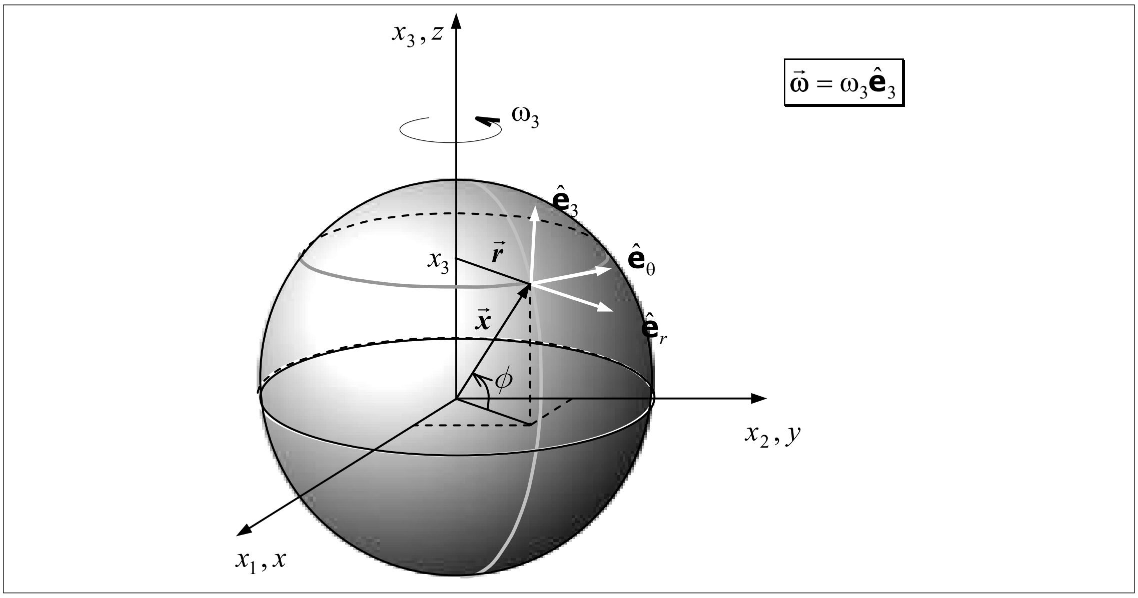


Figure 4.71

NOTE 2: The term $2(\vec{\omega} \wedge \vec{v}_r)$ was established by Gustave-Gaspard Coriolis in 1835, and is associated with the fictitious force called *Coriolis force*. Next, we will represent $2(\vec{\omega} \wedge \vec{v}_r)$ in the system $\hat{\mathbf{e}}'_i$, (see Figure 4.71).

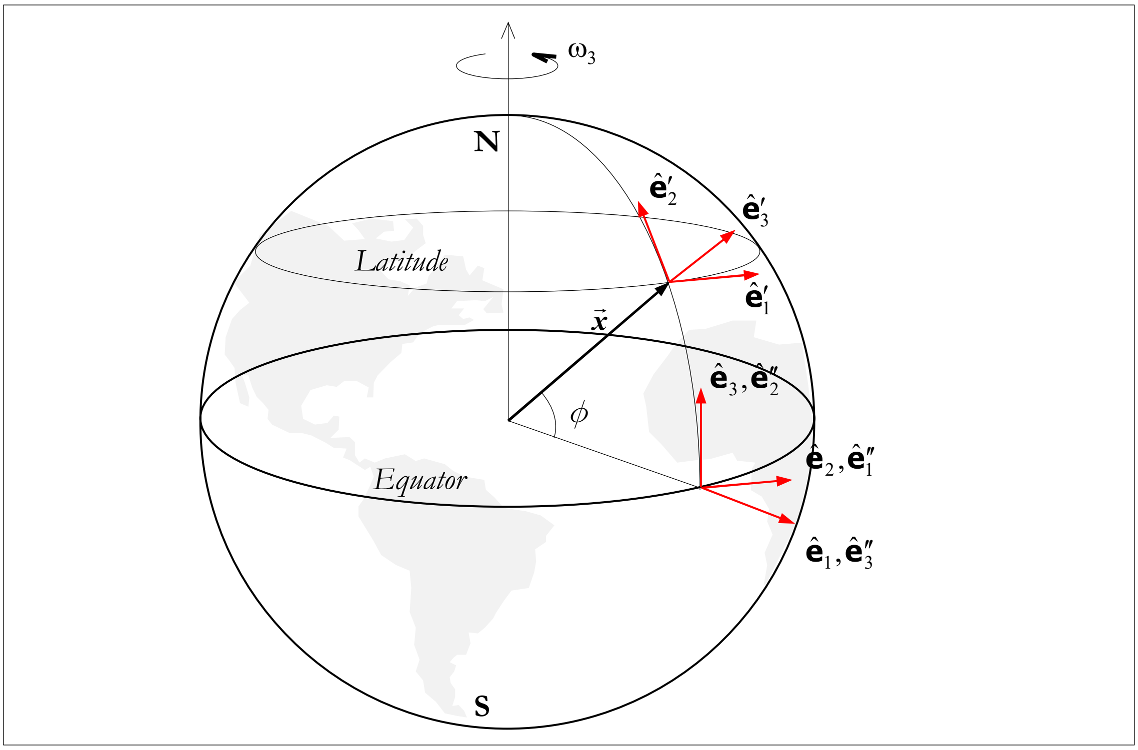


Figure 4.72

The transformation law from $\hat{\mathbf{e}}_i$ to $\hat{\mathbf{e}}'_i$ is given by:

$$\begin{Bmatrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}'_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix} \quad \Rightarrow \quad \mathcal{B} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \tag{4.176}$$

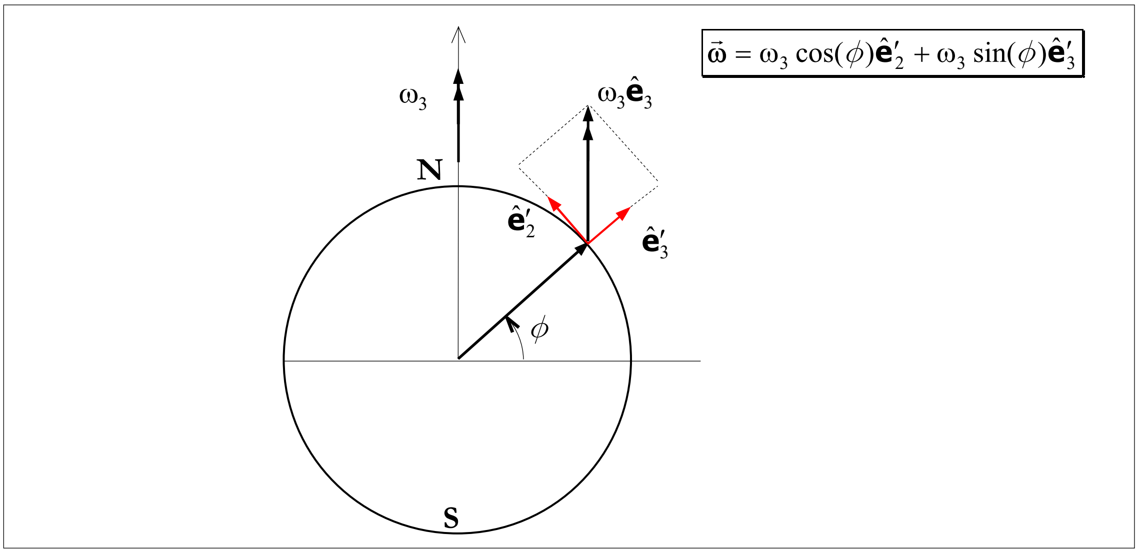


Figure 4.73

The term $2(\vec{\omega} \wedge \vec{v}_r)$ can be obtained as follows:

$$
\begin{aligned}
2(\vec{\omega} \wedge \vec{v}_r) &= \begin{vmatrix} \hat{\mathbf{e}}'_1 & \hat{\mathbf{e}}'_2 & \hat{\mathbf{e}}'_3 \\ 0 & \omega_3 \cos(\phi) & \omega_3 \sin(\phi) \\ v_{r1} & v_{r2} & v_{r3} \end{vmatrix} \\
&= 2\hat{\mathbf{e}}'_1[\omega_3 \cos(\phi) v_{r3} - \omega_3 \sin(\phi) v_{r2}] - 2\hat{\mathbf{e}}'_2[-\omega_3 \sin(\phi) v_{r1}] + 2\hat{\mathbf{e}}'_3[-\omega_3 \cos(\phi) v_{r1}] \\
&= 2[\omega_3 \cos(\phi) v_{r3} - \omega_3 \sin(\phi) v_{r2}]\hat{\mathbf{e}}'_1 + 2[\omega_3 \sin(\phi) v_{r1}]\hat{\mathbf{e}}'_2 - 2[\omega_3 \cos(\phi) v_{r1}]\hat{\mathbf{e}}'_3
\end{aligned}
\tag{4.177}
$$

The term $f = 2\omega_3 \sin(\phi)$ is known as *Coriolis parameter*. To small value of v_{r3} the above equation reduce to:

$$
\left(\frac{D\vec{v}_r}{Dt}\right)_r = -2(\vec{\omega} \wedge \vec{v}_r) = [2\omega_3 \sin(\phi) v_{r2}]\hat{\mathbf{e}}'_1 + [-2\omega_3 \sin(\phi) v_{r1}]\hat{\mathbf{e}}'_2 = [f\, v_{r2}]\hat{\mathbf{e}}'_1 + [-f\, v_{r1}]\hat{\mathbf{e}}'_2
$$

$$
\Rightarrow \begin{cases} \dfrac{Dv_{r1}}{Dt} = f\, v_{r2} \\ \dfrac{Dv_{r2}}{Dt} = -f\, v_{r1} \end{cases}
$$

Figure 4.74: Coriolis effect (**Ref.:** Wikipedia "Coriolis effect").

NOTE 3: Deflection of vertically falling body

A very simple application of the Coriolis effect is presented next. Let us consider an observer on the surface of the Earth. Let us consider also that a body of mass m is free-falling from rest with the following initial conditions: at $t=0$. $(x'_3 = h)$, $(x'_1 = 0)$, $(\frac{d}{dt}x'_3 = v_3 = 0)$, $(v_1 = 0)$, $(v_2 = 0)$. As the body is falling we will calculate the deflection of the body, i.e. we will obtain x'_1 related to the observer which is attached to a system which is rotating with the Earth. We will adopt the system used in Figure 4.72.

The Newton's Second Law ($\vec{F} = m\vec{a}_f$) (apply to an inertial reference frame), then

$$\vec{F} = m\left[\vec{a}_r + 2(\vec{\omega}\wedge\vec{v}_r) + \vec{\omega}\wedge(\vec{\omega}\wedge\vec{x})\right] \quad\Rightarrow\quad m\vec{a}_r = \vec{F} - 2m(\vec{\omega}\wedge\vec{v}_r) = -mg\hat{\mathbf{e}}'_3 - 2m(\vec{\omega}\wedge\vec{v}_r)$$

$$\Rightarrow \vec{a}_r = -g\hat{\mathbf{e}}' - 2(\vec{\omega}\wedge\vec{v}_r) \quad\Rightarrow\quad (\vec{a}_r)_i = \begin{Bmatrix} -2\left[\omega_3\cos(\phi)v_{r3} - \omega_3\sin(\phi)v_{r2}\right] \\ -2\left[\omega_3\sin(\phi)v_{r1}\right] \\ 2\left[\omega_3\cos(\phi)v_{r1}\right] - g \end{Bmatrix} = \begin{Bmatrix} -2\omega_3\cos(\phi)v_{r3} \\ 0 \\ -g \end{Bmatrix}$$

where the acceleration $\vec{a}_f$ is given by (4.175), and we are considering that the term $\vec{\omega}\wedge(\vec{\omega}\wedge\vec{x})$ is very small when compared with the term $2(\vec{\omega}\wedge\vec{v}_r)$ whose components are given by (4.177). Then

$$(\vec{a}_r)_i = \begin{Bmatrix} a_{r1} \\ a_{r1} \\ a_{r1} \end{Bmatrix} = \begin{Bmatrix} \dfrac{d^2x'_1}{dt^2} \\ \dfrac{d^2x'_2}{dt^2} \\ \dfrac{d^2x'_3}{dt^2} \end{Bmatrix} = \begin{Bmatrix} -2\omega_3\cos(\phi)v_{r3} \\ 0 \\ -g \end{Bmatrix} \tag{4.178}$$

Note that

$$\frac{d^2x'_3}{dt^2} = -g \xrightarrow{\text{integrating}} \frac{dx'_3}{dt} = -gt + C_1 \quad\Rightarrow\quad v_{r3} = -gt$$

$$\Rightarrow \frac{dx'_3}{dt} = -gt \xrightarrow{\text{integrating}} x'_3 = -g\frac{t^2}{2} + C_2 \quad\Rightarrow\quad x'_3 = -g\frac{t^2}{2} + h$$

where we have considered the initial conditions, i.e. at $t=0 \Rightarrow (v_{r3}=0) \quad\Rightarrow\quad C_1 = 0$, and $t=0 \Rightarrow (x'_3 = h) \quad\Rightarrow\quad C_2 = h$. Note that $x'_3 = -g\frac{t^2}{2} + h = 0 \Rightarrow h = \frac{gt^2}{2}$.

Considering the equation $v_{r3} = -gt$ into the first component of (4.178) we can obtain:

$$\frac{d^2x'_1}{dt^2} = -2\omega_3\cos(\phi)v_{r3} = 2\omega_3 gt\cos(\phi) \xrightarrow{\text{integrating}} \frac{dx'_1}{dt} = 2\omega_3 g\cos(\phi)\frac{t^2}{2} + C_1 = v_{r1}$$

where the constant of integration is obtained with the initial condition

$at(t=0) \Rightarrow \{v'_{r1} = 0 \quad\Rightarrow\quad C_1 = 0$

$$\frac{dx'_1}{dt} = v_{r1} = \omega_3 g\cos(\phi)t^2 \xrightarrow{\text{integrating}} x'_1 = \omega_3 g\cos(\phi)\frac{t^3}{3} + C_2$$

Note also that $C_2 = 0$, so, the above equation becomes: $x'_1 = \frac{1}{3}\omega_3 g\cos(\phi)t^3$.

As the body is falling from height h we can state that $h=\frac{1}{2}gt^2 \Rightarrow t=\sqrt{\frac{2h}{g}}$, with that the above equation becomes:

$$x'_1=\frac{1}{3}\omega_3 g\cos(\phi)t^3=\frac{\omega_3 g}{3}\left(\frac{2h}{g}\right)^{\frac{3}{2}}\cos(\phi)$$

NOTE 4: ***Acceleration due to sphericity***

Local system $\hat{\mathbf{e}}'_1$ (east)- $\hat{\mathbf{e}}'_2$ (north)- $\hat{\mathbf{e}}'_3$ (radially upward)

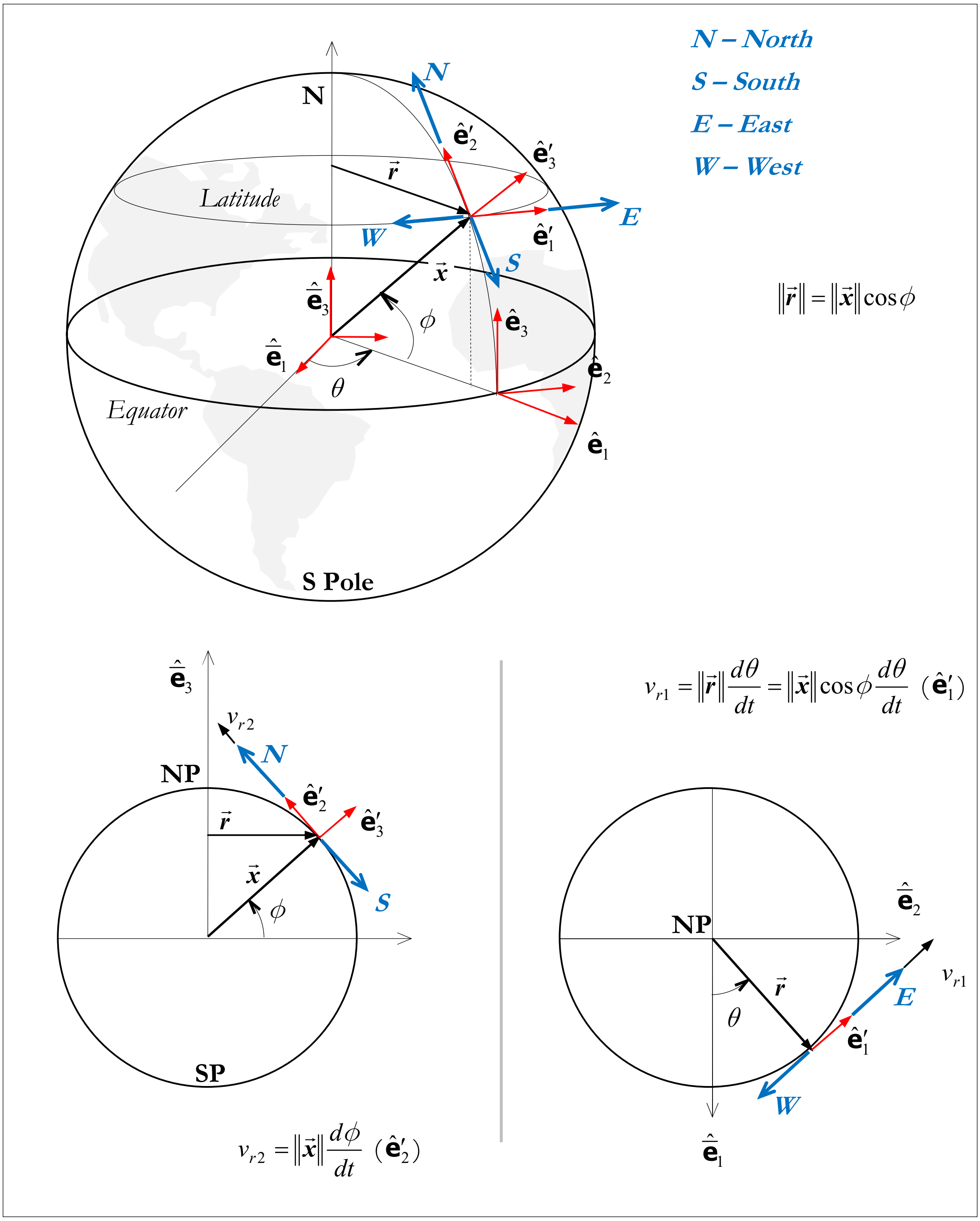


Figure 4.75

Previously we have obtained the transformation matrix from $\hat{\hat{\mathbf{e}}}_i$ to $\hat{\mathbf{e}}_i$, (see equation (4.173)):

$$\mathcal{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.179}$$

and the transformation matrix from $\hat{\mathbf{e}}_i$ to $\hat{\mathbf{e}}'_i$, (see equation (4.176)), is given by:

$$\mathcal{B} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \tag{4.180}$$

Then the transformation matrix from $\hat{\hat{\mathbf{e}}}_i$ to $\hat{\mathbf{e}}'_i$ is given by:

$$\mathcal{C} = \mathcal{B}\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix}$$

The rate of change of $\mathcal{C}$ is given by:

$$\frac{d(\mathcal{C})}{dt} \equiv \dot{\mathcal{C}} = \begin{bmatrix} -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ (-\dot{\phi}\cos\phi\cos\theta + \dot{\theta}\sin\phi\sin\theta) & (-\dot{\phi}\cos\phi\sin\theta - \dot{\theta}\sin\phi\cos\theta) & -\dot{\phi}\sin\phi \\ (-\dot{\phi}\sin\phi\cos\theta - \dot{\theta}\cos\phi\sin\theta) & (-\dot{\phi}\sin\phi\sin\theta + \dot{\theta}\cos\phi\cos\theta) & \dot{\phi}\cos\phi \end{bmatrix}$$

After the algebraic operation $\mathit{\Omega} = \dot{\mathcal{C}}\mathcal{C}^T$ is taken place we can obtain:

$$\mathit{\Omega} = \dot{\mathcal{C}}\mathcal{C}^T = \begin{bmatrix} 0 & \dot{\theta}\sin\phi & -\dot{\theta}\cos\phi \\ -\dot{\theta}\sin\phi & 0 & -\dot{\phi} \\ \dot{\theta}\cos\phi & \dot{\phi} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{v_{r1}}{\|\vec{x}\|\cos\phi}\sin\phi & -\dfrac{v_{r1}}{\|\vec{x}\|\cos\phi}\cos\phi \\ -\dfrac{v_{r1}}{\|\vec{x}\|\cos\phi}\sin\phi & 0 & -\dfrac{v_{r2}}{\|\vec{x}\|} \\ \dfrac{v_{r1}}{\|\vec{x}\|\cos\phi}\cos\phi & \dfrac{v_{r2}}{\|\vec{x}\|} & 0 \end{bmatrix}$$

$$= \frac{1}{\|\vec{x}\|} \begin{bmatrix} 0 & v_{r1}\tan\phi & -v_{r1} \\ -v_{r1}\tan\phi & 0 & -v_{r2} \\ v_{r1} & v_{r2} & 0 \end{bmatrix}$$

which is an antisymmetric matrix, as expected. Notice that according to Figure 4.75 the following relationships $\dot{\theta} \equiv \dfrac{d\theta}{dt} = \dfrac{v_{r1}}{\|\vec{x}\|\cos\phi}$ and $\dot{\phi} \equiv \dfrac{d\phi}{dt} = \dfrac{v_{r2}}{\|\vec{x}\|}$ hold.

We apply the definition, (see equation (4.171)),

$$\left(\frac{D\vec{\mathbf{v}}}{Dt}\right)_f = \left(\frac{D\vec{\mathbf{v}}}{Dt}\right)_r + \vec{\varphi} \wedge \vec{\mathbf{v}}_r$$

Note also that $\mathit{\Omega}^T \cdot \vec{\mathbf{v}}_r = \vec{\varphi} \wedge \vec{\mathbf{v}}_r$ holds, so:

$$\begin{gathered}\Omega^T \cdot \vec{v}_r = \frac{1}{\|\vec{x}\|}\begin{bmatrix} 0 & -v_{r1}\tan\phi & v_{r1} \\ v_{r1}\tan\phi & 0 & v_{r2} \\ -v_{r1} & -v_{r2} & 0 \end{bmatrix}\begin{Bmatrix} v_{r1} \\ v_{r2} \\ v_{r3} \end{Bmatrix} = \frac{1}{\|\vec{x}\|}\begin{Bmatrix} -v_{r1}v_{r2}\tan\phi + v_{r1}v_{r3} \\ v_{r1}^2\tan\phi + v_{r2}v_{r3} \\ -v_{r1}^2 - v_{r2}^2 \end{Bmatrix} \\ \Rightarrow \vec{a}_f = \vec{a}_r + \frac{1}{\|\vec{x}\|}\begin{Bmatrix} -v_{r1}v_{r2}\tan\phi + v_{r1}v_{r3} \\ v_{r1}^2\tan\phi + v_{r2}v_{r3} \\ -v_{r1}^2 - v_{r2}^2 \end{Bmatrix} \quad \Rightarrow \quad \vec{a}_r = \vec{a}_f - \frac{1}{\|\vec{x}\|}\begin{Bmatrix} -v_{r1}v_{r2}\tan\phi + v_{r1}v_{r3} \\ v_{r1}^2\tan\phi + v_{r2}v_{r3} \\ -v_{r1}^2 - v_{r2}^2 \end{Bmatrix}\end{gathered} \tag{4.181}$$

NOTE 5: ***Coriolis + Curvature acceleration***

The acceleration related to the Coriolis terms, (see equations (4.177) and (4.175)), and curvature is given by:

$$\vec{a}_f = \vec{a}_r + 2(\vec{\omega} \wedge \vec{v}_r) + \Omega^T \cdot \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}) \tag{4.182}$$

where

$$2(\vec{\omega} \wedge \vec{v}_r) + \Omega^T \cdot \vec{v}_r = \begin{Bmatrix} 2[\omega_3 v_{r3}\cos(\phi) - \omega_3 v_{r2}\sin(\phi)] \\ 2[\omega_3 v_{r1}\sin(\phi)] \\ -2[\omega_3 v_{r1}\cos(\phi)] \end{Bmatrix} + \frac{1}{\|\vec{x}\|}\begin{Bmatrix} -v_{r1}v_{r2}\tan(\phi) + v_{r1}v_{r3} \\ v_{r1}^2\tan(\phi) + v_{r2}v_{r3} \\ -v_{r1}^2 - v_{r2}^2 \end{Bmatrix} \tag{4.183}$$

Problem 4.47

Consider the rigid body in motion in which there are no forces acting on the body and also consider a torque-free motion. a) Show the Euler's equations of motion:

$$\boxed{\begin{cases} \mathrm{I}_1\dot{\omega}_1 = \omega_2\omega_3(\mathrm{I}_2 - \mathrm{I}_3) \\ \mathrm{I}_2\dot{\omega}_2 = \omega_1\omega_3(\mathrm{I}_3 - \mathrm{I}_1) \\ \mathrm{I}_3\dot{\omega}_3 = \omega_1\omega_2(\mathrm{I}_1 - \mathrm{I}_2) \end{cases}} \quad \textit{Euler's equations of motion} \tag{4.184}$$

where I_i are the principal moment of inertia related to the system G_{xyz} whose origin is at the center of mass G, ω_i are the components of the body angular velocity ($\vec{\omega}$), and $\dot{\omega}_i \equiv \frac{D\omega_i}{Dt}$ denotes the time derivative of the angular velocity.

b) Show that the kinetic energy is constant.

Solution:

The governing equations for a rigid body motion, (see **Problem 4.38**), are:

$$\sum \vec{F} = m\vec{a} \qquad and \qquad \sum \vec{M}_G = \dot{\vec{H}}_G$$

If the body is free of forces and torque we have that:

$$\sum \vec{F} = \vec{0} \qquad and \qquad \sum \vec{M}_G = \vec{0} = \dot{\vec{H}}_G$$

Next we will evaluate the term $\dot{\vec{H}}_G$.

We will consider a mobile system G_{xyz} attached to the body, (see Figure 4.76), so, in this situation we have that $\vec{\varphi} = \vec{\omega}$.

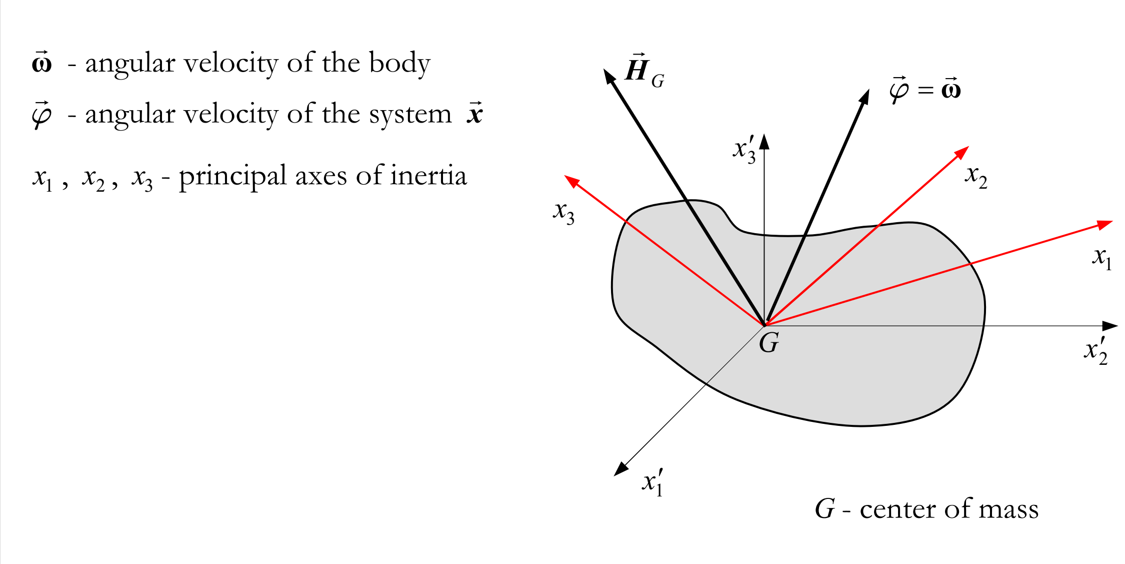


Figure 4.76

In **Problem 4.45** we have obtained an efficient equation in order to calculate $\dot{\vec{\boldsymbol{H}}}_G$, (see equation (4.171)), and by considering $\vec{\varphi} = \vec{\boldsymbol{\omega}}$ we can obtain:

$$\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_f = \left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_r + \vec{\varphi} \wedge \vec{\boldsymbol{H}}_G = \left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_r + \vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{Gxyz}$$

For this problem we have:

$$(\mathbf{I}_{Gxyz})_{ij} = \begin{bmatrix} \mathrm{I}_1 & 0 & 0 \\ 0 & \mathrm{I}_2 & 0 \\ 0 & 0 & \mathrm{I}_3 \end{bmatrix} \quad ; \quad (\vec{\boldsymbol{\omega}})_i = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

Note that we are already considering that the system $Gxyz$ is the principal inertia axis.

The angular momentum:

$$\vec{\boldsymbol{H}}_{Gxyz} = \mathbf{I}_{Gxyz} \cdot \vec{\boldsymbol{\omega}} \xrightarrow{\text{components}} (\vec{\boldsymbol{H}}_{Gxyz})_i = (\mathbf{I}_{Gxyz})_{ij} (\vec{\boldsymbol{\omega}})_j$$

$$\begin{Bmatrix} (\vec{\boldsymbol{H}}_{Gxyz})_1 \\ (\vec{\boldsymbol{H}}_{Gxyz})_2 \\ (\vec{\boldsymbol{H}}_{Gxyz})_3 \end{Bmatrix} = \begin{bmatrix} \mathrm{I}_1 & 0 & 0 \\ 0 & \mathrm{I}_2 & 0 \\ 0 & 0 & \mathrm{I}_3 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \mathrm{I}_1\omega_1 \\ \mathrm{I}_2\omega_2 \\ \mathrm{I}_3\omega_3 \end{Bmatrix}$$

The rate of change of the angular momentum:

Note that, since the system G_{xyz} is attached to the body the mass distribution respect to this system does not change during motion, so, $\mathbf{I}_{Gxyz}$ does not change as well, i.e. $\dot{\mathbf{I}}_{Gxyz} = \mathbf{0}$. With that we can obtain:

$$\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_r = \left(\frac{D\vec{\boldsymbol{H}}_{Gxyz}}{Dt}\right)_{Gxyz} \equiv \dot{\vec{\boldsymbol{H}}}_{Gxyz} \xrightarrow{\text{components}} \begin{Bmatrix} (\dot{\vec{\boldsymbol{H}}}_{Gxyz})_1 \\ (\dot{\vec{\boldsymbol{H}}}_{Gxyz})_2 \\ (\dot{\vec{\boldsymbol{H}}}_{Gxyz})_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\mathrm{I}}_1\omega_1 + \mathrm{I}_1\dot{\omega}_1 \\ \dot{\mathrm{I}}_2\omega_2 + \mathrm{I}_2\dot{\omega}_2 \\ \dot{\mathrm{I}}_3\omega_3 + \mathrm{I}_3\dot{\omega}_3 \end{Bmatrix} = \begin{Bmatrix} \mathrm{I}_1\dot{\omega}_1 \\ \mathrm{I}_2\dot{\omega}_2 \\ \mathrm{I}_3\dot{\omega}_3 \end{Bmatrix}$$

$$
\begin{aligned}
\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{Gxyz} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \mathrm{I}_1\omega_1 & \mathrm{I}_2\omega_2 & \mathrm{I}_3\omega_3 \end{vmatrix} \\
&= (\omega_2 \mathrm{I}_3 \omega_3 - \omega_3 \mathrm{I}_2 \omega_2)\hat{\mathbf{e}}_1 + (\omega_3 \mathrm{I}_1 \omega_1 - \omega_1 \mathrm{I}_3 \omega_3)\hat{\mathbf{e}}_2 + (\omega_1 \mathrm{I}_2 \omega_2 - \omega_2 \mathrm{I}_1 \omega_1)\hat{\mathbf{e}}_3 \\
&= \omega_2 \omega_3 (\mathrm{I}_3 - \mathrm{I}_2)\hat{\mathbf{e}}_1 + \omega_1 \omega_3 (\mathrm{I}_1 - \mathrm{I}_3)\hat{\mathbf{e}}_2 + \omega_1 \omega_2 (\mathrm{I}_2 - \mathrm{I}_1)\hat{\mathbf{e}}_3
\end{aligned}
$$

Components:

$$
\{\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{Gxyz}\}_i = \begin{Bmatrix} \omega_2\omega_3(\mathrm{I}_3 - \mathrm{I}_2) \\ \omega_1\omega_3(\mathrm{I}_1 - \mathrm{I}_3) \\ \omega_1\omega_2(\mathrm{I}_2 - \mathrm{I}_1) \end{Bmatrix}
$$

With that we can calculate

$$
\left(\frac{D\vec{\boldsymbol{H}}_G}{Dt}\right)_f = \dot{\vec{\boldsymbol{H}}}_{Gxyz} + \vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{Gxyz} = \vec{\boldsymbol{0}}
$$

whose components are:

$$
\{\dot{\vec{\boldsymbol{H}}}_{Gxyz}\}_i + \{\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{Gxyz}\}_i = \{\vec{\boldsymbol{0}}\}_i \quad \Rightarrow \quad \begin{Bmatrix} \mathrm{I}_1\dot{\omega}_1 \\ \mathrm{I}_2\dot{\omega}_2 \\ \mathrm{I}_3\dot{\omega}_3 \end{Bmatrix} + \begin{Bmatrix} \omega_2\omega_3(\mathrm{I}_3 - \mathrm{I}_2) \\ \omega_1\omega_3(\mathrm{I}_1 - \mathrm{I}_3) \\ \omega_1\omega_2(\mathrm{I}_2 - \mathrm{I}_1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
$$

$$
\Rightarrow \begin{cases} \mathrm{I}_1\dot{\omega}_1 = \omega_2\omega_3(\mathrm{I}_2 - \mathrm{I}_3) \\ \mathrm{I}_2\dot{\omega}_2 = \omega_1\omega_3(\mathrm{I}_3 - \mathrm{I}_1) \\ \mathrm{I}_3\dot{\omega}_3 = \omega_1\omega_2(\mathrm{I}_1 - \mathrm{I}_2) \end{cases}
$$

b) The kinetic energy for rigid body motion, (see equation (4.161) in **Problem 4.42**), is given by:

$$
\mathcal{K}(t) = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\vec{\boldsymbol{\omega}} \cdot \bar{\bar{\mathbf{I}}} \cdot \vec{\boldsymbol{\omega}}
$$

Since the origin of the adopted system is at G (mass center) we have $\bar{v} = 0$, with that we can obtain:

$$
\mathcal{K}(t) = \frac{1}{2}\omega_k \mathrm{I}_{kj}\omega_j = \frac{1}{2}\begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} \mathrm{I}_1 & 0 & 0 \\ 0 & \mathrm{I}_2 & 0 \\ 0 & 0 & \mathrm{I}_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{2}\left[\mathrm{I}_1\omega_1^2 + \mathrm{I}_2\omega_2^2 + \mathrm{I}_3\omega_3^2\right]
$$

And the rate of change of the kinetic energy becomes:

$$
\begin{aligned}
\frac{D}{Dt}\mathcal{K}(t) = \dot{\mathcal{K}}(t) &= \frac{1}{2}\frac{D}{Dt}\left[\mathrm{I}_1\omega_1^2 + \mathrm{I}_2\omega_2^2 + \mathrm{I}_3\omega_3^2\right] = \frac{1}{2}\left[2\omega_1\mathrm{I}_1\dot{\omega}_1 + 2\omega_2\mathrm{I}_2\dot{\omega}_2 + 2\omega_3\mathrm{I}_3\dot{\omega}_3\right] \\
&= \omega_1\mathrm{I}_1\dot{\omega}_1 + \omega_2\mathrm{I}_2\dot{\omega}_2 + \omega_3\mathrm{I}_3\dot{\omega}_3
\end{aligned}
$$

If we consider the Euler's equation (4.184) the above equation becomes:

$$
\begin{aligned}
\dot{\mathcal{K}}(t) &= \omega_1\mathrm{I}_1\dot{\omega}_1 + \omega_2\mathrm{I}_2\dot{\omega}_2 + \omega_3\mathrm{I}_3\dot{\omega}_3 = \omega_1\omega_2\omega_3(\mathrm{I}_2 - \mathrm{I}_3) + \omega_2\omega_1\omega_3(\mathrm{I}_3 - \mathrm{I}_1) + \omega_3\omega_1\omega_2(\mathrm{I}_1 - \mathrm{I}_2) \\
&= \omega_1\omega_2\omega_3(\mathrm{I}_2 - \mathrm{I}_3 + \mathrm{I}_3 - \mathrm{I}_1 + \mathrm{I}_1 - \mathrm{I}_2) \\
&= 0
\end{aligned}
$$

with that we have shown that the kinetic energy is constant for any problem which is governed by Euler's equations of motion.

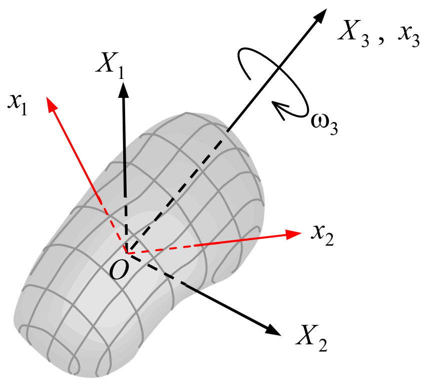
Problem 4.48

Obtain a simplified form of the rigid body governing equations for the particular case:

a) Rigid body rotation around a fixed axis without forces.

Solution:

We will consider the fixed system $OX_1X_2X_3$ and we will adopt the rotation axis by the X_3-axis, (see Figure 4.77), and the mobile system $Ox_1x_2x_3$ is attached to the body.



$$\vec{\varphi} = \vec{\boldsymbol{\omega}} = \omega_3 \hat{\mathbf{e}}_3 = \omega_3 \hat{\mathbf{E}}_3$$

$\vec{\boldsymbol{\omega}}$ - angular velocity of the body

$\vec{\varphi}$ - angular velocity of the system $\vec{\boldsymbol{x}}$

system $OX_1X_2X_3$ => orthonormal base $(\hat{\mathbf{E}}_1, \hat{\mathbf{E}}_2, \hat{\mathbf{E}}_3)$ or $(\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$

system $Ox_1x_2x_3$ => orthonormal base $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ or $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$

Figure 4.77

If the body is free of forces the governing equations becomes:

$$\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \qquad and \qquad \sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O$$

where $\dot{\vec{\boldsymbol{H}}}_O$ can be calculated by means of

$$\dot{\vec{\boldsymbol{H}}}_O \equiv \left(\frac{D\vec{\boldsymbol{H}}_O}{Dt}\right)_{O\vec{x}} = \left(\frac{D\vec{\boldsymbol{H}}_{O\vec{x}}}{Dt}\right)_{O\vec{x}} + \vec{\varphi} \wedge \vec{\boldsymbol{H}}_{O\vec{x}} = \left(\frac{D\vec{\boldsymbol{H}}_{O\vec{x}}}{Dt}\right)_{O\vec{x}} + \vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{O\vec{x}}$$

The angular momentum:

$$\vec{\boldsymbol{H}}_{O\vec{x}} = \mathbf{I}_{O\vec{x}} \cdot \vec{\boldsymbol{\omega}} \xrightarrow{\text{components}} (\vec{\boldsymbol{H}}_{O\vec{x}})_i = (\mathbf{I}_{O\vec{x}})_{ij} (\vec{\boldsymbol{\omega}})_j$$

$$\begin{Bmatrix} (\vec{\boldsymbol{H}}_{O\vec{x}})_1 \\ (\vec{\boldsymbol{H}}_{O\vec{x}})_2 \\ (\vec{\boldsymbol{H}}_{O\vec{x}})_3 \end{Bmatrix} = \begin{bmatrix} \mathrm{I}_{O11} & -\mathcal{I}_{O12} & -\mathcal{I}_{O13} \\ -\mathcal{I}_{O12} & \mathrm{I}_{O22} & -\mathcal{I}_{O23} \\ -\mathcal{I}_{O13} & -\mathcal{I}_{O23} & \mathrm{I}_{O33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} -\mathcal{I}_{O13}\omega_3 \\ -\mathcal{I}_{O23}\omega_3 \\ \mathrm{I}_{O33}\omega_3 \end{Bmatrix}$$

And its rate of change:

$$\begin{Bmatrix} (\dot{\vec{\boldsymbol{H}}}_{O\vec{x}})_1 \\ (\dot{\vec{\boldsymbol{H}}}_{O\vec{x}})_2 \\ (\dot{\vec{\boldsymbol{H}}}_{O\vec{x}})_3 \end{Bmatrix} = \begin{Bmatrix} -\mathcal{I}_{O13}\dot{\omega}_3 \\ -\mathcal{I}_{O23}\dot{\omega}_3 \\ \mathrm{I}_{O33}\dot{\omega}_3 \end{Bmatrix}$$

And we need to calculate the vector $\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{O\bar{x}}$:

$$\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{H}}_{O\bar{x}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & \omega_3 \\ -\mathcal{I}_{O13}\omega_3 & -\mathcal{I}_{O23}\omega_3 & \mathrm{I}_{O33}\omega_3 \end{vmatrix} = \mathcal{I}_{O23}\omega_3^2\hat{\mathbf{e}}_1 - \mathcal{I}_{O13}\omega_3^2\hat{\mathbf{e}}_2$$

thus

$$(\dot{\vec{\boldsymbol{H}}}_O)_i = \begin{Bmatrix} -\mathcal{I}_{O13}\dot{\omega}_3 \\ -\mathcal{I}_{O23}\dot{\omega}_3 \\ \mathrm{I}_{O33}\dot{\omega}_3 \end{Bmatrix} + \begin{Bmatrix} \mathcal{I}_{O23}\omega_3^2 \\ -\mathcal{I}_{O13}\omega_3^2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathcal{I}_{O23}\omega_3^2 - \mathcal{I}_{O13}\dot{\omega}_3 \\ -\mathcal{I}_{O13}\omega_3^2 - \mathcal{I}_{O23}\dot{\omega}_3 \\ \mathrm{I}_{O33}\dot{\omega}_3 \end{Bmatrix}$$

By applying $\sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O$ we can obtain the following set of equations:

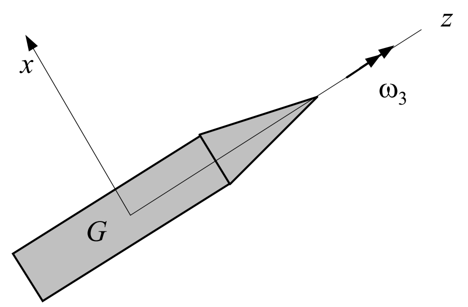
$$\begin{cases} \sum M_{O1} \equiv \sum M_X = \mathcal{I}_{O23}\omega_3^2 - \mathcal{I}_{O13}\dot{\omega}_3 \\ \sum M_{O2} \equiv \sum M_Y = -\mathcal{I}_{O13}\omega_3^2 - \mathcal{I}_{O23}\dot{\omega}_3 \\ \sum M_{O3} \equiv \sum M_Z = \mathrm{I}_{O33}\dot{\omega}_3 \end{cases}$$

where $\dot{\omega} = \alpha$ stands for angular acceleration.

NOTE: If the body is prismatic and if we adopt the prismatic axis the same as the rotating axis the above equations reduce to:

$$\begin{cases} \sum M_{O1} \equiv \sum M_X = 0 \\ \sum M_{O2} \equiv \sum M_Y = 0 \\ \sum M_{O3} \equiv \sum M_Z = \mathrm{I}_{O3}\dot{\omega}_3 \end{cases}$$

since the system $Ox_1x_2x_3$ is principal axes of inertia, (see Figure 4.78).



$$I_{Gxyz} = \begin{bmatrix} \mathrm{I}_{O1} & 0 & 0 \\ 0 & \mathrm{I}_{O2} & 0 \\ 0 & 0 & \mathrm{I}_{O3} \end{bmatrix}$$

$$\mathrm{I}_{O1} = \mathrm{I}_{O2}$$

Figure 4.78

Problem 4.49

A rigid body consists of two masses m at each extremity of the weightless rod of length 2ℓ. The rod is inclined about θ respect to the vertical line and rotates with angular velocity ω as indicated in Figure 4.79.

a) Find the angular momentum of the body;

b) Find the torque ($\sum \vec{\mathbf{M}}$) in order to maintain the rotation.

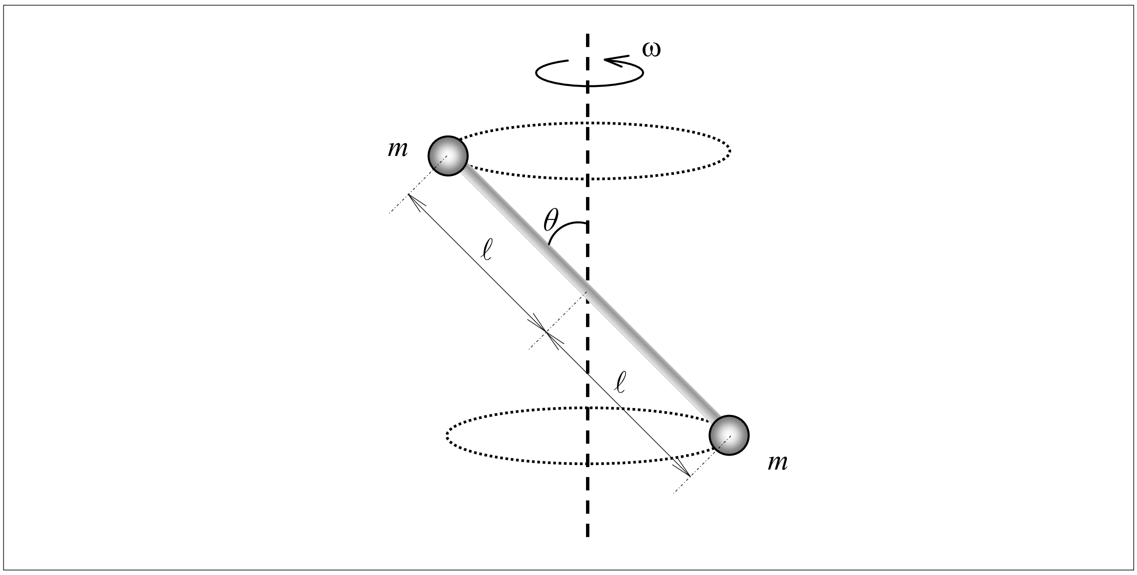


Figure 4.79

Solution:

We apply the governing equations for a rigid solid motion, (see **Problem 4.38**). We will adopt the fixed system in space $OXYZ$ and a mobile system $Oxyz$ which is attached to the body, (see Figure 4.80).

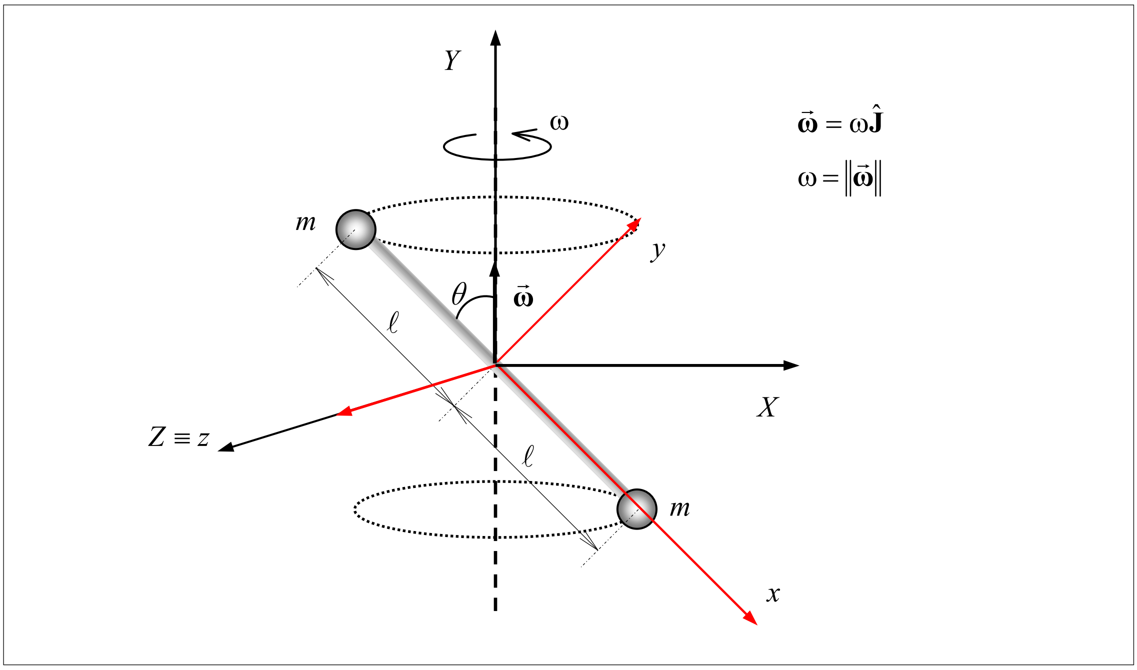


Figure 4.80

The inertia tensor $\mathbf{I}$ (system $Oxyz$) is given by:

$$\mathbf{I}_{Oxyz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2m\ell^2 & 0 \\ 0 & 0 & 2m\ell^2 \end{bmatrix}$$

The angular velocity $\vec{\boldsymbol{\omega}}$ (system $Oxyz$):

$$\vec{\boldsymbol{\omega}} = -\omega\cos(\theta)\hat{\mathbf{i}} + \omega\sin(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

where ω is the module of $\vec{\boldsymbol{\omega}}$.

The angular momentum $\vec{\mathbf{H}}_O$:

$$\vec{\mathbf{H}}_O = \mathbf{I}\cdot\vec{\boldsymbol{\omega}}$$

$$\Rightarrow \begin{bmatrix} \mathrm{H}_{Ox} \\ \mathrm{H}_{Oy} \\ \mathrm{H}_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2m\ell^2 & 0 \\ 0 & 0 & 2m\ell^2 \end{bmatrix} \begin{bmatrix} -\omega\cos(\theta) \\ \omega\sin(\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2m\ell^2\omega\sin(\theta) \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{\mathbf{H}}_O = 0\hat{\mathbf{i}} + [2m\ell^2\omega\sin(\theta)]\ \hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

The torque $\sum\vec{\mathbf{M}}$ can be evaluated as follows:

$$\sum\vec{\mathbf{M}} = \dot{\vec{\mathbf{H}}}_O = (\dot{\vec{\mathbf{H}}}_O)_{Oxyz} + \vec{\varphi}\wedge\vec{\mathbf{H}}_O$$

We can observe that $(\dot{\vec{\mathbf{H}}}_O)_{Oxyz} = \vec{\mathbf{0}}$ and $\vec{\varphi} = \vec{\boldsymbol{\omega}}$ hold, then:

$$\sum\vec{\mathbf{M}} = \dot{\vec{\mathbf{H}}}_O = \vec{\boldsymbol{\omega}}\wedge\vec{\mathbf{H}}_O = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\omega\cos(\theta) & \omega\sin(\theta) & 0 \\ 0 & 2m\ell^2\omega\sin(\theta) & 0 \end{vmatrix}$$

$$\Rightarrow \sum\vec{\mathbf{M}} = \dot{\vec{\mathbf{H}}}_O = -\omega\cos(\theta)2m\ell^2\omega\sin(\theta)\hat{\mathbf{k}} = -\omega^2 m\ell^2\sin(2\theta)\hat{\mathbf{k}}$$

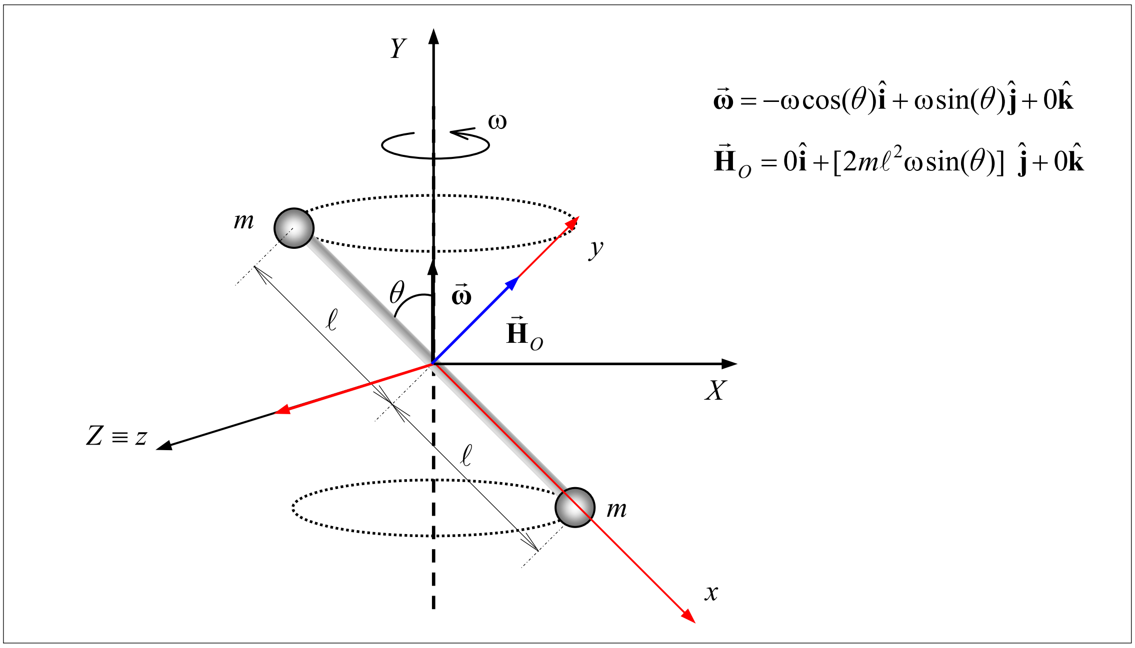


Figure 4.81

Solution using the system OXYZ

The transformation matrix from $OXYZ$ to $Oxyz$ is given by:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \cos\left(\frac{\pi}{2}-\theta\right) & \sin\left(\frac{\pi}{2}-\theta\right) & 0 \\ -\sin\left(\frac{\pi}{2}-\theta\right) & \cos\left(\frac{\pi}{2}-\theta\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inertia tensor for the system $OXYZ$ is:

$$\mathbf{I}_{OXYZ} = \boldsymbol{\mathcal{A}}^T \, \mathbf{I}_{Oxyz} \, \boldsymbol{\mathcal{A}}$$

$$\mathbf{I}_{OXYZ} = \begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2m\ell^2 & 0 \\ 0 & 0 & 2m\ell^2 \end{bmatrix} \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2m\ell^2\cos^2(\theta) & 2m\ell^2\sin(\theta)\cos(\theta) & 0 \\ 2m\ell^2\sin(\theta)\cos(\theta) & 2m\ell^2\sin^2(\theta) & 0 \\ 0 & 0 & 2m\ell^2 \end{bmatrix}$$

The angular momentum becomes:

$$\vec{\mathbf{H}}_{OXYZ} = \boldsymbol{\mathcal{A}}^T \vec{\mathbf{H}}_{Oxyz}$$

$$\begin{bmatrix} \mathrm{H}_{Ox} \\ \mathrm{H}_{Oy} \\ \mathrm{H}_{Oz} \end{bmatrix} = \begin{bmatrix} 0 \\ 2m\ell^2\omega\sin(\theta) \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \mathrm{H}_{OX} \\ \mathrm{H}_{OY} \\ \mathrm{H}_{OZ} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2m\ell^2\omega\sin(\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} 2m\ell^2\omega\cos(\theta)\sin(\theta) \\ 2m\ell^2\omega\sin^2(\theta) \\ 0 \end{bmatrix}$$

The torque:

$$\sum \vec{\mathbf{M}} = -\omega^2 m\ell^2 \sin(2\theta)\hat{\mathbf{K}}$$

Problem 4.50

A gyroscope consists of an outer gimbal, inner gimbal and a rotor with mass m, (see Figure 4.82). The outer gimbal can rotate about the Z-axis defining the angle ϕ (precession angle), the inner gimbal can rotate about the y-axis defining the angle θ (nutation angle), the rotor can rotate about the z-axis defining the angle ψ (rotation angle). The angles (ϕ, θ, ψ) are called Euler angles.

Obtain the governing equations for the gyroscope.

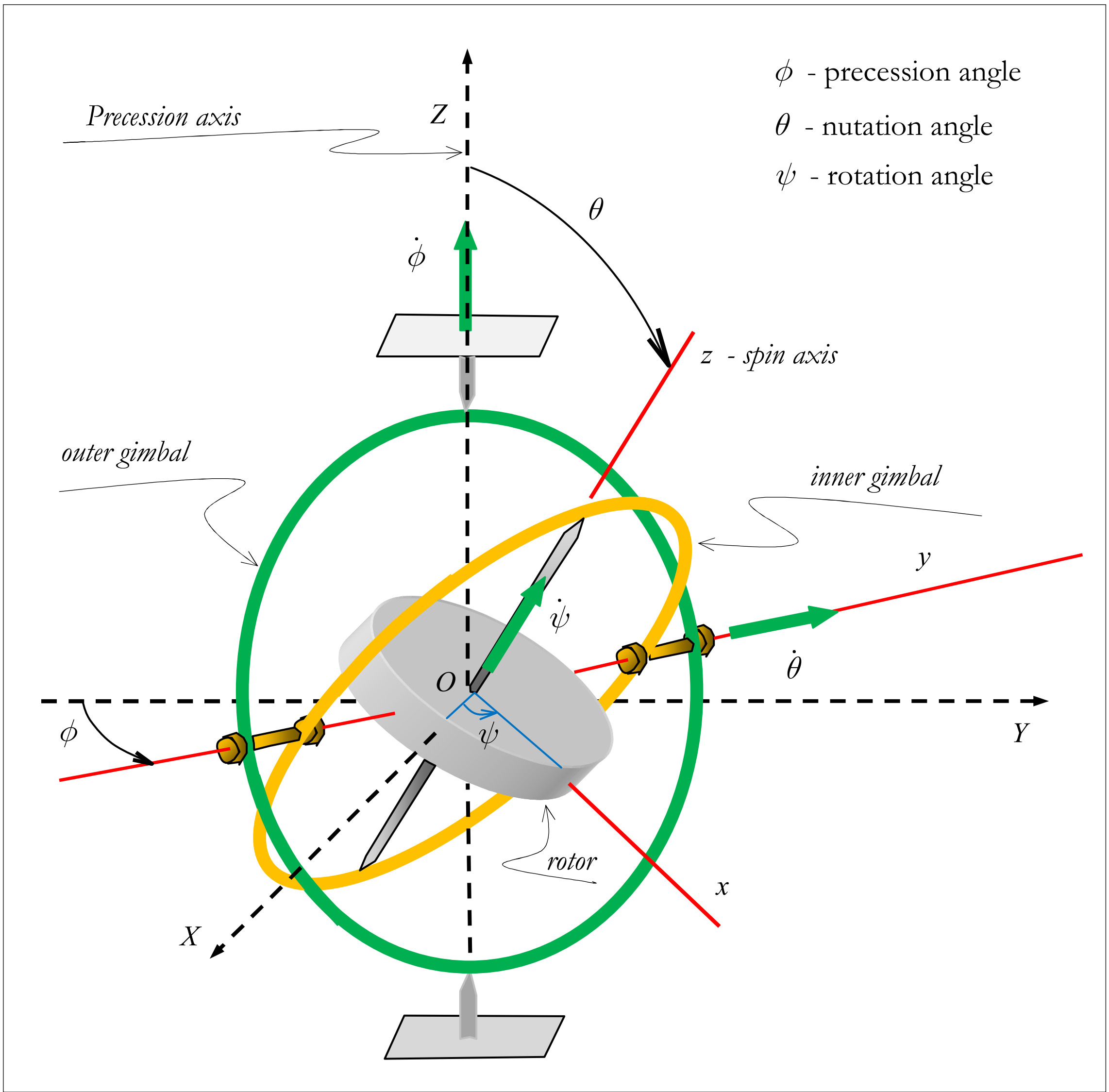


Figure 4.82

Consider the inertia tensor components of the rotor related to the system $Oxyz$ as follows:

$$(\mathbf{I}_{O\bar{x}})_{ij} = \begin{bmatrix} \mathrm{I}' & 0 & 0 \\ 0 & \mathrm{I}' & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix}$$

Solution:

We will adopt the orthonormal basis of the fixed system $OXYZ$ by $(\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, and for the mobile system $Oxyz$ we will adopt the orthonormal basis $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$.

Angular velocity of the rotor is given by, (see Figure 4.83):

$$\begin{aligned}\vec{\boldsymbol{\omega}} &= \dot{\phi}\,\hat{\mathbf{K}} + \dot{\theta}\,\hat{\mathbf{j}} + \dot{\psi}\,\hat{\mathbf{k}} \\ &= [-\dot{\phi}\sin(\theta)\,\hat{\mathbf{i}} + +\dot{\phi}\cos(\theta)]\,\hat{\mathbf{k}}] + \dot{\theta}\,\hat{\mathbf{j}} + \dot{\psi}\,\hat{\mathbf{k}} \\ &= -\dot{\phi}\sin(\theta)\,\hat{\mathbf{i}} + \dot{\theta}\,\hat{\mathbf{j}} + [\dot{\psi} + \dot{\phi}\cos(\theta)]\,\hat{\mathbf{k}}\end{aligned}$$

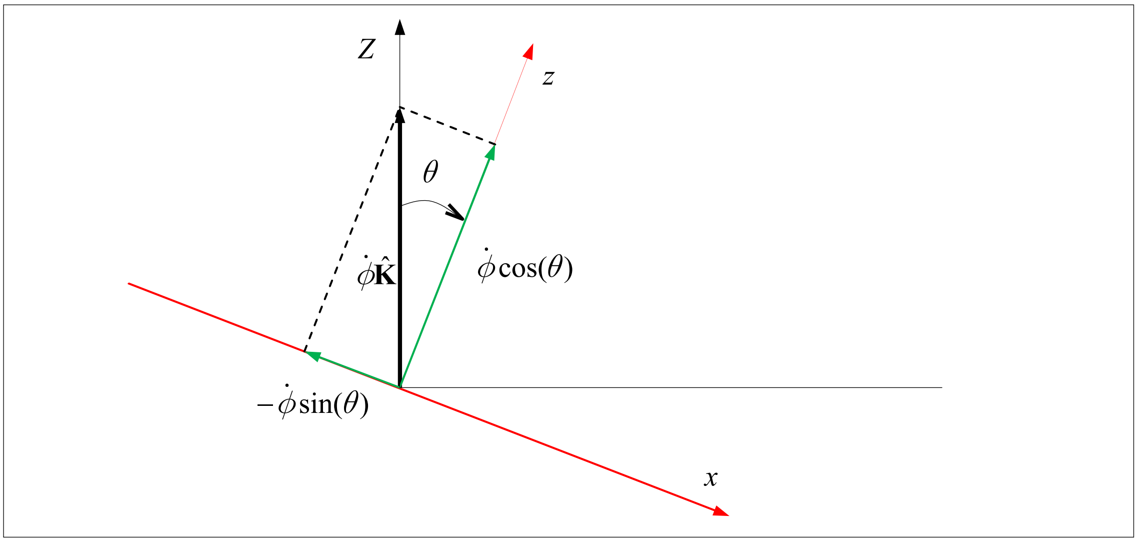


Figure 4.83

The governing equations for a rigid solid motion are given by:

$$\sum \vec{F} = m\,\vec{a} \qquad \text{and} \qquad \sum \vec{M}_O = \dot{\vec{H}}_O$$

where $\dot{\vec{H}}_O$ can be calculated by means of

$$\dot{\vec{H}}_O \equiv \left(\frac{D\vec{H}_O}{Dt}\right)_{O\bar{x}} = \left(\frac{D\vec{H}_{O\bar{x}}}{Dt}\right)_{O\bar{x}} + \vec{\varphi}\wedge\vec{H}_{O\bar{x}}$$

Angular momentum:

$$\vec{H}_{O\bar{x}} = \mathbf{I}_{O\bar{x}}\cdot\vec{\boldsymbol{\omega}} \xrightarrow{\text{components}} (\vec{H}_{O\bar{x}})_i = (\mathbf{I}_{O\bar{x}})_{ij}(\vec{\boldsymbol{\omega}})_j$$

$$\begin{Bmatrix} (\vec{H}_{O\bar{x}})_1 \\ (\vec{H}_{O\bar{x}})_2 \\ (\vec{H}_{O\bar{x}})_3 \end{Bmatrix} = \begin{bmatrix} \mathrm{I}' & 0 & 0 \\ 0 & \mathrm{I}' & 0 \\ 0 & 0 & \mathrm{I} \end{bmatrix} \begin{Bmatrix} -\dot{\phi}\sin(\theta) \\ \dot{\theta} \\ [\dot{\psi}+\dot{\phi}\cos(\theta)] \end{Bmatrix} = \begin{Bmatrix} -\mathrm{I}'\dot{\phi}\sin(\theta) \\ \mathrm{I}'\dot{\theta} \\ \mathrm{I}[\dot{\psi}+\dot{\phi}\cos(\theta)] \end{Bmatrix}$$

and its rate of change can be obtained as follows:

$$\begin{Bmatrix} (\dot{\vec{H}}_{O\bar{x}})_1 \\ (\dot{\vec{H}}_{O\bar{x}})_2 \\ (\dot{\vec{H}}_{O\bar{x}})_3 \end{Bmatrix} = \frac{D}{Dt}\begin{Bmatrix} -\mathrm{I}'\dot{\phi}\sin(\theta) \\ \mathrm{I}'\dot{\theta} \\ \mathrm{I}[\dot{\psi}+\dot{\phi}\cos(\theta)] \end{Bmatrix} = \begin{Bmatrix} -\mathrm{I}'[\ddot{\phi}\sin(\theta)+\dot{\phi}\cos(\theta)\dot{\theta}] \\ \mathrm{I}'\ddot{\theta} \\ \mathrm{I}\dfrac{D}{Dt}[\dot{\psi}+\dot{\phi}\cos(\theta)] \end{Bmatrix}$$

Note that due to the symmetry of the rotor, the inertia tensor does not change over time respected to the system $Oxyz$.

We need to calculate the vector $\vec{\varphi}\wedge\vec{H}_{O\bar{x}}$, where $\vec{\varphi}$ is the angular velocity of the rotating system $Oxyz$. Note that the mobile system can rotate about the $\hat{\mathbf{K}}$-axis and about the $\hat{\mathbf{j}}$-axis, and cannot rotate about the $\hat{\mathbf{k}}$-axis, (see Figure 4.83), then, the angular velocity of the mobile system can be given by:

$$
\begin{aligned}
\vec{\varphi} &= \dot{\phi}\,\hat{\mathbf{K}} + \dot{\theta}\,\hat{\mathbf{j}} \\
&= [-\dot{\phi}\sin(\theta)\,\hat{\mathbf{i}} + \dot{\phi}\cos(\theta)]\,\hat{\mathbf{k}}] + \dot{\theta}\,\hat{\mathbf{j}} \\
&= -\dot{\phi}\sin(\theta)\,\hat{\mathbf{i}} + \dot{\theta}\,\hat{\mathbf{j}} + \dot{\phi}\cos(\theta)\,\hat{\mathbf{k}}
\end{aligned}
$$

$$
\vec{\varphi} \wedge \vec{\boldsymbol{H}}_{O\bar{x}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\dot{\phi}\sin(\theta) & \dot{\theta} & \dot{\phi}\cos(\theta) \\ -\mathrm{I}'\dot{\phi}\sin(\theta) & \mathrm{I}'\dot{\theta} & \mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] \end{vmatrix}
$$

$$
\begin{aligned}
&= \{\dot{\theta}\,\mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}\,\mathrm{I}'\dot{\theta}\cos(\theta)\}\hat{\mathbf{i}} - \{-\mathrm{I}\dot{\phi}\sin(\theta)[\dot{\psi} + \dot{\phi}\cos(\theta)] + \dot{\phi}\mathrm{I}'\dot{\phi}\sin(\theta)\cos(\theta)\}\hat{\mathbf{j}} \\
&\quad + \{-\mathrm{I}'\dot{\theta}\,\dot{\phi}\sin(\theta) + \mathrm{I}'\dot{\theta}\,\dot{\phi}\sin(\theta)\}\hat{\mathbf{k}}
\end{aligned}
$$

$$
(\vec{\varphi} \wedge \vec{\boldsymbol{H}}_{O\bar{x}})_i = \begin{Bmatrix} \dot{\theta}\,\mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}\,\mathrm{I}'\dot{\theta}\cos(\theta) \\ \mathrm{I}\dot{\phi}\sin(\theta)[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}^2\,\mathrm{I}'\sin(\theta)\cos(\theta) \\ 0 \end{Bmatrix}
$$

thus

$$
\begin{aligned}
(\dot{\vec{\boldsymbol{H}}}_O)_i &= \left\{\left(\frac{D\vec{\boldsymbol{H}}_{O\bar{x}}}{Dt}\right)_{O\bar{x}}\right\}_i + (\vec{\varphi} \wedge \vec{\boldsymbol{H}}_{O\bar{x}})_i \\
&= \begin{Bmatrix} -\mathrm{I}'[\ddot{\phi}\sin(\theta) + \dot{\phi}\,\dot{\theta}\cos(\theta)] \\ \mathrm{I}'\ddot{\theta} \\ \mathrm{I}\dfrac{D}{Dt}[\dot{\psi} + \dot{\phi}\cos(\theta)] \end{Bmatrix} + \begin{Bmatrix} \dot{\theta}\,\mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}\,\mathrm{I}'\dot{\theta}\cos(\theta) \\ \mathrm{I}\dot{\phi}\sin(\theta)[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}^2\,\mathrm{I}'\sin(\theta)\cos(\theta) \\ 0 \end{Bmatrix} \\
&= \begin{Bmatrix} -\mathrm{I}'[\ddot{\phi}\sin(\theta) + 2\dot{\phi}\,\dot{\theta}\cos(\theta)] + \dot{\theta}\,\mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] \\ \mathrm{I}\dot{\phi}\sin(\theta)[\dot{\psi} + \dot{\phi}\cos(\theta)] - \dot{\phi}^2\,\mathrm{I}'\sin(\theta)\cos(\theta) + \mathrm{I}'\ddot{\theta} \\ \mathrm{I}\dfrac{D}{Dt}[\dot{\psi} + \dot{\phi}\cos(\theta)] \end{Bmatrix}
\end{aligned}
$$

Applying $\sum \vec{\boldsymbol{M}}_O = \dot{\vec{\boldsymbol{H}}}_O$ we can obtain the following set of equations:

$$
\begin{cases} \sum M_{O1} \equiv \sum M_x = -\mathrm{I}'[\ddot{\phi}\sin(\theta) + 2\dot{\phi}\,\dot{\theta}\cos(\theta)] + \mathrm{I}\,\dot{\theta}\;[\dot{\psi} + \dot{\phi}\cos(\theta)] \\ \sum M_{O2} \equiv \sum M_y = \mathrm{I}'[\ddot{\theta} - \dot{\phi}^2\sin(\theta)\cos(\theta)] + \mathrm{I}\dot{\phi}\sin(\theta)[\dot{\psi} + \dot{\phi}\cos(\theta)] \\ \sum M_{O3} \equiv \sum M_z = \mathrm{I}\dfrac{D}{Dt}[\dot{\psi} + \dot{\phi}\cos(\theta)] \end{cases} \tag{4.185}
$$

NOTE: Particular case: Steady precession.

In this case the variables θ, $\dot{\phi}$ and $\dot{\psi}$ are constant, the following equations are true: $\dot{\theta} = 0$, $\ddot{\phi} = 0$ and $\ddot{\psi} = 0$, so

$$
\begin{Bmatrix} (\vec{\boldsymbol{H}}_{O\bar{x}})_1 \\ (\vec{\boldsymbol{H}}_{O\bar{x}})_2 \\ (\vec{\boldsymbol{H}}_{O\bar{x}})_3 \end{Bmatrix} = \begin{Bmatrix} -\mathrm{I}'\dot{\phi}\sin(\theta) \\ 0 \\ \mathrm{I}[\dot{\psi} + \dot{\phi}\cos(\theta)] \end{Bmatrix}; \quad \vec{\varphi} = \dot{\phi}\,\hat{\mathbf{K}} \xrightarrow{\text{components}} \begin{Bmatrix} (\vec{\varphi})_1 \\ (\vec{\varphi})_2 \\ (\vec{\varphi})_3 \end{Bmatrix} = \begin{Bmatrix} -\dot{\phi}\sin(\theta) \\ 0 \\ \dot{\phi}\cos(\theta) \end{Bmatrix};
$$

$$\begin{Bmatrix} (\vec{\boldsymbol{\omega}})_1 \\ (\vec{\boldsymbol{\omega}})_2 \\ (\vec{\boldsymbol{\omega}})_3 \end{Bmatrix} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} -\dot{\phi}\sin(\theta) \\ 0 \\ [\dot{\psi} + \dot{\phi}\cos(\theta)] \end{Bmatrix}$$

The equations in (4.185) become:

$$\begin{cases} \sum M_{O1} \equiv \sum M_x = 0 \\ \sum M_{O2} \equiv \sum M_y = -\mathrm{I}'\,\dot{\phi}^2 \sin(\theta)\cos(\theta) + \mathrm{I}\dot{\phi}\sin(\theta)\underbrace{[\dot{\psi} + \dot{\phi}\cos(\theta)]}_{=\omega_z} \\ \sum M_{O3} \equiv \sum M_z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sum M_{O1} \equiv \sum M_x = 0 \\ \sum M_{O2} \equiv \sum M_y = [\mathrm{I}\,\omega_z - \mathrm{I}'\,\dot{\phi}\cos(\theta)]\dot{\phi}\sin(\theta) \\ \sum M_{O3} \equiv \sum M_z = 0 \end{cases}$$

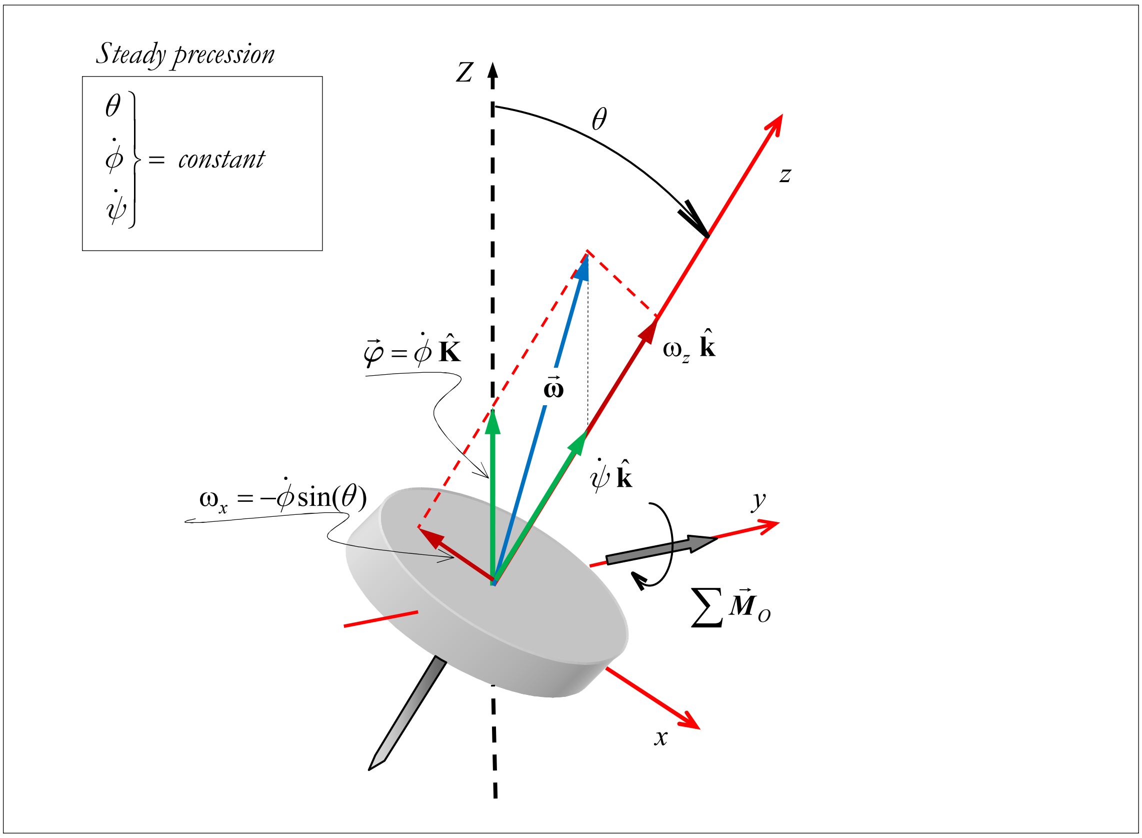


Figure 4.84: Steady precession.

Rigid Solid Motion References

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5 Introduction to: Constitutive Equations, IBVP Statement, and IBVP Solution Strategies

Problem 5.1

Describe the constitutive equations and the free variables for simple thermoelastic materials when we are considering the specific Helmholtz free energy ψ.

Solution:

The constitutive equations for a simple material are in function of the following free variables:

$$
\begin{array}{r|l|}
\textit{Constitutive equation for energy} & \psi = \psi(\boldsymbol{F}, T) \\
\textit{Constitutive equation for stress} & \mathbf{P}(\boldsymbol{F}, T) = \rho_0 \dfrac{\partial \psi(\boldsymbol{F}, T)}{\partial \boldsymbol{F}} \\
\textit{Constitutive equation for entropy} & \eta(\boldsymbol{F}, T) = -\dfrac{\partial \psi(\boldsymbol{F}, T)}{\partial T} \\
\textit{Constitutive equation for heat conduction} & \vec{\mathbf{q}}_0 = \vec{\mathbf{q}}_0(\boldsymbol{F}, T, \nabla_{\vec{X}} T)
\end{array}
$$

The free variables are $\boldsymbol{F}$-deformation gradient, T-temperature, $\nabla_{\vec{X}} T$-temperature gradient, (see Chaves 2013 – Chapter 6). The constitutive equations can also be expressed as follows

$$
\boxed{\begin{aligned}
&\hat{\psi} = \psi(\boldsymbol{E}, T) \\
&\mathbf{S} = \rho_0 \frac{\partial \psi(\boldsymbol{E}, T)}{\partial \boldsymbol{E}} \\
&\eta(\boldsymbol{E}, T) = -\frac{\partial \psi(\boldsymbol{E}, T)}{\partial T} \\
&\vec{\hat{\mathbf{q}}}_0 = \vec{\mathbf{q}}_0(\boldsymbol{E}, T, \nabla_{\vec{X}} T)
\end{aligned}}
\quad ; \quad
\boxed{\begin{aligned}
&\psi = \psi(\boldsymbol{F}, T) \\
&\boldsymbol{\sigma} = \rho \frac{\partial \psi(\boldsymbol{F}, T)}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T \\
&\eta(\boldsymbol{F}, T) = -\frac{\partial \psi(\boldsymbol{F}, T)}{\partial T} \\
&\vec{\mathbf{q}} \;= J^{-1} \vec{\mathbf{q}}_0(\boldsymbol{F}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F}^T \\
&\quad\;\; = J^{-1} \boldsymbol{F} \cdot \vec{\mathbf{q}}_0(\boldsymbol{F}, T, \nabla_{\vec{X}} T)
\end{aligned}}
$$

Problem 5.2

Consider an elastic material in which the energy density (energy per unit volume) is known and is given by:

$$\Psi(I_{\boldsymbol{E}}, I\!I_{\boldsymbol{E}}) = \frac{1}{2}(\lambda + 2\mu) I_{\boldsymbol{E}}^2 - 2\mu I\!I_{\boldsymbol{E}}$$

where λ and μ are material constants, $I_{\boldsymbol{E}} = I_{\boldsymbol{E}}(\boldsymbol{E})$ and $I\!I_{\boldsymbol{E}} = I\!I_{\boldsymbol{E}}(\boldsymbol{E})$ are, respectively, the first and second principal invariants of the Green-Lagrange strain tensor. Obtain the constitutive equations for this problem. Also obtain the explicit expression for the constitutive equations in terms of λ, μ, $I_{\boldsymbol{E}}$ and $I\!I_{\boldsymbol{E}}$.

Formulary

$$I_{\boldsymbol{E}} = I_{\boldsymbol{E}}(\boldsymbol{E}) = \mathrm{Tr}(\boldsymbol{E}) \,;\; I\!I_{\boldsymbol{E}} = I\!I_{\boldsymbol{E}}(\boldsymbol{E}) = \frac{1}{2}\left[(\mathrm{Tr}\boldsymbol{E})^2 - \mathrm{Tr}(\boldsymbol{E}^2)\right];\; \frac{\partial I_{\boldsymbol{E}}}{\partial \boldsymbol{E}} = \mathbf{1} \,;\; \frac{\partial I\!I_{\boldsymbol{E}}}{\partial \boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E})\mathbf{1} - \boldsymbol{E}^T .$$

Solution:

According to the problem, the energy density is only a function of the Green-Lagrange strain tensor. We know that the general expressions for the constitutive equations for a simple thermoelastic material are:

$$\boxed{\begin{aligned} &\hat{\psi} = \psi(\boldsymbol{E}, T) \\ &\mathbf{S} = \rho_0 \frac{\partial \psi(\boldsymbol{E}, T)}{\partial \boldsymbol{E}} \\ &\eta(\boldsymbol{E}, T) = -\frac{\partial \psi(\boldsymbol{E}, T)}{\partial T} \\ &\vec{\mathbf{q}}_0 = \vec{\mathbf{q}}_0(\boldsymbol{E}, T, \nabla_{\vec{X}} T) \end{aligned}}$$

Considering the equation for the given energy density, we can conclude that the problem is independent of temperature, since the energy density equation is not a function of temperature. Then, the remaining constitutive equation is the one related to stress, i.e.:

$$\begin{aligned} \mathbf{S} &= \rho_0 \frac{\partial \psi(\boldsymbol{E})}{\partial \boldsymbol{E}} = \frac{\partial \Psi(I_{\boldsymbol{E}}, I\!I_{\boldsymbol{E}})}{\partial \boldsymbol{E}} = \frac{\partial \Psi(I_{\boldsymbol{E}}, I\!I_{\boldsymbol{E}})}{\partial I_{\boldsymbol{E}}} \frac{\partial I_{\boldsymbol{E}}}{\partial \boldsymbol{E}} + \frac{\partial \Psi(I_{\boldsymbol{E}}, I\!I_{\boldsymbol{E}})}{\partial I\!I_{\boldsymbol{E}}} \frac{\partial I\!I_{\boldsymbol{E}}}{\partial \boldsymbol{E}} \\ &= \left(\frac{2}{2}(\lambda + 2\mu) I_{\boldsymbol{E}}\right)(\mathbf{1}) + (-2\mu)\left(\mathrm{Tr}(\boldsymbol{E})\mathbf{1} - \boldsymbol{E}^T\right) \end{aligned}$$

By simplifying the above equation, and taking into account that $\boldsymbol{E}^T = \boldsymbol{E}$, $I_{\boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E})$, we can obtain:

$$\mathbf{S} = \lambda I_{\boldsymbol{E}} \mathbf{1} + 2\mu \boldsymbol{E}$$

Problem 5.3

Consider the specific Gibbs free energy $\mathsf{G}(\mathbf{S}, T) = \psi(\boldsymbol{E}, T) - \frac{1}{\rho_0} \mathbf{S} : \boldsymbol{E}$ as constitutive equation for energy for thermoelastic materials. Obtain the remaining constitutive equations for thermoelastic materials, which is based on the principle that $\mathsf{G}(\mathbf{S}, T)$ does not depend on the temperature gradient.

Solution:

We start from the Clausius-Duhem inequality in terms of specific Helmholtz free energy in the reference configuration:

$$\mathbf{S} : \dot{\boldsymbol{E}} - \rho_0[\dot{\psi} + \dot{T}\eta] - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{5.1}$$

Taking into account the specific Gibbs free energy we can obtain the rate of change:

$$
\begin{aligned}
&\dot{\mathsf{G}}(\mathbf{S},T)=\dot{\psi}(\boldsymbol{E},T)-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}-\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}\\
&\Rightarrow \dot{\psi}(\boldsymbol{E},T)=\dot{\mathsf{G}}(\mathbf{S},T)+\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}+\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}
\end{aligned}
$$

and by replacing the above equation into the inequality (5.1) we can obtain:

$$
\begin{aligned}
&\mathbf{S}:\dot{\boldsymbol{E}}-\rho_0\left[\dot{\mathsf{G}}(\mathbf{S},T)+\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}+\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}+\dot{T}\eta\right]-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\
&\Rightarrow -\rho_0\dot{\mathsf{G}}(\mathbf{S},T)-\dot{\mathbf{S}}:\boldsymbol{E}-\rho_0\dot{T}\eta-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0
\end{aligned}
\tag{5.2}
$$

Note that $\dot{\mathbf{S}}:\boldsymbol{E}=\boldsymbol{E}:\dot{\mathbf{S}}$ holds. The above inequality suggests that for a variation of Gibbs free energy we must have the following relationships: Strain for "variation" of stress, Entropy for a variation of temperature, and heat conduction in terms of temperature gradient.

The term $\dot{\mathsf{G}}(\mathbf{S},T)$ can also be expressed as follows:

$$
\frac{D\mathsf{G}(\mathbf{S},T)}{Dt}\equiv\dot{\mathsf{G}}(\mathbf{S},T)=\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial T}\dot{T}
$$

and by replacing the above equation into the equation in (5.2) we can obtain:

$$
\begin{aligned}
&-\rho_0\dot{\mathsf{G}}(\mathbf{S},T)-\boldsymbol{E}:\dot{\mathbf{S}}-\rho_0\dot{T}\eta-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\
&\Rightarrow -\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}:\dot{\mathbf{S}}-\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial T}\dot{T}-\boldsymbol{E}:\dot{\mathbf{S}}-\rho_0\dot{T}\eta-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\
&\Rightarrow -\left(\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}+\boldsymbol{E}\right):\dot{\mathbf{S}}-\rho_0\left(\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial T}+\eta\right)\dot{T}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0
\end{aligned}
\tag{5.3}
$$

The above inequality must be satisfied for any admissible thermodynamic process. Let us now consider the process such that $\dot{T}=0$ (isothermal process), and $\vec{\mathbf{q}}_0=\vec{\mathbf{0}}$ (adiabatic process), then the above entropy inequality becomes:

$$
-\left(\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}+\boldsymbol{E}\right):\dot{\mathbf{S}}\geq 0
\tag{5.4}
$$

Note that the above inequality must also be met for any thermodynamic process. Then if in the current process the condition in (5.4) is met, we can apply another process such that $\dot{\mathbf{S}}=-\dot{\mathbf{S}}$, in which the entropy inequality (5.4) is violated. Thus, the only way in which the inequality in (5.4) is satisfied is when:

$$
\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}+\boldsymbol{E}=0\qquad\Rightarrow\qquad\boldsymbol{E}=-\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}
$$

Then if we take into account the above equation into the inequality (5.3), we can obtain:

$$
\begin{aligned}
&-\left(\rho_0\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial\mathbf{S}}+\boldsymbol{E}\right):\dot{\mathbf{S}}-\rho_0\left(\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial T}+\eta\right)\dot{T}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\
&\Rightarrow -\rho_0\left(\frac{\partial\mathsf{G}(\mathbf{S},T)}{\partial T}+\eta\right)\dot{T}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0
\end{aligned}
\tag{5.5}
$$

Now let us consider a process where $\nabla_{\vec{X}} T = \vec{\mathbf{0}}$ (uniform temperature field), then the inequality becomes:

$$-\rho_0 \left(\frac{\partial \mathsf{G}(\mathbf{S}, T)}{\partial T} + \eta \right) \dot{T} \geq 0$$

Starting from this point, we could apply another process where $\dot{T} = -\dot{T}$, in which the entropy inequality is violated. Thus, the only way in which the above inequality is satisfied is when:

$$\frac{\partial \mathsf{G}(\mathbf{S}, T)}{\partial T} + \eta = 0 \quad \Rightarrow \quad \eta = -\frac{\partial \mathsf{G}(\mathbf{S}, T)}{\partial T}$$

Then, the constitutive equations are:

$$\begin{aligned}
&\textit{Constitutive equation for energy} && \mathsf{G} = \mathsf{G}(\mathbf{S}, T) \\
&\textit{Constitutive equation for strain} && \boldsymbol{E} = -\rho_0 \frac{\partial \mathsf{G}(\mathbf{S}, T)}{\partial \mathbf{S}} = \frac{\partial \mathsf{g}(\mathbf{S}, T)}{\partial \mathbf{S}} \\
&\textit{Constitutive equation for entropy} && \eta = -\frac{\partial \mathsf{G}(\mathbf{S}, T)}{\partial T} \\
&\textit{Constitutive equation for heat conduction} && \vec{\mathbf{q}}_0 = \vec{\mathbf{q}}_0(\nabla_{\vec{X}} T)
\end{aligned} \tag{5.6}$$

where $\mathsf{g}(\mathbf{S}, T) = -\rho_0 \mathsf{G}(\mathbf{S}, T)$. Note that the free variables are $(\mathbf{S}, T)$.

Problem 5.4

Show that for an isothermal adiabatic process and with no rate of change of stress the specific Gibbs free energy cannot increase.

Solution:

We start directly from the inequality in (5.3):

$$-\rho_0 \dot{\mathsf{G}}(\mathbf{S}, T) - \boldsymbol{E} : \dot{\mathbf{S}} - \rho_0 \dot{T} \eta - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{5.7}$$

Taking into account the isothermal adiabatic process we have $\dot{T} = 0$, $\vec{\mathbf{q}}_0 = \vec{\mathbf{0}}$, and with no rate of change of stress the equation $\dot{\mathbf{S}} = \mathbf{0}$ holds. With that the inequality in (5.7) becomes:

$$-\rho_0 \dot{\mathsf{G}}(\mathbf{S}, T) \geq 0 \tag{5.8}$$

Note that $\rho_0 > 0$ is always positive, then to satisfy the above inequality the condition $\dot{\mathsf{G}}(\mathbf{S}, T) \leq 0$ must hold.

Problem 5.5

Find the governing equations for a continuum solid which has the following features: Isothermal and adiabatic processes; an infinitesimal strain regime and a linear elastic relationship between stress and strain.

b) Once the stress-strain linear relationship has been established, find the equation in which $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is a tensor-valued isotropic tensor function.

Solution:

When we are dealing with isothermal and adiabatic processes, temperature and entropy play no role.

In an infinitesimal strain regime, the following is satisfied:

Strain tensors: $\boldsymbol{E} \approx \boldsymbol{e} \approx \boldsymbol{\varepsilon} = \nabla^{sym}\vec{\mathbf{u}}$

Stress tensors: $\mathbf{P} \approx \mathbf{S} \approx \boldsymbol{\sigma}$

$\boldsymbol{F} \approx \mathbf{1}$; $\nabla_{\vec{X}} \approx \nabla_{\vec{x}} \approx \nabla$; $\rho \approx \rho_0$. If we take this approach, mass density is no longer unknown ($\dot{\rho} = 0$).

Then, taking into account the fundamental equations:

The Fundamental Equations of Continuum Mechanics (Current configuration)		
The Mass Continuity Equation (The principle of conservation of mass)	$\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{v}) = 0$	(5.9)
The Equations of Motion (The principle of conservation of linear momentum)	$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{v}}$	(5.10)
Cauchy Stress Tensor symmetry (The principle of conservation of angular momentum)	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	(5.11)
The Energy Equation (The principle of conservation of energy)	$\rho\,\dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$	(5.12)
The Entropy Inequality (The principle of irreversibility)	$\rho\dot{\eta}(\vec{x},t) + \frac{1}{T}\boldsymbol{\sigma} : \mathbf{D} - \frac{1}{T}\rho\,\dot{u} - \frac{1}{T^2}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0$	(5.13)

the remaining equations for the proposed problem are:

1) The equations of motion

$$\nabla \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{v}}$$

2) The energy equation (reference configuration):

$$\rho_0\dot{u}(\vec{X},t) = \mathbf{S} : \dot{\boldsymbol{E}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{X},t) \quad \Rightarrow \quad \rho\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$

where u is the specific internal energy, and the relationship $\frac{Du}{Dt} = \frac{D}{Dt}[\psi + T\eta] = \dot{\psi}$ holds, where ψ is the specific Helmholtz free energy. Note also that

$$\rho\dot{\psi} = \dot{\Psi}^e = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$

where Ψ^e is the strain energy density, in which $\dot{\Psi}^e = \dot{\rho}\psi + \rho\dot{\psi} = \rho\dot{\psi}$. Then if we bear in mind the entropy inequality, we can observe that the proposed problem is characterized by a process without any energy dissipation (an *elastic process*), i.e. all stored energy caused by $\boldsymbol{\varepsilon}$ will recover when $\boldsymbol{\varepsilon} = \mathbf{0}$.

3) For this problem, the constitutive equations described in **Problem 5.1** become:

$$\psi = \psi(\boldsymbol{\varepsilon})$$

$$\mathbf{S} \approx \boldsymbol{\sigma} = \rho\frac{\partial\psi(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} = \frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$$

Energy (ψ) and stress are only functions of strain. Then, if we calculate the rate of change of the Helmholtz free energy, i.e. $\dot{\psi}(\boldsymbol{\varepsilon}) = \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}$, and by substituting it into the equation $\rho\dot{\psi} = \frac{D(\rho\psi)}{Dt} = \frac{D(\Psi^e)}{Dt} = \dot{\Psi}^e = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$, we can obtain:

$$\rho \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial \dot{\Psi}^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

Thus, we can conclude that the energy equation is a redundant one, i.e. if the stress is known the energy can be evaluated and vice-versa. So, we can summarize the governing equations for the problem proposed as follows:

The equations of motion:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \ddot{\vec{\mathbf{u}}} \quad \text{(3 equations)}$$

The constitutive equations for stress:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad \text{(6 equations)} \qquad (5.14)$$

Kinematic equations:

$$\boldsymbol{\varepsilon} = \nabla^{sym} \vec{\mathbf{u}} \quad \text{(6 equations)}$$

The unknowns of the proposed problem are: $\boldsymbol{\sigma}$ (6), $\vec{\mathbf{u}}$ (3) and $\boldsymbol{\varepsilon}$ (6), making a total of 15 unknowns and 15 equations, so the problem is well-posed. Then, to achieve the unique solution of the set of partial differential equations given by (5.14) one must introduce the initial and boundary conditions, hence defining the *Initial Boundary Value Problem (IBVP)* for the *linear elasticity problem*. The initial and boundary conditions for this problem are:

The displacement boundary condition, on $S_{\mathbf{u}}$*:*

$$\vec{\mathbf{u}}(\vec{x},t) = \vec{\mathbf{u}}^*(\vec{x},t) \qquad \Big| \qquad \mathrm{u}_i(\vec{x},t) = \mathrm{u}_i^*(\vec{x},t) \qquad (5.15)$$

The stress boundary condition, on $S_{\boldsymbol{\sigma}}$*:*

$$\boldsymbol{\sigma}(\vec{x},t) \cdot \hat{\mathbf{n}} = \vec{\mathbf{t}}^*(\vec{x},\hat{\mathbf{n}},t) \qquad \Big| \qquad \sigma_{jk}\hat{\mathrm{n}}_k = \mathrm{t}_j^*(\vec{x},t) \qquad (5.16)$$

The initial conditions ($t=0$)*:*

$$\vec{\mathbf{u}}(\vec{x},t=0) = \vec{\mathbf{u}}_0 \qquad \Big| \qquad \mathrm{u}_i(\vec{x},t=0) = \mathrm{u}_{0i}(\vec{x})$$

$$\left.\frac{\partial \vec{\mathbf{u}}_0(\vec{x},t)}{\partial t}\right|_{t=0} = \dot{\vec{\mathbf{u}}}_0(\vec{x},t) = \vec{v}_0(\vec{x}) \qquad \Big| \qquad \dot{\mathrm{u}}_{0i}(\vec{x}) = v_{0i} \qquad (5.17)$$

In the particular case when we are dealing with a static or quasi-static problem, the equations of motion become the equilibrium equations ($\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$), and the initial conditions become redundant.

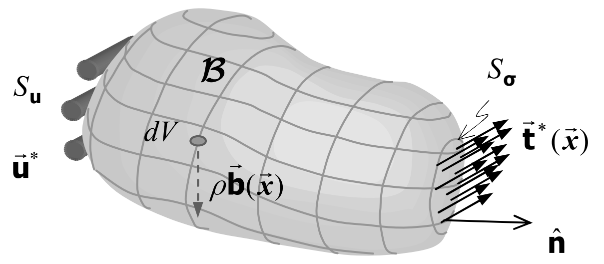


Figure 5.1: Solid under external actions.

In subsection 1.6.1 The Tensor Series (Chapter 1-textbook, Chaves (2013)), we have seen that we can approach a tensor-valued tensor function by means of the following series:

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) &\approx \frac{1}{0!}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)+\frac{1}{1!}\frac{\partial\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\frac{1}{2!}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0):\frac{\partial^2\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\cdots\\ &\approx \boldsymbol{\sigma}_0+\frac{\partial\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0):\frac{\partial^2\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\cdots\end{aligned}$$

Then, by considering the application point $\boldsymbol{\varepsilon}_0=\mathbf{0}$ and $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)=\boldsymbol{\sigma}_0=\mathbf{0}$, and also taking into account that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is linear, higher order terms can be discarded, thus:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon})=\frac{\partial\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}}:\boldsymbol{\varepsilon}=\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:\boldsymbol{\varepsilon}=\mathbb{C}^e:\boldsymbol{\varepsilon} \qquad \sigma_{ij}=\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\varepsilon_{kl}=\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\varepsilon_{kl}=\mathbb{C}^e_{ijkl}\varepsilon_{kl}$$

where $\mathbb{C}^e=\dfrac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}$ is a symmetric fourth-order tensor which is known as the *elasticity tensor*, which contains the material mechanical properties.

Note that, the energy equation has to be quadratic with which we can guarantee that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is linear, since $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})=\dfrac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}$. We can also use series expansion to represent the strain energy density as follows:

$$\begin{aligned}\Psi^e(\boldsymbol{\varepsilon}) &= \frac{1}{0!}\Psi^e(\boldsymbol{\varepsilon}_0)+\frac{1}{1!}\frac{\partial\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\frac{1}{2!}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0):\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\cdots\\ &= \Psi^e_0+\boldsymbol{\sigma}_0:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0):\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_0)+\cdots\\ &= \frac{1}{2}\boldsymbol{\varepsilon}:\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon}_0)}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:\boldsymbol{\varepsilon}=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}^e:\boldsymbol{\varepsilon}\end{aligned}$$

where we have also considered that $\boldsymbol{\varepsilon}_0=\mathbf{0}\Rightarrow\Psi^e_0=0$, $\boldsymbol{\sigma}_0=\mathbf{0}$.

NOTE 1: Although the energy equation is a redundant one, at the time of establishing an analytical or numerical method to solve the problem, we will always start from energy principles, hence the importance of studying the energy equation in a system.

NOTE 2: Analyzing $\mathbb{C}^e$:

Note that, according to the equation $\sigma_{ij}=\mathbb{C}^e_{ijkl}\varepsilon_{kl}$ and due to the symmetry of $\sigma_{ij}=\sigma_{ji}$ and $\varepsilon_{kl}=\varepsilon_{lk}$, the tensor $\mathbb{C}^e$ has minor symmetry, i.e. $\mathbb{C}^e_{ijkl}=\mathbb{C}^e_{jikl}=\mathbb{C}^e_{ijlk}=\mathbb{C}^e_{jilk}$. Note also that:

$$\mathbb{C}^e_{ijkl}=\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}=\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}}=\mathbb{C}^e_{klij}\quad\text{(major symmetry)}$$

NOTE 3: To better illustration of the problem established here, let us consider a particular case (a one-dimensional case) in which the stress and strain components are given by:

$$\sigma_{ij}=\begin{bmatrix}\sigma&0&0\\0&0&0\\0&0&0\end{bmatrix}\quad;\quad\varepsilon_{ij}=\begin{bmatrix}\varepsilon&0&0\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow\sigma_{11}=\mathbb{C}^e_{1111}\varepsilon_{11}\Rightarrow\sigma=E\varepsilon$$

In this case, the stress-strain linear relationship becomes $\sigma = E\varepsilon$ (Hooke's law) and the strain energy density is given by $\Psi^e = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\varepsilon E\varepsilon$, and $\frac{\partial^2\Psi^e}{\partial\varepsilon\partial\varepsilon} = \frac{\partial\sigma}{\partial\varepsilon} = E$, (see Figure 5.2).

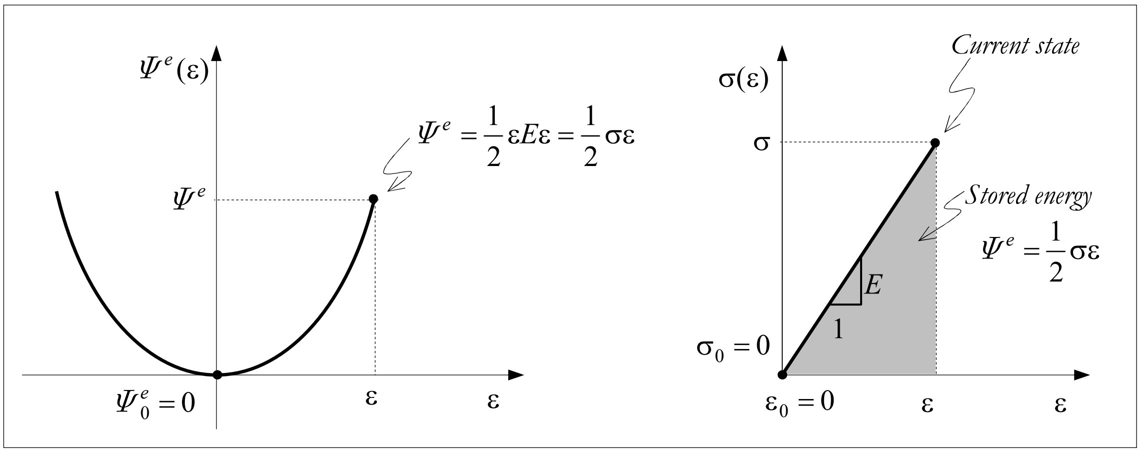


Figure 5.2: Stress-strain relationship (one-dimensional case).

NOTE 4: Here it should be pointed out that in the case of elastic processes the constitutive equation $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is only dependent on the current value of $\boldsymbol{\varepsilon}$, i.e. it is independent of the deformation history. ■

b) The tensor-valued tensor function $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is isotropic if the following is satisfied:

$$\Psi^e(\varepsilon'_{kl}) = \Psi^e(\varepsilon_{kl}) \quad \Rightarrow \quad \sigma'_{ij}(\varepsilon'_{kl}) = \sigma_{ij}(\varepsilon'_{kl})$$

Then, taking into account that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is given by $\sigma_{ij}(\boldsymbol{\varepsilon}) = \mathbb{C}^e_{ijkl}\varepsilon_{kl}$ (indicial notation), we can conclude that:

$$\sigma'_{ij}(\varepsilon'_{kl}) = \sigma_{ij}(\varepsilon'_{kl}) \quad \Rightarrow \quad \mathbb{C}'^e_{ijkl}\varepsilon'_{kl} = \mathbb{C}^e_{ijkl}\varepsilon'_{kl} \quad \Rightarrow \quad \mathbb{C}'^e_{ijkl} = \mathbb{C}^e_{ijkl}$$

That is, the fourth-order tensor $\mathbb{C}^e$ is isotropic. An isotropic symmetric fourth-order tensor has the form $\mathbb{C}^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ or $\mathbb{C}^e = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}$, (see Chapter 1), and here the parameters λ and μ are known as *Lamé constants*. As we have seen in Chapter 1, a symmetric isotropic fourth-order tensor is a function of two variables (λ, μ). We will see that it is possible to express $\mathbb{C}^e$ in terms of other parameters, e.g. (E, ν), (κ, G), where E is the *Young's modulus* (or *longitudinal elastic modulus*), ν is the *Poisson's ratio*, κ is the *bulk modulus*, and $G = \mu$ is the *shear modulus* (or *transversal elastic modulus*).

NOTE 5: Figure 5.3 shows the stress-strain relationship for an isotropic material. Note that, for an isotropic linear elastic material in an infinitesimal strain regime the constitutive equation for stress becomes $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = (\lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}) : \boldsymbol{\varepsilon} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$:

$$\underbrace{\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}}_{\textbf{Elastic}} \quad \xrightarrow{\textbf{linear}} \quad \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \mathbb{C}^e : \boldsymbol{\varepsilon} \quad \xrightarrow{\textbf{isotropic}} \quad \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

It should be emphasized here that due to the fact that the $\mathbb{C}^e$-components are independent of the coordinate system, the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ share the same principal space (eigenvectors), (see Figure 5.3).

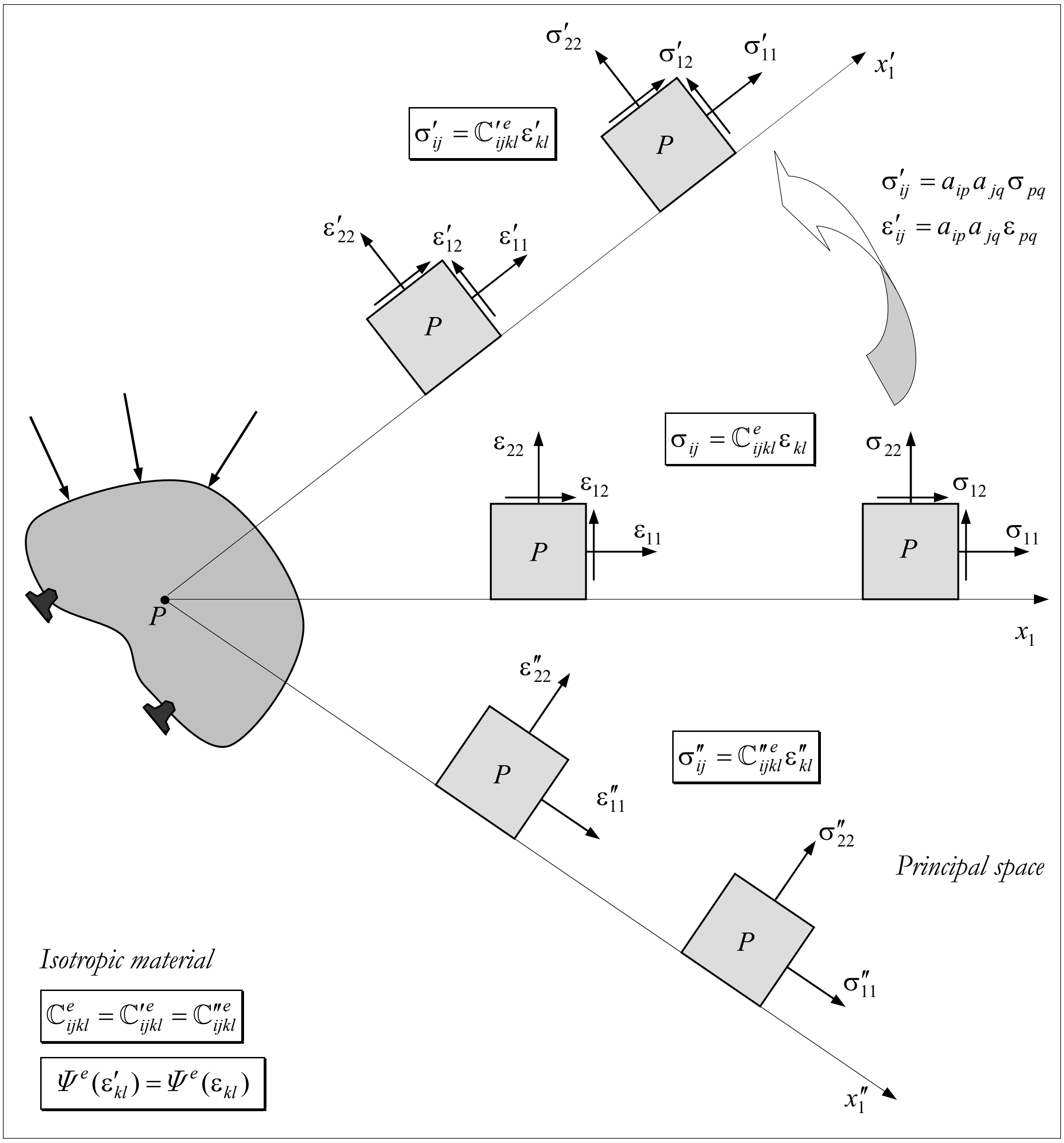


Figure 5.3: Stress-strain relationship (isotropic material).

NOTE 6: We denote the *complementary strain energy density* by $\overline{\Psi}^e(\boldsymbol{\sigma})$ which is a function of $\boldsymbol{\sigma}$, (see Figure 5.4), and is given by:

$$\begin{aligned}
\overline{\Psi}^e(\boldsymbol{\sigma}) &= \frac{1}{0!}\overline{\Psi}^e(\boldsymbol{\sigma}_0) + \frac{1}{1!}\frac{\partial \overline{\Psi}^e(\boldsymbol{\sigma}_0)}{\partial \boldsymbol{\sigma}} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \frac{1}{2!}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) : \frac{\partial^2 \overline{\Psi}^e(\boldsymbol{\sigma}_0)}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \cdots \\
&= \overline{\Psi}^e_0 + \boldsymbol{\sigma}_0 : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) : \frac{\partial^2 \overline{\Psi}^e(\boldsymbol{\sigma}_0)}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \cdots \\
&= \frac{1}{2}\boldsymbol{\sigma} : \frac{\partial^2 \overline{\Psi}^e(\boldsymbol{\sigma}_0)}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{D}^e : \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}
\end{aligned}$$

Note that if we are dealing with linear elastic material $\overline{\Psi}^e(\boldsymbol{\sigma}) = \Psi^e(\boldsymbol{\varepsilon})$ holds, and $\boldsymbol{\varepsilon} = \dfrac{\partial \overline{\Psi}^e(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$.

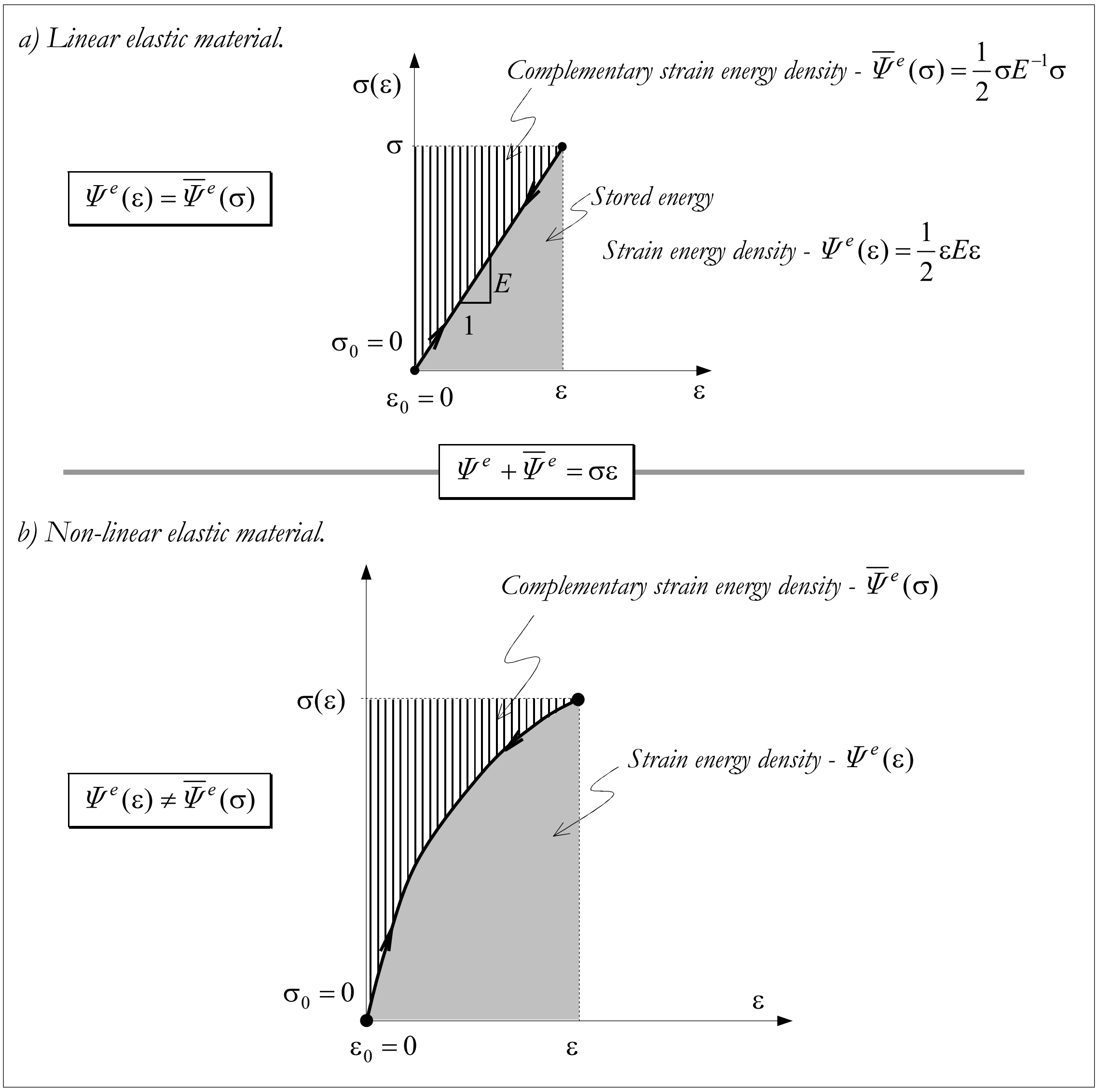


Figure 5.4: Complementary strain energy density (one dimensional case).

NOTE 7: Note that $\overline{\Psi}^e(\sigma) = \sigma\varepsilon - \Psi^e(\varepsilon) \xrightarrow{tensorial} \overline{\Psi}^e(\boldsymbol{\sigma}) = \mathrm{g} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Psi^e(\boldsymbol{\varepsilon}) = -\rho_0 \mathrm{G}(\boldsymbol{\sigma})$, where $\mathrm{g}(\boldsymbol{\sigma}) = -\rho_0 \mathrm{G}(\boldsymbol{\sigma})$ is the Gibbs free energy density (per unit volume) with reversed sign, (see equations in (5.6) in **Problem 5.3**).

NOTE 8: Taking into account the constitutive equation for stress for an isotropic linear elastic material $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ and considering the additive decomposition of the tensor into a spherical and deviatoric parts, i.e. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev} = \dfrac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} + \boldsymbol{\varepsilon}^{dev}$, we can obtain:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu(\boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\left(\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} + \boldsymbol{\varepsilon}^{dev}\right)$$

$$= \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} = \kappa\, \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev}$$

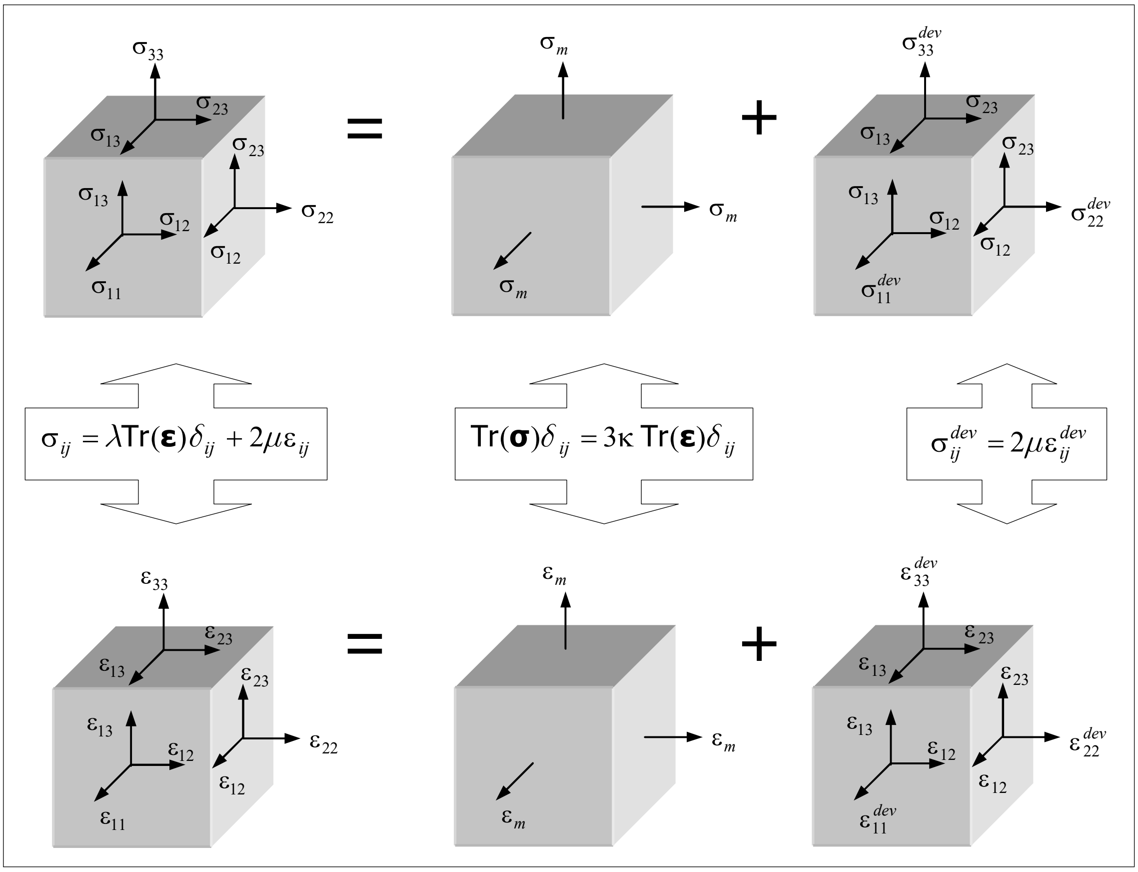


Figure 5.5: Additive decomposition of the constitutive equation.

Recall that, if we are dealing with small deformation regime, the volume ratio (dilatation) is given by:

$$\varepsilon_v = \frac{dV - dV_0}{dV_0} = \frac{\Delta V}{dV_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = I_{\boldsymbol{\varepsilon}}$$

And if we take the trace of $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \left(\lambda + \dfrac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev}$ we can obtain:

$$\boldsymbol{\sigma} : \mathbf{1} = \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} : \mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} : \mathbf{1}$$

$$\Rightarrow \mathrm{Tr}(\boldsymbol{\sigma}) = 3\left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3} = \sigma_m = \left(\lambda + \frac{2\mu}{3}\right)\varepsilon_v$$

where we have considered that $\mathbf{1}:\mathbf{1}=3$ and $\mathrm{Tr}(\boldsymbol{\varepsilon}^{dev})=0$. If we are dealing with a compression stress state ($p>0$) we have:

$$\sigma_{ij}=\begin{bmatrix}-p & 0 & 0\\ 0 & -p & 0\\ 0 & 0 & -p\end{bmatrix} \quad \therefore \quad 3\sigma_m=\mathrm{Tr}(\boldsymbol{\sigma})=-3p<0 \quad \Rightarrow \quad -p=\left(\lambda+\frac{2\mu}{3}\right)\varepsilon_v=\kappa\,\varepsilon_v$$

For these reason, the parameter κ is called *bulk modulus* (or *modulus of compression*), (see Figure 5.6), and is given by:

$$\kappa=\lambda+\frac{2\mu}{3} \tag{5.18}$$

Just as the spherical part of the tensor ($\boldsymbol{\sigma}^{sph}=\kappa\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$) is associated with the volume change, the deviatoric part ($\boldsymbol{\sigma}^{dev}=2\mu\boldsymbol{\varepsilon}^{dev}$) is associated with the shape change, and the parameter $\mu=G$ defines the stiffness to the shape change, where G is known as *shear modulus* or *transversal elastic modulus*, (see NOTE 9).

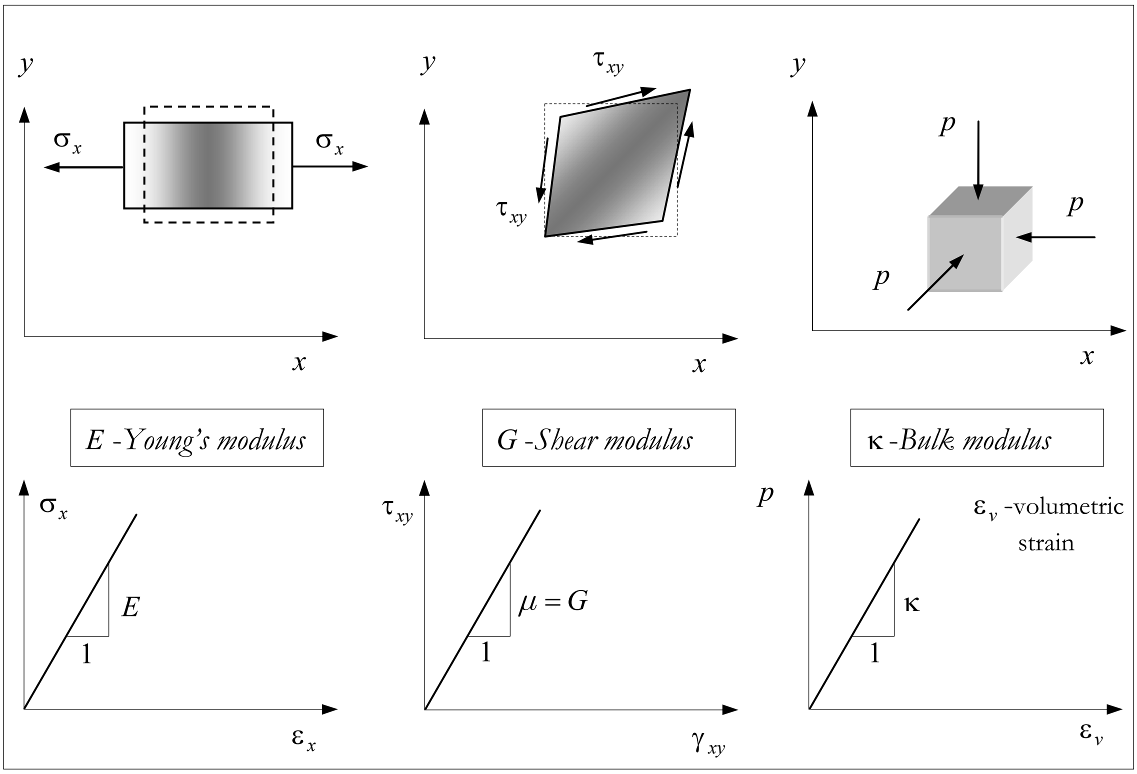


Figure 5.6: Some material mechanical properties.

NOTE 9: In the laboratory the parameters (λ,μ) are not the more appropriated to be obtained. Next we try to rewrite the constitutive equation in terms of other parameters. Recall that the reverse form of the constitutive equation $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}$ was obtained in **Problem 1.98** which is:

$$\boldsymbol{\varepsilon}=\frac{1}{2\mu}\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} \xrightarrow{indicial} \varepsilon_{ij}=\frac{1}{2\mu}\sigma_{ij}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{ij}$$

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}=\frac{1}{2\mu}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{bmatrix}-\frac{\lambda(\sigma_{11}+\sigma_{22}+\sigma_{33})}{2\mu(2\mu+3\lambda)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{5.19}$$

Notice also that the normal stress components σ_{11}, σ_{22}, and σ_{33} only produce normal strain components. Let us consider a particular case in which we only have the normal stress σ_{11}, $\sigma_{22}=0$, $\sigma_{33}=0$, then:

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}=\frac{1}{2\mu}\begin{bmatrix}\sigma_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\frac{\lambda(\sigma_{11})}{2\mu(2\mu+3\lambda)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

with that the normal strain components are:

$$\varepsilon_{11}=\frac{1}{2\mu}\sigma_{11}-\frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11})\delta_{11}=\left(\frac{1}{2\mu}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\right)\sigma_{11}=\left(\frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\right)\sigma_{11} \tag{5.20}$$

$$\varepsilon_{22}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11})\delta_{22}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\sigma_{11} \tag{5.21}$$

$$\varepsilon_{33}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}(\sigma_{11})\delta_{33}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\sigma_{11} \tag{5.22}$$

From the equation for ε_{11} given by the equation in (5.20) we can obtain:

$$\varepsilon_{11}=\left(\frac{(\mu+\lambda)}{\mu(2\mu+3\lambda)}\right)\sigma_{11} \quad\Rightarrow\quad \sigma_{11}=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\varepsilon_{11} \quad\Rightarrow\quad \sigma_{11}=E\varepsilon_{11}$$

where we have denoted by $E=\dfrac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$, which is known as *Young's modulus*, or *longitudinal elastic modulus*.

As expected, due to the material isotropy, the influence of σ_{11} upon ε_{22} and ε_{33} is the same, and we can also obtain:

$$\varepsilon_{22}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\sigma_{11}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\left[\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\varepsilon_{11}\right]=\frac{-\lambda}{2(\lambda+\mu)}\varepsilon_{11}=-\nu\,\varepsilon_{11}$$

$$\varepsilon_{33}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\sigma_{11}=\frac{-\lambda}{2\mu(2\mu+3\lambda)}\left[\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\varepsilon_{11}\right]=\frac{-\lambda}{2(\lambda+\mu)}\varepsilon_{11}=-\nu\,\varepsilon_{11}$$

where we have denoted by $\nu=\dfrac{\lambda}{2(\lambda+\mu)}$, which is known as *Poisson's ratio*. And the Poisson's ratio can assume $-1.0<\nu<0.5$, (see **Problem 1.92**). Note that $\nu=\dfrac{\lambda}{2(\lambda+\mu)}\Rightarrow\lambda=\dfrac{2\nu\mu}{(1-2\nu)}$ and if we replace it into the equation of E we can obtain:

$$E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}=\mu\frac{\left[3\left(\frac{2\nu\mu}{(1-2\nu)}\right)+2\mu\right]}{\left[\left(\frac{2\nu\mu}{(1-2\nu)}\right)+\mu\right]}=\mu\frac{\left[3\left(\frac{2\nu}{(1-2\nu)}\right)+2\right]\mu}{\left[\left(\frac{2\nu}{(1-2\nu)}\right)+1\right]\mu}=\mu\frac{\left[\frac{6\nu}{(1-2\nu)}+2\right]}{\left[\frac{2\nu}{(1-2\nu)}+1\right]}$$

$$=\mu\frac{\left[\frac{6\nu+2(1-2\nu)}{(1-2\nu)}\right]}{\left[\frac{2\nu+(1-2\nu)}{(1-2\nu)}\right]}=2\mu(1+\nu)$$

thus:

$$G=\mu=\frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda=\frac{2\nu\mu}{(1-2\nu)}=\frac{\nu E}{(1+\nu)(1-2\nu)}$$

The physical interpretation of μ comes next, (see Figure 5.6). Let us suppose a stress state in which is acting just the component σ_{12}, with that and according to the equation in (5.19) we can obtain the only strain not equal to zero:

$$\varepsilon_{12}=\varepsilon_{21}=\frac{1}{2\mu}\sigma_{12} \quad \Rightarrow \quad \underbrace{\sigma_{12}}_{\tau_{xy}}=\mu\,\underbrace{2\varepsilon_{12}}_{\gamma_{xy}} \quad \Rightarrow \quad \tau_{xy}=\mu\,\gamma_{xy}=G\,\gamma_{xy} \quad \therefore \quad G=\mu$$

We can also express the bulk modulus in function of (E,ν):

$$\kappa=\lambda+\frac{2\mu}{3}=\frac{\nu E}{(1+\nu)(1-2\nu)}+\frac{2}{3}\frac{E}{[2(1+\nu)]}=\frac{3\nu E+E(1-2\nu)}{3(1+\nu)(1-2\nu)}=\frac{E(1+\nu)}{3(1+\nu)(1-2\nu)}=\frac{E}{3(1-2\nu)}$$

So, we can obtain the relationships between these mechanical properties:

	$G=\mu=$	$E=$	$\kappa=$	$\lambda=$	$\nu=$
$f(G;E)$	G	E	$\dfrac{GE}{9G-3E}$	$\dfrac{G(E-2G)}{3G-E}$	$\dfrac{E-2G}{2G}$
$f(G;\kappa)$	G	$\dfrac{9G\kappa}{3\kappa+G}$	κ	$\kappa-\dfrac{2G}{3}$	$\dfrac{3\kappa-2G}{2(3\kappa+G)}$
$f(G;\lambda)$	G	$\dfrac{G(3\lambda+2G)}{\lambda+G}$	$\lambda+\dfrac{2}{3}G$	λ	$\dfrac{\lambda}{2(\lambda+G)}$
$f(G;\nu)$	G	$2G(1+\nu)$	$\dfrac{2G(1+\nu)}{3(1-2\nu)}$	$\dfrac{2G\nu}{1-2\nu}$	ν
$f(E;\kappa)$	$\dfrac{3\kappa E}{9\kappa-E}$	E	κ	$\dfrac{\kappa(9\kappa-3E)}{9\kappa-E}$	$\dfrac{3\kappa-E}{6\kappa}$
$f(E;\nu)$	$\dfrac{E}{2(1+\nu)}$	E	$\dfrac{E}{3(1-2\nu)}$	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$	ν
$f(\kappa;\lambda)$	$\dfrac{3(\kappa-\lambda)}{2}$	$\dfrac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	κ	λ	$\dfrac{\lambda}{3\kappa-\lambda}$
$f(\kappa;\nu)$	$\dfrac{3\kappa(1-2\nu)}{2(1+\nu)}$	$3\kappa(1-2\nu)$	κ	$\dfrac{3\kappa\nu}{1+\nu}$	ν

We leave the reader to show:

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\sigma}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}}$	$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$	(5.23)
$\boxed{\boldsymbol{\sigma}=\dfrac{\nu E}{(1+\nu)(1-2\nu)}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+\dfrac{E}{(1+\nu)}\boldsymbol{\varepsilon}}$	$\sigma_{ij}=\dfrac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}+\dfrac{E}{(1+\nu)}\varepsilon_{ij}$	(5.24)
$\boxed{\boldsymbol{\sigma}=\left(\kappa-\dfrac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}}$	$\sigma_{ij}=\left(\kappa-\dfrac{2\mu}{3}\right)\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$	(5.25)

and

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\varepsilon} = \dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \dfrac{1}{2\mu}\boldsymbol{\sigma}}$	$\varepsilon_{ij} = \dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} + \dfrac{1}{2\mu}\sigma_{ij}$	(5.26)
$\boxed{\boldsymbol{\varepsilon} = \dfrac{-\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \dfrac{1+\nu}{E}\boldsymbol{\sigma}}$	$\varepsilon_{ij} = \dfrac{-\nu}{E}\sigma_{kk}\delta_{ij} + \dfrac{1+\nu}{E}\sigma_{ij}$	(5.27)
$\boxed{\boldsymbol{\varepsilon} = \left(\dfrac{2\mu-3\kappa}{18\kappa\mu}\right)\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} - \dfrac{1}{2\mu}\boldsymbol{\sigma}}$	$\varepsilon_{ij} = \left(\dfrac{2\mu-3\kappa}{18\kappa\mu}\right)\sigma_{kk}\delta_{ij} - \dfrac{1}{2\mu}\sigma_{ij}$	(5.28)

and that the elasticity tensor for isotropic material can be written as follows:

$$\boxed{\begin{aligned}\mathbb{C}^e &= \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}\\ \mathbb{C}^e &= \frac{\nu E}{(1+\nu)(1-2\nu)}\mathbf{1}\otimes\mathbf{1} + \frac{E}{(1+\nu)}\mathbf{I}\\ \mathbb{C}^e &= \kappa\mathbf{1}\otimes\mathbf{1} + 2\mu\left[\mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right]\end{aligned}} \quad \textit{Elasticity tensor} \qquad (5.29)$$

and:

$$\boxed{\begin{aligned}\mathbb{C}^{e^{-1}} \equiv \mathbb{D}^e &= \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\mathbf{I}\\ \mathbb{C}^{e^{-1}} \equiv \mathbb{D}^e &= \frac{-\nu}{E}\mathbf{1}\otimes\mathbf{1} + \frac{(1+\nu)}{E}\mathbf{I}\\ \mathbb{C}^{e^{-1}} \equiv \mathbb{D}^e &= \frac{1}{9\kappa}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\left[\mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right]\end{aligned}} \quad \textit{Elastic compliance tensor} \qquad (5.30)$$

where $\mathbf{I} \equiv \mathbb{I}^{sym}$ is the symmetric unit fourth-order tensor. In the International System of Units we have $[G]=[\mu]=[\lambda]=[\kappa]=[E]=Pa$, and ν is a dimensionless quantity.

Problem 5.6

In tensile testing we have evaluated the following points:

Point	$\sigma(Pa)$	$\varepsilon(\times 10^{-3})$
1	6.67	0.667
2	13.3	1.33
3	20	2
4	24	3
5	22	3.6

Calculate Young's modulus (E) and define the stress-strain curve limit points.

Solution: First, we verify that the first three points maintain the same proportionalities:

$$E = \frac{\sigma^{(1)}}{\varepsilon^{(1)}} = \frac{\sigma^{(2)}}{\varepsilon^{(2)}} = \frac{\sigma^{(3)}}{\varepsilon^{(3)}} = \frac{20}{2\times 10^{-3}} = 10\,000\,Pa = 10kPa$$

The stress-strain curve can be appreciated in Figure 5.7, in which we define the following points: σ^e - the proportionality point; σ_Y - the yield point; σ^u - the ultimate strength point; and σ^r - the rupture strength point.

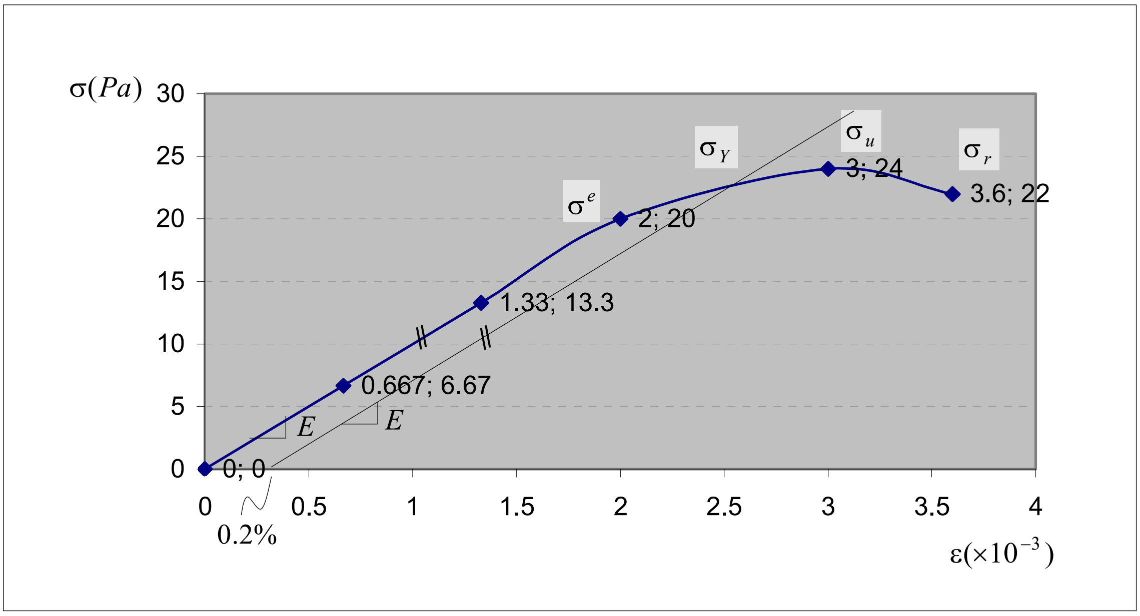


Figure 5.7: Stress-strain curve.

Problem 5.7

Show that the strain energy density, for an isotropic linear elastic material, can be written as follows:

a)
$$\boxed{\Psi^e(\boldsymbol{\varepsilon}) = \frac{1}{2}(\lambda+2\mu)I_{\boldsymbol{\varepsilon}}^2 - 2\mu II_{\boldsymbol{\varepsilon}}} \tag{5.31}$$

or

b)
$$\boxed{\overline{\Psi}^e(\boldsymbol{\sigma}) = \frac{(\lambda+\mu)}{2\mu(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2 - \frac{1}{2\mu} II_{\boldsymbol{\sigma}}} \tag{5.32}$$

or

c)
$$\boxed{\Psi^e(\boldsymbol{\varepsilon}) = \underbrace{\frac{\kappa}{2}[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}_{\text{purely volumetric energy}} + \underbrace{\mu\,\boldsymbol{\varepsilon}^{dev} : \boldsymbol{\varepsilon}^{dev}}_{\text{purely distortional energy}}} \tag{5.33}$$

or

d)
$$\boxed{\overline{\Psi}^e(\boldsymbol{\sigma}) = \underbrace{\frac{1}{6(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2}_{\text{purely volumetric energy}} + \underbrace{\frac{1}{2\mu} J_2}_{\text{purely distortional energy}}} \tag{5.34}$$

where $I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon})$ is the trace of $\boldsymbol{\varepsilon}$ (infinitesimal strain tensor), $I_{\boldsymbol{\sigma}} = \mathrm{Tr}(\boldsymbol{\sigma})$ is the first invariant of the Cauchy stress tensor $\boldsymbol{\sigma}$, and $II_{\boldsymbol{\sigma}^{dev}} = -J_2$ is the second invariant of the deviatoric Cauchy stress tensor. Note that, for linear elastic material the relationship $\overline{\Psi}^e(\boldsymbol{\sigma}) = \Psi^e(\boldsymbol{\varepsilon})$ holds, (see Figure 5.4).

Solution:

a) Taking into account the strain energy $\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$, (see equation (5.23)), we can obtain:

$$\begin{aligned}
\Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}:\left[\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\right] \\
&= \frac{1}{2}\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\underbrace{\boldsymbol{\varepsilon}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{1}{2}\lambda\left[\mathrm{Tr}(\boldsymbol{\varepsilon})\right]^2 + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{1}{2}\lambda I_{\boldsymbol{\varepsilon}}^2 + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}
\end{aligned} \tag{5.35}$$

Taking into account the definition of the second invariant and the symmetry of $\boldsymbol{\varepsilon}$ we can obtain:

$$\begin{aligned}
&I\!I_{\boldsymbol{\varepsilon}} = \frac{1}{2}\left[[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right] = \frac{1}{2}\left[I_{\boldsymbol{\varepsilon}}^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon})\right] = \frac{1}{2}\left[I_{\boldsymbol{\varepsilon}}^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^T)\right] = \frac{1}{2}\left[I_{\boldsymbol{\varepsilon}}^2 - \boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\right] \\
&\Rightarrow \boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = I_{\boldsymbol{\varepsilon}}^2 - 2\,I\!I_{\boldsymbol{\varepsilon}}
\end{aligned} \tag{5.36}$$

Then, the equation in (5.35) can be rewritten as follows:

$$\Psi^e = \frac{1}{2}\lambda I_{\boldsymbol{\varepsilon}}^2 + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{1}{2}\lambda I_{\boldsymbol{\varepsilon}}^2 + \mu\,(I_{\boldsymbol{\varepsilon}}^2 - 2\,I\!I_{\boldsymbol{\varepsilon}}) = \frac{1}{2}(\lambda + 2\mu)I_{\boldsymbol{\varepsilon}}^2 - 2\mu I\!I_{\boldsymbol{\varepsilon}}$$

b) Taking into account the strain energy density $\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma}$ and

$\boldsymbol{\varepsilon} = \dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \dfrac{1}{2\mu}\boldsymbol{\sigma}$, (see equation (5.26)), we can obtain:

$$\begin{aligned}
\Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\left[\frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}\right]:\boldsymbol{\sigma} \\
&= \frac{-\lambda}{4\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\underbrace{\boldsymbol{\sigma}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\sigma})} + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} = \frac{-\lambda}{4\mu(3\lambda+2\mu)}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^2 + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} \\
&= \frac{-\lambda}{4\mu(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^2 + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma}
\end{aligned} \tag{5.37}$$

According to the equation (5.36) we can conclude that $\boldsymbol{\sigma}:\boldsymbol{\sigma} = I_{\boldsymbol{\sigma}}^2 - 2\,I\!I_{\boldsymbol{\sigma}}$, with that the above equation becomes:

$$\begin{aligned}
\Psi^e &= \frac{-\lambda}{4\mu(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^2 + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} = \frac{-\lambda}{4\mu(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^2 + \frac{1}{4\mu}(I_{\boldsymbol{\sigma}}^2 - 2\,I\!I_{\boldsymbol{\sigma}}) \\
&= \frac{(\lambda+\mu)}{2\mu(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^2 - \frac{1}{2\mu}I\!I_{\boldsymbol{\sigma}}
\end{aligned}$$

c) Taking into account the strain energy $\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma}$ and $\boldsymbol{\sigma} = \left(\kappa - \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$, (see equation (5.25)), we can obtain:

$$\begin{aligned}
\Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}:\left[\left(\kappa - \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\right] \\
&= \frac{1}{2}\left(\kappa - \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\underbrace{\boldsymbol{\varepsilon}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \left(\frac{\kappa}{2} - \frac{\mu}{3}\right)\left[\mathrm{Tr}(\boldsymbol{\varepsilon})\right]^2 + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}
\end{aligned} \tag{5.38}$$

If we consider that a second-order tensor can be split additively into a spherical and deviatoric parts, i.e. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev} = \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} + \boldsymbol{\varepsilon}^{dev}$, the expression $\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}$ can be written as:

$$
\begin{aligned}
\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} &= \left(\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} + \boldsymbol{\varepsilon}^{dev}\right):\left(\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} + \boldsymbol{\varepsilon}^{dev}\right) \\
&= \left(\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\right)^2 \mathbf{1}:\mathbf{1} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}:\boldsymbol{\varepsilon}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\boldsymbol{\varepsilon}^{dev}:\mathbf{1} + \boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev} \\
&= \frac{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{3} + \boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev}
\end{aligned}
\tag{5.39}
$$

where we have applied that $\mathbf{1}:\mathbf{1} = 3$, $\mathbf{1}:\boldsymbol{\varepsilon}^{dev} = \boldsymbol{\varepsilon}^{dev}:\mathbf{1} = \mathrm{Tr}(\boldsymbol{\varepsilon}^{dev}) = 0$ (the trace of any deviatoric tensor is zero). With that the equation in (5.38) becomes:

$$
\begin{aligned}
\Psi^e &= \left(\frac{\kappa}{2} - \frac{\mu}{3}\right)[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \left(\frac{\kappa}{2} - \frac{\mu}{3}\right)[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 + \mu\left(\frac{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{3} + \boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev}\right) \\
&= \frac{\kappa}{2}[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 + \mu\,\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev}
\end{aligned}
$$

d) To show the equation (5.34) we will use the strain tensor defined in (5.26), $\boldsymbol{\varepsilon} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}$. Then, the strain energy density can be written as:

$$
\begin{aligned}
\Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\left[\frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}\right]:\boldsymbol{\sigma} \\
&= \frac{-\lambda}{4\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\underbrace{\mathbf{1}:\boldsymbol{\sigma}}_{\mathrm{Tr}(\boldsymbol{\sigma})} + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} = \frac{-\lambda}{4\mu(3\lambda+2\mu)}[\mathrm{Tr}(\boldsymbol{\sigma})]^2 + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma}
\end{aligned}
\tag{5.40}
$$

Note that $\boldsymbol{\sigma}:\boldsymbol{\sigma} = \frac{[\mathrm{Tr}(\boldsymbol{\sigma})]^2}{3} + \boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}$ holds, (see equation (5.39)). Taking into account the equation of the second invariant of a second-order tensor we can obtain:

$$
\begin{aligned}
I\!I_{\boldsymbol{\sigma}^{dev}} &= \frac{1}{2}\left[[\mathrm{Tr}(\boldsymbol{\sigma}^{dev})]^2 - \mathrm{Tr}(\boldsymbol{\sigma}^{dev^2})\right] = \frac{-1}{2}\mathrm{Tr}(\boldsymbol{\sigma}^{dev^2}) \\
&= \frac{-1}{2}\mathrm{Tr}(\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{\sigma}^{dev}) = \frac{-1}{2}\mathrm{Tr}(\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{\sigma}^{dev^T}) = \frac{-1}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}
\end{aligned}
$$

where we have used that: the trace of the deviatoric tensor is zero $\mathrm{Tr}(\boldsymbol{\sigma}^{dev}) = 0$, the symmetry of the tensor $\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma}^{dev^T}$, and trace property $\mathrm{Tr}(\mathbf{A}\cdot\mathbf{B}^T) = \mathbf{A}:\mathbf{B}$. Then, we can obtain:

$$
\boldsymbol{\sigma}:\boldsymbol{\sigma} = \frac{[\mathrm{Tr}(\boldsymbol{\sigma})]^2}{3} + \boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev} = \frac{[\mathrm{Tr}(\boldsymbol{\sigma})]^2}{3} - 2\,I\!I_{\boldsymbol{\sigma}^{dev}} = \frac{[\mathrm{Tr}(\boldsymbol{\sigma})]^2}{3} + 2\mathrm{J}_2
$$

By substituting the above equation into the equation in (5.40), we can obtain:

$$
\begin{aligned}
\Psi^{e} &= \frac{-\lambda}{4\mu(3\lambda+2\mu)}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^{2}+\frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} \\
&= \frac{-\lambda}{4\mu(3\lambda+2\mu)}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^{2}+\frac{1}{4\mu}\left(\frac{\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^{2}}{3}+2\mathrm{J}_{2}\right) \\
&= \left(\frac{-\lambda}{4\mu(3\lambda+2\mu)}+\frac{1}{12\mu}\right)\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^{2}+\frac{1}{2\mu}\mathrm{J}_{2} \\
&= \frac{1}{6(3\lambda+2\mu)}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]^{2}+\frac{1}{2\mu}\mathrm{J}_{2}
\end{aligned}
$$

Problem 5.8

Write in Voigt notation: a.1) the strain energy density and, a.2) the constitutive equations in stress for an isotropic linear elastic material: a.2.1) in terms of (λ,μ) and, a.2.2) in terms of (E,ν) where $\lambda=\dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu=\dfrac{E}{2(1+\nu)}$. b) Write the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ in Voigt notation such as $\{\boldsymbol{\varepsilon}\}=[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\}$ where $\{\boldsymbol{u}\}$ is the displacement field, obtain the matrix $[\boldsymbol{L}^{(1)}]$.

c) Write the equations of motion in Voigt notation.

Solution:

a.1) The strain energy density ($\Psi^{e}(\boldsymbol{\varepsilon})$-scalar) can be expressed as follows:

$$
\Psi^{e}(\boldsymbol{\varepsilon})=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}^{e}:\boldsymbol{\varepsilon}=\frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma}=\frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}
$$

where we have used $\boldsymbol{\sigma}=\mathbb{C}^{e}:\boldsymbol{\varepsilon}$. Note that

$$
\sigma_{ij}\varepsilon_{ij} = \underbrace{\sigma_{1j}\varepsilon_{1j}}_{\substack{\sigma_{11}\varepsilon_{11}\\+\\\sigma_{12}\varepsilon_{12}\\+\\\sigma_{13}\varepsilon_{13}}} + \underbrace{\sigma_{2j}\varepsilon_{2j}}_{\substack{\sigma_{21}\varepsilon_{21}\\+\\\sigma_{22}\varepsilon_{22}\\+\\\sigma_{23}\varepsilon_{23}}} + \underbrace{\sigma_{3j}\varepsilon_{3j}}_{\substack{\sigma_{31}\varepsilon_{31}\\+\\\sigma_{32}\varepsilon_{32}\\+\\\sigma_{33}\varepsilon_{33}}}
$$

thus

$$
\Psi^{e}(\boldsymbol{\varepsilon})=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{33}\varepsilon_{33}+2\sigma_{12}\varepsilon_{12}+2\sigma_{23}\varepsilon_{23}+2\sigma_{13}\varepsilon_{13})
$$

and

$$
\Psi^{e}(\boldsymbol{\varepsilon})=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}\begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix}=\frac{1}{2}\{\boldsymbol{\sigma}\}^{T}\{\boldsymbol{\varepsilon}\}
$$

Then, the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in Voigt notation are stored as follows:

$$\{\boldsymbol{\sigma}\} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \quad ; \quad \{\boldsymbol{\varepsilon}\} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix}$$

a.2.1) The constitutive equation for stress in Voigt notation is:

$$\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon} \xrightarrow{Voigt} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} \Rightarrow \{\boldsymbol{\sigma}\} = [\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}\} \tag{5.41}$$

More detail about this formulation is provided in **Problem 1.98** in Chapter 1 where we have also obtained

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

and

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{-\lambda}{2\mu(2\mu+3\lambda)} & \frac{\mu+\lambda}{\mu(2\mu+3\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} \tag{5.42}$$

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{\mathcal{C}}]^{-1}\{\boldsymbol{\sigma}\}$$

a.2.2) Note that

$$\lambda+2\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} + 2\frac{E}{2(1+\nu)} = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu)$$

$$\lambda = \frac{E}{(1+\nu)(1-2\nu)}\nu$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{E}{(1+\nu)(1-2\nu)}\frac{(1-2\nu)}{2}$$

then, the equation (5.41) can be rewritten as follows:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} \tag{5.43}
$$

Note that

$$
\lambda+\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{2(1+\nu)} = \frac{E}{2(1+\nu)(1-2\nu)}
$$

$$
\mu(2\mu+3\lambda) = \frac{E}{2(1+\nu)}\left[2\frac{E}{2(1+\nu)} + 3\frac{E\nu}{(1+\nu)(1-2\nu)}\right] = \frac{E^2}{2(1+\nu)(1-2\nu)}
$$

$$
\frac{\lambda+\mu}{\mu(2\mu+3\lambda)} = \frac{E}{2(1+\nu)(1-2\nu)}\frac{2(1+\nu)(1-2\nu)}{E^2} = \frac{1}{E}
$$

$$
\frac{\lambda}{2\mu(2\mu+3\lambda)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{(1+\nu)(1-2\nu)}{E^2} = \frac{\nu}{E}
$$

$$
\frac{1}{\mu} = \frac{2(1+\nu)}{E} = \frac{1}{E}2(1+\nu)
$$

Then, the equation (5.42) becomes:

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} \tag{5.44}
$$

b) According to the definition $\varepsilon_{ij} = \frac{1}{2}(\mathsf{u}_{i,j} + \mathsf{u}_{j,i})$ we can obtain:

$$
\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathsf{u}_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial \mathsf{u}_1}{\partial x_2} + \frac{\partial \mathsf{u}_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial \mathsf{u}_1}{\partial x_3} + \frac{\partial \mathsf{u}_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial \mathsf{u}_1}{\partial x_2} + \frac{\partial \mathsf{u}_2}{\partial x_1}\right) & \frac{\partial \mathsf{u}_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial \mathsf{u}_2}{\partial x_3} + \frac{\partial \mathsf{u}_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial \mathsf{u}_1}{\partial x_3} + \frac{\partial \mathsf{u}_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial \mathsf{u}_2}{\partial x_3} + \frac{\partial \mathsf{u}_3}{\partial x_2}\right) & \frac{\partial \mathsf{u}_3}{\partial x_3} \end{bmatrix}
$$

$$\{\boldsymbol{\varepsilon}\}=\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial \mathsf{u}_1}{\partial x_1}\\ \dfrac{\partial \mathsf{u}_2}{\partial x_2}\\ \dfrac{\partial \mathsf{u}_3}{\partial x_3}\\ \dfrac{\partial \mathsf{u}_1}{\partial x_2}+\dfrac{\partial \mathsf{u}_2}{\partial x_1}\\ \dfrac{\partial \mathsf{u}_2}{\partial x_3}+\dfrac{\partial \mathsf{u}_3}{\partial x_2}\\ \dfrac{\partial \mathsf{u}_1}{\partial x_3}+\dfrac{\partial \mathsf{u}_3}{\partial x_1}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial}{\partial x_1} & 0 & 0\\ 0 & \dfrac{\partial}{\partial x_2} & 0\\ 0 & 0 & \dfrac{\partial}{\partial x_3}\\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0\\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1}\end{bmatrix}\begin{Bmatrix}\mathsf{u}_1\\ \mathsf{u}_2\\ \mathsf{u}_3\end{Bmatrix} \quad\Rightarrow\quad \{\boldsymbol{\varepsilon}(\vec{x})\}=[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}(\vec{x})\}$$

NOTE: If we adopt the engineering notation, i.e. $x_1=x$, $x_2=y$, $x_3=z$, $\mathsf{u}_1=u$, $\mathsf{u}_2=v$, $\mathsf{u}_3=w$, $\varepsilon_{11}=\varepsilon_x$, $\varepsilon_{22}=\varepsilon_y$, $\varepsilon_{33}=\varepsilon_z$, $2\varepsilon_{12}=\gamma_{xy}$, $2\varepsilon_{23}=\gamma_{yz}$, $2\varepsilon_{13}=\gamma_{xz}$, the above equation becomes:

$$\{\boldsymbol{\varepsilon}\}=\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial z}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0 & 0\\ 0 & \dfrac{\partial}{\partial y} & 0\\ 0 & 0 & \dfrac{\partial}{\partial z}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\end{Bmatrix} \quad\Rightarrow\quad \{\boldsymbol{\varepsilon}(\vec{x})\}=[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}(\vec{x})\} \qquad (5.45)$$

c) Let us consider the equations of motion, $\nabla\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\rho\dot{\vec{v}}=\rho\ddot{\vec{\mathbf{u}}}$, (see equation (5.14)), in indicial notation $\sigma_{ij,j}+\rho\mathsf{b}_i=\rho\ddot{\mathsf{u}}_i$ and its explicit form:

$$\sigma_{ij,j}+\rho\mathsf{b}_i=\sigma_{i1,1}+\sigma_{i2,2}+\sigma_{i3,3}+\rho\mathsf{b}_i=\rho\ddot{\mathsf{u}}_i$$

$$\Rightarrow\begin{cases}\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}+\rho\mathsf{b}_1=\rho\ddot{\mathsf{u}}_1\\ \sigma_{21,1}+\sigma_{22,2}+\sigma_{23,3}+\rho\mathsf{b}_2=\rho\ddot{\mathsf{u}}_2\\ \sigma_{31,1}+\sigma_{32,2}+\sigma_{33,3}+\rho\mathsf{b}_3=\rho\ddot{\mathsf{u}}_3\end{cases} \quad\Rightarrow\quad \begin{cases}\dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma_{12}}{\partial x_2}+\dfrac{\partial\sigma_{13}}{\partial x_3}+\rho\mathsf{b}_1=\rho\ddot{\mathsf{u}}_1\\ \dfrac{\partial\sigma_{21}}{\partial x_1}+\dfrac{\partial\sigma_{22}}{\partial x_2}+\dfrac{\partial\sigma_{23}}{\partial x_3}+\rho\mathsf{b}_2=\rho\ddot{\mathsf{u}}_2\\ \dfrac{\partial\sigma_{31}}{\partial x_1}+\dfrac{\partial\sigma_{32}}{\partial x_2}+\dfrac{\partial\sigma_{33}}{\partial x_3}+\rho\mathsf{b}_3=\rho\ddot{\mathsf{u}}_3\end{cases}$$

Then, if we consider the stress tensor in Voigt notation, the above set of equations becomes:

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} + \begin{Bmatrix} \rho \mathsf{b}_1 \\ \rho \mathsf{b}_2 \\ \rho \mathsf{b}_3 \end{Bmatrix} = \begin{Bmatrix} \rho \ddot{\mathsf{u}}_1 \\ \rho \ddot{\mathsf{u}}_2 \\ \rho \ddot{\mathsf{u}}_3 \end{Bmatrix} \tag{5.46}$$

$$\Rightarrow [\boldsymbol{L}^{(1)}]^T \{\boldsymbol{\sigma}\} + \{\rho \boldsymbol{b}\} = \{\rho \ddot{\boldsymbol{u}}\}$$

Problem 5.9

Consider an isotropic homogeneous linear elastic material described in **Problem 5.5**. Obtain the governing equation so as to result in a system of three equations and three unknowns, namely: u_1, u_2, u_3, *(Displacement Formulation* – established by Navier (1827)).

Solution:

As we have seen in **Problem 5.5**, the governing equations, for an isotropic linear elastic material in small deformation regime, are:

Tensorial notation	Indicial notation
The equations of motion: $\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \ddot{\vec{\mathbf{u}}}$ (3 equations)	*The equations of motion:* $\sigma_{ij,j} + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i$ (3 equations)
The constitutive equations for stress: $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu \boldsymbol{\varepsilon}$ (6 equations)	*The constitutive equations for stress:* $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$ (6 equations)
The kinematic equations: $\boldsymbol{\varepsilon} = \nabla^{sym} \vec{\mathbf{u}}$ (6 equations)	*The kinematic equations:* $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathsf{u}_i}{\partial x_j} + \frac{\partial \mathsf{u}_j}{\partial x_i} \right)$ (6 equations)

(5.47)

which results in a system with 15 equations and 15 unknowns $(\mathsf{u}_i, \sigma_{ij}, \varepsilon_{ij})$.

The divergence of the Cauchy stress tensor ($\nabla \cdot \boldsymbol{\sigma}$) can be obtained by means of the constitutive equations for stress, i.e.:

$$\begin{aligned} &\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu\, \varepsilon_{ij} \quad \Rightarrow \quad \sigma_{ij,j} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu\, \varepsilon_{ij})_{,j} \\ &\Rightarrow \sigma_{ij,j} = \underbrace{\lambda_{,j}}_{=0_j} \varepsilon_{kk} \delta_{ij} + \lambda \varepsilon_{kk,j} \delta_{ij} + \lambda \varepsilon_{kk} \underbrace{\delta_{ij,j}}_{=0_i} + 2\underbrace{\mu_{,j}}_{=0_j} \varepsilon_{ij} + 2\mu\, \varepsilon_{ij,j} \\ &\Rightarrow \sigma_{ij,j} = \lambda \varepsilon_{kk,j} \delta_{ij} + 2\mu\, \varepsilon_{ij,j} \quad \Rightarrow \quad \sigma_{ij,j} = \lambda \varepsilon_{kk,i} + 2\mu\, \varepsilon_{ij,j} \end{aligned} \tag{5.48}$$

Note that, if the mechanical properties λ and μ are constants throughout the medium, i.e. if they do not vary with $\vec{x}$ (*homogeneous material*) we can obtain $\lambda_{,j} \equiv \frac{\partial \lambda}{\partial x_j} = 0_j$ and $\mu_{,j} \equiv \frac{\partial \mu}{\partial x_j} = 0_j$. We can also express the terms $\varepsilon_{kk,i}$ and $\varepsilon_{ij,j}$ in function of displacements. For this, we use the kinematic equations:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathsf{u}_i}{\partial x_j} + \frac{\partial \mathsf{u}_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(\mathsf{u}_{i,j} + \mathsf{u}_{j,i}\right) \xrightarrow{divergence} \varepsilon_{ij,j} = \frac{1}{2}\left(\mathsf{u}_{i,jj} + \mathsf{u}_{j,ij}\right)$$

Note that

$$\frac{\partial^2 \mathsf{u}_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial \mathsf{u}_i}{\partial x_j}\right) \equiv \mathsf{u}_{i,jj} \equiv [\nabla \cdot (\nabla \vec{\mathbf{u}})]_i \equiv [\nabla^2 \vec{\mathbf{u}}]_i \quad \text{(Laplacian of the vector } \vec{\mathbf{u}}\text{)}$$

$$\mathsf{u}_{j,ij} \equiv \frac{\partial^2 \mathsf{u}_j}{\partial x_j \partial x_i} = \frac{\partial^2 \mathsf{u}_j}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial \mathsf{u}_j}{\partial x_j}\right) \equiv \mathsf{u}_{j,ji} \equiv [\nabla(\nabla \cdot \vec{\mathbf{u}})]_i$$

$$\varepsilon_{kk} = \frac{1}{2}\left(\frac{\partial \mathsf{u}_k}{\partial x_k} + \frac{\partial \mathsf{u}_k}{\partial x_k}\right) = \frac{\partial \mathsf{u}_k}{\partial x_k} \equiv \mathsf{u}_{k,k} \xrightarrow{gradient} \varepsilon_{kk,i} = \mathsf{u}_{k,ki} = \mathsf{u}_{j,ji}$$

With that the equation in (5.48) can be rewritten as:

$$\sigma_{ij,j} = \lambda \varepsilon_{kk,i} + 2\mu \varepsilon_{ij,j} = \lambda \mathsf{u}_{j,ji} + 2\mu \frac{1}{2}\left(\mathsf{u}_{i,jj} + \mathsf{u}_{j,ji}\right) = (\lambda + \mu)\mathsf{u}_{j,ji} + \mu \mathsf{u}_{i,jj}$$

By replacing the above equation into $\sigma_{ij,j} + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i$ (equations of motion), we obtain:

$$\begin{aligned} &\sigma_{ij,j} + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i \\ &\Rightarrow (\lambda + \mu)\mathsf{u}_{j,ji} + \mu \mathsf{u}_{i,jj} + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i \end{aligned}$$

Thus resulting in 3 equations and 3 unknowns ($\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3$):

$$\boxed{\begin{aligned} &(\lambda + \mu)\mathsf{u}_{j,ji} + \mu \mathsf{u}_{i,jj} + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i \\ \\ &(\lambda + \mu)\left[\nabla(\nabla \cdot \vec{\mathbf{u}})\right] + \mu\left[\nabla \cdot (\nabla \vec{\mathbf{u}})\right] + \rho \vec{\mathbf{b}} = \rho \ddot{\vec{\mathbf{u}}} \end{aligned}} \quad \textit{Navier's equations} \qquad (5.49)$$

where $\nabla^2 \blacklozenge \equiv \nabla \cdot (\nabla \blacklozenge)$ stands for the Laplacian of $\blacklozenge$.

NOTE 1: The above equations are known as the *Navier's equations* also known as Navier-Lamé equations. The explicit form of the equation (5.49) is presented as follows:

$$(\lambda + \mu)\mathsf{u}_{j,ji} + \mu \mathsf{u}_{i,jj} + \rho \mathsf{b}_i = (\lambda + \mu)(\mathsf{u}_{1,1i} + \mathsf{u}_{2,2i} + \mathsf{u}_{3,3i}) + \mu(\mathsf{u}_{i,11} + \mathsf{u}_{i,22} + \mathsf{u}_{i,33}) + \rho \mathsf{b}_i = \rho \ddot{\mathsf{u}}_i$$

$$\begin{cases} (\lambda + \mu)(\mathsf{u}_{1,11} + \mathsf{u}_{2,21} + \mathsf{u}_{3,31}) + \mu(\mathsf{u}_{1,11} + \mathsf{u}_{1,22} + \mathsf{u}_{1,33}) + \rho \mathsf{b}_1 = \rho \ddot{\mathsf{u}}_1 \\ (\lambda + \mu)(\mathsf{u}_{1,12} + \mathsf{u}_{2,22} + \mathsf{u}_{3,32}) + \mu(\mathsf{u}_{2,11} + \mathsf{u}_{2,22} + \mathsf{u}_{2,33}) + \rho \mathsf{b}_2 = \rho \ddot{\mathsf{u}}_2 \\ (\lambda + \mu)(\mathsf{u}_{1,13} + \mathsf{u}_{2,23} + \mathsf{u}_{3,33}) + \mu(\mathsf{u}_{3,11} + \mathsf{u}_{3,22} + \mathsf{u}_{3,33}) + \rho \mathsf{b}_3 = \rho \ddot{\mathsf{u}}_3 \end{cases}$$

or:

$$\begin{cases} (\lambda + \mu)\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial \mathsf{u}_1}{\partial x_1} + \dfrac{\partial \mathsf{u}_2}{\partial x_2} + \dfrac{\partial \mathsf{u}_3}{\partial x_3}\right) + \mu\left(\dfrac{\partial^2 \mathsf{u}_1}{\partial x_1^2} + \dfrac{\partial^2 \mathsf{u}_1}{\partial x_2^2} + \dfrac{\partial^2 \mathsf{u}_1}{\partial x_3^2}\right) + \rho \mathsf{b}_1 = \rho \ddot{\mathsf{u}}_1 \\ (\lambda + \mu)\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \mathsf{u}_1}{\partial x_1} + \dfrac{\partial \mathsf{u}_2}{\partial x_2} + \dfrac{\partial \mathsf{u}_3}{\partial x_3}\right) + \mu\left(\dfrac{\partial^2 \mathsf{u}_2}{\partial x_1^2} + \dfrac{\partial^2 \mathsf{u}_2}{\partial x_2^2} + \dfrac{\partial^2 \mathsf{u}_2}{\partial x_3^2}\right) + \rho \mathsf{b}_2 = \rho \ddot{\mathsf{u}}_2 \\ (\lambda + \mu)\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \mathsf{u}_1}{\partial x_1} + \dfrac{\partial \mathsf{u}_2}{\partial x_2} + \dfrac{\partial \mathsf{u}_3}{\partial x_3}\right) + \mu\left(\dfrac{\partial^2 \mathsf{u}_3}{\partial x_1^2} + \dfrac{\partial^2 \mathsf{u}_3}{\partial x_2^2} + \dfrac{\partial^2 \mathsf{u}_3}{\partial x_3^2}\right) + \rho \mathsf{b}_3 = \rho \ddot{\mathsf{u}}_3 \end{cases}$$

NOTE 2: The above set of equations in matrix form becomes $[\mathbf{A}]\{\mathbf{u}\}=\{\mathbf{p}\}$, where:

$$[\mathbf{A}]=\begin{bmatrix} (\lambda+\mu)\dfrac{\partial^2}{\partial x_1^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_1} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_1} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_3^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} \end{bmatrix},$$

$\{\mathbf{u}\}=\begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \mathrm{u}_3 \end{Bmatrix}$, and $\{\mathbf{p}\}=\begin{Bmatrix} \rho\mathrm{b}_1 \\ \rho\mathrm{b}_2 \\ \rho\mathrm{b}_3 \end{Bmatrix}$.

Note that $\nabla^2=(\vec{\nabla})\cdot(\vec{\nabla})=\dfrac{\partial}{\partial x_k}\dfrac{\partial}{\partial x_k}=\dfrac{\partial^2}{\partial x_1\partial x_1}+\dfrac{\partial^2}{\partial x_2\partial x_2}+\dfrac{\partial^2}{\partial x_3\partial x_3}=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}+\dfrac{\partial^2}{\partial x_3^2}$. The matrix $[\mathbf{A}]$ can also be written as follows:

$$[\mathbf{A}]=\begin{bmatrix} (\lambda+\mu)\dfrac{\partial^2}{\partial x_1^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_1} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_1} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_3\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_3^2}+\mu\nabla^2-\rho\dfrac{D^2}{Dt^2} \end{bmatrix}$$

$$=(\lambda+\mu)\begin{bmatrix} \dfrac{\partial^2}{\partial x_1^2} & \dfrac{\partial^2}{\partial x_1\partial x_2} & \dfrac{\partial^2}{\partial x_1\partial x_3} \\ \dfrac{\partial^2}{\partial x_2\partial x_1} & \dfrac{\partial^2}{\partial x_2^2} & \dfrac{\partial^2}{\partial x_2\partial x_3} \\ \dfrac{\partial^2}{\partial x_3\partial x_1} & \dfrac{\partial^2}{\partial x_3\partial x_2} & \dfrac{\partial^2}{\partial x_3^2} \end{bmatrix}+\left(\mu\frac{\partial^2}{\partial x_k\partial x_k}-\rho\frac{D^2}{Dt^2}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the indicial and tensorial notations the above equation can be written as follows:

$$\mathrm{A}_{ij}=(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\left(\mu\frac{\partial^2}{\partial x_k\partial x_k}-\rho\frac{D^2}{Dt^2}\right)\delta_{ij} \quad \text{and} \quad \mathbf{A}=(\lambda+\mu)[(\vec{\nabla})\otimes(\vec{\nabla})]+\left(\mu\nabla^2-\rho\frac{D^2}{Dt^2}\right)\mathbf{1}$$

Then, we can also express the Navier's equation as follows:

$$\left[(\lambda+\mu)[(\vec{\nabla})\otimes(\vec{\nabla})]+\left(\mu\nabla^2-\rho\frac{D^2}{Dt^2}\right)\mathbf{1}\right]\cdot\vec{\mathbf{u}}=-\rho\vec{\mathbf{b}}$$

or

$$\left[(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\left(\mu\frac{\partial^2}{\partial x_k\partial x_k}-\rho\frac{D^2}{Dt^2}\right)\delta_{ij}\right]\mathrm{u}_j=-\rho\mathrm{b}_i$$

The above equation could have been easily obtained by means of the equation in (5.49), i.e.:

$$
\begin{aligned}
&(\lambda+\mu)\mathrm{u}_{j,ji}+\mu\mathrm{u}_{i,jj}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i\\
&\Rightarrow(\lambda+\mu)\mathrm{u}_{k,ki}+\mu\mathrm{u}_{k,jj}\delta_{ik}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_k\delta_{ik}\\
&\Rightarrow(\lambda+\mu)\mathrm{u}_{k,ki}+\mu\mathrm{u}_{k,jj}\delta_{ik}-\rho\ddot{\mathrm{u}}_k\delta_{ik}=-\rho\mathrm{b}_i\\
&\Rightarrow(\lambda+\mu)\mathrm{u}_{k,ki}+(\mu\mathrm{u}_{k,jj}-\rho\ddot{\mathrm{u}}_k)\delta_{ik}=-\rho\mathrm{b}_i\\
&\Rightarrow(\lambda+\mu)\frac{\partial\mathrm{u}_k}{\partial x_k\partial x_i}+\left(\mu\frac{\partial^2\mathrm{u}_k}{\partial x_k\partial x_k}-\rho\frac{D^2\mathrm{u}_k}{Dt^2}\right)\delta_{ik}=-\rho\mathrm{b}_i\\
&\Rightarrow\left[(\lambda+\mu)\frac{\partial}{\partial x_k\partial x_i}+\left(\mu\frac{\partial^2}{\partial x_k\partial x_k}-\rho\frac{D^2}{Dt^2}\right)\delta_{ik}\right]\mathrm{u}_k=-\rho\mathrm{b}_i
\end{aligned}
$$

NOTE 3: We have shown in **Problem 1.106** (Chapter 1) that the following is true:

$$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{a}})=\nabla(\nabla\cdot\vec{\mathbf{a}})-\nabla^2\vec{\mathbf{a}}\quad\xrightarrow{indicial}\quad\epsilon_{ilq}\epsilon_{qjk}\mathrm{a}_{k,jl}=\mathrm{a}_{j,ji}-\mathrm{a}_{i,jj}$$

Then, we can obtain

$$\nabla\cdot(\nabla\vec{\mathbf{u}})\equiv\nabla^2\vec{\mathbf{u}}=\nabla(\nabla\cdot\vec{\mathbf{u}})-\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})\quad\xrightarrow{indicial}\quad\mathrm{u}_{i,jj}=\mathrm{u}_{j,ji}-\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jl}$$

with which the equation (5.49) can also be written as follows:

$$
\begin{aligned}
&(\lambda+\mu)\mathrm{u}_{j,ji}+\mu\mathrm{u}_{i,jj}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i\\
&\Rightarrow(\lambda+\mu)\mathrm{u}_{j,ji}+\mu(\mathrm{u}_{j,ji}-\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jl})+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i\\
&\Rightarrow(\lambda+2\mu)\mathrm{u}_{j,ji}-\mu\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jl}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i
\end{aligned}
$$

and the equivalent in tensorial notation:

$$
\begin{aligned}
&(\lambda+\mu)[\nabla(\nabla\cdot\vec{\mathbf{u}})]+\mu[\nabla\cdot(\nabla\vec{\mathbf{u}})]+\rho\vec{\mathbf{b}}=\rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow(\lambda+\mu)[\nabla(\nabla\cdot\vec{\mathbf{u}})]+\mu[\nabla(\nabla\cdot\vec{\mathbf{u}})-\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})]+\rho\vec{\mathbf{b}}=\rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow(\lambda+2\mu)[\nabla(\nabla\cdot\vec{\mathbf{u}})]-\mu[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})]+\rho\vec{\mathbf{b}}=\rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow(\lambda+2\mu)[\nabla(\nabla\cdot\vec{\mathbf{u}})]-\mu[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})]+\rho\vec{\mathbf{b}}=\rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow(\lambda+2\mu)\mathrm{u}_{j,ji}-\mu\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jl}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i
\end{aligned}
\tag{5.50}
$$

In the Cartesian System we have:

$$\vec{\mathbf{u}}=\mathrm{u}_i\hat{\mathbf{e}}_i=\mathrm{u}_1\hat{\mathbf{e}}_1+\mathrm{u}_2\hat{\mathbf{e}}_2+\mathrm{u}_3\hat{\mathbf{e}}_3$$

$$(\vec{\nabla}\wedge\vec{\mathbf{u}})\equiv\mathrm{rot}(\vec{\mathbf{u}})=(\mathrm{rot}(\vec{\mathbf{u}}))_i\hat{\mathbf{e}}_i=\underbrace{\left(\frac{\partial\mathrm{u}_3}{\partial x_2}-\frac{\partial\mathrm{u}_2}{\partial x_3}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_1}\hat{\mathbf{e}}_1+\underbrace{\left(\frac{\partial\mathrm{u}_1}{\partial x_3}-\frac{\partial\mathrm{u}_3}{\partial x_1}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_2}\hat{\mathbf{e}}_2+\underbrace{\left(\frac{\partial\mathrm{u}_2}{\partial x_1}-\frac{\partial\mathrm{u}_1}{\partial x_2}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_3}\hat{\mathbf{e}}_3$$

$$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})=\left(\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_2}-\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_3}\right)\hat{\mathbf{e}}_1+\left(\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_3}-\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_1}\right)\hat{\mathbf{e}}_2+\left(\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_1}-\frac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_2}\right)\hat{\mathbf{e}}_3$$

$$[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})]_i=\begin{Bmatrix}\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_2}-\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_3}\\[2mm]\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_3}-\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_3}{\partial x_1}\\[2mm]\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_2}{\partial x_1}-\dfrac{\partial(\mathrm{rot}(\vec{\mathbf{u}}))_1}{\partial x_2}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial\mathrm{u}_2}{\partial x_1}-\dfrac{\partial\mathrm{u}_1}{\partial x_2}\right)-\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial\mathrm{u}_1}{\partial x_3}-\dfrac{\partial\mathrm{u}_3}{\partial x_1}\right)\\[2mm]\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial\mathrm{u}_3}{\partial x_2}-\dfrac{\partial\mathrm{u}_2}{\partial x_3}\right)-\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial\mathrm{u}_2}{\partial x_1}-\dfrac{\partial\mathrm{u}_1}{\partial x_2}\right)\\[2mm]\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial\mathrm{u}_1}{\partial x_3}-\dfrac{\partial\mathrm{u}_3}{\partial x_1}\right)-\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial\mathrm{u}_3}{\partial x_2}-\dfrac{\partial\mathrm{u}_2}{\partial x_3}\right)\end{Bmatrix}$$

NOTE 4: If we are dealing with heterogeneous material, the equations in (5.48) must be treated as follows:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$
$$\Rightarrow \sigma_{ij,j} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij})_{,j}$$
$$\Rightarrow \sigma_{ij,j} = (\lambda \varepsilon_{kk})_{,j} \delta_{ij} + (2\mu \varepsilon_{ij})_{,j} = (\lambda \varepsilon_{kk})_{,i} + (2\mu \varepsilon_{ij})_{,j}$$

Taking into account that $2\varepsilon_{ij} = \mathrm{u}_{i,j} + \mathrm{u}_{j,i}$ and $\varepsilon_{kk} = \mathrm{u}_{k,k}$, the above equation becomes:

$$\sigma_{ij,j} = (\lambda \varepsilon_{kk})_{,i} + (2\mu \varepsilon_{ij})_{,j}$$
$$\Rightarrow \sigma_{ij,j} = (\lambda \mathrm{u}_{k,k})_{,i} + \left[\mu(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})\right]_{,j}$$

whereby

$$\sigma_{ij,j} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i \qquad \Rightarrow \qquad (\lambda \mathrm{u}_{k,k})_{,i} + \left[\mu(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})\right]_{,j} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i \tag{5.51}$$

Note that

$\mathrm{u}_{k,k} = \mathrm{Tr}(\nabla \vec{\mathbf{u}}) = (\nabla \cdot \vec{\mathbf{u}})$, and

$\ddot{\mathrm{u}}_i \equiv \dfrac{D\dot{\mathrm{u}}_i}{Dt} = \dfrac{\partial \dot{\mathrm{u}}_i}{\partial t} + \dfrac{\partial \dot{\mathrm{u}}_i}{\partial x_j} v_j = \dfrac{\partial \dot{\mathrm{u}}_i}{\partial t} + \dfrac{\partial \dot{\mathrm{u}}_i}{\partial x_1} v_1 + \dfrac{\partial \dot{\mathrm{u}}_i}{\partial x_2} v_2 + \dfrac{\partial \dot{\mathrm{u}}_i}{\partial x_3} v_3$, and its components

$$\ddot{\mathrm{u}}_i = \begin{Bmatrix} \dfrac{\partial \dot{\mathrm{u}}_1}{\partial t} + \dfrac{\partial \dot{\mathrm{u}}_1}{\partial x_1} v_1 + \dfrac{\partial \dot{\mathrm{u}}_1}{\partial x_2} v_2 + \dfrac{\partial \dot{\mathrm{u}}_1}{\partial x_3} v_3 \\ \dfrac{\partial \dot{\mathrm{u}}_2}{\partial t} + \dfrac{\partial \dot{\mathrm{u}}_2}{\partial x_1} v_1 + \dfrac{\partial \dot{\mathrm{u}}_2}{\partial x_2} v_2 + \dfrac{\partial \dot{\mathrm{u}}_2}{\partial x_3} v_3 \\ \dfrac{\partial \dot{\mathrm{u}}_3}{\partial t} + \dfrac{\partial \dot{\mathrm{u}}_3}{\partial x_1} v_1 + \dfrac{\partial \dot{\mathrm{u}}_3}{\partial x_2} v_2 + \dfrac{\partial \dot{\mathrm{u}}_3}{\partial x_3} v_3 \end{Bmatrix}$$

$$\left[\mu(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})\right]_{,j} = \frac{\partial}{\partial x_j}\left[\mu(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})\right]$$
$$= \frac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{i,1} + \mathrm{u}_{1,i})\right] + \frac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{i,2} + \mathrm{u}_{2,i})\right] + \frac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{i,3} + \mathrm{u}_{3,i})\right]$$

$$\left[\mu(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})\right]_{,j} = \begin{Bmatrix} \dfrac{\partial}{\partial x_1}\left[2\mu(\mathrm{u}_{1,1})\right] + \dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{1,2} + \mathrm{u}_{2,1})\right] + \dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{1,3} + \mathrm{u}_{3,1})\right] \\ \dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{2,1} + \mathrm{u}_{1,2})\right] + \dfrac{\partial}{\partial x_2}\left[2\mu(\mathrm{u}_{2,2})\right] + \dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{2,3} + \mathrm{u}_{3,2})\right] \\ \dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{3,1} + \mathrm{u}_{1,3})\right] + \dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{3,2} + \mathrm{u}_{2,3})\right] + \dfrac{\partial}{\partial x_3}\left[2\mu(\mathrm{u}_{3,3})\right] \end{Bmatrix}$$

The three equations in (5.51), ($i = 1,2,3$), are explicitly given by:

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left[\lambda(\nabla \cdot \vec{\mathbf{u}})\right] + \dfrac{\partial}{\partial x_1}\left[2\mu(\mathrm{u}_{1,1})\right] + \dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{1,2} + \mathrm{u}_{2,1})\right] + \dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{1,3} + \mathrm{u}_{3,1})\right] + \rho \mathrm{b}_1 = \rho \ddot{\mathrm{u}}_1 \\ \dfrac{\partial}{\partial x_2}\left[\lambda(\nabla \cdot \vec{\mathbf{u}})\right] + \dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{2,1} + \mathrm{u}_{1,2})\right] + \dfrac{\partial}{\partial x_2}\left[2\mu(\mathrm{u}_{2,2})\right] + \dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{2,3} + \mathrm{u}_{3,2})\right] + \rho \mathrm{b}_2 = \rho \ddot{\mathrm{u}}_2 \\ \dfrac{\partial}{\partial x_3}\left[\lambda(\nabla \cdot \vec{\mathbf{u}})\right] + \dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{3,1} + \mathrm{u}_{1,3})\right] + \dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{3,2} + \mathrm{u}_{2,3})\right] + \dfrac{\partial}{\partial x_3}\left[2\mu(\mathrm{u}_{3,3})\right] + \rho \mathrm{b}_3 = \rho \ddot{\mathrm{u}}_3 \end{cases}$$

or

$$\boxed{\begin{cases} \dfrac{\partial}{\partial x_1}\left[\lambda(\nabla\cdot\vec{\mathbf{u}})+2\mu(\mathrm{u}_{1,1})\right]+\dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{1,2}+\mathrm{u}_{2,1})\right]+\dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{1,3}+\mathrm{u}_{3,1})\right]+\rho\mathrm{b}_1=\rho\ddot{\mathrm{u}}_1 \\ \dfrac{\partial}{\partial x_2}\left[\lambda(\nabla\cdot\vec{\mathbf{u}})+2\mu(\mathrm{u}_{2,2})\right]+\dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{2,1}+\mathrm{u}_{1,2})\right]+\dfrac{\partial}{\partial x_3}\left[\mu(\mathrm{u}_{2,3}+\mathrm{u}_{3,2})\right]+\rho\mathrm{b}_2=\rho\ddot{\mathrm{u}}_2 \\ \dfrac{\partial}{\partial x_3}\left[\lambda(\nabla\cdot\vec{\mathbf{u}})+2\mu(\mathrm{u}_{3,3})\right]+\dfrac{\partial}{\partial x_1}\left[\mu(\mathrm{u}_{3,1}+\mathrm{u}_{1,3})\right]+\dfrac{\partial}{\partial x_2}\left[\mu(\mathrm{u}_{3,2}+\mathrm{u}_{2,3})\right]+\rho\mathrm{b}_3=\rho\ddot{\mathrm{u}}_3 \end{cases}} \tag{5.52}$$

NOTE 5: Wave equations

If we apply the divergence to the equation (5.50) we can obtain:

$$(\lambda+2\mu)\nabla\cdot[\nabla(\nabla\cdot\vec{\mathbf{u}})]-\mu\underbrace{\nabla\cdot[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})]}_{=0}+\rho\nabla\cdot\vec{\mathbf{b}}=\rho\nabla\cdot\ddot{\vec{\mathbf{u}}}$$

which in indicial notation becomes

$$(\lambda+2\mu)\mathrm{u}_{j,jii}-\mu\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jli}+\rho\mathrm{b}_{i,i}=\rho\ddot{\mathrm{u}}_{i,i}$$

then, we can obtain

Tensorial notation	*Indicial notation*
$\Rightarrow(\lambda+2\mu)\nabla\cdot[\nabla(\nabla\cdot\vec{\mathbf{u}})]+\rho\nabla\cdot\vec{\mathbf{b}}=\rho\nabla\cdot\ddot{\vec{\mathbf{u}}}$	$(\lambda+2\mu)\mathrm{u}_{j,jii}-\mu\epsilon_{ilq}\epsilon_{qjk}\mathrm{u}_{k,jli}+\rho\mathrm{b}_{i,i}=\rho\ddot{\mathrm{u}}_{i,i}$
$\Rightarrow(\lambda+2\mu)\nabla^2(\nabla\cdot\vec{\mathbf{u}})+\rho\nabla\cdot\vec{\mathbf{b}}=\rho\nabla\cdot\ddot{\vec{\mathbf{u}}}$	$\Rightarrow(\lambda+2\mu)\mathrm{u}_{j,jii}+\rho\mathrm{b}_{i,i}=\rho\ddot{\mathrm{u}}_{i,i}$
$\Rightarrow\nabla\cdot\ddot{\vec{\mathbf{u}}}=\dfrac{(\lambda+2\mu)}{\rho}\nabla^2(\nabla\cdot\vec{\mathbf{u}})+\nabla\cdot\vec{\mathbf{b}}$	$\Rightarrow\ddot{\mathrm{u}}_{i,i}=\dfrac{(\lambda+2\mu)}{\rho}\mathrm{u}_{j,jii}+\mathrm{b}_{i,i}$
$\Rightarrow\dfrac{D^2}{Dt^2}(\nabla\cdot\vec{\mathbf{u}})=\dfrac{(\lambda+2\mu)}{\rho}\nabla^2(\nabla\cdot\vec{\mathbf{u}})+\nabla\cdot\vec{\mathbf{b}}$	$\Rightarrow\dfrac{D^2}{Dt^2}\left(\dfrac{\partial\mathrm{u}_i}{\partial x_i}\right)=\dfrac{(\lambda+2\mu)}{\rho}\dfrac{\partial^2}{\partial x_i\partial x_i}\left(\dfrac{\partial\mathrm{u}_j}{\partial x_j}\right)+\dfrac{\partial\mathrm{b}_i}{\partial x_i}$
$\Rightarrow\dfrac{D^2\theta}{Dt^2}=\dfrac{(\lambda+2\mu)}{\rho}\nabla^2\theta+\nabla\cdot\vec{\mathbf{b}}$	$\Rightarrow\dfrac{D^2\theta}{Dt^2}=\alpha^2\dfrac{\partial^2\theta}{\partial x_i\partial x_i}+\dfrac{\partial\mathrm{b}_i}{\partial x_i}$
$\Rightarrow\dfrac{D^2\theta}{Dt^2}=\alpha^2\nabla^2\theta+\nabla\cdot\vec{\mathbf{b}}$	
$\Rightarrow\ddot{\theta}=\alpha^2\nabla^2\theta+\nabla\cdot\vec{\mathbf{b}}$	

(5.53)

where we have considered that $\theta=\nabla\cdot\vec{\mathbf{u}}$ and $\nabla\cdot(\vec{\nabla}\wedge\vec{v})=0$, (see **Problem 1.108**), and

$$\boxed{\alpha=\sqrt{\frac{(\lambda+2\mu)}{\rho}}}\quad \textit{P-wave velocity} \tag{5.54}$$

If the body forces do not change in space we have that $\nabla\cdot\vec{\mathbf{b}}=0$, thus the equation in (5.53) becomes:

$$\boxed{\frac{D^2\theta}{Dt^2}=\alpha^2\nabla^2\theta}\quad \textit{P- wave equation} \tag{5.55}$$

P-waves have no rotation.

Now if we apply the curl ($\vec{\nabla}\wedge$) to the equation (5.50) we obtain:

$$
\begin{aligned}
&(\lambda+2\mu)\vec{\nabla}\wedge\left[\nabla(\nabla\cdot\vec{\mathbf{u}})\right]-\mu\vec{\nabla}\wedge\left[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})\right]+\rho(\vec{\nabla}\wedge\vec{\mathbf{b}})=\rho(\vec{\nabla}\wedge\ddot{\vec{\mathbf{u}}})\\
&\Rightarrow-\mu\vec{\nabla}\wedge\left[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})\right]+\rho(\vec{\nabla}\wedge\vec{\mathbf{b}})=\rho(\vec{\nabla}\wedge\ddot{\vec{\mathbf{u}}})\\
&\Rightarrow-\mu\vec{\nabla}\wedge\left[\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{u}})\right]+\rho(\vec{\nabla}\wedge\vec{\mathbf{b}})=\rho\frac{D^2}{Dt^2}(\vec{\nabla}\wedge\vec{\mathbf{u}})\\
&\Rightarrow-\mu\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})+\rho(\vec{\nabla}\wedge\vec{\mathbf{b}})=\rho\frac{D^2\vec{\varphi}}{Dt^2}\\
&\Rightarrow-\mu\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})=\rho\frac{D^2\vec{\varphi}}{Dt^2}\quad\Rightarrow\quad\rho\frac{D^2\vec{\varphi}}{Dt^2}=-\mu\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})\\
&\Rightarrow\frac{D^2\vec{\varphi}}{Dt^2}=-\frac{\mu}{\rho}\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})\\
&\Rightarrow\frac{D^2\vec{\varphi}}{Dt^2}=-\beta^2\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})
\end{aligned}
\tag{5.56}
$$

where we have considered $\vec{\varphi}=\vec{\nabla}\wedge\vec{\mathbf{u}}$, and that the $\vec{\mathbf{b}}$-field is conservative thus $\vec{\nabla}\wedge\vec{\mathbf{b}}=\vec{\mathbf{0}}$. Note that $\vec{\nabla}\wedge\left[\nabla(\nabla\cdot\vec{\mathbf{u}})\right]=\vec{\nabla}\wedge\left[\nabla\phi\right]=\vec{\mathbf{0}}$, (see **Problem 1.108**), and

$$\boxed{\beta=\sqrt{\frac{\mu}{\rho}}}\quad\textit{Shear wave velocity}\tag{5.57}$$

Note that $\nabla^2\vec{\varphi}=\nabla(\nabla\cdot\vec{\varphi})-\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})\Rightarrow\nabla^2\vec{\varphi}=-\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})$, since $\nabla\cdot\vec{\varphi}=\nabla\cdot(\vec{\nabla}\wedge\vec{\mathbf{u}})=0$ holds for any vector, so $\nabla(\nabla\cdot\vec{\varphi})=\nabla(\nabla\cdot(\vec{\nabla}\wedge\vec{\mathbf{u}}))=\vec{\mathbf{0}}$, (see **Problem 1.108**). With that the equation in (5.56) becomes:

$$\boxed{\frac{D^2\vec{\varphi}}{Dt^2}=\beta^2\nabla^2\vec{\varphi}}\quad\begin{array}{l}\textit{Shear wave equation}\\\textit{(S-wave equation)}\end{array}\tag{5.58}$$

Shear waves have no change in volume.
In the case when $\mu=0$ the equation in (5.55) becomes the acoustic wave equations:

$$\boxed{\frac{D^2\theta}{Dt^2}=c^2\nabla^2\theta}\quad\textit{Acoustic wave equation}\tag{5.59}$$

with

$$\boxed{c=\sqrt{\frac{\lambda}{\rho}}}\quad\textit{Speed of propagation}\tag{5.60}$$

Note that the displacement field was split up into: $\vec{\mathbf{u}}=\nabla\theta+\vec{\nabla}\wedge\vec{\varphi}$ where $\nabla\cdot\vec{\varphi}=0$. We can prove this by means of the identity $\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\mathbf{a}})=\nabla(\nabla\cdot\vec{\mathbf{a}})-\nabla^2\vec{\mathbf{a}}$. If we consider the vectors $\vec{\mathbf{u}}=\nabla^2\vec{\mathbf{a}}$ and $\vec{\varphi}=-\vec{\nabla}\wedge\vec{\mathbf{a}}$, and the scalar $\theta=\nabla\cdot\vec{\mathbf{a}}$, we obtain $\vec{\mathbf{u}}=\nabla\theta+\vec{\nabla}\wedge\vec{\varphi}$, with that we can obtain:

$$\nabla\cdot\vec{\mathbf{u}}=\nabla\cdot(\nabla\theta)+\nabla\cdot(\vec{\nabla}\wedge\vec{\varphi})=\nabla\cdot(\nabla\theta)\quad\text{and}\quad\vec{\nabla}\wedge\vec{\mathbf{u}}=\vec{\nabla}\wedge(\nabla\theta)+\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})=\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{\varphi})$$

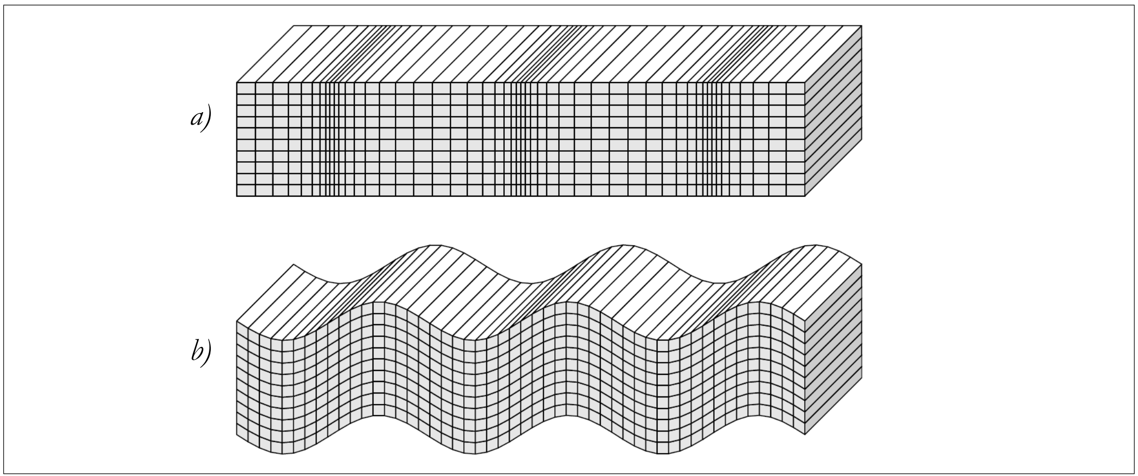


Figure 5.8: Displacement occurring from a harmonic plane P-wave (*a*) and S-wave (*b*). P-wave has no rotation and S-wave no volume change.

NOTE 5.1: If we consider $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \dfrac{E}{2(1+\nu)}$ we can obtain:

$$\frac{\alpha}{\beta} = \frac{\sqrt{\dfrac{(\lambda+2\mu)}{\rho}}}{\sqrt{\dfrac{\mu}{\rho}}} = \sqrt{\frac{(\lambda+2\mu)}{\mu}} = \sqrt{\frac{\dfrac{E\nu}{(1+\nu)(1-2\nu)} + 2\dfrac{E}{2(1+\nu)}}{\dfrac{E}{2(1+\nu)}}} = \sqrt{\frac{(2-2\nu)}{(1-2\nu)}}$$

$$\Rightarrow \alpha = \beta \underbrace{\sqrt{\frac{(2-2\nu)}{(1-2\nu)}}}_{>1}$$

With that we can conclude that the ratio of P- to S-wave velocities depends only on Poisson's ratio. Note that *P-wave* travels faster than *S-wave*, since for isotropic material $-1 < \nu < 0.5$, then $\sqrt{\dfrac{(2-2\nu)}{(1-2\nu)}} > 1$, (see Figure 5.9).

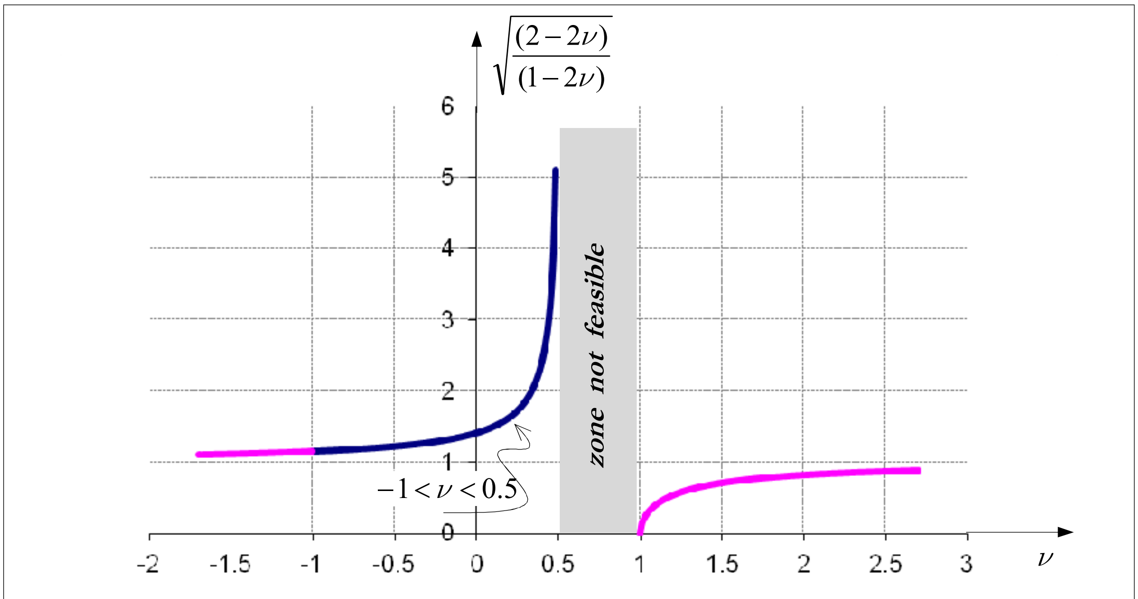


Figure 5.9

NOTE 6: In the previous note (NOTE 5) it was shown that the displacement field $\vec{\mathbf{u}}$ can be split up into $\vec{\mathbf{u}} = \nabla\theta + \vec{\nabla}\wedge\vec{\varphi}$, which is applied to any vector field, i.e. given a vector field $\vec{\mathbf{F}}$, the following is true:

$$\vec{\mathbf{F}} = \nabla\theta + \vec{\nabla}\wedge\vec{\varphi} \qquad \textit{Helmholtz theorem or Helmholtz decomposition} \tag{5.61}$$

which is known as Helmholtz theorem, where θ is a scalar potential field and $\vec{\varphi}$ is a vector potential field, in which the relationships $\nabla\cdot\vec{\varphi} \equiv \mathrm{div}(\vec{\varphi}) = 0$ and $\vec{\nabla}\wedge(\nabla\theta) \equiv \mathrm{rot}(\nabla\theta) = 0$ hold. Note also that the SI units of $[\theta] = [\vec{\varphi}] = m^2$, since $[\vec{\mathbf{u}}] = m(meter)$.

Then, by substituting $\vec{\mathbf{u}} = \nabla\theta + \vec{\nabla}\wedge\vec{\varphi}$, ($\mathrm{u}_i = \theta_{,i} + \epsilon_{ipq}\varphi_{q,p}$), into the Navier's equations given by (5.49) we can obtain:

$$
\begin{aligned}
&(\lambda+\mu)\left[\nabla(\nabla\cdot\vec{\mathbf{u}})\right] + \mu\left[\nabla\cdot(\nabla\vec{\mathbf{u}})\right] + \rho\vec{\mathbf{b}} = \rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow (\lambda+\mu)\left[\nabla(\nabla\cdot(\nabla\theta+\vec{\nabla}\wedge\vec{\varphi}))\right] + \mu\left[\nabla\cdot(\nabla(\nabla\theta+\vec{\nabla}\wedge\vec{\varphi}))\right] + \rho\vec{\mathbf{b}} = \rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow (\lambda+\mu)\left[\nabla(\nabla\cdot(\nabla\theta)) + \nabla(\underbrace{\nabla\cdot(\vec{\nabla}\wedge\vec{\varphi})}_{=0})\right] + \mu\left[\underbrace{\nabla\cdot(\nabla(\nabla\theta))}_{=\nabla(\nabla\cdot(\nabla\theta))} + \underbrace{\nabla\cdot(\nabla(\vec{\nabla}\wedge\vec{\varphi}))}_{\vec{\nabla}\wedge(\nabla^2\vec{\varphi})}\right] + \rho\vec{\mathbf{b}} = \rho\ddot{\vec{\mathbf{u}}}\\
&\Rightarrow (\lambda+2\mu)\left[\nabla(\underbrace{\nabla\cdot(\nabla\theta)}_{=\nabla^2\theta})\right] + \mu\left[\vec{\nabla}\wedge(\nabla^2\vec{\varphi})\right] + \rho\vec{\mathbf{b}} = \rho\nabla\ddot{\theta} + \rho\vec{\nabla}\wedge\ddot{\vec{\varphi}}\\
&\Rightarrow (\lambda+2\mu)\left[\nabla(\nabla^2\theta)\right] + \mu\left[\vec{\nabla}\wedge(\nabla^2\vec{\varphi})\right] + \rho\vec{\mathbf{b}} = \rho(\nabla\ddot{\theta}) + \rho(\vec{\nabla}\wedge\ddot{\vec{\varphi}})
\end{aligned}
\tag{5.62}
$$

The above algebraic manipulations in indicial notation are given by:

$$
\begin{aligned}
&(\lambda+\mu)\mathrm{u}_{j,ji} + \mu\mathrm{u}_{i,jj} + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i\\
&\Rightarrow (\lambda+\mu)(\theta_{,j} + \epsilon_{jpq}\varphi_{q,p})_{,ji} + \mu(\theta_{,i} + \epsilon_{ipq}\varphi_{q,p})_{,jj} + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i\\
&\Rightarrow (\lambda+\mu)(\theta_{,jji} + \epsilon_{jpq}\varphi_{q,pji}) + \mu(\theta_{,ijj} + \epsilon_{ipq}\varphi_{q,pjj}) + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i
\end{aligned}
\tag{5.63}
$$

Note that $\epsilon_{jpq}\varphi_{q,pji} = \epsilon_{jpq}\varphi_{q,ijp} = \epsilon_{jpq}\varphi_{q,ijp} = -\epsilon_{pjq}\varphi_{q,ijp} = 0_i$, since $\epsilon_{jpq} = -\epsilon_{pjq}$ is antisymmetric in jp and $\varphi_{q,ijp}$ is symmetric in jp. Note also that

$$\epsilon_{ipq}\varphi_{q,pjj} = \epsilon_{ipq}\varphi_{q,jjp} = \epsilon_{ipq}(\varphi_{q,jj})_{,p} = \epsilon_{ipq}([\nabla^2\vec{\varphi}]_q)_{,p} = \epsilon_{ipq}([\nabla^2\vec{\varphi}]_q)_{,p} = [\vec{\nabla}\wedge(\nabla^2\vec{\varphi})]_i$$

and $\theta_{,jji} = \theta_{,ijj}$. With the above considerations the equation (5.63) becomes:

$$
\begin{aligned}
&(\lambda+\mu)(\theta_{,jji} + \epsilon_{jpq}\varphi_{q,pji}) + \mu(\theta_{,ijj} + \epsilon_{ipq}\varphi_{q,pjj}) + \rho\mathrm{b}_i = \rho\ddot{\theta}_{,i} + \rho\epsilon_{ipq}\ddot{\varphi}_{q,p}\\
&\Rightarrow (\lambda+2\mu)(\theta_{,jji}) + \mu(\epsilon_{ipq}\varphi_{q,pjj}) + \rho\mathrm{b}_i = \rho\ddot{\theta}_{,i} + \rho\epsilon_{ipq}\ddot{\varphi}_{q,p}
\end{aligned}
$$

NOTE 7: Galerkin Vector

The displacement field can be expressed as follows:

$$\vec{\mathbf{u}} = c(\nabla^2\vec{\mathbf{g}}) - \nabla(\nabla\cdot\vec{\mathbf{g}}) \qquad \Big| \qquad \mathrm{u}_i = c\mathrm{g}_{i,pp} - \mathrm{g}_{p,pi} \tag{5.64}$$

where $\vec{\mathbf{g}}$ is the Galerkin vector with the SI unit $[\vec{\mathbf{g}}] = m^3$, and c is a constant to be determined which is dimensionless. If the Galerkin vector for a problem is known the problem is solved.

Let us consider a static linear elastic problem, then the Navier's equations (5.49) can be expressed as follows $(\lambda+\mu)\mathrm{u}_{j,ji} + \mu\mathrm{u}_{i,jj} + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i = \mathrm{0}_i$, and taking into account that displacement field (5.64) we can obtain:

$$
\begin{aligned}
&(\lambda+\mu)\mathrm{u}_{j,ji}+\mu\mathrm{u}_{i,jj}+\rho\mathrm{b}_i=0_i\\
&\Rightarrow(\lambda+\mu)(c\mathrm{g}_{j,pp}-\mathrm{g}_{p,pj})_{,ji}+\mu(c\mathrm{g}_{i,pp}-\mathrm{g}_{p,pi})_{,jj}+\rho\mathrm{b}_i=0_i\\
&\Rightarrow(\lambda+\mu)(c\mathrm{g}_{j,ppji}-\mathrm{g}_{p,pjji})+\mu(c\mathrm{g}_{i,ppjj}-\mathrm{g}_{p,pijj})+\rho\mathrm{b}_i=0_i
\end{aligned}
\tag{5.65}
$$

Note that $\mathrm{g}_{j,ppji}=\mathrm{g}_{p,pjji}=\mathrm{g}_{p,pijj}$, then the above equation becomes:

$$
\begin{aligned}
&(\lambda+\mu)(c\mathrm{g}_{j,ppji}-\mathrm{g}_{p,pjji})+\mu(c\mathrm{g}_{i,ppjj}-\mathrm{g}_{p,pijj})+\rho\mathrm{b}_i=0_i\\
&\Rightarrow[(\lambda+\mu)(c-1)-\mu]\mathrm{g}_{p,pjji}+c\mu\mathrm{g}_{i,ppjj}+\rho\mathrm{b}_i=0_i
\end{aligned}
\tag{5.66}
$$

The constant c can be obtained by taking the term between brackets equal to zero, i.e.:

$$
[(\lambda+\mu)(c-1)-\mu]=0 \qquad\Rightarrow\qquad c=\frac{\lambda+2\mu}{\lambda+\mu}=2(1-\nu)
\tag{5.67}
$$

Then, the displacement field (5.64) becomes:

$$
\begin{aligned}
\vec{\mathbf{u}}&=\frac{\lambda+2\mu}{\lambda+\mu}(\nabla^2\vec{\mathbf{g}})-\nabla(\nabla\cdot\vec{\mathbf{g}})\\
&=2\mu\left(\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}(\nabla^2\vec{\mathbf{g}})-\frac{1}{2\mu}\nabla(\nabla\cdot\vec{\mathbf{g}})\right)
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\mathrm{u}_i&=\frac{\lambda+2\mu}{\lambda+\mu}\mathrm{g}_{i,pp}-\mathrm{g}_{p,pi}\\
&=\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial^2(\mathrm{g}_i)}{\partial x_p\partial x_p}-\frac{\partial(\mathrm{g}_{p,p})}{\partial x_i}\\
&=\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2(\mathrm{g}_i)-\frac{\partial(\nabla\cdot\vec{\mathbf{g}})}{\partial x_i}\\
&=2\mu\left(\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\nabla^2(\mathrm{g}_i)-\frac{1}{2\mu}\frac{\partial(\nabla\cdot\vec{\mathbf{g}})}{\partial x_i}\right)
\end{aligned}
\tag{5.68}
$$

And the Navier's equation (5.66) in terms of Galerkin vector becomes

$$
\underbrace{[(\lambda+\mu)(c-1)-\mu]}_{=0}\mathrm{g}_{p,pjji}+c\mu\mathrm{g}_{i,ppjj}+\rho\mathrm{b}_i=0_i \qquad\Rightarrow\qquad \mathrm{g}_{i,ppjj}=\frac{-1}{c\mu}\rho\mathrm{b}_i
$$

$$
\Rightarrow\mathrm{g}_{i,ppjj}=\frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\mathrm{b}_i
$$

$$
\Rightarrow\frac{\partial^2}{\partial x_p\partial x_p}\left(\frac{\partial^2(\mathrm{g}_i)}{\partial x_j\partial x_j}\right)=\nabla^2\nabla^2(\mathrm{g}_i)=\nabla^4(\mathrm{g}_i)=\frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\mathrm{b}_i
$$

Thus

$$
\nabla^4(\vec{\mathbf{g}})=\frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\vec{\mathbf{b}}=\frac{-1}{2\mu(1-\nu)}\rho\vec{\mathbf{b}}
\qquad\Bigg|\qquad
\nabla^4(\mathrm{g}_i)=\frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\mathrm{b}_i=\frac{-1}{2\mu(1-\nu)}\rho\mathrm{b}_i
\tag{5.69}
$$

Note that in the absence of body force, each component of the Galerkin vector (g_i) is biharmonic function, i.e. $\nabla^4(\mathrm{g}_i)\equiv\mathrm{g}_{i,kkjj}=0_i$.

The infinitesimal strain tensor ($\boldsymbol{\varepsilon}=(\nabla\vec{\mathbf{u}})^{sym}$) in terms of Galerkin vector becomes:

$$
\begin{aligned}
\nabla\vec{\mathbf{u}}&=\nabla\left(\frac{\lambda+2\mu}{\lambda+\mu}(\nabla^2\vec{\mathbf{g}})-\nabla(\nabla\cdot\vec{\mathbf{g}})\right)=\left(\frac{\lambda+2\mu}{\lambda+\mu}\nabla(\nabla^2\vec{\mathbf{g}})-\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\right)\\
&=\left(\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2(\nabla\vec{\mathbf{g}})-\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\right)
\end{aligned}
$$

Note that $[\nabla^2(\nabla\vec{\mathbf{g}})]^{sym}=\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]$, $\{\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\}^{sym}=\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]$. Then,

$$\boldsymbol{\varepsilon} = (\nabla\vec{\mathbf{u}})^{sym} = \left(\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2(\nabla\vec{\mathbf{g}}) - \nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\right)^{sym} = \frac{\lambda+2\mu}{\lambda+\mu}\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}] - \nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]$$

$$\boxed{\begin{aligned}&\boldsymbol{\varepsilon} = 2\mu\left(\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}] - \frac{1}{2\mu}\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\right)\\ &or\\ &\boldsymbol{\varepsilon} = 2\mu\left(\frac{(1-\nu)}{\mu}\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}] - \frac{1}{2\mu}\nabla[\nabla(\nabla\cdot\vec{\mathbf{g}})]\right)\end{aligned}} \tag{5.70}$$

In indicial notation we have:

$$\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}(\mathsf{u}_{i,j}+\mathsf{u}_{j,i}) = \frac{1}{2}\left[\left(\frac{\lambda+2\mu}{\lambda+\mu}\mathsf{g}_{i,pp} - \mathsf{g}_{p,pi}\right)_{,j} + \left(\frac{\lambda+2\mu}{\lambda+\mu}\mathsf{g}_{j,pp} - \mathsf{g}_{p,pj}\right)_{,i}\right]\\
&= \frac{1}{2}\left[\left(\frac{\lambda+2\mu}{\lambda+\mu}\mathsf{g}_{i,ppj} - \mathsf{g}_{p,pij}\right) + \left(\frac{\lambda+2\mu}{\lambda+\mu}\mathsf{g}_{j,ppi} - \mathsf{g}_{p,pji}\right)\right]\\
&= \frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\left(\mathsf{g}_{i,jpp}+\mathsf{g}_{j,ipp}\right)\right] - \frac{1}{2}\left(\mathsf{g}_{p,pji}+\mathsf{g}_{p,pij}\right) = \frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\left(\mathsf{g}_{i,j}+\mathsf{g}_{j,i}\right)\right]_{,pp} - \mathsf{g}_{p,pij}\\
&= \frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\mathsf{g}_{i,j}+\mathsf{g}_{j,i}\right]_{,pp} - (\mathsf{g}_{p,p})_{,ij}
\end{aligned}$$

The stress tensor field for isotropic linear elastic material ($\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$) in terms of Galerkin vector becomes:

$$\begin{aligned}
\mathrm{Tr}(\boldsymbol{\varepsilon}) = \varepsilon_{kk} &= \frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\left(\mathsf{g}_{k,k}+\mathsf{g}_{k,k}\right)\right]_{,pp} - (\mathsf{g}_{p,p})_{,kk} = \frac{\lambda+2\mu}{2(\lambda+\mu)}\left(\mathsf{g}_{k,kpp}+\mathsf{g}_{k,kpp}\right) - \mathsf{g}_{p,pkk}\\
&= \left(\frac{\lambda+2\mu}{(\lambda+\mu)} - 1\right)\mathsf{g}_{k,kpp} = \frac{\mu}{\lambda+\mu}\mathsf{g}_{k,kpp}
\end{aligned}$$

Note that $\mathsf{g}_{k,kpp} = \mathsf{g}_{p,pkk}$. Then,

$$\begin{aligned}
\sigma_{ij} &= \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij} = \lambda\left(\frac{\mu}{\lambda+\mu}\mathsf{g}_{k,kpp}\right)\delta_{ij} + 2\mu\left(\frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\mathsf{g}_{i,j}+\mathsf{g}_{j,i}\right]_{,pp} - (\mathsf{g}_{p,p})_{,ij}\right)\\
&= 2\mu\left\{\frac{\lambda}{2(\lambda+\mu)}\mathsf{g}_{k,kpp}\delta_{ij} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\mathsf{g}_{i,j}+\mathsf{g}_{i,j}\right]_{,pp} - (\mathsf{g}_{p,p})_{,ij}\right\}\\
&= 2\mu\left\{\nu\,\mathsf{g}_{k,kpp}\delta_{ij} + (1-\nu)\left[\mathsf{g}_{i,j}+\mathsf{g}_{j,i}\right]_{,pp} - (\mathsf{g}_{p,p})_{,ij}\right\}
\end{aligned} \tag{5.71}$$

The above equation in tensorial notation becomes:

$$\boxed{\begin{aligned}&\boldsymbol{\sigma} = 2\mu\left\{\frac{\lambda}{2(\lambda+\mu)}[\nabla^2(\nabla\cdot\vec{\mathbf{g}})]\mathbf{1} + \frac{\lambda+2\mu}{(\lambda+\mu)}\nabla^2\left[(\nabla\vec{\mathbf{g}})^{sym}\right] - \nabla(\nabla(\nabla\cdot\vec{\mathbf{g}}))\right\}\\ &or\\ &\boldsymbol{\sigma} = 2\mu\left\{\nu[\nabla^2(\nabla\cdot\vec{\mathbf{g}})]\mathbf{1} + 2(1-\nu)\nabla^2\left[(\nabla\vec{\mathbf{g}})^{sym}\right] - \nabla(\nabla(\nabla\cdot\vec{\mathbf{g}}))\right\}\end{aligned}} \tag{5.72}$$

where we have considered that

$$\frac{\lambda}{2(\lambda+\mu)}=\frac{E\nu}{2(1+\nu)(1-2\nu)}\frac{2(1+\nu)(1-2\nu)}{E}=\nu$$

$$\frac{\lambda+2\mu}{2(\lambda+\mu)}=\frac{1}{2}\frac{2(1+\nu)(1-2\nu)}{E}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}=(1-\nu)$$

NOTE 8: Love's Strain Function

The Love's strain function is a particular case of Galerkin vector in which

$$\mathsf{g}_1=0 \quad ; \quad \mathsf{g}_2=0 \quad ; \quad \mathsf{g}_3=L$$

where L is the Love's strain function. With that, the equation in (5.69) becomes

$$\nabla^4(L)=\frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\mathsf{b}_3=\frac{-1}{2\mu(1-\nu)}\rho\mathsf{b}_3 \tag{5.73}$$

where we have considered that $\mathsf{b}_1=\mathsf{b}_2=0$. The Love's strain function can be applied to axially symmetric problem.

Taking into account that

$\nabla\cdot\vec{\mathbf{g}}=\mathsf{g}_{i,i}=\mathsf{g}_{1,1}+\mathsf{g}_{2,2}+\mathsf{g}_{3,3}=L_{,3}\equiv\frac{\partial L}{\partial x_3}$, $\nabla^2(\mathsf{g}_i)=\begin{Bmatrix}\nabla^2(\mathsf{g}_1)\\ \nabla^2(\mathsf{g}_2)\\ \nabla^2(\mathsf{g}_3)\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ \nabla^2 L\end{Bmatrix}$, the displacement field (5.68) becomes:

$$\mathsf{u}_i=2\mu\left(\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\nabla^2(\mathsf{g}_i)-\frac{1}{2\mu}\frac{\partial(\nabla\cdot\vec{\mathbf{g}})}{\partial x_i}\right) \quad\Rightarrow\quad \mathsf{u}_i=\begin{Bmatrix}\dfrac{-\partial^2 L}{\partial x_1\partial x_3}\\ \dfrac{-\partial^2 L}{\partial x_2\partial x_3}\\ \dfrac{\lambda+2\mu}{\lambda+\mu}\nabla^2 L-\dfrac{\partial^2 L}{\partial x_3\partial x_3}\end{Bmatrix} \tag{5.74}$$

Problem 5.10

a) Obtain the stress field correspondent to the Galerkin vector:

$$\vec{\mathbf{g}}=\underbrace{2x_1^4}_{=\mathsf{g}_1}\hat{\mathbf{e}}_1+\underbrace{x_2^4}_{=\mathsf{g}_2}\hat{\mathbf{e}}_2+\underbrace{(-8x_1^3x_3-4x_2^3x_3)}_{=\mathsf{g}_3}\hat{\mathbf{e}}_3$$

b) Obtain the body force density.

Solution:

The stress in terms of Galerkin vector is given by (5.71), i.e.:

$$\begin{aligned}\sigma_{ij}&=2\mu\{\nu\,\mathsf{g}_{k,kpp}\delta_{ij}+(1-\nu)[\mathsf{g}_{i,j}+\mathsf{g}_{j,i}]_{,pp}-(\mathsf{g}_{p,p})_{,ij}\}\\ &=2\mu\{\nu[\nabla^2(\nabla\cdot\vec{\mathbf{g}})]\mathbf{1}+2(1-\nu)\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]-\nabla(\nabla(\nabla\cdot\vec{\mathbf{g}}))\}_{ij}\end{aligned} \tag{5.75}$$

The gradient of the Galerkin vector is given by:

$$(\nabla\vec{\mathbf{g}})_{ij} = \mathsf{g}_{i,j} = \begin{bmatrix} \frac{\partial \mathsf{g}_1}{\partial x_1} & \frac{\partial \mathsf{g}_1}{\partial x_2} & \frac{\partial \mathsf{g}_1}{\partial x_3} \\ \frac{\partial \mathsf{g}_2}{\partial x_1} & \frac{\partial \mathsf{g}_2}{\partial x_2} & \frac{\partial \mathsf{g}_2}{\partial x_3} \\ \frac{\partial \mathsf{g}_3}{\partial x_1} & \frac{\partial \mathsf{g}_3}{\partial x_2} & \frac{\partial \mathsf{g}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 8x_1^3 & 0 & 0 \\ 0 & 4x_2^3 & 0 \\ -24x_1^2 x_3 & -12x_2^2 x_3 & -8x_1^3 - 4x_2^3 \end{bmatrix}$$

Note that $(\nabla\cdot\vec{\mathbf{g}}) = \mathrm{Tr}(\nabla\vec{\mathbf{g}}) = 8x_1^3 + 4x_2^3 + (-8x_1^3 - 4x_2^3) = 0$

By applying the Laplacian to $(\nabla\vec{\mathbf{g}})$ we can obtain $\nabla^2(\nabla\vec{\mathbf{g}})$, which in indicial notation becomes:

$$[\nabla^2(\nabla\vec{\mathbf{g}})]_{ij} = (\mathsf{g}_{i,j})_{,kk} = \frac{\partial^2(\mathsf{g}_{i,j})}{\partial x_k \partial x_k} = \mathsf{g}_{i,jkk}$$

$$= \frac{\partial^2(\mathsf{g}_{i,j})}{\partial x_1 \partial x_1} + \frac{\partial^2(\mathsf{g}_{i,j})}{\partial x_2 \partial x_2} + \frac{\partial^2(\mathsf{g}_{i,j})}{\partial x_3 \partial x_3} = \begin{bmatrix} 48x_1 & 0 & 0 \\ 0 & 24x_2 & 0 \\ -48x_3 & -24x_3 & -48x_1 - 24x_2 \end{bmatrix}$$

$$\{\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]\}_{ij} = \frac{1}{2}\left(\begin{bmatrix} 48x_1 & 0 & 0 \\ 0 & 24x_2 & 0 \\ -48x_3 & -24x_3 & -48x_1 - 24x_2 \end{bmatrix} + \begin{bmatrix} 48x_1 & 0 & -48x_3 \\ 0 & 24x_2 & -24x_3 \\ 0 & 0 & -48x_1 - 24x_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 48x_1 & 0 & -24x_3 \\ 0 & 24x_2 & -12x_3 \\ -24x_3 & -12x_3 & -48x_1 - 24x_2 \end{bmatrix}$$

Note that $[\nabla^2(\nabla\vec{\mathbf{g}})]^{sym} = \nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]$. Then, the equation for stress (5.75) becomes:

$$\sigma_{ij} = 2\mu\{\nu[\nabla^2(\nabla\cdot\vec{\mathbf{g}})]\mathbf{1} + 2(1-\nu)\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}] - \nabla(\nabla(\nabla\cdot\vec{\mathbf{g}}))\}_{ij} = 2\mu\{2(1-\nu)\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]\}_{ij}$$

$$\sigma_{ij} = 2\mu\{2(1-\nu)\nabla^2[(\nabla\vec{\mathbf{g}})^{sym}]\}_{ij} = 2\mu\left\{2(1-\nu)\begin{bmatrix} 48x_1 & 0 & -24x_3 \\ 0 & 24x_2 & -12x_3 \\ -24x_3 & -12x_3 & -48x_1 - 24x_2 \end{bmatrix}\right\}$$

$$= 2\mu(1-\nu)\begin{bmatrix} 96x_1 & 0 & -48x_3 \\ 0 & 48x_2 & -24x_3 \\ -48x_3 & -24x_3 & -48(2x_1 + x_2) \end{bmatrix}$$

b) According to the equation in (5.69) the body force density ($\rho\vec{\mathbf{b}}$) and the Galerkin vector ($\vec{\mathbf{g}}$) are related to each other by

$$\nabla^4(\mathsf{g}_i) = \frac{\partial^2}{\partial x_k \partial x_k}\left(\frac{\partial^2(\mathsf{g}_i)}{\partial x_j \partial x_j}\right) = \frac{-(\lambda+\mu)}{\mu(\lambda+2\mu)}\rho\mathsf{b}_i$$
$$\Rightarrow \rho\mathsf{b}_i = \frac{-\mu(\lambda+2\mu)}{(\lambda+\mu)}\nabla^4(\mathsf{g}_i) = -2\mu\left(\frac{(\lambda+2\mu)}{2(\lambda+\mu)}\right)\nabla^4(\mathsf{g}_i) = -2\mu(1-\nu)\nabla^4(\mathsf{g}_i) \tag{5.76}$$

in which

$$\mathsf{g}_1 = 2x_1^4 \quad ; \quad \mathsf{g}_2 = x_2^4 \quad ; \quad \mathsf{g}_3 = -8x_1^3 x_3 - 4x_2^3 x_3$$

$$\frac{\partial^2(\mathsf{g}_i)}{\partial x_j \partial x_j} = \frac{\partial^2(\mathsf{g}_i)}{\partial x_1 \partial x_1} + \frac{\partial^2(\mathsf{g}_i)}{\partial x_2 \partial x_2} + \frac{\partial^2(\mathsf{g}_i)}{\partial x_3 \partial x_3}$$

$$\frac{\partial^2(\mathsf{g}_i)}{\partial x_j \partial x_j} = \mathsf{a}_i = \begin{cases} i=1 \quad \Rightarrow \quad \dfrac{\partial^2(2x_1^4)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(2x_1^4)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(2x_1^4)}{\partial x_3 \partial x_3} = 24x_1^2 = \mathsf{a}_1 \\ i=2 \quad \Rightarrow \quad \dfrac{\partial^2(x_2^4)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(x_2^4)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(x_2^4)}{\partial x_3 \partial x_3} = 12x_2^2 = \mathsf{a}_2 \\ i=3 \quad \Rightarrow \quad \dfrac{\partial^2(\mathsf{g}_3)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(\mathsf{g}_3)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(\mathsf{g}_3)}{\partial x_3 \partial x_3} = -48x_1 x_3 - 24x_2 x_3 = \mathsf{a}_3 \end{cases}$$

$$\nabla^4(\mathsf{g}_i) = \frac{\partial^2}{\partial x_k \partial x_k}\left(\frac{\partial^2(\mathsf{g}_i)}{\partial x_j \partial x_j}\right) = \frac{\partial^2(\mathsf{a}_i)}{\partial x_k \partial x_k} = \begin{cases} i=1 \quad \Rightarrow \quad \dfrac{\partial^2(24x_1^2)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(24x_1^2)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(24x_1^2)}{\partial x_3 \partial x_3} = 48x_1 \\ i=2 \quad \Rightarrow \quad \dfrac{\partial^2(12x_2^2)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(12x_2^2)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(12x_2^2)}{\partial x_3 \partial x_3} = 24x_2 \\ i=3 \quad \Rightarrow \quad \dfrac{\partial^2(\mathsf{a}_3)}{\partial x_1 \partial x_1} + \dfrac{\partial^2(\mathsf{a}_3)}{\partial x_2 \partial x_2} + \dfrac{\partial^2(\mathsf{a}_3)}{\partial x_3 \partial x_3} = 0 \end{cases}$$

Then, the equation in (5.76) becomes:

$$\rho \mathsf{b}_i = -2\mu(1-\nu)\nabla^4(\mathsf{g}_i) = -2\mu(1-\nu)\begin{Bmatrix} 48x_1 \\ 24x_2 \\ 0 \end{Bmatrix}$$

Problem 5.11

a) Show that:

$$\boxed{\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \mathbf{0}} \quad \xrightarrow{\textit{Indicial}} \quad \boxed{\epsilon_{qjk}\epsilon_{til}\varepsilon_{ij,kl} = 0_{qt}} \tag{5.77}$$

where ϵ_{ijk} is the permutation symbol, and $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor.

b) Show also that:

$$\boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} - \varepsilon_{jk,il} = \mathbb{O}_{ijkl}} \tag{5.78}$$

c) Express the explicit form of the equations in (5.77).

Solution:

The infinitesimal strain tensor is given by $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \mathsf{u}_j}{\partial x_i} + \dfrac{\partial \mathsf{u}_i}{\partial x_j}\right) = \dfrac{1}{2}(\mathsf{u}_{j,i} + \mathsf{u}_{i,j})$, and if we take the derivative with respect to $(\vec{x})$ we can obtain:

$$\frac{\partial \varepsilon_{ij}}{\partial x_k} \equiv \varepsilon_{ij,k} = \frac{1}{2}(\mathsf{u}_{j,ik} + \mathsf{u}_{i,jk})$$

Note that $\mathsf{u}_{i,jk} = \mathsf{u}_{i,kj}$ is symmetric in jk, and if we multiply by the antisymmetric tensor in jk, i.e. $\epsilon_{qjk} = -\epsilon_{qkj}$, we can obtain: $\mathsf{u}_{i,jk}\epsilon_{qjk} = 0_{iq}$, thus

$$\epsilon_{qjk}\varepsilon_{ij,k} = \frac{1}{2}(\mathsf{u}_{j,ik} + \mathsf{u}_{i,jk})\epsilon_{qjk} = \frac{1}{2}\mathsf{u}_{j,ik}\epsilon_{qjk} + \frac{1}{2}\underbrace{\mathsf{u}_{i,jk}\epsilon_{qjk}}_{=0_{iq}} = \frac{1}{2}\mathsf{u}_{j,ik}\epsilon_{qjk}$$

once again we take the derivative with respect to $(\vec{x})$ and we can obtain:

$$\frac{\partial(\epsilon_{qjk}\varepsilon_{ij,k})}{\partial x_l} = \epsilon_{qjk}\varepsilon_{ij,kl} = \frac{1}{2}\mathsf{u}_{j,ikl}\epsilon_{qjk}$$

Note that $\mathsf{u}_{j,ikl} = \mathsf{u}_{j,kil} = \mathsf{u}_{j,kli}$ is symmetric in il and $\epsilon_{til} = -\epsilon_{tli}$ is antisymmetic in il. With that, if we multiply both sides of the equation by ϵ_{til} we can obtain the equation in (5.77), i.e.:

$$\epsilon_{til}\epsilon_{qjk}\varepsilon_{ij,kl} = \frac{1}{2}\mathsf{u}_{j,ikl}\epsilon_{til}\epsilon_{qjk} = 0_{jkt}\epsilon_{qjk} = 0_{qt} \qquad \textit{Q.E.D.}$$

b) Now, if we multiply both sides of the above equation by $\epsilon_{tab}\epsilon_{qmn}$, we can obtain:

$$\epsilon_{tab}\epsilon_{qmn}\epsilon_{til}\epsilon_{qjk}\varepsilon_{ij,kl} = 0_{qt}\epsilon_{tab}\epsilon_{qmn} = \mathbb{O}_{abmn}$$

Remember that the relationships $\epsilon_{tab}\epsilon_{til} = \delta_{ai}\delta_{bl} - \delta_{al}\delta_{bi}$ and $\epsilon_{qmn}\epsilon_{qjk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}$ hold, thus:

$$\begin{aligned}
&\epsilon_{tab}\epsilon_{qmn}\epsilon_{til}\epsilon_{qjk}\varepsilon_{ij,kl} = \mathbb{O}_{abmn}\\
&\Rightarrow (\delta_{ai}\delta_{bl} - \delta_{al}\delta_{bi})(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})\varepsilon_{ij,kl} = \mathbb{O}_{abmn}\\
&\Rightarrow (\delta_{ai}\delta_{bl}\delta_{mj}\delta_{nk} - \delta_{ai}\delta_{bl}\delta_{mk}\delta_{nj} - \delta_{al}\delta_{bi}\delta_{mj}\delta_{nk} + \delta_{al}\delta_{bi}\delta_{mk}\delta_{nj})\varepsilon_{ij,kl} = \mathbb{O}_{abmn}
\end{aligned}$$

Then we can obtain $\varepsilon_{am,nb} - \varepsilon_{an,mb} - \varepsilon_{bm,na} + \varepsilon_{bn,ma} = \mathbb{O}_{abmn}$, which is the same as:

$$\varepsilon_{am,bn} + \varepsilon_{bn,am} - \varepsilon_{an,mb} - \varepsilon_{mb,an} = \mathbb{O}_{ambn} \qquad \textit{Q.E.D.}$$

Note that, if we multiply the above equation by δ_{bn} we can also express (5.78) as follows:

$$\begin{aligned}
&\varepsilon_{am,bn}\delta_{bn} + \varepsilon_{bn,am}\delta_{bn} - \varepsilon_{an,mb}\delta_{bn} - \varepsilon_{mb,an}\delta_{bn} = \mathbb{O}_{ambn}\delta_{bn}\\
&\Rightarrow \varepsilon_{am,bb} + \varepsilon_{bb,am} - \varepsilon_{ab,bm} - \varepsilon_{mb,ba} = 0_{ambb}\\
&\Rightarrow [\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon})]_{am} + [\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\varepsilon})]]]_{am} - [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{am} - [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{ma} = 0_{ambb}\\
&\Rightarrow [\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon})]_{am} + [\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\varepsilon})]]]_{am} = [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{am} + [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{ma}\\
&\Rightarrow [\nabla^2_{\vec{x}}\boldsymbol{\varepsilon}]_{am} + [\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\varepsilon})]]]_{am} = [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{am} + [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]_{ma}
\end{aligned}$$

which in tensorial notation becomes:

$$\boxed{\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon}) + \nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\varepsilon})]] = \nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon}) + [\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]^T = 2[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})]^{sym}} \qquad (5.79)$$

c) Note that the equation in (5.77) is symmetric, and has 6 independent equations.

For the case when $q=1, t=1$ we can obtain $\epsilon_{1jk}\epsilon_{1il}\varepsilon_{ij,kl}$ and by expanding the index l we can obtain:

$$\epsilon_{1jk}\epsilon_{1il}\varepsilon_{ij,kl} = \epsilon_{1jk}\epsilon_{1i1}\varepsilon_{ij,k1} + \epsilon_{1jk}\epsilon_{1i2}\varepsilon_{ij,k2} + \epsilon_{1jk}\epsilon_{1i3}\varepsilon_{ij,k3} = \epsilon_{1jk}\epsilon_{1i2}\varepsilon_{ij,k2} + \epsilon_{1jk}\epsilon_{1i3}\varepsilon_{ij,k3}$$

Expanding the index i the above equation becomes:

$$\epsilon_{1jk}\epsilon_{1il}\varepsilon_{ij,kl} = \epsilon_{1jk}\epsilon_{1i2}\varepsilon_{ij,k2} + \epsilon_{1jk}\epsilon_{1i3}\varepsilon_{ij,k3} = \epsilon_{1jk}\epsilon_{132}\varepsilon_{3j,k2} + \epsilon_{1jk}\epsilon_{123}\varepsilon_{2j,k3} = -\epsilon_{1jk}\varepsilon_{3j,k2} + \epsilon_{1jk}\varepsilon_{2j,k3}$$

and by expanding the remaining indices we can obtain:

$$\begin{aligned}\epsilon_{1jk}\epsilon_{1il}\varepsilon_{ij,kl} &= -\epsilon_{1jk}\varepsilon_{3j,k2} + \epsilon_{1jk}\varepsilon_{2j,k3} = -\epsilon_{123}\varepsilon_{32,32} - \epsilon_{132}\varepsilon_{33,22} + \epsilon_{123}\varepsilon_{22,33} + \epsilon_{132}\varepsilon_{23,23} \\ &= -\varepsilon_{32,32} + \varepsilon_{33,22} + \varepsilon_{22,33} - \varepsilon_{23,23} = \varepsilon_{33,22} + \varepsilon_{22,33} - 2\varepsilon_{23,23} = 0 \\ &= \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0\end{aligned}$$

note that $\varepsilon_{23,23} = \varepsilon_{32,32}$.

We leave the reader with the following demonstrations:

when $q = 2, t = 2$

$$\begin{aligned}\epsilon_{2jk}\epsilon_{2il}\varepsilon_{ij,kl} &= -\varepsilon_{31,31} + \varepsilon_{33,11} + \varepsilon_{11,33} - \varepsilon_{13,13} = \varepsilon_{33,11} + \varepsilon_{11,33} - 2\varepsilon_{13,13} = 0 \\ &= \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0\end{aligned}$$

when $q = 3, t = 3$

$$\begin{aligned}\epsilon_{3jk}\epsilon_{3il}\varepsilon_{ij,kl} &= \varepsilon_{11,22} - \varepsilon_{12,12} - \varepsilon_{21,21} + \varepsilon_{22,11} = \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0 \\ &= \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0\end{aligned}$$

when $q = 1, t = 2$

$$\begin{aligned}\epsilon_{1jk}\epsilon_{2il}\varepsilon_{ij,kl} &= -\varepsilon_{12,33} + \varepsilon_{13,23} + \varepsilon_{32,31} - \varepsilon_{33,21} = \varepsilon_{13,23} + \varepsilon_{23,13} - \varepsilon_{33,12} - \varepsilon_{12,33} = 0 \\ &= \frac{\partial^2 \varepsilon_{13}}{\partial x_2 \partial x_3} + \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_3} - \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} - \frac{\partial^2 \varepsilon_{12}}{\partial x_3 \partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) - \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0\end{aligned}$$

when $q = 2, t = 3$

$$\begin{aligned}\epsilon_{2jk}\epsilon_{3il}\varepsilon_{ij,kl} &= -\varepsilon_{11,32} + \varepsilon_{13,12} + \varepsilon_{21,31} - \varepsilon_{23,11} = \varepsilon_{13,12} + \varepsilon_{12,13} - \varepsilon_{23,11} - \varepsilon_{11,23} = 0 \\ &= \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_1} - \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{23}}{\partial x_1}\right) - \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = 0\end{aligned}$$

and when $q = 1, t = 3$

$$\begin{aligned}\epsilon_{1jk}\epsilon_{3il}\varepsilon_{ij,kl} &= \varepsilon_{12,32} - \varepsilon_{13,22} - \varepsilon_{22,31} + \varepsilon_{23,21} = \varepsilon_{12,23} - \varepsilon_{13,22} - \varepsilon_{22,13} + \varepsilon_{23,12} = 0 \\ &= \frac{\partial^2 \varepsilon_{12}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{13}}{\partial x_2 \partial x_2} - \frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} + \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right) - \frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = 0\end{aligned}$$

By regrouping the previous 6 equations we can obtain:

$$\boxed{\begin{cases} S_{11} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \\ S_{22} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0 \\ S_{33} = \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \\ S_{12} = \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 \\ S_{23} = \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = 0 \\ S_{13} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = 0 \end{cases}} \quad \textit{Compatibility equations for 3D} \qquad (5.80)$$

or in tensorial notation

$$\boxed{\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \mathbf{0}}$$

The above equations in Voigt notation become:

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{13} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0 & \dfrac{\partial^2}{\partial x_3^2} & \dfrac{\partial^2}{\partial x_2^2} & 0 & \dfrac{-\partial^2}{\partial x_2 \partial x_3} & 0 \\ \dfrac{\partial^2}{\partial x_3^2} & 0 & \dfrac{\partial^2}{\partial x_1^2} & 0 & 0 & \dfrac{-\partial^2}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2}{\partial x_2^2} & \dfrac{\partial^2}{\partial x_1^2} & 0 & \dfrac{-\partial^2}{\partial x_1 \partial x_2} & 0 & 0 \\ \hline 0 & 0 & \dfrac{-\partial^2}{\partial x_1 \partial x_2} & -\frac{1}{2}\dfrac{\partial^2}{\partial x_3^2} & \frac{1}{2}\dfrac{\partial^2}{\partial x_1 \partial x_3} & \frac{1}{2}\dfrac{\partial^2}{\partial x_2 \partial x_3} \\ \dfrac{-\partial^2}{\partial x_2 \partial x_3} & 0 & 0 & \frac{1}{2}\dfrac{\partial^2}{\partial x_1 \partial x_3} & -\frac{1}{2}\dfrac{\partial^2}{\partial x_1^2} & \frac{1}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} \\ 0 & \dfrac{-\partial^2}{\partial x_1 \partial x_3} & 0 & \frac{1}{2}\dfrac{\partial^2}{\partial x_2 \partial x_3} & \frac{1}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & -\frac{1}{2}\dfrac{\partial^2}{\partial x_2^2} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (5.81)$$

$$\{\boldsymbol{S}\} = [\boldsymbol{L}^{(2)}]\ \{\boldsymbol{\varepsilon}\} = \{\mathbf{0}\}$$

NOTE 1: The equations in (5.80) are known as the compatibility equations. The compatibility equations guarantee that the displacement field is unique and continuous, (see Figure 5.10). In other words, the 6 components of the strain tensor are not independent and cannot be arbitrary.

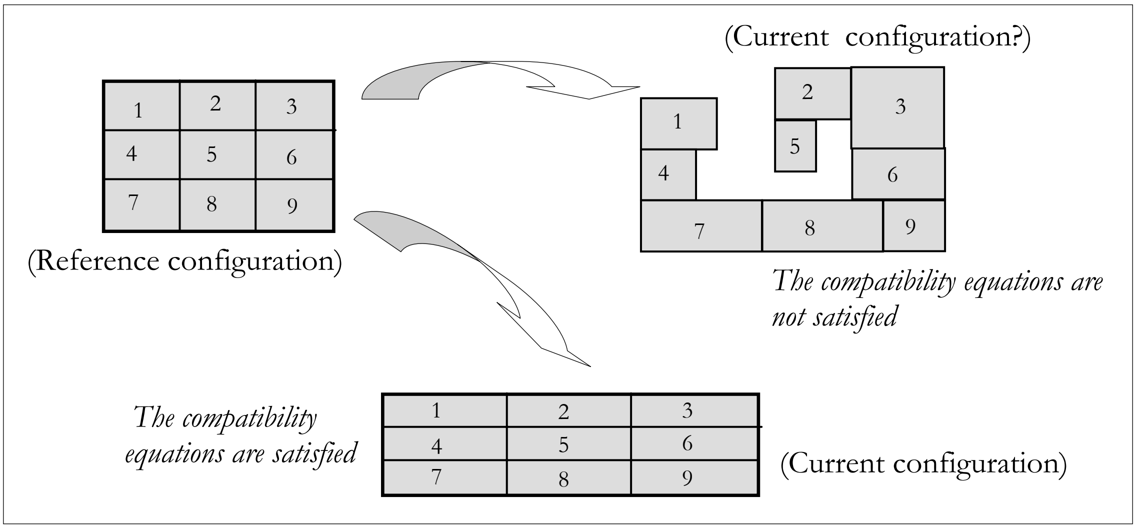


Figure 5.10

NOTE 2: When using numerical method for obtaining the solution, e.g. finite element method, the way to ensure the compatibility equations is by means of the continuity of the displacement field. With regards the finite element method, when we assembly the elements (tie nodes), in general, we are ensuring that the compatibility equations are satisfied.

NOTE 3: When the displacement field is independent of one direction, e.g. $\vec{\mathbf{u}} = \vec{\mathbf{u}}(x_1, x_2)$, the compatibility equations reduce to:

$$\boxed{S_{33} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0} \quad \text{Compatibility equation for 2D} \tag{5.82}$$

since $\varepsilon_{i3} = \varepsilon_{3i} = 0$. The above equation in Engineering notation becomes:

$$\boxed{S_z = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0} \quad \text{Compatibility equation for 2D (Engineering notation)} \tag{5.83}$$

NOTE 4: To understand the compatibility condition let us consider an example in two dimensional case (2D), where we have the scalar field $\phi = \phi(x_1, x_2)$ and we know the following derivatives: $\frac{\partial \phi}{\partial x_1} = x_1 + 3x_2$ and $\frac{\partial \phi}{\partial x_2} = x_1^2$, we can see clearly that the scalar field ϕ is incompatible since

$$\left.\begin{array}{l} \dfrac{\partial \phi}{\partial x_1} = x_1 + 3x_2 = \mathsf{F}_1 \quad \Rightarrow \quad \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \phi}{\partial x_1}\right) = \dfrac{\partial^2 \phi}{\partial x_2 \partial x_1} = \dfrac{\partial (x_1 + 3x_2)}{\partial x_2} = 3 \\ \dfrac{\partial \phi}{\partial x_2} = x_1^2 = \mathsf{F}_2 \quad \Rightarrow \quad \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial \phi}{\partial x_2}\right) = \dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} = \dfrac{\partial (x_1^2)}{\partial x_2} = 2x_1 \end{array}\right\} \Rightarrow \underbrace{\frac{\partial^2 \phi}{\partial x_2 \partial x_1} \neq \frac{\partial^2 \phi}{\partial x_1 \partial x_2}}_{\text{incompatible}}$$

The scalar field $\phi = \phi(x_1, x_2)$ will be compatible if and only if:

$$\left.\begin{array}{l} \dfrac{\partial \phi}{\partial x_1} = \mathsf{F}_1(x_1, x_2) \\ \dfrac{\partial \phi}{\partial x_2} = \mathsf{F}_2(x_1, x_2) \end{array}\right\} \xrightarrow{\phi \text{ is compatible field iff}} \frac{\partial \mathsf{F}_1}{\partial x_2} = \frac{\partial \mathsf{F}_2}{\partial x_1} \tag{5.84}$$

If we consider the Green's theorem, (Chaves(2013)-Chapter), which states:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{e}}_3 dS \xrightarrow{components} \oint_{\Gamma} \mathsf{F}_1 dx_1 + \mathsf{F}_2 dx_2 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) dS_3$$

and also considering the equation in (5.84), we conclude that: if $\vec{\mathbf{F}} = \nabla_{\vec{x}} \phi$, ϕ is compatible if and only if $\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{e}}_3 dS = 0 \quad \Rightarrow \quad \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}} = \vec{\mathbf{0}}$.

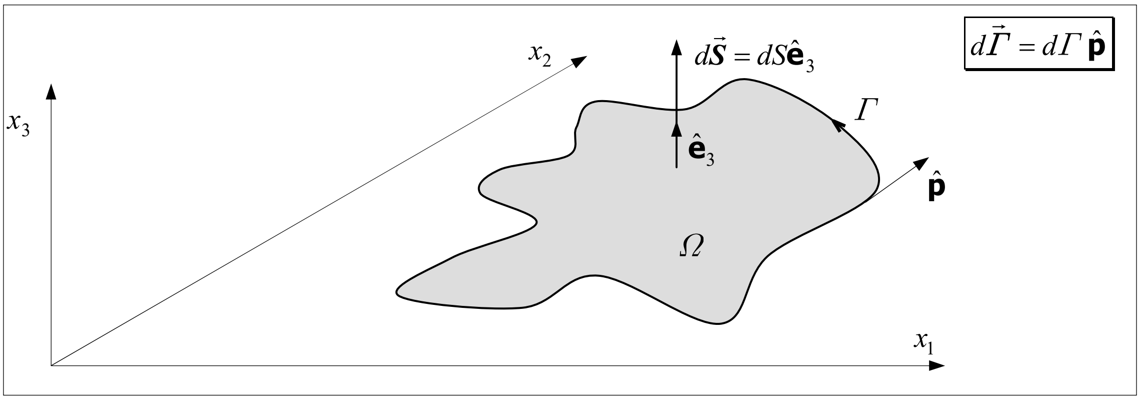


Figure 5.11: Green's theorem.

NOTE 5: Let us consider that $\vec{\mathbf{F}} = (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) \cdot \vec{\mathbf{a}} = \vec{\mathbf{a}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T$, where $\boldsymbol{\varepsilon}$ is a second-order tensor field and $\vec{\mathbf{a}}$ is an arbitrary vector independent of $\vec{x}$ (constant). Note also that the following relations are true:

$$\text{(a)} \quad \oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\Gamma} = \oint_{\Gamma} \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) \cdot \vec{\mathbf{a}} \right] \cdot d\vec{\Gamma} = \oint_{\Gamma} \left[\vec{\mathbf{a}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right] \cdot d\vec{\Gamma} = \vec{\mathbf{a}} \cdot \oint_{\Gamma} (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \cdot d\vec{\Gamma}$$

and

$$\text{(b)} \quad \begin{aligned} \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{S} &= \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) \cdot \vec{\mathbf{a}} \right] \right\} \cdot d\vec{S} = \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge \left[\vec{\mathbf{a}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right] \right\} \cdot d\vec{S} \\ &= \int_{\Omega} \vec{\mathbf{a}} \cdot \left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T \cdot d\vec{S} = \vec{\mathbf{a}} \cdot \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T \cdot d\vec{S} \end{aligned}$$

In indicial notation becomes

$$\text{(a)} \quad \oint_{\Gamma} \mathsf{F}_i \, (d\vec{\Gamma})_i = \oint_{\Gamma} (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})_{ij} \, \mathsf{a}_j \, (d\vec{\Gamma})_i = \oint_{\Gamma} \mathsf{a}_j \, (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})_{ij} \, (d\vec{\Gamma})_i = \mathsf{a}_j \oint_{\Gamma} (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})_{ij} \, (d\vec{\Gamma})_i$$

(b)

$$\begin{aligned} \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}})_i \, (d\vec{S})_i &= \int_{\Omega} \epsilon_{ijk} \mathsf{F}_{k,j} \, (d\vec{S})_i = \int_{\Omega} \epsilon_{ijk} \left[\mathsf{a}_p \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{kp} \right]_{,j} (d\vec{S})_i \\ &= \int_{\Omega} \epsilon_{ijk} \left[\underbrace{\mathsf{a}_{p,j}}_{=0_{pj}} \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{kp} + \mathsf{a}_p \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{kp,j} \right] (d\vec{S})_i \\ &= \int_{\Omega} \epsilon_{ijk} \mathsf{a}_p \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{kp,j} (d\vec{S})_i = \mathsf{a}_p \int_{\Omega} \epsilon_{ijk} \left[(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{kp,j} (d\vec{S})_i \\ &= \mathsf{a}_p \int_{\Omega} \epsilon_{ijk} \left[\epsilon_{psq} \varepsilon_{qk,s} \right]_{,j} (d\vec{S})_i = \mathsf{a}_p \int_{\Omega} \epsilon_{ijk} \, \epsilon_{psq} \, \varepsilon_{qk,sj} \, (d\vec{S})_i \\ &= \mathsf{a}_p \int_{\Omega} \left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right]_{ip} (d\vec{S})_i = \vec{\mathbf{a}} \cdot \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T \cdot d\vec{S} \end{aligned}$$

It would be worth reviewing **Problem 1.110**, in which we have shown that the following relationship $(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) = \epsilon_{ksq}\varepsilon_{qp,s}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_p$ holds, thus $(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \epsilon_{psq}\varepsilon_{qk,s}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_p$. Also in **Problem 1.110** we have shown that $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \overline{\boldsymbol{\varepsilon}})^T = \epsilon_{ipq}\epsilon_{tsj}\overline{\varepsilon}_{qj,ps}\hat{\mathbf{e}}_t \otimes \hat{\mathbf{e}}_i$, which is equivalent to $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \overline{\boldsymbol{\varepsilon}})^T = \epsilon_{psq}\epsilon_{ijk}\overline{\varepsilon}_{qk,sj}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_p$.

And by consider the Stokes' Theorem, (see Chaves (2013)-Chapter 1), we conclude:

$$\underbrace{\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{S}}$$

$$\Downarrow$$

$$\vec{\mathbf{a}} \cdot \oint_{\Gamma} (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \cdot d\vec{\Gamma} = \vec{\mathbf{a}} \cdot \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T \cdot d\vec{S}$$

$$\Downarrow$$

$$\oint_{\Gamma} (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \cdot d\vec{\Gamma} = \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T \cdot d\vec{S}$$

Then, for a compatible field it must fulfill:

$$\left\{ \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T \right\}^T = \mathbf{0} \quad \Rightarrow \quad \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \mathbf{0}$$

Now let us consider $\vec{\mathbf{A}} = \boldsymbol{F} \cdot \vec{\mathbf{a}}$ where $\boldsymbol{F}$ is the gradient deformation, $\boldsymbol{F} = \dfrac{\partial \vec{\boldsymbol{x}}}{\partial \vec{\boldsymbol{X}}}$, and $\vec{\mathbf{a}}$ is an arbitrary vector. By apply the Stokes' theorem we can obtain:

$$\underbrace{\oint_{\Gamma} \vec{\mathbf{A}} \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{A}}) \cdot d\vec{S} \quad \Rightarrow \quad \oint_{\Gamma} (\boldsymbol{F} \cdot \vec{\mathbf{a}}) \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge (\boldsymbol{F} \cdot \vec{\mathbf{a}})) \cdot d\vec{S}}$$

$$\Downarrow \tag{5.85}$$

$$\vec{\mathbf{a}} \cdot \oint_{\Gamma} \boldsymbol{F}^T \cdot d\vec{\Gamma} = \vec{\mathbf{a}} \cdot \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F} \right\}^T \cdot d\vec{S} \quad \Rightarrow \quad \oint_{\Gamma} \boldsymbol{F}^T \cdot d\vec{\Gamma} = \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F} \right\}^T \cdot d\vec{S}$$

Then, for a compatible field it must fulfill:

$$\left\{ \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F} \right\}^T = \mathbf{0} \quad \Rightarrow \quad \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{F} = \mathbf{0}$$

More detail about these algebraic manipulations is provided in **Problem 1.110**.

NOTE 6: Note that if:

$$\oint_{\Gamma} \boldsymbol{T} \cdot d\vec{\Gamma} = \int_{\Omega} \left\{ \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{T}^T \right\}^T \cdot d\vec{S} \tag{5.86}$$

Then, for a compatible field it must fulfill:

$$\left\{ \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{T}^T \right\}^T = \mathbf{0} \quad \Rightarrow \quad \vec{\nabla}_{\vec{x}} \wedge \boldsymbol{T}^T = \mathbf{0}$$

And if we use indicial notation we can obtain:

$$\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{T}^T = \epsilon_{ipq} T_{jq,p} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = 0_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

By multiply by ϵ_{ikt} we can obtain:

$$\epsilon_{ipq}\epsilon_{ikt} T_{jq,p} = \epsilon_{ikt} 0_{ij} = 0_{ktj}$$

Considering $\epsilon_{ipq}\epsilon_{ikt} = (\delta_{pk}\delta_{qt} - \delta_{pt}\delta_{qk})$ the above equation can be written as follows:

$$\epsilon_{ipq}\epsilon_{ikt}T_{jq,p} = 0_{ktj} \quad \Rightarrow \quad T_{jq,p}(\delta_{pk}\delta_{qt} - \delta_{pt}\delta_{qk}) = 0_{ktj} \quad \Rightarrow \quad \Rightarrow T_{jt,k} - T_{jk,t} = 0_{ktj} \tag{5.87}$$

In other words, the above equation is a necessary and sufficient condition that the integrands of $\oint_{\Gamma} \boldsymbol{T} \cdot d\vec{\Gamma}$ be exact differentials.

NOTE 7: In this note we will demonstrate the compatibility equations, for small deformation regime, using the demonstration described by E. Cesàro, (see Sokolnikoff (1956), Love(1944)).

Let us consider the material point $P^0(\vec{x}^0)$ in which the displacement $\mathrm{u}_i^0(\vec{x}^0)$ and the infinitesimal spin tensor $\omega_{ij}^0(\vec{x}^0)$ are known. Next we will determine the displacement at any other material point $P'(\vec{x}')$ in terms of $\boldsymbol{\varepsilon}$, (see Figure 5.12).

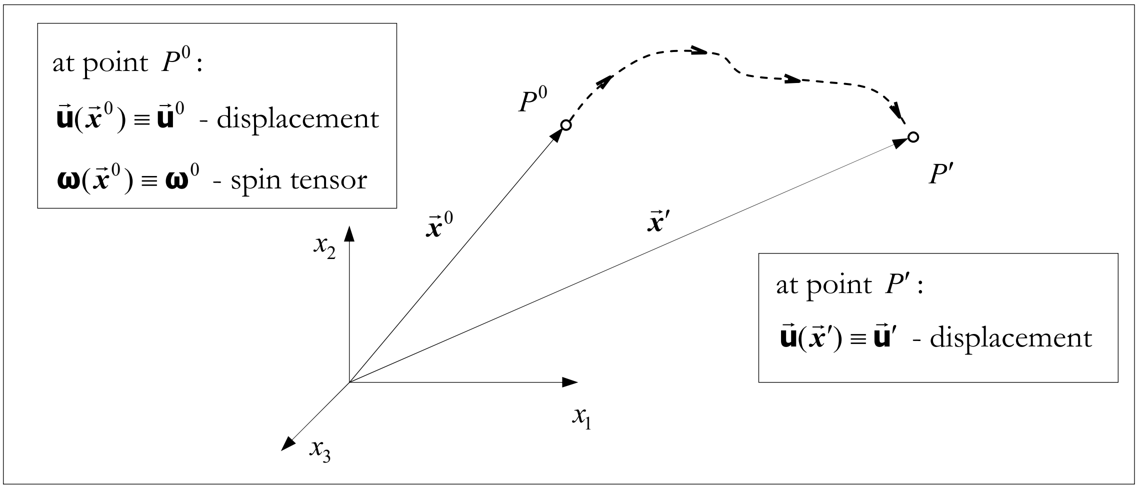


Figure 5.12.

Consider the displacement differential element

$$d\vec{\mathbf{u}} = (\nabla_{\vec{x}}\vec{\mathbf{u}}) \cdot d\vec{x} = ((\nabla_{\vec{x}}\vec{\mathbf{u}})^{sym} + (\nabla_{\vec{x}}\vec{\mathbf{u}})^{skew}) \cdot d\vec{x} = (\boldsymbol{\varepsilon} + \boldsymbol{\omega}) \cdot d\vec{x} = \boldsymbol{\varepsilon} \cdot d\vec{x} + \boldsymbol{\omega} \cdot d\vec{x},$$

which in indicial notation becomes $d\mathrm{u}_i = \varepsilon_{ij}dx_j + \omega_{ij}dx_j$, and by integrating it along the path ($P^0 \to P'$) we can obtain:

$$\begin{aligned}
&\int_{P^0}^{P'} d\mathrm{u}_i = \int_{P^0}^{P'} \varepsilon_{ij}dx_j + \int_{P^0}^{P'} \omega_{ij}dx_j \quad \Rightarrow \quad \mathrm{u}_i\Big|_{P^0}^{P'} = \int_{P^0}^{P'} \varepsilon_{ij}dx_j + \int_{P^0}^{P'} \omega_{ij}dx_j \\
&\Rightarrow \mathrm{u}'_i(\vec{x}') - \mathrm{u}_i^0(\vec{x}^0) = \int_{P^0}^{P'} \varepsilon_{ij}dx_j + \int_{P^0}^{P'} \omega_{ij}dx_j \\
&\Rightarrow \mathrm{u}'_i(\vec{x}') = \mathrm{u}_i^0(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij}dx_j + \int_{P^0}^{P'} \omega_{ij}dx_j
\end{aligned} \tag{5.88}$$

Note that

$$\int_{P^0}^{P'} \omega_{ij}dx_j = \int_{P^0}^{P'} \omega_{ij}\delta_{jk}dx_k = \int_{P^0}^{P'} \omega_{ij}\frac{\partial(x_j - x'_j)}{\partial x_k}dx_k = [\omega_{ij}(x_j - x'_j)]\Big|_{P^0}^{P'} - \int_{P^0}^{P'} \frac{\partial\omega_{ij}}{\partial x_k}(x_j - x'_j)dx_k$$

where we have applied the integration by parts. The above equation can also be expressed as follows:

$$\begin{aligned}\int_{P^0}^{P'} \omega_{ij} dx_j &= [\omega_{ij}(x_j - x'_j)]\Big|_{P^0}^{P'} - \int_{P^0}^{P'} \frac{\partial \omega_{ij}}{\partial x_k}(x_j - x'_j)dx_k \\ &= [\omega'_{ij}(\vec{x}')(x'_j - x'_j)] - [\omega^0_{ij}(\vec{x}^0)(x^0_j - x'_j)] - \int_{P^0}^{P'} \omega_{ij,k}(x_j - x'_j)dx_k \\ &= -[\omega^0_{ij}(x^0_j - x'_j)] - \int_{P^0}^{P'} \omega_{ij,k}(x_j - x'_j)dx_k\end{aligned}$$

Then, the equation (5.88) can be rewritten as follows:

$$\begin{aligned}&\mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij} dx_j + \int_{P^0}^{P'} \omega_{ij} dx_j \\ &\Rightarrow \mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij} dx_j - [\omega^0_{ij}(x^0_j - x'_j)] - \int_{P^0}^{P'} \omega_{ij,k}(x_j - x'_j)dx_k\end{aligned} \tag{5.89}$$

Considering that $\vec{\varphi}$ is the axial vector associated with the antisymmetric tensor $\boldsymbol{\omega}$ the following is true $\omega_{ij} = -\varphi_q \epsilon_{qij} = \varphi_q \epsilon_{iqj}$, then $\omega_{ij,k} = -\varphi_{q,k} \epsilon_{qij} = \varphi_{q,k} \epsilon_{iqj}$.

We can prove that $\dfrac{\partial \varphi_q}{\partial x_k} = [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]_{qk}$, (see **Problem 5.12**, Eq. (5.100)). Then, we can say that

$$\begin{aligned}\omega_{ij,k}(x_j - x'_j)dx_k &= \varphi_{q,k} \epsilon_{iqj}(x_j - x'_j)dx_k = [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]_{qk} \epsilon_{iqj}(x_j - x'_j)dx_k \\ &= \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}_{ki} dx_k \\ &= \{\{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}^T \cdot d\vec{x}\}_i\end{aligned} \tag{5.90}$$

Taking into account the above equation, the equation in (5.89) becomes

$$\begin{aligned}&\mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij} dx_j - [\omega^0_{ij}(x^0_j - x'_j)] - \int_{P^0}^{P'} \omega_{ij,k}(x_j - x'_j)dx_k \\ &\Rightarrow \mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij} dx_j - [\omega^0_{ij}(x^0_j - x'_j)] - \int_{P^0}^{P'} \{\{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}^T \cdot d\vec{x}\}_i\end{aligned} \tag{5.91}$$

or in tensorial notation:

$$\vec{\mathbf{u}}' = \vec{\mathbf{u}}^0 + \int_{P^0}^{P'} \boldsymbol{\varepsilon} \cdot d\vec{x} - [\boldsymbol{\omega}^0 \cdot (\vec{x}^0 - \vec{x}')] - \int_{P^0}^{P'} \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}^T \cdot d\vec{x}\text{, thus}$$

$$\boxed{\vec{\mathbf{u}}' = \vec{\mathbf{u}}^0 - [\boldsymbol{\omega}^0 \cdot (\vec{x}^0 - \vec{x}')] + \int_{P^0}^{P'} [\boldsymbol{\varepsilon} - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}^T] \cdot d\vec{x}} \tag{5.92}$$

Note that the line integral (from P^0 to P') must be path-independent, hence the line integral vanish to a closed path (conservative system), i.e.:

$$\oint_{\Gamma} [\boldsymbol{\varepsilon} - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}^T] \cdot d\vec{\Gamma} = \vec{\mathbf{0}}$$
$$\Rightarrow \oint_{\Gamma} [\boldsymbol{\varepsilon}^T - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}]^T \cdot d\vec{\Gamma} = \vec{\mathbf{0}} \tag{5.93}$$

And by applying the Stokes theorem, (see equation (5.85)), we can conclude that:

$$\oint_{\Gamma} [\boldsymbol{\varepsilon}^T - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}]^T \cdot d\vec{\Gamma} = \int_{\Omega} \left\{\vec{\nabla}_{\vec{x}} \wedge [\boldsymbol{\varepsilon}^T - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}]\right\}^T \cdot d\vec{S} = \vec{\mathbf{0}} \tag{5.94}$$

In **Problem 1.115** we have shown that

$$\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge \vec{x}\} = \{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge \vec{x} + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]$$

Then, taking into account the above equation and $\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}$, the equation (5.94) becomes:

$$\int_{\Omega} \left\{\vec{\nabla}_{\vec{x}} \wedge [\boldsymbol{\varepsilon}^T - \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}]\right\}^T \cdot d\vec{S} = \vec{\mathbf{0}}$$
$$\Rightarrow \int_{\Omega} \left\{ [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}^T] - [\vec{\nabla}_{\vec{x}} \wedge \{[\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T \wedge (\vec{x} - \vec{x}')\}] \right\}^T \cdot d\vec{S} = \vec{\mathbf{0}}$$
$$\Rightarrow \int_{\Omega} \left\{ [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}] - [\{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge (\vec{x} - \vec{x}') + [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]] \right\}^T \cdot d\vec{S} = \vec{\mathbf{0}} \tag{5.95}$$
$$\Rightarrow \int_{\Omega} \left\{ -\{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge (\vec{x} - \vec{x}') \right\}^T \cdot d\vec{S} = \vec{\mathbf{0}}$$

with that we can conclude that

$$\left\{ -\{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge (\vec{x} - \vec{x}') \right\}^T = \mathbf{0}$$
$$\Rightarrow \{\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T\} \wedge (\vec{x} - \vec{x}') = \mathbf{0}$$

Since the vector $(\vec{x} - \vec{x}')$ must be arbitrary we can conclude that $\vec{\nabla}_{\vec{x}} \wedge [\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}]^T = \mathbf{0}$.

Note that, if we take into account that $\dfrac{\partial \varphi_q}{\partial x_k} = \dfrac{-1}{2}\epsilon_{qst}\left(\dfrac{\partial \varepsilon_{sk}}{\partial x_t} - \dfrac{\partial \varepsilon_{tk}}{\partial x_s}\right)$, (see equation in (5.100)), the equation in (5.90) can be rewritten as follows:

$$\begin{aligned}
\boldsymbol{\omega}_{ij,k}(x_j - x'_j)dx_k &= \varphi_{q,k}\epsilon_{iqj}(x_j - x'_j)dx_k = \frac{-1}{2}\epsilon_{qst}\left(\frac{\partial \varepsilon_{sk}}{\partial x_t} - \frac{\partial \varepsilon_{tk}}{\partial x_s}\right)\epsilon_{iqj}(x_j - x'_j)dx_k \\
&= \frac{1}{2}\epsilon_{qst}\epsilon_{qij}(\varepsilon_{sk,t} - \varepsilon_{tk,s})(x_j - x'_j)dx_k \\
&= \frac{1}{2}(\delta_{si}\delta_{tj} - \delta_{sj}\delta_{ti})(\varepsilon_{sk,t} - \varepsilon_{tk,s})(x_j - x'_j)dx_k \\
&= \frac{1}{2}(\delta_{si}\delta_{tj}\varepsilon_{sk,t} - \delta_{si}\delta_{tj}\varepsilon_{tk,s} - \delta_{sj}\delta_{ti}\varepsilon_{sk,t} + \delta_{sj}\delta_{ti}\varepsilon_{tk,s})(x_j - x'_j)dx_k \\
&= \frac{1}{2}(\varepsilon_{ik,j} - \varepsilon_{jk,i} - \varepsilon_{jk,i} + \varepsilon_{ik,j})(x_j - x'_j)dx_k \\
&= (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j)dx_k
\end{aligned} \tag{5.96}$$

Then, the equation in (5.89) can also be written as follows:

$$
\begin{aligned}
&\mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) + \int_{P^0}^{P'} \varepsilon_{ij} dx_j - [\omega^0_{ij}(x^0_j - x'_j)] - \int_{P^0}^{P'} \omega_{ij,k}(x_j - x'_j) dx_k \\
&\Rightarrow \mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) - [\omega^0_{ij}(x^0_j - x'_j)] + \int_{P^0}^{P'} \varepsilon_{ik} dx_k - \int_{P^0}^{P'} (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j) dx_k \\
&\Rightarrow \mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) - [\omega^0_{ij}(x^0_j - x'_j)] + \int_{P^0}^{P'} [\varepsilon_{ik} - (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j)] dx_k \\
&\Rightarrow \mathsf{u}'_i(\vec{x}') = \mathsf{u}^0_i(\vec{x}^0) - [\omega^0_{ij}(x^0_j - x'_j)] + \int_{P^0}^{P'} T_{ik} dx_k
\end{aligned}
\tag{5.97}
$$

where $T_{ik} = [\varepsilon_{ik} - (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j)]$. Since the displacement must be independent of the path of integration, the integrands $T_{ik} dx_k$ must be exact differentials, (see equation (5.87)). Hence, applying a necessary and sufficient condition that the integrands be exact differentials we can obtain $T_{jt,k} - T_{jk,t} = 0_{ktj}$, in which:

$T_{jt} = [\varepsilon_{jt} - (\varepsilon_{jt,p} - \varepsilon_{pt,j})(x_p - x'_p)]$ and $T_{jk} = [\varepsilon_{jk} - (\varepsilon_{jk,p} - \varepsilon_{pk,j})(x_p - x'_p)]$,

thus

$$
\begin{aligned}
&T_{jt,k} - T_{jk,t} = 0_{ktj} \\
&\Rightarrow [\varepsilon_{jt} - (\varepsilon_{jt,p} - \varepsilon_{pt,j})(x_p - x'_p)]_{,k} - [\varepsilon_{jk} - (\varepsilon_{jk,p} - \varepsilon_{pk,j})(x_p - x'_p)]_{,t} = 0_{ktj} \\
&\Rightarrow \varepsilon_{jt,k} - (\varepsilon_{jt,p} - \varepsilon_{pt,j})_{,k}(x_p - x'_p) - (\varepsilon_{jt,p} - \varepsilon_{pt,j})(x_p - x'_p)_{,k} \\
&\qquad\qquad - \varepsilon_{jk,t} + (\varepsilon_{jk,p} - \varepsilon_{pk,j})_{,t}(x_p - x'_p) + (\varepsilon_{jk,p} - \varepsilon_{pk,j})(x_p - x'_p)_{,t} = 0_{ktj} \\
&\Rightarrow \varepsilon_{jt,k} - (\varepsilon_{jt,pk} - \varepsilon_{pt,jk})(x_p - x'_p) - (\varepsilon_{jt,p} - \varepsilon_{pt,j})\delta_{pk} \\
&\qquad\qquad - \varepsilon_{jk,t} + (\varepsilon_{jk,pt} - \varepsilon_{pk,jt})(x_p - x'_p) + (\varepsilon_{jk,p} - \varepsilon_{pk,j})\delta_{pt} = 0_{ktj} \\
&\Rightarrow \varepsilon_{jt,k} - (\varepsilon_{jt,p} - \varepsilon_{pt,j})\delta_{pk} - \varepsilon_{jk,t} + (\varepsilon_{jk,p} - \varepsilon_{pk,j})\delta_{pt} \\
&\qquad\qquad + (\varepsilon_{jk,pt} - \varepsilon_{pk,jt} - \varepsilon_{jt,pk} + \varepsilon_{pt,jk})(x_p - x'_p) = 0_{ktj} \\
&\Rightarrow \varepsilon_{jt,k} - \varepsilon_{jt,k} + \varepsilon_{kt,j} - \varepsilon_{jk,t} + \varepsilon_{jk,t} - \varepsilon_{tk,j} \\
&\qquad\qquad + (\varepsilon_{jk,pt} - \varepsilon_{pk,jt} - \varepsilon_{jt,pk} + \varepsilon_{pt,jk})(x_p - x'_p) = 0_{ktj}
\end{aligned}
$$

with that we can obtain:

$$
(\varepsilon_{jk,pt} - \varepsilon_{pk,jt} - \varepsilon_{jt,pk} + \varepsilon_{pt,jk})(x_p - x'_p) = 0_{ktj} \tag{5.98}
$$

Since the vector $(x_p - x'_p)$ is arbitrary we can conclude that:

$$
\varepsilon_{jk,pt} - \varepsilon_{pk,jt} - \varepsilon_{jt,pk} + \varepsilon_{pt,jk} = 0_{jkpt} \tag{5.99}
$$

which matches the equation in (5.78).

Problem 5.12

Given the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, and the displacement field $\vec{\mathbf{u}}$, (a) show that:

$$
\boxed{(\boldsymbol{J})_{ij} \equiv (\nabla_{\vec{x}} \vec{\mathbf{u}})_{ij} = \frac{\partial \mathsf{u}_i}{\partial x_j} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} - \varphi_3 & \varepsilon_{13} + \varphi_2 \\ \varepsilon_{12} + \varphi_3 & \varepsilon_{22} & \varepsilon_{23} - \varphi_1 \\ \varepsilon_{13} - \varphi_2 & \varepsilon_{23} + \varphi_1 & \varepsilon_{33} \end{bmatrix}}
$$

where φ_i are the rotation vector components.

b) Show also that:

$$\boxed{\begin{aligned}\frac{\partial\varphi_k}{\partial x_p} &= \frac{-1}{2}\epsilon_{kij}\omega_{ij,p} = \frac{-1}{2}\epsilon_{kij}\left(\frac{\partial\varepsilon_{ip}}{\partial x_j}-\frac{\partial\varepsilon_{jp}}{\partial x_i}\right) = \frac{-1}{2}\left(-[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon}]_{kp}-[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon}]_{kp}\right)=[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon}]_{kp}\\ &= -\begin{bmatrix}\frac{\partial\omega_{23}}{\partial x_1} & \frac{\partial\omega_{23}}{\partial x_2} & \frac{\partial\omega_{23}}{\partial x_3}\\ \frac{\partial\omega_{31}}{\partial x_1} & \frac{\partial\omega_{31}}{\partial x_2} & \frac{\partial\omega_{31}}{\partial x_3}\\ \frac{\partial\omega_{12}}{\partial x_1} & \frac{\partial\omega_{12}}{\partial x_2} & \frac{\partial\omega_{12}}{\partial x_3}\end{bmatrix} = \begin{bmatrix}\left(\frac{\partial\varepsilon_{13}}{\partial x_2}-\frac{\partial\varepsilon_{12}}{\partial x_3}\right) & \left(\frac{\partial\varepsilon_{23}}{\partial x_2}-\frac{\partial\varepsilon_{22}}{\partial x_3}\right) & \left(\frac{\partial\varepsilon_{33}}{\partial x_2}-\frac{\partial\varepsilon_{23}}{\partial x_3}\right)\\ \left(\frac{\partial\varepsilon_{11}}{\partial x_3}-\frac{\partial\varepsilon_{13}}{\partial x_1}\right) & \left(\frac{\partial\varepsilon_{12}}{\partial x_3}-\frac{\partial\varepsilon_{23}}{\partial x_1}\right) & \left(\frac{\partial\varepsilon_{13}}{\partial x_3}-\frac{\partial\varepsilon_{33}}{\partial x_1}\right)\\ \left(\frac{\partial\varepsilon_{12}}{\partial x_1}-\frac{\partial\varepsilon_{11}}{\partial x_2}\right) & \left(\frac{\partial\varepsilon_{22}}{\partial x_1}-\frac{\partial\varepsilon_{12}}{\partial x_2}\right) & \left(\frac{\partial\varepsilon_{23}}{\partial x_1}-\frac{\partial\varepsilon_{13}}{\partial x_2}\right)\end{bmatrix}\end{aligned}}$$

(5.100)

where $(\nabla_{\vec{x}}\vec{\varphi})_{kp} = \frac{\partial\varphi_k}{\partial x_p}$, and $\boldsymbol{\omega}$ is the infinitesimal spin tensor. And the relationship $\nabla_{\vec{x}}\vec{\varphi} = \vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon}$ holds, (see **Problem 1.110**).

Solution:

a) The displacement gradient $\boldsymbol{J}\equiv\nabla_{\vec{x}}\vec{\mathbf{u}}$ can be split additively into a symmetric and an antisymmetric part:

$$\boldsymbol{J}\equiv\nabla_{\vec{x}}\vec{\mathbf{u}} = \underbrace{\frac{1}{2}\left[(\nabla_{\vec{x}}\vec{\mathbf{u}})+(\nabla_{\vec{x}}\vec{\mathbf{u}})^T\right]}_{=(\nabla_{\vec{x}}\vec{\mathbf{u}})^{sym}}+\underbrace{\frac{1}{2}\left[(\nabla_{\vec{x}}\vec{\mathbf{u}})-(\nabla_{\vec{x}}\vec{\mathbf{u}})^T\right]}_{=(\nabla_{\vec{x}}\vec{\mathbf{u}})^{skew}} = \underbrace{(\nabla_{\vec{x}}\vec{\mathbf{u}})^{sym}}_{=\boldsymbol{\varepsilon}}+\underbrace{(\nabla_{\vec{x}}\vec{\mathbf{u}})^{skew}}_{=\boldsymbol{\omega}} = \boldsymbol{\varepsilon}+\boldsymbol{\omega}$$

where the symmetric part $\boldsymbol{\varepsilon}=(\nabla_{\vec{x}}\vec{\mathbf{u}})^{sym}$ represents the infinitesimal strain tensor, and the antisymmetric part $\boldsymbol{\omega}=(\nabla_{\vec{x}}\vec{\mathbf{u}})^{skew}$ represents the infinitesimal spin tensor (rotation tensor). If we consider that $\vec{\varphi}$ is the axial vector associated with the antisymmetric tensor $\boldsymbol{\omega}$ we can conclude that:

$$\omega_{ij} = \begin{bmatrix}0 & \omega_{12} & \omega_{13}\\ \omega_{21} & 0 & \omega_{23}\\ \omega_{31} & \omega_{32} & 0\end{bmatrix} = \begin{bmatrix}0 & \omega_{12} & \omega_{13}\\ -\omega_{12} & 0 & \omega_{23}\\ -\omega_{13} & -\omega_{23} & 0\end{bmatrix} = \begin{bmatrix}0 & -\varphi_3 & \varphi_2\\ \varphi_3 & 0 & -\varphi_1\\ -\varphi_2 & \varphi_1 & 0\end{bmatrix}$$

with that

$$(\nabla_{\vec{x}}\vec{\mathbf{u}})_{ij} = \frac{\partial\mathrm{u}_i}{\partial x_j} = \begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix} + \begin{bmatrix}0 & -\varphi_3 & \varphi_2\\ \varphi_3 & 0 & -\varphi_1\\ -\varphi_2 & \varphi_1 & 0\end{bmatrix} = \begin{bmatrix}\varepsilon_{11} & \varepsilon_{12}-\varphi_3 & \varepsilon_{13}+\varphi_2\\ \varepsilon_{12}+\varphi_3 & \varepsilon_{22} & \varepsilon_{23}-\varphi_1\\ \varepsilon_{13}-\varphi_2 & \varepsilon_{23}+\varphi_1 & \varepsilon_{33}\end{bmatrix}$$

b) Recall from chapter on Tensors that an antisymmetric tensor ($\boldsymbol{\omega}$) and its axial vector ($\vec{\varphi}$) are related to each other, in indicial notation, by means of $\omega_{ij}=-\varphi_k\epsilon_{kij}$ or $\varphi_k = -\frac{1}{2}\epsilon_{kij}\omega_{ij}$. And the gradient of $\vec{\varphi}$ can be obtained as follows:

$$\frac{\partial\varphi}{\partial x_p}\equiv\varphi_{k,p} = -\left(\frac{1}{2}\epsilon_{kij}\omega_{ij}\right)_{,p} = -\frac{1}{2}\epsilon_{kij}\omega_{ij,p}$$

By expanding the dummy indices i,j, we can obtain the following terms:

$$\varphi_{k,p} = -\frac{1}{2}\epsilon_{kij}\omega_{ij,p} = -\frac{1}{2}(\epsilon_{k12}\omega_{12,p}+\epsilon_{k13}\omega_{13,p}+\epsilon_{k21}\omega_{21,p}+\epsilon_{k23}\omega_{23,p}+\epsilon_{k31}\omega_{31,p}+\epsilon_{k32}\omega_{32,p})$$

Note that the rows of $\varphi_{k,p}$ $(k=1,2,3)$ are given by:

$$\varphi_{k,p}=\frac{-1}{2}\epsilon_{kij}\omega_{ij,p}$$

$$\Rightarrow\begin{cases}(k=1)\Rightarrow\varphi_{1,p}=\dfrac{-1}{2}(\epsilon_{123}\omega_{23,p}+\epsilon_{132}\omega_{32,p})=\dfrac{-1}{2}(\omega_{23,p}-\omega_{32,p})=-\omega_{23,p}\\(k=2)\Rightarrow\varphi_{2,p}=\dfrac{-1}{2}(\epsilon_{213}\omega_{13,p}+\epsilon_{231}\omega_{31,p})=\dfrac{-1}{2}(-\omega_{13,p}+\omega_{31,p})=-\omega_{31,p}\\(k=3)\Rightarrow\varphi_{3,p}=\dfrac{-1}{2}(\epsilon_{312}\omega_{12,p}+\epsilon_{321}\omega_{21,p})=\dfrac{-1}{2}(\omega_{12,p}-\omega_{21,p})=-\omega_{12,p}\end{cases}$$

where we have used the antisymmetric tensor definition $\omega_{ij}=-\omega_{ji}$. Taking into account the above equation we can conclude that:

$$\varphi_{k,p}=\frac{-1}{2}\epsilon_{kij}\omega_{ij,p}=-\begin{bmatrix}\omega_{23,1}&\omega_{23,2}&\omega_{23,3}\\\omega_{31,1}&\omega_{31,2}&\omega_{31,3}\\\omega_{12,1}&\omega_{12,2}&\omega_{12,3}\end{bmatrix}=-\begin{bmatrix}\dfrac{\partial\omega_{23}}{\partial x_1}&\dfrac{\partial\omega_{23}}{\partial x_2}&\dfrac{\partial\omega_{23}}{\partial x_3}\\\dfrac{\partial\omega_{31}}{\partial x_1}&\dfrac{\partial\omega_{31}}{\partial x_2}&\dfrac{\partial\omega_{31}}{\partial x_3}\\\dfrac{\partial\omega_{12}}{\partial x_1}&\dfrac{\partial\omega_{12}}{\partial x_2}&\dfrac{\partial\omega_{12}}{\partial x_3}\end{bmatrix}\tag{5.101}$$

To obtain the derivative of ω_{ij} with respect to x_p we will start from the definition $\omega_{ij}=\left[(\nabla_{\vec{x}}\vec{\mathbf{u}})^{skew}\right]_{ij}=\dfrac{1}{2}\left(\dfrac{\partial\mathrm{u}_i}{\partial x_j}-\dfrac{\partial\mathrm{u}_j}{\partial x_i}\right)=\dfrac{1}{2}(\mathrm{u}_{i,j}-\mathrm{u}_{j,i})$, thus:

$$\frac{\partial\omega_{ij}}{\partial x_p}\equiv\omega_{ij,p}=\frac{1}{2}(\mathrm{u}_{i,j}-\mathrm{u}_{j,i})_{,p}=\frac{1}{2}(\mathrm{u}_{i,jp}-\mathrm{u}_{j,ip})$$

The value of the above equation is not altered if we add and subtract the term $\dfrac{1}{2}\mathrm{u}_{p,ij}$, thus:

$$\begin{aligned}\omega_{ij,p}&=\frac{1}{2}(\mathrm{u}_{i,jp}-\mathrm{u}_{j,ip}+\mathrm{u}_{p,ij}-\mathrm{u}_{p,ij})=\frac{1}{2}(\mathrm{u}_{i,jp}+\mathrm{u}_{p,ij})-\frac{1}{2}(\mathrm{u}_{j,ip}+\mathrm{u}_{p,ij})\\&=\frac{1}{2}(\mathrm{u}_{i,pj}+\mathrm{u}_{p,ij})-\frac{1}{2}(\mathrm{u}_{j,pi}+\mathrm{u}_{p,ji})=\left(\frac{1}{2}(\mathrm{u}_{i,p}+\mathrm{u}_{p,i})\right)_{,j}-\left(\frac{1}{2}(\mathrm{u}_{j,p}+\mathrm{u}_{p,j})\right)_{,i}\\&=\varepsilon_{ip,j}-\varepsilon_{jp,i}=\frac{\partial\varepsilon_{ip}}{\partial x_j}-\frac{\partial\varepsilon_{jp}}{\partial x_i}\end{aligned}$$

Substituting the above equation into the equation (5.101) and by expanding the dummy indices i,j we can obtain:

$$\begin{aligned}\varphi_{k,p}&=\frac{-1}{2}\epsilon_{kij}(\varepsilon_{ip,j}-\varepsilon_{jp,i})=\frac{-1}{2}(\epsilon_{kij}\varepsilon_{ip,j}-\epsilon_{kij}\varepsilon_{jp,i})\\&=\frac{-1}{2}(\epsilon_{k12}\varepsilon_{1p,1}+\epsilon_{k13}\varepsilon_{1p,3}+\epsilon_{k21}\varepsilon_{2p,1}+\epsilon_{k23}\varepsilon_{2p,3}+\epsilon_{k31}\varepsilon_{3p,1}+\epsilon_{k32}\varepsilon_{3p,2}\\&\qquad-\epsilon_{k12}\varepsilon_{2p,1}-\epsilon_{k13}\varepsilon_{3p,1}-\epsilon_{k21}\varepsilon_{1p,2}-\epsilon_{k23}\varepsilon_{3p,2}-\epsilon_{k31}\varepsilon_{1p,3}-\epsilon_{k32}\varepsilon_{2p,3})\end{aligned}$$

Note that the rows of $\varphi_{k,p}$ $(k=1,2,3)$ can be obtained as follows:

$$\varphi_{k,p}=\begin{cases}(k=1)\Rightarrow \varphi_{1,p}=\dfrac{-1}{2}(\epsilon_{123}\varepsilon_{2p,3}+\epsilon_{132}\varepsilon_{3p,2}-\epsilon_{123}\varepsilon_{3p,2}-\epsilon_{132}\varepsilon_{2p,3})=\varepsilon_{3p,2}-\varepsilon_{2p,3}\\ (k=2)\Rightarrow \varphi_{2,p}=\dfrac{-1}{2}(\epsilon_{k13}\varepsilon_{1p,3}+\epsilon_{k31}\varepsilon_{3p,1}+\epsilon_{k13}\varepsilon_{3p,1}-\epsilon_{k31}\varepsilon_{1p,3})=\varepsilon_{1p,3}-\varepsilon_{3p,1}\\ (k=3)\Rightarrow \varphi_{3,p}=\dfrac{-1}{2}(\epsilon_{312}\varepsilon_{1p,1}+\epsilon_{321}\varepsilon_{2p,1}-\epsilon_{312}\varepsilon_{2p,1}-\epsilon_{321}\varepsilon_{1p,2})=\varepsilon_{2p,1}-\varepsilon_{1p,2}\end{cases}$$

Then:

$$\varphi_{k,p}=\begin{bmatrix}(\varepsilon_{31,2}-\varepsilon_{21,3}) & (\varepsilon_{32,2}-\varepsilon_{22,3}) & (\varepsilon_{33,2}-\varepsilon_{23,3})\\ (\varepsilon_{11,3}-\varepsilon_{31,1}) & (\varepsilon_{12,3}-\varepsilon_{32,1}) & (\varepsilon_{13,3}-\varepsilon_{33,1})\\ (\varepsilon_{21,1}-\varepsilon_{11,2}) & (\varepsilon_{22,1}-\varepsilon_{12,2}) & (\varepsilon_{23,1}-\varepsilon_{13,2})\end{bmatrix}=[\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon}]_{kp} \tag{5.102}$$

Then, taking into account the equation (5.101) and (5.102) we can conclude that:

$$\varphi_{k,p}=\frac{-1}{2}\epsilon_{kij}\omega_{ij,p}=\frac{-1}{2}\epsilon_{kij}\left(\frac{\partial\varepsilon_{ip}}{\partial x_j}-\frac{\partial\varepsilon_{jp}}{\partial x_i}\right)$$

$$=-\begin{bmatrix}\dfrac{\partial\omega_{23}}{\partial x_1} & \dfrac{\partial\omega_{23}}{\partial x_2} & \dfrac{\partial\omega_{23}}{\partial x_3}\\ \dfrac{\partial\omega_{31}}{\partial x_1} & \dfrac{\partial\omega_{31}}{\partial x_2} & \dfrac{\partial\omega_{31}}{\partial x_3}\\ \dfrac{\partial\omega_{12}}{\partial x_1} & \dfrac{\partial\omega_{12}}{\partial x_2} & \dfrac{\partial\omega_{12}}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\left(\dfrac{\partial\varepsilon_{13}}{\partial x_2}-\dfrac{\partial\varepsilon_{12}}{\partial x_3}\right) & \left(\dfrac{\partial\varepsilon_{23}}{\partial x_2}-\dfrac{\partial\varepsilon_{22}}{\partial x_3}\right) & \left(\dfrac{\partial\varepsilon_{33}}{\partial x_2}-\dfrac{\partial\varepsilon_{23}}{\partial x_3}\right)\\ \left(\dfrac{\partial\varepsilon_{11}}{\partial x_3}-\dfrac{\partial\varepsilon_{13}}{\partial x_1}\right) & \left(\dfrac{\partial\varepsilon_{12}}{\partial x_3}-\dfrac{\partial\varepsilon_{23}}{\partial x_1}\right) & \left(\dfrac{\partial\varepsilon_{13}}{\partial x_3}-\dfrac{\partial\varepsilon_{33}}{\partial x_1}\right)\\ \left(\dfrac{\partial\varepsilon_{12}}{\partial x_1}-\dfrac{\partial\varepsilon_{11}}{\partial x_2}\right) & \left(\dfrac{\partial\varepsilon_{22}}{\partial x_1}-\dfrac{\partial\varepsilon_{12}}{\partial x_2}\right) & \left(\dfrac{\partial\varepsilon_{23}}{\partial x_1}-\dfrac{\partial\varepsilon_{13}}{\partial x_2}\right)\end{bmatrix}$$

where we have used the symmetry of $\boldsymbol{\varepsilon}$, i.e. $\varepsilon_{ij}=\varepsilon_{ji}$.

Example: Let us suppose that we know the infinitesimal strain tensor which is given by

$$\varepsilon_{ij}=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}=\begin{bmatrix}8x_1 & \dfrac{-x_2}{2} & \dfrac{3}{2}x_1^2x_3\\ \dfrac{-x_2}{2} & x_1 & 0\\ \dfrac{3}{2}x_1^2x_3 & 0 & x_1^3\end{bmatrix} \tag{5.103}$$

with the following boundary conditions:

$$\mathrm{u}_i(\vec{x}=\vec{\mathbf{0}},t)=\begin{bmatrix}3t\\0\\0\end{bmatrix}\qquad and\qquad \boldsymbol{\omega}(\vec{x}=\vec{\mathbf{0}},t)=\mathbf{0}\qquad\Rightarrow\qquad \vec{\varphi}(\vec{x}=\vec{\mathbf{0}},t)=\vec{\mathbf{0}}$$

To this example, we can obtain:

$$\varphi_{k,p}=\begin{bmatrix}\left(\dfrac{\partial\varepsilon_{13}}{\partial x_2}-\dfrac{\partial\varepsilon_{12}}{\partial x_3}\right) & \left(\dfrac{\partial\varepsilon_{23}}{\partial x_2}-\dfrac{\partial\varepsilon_{22}}{\partial x_3}\right) & \left(\dfrac{\partial\varepsilon_{33}}{\partial x_2}-\dfrac{\partial\varepsilon_{23}}{\partial x_3}\right)\\ \left(\dfrac{\partial\varepsilon_{11}}{\partial x_3}-\dfrac{\partial\varepsilon_{13}}{\partial x_1}\right) & \left(\dfrac{\partial\varepsilon_{12}}{\partial x_3}-\dfrac{\partial\varepsilon_{23}}{\partial x_1}\right) & \left(\dfrac{\partial\varepsilon_{13}}{\partial x_3}-\dfrac{\partial\varepsilon_{33}}{\partial x_1}\right)\\ \left(\dfrac{\partial\varepsilon_{12}}{\partial x_1}-\dfrac{\partial\varepsilon_{11}}{\partial x_2}\right) & \left(\dfrac{\partial\varepsilon_{22}}{\partial x_1}-\dfrac{\partial\varepsilon_{12}}{\partial x_2}\right) & \left(\dfrac{\partial\varepsilon_{23}}{\partial x_1}-\dfrac{\partial\varepsilon_{13}}{\partial x_2}\right)\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ -x_1x_3 & 0 & \dfrac{-3}{2}x_1^2\\ 0 & \dfrac{-3}{2} & 0\end{bmatrix}$$

Note also that the following holds:

$$\varphi_{k,p} = \begin{bmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_1}{\partial x_3} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_3} \\ \frac{\partial\varphi_3}{\partial x_1} & \frac{\partial\varphi_3}{\partial x_2} & \frac{\partial\varphi_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_1 x_3 & 0 & \frac{-3}{2}x_1^2 \\ 0 & \frac{3}{2} & 0 \end{bmatrix}$$

with that and by means of the integration we can obtain the components φ_i:

$$\left.\begin{aligned} \frac{\partial\varphi_1}{\partial x_1} &= 0 \\ \frac{\partial\varphi_1}{\partial x_2} &= 0 \\ \frac{\partial\varphi_1}{\partial x_3} &= 0 \end{aligned}\right\} \Rightarrow \varphi_1 = C_1(t) \quad ; \quad \left.\begin{aligned} \frac{\partial\varphi_2}{\partial x_1} &= -x_1 x_3 \\ \frac{\partial\varphi_2}{\partial x_2} &= 0 \\ \frac{\partial\varphi_2}{\partial x_3} &= \frac{-3}{2}x_1^2 \end{aligned}\right\} \Rightarrow \varphi_2 = \frac{-3}{2}x_1^2 x_3 + C_2(t)$$

$$\left.\begin{aligned} \frac{\partial\varphi_3}{\partial x_1} &= 0 \\ \frac{\partial\varphi_3}{\partial x_2} &= \frac{3}{2} \\ \frac{\partial\varphi_3}{\partial x_3} &= 0 \end{aligned}\right\} \Rightarrow \varphi_3 = \frac{3}{2}x_2 + C_3(t)$$

By applying the boundary condition $\vec{\varphi}(\vec{x} = \vec{\mathbf{0}}, t) = \vec{\mathbf{0}}$, we can conclude that $C_i(t) = 0$. Then:

$$\varphi_i = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-3}{2}x_1^2 x_3 \\ \frac{3}{2}x_2 \end{bmatrix}$$

And the infinitesimal spin tensor becomes:

$$\omega_{ij} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-3}{2}x_2 & \frac{-3}{2}x_1^2 x_3 \\ \frac{3}{2}x_2 & 0 & 0 \\ \frac{3}{2}x_1^2 x_3 & 0 & 0 \end{bmatrix}$$

The displacement field can be obtained by means of

$$\frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} + \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8x_1 & \frac{-x_2}{2} & \frac{3}{2}x_1^2 x_3 \\ \frac{-x_2}{2} & x_1 & 0 \\ \frac{3}{2}x_1^2 x_3 & 0 & x_1^3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-3}{2}x_2 & \frac{-3}{2}x_1^2 x_3 \\ \frac{3}{2}x_2 & 0 & 0 \\ \frac{3}{2}x_1^2 x_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 8x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ 3x_1^2 x_3 & 0 & x_1^3 \end{bmatrix}$$

or

$$\frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 8x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ 3x_1^2 x_3 & 0 & x_1^3 \end{bmatrix}$$

with that we can obtain

$$\left.\begin{aligned} \frac{\partial \mathrm{u}_1}{\partial x_1} &= 8x_1 \\ \frac{\partial \mathrm{u}_1}{\partial x_2} &= -2x_2 \\ \frac{\partial \mathrm{u}_1}{\partial x_3} &= 0 \end{aligned}\right\} \Rightarrow \mathrm{u}_1 = 4x_1^2 - x_2^2 + K_1(t) \quad ; \quad \left.\begin{aligned} \frac{\partial \mathrm{u}_2}{\partial x_1} &= x_2 \\ \frac{\partial \mathrm{u}_2}{\partial x_2} &= x_1 \\ \frac{\partial \mathrm{u}_2}{\partial x_3} &= 0 \end{aligned}\right\} \Rightarrow \mathrm{u}_2 = x_1 x_2 + K_2(t)$$

$$\left.\begin{aligned} \frac{\partial \mathrm{u}_3}{\partial x_1} &= 3x_1^2 x_3 \\ \frac{\partial \mathrm{u}_3}{\partial x_2} &= 0 \\ \frac{\partial \mathrm{u}_1}{\partial x_3} &= x_1^3 \end{aligned}\right\} \Rightarrow \mathrm{u}_3 = x_1^3 x_3 + K_3(t)$$

The constants of integration can be obtained by means of the boundary condition:

$$\mathrm{u}_i(\vec{\boldsymbol{x}}, t) = \begin{bmatrix} 4x_1^2 - x_2^2 + K_1(t) \\ x_1 x_2 + K_2(t) \\ x_1^3 x_3 + K_3(t) \end{bmatrix} \xrightarrow{\vec{x}=\vec{\mathbf{0}}} \mathrm{u}_i(\vec{\boldsymbol{x}} = \vec{\mathbf{0}}, t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix} = \begin{bmatrix} 3t \\ 0 \\ 0 \end{bmatrix}$$

Then, the displacement field becomes:

$$\mathrm{u}_i(\vec{\boldsymbol{x}}, t) = \begin{bmatrix} 4x_1^2 - x_2^2 + 3t \\ x_1 x_2 \\ x_1^3 x_3 \end{bmatrix}$$

It is interesting to verify that the displacement field is compatible, since the infinitesimal strain tensor field, (see equation (5.103)), fulfills the compatible equations (see equations in (5.80)). We leave the reader to verify this fact.

Problem 5.13

Consider a cantilever beam in which the infinitesimal strain tensor is given by

$$\varepsilon_{ik}(x_1, x_3) = \begin{bmatrix} \kappa_{x_2} x_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \boldsymbol{\varepsilon} = \kappa_{x2} x_3 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \tag{5.104}$$

where $\kappa_{x_2} = \kappa_{x_2}(x_1)$ is the curvature of the beam which is constant on cross section, (see Figure 5.13). a) Check whether the compatibility equations are fulfilled or not. b) Obtain the displacement field.

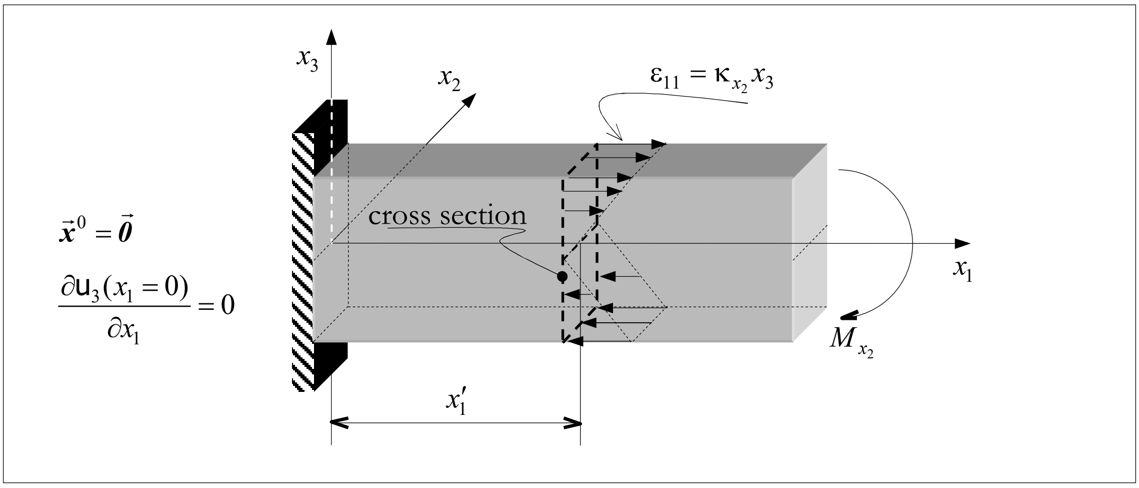


Figure 5.13: Cantilever beam.

Solution:

a) We can use the compatibility equations in (5.80) to verify that:

$$
\begin{cases}
\dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \quad \checkmark \\
\dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = \dfrac{\partial^2 (\kappa_{x_2} x_3)}{\partial x_3^2} = -\dfrac{\partial(\kappa_{x_2})}{\partial x_3} = 0 \quad \checkmark \\
\dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \dfrac{\partial^2 (\kappa_{x_2} x_3)}{\partial x_2^2} = 0 \quad \checkmark \\
\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 \quad \checkmark \\
\dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = -\dfrac{\partial^2 (\kappa_{x_2} x_3)}{\partial x_2 \partial x_3} = -\dfrac{\partial(\kappa_{x_2})}{\partial x_2} = 0 \quad \checkmark \\
\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = 0 \quad \checkmark
\end{cases}
$$

Note that, the compatibility equations are satisfied if κ_{x_2} is neither a function of x_2 nor x_3, i.e. κ_{x_2} must be constant on the cross-section.

We can also verify the compatibility equations by means of $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \mathbf{0}$, i.e.:

$$
(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon}) = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \wedge (\varepsilon_{11}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) = \left(\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3\right) \wedge [\kappa_{x_2} x_3 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1]
$$

$$
= \frac{\partial(\kappa_{x_2} x_3)}{\partial x_1}\underbrace{\hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_1}_{=\vec{\mathbf{0}}} \otimes \hat{\mathbf{e}}_1 + \underbrace{\frac{\partial(\kappa_{x_2} x_3)}{\partial x_2}}_{=0}\underbrace{\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_1}_{=-\hat{\mathbf{e}}_3} \otimes \hat{\mathbf{e}}_1 + \frac{\partial(\kappa_{x_2} x_3)}{\partial x_3}\underbrace{\hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_1}_{=\hat{\mathbf{e}}_2} \otimes \hat{\mathbf{e}}_1
$$

$$
= \kappa_{x2}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1
$$

$$
(\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = [\kappa_{x_2}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1]^T = \kappa_{x_2}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2
$$

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T = \left(\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3 \right) \wedge [\kappa_{x_2}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2]$$

$$= \frac{\partial(\kappa_{x_2})}{\partial x_1}\underbrace{\hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_1}_{=\vec{\mathbf{0}}} \otimes \hat{\mathbf{e}}_2 + \underbrace{\frac{\partial(\kappa_{x_2})}{\partial x_2}}_{=0}\underbrace{\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_1}_{=-\hat{\mathbf{e}}_3} \otimes \hat{\mathbf{e}}_2 + \underbrace{\frac{\partial(\kappa_{x_2})}{\partial x_3}}_{=0}\underbrace{\hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_1}_{=\hat{\mathbf{e}}_2} \otimes \hat{\mathbf{e}}_2$$

$$= \mathbf{0} \quad \checkmark$$

b) The displacement field can be obtained by means of the equation in (5.97), i.e.:

$$\mathrm{u}'_i(\vec{x}') = \mathrm{u}^0_i(\vec{x}^0) - [\omega^0_{ij}(x^0_j - x'_j)] + \int_{P^0}^{P'} [\varepsilon_{ik} - (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j)]dx_k \tag{5.105}$$

and by applying the boundary conditions, (see Figure 5.13), we can obtain:

$$\begin{aligned} \mathrm{u}'_i(\vec{x}') &= \int_{\vec{0}}^{\vec{x}'} T_{ik}\,dx_k = \int_{\vec{0}}^{\vec{x}'} [\varepsilon_{ik} - (\varepsilon_{ik,j} - \varepsilon_{jk,i})(x_j - x'_j)]dx_k \\ &= \int_{\vec{0}}^{\vec{x}'} [\varepsilon_{ik} - \varepsilon_{ik,j}(x_j - x'_j) + \varepsilon_{jk,i}(x_j - x'_j)]dx_k = \int_{\vec{0}}^{\vec{x}'} [\overline{T}^{(1)}_{ik} - \overline{T}^{(2)}_{ik} + \overline{T}^{(3)}_{ik}]dx_k \end{aligned} \tag{5.106}$$

where $\overline{T}^{(1)}_{ik} = \varepsilon_{ik}$,

$$\overline{T}^{(2)}_{ik} = \varepsilon_{ik,j}(x_j - x'_j) = \varepsilon_{ik,1}(x_1 - x'_1) + \varepsilon_{ik,2}(x_2 - x'_2) + \varepsilon_{ik,3}(x_3 - x'_3) = \varepsilon_{ik,1}(x_1 - x'_1) + \varepsilon_{ik,3}(x_3 - x'_3)$$

$$= \begin{bmatrix} \kappa_{x_2,1}x_3(x_1 - x'_1) + \kappa_{x_2}(x_3 - x'_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\kappa_{x_2,1} \equiv \dfrac{\partial \kappa_{x_2}}{\partial x_1}$ and

$$\overline{T}^{(3)}_{ik} = \varepsilon_{jk,i}(x_j - x'_j) = \varepsilon_{1k,i}(x_1 - x'_1) + \varepsilon_{2k,i}(x_2 - x'_2) + \varepsilon_{3k,i}(x_3 - x'_3) = \varepsilon_{1k,i}(x_1 - x'_1)$$

$$= \begin{bmatrix} \kappa_{x_2,1}x_3(x_1 - x'_1) & 0 & 0 \\ 0 & 0 & 0 \\ \kappa_{x_2}(x_1 - x'_1) & 0 & 0 \end{bmatrix}$$

Then,

$$T_{ik} = \begin{bmatrix} \kappa_{x_2}x_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \kappa_{x_2,1}x_3(x_1 - x'_1) + \kappa_{x_2}(x_3 - x'_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \kappa_{x_2,1}x_3(x_1 - x'_1) & 0 & 0 \\ 0 & 0 & 0 \\ \kappa_{x_2}(x_1 - x'_1) & 0 & 0 \end{bmatrix}$$

$$T_{ik} = \begin{bmatrix} \kappa_{x_2}x'_3 & 0 & 0 \\ 0 & 0 & 0 \\ \kappa_{x_2}(x_1 - x'_1) & 0 & 0 \end{bmatrix}$$

and $T_{ik}dx_k = \begin{bmatrix} \kappa_{x_2}x'_3 & 0 & 0 \\ 0 & 0 & 0 \\ \kappa_{x_2}(x_1 - x'_1) & 0 & 0 \end{bmatrix} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{Bmatrix} \kappa_{x_2}x'_3 dx_1 \\ 0 \\ \kappa_{x_2}(x_1 - x'_1)dx_1 \end{Bmatrix}$

Then, by substituting the above equation into the equation (5.106) we can obtain:

$$\mathrm{u}_i'(\vec{x}') = \int_{\vec{0}}^{\vec{x}'} T_{ik}\,dx_k \quad \Rightarrow \quad \begin{Bmatrix} \mathrm{u}_1' \\ \mathrm{u}_2' \\ \mathrm{u}_3' \end{Bmatrix} = \begin{Bmatrix} \int_{\vec{0}}^{\vec{x}'} \kappa_{x_2} x_3' dx_1 \\ 0 \\ \int_{\vec{0}}^{\vec{x}'} \kappa_{x_2} (x_1 - x_1') dx_1 \end{Bmatrix} = \begin{Bmatrix} x_3' \int_{\vec{0}}^{\vec{x}'} \kappa_{x_2} dx_1 \\ 0 \\ \int_{\vec{0}}^{\vec{x}'} \kappa_{x_2} (x_1 - x_1') dx_1 \end{Bmatrix}$$

Note that, if we consider that κ is constant, we can obtain:

$$\begin{Bmatrix} \mathrm{u}_1' \\ \mathrm{u}_2' \\ \mathrm{u}_3' \end{Bmatrix} = \begin{Bmatrix} \kappa_{x_2} x_3' \int_{\vec{0}}^{\vec{x}'} dx_1 \\ 0 \\ \kappa_{x_2} \int_{\vec{0}}^{\vec{x}'} (x_1 - x_1') dx_1 \end{Bmatrix} = \begin{Bmatrix} \kappa_{x_2} x_3' (x_1)\Big|_0^{x_1'} \\ 0 \\ \kappa_{x_2} \left(\dfrac{x_1^2}{2} - x_1' x_1\right)\Big|_0^{x_1'} \end{Bmatrix} = \begin{Bmatrix} \kappa_{x_2} x_3' x_1' \\ 0 \\ -\kappa_{x_2} \dfrac{x_1'^2}{2} \end{Bmatrix}$$

Then

$$\frac{\partial \mathrm{u}_1'}{\partial x_1} = x_3' \kappa_{x_2} \quad ; \quad \frac{\partial^2 \mathrm{u}_3'}{\partial x_1 \partial x_1} = -\kappa_{x_2}$$

Note that for the neutral line (line at x_3) there is no u_1-displacement, there is only deflection (u_3-displacement).

The stress field (with $\nu = 0$), (see **Problem 5.5** – NOTE 9), can be obtained

$$\sigma_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij} + \frac{E}{(1+\nu)} \varepsilon_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{E}{(1+\nu)} \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \varepsilon_{11}(1-\nu) & 0 & 0 \\ 0 & \nu\varepsilon_{11} & 0 \\ 0 & 0 & \nu\varepsilon_{11} \end{bmatrix} = \begin{bmatrix} E\varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The resultant force on a cross-section can be obtained as follows:

$$F = \int_A \sigma_{11} dA = \int_A E\varepsilon_{11} dA = \int_A E\kappa_{x_2} x_3 dA = E\kappa_{x_2} \underbrace{\int_A x_3 dA}_{=0} = 0$$

Note that the first moment of the area about the x_2-axis ($\int_A x_3 dA x_2$) is equal to zero, since the system is located at the geometrical center.

The bending moment on the cross-section can be obtained as follows:

$$M_{x_2} = \int_A \sigma_{11} x_3 dA = \int_A E\varepsilon_{11} x_3 dA = \int_A E\kappa_{x_2} x_3^2 dA = E\kappa_{x_2} \underbrace{\int_A x_3^2 dA}_{=I_{x2}} = E\kappa_{x_2} I_{x_2}$$

where $I_{x_2} = \int_A x_3^2 dA$ is the second moment of the area about the x_2-axis. With that we can conclude that on the cross-section the following is true:

$$\kappa_{x_2} = \frac{M_{x_2}}{EI_{x_2}} \quad ; \quad \sigma_{11}(x_3) = E\varepsilon_{11} = E\kappa_{x_2} x_3 = \frac{M_{x_2}}{I_{x_2}} x_3$$

Problem 5.14

Consider the infinitesimal strain tensor field

$$\varepsilon_{ik}(x_1,x_3)=\begin{bmatrix}\kappa_{x_2}x_3 & 0 & 0\\ 0 & -\nu\kappa_{x_2}x_3 & 0\\ 0 & 0 & -\nu\kappa_{x_2}x_3\end{bmatrix}$$
$$\boldsymbol{\varepsilon}=\kappa_{x_2}x_3\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3 \tag{5.107}$$

where κ_{x_2} is a function of x_1, i.e. $\kappa_{x_2}=\kappa_{x_2}(x_1)$ and ν (Poisson's ratio) is a constant. a) Check whether the compatibility equations are fulfilled or not. b) In the case that the compatibility equations are not satisfied, what should be met to ensure the continuity of the displacement field?

Solution:

We can verify the compatibility equations by means of $[\vec{\nabla}_{\vec{x}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon})^T]=\mathbf{0}$, i.e.:

$$(\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon})=\frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i\wedge(\kappa_{x_2}x_3\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3)$$
$$=\left(\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3\right)\wedge[\kappa_{x_2}x_3\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2-\nu\kappa_{x_2}x_3\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3]$$
$$=-\frac{\partial(\nu\kappa_{x_2}x_3)}{\partial x_1}\underbrace{\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_2}_{=\hat{\mathbf{e}}_3}\otimes\hat{\mathbf{e}}_2-\frac{\partial(\nu\kappa_{x_2}x_3)}{\partial x_1}\underbrace{\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_3}_{=-\hat{\mathbf{e}}_2}\otimes\hat{\mathbf{e}}_3$$
$$+\underbrace{\frac{\partial(\kappa_{x_2}x_3)}{\partial x_2}}_{=0}\underbrace{\hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_1}_{=-\hat{\mathbf{e}}_3}\otimes\hat{\mathbf{e}}_1-\underbrace{\frac{\partial(\nu\kappa_{x_2}x_3)}{\partial x_2}}_{=0}\underbrace{\hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_3}_{=\hat{\mathbf{e}}_1}\otimes\hat{\mathbf{e}}_3$$
$$+\frac{\partial(\kappa_{x_2}x_3)}{\partial x_3}\underbrace{\hat{\mathbf{e}}_3\wedge\hat{\mathbf{e}}_1}_{=\hat{\mathbf{e}}_2}\otimes\hat{\mathbf{e}}_1-\frac{\partial(\nu\kappa_{x_2}x_3)}{\partial x_3}\underbrace{\hat{\mathbf{e}}_3\wedge\hat{\mathbf{e}}_2}_{=-\hat{\mathbf{e}}_1}\otimes\hat{\mathbf{e}}_2$$
$$=-\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_2+\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_3+\kappa_{x_2}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_1+\nu\kappa_{x_2}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_2$$

thus

$$(\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon})^T=-\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_3+\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_2+\kappa_{x_2}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_2+\nu\kappa_{x_2}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_1$$

$$\vec{\nabla}_{\vec{x}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon})^T=\left(\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3\right)\wedge[(\vec{\nabla}_{\vec{x}}\wedge\boldsymbol{\varepsilon})^T]$$
$$=\left(\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3\right)\wedge[-\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_3+\nu x_3\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_2$$
$$+\kappa_{x_2}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_2+\nu\kappa_{x_2}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_1]$$
$$=-\nu x_3\frac{\partial^2\kappa_{x_2}}{\partial x_1^2}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3-\nu x_3\frac{\partial^2\kappa_{x_2}}{\partial x_1^2}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2+\nu\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_1+\nu\frac{\partial\kappa_{x_2}}{\partial x_1}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_3$$

$$
S_{ij} = [\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T]_{ij} = \begin{bmatrix} 0 & 0 & \nu \dfrac{\partial \kappa_{x_2}}{\partial x_1} \\ 0 & -\nu x_3 \dfrac{\partial^2 \kappa_{x_2}}{\partial x_1^2} & 0 \\ \nu \dfrac{\partial \kappa_{x_2}}{\partial x_1} & 0 & -\nu x_3 \dfrac{\partial^2 \kappa_{x_2}}{\partial x_1^2} \end{bmatrix} \neq \mathbf{0}_{ij}
$$

Note that the compatibility equations are not satisfied. One possibility to guarantee the continuity of the displacement field ($[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{\varepsilon})^T] = \mathbf{0}$), related to the strain field (5.107), is when κ_{x_2} is a constant, another possibility is when $\nu = 0$, (see **Problem 5.13**).

Note also that the above equation could have been obtained by means of the equation in (5.80), i.e:

$$
\begin{cases}
S_{11} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \quad \checkmark \\
S_{22} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = -\nu x_3 \dfrac{\partial^2 \kappa_{x_2}}{\partial x_1^2} \neq 0 \quad \text{✗ it fails} \\
S_{33} = \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = -\nu x_3 \dfrac{\partial^2 \kappa_{x_2}}{\partial x_1^2} \neq 0 \quad \text{✗ it fails} \\
S_{12} = \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 \quad \checkmark \\
S_{23} = \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = 0 \quad \checkmark \\
S_{13} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = \nu \dfrac{\partial \kappa_{x_2}}{\partial x_1} \neq 0 \quad \text{✗ it fails}
\end{cases}
$$

Problem 5.15

By considering a homogeneous isotropic linear elastic material, and a static problem without body forces in which the stress field is given by:

$$
[\boldsymbol{\sigma}(\vec{x})]_{ij} = \begin{bmatrix} \dfrac{C}{2(\lambda+\mu)}\left[(\lambda+2\mu)x_2^2 + \lambda x_1^2\right] & \dfrac{-C\lambda}{(\lambda+\mu)} x_1 x_2 & 0 \\ \dfrac{-C\lambda}{(\lambda+\mu)} x_1 x_2 & \dfrac{C}{2(\lambda+\mu)}\left[(\lambda+2\mu)x_1^2 + \lambda x_2^2\right] & 0 \\ 0 & 0 & \dfrac{C\lambda}{2(\lambda+\mu)}(x_1^2 + x_2^2) \end{bmatrix}
$$

where $C \neq 0$

a) Check if the equations of motion are satisfied;

b) Check if the stress field is appropriated to represent any continuum.

Solution:

a) For the static problem, the equations of motions ($\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \vec{a}$) become the equilibrium equations ($\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$). And the equilibrium equations without body forces ($\rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$) is represented by:

$$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \underbrace{\rho \vec{\mathbf{b}}}_{=\vec{\mathbf{0}}} = \vec{\mathbf{0}} \xrightarrow{indicial} \sigma_{ij,j} = 0_i \tag{5.108}$$

thus

$$\sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0_i \quad \Rightarrow \quad \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0 \end{cases}$$

Then, by substituting the given stress field components we can obtain:

$$\begin{cases} \dfrac{C}{2(\lambda+\mu)} \dfrac{\partial[(\lambda+2\mu)x_2^2 + \lambda x_1^2]}{\partial x_1} + \dfrac{-C\lambda}{(\lambda+\mu)} \dfrac{\partial(x_1 x_2)}{\partial x_2} + 0 = \dfrac{C}{2(\lambda+\mu)}[2\lambda x_1] + \dfrac{-C\lambda}{(\lambda+\mu)}(x_1) = 0 \ \checkmark \\ \dfrac{-C\lambda}{(\lambda+\mu)} \dfrac{\partial(x_1 x_2)}{\partial x_1} + \dfrac{C}{2(\lambda+\mu)} \dfrac{\partial[(\lambda+2\mu)x_1^2 + \lambda x_2^2]}{\partial x_2} + 0 = \dfrac{-C\lambda}{(\lambda+\mu)}(x_2) + \dfrac{C}{2(\lambda+\mu)}[2\lambda x_2] = 0 \ \checkmark \\ 0 + 0 + \dfrac{C\lambda}{2(\lambda+\mu)} \dfrac{\partial(x_1^2 + x_2^2)}{\partial x_3} = 0 = 0 \ \checkmark \end{cases}$$

Then, the three equations are satisfied.

b) Any continuum must satisfy the compatibility equations, so, for a given stress field if the correspondent strain field satisfies the compatibility equations, the stress field is acceptable to represent the continuum stress state. In **Problem 5.5**, (see equation (5.26)), we have shown that:

$$\boldsymbol{\varepsilon} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} \qquad \Bigg| \qquad \varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij}$$

For this problem we have:

$$\begin{aligned} \mathrm{Tr}(\boldsymbol{\sigma}) &= \sigma_{11} + \sigma_{22} + \sigma_{33} = \frac{C}{2(\lambda+\mu)} \{[(\lambda+2\mu)x_2^2 + \lambda x_1^2] + [(\lambda+2\mu)x_1^2 + \lambda x_2^2] + [\lambda(x_1^2 + x_2^2)]\} \\ &= \frac{C(3\lambda+2\mu)}{2(\lambda+\mu)}(x_1^2 + x_2^2) \end{aligned}$$

then

$$\begin{aligned} \boldsymbol{\varepsilon} &= \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \frac{C(3\lambda+2\mu)}{2(\lambda+\mu)}(x_1^2 + x_2^2)\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} \\ &= \frac{-\lambda C}{4\mu(\lambda+\mu)}(x_1^2 + x_2^2)\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} \end{aligned}$$

whose components are:

$$
\varepsilon_{ij} = \frac{-\lambda C}{4\mu(\lambda+\mu)}(x_1^2+x_2^2)\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \Rightarrow
\begin{cases}
\varepsilon_{11} = \dfrac{-\lambda C}{4\mu(\lambda+\mu)}(x_1^2+x_2^2) + \dfrac{1}{2\mu}\sigma_{11} = \dfrac{Cx_2^2}{2(\lambda+\mu)} \\
\varepsilon_{22} = \dfrac{-\lambda C}{4\mu(\lambda+\mu)}(x_1^2+x_2^2) + \dfrac{1}{2\mu}\sigma_{22} = \dfrac{Cx_1^2}{2(\lambda+\mu)} \\
\varepsilon_{33} = \dfrac{-\lambda C}{4\mu(\lambda+\mu)}(x_1^2+x_2^2) + \dfrac{1}{2\mu}\sigma_{33} = 0 \\
\varepsilon_{12} = \dfrac{1}{2\mu}\sigma_{12} = \dfrac{-C\lambda}{2\mu(\lambda+\mu)}x_1x_2 \\
\varepsilon_{23} = \dfrac{1}{2\mu}\sigma_{23} = 0 \\
\varepsilon_{13} = \dfrac{1}{2\mu}\sigma_{13} = 0
\end{cases}
$$

Now, we check the compatibility equations:

$$
\begin{cases}
S_{11} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 + 0 - 0 = 0 \quad \checkmark \\
S_{22} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0 + 0 - 0 = 0 \quad \checkmark \\
S_{33} = \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \dfrac{C}{(\lambda+\mu)} + \dfrac{C}{(\lambda+\mu)} - 2\left(\dfrac{-C\lambda}{2\mu(\lambda+\mu)}\right) = \dfrac{C(\lambda+2\mu)}{\mu(\lambda+\mu)} \neq 0 \quad \text{✗} \\
S_{12} = \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 + 0 - 0 - 0 = 0 \quad \checkmark \\
S_{23} = \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = -0 + 0 + 0 - 0 = 0 \quad \checkmark \\
S_{13} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} 0 - 0 + 0 - 0 = \quad \checkmark
\end{cases}
$$

As we can see the equation $S_{33} \neq 0$ fails. So, the given stress field is not appropriated to represent any continuum.

In **Problem 5.16** we will derive a set of equations which is taking into account *the equations of motion*, *constitutive equations* and *the compatibility equations* simultaneously. This formulation is called *Stress Formulation*. And for the particular case in which the problem is static and without body force the stress formulation is called *Beltrami's equation*.

Problem 5.16

a) Show that the governing equations for a homogeneous isotropic linear elastic material, (see equations in (5.47)), can be replaced by six equations and six unknowns (σ_{ij}), *(Stress Formulation)*, i.e.:

Indicial notation

$$\boxed{\sigma_{ij,kk}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\sigma_{kk,ij}-\frac{\lambda}{(2\mu+3\lambda)}\sigma_{ll,kk}\delta_{ij}=2[(\rho\ddot{u}_i)_{,j}]^{sym}-2[(\rho b_i)_{,j}]^{sym}}$$

Tensorial notation

$$\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]-\frac{\lambda}{(2\mu+3\lambda)}\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\mathbf{1}=2[\nabla_{\vec{x}}(\rho\ddot{\vec{\mathbf{u}}})][\ ]^{sym}-2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym}}$$

(5.109)

where $\nabla^2_{\vec{x}}\boldsymbol{\sigma}\equiv\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\sigma})$ and $\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\equiv\nabla_{\vec{x}}\cdot[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]$.

b) or by:

Indicial notation

$$\boxed{\sigma_{ij,kk}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\sigma_{kk,ij}=\frac{-\lambda}{(2\mu+\lambda)}[(\rho b_k)_{,k}-(\rho\ddot{u}_k)_{,k}]\delta_{ij}+2[(\rho\ddot{u}_i)_{,j}]^{sym}-2[(\rho b_i)_{,j}]^{sym}}$$

Tensorial notation

$$\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]=\frac{-\lambda}{(2\mu+\lambda)}\nabla_{\vec{x}}\cdot[(\rho\vec{\mathbf{b}})-(\rho\ddot{\vec{\mathbf{u}}})]\mathbf{1}+2[\nabla_{\vec{x}}(\rho\ddot{\vec{\mathbf{u}}})]^{sym}-2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym}}$$

(5.110)

c) Considering that $\lambda=\dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu=\dfrac{E}{2(1+\nu)}$, express the equations (5.109) and (5.110) in function of (E,ν).

Hint: The *kinematic equations* $\boldsymbol{\varepsilon}=\nabla^{sym}\vec{\mathbf{u}}$ can be considered by using the equation, (see **Problem 5.11**):

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{il,jk}-\varepsilon_{jk,il}=\mathbb{O}_{ijkl} \tag{5.111}$$

Solution: a) We can obtain the inverse of the *constitutive equation in stress* ($\boldsymbol{\sigma}=\mathbb{C}^e:\boldsymbol{\varepsilon}$):

$$\mathbb{C}^{e^{-1}}:\boldsymbol{\sigma}=\mathbb{C}^{e^{-1}}:\mathbb{C}^e:\boldsymbol{\varepsilon}=\mathbb{I}^{sym}:\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{sym}=\boldsymbol{\varepsilon}\quad\Rightarrow\quad\boldsymbol{\varepsilon}=\mathbb{C}^{e^{-1}}:\boldsymbol{\sigma}$$

For isotropic materials, (see equation (5.26)), the strain tensor can be obtained as follows:

$$\boldsymbol{\varepsilon}=\frac{1}{2\mu}\boldsymbol{\sigma}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}\xrightarrow{\text{indicial}}\varepsilon_{ij}=\frac{1}{2\mu}\sigma_{ij}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq}\delta_{ij}.$$

As we are considering a homogeneous material, the mechanical properties do not vary with $\vec{\boldsymbol{x}}$, i.e. $\lambda_{,i}\equiv\dfrac{\partial\lambda}{\partial x_i}=0_i$ and $\mu_{,i}\equiv\dfrac{\partial\mu}{\partial x_i}=0_i$, then:

$$\frac{\partial^2\varepsilon_{ij}}{\partial x_k\partial x_l}\equiv\varepsilon_{ij,kl}=\left(\frac{1}{2\mu}\sigma_{ij}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq}\delta_{ij}\right)_{,kl}=\frac{1}{2\mu}\sigma_{ij,kl}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,kl}\delta_{ij} \tag{5.112}$$

Moreover, if we multiply the equation in (5.111) ("*kinematic equations*") by δ_{jk} we can obtain:

$$\begin{aligned}&\varepsilon_{ij,kl}\delta_{jk}+\varepsilon_{kl,ij}\delta_{jk}-\varepsilon_{il,jk}\delta_{jk}-\varepsilon_{jk,il}\delta_{jk}=\mathbb{O}_{ijkl}\delta_{jk}\\&\Rightarrow\varepsilon_{ik,kl}+\varepsilon_{kl,ik}-\varepsilon_{il,kk}-\varepsilon_{kk,il}=0_{il}\end{aligned}\tag{5.113}$$

Note that, according to equation (5.112) the following is true:

$$\varepsilon_{ik,kl}=\frac{1}{2\mu}\sigma_{ik,kl}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,kl}\delta_{ik}=\frac{1}{2\mu}\sigma_{ik,kl}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}$$

$$\varepsilon_{kl,ik}=\frac{1}{2\mu}\sigma_{kl,ik}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,ik}\delta_{kl}=\frac{1}{2\mu}\sigma_{lk,ki}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}$$

$$\varepsilon_{il,kk}=\frac{1}{2\mu}\sigma_{il,kk}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}$$

$$\varepsilon_{kk,il}=\frac{1}{2\mu}\sigma_{kk,il}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}\underbrace{\delta_{kk}}_{=3}=\frac{1}{2\mu}\sigma_{kk,il}-\frac{3\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}$$

$$=\frac{1}{2\mu}\sigma_{qq,il}-\frac{3\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}=\left(\frac{1}{2\mu}-\frac{3\lambda}{2\mu(2\mu+3\lambda)}\right)\sigma_{qq,il}=\frac{2\mu}{2\mu(2\mu+3\lambda)}\sigma_{qq,il}$$

$$\frac{\partial^2\varepsilon_{ij}}{\partial x_k\partial x_l}\equiv\varepsilon_{ij,kl}=\frac{1}{2\mu}\sigma_{ij,kl}-\frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,kl}\delta_{ij}$$

With that the equation in (5.113) becomes:

$$\varepsilon_{ik,kl}+\varepsilon_{kl,ik}-\varepsilon_{il,kk}-\varepsilon_{kk,il}=0_{il}$$

$$\frac{1}{2\mu}\left(\sigma_{ik,kl}-\frac{2\lambda}{(2\mu+3\lambda)}\sigma_{qq,il}+\sigma_{lk,ki}-\sigma_{il,kk}+\frac{\lambda}{(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}-\frac{2\mu}{(2\mu+3\lambda)}\sigma_{qq,il}\right)=0_{il}$$

$$\Rightarrow\sigma_{ik,kl}-\left(\frac{2\lambda}{(2\mu+3\lambda)}+\frac{2\mu}{(2\mu+3\lambda)}\right)\sigma_{qq,il}+\sigma_{lk,ki}-\sigma_{il,kk}+\frac{\lambda}{(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}=0_{il}$$

$$\Rightarrow\sigma_{ik,kl}-\frac{2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{qq,il}+\sigma_{lk,ki}-\sigma_{il,kk}+\frac{\lambda}{(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}=0_{il}$$

$$\Rightarrow\frac{-2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{qq,il}-\sigma_{il,kk}+\frac{\lambda}{(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}=-\sigma_{ik,kl}-\sigma_{lk,ki}\tag{5.114}$$

From the *equations of motion* $\sigma_{ij,j}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i$ we can obtain:

$$\sigma_{ij,jk}+(\rho\mathrm{b}_i)_{,k}=(\rho\ddot{\mathrm{u}}_i)_{,k}$$

with that the following is true:

$$\sigma_{ik,kl}+(\rho\mathrm{b}_i)_{,l}=(\rho\ddot{\mathrm{u}}_i)_{,l}\quad\Rightarrow\quad-\sigma_{ik,kl}=(\rho\mathrm{b}_i)_{,l}-(\rho\ddot{\mathrm{u}}_i)_{,l}$$

$$\sigma_{lk,ki}+(\rho\mathrm{b}_l)_{,i}=(\rho\ddot{\mathrm{u}}_l)_{,i}\quad\Rightarrow\quad-\sigma_{lk,ki}=(\rho\mathrm{b}_l)_{,i}-(\rho\ddot{\mathrm{u}}_l)_{,i}\,.$$

And note that $-\sigma_{ik,kl}-\sigma_{lk,ki}=(\rho\mathrm{b}_i)_{,l}-(\rho\ddot{\mathrm{u}}_i)_{,l}+(\rho\mathrm{b}_l)_{,i}-(\rho\ddot{\mathrm{u}}_l)_{,i}=2[(\rho\mathrm{b}_i)_{,l}]^{sym}-2[(\rho\ddot{\mathrm{u}}_i)_{,l}]^{sym}$

By replacing the above equation into the equation (5.114) we can obtain:

$$\frac{-2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{qq,il}-\sigma_{il,kk}+\frac{\lambda}{(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{il}=2[(\rho\mathrm{b}_i)_{,l}]^{sym}-2[(\rho\ddot{\mathrm{u}}_i)_{,l}]^{sym}$$

Restructuring the above and considering that $(l=j)$ we can obtain:

$$\sigma_{ij,kk} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{kk,ij} - \frac{\lambda}{(2\mu+3\lambda)}\sigma_{ll,kk}\delta_{ij} = 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym} \tag{5.115}$$

Q.E.D.

which matches the equation in (5.109). The above equation could have been obtained by means of equation in (5.79):

$$\boxed{\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon}) + \nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\varepsilon})\right]\right] = \nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon}) + \left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})\right]^T} \tag{5.116}$$

where $\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$, and is also true that

$$\begin{aligned}\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon}) &= \nabla_{\vec{x}}\cdot\left[\nabla_{\vec{x}}\left(\frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}\right)\right] \\ &= \frac{1}{2\mu}\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\sigma}) - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}\cdot\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\mathbf{1}\end{aligned} \quad ;$$

$$\mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{1}{(2\mu+3\lambda)}\mathrm{Tr}(\boldsymbol{\sigma}) \quad \Rightarrow \quad \nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\varepsilon})\right]\right] = \frac{1}{(2\mu+3\lambda)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right];$$

$$\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon} = \frac{1}{2\mu}\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}\cdot(\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}) = \frac{1}{2\mu}\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))$$

Note that $[\nabla_{\vec{x}}\cdot(\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1})]_i = (\sigma_{kk}\delta_{ij})_{,j} = \sigma_{kk,j}\delta_{ij} + \sigma_{kk}\delta_{ij,j} = \sigma_{kk,j}\delta_{ij} = \sigma_{kk,i} = [\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]_i$, and if we consider $\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\ddot{\vec{\mathbf{u}}}$ we can obtain:

$$\begin{aligned}&\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon} = \frac{1}{2\mu}\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma})) = \frac{1}{2\mu}(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}}) - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma})) \\ &\Rightarrow \nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon}) = \frac{1}{2\mu}\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})] - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\end{aligned}$$

with that we can obtain:

$$\begin{aligned}\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon}) + \left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})\right]^T &= 2\left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})\right]^{sym} \\ &= \frac{2}{2\mu}\{\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})]\}^{sym} - \frac{2\lambda}{2\mu(2\mu+3\lambda)}\{\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\}^{sym} \\ &= \frac{2}{2\mu}\{\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})]\}^{sym} - \frac{2\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\end{aligned}$$

$\{\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\}_{ij} = \sigma_{kk,ij} = \sigma_{kk,ji} = \{\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\}_{ji}$ (symmetric). Taking into account the above equations into the equation (5.116) we can obtain:

$$\begin{aligned}&\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\varepsilon}) + \nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\varepsilon})\right]\right] = \nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon}) + \left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\boldsymbol{\varepsilon})\right]^T \\ &\Rightarrow \frac{1}{2\mu}\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\boldsymbol{\sigma}) - \frac{\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}\cdot\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\mathbf{1} + \frac{1}{(2\mu+3\lambda)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] = \\ &\qquad \frac{2}{2\mu}\{\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})]\}^{sym} - \frac{2\lambda}{2\mu(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]\end{aligned}$$

$$\Rightarrow \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \boldsymbol{\sigma}) - \frac{\lambda}{(2\mu+3\lambda)} (\nabla_{\vec{x}} \cdot \nabla_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]) \mathbf{1} + \frac{2\mu}{(2\mu+3\lambda)} \nabla_{\vec{x}} \left[\nabla_{\vec{x}} \left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] =$$
$$2\{\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})]\}^{sym} - \frac{2\lambda}{(2\mu+3\lambda)} \nabla_{\vec{x}}[\nabla_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\sigma}))]$$
$$\Rightarrow \nabla^2_{\vec{x}} \boldsymbol{\sigma} - \frac{\lambda}{(2\mu+3\lambda)} \nabla^2_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})] \mathbf{1} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)} \nabla_{\vec{x}} \left[\nabla_{\vec{x}} \left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] = 2\{\nabla_{\vec{x}}[(\rho\ddot{\vec{\mathbf{u}}} - \rho\vec{\mathbf{b}})]\}^{sym}$$

which matches the equation in (5.115) or (5.109).

b) Starting from the equation (5.115):

$$\sigma_{ij,kk} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)} \sigma_{kk,ij} = \frac{\lambda}{(2\mu+3\lambda)} \sigma_{ll,kk} \delta_{ij} + 2[(\rho\ddot{\mathrm{u}}_i)_{,j}]^{sym} - 2[(\rho\mathrm{b}_i)_{,j}]^{sym} \tag{5.117}$$

Our goal now is to obtain an expression for $\sigma_{ll,kk}$. If we multiply equation (5.111) by $\delta_{jk}\delta_{li}$ we can obtain:

$$\begin{aligned}
&\varepsilon_{ij,kl}\delta_{jk}\delta_{li} + \varepsilon_{kl,ij}\delta_{jk}\delta_{li} - \varepsilon_{il,jk}\delta_{jk}\delta_{li} - \varepsilon_{jk,il}\delta_{jk}\delta_{li} = \mathbb{O}_{ijkl}\delta_{jk}\delta_{li} \\
&\Rightarrow \varepsilon_{ij,ji} + \varepsilon_{ji,ij} - \varepsilon_{ii,jj} - \varepsilon_{jj,ii} = 2\varepsilon_{ij,ij} - 2\varepsilon_{ii,jj} = 0 \\
&\Rightarrow \varepsilon_{ij,ij} - \varepsilon_{ii,jj} = 0
\end{aligned} \tag{5.118}$$

If we use the inverse of the constitutive equation, (see equation (5.112)), we can obtain:

$$\begin{aligned}
\varepsilon_{ij,ij} &= \frac{1}{2\mu}\sigma_{ij,ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,ij}\delta_{ij} = \frac{1}{2\mu}\sigma_{ij,ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,ii} \\
\varepsilon_{ii,kk} &= \frac{1}{2\mu}\sigma_{ii,kk} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,kk}\delta_{ii} = \left(\frac{2\mu}{2\mu(2\mu+3\lambda)}\right)\sigma_{ii,kk}
\end{aligned} \tag{5.119}$$

With that the equation in (5.118) becomes:

$$\begin{aligned}
&\Rightarrow \varepsilon_{ij,ij} - \varepsilon_{ii,jj} = 0 \\
&\Rightarrow \frac{1}{2\mu}\sigma_{ij,ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{qq,ii} - \left(\frac{2\mu}{2\mu(2\mu+3\lambda)}\right)\sigma_{ii,kk} = 0 \\
&\Rightarrow \sigma_{ij,ij} - \left(\frac{\lambda}{(2\mu+3\lambda)} + \frac{2\mu}{(2\mu+3\lambda)}\right)\sigma_{ii,kk} = 0 \\
&\Rightarrow \sigma_{ij,ij} = \left(\frac{2\mu+\lambda}{(2\mu+3\lambda)}\right)\sigma_{ii,kk}
\end{aligned} \tag{5.120}$$

The above equation can also be written in terms of ν

$$\boxed{\begin{array}{ccc}
\sigma_{ij,ij} = \dfrac{2\mu+\lambda}{2\mu+3\lambda}\sigma_{ii,kk} & ; & \sigma_{ij,ij} = \dfrac{1-\nu}{1+\nu}\sigma_{ii,kk} \\
\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) = \dfrac{2\mu+\lambda}{2\mu+3\lambda} \nabla_{\vec{x}} \cdot \left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] & ; & \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) = \dfrac{1-\nu}{1+\nu} \nabla_{\vec{x}} \cdot \left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right]
\end{array}} \tag{5.121}$$

Now, by means of the equations of motion $\sigma_{ij,j} + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i$ we can obtain:

$$\sigma_{ij,ji} + (\rho\mathrm{b}_i)_{,i} = (\rho\ddot{\mathrm{u}}_i)_{,i} \qquad \Rightarrow \qquad \sigma_{ij,ji} = (\rho\ddot{\mathrm{u}}_i)_{,i} - (\rho\mathrm{b}_i)_{,i}$$

With that the equation in (5.120) becomes:

$$
\begin{aligned}
&\sigma_{ij,ij} = \left(\frac{2\mu+\lambda}{(2\mu+3\lambda)}\right)\sigma_{ii,kk} \\
&\Rightarrow (\rho\ddot{u}_i)_{,i} - (\rho b_i)_{,i} = \left(\frac{2\mu+\lambda}{(2\mu+3\lambda)}\right)\sigma_{ii,kk} \\
&\Rightarrow \sigma_{ii,kk} = \sigma_{ll,kk} = \frac{(2\mu+3\lambda)}{2\mu+\lambda}\left[(\rho\ddot{u}_k)_{,k} - (\rho b_k)_{,k}\right] = -\frac{(2\mu+3\lambda)}{2\mu+\lambda}\left[(\rho b_k)_{,k} - (\rho\ddot{u}_k)_{,k}\right]
\end{aligned}
\tag{5.122}
$$

Replacing equation (5.122) into (5.117), we can obtain:

$$
\begin{aligned}
&\sigma_{ij,kk} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{kk,ij} = \frac{\lambda}{(2\mu+3\lambda)}\sigma_{ll,kk}\delta_{ij} + 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym} \\
&\sigma_{ij,kk} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{kk,ij} = \frac{-\lambda}{(2\mu+3\lambda)}\frac{(2\mu+3\lambda)}{2\mu+\lambda}[(\rho b_k)_{,k} - (\rho\ddot{u}_k)_{,k}]\delta_{ij} + 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym} \\
&\Rightarrow \sigma_{ij,kk} + \frac{2(\mu+\lambda)}{(2\mu+3\lambda)}\sigma_{kk,ij} = \frac{-\lambda}{(2\mu+\lambda)}[(\rho b_k)_{,k} - (\rho\ddot{u}_k)_{,k}]\delta_{ij} + 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym}
\end{aligned}
$$

$$
Q.E.D. \tag{5.123}
$$

Thus obtaining the equation in (5.110)

c) After some algebraic manipulations we can obtain:

$$
\frac{1}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E}; \qquad \frac{\lambda}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E}\frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{\nu}{(1+\nu)};
$$

$$
\frac{\mu}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E}\frac{E}{2(1+\nu)} = \frac{(1-2\nu)}{2(1+\nu)}; \qquad \frac{2(\mu+\lambda)}{(2\mu+3\lambda)} = 2\frac{\nu}{(1+\nu)} + 2\frac{(1-2\nu)}{2(1+\nu)} = \frac{1}{(1+\nu)},
$$

$$
(2\mu+\lambda) = 2\frac{E}{2(1+\nu)} + \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)};
$$

$$
\frac{\lambda}{(2\mu+\lambda)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{(1+\nu)(1-2\nu)}{E(1-\nu)} = \frac{\nu}{(1-\nu)}; \qquad \frac{2\mu+\lambda}{2\mu+3\lambda} = \frac{1-\nu}{1+\nu},
$$

whereby the equation (5.109) becomes:

$$
\boxed{\sigma_{ij,kk} + \frac{1}{(1+\nu)}\sigma_{kk,ij} - \frac{\nu}{(1+\nu)}\sigma_{ll,kk}\delta_{ij} = 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym}}
$$

Tensorial notation

$$
\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma} + \frac{1}{(1+\nu)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] - \frac{\nu}{(1+\nu)}\nabla^2_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\mathbf{1} = 2[\nabla_{\vec{x}}(\rho\ddot{\vec{\mathbf{u}}})]^{sym} - 2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym}}
\tag{5.124}
$$

and the equation (5.110) becomes:

$$
\boxed{\sigma_{ij,kk} + \frac{1}{(1+\nu)}\sigma_{kk,ij} = \frac{-\nu}{(1-\nu)}[(\rho b_k)_{,k} - (\rho\ddot{u}_k)_{,k}]\delta_{ij} + 2[(\rho\ddot{u}_i)_{,j}]^{sym} - 2[(\rho b_i)_{,j}]^{sym}}
$$

Tensorial notation

$$
\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma} + \frac{1}{(1+\nu)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] = \frac{-\nu}{(1-\nu)}\nabla_{\vec{x}}\cdot[(\rho\vec{\mathbf{b}}) - (\rho\ddot{\vec{\mathbf{u}}})]\mathbf{1} + 2[\nabla_{\vec{x}}(\rho\ddot{\vec{\mathbf{u}}})]^{sym} - 2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym}}
\tag{5.125}
$$

NOTE 1: For a static problem the above equation becomes:

$$\boxed{\sigma_{ij,kk}+\frac{1}{(1+\nu)}\sigma_{kk,ij}=\frac{-\nu}{(1-\nu)}[(\rho \mathrm{b}_k)_{,k}]\delta_{ij}-2[(\rho \mathrm{b}_i)_{,j}]^{sym}}$$
$$\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma}+\frac{1}{(1+\nu)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right]=\frac{-\nu}{(1-\nu)}[\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]\mathbf{1}-2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym}}$$ *Michell's equations* (5.126)

which are known as Michell's equations, which were obtained by Michell in 1900.

If the body forces do not vary with $\vec{x}$, the Michell's equations (5.126) reduce to:

$$\boxed{\sigma_{ij,kk}+\frac{1}{(1+\nu)}\sigma_{kk,ij}=0_{ij}}$$
$$\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma}+\frac{1}{(1+\nu)}\nabla_{\vec{x}}\left[\nabla_{\vec{x}}\left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right]=\mathbf{0}}$$ *Beltrami's equations* (5.127)

which are known as Beltrami's equations, which were obtained by Beltrami in 1892, (see Sokolnikoff (1956) first edition in (1946)).

If we take into account that $\sigma_{ij,kk}=\dfrac{\partial^2\sigma_{ij}}{\partial x_k\partial x_k}=\nabla^2_{\vec{x}}\sigma_{ij}$, $[(\rho \mathrm{b}_i)_{,j}]^{sym}=\dfrac{1}{2}\left(\dfrac{\partial(\rho \mathrm{b}_i)}{\partial x_j}+\dfrac{\partial(\rho \mathrm{b}_j)}{\partial x_i}\right)$, and $[(\rho \mathrm{b}_k)_{,k}]\delta_{ij}=[][\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]\delta_{ij}$, the Michell's equations can be rewritten explicitly as follows:

$$\sigma_{ij,kk}+\frac{1}{(1+\nu)}\sigma_{kk,ij}=\frac{-\nu}{(1-\nu)}[(\rho \mathrm{b}_k)_{,k}]\delta_{ij}-2[(\rho \mathrm{b}_i)_{,j}]^{sym}$$
$$\Rightarrow\nabla^2_{\vec{x}}\sigma_{ij}+\frac{1}{(1+\nu)}\frac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_i\partial x_j}=\frac{-\nu}{(1-\nu)}[\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]\delta_{ij}-\left(\frac{\partial(\rho \mathrm{b}_i)}{\partial x_j}+\frac{\partial(\rho \mathrm{b}_j)}{\partial x_i}\right)$$

Then, the above six equations are:

$$\begin{cases}
\nabla^2_{\vec{x}}\sigma_{11}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_1^2}=\dfrac{-\nu}{(1-\nu)}[\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]-2\dfrac{\partial(\rho \mathrm{b}_1)}{\partial x_1}\\
\nabla^2_{\vec{x}}\sigma_{22}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_2^2}=\dfrac{-\nu}{(1-\nu)}[\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]-2\dfrac{\partial(\rho \mathrm{b}_2)}{\partial x_2}\\
\nabla^2_{\vec{x}}\sigma_{33}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_3^2}=\dfrac{-\nu}{(1-\nu)}[\nabla_{\vec{x}}\cdot(\rho\vec{\mathbf{b}})]-2\dfrac{\partial(\rho \mathrm{b}_3)}{\partial x_3}\\
\nabla^2_{\vec{x}}\sigma_{12}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_1\partial x_2}=-\left(\dfrac{\partial(\rho \mathrm{b}_1)}{\partial x_2}+\dfrac{\partial(\rho \mathrm{b}_2)}{\partial x_1}\right)\\
\nabla^2_{\vec{x}}\sigma_{23}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_2\partial x_3}=-\left(\dfrac{\partial(\rho \mathrm{b}_2)}{\partial x_3}+\dfrac{\partial(\rho \mathrm{b}_3)}{\partial x_2}\right)\\
\nabla^2_{\vec{x}}\sigma_{13}+\dfrac{1}{(1+\nu)}\dfrac{\partial^2[\mathrm{Tr}(\boldsymbol{\sigma})]}{\partial x_1\partial x_3}=-\left(\dfrac{\partial(\rho \mathrm{b}_1)}{\partial x_3}+\dfrac{\partial(\rho \mathrm{b}_3)}{\partial x_1}\right)
\end{cases}\tag{5.128}$$

NOTE 2: For a static problem $(\ddot{u}_k = 0_k)$, the equation in (5.122) becomes:

$$\sigma_{ll,kk} = -\frac{(2\mu+3\lambda)}{2\mu+\lambda}(\rho b_k)_{,k} = -\frac{(1+\nu)}{(1-\nu)}(\rho b_k)_{,k} \quad \Bigg| \quad \begin{aligned} &\nabla_{\vec{x}} \cdot \{\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\} = \frac{-(1+\nu)}{(1-\nu)}\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}}) \\ &\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] = \frac{-(1+\nu)}{(1-\nu)}\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}}) \end{aligned} \tag{5.129}$$

The above equation can also be obtained by means of equation (5.126) with $(i = j)$, i.e.: we are obtaining the trace of (5.126):

$$\begin{aligned} &\sigma_{ii,kk} + \frac{1}{(1+\nu)}\sigma_{kk,ii} = \frac{-\nu}{(1-\nu)}[(\rho b_k)_{,k}]\underbrace{\delta_{ii}}_{=3} - 2[(\rho b_i)_{,i}] \\ &\Rightarrow \left(1 + \frac{1}{(1+\nu)}\right)\sigma_{ii,kk} = \left(\frac{-3\nu}{(1-\nu)} - 2\right)[(\rho b_k)_{,k}] \\ &\Rightarrow \left(\frac{(2+\nu)}{(1+\nu)}\right)\sigma_{ii,kk} = -\left(\frac{(2+\nu)}{(1-\nu)}\right)[(\rho b_k)_{,k}] \quad \Rightarrow \quad \sigma_{ii,kk} = -\frac{(1+\nu)}{(1-\nu)}[(\rho b_k)_{,k}] \end{aligned} \tag{5.130}$$

Note that $\sigma_{ii,kk} = \sigma_{kk,ii}$ and $(\rho b_k)_{,k} = (\rho b_i)_{,i}$. The above equation in tensorial notation becomes:

$$(\nabla^2_{\vec{x}}\boldsymbol{\sigma}) : \mathbf{1} + \frac{1}{(1+\nu)}\{\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]\} : \mathbf{1} = \frac{-\nu}{(1-\nu)}[\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}})]\mathbf{1} : \mathbf{1} - 2[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})]^{sym} : \mathbf{1} \tag{5.131}$$

Note that

$$\begin{aligned} &(\nabla^2_{\vec{x}}\boldsymbol{\sigma}) : \mathbf{1} \equiv \{\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}}\boldsymbol{\sigma})\} : \mathbf{1} = \nabla_{\vec{x}} \cdot [\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]] \equiv \nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] \\ &\{\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]\} : \mathbf{1} = \nabla_{\vec{x}} \cdot [\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]] \equiv \nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] \\ &\mathbf{1} : \mathbf{1} = 3 \\ &\left[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})\right]^{sym} : \mathbf{1} = \nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}}) \end{aligned}$$

with that the equation (5.131) becomes:

$$\begin{aligned} &\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] + \frac{1}{(1+\nu)}\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] = \frac{-3\nu}{(1-\nu)}[\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}})] - 2[\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}})] \\ &\Rightarrow \nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})] = \frac{-(1+\nu)}{(1-\nu)}[\nabla_{\vec{x}} \cdot (\rho\vec{\mathbf{b}})] \end{aligned}$$

NOTE 3: For the two-dimensional elasticity case, the stress formulation is provided in **Problem 6.34**.

Problem 5.17

Consider a static linear elastic problem, and also that the mass density (ρ) and the mechanical properties (λ, μ) are homogeneous fields, and that the specific body force $\vec{\mathbf{b}}$ is a conservative and homogeneous field. Show that the Cauchy stress tensor, the infinitesimal strain tensor, and the displacement components are biharmonic functions.

Solution:

Taking into account the static problem, the equations of motion $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \ddot{\vec{\mathbf{u}}}$ becomes the equilibrium equations $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$, and by applying the divergence to it we can obtain:

Tensorial notation	*Indicial notation*	
$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$ $\Rightarrow \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) + \underbrace{\nabla_{\vec{x}} \cdot (\rho \vec{\mathbf{b}})}_{=0} = 0$ $\Rightarrow \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) = 0$	$\sigma_{ij,j} + \rho \mathrm{b}_i = 0_i$ $\Rightarrow \sigma_{ij,ji} + \underbrace{(\rho \mathrm{b}_i)_{,i}}_{=0} = 0$ $\Rightarrow \sigma_{ij,ij} = 0$	(5.132)

where we have considered that the body force density $(\rho \vec{\mathbf{b}})$ does not change with $\vec{x}$ (homogeneous field).

If we take into account the equation in (5.121) we can conclude that:

Tensorial notation	*Indicial notation*	
$\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) = \dfrac{1-\nu}{1+\nu} \nabla_{\vec{x}} \cdot [\nabla_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]] = 0$ $\Rightarrow \nabla_{\vec{x}} \cdot [\nabla_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]] \equiv \nabla^2_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})] = 0$	$\sigma_{ij,ij} = \dfrac{1-\nu}{1+\nu} \sigma_{ii,kk} = 0$ $\Rightarrow \sigma_{ii,kk} = \dfrac{\partial^2 \sigma_{ii}}{\partial x_k \partial x_k} = \nabla^2_{\vec{x}} (\sigma_{ii}) = 0$	(5.133)

with that we show that $[\mathrm{Tr}(\boldsymbol{\sigma})]$ is harmonic function. Then it is easy to show that $[\mathrm{Tr}(\boldsymbol{\varepsilon})]$ is also harmonic function, since $[\mathrm{Tr}(\boldsymbol{\sigma})] = 3\left(\lambda + \dfrac{2\mu}{3}\right)[\mathrm{Tr}(\boldsymbol{\varepsilon})]$, (see **Problem 5.5** NOTE 8):

$$\nabla^2_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})] = \nabla^2_{\vec{x}} \left[3\left(\lambda + \frac{2\mu}{3}\right)[\mathrm{Tr}(\boldsymbol{\varepsilon})] \right] = 0 \quad \Rightarrow \quad \nabla^2_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\varepsilon})] = 0 \tag{5.134}$$

If we apply the Laplacian to the Beltrami's equations (5.127) we can obtain:

Tensorial notation	*Indicial notation*	
$\nabla^2_{\vec{x}} \boldsymbol{\sigma} + \dfrac{1}{(1+\nu)} \nabla_{\vec{x}} [\nabla_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]] = \mathbf{0}$ $\Rightarrow \nabla^2_{\vec{x}} \nabla^2_{\vec{x}} \boldsymbol{\sigma} + \dfrac{1}{(1+\nu)} \nabla^2_{\vec{x}} \{\nabla_{\vec{x}} [\nabla_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]]\} = \mathbf{0}$ $\Rightarrow \nabla^2_{\vec{x}} \nabla^2_{\vec{x}} \boldsymbol{\sigma} + \dfrac{1}{(1+\nu)} \nabla_{\vec{x}} \left\{ \nabla_{\vec{x}} [\underbrace{\nabla^2_{\vec{x}} [\mathrm{Tr}(\boldsymbol{\sigma})]}_{=0}] \right\} = \mathbf{0}$ $\Rightarrow \nabla^2_{\vec{x}} \nabla^2_{\vec{x}} \boldsymbol{\sigma} \equiv \nabla^4_{\vec{x}} \boldsymbol{\sigma} = \mathbf{0}$	$\sigma_{ij,kk} + \dfrac{1}{(1+\nu)} \sigma_{kk,ij} = 0_{ij}$ $\Rightarrow \sigma_{ij,kkpp} + \dfrac{1}{(1+\nu)} \sigma_{kk,ijpp} = 0_{ij}$ $\Rightarrow \sigma_{ij,kkpp} + \dfrac{1}{(1+\nu)} (\underbrace{\sigma_{kk,pp}}_{=0})_{,ij} = 0_{ij}$ $\Rightarrow \sigma_{ij,kkpp} = \dfrac{\partial^2}{\partial x_k \partial x_k} \left(\dfrac{\partial^2 (\sigma_{ij})}{\partial x_p \partial x_p} \right) = 0$ $\Rightarrow \nabla^2_{\vec{x}} \nabla^2_{\vec{x}} (\sigma_{ij}) \equiv \nabla^4_{\vec{x}} (\sigma_{ij}) = 0_{ij}$	(5.135)

With that we show that the Cauchy stress tensor is biharmonic function, where the operator $\nabla^4_{\vec{x}} \equiv \nabla^2_{\vec{x}} \nabla^2_{\vec{x}}$ is known as the bilaplacian. We can show that the infinitesimal strain tensor is also biharmonic function, i.e.: $\nabla^4_{\vec{x}} \boldsymbol{\varepsilon} = \mathbf{0}$. Taking into account the above equation and the constitutive equation in stress for isotropic linear elastic material $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}$ we obtain:

$$\nabla^4_{\vec{x}}\boldsymbol{\sigma} = \mathbf{0}$$
$$\Rightarrow \nabla^4_{\vec{x}}(\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}) = \nabla^4_{\vec{x}}(\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}) + \nabla^4_{\vec{x}}(2\mu\boldsymbol{\varepsilon}) = \underbrace{\nabla^4_{\vec{x}}(\mathrm{Tr}(\boldsymbol{\varepsilon}))}_{=0}\lambda\mathbf{1} + 2\mu\nabla^4_{\vec{x}}(\boldsymbol{\varepsilon}) = \mathbf{0}$$
$$\Rightarrow \nabla^4_{\vec{x}}\boldsymbol{\varepsilon} = \mathbf{0}$$

To show that the displacement components are biharmonic function, we will start with the Navier's equations (5.49) for a static case, $(\lambda+\mu)\mathrm{u}_{j,ji} + \mu\mathrm{u}_{i,jj} + \rho\mathrm{b}_i = \rho\ddot{\mathrm{u}}_i = 0_i$, and if we apply the Laplacian to it we can obtain:

$$(\lambda+\mu)\mathrm{u}_{j,jikk} + \mu\mathrm{u}_{i,jjkk} + \underbrace{(\rho\mathrm{b}_i)_{,kk}}_{=0_i} = 0_i \tag{5.136}$$

where we have considered that λ, μ and $(\rho\mathrm{b}_i)$ do not change with $\vec{x}$. Note also that

$$\mathrm{u}_{i,jjk} = \mathrm{u}_{i,kjj} = \nabla^2_{\vec{x}}(\mathrm{u}_{i,k}) = \{\nabla^2_{\vec{x}}[\nabla_{\vec{x}}\vec{\mathbf{u}}]\}_{ik}$$

and if we consider the infinitesimal strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}(\mathrm{u}_{i,j} + \mathrm{u}_{j,i}) \xrightarrow{trace} \varepsilon_{kk} = \frac{1}{2}(\mathrm{u}_{k,k} + \mathrm{u}_{k,k}) = \mathrm{u}_{k,k}$$
$$\Rightarrow \varepsilon_{kk,ij} = \mathrm{u}_{k,kij} \xrightarrow{trace} \varepsilon_{kk,pp} = \mathrm{u}_{k,kpp} = 0 \xrightarrow{tensorial} \nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\varepsilon})] = \nabla^2_{\vec{x}}[\,\mathrm{Tr}(\nabla_{\vec{x}}\vec{\mathbf{u}})\,] = 0$$

where we have used the equation (5.134). Then, the equation in (5.136) can be written as follows:

$$(\lambda+\mu)\mathrm{u}_{j,jikk} + \mu\mathrm{u}_{i,jjkk} = 0_i \quad \Rightarrow \quad (\lambda+\mu)(\underbrace{\mathrm{u}_{j,jkk}}_{=0})_{,i} + \mu\mathrm{u}_{i,jjkk} = 0_i$$
$$\Rightarrow \mathrm{u}_{i,jjkk} = \nabla^4_{\vec{x}}\mathrm{u}_i = 0_i \quad \Rightarrow \quad \begin{cases} \nabla^4_{\vec{x}}\mathrm{u}_1 = 0 \\ \nabla^4_{\vec{x}}\mathrm{u}_2 = 0 \\ \nabla^4_{\vec{x}}\mathrm{u}_3 = 0 \end{cases} \tag{5.137}$$

with that we show that the displacement components are biharmonic functions.

Problem 5.18

a) Given a scalar field Φ such as:

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2} \quad ; \quad \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} \quad ; \quad \sigma_{12} = \sigma_{21} = \frac{-\partial^2\Phi}{\partial x_1\,\partial x_2} \tag{5.138}$$

Show that

Indicial notation	*Tensorial notation*
$\Phi_{,iijj} = 0 \quad (i,j=1,2)$	$\nabla\cdot\{\nabla[\nabla\cdot(\nabla\Phi)]\} = 0$
$\Rightarrow \frac{\partial^4\Phi}{\partial x_1^4} + 2\frac{\partial^4\Phi}{\partial x_1^2 x_2^2} + \frac{\partial^4\Phi}{\partial x_2^4} = 0$	$\Rightarrow \nabla^2\nabla^2\Phi = 0$
	$\Rightarrow \nabla^4\Phi = 0$

(5.139)

Consider a linear elastic material, a static problem, and with no body forces. Consider also that the Cauchy stress tensor is only dependent of x_1 and x_2, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2)$.

b) Show whether the equilibrium equations are satisfied or not.

Solution:

a) In **Problem 5.16**, (see equation (5.130)), we have shown that:

$$\sigma_{ii,kk} = \frac{-(1+\nu)}{(1-\nu)}\left[(\rho \mathrm{b}_k)_{,k}\right] = 0$$

where we have considered that $(\rho \mathrm{b}_k)_{,k} = 0$. For the proposed problem we have $i,k = 1,2$, with which:

$$\sigma_{ii,kk} = 0 \quad \Rightarrow \quad \sigma_{ii,11} + \sigma_{ii,22} = 0 \quad \Rightarrow \quad \sigma_{11,11} + \sigma_{22,11} + \sigma_{11,22} + \sigma_{22,22} = 0$$
$$\Rightarrow \frac{\partial^2 \sigma_{11}}{\partial x_1^2} + \frac{\partial^2 \sigma_{22}}{\partial x_1^2} + \frac{\partial^2 \sigma_{11}}{\partial x_2^2} + \frac{\partial^2 \sigma_{22}}{\partial x_2^2} = 0$$

Using the definition given by (5.138), we can conclude that:

$$\frac{\partial^2 \sigma_{11}}{\partial x_1^2} + \frac{\partial^2 \sigma_{22}}{\partial x_1^2} + \frac{\partial^2 \sigma_{11}}{\partial x_2^2} + \frac{\partial^2 \sigma_{22}}{\partial x_2^2} = 0$$
$$\Rightarrow \frac{\partial^2}{\partial x_1^2}\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2}{\partial x_1^2}\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2}{\partial x_2^2}\frac{\partial^2 \Phi}{\partial x_1^2} = 0$$
$$\Rightarrow \frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0$$

Q.E.D.

b) For the bidimensional case (2D), the equilibrium equations (without body forces) reduce to:

$$\sigma_{ij,j} = 0_i \xrightarrow{(i,j=1,2)} \sigma_{i1,1} + \sigma_{i2,2} = 0_i \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} = 0 \end{cases}$$

Using the definition (5.138), we can obtain:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial}{\partial x_1}\dfrac{\partial^2 \Phi}{\partial x_2^2} - \dfrac{\partial}{\partial x_2}\dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} = \dfrac{\partial^3 \Phi}{\partial x_1 \partial x_2^2} - \dfrac{\partial^3 \Phi}{\partial x_1 \partial x_2^2} = 0 \quad \checkmark \\ -\dfrac{\partial}{\partial x_1}\dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} + \dfrac{\partial}{\partial x_2}\dfrac{\partial^2 \Phi}{\partial x_1^2} = -\dfrac{\partial^3 \Phi}{\partial x_1^2 \partial x_2} + \dfrac{\partial^3 \Phi}{\partial x_2 \partial x_1^2} = 0 \quad \checkmark \end{cases}$$

With this, we show that the expressions for stresses given by (5.138) satisfy the equilibrium equations.

NOTE: In the literature, Φ is known as the *Airy stress function*, (see **Problem 6.34**), and the SI unit of Φ is $[\Phi] = N(Newton)$.

Problem 5.19

Let us consider that the Cauchy stress tensor field can be obtained as follows:

$$\boldsymbol{\sigma} = \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})^T \qquad [\boldsymbol{\sigma}] = \frac{N}{m^2} = Pa(Pascal) \qquad (5.140)$$

where the second-order tensor $\boldsymbol{P}$ has the following Cartesian components:

$$P_{ij} = \begin{bmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{bmatrix} \qquad \therefore \quad [\boldsymbol{P}] = N \quad (Newton)$$

a) Obtain the explicit components of the stress tensor in function of χ_i. b) Check whether the body is in equilibrium by considering the static state and without body force.

Solution:

a) In **Problem 1.110** we have shown that the following is true:

$$\sigma_{qt} = [\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})^T]_{qt} = \epsilon_{qjk}\epsilon_{til} P_{ij,kl}$$

Note also that the explicit equations for σ_{qt}, (given by (5.140)), have the same structure as the one used to obtain the components S_{ij} in equation (5.80), (see **Problem 5.11**), so

$$\begin{cases} \sigma_{11} = \dfrac{\partial^2 P_{33}}{\partial x_2^2} + \dfrac{\partial^2 P_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 P_{23}}{\partial x_2 \partial x_3} \\ \sigma_{22} = \dfrac{\partial^2 P_{33}}{\partial x_1^2} + \dfrac{\partial^2 P_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 P_{13}}{\partial x_1 \partial x_3} \\ \sigma_{33} = \dfrac{\partial^2 P_{11}}{\partial x_2^2} + \dfrac{\partial^2 P_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 P_{12}}{\partial x_1 \partial x_2} \\ \sigma_{12} = \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial P_{23}}{\partial x_1} + \dfrac{\partial P_{13}}{\partial x_2} - \dfrac{\partial P_{12}}{\partial x_3}\right) - \dfrac{\partial^2 P_{33}}{\partial x_1 \partial x_2} \\ \sigma_{23} = \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial P_{23}}{\partial x_1} + \dfrac{\partial P_{13}}{\partial x_2} + \dfrac{\partial P_{12}}{\partial x_3}\right) - \dfrac{\partial^2 P_{11}}{\partial x_2 \partial x_3} \\ \sigma_{13} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial P_{23}}{\partial x_1} - \dfrac{\partial P_{13}}{\partial x_2} + \dfrac{\partial P_{12}}{\partial x_3}\right) - \dfrac{\partial^2 P_{22}}{\partial x_1 \partial x_3} \end{cases} \tag{5.141}$$

Taking into account that $P_{11} = \chi_1$, $P_{22} = \chi_2$, $P_{33} = \chi_3$ and $P_{12} = P_{23} = P_{13} = 0$, the stress components become:

$$\sigma_{qt} = \begin{bmatrix} \left(\dfrac{\partial^2 \chi_3}{\partial x_2^2} + \dfrac{\partial^2 \chi_2}{\partial x_3^2}\right) & -\dfrac{\partial^2 \chi_3}{\partial x_1 \partial x_2} & -\dfrac{\partial^2 \chi_2}{\partial x_1 \partial x_3} \\ -\dfrac{\partial^2 \chi_3}{\partial x_1 \partial x_2} & \left(\dfrac{\partial^2 \chi_3}{\partial x_1^2} + \dfrac{\partial^2 \chi_1}{\partial x_3^2}\right) & -\dfrac{\partial^2 \chi_1}{\partial x_2 \partial x_3} \\ -\dfrac{\partial^2 \chi_2}{\partial x_1 \partial x_3} & -\dfrac{\partial^2 \chi_1}{\partial x_2 \partial x_3} & \left(\dfrac{\partial^2 \chi_1}{\partial x_2^2} + \dfrac{\partial^2 \chi_2}{\partial x_1^2}\right) \end{bmatrix} \tag{5.142}$$

b) We start from the equations of motion $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{v}} = \rho\ddot{\vec{\mathbf{u}}}$, (see equation (5.14)), in indicial notation $\sigma_{ij,j} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i$ and by considering the static state ($\ddot{\mathrm{u}}_i = 0_i$) and without body force ($\mathrm{b}_i = 0_i$) the equations of motion become the equilibrium equations, namely:

$$\sigma_{ij,j} = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0_i$$

$$\Rightarrow \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3_3} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0 \end{cases}$$

And by substituting the stress components given by Eq. (5.142) we can conclude that

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial^2 \chi_3}{\partial x_2^2} + \dfrac{\partial^2 \chi_2}{\partial x_3^2}\right) - \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2 \chi_3}{\partial x_1 \partial x_2}\right) - \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial^2 \chi_2}{\partial x_1 \partial x_3}\right) = 0 \quad \checkmark \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = -\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial^2 \chi_3}{\partial x_1 \partial x_2}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2 \chi_3}{\partial x_1^2} + \dfrac{\partial^2 \chi_1}{\partial x_3^2}\right) - \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial^2 \chi_1}{\partial x_2 \partial x_3}\right) = 0 \quad \checkmark \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = -\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial^2 \chi_2}{\partial x_1 \partial x_3}\right) - \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2 \chi_1}{\partial x_2 \partial x_3}\right) + \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial^2 \chi_1}{\partial x_2^2} + \dfrac{\partial^2 \chi_2}{\partial x_1^2}\right) = 0 \quad \checkmark \end{cases}$$

NOTE 1: In the literature χ_i are known as *stress functions*. In the particular case when $\chi_3 = \Phi$ and $\chi_1 = \chi_2 = 0$ we fall back into the two-dimensional problem discussed in **Problem 5.18**, where $\Phi = \Phi(x_1, x_2)$ is the *Airy stress function*. In this case the Eq. (5.142) becomes:

$$\sigma_{qt} = \begin{bmatrix} \dfrac{\partial^2 \Phi}{\partial x_2^2} & -\dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} & 0 \\ -\dfrac{\partial^2 \Phi}{\partial x_1 \partial x_2} & \dfrac{\partial^2 \Phi}{\partial x_1^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

NOTE 2: Note that the stress field can also be expressed by other stress function ξ, (see Love (1944)). In this case the $\boldsymbol{P}$-components are:

$$P_{ij} = \frac{-1}{2}\begin{bmatrix} 0 & \xi_3 & \xi_2 \\ \xi_3 & 0 & \xi_1 \\ \xi_2 & \xi_1 & 0 \end{bmatrix} \qquad \therefore \quad [\boldsymbol{P}] = N \quad (Newton)$$

By substituting these components into the equation in (5.141) we can obtain:

$$\sigma_{qt} = \begin{bmatrix} \dfrac{\partial^2 \xi_1}{\partial x_2 \partial x_3} & \dfrac{-1}{2}\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \xi_1}{\partial x_1} + \dfrac{\partial \xi_2}{\partial x_2} - \dfrac{\partial \xi_3}{\partial x_3}\right) & \dfrac{-1}{2}\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \xi_1}{\partial x_1} - \dfrac{\partial \xi_2}{\partial x_2} + \dfrac{\partial \xi_3}{\partial x_3}\right) \\ \dfrac{-1}{2}\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \xi_1}{\partial x_1} + \dfrac{\partial \xi_2}{\partial x_2} - \dfrac{\partial \xi_3}{\partial x_3}\right) & \dfrac{\partial^2 \xi_2}{\partial x_1 \partial x_3} & \dfrac{-1}{2}\dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \xi_1}{\partial x_1} + \dfrac{\partial \xi_2}{\partial x_2} + \dfrac{\partial \xi_3}{\partial x_3}\right) \\ \dfrac{-1}{2}\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \xi_1}{\partial x_1} - \dfrac{\partial \xi_2}{\partial x_2} + \dfrac{\partial \xi_3}{\partial x_3}\right) & \dfrac{-1}{2}\dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \xi_1}{\partial x_1} + \dfrac{\partial \xi_2}{\partial x_2} + \dfrac{\partial \xi_3}{\partial x_3}\right) & \dfrac{\partial^2 \xi_3}{\partial x_1 \partial x_2} \end{bmatrix} \tag{5.143}$$

We leave the reader to check whether the equilibrium equations are satisfied or not.

Let us suppose that $\xi_3 = \xi_3(x_2, x_3)$ and $\xi_1 = \xi_2 = 0$, with that the stress field becomes:

$$\sigma_{qt} = \begin{bmatrix} 0 & \frac{-1}{2}\frac{\partial}{\partial x_3}\left(-\frac{\partial \xi_3}{\partial x_3}\right) & \frac{-1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial \xi_3}{\partial x_3}\right) \\ \frac{-1}{2}\frac{\partial}{\partial x_3}\left(-\frac{\partial \xi_3}{\partial x_3}\right) & 0 & 0 \\ \frac{-1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial \xi_3}{\partial x_3}\right) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial \phi}{\partial x_3} & -\frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} & 0 & 0 \\ -\frac{\partial \phi}{\partial x_2} & 0 & 0 \end{bmatrix}$$

where we have considered that $\phi = \phi(x_2, x_3) = \frac{1}{2}\frac{\partial \xi_3}{\partial x_3}$. In the literature ϕ is known as the *Prandtl's stress function*, (see 6.4 Introduction to Torsion – **Problem 6.44** – NOTE 2).

Additional NOTE: The components of $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})$ (not symmetric) were obtained in **Problem 1.110** and are given by:

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})]_{tj} = P_{sj,ts} - P_{tj,ss}$$

then

$$\begin{aligned}
&P_{sj,ts} - P_{tj,ss} = P_{1j,t1} + P_{2j,t2} + P_{3j,t3} - (P_{tj,11} + P_{tj,22} + P_{tj,33}) \\
&(t=1, j=1) \quad \Rightarrow \quad P_{11,11} + P_{21,12} + P_{31,13} - (P_{11,11} + P_{11,22} + P_{11,33}) = -(\chi_{1,22} + \chi_{1,33}) \\
&(t=2, s=2) \quad \Rightarrow \quad P_{12,21} + P_{22,22} + P_{32,23} - (P_{22,11} + P_{22,22} + P_{22,33}) = -(\chi_{2,11} + \chi_{2,33}) \\
&(t=3, j=3) \quad \Rightarrow \quad P_{13,31} + P_{23,32} + P_{33,33} - (P_{33,11} + P_{33,22} + P_{33,33}) = -(\chi_{3,11} + \chi_{3,22}) \\
&(t=1, j=2) \quad \Rightarrow \quad P_{12,11} + P_{22,12} + P_{32,13} - (P_{12,11} + P_{12,22} + P_{12,33}) = \chi_{2,12} \\
&(t=2, j=3) \quad \Rightarrow \quad P_{13,21} + P_{23,22} + P_{33,23} - (P_{23,11} + P_{23,22} + P_{23,33}) = \chi_{3,23} \\
&(t=1, j=3) \quad \Rightarrow \quad P_{13,11} + P_{23,12} + P_{33,13} - (P_{13,11} + \chi_{13,22} + P_{13,33}) = \chi_{3,13} \\
&\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

Thus

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})]_{tj} = \begin{bmatrix} -\left(\frac{\partial^2 \chi_1}{\partial x_2^2} + \frac{\partial^2 \chi_1}{\partial x_3^2}\right) & \frac{\partial^2 \chi_2}{\partial x_1 \partial x_2} & \frac{\partial^2 \chi_3}{\partial x_1 \partial x_3} \\ \frac{\partial^2 \chi_1}{\partial x_1 \partial x_2} & -\left(\frac{\partial^2 \chi_2}{\partial x_1^2} + \frac{\partial^2 \chi_2}{\partial x_3^2}\right) & \frac{\partial^2 \chi_3}{\partial x_2 \partial x_3} \\ \frac{\partial^2 \chi_1}{\partial x_1 \partial x_3} & \frac{\partial^2 \chi_2}{\partial x_2 \partial x_3} & -\left(\frac{\partial^2 \chi_3}{\partial x_1^2} + \frac{\partial^2 \chi_3}{\partial x_2^2}\right) \end{bmatrix}$$

Note that

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})^T] \neq [\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})]$$

In the case when $\boldsymbol{P}$ is a spherical tensor, e.g. $\boldsymbol{P} = \alpha \mathbf{1}$, in which $\alpha = \alpha(x_1, x_2, x_3)$, we can conclude that

$$\boldsymbol{P} = \alpha \mathbf{1} \quad \Rightarrow \quad [\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})^T] = -[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \boldsymbol{P})]$$

Problem 5.20

Consider the governing equation for the linear elastic problem described in **Problem 5.5**. Obtain an equivalent formulation such as the unknowns are displacement $\vec{\mathbf{u}}$ and stress $\boldsymbol{\sigma}$ *(Mixed Formulation)*. Use Voigt notation.

Solution:

Taking into account the governing equations for the elastic linear problem:

Tensorial notation	Voigt notation
The equations of motion: $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \ddot{\vec{\mathbf{u}}}$ (3 equations)	*The equations of motion:* $[\boldsymbol{L}^{(1)}]^T \{\boldsymbol{\sigma}\} + \{\rho \boldsymbol{b}\} = \{\rho \ddot{\boldsymbol{u}}\}$ (3 equations)
The constitutive equations for stress: $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \mathbb{C}^e : \boldsymbol{\varepsilon}$ (6 equations)	*The constitutive equations for stress:* $\{\boldsymbol{\sigma}\} = [\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}\}$ (6 equations)
The kinematic equations: $\boldsymbol{\varepsilon} = \nabla^{sym}_{\vec{x}} \vec{\mathbf{u}}$ (6 equations)	*The kinematic equations:* $\{\boldsymbol{\varepsilon}\} = [\boldsymbol{L}^{(1)}]\ \{\boldsymbol{u}\}$ (6 equations)

(5.144)

where the equations in Voigt notation were obtained in **Problem 5.8**, where

$$[\boldsymbol{L}^{(1)}]^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}$$

To eliminate the strain from the governing equations, we replace the kinematic equation into the constitutive equations for stress, i.e.:

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}\} \quad \Rightarrow \quad \{\boldsymbol{\sigma}\} = [\boldsymbol{\mathcal{C}}][\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\}$$

$$\Rightarrow [\boldsymbol{\mathcal{C}}]^{-1}\{\boldsymbol{\sigma}\} = \underbrace{[\boldsymbol{\mathcal{C}}]^{-1}[\boldsymbol{\mathcal{C}}]}_{=[\mathit{1}]}[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\}$$

$$\Rightarrow [\boldsymbol{\mathcal{C}}]^{-1}\{\boldsymbol{\sigma}\} - [\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\} = \{\boldsymbol{0}\}$$

whereby the system (5.144) becomes:

$$\begin{cases} [\boldsymbol{L}^{(1)}]^T \{\boldsymbol{\sigma}\} + \{\rho \boldsymbol{b}\} = \{\rho \ddot{\boldsymbol{u}}\} \\ [\boldsymbol{\mathcal{C}}]^{-1}\{\boldsymbol{\sigma}\} - [\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\} = \{\boldsymbol{0}\} \end{cases}$$

which is also possible to write the above set of equations as follows:

$$\begin{bmatrix} [\boldsymbol{0}] & [\boldsymbol{L}^{(1)}]^T \\ -[\boldsymbol{L}^{(1)}] & [\boldsymbol{\mathcal{C}}]^{-1} \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{u}\} \\ \{\boldsymbol{\sigma}\} \end{Bmatrix} = \begin{Bmatrix} -\{\rho \boldsymbol{b}\} + \{\rho \ddot{\boldsymbol{u}}\} \\ \{\boldsymbol{0}\} \end{Bmatrix}$$

NOTE 1: The above formulation is known as *Mixed Formulation*. It is interesting to note that in the formulations either in displacement or in stress, (see **Problem 5.9** and **Problem 5.16**), we have second derivative of the unknowns, meanwhile in the mixed formulation we deal only with the first derivative of the unknowns, and moreover this formulation does not deal with the derivative of the mechanical properties.

NOTE 2: We can summarize that the linear elastic problem, considering an isotropic homogenous linear elastic material, is governed by the set of partial differential equations:

Tensorial notation	Voigt notation
The equations of motion: $\nabla\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\rho\dot{\vec{v}}=\rho\ddot{\vec{\mathbf{u}}}$ (3 equations)	*The equations of motion:* $[\boldsymbol{L}^{(1)}]^T\{\boldsymbol{\sigma}\}+\{\rho\boldsymbol{b}\}=\{\rho\ddot{\boldsymbol{u}}\}$ (3 equations)
The constitutive equations for stress: $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})=\mathbb{C}^e:\boldsymbol{\varepsilon}$ (6 equations)	*The constitutive equations for stress:* $\{\boldsymbol{\sigma}\}=[\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}\}$ (6 equations)
The kinematic equations: $\boldsymbol{\varepsilon}=\nabla^{sym}\vec{\mathbf{u}}$ (6 equations)	*The kinematic equations:* $\{\boldsymbol{\varepsilon}\}=[\boldsymbol{L}^{(1)}]\ \{\boldsymbol{u}\}$ (6 equations)

(5.145)

making a total of 15 equations and 15 unknowns, namely $(\mathrm{u}_i,\sigma_{ij},\varepsilon_{ij})$.

This set of equations can also be represented by:

1) Displacement Formulation, (see **Problem 5.9**):

$$\boxed{\begin{gathered}(\lambda+\mu)\mathrm{u}_{j,ji}+\mu\mathrm{u}_{i,jj}+\rho\mathrm{b}_i=\rho\ddot{\mathrm{u}}_i\\[2ex](\lambda+\mu)[\nabla(\nabla\cdot\vec{\mathbf{u}})]+\mu[\nabla\cdot(\nabla\vec{\mathbf{u}})]+\rho\vec{\mathbf{b}}=\rho\ddot{\vec{\mathbf{u}}}\end{gathered}}\quad\textit{Navier's equations}\tag{5.146}$$

in which we have 3 equations and 3 unknowns (u_i).

2) Stress Formulation, (see **Problem 5.16**):

Indicial notation

$$\boxed{\sigma_{ij,kk}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\sigma_{kk,ij}-\frac{\lambda}{(2\mu+3\lambda)}\sigma_{ll,kk}\delta_{ij}=2\left[(\rho\ddot{\mathrm{u}}_i)_{,j}\right]^{sym}-2\left[(\rho\mathrm{b}_i)_{,j}\right]^{sym}}$$

Tensorial notation

$$\boxed{\nabla^2_{\vec{x}}\boldsymbol{\sigma}+\frac{2(\lambda+\mu)}{(2\mu+3\lambda)}\nabla_{\vec{x}}[\nabla_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]]-\frac{\lambda}{(2\mu+3\lambda)}\nabla^2_{\vec{x}}[\mathrm{Tr}(\boldsymbol{\sigma})]\mathbf{1}=2\left[\nabla_{\vec{x}}(\rho\ddot{\vec{\mathbf{u}}})\right]^{sym}-2\left[\nabla_{\vec{x}}(\rho\vec{\mathbf{b}})\right]^{sym}}$$

(5.147)

in which we have 6 equations and 6 unknowns (σ_{ij}).

3) Mixed Formulation, (see **Problem 5.20**):

$$\begin{cases}[\boldsymbol{L}^{(1)}]^T\{\boldsymbol{\sigma}\}+\{\rho\boldsymbol{b}\}=\{\rho\ddot{\boldsymbol{u}}\} & \text{(3 equations)}\\ [\boldsymbol{\mathcal{C}}]^{-1}\{\boldsymbol{\sigma}\}-[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}\}=\{\boldsymbol{0}\} & \text{(6 equations)}\end{cases}\tag{5.148}$$

in which we have 9 equations and 9 unknowns $(\mathrm{u}_i,\sigma_{ij})$.

Problem 5.21

Let us consider two systems made up by the same linear elastic material but with different load conditions as indicated in Figure 5.14.

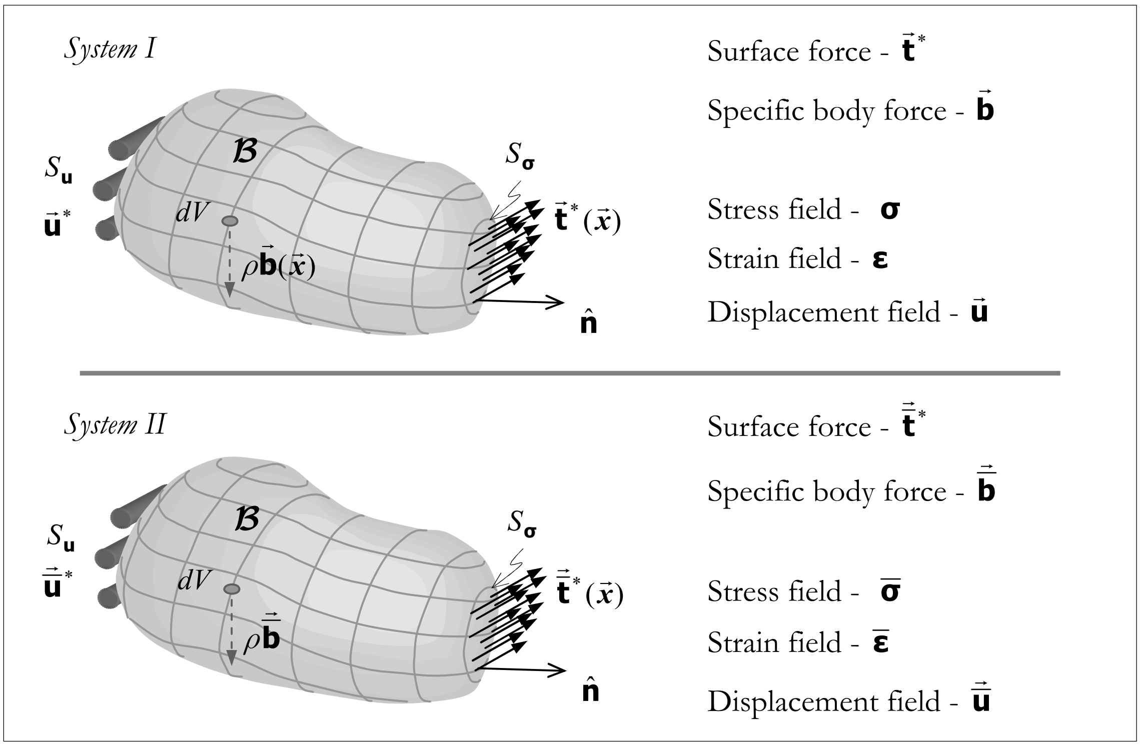


Figure 5.14: Two systems under external actions.

Show the Betti's theorem also known as Betti's reciprocal theorem:

$$\boxed{\int_V \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} \, dV = \int_V \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}} dV} \quad \textit{Betti's theorem} \tag{5.149}$$

Solution:

Taking into account the constitutive equation for stress, $\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}$, in indicial notation:

$$\sigma_{ij} = \mathbb{C}^e_{ijkl}\varepsilon_{kl}$$

And by multiplying both sides of the equation by the field $\bar{\boldsymbol{\varepsilon}}$ we can obtain:

$$\sigma_{ij}\bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij}\mathbb{C}^e_{ijkl}\varepsilon_{kl} \xrightarrow{\text{Major Simmetry of } \mathbb{C}} \sigma_{ij}\bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij}\mathbb{C}^e_{ijkl}\varepsilon_{kl} = \varepsilon_{kl}\mathbb{C}^e_{klij}\bar{\varepsilon}_{ij}$$

where we have applied the major symmetry of the elasticity tensor ($\mathbb{C}^e_{ijkl} = \mathbb{C}^e_{klij}$). Since the both systems are made up by the same material the relationship $\bar{\boldsymbol{\sigma}} = \mathbb{C}^e : \bar{\boldsymbol{\varepsilon}}$ holds. With that the above equation becomes:

$$\sigma_{ij}\bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij}\mathbb{C}^e_{ijkl}\varepsilon_{kl} = \varepsilon_{kl}\mathbb{C}^e_{klij}\bar{\varepsilon}_{ij} = \varepsilon_{kl}\bar{\sigma}_{kl} \xrightarrow{\text{Tensorial notation}} \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}}$$

If now we integrate over the whole volume we can obtain the Betti's theorem:

$$\int_V \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} \, dV = \int_V \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}} \, dV \tag{5.150}$$

NOTE 1: The above equation is only valid if $\mathbb{C}^e_{ijkl} = \mathbb{C}^e_{klij}$ holds, i.e. if $\mathbb{C}^e$ has major symmetry. In other words, the condition $\mathbb{C}^e_{ijkl} = \mathbb{C}^e_{klij}$ enforces the existence of the stored-energy function (Ψ^e), such as:

$$\mathbb{C}^e_{ijkl} = \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon})}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = \mathbb{C}^e_{klij}$$

NOTE 2: The Betti's theorem is the start point to obtain the formulation of the *Boundary Element Method.*

NOTE 3: The Betti's theorem can also be expressed in another form which we show below.

Recall that $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \frac{1}{2}(\mathrm{u}_{i,j} + \mathrm{u}_{j,i})$, which is also valid for the *system II*, i.e. $\bar{\varepsilon}_{ij} = \frac{1}{2}(\bar{\mathrm{u}}_{i,j} + \bar{\mathrm{u}}_{j,i})$. Then:

$$\begin{gathered}
\int_V \bar{\sigma}_{ij} \varepsilon_{ij} dV = \int_V \sigma_{ij} \bar{\varepsilon}_{ij} dV \\
\frac{1}{2}\int_V \bar{\sigma}_{ij}(\mathrm{u}_{i,j} + \mathrm{u}_{j,i}) dV = \frac{1}{2}\int_V \sigma_{ij}(\bar{\mathrm{u}}_{i,j} + \bar{\mathrm{u}}_{j,i}) dV \\
\int_V \bar{\sigma}_{ij} \mathrm{u}_{i,j} dV = \int_V \sigma_{ij} \bar{\mathrm{u}}_{i,j} dV
\end{gathered} \tag{5.151}$$

where $\bar{\sigma}_{ij}\mathrm{u}_{i,j} = \bar{\sigma}_{ij}\mathrm{u}_{j,i}$ and $\sigma_{ij}\bar{\mathrm{u}}_{i,j} = \sigma_{ij}\bar{\mathrm{u}}_{j,i}$ hold due to the symmetry of $\bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$, respectively. Also note that:

$$\begin{aligned}
(\bar{\sigma}_{ij}\mathrm{u}_i)_{,j} &= \bar{\sigma}_{ij,j}\mathrm{u}_i + \bar{\sigma}_{ij}\mathrm{u}_{i,j} \quad \Rightarrow \quad \bar{\sigma}_{ij}\mathrm{u}_{i,j} = (\bar{\sigma}_{ij}\mathrm{u}_i)_{,j} - \bar{\sigma}_{ij,j}\mathrm{u}_i \\
(\sigma_{ij}\bar{\mathrm{u}}_i)_{,j} &= \sigma_{ij,j}\bar{\mathrm{u}}_i + \sigma_{ij}\bar{\mathrm{u}}_{i,j} \quad \Rightarrow \quad \sigma_{ij}\bar{\mathrm{u}}_{i,j} = (\sigma_{ij}\bar{\mathrm{u}}_i)_{,j} - \sigma_{ij,j}\bar{\mathrm{u}}_i
\end{aligned}$$

With that the equation in (5.151) becomes:

$$\begin{aligned}
&\int_V \bar{\sigma}_{ij}\mathrm{u}_{i,j} dV = \int_V \sigma_{ij}\bar{\mathrm{u}}_{i,j} dV \\
&\int_V (\bar{\sigma}_{ij}\mathrm{u}_i)_{,j} - \bar{\sigma}_{ij,j}\mathrm{u}_i dV = \int_V (\sigma_{ij}\bar{\mathrm{u}}_i)_{,j} - \sigma_{ij,j}\bar{\mathrm{u}}_i dV \\
&\int_V (\bar{\sigma}_{ij}\mathrm{u}_i)_{,j}\, dV - \int_V \bar{\sigma}_{ij,j}\mathrm{u}_i dV = \int_V (\sigma_{ij}\bar{\mathrm{u}}_i)_{,j}\, dV - \int_V \sigma_{ij,j}\bar{\mathrm{u}}_i dV
\end{aligned} \tag{5.152}$$

Applying the divergence theorem to the first one integral on both sides of the equation, we can obtain:

$$\begin{aligned}
&\int_S \bar{\sigma}_{ij}\mathrm{u}_i\hat{\mathrm{n}}_j dS - \int_V \bar{\sigma}_{ij,j}\mathrm{u}_i dV = \int_S \sigma_{ij}\bar{\mathrm{u}}_i\hat{\mathrm{n}}_j dS - \int_V \sigma_{ij,j}\bar{\mathrm{u}}_i dV \\
\Rightarrow &\int_S \bar{\mathrm{t}}_i\,\mathrm{u}_i dS - \int_V \bar{\sigma}_{ij,j}\mathrm{u}_i dV = \int_S \mathrm{t}_i\,\bar{\mathrm{u}}_i dS - \int_V \sigma_{ij,j}\bar{\mathrm{u}}_i dV
\end{aligned} \tag{5.153}$$

where we have applied the definition $\bar{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} = \bar{\vec{\mathbf{t}}}$ and $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \vec{\mathbf{t}}$. The above equation in tensorial notation becomes:

$$\int_S \bar{\vec{\mathbf{t}}} \cdot \vec{\mathbf{u}} dS - \int_V (\nabla \cdot \bar{\boldsymbol{\sigma}}) \cdot \vec{\mathbf{u}} dV = \int_S \vec{\mathbf{t}} \cdot \bar{\vec{\mathbf{u}}} dS - \int_V (\nabla \cdot \boldsymbol{\sigma}) \cdot \bar{\vec{\mathbf{u}}} dV \tag{5.154}$$

If we resort to the equations of motion, it is satisfied that:

$$\nabla \cdot \bar{\boldsymbol{\sigma}} + \rho \vec{\bar{\mathbf{b}}} = \rho \ddot{\vec{\bar{\mathbf{u}}}} \quad \Rightarrow \quad -\nabla \cdot \bar{\boldsymbol{\sigma}} = \rho(\vec{\bar{\mathbf{b}}} - \ddot{\vec{\bar{\mathbf{u}}}})$$

$$\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \ddot{\vec{\mathbf{u}}} \quad \Rightarrow \quad -\nabla \cdot \boldsymbol{\sigma} = \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}})$$

with that the equation in (5.154) becomes:

$$\boxed{\int_S \vec{\bar{\mathbf{t}}} \cdot \vec{\mathbf{u}} dS + \int_V \rho(\vec{\bar{\mathbf{b}}} - \ddot{\vec{\bar{\mathbf{u}}}}) \cdot \vec{\mathbf{u}} dV = \int_S \vec{\mathbf{t}} \cdot \vec{\bar{\mathbf{u}}} dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \vec{\bar{\mathbf{u}}} dV} \quad \textit{Betti's theorem} \tag{5.155}$$

Note that, if we consider $S = S_{\mathbf{u}} + S_{\boldsymbol{\sigma}}$ we have:

$$\begin{aligned} \int_S \vec{\bar{\mathbf{t}}} \cdot \vec{\mathbf{u}} dS &= \int_{S_{\boldsymbol{\sigma}}} \vec{\bar{\mathbf{t}}}^* \cdot \vec{\mathbf{u}} dS + \int_{S_{\mathbf{u}}} \vec{\bar{\mathbf{t}}} \cdot \vec{\mathbf{u}}^* dS \\ \int_S \vec{\mathbf{t}} \cdot \vec{\bar{\mathbf{u}}} dS &= \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\bar{\mathbf{u}}} dS + \int_{S_{\mathbf{u}}} \vec{\mathbf{t}} \cdot \vec{\bar{\mathbf{u}}}^* dS \end{aligned} \tag{5.156}$$

For the particular case when the system is in equilibrium and in the absence of body force, the equation (5.155) becomes:

$$\int_S \vec{\bar{\mathbf{t}}} \cdot \vec{\mathbf{u}} dS = \int_S \vec{\mathbf{t}} \cdot \vec{\bar{\mathbf{u}}} dS \tag{5.157}$$

In addition, if we have concentrated forces instead of surface force, the above equation becomes:

$$\bar{F}_i^{loc} \mathrm{u}_i^{loc} = F_i^{loc} \bar{\mathrm{u}}_i^{loc} \qquad \Big| \qquad \vec{\bar{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc} = \vec{F}^{loc} \cdot \vec{\bar{\mathbf{u}}}^{loc} \tag{5.158}$$

Problem 5.22

Let us consider two systems as described in Figure 5.14. Show the Principle of Virtual Work which states that:

$$\boxed{\underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\bar{\mathbf{u}}} dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \vec{\bar{\mathbf{u}}} dV}_{\text{Total external virtual work}} = \underbrace{\int_V \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}} \, dV}_{\text{Total internal virtual work}}} \quad \textit{Principle of Virtual Work} \tag{5.159}$$

where $\vec{\mathbf{u}} = \vec{\mathbf{u}}^*$ on $S_{\mathbf{u}}$ is known (prescribed).

Solution:

We can prove the Principle of Virtual Work by starting directly from the relationship:

$$\int_V \sigma_{ij} \bar{\varepsilon}_{ij} dV = \frac{1}{2} \int_V \sigma_{ij} (\bar{\mathrm{u}}_{i,j} + \bar{\mathrm{u}}_{j,i}) dV = \int_V \sigma_{ij} \bar{\mathrm{u}}_{i,j} dV \tag{5.160}$$

Note that $(\sigma_{ij} \bar{\mathrm{u}}_i)_{,j} = \sigma_{ij,j} \bar{\mathrm{u}}_i + \sigma_{ij} \bar{\mathrm{u}}_{i,j} \quad \Rightarrow \quad \sigma_{ij} \bar{\mathrm{u}}_{i,j} = (\sigma_{ij} \bar{\mathrm{u}}_i)_{,j} - \sigma_{ij,j} \bar{\mathrm{u}}_i$, thus:

$$\begin{aligned} &\int_V \sigma_{ij} \bar{\varepsilon}_{ij} dV = \int_V \sigma_{ij} \bar{\mathrm{u}}_{i,j} dV = \int_V (\sigma_{ij} \bar{\mathrm{u}}_i)_{,j} - \sigma_{ij,j} \bar{\mathrm{u}}_i dV \\ &\Rightarrow \int_V \sigma_{ij} \bar{\varepsilon}_{ij} dV = \int_V (\sigma_{ij} \bar{\mathrm{u}}_i)_{,j} \, dV - \int_V \sigma_{ij,j} \bar{\mathrm{u}}_i dV \end{aligned} \tag{5.161}$$

by applying the divergence theorem to the first volume integral on the right side of the equation, we can obtain:

$$\begin{aligned}\int_V \sigma_{ij}\bar{\varepsilon}_{ij}dV &= \int_V (\sigma_{ij}\bar{\mathsf{u}}_i)_{,j}\,dV - \int_V \sigma_{ij,j}\bar{\mathsf{u}}_i dV = \int_{S_{\boldsymbol{\sigma}}} \sigma_{ij}\bar{\mathsf{u}}_i\hat{\mathsf{n}}_j dS_\sigma - \int_V \sigma_{ij,j}\bar{\mathsf{u}}_i dV \\ &= \int_{S_{\boldsymbol{\sigma}}} \mathsf{t}_i^*\bar{\mathsf{u}}_i dS - \int_V \sigma_{ij,j}\bar{\mathsf{u}}_i dV\end{aligned} \tag{5.162}$$

where we have applied the definition $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \vec{\mathbf{t}}^*$. The above equation in tensorial notation becomes:

$$\int_V \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,dV = \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\vec{\bar{\mathbf{u}}}dS - \int_V (\nabla\cdot\boldsymbol{\sigma})\cdot\vec{\bar{\mathbf{u}}}dV \tag{5.163}$$

If we use the equations of motion we can obtain $\nabla\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\ddot{\vec{\mathbf{u}}} \Rightarrow -\nabla\cdot\boldsymbol{\sigma} = \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}})$, with that the equation in (5.163) becomes:

$$\underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\vec{\bar{\mathbf{u}}}dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}})\cdot\vec{\bar{\mathbf{u}}}dV}_{\text{Total external virtual work}} = \underbrace{\int_V \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,dV}_{\substack{\text{Total internal}\\ \text{virtual work}}}$$

which is known as the Principle of Virtual Work. Note that, for the demonstration, we have not used the major symmetry of $\mathbb{C}^e$.

For the particular case when the system is in equilibrium and in the absence of body force, the above equation becomes:

$$\int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\vec{\bar{\mathbf{u}}}(\vec{x})dS = \int_V \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,dV \tag{5.164}$$

In addition, if we have concentrated forces instead of surface force, the above equation becomes:

Tensorial notation	*Voigt notation*	
$\vec{\boldsymbol{F}}^{loc}\cdot\vec{\bar{\mathbf{u}}}^{loc} = \int_V \boldsymbol{\sigma}:\bar{\boldsymbol{\varepsilon}}\,dV$	$\{\boldsymbol{F}^{loc}\}^T\{\bar{\mathbf{u}}^{loc}\} = \int_V \{\boldsymbol{\sigma}\}^T\{\bar{\boldsymbol{\varepsilon}}\}\,dV$	(5.165)

where the direction of $\bar{\mathsf{u}}_i^{loc}$-component is the same as the F_i^{loc}-component direction, where $\{\boldsymbol{F}^{loc}\} = \{F_1, F_2, ..., F_n\}^T$, $\{\bar{\mathbf{u}}^{loc}\} = \{\mathsf{U}_1, \mathsf{U}_2, ..., \mathsf{U}_n\}^T$.

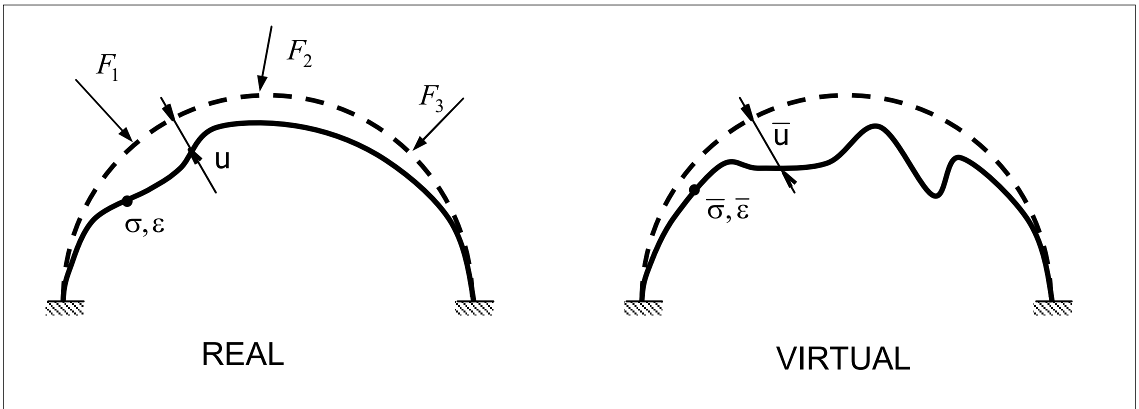


Figure 5.15

NOTE 1: The Principle of Virtual Work states: "A structure is in equilibrium, under a system of external forces, if and only if the *total external virtual work* equals the *total internal virtual work* for every virtual displacement field ($\vec{\bar{\mathbf{u}}}$)".

NOTE 2: The Principle of Virtual Work is used to discretization techniques of the problem such as the *Finite Element Technique*, in which the fundamental unknown is the displacement.

NOTE 3: It is easy to show that the equation in (5.159) is also valid for rate of change of virtual fields $\dot{\vec{\bar{\mathbf{u}}}}$, $\dot{\bar{\boldsymbol{\varepsilon}}}$, i.e.:

$$\boxed{\underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \dot{\vec{\bar{\mathbf{u}}}} dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \dot{\vec{\bar{\mathbf{u}}}} dV}_{\text{Total external virtual work}} = \underbrace{\int_V \boldsymbol{\sigma} : \dot{\bar{\boldsymbol{\varepsilon}}}\, dV}_{\text{Total internal virtual work}}} \quad \textit{Principle of Virtual Work} \tag{5.166}$$

Also it is valid for a variation of the virtual field $\delta\vec{\bar{\mathbf{u}}} \Rightarrow \delta\bar{\boldsymbol{\varepsilon}}$, i.e.:

$$\boxed{\underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \delta\vec{\bar{\mathbf{u}}} dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \delta\vec{\bar{\mathbf{u}}} dV}_{\text{Total external virtual work}} = \underbrace{\int_V \boldsymbol{\sigma} : \delta\bar{\boldsymbol{\varepsilon}}\, dV}_{\text{Total internal virtual work}}} \quad \textit{Principle of Virtual Work} \tag{5.167}$$

NOTE 4: We can also define the *Principle of complementary virtual work* as follows:

$$\boxed{\underbrace{\int_{S_{\mathbf{u}}} \vec{\bar{\mathbf{t}}} \cdot \vec{\mathbf{u}}^* dS + \int_V \rho(\vec{\bar{\mathbf{b}}} - \ddot{\vec{\bar{\mathbf{u}}}}) \cdot \vec{\mathbf{u}} dV}_{\text{Total external complementary virtual work}} = \underbrace{\int_V \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}\, dV}_{\text{Total internal complementary virtual work}}} \quad \textit{Principle of Complementary Virtual Work} \tag{5.168}$$

with $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \vec{\mathbf{t}}^*$ on $S_{\boldsymbol{\sigma}}$. Considering a static case without body forces and that the external action is characterized by concentrated forces, the principle of complementary virtual work becomes:

$$\underbrace{\vec{\bar{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc}}_{\substack{\text{Total external complementary virtual work} \\ \text{(due to concentrated forces)}}} = \underbrace{\int_V \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}\, dV}_{\text{Total internal complementary virtual work}} \quad \textit{Principle of Complementary Virtual Work (static case without body forces and with concentrated forces)} \tag{5.169}$$

NOTE 5: Note that, if we are using the Principle of Virtual Work the fundamental unknowns are displacements (strains), if we are using the Principles of Complementary Virtual Work the fundamental unknowns are forces (stresses), and if we are using the Betti's reciprocal theorem the fundamental unknowns are displacements and forces simultaneously, (see equation (5.154)).

Problem 5.23

Consider a sub-domain (Ω) made up by a homogeneous, isotropic linear elastic material. Consider also that at some points of the sub-domain boundary there are concentrated forces $\{\boldsymbol{F}^{(e)}\} \equiv \{\boldsymbol{F}^{loc}\}$, and that the displacement field into the sub-domain is approximated by $\{\mathbf{u}(\vec{x})\} = [\boldsymbol{N}(\vec{x})]\{\mathbf{u}^{(e)}\}$ where $\{\mathbf{u}^{(e)}\} \equiv \{\mathbf{u}^{loc}\}$ are the displacements at the points where concentrated forces are applied. Prove that the governing equations for a linear elastic problem in static equilibrium can be replaced by:

$$\boxed{\{\boldsymbol{F}^{(e)}\} = [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} \quad with \quad [\boldsymbol{K}^{(e)}] = \int_V [\mathbf{B}]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}]\, dV} \tag{5.170}$$

where $[\boldsymbol{\mathcal{C}}]$ is the elasticity tensor in Voigt notation, and obtain an expression for $[\mathbf{B}(\vec{x})]$.

Hint: Use the Principle of Virtual Work and use the same approximate used by $\{\mathbf{u}(\vec{x})\}$ to approach the virtual field $\{\overline{\mathbf{u}}(\vec{x})\}$.

Solution:

We can start directly from the equation in (5.165), which is equivalent to:

$$\vec{F} \cdot \vec{\overline{\mathbf{u}}} = \int_V \boldsymbol{\sigma} : \overline{\boldsymbol{\varepsilon}}\, dV = \int_V \boldsymbol{\sigma} : (\nabla^{sym} \vec{\overline{\mathbf{u}}})\, dV \quad \Rightarrow \quad \vec{\overline{\mathbf{u}}} \cdot \vec{F} = \int_V (\nabla^{sym} \vec{\overline{\mathbf{u}}}) : \boldsymbol{\sigma}\, dV \tag{5.171}$$

The above equation in Voigt notation becomes:

$$\vec{\overline{\mathbf{u}}} \cdot \vec{F} = \int_V (\nabla^{sym} \vec{\overline{\mathbf{u}}}) : \boldsymbol{\sigma}\, dV \xrightarrow{Voigt} \{\overline{\mathbf{u}}^{(e)}\}^T \{\boldsymbol{F}^{(e)}\} = \int_V \{\overline{\boldsymbol{\varepsilon}}\}^T \{\boldsymbol{\sigma}\}\, dV \tag{5.172}$$

Note that, the above equation is already considering the *equilibrium equations,* (see equations (5.163)-(5.165)). The *constitutive equations in stress*, in Voigt notation is given by $\{\boldsymbol{\sigma}(\vec{x})\} = [\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}(\vec{x})\}$, where the strain tensor field is given by the *kinematic equations* $\boldsymbol{\varepsilon}(\vec{x}) = \nabla^{sym} \vec{\mathbf{u}}$. In **Problem 5.8** we have obtained the symmetric part of the displacement field gradient, $\varepsilon_{ij} = \frac{1}{2}(\mathsf{u}_{i,j} + \mathsf{u}_{j,i})$, in Voigt notation, i.e.:

$$\{\boldsymbol{\varepsilon}(\vec{x})\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \mathsf{u}_1}{\partial x_1} \\ \dfrac{\partial \mathsf{u}_2}{\partial x_2} \\ \dfrac{\partial \mathsf{u}_3}{\partial x_3} \\ \dfrac{\partial \mathsf{u}_1}{\partial x_2} + \dfrac{\partial \mathsf{u}_2}{\partial x_1} \\ \dfrac{\partial \mathsf{u}_2}{\partial x_3} + \dfrac{\partial \mathsf{u}_3}{\partial x_2} \\ \dfrac{\partial \mathsf{u}_1}{\partial x_3} + \dfrac{\partial \mathsf{u}_3}{\partial x_1} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{Bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \mathsf{u}_3 \end{Bmatrix} \quad \Rightarrow \quad \{\boldsymbol{\varepsilon}(\vec{x})\} = [\boldsymbol{L}^{(1)}]\{\mathbf{u}(\vec{x})\}$$

Then, taking into account that $\{\mathbf{u}(\vec{x})\} = [\boldsymbol{N}(\vec{x})]\{\mathbf{u}^{(e)}\}$ the above equation becomes:

$$\{\boldsymbol{\varepsilon}(\vec{x})\} = [\boldsymbol{L}^{(1)}]\{\mathbf{u}(\vec{x})\} = [\boldsymbol{L}^{(1)}][\boldsymbol{N}(\vec{x})]\{\mathbf{u}^{(e)}\} = [\mathbf{B}(\vec{x})]\{\mathbf{u}^{(e)}\}$$

where

$$[\mathbf{B}(\vec{x})] = [\boldsymbol{L}^{(1)}]\ [\boldsymbol{N}(\vec{x})] \tag{5.173}$$

The stress field can be expressed as follows:

$$\{\boldsymbol{\sigma}(\vec{x})\} = [\boldsymbol{\mathcal{C}}]\{\boldsymbol{\varepsilon}(\vec{x})\} = [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\{\mathbf{u}^{(e)}\}$$

We can adopt the same displacement field approach to approximate the virtual displacement field, with which we can obtain:

$$\{\overline{\mathbf{u}}(\vec{x})\} = [\boldsymbol{N}(\vec{x})]\{\overline{\mathbf{u}}^{(e)}\} \quad \Rightarrow \quad \{\overline{\boldsymbol{\varepsilon}}(\vec{x})\} = [\mathbf{B}(\vec{x})]\{\overline{\mathbf{u}}^{(e)}\}$$

Then, the equation in (5.172) becomes:

$$\{\overline{\mathbf{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \int_V \{\overline{\boldsymbol{\varepsilon}}\}^T\{\boldsymbol{\sigma}\}\,dV = \int_V \left\{[\mathbf{B}(\vec{x})]\{\overline{\mathbf{u}}^{(e)}\}\right\}^T[\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\{\mathbf{u}^{(e)}\}\,dV$$

or:

$$\{\overline{\mathbf{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \int_V \{\overline{\mathbf{u}}^{(e)}\}^T[\mathbf{B}(\vec{x})]^T[\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\{\mathbf{u}^{(e)}\}\,dV \tag{5.174}$$

Note that neither $\{\mathbf{u}^{(e)}\}$ nor $\{\overline{\mathbf{u}}^{(e)}\}$ depend on $\vec{x}$, then:

$$\{\overline{\mathbf{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T\left(\int_V [\mathbf{B}(\vec{x})]^T[\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\,dV\right)\{\mathbf{u}^{(e)}\}$$

Since the vector $\{\overline{\mathbf{u}}^{(e)}\}$ is arbitrary, we can conclude that

$$\Rightarrow \{\boldsymbol{F}^{(e)}\} = \left(\int_V [\mathbf{B}(\vec{x})]^T[\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\,dV\right)\{\mathbf{u}^{(e)}\} \quad \Rightarrow \quad \{\boldsymbol{F}^{(e)}\} = [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} \tag{5.175}$$

NOTE: $[\boldsymbol{K}^{(e)}]$ is known as the *stiffness matrix* of the sub-domain (finite element), and the matrix $[\boldsymbol{N}(\vec{x})]$ from the relationship $\{\mathbf{u}(\vec{x})\} = [\boldsymbol{N}(\vec{x})]\{\mathbf{u}^{(e)}\}$ is known as the shape function matrix. The shape functions are functions defined into the domain that allows us to obtain $\{\mathbf{u}(\vec{x})\}$ at any point of the domain through the nodal values of the function $\{\mathbf{u}^{(e)}\}$. A special emphasis about shape functions is taken place in **Problem 6.40** in Chapter 6.

Problem 5.24

a) Consider a sub-domain (Ω) made up by a homogeneous, isotropic linear elastic material. Consider also that at some points of the sub-domain boundary there are concentrated forces $\{\boldsymbol{F}^{(e)}\} \equiv \{\boldsymbol{F}^{loc}\}$ (nodal forces), and that the displacement field is approximated by $\{\mathbf{u}(\vec{x})\} = [\boldsymbol{N}(\vec{x})]\{\mathbf{u}^{(e)}\}$ where the nodal displacements $\{\mathbf{u}^{(e)}\} \equiv \{\mathbf{u}^{loc}\}$ are the displacements at the points where concentrated forces are applied. Prove that the governing equations for a linear elastic problem can be replaced by:

$$\boxed{[\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\} = \{\boldsymbol{F}^{(e)}\}} \tag{5.176}$$

where $[\boldsymbol{K}^{(e)}] = \int_V [\mathbf{B}]^T[\boldsymbol{\mathcal{C}}][\mathbf{B}]\,dV$ (stiffness matrix), $[\boldsymbol{M}^{(e)}] = \int_V \rho[\boldsymbol{N}]^T[\boldsymbol{N}]\,dV$ (mass matrix), $[\boldsymbol{\mathcal{C}}]$ is the elasticity tensor in Voigt notation.

b) Show that the equation in (5.176) represents a conservative system.

Hint: Use the Principle of Virtual Work and consider the following approximations for the fields:

$$\begin{cases} \{\mathbf{u}(\vec{x})\} = [N(\vec{x})]\{\mathbf{u}^{(e)}\} & \text{(displacement field)} \\ \{\dot{\mathbf{u}}(\vec{x})\} = [N(\vec{x})]\{\dot{\mathbf{u}}^{(e)}\} & \text{(velocity field)} \\ \{\ddot{\mathbf{u}}(\vec{x})\} = [N(\vec{x})]\{\ddot{\mathbf{u}}^{(e)}\} & \text{(acceleration field)} \\ \{\boldsymbol{\varepsilon}(\vec{x})\} = [L^{(1)}][N(\vec{x})]\{\mathbf{u}^{(e)}\} = [\mathbf{B}(\vec{x})]\{\mathbf{u}^{(e)}\} & \text{(strain field)} \\ \{\boldsymbol{\sigma}(\vec{x})\} = [\mathcal{C}]\{\boldsymbol{\varepsilon}(\vec{x})\} = [\mathcal{C}][\mathbf{B}]\{\mathbf{u}^{(e)}\} & \text{(stress field)} \\ \{\mathbf{t}^{*}(\vec{x})\} = [\mathbf{N}^{\vec{\mathbf{t}}}(\vec{x})]\{f_{\vec{\mathbf{t}}}^{(e)}\} & \text{(vector traction field)} \\ \{\mathbf{b}(\vec{x})\} = [\mathbf{N}^{\vec{\mathbf{b}}}(\vec{x})]\{f_{\vec{\mathbf{b}}}^{(e)}\} & \text{(body force field)} \end{cases} \tag{5.177}$$

Use the same approximations for the respective virtual fields.

Solution:

The Principle of Virtual Work states that:

$$\underbrace{\int_V \boldsymbol{\sigma} : \overline{\boldsymbol{\varepsilon}}\, dV}_{\substack{\text{Total internal} \\ \text{virtual work}}} = \underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^{*} \cdot \vec{\overline{\mathbf{u}}}\, dS_{\sigma} + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \vec{\overline{\mathbf{u}}}\, dV}_{\text{Total external virtual work}}$$

We rewrite the above equation using Voigt notation:

$$\begin{aligned} \int_V \{\boldsymbol{\sigma}\}^T \{\overline{\boldsymbol{\varepsilon}}\}\, dV &= \int_{S_{\boldsymbol{\sigma}}} \{\mathbf{t}\}^T \{\overline{\mathbf{u}}\}\, dS + \int_V \{\rho \mathbf{b}\}^T \{\overline{\mathbf{u}}\}\, dV - \int_V \{\rho \ddot{\mathbf{u}}\}^T \{\overline{\mathbf{u}}\}\, dV \\ &= \int_{S_{\boldsymbol{\sigma}}} \{\overline{\mathbf{u}}\}^T \{\mathbf{t}\}\, dS + \int_V \{\overline{\mathbf{u}}\}^T \{\rho \mathbf{b}\}\, dV - \int_V \{\overline{\mathbf{u}}\}^T \{\rho \ddot{\mathbf{u}}\}\, dV \end{aligned} \tag{5.178}$$

Using the adopted approximations, (see equations in (5.177)), we can obtain the following terms:

$$\begin{aligned} \int_V \{\boldsymbol{\sigma}\}^T \{\overline{\boldsymbol{\varepsilon}}\}\, dV &= \int_V \{\overline{\boldsymbol{\varepsilon}}\}^T \{\boldsymbol{\sigma}\}\, dV = \int_V \left\{[\mathbf{B}]\{\overline{\mathbf{u}}^{(e)}\}\right\}^T \left\{[\mathcal{C}][\mathbf{B}]\{\mathbf{u}^{(e)}\}\right\} dV = \\ &= \int_V \{\overline{\mathbf{u}}^{(e)}\}^T [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}]\{\mathbf{u}^{(e)}\}\, dV = \{\overline{\mathbf{u}}^{(e)}\}^T \left(\int_V [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}]\, dV \right) \{\mathbf{u}^{(e)}\} \\ &= \{\overline{\mathbf{u}}^{(e)}\}^T [\boldsymbol{K}^{(e)}] \{\mathbf{u}^{(e)}\} \end{aligned}$$

$$\begin{aligned} \int_{S_{\boldsymbol{\sigma}}} \{\overline{\mathbf{u}}\}^T \{\mathbf{t}\}\, dS &= \int_{S_{\boldsymbol{\sigma}}} \left\{[N]\{\overline{\mathbf{u}}^{(e)}\}\right\}^T [\mathbf{N}^{\vec{\mathbf{t}}}]\{f_{\vec{\mathbf{t}}}^{(e)}\}\, dS = \int_{S_{\boldsymbol{\sigma}}} \{\overline{\mathbf{u}}^{(e)}\}^T [N]^T [\mathbf{N}^{\vec{\mathbf{t}}}]\{f_{\vec{\mathbf{t}}}^{(e)}\}\, dS \\ &= \{\overline{\mathbf{u}}^{(e)}\}^T \left(\int_{S_{\boldsymbol{\sigma}}} [N]^T [\mathbf{N}^{\vec{\mathbf{t}}}]\, dS \right) \{f_{\vec{\mathbf{t}}}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T [\mathbf{G}^{\vec{\mathbf{t}}}]\{f_{\vec{\mathbf{t}}}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T \{F_{\vec{\mathbf{t}}}^{(e)}\} \end{aligned}$$

$$\begin{aligned} \int_V \{\overline{\mathbf{u}}\}^T \{\rho \mathbf{b}\}\, dV &= \int_V \left\{[N]\{\overline{\mathbf{u}}^{(e)}\}\right\}^T \rho [\mathbf{N}^{\vec{\mathbf{b}}}]\{f_{\vec{\mathbf{b}}}^{(e)}\}\, dV = \{\overline{\mathbf{u}}^{(e)}\}^T \left(\int_V \rho [N]^T [\mathbf{N}^{\vec{\mathbf{b}}}]\, dV \right) \{f_{\vec{\mathbf{b}}}^{(e)}\} \\ &= \{\overline{\mathbf{u}}^{(e)}\}^T [\mathbf{G}^{\vec{\mathbf{b}}}]\{f_{\vec{\mathbf{b}}}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T \{\vec{F}_{\vec{\mathbf{b}}}^{(e)}\} \end{aligned}$$

$$\int_V \{\overline{\mathbf{u}}\}^T \{\rho \ddot{\mathbf{u}}\}\, dV = \int_V \left\{[\boldsymbol{N}]\{\overline{\mathbf{u}}^{(e)}\}\right\}^T \rho [\boldsymbol{N}]\{\ddot{\mathbf{u}}^{(e)}\}\, dV = \int_V \rho \{\overline{\mathbf{u}}^{(e)}\}^T [\boldsymbol{N}]^T [\boldsymbol{N}]\{\ddot{\mathbf{u}}^{(e)}\}\, dV$$
$$= \{\overline{\mathbf{u}}^{(e)}\}^T \left(\int_V \rho [\boldsymbol{N}]^T [\boldsymbol{N}]\, dV \right) \{\ddot{\mathbf{u}}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}$$

Taking into account the above relationships into the equation (5.178) we can obtain:

$$\{\overline{\mathbf{u}}^{(e)}\}^T [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} = \{\overline{\mathbf{u}}^{(e)}\}^T \{\boldsymbol{F}_{\bar{\mathbf{t}}}^{(e)}\} + \{\overline{\mathbf{u}}^{(e)}\}^T \{\boldsymbol{F}_{\bar{\mathbf{b}}}^{(e)}\} - \{\overline{\mathbf{u}}^{(e)}\}^T [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}$$
$$\Rightarrow \{\overline{\mathbf{u}}^{(e)}\}^T \left([\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\}\right) = \{\overline{\mathbf{u}}^{(e)}\}^T \left(\{\boldsymbol{F}_{\bar{\mathbf{t}}}^{(e)}\} + \{\boldsymbol{F}_{\bar{\mathbf{b}}}^{(e)}\} - [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}\right)$$

Since the virtual displacement $\{\overline{\mathbf{u}}^{(e)}\}$ is arbitrary we can conclude that:

$$[\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} = \{\boldsymbol{F}_{\bar{\mathbf{t}}}^{(e)}\} + \{\boldsymbol{F}_{\bar{\mathbf{b}}}^{(e)}\} - [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}$$
$$\Rightarrow [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\} = \{\boldsymbol{F}_{\bar{\mathbf{t}}}^{(e)}\} + \{\boldsymbol{F}_{\bar{\mathbf{b}}}^{(e)}\}$$
$$\Rightarrow [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\} = \{\boldsymbol{F}^{(e)}\}$$

Q.E.D.

b) To show that the above system is conservative we will consider the discretization of time where the current time we denote by t and the next time by $t+\Delta t$, where Δt is the time increment. In any time the above equation must be true, so:

$$\begin{cases} [\boldsymbol{K}^{(e)}]_t \{\mathbf{u}^{(e)}\}_t + [\boldsymbol{M}^{(e)}]_t \{\ddot{\mathbf{u}}^{(e)}\}_t = \{\boldsymbol{F}^{(e)}\}_t \\ [\boldsymbol{K}^{(e)}]_{t+\Delta t} \{\mathbf{u}^{(e)}\}_{t+\Delta t} + [\boldsymbol{M}^{(e)}]_{t+\Delta t} \{\ddot{\mathbf{u}}^{(e)}\}_{t+\Delta t} = \{\boldsymbol{F}^{(e)}\}_{t+\Delta t} = \{\boldsymbol{F}^{(e)}\}_t \end{cases}$$

$$\Rightarrow [\boldsymbol{K}^{(e)}]_{t+\Delta t} \{\mathbf{u}^{(e)}\}_{t+\Delta t} + [\boldsymbol{M}^{(e)}]_{t+\Delta t} \{\ddot{\mathbf{u}}^{(e)}\}_{t+\Delta t} = [\boldsymbol{K}^{(e)}]_t \{\mathbf{u}^{(e)}\}_t + [\boldsymbol{M}^{(e)}]_t \{\ddot{\mathbf{u}}^{(e)}\}_t$$

where the force $\{\boldsymbol{F}^{(e)}\}$ is constant over time. Notice that, if the vector $\{\boldsymbol{F}^{(e)}\}$ is constant over time we can obtain:

$$\frac{D}{Dt}\left([\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}\right) = \frac{D}{Dt}\{\boldsymbol{F}^{(e)}\} = \{\boldsymbol{0}\}$$
$$\Rightarrow \left([\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}\right) = \text{constant over time}$$

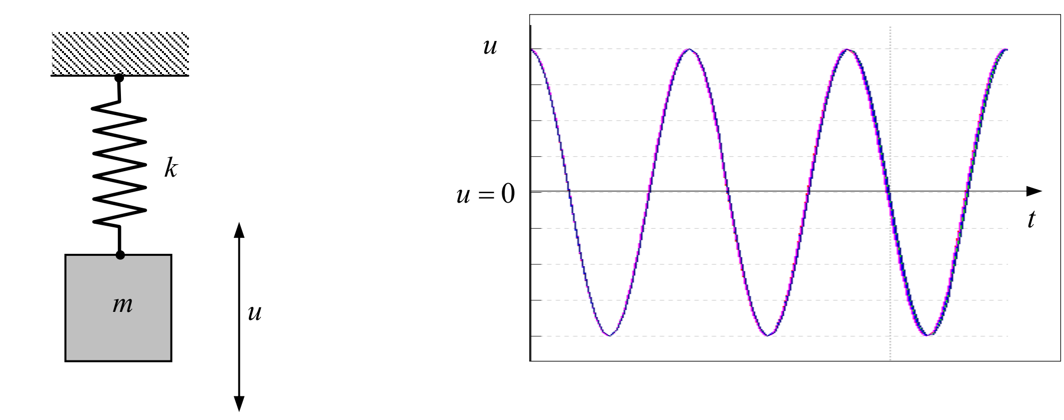
NOTE 1: The equation in (5.176) is a forced harmonic motion. Let us consider the one-dimensional case where $[\boldsymbol{K}^{(e)}]$ represents the spring constant k, $[\boldsymbol{M}^{(e)}]$ represents the mass m, and the displacement and acceleration are represented by u and $\ddot{u}$ respectively, (see Figure 5.16). With that the equation in (5.176), without applied force, becomes:

$$ku + m\ddot{u} = 0 \quad \Rightarrow \quad ku = -m\ddot{u}$$

Note that the energy is conserved. Considering the internal energy for the spring ($\frac{1}{2}uku$) and the kinetic energy ($\frac{1}{2}\dot{u}m\dot{u} = \frac{1}{2}mv^2$) for the particle of mass m, and by apply the energy equation we can obtain:

$$\frac{D\mathcal{K}}{Dt} + \frac{DU}{Dt} = \underbrace{\frac{\text{Đ}\mathcal{W}}{Dt} + \frac{\text{Đ}Q}{Dt}}_{=0} = 0 \quad \Rightarrow \quad \frac{D}{Dt}\left(\frac{1}{2}\dot{u}m\dot{u}\right) + \frac{D}{Dt}\left(\frac{1}{2}uku\right) = 0 \qquad \left[\frac{J}{s} = W\right] \tag{5.179}$$
$$\Rightarrow m\ddot{u}\dot{u} + ku\dot{u} = 0 \quad \Rightarrow \quad \Rightarrow (m\ddot{u} + ku)\dot{u} = 0 \quad \Rightarrow \quad m\ddot{u} + ku = 0$$

where $\mathcal{K}$ is the kinetic energy, U is the internal energy. The equation $m\ddot{u} + ku = 0$ is denoted by the simple harmonic motion, (see Figure 5.16).



b) Mechanical model (mass+spring) b) Displacement vs. time

Figure 5.16: Mechanical model for simple harmonic motion.

NOTE 2: If we do the experiment using the mechanical model described in Figure 5.16 we will observe that the motion in reality is not conservative, i.e. there is dissipation of energy. In other word, there is damping of the system until the rest is achieved, (see Figure 5.18). This phenomenon occurs due to the internal mechanisms of the structures. Traditionally, this damping intrinsic of the structures can be dealt by means of the parameter d (damping) multiply by velocity, (see Figure 5.17).

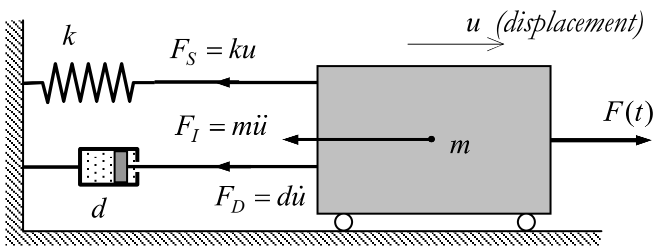


Figure 5.17: Mechanical model (mass+dashpot+spring).

Taking into account the discrete mechanical model described in Figure 5.17 and by applying the force equilibrium we can obtain:

$$F_I + F_D + F_S = F(t) \qquad \Rightarrow \qquad m\ddot{u} + d\dot{u} + ku = F(t)$$

And the equation in (5.176) can be rewritten in order to take the damping effect as follows:

$$\boxed{[\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} + [\boldsymbol{D}^{(e)}]\{\dot{\mathbf{u}}^{(e)}\} + [\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\} = \{\boldsymbol{F}^{(e)}\}} \tag{5.180}$$

where $[\boldsymbol{D}^{(e)}]$ is the damping matrix. Note that to solve the equation (5.180) we need to integrate over time. To solve (5.180) we must transform the equation in (5.180) into an equivalent system as follows:

$$[\boldsymbol{K}^{eff}]\{\mathbf{u}^{(e)}\}_{t+\Delta t} = \{\boldsymbol{F}^{eff}\} \tag{5.181}$$

where $[\boldsymbol{K}^{eff}]$ is the effective stiffness matrix, and $\{\boldsymbol{F}^{eff}\}$ is the effective nodal force vector. For more details about this the reader is referred to **Annex A** at the end of the Book.

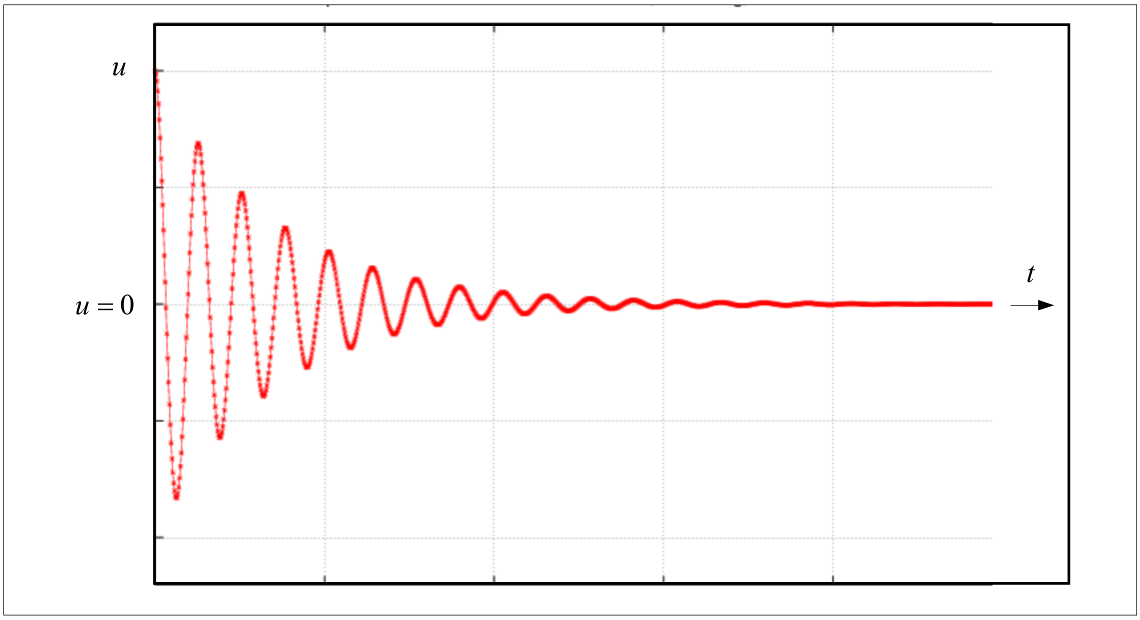


Figure 5.18: Displacement vs. time – Mechanical model with damping.

NOTE 3: Although the matrices $[\boldsymbol{K}^{(e)}]$ and $[\boldsymbol{M}^{(e)}]$ are obtained by means of material and geometrical properties, the "viscous damping" matrix $[\boldsymbol{D}^{(e)}]$ there is no universal formula, and estimating a suitable damping matrix is still a challenging task. Whereas the equation $[\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\}+[\boldsymbol{M}^{(e)}]\{\ddot{\mathbf{u}}^{(e)}\}=\{\boldsymbol{F}^{(e)}\}$ (forced harmonic motion) can be obtained from the fundamental principles of Continuum Mechanics, the equation (5.180) cannot, in other words, from a Continuum Mechanics point of view there is no matrix $[\boldsymbol{D}^{(e)}]$. Some authors adopt $[\boldsymbol{D}^{(e)}]$ as a function of $[\boldsymbol{K}^{(e)}]$, or a function of $[\boldsymbol{M}^{(e)}]$, or Rayleigh damping which is a linear combination between the two, i.e.: $[\boldsymbol{D}^{(e)}]=\alpha[\boldsymbol{M}^{(e)}]+\xi[\boldsymbol{K}^{(e)}]$, so as to guarantee definite positiveness of the matrix $[\boldsymbol{D}^{(e)}]$. Moreover, identification of the valid damping coefficients α and ξ, for large systems, is highly complicated. Nowadays, characterization of damping force has been an active area of research in structural dynamics, (see Clough&Penzien (1975), Chaves (2015)).

Structural Dynamics References

CLOUGH, R.W. & PENZIEN, J. (1975). *Dynamic of Structures*. McGraw-Hill Companies.

TEDESCO, J.M.; MCDOUGAL, W.G. & ROSS, C.A.(1998). *Structural dynamics: theory and applications*. Addison Wesley Longman, Inc.

CHAVES, E.W.V. (2015). Dynamic analysis: a new point of view. *Continuum Mechanics and Thermodynamics,* Springer, DOI 10.1007/s00161-015-0419-4.

Problem 5.25

For an equilibrium system let us consider the total potential energy Π defined as follows:

$$\boxed{\Pi(\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV - \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \quad \textit{The total potential energy} \tag{5.182}$$

where

$$\boxed{U^{int} = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV = \int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV} \quad \textit{The internal potential energy} \tag{5.183}$$

and

$$\boxed{U^{ext} = \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS + \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \quad \textit{The external potential energy} \tag{5.184}$$

Also let us consider that the first variation of Π, denoted by $\delta\Pi$, equals zero for a stationary value of Π. Show that, if $\delta\Pi = 0$ is equivalent to a stationary value of Π, so $\Pi(\vec{\mathbf{u}})$ assume a minimum value.

Obs.: Consider that during the deformation process, the external actions ($\vec{\mathbf{t}}^*, \vec{\mathbf{b}}$) do not vary, and also consider a linear elastic material.

Solution:

The first variation ($\delta\Pi$) can be obtained as follows:

$$\begin{aligned} \delta\Pi &= \delta\left(\int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV - \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV \right) \\ &= \delta \int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV - \delta \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \delta \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV \\ &= \int_V \frac{1}{2} \delta(\boldsymbol{\sigma} : \boldsymbol{\varepsilon})\, dV - \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \delta\vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \delta\vec{\mathbf{u}} dV \end{aligned} \tag{5.185}$$

Note that:

$$\begin{aligned} \delta\Psi^e(\boldsymbol{\varepsilon}) &= \frac{1}{2}\delta(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}) = \frac{1}{2}(\delta\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}) = \frac{1}{2}\left[\delta(\mathbb{C}^e : \boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\right] \\ &= \frac{1}{2}\left[(\mathbb{C}^e : \delta\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\right] = \frac{1}{2}\left[\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\right] = \frac{1}{2}\left[\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\right] \\ &= \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} \\ &= \frac{\partial \Psi^e}{\partial \boldsymbol{\varepsilon}} : \delta\boldsymbol{\varepsilon} \end{aligned} \tag{5.186}$$

where we have considered $\boldsymbol{\sigma} = \frac{\partial \Psi^e}{\partial \boldsymbol{\varepsilon}}$, (see **Problem 5.5**). For small deformation regime we can also write the above equation as follows:

$$\delta\Psi^e(\boldsymbol{\varepsilon}) = \frac{\partial \Psi^e}{\partial \boldsymbol{\varepsilon}} : \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \delta(\nabla^{sym}\vec{\mathbf{u}}) = \boldsymbol{\sigma} : (\nabla^{sym}\delta\vec{\mathbf{u}}) = \boldsymbol{\sigma} : (\nabla\delta\vec{\mathbf{u}}) \tag{5.187}$$

where we have used the property $\mathbf{A}^{sym}:\mathbf{B}=\mathbf{A}^{sym}:(\mathbf{B}^{sym}+\mathbf{B}^{skew})=\mathbf{A}^{sym}:\mathbf{B}^{sym}$. Then, the equation in (5.185) becomes:

$$
\begin{aligned}
\delta\Pi &= \int_V \frac{1}{2}\delta(\boldsymbol{\sigma}:\boldsymbol{\varepsilon})\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}dV \\
&= \int_V \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}dV \\
&= \int_V \delta\Psi^e\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}dV
\end{aligned}
$$

The expression $\Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})$ can be obtained as follows, (see equation (5.182)):

$$
\Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon}+\delta\boldsymbol{\varepsilon})\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dS - \int_V (\rho\vec{\mathbf{b}})\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dV \tag{5.188}
$$

By using the Taylor series to approach $\Psi^e(\boldsymbol{\varepsilon}+\delta\boldsymbol{\varepsilon})$ we can obtain:

$$
\Psi^e(\boldsymbol{\varepsilon}+\delta\boldsymbol{\varepsilon}) = \Psi^e(\boldsymbol{\varepsilon}) + \frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}:\delta\boldsymbol{\varepsilon} + \frac{1}{2}\delta\boldsymbol{\varepsilon}:\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:\delta\boldsymbol{\varepsilon}+\dots \tag{5.189}
$$

Note that $\frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}:\delta\boldsymbol{\varepsilon}=\delta\Psi^e$, (see equation (5.186)), and $\mathbb{C}^e=\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}$, (see **Problem 5.5**), with which the equation in (5.189) becomes:

$$
\begin{aligned}
\Psi^e(\boldsymbol{\varepsilon}+\delta\boldsymbol{\varepsilon}) &= \Psi^e(\boldsymbol{\varepsilon}) + \frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}:\delta\boldsymbol{\varepsilon} + \frac{1}{2}\delta\boldsymbol{\varepsilon}:\frac{\partial^2\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}\otimes\partial\boldsymbol{\varepsilon}}:\delta\boldsymbol{\varepsilon}+\dots \\
&\approx \Psi^e(\boldsymbol{\varepsilon}) + \delta\Psi^e + \frac{1}{2}\delta\boldsymbol{\varepsilon}:\mathbb{C}^e:\delta\boldsymbol{\varepsilon}
\end{aligned}
$$

and by replace the above equation into the equation (5.188) we can obtain:

$$
\begin{aligned}
\Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) &= \int_V \Psi^e(\boldsymbol{\varepsilon}+\delta\boldsymbol{\varepsilon})\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dS - \int_V (\rho\vec{\mathbf{b}})\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dV \\
&= \int_V \Psi^e(\boldsymbol{\varepsilon})\,dV + \int_V \delta\Psi^e\,dV + \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon}:\mathbb{C}^e:\delta\boldsymbol{\varepsilon}\,dV \\
&\qquad - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dS - \int_V (\rho\vec{\mathbf{b}})\cdot(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}})dV \\
&= \int_V \Psi^e(\boldsymbol{\varepsilon})\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\vec{\mathbf{u}}dV + \\
&\qquad + \int_V \delta\Psi^e\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS_\sigma - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}dV + \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon}:\mathbb{C}^e:\delta\boldsymbol{\varepsilon}\,dV
\end{aligned} \tag{5.190}
$$

Note that:

$$
\Pi(\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon})\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\vec{\mathbf{u}}dV
$$

and

$$
\delta\Pi = \int_V \delta\Psi^e\,dV - \int_{S_\sigma} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}dV = 0
$$

Taking into account the previous equations into the equation in (5.190) we can conclude that:

$$\Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) = \Pi(\vec{\mathbf{u}}) + \delta\Pi + \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon}\, dV$$

$$\Rightarrow \Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) - \Pi(\vec{\mathbf{u}}) = \delta\Pi + \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon}\, dV$$

$$\Rightarrow \Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) - \Pi(\vec{\mathbf{u}}) = \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon}\, dV$$

where we have considered $\delta\Pi = 0$. Note that the term $\delta\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon} > 0$ is always positive for any valor of $\delta\boldsymbol{\varepsilon}$ since $\mathbb{C}^e$ is a positive definite tensor, (see Chapter 1). Then, we can guarantee that:

$$\Delta\Pi = \Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) - \Pi(\vec{\mathbf{u}}) = \int_V \frac{1}{2}\delta\boldsymbol{\varepsilon} : \mathbb{C}^e : \delta\boldsymbol{\varepsilon}\, dV > 0 \quad \Rightarrow \quad \Pi(\vec{\mathbf{u}}+\delta\vec{\mathbf{u}}) > \Pi(\vec{\mathbf{u}})$$

$$So, \qquad \delta\Pi = 0 \quad \Rightarrow \quad \Pi(\vec{\mathbf{u}}) \text{ is a minimum}$$

NOTE 1: For a system characterized by a *linear elastic problem*, the equilibrium point corresponds to the minimum value of Π, (see Figure 5.19). This is known as the *principle of minimum potential energy*.

NOTE 2: When the external action is characterized by concentrated forces and in the absence of body forces, the equation (5.182) becomes:

$$\boxed{\Pi(\vec{\mathbf{u}}) = U^{int} + U^{ext} = \int_V \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV - \{\boldsymbol{F}^{loc}\}^T\{\mathbf{u}^{loc}\}} \quad \textit{The total potential energy} \qquad (5.191)$$

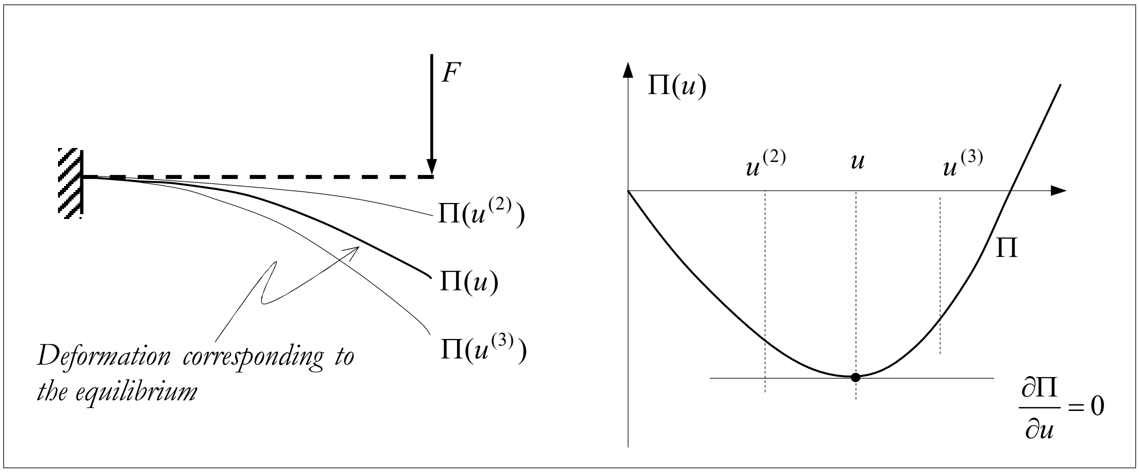


Figure 5.19

NOTE 3: By means of equation (5.191), it is easy to show the *Castigliano's Theorem – Part I:*

$$\frac{\partial \Pi(\vec{\mathbf{u}})}{\partial\{\mathbf{u}^{loc}\}} = \frac{\partial U^{int}}{\partial\{\mathbf{u}^{loc}\}} + \frac{\partial U^{ext}}{\partial\{\mathbf{u}^{loc}\}} = \frac{\partial U^{int}}{\partial\{\mathbf{u}^{loc}\}} - \frac{\partial\left[\{\boldsymbol{F}^{loc}\}^T\{\mathbf{u}^{loc}\}\right]}{\partial\{\mathbf{u}^{loc}\}} = 0$$

$$\Rightarrow \{\boldsymbol{F}^{loc}\} = \frac{\partial U^{int}}{\partial\{\mathbf{u}^{loc}\}}$$

where $\{\boldsymbol{F}^{loc}\} = \{F_1, F_2, ..., F_n\}^T$, $\{\overline{\mathbf{u}}^{loc}\} = \{\mathsf{U}_1, \mathsf{U}_2, ..., \mathsf{U}_n\}^T$. Note that the term U^{int} must be in function of $\{\mathbf{u}^{loc}\}$.

Note also that the Potential can also be expressed in terms of forces $\Pi(\{\boldsymbol{F}^{loc}\})$, so, we can define the *Castigliano's Theorem – Part II*

$$\frac{\partial \Pi(\{\boldsymbol{F}^{loc}\})}{\partial\{\boldsymbol{F}^{loc}\}} = \frac{\partial U^{int}}{\partial\{\boldsymbol{F}^{loc}\}} + \frac{\partial U^{ext}}{\partial\{\boldsymbol{F}^{loc}\}} = \frac{\partial U^{int}}{\partial\{\boldsymbol{F}^{loc}\}} - \frac{\partial\left[\{\boldsymbol{F}^{loc}\}^T\{\mathbf{u}^{loc}\}\right]}{\partial\{\boldsymbol{F}^{loc}\}} = 0$$

$$\Rightarrow \{\mathbf{u}^{loc}\} = \frac{\partial U^{int}}{\partial\{\boldsymbol{F}^{loc}\}}$$

in this U^{int} must be expressed in terms of $\{\boldsymbol{F}^{loc}\}$.

NOTE 4: For better illustration of the proposed problem, we will consider a rod of length L and cross-sectional area A. Consider also that the stress and strain fields are homogeneous and given by:

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \sigma_{11} = \mathbb{C}^e_{1111}\varepsilon_{11} \Rightarrow \sigma = E\varepsilon$$

Consider also that the displacement field is approached by a linear function ($u(x) = a_1 + a_2 x$), and that on the extremities of the rod, we have the forces $F^{(1)}$, $F^{(2)}$, and the nodal displacements $\mathsf{U}^{(1)}, \mathsf{U}^{(2)}$, (see Figure 5.20).

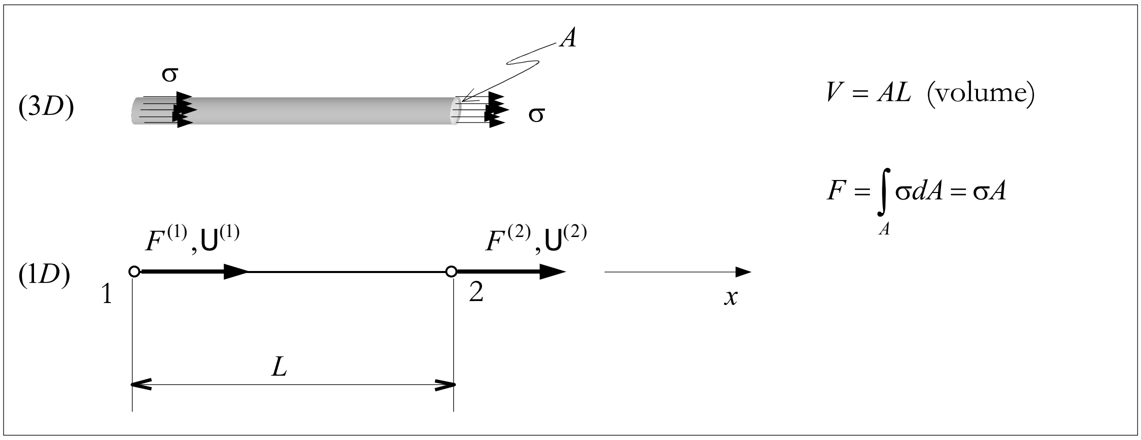


Figure 5.20: Rod under axial force.

The goal now is to express the total potential energy in terms of $\mathsf{U}^{(1)}, \mathsf{U}^{(2)}$. Note that, due to the concentrated forces we have:

$$U^{ext} = \{\boldsymbol{F}^{loc}\}^T \{\mathbf{u}^{loc}\} = \{F^{(1)} \quad F^{(2)}\}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = F^{(1)}\mathsf{U}^{(1)} + F^{(2)}\mathsf{U}^{(2)} \tag{5.192}$$

For this case, (see **Problem 5.5** - **NOTE 3**), the linear stress-strain relationship is given by $\sigma = E\varepsilon$, and the strain energy density by $\Psi^e = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\varepsilon E\varepsilon$. Then, the total internal energy is given by:

$$U^{int} = \int_V \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV \xrightarrow{1D} U^{int} = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V E\varepsilon\varepsilon\, dV = \frac{1}{2}\int_V E\varepsilon^2\, dV$$

where $\varepsilon_{11} = \frac{\partial \mathsf{u}_1}{\partial x_1} = \frac{\partial u(x)}{\partial x} = \varepsilon$, thus

$$U^{int} = \frac{1}{2}\int_V E\varepsilon^2\, dV = \frac{1}{2}\int_V E\left(\frac{\partial u(x)}{\partial x}\right)^2 dV \tag{5.193}$$

Now we will express the displacement field in terms of their nodal values $\mathsf{U}^{(1)}, \mathsf{U}^{(2)}$. To do this we will use the approach adopted $u(x) = a_1 + a_2 x$, where:

$$\left.\begin{aligned} u(x=0) &= \mathsf{U}^{(1)} = a_1 \\ u(x=L) &= \mathsf{U}^{(2)} = a_1 + a_2 L \end{aligned}\right\} \Rightarrow \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Next we evaluate the coefficients a_1 and a_2. To do this, we obtain the reverse form of the above relationship, i.e.:

$$\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \xrightarrow{reverse} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} \Rightarrow \begin{cases} a_1 = \mathsf{U}^{(1)} \\ a_2 = \frac{1}{L}\left(\mathsf{U}^{(2)} - \mathsf{U}^{(1)}\right) \end{cases}$$

with which we can obtain the displacement field in terms of its nodal values:

$$\begin{aligned} u(x) &= a_1 + a_2 x = \mathsf{U}^{(1)} + \frac{1}{L}\left(\mathsf{U}^{(2)} - \mathsf{U}^{(1)}\right)x = \left(1 - \frac{x}{L}\right)\mathsf{U}^{(1)} + \frac{x}{L}\mathsf{U}^{(2)} \\ \Rightarrow u(x) &= \left[\left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right)\right]\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = [\boldsymbol{N}(\vec{\boldsymbol{x}})]\{\mathbf{u}^{(e)}\} \end{aligned} \tag{5.194}$$

and the equation in (5.193) becomes:

$$U^{int} = \frac{1}{2}\int_V E\left(\frac{\partial u(x)}{\partial x}\right)^2 dV = \frac{E}{2}\int_V \left(\frac{1}{L}\left(\mathsf{U}^{(2)} - \mathsf{U}^{(1)}\right)\right)^2 dV = \frac{E}{2L^2}\int_V \left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) dV$$

Note that $\mathsf{U}^{(1)}$ and $\mathsf{U}^{(2)}$ are independent of x, then:

$$\begin{aligned} U^{int} &= \frac{E}{2L^2}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right)\int_V dV = \frac{E}{2L^2}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right)V \\ &= \frac{EAL}{2L^2}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) = \frac{EA}{2L}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) \end{aligned} \tag{5.195}$$

Then, the total potential energy, (see equation (5.191)), is given by equations (5.192) and (5.195), i.e.:

$$\Pi(\vec{\mathbf{u}}) = U^{int} - U^{ext} = \frac{EA}{2L}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) - \left(F^{(1)}\mathsf{U}^{(1)} + F^{(2)}\mathsf{U}^{(2)}\right) = \Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)})$$

As we are looking for a stationary state, the following must be true:

$$\begin{cases} \dfrac{\partial \Pi(\mathsf{U}^{(1)},\mathsf{U}^{(2)})}{\partial \mathsf{U}^{(1)}} = \dfrac{\partial}{\partial \mathsf{U}^{(1)}}\left[\dfrac{EA}{2L}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) - \left(F^{(1)}\mathsf{U}^{(1)} + F^{(2)}\mathsf{U}^{(2)}\right)\right] = 0 \\ \qquad = \dfrac{EA}{2L}\left(-2\mathsf{U}^{(2)} + 2\mathsf{U}^{(1)}\right) - F^{(1)} = 0 \\ \qquad = \dfrac{EA}{L}\left(\mathsf{U}^{(1)} - \mathsf{U}^{(2)}\right) - F^{(1)} = 0 \\ \dfrac{\partial \Pi(\mathsf{U}^{(1)},\mathsf{U}^{(2)})}{\partial \mathsf{U}^{(2)}} = \dfrac{\partial}{\partial \mathsf{U}^{(2)}}\left[\dfrac{EA}{2L}\left(\mathsf{U}^{(2)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(1)2}\right) - \left(F^{(1)}\mathsf{U}^{(1)} + F^{(2)}\mathsf{U}^{(2)}\right)\right] = 0 \\ \qquad = \dfrac{EA}{2L}\left(2\mathsf{U}^{(2)} - 2\mathsf{U}^{(1)}\right) - F^{(2)} = 0 \\ \qquad = \dfrac{EA}{L}\left(\mathsf{U}^{(2)} - \mathsf{U}^{(1)}\right) - F^{(2)} = 0 \end{cases}$$

Rearranging the above equations in matrix form we can obtain:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F^{(1)} \\ F^{(2)} \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} = \{\boldsymbol{F}^{(e)}\} \tag{5.196}$$

Note that $[\boldsymbol{K}^{(e)}]$ has no inverse, since $\det[\boldsymbol{K}^{(e)}] = 0$, which indicates that the problem has infinity solution since we have not imposed any restriction to motion. To solve the problem we have to introduce the boundary conditions in order to guarantee the unique solution.

Note that the matrix $[\boldsymbol{K}^{(e)}]$ of the above equation could have been obtained by means of the equation (5.175), (see **Problem 5.23**), and for this particular case we have $[\boldsymbol{\mathcal{C}}] = E$, then, the equation (5.194) becomes:

$$[\mathbf{B}(\vec{x})] = [\boldsymbol{L}^{(1)}][\boldsymbol{N}(\vec{x})] = \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right)\right] = \left[\left(\frac{-1}{L}\right) \quad \left(\frac{1}{L}\right)\right]$$

thus

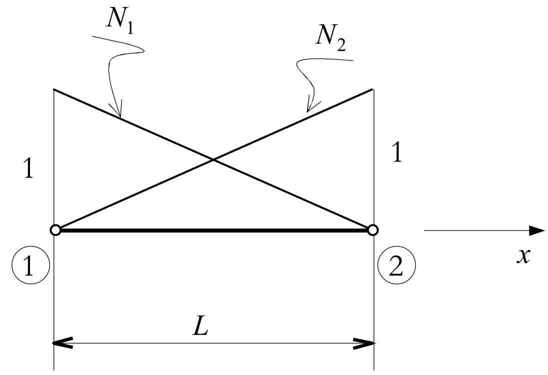
$$[\boldsymbol{K}^{(e)}] = \int_V [\mathbf{B}(\vec{x})]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\, dV = \int_V \begin{bmatrix} \left(\frac{-1}{L}\right) \\ \left(\frac{1}{L}\right) \end{bmatrix} E \left[\left(\frac{-1}{L}\right) \quad \left(\frac{1}{L}\right)\right] dV = E\int_V \frac{1}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dV$$

$$= \frac{E}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_V dV = \frac{E}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} V = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

NOTE 5: It is interesting to note that, initially we had a continuum problem (infinity of material points) represented by its governing equations and we have transformed this continuum problem into a set of discrete equations (5.196). In other words, we have applied a numerical technique to solve the problem. The main goal of any numerical technique is to transform the continuum governing equations into a set of discrete equations. Among these techniques we can quote: Finite Differences, Finite Element, Boundary Element, Finite Volume, etc.

NOTE 6: Analyzing $[\boldsymbol{N}(\vec{\boldsymbol{x}})]$

Note that the shape functions are $[\boldsymbol{N}(\vec{\boldsymbol{x}})] = \left[N_1(\vec{\boldsymbol{x}}) \quad N_2(\vec{\boldsymbol{x}})\right]^T = \left[\left(1-\dfrac{x}{L}\right) \quad \left(\dfrac{x}{L}\right)\right]^T$. And these functions are drawn as indicated in Figure 5.21.



$$N_1(x) = 1 - \frac{x}{L} \Rightarrow \begin{cases} N_1(x=0) = 1 \\ N_1(x=L) = 0 \end{cases}$$

$$N_2(x) = \frac{x}{L} \Rightarrow \begin{cases} N_2(x=0) = 0 \\ N_2(x=L) = 1 \end{cases}$$

$$N_1(x) + N_2(x) = 1$$

Figure 5.21: Shape function (linear approach)

The adopted approximation for $[\boldsymbol{N}(\vec{\boldsymbol{x}})]$ will depend on the problem. For the previous problem we have that the strain is constant into the domain, so, it is sufficient to adopt a linear approximation for displacement since by definition $\varepsilon = \dfrac{\partial u(x)}{\partial x}$. As consequence we need only two points on the boundary to define $[\boldsymbol{N}(\vec{\boldsymbol{x}})]$. If a problem requires a quadratic function for displacement approximation, so, we will need three points to define $[\boldsymbol{N}(\vec{\boldsymbol{x}})]$, and so on.

NOTE 7:

Principle of the Stationarity of Potential Energy

In this problem we have establish the principle of the stationarity of Potential Energy, (see equation in (5.182)):

$$\boxed{\Pi(\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \tag{5.197}$$

where we have considered $\Psi^e(\boldsymbol{\varepsilon}) = \dfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$. The functional is stationary if and only if $\delta_{\vec{\mathbf{u}}}\Pi(\vec{\mathbf{u}}) = 0$.

Hellinger-Reissner's Variational Principle

In **Problem 5.5**, (see **NOTE 7)** have established that

$$\begin{aligned} &\overline{\Psi}^e(\sigma) = \sigma\varepsilon - \Psi^e(\varepsilon) \xrightarrow{tensorial} \overline{\Psi}^e(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Psi^e(\boldsymbol{\varepsilon}) = -\rho_0 \mathsf{G}(\boldsymbol{\sigma}) = \mathsf{g}(\boldsymbol{\sigma}) \\ &\Rightarrow \Psi^e(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \mathsf{g}(\boldsymbol{\sigma}) \end{aligned} \tag{5.198}$$

where $\overline{\mathsf{g}}(\boldsymbol{\sigma})$ is the Gibbs free energy density with reversed sign.

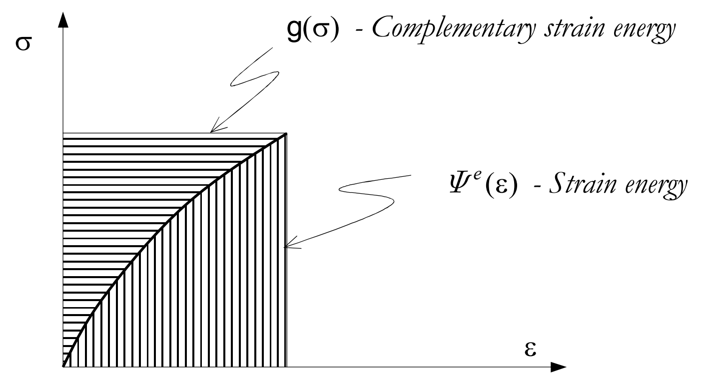


Figure 5.22: Strain energies.

By replacing $\Psi^e(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \mathsf{g}(\boldsymbol{\sigma})$ into the functional (5.197) we can obtain:

$$\begin{aligned}
&\Pi(\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV \\
&\Rightarrow \Pi_{HR}(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = \int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \mathsf{g}(\boldsymbol{\sigma})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV
\end{aligned} \tag{5.199}$$

Note that $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} : (\nabla^{sym}\vec{\mathbf{u}}) = \boldsymbol{\sigma} : (\nabla\vec{\mathbf{u}})$. Then, we can obtain:

$$\boxed{\Pi_{HR}(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = \int_V \boldsymbol{\sigma} : (\nabla\vec{\mathbf{u}}) - \mathsf{g}(\boldsymbol{\sigma})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \tag{5.200}$$

The functional (5.200) is stationary for variation of $\vec{\mathbf{u}}$ vanishing on $S_{\vec{\mathbf{u}}}$ if and only if $\boldsymbol{\sigma}$ satisfies the equilibrium equations, and is stationary for variation of $\boldsymbol{\sigma}$ if and only if they satisfy the constitutive equation (strain-stress).

$$\begin{aligned}
\delta_{\vec{\mathbf{u}}}\Pi_{HR}(\vec{\mathbf{u}}, \boldsymbol{\sigma}) &= \int_V \boldsymbol{\sigma} : (\nabla\delta\vec{\mathbf{u}})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \delta\vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \delta\vec{\mathbf{u}} dV = 0 \\
&= \int_V \sigma_{ij} (\delta \mathsf{u})_{i,j}\, dV - \int_{S_{\boldsymbol{\sigma}}} \mathsf{t}_i^* (\delta\mathsf{u})_i dS - \int_V (\rho\mathsf{b})_i (\delta\mathsf{u})_i dV = 0 \\
&= \int_{S_{\boldsymbol{\sigma}}} \sigma_{ij} (\delta\mathsf{u})_i \hat{\mathsf{n}}_j\, dS - \int_V \sigma_{ij,j} (\delta\mathsf{u})_i\, dV - \int_{S_{\boldsymbol{\sigma}}} \mathsf{t}_i^* (\delta\mathsf{u})_i dS_\sigma - \int_V (\rho\mathsf{b})_i (\delta\mathsf{u})_i dV = 0 \\
&= -\int_V \left[\sigma_{ij,j} + (\rho\mathsf{b})_i\right] (\delta\mathsf{u})_i\, dV + \int_{S_{\boldsymbol{\sigma}}} \left[\sigma_{ij}\hat{\mathsf{n}}_j - \mathsf{t}_i^*\right] (\delta\mathsf{u})_i\, dS = 0
\end{aligned} \tag{5.201}$$

In the volume we can obtain the equilibrium equations: $\sigma_{ij,j} + (\rho\mathsf{b})_i = 0_i$.

On surface $S_{\boldsymbol{\sigma}}$ we can obtain the boundary condition in stress: $\sigma_{ij}\hat{\mathsf{n}}_j - \mathsf{t}_i^* = 0_i$

$$\begin{aligned}
\delta_{\boldsymbol{\sigma}}\Pi_{HR}(\vec{\mathbf{u}}, \boldsymbol{\sigma}) &= \int_V \delta\boldsymbol{\sigma} : (\nabla^{sym}\vec{\mathbf{u}}) - \delta\mathsf{g}(\boldsymbol{\sigma})\, dV = 0 \\
&= \int_V (\nabla^{sym}\vec{\mathbf{u}}) : \delta\boldsymbol{\sigma} - \frac{\partial \mathsf{g}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} : \delta\boldsymbol{\sigma}\, dV = 0 \\
&= \int_V \left[(\nabla^{sym}\vec{\mathbf{u}}) - \frac{\partial \mathsf{g}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right] : \delta\boldsymbol{\sigma}\, dV = 0
\end{aligned} \tag{5.202}$$

In the volume we can obtain the constitutive equation for strain: $(\nabla^{sym}\vec{\mathbf{u}}) - \dfrac{\partial \mathsf{g}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \mathbf{0}$.

Hu-Washizu's Variational Principle

The Hu-Washizu's principle is a generalization of the Hellinger-Reissner's principle, in which the functional, in addition of the independent fields $(\vec{\mathbf{u}},\boldsymbol{\sigma})$, also depends on $\boldsymbol{\varepsilon}$-field:

$$\boxed{\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V [\Psi^e(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^{sym}\vec{\mathbf{u}}) - (\rho\vec{\mathbf{b}})\cdot\vec{\mathbf{u}}]\, dV - \int_{S_{\vec{\mathbf{u}}}} (\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\cdot(\vec{\mathbf{u}}^* - \vec{\mathbf{u}})dS - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\vec{\mathbf{u}}dS} \tag{5.203}$$

and is stationary if and only if:

$$\begin{cases}\delta_{\vec{\mathbf{u}}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0 & \Rightarrow \quad \text{Equilibrium equations} \\ \delta\vec{\mathbf{u}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \text{on } S_{\vec{\mathbf{u}}} & \end{cases}$$

$$\left\{\delta_{\boldsymbol{\sigma}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0 \quad \Rightarrow \quad \begin{cases}\text{Kinematic Equations} \\ \text{Boundary condition on } S_{\vec{\mathbf{u}}}\end{cases}\right.$$

$$\left\{\delta_{\boldsymbol{\varepsilon}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0 \quad \Rightarrow \quad \text{Constitutive equations for stress}\right.$$

That is:

- $\delta_{\vec{\mathbf{u}}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0$

$$\delta_{\vec{\mathbf{u}}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[\boldsymbol{\sigma} : (\nabla^{sym}\delta\vec{\mathbf{u}}) - (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}\right] dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS$$

$$\Rightarrow \delta_{\vec{\mathbf{u}}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[(\nabla\cdot\boldsymbol{\sigma})\cdot(\delta\vec{\mathbf{u}}) - (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}\right] dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS$$

$$\Rightarrow \delta_{\vec{\mathbf{u}}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[(\nabla\cdot\boldsymbol{\sigma}) - (\rho\vec{\mathbf{b}})\right]\cdot(\delta\vec{\mathbf{u}})\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^*\cdot\delta\vec{\mathbf{u}}dS$$

- $\delta_{\boldsymbol{\sigma}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0$

$$\delta_{\boldsymbol{\sigma}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[-\delta\boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^{sym}\vec{\mathbf{u}})\right] dV - \int_{S_{\vec{\mathbf{u}}}} (\delta\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\cdot(\vec{\mathbf{u}}^* - \vec{\mathbf{u}})dS = 0$$

$$\Rightarrow \delta_{\boldsymbol{\sigma}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V -\left[(\boldsymbol{\varepsilon} - \nabla^{sym}\vec{\mathbf{u}})\right] : \delta\boldsymbol{\sigma}\, dV - \int_{S_{\vec{\mathbf{u}}}} \left[\hat{\mathbf{n}}\otimes(\vec{\mathbf{u}}^* - \vec{\mathbf{u}})\right] : \delta\boldsymbol{\sigma}dS = 0$$

- $\delta_{\boldsymbol{\varepsilon}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = 0$

$$\delta_{\boldsymbol{\varepsilon}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[\delta_{\boldsymbol{\varepsilon}}\Psi^e(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma} : (\delta_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon})\right] dV = 0$$

$$\Rightarrow \delta_{\boldsymbol{\varepsilon}}\Pi_{HW}(\vec{\mathbf{u}},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \int_V \left[\frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} : \delta_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon} - \boldsymbol{\sigma} : (\delta_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon})\right] dV = \int_V \left[\frac{\partial\Psi^e(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}\right] : \delta_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}\, dV = 0$$

NOTE 8: ***Discretization of the Fields***

The variation of the Hu-Washizu's principle can be written as follows

$$
\begin{aligned}
\delta\Pi_{HW} &= \delta\int_V \left[\Psi^e(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^{sym}\vec{\mathbf{u}}) - (\rho\vec{\mathbf{b}})\cdot\vec{\mathbf{u}}\right] dV - \delta\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS = 0 \\
&= \int_V \Psi^e(\boldsymbol{\varepsilon}) : \delta\boldsymbol{\varepsilon}\, dV - \delta\int_V \left[\boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^{sym}\vec{\mathbf{u}})\right] dV - \int_V (\rho\vec{\mathbf{b}})\cdot\delta\vec{\mathbf{u}}\, dV - \int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \delta\vec{\mathbf{u}} dS = 0 \\
&= \int_V \delta\boldsymbol{\varepsilon} : \Psi^e(\boldsymbol{\varepsilon})\, dV + \delta\int_V \left[\boldsymbol{\sigma} : (\nabla^{sym}\vec{\mathbf{u}} - \boldsymbol{\varepsilon})\right] dV - \int_V \delta\vec{\mathbf{u}}\cdot(\rho\vec{\mathbf{b}}) dV - \int_{S_{\boldsymbol{\sigma}}} \delta\vec{\mathbf{u}}\cdot\vec{\mathbf{t}}^* dS = 0 \\
&= \int_V \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}\, dV + \delta\int_V \left[\boldsymbol{\sigma} : (\nabla^{sym}\vec{\mathbf{u}} - \boldsymbol{\varepsilon})\right] dV - \int_V \delta\vec{\mathbf{u}}\cdot(\rho\vec{\mathbf{b}}) dV - \int_{S_{\boldsymbol{\sigma}}} \delta\vec{\mathbf{u}}\cdot\vec{\mathbf{t}}^* dS = 0
\end{aligned}
$$

In the implementation of finite element methods we often use Voigt notation when we are dealing with symmetric matrix. Using Voigt notation the above equation becomes:

$$
\begin{aligned}
&\delta\Pi_{HW} = \int_V \{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}\}\, dV + \delta\int_V \{\boldsymbol{\sigma}\}^T\{\nabla^{sym}\mathbf{u} - \boldsymbol{\varepsilon}\} dV - \int_V \{\delta\mathbf{u}\}^T\{\rho\mathbf{b}\}\, dV - \int_{S_{\boldsymbol{\sigma}}} \{\delta\mathbf{u}\}^T\{\mathbf{t}^*\}\, dS = 0 \\
&\Rightarrow \int_V \{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}\}\, dV + \delta\int_V \{\boldsymbol{\sigma}\}^T\{\nabla^{sym}\mathbf{u} - \boldsymbol{\varepsilon}\} dV = \int_V \{\delta\mathbf{u}\}^T\{\rho\mathbf{b}\}\, dV + \int_{S_{\boldsymbol{\sigma}}} \{\delta\mathbf{u}\}^T\{\mathbf{t}^*\}\, dS
\end{aligned}
\tag{5.204}
$$

Consider as approximation for displacement, strain, and stress fields, respectively, (see Jirásek (1998)), as follows:

$$
\begin{cases}
\{\mathbf{u}\} \approx [\mathbb{N}]\{\mathbf{d}\} + [\mathbb{N}_c]\{\mathbf{d}_c\} \\
\{\boldsymbol{\varepsilon}\} \approx [\mathbf{B}]\{\mathbf{d}\} + [\mathbf{G}]\{e\} \\
\{\boldsymbol{\sigma}\} \approx [\mathbf{S}]\{\mathbf{s}\}
\end{cases}
\quad \textit{and} \quad
\begin{cases}
\{\delta\mathbf{u}\} \approx [\mathbb{N}]\{\delta\mathbf{d}\} + [\mathbb{N}_c]\{\delta\mathbf{d}_c\} \\
\{\delta\boldsymbol{\varepsilon}\} \approx [\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{G}]\{\delta e\} \\
\{\delta\boldsymbol{\sigma}\} \approx [\mathbf{S}]\{\delta\mathbf{s}\}
\end{cases}
\tag{5.205}
$$

where the matrices $[\mathbb{N}]$ and $[\mathbf{B}]$ contain the displacement interpolation functions and their derivatives (strain interpolation matrix), respectively. $[\mathbb{N}_c]$ and $[\mathbf{G}]$ are matrices containing some enrichment terms for displacement and strain respectively. $[\mathbf{S}]$ is a stress interpolation matrix. $\{\mathbf{d}\}$, $\{\mathbf{d}_c\}$, $\{e\}$ and $\{\mathbf{s}\}$ collect the degrees of freedom corresponding to nodal displacement, enhanced displacement modes, enhanced strain modes, and stress parameters, respectively. If we consider the variation of the Hu-Washizu's principle:

$$
\underbrace{\int_V \{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}\}\, dV}_{\boxed{1}} + \underbrace{\delta\int_V \{\boldsymbol{\sigma}\}^T\{\nabla^{sym}\mathbf{u} - \boldsymbol{\varepsilon}\} dV}_{\boxed{2}} = \underbrace{\int_V \{\delta\mathbf{u}\}^T\{\rho\mathbf{b}\}\, dV + \int_{S_{\boldsymbol{\sigma}}} \{\delta\mathbf{u}\}^T\{\mathbf{t}^*\}\, dS}_{\boxed{3}}
\tag{5.206}
$$

we can obtain:

$$
\begin{aligned}
&(1) \Rightarrow \int_V \{\delta\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}\}\, dV = \int_V \{[\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{G}]\{\delta e\}\}^T\{\boldsymbol{\sigma}\}\, dV = \int_V \{[\mathbf{B}]\{\delta\mathbf{d}\}\}^T\{\boldsymbol{\sigma}\}\, dV + \int_V \{[\mathbf{G}]\{\delta e\}\}^T\{\boldsymbol{\sigma}\}\, dV \\
&= \{\delta\mathbf{d}\}^T \int_V [\mathbf{B}]^T\{\boldsymbol{\sigma}\}\, dV + \{\delta e\}^T \int_V [\mathbf{G}]^T\{\boldsymbol{\sigma}\}\, dV
\end{aligned}
$$

$$(2) \Rightarrow \delta\int_V \{\boldsymbol{\sigma}\}^T \left\{\nabla^{sym}\mathbf{u} - \boldsymbol{\varepsilon}\right\} dV = \underbrace{\int_V \{\delta\boldsymbol{\sigma}\}^T \left\{\nabla^{sym}\mathbf{u} - \boldsymbol{\varepsilon}\right\} dV}_{\boxed{2.1}} + \underbrace{\int_V \{\boldsymbol{\sigma}\}^T \left\{\nabla^{sym}\delta\mathbf{u} - \delta\boldsymbol{\varepsilon}\right\} dV}_{\boxed{2.2}}$$

$$(2.1) \Rightarrow \int_V \{\delta\boldsymbol{\sigma}\}^T \left\{\nabla^{sym}\vec{\mathbf{u}} - \boldsymbol{\varepsilon}\right\} dV = \int_V \left\{[\mathbf{S}]\{\delta\mathbf{s}\}\right\}^T \left\{\nabla^{sym}\left([\mathbb{N}]\{\mathbf{d}\} + [\mathbb{N}_c]\{\mathbf{d}_c\}\right) - \left([\mathbf{B}]\{\mathbf{d}\} + [\mathbf{G}]\{e\}\right)\right\} dV$$

$$= \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T \left\{\nabla^{sym}\left([\mathbb{N}]\{\mathbf{d}\} + [\mathbb{N}_c]\{\mathbf{d}_c\}\right)\right\} dV - \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T \left\{\left([\mathbf{B}]\{\mathbf{d}\} + [\mathbf{G}]\{e\}\right)\right\} dV$$

$$= \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T \left\{\nabla^{sym}\left([\mathbb{N}]\{\mathbf{d}\}\right)\right\} + [\mathbf{S}]^T \left\{\nabla^{sym}\left([\mathbb{N}_c]\{\mathbf{d}_c\}\right)\right\} dV - \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T \left\{\left([\mathbf{B}]\{\mathbf{d}\} + [\mathbf{G}]\{e\}\right)\right\} dV$$

$$= \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T [\mathbf{B}]\{\mathbf{d}\} + [\mathbf{S}]^T [\mathbf{B}_c]\{\mathbf{d}_c\} dV - \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T [\mathbf{B}]\{\mathbf{d}\} + [\mathbf{S}]^T [\mathbf{G}]\{e\} dV$$

$$= \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T [\mathbf{B}_c]\{\mathbf{d}_c\} dV - \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T [\mathbf{G}]\{e\} dV$$

$$= \{\delta\mathbf{s}\}^T \int_V [\mathbf{S}]^T \left\{ [\mathbf{B}_c]\{\mathbf{d}_c\} - [\mathbf{G}]\{e\} \right\} dV$$

where we have considered $\left\{\nabla^{sym}\left([\mathbb{N}]\{\mathbf{d}\}\right)\right\} = [\mathbf{B}]\{\mathbf{d}\}$ and $\left\{\nabla^{sym}\left([\mathbb{N}_c]\{\mathbf{d}_c\}\right)\right\} = [\mathbf{B}_c]\{\mathbf{d}_c\}$.

$$(2.2) \Rightarrow \int_V \{\boldsymbol{\sigma}\}^T \left\{\nabla^{sym}\delta\vec{\mathbf{u}} - \delta\boldsymbol{\varepsilon}\right\} dV = \int_V \left\{\nabla^{sym}\delta\mathbf{u} - \delta\boldsymbol{\varepsilon}\right\}^T \{\boldsymbol{\sigma}\} dV$$

$$= \int_V \left\{ \nabla^{sym}\left([\mathbb{N}]\{\delta\mathbf{d}\} + [\mathbb{N}_c]\{\delta\mathbf{d}_c\}\right) - \left([\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{G}]\{\delta e\}\right) \right\}^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV$$

$$= \int_V \left\{ \nabla^{sym}\left([\mathbb{N}]\{\delta\mathbf{d}\}\right) + \nabla^{sym}\left([\mathbb{N}_c]\{\delta\mathbf{d}_c\}\right) - \left([\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{G}]\{\delta e\}\right) \right\}^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV$$

$$= \int_V \left\{ [\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{B}_c]\{\delta\mathbf{d}_c\} - \left([\mathbf{B}]\{\delta\mathbf{d}\} + [\mathbf{G}]\{\delta e\}\right) \right\}^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV$$

$$= \int_V \left\{ [\mathbf{B}_c]\{\delta\mathbf{d}_c\} - [\mathbf{G}]\{\delta e\} \right\}^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV = \int_V \left\{ \{\delta\mathbf{d}_c\}^T [\mathbf{B}_c]^T - \{\delta e\}^T [\mathbf{G}]^T \right\} \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV$$

$$= \{\delta\mathbf{d}_c\}^T \int_V [\mathbf{B}_c]^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV - \{\delta e\}^T \int_V [\mathbf{G}]^T \left\{[\mathbf{S}]\{\mathbf{s}\}\right\} dV$$

$$(3) \Rightarrow \int_V \{\delta\mathbf{u}\}^T \{\rho\mathbf{b}\}\, dV + \int_{S_{\boldsymbol{\sigma}}} \{\delta\mathbf{u}\}^T \{\mathbf{t}^*\}\, dS = \{\delta\mathbf{d}\}^T \{f_{ext}\} + \{\delta\mathbf{d}_c\}^T \{f_c\}$$

Taking into account the previous terms, the equation in (5.206) becomes:

$$\{\delta\mathbf{d}\}^T\int_V[\mathbf{B}]^T\{\boldsymbol{\sigma}\}\,dV+\{\delta\boldsymbol{e}\}^T\int_V[\mathbf{G}]^T\{\{\boldsymbol{\sigma}\}-\{[\mathbf{S}]\{\mathbf{s}\}\}\}\,dV+\{\delta\mathbf{s}\}^T\int_V[\mathbf{S}]^T\{\;[\mathbf{B}_c]\{\mathbf{d}_c\}-[\mathbf{G}]\{\boldsymbol{e}\}\;\}dV+$$
$$+\{\delta\mathbf{d}_c\}^T\int_V[\mathbf{B}_c]^T\{[\mathbf{S}]\{\mathbf{s}\}\}dV=\{\delta\mathbf{d}\}^T\{\boldsymbol{f}_{ext}\}+\{\delta\mathbf{d}_c\}^T\{\boldsymbol{f}_c\}\tag{5.207}$$

Since $\{\vec{\mathbf{u}}\}$, $\{\boldsymbol{\varepsilon}\}$ and $\{\boldsymbol{\sigma}\}$ are variables of the independent fields, so, we can say that:

$$\begin{cases}\{\delta\mathbf{d}\}^T\int_V[\mathbf{B}]^T\{\boldsymbol{\sigma}\}\,dV=\{\delta\mathbf{d}\}^T\{\boldsymbol{f}_{ext}\}\\[2ex] \{\delta\boldsymbol{e}\}^T\int_V[\mathbf{G}]^T\{\{\boldsymbol{\sigma}\}-\{[\mathbf{S}]\{\mathbf{s}\}\}\}\,dV=\{\mathbf{0}\}\\[2ex] \{\delta\mathbf{s}\}^T\int_V[\mathbf{S}]^T\{\;[\mathbf{B}_c]\{\mathbf{d}_c\}-[\mathbf{G}]\{\boldsymbol{e}\}\;\}dV=\{\mathbf{0}\}\\[2ex] \{\delta\mathbf{d}_c\}^T\int_V[\mathbf{B}_c]^T\{[\mathbf{S}]\{\mathbf{s}\}\}dV=\{\delta\mathbf{d}_c\}^T\{\boldsymbol{f}_c\}=\{\mathbf{0}\}\end{cases}\tag{5.208}$$

If we consider $\{\boldsymbol{f}_c\}=\{\mathbf{0}\}$, the above equations can be rewritten as follows:

$$\begin{cases}\int_V[\mathbf{B}]^T\{\boldsymbol{\sigma}\}\,dV=\{\boldsymbol{f}_{ext}\}\\[2ex] \int_V[\mathbf{G}]^T\{\{\boldsymbol{\sigma}\}-\{[\mathbf{S}]\{\mathbf{s}\}\}\}\,dV=\{\mathbf{0}\}\\[2ex] \int_V[\mathbf{S}]^T\{\;[\mathbf{B}_c]\{\mathbf{d}_c\}-[\mathbf{G}]\{\boldsymbol{e}\}\;\}dV=\{\mathbf{0}\}\\[2ex] \int_V[\mathbf{B}_c]^T\{[\mathbf{S}]\{\mathbf{s}\}\}dV=\{\mathbf{0}\}\end{cases}\tag{5.209}$$

Taking into account that the stress-strain relationship is given by the following expression:

$$\{\boldsymbol{\sigma}\}=[\mathcal{C}]\{\boldsymbol{\varepsilon}\}=[\mathcal{C}]\{[\mathbf{B}]\{\mathbf{d}\}+[\mathbf{G}]\{\boldsymbol{e}\}\}\tag{5.210}$$

and by substituting into the equation in (5.209) we can obtain:

$$\begin{cases}\int_V[\mathbf{B}]^T[\mathcal{C}][\mathbf{B}]\,dV\;\;\{\mathbf{d}\}+\int_V[\mathbf{B}]^T[\mathcal{C}][\mathbf{G}]\,dV\;\;\{\boldsymbol{e}\}=\{\boldsymbol{f}_{ext}\} & \text{(a)}\\[2ex] \int_V[\mathbf{G}]^T[\mathcal{C}][\mathbf{B}]\,dV\;\;\{\mathbf{d}\}+\int_V[\mathbf{G}]^T[\mathcal{C}][\mathbf{G}]\,dV\;\;\{\boldsymbol{e}\}-\int_V[\mathbf{G}]^T[\mathbf{S}]\,dV\;\;\{\mathbf{s}\}=\{\mathbf{0}\} & \text{(b)}\\[2ex] \int_V[\mathbf{S}]^T[\mathbf{B}_c]\,dV\;\;\{\mathbf{d}_c\}-\int_V[\mathbf{S}]^T[\mathbf{G}]\;dV\;\;\{\boldsymbol{e}\}=\{\mathbf{0}\} & \text{(c)}\\[2ex] \int_V[\mathbf{B}_c]^T[\mathbf{S}]\,dV\;\;\{\mathbf{s}\}=\{\mathbf{0}\} & \text{(d)}\end{cases}\tag{5.211}$$

Rewriting the above equation in matrix form we can obtain:

$$\int_V \begin{bmatrix} [\mathbf{B}]^T[\mathcal{C}][\mathbf{B}] & [\mathbf{B}]^T[\mathcal{C}][\mathbf{G}] & 0 & 0 \\ [\mathbf{G}]^T[\mathcal{C}][\mathbf{B}] & [\mathbf{G}]^T[\mathcal{C}][\mathbf{G}] & -[\mathbf{G}]^T[\mathbf{S}] & 0 \\ 0 & -[\mathbf{S}]^T[\mathbf{G}] & 0 & [\mathbf{S}]^T[\mathbf{B}_c] \\ 0 & 0 & [\mathbf{B}_c]^T[\mathbf{S}] & 0 \end{bmatrix} dV \begin{Bmatrix} \{\mathbf{d}\} \\ \{e\} \\ \{\mathbf{s}\} \\ \{\mathbf{d}_c\} \end{Bmatrix} = \begin{Bmatrix} \{\boldsymbol{f}_{ext}\} \\ \{\mathbf{0}\} \\ \{\mathbf{0}\} \\ \{\mathbf{0}\} \end{Bmatrix} \tag{5.212}$$

Let us suppose that we do not introduce any displacement enhancement terms, thus $\{\mathbf{d}_c\} = \{\mathbf{0}\} \rightarrow [\mathbf{B}_c] = [\mathbf{0}]$, with that the equation in (5.211)(c) becomes:

$$\int_V [\mathbf{S}]^T[\mathbf{G}] \; dV \; \{e\} = \{\mathbf{0}\} \tag{5.213}$$

Thus, piecewise constant stress functions $\{\boldsymbol{\sigma}\}$ will require $[\mathbf{S}] = [\mathbf{1}]$ (unit matrix). The compatibility conditions (5.213) now read:

$$\boxed{\int_V [\mathbf{G}] \; dV = [\mathbf{0}]} \tag{5.214}$$

Discontinuity on displacement and strain fields – Applying the Principle of Virtual Work

As we have seen before, virtual work is the work done by real force acting through virtual displacements. A virtual displacement is any displacement consistent with the constraints of the structure, i.e. which satisfies the boundary conditions.

The principle states that the virtual work of the internal forces must be equal to the virtual work of the external forces:

$$\boxed{\underbrace{\int_V \boldsymbol{\sigma} : \bar{\boldsymbol{\varepsilon}} \, dV}_{\substack{\text{Total internal} \\ \text{virtual work} \\ W_{int}}} = \underbrace{\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* \cdot \vec{\bar{\mathbf{u}}} dS + \int_V \rho(\vec{\mathbf{b}} - \ddot{\vec{\mathbf{u}}}) \cdot \vec{\bar{\mathbf{u}}} dV}_{\substack{\text{Total external virtual work} \\ W_{ext}}}} \tag{5.215}$$

for all the admissible virtual displacements $\vec{\bar{\mathbf{u}}}$.

Let us consider a discretized system where we can say that all the forces are applied in the nodes of the finite element (CST-Constant Strain Triangle), (see Figure 5.23).

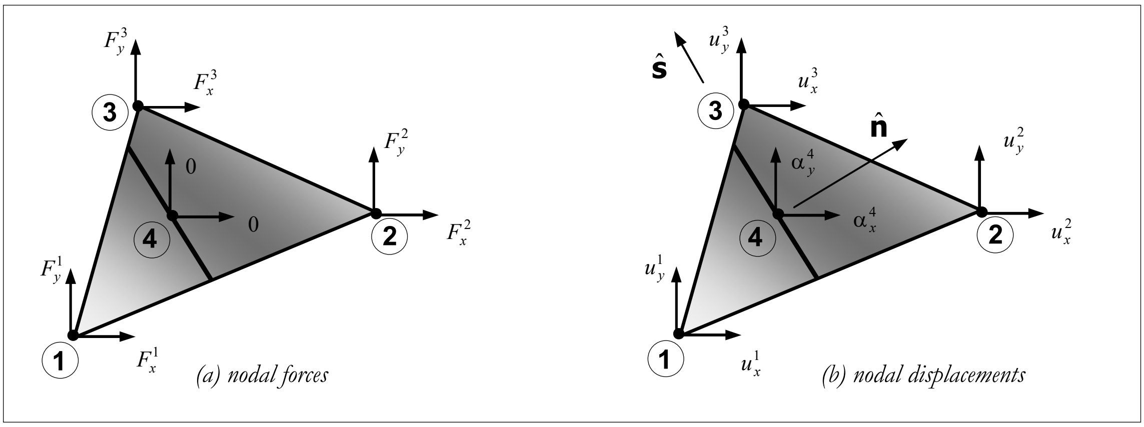


Figure 5.23: Discontinuous CST finite element.

The nodal forces and nodal displacements can be expressed as follows:

$$\left\{\begin{array}{c}\{\mathbf{F}\}\\ \hline \{\overline{\mathbf{F}}\}\end{array}\right\}=\left\{\begin{array}{c}F_x^1\\F_y^1\\F_x^2\\F_y^2\\F_x^3\\F_y^3\\ \hline 0\\0\end{array}\right\} \quad ; \quad \left\{\begin{array}{c}\{\mathbf{a}_e\}\\ \hline \{\boldsymbol{\alpha}_e\}\end{array}\right\}=\left\{\begin{array}{c}u_x^1\\u_y^1\\u_x^2\\u_y^2\\u_x^3\\u_y^3\\ \hline \alpha_x^4\\\alpha_y^4\end{array}\right\} \tag{5.216}$$

Hence, the external virtual work becomes:

$$W_{ext}=\{\mathbf{a}_e^*\}^T\{\mathbf{F}\}+\{\boldsymbol{\alpha}_e^*\}^T\{\overline{\mathbf{F}}\}=\left\{\{\mathbf{a}_e^*\}^T \quad \{\boldsymbol{\alpha}_e^*\}^T\right\}\left\{\begin{array}{c}\{\mathbf{F}\}\\\{\overline{\mathbf{F}}\}\end{array}\right\} \tag{5.217}$$

We consider the strain field and the virtual strain field are compound by two parts:

$$\{\boldsymbol{\varepsilon}\}=\{\overline{\boldsymbol{\varepsilon}}\}+\{\widetilde{\boldsymbol{\varepsilon}}\} \quad ; \quad \{\boldsymbol{\varepsilon}^*\}=\{\overline{\boldsymbol{\varepsilon}}^*\}+\{\widetilde{\boldsymbol{\varepsilon}}^*\} \tag{5.218}$$

thus the internal virtual work becomes:

$$W_{int}=\int_V\{\boldsymbol{\sigma}\}^T\{\boldsymbol{\varepsilon}\}\,dV=\int_V\{\boldsymbol{\sigma}\}^T\left\{\{\overline{\boldsymbol{\varepsilon}}^*\}+\{\widetilde{\boldsymbol{\varepsilon}}^*\}\right\}dV=\int_V\left\{\{\overline{\boldsymbol{\varepsilon}}^*\}+\{\widetilde{\boldsymbol{\varepsilon}}^*\}\right\}^T\{\boldsymbol{\sigma}\}\,dV \tag{5.219}$$

Symmetric formulation

The discretization for the first approximation is:

$$\{\boldsymbol{\varepsilon}\}=\underbrace{[\mathbf{B}]\{\mathbf{a}\}}_{\{\overline{\boldsymbol{\varepsilon}}\}}+\underbrace{[\mathbf{G}_e]\{\boldsymbol{\alpha}_e\}}_{\{\widetilde{\boldsymbol{\varepsilon}}\}}=\left\{[\mathbf{B}] \quad [\mathbf{G}_e]\right\}\left\{\begin{array}{c}\{\mathbf{a}_e\}\\\{\boldsymbol{\alpha}_e\}\end{array}\right\}; \qquad \{\boldsymbol{\varepsilon}^*\}=\underbrace{[\mathbf{B}]\{\mathbf{a}_e^*\}}_{\{\overline{\boldsymbol{\varepsilon}}^*\}}+\underbrace{[\mathbf{G}_e]\{\boldsymbol{\alpha}_e^*\}}_{\{\widetilde{\boldsymbol{\varepsilon}}^*\}}=\left\{[\mathbf{B}] \quad [\mathbf{G}_e]\right\}\left\{\begin{array}{c}\{\mathbf{a}_e^*\}\\\{\boldsymbol{\alpha}_e^*\}\end{array}\right\} \tag{5.220}$$

Notice that we have used the same approximation function $[\mathbf{B}]$, $[\mathbf{G}_e]$ for virtual and real strains. Then, the stress field can be written as follows:

$$\{\boldsymbol{\sigma}\}=[\mathcal{C}]\{\boldsymbol{\varepsilon}\}=[\mathcal{C}]\left\{[\mathbf{B}] \quad [\mathbf{G}_e]\right\}\left\{\begin{array}{c}\{\mathbf{a}_e\}\\\{\boldsymbol{\alpha}_e\}\end{array}\right\} \tag{5.221}$$

By replace the approximations (5.220) and (5.221) into the equation (5.219), the internal virtual work becomes:

$$\begin{aligned} W_{int} &= \int_V \left\{ \{\bar{\boldsymbol{\varepsilon}}^*\} + \{\tilde{\boldsymbol{\varepsilon}}^*\} \right\}^T \{\boldsymbol{\sigma}\}\, dV = \int_V \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \begin{Bmatrix} \{\mathbf{a}_e^*\} \\ \{\boldsymbol{\alpha}_e^*\} \end{Bmatrix} \right\}^T [\mathcal{C}] \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \right\} \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} dV \\ &= \int_V \begin{Bmatrix} \{\mathbf{a}_e^*\} \\ \{\boldsymbol{\alpha}_e^*\} \end{Bmatrix}^T \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \right\}^T [\mathcal{C}] \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \right\} \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} dV \\ &= \left[\{\mathbf{a}_e^*\}^T \quad \{\boldsymbol{\alpha}_e^*\}^T \right] \int_V \begin{Bmatrix} [\mathbf{B}]^T \\ [\mathbf{G}_e]^T \end{Bmatrix} [\mathcal{C}] \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \right\} dV \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \end{aligned} \tag{5.222}$$

By apply $W_{ext} = W_{int}$, (see Eq. (5.217) and (5.222)), we can obtain

$$\begin{Bmatrix} \{\mathbf{F}\} \\ \{\overline{\mathbf{F}}\} \end{Bmatrix} = \int_{V_e} \begin{bmatrix} [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{B}]^T [\mathcal{C}][\mathbf{G}_e] \\ [\mathbf{G}_e]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{G}_e]^T [\mathcal{C}][\mathbf{G}_e] \end{bmatrix} dV \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \tag{5.223}$$

and considering the traction vector continuity, i.e. $\{\overline{\mathbf{F}}\} = \{\mathbf{0}\}$, we obtain:

$$\begin{Bmatrix} \{\mathbf{F}\} \\ \{\mathbf{0}\} \end{Bmatrix} = \underbrace{\int_{V_e} \begin{bmatrix} [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{B}]^T [\mathcal{C}][\mathbf{G}_e] \\ [\mathbf{G}_e]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{G}_e]^T [\mathcal{C}][\mathbf{G}_e] \end{bmatrix} dV}_{=[\mathbf{K}_e]} \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \tag{5.224}$$

Anti-symmetric formulation

Now consider the real and virtual strain approximation by:

$$\begin{aligned} \{\boldsymbol{\varepsilon}\} &= \underbrace{[\mathbf{B}]\{\mathbf{a}_e\}}_{\{\bar{\boldsymbol{\varepsilon}}\}} + \underbrace{[\mathbf{G}_e]\{\boldsymbol{\alpha}_e\}}_{\{\tilde{\boldsymbol{\varepsilon}}\}} = \left[[\mathbf{B}] \quad [\mathbf{G}_e] \right] \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \\ \{\boldsymbol{\varepsilon}^*\} &= [\mathbf{B}]\{\mathbf{a}_e^*\} + [\mathbf{G}_e^*]\{\boldsymbol{\alpha}_e^*\} = \left[[\mathbf{B}] \quad [\mathbf{G}_e^*] \right] \begin{Bmatrix} \{\mathbf{a}_e^*\} \\ \{\boldsymbol{\alpha}_e^*\} \end{Bmatrix} \end{aligned} \tag{5.225}$$

where we are considering different approximation functions for virtual and real strains i.e. $[\mathbf{G}_e] \neq [\mathbf{G}_e^*]$.

Using equation (5.219), and discretization (5.225) we can obtain:

$$W_{int} = \left\{ \{\mathbf{a}_e^*\}^T \quad \{\boldsymbol{\alpha}_e^*\}^T \right\} \int_{V_e} \begin{Bmatrix} [\mathbf{B}]^T \\ [\mathbf{G}_e^*]^T \end{Bmatrix} [\mathcal{C}] \left\{ [\mathbf{B}] \quad [\mathbf{G}_e] \right\} dV \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \tag{5.226}$$

Considering $W_{ext} = W_{int}$ and considering the traction vector continuity, we can obtain:

$$\begin{Bmatrix} \{\mathbf{F}\} \\ \{\mathbf{0}\} \end{Bmatrix} = \underbrace{\int_{\mathcal{B}_e} \begin{bmatrix} [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{B}]^T [\mathcal{C}][\mathbf{G}_e] \\ [\mathbf{G}_e^*]^T [\mathcal{C}][\mathbf{B}] & [\mathbf{G}_e^*]^T [\mathcal{C}][\mathbf{G}_e] \end{bmatrix} dV}_{=[\mathbf{K}_e]} \begin{Bmatrix} \{\mathbf{a}_e\} \\ \{\boldsymbol{\alpha}_e\} \end{Bmatrix} \tag{5.227}$$

According to Jirásek(1998) there are three major classes of these models:

- SOS (Statically Optimal Symmetric) formulation cannot properly reflect the kinematics of a completely open crack but it gives a natural stress continuity condition;
- KOS (Kinematically Optimal Symmetric) formulation describes the kinematic aspects satisfactorily but leads to an awkward relationship between the stress in the continuous part of the element and the tractions across the discontinuity line. These findings motivate the development of the nonsymmetric;
- SKON (Statically and Kinematically Optimal Nonsymmetric) formulation, which combines the strong points of each of the symmetric formulations.

Reference

JIRÁSEK, M. (1998). Finite elements with embedded cracks. *LSC Internal Report 98/01*, April.

Problem 5.26

Consider a rod of length L and cross-sectional area A which undergoes deformation because of its own weight, (see Figure 5.24 (a)). The rod is fixed at the top and is in static equilibrium. Use the total potential energy to obtain an analogous equation as the one obtained in **Problem 5.25** in **NOTE 4**, i.e. obtain an equivalent equation $[\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} = \{\boldsymbol{F}^{(e)}\}$ associated with this problem. Obtain also the displacement field.

Hypothesis: Homogeneous isotropic linear elastic material, small deformation regime.

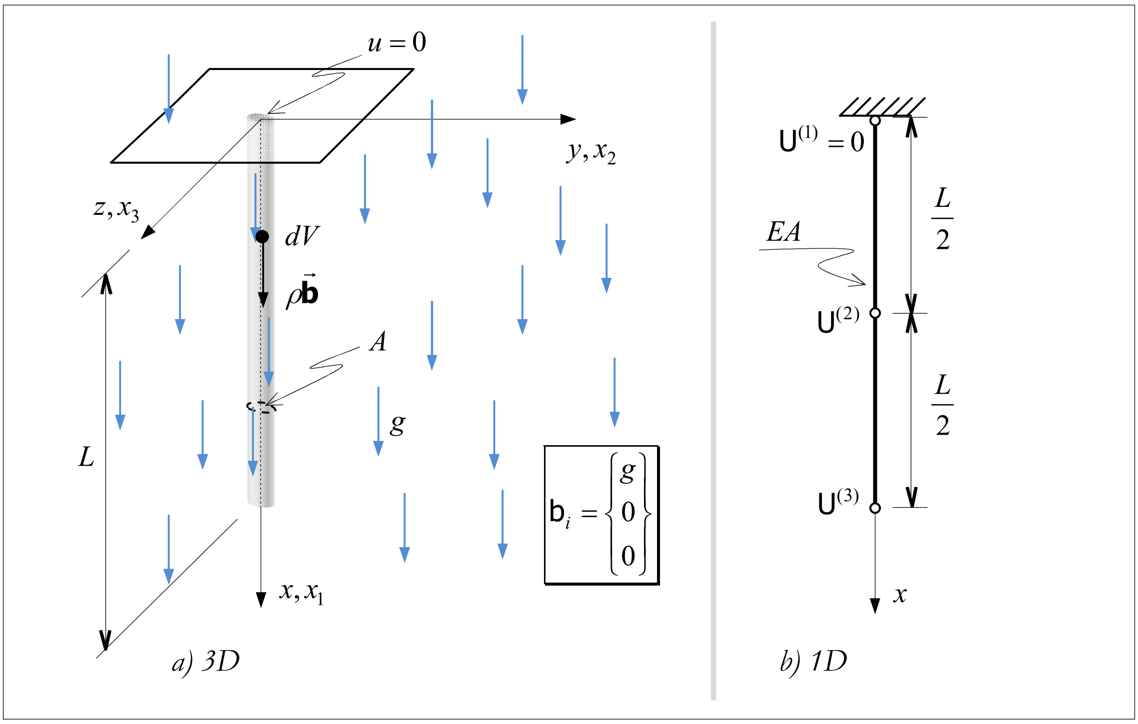


Figure 5.24

Solution:

To find out which displacement approach we must adopt, we will analyze the equilibrium equations ($\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$):

$$\sigma_{ij,j} + \rho \mathrm{b}_i = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i = 0_i$$

$$\Rightarrow \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + \rho \mathrm{b}_1 = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + \rho \mathrm{b}_2 = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + \rho \mathrm{b}_3 = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = -\rho \mathrm{b}_1 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = -\rho \mathrm{b}_2 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = -\rho \mathrm{b}_3 \end{cases}$$

and for this problem we have:

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \sigma_{11} = \mathbb{C}^e_{1111}\varepsilon_{11} \Rightarrow \sigma = E\varepsilon$$

Then, the equilibrium equations reduce to:

$$\Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} = -\rho g \quad \xrightarrow{\text{Engineering notation}} \quad \dfrac{\partial \sigma}{\partial x} = -\rho g \\ 0 = 0 \\ 0 = 0 \end{cases} \tag{5.228}$$

Note that the term ρg is constant in the rod, and according to the above equilibrium equation, the stress σ must be a linear function in x. And if we consider that $\sigma = E\varepsilon$, ε also requires a linear function in x, and as a consequence the displacement u must be a quadratic function in x since $\varepsilon = \dfrac{\partial u}{\partial x}$.

Then, the displacement field will be approached by the quadratic function ($u(x) = a_1 + a_2 x + a_3 x^2$), hence we will need three points to be able to define this function. We will adopted the points: $x = 0$, $x = \dfrac{L}{2}$ and $x = L$. With that we can obtain:

$$\left.\begin{aligned} & u(x=0) = \mathsf{U}^{(1)} = a_1 \\ & u(x = \frac{L}{2}) = \mathsf{U}^{(2)} = a_1 + a_2 \frac{L}{2} + a_3 \frac{L^2}{4} \\ & u(x = L) = \mathsf{U}^{(3)} = a_1 + a_2 L + a_3 L^2 \end{aligned}\right\} \quad \Rightarrow \quad \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Taking the reverse form of the above equation we can obtain:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} L^2 & 0 & 0 \\ -3L & 4L & -L \\ 2 & -4 & 2 \end{bmatrix} \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} a_1 = \mathsf{U}^{(1)} \\ a_2 = \dfrac{-3}{4}\mathsf{U}^{(1)} + \dfrac{4}{L}\mathsf{U}^{(2)} - \dfrac{1}{L}\mathsf{U}^{(3)} \\ a_3 = \dfrac{2}{L^2}\mathsf{U}^{(1)} - \dfrac{4}{L^2}\mathsf{U}^{(2)} + \dfrac{2}{L^2}\mathsf{U}^{(3)} \end{cases}$$

With that the displacement field in terms of $\mathsf{U}^{(1)}$, $\mathsf{U}^{(2)}$, and $\mathsf{U}^{(3)}$ is given by:

$$u = a_1 + a_2 x + a_3 x^2 = \mathsf{U}^{(1)} + \left(\frac{-3}{4}\mathsf{U}^{(1)} + \frac{4}{L}\mathsf{U}^{(2)} - \frac{1}{L}\mathsf{U}^{(3)} \right) x + \left(\frac{2}{L^2}\mathsf{U}^{(1)} - \frac{4}{L^2}\mathsf{U}^{(2)} + \frac{2}{L^2}\mathsf{U}^{(3)} \right) x^2$$

by simplifying the above equation we can obtain:

$$\begin{aligned} u(x) &= \left(1 - \frac{3x}{L} + \frac{2x^2}{L^2} \right)\mathsf{U}^{(1)} + \left(\frac{4x}{L} - \frac{4x^2}{L^2} \right)\mathsf{U}^{(2)} + \left(\frac{-x}{L} + \frac{2x^2}{L^2} \right)\mathsf{U}^{(3)} \\ &= N_1 \mathsf{U}^{(1)} + N_2 \mathsf{U}^{(2)} + N_3 \mathsf{U}^{(3)} \\ &= \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} \\ &= [\boldsymbol{N}(x)]\{\mathbf{u}^{(e)}\} \end{aligned} \tag{5.229}$$

where $N_1(x)$, $N_2(x)$ and $N_3(x)$ are the shape functions.

The goal now is to express the total potential energy in terms of $\mathsf{U}^{(1)}$, $\mathsf{U}^{(2)}$ and $\mathsf{U}^{(3)}$.

The term U^{ext}, (see equation (5.184)), becomes:

$$U^{ext} = \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS_\sigma + \int_V (\rho \vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV = \int_V (\rho \vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV = \int_x \rho g u(x) A dx = \rho g A \int_x u(x) dx$$
$$= \rho g A \int_0^L \left[\left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right) \mathsf{U}^{(1)} + \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right) \mathsf{U}^{(2)} + \left(\frac{-x}{L} + \frac{2x^2}{L^2}\right) \mathsf{U}^{(3)} \right] dx$$

After the integration is taken place we can obtain:

$$U^{ext}(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)}) = \rho g A \left[\frac{L}{6} \mathsf{U}^{(1)} + \frac{2L}{3} \mathsf{U}^{(2)} + \frac{L}{6} \mathsf{U}^{(3)} \right] \tag{5.230}$$

The term U^{int} for this problem is the same as the one given by the equation in (5.193), i.e.

$$U^{int} = \frac{1}{2} \int_V E \varepsilon^2 \, dV = \frac{1}{2} \int_V E \left(\frac{\partial u(x)}{\partial x} \right)^2 dV = \frac{1}{2} \int_0^L AE \left(\frac{\partial u(x)}{\partial x} \right)^2 dx \tag{5.231}$$

where

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left[\left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right) \mathsf{U}^{(1)} + \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right) \mathsf{U}^{(2)} + \left(\frac{-x}{L} + \frac{2x^2}{L^2}\right) \mathsf{U}^{(3)} \right]$$
$$= \left(\frac{-3}{L} + \frac{4x}{L^2} \right) \mathsf{U}^{(1)} + \left(\frac{4}{L} - \frac{8x}{L^2} \right) \mathsf{U}^{(2)} + \left(\frac{-1}{L} + \frac{4x}{L^2} \right) \mathsf{U}^{(3)}$$

thus

$$U^{int} = \frac{EA}{2} \int_0^L \left[\left(\frac{-3}{L} + \frac{4x}{L^2} \right) \mathsf{U}^{(1)} + \left(\frac{4}{L} - \frac{8x}{L^2} \right) \mathsf{U}^{(2)} + \left(\frac{-1}{L} + \frac{4x}{L^2} \right) \mathsf{U}^{(3)} \right]^2 dx$$

By solving the above integral we can obtain:

$$U^{int}(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)}) = \frac{EA}{6L} \left[7\mathsf{U}^{(1)2} + 16\mathsf{U}^{(2)2} + 7\mathsf{U}^{(3)2} + 2\mathsf{U}^{(1)}\mathsf{U}^{(3)} - 16\mathsf{U}^{(1)}\mathsf{U}^{(2)} - 16\mathsf{U}^{(2)}\mathsf{U}^{(3)} \right] \tag{5.232}$$

The total potential energy is given by:

$$\Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)}) = U^{int} - U^{ext}$$
$$= \frac{EA}{6L} \left[7\mathsf{U}^{(1)2} + 16\mathsf{U}^{(2)2} + 7\mathsf{U}^{(3)2} + 2\mathsf{U}^{(1)}\mathsf{U}^{(3)} - 16\mathsf{U}^{(1)}\mathsf{U}^{(2)} - 16\mathsf{U}^{(2)}\mathsf{U}^{(3)} \right]$$
$$- \rho g A \left[\frac{L}{6} \mathsf{U}^{(1)} + \frac{2L}{3} \mathsf{U}^{(2)} + \frac{L}{6} \mathsf{U}^{(3)} \right]$$

As we are looking for the stationary state, the following must be fulfilled:

$$\begin{cases} \dfrac{\partial \Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)})}{\partial \mathsf{U}^{(1)}} = 0 \quad \Rightarrow \quad \dfrac{EA}{3L}\left(7\mathsf{U}^{(1)} - 8\mathsf{U}^{(2)} + \mathsf{U}^{(3)}\right) - \dfrac{\rho g A L}{6} = 0 \\ \dfrac{\partial \Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)})}{\partial \mathsf{U}^{(2)}} = 0 \quad \Rightarrow \quad \dfrac{EA}{3L}\left(-8\mathsf{U}^{(1)} + 16\mathsf{U}^{(2)} - 8\mathsf{U}^{(3)}\right) - \dfrac{2\rho g A L}{3} = 0 \\ \dfrac{\partial \Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}, \mathsf{U}^{(3)})}{\partial \mathsf{U}^{(3)}} = 0 \quad \Rightarrow \quad \dfrac{EA}{3L}\left(\mathsf{U}^{(1)} - 8\mathsf{U}^{(2)} + 7\mathsf{U}^{(3)}\right) - \dfrac{\rho g A L}{6} = 0 \end{cases}$$

Rearranging the above set of equations in matrix form we can obtain:

$$\frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\} = \{\boldsymbol{F}^{(e)}\} \tag{5.233}$$

Note that $[\boldsymbol{K}^{(e)}]$ has no inverse, since $\det[\boldsymbol{K}^{(e)}] = 0$. To solve the problem we have to introduce the boundary conditions. According to the problem statement, the displacement at $x = 0$ is equal to zero, i.e. $\mathsf{U}^{(1)} = 0$. We apply this boundary condition by eliminate the first line and column of the system (5.233), in other words we eliminate the terms associated with the degree-of-freedom $\mathsf{U}^{(1)}$, i.e.:

$$\frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \quad or \quad \frac{EA}{3L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & -8 \\ 0 & -8 & 7 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \frac{EA}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \left(\frac{EA}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\right)^{-1}\left(\frac{EA}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\right)\begin{Bmatrix} \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\left(\frac{EA}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\right)^{-1}\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\left(\frac{EA}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\right)^{-1}\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g A L}{6}\left(\frac{L}{16EA}\begin{bmatrix} 7 & 8 \\ 8 & 1 \end{bmatrix}\right)\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \mathsf{U}^{(2)} \\ \mathsf{U}^{(3)} \end{Bmatrix} = \frac{\rho g L^2}{8E}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$

Now, if we substitute the values of $\mathsf{U}^{(1)}$, $\mathsf{U}^{(2)}$, and $\mathsf{U}^{(3)}$ into the displacement field, (see equation (5.229)), we can obtain:

$$\begin{aligned} u &= \left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right)\mathsf{U}^{(1)} + \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right)\mathsf{U}^{(2)} + \left(\frac{-x}{L} + \frac{2x^2}{L^2}\right)\mathsf{U}^{(3)} \\ &= \left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right)0 + \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right)\left(\frac{3\rho g L^2}{8E}\right) + \left(\frac{-x}{L} + \frac{2x^2}{L^2}\right)\left(\frac{4\rho g L^2}{8E}\right) \end{aligned}$$

By simplifying the above equation we can obtain:

$$u^{(Q)} = \frac{\rho g}{2E}(2Lx - x^2) \tag{5.234}$$

which is also the exact solution for the proposed problem. Then, the strain and stress field can be obtained as follows:

$$\varepsilon^{(Q)} = \frac{\partial u(x)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\rho g}{2E}(2Lx - x^2)\right] = \frac{\rho g}{E}(L - x) \tag{5.235}$$

and

$$\sigma^{(Q)} = \varepsilon^{(Q)} E = \rho g (L - x) \tag{5.236}$$

If we replace the nodal displacement into the total potential energy we can obtain:

$$\Pi^{(Q)} = -\frac{1}{6}\frac{(\rho g)^2 A L^3}{E}$$

NOTE 1: The shape functions $[\boldsymbol{N}(\vec{\boldsymbol{x}})]$

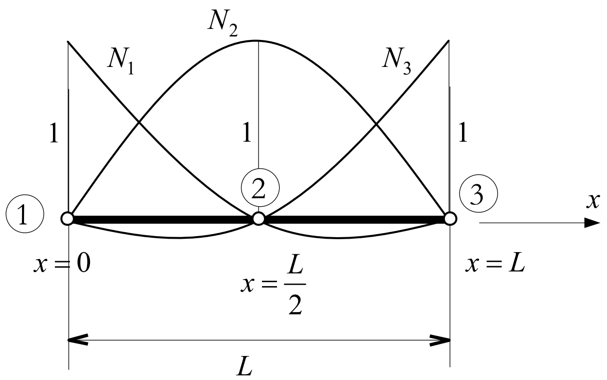
Note that the shape functions obtained (5.229) are:

$$N_1(x) = 1 - \frac{3x}{L} + \frac{2x^2}{L^2} \quad \Rightarrow \quad \begin{cases} N_1(x=0)=1 \\ N_1(x=\frac{L}{2})=0 \\ N_1(x=L)=0 \end{cases}$$

$$N_2(x) = \frac{4x}{L} - \frac{4x^2}{L^2} \quad \Rightarrow \quad \begin{cases} N_2(x=0)=0 \\ N_2(x=\frac{L}{2})=1 \\ N_2(x=L)=0 \end{cases}$$

$$N_3(x) = \frac{-x}{L} + \frac{2x^2}{L^2} \quad \Rightarrow \quad \begin{cases} N_3(x=0)=0 \\ N_3(x=\frac{L}{2})=0 \\ N_3(x=L)=1 \end{cases}$$

Note also that $N_1(x)+N_2(x)+N_3(x)=1$ holds. And these functions are drawn into the domain as indicated in Figure 5.25.



$$\begin{cases} N_1 = 1 - \frac{3x}{L} + \frac{2x^2}{L^2} \\ N_2 = \frac{4x}{L} - \frac{4x^2}{L^2} \\ N_3 = \frac{-x}{L} + \frac{2x^2}{L^2} \end{cases}$$

$$N_1(x)+N_2(x)+N_3(x)=1$$

Figure 5.25: Shape functions (1D) – quadratic function.

NOTE 2: Analytical solution (the exact one) – by using the direct integration

We start from the equilibrium equation $\frac{\partial \sigma}{\partial x} = -\rho g$, (see equation (5.228)), and by integrating the equation over x we can obtain:

$$\frac{\partial \sigma}{\partial x} = -\rho g \xrightarrow{\text{integrating}} \int \partial\sigma = -\int \rho g \partial x \quad \Rightarrow \quad \sigma = -\rho g x + C_1$$

The constant of integration can be obtained at $x=0$. In this situation the total force at $x=0$ is given by $F=\rho g V = \rho g A L$, and the stress can be obtained by $\sigma(x=0) = \frac{F}{A} = \frac{\rho g A L}{A} = \rho g L$. Then, the constant of integration becomes $\sigma(x=0) = C_1 = \rho g L$. Hence, the stress field becomes:

$$\sigma = -\rho g x + \rho g L = \rho g (L - x)$$

Using the *constitutive equation*, the strain field can be obtained as follows:

$$\sigma = E\varepsilon \qquad \Rightarrow \qquad \varepsilon = \frac{\sigma}{E} = \frac{\rho g(L-x)}{E}$$

Taking into account the relationship $\varepsilon = \dfrac{\partial u}{\partial x}$ *(kinematic equation)*, and by integrating the equation over x we can obtain:

$$\varepsilon = \frac{\partial u}{\partial x} \xrightarrow{\text{integrating}} \int \partial u = \int \varepsilon \partial x = \int \left(\frac{\rho g(L-x)}{E} \right) \partial x \qquad \Rightarrow \qquad u(x) = \frac{\rho g}{E}\left(Lx - \frac{x^2}{2} \right) + C_2$$

At $x = 0$ there is no displacement, so, $u(x=0) = 0 \Rightarrow C_2 = 0$, then, the displacement field, (see Figure 5.26), becomes:

$$u(x) = \frac{\rho g}{E}\left(Lx - \frac{x^2}{2} \right) = \frac{\rho g}{2E}(2Lx - x^2) \xrightarrow{x=L} u(x=L) = \frac{\rho g L^2}{2E} \tag{5.237}$$

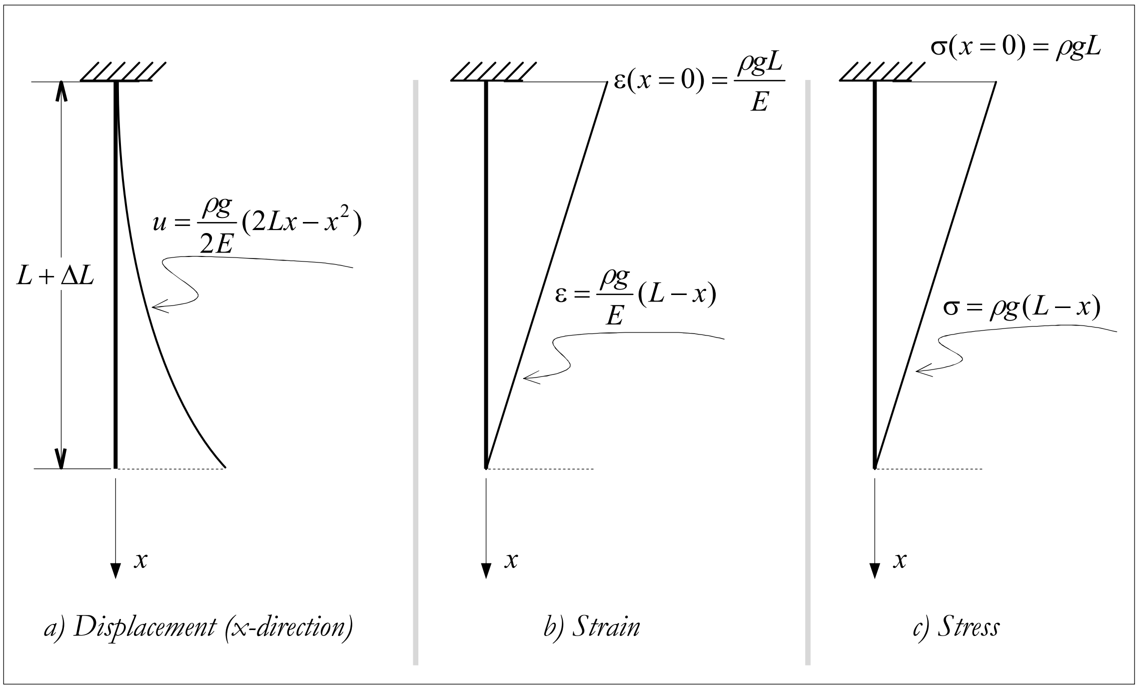


Figure 5.26

NOTE 3: Note that for simple problems the analytical solution is very easy to be obtained, and this solution serves to indicate how good is the technique employed. Note that the analytical solution (equation (5.237)), is the same as the numerical solution (5.234) in which we have used the quadratic function to approach the displacement field, and if we consider a cubic function to approach the displacement field the solution must be the same. Let us check this fact:

Displacement field (cubic function): $u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 = a_2 x + a_3 x^2 + a_4 x^3$.

Note the, at $x = 0$ there is no displacement, so, $u(x=0) = a_1 = 0$.

Then, $u(x) = a_2 x + a_3 x^2 + a_4 x^3 \qquad \Rightarrow \qquad \dfrac{\partial u}{\partial x} = a_2 + 2a_3 x + 3a_4 x^2$

Internal Potential in function of (a_2, a_3, a_4):

$$U^{int} = \frac{AE}{2}\int_0^L \left(\frac{\partial u(x)}{\partial x}\right)^2 dx = \frac{AE}{2}\int_0^L (a_2 + 2a_3x + 3a_4x^2)^2 \ dx$$
$$= \frac{AEL}{30}(27L^4a_4^2 + 45L^3a_4a_3 + 30L^2a_2a_4 + 20L^2a_3^2 + 30La_2a_3 + 15a_2^2)$$

External Potential in function of (a_2, a_3, a_4):

$$U^{ext} = \rho gA\int_x u(x)dx = \rho gA\int_0^L (a_2x + a_3x^2 + a_4x^3)dx = \rho gA\left(\frac{1}{2}L^2a_2 + \frac{1}{3}L^3a_3 + \frac{1}{4}L^4a_4\right)$$

The total potential energy in function of (a_2, a_3, a_4):

$$\Pi(a_2, a_3, a_4) = U^{int} - U^{ext}$$
$$= \frac{AEL}{30}(27L^4a_4^2 + 45L^3a_4a_3 + 30L^2a_2a_4 + 20L^2a_3^2 + 30La_2a_3 + 15a_2^2)$$
$$- \rho gA\left(\frac{1}{2}L^2a_2 + \frac{1}{3}L^3a_3 + \frac{1}{4}L^4a_4\right)$$

As we are looking for the stationary state, the following must be fulfilled:

$$\begin{cases} \dfrac{\partial \Pi(a_2,a_3,a_4)}{\partial a_2} = 0 & \Rightarrow \quad \dfrac{EAL}{30}(30L^2a_4 + 30La_3 + 30a_2) - \dfrac{\rho gAL^2}{2} = 0 \\ \dfrac{\partial \Pi(a_2,a_3,a_4)}{\partial a_3} = 0 & \Rightarrow \quad \dfrac{EAL}{30}(45L^3a_4 + 40L^2a_3 + 30La_2) - \dfrac{\rho gAL^3}{3} = 0 \\ \dfrac{\partial \Pi(a_2,a_3,a_4)}{\partial a_4} = 0 & \Rightarrow \quad \dfrac{EAL}{30}(54L^4a_4 + 45L^3a_3 + 30L^2a_2) - \dfrac{\rho gAL^4}{4} = 0 \end{cases}$$

Simplifying and rearranging the above set of equations in matrix form we can obtain:

$$\frac{E}{30}\begin{bmatrix} 30 & 30L & 30L^2 \\ 30L & 40L^2 & 45L^3 \\ 30L^2 & 45L^3 & 54L^4 \end{bmatrix}\begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{\rho g}{2}\begin{bmatrix} 6L \\ 4L^2 \\ 3L^3 \end{bmatrix} \xrightarrow{\text{Solve}} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{1}{2E}\begin{bmatrix} 2\rho gL \\ -\rho g \\ 0 \end{bmatrix}$$

Then, the displacement field $(u(x) = a_2x + a_3x^2 + a_4x^3)$ becomes:

$$u(x) = a_2x + a_3x^2 + a_4x^3 = \frac{1}{2E}(2\rho gLx - \rho gx^2) = \frac{\rho g}{2E}(2Lx - x^2)$$

which is the same solution as the one provided by the quadratic function used to approach the displacement field.

Next, let us adopt a linear function to approach the displacement field $u(x) = a_2x$, then

Internal Potential:

$$U^{int} = \frac{AE}{2}\int_0^L \left(\frac{\partial u(x)}{\partial x}\right)^2 dx = \frac{AE}{2}\int_0^L (a_2)^2 \ dx = \frac{AEL}{2}a_2^2$$

External Potential:

$$U^{ext} = \rho g A \int_x u(x) dx = \rho g A \int_0^L (a_2 x) dx = \frac{1}{2} \rho g A L^2 a_2$$

The total potential energy:

$$\Pi(a_2) = U^{int} - U^{ext} = \frac{AEL}{2} a_2^2 - \frac{1}{2} \rho g A L^2 a_2$$

The equilibrium point:

$$\frac{\partial \Pi(a_2)}{\partial a_2} = 0 \quad \Rightarrow \quad EALa_2 - \frac{\rho g A L^2}{2} = 0 \quad \Rightarrow \quad a_2 = \frac{\rho g L}{2E}$$

Then

$$u^{(L)} = a_2 x = \frac{\rho g L}{2E} x \quad ; \quad \varepsilon^{(L)} = \frac{\partial u^{(L)}}{\partial x} = \frac{\rho g L}{2E} \quad ; \quad \sigma^{(L)} = E\varepsilon = \frac{1}{2} \rho g L$$

Note that, for this case a linear function is not a good approach.

The total $\Pi^{(L)} = \frac{AEL}{2} a_2^2 - \frac{1}{2} \rho g A L^2 a_2 = -\frac{1}{8} \frac{(\rho g)^2 A L^3}{E}$

NOTE 4:

Next, we will establish the stiffness matrix $[\boldsymbol{K}^{(e)}]$ of the rod element by considering the linear approximation for the displacement field. By adopting the linear function $u(x) = a_1 + a_2 x$ we can obtain:

$$\left.\begin{array}{l} u(x=0) = \mathsf{U}^{(1)} = a_1 \\ u(x=L) = \mathsf{U}^{(2)} = a_1 + a_2 L \end{array}\right\} \Rightarrow \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \xrightarrow{\text{Reverse}} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix}$$

Then, the displacement field becomes:

$$\begin{aligned} u(x) = a_1 + a_2 x &= \mathsf{U}^{(1)} + \left(\frac{-1}{L} \mathsf{U}^{(1)} + \frac{1}{L} \mathsf{U}^{(2)} \right) x = \left(1 - \frac{x}{L}\right) \mathsf{U}^{(1)} + \frac{x}{L} \mathsf{U}^{(2)} \\ &= \left[\left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right) \right] \begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix} \\ &= [\boldsymbol{N}]\{\mathbf{u}^{(e)}\} \end{aligned}$$

whose equation has already been obtained in **Problem 5.25** (NOTE 4). Then, the stiffness matrix will be the same, but the nodal forces will be not the same.

Internal Potential:

$$U^{int} = \frac{AE}{2} \int_0^L \left(\frac{\partial u(x)}{\partial x} \right)^2 dx = \frac{AE}{2} \int_0^L \left(\frac{-1}{L} \mathsf{U}^{(1)} + \frac{1}{L} \mathsf{U}^{(2)} \right)^2 dx = \frac{AE}{2L} \left(\mathsf{U}^{(1)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(2)2} \right)$$

External Potential:

$$U^{ext} = \rho g A \int_x u(x) dx = \rho g A \int_0^L \left[\left(1 - \frac{x}{L}\right) \mathsf{U}^{(1)} + \frac{x}{L} \mathsf{U}^{(2)} \right] dx = \frac{1}{2} \rho g A L (\mathsf{U}^{(1)} + \mathsf{U}^{(2)})$$

The total potential energy:

$$\Pi(\mathsf{U}^{(1)}, \mathsf{U}^{(2)}) = U^{int} - U^{ext} = \frac{AE}{2L} \left(\mathsf{U}^{(1)2} - 2\mathsf{U}^{(1)}\mathsf{U}^{(2)} + \mathsf{U}^{(2)2} \right) - \frac{1}{2} \rho g A L (\mathsf{U}^{(1)} + \mathsf{U}^{(2)})$$

The equilibrium point:

$$\frac{\partial \Pi(\mathsf{U}^{(1)},\mathsf{U}^{(2)})}{\partial \mathsf{U}^{(1)}}=0 \quad \Rightarrow \quad \frac{AE}{2L}(2\mathsf{U}^{(1)}-2\mathsf{U}^{(2)})-\frac{1}{2}\rho g A L=0 \quad \Rightarrow \quad \frac{AE}{L}(\mathsf{U}^{(1)}-\mathsf{U}^{(2)})=\frac{1}{2}\rho g A L$$

$$\frac{\partial \Pi(\mathsf{U}^{(1)},\mathsf{U}^{(2)})}{\partial \mathsf{U}^{(2)}}=0 \quad \Rightarrow \quad \frac{AE}{2L}(2\mathsf{U}^{(2)}-2\mathsf{U}^{(1)})-\frac{1}{2}\rho g A L=0 \quad \Rightarrow \quad \frac{AE}{L}(\mathsf{U}^{(2)}-\mathsf{U}^{(1)})=\frac{1}{2}\rho g A L$$

Simplifying and rearranging the above set of equations in matrix form we can obtain:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(1)} \\ \mathsf{U}^{(2)} \end{Bmatrix}=\frac{1}{2}\rho g A L\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{K}^{(e)}]\{\mathbf{u}^{(e)}\}=\{\boldsymbol{F}^{(e)}\}$$

As we have seen in the previous NOTE, there is an error when we use the linear approximation for the displacement field. Next, we will divide the domain in sub-domain. To establish the displacement field we will adopt a generic element, where the initial point is $x^{(i)}$ and the final point is $x^{(f)}$, (see Figure 5.27 (a)).

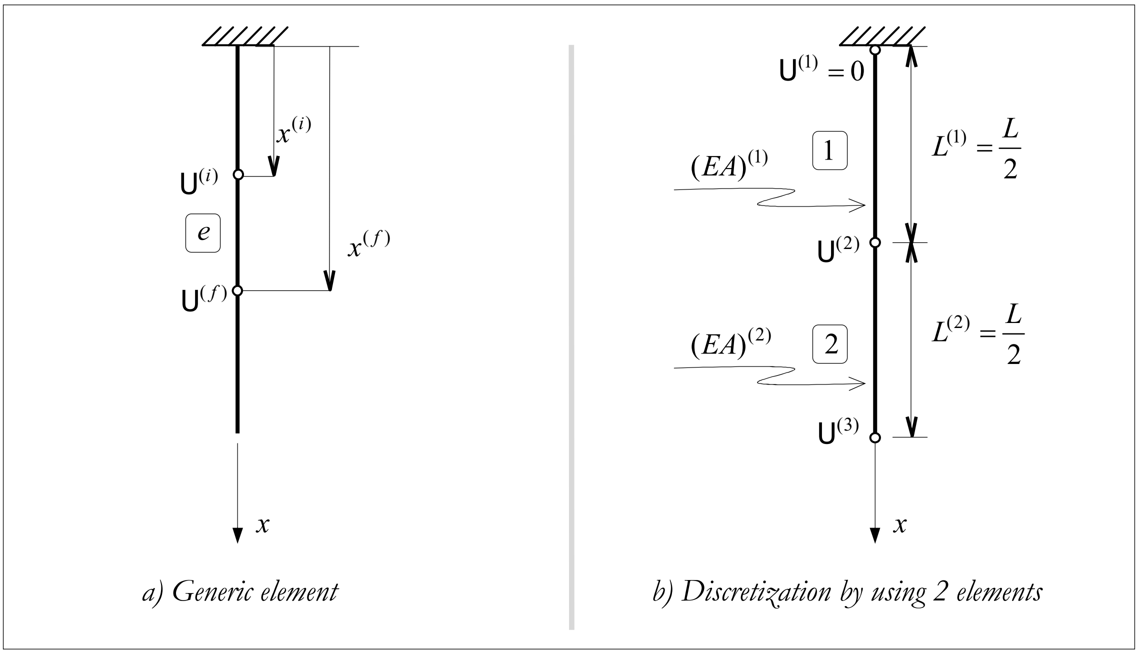


Figure 5.27

Taking into account the linear approximation $u(x)=a_1+a_2x$ we can obtain:

$$\left.\begin{aligned} u(x=x^{(i)})=\mathsf{U}^{(i)}=a_1+a_2x^{(i)} \\ u(x=x^{(f)})=\mathsf{U}^{(f)}=a_1+a_2x^{(f)} \end{aligned}\right\} \quad \Rightarrow \quad \begin{Bmatrix} \mathsf{U}^{(i)} \\ \mathsf{U}^{(f)} \end{Bmatrix}=\begin{bmatrix} 1 & x^{(i)} \\ 1 & x^{(f)} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\xrightarrow{Inverse}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=\frac{1}{(x^{(f)}-x^{(i)})}\begin{bmatrix} x^{(f)} & -x^{(i)} \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \mathsf{U}^{(i)} \\ \mathsf{U}^{(f)} \end{Bmatrix}$$

Then, the displacement field becomes:

$$u(x)=a_1+a_2x \quad \Rightarrow \quad u(x)=\frac{1}{(x^{(f)}-x^{(i)})}(x^{(f)}-x)\mathsf{U}^{(i)}+\frac{1}{(x^{(f)}-x^{(i)})}(x-x^{(i)})\mathsf{U}^{(f)}$$

and its derivative:

$$\frac{\partial u(x)}{\partial x}=\frac{-1}{(x^{(f)}-x^{(i)})}\mathsf{U}^{(i)}+\frac{1}{(x^{(f)}-x^{(i)})}\mathsf{U}^{(f)}$$

Let us divide the domain into 2 sub-domains (2 finite elements), (see Figure 5.27 (b)), where:

Element $e=1$:

$$\left.\begin{array}{l} x^{(i)}=0 \\ x^{(f)}=\dfrac{L}{2} \\ \mathsf{U}^{(i)}=\mathsf{U}^{(1)} \\ \mathsf{U}^{(f)}=\mathsf{U}^{(2)} \end{array}\right\} \Rightarrow \begin{cases} u^{(1)}=\left(1-\dfrac{2x}{L}\right)\mathsf{U}^{(1)}+\left(1-\dfrac{2x}{L}\right)\mathsf{U}^{(2)} \\ \dfrac{\partial u^{(1)}}{\partial x}=\left(\dfrac{-2}{L}\right)\mathsf{U}^{(1)}+\left(\dfrac{2}{L}\right)\mathsf{U}^{(2)} \end{cases}$$

Element $e=2$:

$$\left.\begin{array}{l} x^{(i)}=\dfrac{L}{2} \\ x^{(f)}=L \\ \mathsf{U}^{(i)}=\mathsf{U}^{(2)} \\ \mathsf{U}^{(f)}=\mathsf{U}^{(3)} \end{array}\right\} \Rightarrow \begin{cases} u^{(2)}=\left(2-\dfrac{2x}{L}\right)\mathsf{U}^{(2)}+\left(\dfrac{2x}{L}-1\right)\mathsf{U}^{(3)} \\ \dfrac{\partial u^{(2)}}{\partial x}=\left(\dfrac{-2}{L}\right)\mathsf{U}^{(2)}+\left(\dfrac{2}{L}\right)\mathsf{U}^{(3)} \end{cases}$$

Internal Potential:

$$\begin{aligned} U^{int} &=\frac{1}{2}\int_0^L AE\left(\frac{\partial u(x)}{\partial x}\right)^2 dx=\frac{1}{2}\int_0^{\frac{L}{2}}(AE)^{(1)}\left(\frac{\partial u^{(1)}}{\partial x}\right)^2 dx+\frac{1}{2}\int_{\frac{L}{2}}^{L}(AE)^{(2)}\left(\frac{\partial u^{(2)}}{\partial x}\right)^2 dx \\ &=\frac{(AE)^{(1)}}{L}\left(\mathsf{U}^{(1)2}-2\mathsf{U}^{(1)}\mathsf{U}^{(2)}+\mathsf{U}^{(2)2}\right)+\frac{(AE)^{(2)}}{L}\left(\mathsf{U}^{(2)2}-2\mathsf{U}^{(2)}\mathsf{U}^{(3)}+\mathsf{U}^{(3)2}\right) \end{aligned}$$

External Potential:

$$\begin{aligned} U^{ext} &=\rho g\int_x Au(x)dx=\rho g\int_0^{\frac{L}{2}}A^{(1)}u^{(1)}dx+\rho g\int_{\frac{L}{2}}^{L}A^{(2)}u^{(2)}dx \\ &=\frac{1}{4}\rho gA^{(1)}L\mathsf{U}^{(1)}+\frac{1}{4}\rho gL(A^{(1)}+A^{(2)})\mathsf{U}^{(2)}+\frac{1}{4}\rho gA^{(2)}L\mathsf{U}^{(3)} \end{aligned}$$

The total potential energy

$$\begin{aligned} \Pi(\mathsf{U}^{(1)},\mathsf{U}^{(2)},\mathsf{U}^{(3)}) &=U^{int}-U^{ext} \\ &=\frac{(AE)^{(1)}}{L}\left(\mathsf{U}^{(1)2}-2\mathsf{U}^{(1)}\mathsf{U}^{(2)}+\mathsf{U}^{(2)2}\right)+\frac{(AE)^{(2)}}{L}\left(\mathsf{U}^{(2)2}-2\mathsf{U}^{(2)}\mathsf{U}^{(3)}+\mathsf{U}^{(3)2}\right) \\ &\quad-\left(\frac{1}{4}\rho gA^{(1)}L\mathsf{U}^{(1)}+\frac{1}{4}\rho gL(A^{(1)}+A^{(2)})\mathsf{U}^{(2)}+\frac{1}{4}\rho gA^{(2)}L\mathsf{U}^{(3)}\right) \end{aligned}$$

The equilibrium point

$$\frac{\partial \Pi(\mathsf{U}^{(a)})}{\partial \mathsf{U}^{(1)}}=0 \quad \Rightarrow \quad \frac{2(AE)^{(1)}}{L}(\mathsf{U}^{(1)}-\mathsf{U}^{(2)})-\frac{1}{4}\rho g L A^{(1)}=0$$

$$\frac{\partial \Pi(\mathsf{U}^{(a)})}{\partial \mathsf{U}^{(2)}}=0 \quad \Rightarrow \quad \frac{2(AE)^{(2)}}{L}(\mathsf{U}^{(2)}-\mathsf{U}^{(1)})+\frac{2(AE)^{(2)}}{L}(\mathsf{U}^{(2)}-\mathsf{U}^{(3)})-\frac{1}{4}\rho g L(A^{(1)}+A^{(2)})=0$$

$$\frac{\partial \Pi(\mathsf{U}^{(a)})}{\partial \mathsf{U}^{(3)}}=0 \quad \Rightarrow \quad \frac{2(AE)^{(2)}}{L}(\mathsf{U}^{(3)}-\mathsf{U}^{(2)})-\frac{1}{4}\rho g L A^{(2)}=0$$

Rearranging the above set of equations in matrix form we can obtain:

$$\frac{2}{L}\begin{bmatrix} [(AE)^{(1)}] & [-(AE)^{(1)}] & \\ [-(AE)^{(1)}] & [(AE)^{(1)} + (AE)^{(2)}] & [-(AE)^{(2)}] \\ & [-(AE)^{(2)}] & [(AE)^{(2)}] \end{bmatrix}\begin{Bmatrix}\mathsf{U}^{(1)}\\ \mathsf{U}^{(2)}\\ \mathsf{U}^{(3)}\end{Bmatrix}=\frac{1}{4}\rho g L\begin{Bmatrix}[A^{(1)}] & \\ [A^{(1)}+ & A^{(2)}]\\ & [A^{(2)}]\end{Bmatrix}$$

$$[\boldsymbol{K}]\{\mathbf{u}\}=\{\boldsymbol{F}\}$$

Note that the above matrix could have been obtained directly if we consider the stiffness matrix of the element and the nodal force vector of the element e:

$$[\boldsymbol{k}^{(e)}]=\frac{(EA)^{(e)}}{L^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad ; \quad \{\boldsymbol{f}^{(e)}\}=\frac{1}{2}\rho g (LA)^{(e)}\begin{Bmatrix}1\\1\end{Bmatrix}$$

Element $e=1$, $(L^{(1)}=\frac{L}{2})$:

$$[\boldsymbol{k}^{(1)}]=\frac{2(EA)^{(1)}}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad ; \quad \{\boldsymbol{f}^{(1)}\}=\frac{1}{4}\rho g L A^{(1)}\begin{Bmatrix}1\\1\end{Bmatrix} \quad ; \quad \{\boldsymbol{u}^{(1)}\}=\begin{Bmatrix}\mathsf{U}^{(1)}\\ \mathsf{U}^{(2)}\end{Bmatrix}$$

Element $e=2$, $(L^{(2)}=\frac{L}{2})$:

$$[\boldsymbol{k}^{(2)}]=\frac{2(EA)^{(2)}}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad ; \quad \{\boldsymbol{f}^{(2)}\}=\frac{1}{4}\rho g L A^{(2)}\begin{Bmatrix}1\\1\end{Bmatrix} \quad ; \quad \{\boldsymbol{u}^{(2)}\}=\begin{Bmatrix}\mathsf{U}^{(2)}\\ \mathsf{U}^{(3)}\end{Bmatrix}$$

Then the global stiffness matrix $[\boldsymbol{K}]$ can be obtained by adding the contribution of each stiffness matrix of the element into $[\boldsymbol{K}]$, and the same to the global nodal vector. This process is called *the assemble process*.

Considering the $(EA)^{(1)}=(EA)^{(2)}=EA$ the set of discrete equations becomes:

$$\frac{2AE}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\mathsf{U}^{(1)}\\ \mathsf{U}^{(2)}\\ \mathsf{U}^{(3)}\end{Bmatrix}=\frac{1}{4}\rho g L A\begin{Bmatrix}1\\2\\1\end{Bmatrix}$$

Applying the boundary condition and solving the system the nodal displacement can be obtained:

$$\frac{2AE}{L}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\mathsf{U}^{(1)}\\ \mathsf{U}^{(2)}\\ \mathsf{U}^{(3)}\end{Bmatrix}=\frac{1}{4}\rho g L A\begin{Bmatrix}0\\2\\1\end{Bmatrix} \xrightarrow{\text{Solve}} \begin{Bmatrix}\mathsf{U}^{(1)}\\ \mathsf{U}^{(2)}\\ \mathsf{U}^{(3)}\end{Bmatrix}=\frac{\rho g L^2}{8E}\begin{Bmatrix}0\\3\\4\end{Bmatrix}$$

which matches the exact solution.

The procedure we have developed is the basis of the Finite Element Method which basically consists of:

- Adopt an approach to the unknown field;
- Split (discretize) the domain into sub-domain (finite element);
- Set the stiffness matrix of each sub-domain and the nodal force vector;
- Assemble the global stiffness matrix of the structure;
- Apply the boundary condition;
- Solve the system.

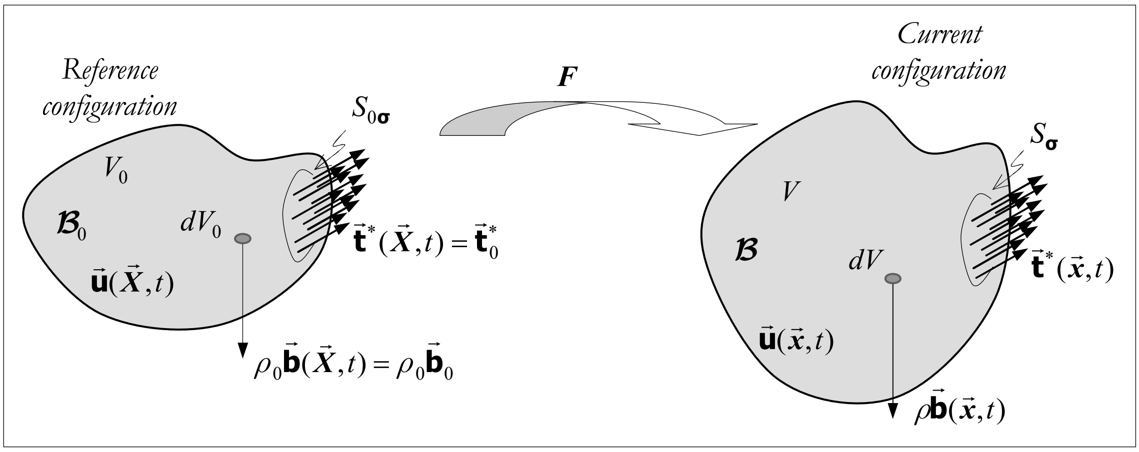
For more detail about Finite Element Method the reader is referred to Zienkiewicz & Taylor (1994), Bathe(1996).

Problem 5.27

Show that:

$$\boxed{\int_{S_\sigma} \vec{\mathbf{t}}^*(\vec{X},t)\cdot\dot{\overline{\vec{\mathbf{u}}}}\,dS_{0\sigma} + \int_{V_0}\rho_0\left[\vec{\mathbf{b}}(\vec{X},t)-\ddot{\vec{\mathbf{u}}}(\vec{X},t)\right]\cdot\dot{\overline{\vec{\mathbf{u}}}}\,dV_0 = \int_{V_0}\mathbf{P}:\nabla_{\vec{X}}\dot{\overline{\vec{\mathbf{u}}}}\,dV_0} \tag{5.238}$$

where $\dot{\overline{\vec{\mathbf{u}}}}$ is the virtual displacement field, and $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor.



Solution:

Although the variables $\vec{\mathbf{t}}^*(\vec{X},t)$ and $\vec{\mathbf{b}}(\vec{X},t)$ are not intrinsic variables of the reference configuration like the variables ρ_0, S_0, V_0, for simplicity, we denote $\vec{\mathbf{t}}^*(\vec{X},t)=\vec{\mathbf{t}}_0^*$ and $\vec{\mathbf{b}}(\vec{X},t)=\vec{\mathbf{b}}_0$.

Remember also, (see Chapter 2 of the textbook), that:

$$\frac{D}{Dt}F_{ij}\equiv\dot{F}_{ij}=\frac{\partial}{\partial t}\frac{\partial x_i(\vec{X},t)}{\partial X_j}=\frac{\partial}{\partial X_j}\underbrace{\frac{\partial x_i(\vec{X},t)}{\partial t}}_{\dot{x}_i}=\frac{\partial\dot{\mathrm{u}}_i(\vec{X},t)}{\partial X_j}=\dot{\mathrm{u}}_{i,J}(\vec{X},t)$$

or $\dot{\boldsymbol{F}}=\boldsymbol{\ell}\cdot\boldsymbol{F}=\nabla_{\vec{X}}\dot{\vec{\mathbf{u}}}(\vec{X},t)=\dfrac{\partial\dot{\vec{\mathbf{u}}}(\vec{X},t)}{\partial\vec{X}}$

and $\boldsymbol{\ell}=\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}=\nabla_{\vec{X}}\dot{\vec{\mathbf{u}}}(\vec{X},t)\cdot\boldsymbol{F}^{-1}$

$$\dot{\boldsymbol{F}}^{-1}=-\boldsymbol{F}^{-1}\cdot\boldsymbol{\ell}=-\boldsymbol{F}^{-1}\cdot\nabla_{\vec{X}}\dot{\vec{\mathbf{u}}}(\vec{X},t)\cdot\boldsymbol{F}^{-1}=-\boldsymbol{F}^{-1}\cdot\nabla_{\vec{x}}\dot{\vec{\mathbf{u}}}(\vec{x},t)$$

Taking into account the above relations, it is also valid that:

$$\dot{\overline{\boldsymbol{F}}}=\nabla_{\vec{X}}\dot{\overline{\vec{\mathbf{u}}}}(\vec{X},t)\quad\text{y}\quad\dot{\overline{\boldsymbol{F}}}^{-1}=-\overline{\boldsymbol{F}}^{-1}\cdot\nabla_{\vec{X}}\dot{\overline{\vec{\mathbf{u}}}}(\vec{X},t)\cdot\overline{\boldsymbol{F}}^{-1}=-\overline{\boldsymbol{F}}^{-1}\cdot\nabla_{\vec{x}}\dot{\overline{\vec{\mathbf{u}}}}(\vec{x},t)$$

With that we can obtain:

$$\int_{V_0}\mathbf{P}:\dot{\overline{\boldsymbol{F}}}\,dV_0=\int_{V_0}\mathrm{P}_{iJ}\dot{\overline{F}}_{iJ}\,dV_0=\int_{V_0}\mathrm{P}_{iJ}\dot{\overline{\mathrm{u}}}_{i,J}(\vec{X},t)\,dV_0$$

$$(\mathrm{P}_{iJ}\dot{\overline{\mathrm{u}}}_i)_{,J}=\mathrm{P}_{iJ,J}\dot{\overline{\mathrm{u}}}_i+\mathrm{P}_{iJ}\dot{\overline{\mathrm{u}}}_{i,J}\qquad\Rightarrow\qquad\mathrm{P}_{iJ}\dot{\overline{\mathrm{u}}}_{i,J}=(\mathrm{P}_{iJ}\dot{\overline{\mathrm{u}}}_i)_{,J}-\mathrm{P}_{iJ,J}\dot{\overline{\mathrm{u}}}_i$$

thus:

$$\int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{V_0} \mathsf{P}_{iJ} \dot{\vec{\mathsf{u}}}_{i,J} (\vec{X},t)\, dV_0 = \int_{V_0} (\mathsf{P}_{iJ} \dot{\vec{\mathsf{u}}}_i (\vec{X},t))_{,J} - \mathsf{P}_{iJ,J} \dot{\vec{\mathsf{u}}}_i (\vec{X},t)\, dV_0$$

$$\int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{V_0} (\mathsf{P}_{iJ} \dot{\vec{\mathsf{u}}}_i (\vec{X},t))_{,J}\, dV_0 - \int_{V_0} \mathsf{P}_{iJ,J} \dot{\vec{\mathsf{u}}}_i (\vec{X},t)\, dV_0$$

$$\int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{S_0} \mathsf{P}_{iJ} \dot{\vec{\mathsf{u}}}_i (\vec{X},t) \hat{\mathsf{n}}_J\, dS_0 - \int_{V_0} \mathsf{P}_{iJ,J} \dot{\vec{\mathsf{u}}}_i (\vec{X},t)\, dV_0$$

where we have applied the divergence theorem. The above in tensorial notation becomes:

$$\int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{S_0} (\mathbf{P} \cdot \hat{\mathbf{n}}) \cdot \dot{\vec{\mathbf{u}}}_i (\vec{X},t)\, dS_0 - \int_{V_0} (\nabla_{\vec{X}} \cdot \mathbf{P}) \cdot \dot{\vec{\mathbf{u}}}_i (\vec{X},t)\, dV_0$$

Remember that the equations of motion in the reference configuration are given by:

$$\nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0 \vec{\mathbf{b}}_0 = \rho_0 \ddot{\vec{\mathbf{u}}}(\vec{X},t) \quad \Rightarrow \quad -\nabla_{\vec{X}} \cdot \mathbf{P} = \rho_0 [\vec{\mathbf{b}}_0 - \ddot{\vec{\mathbf{u}}}(\vec{X},t)]$$

and taking into account that $\dot{\vec{F}} = \nabla_{\vec{X}} \dot{\vec{\mathbf{u}}}(\vec{X},t)$ and $\vec{\mathbf{t}}_0^* = \mathbf{P} \cdot \hat{\mathbf{n}}$ we can obtain:

$$\int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{S_0} (\mathbf{P} \cdot \hat{\mathbf{n}}) \cdot \dot{\vec{\mathbf{u}}}_i (\vec{X},t)\, dS_0 - \int_{V_0} (\nabla_{\vec{X}} \cdot \mathbf{P}) \cdot \dot{\vec{\mathbf{u}}}_i (\vec{X},t)\, dV_0$$

$$\int_{V_0} \mathbf{P} : \nabla_{\vec{X}} \dot{\vec{\mathbf{u}}}(\vec{X},t)\, dV_0 = \int_{S_0} \vec{\mathbf{t}}_0^* \cdot \dot{\vec{\mathbf{u}}}(\vec{X},t)\, dS_0 + \int_{V_0} \rho_0 [\vec{\mathbf{b}}_0 - \ddot{\vec{\mathbf{u}}}(\vec{X},t)] \cdot \dot{\vec{\mathbf{u}}}_i (\vec{X},t)\, dV_0$$

Reminder: Recall from Chapter 5 of the textbook that the stress power can be expressed in different ways, namely:

$$\mathsf{W}_{int}(t) = \int_{V_0} \mathbf{P} : \dot{F}\, dV_0 = \int_{V_0} \mathbf{S} : \dot{E}\, dV_0 = \int_{V_0} \mathbf{P} : \dot{F}\, dV_0 = \frac{1}{2} \int_{V_0} \mathbf{S} : \dot{C}\, dV_0 = \int_{V} \frac{1}{J} \mathbf{P} : \dot{F}\, dV$$

$$= \int_{V} \frac{\rho}{\rho_0} \mathbf{P} : \dot{F}\, dV = \int_{V} \boldsymbol{\sigma} : \mathbf{D}\, dV = \int_{V_0} \underbrace{J\boldsymbol{\sigma}}_{\boldsymbol{\tau}} : \mathbf{D}\, dV_0 = \int_{V_0} \boldsymbol{\tau} : \mathbf{D}\, dV_0$$

NOTE 1: Remember that neither $\mathbf{P}$ nor $\dot{F}$ are in any configuration, but the scalar $\mathbf{P} : \dot{F}$ is in the reference configuration.

NOTE 2: Taking into account the above. The total external virtual work can also be expressed as follows:

$$\int_{V} \boldsymbol{\sigma} : \overline{\mathbf{D}}\, dV = \int_{V} \boldsymbol{\sigma} : \left[\nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{x},t) \right]^{sym} dV = \int_{V} \boldsymbol{\sigma} : \nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{x},t)\, dV = \int_{V_0} \mathbf{P} : \dot{\vec{F}}\, dV_0 = \int_{V_0} \mathbf{P} : \nabla_{\vec{X}} \dot{\vec{\mathbf{u}}}(\vec{X},t)\, dV_0$$

where we have used that $\mathbf{D} = \boldsymbol{\ell}^{sym} = [\nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{x},t)]^{sym}$, (see **Problem 2.35**). Note that, due to the symmetry of $\boldsymbol{\sigma}$ the relationship $\boldsymbol{\sigma} : [\nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{x},t)]^{sym} = \boldsymbol{\sigma} : \nabla_{\vec{x}} \dot{\vec{\mathbf{u}}}(\vec{x},t)$ holds.

NOTE 3: From a Variational Principle point of view, (see Holzapfel (2000)), the equation in (5.238) is also valid for a variation of the virtual field:

$$\boxed{\int_{S_\sigma} \vec{\mathbf{t}}^* (\vec{X},t) \cdot \delta \vec{\mathbf{u}}\, dS_{0\sigma} + \int_{V_0} \rho_0 [\vec{\mathbf{b}}(\vec{X},t) - \ddot{\vec{\mathbf{u}}}(\vec{X},t)] \cdot \delta \vec{\mathbf{u}}\, dV_0 = \int_{V_0} \mathbf{P} : \nabla_{\vec{X}} \delta \vec{\mathbf{u}}\, dV_0} \tag{5.239}$$

Problem 5.28

a) Show that the symmetric second-order tensor $\mathbf{A}=\mathbf{A}^{sym}$ can be split up into $\mathbf{A}=\mathbf{A}^{P}+\mathbf{A}^{S}$ where $\mathbf{A}^{P}=\mathbb{P}^{P}:\mathbf{A}$, $\mathbf{A}^{S}=\mathbb{P}^{S}:\mathbf{A}$, with $\mathbb{P}^{P}=(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\otimes(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})$ and $\mathbb{P}^{S}=\mathbb{I}^{sym}-(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\otimes(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})$, where $\hat{\mathbf{b}}$ is a unit vector according to certain direction, and $\mathbb{I}^{sym}$ is the symmetric part of the fourth-order unit tensor. b) Show that the constitutive equation for stress $\boldsymbol{\sigma}=\mathbb{C}^{e}:\boldsymbol{\varepsilon}$ can be written as follows:

$$\begin{Bmatrix}\boldsymbol{\sigma}^{P}\\ \boldsymbol{\sigma}^{S}\end{Bmatrix}=\begin{bmatrix}\mathbb{C}^{PP} & \mathbb{C}^{PS}\\ \mathbb{C}^{SP} & \mathbb{C}^{SS}\end{bmatrix}:\begin{Bmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon}\end{Bmatrix} \qquad \begin{Bmatrix}\sigma^{P}_{ij}\\ \sigma^{S}_{ij}\end{Bmatrix}=\begin{bmatrix}\mathbb{C}^{PP}_{ijkl} & \mathbb{C}^{PS}_{ijkl}\\ \mathbb{C}^{SP}_{ijkl} & \mathbb{C}^{SS}_{ijkl}\end{bmatrix}\begin{Bmatrix}\varepsilon_{kl}\\ \varepsilon_{kl}\end{Bmatrix} \tag{5.240}$$

where

$$\begin{aligned}\mathbb{C}^{PP}&=\mathbb{P}^{P}:\mathbb{C}^{e}:\mathbb{P}^{P}\\ \mathbb{C}^{PS}&=\mathbb{P}^{P}:\mathbb{C}^{e}:\mathbb{P}^{S}\\ \mathbb{C}^{SP}&=\mathbb{P}^{S}:\mathbb{C}^{e}:\mathbb{P}^{P}\\ \mathbb{C}^{SS}&=\mathbb{P}^{S}:\mathbb{C}^{e}:\mathbb{P}^{S}\end{aligned} \qquad \begin{aligned}\mathbb{C}^{PP}_{ijkl}&=\mathbb{P}^{P}_{ijpq}\mathbb{C}^{e}_{pqst}\mathbb{P}^{P}_{stkl}\\ \mathbb{C}^{PS}_{ijkl}&=\mathbb{P}^{P}_{ijpq}\mathbb{C}^{e}_{pqst}\mathbb{P}^{S}_{stkl}\\ \mathbb{C}^{SP}_{ijkl}&=\mathbb{P}^{S}_{ijpq}\mathbb{C}^{e}_{pqst}\mathbb{P}^{P}_{stkl}\\ \mathbb{C}^{SS}_{ijkl}&=\mathbb{P}^{S}_{ijpq}\mathbb{C}^{e}_{pqst}\mathbb{P}^{S}_{stkl}\end{aligned} \tag{5.241}$$

Solution:

a) Bu using the Cartesian system the tensor $\mathbf{A}$ can be represented as follows:

$$\begin{aligned}\mathbf{A}&=\mathsf{A}_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)=\mathsf{A}_{i1}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_1)+\mathsf{A}_{i2}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_2)+\mathsf{A}_{i3}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_3)\\ &=\mathsf{A}_{11}(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)+\mathsf{A}_{21}(\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_1)+\mathsf{A}_{31}(\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_1)+\mathsf{A}_{12}(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_2)+\mathsf{A}_{22}(\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2)+\mathsf{A}_{32}(\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_2)\\ &\quad+\mathsf{A}_{13}(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_3)+\mathsf{A}_{23}(\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_3)+\mathsf{A}_{33}(\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3)\end{aligned}$$

and its components in matrix form are given by:

$$\mathsf{A}_{ij}=\begin{bmatrix}\mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13}\\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23}\\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33}\end{bmatrix}=\begin{bmatrix}0 & \mathsf{A}_{12} & \mathsf{A}_{13}\\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23}\\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33}\end{bmatrix}+\begin{bmatrix}\mathsf{A}_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\overline{\mathsf{A}}_{ij}+\overline{\overline{\mathsf{A}}}_{ij}$$

Note also that the normal component $\mathsf{A}_{11}=\mathsf{A}_{N}^{(\hat{\mathbf{e}}_1)}$ (according to $\hat{\mathbf{e}}_1$-direction) can also be obtained by $\mathsf{A}_{11}=\mathbf{A}:(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)=(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1):\mathbf{A}$, so the tensor $\overline{\overline{\mathbf{A}}}=\mathsf{A}_{11}(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)$ can be written as follows:

$$\overline{\overline{\mathbf{A}}}=(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\mathsf{A}_{11}\equiv(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\otimes\mathsf{A}_{11}=(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\otimes(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1):\mathbf{A}$$

thus

$$\begin{aligned}\overline{\mathbf{A}}&=\mathbf{A}-\overline{\overline{\mathbf{A}}}=\mathbf{A}-(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\otimes(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1):\mathbf{A}=\mathbb{I}^{sym}:\mathbf{A}-(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\otimes(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1):\mathbf{A}\\ &=\left[\mathbb{I}^{sym}-(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\otimes(\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1)\right]:\mathbf{A}\end{aligned}$$

Although we have shown the equation by considering the unit vector $\hat{\mathbf{e}}_1$, the above equation is also valid for any unit vector, i.e.:

$$\begin{aligned}\mathbf{A}^{P}&=(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\otimes(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}}):\mathbf{A}=\mathbb{P}^{P}:\mathbf{A}\\ \mathbf{A}^{S}&=\left[\mathbb{I}^{sym}-(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\otimes(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\right]:\mathbf{A}=\mathbb{P}^{S}:\mathbf{A}\end{aligned}$$

Note that $\mathsf{A}_{N}^{(\hat{\mathbf{b}})}=(\hat{\mathbf{b}}\otimes\hat{\mathbf{b}}):\mathbf{A}=\hat{\mathbf{b}}\cdot\mathbf{A}\cdot\hat{\mathbf{b}}$ is the normal component according to $\hat{\mathbf{b}}$-direction, i.e. it is parallel to $\hat{\mathbf{b}}$. It is interesting to review **Problem 1.119**.

b) We can apply the above definition in order to obtain:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^P + \boldsymbol{\sigma}^S = \mathbb{P}^P : \boldsymbol{\sigma} + \mathbb{P}^S : \boldsymbol{\sigma}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^P + \boldsymbol{\varepsilon}^S = \mathbb{P}^P : \boldsymbol{\varepsilon} + \mathbb{P}^S : \boldsymbol{\varepsilon}$$

with that, and by considering that $\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}$, we can obtain:

$$\begin{aligned}\boldsymbol{\sigma}^P &= \mathbb{P}^P : \boldsymbol{\sigma} = \mathbb{P}^P : \mathbb{C}^e : \boldsymbol{\varepsilon} = \mathbb{P}^P : \mathbb{C}^e : (\boldsymbol{\varepsilon}^P + \boldsymbol{\varepsilon}^S) = \mathbb{P}^P : \mathbb{C}^e : \boldsymbol{\varepsilon}^P + \mathbb{P}^P : \mathbb{C}^e : \boldsymbol{\varepsilon}^S \\ &= \mathbb{P}^P : \mathbb{C}^e : \mathbb{P}^P : \boldsymbol{\varepsilon} + \mathbb{P}^P : \mathbb{C}^e : \mathbb{P}^S : \boldsymbol{\varepsilon}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\sigma}^S &= \mathbb{P}^S : \boldsymbol{\sigma} = \mathbb{P}^S : \mathbb{C}^e : \boldsymbol{\varepsilon} = \mathbb{P}^S : \mathbb{C}^e : (\boldsymbol{\varepsilon}^P + \boldsymbol{\varepsilon}^S) = \mathbb{P}^S : \mathbb{C}^e : \boldsymbol{\varepsilon}^P + \mathbb{P}^S : \mathbb{C}^e : \boldsymbol{\varepsilon}^S \\ &= \mathbb{P}^S : \mathbb{C}^e : \mathbb{P}^P : \boldsymbol{\varepsilon} + \mathbb{P}^S : \mathbb{C}^e : \mathbb{P}^S : \boldsymbol{\varepsilon}\end{aligned}$$

thus

$$\begin{Bmatrix} \boldsymbol{\sigma}^P \\ \boldsymbol{\sigma}^S \end{Bmatrix} = \begin{bmatrix} \mathbb{P}^P : \mathbb{C}^e & \mathbb{P}^P : \mathbb{C}^e \\ \mathbb{P}^S : \mathbb{C}^e & \mathbb{P}^S : \mathbb{C}^e \end{bmatrix} : \begin{Bmatrix} \boldsymbol{\varepsilon}^P \\ \boldsymbol{\varepsilon}^S \end{Bmatrix}$$

or

$$\begin{Bmatrix} \boldsymbol{\sigma}^P \\ \boldsymbol{\sigma}^S \end{Bmatrix} = \begin{bmatrix} \mathbb{P}^P : \mathbb{C}^e : \mathbb{P}^P & \mathbb{P}^P : \mathbb{C}^e : \mathbb{P}^S \\ \mathbb{P}^S : \mathbb{C}^e : \mathbb{P}^P & \mathbb{P}^S : \mathbb{C}^e : \mathbb{P}^S \end{bmatrix} : \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} \end{Bmatrix}$$

Appendix 5A

$$\lambda = \frac{E}{(1+\nu)(1-2\nu)}\nu$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{E}{(1+\nu)(1-2\nu)}\frac{(1-2\nu)}{2}$$

$$\lambda + 2\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} + 2\frac{E}{2(1+\nu)} = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu)$$

$$\lambda + \mu = \frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{2(1+\nu)} = \frac{E}{2(1+\nu)(1-2\nu)}$$

$$\mu(2\mu+3\lambda) = \frac{E}{2(1+\nu)}\left[2\frac{E}{2(1+\nu)} + 3\frac{E\nu}{(1+\nu)(1-2\nu)}\right] = \frac{E^2}{2(1+\nu)(1-2\nu)}$$

$$\frac{\lambda+\mu}{\mu(2\mu+3\lambda)} = \frac{E}{2(1+\nu)(1-2\nu)}\frac{2(1+\nu)(1-2\nu)}{E^2} = \frac{1}{E}$$

$$\frac{\lambda}{2\mu(2\mu+3\lambda)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{(1+\nu)(1-2\nu)}{E^2} = \frac{\nu}{E}$$

$$\frac{1}{\mu} = \frac{2(1+\mathrm{v})}{E} = \frac{1}{E}2(1+\nu)$$

$$\frac{1}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E},$$

$$\frac{\lambda}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E}\frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{\nu}{(1+\nu)},$$

$$\frac{\mu}{(2\mu+3\lambda)} = \frac{(1-2\nu)}{E}\frac{E}{2(1+\nu)} = \frac{(1-2\nu)}{2(1+\nu)},$$

$$\frac{2(\mu+\lambda)}{(2\mu+3\lambda)} = 2\frac{\nu}{(1+\nu)} + 2\frac{(1-2\nu)}{2(1+\nu)} = \frac{1}{(1+\nu)},$$

$$(2\mu+\lambda) = 2\frac{E}{2(1+\nu)} + \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)},$$

$$\frac{\lambda}{(2\mu+\lambda)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{(1+\nu)(1-2\nu)}{E(1-\nu)} = \frac{\nu}{(1-\nu)},$$

$$\frac{\lambda}{(\lambda+\mu)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{2(1+\nu)(1-2\nu)}{E} = 2\nu$$

$$\frac{2\mu+\lambda}{2\mu+3\lambda} = \frac{1-\nu}{1+\nu},$$

6 Linear Elasticity

6.1 Three-Dimensional Elasticity (3D)

Problem 6.1

The cylinder described in Figure 6.1 is made up of an isotropic linear elastic material, and is subjected to a strain state (in cylindrical coordinates) as follows:

$$
\begin{aligned}
e_{rr} &= e_{\theta\theta} = a \sin\theta \\
e_{r\theta} &= \frac{a\cos\theta}{2} \\
e_{zz} &= e_{\theta z} = e_{rz} = 0
\end{aligned}
\tag{6.1}
$$

where e_{ij} are the Almansi strain tensor components.

Calculate the traction vector $\vec{\mathbf{t}}$ on the boundary by using the cylindrical coordinates.

Hypothesis: Assumptions: Small deformation regime and consider the Lamé constants λ, μ.

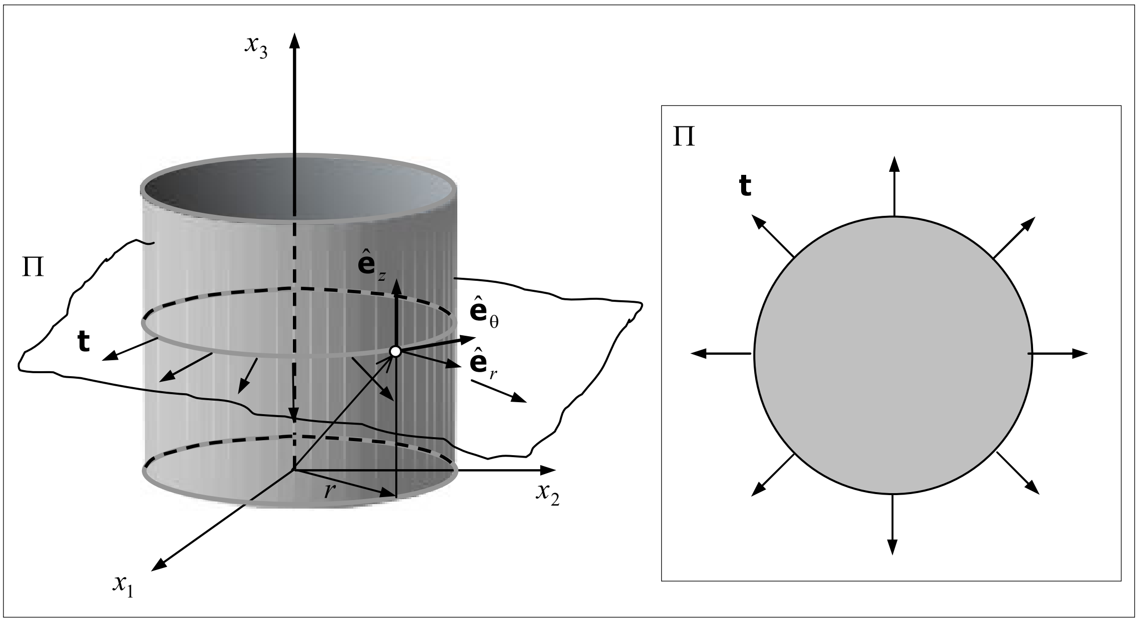


Figure 6.1

Solution:

Small deformation regime: $\boldsymbol{e} \approx \boldsymbol{E} \approx \boldsymbol{\varepsilon}$

$$\varepsilon(r,\theta,z) = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} a\sin\theta & \frac{a\cos\theta}{2} & 0 \\ \frac{a\cos\theta}{2} & a\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \tag{6.3}$$

$$\mathrm{Tr}(\boldsymbol{\varepsilon}) = 2a\sin\theta \tag{6.4}$$

thus,

$$\boldsymbol{\sigma} = \lambda 2a\sin\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} a\sin\theta & \frac{a\cos\theta}{2} & 0 \\ \frac{a\cos\theta}{2} & a\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.5}$$

$$\boldsymbol{\sigma}_{(r,\theta r,\theta} = \begin{bmatrix} \lambda 2a\sin\theta + 2\mu a\sin\theta & \mu a\cos\theta & 0 \\ \mu a\cos\theta & \lambda 2a\sin\theta + 2\mu a\sin\theta & 0 \\ 0 & 0 & \lambda 2a\sin\theta \end{bmatrix} \tag{6.6}$$

The traction vector $\vec{\mathbf{t}}$:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \tag{6.7}$$

$\hat{\mathbf{n}} = (1,0,0)$

$$\begin{bmatrix} \mathrm{t}_1^{(\hat{\mathbf{n}})} \\ \mathrm{t}_2^{(\hat{\mathbf{n}})} \\ \mathrm{t}_3^{(\hat{\mathbf{n}})} \end{bmatrix} = \begin{bmatrix} 2\lambda a\sin\theta + 2\mu a\sin\theta \\ \mu a\cos\theta \\ 0 \end{bmatrix} \tag{6.8}$$

Problem 6.2

The parallelepiped described in Figure 6.2 is deformed as indicated by the dashed lines. The displacement components are given as follows:

$$u = C_1 xyz \quad ; \quad v = C_2 xyz \quad ; \quad w = C_3 xyz \tag{6.9}$$

a) Obtain the strain state at the point E. In the current reference the point is represented by E' whose coordinates are $E'(1.503;\ \ 1.001;\ \ 1.997)$;

b) Obtain the normal strain at the point E in the direction of the line EA;

c) Calculate the angular distortion at the point E that undergoes the angle formed by the lines EA and EF.

d) Find the volume variation and the average volumetric deformation.

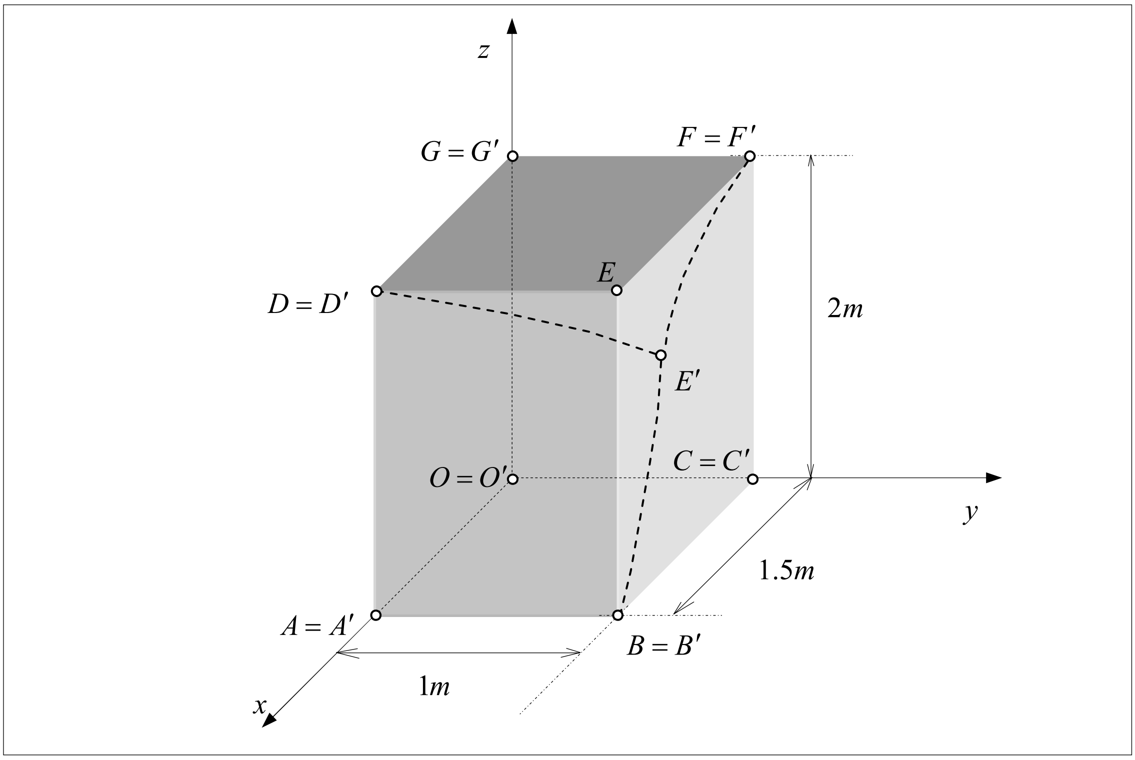


Figure 6.2

Solution:

a) The strain state in function of the displacements is given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) \tag{6.10}$$

which in engineering notation becomes:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{6.11}$$

To calculate the strain state at any point we need a priori to calculate the displacement field.

Calculation of the constants:

By substituting the values given by the point $E(1.5;\ 1.0;\ 2.0)$, we can obtain:

$$\begin{aligned} u^{(E)} &= X_1^{(E)} - x_1^{(E)} = 1.503 - 1.5 = C_1(1.5)(1.0)(2.0) \Rightarrow C_1 = 0.001 \\ v^{(E)} &= X_2^{(E)} - x_2^{(E)} = 1.001 - 1.0 = C_2(1.5)(1.0)(2.0) \Rightarrow C_2 = \frac{0.001}{3} \\ w^{(E)} &= X_3^{(E)} - x_3^{(E)} = 1.997 - 2.0 = C_3(1.5)(1.0)(2.0) \Rightarrow C_3 = -0.001 \end{aligned} \tag{6.12}$$

Then, the displacement field becomes:

Engineering notation	Scientific notation
$u = 0.001\,xyz$	$\mathsf{u}_1 = 0.001\,X_1X_2X_3$
$v = \dfrac{0.001}{3}\,xyz$	$\mathsf{u}_2 = \dfrac{0.001}{3}\,X_1X_2X_3$
$w = -0.001\,xyz$	$\mathsf{u}_3 = -0.001\,X_1X_2X_3$

(6.13)

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0.001yz = 0.002 = \varepsilon_{11}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{0.001}{3} xz = 0.001 = \varepsilon_{22}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = -0.001xy = -0.0015 = \varepsilon_{33}$$

$$\gamma_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{0.001}{3} yz + 0.001xz = \frac{0.011}{3} = 2\varepsilon_{12}$$

$$\gamma_{xz} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = -0.001yz + 0.001xy = -0.0005 = 2\varepsilon_{13}$$

$$\gamma_{yz} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = -0.001xz + \frac{0.001}{3} xy = -0.0025 = 2\varepsilon_{23}$$

The strain field becomes:

$$\varepsilon_{ij} = 0.001\begin{bmatrix} yz & \frac{1}{2}\left(\frac{yz}{3} + xz\right) & \frac{1}{2}(xy - yz) \\ \frac{1}{2}\left(\frac{yz}{3} + xz\right) & \frac{xz}{3} & \frac{1}{2}\left(\frac{xy}{3} - xz\right) \\ \frac{1}{2}(xy - yz) & \frac{1}{2}\left(\frac{xy}{3} - xz\right) & -xy \end{bmatrix}$$

The strain state at the point $E(x = 1.5;\ y = 1.0;\ z = 2.0)$ is:

$$\varepsilon_{ij}\big|_E = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 0.002 & \left(\frac{0.011}{6}\right) & -0.00025 \\ \left(\frac{0.011}{6}\right) & 0.001 & -0.00125 \\ -0.00025 & -0.00125 & -0.0015 \end{bmatrix} \tag{6.14}$$

b) The normal strain component associated with the direction $\hat{\boldsymbol{M}}$ is obtained as follows:

$$\varepsilon_{\hat{M}} = \hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}} \quad \xrightarrow{indicial} \quad \varepsilon_{\hat{M}} = \varepsilon_{ij}\,\hat{M}_i\,\hat{M}_j \tag{6.15}$$

By expanding the above equation and by considering the symmetry of the strain tensor we can obtain:

$$\varepsilon_{\hat{M}} = \varepsilon_{11}\,\hat{M}_1^2 + \varepsilon_{22}\,\hat{M}_2^2 + \varepsilon_{33}\,\hat{M}_3^2 + 2\varepsilon_{12}\,\hat{M}_1\hat{M}_2 + 2\varepsilon_{13}\,\hat{M}_1\hat{M}_3 + 2\varepsilon_{23}\,\hat{M}_2\hat{M}_3 \tag{6.16}$$

or by using the engineering notation:

$$\varepsilon_{\hat{M}} = \varepsilon_x\,\hat{M}_1^2 + \varepsilon_y\,\hat{M}_2^2 + \varepsilon_z\,\hat{M}_3^2 + \gamma_{xy}\,\hat{M}_1\hat{M}_2 + \gamma_{xz}\,\hat{M}_1\hat{M}_3 + \gamma_{yz}\,\hat{M}_2\hat{M}_3 \tag{6.17}$$

The unit vector components $\hat{M}_i$ is given by the direction cosines of the direction of the line EA:

$$\hat{M}_1 = 0; \qquad \hat{M}_2 = \frac{-1}{\sqrt{5}}; \qquad \hat{M}_3 = \frac{-2}{\sqrt{5}} \tag{6.18}$$

By substituting the corresponding values into the equation (6.17), we can obtain:

$$\begin{aligned} \varepsilon_{\hat{M}} &= \varepsilon_y \hat{M}_2^2 + \varepsilon_z \hat{M}_3^2 + \gamma_{yz} \hat{M}_2 \hat{M}_3 \\ \varepsilon_{\hat{M}} &= 0.001\frac{1}{5} + (-0.0015)\frac{4}{5} + (-0.0025)\frac{2}{5} = -2\times 10^{-3} \end{aligned} \tag{6.19}$$

c) For small deformation, the distortion of the angle at the point E formed by the lines EA and EF, with $\Theta = 90^{\circ}$, becomes:

$$\varepsilon_{\hat{M}\hat{N}} = \frac{-1}{2}\Delta\theta_{\hat{M}\hat{N}} = -\frac{1}{2}\left(\frac{-2\hat{\boldsymbol{M}}\cdot\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{N}}}{\sin\Theta}\right) = \hat{\boldsymbol{M}}\cdot\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{N}} \xrightarrow{\text{components}} \varepsilon_{\hat{M}\hat{N}} = \varepsilon_{ij}\hat{M}_i\hat{N}_j \tag{6.20}$$

More details about the above equation are provided in the textbook in Chapter 2 – Continuum Kinematics (in the sub-section small deformation regime). Expanding the above expression and by considering the symmetry of the strain tensor we can obtain:

$$\begin{aligned} \varepsilon_{\hat{M}\hat{N}} = {} & \varepsilon_{11}\hat{M}_1\hat{N}_1 + \varepsilon_{22}\hat{M}_2\hat{N}_2 + \varepsilon_{33}\hat{M}_3\hat{N}_3 + \varepsilon_{12}\left(\hat{M}_1\hat{N}_2 + \hat{M}_2\hat{N}_1\right) + \\ & \varepsilon_{13}\left(\hat{M}_1\hat{N}_3 + \hat{M}_3\hat{N}_1\right) + \varepsilon_{23}\left(\hat{M}_2\hat{N}_3 + \hat{M}_3\hat{N}_2\right) \end{aligned} \tag{6.21}$$

or in engineering notation:

$$\begin{aligned} \frac{\gamma_{\hat{M}\hat{N}}}{2} = {} & \varepsilon_x\hat{M}_1\hat{N}_1 + \varepsilon_y\hat{M}_2\hat{N}_2 + \varepsilon_z\hat{M}_3\hat{N}_3 + \frac{\gamma_{xy}}{2}\left(\hat{M}_1\hat{N}_2 + \hat{M}_2\hat{N}_1\right) + \\ & \frac{\gamma_{xz}}{2}\left(\hat{M}_1\hat{N}_3 + \hat{M}_3\hat{N}_1\right) + \frac{\gamma_{yz}}{2}\left(\hat{M}_2\hat{N}_3 + \hat{M}_3\hat{N}_2\right) \end{aligned} \tag{6.22}$$

and by considering the following unit vectors according to EA and EF direcctions respectively:

$$\hat{M}_i = \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \quad ; \quad \hat{N}_i = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \tag{6.23}$$

we can obtain:

$$\begin{aligned} \frac{\gamma_{\hat{M}\hat{N}}}{2} &= \varepsilon_{12}\hat{M}_2\hat{N}_1 + \varepsilon_{13}\hat{M}_3\hat{N}_1 = \left(\frac{0.011}{6}\right)(-1)\left(\frac{-1}{\sqrt{5}}\right) + (-0.00025)(-1)\left(\frac{-2}{\sqrt{5}}\right) \\ \frac{\gamma_{\hat{M}\hat{N}}}{2} &= 5.96284793998\times 10^{-4} \quad \Rightarrow \quad \gamma_{\hat{M}\hat{N}} = 1.1925696\times 10^{-3} \end{aligned} \tag{6.24}$$

Alternative Solution

We can construct an orthogonal basis associated with the unit vectors $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$ by means of the cross product $\hat{\boldsymbol{P}} = \hat{\boldsymbol{M}} \wedge \hat{\boldsymbol{N}}$. Then, we can obtain the components of the unit vector $\hat{\boldsymbol{P}}$:

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{M}} \wedge \hat{\boldsymbol{N}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ -1 & 0 & 0 \end{vmatrix} = \frac{2}{\sqrt{5}}\hat{\mathbf{e}}_2 - \frac{1}{\sqrt{5}}\hat{\mathbf{e}}_3 \quad \Rightarrow \quad \hat{P}_i = \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \tag{6.25}$$

Then, the transformation matrix from the system $X_1X_2X_3$ to the system made up by the unit vectors $\hat{\boldsymbol{M}}$, $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{P}}$ are given by:

$$\mathcal{A} = a_{ij} = \begin{bmatrix} \hat{M}_1 & \hat{M}_2 & \hat{M}_3 \\ \hat{N}_1 & \hat{N}_2 & \hat{N}_3 \\ \hat{P}_1 & \hat{P}_2 & \hat{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ -1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \tag{6.26}$$

By applying the component transformation law for a second-order tensor components, i.e. $\varepsilon_{ij} = a_{ik} a_{jl} \varepsilon_{kl}$ or in matrix form $\boldsymbol{\varepsilon}' = \mathcal{A}\,\boldsymbol{\varepsilon}\,\mathcal{A}^T$, we can obtain:

$$\boldsymbol{\varepsilon}' = \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ -1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0.002 & \left(\frac{0.011}{6}\right) & -0.00025 \\ \left(\frac{0.011}{6}\right) & 0.001 & -0.00125 \\ -0.00025 & -0.00125 & -0.0015 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ -1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}^T \tag{6.27}$$

Thus:

$$\varepsilon_{\hat{M}} \qquad\qquad \varepsilon_{\hat{M}\hat{N}} = \frac{\gamma_{\hat{M}\hat{N}}}{2}$$

$$\varepsilon'_{ij} = \begin{bmatrix} -2\times10^{-3} & 5.96284794\times10^{-4} & -2.5\times10^{-4} \\ 5.96284794\times10^{-4} & 2\times10^{-3} & -1.75158658\times10^{-3} \\ -2.5\times10^{-4} & -1.75158658\times10^{-3} & 1.5\times10^{-3} \end{bmatrix} \tag{6.28}$$

NOTE: Note that this example is not a case of homogeneous deformation, i.e. a straight edge in the reference configuration is no longer straight line in the current configuration. To obtain the deformed unit vector we must apply the linear transformation $\hat{\boldsymbol{m}} = \boldsymbol{F}\cdot\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{n}} = \boldsymbol{F}\cdot\hat{\boldsymbol{N}}$, where $\boldsymbol{F}$ is the deformation gradient.

d) The volume ratio (dilatation) by definition is $\varepsilon_V = \frac{\Delta(dV)}{dV}$ where dV is the differential volume.

For small deformation regime we have:

$$\varepsilon_V = \frac{\Delta(dV)}{dV} = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad \Rightarrow \qquad \Delta(dV) = (\varepsilon_x + \varepsilon_y + \varepsilon_z)dV \tag{6.29}$$

by integrating the above equation over the volume we can obtain the volume variation:

$$\Delta V = \int_V (\varepsilon_x + \varepsilon_y + \varepsilon_z) dV = 0.001 \int_{z=0}^{2.0}\int_{y=0}^{1}\int_{x=0}^{1.5} \left(yz + \frac{xz}{3} - xy \right) dxdydz \tag{6.30}$$

thus:

$$\Delta V = 1.125\times10^{-3}\, m^3 \tag{6.31}$$

Then:

$$\varepsilon_V = \frac{\Delta(dV)}{dV} = \frac{1.125\times10^{-3}}{1.5\times1.0\times2.0} = 0.375\times10^{-3} \tag{6.32}$$

Problem 6.3

The stress state at one point of the structure, which is made up of an isotropic linear elastic material, is given by:

$$\sigma_{ij} = \begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

a) Obtain the engineering strain tensor components. Consider the Young's modulus ($E = 207GPa$) and the shear modulus ($G = 80GPa$).

b) Consider that a cube of side $5cm$ is subjected to this stress state. Obtain the volume variation in the cube.

Solution:

The strain components can be obtained by means of the equations:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = 3.333\times10^{-5}; & \gamma_{xy} &= \frac{1}{G}\tau_{xy} = 2.5\times10^{-5} \\
\varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] = -2.318\times10^{-5}; & \gamma_{yz} &= \frac{1}{G}\tau_{yz} = 0 \\
\varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = -4.348\times10^{-6}; & \gamma_{xz} &= \frac{1}{G}\tau_{xz} = 0
\end{aligned} \tag{6.33}$$

where the Poisson's ratio can be obtained by:

$$G = \frac{E}{2(1+\nu)} \quad \Rightarrow \quad \nu = \frac{E}{2G} - 1 = \frac{207}{160} - 1 \approx 0.29375$$

Thus:

$$\varepsilon_{ij} = \begin{bmatrix} 33.24 & 12.5 & 0 \\ 12.5 & -23.01 & 0 \\ 0 & 0 & -4.257 \end{bmatrix} \times 10^{-6}$$

Alternative solution:

In the textbook (Chaves(2013)) we have shown that $\mathbb{C}^{e^{-1}} = \frac{(1+\nu)}{E}\mathbf{I} - \frac{\nu}{E}\mathbf{1}\otimes\mathbf{1}$, with that we can obtain:

$$\boldsymbol{\varepsilon} = \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma} = \left[\frac{(1+\nu)}{E}\mathbf{I} - \frac{\nu}{E}\mathbf{1}\otimes\mathbf{1}\right] : \boldsymbol{\sigma} = \frac{(1+\nu)}{E}\mathbf{I} : \boldsymbol{\sigma} - \frac{\nu}{E}\mathbf{1}\otimes\mathbf{1} : \boldsymbol{\sigma} = \frac{(1+\nu)}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

And its components are:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\varepsilon_{ij} = 6.251\times10^{-6}\begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 4.2609\times10^{-6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 33.24 & 12.5 & 0 \\ 12.5 & -23.01 & 0 \\ 0 & 0 & -4.257 \end{bmatrix}\times10^{-6}$$

where we have considered $\frac{(1+\nu)}{E} = 6.25\times10^{-6}\left[\frac{1}{MPa}\right]$ and $\frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma}) = 4.25725\times10^{-6}$.

In the small deformation regime the volumetric deformation (linear) is equal to the trace of the strain tensor:

$$D_V^L \equiv \varepsilon_V = I_{\boldsymbol{\varepsilon}} = (33.24 - 23.01 - 4.257)\times10^{-6} = 5.973\times10^{-6}$$

Then, the volume variation can be evaluated as follows:

$$\Delta V = \varepsilon_V V_0 = 5.973\times10^{-6}(5\times5\times5) = 7.466\times10^{-4}\,cm^3$$

Problem 6.4

A parallelepiped (elastic body) of dimensions $a = 3cm$, $b = 3cm$, $c = 4cm$, is made up of an isotropic homogeneous linear elastic material, which is accommodated into a cavity of the same shape and dimensions as the parallelepiped, (see Figure 6.3). The cavity walls are made up of a very rigid material (undeformable), (Ortiz Berrocal (1985)).

Via a rigid plate (dimensions $a\times b$) of negligible weight and negligible friction we apply a perpendicular compression force equal to $F = 200N$ which compresses the elastic block.

Consider that the elastic body properties are: $E = 2\times10^4\,N/cm^2$ (Young's modulus); $\nu = 0.3$ (Poisson's ratio).

a) Calculate the lateral force exerted by the cavity wall on the parallelepiped;
b) Calculate the height variation of the elastic body, i.e. find Δc.

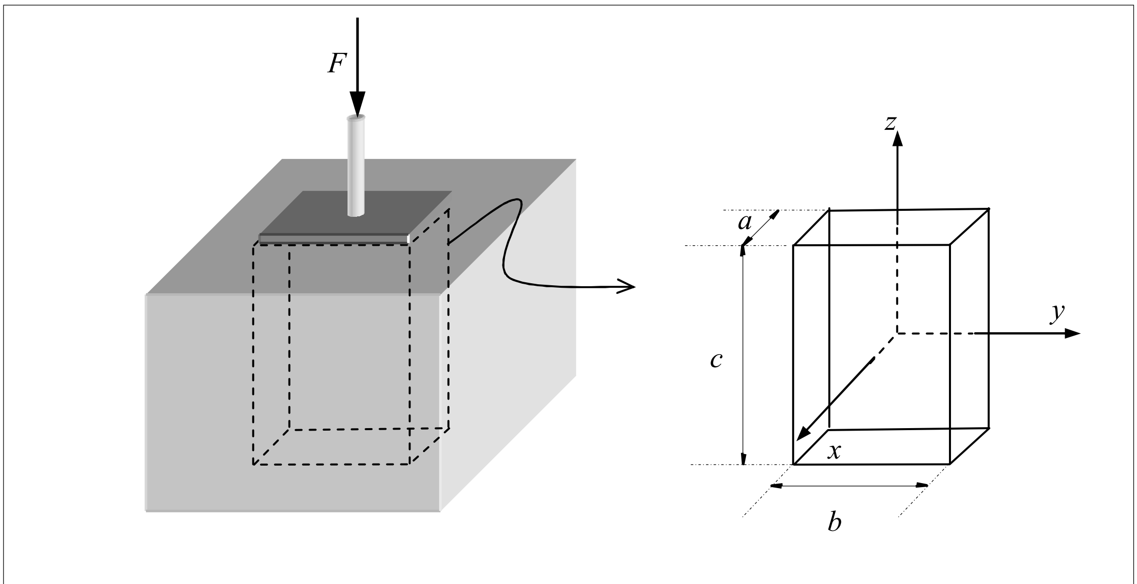


Figure 6.3

Solution:

At any point of the elastic body the stress state is only characterized by normal components σ_x, σ_y and σ_z. The stress σ_z is given by:

$$\sigma_z = -\frac{200}{ab} = -\frac{200}{3\times 3} = -\frac{200}{9}\frac{N}{cm^2} \tag{6.34}$$

Note that, due to the problem symmetry the stresses σ_x and σ_y are equal, then:

$$\begin{aligned}
&\varepsilon_x = \varepsilon_y = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = 0 \quad \Rightarrow \quad \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_x + \sigma_z\right)\right] = 0 \\
&\Rightarrow \sigma_x - \nu\left(\sigma_x + \sigma_z\right) = 0 \\
&\Rightarrow \sigma_x = \frac{\nu\,\sigma_z}{(1-\nu)}
\end{aligned} \tag{6.35}$$

thus:

$$\sigma_x = \frac{\nu\,\sigma_z}{(1-\nu)} = \frac{0.3}{(1-0.3)}\left(\frac{-200}{9}\right) = -\frac{200}{21}\frac{N}{cm^2} \tag{6.36}$$

The force applied by the wall on the elastic body is given by:

$$F_y = \sigma_y\, a\, c = \frac{-200}{21}\times 3\times 4 = -114.28N$$

$$F_x = \sigma_x\, b\, c = \frac{-200}{21}\times 3\times 4 = -114.28N$$

The strain ε_z can be obtained as follows:

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{E}\left[\sigma_z - 2\nu\,\sigma_x\right] = \frac{1}{2\times 10^4}\left[-\frac{200}{9} + 2\times 0.3\times\frac{200}{21}\right] = -8.25\times 10^{-4}$$

Then, the height variation is given by:

$$\Delta c = \varepsilon_z\, c = -8.25\times 10^{-4}\times 4 = -0.0033cm \tag{6.37}$$

Problem 6.5

Under the approximation of small deformation theory, the displacement field is given by:

$$\vec{\mathbf{u}} = (x_1 - x_3)^2\times 10^{-3}\,\hat{\mathbf{e}}_1 + (x_2 + x_3)^2\times 10^{-3}\,\hat{\mathbf{e}}_2 - x_1 x_2\;\times 10^{-3}\,\hat{\mathbf{e}}_3$$

Obtain the infinitesimal strain tensor, the infinitesimal spin tensor at the point $P(0,2,-1)$.

Solution:

Displacement gradient field components:

$$\frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2(x_1 - x_3) & 0 & -2(x_1 - x_3) \\ 0 & 2(x_2 + x_3) & 2(x_2 + x_3) \\ -x_2 & -x_1 & 0 \end{bmatrix}\times 10^{-3}$$

and at the particular point $P(0,2-1)$ the above equation becomes

$$\left.\frac{\partial \mathrm{u}_i}{\partial x_j}\right|_P = \begin{bmatrix} 2(x_1 - x_3) & 0 & -2(x_1 - x_3) \\ 0 & 2(x_2 + x_3) & 2(x_2 + x_3) \\ -x_2 & -x_1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix}\times 10^{-3}$$

The second-order tensor $(\nabla_{\vec{x}}\vec{\mathbf{u}})$ can be split additively into a symmetric ($\boldsymbol{\varepsilon}$) and an antisymmetric part ($\boldsymbol{\omega}$):

$$(\nabla_{\vec{x}}\vec{\mathbf{u}})_{ij} = \frac{\partial \mathrm{u}_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij}$$

where

Infinitesimal strain tensor	Infinitesimal spin tensor
$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \times 10^{-3}$	$\omega_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} - \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \times 10^{-3}$

(6.38)

Problem 6.6

Under the restriction of small deformation theory, the displacement field is given by:

$$\vec{\mathbf{u}} = a(x_1^2 - 5x_2^2)\ \hat{\mathbf{e}}_1 + (2\,a\,x_1 x_2)\hat{\mathbf{e}}_2 - (0)\ \hat{\mathbf{e}}_3$$

a) Obtain the linear strain tensor and the linear spin tensor;
b) Obtain the principal strains and principal stresses;
c) Given the shear modulus G, obtain the value of the Young's modulus E to guarantee the balance at any point of the continuum.

Obs.: The body forces can be discarded.

Solution:

a) Considering that $\mathrm{u}_1 = a(x_1^2 - 5x_2^2)$, $\mathrm{u}_2 = 2\,a\,x_1 x_2$, $\mathrm{u}_3 = 0$, the displacement gradient components are given by:

$$(\nabla_{\vec{x}}\vec{\mathbf{u}})_{ij} = \frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 a & -10ax_2 & 0 \\ 2ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Decomposing additively the displacement gradient in a symmetric part (the linear strain tensor - ε_{ij}) and an antisymmetric part (the infinitesimal spin tensor- ω_{ij}) we can obtain:

$$\frac{\partial \mathrm{u}_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\begin{bmatrix} 2x_1 a & -10ax_2 & 0 \\ 2ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2x_1 a & 2ax_2 & 0 \\ -10ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 a & -4ax_2 & 0 \\ -4ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\begin{bmatrix} 2x_1 a & -10ax_2 & 0 \\ 2ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2x_1 a & 2ax_2 & 0 \\ -10ax_2 & 2ax_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -6ax_2 & 0 \\ 6ax_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) Note that since $\varepsilon_{i3} = \varepsilon_{3i} = 0$, $\varepsilon_{33} = 0$ is already a principal value which is associated with the unit vector $\hat{n}_i = 0[0,01]$. The remaining principal values can be obtained as follows:

$$\begin{vmatrix} 2x_1 a - \bar{\lambda} & -4ax_2 \\ -4ax_2 & 2ax_1 - \bar{\lambda} \end{vmatrix} = 0 \quad \Rightarrow \quad \left(2x_1 a - \bar{\lambda}\right)^2 - \left(4ax_2\right)^2 = 0$$

$$\Rightarrow \left(2x_1 a - \bar{\lambda}\right)^2 = \left(4ax_2\right)^2 \quad \Rightarrow \quad 2x_1 a - \bar{\lambda} = \pm 4ax_2 \Rightarrow \begin{cases} \bar{\lambda}_1 = 2x_1 a - 4ax_2 \\ \bar{\lambda}_2 = 2x_1 a + 4ax_2 \end{cases}$$

$$\varepsilon'_{ij} = \begin{bmatrix} 2x_1 a + 4ax_2 & 0 & 0 \\ 0 & 2x_1 a - 4ax_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(principal strains)}$$

Since the strain and stress share the same principal space we can use the equation $\sigma_{ij} = \lambda 4x_1 a\delta_{ij} + 2\mu\varepsilon_{ij}$ to obtain the principal stresses:

$$\sigma'_{ij} = \lambda 4x_1 a\delta_{ij} + 2\mu\varepsilon'_{ij} = \lambda 4x_1 a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu\begin{bmatrix} 2x_1 a + 4ax_2 & 0 & 0 \\ 0 & 2x_1 a - 4ax_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda 4x_1 a + 2\mu(2x_1 a + 4ax_2) & 0 & 0 \\ 0 & \lambda 4x_1 a + 2\mu(2x_1 a - 4ax_2) & 0 \\ 0 & 0 & \lambda 4x_1 a \end{bmatrix}$$

c) The equilibrium equations without body forces ($\rho\vec{\mathbf{b}} = \vec{\mathbf{0}}$) become:

$$\nabla \cdot \boldsymbol{\sigma} + \underbrace{\rho\vec{\mathbf{b}}}_{=\vec{\mathbf{0}}} = \vec{\mathbf{0}} \xrightarrow{Indicial} \sigma_{ij,j} = 0_i$$

and by expanding the above equation, we can obtain:

$$\sigma_{ij,j} = 0_i \quad \Rightarrow \quad \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} + \dfrac{\partial\sigma_{13}}{\partial x_3} = 0 \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} = 0 \\ \dfrac{\partial\sigma_{31}}{\partial x_1} + \dfrac{\partial\sigma_{32}}{\partial x_2} + \dfrac{\partial\sigma_{33}}{\partial x_3} = 0 \end{cases}$$

and by considering that $\varepsilon_{kk} = 4x_1 a$, the stress tensor components $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ become $\sigma_{ij} = \lambda 4x_1 a\delta_{ij} + 2\mu\varepsilon_{ij}$, thus

$$\sigma_{11} = \lambda 4x_1 a\delta_{11} + 2\mu\varepsilon_{11} = \lambda 4x_1 a + 2\mu(2x_1 a) = 4x_1 a(\lambda + \mu)$$
$$\sigma_{12} = \lambda 4x_1 a\delta_{12} + 2\mu\varepsilon_{12} = 2\mu(-4ax_2) = -8\mu a x_2$$
$$\sigma_{13} = 0$$

W can also use:

$$\sigma_{ij} = \lambda 4x_1 a\delta_{ij} + 2\mu\varepsilon_{ij} = \lambda 4x_1 a\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 2\mu\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}$$

$$= \lambda 4x_1 a\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 2\mu\begin{bmatrix}2x_1 a & -4ax_2 & 0\\-4ax_2 & 2ax_1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}4x_1 a(\lambda+\mu) & -8\mu a x_2 & 0\\-8\mu a x_2 & 4x_1 a(\lambda+\mu) & 0\\0 & 0 & \lambda 4x_1 a\end{bmatrix}$$

Then, the first equilibrium equation becomes:

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = 0 \quad\Rightarrow\quad 4a(\lambda+\mu) - 8\mu a = 0 \quad\Rightarrow\quad \lambda+\mu = 2\mu \Rightarrow \lambda = \mu = G$$

In addition, note that $E = \dfrac{G(3\lambda+2G)}{\lambda+G}$, which was obtained by means of the relationships $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = G = \dfrac{E}{2(1+\nu)}$. Then, we can conclude that:

$$E = \frac{G(3\lambda+2G)}{\lambda+G} = \frac{G(3G+2G)}{G+G} = 2.5G$$

Problem 6.7

The stress state at a point of the continuum is represented by the Cauchy stress tensor components:

$$\sigma_{ij} = \begin{bmatrix}-26 & 6 & 0\\6 & 9 & 0\\0 & 0 & 29\end{bmatrix} kPa$$

By considering an isotropic linear elastic material: a) Obtain the principal invariants of $\boldsymbol{\sigma}$;

b) Obtain the spherical and deviatoric parts of $\boldsymbol{\sigma}$;) Obtain the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$; d) Draw the Mohr's circle in stress. Obtain the maximum normal and tangential stress; e) Considering a small deformation regime and taking into account that the elastic mechanical properties are $\lambda = 20000kPa$ and $\mu = 20000kPa$ (λ, μ are the Lamé constants), obtain the infinitesimal strain tensor; f) Obtain the eigenvalues and eigenvectors of $\boldsymbol{\varepsilon}$.

Solution:

a) The principal invariants of $\boldsymbol{\sigma}$

$$I_{\boldsymbol{\sigma}} = 12\times 10^3 \qquad (Pa)$$

$$II_{\boldsymbol{\sigma}} = \begin{vmatrix}9 & 0\\0 & 29\end{vmatrix}\times 10^6 + \begin{vmatrix}-26 & 0\\0 & 29\end{vmatrix}\times 10^6 + \begin{vmatrix}-26 & 6\\6 & 9\end{vmatrix}\times 10^6 = -763\times 10^6 \qquad (Pa)^2$$

$$III_{\boldsymbol{\sigma}} = \det(\boldsymbol{\sigma}) = -7830\times 10^9 \qquad (Pa)^3$$

The spherical and deviatoric parts are related to tensor by $\sigma_{ij} = \sigma_{ij}^{dev} + \sigma_{ij}^{sph}$:

The mean stress: $\sigma_m = \dfrac{1}{3}\sigma_{ii} = \dfrac{1}{3} I_{\boldsymbol{\sigma}} = \dfrac{(29-26+9)}{3} = 4\times 10^3\, Pa$

$$\sigma_{ij}^{hyd} \equiv \sigma_{ij}^{sph} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} kPa$$

The deviatoric part $\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{sph}$ becomes:

$$\sigma_{ij}^{dev} = \begin{bmatrix} -26-4 & 6 & 0 \\ 6 & 9-4 & 0 \\ 0 & 0 & 29-4 \end{bmatrix} = \begin{bmatrix} -30 & 6 & 0 \\ 6 & 5 & 0 \\ 0 & 0 & 25 \end{bmatrix} kPa$$

By solving the characteristic equation we can obtain the eigenvalues:

$$\sigma_I = 29kPa \quad ; \quad \sigma_{II} = 10kPa \quad ; \quad \sigma_{III} = -27kPa:$$

The eigenvectors:

$$\sigma_I = 29kPa \xrightarrow{\text{principal direction}} \hat{\mathrm{n}}_i^{(1)} = [0 \quad 0 \quad 1]$$
$$\sigma_{II} = 10kPa \xrightarrow{\text{principal direction}} \hat{\mathrm{n}}_i^{(2)} = [0.1644 \quad 0.98639 \quad 0]$$
$$\sigma_{III} = -27kPa \xrightarrow{\text{principal direction}} \hat{\mathrm{n}}_i^{(3)} = [0.98639 \quad -0.1644 \quad 0]$$

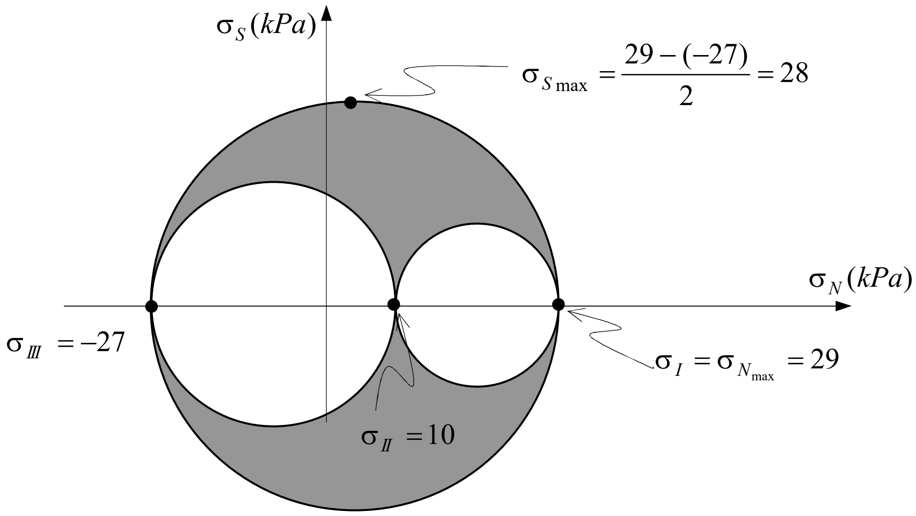


Figure 6.4

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij} \xrightarrow{\text{reverse form}} \varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma_{ij}$$

where $\frac{-\lambda}{2\mu(3\lambda+2\mu)} = -5\times10^{-9}(Pa)^{-1}$, $\mathrm{Tr}(\boldsymbol{\sigma}) = 1.2\times10^4 (Pa)$

$$\varepsilon_{ij} = (-5\times10^{-9})(1.2\times10^4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2.5\times10^{-8}\begin{bmatrix} -26 & 6 & 0 \\ 6 & 9 & 0 \\ 0 & 0 & 29 \end{bmatrix}\times10^3$$
$$= -6\times10^{-5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2.5\times10^{-5}\begin{bmatrix} -26 & 6 & 0 \\ 6 & 9 & 0 \\ 0 & 0 & 29 \end{bmatrix} = \begin{bmatrix} -7.1 & 1.5 & 0 \\ 1.5 & 1.65 & 0 \\ 0 & 0 & 6.65 \end{bmatrix}\times10^{-4}$$

As the material is isotropic, the stress and strain have the same principal directions, so, we can work in the principal space in order to obtain the principal strains:

$$\varepsilon'_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma'_{ij}$$

$$= -6\times10^{-5}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + 2.5\times10^{-5}\begin{bmatrix}29 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & -27\end{bmatrix} = \begin{bmatrix}66.5 & 0 & 0\\ 0 & 19 & 0\\ 0 & 0 & 73.5\end{bmatrix}\times10^{-5}$$

Problem 6.8

Show that the constitutive equations in stress, for an isotropic linear elastic material, can be represented by the set of equations:

$$\begin{cases}\boldsymbol{\sigma}^{dev} = 2\mu\boldsymbol{\varepsilon}^{dev}\\ \mathrm{Tr}(\boldsymbol{\sigma}) = 3\kappa\mathrm{Tr}(\boldsymbol{\varepsilon})\end{cases}$$

where $\mu = G$ is the shear modulus, and κ is the bulk modulus.

Solution:

$$\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon} = [\lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}] : \boldsymbol{\varepsilon} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} + \boldsymbol{\sigma}^{sph} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu(\boldsymbol{\varepsilon}^{dev} + \boldsymbol{\varepsilon}^{sph})$$

$$\Rightarrow \boldsymbol{\sigma}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu(\boldsymbol{\varepsilon}^{dev} + \boldsymbol{\varepsilon}^{sph})$$

$$\Rightarrow \boldsymbol{\sigma}^{dev} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} + 2\mu\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} - \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1}$$

$$\Rightarrow \boldsymbol{\sigma}^{dev} = \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} - \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1}$$

The trace of the stress tensor:

$$\mathrm{Tr}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \mathbf{1} = [\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}] : \mathbf{1} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})3 + 2\mu\mathrm{Tr}(\boldsymbol{\varepsilon}) = (3\lambda + 2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon})$$

with that we can obtain:

$$\Rightarrow \boldsymbol{\sigma}^{dev} = \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} - \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1}$$

$$\Rightarrow \boldsymbol{\sigma}^{dev} = \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} - \frac{(3\lambda + 2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}$$

$$\Rightarrow \boldsymbol{\sigma}^{dev} = \underbrace{\left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} - \frac{(3\lambda + 2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}}_{=\mathbf{0}} + 2\mu\boldsymbol{\varepsilon}^{dev}$$

To the equations $\boldsymbol{\sigma}^{dev} = 2\mu\boldsymbol{\varepsilon}^{dev}$ we must add the constraint:

$$\left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} - \frac{(3\lambda + 2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} = \mathbf{0} \quad \Rightarrow \quad \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} - \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1} = \mathbf{0}$$

$$\Rightarrow \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} = 3\left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} \quad \Rightarrow \quad \Rightarrow \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} = 3\kappa\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$$

or $\mathrm{Tr}(\boldsymbol{\sigma}) = 3\kappa\,\mathrm{Tr}(\boldsymbol{\varepsilon})$.

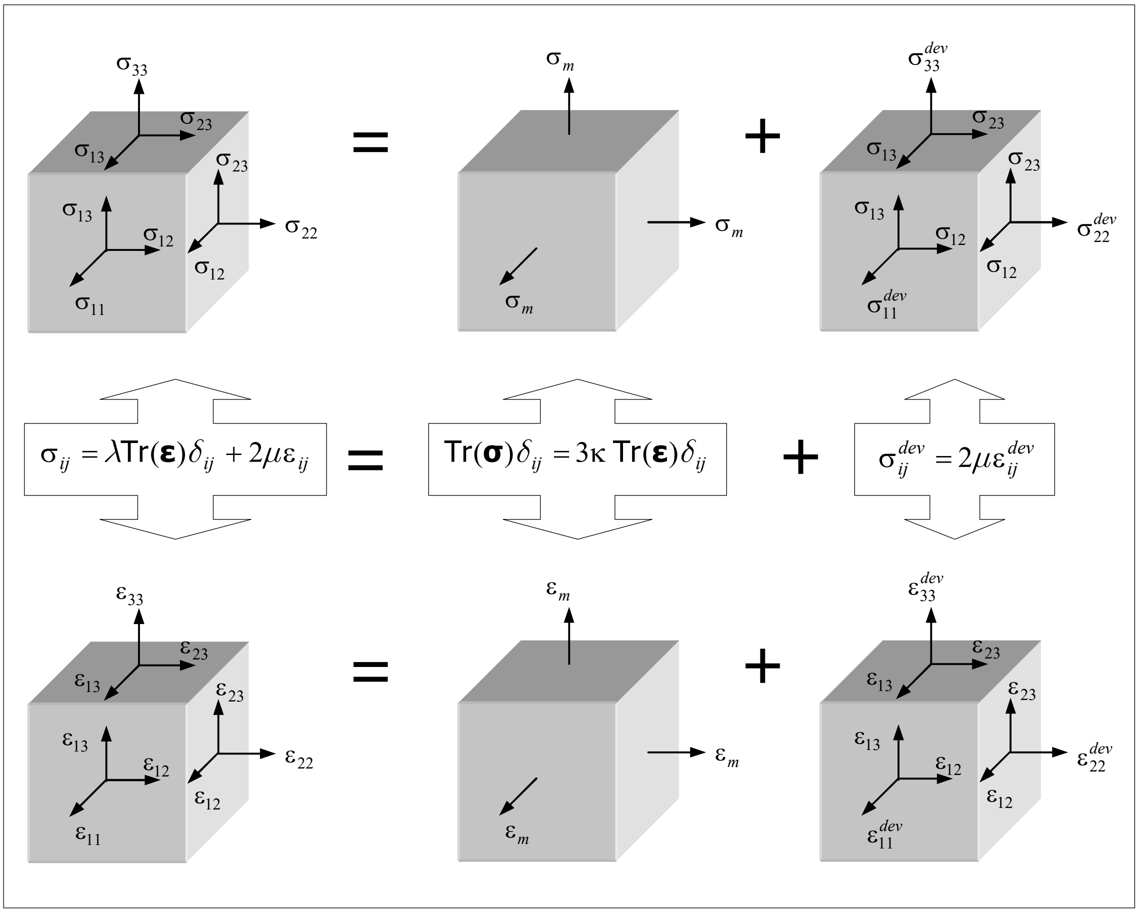


Figure 6.5: Constitutive equations for isotropic material.

Alternative solution:

Starting from the constitutive equation in stress for an isotropic linear elastic material $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$, and by considering the linear regime the relationship $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}) = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{sph}) + \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{dev})$ holds, where:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{sph}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}^{sph})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{sph}$$

$$\boldsymbol{\sigma}^{sph} = \lambda \mathrm{Tr}\left[\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}\right]\mathbf{1} + 2\mu\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1} = \left(\lambda + \frac{2\mu}{3}\right)\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} = \kappa\, \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$$

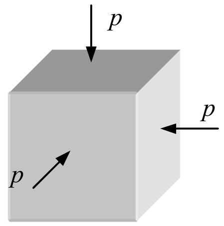
$$\frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1} = \kappa\, \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$$

$$\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} = 3\kappa\, \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$$

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{dev}) = \lambda \underbrace{\mathrm{Tr}(\boldsymbol{\varepsilon}^{dev})}_{=0}\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} = 2\mu\boldsymbol{\varepsilon}^{dev}$$

Note that $\mathrm{Tr}[\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{sph})] = \mathrm{Tr}[\boldsymbol{\sigma}^{sph}] = \mathrm{Tr}[\boldsymbol{\sigma}]$ holds.

NOTE: Note that for an isotropic material if we have a purely spherical state of compression:



$$p > 0$$

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \quad \therefore \quad \mathrm{Tr}(\boldsymbol{\sigma}) = -3p < 0$$

we have that $\mathrm{Tr}(\boldsymbol{\sigma}) = 3\kappa\, \mathrm{Tr}(\boldsymbol{\varepsilon}) < 0$, and considering that $\kappa = \dfrac{E}{3(1-2\nu)}$, we can conclude that: if $\nu > 0.5$ this implies that $\kappa < 0$ and as consequence $\mathrm{Tr}(\boldsymbol{\varepsilon}) > 0$, i.e. an expansion, which has no physical meaning for a compression state in isotropic materials. With that we can conclude that $\nu < 0.5$.

Problem 6.9

A parallelepiped of dimensions $a = 0.10m$, $b = 0.20m$, $c = 0.30m$, (see Figure 6.6), is made up of an elastic material whose mechanical properties are: Poisson's ration $\nu = 0.3$ and Young's modulus $E = 2\times 10^6 N/m^2$. Said parallelepiped is introduced into a cavity of width b whose walls are very rigid, so that two opposite faces of the parallelepiped are in contact with the cavity walls. Once the parallelepiped is this position the temperature is raised by the increment $\Delta T = 30^{\mathrm{o}} C$.

a) Calculate the values of the principal stresses at any point of the parallelepiped.
b) Find the strain components.

Consider that the thermal expansion coefficient of the material is $1.25\times 10^{-5}\,{}^{\mathrm{o}}C^{-1}$.

Solution:

For an isotropic material the temperature variation (ΔT) only affects the normal strain components $\left(\boldsymbol{\varepsilon} = \dfrac{1+\nu}{E}\boldsymbol{\sigma} - \dfrac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \alpha\Delta T\mathbf{1}\right)$, so, the solid will be only subjected by normal stresses. Note also that the solid can deform freely according to the directions x and z, hence the normal stresses are $\sigma_x = \sigma_z = 0$. The solid is restricted to move according to the y-direction, hence $\varepsilon_y = 0$:

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \alpha\Delta T = \frac{1}{E}\sigma_y + \alpha\Delta T = 0 \quad \Rightarrow \quad \sigma_y = -E\alpha\Delta T$$

By means of the problem data , (Figure 6.6), we can obtain:

$$\sigma_y = -E\alpha\Delta T = -2\times 10^6 \times 1.25\times 10^{-5}(30) = -750\frac{N}{m^2}$$

The Cauchy stress tensor filed is constant, so, the stress components at any point of the body are:

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -750 & 0 \\ 0 & 0 & 0 \end{bmatrix} Pa$$

b)

$$\varepsilon_x = \varepsilon_z = \frac{-\nu\sigma_y}{E} + \alpha\Delta T = 1.125\times10^{-4} + 3.75\times10^{-4} = 4.875\times10^{-4}$$

The strain tensor components:

$$\varepsilon_{ij} = \begin{bmatrix} 4.875 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4.875 \end{bmatrix}\times10^{-4}$$

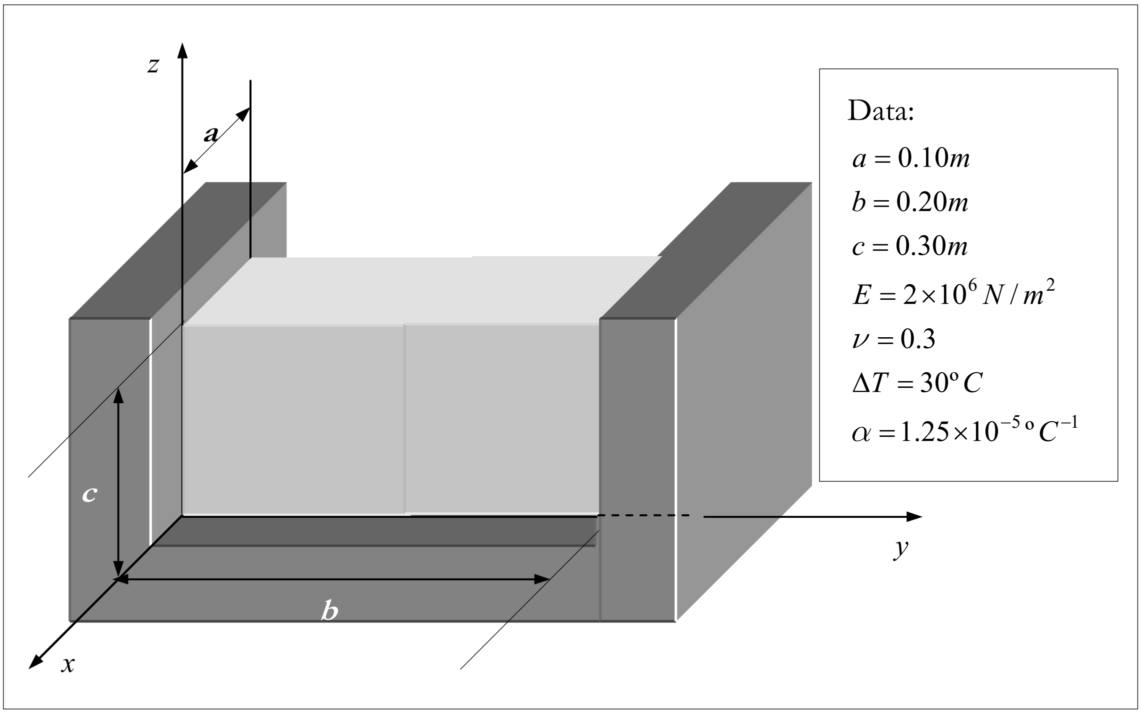


Figure 6.6

Problem 6.10

Consider a container whose squared cross section has dimensions $0.10m\times0.10m$, consider also that the container walls are very rigid. In the interior of said container is placed a synthetic rubber block whose dimensions are $0.10m\times0.10m\times0.5m$, (see Figure 6.7(a)). The rubber block fits perfectly into the rigid container.

Consider that the mechanical properties of the rubber are $E = 2.94\times10^6 N/m^2$ (Young's modulus) and $\nu = 0.1$ (Poisson's ratio).

Above the rubber is poured $0.004m^3$ of mercury, whose mass density is $13580 kg/m^3$.

a) Obtain the height H that reach the mercury, (see Figure 6.7(b));
b) Obtain the stress state at any point of the rubber block.

Hypothesis: 1) the weight of the rubber is negligible. 2) Consider the acceleration of gravity equal to $g = 10m/s^2$, and also consider that between the rubber block faces and the container walls there are no friction.

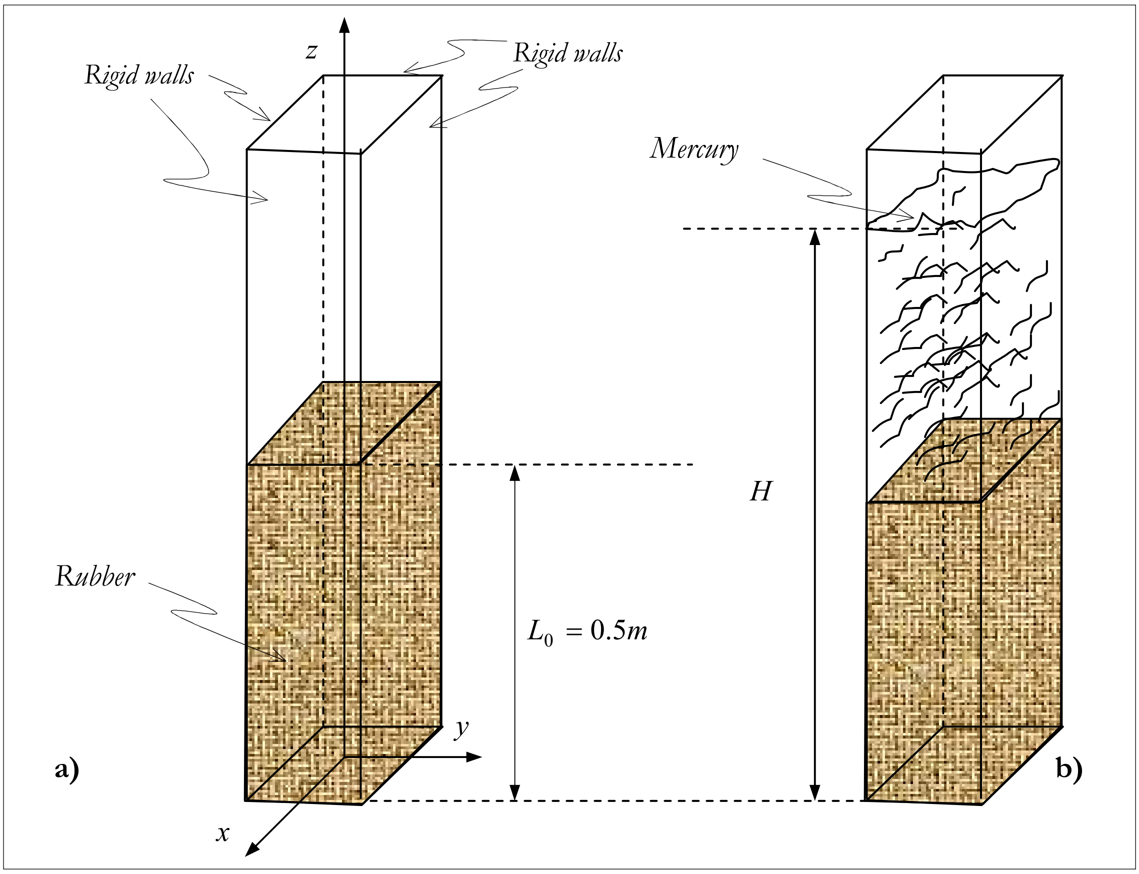


Figure 6.7

Solution:

The total force acting in the rubber, due to the weight of mercury, can be calculated as follows:

$$F = ma = V_{mer}\rho_{mer} g = 0.004(m^3)\times 13580\left(\frac{kg}{m^3}\right)\times 10\left(\frac{m}{s^2}\right) = 543.20\left(\frac{kgm}{s^2} \equiv N\right)$$

Then, the normal stress according to the z-direction is:

$$\sigma_z = -\frac{F}{A} = -\frac{543.20}{(0.1\times 0.1)} = -54.320\times 10^3 \frac{N}{m^2}$$

According to the directions x and y the rubber does not deform, hence $\varepsilon_x = \varepsilon_y = 0$, and by using these restrictions we can conclude that:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = 0 \quad \Rightarrow \quad \sigma_x = \nu(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] = 0 \quad \Rightarrow \quad \sigma_y = \nu(\sigma_x + \sigma_z)$$

$$\sigma_y = \nu(\sigma_x + \sigma_z) = \nu\{[\nu(\sigma_y + \sigma_z)] + \sigma_z\} = \nu^2\sigma_y + \nu^2\sigma_z + \nu\sigma_z = \nu^2\sigma_y + (\nu^2 + \nu)\sigma_z$$

$$\Rightarrow \sigma_y = \frac{(\nu^2 + \nu)}{(1-\nu^2)}\sigma_z = \frac{\nu}{(1-\nu)}\sigma_z = -6035.55Pa = \sigma_x$$

The normal strain according to the z-direction is given by:

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = \frac{1}{2.94\times 10^6}\{-54320 - 0.1[2(-6035.55)]\} = -0.0180656$$

b) The height H reached by the mercury is given by:

$$H = h_{mer} + (L_0 - \Delta L)$$

where the length variation of the rubber block is:

$$\Delta L = L_0 \varepsilon_z = 0.5 \times (-0.018656) = -0.00903m$$

By considering the mercury incompressible, the parameter h_{mer} can be calculated as:

$$V_{mer} = b^2 \times h_{mer} = 0.004 \quad \Rightarrow \quad h_{mer} = \frac{0.004}{0.1\times 0.1} = 0.4m$$

thus,

$$H = h_{mer} + (L_0 - \Delta L) = 0.4 + (0.5 - 0.00903) = 0.891m$$

Problem 6.11

By means of a material test in the laboratory, it was obtained the following relationships:

$$\begin{aligned}
\varepsilon_x &= \left(\frac{1}{E_1}\right)\sigma_x + \left(\frac{-\nu_{21}}{E_2}\right)\sigma_y + \left(\frac{-\nu_{31}}{E_3}\right)\sigma_z \\
\varepsilon_y &= \left(\frac{-\nu_{12}}{E_1}\right)\sigma_x + \left(\frac{1}{E_2}\right)\sigma_y + \left(\frac{-\nu_{32}}{E_3}\right)\sigma_z \\
\varepsilon_z &= \left(\frac{-\nu_{13}}{E_1}\right)\sigma_x + \left(\frac{-\nu_{23}}{E_2}\right)\sigma_y + \left(\frac{1}{E_3}\right)\sigma_z
\end{aligned} \tag{6.39}$$

where $\nu_{12} = 0.2$, $\nu_{13} = 0.3$, $\nu_{23} = 0.25$, $E_1 = 1000MPa$, $E_2 = 2000MPa$, $E_3 = 1500MPa$.

Knowing that the analyzed material is orthotropic, obtain the values of ν_{21}, ν_{31} and ν_{32}.

Solution:

The elasticity matrix for orthotropic materials has the following format:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{66} \end{bmatrix} \quad \begin{array}{l} \textit{Orthotropic symmetry} \\ \text{9 independent constants} \end{array} \tag{6.40}$$

By restructuring the relationships given by (6.39) in matrix form we can obtain:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{E_1}\right) & \left(\frac{-\nu_{21}}{E_2}\right) & \left(\frac{-\nu_{31}}{E_3}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{12}}{E_1}\right) & \left(\frac{1}{E_2}\right) & \left(\frac{-\nu_{32}}{E_3}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{13}}{E_1}\right) & \left(\frac{-\nu_{23}}{E_2}\right) & \left(\frac{1}{E_3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} \qquad (6.41)$$

Then, for orthotropic material it must fulfill that:

$$\left(\frac{-\nu_{21}}{E_2}\right) = \left(\frac{-\nu_{12}}{E_1}\right) \quad ; \quad \left(\frac{-\nu_{31}}{E_3}\right) = \left(\frac{-\nu_{13}}{E_1}\right) \quad ; \quad \left(\frac{-\nu_{32}}{E_3}\right) = \left(\frac{-\nu_{23}}{E_2}\right)$$

with that we can obtain

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} \quad \Rightarrow \quad \nu_{21} = \frac{E_2\nu_{12}}{E_1} = \frac{2000\times 0.2}{1000} = 0.4$$

$$\frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \quad \Rightarrow \quad \nu_{31} = \frac{E_3\nu_{13}}{E_1} = \frac{1500\times 0.3}{1000} = 0.45$$

$$\frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \quad \Rightarrow \quad \nu_{32} = \frac{E_3\nu_{23}}{E_2} = \frac{1500\times 0.25}{2000} = 0.1875$$

Problem 6.12

Given an isotropic linear elastic material whose mechanical properties are $E = 71\,GPa$ (Young's modulus), $G = 26.6\,GPa$ (shear modulus), find the strain tensor components and the strain energy density at the point in which the stress state, in Cartesian basis, is represented by:

$$\sigma_{ij} = \begin{bmatrix} 20 & -4 & 5 \\ -4 & 0 & 10 \\ 5 & 10 & 15 \end{bmatrix} MPa$$

Solution: Poisson's ratio can be obtained by means of the equation:

$$\mu = G = \frac{E}{2(1+\nu)} \Rightarrow \nu = \frac{E}{2G} - 1 \approx 0.335 \qquad \textit{and} \qquad \frac{(1+\nu)}{E} = \frac{1}{2G} = \frac{1}{53.2}(GPa)^{-1}$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] = \frac{1}{71\times 10^9}\left[20 - 0.335\left(0+15\right)\right]10^6 = 211\times 10^{-6}$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right] = \frac{1}{71\times 10^9}\left[0 - 0.335\left(20+15\right)\right]10^6 = -165\times 10^{-6}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right] = \frac{1}{71\times 10^9}\left[15 - 0.335\left(20+0\right)\right]10^6 = 117\times 10^{-6}$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+0.335}{71\times 10^9}(-4\times 10^6) = 75\times 10^{-6}$$

$$\varepsilon_{13} = \frac{1+\nu}{E}\sigma_{13} = \frac{1+0.335}{71\times 10^9}(5\times 10^6) = 94\times 10^{-6}$$

$$\varepsilon_{23} = \frac{1+\nu}{E}\sigma_{23} = \frac{1+0.335}{71\times 10^{9}}(10\times 10^{6}) = 188\times 10^{-6}$$

thus:

$$\varepsilon_{ij} = \begin{bmatrix} 211 & -75 & 94 \\ -75 & -165 & 188 \\ 94 & 188 & 117 \end{bmatrix}\times 10^{-6}$$

We can also use the equation

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}$$

where $\frac{(1+\nu)}{E} = \frac{1}{53200}(MPa)^{-1}$, $\mathrm{Tr}(\boldsymbol{\sigma}) = 35(MPa)$, $\frac{\nu}{E} = 4.71831\times 10^{-6}(MPa)^{-1}$, then

$$\varepsilon_{ij} = \frac{1}{53200MPa}\begin{bmatrix} 20 & -4 & 5 \\ -4 & 0 & 10 \\ 5 & 10 & 15 \end{bmatrix} MPa - (4.71831\times 10^{-6})(MPa)^{-1}(35MPa)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 211 & -75 & 94 \\ -75 & -165 & 188 \\ 94 & 188 & 117 \end{bmatrix}\times 10^{-6}$$

Then, the strain energy density for a linear elastic material is obtained by the equation:

$$\Psi^{e}(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}^{e}:\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} \xrightarrow{indicial} \Psi^{e} = \frac{1}{2}\varepsilon_{ij}\sigma_{ij}$$

Next, by considering the symmetry of the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the strain energy density can be calculated as follows:

$$\Psi^{e} = \frac{1}{2}\left[\varepsilon_{11}\sigma_{11} + \varepsilon_{22}\sigma_{22} + \varepsilon_{33}\sigma_{33} + 2\varepsilon_{12}\sigma_{12} + 2\varepsilon_{23}\sigma_{23} + 2\varepsilon_{13}\sigma_{13}\right]$$

$$= \frac{1}{2}\left[(211)(20) + (-165)(0) + (117)(15) + 2(-75)(-4) + 2(188)(10) + 2(94)(5)\right] = 5637.5\ J/m^{3}$$

We can also obtain the strain energy density by using the equation:

$$\overline{\Psi}^{e}(\boldsymbol{\sigma}) = \frac{1}{6(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^{2} - \frac{1}{2\mu}II_{\boldsymbol{\sigma}^{dev}} = \frac{1}{6(3\lambda+2\mu)}I_{\boldsymbol{\sigma}}^{2} + \frac{1}{2\mu}J_{2}$$

and if we consider that $I_{\boldsymbol{\sigma}} = 3.5\times 10^{7}$; $II_{\boldsymbol{\sigma}} = -2.4933\times 10^{14}$; $\lambda \approx 5.3804\times 10^{10}\,Pa$; $\mu = G$, we can obtain $\Psi^{e} \approx 5638.03\ J/m^{3}$. Note that $\overline{\Psi}^{e}(\boldsymbol{\sigma}) = \Psi^{e}(\boldsymbol{\varepsilon})$ since we are dealing with linear elastic material, (see **Problem 5.5**). And any discrepancies in the numerical results of Ψ^{e} are due to numerical approximations.

Problem 6.13

Find the strain energy density in terms of the principal invariants of $\boldsymbol{\varepsilon}$.

Solution:

$$\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}:[\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}] = \frac{\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2}\underbrace{\boldsymbol{\varepsilon}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}$$

$$= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^T) = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon})$$

$$= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}^2)$$

We can add and subtract the term $\mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2$ without altering the above outcome:

$$\Psi^e = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}^2) - \mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 = \frac{1}{2}(\lambda+2\mu)[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mu\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\}$$

Finally, if we consider that the principal invariants of the strain tensor $\boldsymbol{\varepsilon}$ are $I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon})$, $II_{\boldsymbol{\varepsilon}} = \frac{1}{2}\{I_{\boldsymbol{\varepsilon}}^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\}$, we can obtain:

$$\boxed{\Psi^e = \frac{1}{2}(\lambda+2\mu)I_{\boldsymbol{\varepsilon}}^2 - 2\mu II_{\boldsymbol{\varepsilon}} = \Psi^e(I_{\boldsymbol{\varepsilon}}, II_{\boldsymbol{\varepsilon}})}$$

Problem 6.14

The responses of a liner thermoelastic solid due to two actions are known, namely: $I(\vec{\mathbf{b}}^{(I)}, \vec{\mathbf{t}}^{*(I)}\ on\ S_{\boldsymbol{\sigma}}; \vec{\mathbf{u}}^{*(I)}\ on\ S_{\vec{\mathbf{u}}}; \Delta T^{(I)})$ and $II(\vec{\mathbf{b}}^{(II)},\ \vec{\mathbf{t}}^{*(II)}\ on\ S_{\boldsymbol{\sigma}};\ \vec{\mathbf{u}}^{*(II)}\ on\ S_{\vec{\mathbf{u}}}; \Delta T^{(II)})$. Obtain the response of the system formed by $I + II$ and justify, (see Oliver (2000)).

Solution:

As we are dealing with a linear regime the following is satisfied:

$$\vec{\mathbf{b}} = \vec{\mathbf{b}}^{(I)} + \vec{\mathbf{b}}^{(II)} \quad ; \quad \Delta T = \Delta T^{(I)} + \Delta T^{(II)} \quad ; \quad \vec{\mathbf{t}}^* = \vec{\mathbf{t}}^{*(I)} + \vec{\mathbf{t}}^{*(II)} \quad ; \quad \vec{\mathbf{u}}^* = \vec{\mathbf{u}}^{*(I)} + \vec{\mathbf{u}}^{*(II)}$$

The same is true for the fields:

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}^{(I)} + \vec{\mathbf{u}}^{(II)} \quad ; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(I)} + \boldsymbol{\varepsilon}^{(II)} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{(I)} + \boldsymbol{\sigma}^{(II)}$$

Starting from the governing equations of linear thermoelastic equilibrium we have:

- *The equilibrium equations:*

$$\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \nabla_{\vec{x}}\cdot(\boldsymbol{\sigma}^{(I)} + \boldsymbol{\sigma}^{(II)}) + \rho(\vec{\mathbf{b}}^{(I)} + \vec{\mathbf{b}}^{(II)}) = [\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}^{(I)} + \rho\vec{\mathbf{b}}^{(I)}] + [\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}^{(II)} + \rho\vec{\mathbf{b}}^{(II)}] = \vec{\mathbf{0}}$$

- *The kinematic equations:*

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(I)} + \boldsymbol{\varepsilon}^{(II)} = \frac{1}{2}\left[\nabla_{\vec{x}}\vec{\mathbf{u}}^{(I)} + (\nabla_{\vec{x}}\vec{\mathbf{u}}^{(I)})^T\right] + \frac{1}{2}\left[\nabla_{\vec{x}}\vec{\mathbf{u}}^{(II)} + (\nabla_{\vec{x}}\vec{\mathbf{u}}^{(II)})^T\right]$$

$$= \frac{1}{2}\left\{\left[\nabla_{\vec{x}}\vec{\mathbf{u}}^{(I)} + \nabla_{\vec{x}}\vec{\mathbf{u}}^{(II)}\right] + \left[\nabla_{\vec{x}}\vec{\mathbf{u}}^{(I)} + \nabla_{\vec{x}}\vec{\mathbf{u}}^{(II)}\right]^T\right\}$$

$$= \frac{1}{2}\left\{\nabla_{\vec{x}}\left[\vec{\mathbf{u}}^{(I)} + \vec{\mathbf{u}}^{(II)}\right] + \left[\nabla_{\vec{x}}(\vec{\mathbf{u}}^{(I)} + \vec{\mathbf{u}}^{(II)})\right]^T\right\} = \frac{1}{2}\left\{\nabla_{\vec{x}}\vec{\mathbf{u}} + [\nabla_{\vec{x}}\vec{\mathbf{u}}]^T\right\} = \boldsymbol{\varepsilon}$$

- *The constitutive equations in stress:*

$\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon} + \boldsymbol{M}\Delta T$

where $\boldsymbol{M}$ is the thermal stress tensor

$$\begin{aligned}\boldsymbol{\sigma} &= \mathbb{C}^e : \boldsymbol{\varepsilon} + \boldsymbol{M}\Delta T = \mathbb{C}^e : (\boldsymbol{\varepsilon}^{(I)} + \boldsymbol{\varepsilon}^{(II)}) + \boldsymbol{M}(\Delta T^{(I)} + \Delta T^{(II)}) \\ &= (\mathbb{C}^e : \boldsymbol{\varepsilon}^{(I)} + \boldsymbol{M}\Delta T^{(I)}) + (\mathbb{C}^e : \boldsymbol{\varepsilon}^{(II)} + \boldsymbol{M}\Delta T^{(II)}) \\ &= \boldsymbol{\sigma}^{(I)} + \boldsymbol{\sigma}^{(II)}\end{aligned}$$

Then, we can conclude that all the conditions are met. Then, we can apply the principle of superposition to the linear thermoelastic problem, since we are dealing with linear regime.

Problem 6.15

Let us consider the rod of length $L = 7.5m$, whose cross sectional diameter is equal to $0.1m$. The rod is made up of a material whose thermo-mechanical properties are: $E = 2.0\times 10^{11} Pa$ (Young's modulus) and $\alpha = 20\times 10^{-6}\dfrac{1}{{}^{\circ}C}$ (coefficient of thermal expansion). Initially the rod has a temperature equal to $15^{\circ}C$ which later rises to $50^{\circ}C$.

a) Considering that the rod can expand freely, calculate the total elongation of the rod, ΔL;

b) Now assume that the rod cannot expand freely because concrete blocks have been placed at its ends, (see Figure 6.8(b)). Find the stress in the rod.

Hint: Consider the problem in one dimension.

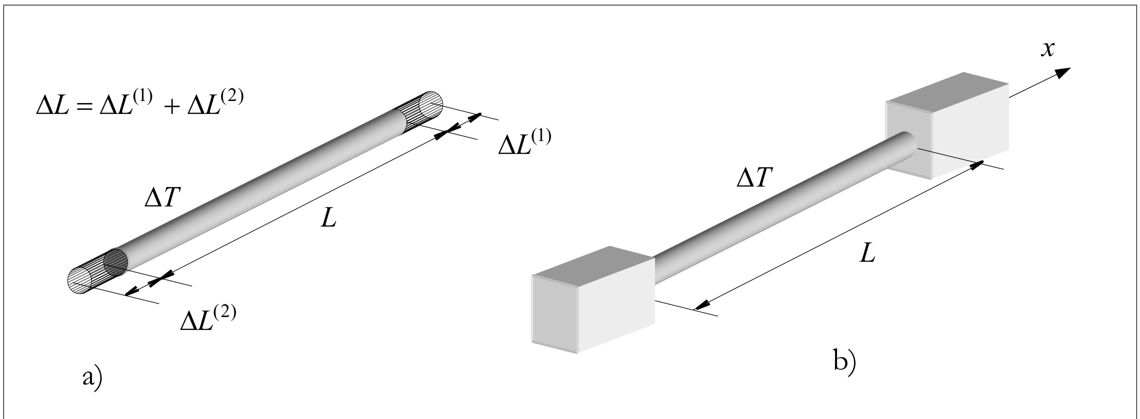


Figure 6.8: Rod under thermal effect.

Solution: a) To obtain the elongation, we pre-calculate the thermal strain according to the rod axis direction $\varepsilon_{ij} = \alpha\,\Delta T\,\delta_{ij}$. Since this is a one-dimensional case, we need only consider the normal strain component according to the x-direction, $\varepsilon_{11} = \varepsilon_x$, then:

$$\varepsilon_{11} = \varepsilon_x = 20\times 10^{-6}(50-15) = 7\times 10^{-4}$$

Then, the total elongation, $\Delta L = \Delta L^{(1)} + \Delta L^{(2)}$, is obtained by solving the integral:

$$\Delta L = \int_0^L \varepsilon_x dx = \varepsilon_x L = 7\times 10^{-4}\times 7.5 = 5.25\times 10^{-3}\,m$$

Note that as the rod can expand freely, it is stress-free.

b) If the ends cannot move, there will be a homogeneous stress field equal to:

$$\sigma_x = -E\alpha\,\Delta T = -E\,"\varepsilon_x" = -2.0\times 10^{11}\times 7\times 10^{-4} = -1.4\times 10^{8}\,Pa$$

Note that in the case 2) there is no strain, since $\Delta L = 0$. Moreover, it is the same as when the initial length is equal to $L+\Delta L$ in which we apply compression stress in order to obtain a final length equal to L.

Problem 6.16

Consider an isotropic linear elastic material with the following thermo mechanical properties $E = 10^6\,Pa$ (Young's modulus), $\nu = 0.25$ (Poisson's ratio), $\alpha = 20\times 10^{-6}\,{}^{\mathrm{o}}C^{-1}$ (Coefficient of thermal expansion).

Consider that that at one point of the solid the stress tensor components are given by:

$$\sigma_{ij} = \begin{bmatrix} 12 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 6 \end{bmatrix} Pa$$

a) Obtain the principal stresses and principal directions of the stress tensor; Obtain the maximum shear stress.

b) Obtain the strain tensor components. And find the principal strains and directions.

c) Obtain the strain energy density.

d) If the solid undergoes a change in temperature $\Delta T = 50^{\mathrm{o}}\,C$, obtain the final strain state at this point.

e) We can say that we are dealing with a state of plane stress?

Solution:

a) We obtain the eigenvalues by solving the characteristic determinant. Note that we already know an eigenvalue $\sigma_2 = 0$ which is associated with the direction $\hat{n}_i^{(2)} = [0 \quad \pm 1 \quad 0]$. Then, to obtain the remaining eigenvalues, it is sufficient to solve:

$$\begin{vmatrix} 12-\sigma & 4 \\ 4 & 6-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad \sigma^2 - 18\sigma + 56 = 0$$

Solving the quadratic equation we can obtain:

$$\sigma_{(1,3)} = \frac{18 \pm \sqrt{324-224}}{2} \quad \Rightarrow \quad \begin{cases} \sigma_1 = 14 \\ \sigma_3 = 4 \end{cases}$$

$$\sigma'_{ij} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} Pa$$

And the eigenvectors (unit vectors) are given by:

$$\sigma_1 = 14 \xrightarrow{eigenvector} \hat{n}_i^{(1)} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} = [0.8944 \quad 0 \quad 0.4472]$$

$$\sigma_2 = 0 \xrightarrow{eigenvector} \hat{n}_i^{(2)} = [0 \quad 1 \quad 0]$$

$$\sigma_3 = 4 \xrightarrow{eigenvector} \hat{n}_i^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} = [0.4472 \quad 0 \quad -0.8944]$$

Making the change of nomenclature such that $\sigma_I > \sigma_{II} > \sigma_{III}$, we have $\sigma_I = 14$, $\sigma_{II} = 4$, $\sigma_{III} = 0$, and the Mohr's circle can be represented as shown in Figure 6.9.

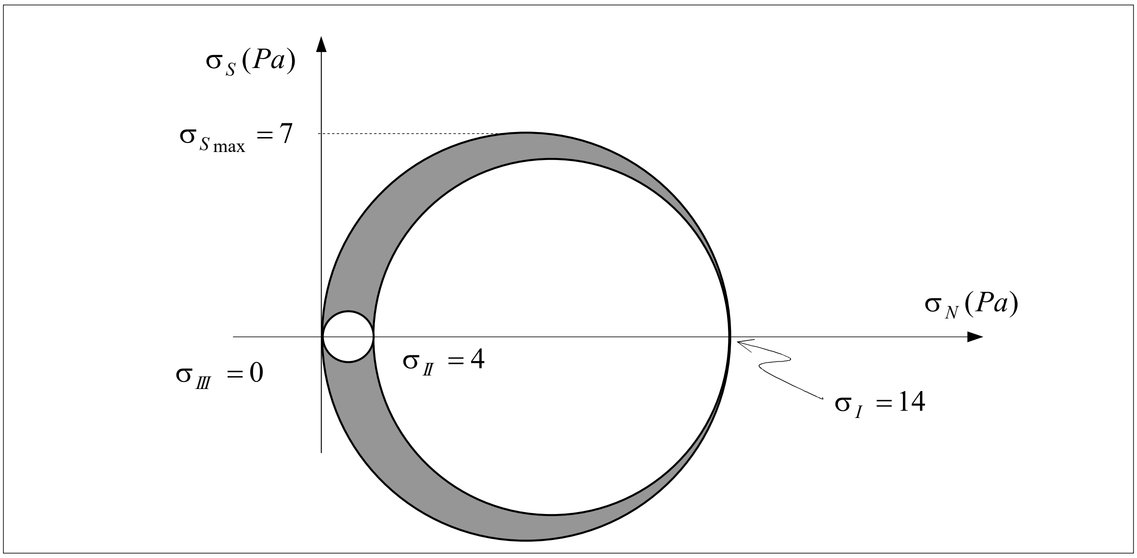


Figure 6.9

We can obtain the maximum shear stress, (see Figure 6.9), as follows:

$$\sigma_{S\max} = \frac{\sigma_I - \sigma_{III}}{2} = \frac{(14)-(0)}{2} = 7Pa$$

b) The Cauchy stress tensor components are given by:

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij} \quad \xrightarrow{inverse} \quad \varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma_{ij}$$

Remember that $\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}$, and the reciprocal form $\boldsymbol{\varepsilon} = \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}$.

where $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} = 4\times 10^5\,Pa$, $\mu = G = \dfrac{E}{2(1+\nu)} = 4\times 10^5\,Pa$, $\dfrac{1}{2\mu} = 1.25\times 10^{-6}$, $\mathrm{Tr}(\boldsymbol{\sigma}) = 18$, $\dfrac{-\lambda}{2\mu(3\lambda+2\mu)} = -2.5\times 10^{-7}\,Pa$, $\dfrac{-\lambda \mathrm{Tr}(\boldsymbol{\sigma})}{2\mu(3\lambda+2\mu)} = -4.5\times 10^{-6}\,Pa$

$$\varepsilon_{ij} = -4.5\times 10^{-6}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + 1.25\times 10^{-6}\begin{bmatrix}12&0&4\\0&0&0\\4&0&6\end{bmatrix} = \begin{bmatrix}10.5&0&5\\0&-4.5&0\\5&0&3\end{bmatrix}\times 10^{-6}$$

For an isotropic linear material the principal directions of the stress and strain match. The principal strains can be obtained by means of $\varepsilon'_{ij} = \dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \dfrac{1}{2\mu}\sigma'_{ij}$ in the principal space, i.e.:

$$\varepsilon'_{ij} = -4.5\times 10^{-6}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + 1.25\times 10^{-6}\begin{bmatrix}14&0&0\\0&0&0\\0&0&4\end{bmatrix} = \begin{bmatrix}13&0&0\\0&-4.5&0\\0&0&0.5\end{bmatrix}\times 10^{-6}$$

The strain energy density is given by $\Psi^e = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$. We can use the principal space to obtain the strain energy density, i.e.:

$$\varepsilon'_{ij} = \begin{bmatrix} 13 & 0 & 0 \\ 0 & -4.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \times 10^{-6} (\text{dimensionless}) \quad ; \quad \sigma'_{ij} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} Pa$$

c) With that we can obtain:

$$\Psi^e = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma'_{ij}\varepsilon'_{ij} = \frac{1}{2}\left[\sigma'_{11}\varepsilon'_{11} + \sigma'_{33}\varepsilon'_{33}\right] = 92 \times 10^{-6} \frac{J}{m^3}$$

d) Using the principle of superposition:

$$\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{\sigma}) + \varepsilon_{ij}(\Delta T) = \varepsilon_{ij}(\boldsymbol{\sigma}) + \alpha \Delta T \delta_{ij}$$

and by substituting the variables values we can obtain:

$$\varepsilon_{ij} = \begin{bmatrix} 10.5 & 0 & 5 \\ 0 & -4.5 & 0 \\ 5 & 0 & 3 \end{bmatrix} \times 10^{-6} + 20 \times 10^{-6}(50) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1010.5 & 0 & 5 \\ 0 & 995.5 & 0 \\ 5 & 0 & 1003 \end{bmatrix} \times 10^{-6}$$

For isotropic materials, the principal directions of the infinitesimal strain tensor are the same as the stress tensor.

e) We cannot say that we are dealing with a state of plane stress, since we do not know any information about how stresses vary in the continuum, i.e. we do not know the *stress field*. Remember that the state of plane stress is considered when the stress tensor field is independent of one direction.

Problem 6.17

Let us consider a bar to which at one end we apply the force $6000N$, (see Figure 6.10). Find $\varepsilon_x, \varepsilon_y, \varepsilon_z$, and the length change of the bar. Consider also that the bar is made up of a material whose mechanical properties are: Young's modulus: $E = 10^7 Pa$; Poisson's ratio: $\nu = 0.3$.

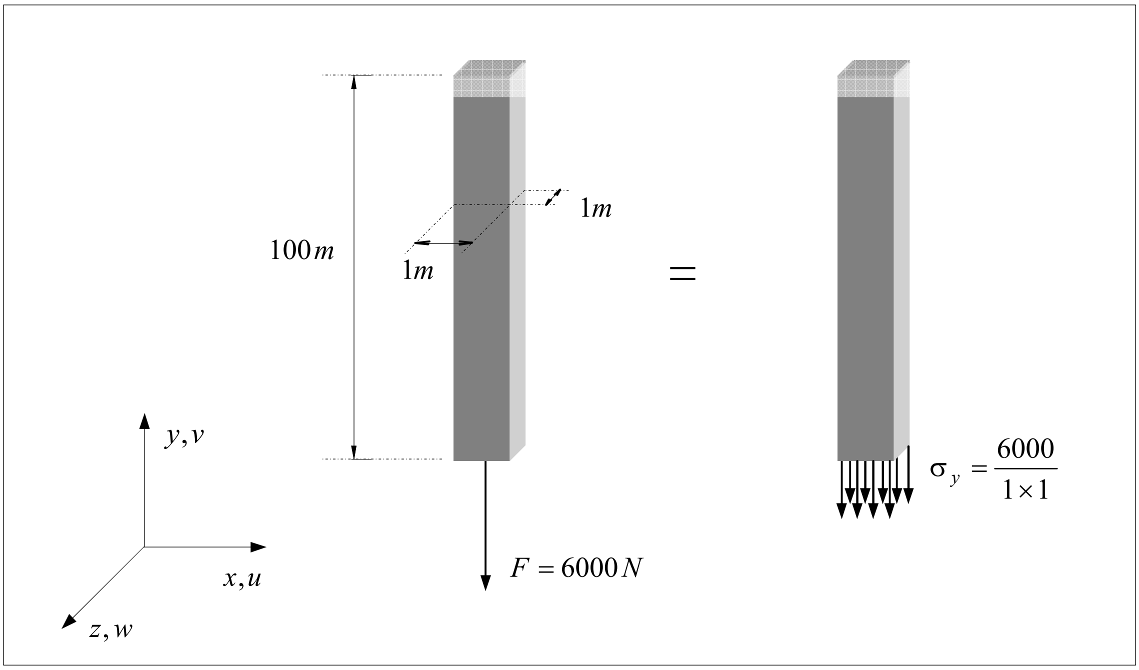


Figure 6.10

Solution: Using the normal strain expressions we can obtain:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = -\frac{\nu}{E}\sigma_y = -\frac{(0.3)(6000)}{10^7} = -0.00018$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] = \frac{\sigma_y}{E} = \frac{6000}{10^7} = 0.0006$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = -\frac{\nu}{E}\sigma_y = -0.00018$$

The total change in cross-sectional dimensions is $u = w = -0.00018 \times 1 = -1.8 \times 10^{-4}\,m$, and the total change in length is $v = 0.0006 \times 100 = 6.0 \times 10^{-2}\,m$.

Problem 6.18

Let us consider a prism (rectangular parallelepiped) whose mechanical properties are: $E_P = 27.44 \times 10^5\,N/cm^2$ (Young's modulus) and $\nu = 0.1$ (Poisson's ratio). The side length of the squared cross section is $a = 20cm$. In both bases of the prism are placed two plates perfectly smooth and rigid, such plates are connected together by four identical cables whose cross section areas are $A_C = 1cm^2$ and they have as mechanical property: Young's modulus ($E_1 = 19.6 \times 10^6\,N/cm^2$). Initially the length of the prism is equal to $\ell = 1m$, (see Figure 6.11). Later, on two opposite sides of the prism we apply a compressive pressure $p = 7350N/cm^2$ as indicated in Figure 6.11.

a) Obtain the stress on the cables σ_C;
b) Obtain the principal stresses in the prism;
c) Obtain the volume variation of the prism.

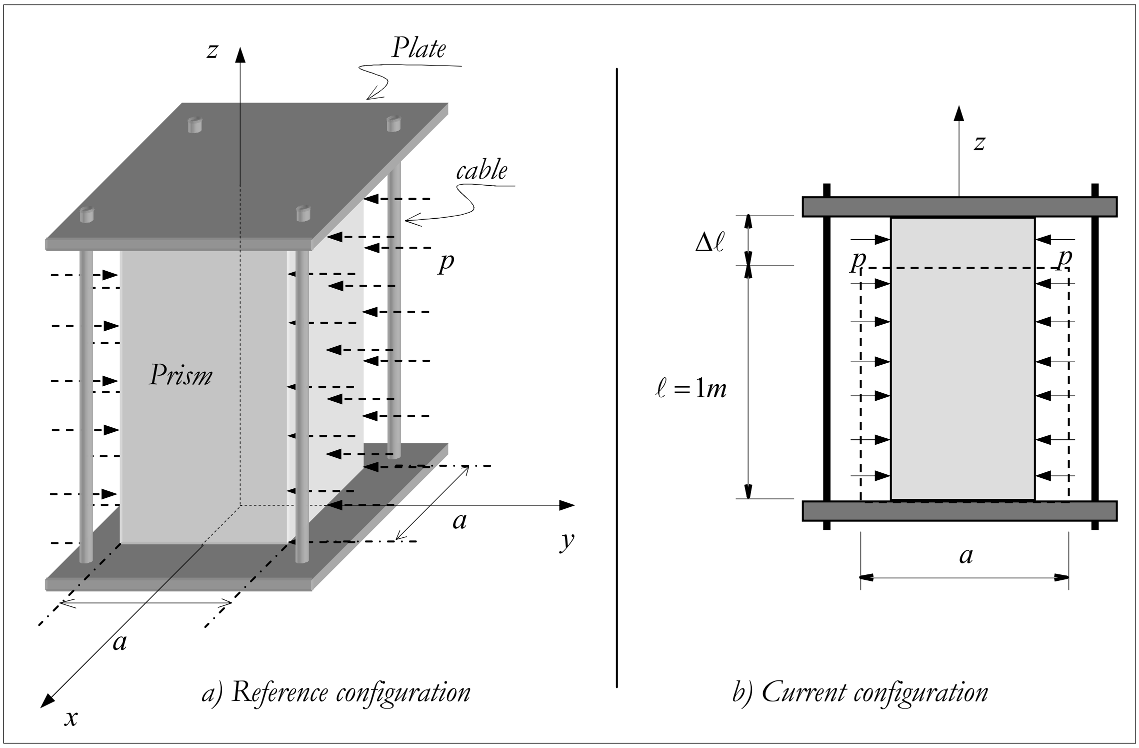


Figure 6.11

Solution:

In Figure 6.12 we show the behavior of the prism with and without the cables.

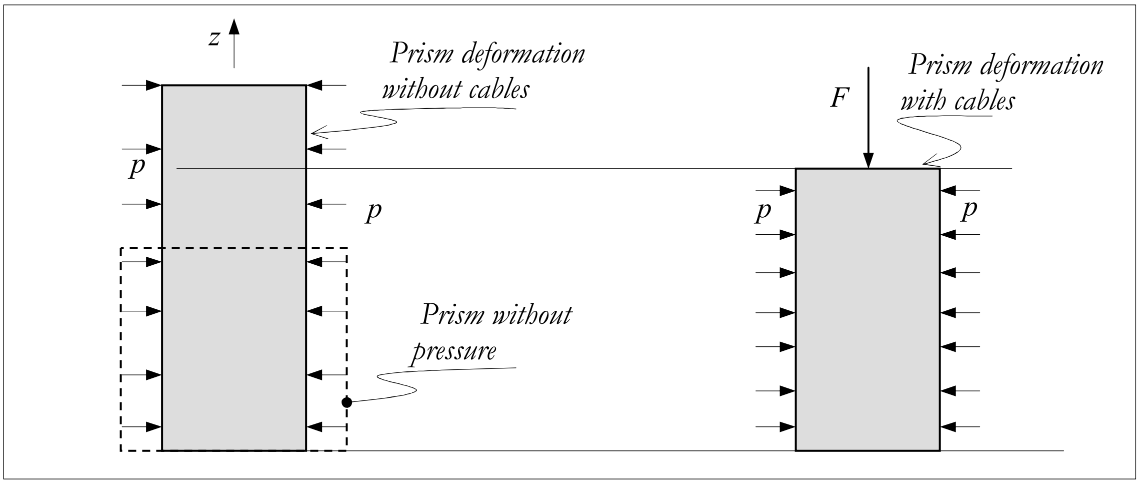


Figure 6.12

Verify that the cable and the prism deform in the same way according to the z-direction, thus:

$$\varepsilon_z^P = \varepsilon_z^C$$

On the cable it fulfills that:

$$\sigma_C = E_C \varepsilon_z^C \qquad \Rightarrow \qquad \varepsilon_z^C = \frac{\sigma_z^C}{E_C}$$

Since the prism has only normal length variation, we will have only normal strain, and in turn normal stress only. The stress field in the prism is given by:

$$\sigma_{ij}^P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & \dfrac{-4\sigma_z^C A_C}{a^2} \end{bmatrix}$$

The strain in the prism according to the direction z:

$$\varepsilon_z^P = \frac{1}{E_P}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{E_P}\left[\frac{-4\sigma_z^C A_C}{a^2} + \nu p\right]$$

By applying the condition $\varepsilon_z^P = \varepsilon_z^C$ we can obtain the equation:

$$\varepsilon_z^P = \varepsilon_z^C \qquad \Rightarrow \qquad \frac{1}{E_P}\left[\frac{-4\sigma_z^C A_C}{a^2} + \nu p\right] = \frac{\sigma_z^C}{E_C}$$

After some algebraic manipulations we can obtain the stress on the cable:

$$\sigma_z^C = \frac{\nu E_C p a^2}{(E_P a^2 + 4E_C A_C)} = \frac{0.1 \times 19.6 \times 10^6 \times 7350 \times 20^2}{(27.44 \times 10^5 \times 20^2 + 4 \times 19.6 \times 10^6 \times 1)} = 4900 \frac{N}{cm^2}$$

The normal stress in the prism according to the z-direction becomes:

$$\sigma_z^P = -\frac{4\sigma_z^C A_C}{a^2} = -\frac{4\times 4900\times 1}{20^2} = -49\frac{N}{cm^2}$$

$$\sigma_{ij}^P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -7350 & 0 \\ 0 & 0 & -49 \end{bmatrix} \frac{N}{cm^2}$$

The volume variation of the prism is obtained as follows:

$$\Delta V = \varepsilon_V V_0$$

where $\varepsilon_V = I_{\boldsymbol{\varepsilon}}$ is the linear volumetric deformation (small deformation regime), in which the relation $\mathrm{Tr}(\boldsymbol{\sigma}) = 3\kappa \mathrm{Tr}(\boldsymbol{\varepsilon})$ holds, where $\kappa = \lambda + \frac{2\mu}{3} = \frac{E}{3(1-2\nu)}$ is the bulk modulus (see **Problem 6.8**), then:

$$\varepsilon_V = I_{\boldsymbol{\varepsilon}} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x + \sigma_y + \sigma_z}{E_P}(1-2\nu) = -2.12857\times 10^{-3}$$

and $V_0 = 4\times 10^4 cm^3$ is the initial prism volume, thus:

$$\Delta V = \varepsilon_V V_0 = (-2.12857\times 10^{-3})(4\times 10^4) = -85.1428 cm^3$$

Problem 6.19

Two rectangular parallelepipeds made up of same material and same shape $a\times b\times c$ are placed on either side of a rigid flat plate attached thereto by their sides $a\times c$. Both parallelepipeds, together with the plate, are introduced into a cavity such as indicated in Figure 6.13. The walls of the cavity are flat, rigid and perfectly smooth. We apply the pressures (force per unit surface area) p_1 and p_2 on the upper faces of the prisms as indicated in Figure 6.13. Consider the Young's modulus E and the Poisson's ratio ν.

a) Obtain the principal stresses in both prisms;
b) Obtain the block edge length variations.

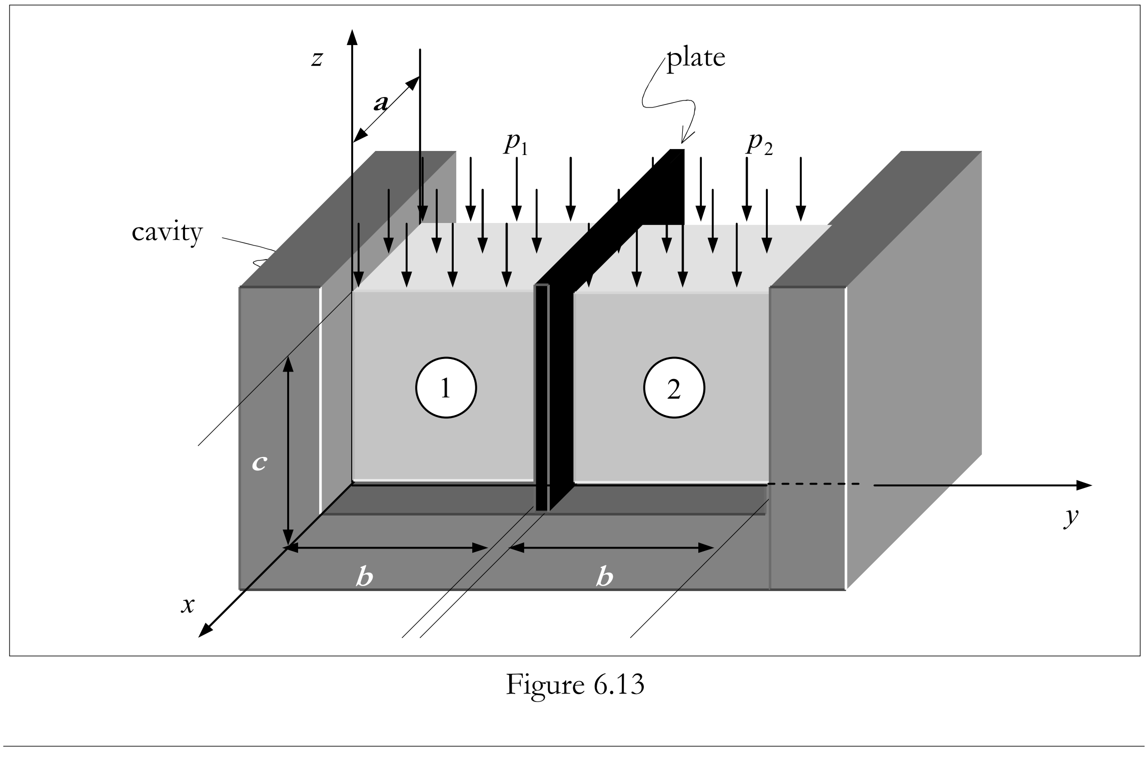


Figure 6.13

Solution:

Prism 1: $\sigma_x^{(1)} = 0 \quad ; \quad \sigma_y^{(1)} \quad ; \quad \sigma_z^{(1)} = -p_1$

Prism 2: $\sigma_x^{(2)} = 0 \quad ; \quad \sigma_y^{(2)} \quad ; \quad \sigma_z^{(2)} = -p_2$

For compatibility of stress:

$$\sigma_y^{(1)} = \sigma_y^{(2)} = \sigma_y$$

$$\varepsilon_y^{(1)} + \varepsilon_y^{(2)} = 0 \quad \Rightarrow \quad \frac{1}{E}\left[\sigma_y^{(1)} - \nu\left(\sigma_x^{(1)} + \sigma_z^{(1)}\right)\right] + \frac{1}{E}\left[\sigma_y^{(2)} - \nu\left(\sigma_x^{(2)} + \sigma_z^{(2)}\right)\right] = 0$$

$$\Rightarrow \left[\sigma_y - \nu\,\sigma_z^{(1)}\right] + \left[\sigma_y - \nu\,\sigma_z^{(2)}\right] = 0 \quad \Rightarrow \quad \left[\sigma_y + \nu\, p_1\right] + \left[\sigma_y + \nu\, p_2\right] = 0$$

thus

$$\sigma_y = \frac{-\nu(p_1 + p_2)}{2}$$

Prism 1: $\sigma_x^{(1)} = 0 \quad ; \quad \sigma_y^{(1)} = -\dfrac{\nu(p_1 + p_2)}{2} \quad ; \quad \sigma_z^{(1)} = -p_1$

Prism 2: $\sigma_x^{(2)} = 0 \quad ; \quad \sigma_y^{(2)} = -\dfrac{\nu(p_1 + p_2)}{2} \quad ; \quad \sigma_z^{(2)} = -p_2$

The strains in each prism are given by:

Prism 1:

$$\begin{aligned}
\varepsilon_x^{(1)} &= \frac{1}{E}\left[\sigma_x^{(1)} - \nu\left(\sigma_y^{(1)} + \sigma_z^{(1)}\right)\right] = \frac{\nu}{2E}\left[\nu(p_1 + p_2) + 2p_1\right] \\
\varepsilon_y^{(1)} &= \frac{1}{E}\left[\sigma_y^{(1)} - \nu\left(\sigma_x^{(1)} + \sigma_z^{(1)}\right)\right] = \frac{\nu}{2E}(p_1 - p_2) \\
\varepsilon_z^{(1)} &= \frac{1}{E}\left[\sigma_z^{(1)} - \nu\left(\sigma_x^{(1)} + \sigma_y^{(1)}\right)\right] = \frac{1}{2E}\left[\nu^2(p_1 + p_2) - 2p_1\right]
\end{aligned}$$

Prism 2:

$$\begin{aligned}
\varepsilon_x^{(2)} &= \frac{1}{E}\left[\sigma_x^{(2)} - \nu\left(\sigma_y^{(2)} + \sigma_z^{(2)}\right)\right] = \frac{\nu}{2E}\left[\nu(p_1 + p_2) + 2p_2\right] \\
\varepsilon_y^{(2)} &= \frac{1}{E}\left[\sigma_y^{(2)} - \nu\left(\sigma_x^{(2)} + \sigma_z^{(2)}\right)\right] = \frac{\nu}{2E}(p_2 - p_1) \\
\varepsilon_z^{(2)} &= \frac{1}{E}\left[\sigma_z^{(2)} - \nu\left(\sigma_x^{(2)} + \sigma_y^{(2)}\right)\right] = \frac{1}{2E}\left[\nu^2(p_1 + p_2) - 2p_2\right]
\end{aligned}$$

The edge variations:

Prism 1	Prism 2	
$\Delta a^{(1)} = \varepsilon_x^{(1)} a = \dfrac{a\nu}{2E}\left[\nu(p_1 + p_2) + 2p_1\right]$	$\Delta a^{(2)} = \varepsilon_x^{(2)} a = \dfrac{\nu a}{2E}\left[\nu(p_1 + p_2) + 2p_2\right]$	
$\Delta b^{(1)} = \varepsilon_y^{(1)} b = \dfrac{\nu b}{2E}(p_1 - p_2)$	$\Delta b^{(2)} = \varepsilon_y^{(2)} b = \dfrac{\nu b}{2E}(p_2 - p_1)$	(6.42)
$\Delta c^{(1)} = \varepsilon_z^{(1)} c = \dfrac{c}{2E}\left[\nu^2(p_1 + p_2) - 2p_1\right]$	$\Delta c^{(2)} = \varepsilon_z^{(2)} c = \dfrac{c}{2E}\left[\nu^2(p_1 + p_2) - 2p_2\right]$	

Problem 6.20

A metallic cube with sides $a = 0.20m$ is immersed in the sea at the depth $z = 400m$.

Knowing the Young's modulus of the metal $E = 21\times10^{10}\,Pa$, and the Poisson's ratio $\nu = 0.3$, calculate the volume variation of the cube. Consider the acceleration of gravity equals to $g = 10m/s^2$.

Hypothesis: Although the mass density varies with temperature, salinity, and pressure (depth), consider that the mass density of seawater equal to $\rho = 1027kg/m^3$ and constant.

Because of the depth and cube dimensions we can take as a good approximation that the whole cube is subjected to the same pressure as indicated in Figure 6.14.

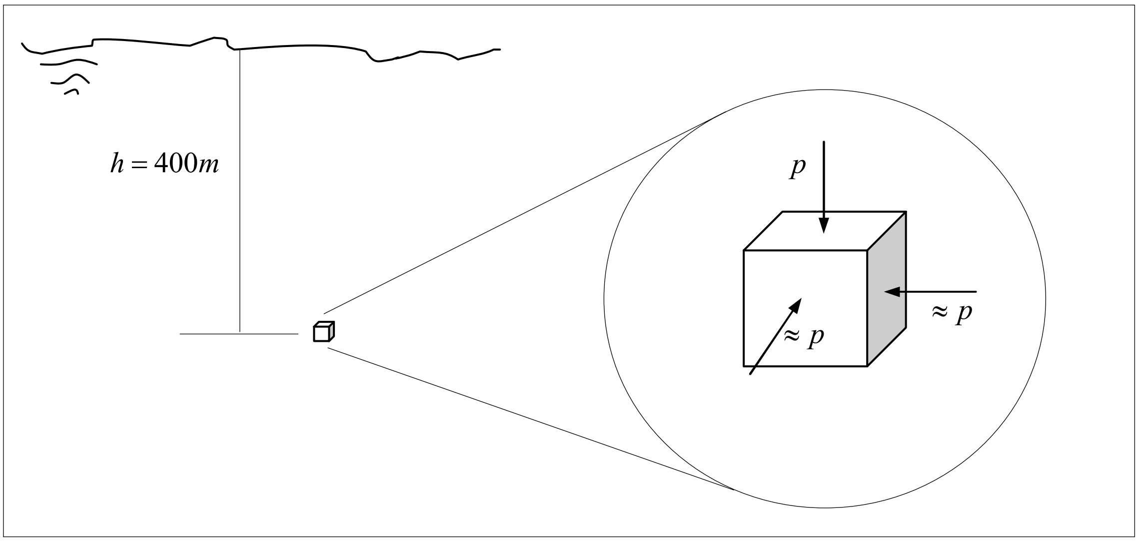


Figure 6.14

Solution:

The pressure can be obtained by $p = \dfrac{F}{A}$, where A is the area and F can be obtained by means of the Newton's second law $F = ma = V\rho g$ (weight of water column). Then:

$$p = \frac{F}{A} = \frac{V\rho g}{A} = \frac{Ah\rho g}{A} = \rho g h = 1027\frac{kg}{m^3}10\frac{m}{s^2}400m = 4.108\times10^6\frac{kg\,m}{m^2 s^2} = 4.108\times10^6\,Pa$$

The stress tensor components in the cube are given by:

$$\sigma_{ij} \approx \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = \begin{bmatrix} -4.108 & 0 & 0 \\ 0 & -4.108 & 0 \\ 0 & 0 & -4.108 \end{bmatrix} MPa$$

As we have only normal stress components and the material is isotropic, only normal strains appear:

$$\varepsilon_z = \varepsilon_y = \varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = \frac{1}{21\times10^{10}}\left[-4.108 - 0.3\left(-4.108 - 4.108\right)\right]\times10^6$$

thus,

$$\varepsilon_z = \varepsilon_y = \varepsilon_x = -7.82\times10^{-6} \quad \Rightarrow \quad \varepsilon_{ij} = \begin{bmatrix} -7.82 & 0 & 0 \\ 0 & -7.82 & 0 \\ 0 & 0 & -7.82 \end{bmatrix}\times10^{-6}$$

For small deformation regime the linear volumetric deformation is equal to the trace of the infinitesimal strain tensor ($D_V^L \equiv \varepsilon_V = \mathrm{Tr}(\boldsymbol{\varepsilon})$), and the volume variation of the cube is

$$\frac{\Delta V}{V_0} = D_V^L \equiv \varepsilon_V = \mathrm{Tr}(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \Delta V = V_0 \mathrm{Tr}(\boldsymbol{\varepsilon}) = 0.2^3 \times (-2.346 \times 10^{-5}) = -1.8768 \times 10^{-7} m^3$$

where we have considered that $\mathrm{Tr}(\boldsymbol{\varepsilon}) = -2.346 \times 10^{-5}$.

Problem 6.21

A solid cylinder of radius $R = 0.05m$ and height $0.25m$ is made up of a material whose mechanical properties are: $E = 3 \times 10^4 MPa$ (Young's modulus) and $\nu = 0.2$ (Poisson's ratio). Said cylinder is placed between two pistons, which can be considered infinitely rigid, and all of this is enclosed in a hermetically sealed container as shown in Figure 6.15.

The container is filled with oil, and by suitable mechanism, the fluid pressure is raised to the value $p = 15MPa$. By operating the mechanical press, we apply a total axial force of $F = 2.35619 \times 10^5 N$ (piston force+pressure) on the bases of the cylinder.

At a generic point of the body:

a) Obtain the stress tensor components;
b) Obtain the strain tensor components;
c) Obtain the displacement field components (u, v, w).

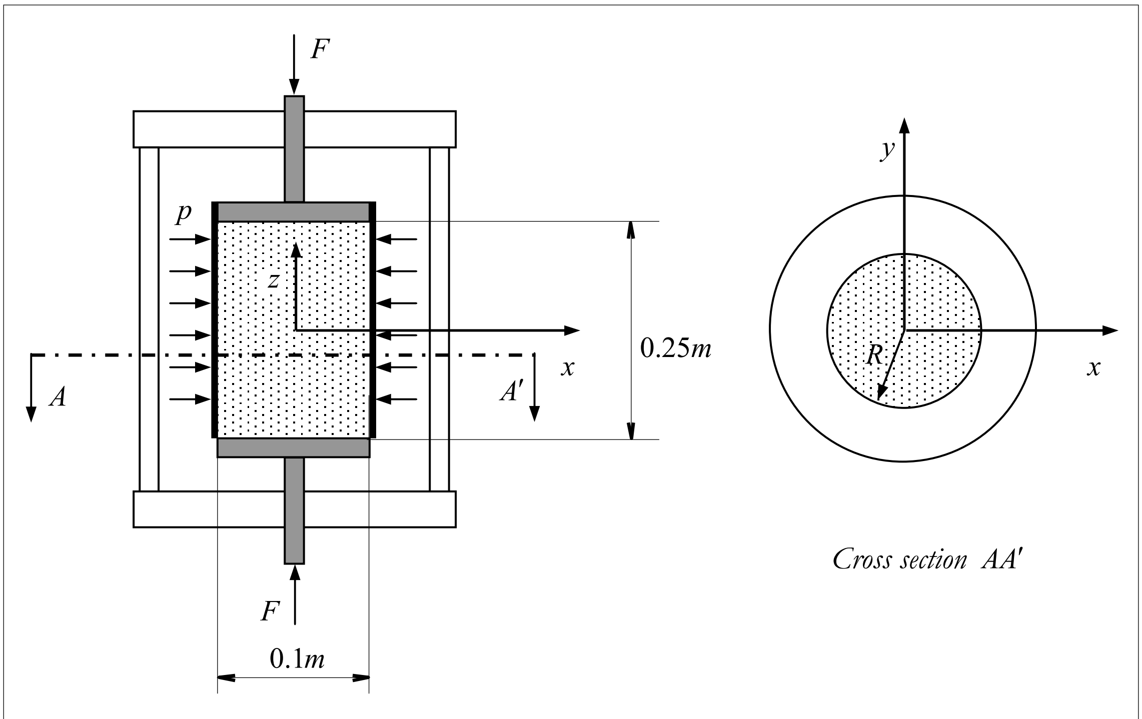


Figure 6.15: Triaxial compression test.

Solution:

a) The stress tensor components

$$\sigma_z = -\frac{F}{A} = -\frac{2.35619 \times 10^5}{\pi (0.05)^2} = -30MPa \quad ; \quad \sigma_x = \sigma_y = -p = -15MPa$$

thus,

$$\sigma_{ij} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -30 \end{bmatrix} MPa$$

b) For an isotropic linear elastic material, the normal stresses only produce normal strains, then:

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \end{cases}$$

By substituting the variable values we can obtain the following strain tensor components:

$$\varepsilon_{ij} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -8 \end{bmatrix} \times 10^{-4}$$

c) The displacement field

As we are considering the small deformation regime, the following is fulfilled:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

Integrating and obtaining the constants of integration we finally obtain the displacement field:

$$u = -2\times 10^{-4}\, x \quad ; \quad v = -2\times 10^{-4}\, y \quad ; \quad w = -8\times 10^{-4}\, z$$

Problem 6.22

The cube of sides $0.1m$ is made up of a material whose mechanical properties are represented by the Lamé constants: $\lambda = 8333.33 MPa$, $\mu = 12500 MPa$.

A deformation is imposed to the material as shown in Figure 6.16, in which all faces remains plane, the faces $AEFB$ and $DHGC$ become parallelograms and the remaining faces continue squares:

a) Obtain the displacement field;
b) Obtain the strain field;
c) Obtain the stress field;
d) Obtain the actions performed by the testing machine on the faces $ABFE$ and $BCGF$.

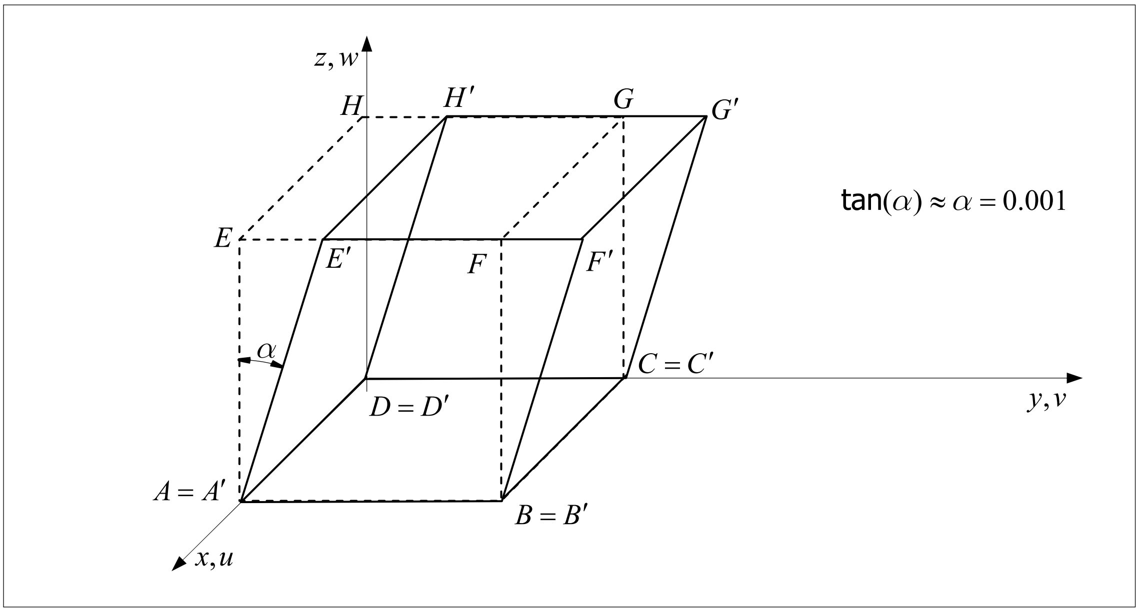


Figure 6.16: The deformed hexahedron.

Solution:

a) According to Figure 6.16 we can verify that there are only shear strain components. Moreover we can also verify that there are no displacements according to the directions x and z, then $u = 0$, $w = 0$. By means of triangle analogy we can obtain the displacement v:

$$\textit{small rotation} \quad \Rightarrow \quad \tan(\alpha) \approx \alpha = 0.001 = \frac{v}{z} \quad \Rightarrow \quad v(z) = 0.001z$$

The displacement field can be appreciated in Figure 6.17.

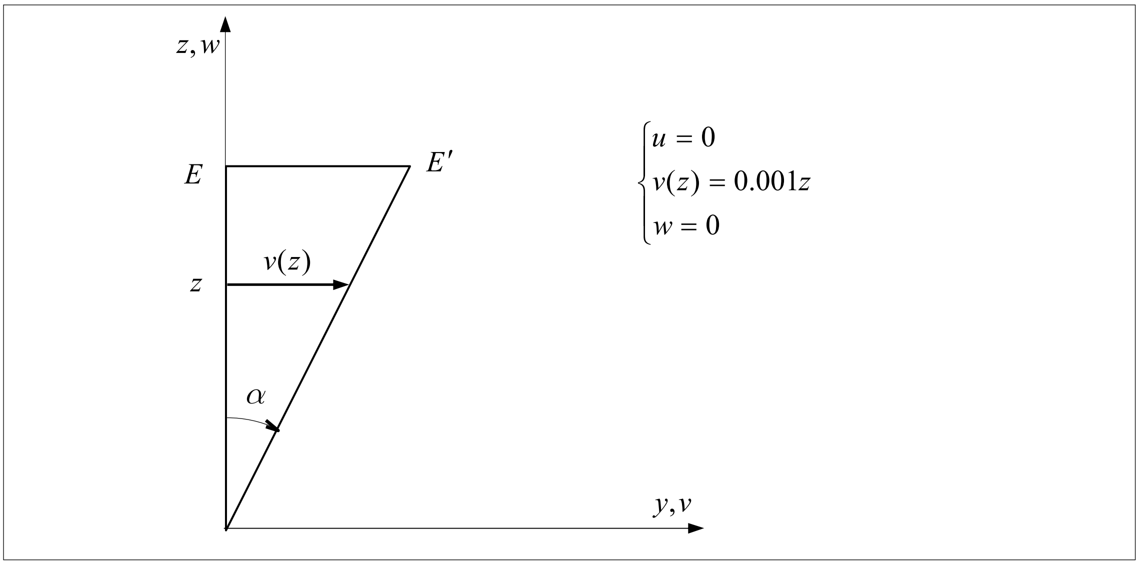


Figure 6.17

b) By considering the strain tensor components:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

we can conclude that $\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{xz} = 0$ and the component γ_{yz} is given by:

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0.001$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0005 \\ 0 & 0.0005 & 0 \end{bmatrix}$$

c) The stress field $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$

Considering $\mathrm{Tr}(\boldsymbol{\varepsilon}) = 0$, $\lambda = 8333.33 MPa$, $\mu = 12500 MPa$, we can obtain:

$$\sigma_{ij} = 2\times(12500)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0005 \\ 0 & 0.0005 & 0 \end{bmatrix} MPa = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12.5 \\ 0 & 12.5 & 0 \end{bmatrix} MPa$$

d) The principal strains:

$$\begin{vmatrix} -\varepsilon & 0.0005 \\ 0.0005 & -\varepsilon \end{vmatrix} = 0 \quad \Rightarrow \quad \varepsilon^2 = 0.0005^2 \quad \Rightarrow \quad \varepsilon = \pm 0.0005 \Rightarrow \begin{cases} \varepsilon_2 = +0.0005 \\ \varepsilon_3 = -0.0005 \end{cases}$$

Remember that in the small deformation regime, the stress and strain share the same principal directions, then we can work in the principal space in order to obtain the principal stresses by using $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$, i.e.:

$$\sigma'_{ij} = 2\times(12500)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.0005 & 0 \\ 0 & 0 & -0.0005 \end{bmatrix} MPa = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12.5 & 0 \\ 0 & 0 & -12.5 \end{bmatrix} MPa$$

e) To obtain the total force acting on one surface, we multiply the surface force by the area of the corresponding face. The force per surface unit is obtained by means of the traction vector $\mathbf{t}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$. For the face $ABFE$ the unit vector is given by $\hat{\mathsf{n}}_i = [1,0,0]$, thus:

$$\begin{bmatrix} \mathsf{t}_1^{(ABFE)} \\ \mathsf{t}_2^{(ABFE)} \\ \mathsf{t}_3^{(ABFE)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12.5 \\ 0 & 12.5 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For the face $BCGF$, the unit vector is given by $\hat{\mathsf{n}}_i = [0,1,0]$, thus

$$\begin{bmatrix} \mathsf{t}_1^{(BCGF)} \\ \mathsf{t}_2^{(BCGF)} \\ \mathsf{t}_3^{(BCGF)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 12.5 \\ 0 & 12.5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12.5 \end{bmatrix} MPa$$

If we do the same procedure for the other faces we obtain the representation of the surface forces on the faces as indicated in Figure 6.18:

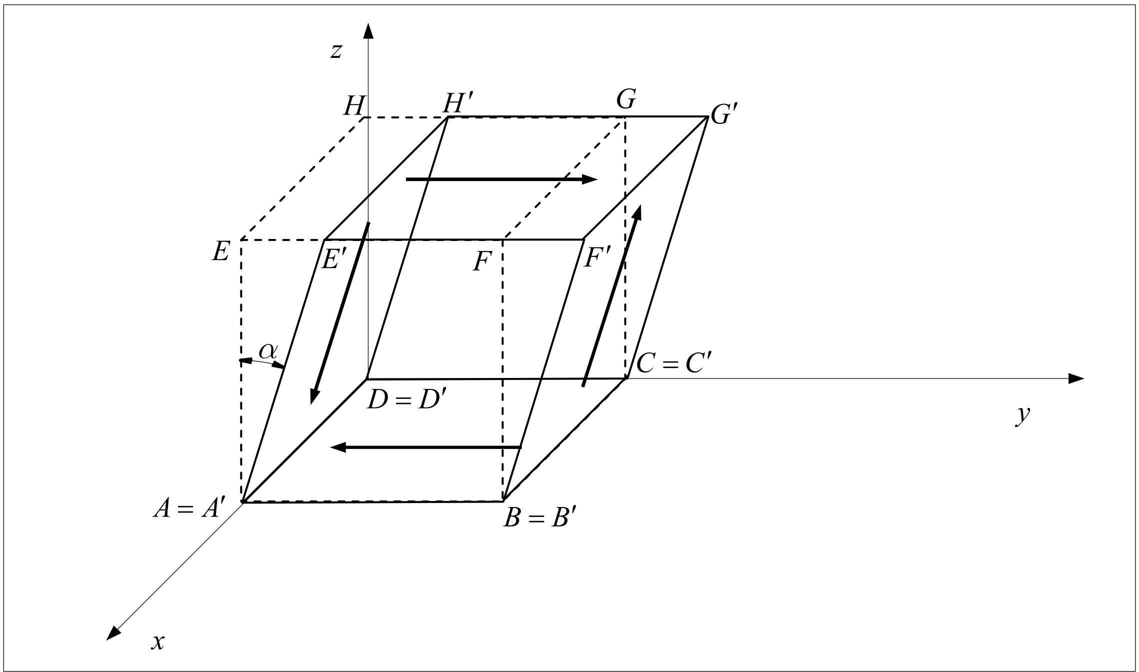


Figure 6.18: The surface forces in the hexahedron.

Problem 6.23

Consider the prism (rectangular parallelepiped) as indicated in Figure 6.19, we apply the forces $F_1 = 10N$ and $F_2 = 2N$ as indicated in Figure 6.19. The prism edge lengths are: $\overline{AB} = 4cm$, $\overline{AD} = \frac{10}{3} cm$, $\overline{AA'} = 2cm$. Consider the following material properties: $E = 2.5 \times 10^6 \frac{N}{cm^2}$ (Young's modulus), $\nu = 0.25$ (Poisson's ratio), and $\alpha = 5 \times 10^{-8} \frac{1}{{}^\circ C}$ (coefficient of thermal expansion).

a) Obtain the principal stresses; b) Obtain the traction vector on the plane Π. Is it on that plane Π where the maximum shear acts? Justify your answer. c) Obtain the values of the forces F_1 and F_2 to be applied to guarantee that in the solid there is no displacement according to the directions x_1 and x_2, when the prism is subjected to a temperature variation of $\Delta T = 20^\circ C$.

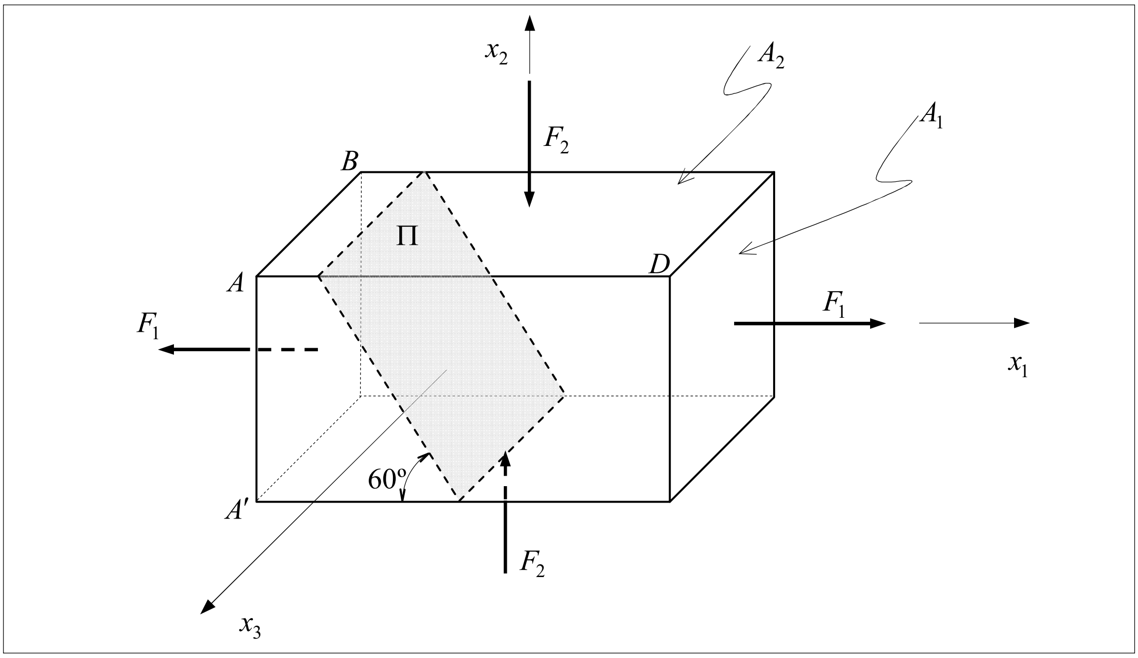


Figure 6.19

a) The stress field

$$A_1 = 8.0\,,\; A_2 = 4 \times \frac{10}{3} \qquad \Rightarrow \qquad \sigma_{ij} = \begin{bmatrix} \frac{F_1}{A_1} & 0 & 0 \\ 0 & -\frac{F_2}{A_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & -0.15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{N}{cm^2}$$

whose values are also the principal stresses, since there is no shear stresses.

b)

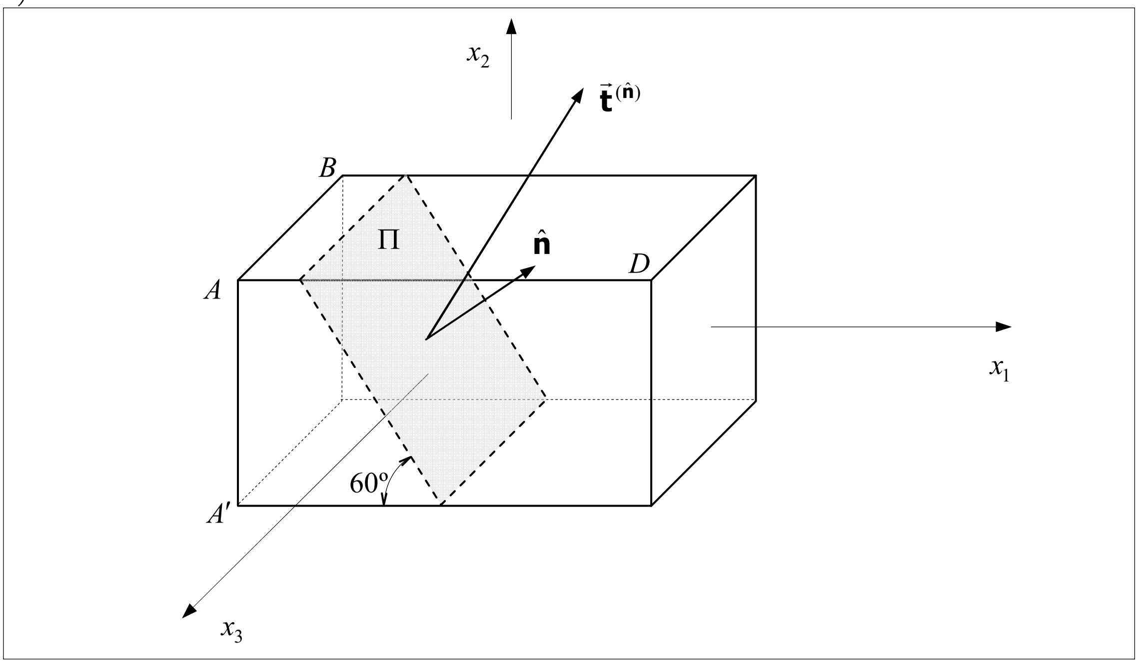


Figure 6.20

The unit vector components are: $\hat{n}_i = \left[\frac{\sqrt{3}}{2};\quad \frac{1}{2};\quad 0\right]$. Then, the traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is given by:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} \quad ; \quad t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{n}_j \quad \Rightarrow \quad t_i^{(\hat{\mathbf{n}})} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & -0.15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0825 \\ -0.075 \\ 0 \end{bmatrix}$$

The normal stress component is:

$$\sigma_N = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\cdot\hat{\mathbf{n}} = t_i^{(\hat{\mathbf{n}})}\hat{n}_i \quad \Rightarrow \quad \sigma_N = \begin{bmatrix} 1.0825 & -0.075 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = 0.9$$

The tangential stress component can be obtained by means of the Pythagorean Theorem:

$$\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 = \sigma_N^2 + \sigma_S^2 \quad \Rightarrow \quad \sigma_S = \sqrt{\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 - \sigma_N^2}$$

where

$$\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\cdot\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = t_i^{(\hat{\mathbf{n}})}t_i^{(\hat{\mathbf{n}})} = \begin{bmatrix} 1.0825 & -0.075 & 0 \end{bmatrix} \begin{bmatrix} 1.0825 \\ -0.075 \\ 0 \end{bmatrix} = 1.1775$$

Thus:

$$\sigma_S = \sqrt{\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 - \sigma_N^2} = \sqrt{1.1775 - 0.9^2} = 0.60621778$$

The Mohr's circle in stress is drawn as described in Figure 6.21.

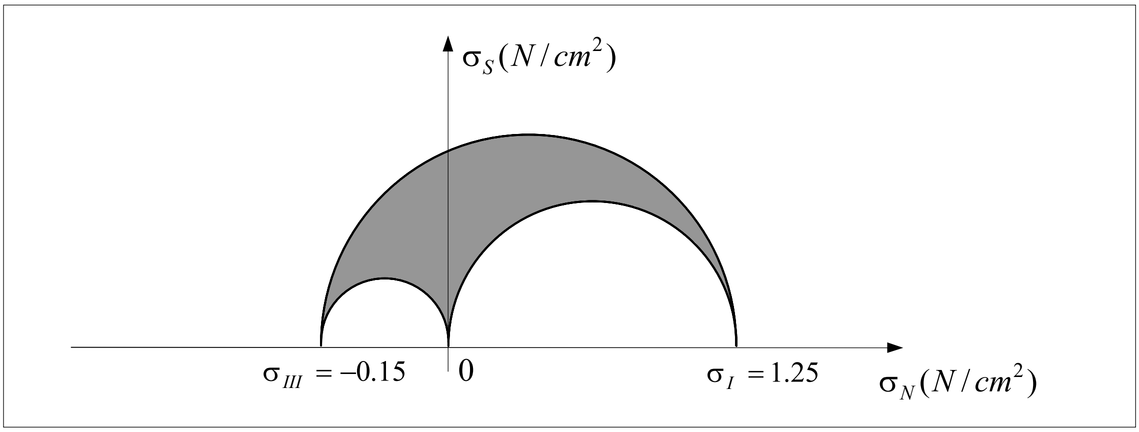


Figure 6.21

We can verify that for any point in the solid, the maximum tangential stress is on the plane defined by the unit vector $\hat{\mathsf{n}}_i = \left[\frac{\sqrt{2}}{2};\quad \frac{\sqrt{2}}{2};\quad 0\right]$ and the maximum tangential stress is:

$$\sigma_{S\max} = \frac{\sigma_I - \sigma_{III}}{2} = 0.7 > \sigma_S$$

c) We consider the following strain field:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \alpha\Delta T\mathbf{1} \quad ; \quad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \alpha\Delta T\delta_{ij}$$

For the particular case $\mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22}$ we have:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left[\alpha\Delta T - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we can construct the following set of equations:

$$\begin{cases} \varepsilon_{11} = 0 = \dfrac{1+\nu}{E}\sigma_{11} + \left[\alpha\Delta T - \dfrac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\right] = \dfrac{1+\nu}{E}\sigma_{11} + \left[\alpha\Delta T - \dfrac{\nu}{E}(\sigma_{11} + \sigma_{22})\right] \\ \varepsilon_{22} = 0 = \dfrac{1+\nu}{E}\sigma_{22} + \left[\alpha\Delta T - \dfrac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\right] = \dfrac{1+\nu}{E}\sigma_{22} + \left[\alpha\Delta T - \dfrac{\nu}{E}(\sigma_{11} + \sigma_{22})\right] \end{cases}$$

By solving the above set of equations we can obtain:

$$\sigma_{11} = \sigma_{22} = -\frac{E\alpha\Delta T}{(1-\nu)} = -3.33333\frac{N}{cm^2}$$

Then, the forces are given by:

$$\begin{cases} F_1 = \sigma_{11}A_1 = -26.66666N \\ F_2 = \sigma_{22}A_2 = -44.44444N \end{cases}$$

6.2 Two-Dimensional Linear Elasticity (2D)

Problem 6.24

a) Define the state of plane stress and the state of plane strain. b) Obtain the relationships for $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ and for $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ by considering both plane states. c) Give practical examples in which we can apply these states.

Solution:

a.1) In the case of plane stress one of the dimensions of the structural elements is very small when compared to the other two, and the load is perpendicular to the direction of smallest dimension. As a result of this the stress tensor field components related to this direction are equal to zero, e.g. $\sigma_{i3}(\vec{\boldsymbol{x}}) = \sigma_{3i}(\vec{\boldsymbol{x}}) = 0$.

a.2) In the case of plane strain, the structural element has a prismatic axis, in which the dimension that corresponds to the direction of the prismatic axis is much larger than the other two dimensions. Additionally, the loads applied are normal to the prismatic axis. Under these conditions the strain tensor field components: ε_{13}, ε_{23} and ε_{33} are zero, i.e. $\varepsilon_{i3}(\vec{\boldsymbol{x}}) = \varepsilon_{3i}(\vec{\boldsymbol{x}}) = 0$.

b.1 – State of plane stress

In this case, the stress tensor field components have the format:

$$\sigma_{ij}(\vec{\boldsymbol{x}}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.43}$$

Let us start from the strain equation:

$$\boldsymbol{\varepsilon} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} \quad ; \quad \varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma_{ij}$$

and its trace can be obtained as follows:

$$\boldsymbol{\varepsilon}:\mathbf{1} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}:\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}:\mathbf{1} = \frac{-3\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma}) + \frac{1}{2\mu}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{1}{(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})$$

$$\Rightarrow \mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{1}{(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma}) \quad \Leftrightarrow \quad \mathrm{Tr}(\boldsymbol{\sigma}) = (3\lambda+2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon})$$

The component ε_{33} is no longer an unknown since:

$$\begin{aligned} \varepsilon_{33} &= \frac{-\lambda}{2\mu(3\lambda+2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})\underbrace{\delta_{33}}_{=1} + \frac{1}{2\mu}\underbrace{\sigma_{33}}_{=0} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma}) \\ \Rightarrow \varepsilon_{33} &= \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{-\lambda}{2\mu(3\lambda+2\mu)}(3\lambda+2\mu)\mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{-\lambda}{2\mu}\mathrm{Tr}(\boldsymbol{\varepsilon}) \\ \Rightarrow \varepsilon_{33} &= \frac{-\lambda}{2\mu}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) \\ \Rightarrow \varepsilon_{33} &= \frac{-\lambda}{(\lambda+2\mu)}(\varepsilon_{11}+\varepsilon_{22}) \end{aligned} \tag{6.44}$$

The stress components $\sigma_{ij} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}$ become:

$$
\begin{aligned}
\sigma_{ij} &= \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+2\mu\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&0\\\varepsilon_{12}&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{bmatrix}\\
&=[\lambda(\varepsilon_{11}+\varepsilon_{22})+\frac{-\lambda^2}{(\lambda+2\mu)}(\varepsilon_{11}+\varepsilon_{22})]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+2\mu\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&0\\\varepsilon_{12}&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{bmatrix}\\
&=\frac{2\lambda\mu}{(\lambda+2\mu)}(\varepsilon_{11}+\varepsilon_{22})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+2\mu\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&0\\\varepsilon_{12}&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{bmatrix}
\end{aligned}
\tag{6.45}
$$

In indicial notation the above equation becomes:

$$
\begin{cases}
\sigma_{ij}=\dfrac{\lambda\mu}{(\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij}+2\mu\varepsilon_{ij} \quad ; \quad (i,j=1,2) \quad \text{with} \quad \mathrm{Tr}(\boldsymbol{\varepsilon})=\varepsilon_{11}+\varepsilon_{22}\\
\varepsilon_{ij}=\dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\dfrac{1}{2\mu}\sigma_{ij} \quad (i,j=1,2,3) \quad (\text{the same as } 3D)
\end{cases}
\tag{6.46}
$$

or

$$
\begin{cases}
\sigma_{ij}=\dfrac{\nu E}{(1-\nu^2)}\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij}+\dfrac{E}{(1+\nu)}\varepsilon_{ij} \quad ; \quad (i,j=1,2) \quad \text{with} \quad \mathrm{Tr}(\boldsymbol{\varepsilon})=\varepsilon_{11}+\varepsilon_{22}\\
\varepsilon_{ij}=\dfrac{-\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\dfrac{(1+\nu)}{E}\sigma_{ij} \quad (i,j=1,2,3) \quad (\text{the same as } 3D)
\end{cases}
\tag{6.47}
$$

The equation in (6.45) can also be written as follows:

$$
\sigma_{ij}=\begin{bmatrix}
\left(\dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}\varepsilon_{11}+\dfrac{2\lambda\mu}{(\lambda+2\mu)}\varepsilon_{22}\right) & 2\mu\varepsilon_{12} & 0\\
2\mu\varepsilon_{12} & \left(\dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}\varepsilon_{22}+\dfrac{2\lambda\mu}{(\lambda+2\mu)}\varepsilon_{11}\right) & 0\\
0 & 0 & 0
\end{bmatrix}
$$

Taking into account the relationships between the mechanical parameters we can obtain: $\dfrac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}=\dfrac{E}{(1-\nu^2)}$, $\dfrac{2\lambda\mu}{(\lambda+2\mu)}=\dfrac{E\nu}{(1-\nu^2)}$, $2\mu=\dfrac{E}{(1+\nu)}=\dfrac{E(1-\nu)}{(1+\nu)(1-\nu)}=\dfrac{E(1-\nu)}{(1-\nu^2)}$, thus:

$$
\sigma_{ij}=\frac{E}{(1-\nu^2)}\begin{bmatrix}(\varepsilon_{11}+\nu\varepsilon_{22}) & (1-\nu)\varepsilon_{12} & 0\\ (1-\nu)\varepsilon_{12} & (\varepsilon_{22}+\nu\varepsilon_{11}) & 0\\ 0&0&0\end{bmatrix} \quad ; \quad \mathrm{Tr}(\boldsymbol{\sigma})=\frac{E(\varepsilon_{11}+\varepsilon_{22})}{(1-\nu)}
$$

Alternative solution: Voigt notation and engineering notation

Taking into account the conditions $\sigma_{i3}=\sigma_{3i}=0$, and the relationship for strain $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ in Voigt notation:

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \cancelto{0}{\sigma_z} \\ \tau_{xy} \\ \cancelto{0}{\tau_{yz}} \\ \cancelto{0}{\tau_{zx}} \end{bmatrix} \tag{6.48}
$$

Then, if we remove the columns and rows associated with the zero stresses, the strain-stress relationship ($\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$) for the plane stress case is given by:

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \xrightarrow{G=\frac{E}{2(1+\nu)}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{6.49}
$$

The reciprocal of the above equation results the Hooke's law ($\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$) for the state of plane stress:

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \Leftrightarrow \quad \{\boldsymbol{\sigma}\} = [\mathcal{C}^{(2D\_1)}]\{\boldsymbol{\varepsilon}\} \tag{6.50}
$$

Note that the normal strain ε_z is not equal to zero, since ε_z is not just dependant on the normal stress σ_z:

$$
\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{E}\left[-\nu\left(\sigma_x + \sigma_y\right)\right] = \frac{-\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{-\nu(\varepsilon_{11} + \varepsilon_{22})}{(1-\nu)} \tag{6.51}
$$

Then, the strain tensor components are represented as follows:

$$
\varepsilon_{ij}(\vec{\boldsymbol{x}}) = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{6.52}
$$

NOTE 1: If we want to be extremely rigorous there is no state of plane stress, in other words, there is no real structure such as the *strain field* has the format presented in Eq. (6.52). Moreover, as we will see later, the compatibility equations, (see **Problem 5.11**), are not satisfied, in general, if the strain field has the format presented by the equation in (6.52). But, when we are dealing with small deformation regime and the thickness (t) is very small compared to the other dimensions (L, h), (see Figure 6.22), the error committed by using the state of plane stress is small. Keep in mind that the state of plane stress is always an approximation, and the only way in which the compatibility equations are satisfied is by discarding completely the third dimension.

b.2 – State of Plane Strain

In this case the strain tensor field components have the format:

$$\varepsilon_{ij}(\vec{x})=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & 0\\ \varepsilon_{12} & \varepsilon_{22} & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}\varepsilon_{x} & \frac{1}{2}\gamma_{xy} & 0\\ \frac{1}{2}\gamma_{xy} & \varepsilon_{y} & 0\\ 0 & 0 & 0\end{bmatrix} \tag{6.53}$$

Let us start from the stress equation:

$$\boldsymbol{\sigma}=\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon} \quad ; \quad \sigma_{ij}=\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij}+2\mu\varepsilon_{ij}$$

and its trace can be obtained as follows:

$$\boldsymbol{\sigma}:\mathbf{1}=\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}:\mathbf{1}+2\mu\boldsymbol{\varepsilon}:\mathbf{1} \quad \Rightarrow \quad \mathrm{Tr}(\boldsymbol{\sigma})=3\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})+2\mu\mathrm{Tr}(\boldsymbol{\varepsilon})=[3\lambda+2\mu]\mathrm{Tr}(\boldsymbol{\varepsilon})$$

$$\Rightarrow \mathrm{Tr}(\boldsymbol{\varepsilon})=\frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3\lambda+2\mu}=\frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3\lambda+2\mu}$$

The component σ_{33} is no longer an unknown since:

$$\sigma_{ij}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij}+2\mu\varepsilon_{ij}\Rightarrow \sigma_{33}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{33}+2\mu\varepsilon_{33}\Rightarrow\sigma_{33}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon}) \tag{6.54}$$

Then, the component σ_{33} is defined as follows:

$$\begin{aligned}
&\sigma_{33}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})=\frac{\lambda}{3\lambda+2\mu}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\
&\Rightarrow \sigma_{33}-\frac{\lambda}{3\lambda+2\mu}\sigma_{33}=\frac{\lambda}{3\lambda+2\mu}(\sigma_{11}+\sigma_{22})\\
&\Rightarrow \sigma_{33}\left(1-\frac{\lambda}{3\lambda+2\mu}\right)=\frac{\lambda}{3\lambda+2\mu}(\sigma_{11}+\sigma_{22})\\
&\Rightarrow \sigma_{33}=\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})
\end{aligned}$$

And the strain components $\varepsilon_{ij}=\dfrac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\dfrac{1}{2\mu}\sigma_{ij}$ become:

$$\begin{aligned}
\varepsilon_{ij}&=\frac{-\lambda}{2\mu(3\lambda+2\mu)}(\sigma_{11}+\sigma_{22}+\sigma_{33})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{1}{2\mu}\begin{bmatrix}\sigma_{11}&\sigma_{12}&0\\\sigma_{12}&\sigma_{22}&0\\0&0&\sigma_{33}\end{bmatrix}\\
&=\frac{-\lambda}{2\mu(3\lambda+2\mu)}\left(\sigma_{11}+\sigma_{22}+\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})\right)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{1}{2\mu}\begin{bmatrix}\sigma_{11}&\sigma_{12}&0\\\sigma_{12}&\sigma_{22}&0\\0&0&\sigma_{33}\end{bmatrix}\\
&=\frac{-\lambda}{4\mu(\lambda+\mu)}(\sigma_{11}+\sigma_{22})\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{1}{2\mu}\begin{bmatrix}\sigma_{11}&\sigma_{12}&0\\\sigma_{12}&\sigma_{22}&0\\0&0&\sigma_{33}\end{bmatrix}
\end{aligned} \tag{6.55}$$

In indicial notation the above equation becomes:

$$\begin{cases}\varepsilon_{ij}=\dfrac{-\lambda}{4\mu(\lambda+\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\dfrac{1}{2\mu}\sigma_{ij} \quad ; \quad (i,j=1,2) \quad \text{with} \quad \mathrm{Tr}(\boldsymbol{\sigma})=\sigma_{11}+\sigma_{22}\\ \sigma_{ij}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij}+2\mu\varepsilon_{ij} \quad (i,j=1,2,3) \quad (\text{the same as } 3D)\end{cases} \tag{6.56}$$

or

$$\begin{cases} \varepsilon_{ij} = \dfrac{-\nu(1+\nu)}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij} + \dfrac{(1+\nu)}{E}\sigma_{ij} \quad ; \quad (i,j=1,2) \quad \text{with} \quad \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{11}+\sigma_{22} \\ \sigma_{ij} = \dfrac{E\nu}{(1+\nu)(1-2\nu)}\mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + \dfrac{E}{(1+\nu)}\varepsilon_{ij} \quad (i,j=1,2,3) \quad \text{(the same as } 3D\text{)} \end{cases} \tag{6.57}$$

The equation in (6.55) can also be written as follows:

$$\varepsilon_{ij} = \begin{bmatrix} \left(\dfrac{(2\lambda+\mu)}{4\mu(\lambda+\mu)}\sigma_{11} + \dfrac{-\lambda}{4\mu(\lambda+\mu)}\sigma_{22}\right) & \dfrac{1}{2\mu}\sigma_{12} & 0 \\ \dfrac{1}{2\mu}\sigma_{12} & \left(\dfrac{(2\lambda+\mu)}{4\mu(\lambda+\mu)}\sigma_{22} + \dfrac{-\lambda}{4\mu(\lambda+\mu)}\sigma_{11}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking into account the relationships between the mechanical parameters we can obtain: $\dfrac{(\lambda+2\mu)}{4\mu(\lambda+\mu)} = \dfrac{(1+\nu)(1-\nu)}{E}$, $\dfrac{-\lambda}{4\mu(\lambda+\mu)} = \dfrac{-\nu(1+\nu)}{E}$, $\dfrac{1}{2\mu} = \dfrac{(1+\nu)}{E}$, thus:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E}\begin{bmatrix} (1-\nu)\sigma_{11} - \nu\sigma_{22} & \sigma_{12} & 0 \\ \sigma_{12} & (1-\nu)\sigma_{22} - \nu\sigma_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Alternative solution: Voigt notation and engineering notation

If we start from the generalized Hooke's law and by deleting the columns and rows associated with the zero strains, *i.e.*:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \cancelto{0}{\varepsilon_z} \\ \gamma_{xy} \\ \cancelto{0}{\gamma_{yz}} \\ \cancelto{0}{\gamma_{zx}} \end{bmatrix} \tag{6.58}$$

we can obtain:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \Leftrightarrow \quad \{\boldsymbol{\sigma}\} = [\mathcal{C}^{(2D\_2)}]\{\boldsymbol{\varepsilon}\} \tag{6.59}$$

Then, the stress according to the direction z is given by:

$$\sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_x + \varepsilon_y\right) \tag{6.60}$$

Additionally, the reciprocal of (6.59) is:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{6.61}$$

c.1 – We can apply the plane stress approximation for the deep beam problems, (see Figure 6.22).

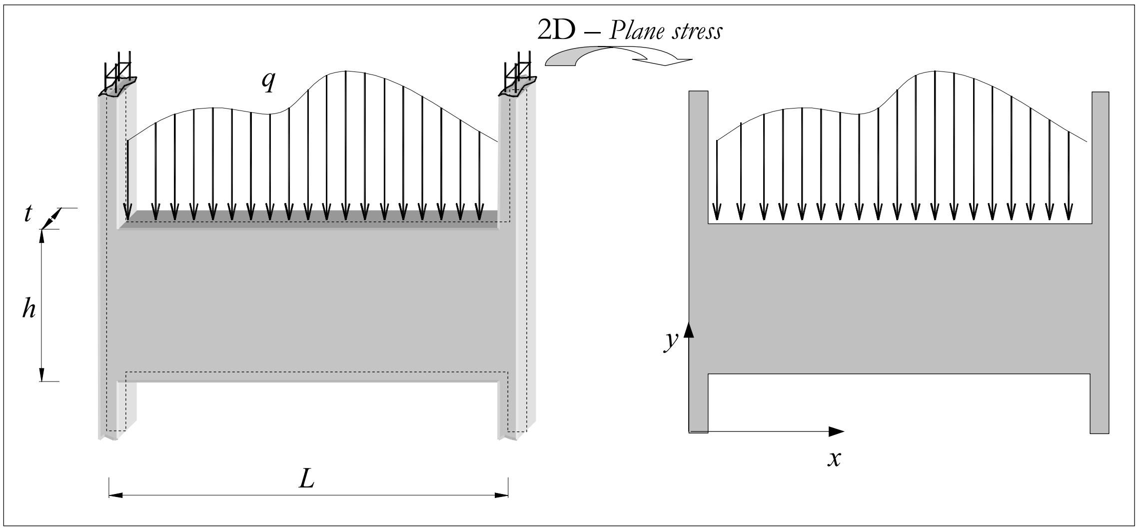


Figure 6.22: Deep beam.

c.2 – We can apply the plane strain approximation for cylinder under pressure, (Figure 6.23), Tunnels, (see Figure 6.24), dams, (see Figure 6.25).

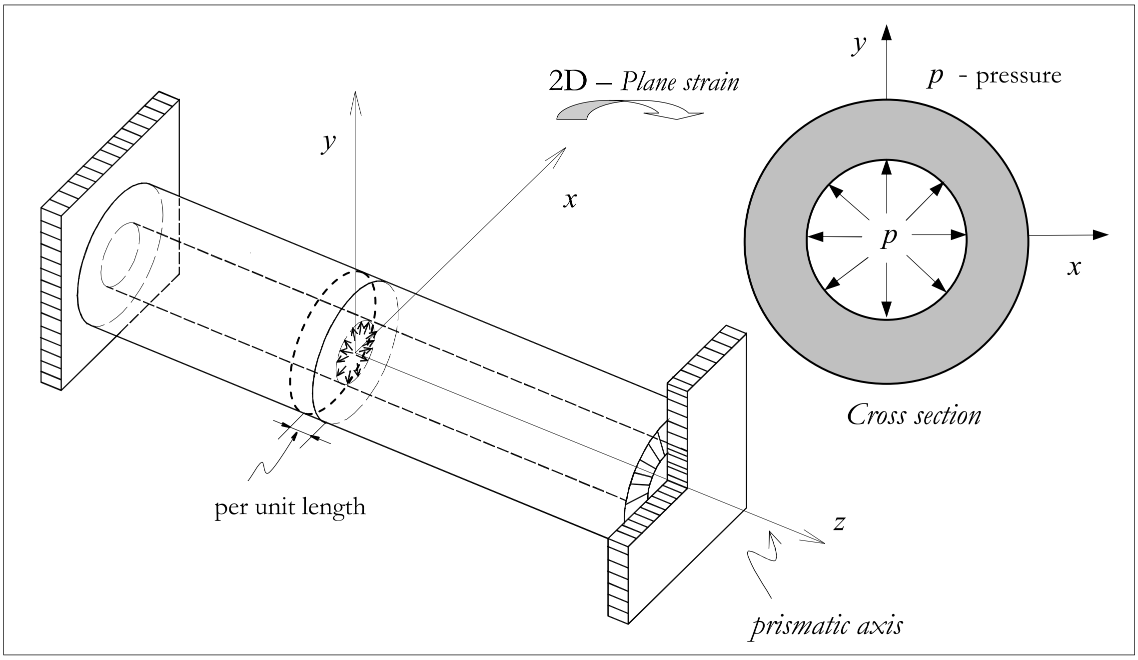


Figure 6.23: Cylinder under pressure.

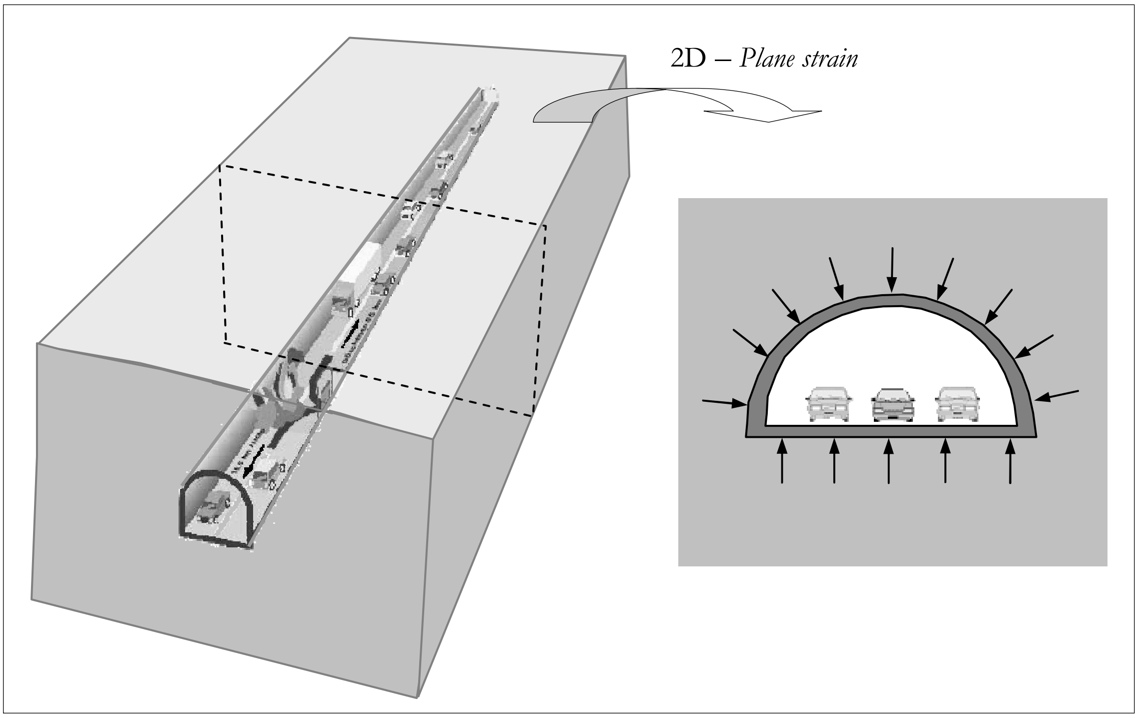


Figure 6.24: Tunnel.

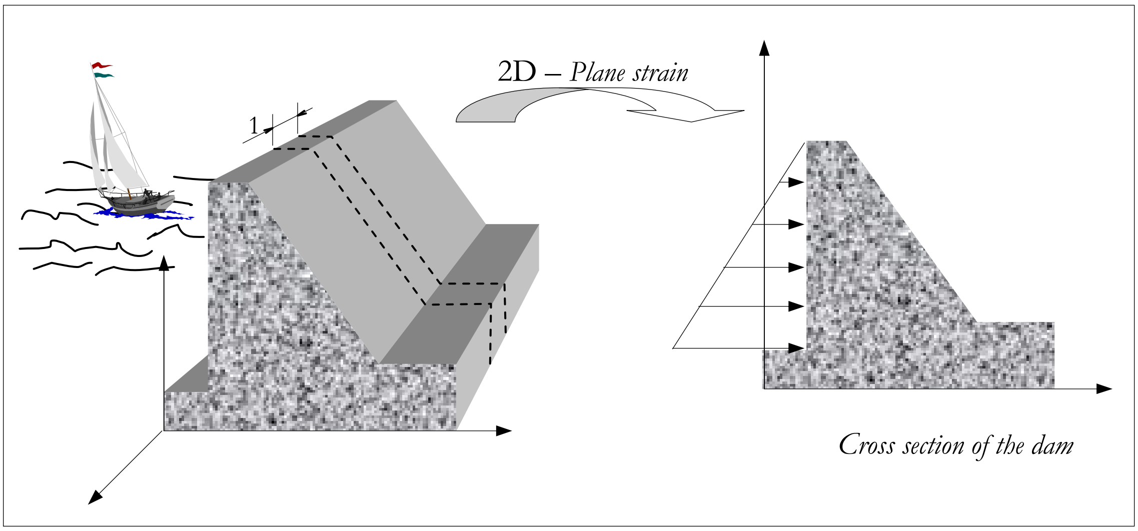


Figure 6.25: Dam.

Note that to adopt the state of plane strain, the cross section properties, e.g. dimensions, mechanical properties, cannot vary along the prismatic axis, otherwise we will have an error associated with it.

Problem 6.25

Consider the stress-strain relationship:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{\overline{E}}{1-\overline{\nu}^2}\begin{bmatrix}1 & \overline{\nu} & 0\\ \overline{\nu} & 1 & 0\\ 0 & 0 & \frac{1-\overline{\nu}}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{6.62}$$

Find the values for $\overline{E}$ and $\overline{\nu}$ in order to achieve the stress-strain relationships for the states of plane stress and plane strain.

Solution:

If we compare the equations (6.62) and (6.50) we can conclude that for state of plane stress we have $\overline{E}=E$ and $\overline{\nu}=\nu$. Now let us consider the strain equations for both states:

Strain for state of plane *stress*, (see equation (6.47)):

$$\begin{aligned}\varepsilon_{ij}&=\frac{-\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\frac{(1+\nu)}{E}\sigma_{ij}\qquad (i,j=1,2,3)\qquad \text{(the same as 3}D\text{)}\\ &=\frac{-\overline{\nu}}{\overline{E}}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\frac{(1+\overline{\nu})}{\overline{E}}\sigma_{ij}\end{aligned} \tag{6.63}$$

Strain for state of plane *strain*, (see equation (6.57)):

$$\varepsilon_{ij}=\frac{-\nu(1+\nu)}{E}\mathrm{Tr}(\boldsymbol{\sigma})\delta_{ij}+\frac{(1+\nu)}{E}\sigma_{ij}\quad ;\quad (i,j=1,2)\quad \text{with}\quad \mathrm{Tr}(\boldsymbol{\sigma})=\sigma_{11}+\sigma_{22} \tag{6.64}$$

Then, by means of the equations (6.63) and (6.64) we can obtain the following equations:

$$\left.\begin{aligned}\frac{-\overline{\nu}}{\overline{E}}=\frac{-\nu(1+\nu)}{E}\quad&\Rightarrow\quad \overline{\nu}=\frac{\nu\overline{E}(1+\nu)}{E}\\ \frac{(1+\overline{\nu})}{\overline{E}}=\frac{(1+\nu)}{E}\quad&\Rightarrow\quad \overline{E}=\frac{E(1+\overline{\nu})}{(1+\nu)}\end{aligned}\right\}\quad\Rightarrow\quad \overline{\nu}=\overline{E}\frac{\nu(1+\nu)}{E}=\frac{E(1+\overline{\nu})}{(1+\nu)}\frac{\nu(1+\nu)}{E}$$

thus, $\overline{\nu}=\dfrac{E(1+\overline{\nu})}{(1+\nu)}\dfrac{\nu(1+\nu)}{E}=(1+\overline{\nu})\nu\quad\Rightarrow\quad\overline{\nu}=\dfrac{\nu}{(1-\nu)}$

and

$$\overline{E}=\frac{E(1+\overline{\nu})}{(1+\nu)}=\frac{E\left(1+\frac{\nu}{(1-\nu)}\right)}{(1+\nu)}=\frac{E}{(1-\nu)(1+\nu)}=\frac{E}{(1-\nu^2)}$$

Then,

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{\overline{E}}{1-\overline{\nu}^2}\begin{bmatrix}1 & \overline{\nu} & 0\\ \overline{\nu} & 1 & 0\\ 0 & 0 & \frac{1-\overline{\nu}}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}\quad\therefore\quad\begin{cases}\text{for state of plane stress}\begin{cases}\overline{E}=E\\ \overline{\nu}=\nu\end{cases}\\[2ex] \text{for state of plane strain}\begin{cases}\overline{E}=\dfrac{E}{(1-\nu^2)}\\ \overline{\nu}=\dfrac{\nu}{(1-\nu)}\end{cases}\end{cases} \tag{6.65}$$

Problem 6.26

Figure 6.26 (a) shows a support device for a machine. Said support apparatus is made up of a neoprene block of dimensions ($50\times 20cm$) which is characterized by the element ABCD described in Figure 6.26(b).

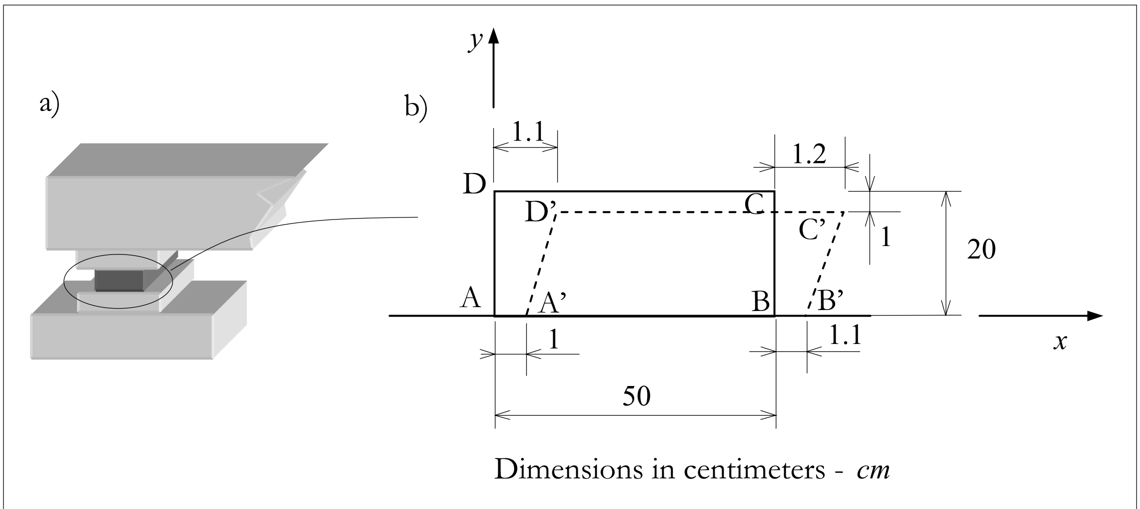


Figure 6.26

Under the action of vertical and horizontal loads the neoprene deforms as shown in Figure 6.26 (b) (A'B'C'D') in which the displacement field (u,v) is represented as follows:

$$u = a_1 x + b_1 y + c_1$$
$$v = a_2 x + b_2 y + c_2$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants to be determined.

a) Calculate the strain tensor components and the volumetric deformation at any point;
b) Calculate the stresses at any point;
c) The maximum normal stress;
d) Obtain the unit extension according to the direction of the diagonal AC.

Hypothesis:

1 – Isotropic linear elastic material with Young's modulus equals to $1000N/cm^2$ and the shear modulus equals to $\dfrac{1}{0.0028}N/cm^2$.

2 – It is assumed a state of plane strain.

Solution:

$$\begin{cases} u = a_1 x + b_1 y + c_1 \\ v = a_2 x + b_2 y + c_2 \end{cases} \tag{6.66}$$

According to Figure 6.26 we can obtain:

$$\begin{aligned} u(0;0) &= 1 = c_1 \\ u(50;0) &= 1.1 = 50a_1 + 1 \Rightarrow a_1 = 0.002 \\ u(0;20) &= 1.1 = 20b_1 + 1 \Rightarrow b_1 = 0.005 \end{aligned} \tag{6.67}$$

thus

$$u = 0.002x + 0.005y + 1 \tag{6.68}$$

For the vertical displacement:

$$
\begin{aligned}
v(0;0) &= 0 = c_2 \\
u(50;0) &= 0 = 50a_2 \Rightarrow a_2 = 0 \\
u(0;20) &= -1 = 20b_2 \Rightarrow b_2 = -0.05
\end{aligned} \tag{6.69}
$$

$$
v = -0.05y \tag{6.70}
$$

Then:

$$
\begin{cases} u = 0.002x + 0.005y + 1 \\ v = -0.05y \end{cases} \tag{6.71}
$$

a) Strains

$$
\varepsilon_x = \frac{\partial u}{\partial x} = 0.002 \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} = -0.05 \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.005 \tag{6.72}
$$

The linear volumetric deformation (small deformation regime):

$$
D_V^L = \boldsymbol{\varepsilon}_V = \varepsilon_x + \varepsilon_y + \varepsilon_z = I_{\boldsymbol{\varepsilon}} = -0.048 \tag{6.73}
$$

b) Stress components

$$
G = \frac{E}{2(1+\nu)} \Rightarrow \nu = \frac{E}{2G} - 1 = 0.4
$$

$$
\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right] \\
&= 3571.4286 \times \left[(0.6)\times 0.002 - 0.4 \times 0.05\right] = -67.1428 \quad (N/cm^2) \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\varepsilon_x\right] \\
&= 3571.4286 \times \left[(0.6)\times(-0.05) + 0.4 \times 0.002\right] = -104.2857 \quad (N/cm^2) \\
\tau_{xy} &= G\gamma_{xy} = \frac{1}{0.0028} \times 0.005 = 1.785714 \quad (N/cm^2)
\end{aligned} \tag{6.74}
$$

As an alternative solution we can use the equation $\sigma_{ij} = \dfrac{\nu E \,\mathrm{Tr}(\boldsymbol{\varepsilon})}{(1+\nu)(1-2\nu)}\delta_{ij} + \dfrac{E}{(1+\nu)}\varepsilon_{ij}$, where:

$$
\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 0.002 & \frac{1}{2}(0.005) & 0 \\ \frac{1}{2}(0.005) & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\frac{\nu E \,\mathrm{Tr}(\boldsymbol{\varepsilon})}{(1+\nu)(1-2\nu)} = -68.571429\frac{N}{cm^2}, \quad \frac{E}{(1+\nu)} = 714.285714\frac{N}{cm^2}
$$

$$
\begin{aligned}
\sigma_{ij} &= -68.571429\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 714.285714\begin{bmatrix} 0.002 & \frac{1}{2}(0.005) & 0 \\ \frac{1}{2}(0.005) & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\approx \begin{bmatrix} -67.1428 & 1.785714 & 0 \\ 1.785714 & -104.2857 & 0 \\ 0 & 0 & -68.571 \end{bmatrix}\frac{N}{cm^2}
\end{aligned}
$$

c) The principal stresses

$$\sigma_{(1,2)} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{6.75}$$

$$\sigma_{(1,2)} = \frac{-67.1428 - 104.2857}{2} \pm \sqrt{\left(\frac{-67.1428 + 104.2857}{2}\right)^2 + 5.35714^2} \tag{6.76}$$
$$= -171.4285 \pm 19.328675$$

$$\begin{cases} \sigma_1 = -152.099824\, N/cm^2 \\ \sigma_2 = -190.757175\, N/cm^2 \end{cases} \tag{6.77}$$

d) The unit extension

The diagonal $(\overline{AC})$ in the reference configuration measures:

$$L_0 = \overline{AC} = \sqrt{50^2 + 20^2} = 53.852 cm \tag{6.78}$$

and the deformed diagonal

$$\overline{A'C'} = \sqrt{50.2^2 + 19^2} = 53.675 cm \quad \Rightarrow \quad \Delta L = \overline{A'C'} - \overline{AC} = -0.177 cm \tag{6.79}$$

The unit extension is:

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{-0.177}{53.852} = -0.0033 \tag{6.80}$$

Problem 6.27

Consider a soil made up of a linear elastic material. At a point in the soil the volumetric deformation is $\varepsilon_V = -2 \times 10^{-3}$, the shear deformation is $\varepsilon_{12} = -\sqrt{3} \times 10^{-3}$ and the normal strain is $\varepsilon_{11} = 0$. The soil is subjected to a state of plane strain according to the plane $x_1 - x_2$.

a) Obtain the Cartesian components of the infinitesimal strain tensor. Obtain the principal strains, and the directions where they occur.

b) Assuming that the mechanical properties are $E = 50 MPa$ (Young's modulus) and $\nu = \frac{1}{4}$ (Poisson's ratio), obtain the stress tensor components and the principal stresses. Obtain the maximum normal and shear stresses.

c) Obtain the strain energy density, i.e. the energy per unit volume.

Solution:

a) By means of the problem data, the infinitesimal strain tensor components are given by:

$$\varepsilon_{ij} = \begin{bmatrix} 0 & -\sqrt{3} \times 10^{-3} & 0 \\ -\sqrt{3} \times 10^{-3} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where we have considered the plane strain hypothesis $\varepsilon_{3i} = \varepsilon_{i3} = 0$. By means of the volumetric deformation: $D_V^L \approx \varepsilon_V = I_{\boldsymbol{\varepsilon}} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -2 \times 10^{-3} \Rightarrow \varepsilon_{22} = -2 \times 10^{-3}$. Then, the strain components are:

$$\varepsilon_{ij} = \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3} \xrightarrow{\text{plane strain}} \varepsilon_{ij} = \begin{bmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{bmatrix} \times 10^{-3}$$

The principal strains

$$\begin{vmatrix} 0-\bar{\lambda} & -\sqrt{3} \\ -\sqrt{3} & -2-\bar{\lambda} \end{vmatrix} = 0 \quad \Rightarrow \quad \bar{\lambda}^2 + 2\bar{\lambda} - 3 = 0 \quad \Rightarrow \quad \begin{cases} \bar{\lambda}_1 = 1 \\ \bar{\lambda}_2 = -3 \end{cases}$$

thus

$$\begin{cases} \varepsilon_1 = 1\times 10^{-3} \\ \varepsilon_2 = -3\times 10^{-3} \end{cases} \quad \Rightarrow \quad \varepsilon'_{ij} = \begin{bmatrix} 1\times 10^{-3} & 0 \\ 0 & -3\times 10^{-3} \end{bmatrix}$$

b)

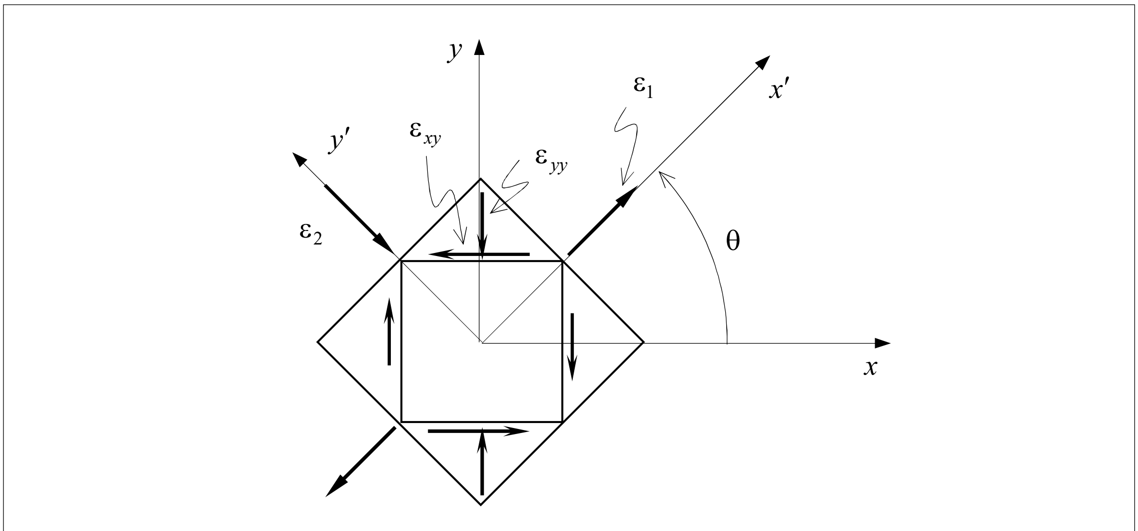


Figure 6.27

The Mohr's circle in strain is drawn in Figure 6.28.

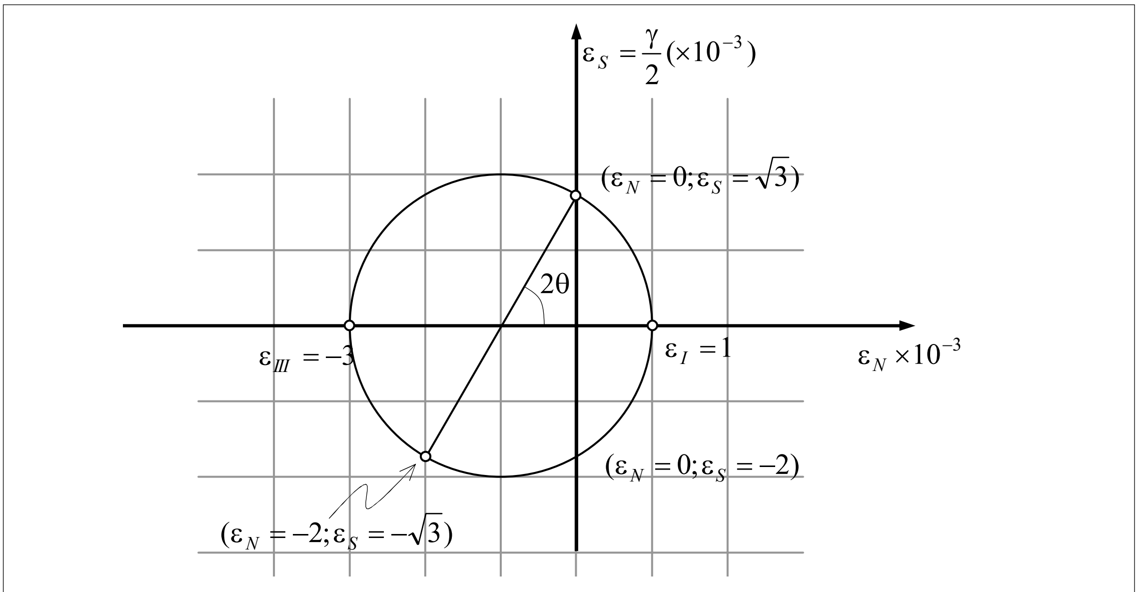


Figure 6.28

Note that the radius is $R = (1-(-3))/2 = 2$. Then:

$$\tan(2\theta) = \frac{\sqrt{3}}{1} \quad \Rightarrow \quad 2\theta = \arctan(\sqrt{3}) \quad \Rightarrow \quad \theta = 30^{\mathrm{o}}$$

b) Applying $\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}$, where $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} = 20MPa$,

$\mu = \dfrac{E}{2(1+\nu)} = 20MPa$, $\mathrm{Tr}(\boldsymbol{\varepsilon}) = -2\times 10^{-3}$. Then:

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{3}$$

$$= \left(\begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} + 40 \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \times \underbrace{10^{-3}\,MPa}_{=10^{3}\,Pa}$$

Thus:

$$\sigma_{ij} = \begin{bmatrix} -40 & -40\sqrt{3} & 0 \\ -40\sqrt{3} & -120 & 0 \\ 0 & 0 & -40 \end{bmatrix} kPa$$

As the material is isotropic, the stress and strain share the same principal space. In addition, the eigenvalues of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ can be related to each other as follows.

By substituting the value of $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ into the definition of the eigenvalue-eigenvector, we can obtain:

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}}$$

$$(\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon})\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \quad \Rightarrow \quad \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\cdot\hat{\mathbf{n}} + 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}}$$

$$\Rightarrow \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}} + 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \quad \Rightarrow \quad 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}}$$

$$\Rightarrow 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = (\gamma_{\boldsymbol{\sigma}} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}))\hat{\mathbf{n}} \quad \Rightarrow \quad \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \left(\frac{\gamma_{\boldsymbol{\sigma}} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\right)\hat{\mathbf{n}}$$

$$\Rightarrow \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}} = \gamma_{\boldsymbol{\varepsilon}}\hat{\mathbf{n}}$$

Then:

$$\gamma_{\boldsymbol{\varepsilon}} = \frac{\gamma_{\boldsymbol{\sigma}} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu} \quad \Rightarrow \quad \gamma_{\boldsymbol{\sigma}} = 2\mu\gamma_{\boldsymbol{\varepsilon}} + \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})$$

And the eigenvalues of $\boldsymbol{\sigma}$ can be obtained as follows:

$$\gamma_{\boldsymbol{\sigma}}^{(1)} \equiv \sigma_{I} = 2\mu\gamma_{\boldsymbol{\varepsilon}}^{(1)} + \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) = (40\times 10^{6})\times(1\times 10^{-3}) + (20\times 10^{6})\times(-2\times 10^{-3}) = 0$$
$$\gamma_{\boldsymbol{\sigma}}^{(2)} \equiv \sigma_{II} = 2\mu\gamma_{\boldsymbol{\varepsilon}}^{(2)} + \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) = (40\times 10^{6})\times(0) + (20\times 10^{6})\times(-2\times 10^{-3}) = -40\times 10^{3}\,Pa$$
$$\gamma_{\boldsymbol{\sigma}}^{(3)} \equiv \sigma_{III} = 2\mu\gamma_{\boldsymbol{\varepsilon}}^{(3)} + \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) = (40\times 10^{6})\times(-3\times 10^{-3}) + (20\times 10^{6})\times(-2\times 10^{-3}) = -160\times 10^{3}\,Pa$$

We can also use the equation $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ in the principal space:

$$\sigma'_{ij} = \left(\begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} + 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \times \underbrace{10^{-3}\,MPa}_{=10^3\,Pa} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -160 & 0 \\ 0 & 0 & -40 \end{bmatrix} kPa$$

The Mohr's circle in stress is described in Figure 6.29.

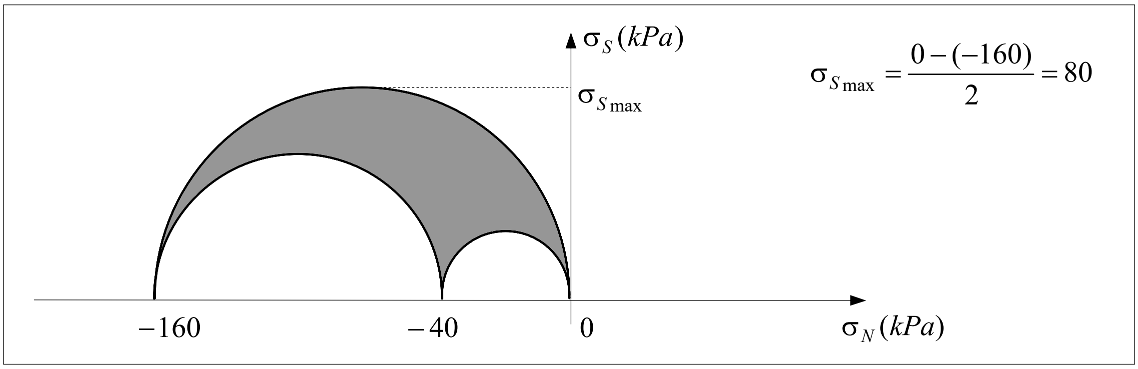


Figure 6.29

c) The strain energy density is $\Psi^e = \frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$. We can use the principal space in order to obtain the strain energy density, i.e.:

$$\begin{aligned} \Psi^e &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) \\ &= \frac{1}{2}\left[0 + (-160\times 10^3)(-3\times 10^{-3}) + 0\right] = 240 Pa\frac{m}{m} = 240\frac{N}{m^2}\frac{m}{m} = 240\frac{J}{m^3} \end{aligned}$$

where

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -160 & 0 \\ 0 & 0 & -40 \end{bmatrix} \times 10^3\,Pa \quad ; \quad \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3} \quad \left[\frac{m}{m}\right]$$

Problem 6.28

A solid, which can be approximated by the state of plane strain, has one point in which the infinitesimal strain tensor components are given by:

$$\varepsilon_{ij} = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

Consider that the material has an isotropic linear elastic behavior defined by the Young's modulus $E = 10MPa$ and Poisson's ratio $\nu = 0.25$.

a) Obtain the volumetric deformation and the deviatoric part of the strain tensor;

b) Obtain the principal strains and the principal directions;

c) Obtain the Cauchy stress tensor components;

d) Obtain the maximum and minimum normal stress;

e) It is known that the material fails when the tangential stress exceeds the value $40\,kPa$. Check whether the material fails or not.

Solution:

a) Volumetric deformation (ε_V):

$$\varepsilon_V = I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = (-2-10)\times 10^{-3} = -12\times 10^{-3}$$

Additive decomposition of the strain tensor into a spherical and deviatoric parts $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}$, where the spherical part is given by:

$$\varepsilon_{ij}^{sph} = \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\delta_{ij} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\times 10^{-3}$$

and the deviatoric part is:

$$\varepsilon_{ij}^{dev} = \varepsilon_{ij} - \varepsilon_{ij}^{sph} = \left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\right)\times 10^{-3} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix}\times 10^{-3}$$

b) The principal strains are obtained by means of the characteristic determinant:

$$\begin{vmatrix} -2-\overline{\lambda} & 3 \\ 3 & -10-\overline{\lambda} \end{vmatrix} = 0 \quad \Rightarrow \quad \overline{\lambda}^2 + 12\overline{\lambda} + 11 = 0$$

By solving the above quadratic equation we can obtain:

$$\overline{\lambda}_{(1,2)} = \frac{-(12)\pm\sqrt{(12)^2 - 4(1)(11)}}{2(1)} = \frac{-12\pm 10}{2} \quad \Rightarrow \quad \begin{cases} \overline{\lambda}_{(1)} = -1.0 \\ \overline{\lambda}_{(2)} = -11 \end{cases}$$

Then, the principal strains are:

$$\varepsilon_1 = -1.0\times 10^{-3} \quad ; \quad \varepsilon_2 = -11.0\times 10^{-3}$$

The principal directions can be obtained by solving $(\varepsilon_{ij} - \overline{\lambda}\delta_{ij})\mathsf{n}_j^{(\lambda)} = 0_i \qquad (i,j=1,2)$

The principal direction associated with the eigenvalue $\overline{\lambda}_{(1)} = -1.0$:

$$\begin{bmatrix} -2-(-1) & 3 \\ 3 & -10-(-1) \end{bmatrix}\begin{bmatrix} \mathsf{n}_1^{(1)} \\ \mathsf{n}_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -\mathsf{n}_1^{(1)} + 3\mathsf{n}_2^{(1)} = 0 \Rightarrow \mathsf{n}_1^{(1)} = 3\mathsf{n}_2^{(1)} \\ 3\mathsf{n}_1^{(1)} - 9\mathsf{n}_2^{(1)} = 0 \end{cases}$$

restriction $\mathsf{n}_1^{(1)2} + \mathsf{n}_2^{(1)2} = 1$, with that we can obtain $(3\mathsf{n}_2^{(1)})^2 + \mathsf{n}_2^{(1)2} = 1 \Rightarrow \mathsf{n}_2^{(1)} = \frac{1}{\sqrt{10}}$, and $\mathsf{n}_1^{(1)} = \frac{3}{\sqrt{10}}$

The principal direction associated with the eigenvalue $\overline{\lambda}_{(1)} = -11.0$:

$$\begin{bmatrix} -2-(-11) & 3 \\ 3 & -10-(-11) \end{bmatrix}\begin{bmatrix} \mathsf{n}_1^{(2)} \\ \mathsf{n}_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 9\mathsf{n}_1^{(2)} + 3\mathsf{n}_2^{(2)} = 0 \\ 3\mathsf{n}_1^{(2)} + \mathsf{n}_2^{(2)} = 0 \Rightarrow \mathsf{n}_2^{(2)} = -3\mathsf{n}_1^{(2)} \end{cases}$$

with the restriction $\mathsf{n}_1^{(2)2} + \mathsf{n}_2^{(2)2} = 1$, we can obtain $\mathsf{n}_1^{(2)} = \frac{1}{\sqrt{10}}$, and $\mathsf{n}_2^{(2)} = \frac{-3}{\sqrt{10}}$

We summarize the eigenvalues and eigenvectors as follows:

$$\varepsilon_1 = -1\times 10^{-3} \xrightarrow{\text{principal direction}} \hat{n}_i^{(1)} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \end{bmatrix}$$

$$\varepsilon_2 = -11\times 10^{-3} \xrightarrow{\text{principal direction}} \hat{n}_i^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} & 0 \end{bmatrix}$$

$$\varepsilon_1 = 0 \xrightarrow{\text{principal direction}} \hat{n}_i^{(3)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

c) The Cauchy stress tensor components are given by:

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 4MPa$, $\mu = G = \frac{E}{2(1+\nu)} = 4MPa$, $\mathrm{Tr}(\boldsymbol{\varepsilon}) = -12\times 10^{-3}$:

$$\sigma_{ij} = \left(4\times(-12)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\times(4)\begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)\times 10^{-3} MPa = \begin{bmatrix} -64 & 24 & 0 \\ 24 & -128 & 0 \\ 0 & 0 & -48 \end{bmatrix} kPa$$

As the material is isotropic the principal directions for the stress and strain are the same. The principal stresses can be obtained by working in the principal space $\sigma'_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon'_{ij}$:

$$\sigma'_{ij} = \left(4\times(-12)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\times(4)\begin{bmatrix} -1 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)\times 10^{-3} MPa = \begin{bmatrix} -56 & 0 & 0 \\ 0 & -136 & 0 \\ 0 & 0 & -48 \end{bmatrix} kPa$$

d) By considering that $\sigma_I = -48kPa$, $\sigma_{II} = -56kPa$, $\sigma_{III} = -136kPa$, the Mohr's circle in stress is described in Figure 6.30.

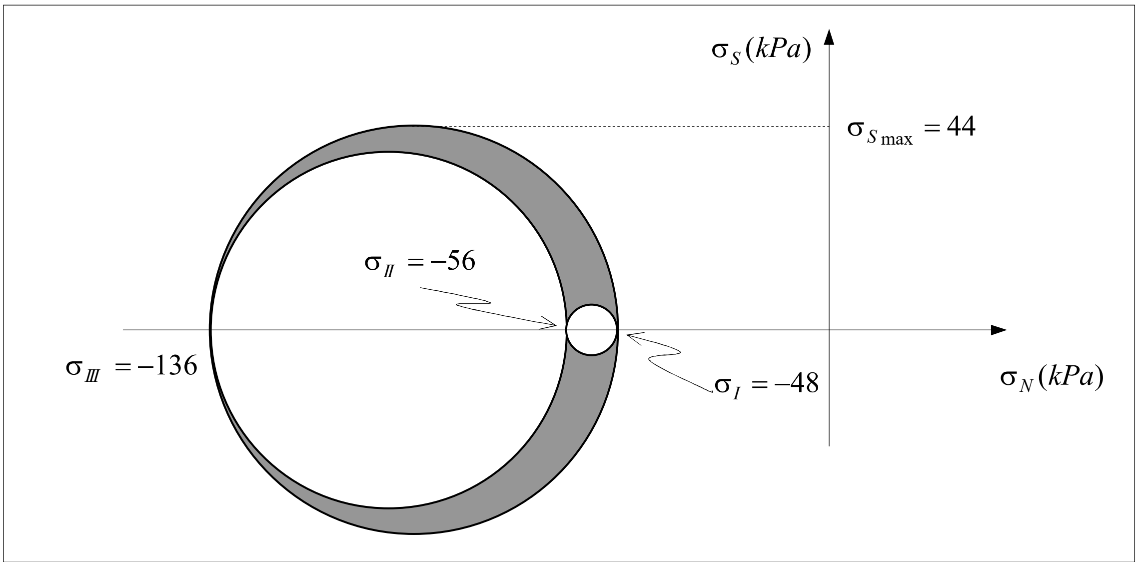


Figure 6.30: Mohr's circle in stress

The maximum shear stress, (see Figure 6.30), can be obtained as follows:

$$\sigma_{S\max} = \frac{\sigma_I - \sigma_{III}}{2} = \frac{(-48)-(-136)}{2} = 44kPa$$

Then, the material fails.

Problem 6.29

A *strain gauge* (or strain gage) is a device used to calculate the strain according to one direction. Consider a *strain rosette* that contains three strain gauges arranged as indicated in Figure 6.31. At one point we have calculated the following strain values:

$$\varepsilon_x = 0.33 \times 10^{-3} \quad ; \quad \varepsilon'_x = 0.22 \times 10^{-3} \quad ; \quad \varepsilon_y = -0.05 \times 10^{-3}$$

Consider an isotropic linear elastic material with the following mechanical properties: $E = 29000\,Pa$ (Young's modulus); $\nu = 0.3$ (Poisson's ratio).

a) Find the maximum shear stress at the point in question.

b) Obtain the eigenvalues (principal strains) and eigenvectors (principal directions) of the strain tensor;

c) Obtain the eigenvalues (principal stresses) and eigenvectors (principal directions) of the stress tensor.

Hypothesis: Consider the state of plane strain.

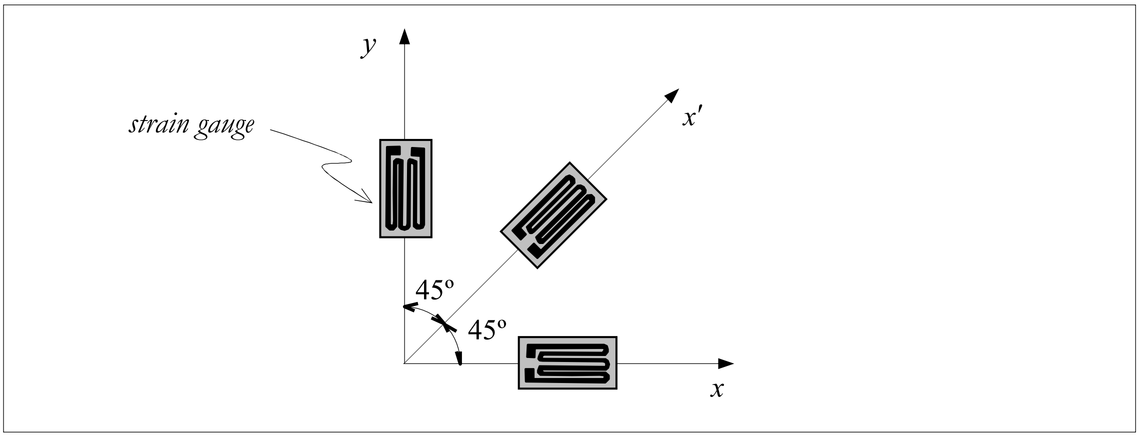


Figure 6.31: Strain rosette.

Solution:

We have to obtain the strain tensor components in the system x, y, z and to do so we will use the coordinate transformation law in order to obtain the component $\gamma_{xy} = 2\varepsilon_{12}$. Remember that in two-dimensional cases the normal component in a new system, (see **Problem 1.99** in Chapter 1), is given by:

$$\varepsilon'_{11} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + \frac{\varepsilon_{11} - \varepsilon_{22}}{2} \cos(2\theta) + \varepsilon_{12} \sin(2\theta)$$

The above equation was obtained by means of the transformation law, (see Chapter 1 of the textbook), which in engineering notation becomes:

$$\varepsilon'_x = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)$$

Then, γ_{xy} can be obtained as follows:

$$\gamma_{xy} = \frac{2}{\sin(2\theta)} \left(\varepsilon'_x - \frac{(\varepsilon_x + \varepsilon_y)}{2} - \frac{(\varepsilon_x - \varepsilon_y)}{2} \cos(2\theta) \right) = 0.16 \times 10^{-3}$$

thus

$$\varepsilon_{ij} = \begin{bmatrix} 0.33 & 0.08 & 0 \\ 0.08 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

Then, the stress components can be evaluated as follows:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu\varepsilon_y\right] = 12.0462\,Pa$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_y + \nu\varepsilon_x\right] = 3.5692\,Pa$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = 1.7846\,Pa \;\; ; \;\; \sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)}\left[\varepsilon_x + \varepsilon_y\right] = 4.684\,Pa$$

Additionally, the maximum shear stress is given by:

$$\sigma_{S\max} = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.5988\,Pa$$

a) The characteristic equation for the strain tensor (2D) is:

$$\varepsilon^2 - 0.28\,\varepsilon - 2.29 \times 10^{-2} = 0 \qquad (\times 10^{-3})$$

Then, by solving the above equation we can find the eigenvalues (principal strains) given by:

$$\varepsilon_1 = 0.346155 \times 10^{-3} \quad ; \quad \varepsilon_2 = -0.06615528 \times 10^{-3}$$

Then, the eigenvectors of the infinitesimal strain tensor are:

$$\begin{array}{r} \xrightarrow{\text{Eigenvector associated with } \varepsilon_1} \\ \xrightarrow{\text{Eigenvector associated with } \varepsilon_2} \\ \xrightarrow{\text{Eigenvector associated with } \varepsilon_3} \end{array} \begin{bmatrix} 0.9802 & 0.1979 & 0 \\ -0.1979 & 0.9802 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) By considering the stress tensor components:

$$\sigma_{ij} = \begin{bmatrix} 12.0462 & 1.7846 & 0 \\ 1.7846 & 3.5692 & 0 \\ 0 & 0 & 4.684 \end{bmatrix} Pa$$

we can obtain the characteristic determinant and in turn the eigenvalues (principal stresses) $\sigma_1 = 12.40654$, $\sigma_2 = 3.208843$. Additionally, the eigenvectors of the stress tensor are:

$$\begin{array}{r} \xrightarrow{\text{Eigenvector associated with } \sigma_1} \\ \xrightarrow{\text{Eigenvector associated with } \sigma_2} \\ \xrightarrow{\text{Eigenvector associated with } \sigma_3} \end{array} \begin{bmatrix} 0.9802 & 0.1979 & 0 \\ -0.1979 & 0.9802 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As expected, the eigenvectors of stress and strain are the same; since we are working with isotropic linear elastic material.

b) Alternative solution for the stress tensor components:

Knowing the strain tensor components:

$$\varepsilon_{ij} = \begin{bmatrix} 0.33 & 0.08 & 0 \\ 0.08 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

We can apply the constitutive equation: $\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}$, where the Lamé constants are given by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 16.7307692 \times 10^3 \, Pa \,, \quad \mu = \frac{E}{2(1+\nu)} = 11.15384615 \times 10^3 \, Pa$$

and $\mathrm{Tr}(\boldsymbol{\varepsilon}) = 0.27999972 \times 10^{-3} \approx 0.28 \times 10^{-3}$, with that $\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}$ becomes:

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} 0.33 & 0.08 & 0 \\ 0.08 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$= \begin{bmatrix} 12.0461 & 1.784615 & 0 \\ 1.784615 & 3.5692 & 0 \\ 0 & 0 & 4.6846 \end{bmatrix} Pa$$

As the material is isotropic, the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ share the same principal directions, then we can use the same equation $\sigma'_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon'_{ij}$ in the principal space, i.e.:

$$\sigma'_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

$$= \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} 0.346155 & 0 & 0 \\ 0 & -0.0662 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3} = \begin{bmatrix} 12.40752 & 0 & 0 \\ 0 & 3.20783 & 0 \\ 0 & 0 & 4.6846 \end{bmatrix} Pa$$

Problem 6.30

A *strain gauge* (or strain gage) is a device used to obtain the strain in only one direction. Consider a *strain rosette* that contains three strain gauges arranged according to a equilateral triangle, (see Figure 6.32), and records the strain values according to the directions x_1, x'_1 and x''_1.

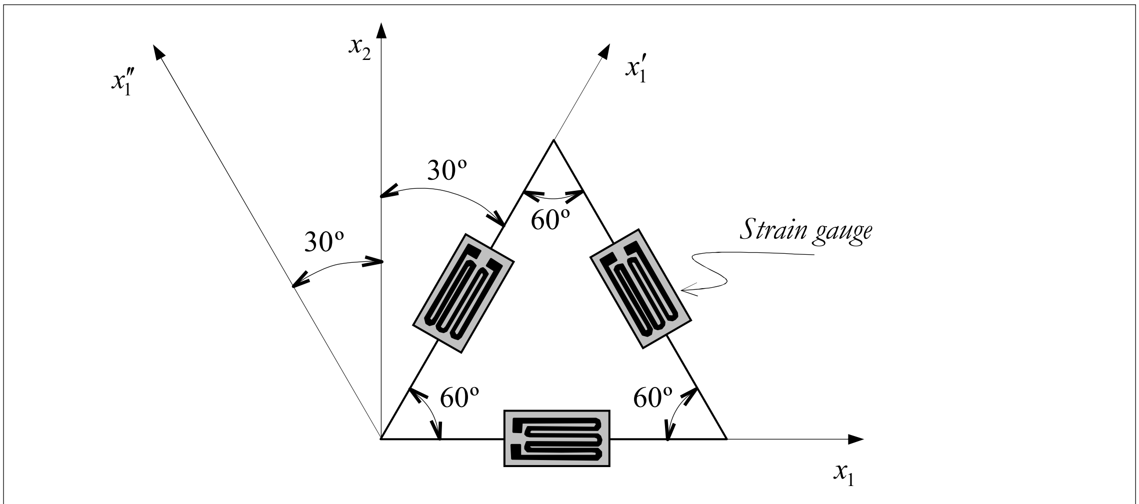


Figure 6.32

The strain calculated according to the directions x_1, x'_1 and x''_1 are respectively:

$$\varepsilon_{11} = -4\times10^{-4} \quad ; \quad \varepsilon'_{11} = 1\times10^{-4} \quad ; \quad \varepsilon''_{11} = 4\times10^{-4}$$

Obtain $\varepsilon_{22} = \varepsilon_y$, $2\varepsilon_{12} = \gamma_{xy}$, $\varepsilon'_{22} \equiv \varepsilon'_y$. Show that $\varepsilon_{11} + \varepsilon_{22} = \varepsilon'_{11} + \varepsilon'_{22}$.

Hypothesis: Consider a state of plane strain.

Solution:

Using the component transformation law for the second-order tensor and considering the plane state, we can obtain that:

$$\begin{cases} \varepsilon'_{11} = \dfrac{\varepsilon_{11}+\varepsilon_{22}}{2} + \dfrac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos(2\theta_1) + \varepsilon_{12}\sin(2\theta_1) & (6.81) \\ \varepsilon''_{11} = \dfrac{\varepsilon_{11}+\varepsilon_{22}}{2} + \dfrac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos(2\theta_2) + \varepsilon_{12}\sin(2\theta_2) & (6.82) \end{cases}$$

where $\theta_1 = 60^\circ$ and $\theta_2 = 120^\circ$, thus $\cos 2\theta_1 = \cos 2\theta_2 = \dfrac{-1}{2}$ and $\sin 2\theta_1 = -\sin 2\theta_2 = \dfrac{\sqrt{3}}{2}$. Then, by combining the two above equations it is possible to eliminate ε_{12}, i.e.:

$$\varepsilon'_{11} + \varepsilon''_{11} = \varepsilon_{11} + \varepsilon_{22} - \frac{\varepsilon_{11}-\varepsilon_{22}}{2} \quad \Rightarrow \quad \varepsilon_{22} = \frac{2}{3}\left(\varepsilon'_{11} + \varepsilon''_{11} - \frac{\varepsilon_{11}}{2}\right) = 4.66667\times10^{-4}$$

Once the value $\varepsilon_{22} = 4.66667\times10^{-4}$ is obtained, we can replace it into the equation (6.81) with which we can obtain:

$$\gamma_{xy} = 2\varepsilon_{12} = \frac{1}{\sqrt{3}}\left(4\varepsilon'_{11} - \varepsilon_{11} - 3\varepsilon_{22}\right) = -3.46410\times10^{-4} \quad \Rightarrow \quad \varepsilon_{12} = -1.73205\times10^{-4}$$

To obtain ε'_{22}, we must obtain the angle formed by x_1 and x'_2, which is $\theta_3 = 60^\circ + 90^\circ = 150^\circ$, thus:

$$\varepsilon'_{22} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos(2\theta_3) + \varepsilon_{12}\sin(2\theta_3) = -0.33333\times10^{-4}$$

Checking that:

$$\varepsilon_{11} + \varepsilon_{22} = \varepsilon'_{11} + \varepsilon'_{22} = 0.66667\times10^{-4} = \mathrm{Tr}(\boldsymbol{\varepsilon})$$

As expected, since the trace is an invariant.

Problem 6.31

Consider the dam cross section, (Figure 6.33), in which the displacement is known and given by:

$$\begin{cases} u(x,y) = -4x^2 - y^2 + 2xy + 2 \\ v(x,y) = -4y^2 - x^2 + 2xy + 5 \end{cases}$$

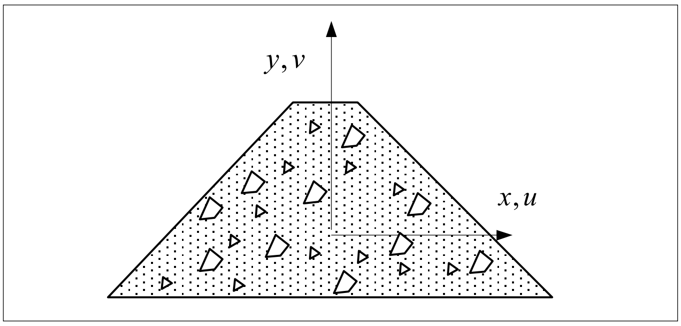


Figure 6.33

This structure is made up of a material with the following mechanical properties: $E = 100MPa$ (Young's modulus), $G = 35.7MPa$ (shear modulus), $\nu = 0.4$ (Poisson'ratio). Assuming that the structure is under a small deformation regime: a) Find the stress field; b)

For the given displacement field, show whether the equilibrium equations are satisfied or not.

Solution:

a) We can calculate the strain tensor components as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -8x + 2y \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} = -8y + 2x \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

which in matrix form is:

$$\varepsilon_{ij} = \begin{bmatrix} -8x + 2y & 0 & 0 \\ 0 & -8y + 2x & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) For the dam, as we have seen, we can adopt the approximation of state of plane strain, so,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = 357.1428 \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} -8x + 2y \\ -8y + 2x \\ 0 \end{bmatrix} MPa$$

$$\Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = 357.1428 \begin{bmatrix} -4x - 2y \\ -2x - 4y \\ 0 \end{bmatrix} MPa$$

$$\sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)} (\varepsilon_x + \varepsilon_y) = 357.1428 \times \left[(-8x + 2y) + (-8y + 2x) \right]$$

Then, the equilibrium equations become:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + \rho \mathsf{b}_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + \rho \mathsf{b}_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + \rho \mathsf{b}_z = 0 \end{cases} \Rightarrow \begin{cases} -4 + 0 + 0 + 0 \neq 0 \quad \text{(it fails)} \ \text{✘} \\ 0 - 4 + 0 + 0 \neq 0 \quad \text{(it fails)} \ \text{✘} \\ 0 + 0 + \dfrac{\partial \sigma_z}{\partial z} + 0 = 0 \end{cases}$$

So, the given displacement field does not satisfy the equilibrium equations.

Problem 6.32

A gravity dam of triangular cross section is made up of concrete with "specific weight" equal to $\frac{5}{2}\gamma$, where γ is the specific weight of water. The shape and dimensions of the cross section are indicated in Figure 6.34, and the stress field in the dam (state of plane strain) is given by:

$$\sigma_{11} = -\gamma x_2 \quad ; \quad \sigma_{22} = \frac{\gamma}{2}(x_1 - 3x_2) \quad ; \quad \sigma_{12} = -\gamma\, x_1$$

Consider: Poisson's ratio: $\nu = \frac{1}{4}$; Young's modulus E.

a) Obtain the graphical representation of the surface force (traction vector) acting on the face $\overline{AB}$ due to the ground reaction;

b) Obtain the principal stresses at the points A and B. Starting from the Mohr's circle in stress, obtain the extreme values of the stresses at the respective points.

c) Obtain the strain field in the dam.

NOTE: Although in the literature γ is known as the **specific weight**, also known as the **unit weight,** in reality γ is the module of the body force per unit volume, i.e. $\gamma = \|\vec{\mathbf{p}}\| = \|\rho\vec{\mathbf{b}}\| = \rho g$, where $\vec{\mathbf{b}}$ is the body force per unit mass $[\vec{\mathbf{b}}] = \frac{N}{kg} = \frac{m}{s^2}$. Recall that in the International System of Units (SI) the term "*specific*" is related to "*per unit mass*", which is not the case of γ, the correct term would be the weight density, since the term "*density*" is related to "*per unit volume*".

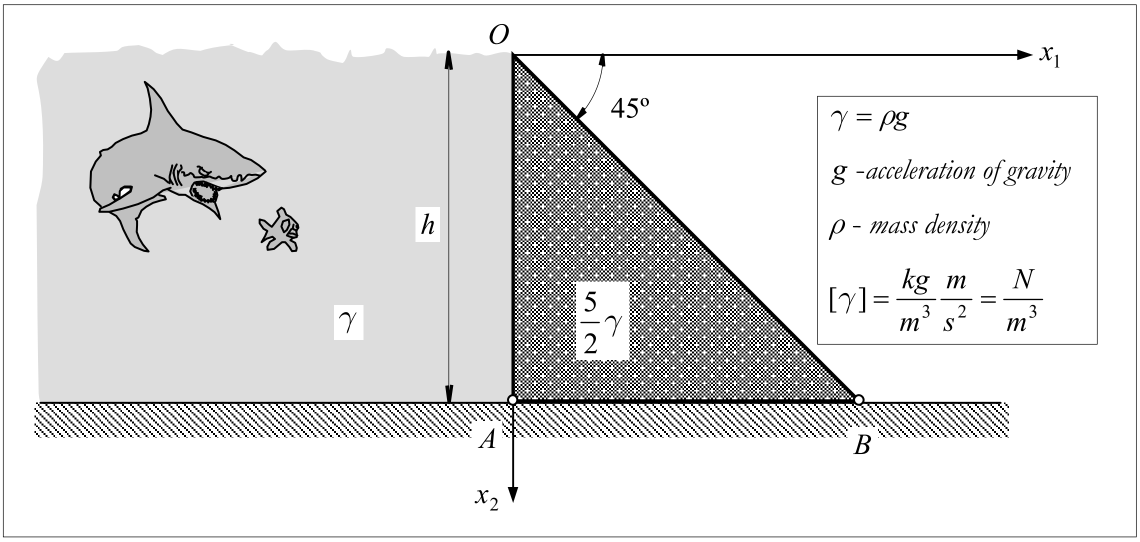


Figure 6.34

Solution:

a) The stress and strain fields in the dam cross section are respectively:

$$\sigma_{ij} = \begin{bmatrix} -\gamma\, x_2 & -\gamma\, x_1 & 0 \\ -\gamma\, x_1 & \frac{\gamma}{2}(x_1 - 3x_2) & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad ; \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can obtain the surface force by means of the traction vector $\mathbf{t}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$. For the side $\overline{AB}$ whose normal unit vector is $\hat{\mathrm{n}}_i = [0,1,0]$, we can obtain:

$$\begin{bmatrix} \mathrm{t}_1^{(AB)} \\ \mathrm{t}_2^{(AB)} \\ \mathrm{t}_3^{(AB)} \end{bmatrix} = \begin{bmatrix} -\gamma x_2 & -\gamma x_1 & 0 \\ -\gamma x_1 & \frac{\gamma}{2}(x_1 - 3x_2) & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\gamma x_1 \\ \frac{\gamma}{2}(x_1 - 3x_2) \\ 0 \end{bmatrix}$$

The surface force on the base can be appreciated in Figure 6.35.

b) Note that σ_{33} is already a principal stress. Starting from $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ we can obtain σ_{33}, i.e.:

$$\sigma_{ij} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij} \Rightarrow \sigma_{33} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\delta_{33} + 2\mu\varepsilon_{33} \Rightarrow \sigma_{33} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})$$

The term $\mathrm{Tr}(\boldsymbol{\varepsilon})$ can be obtained by means of the double scalar product between $\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ and the second-order unit tensor, thus:

$$\boldsymbol{\sigma} : \mathbf{1} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} : \mathbf{1} + 2\mu\boldsymbol{\varepsilon} : \mathbf{1} \quad \Rightarrow \quad \mathrm{Tr}(\boldsymbol{\sigma}) = 3\lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) + 2\mu \mathrm{Tr}(\boldsymbol{\varepsilon}) = [3\lambda + 2\mu]\mathrm{Tr}(\boldsymbol{\varepsilon})$$

$$\Rightarrow \mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3\lambda + 2\mu} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3\lambda + 2\mu}$$

Then, the component σ_{33} is defined as follows:

$$\sigma_{33} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{\lambda}{3\lambda + 2\mu}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad \Rightarrow \quad \sigma_{33} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{33} = \frac{\lambda}{3\lambda + 2\mu}(\sigma_{11} + \sigma_{22})$$

$$\Rightarrow \sigma_{33}\left(1 - \frac{\lambda}{3\lambda + 2\mu}\right) = \frac{\lambda}{3\lambda + 2\mu}(\sigma_{11} + \sigma_{22}) \quad \Rightarrow \quad \sigma_{33} = \frac{\lambda}{2(\lambda + \mu)}(\sigma_{11} + \sigma_{22}) = \nu(\sigma_{11} + \sigma_{22})$$

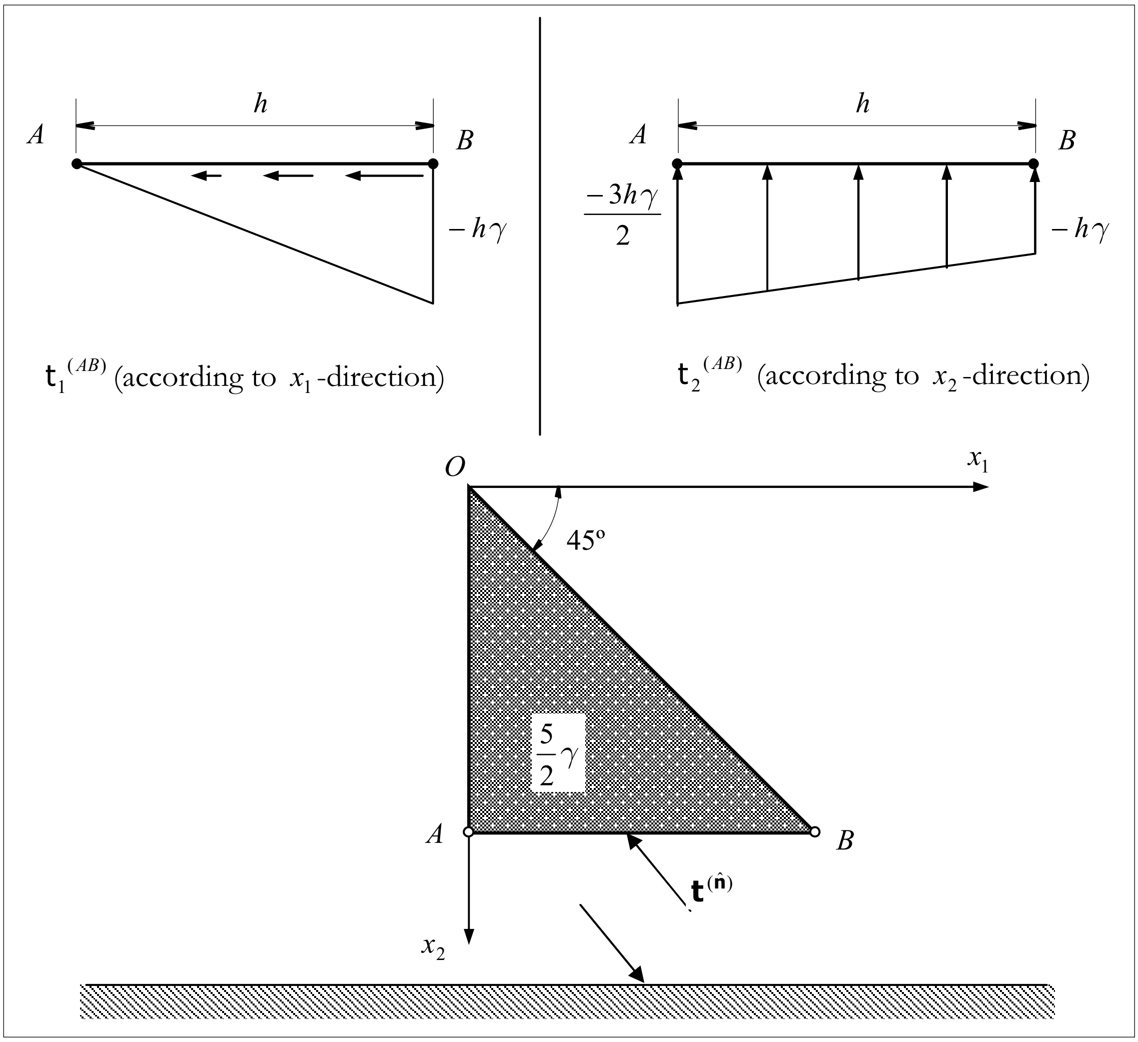


Figure 6.35

By substituting the values of σ_{11} and σ_{22}, we can obtain:

$$\sigma_{33} = \frac{\lambda}{2(\lambda + \mu)}(\sigma_{11} + \sigma_{22}) = \nu\left[-\gamma x_2 + \frac{\gamma}{2}(x_1 - 3x_2)\right] = \frac{\gamma\nu}{2}[x_1 - 5x_2] = \frac{\gamma}{8}[x_1 - 5x_2]$$

where we have considered $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$.

The stress state at the point $A(x_1 = 0; x_2 = h)$ is given by:

$$\sigma_{ij}^{(A)} = \begin{bmatrix} -\gamma\, x_2 & -\gamma\, x_1 & 0 \\ -\gamma\, x_1 & \dfrac{\gamma}{2}(x_1 - 3x_2) & 0 \\ 0 & 0 & \dfrac{\gamma}{8}(x_1 - 5x_2) \end{bmatrix} = \begin{bmatrix} -\gamma\, h & 0 & 0 \\ 0 & \dfrac{-3h\gamma}{2} & 0 \\ 0 & 0 & \dfrac{-5h\gamma}{8} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \dfrac{-3}{2} & 0 \\ 0 & 0 & \dfrac{-5}{8} \end{bmatrix} h\gamma$$

Note that this space is already the principal space. Mohr's circle in stress at the point A is drawn in Figure 6.36.

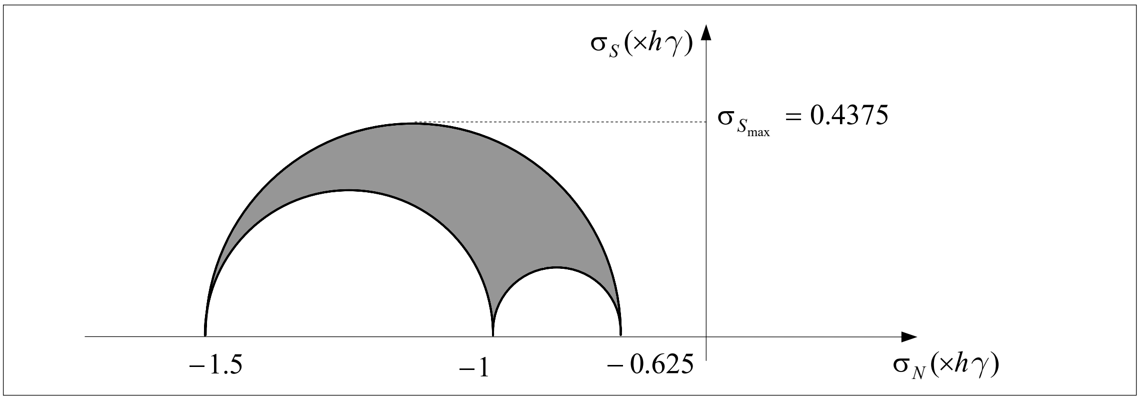


Figure 6.36

The stress state at the point $B(x_1 = h; x_2 = h)$ is given by:

$$\sigma_{ij}^{(B)} = \begin{bmatrix} -\gamma\, x_2 & -\gamma\, x_1 & 0 \\ -\gamma\, x_1 & \dfrac{\gamma}{2}(x_1 - 3x_2) & 0 \\ 0 & 0 & \dfrac{\gamma}{8}(x_1 - 5x_2) \end{bmatrix} = \begin{bmatrix} -\gamma\, h & -\gamma\, h & 0 \\ -\gamma\, h & \dfrac{\gamma}{2}(h - 3h) & 0 \\ 0 & 0 & \dfrac{\gamma}{8}(h - 5h) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & \dfrac{-1}{2} \end{bmatrix} \gamma h$$

The principal stresses at the point $B(x_1 = h; x_2 = h)$ are given by:

$$\begin{vmatrix} -1-\sigma & -1 \\ -1 & -1-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad (-1-\sigma)^2 - 1 = 0 \quad \Rightarrow \quad (-1-\sigma)^2 = 1 \quad \Rightarrow \quad (-1-\sigma) = \pm 1$$

$$\Rightarrow \begin{cases} \sigma_1 = -2 \\ \sigma_2 = 0 \end{cases}$$

The Mohr's circle in stress for the point B is drawn in Figure 6.37.

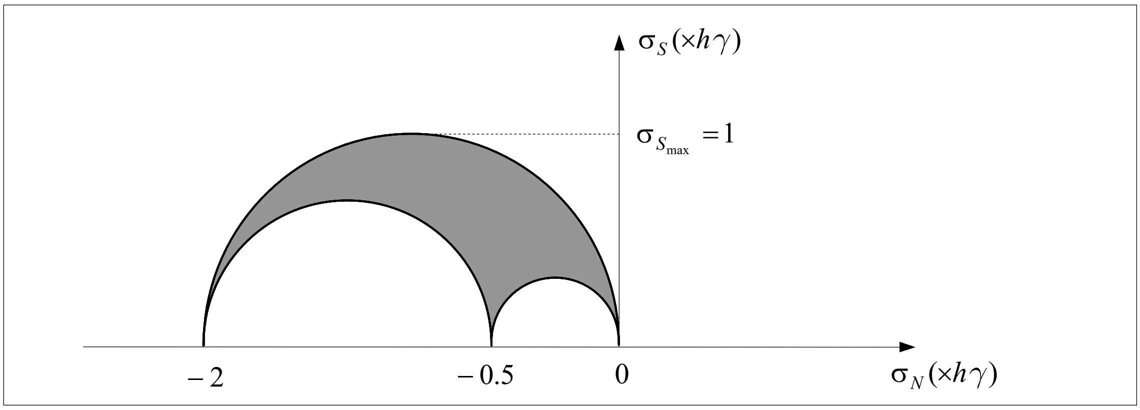


Figure 6.37

c) We can obtain the expression for the strain field by starting from the equation:

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \quad \Rightarrow \quad 2\mu\boldsymbol{\varepsilon} = \boldsymbol{\sigma} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}$$

Remember that we have obtained that $\mathrm{Tr}(\boldsymbol{\varepsilon}) = \dfrac{\mathrm{Tr}(\boldsymbol{\sigma})}{3\lambda + 2\mu}$, then the above equation becomes:

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

We can also express the above equation in terms of E and ν:

$$\mu = G = \frac{E}{2(1+\nu)} \Rightarrow \frac{1}{2\mu} = \frac{(1+\nu)}{E}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \Rightarrow \frac{1}{\mu(3\lambda + 2\mu)} = \frac{1}{E(\lambda + \mu)}$$

$$\frac{\lambda}{2\mu(3\lambda + 2\mu)} = \frac{\lambda}{2}\frac{1}{E(\lambda + \mu)} = \frac{\nu}{E}$$

Then:

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} \quad \text{or} \quad \boldsymbol{\varepsilon} = \frac{(1+\nu)}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

The trace of $\boldsymbol{\sigma}$ is given by:

$$\mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = (-\gamma x_2) + \left[\frac{\gamma}{2}(x_1 - 3x_2)\right] + \left[\frac{\gamma}{8}(x_1 - 5x_2)\right] = \frac{5}{8}\gamma(x_1 - 5x_2)$$

With that we can obtain the strain tensor components, when $\nu = \dfrac{1}{4}$, as follows:

$$\varepsilon_{ij} = \frac{5}{4E}\sigma_{ij} - \frac{5}{32E}\gamma(x_1 - 5x_2)\delta_{ij}$$

$$\varepsilon_{ij} = \frac{5}{4E}\begin{bmatrix} -\gamma x_2 & -\gamma x_1 & 0 \\ -\gamma x_1 & \frac{\gamma}{2}(x_1 - 3x_2) & 0 \\ 0 & 0 & \frac{\gamma}{8}\left[x_1 - 5x_2\right] \end{bmatrix} - \frac{5}{32E}\gamma(x_1 - 5x_2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{5\gamma}{4E}\begin{bmatrix} -\frac{1}{8}(x_1 + 3x_2) & -x_1 & 0 \\ -x_1 & -\frac{1}{8}(-3x_1 + 7x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 6.33

Consider the infinitesimal strain tensor field (2D):

$$\varepsilon_{ij} = \begin{bmatrix} -\kappa_{x_3} x_2 & \varepsilon_{12} \\ \varepsilon_{12} & \nu\kappa_{x_3} x_2 \end{bmatrix} \qquad (i, j = 1,2) \tag{6.83}$$

where $\kappa_{x_3} = \kappa_{x_3}(x_1)$, i.e. κ_{x_3} is a function of x_1 and ν is a constant (Poisson's ratio). Consider the state of plane stress, (see **Problem 6.24**)), with no body force.

a) Obtain ε_{12} in order to achieve the equilibrium and obtain the stress field.

As boundary condition, consider that $\varepsilon_{12}(x_2 = \pm\frac{a}{2}) = 0$.

b) Express the infinitesimal strain tensor and the stress tensor in terms of (P, E, I_{x_3}), where P is the concentrated force at $x_1 = L$, E is the Young's modulus, I_{x_3} is the moment of inertia of the cross-sectional area about the x_3-axis, which for a rectangular section is $I_{x_3} = \int_A x_2^2 dA = \int_{x_3=\frac{-b}{2}}^{x_3=\frac{b}{2}} \int_{x_2=\frac{-a}{2}}^{x_2=\frac{a}{2}} x_2^2 dx_2 dx_3 = \frac{ba^3}{12}$. For boundary condition, consider that at x_1: $P = \int_A \sigma_{12} dA$ and that at $x_1 = 0 \Rightarrow \kappa_{x_3} = \frac{PL}{EI_{x_3}}$, where EI is called modulus of flexural rigidity.

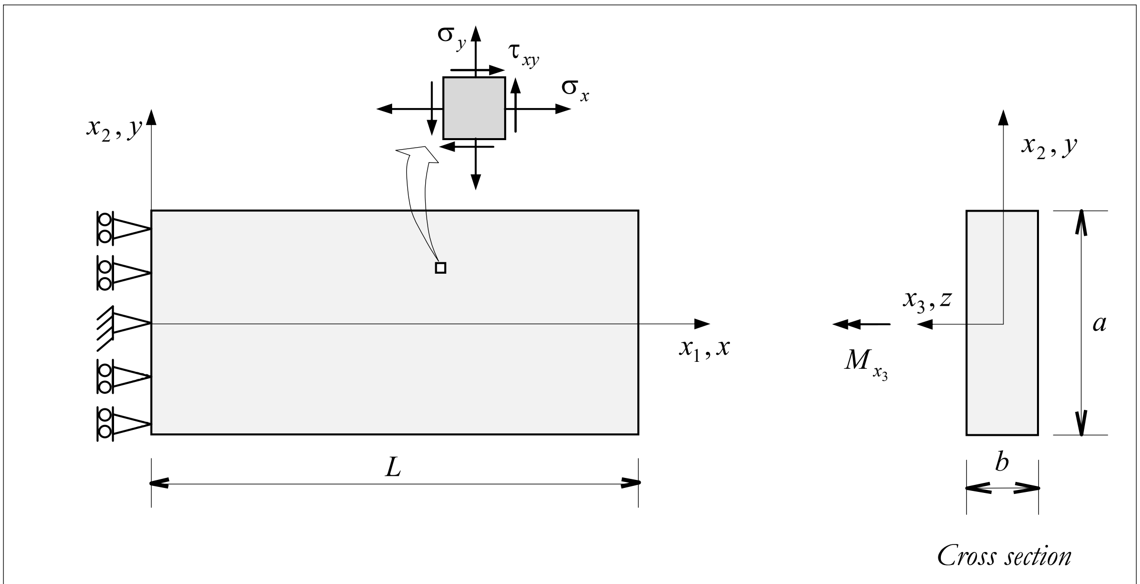


Figure 6.38: Fixed-free beam (boundary conditions).

Solution:

Before applying the equilibrium equations we will need to obtain the stress field. In **Problem 6.24** the stress field for the state of plane stress was obtained and by considering the strain components (6.83) we can obtain:

$$\sigma_{ij} = \frac{E}{(1-\nu^2)}\begin{bmatrix} (\varepsilon_{11}+\nu\varepsilon_{22}) & (1-\nu)\varepsilon_{12} \\ (1-\nu)\varepsilon_{12} & (\varepsilon_{22}+\nu\varepsilon_{11}) \end{bmatrix} = \begin{bmatrix} -E\kappa_{x_3}x_2 & \dfrac{E}{(1+\nu)}\varepsilon_{12} \\ \dfrac{E}{(1+\nu)}\varepsilon_{12} & 0 \end{bmatrix} \tag{6.84}$$

Considering the equilibrium equations without body forces, $\sigma_{ij,j} = 0_i \ (i,j=1,2)$, we can obtain:

$$\sigma_{ij,j} = 0_i \quad \Rightarrow \quad \sigma_{i1,1} + \sigma_{i2,2} = 0_i \quad \Rightarrow \quad \begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} = 0 \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} = 0 \end{cases} \tag{6.85}$$

By substituting the stress components given by equation (6.84) we can obtain:

$$\begin{cases} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} = 0 \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{-\partial(E\kappa_{x_3}x_2)}{\partial x_1} + \dfrac{E}{(1+\nu)}\dfrac{\partial(\varepsilon_{12})}{\partial x_2} = 0 \\ \dfrac{E}{(1+\nu)}\dfrac{\partial(\varepsilon_{12})}{\partial x_1} = 0 \Rightarrow \dfrac{\partial(\varepsilon_{12})}{\partial x_1} = 0 \end{cases} \tag{6.86}$$

Note that ε_{12} does not depend on x_1. From the first equilibrium equation we can obtain:

$$\frac{-\partial(E\kappa_{x_3}x_2)}{\partial x_1} + \frac{E}{(1+\nu)}\frac{\partial(\varepsilon_{12})}{\partial x_2} = 0 \quad \Rightarrow \quad \frac{\partial\varepsilon_{12}}{\partial x_2} = (1+\nu)x_2\frac{\partial\kappa_{x_3}}{\partial x_1} \equiv (1+\nu)x_2\kappa_{x_3,1}$$

By integrating in x_2 the above equation we can obtain:

$$\frac{\partial\varepsilon_{12}}{\partial x_2} = (1+\nu)x_2\kappa_{x_3,1} \xrightarrow{\text{integrating}} \varepsilon_{12} = (1+\nu)\kappa_{x_3,1}\frac{x_2^2}{2} + C \tag{6.87}$$

The constant of integration can be obtained by means of the boundary condition $\varepsilon_{12}(x_2 = \pm\frac{a}{2}) = 0$:

$$\varepsilon_{12} = \frac{1}{2}(1+\nu)\kappa_{x_3,1}x_2^2 + C \xrightarrow{x_2=\pm\frac{a}{2}} \varepsilon_{12} = \frac{1}{2}(1+\nu)\kappa_{x_3,1}\left(\frac{a}{2}\right)^2 + C = 0$$

$$\Rightarrow C = \frac{-(1+\nu)\kappa_{x_3,1}}{2}\frac{a^2}{4}$$

Then, the strain ε_{12}, (see equation (6.87)), becomes:

$$\varepsilon_{12} = (1+\nu)\kappa_{x_3,1}\frac{x_2^2}{2} + C = (1+\nu)\kappa_{x_3,1}\frac{x_2^2}{2} - \frac{(1+\nu)\kappa_{x_3,1}}{2}\frac{a^2}{4} = \frac{(1+\nu)\kappa_{x_3,1}}{2}\left(x_2^2 - \frac{a^2}{4}\right) \tag{6.88}$$

With that the infinitesimal strain field becomes:

$$\varepsilon_{ij}=\begin{bmatrix} -\kappa_{x_3}x_2 & \dfrac{(1+\nu)\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) \\ \dfrac{(1+\nu)\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) & \nu\kappa_{x_3}x_2 \end{bmatrix}$$

And the stress field:

$$\sigma_{ij}=\begin{bmatrix} -E\kappa_{x_3}x_2 & \dfrac{E}{(1+\nu)}\varepsilon_{12} \\ \dfrac{E}{(1+\nu)}\varepsilon_{12} & 0 \end{bmatrix}=\begin{bmatrix} -E\kappa_{x_3}x_2 & \dfrac{E\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) \\ \dfrac{E\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) & 0 \end{bmatrix}$$

b) By applying the condition $P=\int\limits_A \sigma_{12}dA$ we can obtain:

$$\begin{aligned} P&=\int\limits_A \sigma_{12}dA=\int\limits_A \frac{E\kappa_{x_3,1}}{2}\left(x_2^2-\frac{a^2}{4}\right)dA=\frac{E\kappa_{x_3,1}}{2}\int\limits_A\left(x_2^2-\frac{a^2}{4}\right)dA \\ &=\frac{E\kappa_{x_3,1}}{2}\left(\int\limits_A x_2^2 dA-\frac{a^2}{4}\int\limits_A dA\right)=\frac{E\kappa_{x_3,1}}{2}\left(I_{x_3}-\frac{a^2}{4}A\right)=\frac{E\kappa_{x_3,1}}{2}\left(I_{x_3}-\frac{a^2}{4}ba\right) \\ &=\frac{E\kappa_{x_3,1}}{2}\left(I_{x_3}-\frac{3ba^3}{12}\right)=\frac{E\kappa_{x_3,1}}{2}\left(I_{x_3}-3I_{x_3}\right)=-E\kappa_{x_3,1}I_{x_3} \end{aligned} \tag{6.89}$$

with that we can obtain that:

$$P=-E\kappa_{x_3,1}I_{x_3} \quad\Rightarrow\quad \kappa_{x_3,1}=\frac{-P}{EI_{x_3}} \tag{6.90}$$

By integrating we can obtain:

$$\kappa_{x_3,1}\equiv\frac{\partial\kappa_{x_3}}{\partial x_1}=\frac{-P}{EI_{x_3}} \xrightarrow{\text{integrating}} \kappa_{x_3}=\frac{-P}{EI_{x_3}}x_1+C=\frac{P}{EI_{x_3}}(L-x_1) \tag{6.91}$$

where we have applied the boundary condition ($x_1=0\Rightarrow\ \kappa_{x_3}=\dfrac{PL}{EI_{x_3}}$) to obtain $C=\dfrac{PL}{EI_{x_3}}$.

With that the stress components can also be expressed as follows:

$$\sigma_{11}=-E\kappa_{x_3}x_2=\frac{Px_2}{I_{x_3}}(x_1-L) \quad;\quad \sigma_{22}=0 \quad;\quad \sigma_{12}=\frac{P}{2I_{x_3}}\left(\frac{a^2}{4}-x_2^2\right) \tag{6.92}$$

or by using Engineering notation:

$$\sigma_x=-E\kappa_z y=\frac{Py}{I_z}(x-L) \quad;\quad \sigma_y=0 \quad;\quad \tau_{xy}=\frac{P}{2I_z}\left(\frac{a^2}{4}-y^2\right)$$

Note that, the equation for σ_{12}, given by (6.92), is independent of x_1 and σ_{12} has parabolic distribution on the cross section area.

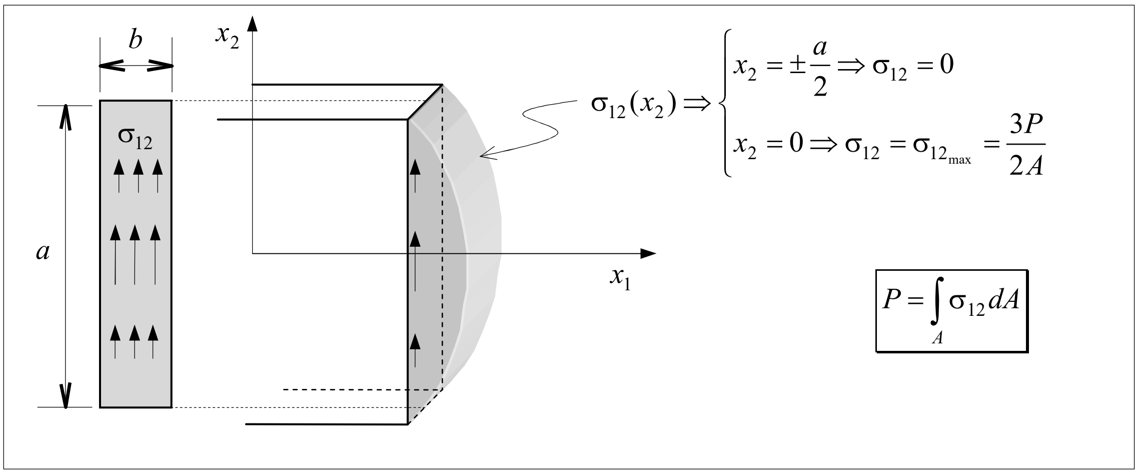


Figure 6.39: Tangential stress distribution on the cross-section area.

The maximum tangential stress is acting at $x_2 = 0$:

$$\sigma_{12} = \frac{P}{2I_{x_3}}\left(\frac{a^2}{4} - x_2^2\right) \xrightarrow{x_2=0} \sigma_{12\,\max} = \frac{Pa^2}{8I_{x_3}} = \frac{3P}{2ab} = \frac{3P}{2A}$$

where A is cross section area.

NOTE 1: The displacement field can be obtained by starting from the strain definition:

$$\varepsilon_{11} = \frac{\partial \mathsf{u}_1}{\partial x_1} = -\frac{P}{EI_{x_3}}(L - x_1)x_2 \xrightarrow{\text{integrating in } x_1} \mathsf{u}_1(x_2, x_1) = \frac{-Px_2}{EI_{x_3}}\left(Lx_1 - \frac{x_1^2}{2}\right) + f_1(x_2) + C_1$$

$$\varepsilon_{22} = \frac{\partial \mathsf{u}_2}{\partial x_2} = \nu\frac{P}{EI_{x_3}}(L - x_1)x_2 \xrightarrow{\text{integrating in } x_2} \mathsf{u}_2(x_2, x_1) = \nu\frac{P}{EI_{x_3}}(L - x_1)\frac{x_2^2}{2} + f_2(x_1) + C_2$$

Applying the boundary condition $\mathsf{u}_2(x_2 = 0, x_1 = 0) = \mathsf{u}_1(x_2, x_1) = 0$, we can conclude that $C_1 = C_2 = 0$, thus

$$\begin{cases} \mathsf{u}_1(x_2, x_1) = \dfrac{-Px_2}{EI_{x_3}}\left(Lx_1 - \dfrac{x_1^2}{2}\right) + f_1(x_2) \\ \mathsf{u}_2(x_2, x_1) = \dfrac{P\nu}{2EI_{x_3}} x_2^2(L - x_1) + f_2(x_1) \end{cases} \tag{6.93}$$

The tangential strain component can be obtained as follows:

$$2\varepsilon_{12} = \frac{\partial \mathsf{u}_1}{\partial x_2} + \frac{\partial \mathsf{u}_2}{\partial x_1} = \frac{-P}{EI_{x_3}}\left(Lx_1 - \frac{x_1^2}{2}\right) + \frac{\partial f_1(x_2)}{\partial x_2} - \frac{P\nu}{2EI_{x_3}} x_2^2 + \frac{\partial f_2(x_1)}{\partial x_1} \tag{6.94}$$

Note that we have obtained previously that ε_{12} is independent of x_1, so, the following must hold:

$$\frac{-P}{EI_{x_3}}\left(Lx_1 - \frac{x_1^2}{2}\right) + \frac{\partial f_2(x_1)}{\partial x_1} = 0 \quad \Rightarrow \quad \frac{\partial f_2(x_1)}{\partial x_1} = \frac{P}{EI_{x_3}}\left(Lx_1 - \frac{x_1^2}{2}\right)$$

$$\xrightarrow{\text{integrating in } x_1} f_2(x_1) = \frac{P}{EI_{x_3}}\left(\frac{Lx_1^2}{2} - \frac{x_1^3}{6}\right)$$

Then, the equation in (6.94) becomes

$$2\varepsilon_{12}=\frac{\partial \mathrm{u}_1}{\partial x_2}+\frac{\partial \mathrm{u}_2}{\partial x_1}=\frac{\partial f_1(x_2)}{\partial x_2}-\frac{P\nu}{2EI_{x_3}}x_2^2=(1+\nu)\kappa_{x_3,1}\left(x_2^2-\frac{a^2}{4}\right)=\frac{-P(1+\nu)}{EI_{x_3}}\left(x_2^2-\frac{a^2}{4}\right)$$

$$\Rightarrow \frac{\partial f_1(x_2)}{\partial x_2}-\frac{P\nu}{2EI_{x_3}}x_2^2=\frac{-P(1+\nu)}{EI_{x_3}}\left(x_2^2-\frac{a^2}{4}\right)$$

$$\Rightarrow \frac{\partial f_1(x_2)}{\partial x_2}=\frac{P\nu}{2EI_{x_3}}x_2^2-\frac{P(1+\nu)}{EI_{x_3}}\left(x_2^2-\frac{a^2}{4}\right)$$

By integrating the above equation in x_2 we can obtain

$$f_1(x_2)=\frac{P\nu}{2EI_{x_3}}\frac{x_2^3}{3}-\frac{P(1+\nu)}{EI_{x_3}}\left(\frac{x_2^3}{3}-\frac{a^2}{4}x_2\right)$$

Then, the displacement field (6.93) becomes:

$$\begin{cases}\mathrm{u}_1(x_2,x_1)=\dfrac{-Px_2}{EI_{x_3}}\left(Lx_1-\dfrac{x_1^2}{2}\right)+\dfrac{P\nu}{6EI_{x_3}}x_2^3-\dfrac{P(1+\nu)}{EI_{x_3}}\left(\dfrac{x_2^3}{3}-\dfrac{a^2}{4}x_2\right)\\[2ex] \mathrm{u}_2(x_2,x_1)=\dfrac{P\nu}{2EI_{x_3}}x_2^2(L-x_1)+\dfrac{P}{EI_{x_3}}\left(\dfrac{Lx_1^2}{2}-\dfrac{x_1^3}{6}\right)\end{cases} \tag{6.95}$$

NOTE 2: We will check the compatibility equation for two-dimensional problem, (see **Problem 5.11** – NOTE 3),

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}-2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}=0+\nu x_2\frac{\partial^2\kappa_{x_3}}{\partial x_1^2}-2(1+\nu)x_2\frac{\partial^2\kappa_{x_3}}{\partial x_1^2}=0 \;\checkmark$$

Note that $\dfrac{\partial^2\kappa_{x_3}}{\partial x_1^2}\equiv\kappa_{x_3,11}=0$, a fact already verified by the equilibrium equations.

The problem presented previously is only valid if we discard completely the dimension x_3. The reason follows.

As we are treating the problem by the state of plane stress we do not have stress σ_{i3} but we have the strain $\varepsilon_{33}\neq 0$, (see **Problem 6.24**). Then, the strain field becomes:

$$\varepsilon_{ij}=\begin{bmatrix}-\kappa_{x_3}x_2 & \dfrac{(1+\nu)\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) & 0\\[2ex] \dfrac{(1+\nu)\kappa_{x_3,1}}{2}\left(x_2^2-\dfrac{a^2}{4}\right) & \nu\kappa_{x_3}x_2 & 0\\[2ex] 0 & 0 & \nu\kappa_{x_3}x_2\end{bmatrix}$$

For the above strain field, the compatibility equations, (see **Problem 5.11**), are not satisfied, i.e.:

$$\begin{cases} \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \quad \checkmark \\ \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0 \quad \checkmark \\ \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \quad \checkmark \\ \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = -\dfrac{\partial^2 (\nu \kappa_{x_3} x_2)}{\partial x_1 \partial x_2} = -\nu \kappa_{x_3,1} \neq 0 \quad \text{✘} \quad \text{(it fails)} \\ \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = 0 \quad \checkmark \\ \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = 0 \quad \checkmark \end{cases}$$

where we have used $\kappa_{x_3,11} \equiv \dfrac{\partial^2 \kappa_{x_3}}{\partial x_1^2} = 0$, since ε_{12} is independent of x_1, (see equation (6.88)), this implies that $\kappa_{x_3,1}$ is a constant.

NOTE 3: Let us consider the case in which $L = 5m$, $a = 3m$, $b = 0.1m$, $P = 1.0\times10^{8} N$, $E = 210\times10^{9} Pa$, $\nu = 0.4$, (see Figure 6.38). With these data the beam deformation by using equations (6.95) is presented in Figure 6.40. Note that a plane cross section before deformation does not remain plane after deformation. Note that the fiber at $x_2 = 0$ there is no displacement according to the x_1-direction (displacement u_1). As academic example, let us change the parameter $a = 3m$ by $a = 1m$. In this case we can appreciate that the cross section remains plane after deformation, (see Figure 6.41).

The assumption in which the plane cross section remains plane and perpendicular to the *neutral line* ($\sigma_{11} = 0$) after deformation is known as *Euler-Bernoulli's hypothesis*, and by means of this assumption it is possible to treat beams as one dimensional case (Classical Beam Theory), (see section 6.5 Introduction to one-dimensional elements).

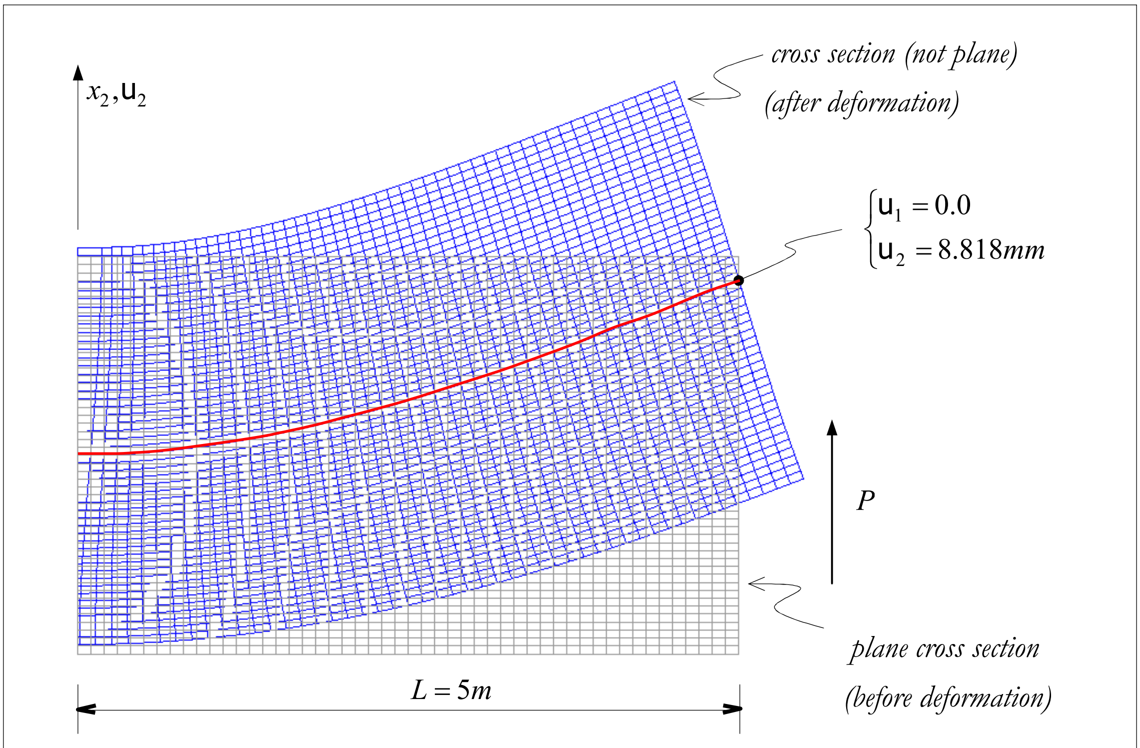


Figure 6.40: Beam deformation (amplified deformation) – for the case $a = 3m$.

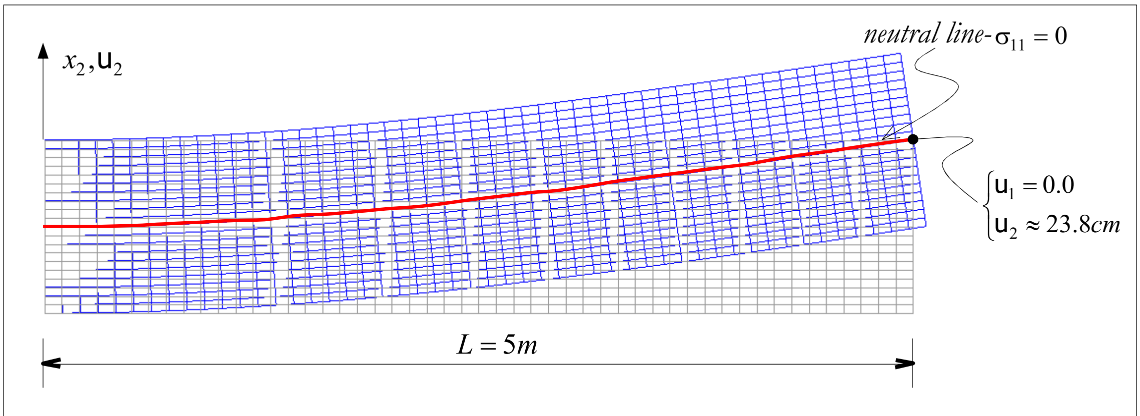


Figure 6.41: Beam deformation for the case $a = 1m$ (amplified deformation).

6.2.1 Using Stress Function to solve 2D Linear Elasticity Problems

Problem 6.34

In **Problem 5.16** we have shown the *Stress Formulation* for three-dimensional elasticity. Obtain the equivalent formulation for two-dimensional elasticity, *i.e.* considering the state of plane stress and strain.

Solution:

As seen in **Problem 5.5**, the governing equations, for an isotropic linear elastic material in small deformation regime, are:

Tensorial notation	Indicial notation
The equations of motion: $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \ddot{\vec{\mathbf{u}}} = \rho \vec{a}$ (2 equations)	*The equations of motion:* $\sigma_{ij,j} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i = \rho a_i$ (2 equations)
The constitutive equations for stress: $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ (3 equations)	*The constitutive equations for stress:* $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ (3 equations)
The kinematic equations: $\boldsymbol{\varepsilon} = \nabla^{sym}_{\vec{x}} \vec{\mathbf{u}}$ (3 equations)	*The kinematic equations:* $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_i}{\partial x_j} + \dfrac{\partial \mathrm{u}_j}{\partial x_i}\right)$ (3 equations)

(6.96)

where we have also considered two-dimensional problem $(i, j = 1,2)$. Note that for two-dimensional problem we have 8 equations and 8 unknowns namely, u_i (2 unknowns), ε_{ij} (3 unknowns) and σ_{ij} (3 unknowns). The *kinematic equations* can be replaced by the compatibility equations, (see **Problem 5.11**):

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} - \varepsilon_{jk,il} = \mathbb{O}_{ijkl}$$

In the two-dimensional case the compatibility equations, (see **Problem 5.11 –** NOTE 3), reduce to:

$$S_{33} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \xrightarrow{\text{Engineering Notation}} S_z = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (6.97)$$

And the *equations of motion* for two-dimensional case become:

$$\sigma_{ij,j} + \rho \mathrm{b}_i = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i \xrightarrow{2D} \sigma_{i1,1} + \sigma_{i2,2} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i \qquad (i = 1,2)$$

$$\Rightarrow \begin{cases} \sigma_{11,1} + \sigma_{12,2} + \rho \mathrm{b}_1 = \rho \ddot{\mathrm{u}}_1 \\ \sigma_{21,1} + \sigma_{22,2} + \rho \mathrm{b}_2 = \rho \ddot{\mathrm{u}}_2 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \rho \mathrm{b}_1 = \rho \ddot{\mathrm{u}}_1 = \rho a_1 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \rho \mathrm{b}_2 = \rho \ddot{\mathrm{u}}_2 = \rho a_2 \end{cases}$$

or in engineering notation:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \rho \mathrm{b}_1 = \rho \ddot{\mathrm{u}}_1 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \rho \mathrm{b}_2 = \rho \ddot{\mathrm{u}}_2 \end{cases} \xrightarrow{\text{Engineering Notation}} \begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \rho \mathrm{b}_x = \rho a_x \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \rho \mathrm{b}_y = \rho a_y \end{cases}$$

We take the derivative of the first equation with respect to x and the second one with respect to y, i.e.:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \rho \mathrm{b}_x = \rho a_x \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \rho \mathrm{b}_y = \rho a_y \end{cases} \begin{matrix} \xrightarrow{\frac{\partial}{\partial x}} \\ \xrightarrow{\frac{\partial}{\partial y}} \end{matrix} \begin{cases} \dfrac{\partial^2 \sigma_x}{\partial x^2} + \dfrac{\partial^2 \tau_{xy}}{\partial x \partial y} = \dfrac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) \\ \dfrac{\partial^2 \tau_{xy}}{\partial x \partial y} + \dfrac{\partial^2 \sigma_y}{\partial y^2} = \dfrac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\dfrac{\partial^2 \sigma_x}{\partial x^2} + \dfrac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) & (1) \\ \dfrac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\dfrac{\partial^2 \sigma_y}{\partial y^2} + \dfrac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) & (2) \end{cases}$$

By adding the both equations, (1)+(2), we can obtain

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) - \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) \tag{6.98}$$

a) The state of plane stress

The constitutive equations for the state of plane stress were obtained in **Problem 6.24** and the strain field, (see equation (6.49)), is given by:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \varepsilon_x = \dfrac{1}{E}\sigma_x - \dfrac{\nu}{E}\sigma_y \\ \varepsilon_y = \dfrac{-\nu}{E}\sigma_x + \dfrac{1}{E}\sigma_y \\ \gamma_{xy} = \dfrac{2(1+\nu)}{E}\tau_{xy} \end{cases}$$

By substituting the above strain components into the compatibility equation ("*kinematic equations*"), and by considering the *homogeneous material*, we can obtain:

$$\begin{aligned} & \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \\ & \Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y\right) + \frac{\partial^2}{\partial x^2}\left(\frac{-\nu}{E}\sigma_x + \frac{1}{E}\sigma_y\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{2(1+\nu)}{E}\tau_{xy}\right) = 0 \\ & \Rightarrow \frac{1}{E}\frac{\partial^2 \sigma_x}{\partial y^2} - \frac{\nu}{E}\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\nu}{E}\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{1}{E}\frac{\partial^2 \sigma_y}{\partial x^2} - \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \\ & \Rightarrow \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \end{aligned} \tag{6.99}$$

To consider simultaneously the two equations of motion we can use the equation in (6.98):

$$\Rightarrow 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = (1+\nu)\left[-\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x)\right] + (1+\nu)\left[-\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y)\right]$$

and by substituting the above equation into the equation (6.99) we can obtain:

$$\frac{\partial^2 \sigma_x}{\partial y^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2} - \nu \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

$$\Rightarrow \frac{\partial^2 \sigma_x}{\partial y^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2} - \nu \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - (1+\nu)\left[-\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) \right]$$
$$-(1+\nu)\left[-\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) \right] = 0$$

By simplifying the above equation we can obtain:

Stress formulation 2D – The state of plane stress

$$\boxed{\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = (1+\nu)\left[\frac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) + \frac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) \right]} \tag{6.100}$$

For the static or quasi-static case the above equation reduces to:

Stress formulation 2D – The state of plane stress (static case)

$$\boxed{\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = -(1+\nu)\left[\frac{\partial}{\partial x}(\rho \mathrm{b}_x) + \frac{\partial}{\partial y}(\rho \mathrm{b}_y) \right]} \tag{6.101}$$

b) The state of plane strain

The constitutive equations for the state of plane strain were obtained in **Problem 6.24** and the strain field, (see equation (6.61)), is given by:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \varepsilon_x = \dfrac{(1+\nu)(1-\nu)}{E}\sigma_x - \dfrac{\nu(1+\nu)}{E}\sigma_y \\ \varepsilon_y = -\dfrac{\nu(1+\nu)}{E}\sigma_x + \dfrac{(1+\nu)(1-\nu)}{E}\sigma_y \\ \gamma_{xy} = \dfrac{2(1+\nu)}{E}\tau_{xy} \end{cases}$$

By substituting the above strain components into the compatibility equation ("*kinematic equations*"), and by considering the homogeneous material, we can obtain:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{(1+\nu)(1-\nu)}{E}\sigma_x - \frac{\nu(1+\nu)}{E}\sigma_y \right) + \frac{\partial^2}{\partial x^2}\left(-\frac{\nu(1+\nu)}{E}\sigma_x + \frac{(1+\nu)(1-\nu)}{E}\sigma_y \right)$$
$$-\frac{\partial^2}{\partial x \partial y}\left(\frac{2(1+\nu)}{E}\tau_{xy} \right) = 0 \tag{6.102}$$

$$\Rightarrow (1-\nu)\frac{\partial^2 \sigma_x}{\partial y^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2} - \nu \frac{\partial^2 \sigma_x}{\partial x^2} + (1-\nu)\frac{\partial^2 \sigma_y}{\partial x^2} - 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

To consider simultaneously the two equations of motion we can use the equation in (6.98):

$$\Rightarrow 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \left[-\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial}{\partial x}(\rho a_x - \rho \mathrm{b}_x) \right] + \left[-\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial}{\partial y}(\rho a_y - \rho \mathrm{b}_y) \right]$$

and by substituting the above equation into the equation (6.102) we can obtain:

$$(1-\nu)\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}+(1-\nu)\frac{\partial^2\sigma_y}{\partial x^2}-2\frac{\partial^2\tau_{xy}}{\partial x\partial y}=0$$

$$\Rightarrow(1-\nu)\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}+(1-\nu)\frac{\partial^2\sigma_y}{\partial x^2}-\left[-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial}{\partial x}(\rho a_x-\rho \mathsf{b}_x)\right]$$
$$-\left[-\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial}{\partial y}(\rho a_y-\rho \mathsf{b}_y)\right]=0$$

By simplifying the above equation we can obtain:

Stress formulation 2D – The state of plane strain

$$\boxed{\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}=\frac{1}{(1-\nu)}\left[\frac{\partial}{\partial x}(\rho a_x-\rho \mathsf{b}_x)+\frac{\partial}{\partial y}(\rho a_y-\rho \mathsf{b}_y)\right]} \tag{6.103}$$

For the static or quasi-static case the above equation reduces to:

Stress formulation 2D – The state of plane strain (static case)

$$\boxed{\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}=\frac{-1}{(1-\nu)}\left[\frac{\partial}{\partial x}(\rho \mathsf{b}_x)+\frac{\partial}{\partial y}(\rho \mathsf{b}_y)\right]} \tag{6.104}$$

NOTE 1: Recall that the body forces can be represented by means of the potential ϕ, i.e. $\vec{\mathbf{b}}=-\nabla_{\vec{x}}\phi$, since $\vec{\mathbf{b}}$ is a conservative field. Then, we can write $\mathsf{b}_x=\frac{-\partial\phi}{\partial x}$ and $\mathsf{b}_y=\frac{-\partial\phi}{\partial y}$. Recall also that in **Problem 5.18** we have defined the Airy stress function Φ. If we take into account the body forces we can write:

$$\sigma_x-\rho\phi=\frac{\partial^2\Phi}{\partial y^2}\quad;\quad\sigma_y-\rho\phi=\frac{\partial^2\Phi}{\partial x^2}\quad;\quad\tau_{xy}=\tau_{yx}=-\frac{\partial^2\Phi}{\partial x\partial y} \tag{6.105}$$

thus

$$\sigma_x=\rho\phi+\frac{\partial^2\Phi}{\partial y^2}\quad;\quad\sigma_y=\rho\phi+\frac{\partial^2\Phi}{\partial x^2} \tag{6.106}$$

Substituting the above stress components into the equation (6.101) and by considering the mass density field homogeneous we can obtain:

$$\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}=-(1+\nu)\left[\frac{\partial}{\partial x}(\rho \mathsf{b}_x)+\frac{\partial}{\partial y}(\rho \mathsf{b}_y)\right]=-\rho(1+\nu)\left[\frac{\partial \mathsf{b}_x}{\partial x}+\frac{\partial \mathsf{b}_y}{\partial y}\right]$$

$$\frac{\partial^2}{\partial x^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial x^2}\right)=\rho(1+\nu)\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]$$

$$\Rightarrow\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}+2\rho\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=\rho(1+\nu)\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]$$

$$\Rightarrow\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=\rho[(1+\nu)-2]\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]$$

thus:

Stress formulation 2D – The state of plane stress (static case)

$$\boxed{\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=-\rho(1-\nu)\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]} \tag{6.107}$$

Now, if we substitute the stress components (6.106) into the equation (6.104) we can obtain:

$$\frac{\partial^2}{\partial x^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left(\rho\phi+\frac{\partial^2\Phi}{\partial x^2}\right)=\frac{-\rho}{(1-\nu)}\left[\frac{\partial \mathrm{b}_x}{\partial x}+\frac{\partial \mathrm{b}_y}{\partial y}\right]$$

$$\Rightarrow\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}+2\rho\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=\frac{\rho}{(1-\nu)}\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]$$

$$\Rightarrow\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=\rho\left(\frac{1}{(1-\nu)}-2\right)\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]$$

thus

Stress formulation 2D – The state of plane strain (static case)

$$\boxed{\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=-\rho\frac{(1-2\nu)}{(1-\nu)}\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right]} \tag{6.108}$$

Near to the Earth surface the body forces can be considered uniform (homogenous field), hence $\frac{\partial^2\phi}{\partial x}\approx\frac{\partial^2\phi}{\partial y}\approx 0$. With that the governing equation for two-dimensional cases, (see equations (6.107) and (6.108)), becomes:

Stress formulation 2D – (static case and homogenous body forces field)

$$\boxed{\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=0}\quad;\quad\boxed{\nabla^4_{\vec{x}}\Phi=0}\quad;\quad\boxed{\Phi_{,iijj}=0\qquad(i,j=1,2)} \tag{6.109}$$

NOTE 2: Note that we have reduced the original problem, 8 equations and 8 unknowns, (see equation (6.96)), to 1 equation (6.109) and 1 unknown (Φ).

Recall that the analytical solution (the exact one) in most practical cases is quite complex and even impossible to be obtained. So we resort to numerical technique, which consists in: given a problem we find the solution. During the era of G.B. Airy (1862) the only possible solution was the analytical one, since the numerical techniques were scarce. Then, they used to address the elastic problem through *inverse method* (Laier&Barreiro (1983)), i.e. for a given solution of the equation (6.109) they seek which problem represents such solution.

The stress function can be adopted, for example, by a polynomial function, (see Figure 6.42):

$$\begin{aligned}\Phi=&K_1+K_2x+K_3y+K_4x^2+K_5xy+K_6y^2+K_7x^3+K_8x^2y+K_9xy^2\\&+K_{10}y^3+K_{11}x^4+K_{12}x^3y+K_{13}x^2y^2+K_{14}xy^3+K_{15}xy^4+K_{16}x^5\\&+K_{17}x^4y+K_{18}x^3y^2+K_{19}x^2y^3+K_{20}xy^4+K_{21}y^5+K_{22}x^6+\cdots\end{aligned} \tag{6.110}$$

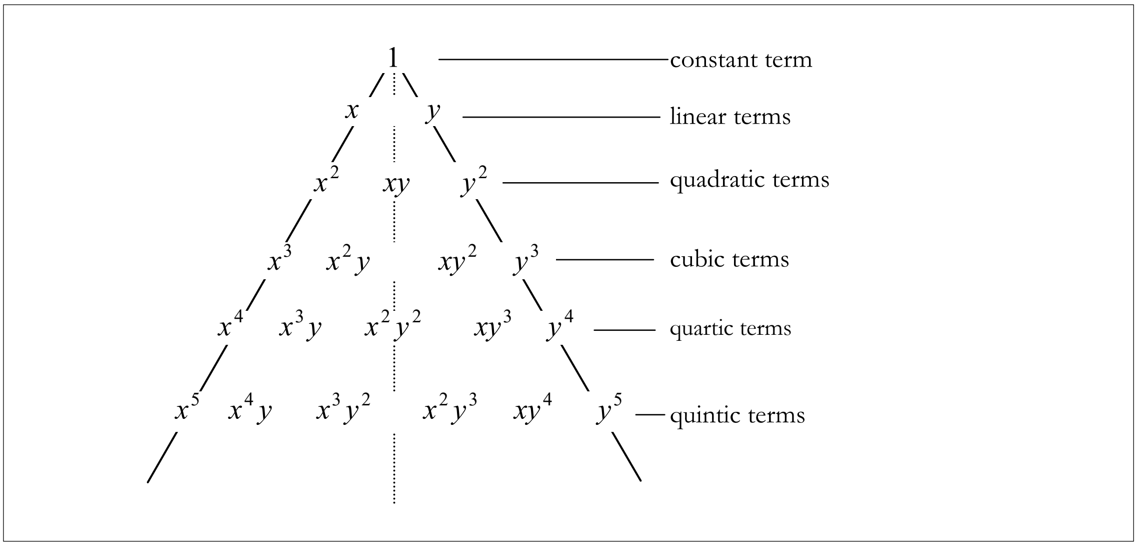


Figure 6.42: Pascal's polynomial for 2D.

Example: Let us assume that the Airy stress function is given by the polynomial:

$$\Phi = K_4 x^2 + K_5 xy + K_6 y^2 \tag{6.111}$$

where K_1, K_2, and K_3 are constants. If we are not considering the body forces the stress field (6.105) becomes:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = 2K_6 \quad ; \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = 2K_4 \quad ; \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -K_5 \tag{6.112}$$

Note that the stress field is homogenous, i.e. it is independent of $\vec{\boldsymbol{x}}$. For the particular case when $K_4 = K_5 = 0$ we can obtain the problem represented by the bar subjected to axial force F at its ends, (see Figure 6.43):

Stress field (Bar subjected to axial force):

$$\begin{cases} \sigma_x = \dfrac{F}{A} = 2K_6 \quad \Rightarrow \quad K_6 = \dfrac{F}{2A} \\ \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

Strain field (Bar subjected to axial force):

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} = \dfrac{2K_6}{E} = \dfrac{F}{EA} \\ \varepsilon_y = -\nu \dfrac{\sigma_x}{E} = -\nu \dfrac{2K_6}{E} = \dfrac{-\nu F}{EA} \\ \gamma_{xy} = 0 \end{cases}$$

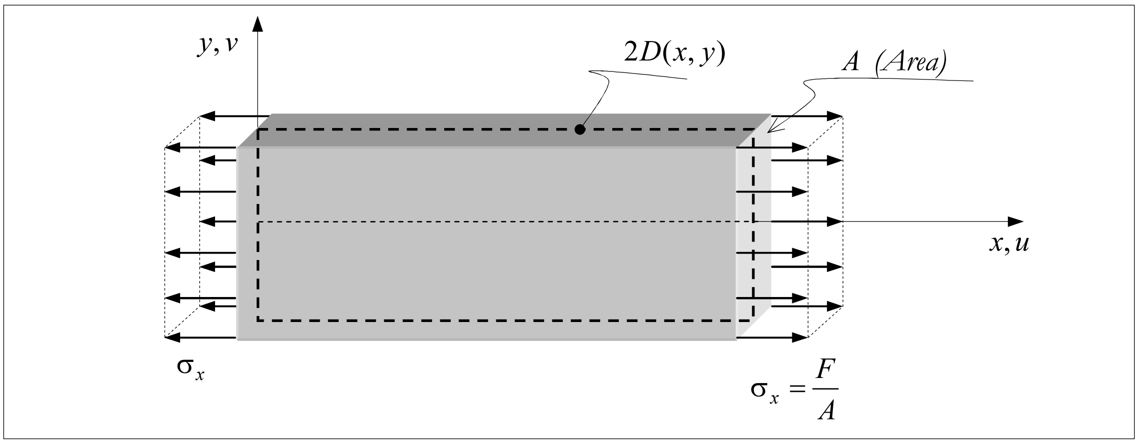


Figure 6.43: Bar subjected to axial force.

Displacement field (Bar subjected to axial force):

$$\begin{cases} \dfrac{\partial u}{\partial x} = \varepsilon_x = \dfrac{F}{EA} \xrightarrow{\text{integrating in x}} u(x,y) = \dfrac{F}{EA} x + f_1(y) + C_1 \\ \dfrac{\partial v}{\partial y} = \varepsilon_y = -\dfrac{\nu F}{EA} \xrightarrow{\text{integrating in y}} v(x,y) = \dfrac{-\nu F}{EA} y + f_2(x) + C_2 \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = \gamma_{xy} = 0 \end{cases}$$

where $f_1(y)$ is a function of y, $f_2(x)$ is a function of x, C_1 and C_2 are constants of integration. By means of the above third equation we can obtain:

$$\begin{aligned} & \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \gamma_{xy} = 0 \\ \Rightarrow & \frac{\partial}{\partial x}\left(\frac{-\nu F}{EA} y + f_2(x) + C_2 \right) + \frac{\partial}{\partial y}\left(\frac{F}{EA} x + f_1(y) + C_1 \right) = 0 \\ \Rightarrow & \frac{\partial f_2(x)}{\partial x} + \frac{\partial f_1(y)}{\partial y} = 0 \\ \Rightarrow & \frac{\partial f_2(x)}{\partial x} = -\frac{\partial f_1(y)}{\partial y} \end{aligned}$$

The only possible solution is when:

$$\frac{\partial f_2(x)}{\partial x} = -C_3 \qquad and \qquad \frac{\partial f_1(y)}{\partial y} = C_3$$

where C_3 is a constant. Then, the displacement field becomes:

$$\begin{cases} u(x,y) = \dfrac{F}{EA} x + C_3 y + C_1 \\ v(x,y) = \dfrac{-\nu F}{EA} y - C_3 x + C_2 \end{cases}$$

The constants C_1, C_2 and C_3 can be obtained by means of the problem boundary condition. Let us assume that the bar has the boundary condition as indicated in Figure 6.44.

According to the boundary conditions, (see Figure 6.44), we can obtain:

$$\begin{cases} u(x=0, y=0)=0 \Rightarrow u(x=0, y=0)=C_1=0 \\ v(x=0, y=0)=0 \Rightarrow u(x=0, y=0)=C_2=0 \\ u(x=0, y)=0 \Rightarrow u(x=0, y=0)=C_3=0 \end{cases}$$

in which the displacement field becomes:

$$\begin{cases} u=\dfrac{F}{EA}x \\ v=\dfrac{-\nu F}{EA}y \end{cases} \tag{6.113}$$

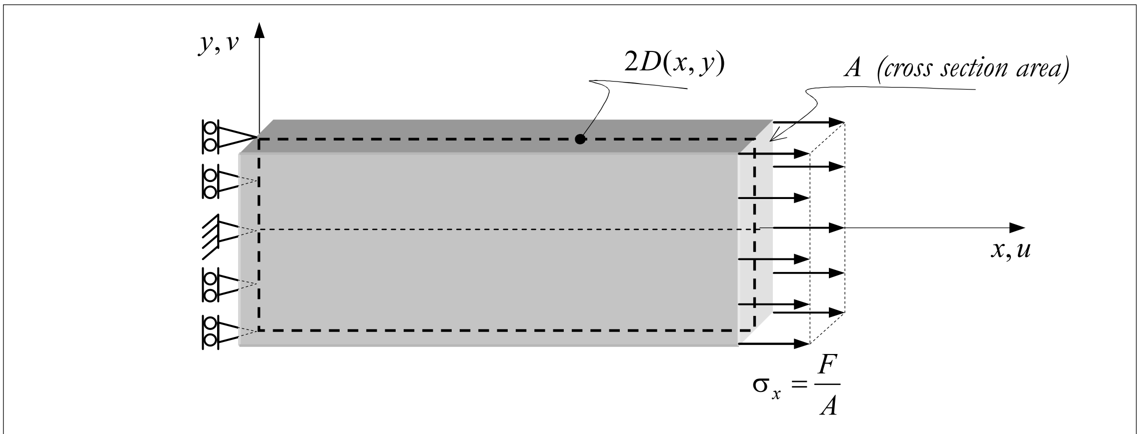


Figure 6.44: Bar subjected to axial force.

Problem 6.35

Obtain the displacement field for a problem (without body force) described in Figure 6.45. As boundary condition consider that at $(x=0, y=0, z=0) \Rightarrow (u=0, v=0, w=0)$.

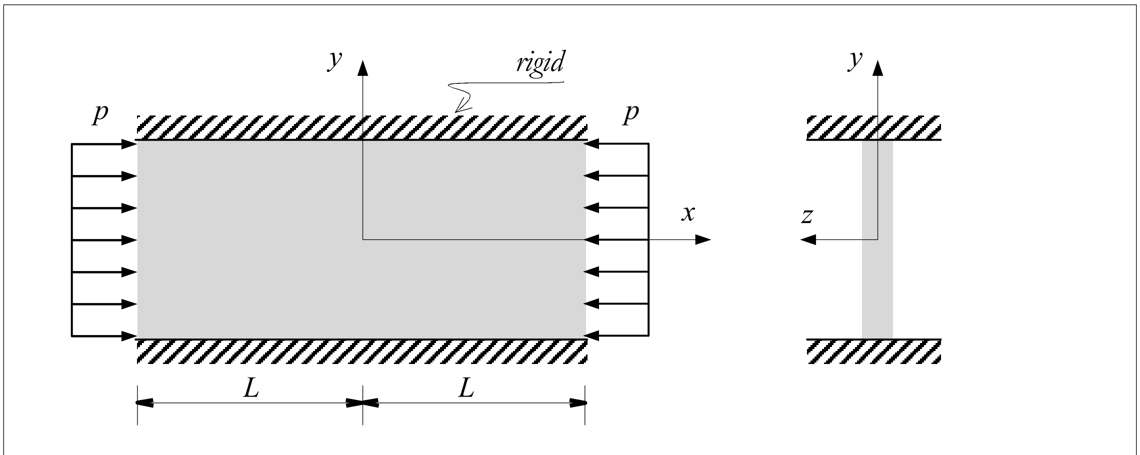


Figure 6.45

Solution:

Let us assume the following Airy stress function:

$$\Phi = K_4x^2 + K_5xy + K_6y^2 \tag{6.114}$$

with that we can obtain the stresses:

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = 2K_6 \quad ; \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2} = 2K_4 \quad ; \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y} = -K_5 = 0 \tag{6.115}$$

Note that $K_5 = 0$ since normal stress only produce normal strain, so, $\gamma_{xy} = 0 \Rightarrow \tau_{xy} = 0$. Note also that according to Figure 6.45 we can conclude that

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = 2K_6 = -p \quad \Rightarrow \quad K_6 = \frac{-p}{2}$$

For this problem we have that $\varepsilon_y = 0$ and by considering the stress by considering the state of plane stress, (see **Problem 6.24**), we can obtain the stress field:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ 0 \\ 0 \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} \varepsilon_x \\ \nu\varepsilon_x \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x \frac{E}{(1-\nu^2)} \\ \nu\varepsilon_x \frac{E}{(1-\nu^2)} \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_x \frac{E}{(1-\nu^2)} \\ \nu\sigma_x \\ 0 \end{bmatrix} = \begin{bmatrix} -p \\ -\nu p \\ 0 \end{bmatrix}$$

By means of the above equation we can obtain the normal strain $\varepsilon_x \frac{-(1-\nu^2)}{E} p$.

Let us check the strain field:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \sigma_x - \nu\sigma_y \\ \sigma_y - \nu\sigma_x \\ 0 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} (-p) - \nu(-\nu p) \\ (-\nu p) - \nu(-p) \\ 0 \end{bmatrix} = \frac{-p}{E} \begin{bmatrix} (1-\nu^2) \\ 0 \\ 0 \end{bmatrix}$$

The normal strain ε_z can be obtained as follows:

$$\varepsilon_z = \frac{-\nu}{E} \mathrm{Tr}(\boldsymbol{\sigma}) = \frac{-\nu(\varepsilon_x + \varepsilon_y)}{(1-\nu)} = \frac{-\nu(1-\nu^2)}{(1-\nu)} \left(\frac{-p}{E} \right) = \frac{\nu(1-\nu^2)}{E(1-\nu)} p = \frac{\nu(1+\nu)}{E} p$$

Taking into account the definition of the normal strain we can obtain:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{-p}{E}(1-\nu^2) \xrightarrow{\text{integrating in x}} u = \frac{-p}{E}(1-\nu^2)x + C_1$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{\nu(1+\nu)}{E} p \xrightarrow{\text{integrating in z}} w = \frac{\nu(1+\nu)p}{E} z + C_2$$

and by applying the boundary condition at $(x=0, y=0, z=0)$ we can conclude that $C_i = 0$, thus

$$\begin{cases} u = \frac{-p}{E}(1-\nu^2)x \\ v = 0 \\ w = \frac{\nu(1+\nu)p}{E} z \end{cases}$$

Problem 6.36

Consider the Airy stress function:

$$\Phi = K_{10} y^3 \tag{6.116}$$

where K_{10} is a constant. What is the problem governed by the Airy stress function (6.116)? Obtain the stress, strain and displacement fields. Consider the state of plane stress.

Solution:

If we are not considering the body forces the stress field (6.105) becomes:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = 6K_{10} y \quad ; \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad ; \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = 0 \tag{6.117}$$

Note that the stress field $\sigma_x = \sigma_x(y)$ depends only on y. For a given cross section of the bar we have:

Resultant force on the cross-section:

$$F = \int_A \sigma_x dA = \int_{z=\frac{-b}{2}}^{z=\frac{b}{2}} \int_{y=\frac{-a}{2}}^{y=\frac{a}{2}} 6K_{10} y dy dz = 0$$

Bending moment on the cross-section:

$$M = \int_A y \sigma_x dA = \int_A 6K_{10} y^2 dA = 6K_{10} \int_A y^2 dA = 6K_{10} I_z$$

where $I_z = \int_A y^2 dA$ is the moment of inertia of the cross-sectional area about the z axis. Note that, this is the case of pure bending, (see Figure 6.46). We can also obtain that $M = 6K_{10} I_z \quad \Rightarrow \quad K_{10} = \frac{M}{6I_z}$.

Let us analyze the sign of M. According to our sign convention the moment is positive if the moment vector has the same sense as the axis, e.g. the vector M_z is positive if it has the same sense as the z-axis, (see Figure 6.46). Note also that according to this sign convention we have $\sigma_x < 0$ for values of $y > 0$, so, for a positive value of M_z we have $K_{10} < 0$:

$$K_{10} = \frac{-M_z}{6I_z}$$

Stress field (Pure bending):

$$\sigma_x = 6K_{10} y = \frac{-M_z}{I_z} y \quad ; \quad \sigma_y = 0 \quad ; \quad \tau_{xy} = 0$$

Strain field (pure bending):

$$
\begin{cases}
\varepsilon_x = \dfrac{\sigma_x}{E} = \dfrac{6K_{10}y}{E} = \dfrac{-M_z}{EI_z}y \\
\varepsilon_y = -\nu\dfrac{\sigma_x}{E} = -\nu\dfrac{6K_{10}y}{E} = \dfrac{\nu M_z}{EI_z}y \\
\gamma_{xy} = 0
\end{cases}
$$

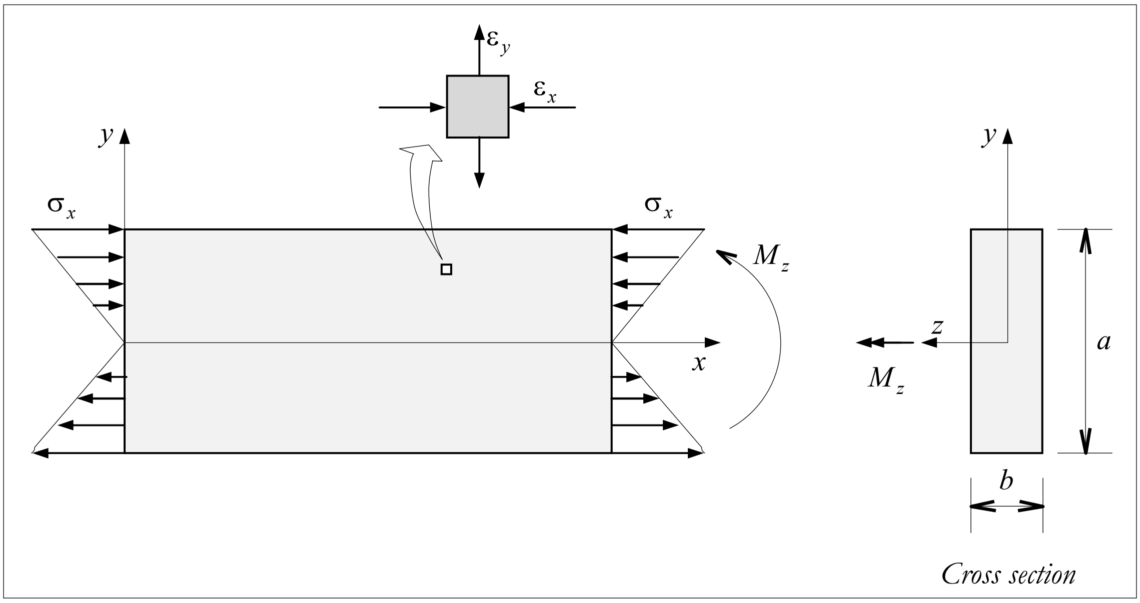


Figure 6.46: Beam under pure bending.

Displacement field (Pure bending):

$$
\begin{cases}
\dfrac{\partial u}{\partial x} = \varepsilon_x = \dfrac{-M_z}{EI_z}y \xrightarrow{\text{integrating in } x} u(x,y) = \dfrac{-M_z}{EI_z}yx + f_1(y) + C_1 \\
\dfrac{\partial v}{\partial y} = \varepsilon_y = \dfrac{\nu M_z}{EI_z}y \xrightarrow{\text{integrating in } y} v(x,y) = \dfrac{\nu M_z}{2EI_z}y^2 + f_2(x) + C_2 \\
\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = \gamma_{xy} = 0
\end{cases}
$$

Taking into account that:

$$
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \gamma_{xy} = 0
$$

$$
\Rightarrow \frac{\partial}{\partial x}\left(\frac{\nu M_z}{2EI_z}y^2 + f_2(x) + C_2\right) + \frac{\partial}{\partial y}\left(\frac{-M_z}{EI_z}yx + f_1(y) + C_1\right) = 0
$$

$$
\Rightarrow \frac{\partial f_2(x)}{\partial x} + \frac{-M_z}{EI_z}x + \frac{\partial f_1(y)}{\partial y} = 0
$$

$$
\Rightarrow \frac{\partial f_2(x)}{\partial x} + \frac{-M_z}{EI_z}x = -\frac{\partial f_1(y)}{\partial y}
$$

Similarly to the previous example we can conclude that:

$$\frac{\partial f_2(x)}{\partial x} - \frac{M_z}{EI_z} x = C_3 \qquad \textit{and} \qquad \frac{\partial f_1(y)}{\partial y} = -C_3$$

By integrate the above equations we can obtain:

$$\frac{\partial f_2(x)}{\partial x} - \frac{M_z}{EI_z} x = C_3 \qquad \Rightarrow \qquad f_2(x) = C_3 x + \frac{M_z}{2EI_z} x^2$$

$$\frac{\partial f_1(y)}{\partial y} = -C_3 \qquad \Rightarrow \qquad f_1(y) = -C_3 y$$

with that the displacement field becomes:

$$\begin{cases} u = \dfrac{-M_z}{EI_z} yx + f_1(y) + C_1 = \dfrac{-M_z}{EI_z} yx - C_3 y + C_1 \\ v = \dfrac{\nu M_z}{2EI_z} y^2 + f_2(x) + C_2 = \dfrac{\nu M_z}{2EI_z} y^2 + C_3 x + \dfrac{M_z}{2EI_z} x^2 + C_2 = \dfrac{M_z}{2EI_z}(\nu y^2 + x^2) + C_3 x + C_2 \end{cases}$$

where the constants C_1, C_2 and C_3 can be obtained by means of the problem boundary conditions. Let us assume that the bar has the boundary condition as indicated in Figure 6.47, in which one end of the beam has a fixed support (clamped or cantileved) and the other end is free.

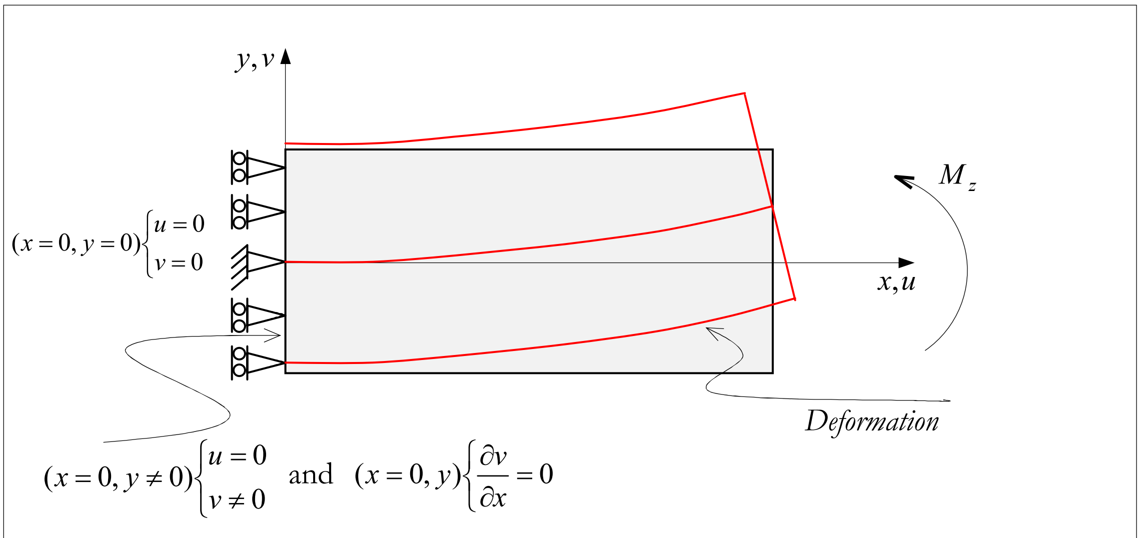


Figure 6.47: Fixed-free beam (boundary conditions).

With these conditions we can obtain:

$$\begin{cases} u(x=0, y=0) = 0 \Rightarrow u(x=0, y=0) = C_1 = 0 \\ v(x=0, y=0) = 0 \Rightarrow u(x=0, y=0) = C_2 = 0 \end{cases}$$

$$v(x,y) = \frac{M_z}{2EI_z}(\nu y^2 + x^2) + C_3 x + C_2 \qquad \Rightarrow \qquad \frac{\partial v(x,y)}{\partial x} = \frac{M_z}{EI_z} x + C_3$$

$$\frac{\partial v(x=0, y=0)}{\partial x} = 0 = C_3$$

Note also that $\dfrac{\partial^2 v(x,y)}{\partial x^2} = \dfrac{M_z}{EI_z}$, the second derivative of the deflection $v(x,y)$ is positive.

Then, the displacement field becomes:

$$\begin{cases} u(x,y) = \dfrac{-M_z}{EI_z} xy \\ v(x,y) = \dfrac{M_z}{2EI_z}(\nu y^2 + x^2) \end{cases}$$

The neutral line corresponds to the line in which $\sigma_x = \frac{-M_z}{I_z} y = 0$. And the defection of the neutral line ($y = 0$) is given by $v = \frac{M_z}{2EI_z} x^2$.

Problem 6.37

Obtain the stress field for the problem, without body force, which is represented by the Airy stress function:

$$\Phi = K_5 xy + K_{10} y^3 + K_{14} xy^3 \tag{6.118}$$

As boundary condition (B.C.) consider that

at $(y = \pm \frac{a}{2}) \Rightarrow \tau_{xy} = 0$

at $x \Rightarrow P = \int_A \tau_{xy} dA$, where $A = ab$ is the area of the rectangular cross section

at $x = 0 \Rightarrow M = -PL$ (Bending moment)

Solution:

For this problem we have

$$\frac{\partial \Phi}{\partial x} = K_5 y + K_{14} y^3 \quad ; \quad \frac{\partial \Phi}{\partial y} = K_5 x + 3K_{10} y^2 + 3K_{14} xy^2$$

Then,

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = 6K_{10} y + 6K_{14} xy \quad ; \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad ; \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -(K_5 + 3K_{14} y^2) \tag{6.119}$$

Applying the boundary condition $(y = \pm \frac{a}{2}) \Rightarrow \tau_{xy} = 0$, we can conclude that

$$\tau_{xy}(y = \pm \frac{a}{2}) = -K_5 - 3K_{14} \frac{a^2}{4} = 0 \quad \Rightarrow \quad K_5 = -3K_{14} \frac{a^2}{4}$$

With that the tangential stress becomes

$$\tau_{xy} = -\left(-3K_{14} \frac{a^2}{4}\right) - 3K_{14} y^2 = 3K_{14}\left(\frac{a^2}{4} - y^2\right)$$

By applying the boundary condition at $x \Rightarrow P = \int_A \tau_{xy} dA$, we can obtain:

$$P = \int_A \tau_{xy} dA = \int_A 3K_{14}\left(\frac{a^2}{4} - y^2\right) dA = 3K_{14}\left(\frac{a^2}{4}\int_A dA - \int_A y^2 dA\right) = 3K_{14}\left(\frac{a^2}{4}A - I_z\right)$$

$$\Rightarrow P = 3K_{14}\left(\frac{a^2}{4}ab - I_z\right) = 3K_{14}\left(\frac{3ba^3}{12} - I_z\right) = 3K_{14}(3I_z - I_z) = 6K_{14}I_z$$

$$\Rightarrow K_{14} = \frac{P}{6I_z}$$

Then, the stresses given by (6.119) become

$$\sigma_x = 6K_{10}y + \frac{P}{I_z}xy \quad ; \quad \sigma_y = 0 \quad ; \quad \tau_{xy} = \frac{P}{2I_z}\left(\frac{a^2}{4} - y^2\right) \tag{6.120}$$

The bending moment M acting at the cross-section can be obtained as follows:

$$M(x) = \int_A \sigma_x y dA = \int_A \left(6K_{10}y + \frac{P}{I_z}xy\right) y dA = 6K_{10}\underbrace{\int_A y^2 dA}_{=I_z} + \frac{P}{I_z}x\underbrace{\int_A y^2 dA}_{=I_z}$$

$$\Rightarrow M(x) = 6K_{10}I_z + \frac{P}{I_z}xI_z$$

$$\Rightarrow M(x) = 6K_{10}I_z + Px$$

The constant K_{10} can be obtained by means of the B.C.: $x = 0 \Rightarrow \; M = -PL$:

$$M(x=0) = 6K_{10}I_z = -PL \quad \Rightarrow \quad K_{10} = \frac{-PL}{6I_z}$$

Then, the stresses given by (6.119) become

$$\sigma_x = \frac{Py}{I_z}(x - L) \quad ; \quad \sigma_y = 0 \quad ; \quad \tau_{xy} = \frac{P}{2I_z}\left(\frac{a^2}{4} - y^2\right) \tag{6.121}$$

The strain-stress relationship ($\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$) for the state of plane stress is given by:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$= \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \frac{Py}{I_z}(x-L) \\ 0 \\ \frac{P}{2I_z}\left(\frac{a^2}{4} - y^2\right) \end{bmatrix} = \begin{bmatrix} \frac{Py}{EI_z}(x-L) \\ -\nu\frac{Py}{EI_z}(x-L) \\ \frac{P(1+\nu)}{EI_z}\left(\frac{a^2}{4} - y^2\right) \end{bmatrix} \tag{6.122}$$

By considering that

$$\kappa_z = \frac{-P}{EI_z}(x - L) \quad ; \quad \frac{\partial \kappa_z}{\partial x} \equiv \kappa_{z,x} = \frac{-P}{EI_z}$$

the equation in (6.122) can be rewritten as follows

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{Py}{EI_z}(x-L) \\ -\nu\dfrac{Py}{EI_z}(x-L) \\ \dfrac{P(1+\nu)}{EI_z}\left(\dfrac{a^2}{4}-y^2\right) \end{bmatrix} = \begin{bmatrix} -\kappa_z y \\ \nu\kappa_z y \\ -(1+\nu)\kappa_{z,x}\left(\dfrac{a^2}{4}-y^2\right) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} -\kappa_z y & \dfrac{(1+\nu)\kappa_{z,x}}{2}\left(y^2-\dfrac{a^2}{4}\right) \\ \dfrac{(1+\nu)\kappa_{z,x}}{2}\left(y^2-\dfrac{a^2}{4}\right) & \nu\kappa_z y \end{bmatrix}$$

Note that this problem was already established in **Problem 6.33**.

Problem 6.38

Obtain the stress field for a problem (without body force) for the problem represented in Figure 6.48.

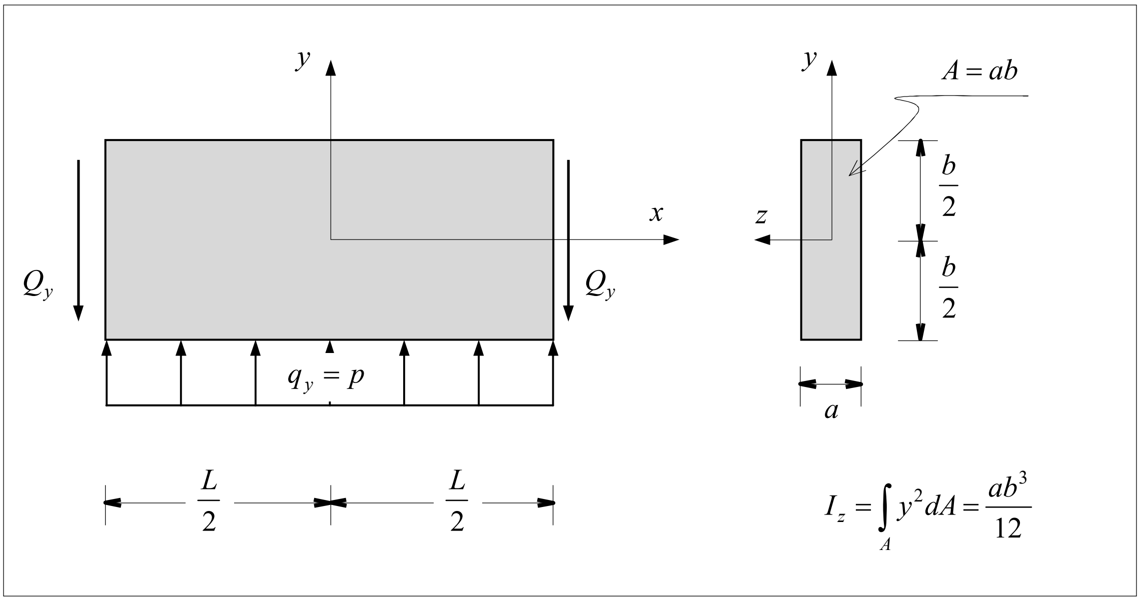


Figure 6.48

Consider the following Airy stress function, (Sechler (1952)):

$$\Phi = K_4 x^2 + K_8 x^2 y + K_{10} y^3 + K_{11} x^4 + K_{17} x^4 y + K_{19} x^2 y^3 + K_{21} y^5 \tag{6.123}$$

and consider that $\sigma_x = \sigma_x(y)$ is not a function of x.

As boundary condition (B.C.) consider that

at $(y = \frac{b}{2}) \Rightarrow \sigma_y = \tau_{xy} = 0$

at $(y=\frac{-b}{2}) \Rightarrow \ \sigma_y = -p \quad ; \quad \tau_{xy}=0$

at $(x=\pm\frac{L}{2}) \Rightarrow \quad \int_A \tau_{xy} dA = Q_y = \mp \frac{pL}{2} \quad ; \quad \int_A \sigma_x dA = F_x = 0 \quad ; \quad \int_A \sigma_x y dA = M_z = 0$

Solution:

The following must hold at any point of the beam:

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2 \partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0 \quad ; \quad \nabla^4_{\vec{x}}\Phi = 0 \quad ; \quad \Phi_{,iijj} = 0 \quad (i,j=1,2) \tag{6.124}$$

Then, let us take the derivatives:

$$\Phi = K_4 x^2 + K_8 x^2 y + K_{10} y^3 + K_{11} x^4 + K_{17} x^4 y + K_{19} x^2 y^3 + K_{21} y^5$$

$$\frac{\partial\Phi}{\partial x} = 2K_4 x + 2K_8 xy + 4K_{11} x^3 + 4K_{17} x^3 y + 2K_{19} xy^3$$

$$\frac{\partial^2\Phi}{\partial x^2} = 2K_4 + 2K_8 y + 12K_{11} x^2 + 12K_{17} x^2 y + 2K_{19} y^3$$

$$\frac{\partial^3\Phi}{\partial x^3} = 24K_{11} x + 24K_{17} xy$$

$$\frac{\partial^4\Phi}{\partial x^4} = 24K_{11} + 24K_{17} y$$

$$\Phi = K_4 x^2 + K_8 x^2 y + K_{10} y^3 + K_{11} x^4 + K_{17} x^4 y + K_{19} x^2 y^3 + K_{21} y^5$$

$$\frac{\partial\Phi}{\partial y} = K_8 x^2 + 3K_{10} y^2 + K_{17} x^4 + 3K_{19} x^2 y^2 + 5K_{21} y^4$$

$$\frac{\partial^2\Phi}{\partial y^2} = 6K_{10} y + 6K_{19} x^2 y + 20K_{21} y^3$$

$$\frac{\partial^3\Phi}{\partial y^3} = 6K_{10} + 6K_{19} x^2 + 60K_{21} y^2$$

$$\frac{\partial^4\Phi}{\partial y^4} = 120K_{21} y$$

$$\frac{\partial}{\partial y^2}\left(\frac{\partial^2\Phi}{\partial x^2}\right) = \frac{\partial^4\Phi}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y^2}(2K_4 + 2K_8 y + 12K_{11} x^2 + 12K_{17} x^2 y + 2K_{19} y^3) = 12K_{19} y$$

$$\frac{\partial}{\partial x^2}\left(\frac{\partial^2\Phi}{\partial y^2}\right) = \frac{\partial^4\Phi}{\partial x^2 \partial y^2} = \frac{\partial}{\partial x^2}(6K_{10} y + 6K_{19} x^2 y + 20K_{21} y^3) = 12K_{19} y = 12K_{19} y$$

Then, the equation in (6.124) becomes:

$$\begin{aligned}&\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=0\\&\Rightarrow 24K_{11}+24K_{17}y+2(12K_{19}y)+120K_{21}y=0\end{aligned} \tag{6.125}$$

Note that the above equation is only fulfilled if and only if $K_{11}=0$, then

$$\begin{aligned}&\Phi=K_4x^2+K_8x^2y+K_{10}y^3+K_{11}x^4+K_{17}x^4y+K_{19}x^2y^3+K_{21}y^5\\&\Rightarrow\Phi=K_4x^2+K_8x^2y+K_{10}y^3+K_{17}x^4y+K_{19}x^2y^3+K_{21}y^5\end{aligned} \tag{6.126}$$

From equation (6.125) we can obtain the following equation:

$$\begin{aligned}&24K_{17}y+2(12K_{19}y)+120K_{21}y=0\\&\Rightarrow y(24K_{17}+24K_{19}+120K_{21})=0\\&\Rightarrow 24K_{17}+24K_{19}+120K_{21}=0\quad\Rightarrow\quad K_{17}+K_{19}+5K_{21}=0\end{aligned} \tag{6.127}$$

The stress field can be expressed as follows:

$$\begin{aligned}&\sigma_x=\frac{\partial^2\Phi}{\partial y^2}=6K_{10}y+6K_{19}x^2y+20K_{21}y^3\\&\sigma_y=\frac{\partial^2\Phi}{\partial x^2}=2K_4+2K_8y+12\underbrace{K_{11}}_{=0}x^2+12K_{17}x^2y+2K_{19}y^3=2K_4+2K_8y+12K_{17}x^2y+2K_{19}y^3\\&\tau_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}=\frac{-\partial}{\partial x}(K_8x^2+3K_{10}y^2+K_{17}x^4+3K_{19}x^2y^2+5K_{21}y^4)=-2K_8x-4K_{17}x^3-6K_{19}xy^2\end{aligned} \tag{6.128}$$

Note that σ_y is not a function of x, thus $K_{17}=0$, then:

$$\begin{cases}\sigma_x=\dfrac{\partial^2\Phi}{\partial y^2}=6K_{10}y+6K_{19}x^2y+20K_{21}y^3\\[2mm]\sigma_y=\dfrac{\partial^2\Phi}{\partial x^2}=2K_4+2K_8y+2K_{19}y^3\\[2mm]\tau_{xy}=-\dfrac{\partial^2\Phi}{\partial x\partial y}=-2K_8x-6K_{19}xy^2\end{cases} \tag{6.129}$$

Taking into account that $K_{17}=0$, the equation (6.127) becomes:

$$K_{17}+K_{19}+5K_{21}=0\quad\Rightarrow\quad K_{19}+5K_{21}=0\quad\Rightarrow\quad K_{19}=-5K_{21} \tag{6.130}$$

With that the stress field (6.129) can be rewritten as follows:

$$\begin{cases}\sigma_x=\dfrac{\partial^2\Phi}{\partial y^2}=6K_{10}y+6(-5K_{21})x^2y+20K_{21}y^3=6K_{10}y+(20y^3-30x^2y)K_{21}\\[2mm]\sigma_y=\dfrac{\partial^2\Phi}{\partial x^2}=2K_4+2K_8y+2(-5K_{21})y^3=2K_4+2K_8y-10K_{21}y^3\\[2mm]\tau_{xy}=-\dfrac{\partial^2\Phi}{\partial x\partial y}=-2K_8x-6(-5K_{21})xy^2=-2K_8x+30K_{21}xy^2\end{cases} \tag{6.131}$$

Applying the Boundary Conditions

1) at $(y=\frac{b}{2}) \Rightarrow \sigma_y = \tau_{xy} = 0$

$$\tau_{xy}(\frac{b}{2}) = -2K_8 x + 30K_{21} x y^2 = -2K_8 x + 30K_{21} x \left(\frac{b}{2}\right)^2 = 0 \quad \Rightarrow \quad -2K_8 = -30K_{21}\frac{b^2}{4}$$
$$\Rightarrow \quad K_{21} = \frac{4}{15b^2}K_8 \quad \Leftrightarrow \quad K_8 = \frac{15b^2}{4}K_{21} \tag{6.132}$$

and

$$\sigma_y(y) = 2K_4 + 2K_8 y - 10K_{21}y^3 = 2K_4 + 2\frac{15b^2}{4}K_{21}y - 10K_{21}y^3$$
$$\Rightarrow \sigma_y(y) = 2K_4 + \left(\frac{15b^2}{2}y - 10y^3\right)K_{21} \tag{6.133}$$

then, when $(y=\frac{b}{2})$

$$\sigma_y(\frac{b}{2}) = 2K_4 + \left(\frac{15b^2}{2}\left(\frac{b}{2}\right) - 10\left(\frac{b}{2}\right)^3\right)K_{21} = 0$$
$$\Rightarrow 2K_4 + \left(\frac{15b^3}{4} - \frac{10b^3}{8}\right)K_{21} = 0 \quad \Rightarrow \quad K_4 + \left(\frac{10b^3}{8}\right)K_{21} = 0 \tag{6.134}$$
$$\Rightarrow K_4 = \frac{-5b^3}{4}K_{21} \quad \Rightarrow \quad K_{21} = \frac{-4}{5b^3}K_4$$

Then, the equation in (6.133):

$$\sigma_y(y) = 2K_4 + \left(\frac{15b^2}{2}y - 10y^3\right)K_{21} \quad \Rightarrow \quad \sigma_y(y) = 2\frac{-5b^3}{4}K_{21} + \left(\frac{15b^2}{2}y - 10y^3\right)K_{21}$$
$$\Rightarrow \sigma_y(y) = \left(\frac{-5b^3}{2} + \frac{15b^2}{2}y - 10y^3\right)K_{21} \tag{6.135}$$

2) at $(y=\frac{-b}{2}) \Rightarrow \sigma_y = -p$. Then:

$$\sigma_y(\frac{-b}{2}) = \left(\frac{-5b^3}{2} + \frac{15b^2}{2}y - 10y^3\right)K_{21} = \left(\frac{-5b^3}{2} + \frac{15b^2}{2}\left(\frac{-b}{2}\right) - 10\left(\frac{-b}{2}\right)^3\right)K_{21} = -p$$
$$\Rightarrow \left(\frac{-10b^3}{4} - \frac{15b^3}{4} + \frac{5b^3}{4}\right)K_{21} = -p \quad \Rightarrow \quad K_{21} = \frac{p}{5b^3} \tag{6.136}$$

Then

$$K_{21} = \frac{p}{5b^3}$$
$$K_8 = \frac{15b^2}{4}K_{21} = \frac{15b^2}{4}\frac{p}{5b^3} = \frac{3p}{4b} \tag{6.137}$$
$$K_4 = \frac{-5b^3}{4}K_{21} = \frac{-5b^3}{4}\frac{p}{5b^3} = \frac{-p}{4}$$

The stress $\sigma_y(y)$, (see equation (6.135)), becomes

$$\sigma_y(y)=\left(\frac{-5b^3}{2}+\frac{15b^2}{2}y-10y^3\right)K_{21} \quad\Rightarrow\quad \sigma_y(y)=\left(\frac{-5b^3}{2}+\frac{15b^2}{2}y-10y^3\right)\frac{p}{5b^3}$$
$$\Rightarrow \sigma_y(y)=\frac{-p}{2}+\frac{3p}{2b}y-\frac{2p}{b^3}y^3 \tag{6.138}$$

and

$$\begin{cases}\sigma_x=6K_{10}y+(20y^3-30x^2y)K_{21}=6K_{10}y+(20y^3-30x^2y)\dfrac{p}{5b^3}=6K_{10}y+(4y^3-6x^2y)\dfrac{p}{b^3}\\ \sigma_y=\dfrac{-p}{2}+\dfrac{3p}{2b}y-\dfrac{2p}{b^3}y^3\\ \tau_{xy}=-2K_8x+30K_{21}xy^2=-2\left(\dfrac{3p}{4b}\right)x+30\left(\dfrac{p}{5b^3}\right)xy^2=\dfrac{-3p}{2b}x+\dfrac{6p}{b^3}xy^2\end{cases} \tag{6.139}$$

To determine the coefficient K_{10} we must apply the boundary condition:

at $(x=\pm\frac{L}{2})\Rightarrow \int_A \sigma_x y dA = M_z = 0$

$$M_z=\int_A\sigma_x ydA=\int_A\left\{6K_{10}y+\left[4y^3-6\left(\frac{L}{2}\right)^2y\right]\frac{p}{b^3}\right\}ydA=0$$

$$\Rightarrow 6K_{10}\int_A y^2dA+\frac{4p}{b^3}\int_A y^4dA-\frac{6p}{b^3}\frac{L^2}{4}\int_A y^2dA=0 \quad\therefore\quad \left(\int_A y^4dA=\int_{\frac{-a}{2}}^{\frac{a}{2}}\int_{\frac{-b}{2}}^{\frac{b}{2}}y^4dydz=\frac{ab^5}{80}\right)$$

$$\Rightarrow 6K_{10}I_z+\frac{4p}{b^3}\left(\frac{ab^5}{80}\right)-\frac{3pL^2}{2b^3}I_z=0 \quad\Rightarrow\quad 6K_{10}I_z+\frac{12p}{20b}\left(\frac{ab^3}{12}\right)-\frac{3pL^2}{2b^3}I_z=0$$

$$\Rightarrow K_{10}+\frac{p}{10b}-\frac{pL^2}{4b^3}=0 \quad\Rightarrow\quad K_{10}=p\left(\frac{L^2}{4b^3}-\frac{1}{10b}\right)$$

Then, the stress field becomes:

$$\begin{cases}\sigma_x=6K_{10}y+(4y^3-6x^2y)\dfrac{p}{b^3}=3p\left(\dfrac{L^2}{2b^3}-\dfrac{1}{5b}\right)y+(4y^3-6x^2y)\dfrac{p}{b^3}\\ \sigma_y=\dfrac{-p}{2}+\dfrac{3p}{2b}y-\dfrac{2p}{b^3}y^3\\ \tau_{xy}=\dfrac{-3p}{2b}x+\dfrac{6p}{b^3}xy^2\end{cases} \tag{6.140}$$

Note that the following equations is true

$$I_z=\frac{ab^3}{12} \quad\Rightarrow\quad b^3=\frac{12I_z}{a} \quad\Rightarrow\quad b^2=\frac{12I_z}{ab} \quad\Rightarrow\quad b=\frac{12I_z}{ab^2}$$

Then we can also express the stress field as follows

$$\sigma_x = 3p\left(\frac{L^2}{2\frac{12I_z}{a}} - \frac{1}{5\frac{12I_z}{ab^2}}\right)y + (4y^3 - 6x^2y)\frac{p}{\frac{12I_z}{a}}$$

$$\Rightarrow \sigma_x = 3p\left(\frac{aL^2}{24I_z} - \frac{ab^2}{60I_z}\right)y + (4y^3 - 6x^2y)\frac{ap}{12I_z}$$

$$\Rightarrow \sigma_x = \frac{pa}{I_z}\left(\frac{L^2}{8} - \frac{b^2}{20}\right)y + \frac{ap}{12I_z}(4y^3 - 6x^2y)$$

$$\Rightarrow \sigma_x = \frac{pa}{160I_z}\left(20L^2 - 8b^2\right)y + \frac{ap}{12I_z}(4y^3 - 6x^2y)$$

We can restructure the above equation to obtain:

$$\sigma_x = \frac{pa}{160I_z}\left(20L^2y\right) - \frac{pa}{160I_z}\left(8b^2y\right) + \frac{ap}{12I_z}(4y^3) - \frac{ap}{12I_z}(6x^2y)$$

$$\Rightarrow \sigma_x = \frac{pa}{8I_z}\left(L^2 - 4x^2\right)y + \frac{pa}{60I_z}\left(20y^3 - 3b^2y\right)$$

$$\sigma_y = \frac{-p}{2} + \frac{3p}{2b}y - \frac{2p}{b^3}y^3 = \frac{-p}{2} + \frac{3p}{2\frac{12I_z}{ab^2}}y - \frac{2p}{\frac{12I_z}{a}}y^3$$

$$\Rightarrow \sigma_y = \frac{pa}{24I_z}\left(-b^3 + 3b^2y - 4y^3\right)$$

$$\tau_{xy} = \frac{-3p}{2b}x + \frac{6p}{b^3}xy^2 = \frac{-3p}{2\frac{12I_z}{ab^2}}x + \frac{6p}{\frac{12I_z}{a}}xy^2$$

$$\Rightarrow \tau_{xy} = \frac{pa}{8I_z}\left(4y^2 - b^2\right)x$$

Then we can also express the stress field as follows, (Sechler (1952)):

$$\begin{cases} \sigma_x = \dfrac{pa}{8I_z}\left(L^2 - 4x^2\right)y + \dfrac{pa}{60I_z}\left(20y^3 - 3b^2y\right) \\ \sigma_y = \dfrac{pa}{24I_z}\left(-b^3 + 3b^2y - 4y^3\right) \\ \tau_{xy} = \dfrac{pa}{8I_z}(4y^2 - b^2)x \end{cases} \tag{6.141}$$

Stress Function References

LAIER, J.E. & BARREIRO, J.C. (1983). *Complementos de Resistências dos Materiais.* Publicação 073/92, São Carlos, Universidade de São Paulo, Escola de Engenharia de São Carlos.

SECHLER, E. (1952). *Elasticity in Engineering.* John Willey & Sons, Inc. New York.

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6.3 Introduction to Finite Element for Linear Elasticity Problems

Problem 6.39

Let us consider a two-dimensional problem such as the infinitesimal strain field, $\{\boldsymbol{\varepsilon}(x,y)\}$, into the domain (Ω) is given by:

$$\{\boldsymbol{\varepsilon}(x,y)\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\left[\begin{array}{cc|cc|cc}\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x}\end{array}\right]\begin{Bmatrix}u^{(1)}\\ v^{(1)}\\ \hline u^{(2)}\\ v^{(2)}\\ \hline u^{(3)}\\ v^{(3)}\end{Bmatrix} \tag{6.142}$$

where $u^{(i)}, v^{(i)}$ are the displacements at the nodes $(i=1,2,3)$ associated with the directions x and y respectively, and $N_1=N_1(x,y)$, $N_2=N_2(x,y)$ and $N_3=N_3(x,y)$ are continuous functions. Check whether the compatibility equation is satisfied or not. Express the displacement fields $(u(x,y), v(x,y))$ in terms of nodal displacements $\{\boldsymbol{u}^{(e)}\}$. Obtain also the stress field for plane stress and plane strain states.

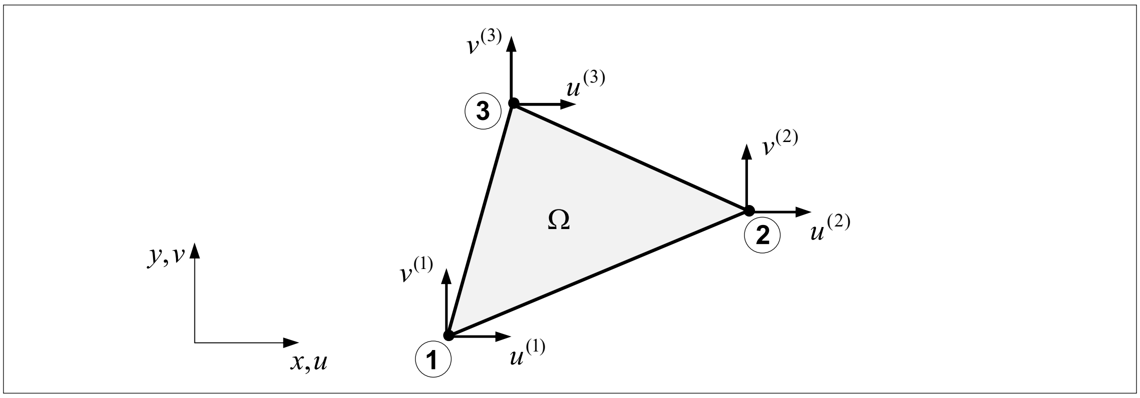


Figure 6.49: Domain Ω.

Solution:

For two-dimensional problem the compatibility equations, (see **Problem 5.11** – NOTE 3), reduce to

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}-2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}=0 \xrightarrow{\text{Engineering notation}} \frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=0 \tag{6.143}$$

By means of equation (6.142) we can obtain:

$$\frac{\partial^2\varepsilon_x}{\partial y^2}=\frac{\partial^2}{\partial y^2}\left(\frac{\partial N_1}{\partial x}u^{(1)}+\frac{\partial N_2}{\partial x}u^{(2)}+\frac{\partial N_3}{\partial x}u^{(3)}\right)=\frac{\partial^3 N_1}{\partial y^2\partial x}u^{(1)}+\frac{\partial^3 N_2}{\partial y^2\partial x}u^{(2)}+\frac{\partial^3 N_3}{\partial y^2\partial x}u^{(3)}$$

$$\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2}{\partial x^2}\left(\frac{\partial N_1}{\partial y}v^{(1)}+\frac{\partial N_2}{\partial y}v^{(2)}+\frac{\partial N_3}{\partial y}v^{(3)}\right)=\frac{\partial^3 N_1}{\partial y\partial x^2}v^{(1)}+\frac{\partial^3 N_2}{\partial y\partial x^2}v^{(2)}+\frac{\partial^3 N_3}{\partial y\partial x^2}v^{(3)}$$

$$
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial N_1}{\partial y} u^{(1)} + \frac{\partial N_1}{\partial x} v^{(1)} + \frac{\partial N_2}{\partial y} u^{(2)} + \frac{\partial N_2}{\partial x} v^{(2)} + \frac{\partial N_3}{\partial y} u^{(3)} + \frac{\partial N_3}{\partial x} v^{(3)} \right)
$$

$$
= \frac{\partial^3 N_1}{\partial y^2 \partial x} u^{(1)} + \frac{\partial^3 N_2}{\partial y^2 \partial x} u^{(2)} + \frac{\partial^3 N_3}{\partial y^2 \partial x} u^{(3)} + \frac{\partial^3 N_1}{\partial x^2 \partial y} v^{(1)} + \frac{\partial^3 N_2}{\partial x^2 \partial y} v^{(2)} + \frac{\partial^3 N_3}{\partial x^2 \partial y} v^{(3)}
$$

Then, by substituting the above derivatives into the equation in (6.143) we can conclude that the compatibility equation is satisfied.

The displacement field can be obtained by means of the normal strain definition, i.e.:

$$
\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u^{(1)} + \frac{\partial N_2}{\partial x} u^{(2)} + \frac{\partial N_3}{\partial x} u^{(3)} = \frac{\partial}{\partial x}\left(N_1 u^{(1)} + N_2 u^{(2)} + N_3 u^{(3)} \right) = \frac{\partial u}{\partial x}
$$

$$
\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v^{(1)} + \frac{\partial N_2}{\partial y} v^{(2)} + \frac{\partial N_3}{\partial y} v^{(3)} = \frac{\partial}{\partial y}\left(N_1 v^{(1)} + N_2 v^{(2)} + N_3 v^{(3)} \right) = \frac{\partial v}{\partial y}
$$

Thus,

$$
\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hline u^{(2)} \\ v^{(2)} \\ \hline u^{(3)} \\ v^{(3)} \end{Bmatrix} \quad \Rightarrow \quad \{\boldsymbol{u}(x,y)\} = [\boldsymbol{N}(x,y)]\{\boldsymbol{u}^{(e)}\}
$$

Note also that

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hline u^{(2)} \\ v^{(2)} \\ \hline u^{(3)} \\ v^{(3)} \end{Bmatrix}
$$

or in compact form:

$$
\{\boldsymbol{\varepsilon}(x,y)\} = [\boldsymbol{L}^{(1)}]\{\boldsymbol{u}(x,y)\} = [\boldsymbol{L}^{(1)}][\boldsymbol{N}(x,y)]\{\boldsymbol{u}^{(e)}\} = [\boldsymbol{B}(x,y)]\{\boldsymbol{u}^{(e)}\} \tag{6.144}
$$

The stress-strain relationship for two-dimensional problem, (see **Problem 6.25**)), can be expressed as follows:

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{\overline{E}}{1-\overline{\nu}^2} \begin{bmatrix} 1 & \overline{\nu} & 0 \\ \overline{\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-\overline{\nu}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad \therefore \quad \begin{cases} \text{if state of plane stress} & \begin{cases} \overline{E} = E \\ \overline{\nu} = \nu \end{cases} \\ \text{if state of plane strain} & \begin{cases} \overline{E} = \dfrac{E}{(1-\nu^2)} \\ \overline{\nu} = \dfrac{\nu}{(1-\nu)} \end{cases} \end{cases} \tag{6.145}
$$

$$
\{\boldsymbol{\sigma}(x,y)\} = [\mathcal{C}^{(2D)}]\{\boldsymbol{\varepsilon}(x,y)\}
$$

Then, if we consider the relationship between the strain field $\{\boldsymbol{\varepsilon}(x,y)\}$ and the nodal displacement $\{\boldsymbol{u}^{(e)}\}$, (see equation (6.144)), we can express the stress in terms of nodal displacement:

$$
\{\boldsymbol{\sigma}(x,y)\} = [\mathcal{C}^{(2D)}]\{\boldsymbol{\varepsilon}(x,y)\} \quad \Rightarrow \quad \{\boldsymbol{\sigma}(x,y)\} = [\mathcal{C}^{(2D)}][\boldsymbol{B}(x,y)]\{\boldsymbol{u}^{(e)}\}
$$

Problem 6.40

Consider a linear elastic problem and let us adopt the linear approximation for the displacement field $\{\boldsymbol{u}(x,y)\}$ for two-dimensional problem, (see Figure 6.50):

$$\{\boldsymbol{u}(x,y)\} = \begin{cases} u(x,y) = a_1 + a_2 x + a_3 y \\ v(x,y) = a_4 + a_5 x + a_6 y \end{cases} \tag{6.146}$$

where a_k $(k = 1,2,3,4,5,6)$ are constants to be determined.

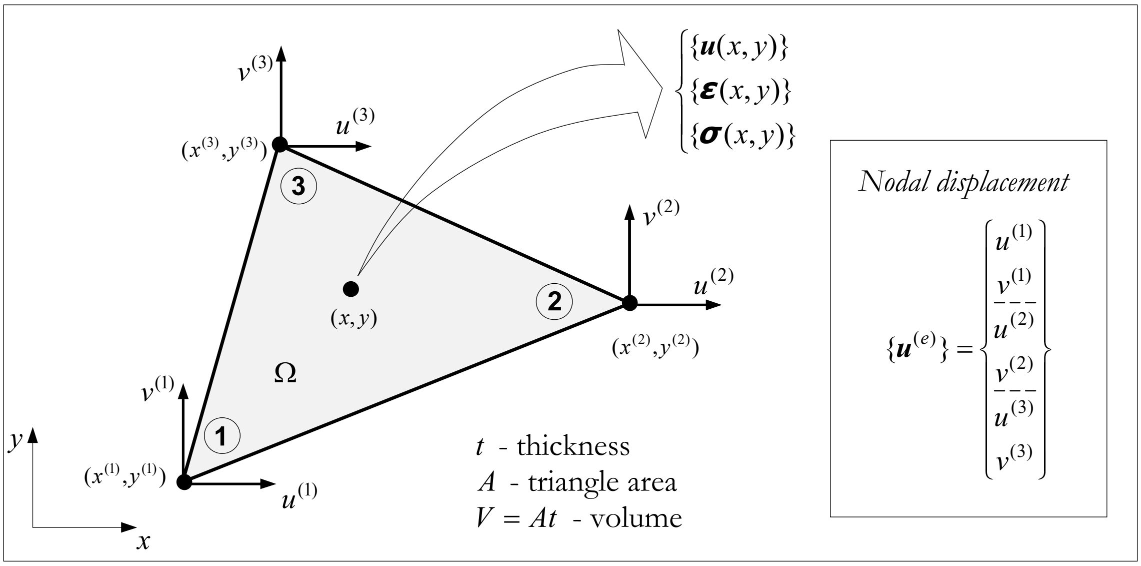


Figure 6.50: Domain Ω.

a) Obtain an explicit expression between displacement field ($\{\boldsymbol{u}(x,y)\}$) and nodal displacement ($\{\boldsymbol{u}^{(e)}\}$), (see Figure 6.50).

b) In **Problem 5.23** it was obtained the equation $\{\boldsymbol{f}^{(e)}\} = [\boldsymbol{k}^{(e)}]\{\boldsymbol{u}^{(e)}\}$ which is equivalent to the governing equation for static linear elastic problem, where $\{\boldsymbol{f}^{(e)}\}$ stands for nodal forces whose directions are coincident with the nodal displacement directions. Obtain the stiffness matrix $[\boldsymbol{k}^{(e)}]$ for the problem established here.

Solution:

The displacement field (6.146) can also be expressed as follows:

$$\{\boldsymbol{u}(x,y)\} = \begin{cases} u(x,y) = a_1 + a_2 x + a_3 y \\ v(x,y) = a_4 + a_5 x + a_6 y \end{cases}$$

$$\Rightarrow \{\boldsymbol{u}(x,y)\} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \quad \Rightarrow \quad \{\boldsymbol{u}(x,y)\} = [\boldsymbol{X}]\{\alpha\} \tag{6.147}$$

The next step is determine the coefficients a_k $(k = 1,2,3,4,5,6)$, i.e. the vector $\{\alpha\}$.

The displacement field given by (6.147) is also valid for the nodes 1, 2 and 3, so:

$$\begin{cases} u(x^{(1)},y^{(1)}) \equiv u^{(1)} = a_1 + a_2 x^{(1)} + a_3 y^{(1)} \\ v(x^{(1)},y^{(1)}) \equiv v^{(1)} = a_4 + a_5 x^{(1)} + a_6 y^{(1)} \\ \hdashline u(x^{(2)},y^{(2)}) \equiv u^{(2)} = a_1 + a_2 x^{(2)} + a_3 y^{(2)} \\ v(x^{(2)},y^{(2)}) \equiv v^{(2)} = a_4 + a_5 x^{(2)} + a_6 y^{(2)} \\ \hdashline u(x^{(3)},y^{(3)}) \equiv u^{(3)} = a_1 + a_2 x^{(3)} + a_3 y^{(3)} \\ v(x^{(3)},y^{(3)}) \equiv v^{(3)} = a_4 + a_5 x^{(3)} + a_6 y^{(3)} \end{cases}$$

or in matrix form:

$$\begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hdashline u^{(2)} \\ v^{(2)} \\ \hdashline u^{(3)} \\ v^{(3)} \end{Bmatrix} = \begin{bmatrix} 1 & x^{(1)} & y^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x^{(3)} & y^{(3)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \quad \Leftrightarrow \quad \{\boldsymbol{u}^{(e)}\}_{6\times 1} = [\mathbf{A}]_{6\times 6}\{\alpha\}_{6\times 1}$$

Then, if the inverse of $[\mathbf{A}]$ is known the vector $\{\alpha\}$ can be determined, i.e.:

$$\{\boldsymbol{u}^{(e)}\} = [\mathbf{A}]\{\alpha\} \quad \Rightarrow \quad [\mathbf{A}]^{-1}\{\boldsymbol{u}^{(e)}\} = [\mathbf{A}]^{-1}[\mathbf{A}]\{\alpha\} \quad \Rightarrow \quad [\mathbf{A}]^{-1}\{\boldsymbol{u}^{(e)}\} = [\boldsymbol{I}]\{\alpha\} = \{\alpha\}$$
$$\Rightarrow \{\alpha\} = [\mathbf{A}]^{-1}\{\boldsymbol{u}^{(e)}\}$$

And by substituting the above equation into the displacement field (6.147) we can obtain:

$$\{\boldsymbol{u}(x,y)\} = [\boldsymbol{X}]\{\alpha\} \quad \Rightarrow \quad \{\boldsymbol{u}(x,y)\} = [\boldsymbol{X}][\mathbf{A}]^{-1}\{\boldsymbol{u}^{(e)}\} = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\} \tag{6.148}$$

Note that by definition the shape function relates the *function field* to the *nodal value* of the function, so, we can conclude that the shape functions to approach the displacement field are:

$$[\boldsymbol{N}]_{2\times 6} = [\boldsymbol{X}]_{2\times 6}[\mathbf{A}]^{-1}_{6\times 6} \tag{6.149}$$

The matrix $[\mathbf{A}]^{-1}$ is given by:

$$[\mathbf{A}]^{-1} = \frac{1}{2A}\begin{bmatrix} \begin{pmatrix} x^{(2)}y^{(3)} \\ -y^{(2)}x^{(3)} \end{pmatrix} & 0 & \begin{pmatrix} x^{(3)}y^{(1)} \\ -y^{(3)}x^{(1)} \end{pmatrix} & 0 & \begin{pmatrix} x^{(1)}y^{(2)} \\ -y^{(1)}x^{(2)} \end{pmatrix} & 0 \\ (y^{(2)}-y^{(3)}) & 0 & (y^{(3)}-y^{(1)}) & 0 & (y^{(1)}-y^{(2)}) & 0 \\ (x^{(3)}-x^{(2)}) & 0 & (x^{(1)}-x^{(3)}) & 0 & (x^{(2)}-x^{(1)}) & 0 \\ 0 & \begin{pmatrix} x^{(2)}y^{(3)} \\ -y^{(2)}x^{(3)} \end{pmatrix} & 0 & \begin{pmatrix} x^{(3)}y^{(1)} \\ -y^{(3)}x^{(1)} \end{pmatrix} & 0 & \begin{pmatrix} x^{(1)}y^{(2)} \\ -y^{(1)}x^{(2)} \end{pmatrix} \\ 0 & (y^{(2)}-y^{(3)}) & 0 & (y^{(3)}-y^{(1)}) & 0 & (y^{(1)}-y^{(2)}) \\ 0 & (x^{(3)}-x^{(2)}) & 0 & (x^{(1)}-x^{(3)}) & 0 & (x^{(2)}-x^{(1)}) \end{bmatrix}$$

where A is the triangle area. Then, after the matrix multiplication in (6.149) is taken place we can obtain:

$$[\boldsymbol{N}] = \left[\begin{array}{cc:cc:cc} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{array}\right]$$

where

$$
\begin{cases}
N_1(x,y) = \dfrac{1}{2A}[x(y^{(2)} - y^{(3)}) + y(x^{(3)} - x^{(2)}) + (x^{(2)}y^{(3)} - y^{(2)}x^{(3)})] \\
N_2(x,y) = \dfrac{1}{2A}[x(y^{(3)} - y^{(1)}) + y(x^{(1)} - x^{(3)}) + (x^{(3)}y^{(1)} - y^{(3)}x^{(1)})] \\
N_3(x,y) = \dfrac{1}{2A}[x(y^{(1)} - y^{(2)}) + y(x^{(2)} - x^{(1)}) + (x^{(1)}y^{(2)} - y^{(1)}x^{(2)})]
\end{cases}
\tag{6.150}
$$

In **Problem 5.23** we have shown that

$$\{\boldsymbol{f}^{(e)}\} = [\boldsymbol{k}^{(e)}]\{\boldsymbol{u}^{(e)}\}$$

where the stiffness matrix $[\boldsymbol{k}^{(e)}]$ can be obtained as follows:

$$[\boldsymbol{k}^{(e)}] = \int_V [\mathbf{B}(\vec{x})]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\, dV \tag{6.151}$$

where the matrix $[\mathbf{B}]$ relates strain field to nodal displacements, (see **Problem 5.23**), i.e.:

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} =
\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} =
\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}
\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} =
\underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}
\left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right]}_{=[\boldsymbol{L}^{(1)}][\boldsymbol{N}(\vec{x})]=[\mathbf{B}(\vec{x})]}
\begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hline u^{(2)} \\ v^{(2)} \\ \hline u^{(3)} \\ v^{(3)} \end{Bmatrix}
$$

or $\{\boldsymbol{\varepsilon}(\vec{x})\} = [\boldsymbol{L}^{(1)}]\{\boldsymbol{u}(\vec{x})\} = [\boldsymbol{L}^{(1)}][\boldsymbol{N}(\vec{x})]\{\boldsymbol{u}^{(e)}\} = [\mathbf{B}(\vec{x})]\{\boldsymbol{u}^{(e)}\}$.

By considering the shape functions (6.150), the matrix $[\mathbf{B}]$ becomes:

$$
[\mathbf{B}(\vec{x})] =
\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}
\left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right] =
\left[\begin{array}{cc|cc|cc}
\dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\
0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\
\dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x}
\end{array}\right]
$$

where

$$
\begin{aligned}
\frac{\partial N_1}{\partial x} &= \frac{1}{2A}\frac{\partial}{\partial x}[x(y^{(2)} - y^{(3)}) + y(x^{(3)} - x^{(2)}) + (x^{(1)}y^{(3)} - y^{(2)}x^{(3)})] = \frac{1}{2A}(y^{(2)} - y^{(3)}) \\
\frac{\partial N_1}{\partial y} &= \frac{1}{2A}\frac{\partial}{\partial y}[x(y^{(2)} - y^{(3)}) + y(x^{(3)} - x^{(2)}) + (x^{(1)}y^{(3)} - y^{(2)}x^{(3)})] = \frac{1}{2A}(x^{(3)} - x^{(2)}) \\
\frac{\partial N_2}{\partial x} &= \frac{1}{2A}\frac{\partial}{\partial x}[x(y^{(3)} - y^{(1)}) + y(x^{(1)} - x^{(3)}) + (x^{(3)}y^{(1)} - y^{(3)}x^{(1)})] = \frac{1}{2A}(y^{(3)} - y^{(1)}) \\
\frac{\partial N_2}{\partial y} &= \frac{1}{2A}\frac{\partial}{\partial y}[x(y^{(3)} - y^{(1)}) + y(x^{(1)} - x^{(3)}) + (x^{(3)}y^{(1)} - y^{(3)}x^{(1)})] = \frac{1}{2A}(x^{(1)} - x^{(3)}) \\
\frac{\partial N_3}{\partial x} &= \frac{1}{2A}\frac{\partial}{\partial x}[x(y^{(1)} - y^{(2)}) + y(x^{(2)} - x^{(1)}) + (x^{(1)}y^{(2)} - y^{(1)}x^{(2)})] = \frac{1}{2A}(y^{(1)} - y^{(2)}) \\
\frac{\partial N_3}{\partial y} &= \frac{1}{2A}\frac{\partial}{\partial y}[x(y^{(1)} - y^{(2)}) + y(x^{(2)} - x^{(1)}) + (x^{(1)}y^{(2)} - y^{(1)}x^{(2)})] = \frac{1}{2A}(x^{(2)} - x^{(1)})
\end{aligned}
$$

Then, the matrix $[\mathbf{B}]$ becomes:

$$[\mathbf{B}]=\left[\begin{array}{cc:cc:cc}\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x}\end{array}\right]$$

$$=\frac{1}{2A}\left[\begin{array}{cc:cc:cc}y^{(2)}-y^{(3)} & 0 & y^{(3)}-y^{(1)} & 0 & y^{(1)}-y^{(2)} & 0\\ 0 & x^{(3)}-x^{(2)} & 0 & x^{(1)}-x^{(3)} & 0 & x^{(2)}-x^{(1)}\\ x^{(3)}-x^{(2)} & y^{(2)}-y^{(3)} & x^{(1)}-x^{(3)} & y^{(3)}-y^{(1)} & x^{(2)}-x^{(1)} & y^{(1)}-y^{(2)}\end{array}\right] \tag{6.152}$$

$$=\frac{1}{2A}\left[\begin{array}{cc:cc:cc}a_1 & 0 & a_2 & 0 & a_3 & 0\\ 0 & b_1 & 0 & b_2 & 0 & b_3\\ b_1 & a_1 & b_2 & a_2 & b_3 & a_3\end{array}\right]$$

$$\Rightarrow [\mathbf{B}]=[[\overline{\mathbf{B}}_1]_{3\times 2}\quad [\overline{\mathbf{B}}_2]_{3\times 2}\quad [\overline{\mathbf{B}}_3]_{3\times 2}] \tag{6.153}$$

Note that, since the displacement field is linear the matrix $[\mathbf{B}]$ is constant into the sub-domain, and as consequence the strain and stress fields are also constant fields into the sub-domain. For this reason this triangular sub-domain is called *Constant Strain Triangle – CST*.

Then, the stiffness matrix can be obtained as follows:

$$[\boldsymbol{k}^{(e)}]_{6\times 6}=\int_V [\mathbf{B}]^T[\mathcal{C}^{(2D)}][\mathbf{B}]\,dV=[\mathbf{B}]^T[\mathcal{C}^{(2D)}][\mathbf{B}]\underbrace{\int_V dV}_{=V=At}=[\mathbf{B}]^T_{6\times 3}[\mathcal{C}^{(2D)}]_{3\times 3}[\mathbf{B}]_{3\times 6}At \tag{6.154}$$

where the matrix $[\mathcal{C}^{(2D)}]$ for 2D case was obtained in **Problem 6.25**, i.e.:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{\overline{E}}{1-\overline{\nu}^2}\begin{bmatrix}1 & \overline{\nu} & 0\\ \overline{\nu} & 1 & 0\\ 0 & 0 & \frac{1-\overline{\nu}}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \quad \therefore \quad \begin{cases}\text{if state of plane stress} & \begin{cases}\overline{E}=E\\ \overline{\nu}=\nu\end{cases}\\ \text{if state of plane strain} & \begin{cases}\overline{E}=\dfrac{E}{(1-\nu^2)}\\ \overline{\nu}=\dfrac{\nu}{(1-\nu)}\end{cases}\end{cases} \tag{6.155}$$

$$\{\boldsymbol{\sigma}(x,y)\}=[\mathcal{C}^{(2D)}]\{\boldsymbol{\varepsilon}(x,y)\}$$

The stress field can also be expressed in terms of nodal displacements as follows:

$$\{\boldsymbol{\sigma}(x,y)\}=[\mathcal{C}^{(2D)}]\{\boldsymbol{\varepsilon}(x,y)\}=[\mathcal{C}^{(2D)}][\mathbf{B}(\vec{x})]\{\boldsymbol{u}^{(e)}\} \tag{6.156}$$

NOTE 1: Note that we can obtain the explicit form of the stiffness matrix by means of the matrix multiplications given by equation in (6.154), but in some cases the explicit form of the stiffness matrix is not so easy to be obtained, then we resort to *Numerical Integration* (also called *Quadrature*) in order to solve numerically the integral (6.151).

The explicit form of (6.154) follows. The matrix $[\overline{\mathbf{B}}_i]_{3\times 2}$ from the equation (6.153) can be rewritten as follows:

$$[\bar{\mathbf{B}}_i]_{3\times 2} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} a_i & 0 \\ 0 & b_i \\ b_i & a_i \end{bmatrix}$$

Then, the equation in (6.154) can be rewritten as follows

$$[\boldsymbol{k}^{(e)}] = At[\mathbf{B}]^T_{6\times 3}[\boldsymbol{\mathcal{C}}^{(2D)}]_{3\times 3}[\mathbf{B}]_{3\times 6}$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = At\left(\begin{bmatrix} [\bar{\mathbf{B}}_1]^T \\ [\bar{\mathbf{B}}_2]^T \\ [\bar{\mathbf{B}}_3]^T \end{bmatrix}[\boldsymbol{\mathcal{C}}^{(2D)}][[\bar{\mathbf{B}}_1] \mid [\bar{\mathbf{B}}_2] \mid [\bar{\mathbf{B}}_3]]\right)$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = At\left(\begin{bmatrix} [\bar{\mathbf{B}}_1]^T[\boldsymbol{\mathcal{C}}^{(2D)}] \\ [\bar{\mathbf{B}}_2]^T[\boldsymbol{\mathcal{C}}^{(2D)}] \\ [\bar{\mathbf{B}}_3]^T[\boldsymbol{\mathcal{C}}^{(2D)}] \end{bmatrix}[[\bar{\mathbf{B}}_1] \mid [\bar{\mathbf{B}}_2] \mid [\bar{\mathbf{B}}_3]]\right)$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = At\begin{bmatrix} [\bar{\mathbf{B}}_1]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_1] & [\bar{\mathbf{B}}_1]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_2] & [\bar{\mathbf{B}}_1]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_3] \\ [\bar{\mathbf{B}}_2]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_1] & [\bar{\mathbf{B}}_2]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_2] & [\bar{\mathbf{B}}_2]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_3] \\ [\bar{\mathbf{B}}_3]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_1] & [\bar{\mathbf{B}}_3]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_2] & [\bar{\mathbf{B}}_3]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_3] \end{bmatrix}$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = At[[\bar{\mathbf{B}}_i]^T[\boldsymbol{\mathcal{C}}^{(2D)}][\bar{\mathbf{B}}_j]] \qquad (i,j = 1,2,3)$$

Let us consider the elasticity tensor components for 2D case:

$$[\boldsymbol{\mathcal{C}}^{(2D)}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{bmatrix}$$

Then, we can obtain:

$$[\bar{\mathbf{B}}_i]^T[\boldsymbol{\mathcal{C}}][\bar{\mathbf{B}}_j] = \frac{1}{4A^2}\begin{bmatrix} a_i & 0 & b_i \\ 0 & b_i & a_i \end{bmatrix}\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{bmatrix}\begin{bmatrix} a_j & 0 \\ 0 & b_j \\ b_j & a_j \end{bmatrix} = \frac{1}{4A^2}[\bar{\mathbf{k}}_{ij}]$$

where

$$[\bar{\mathbf{k}}_{ij}]_{2\times 2} = \begin{bmatrix} a_i\mathcal{C}_{11}a_j + b_i\mathcal{C}_{33}b_j & a_i\mathcal{C}_{12}b_j + b_i\mathcal{C}_{33}a_j \\ b_i\mathcal{C}_{12}a_j + a_i\mathcal{C}_{33}b_j & b_i\mathcal{C}_{22}b_j + a_i\mathcal{C}_{33}a_j \end{bmatrix}$$

Then, the stiffness matrix becomes

$$[\boldsymbol{k}^{(e)}]_{6\times 6} = \frac{t}{4A}\begin{bmatrix} [\bar{\mathbf{k}}_{11}] & [\bar{\mathbf{k}}_{12}] & [\bar{\mathbf{k}}_{13}] \\ [\bar{\mathbf{k}}_{21}] & [\bar{\mathbf{k}}_{22}] & [\bar{\mathbf{k}}_{23}] \\ [\bar{\mathbf{k}}_{31}] & [\bar{\mathbf{k}}_{32}] & [\bar{\mathbf{k}}_{33}] \end{bmatrix} = \frac{t}{4A}\begin{bmatrix} \bar{k}^{(e)}_{11} & \bar{k}^{(e)}_{12} & \bar{k}^{(e)}_{13} & \bar{k}^{(e)}_{14} & \bar{k}^{(e)}_{15} & \bar{k}^{(e)}_{16} \\ & \bar{k}^{(e)}_{22} & \bar{k}^{(e)}_{23} & \bar{k}^{(e)}_{24} & \bar{k}^{(e)}_{25} & \bar{k}^{(e)}_{26} \\ & & \bar{k}^{(e)}_{33} & \bar{k}^{(e)}_{34} & \bar{k}^{(e)}_{35} & \bar{k}^{(e)}_{36} \\ & & & \bar{k}^{(e)}_{44} & \bar{k}^{(e)}_{45} & \bar{k}^{(e)}_{46} \\ & \mathit{symmetric} & & & \bar{k}^{(e)}_{55} & \bar{k}^{(e)}_{56} \\ & & & & & \bar{k}^{(e)}_{66} \end{bmatrix} \tag{6.157}$$

where

$$
\begin{array}{lll}
\bar{k}_{11}^{(e)} = a_1^2 \mathcal{C}_{11} + b_1^2 \mathcal{C}_{33}; & \bar{k}_{12}^{(e)} = a_1 \mathcal{C}_{12} b_1 + b_1 \mathcal{C}_{33} a_1; & \bar{k}_{13}^{(e)} = a_1 \mathcal{C}_{11} a_2 + b_1 \mathcal{C}_{33} b_2; \\
\bar{k}_{14}^{(e)} = a_1 \mathcal{C}_{12} b_2 + b_1 \mathcal{C}_{33} a_2; & \bar{k}_{15}^{(e)} = a_1 \mathcal{C}_{11} a_3 + b_1 \mathcal{C}_{33} b_3; & \bar{k}_{16}^{(e)} = a_1 \mathcal{C}_{12} b_3 + b_1 \mathcal{C}_{33} a_3; \\
\bar{k}_{22}^{(e)} = b_1^2 \mathcal{C}_{22} + a_1^2 \mathcal{C}_{33}; & \bar{k}_{23}^{(e)} = b_1 \mathcal{C}_{12} a_2 + a_1 \mathcal{C}_{33} b_2; & \bar{k}_{24}^{(e)} = b_1 \mathcal{C}_{22} b_2 + a_1 \mathcal{C}_{33} a_2; \\
\bar{k}_{25}^{(e)} = b_1 \mathcal{C}_{12} a_3 + a_1 \mathcal{C}_{33} b_3; & \bar{k}_{26}^{(e)} = b_1 \mathcal{C}_{22} b_3 + a_1 \mathcal{C}_{33} a_3; & \bar{k}_{33}^{(e)} = a_2^2 \mathcal{C}_{11} + b_2^2 \mathcal{C}_{33}; \\
\bar{k}_{34}^{(e)} = a_2 \mathcal{C}_{12} b_2 + b_2 \mathcal{C}_{33} a_2; & \bar{k}_{35}^{(e)} = a_2 \mathcal{C}_{11} a_3 + b_2 \mathcal{C}_{33} b_3; & \bar{k}_{36}^{(e)} = a_2 \mathcal{C}_{12} b_3 + b_2 \mathcal{C}_{33} a_3; \\
\bar{k}_{44}^{(e)} = b_2^2 \mathcal{C}_{22} + a_2^2 \mathcal{C}_{33}; & \bar{k}_{45}^{(e)} = b_2 \mathcal{C}_{12} a_3 + a_2 \mathcal{C}_{33} b_3; & \bar{k}_{46}^{(e)} = b_2 \mathcal{C}_{22} b_3 + a_2 \mathcal{C}_{33} a_3; \\
\bar{k}_{55}^{(e)} = a_3^2 \mathcal{C}_{11} + b_3^2 \mathcal{C}_{33}; & \bar{k}_{56}^{(e)} = a_3 \mathcal{C}_{12} b_3 + b_3 \mathcal{C}_{33} a_3; & \bar{k}_{66}^{(e)} = b_3^2 \mathcal{C}_{22} + a_3^2 \mathcal{C}_{33};
\end{array}
\tag{6.158}
$$

with

$$
\begin{array}{lll}
a_1 = y^{(2)} - y^{(3)}; & a_2 = y^{(3)} - y^{(1)}; & a_3 = y^{(1)} - y^{(2)}; \\
b_1 = x^{(3)} - x^{(2)}; & b_2 = x^{(1)} - x^{(3)}; & b_3 = x^{(2)} - x^{(1)}.
\end{array}
\tag{6.159}
$$

NOTE 2: The shape functions for Strain Constant Triangle can be appreciated in Figure 6.51. Note that the shape function N_1 at node 1 has the value equal to 1 and assumes zero for the remaining nodes:

$$N_1(x,y) = \frac{1}{2A}[x(y^{(2)} - y^{(3)}) + y(x^{(3)} - x^{(2)}) + (x^{(2)} y^{(3)} - y^{(2)} x^{(3)})]$$

$$(x = x^{(1)}, y = y^{(1)}) \Rightarrow N_1(x^{(1)}, y^{(1)}) = \frac{1}{2A}[x^{(1)}(y^{(2)} - y^{(3)}) + y^{(1)}(x^{(3)} - x^{(2)}) + (x^{(2)} y^{(3)} - y^{(2)} x^{(3)})] = 1$$

And the summation of the shape functions must be equal to 1, i.e. $N_1 + N_2 + N_3 = 1$.

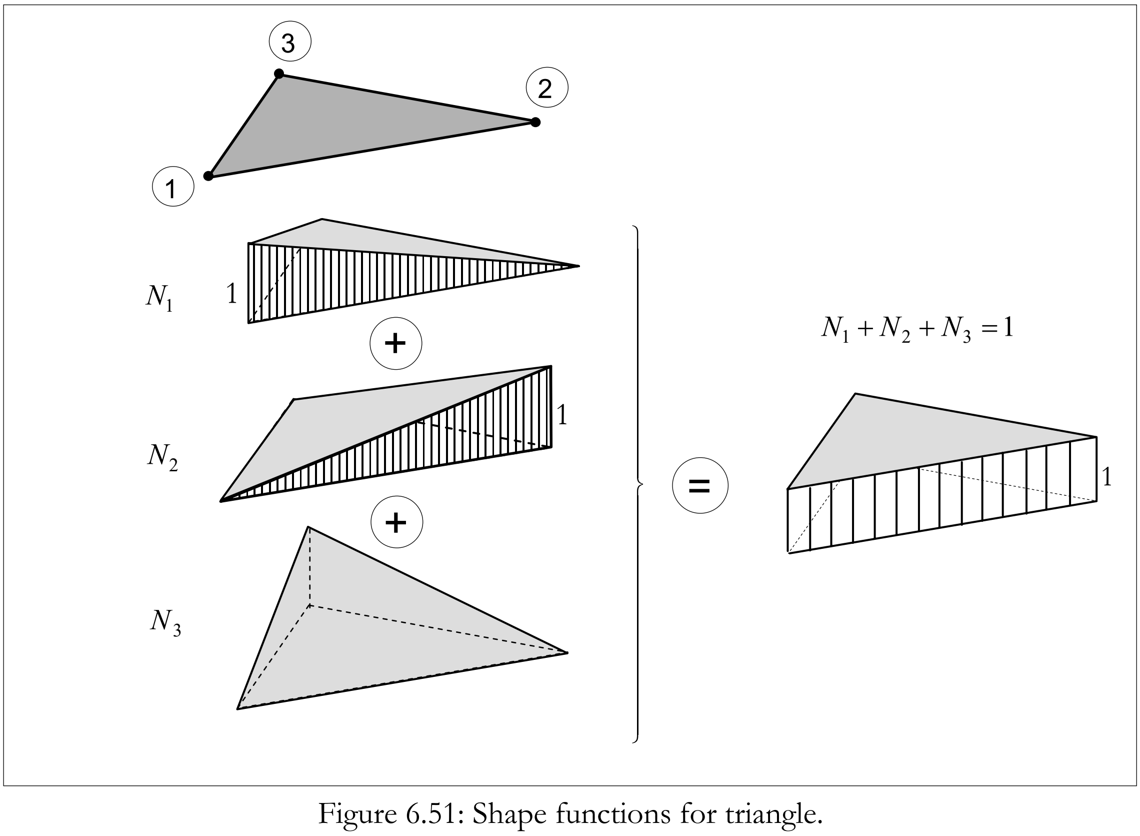


Figure 6.51: Shape functions for triangle.

NOTE 3: Another way to obtain the shape function follows. Let us consider only the displacement according to x-direction:

$$u(x,y) = a_1 + x a_2 + y a_3 = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.160}$$

and its nodal values

$$\left.\begin{aligned} u^{(1)} &= a_1 + x^{(1)} a_2 + y^{(1)} a_3 \\ u^{(2)} &= a_1 + x^{(2)} a_2 + y^{(2)} a_3 \\ u^{(3)} &= a_1 + x^{(3)} a_2 + y^{(3)} a_3 \end{aligned}\right\} \quad \Rightarrow \quad \begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{Bmatrix} = \begin{bmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

In **Problem 1.16** we have used the Cramer's rule to obtain the solution for the above set of equations, i.e.:

$$a_1 = \frac{\begin{vmatrix} u^{(1)} & x^{(1)} & y^{(1)} \\ u^{(2)} & x^{(2)} & y^{(2)} \\ u^{(3)} & x^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}} \quad ; \quad a_2 = \frac{\begin{vmatrix} 1 & u^{(1)} & y^{(1)} \\ 1 & u^{(2)} & y^{(2)} \\ 1 & u^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}} \quad ; \quad a_3 = \frac{\begin{vmatrix} 1 & x^{(1)} & u^{(1)} \\ 1 & x^{(2)} & u^{(2)} \\ 1 & x^{(3)} & u^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}$$

where $|\bullet| \equiv \det(\bullet)$ stands for the determinant of $\bullet$. Note that

$$\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix} \equiv \det\left(\begin{bmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{bmatrix} \right) = 2!A = 2A$$

where A is the triangle area. Note also that

$$\begin{vmatrix} u^{(1)} & x^{(1)} & y^{(1)} \\ u^{(2)} & x^{(2)} & y^{(2)} \\ u^{(3)} & x^{(3)} & y^{(3)} \end{vmatrix} = u^{(1)} \begin{vmatrix} x^{(2)} & y^{(2)} \\ x^{(3)} & y^{(3)} \end{vmatrix} - u^{(2)} \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(3)} & y^{(3)} \end{vmatrix} + u^{(3)} \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(2)} & y^{(2)} \end{vmatrix}$$

$$\begin{vmatrix} 1 & u^{(1)} & y^{(1)} \\ 1 & u^{(2)} & y^{(2)} \\ 1 & u^{(3)} & y^{(3)} \end{vmatrix} = -u^{(1)} \begin{vmatrix} 1 & y^{(2)} \\ 1 & y^{(3)} \end{vmatrix} + u^{(2)} \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(3)} \end{vmatrix} - u^{(3)} \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(2)} \end{vmatrix}$$

$$\begin{vmatrix} 1 & x^{(1)} & u^{(1)} \\ 1 & x^{(2)} & u^{(2)} \\ 1 & x^{(3)} & u^{(3)} \end{vmatrix} = u^{(1)} \begin{vmatrix} 1 & x^{(2)} \\ 1 & x^{(3)} \end{vmatrix} - u^{(2)} \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(3)} \end{vmatrix} + u^{(3)} \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix}$$

Then, the displacement field given by the equation in (6.160) can be expressed as follows:

$$
\begin{aligned}
& u(x,y) = a_1 + x a_2 + y a_3 \\
& \Rightarrow \begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix} u(x,y) = u^{(1)} \begin{vmatrix} x^{(2)} & y^{(2)} \\ x^{(3)} & y^{(3)} \end{vmatrix} - u^{(2)} \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(3)} & y^{(3)} \end{vmatrix} + u^{(3)} \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(2)} & y^{(2)} \end{vmatrix} + \\
& \qquad x\left(-u^{(1)} \begin{vmatrix} 1 & y^{(2)} \\ 1 & y^{(3)} \end{vmatrix} + u^{(2)} \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(3)} \end{vmatrix} - u^{(3)} \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(2)} \end{vmatrix} \right) + y\left(u^{(1)} \begin{vmatrix} 1 & x^{(2)} \\ 1 & x^{(3)} \end{vmatrix} - u^{(2)} \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(3)} \end{vmatrix} + u^{(3)} \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix} \right) \\
& \Rightarrow \begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix} u(x,y) = u^{(1)} \left(\begin{vmatrix} x^{(2)} & y^{(2)} \\ x^{(3)} & y^{(3)} \end{vmatrix} - x \begin{vmatrix} 1 & y^{(2)} \\ 1 & y^{(3)} \end{vmatrix} + y \begin{vmatrix} 1 & x^{(2)} \\ 1 & x^{(3)} \end{vmatrix} \right) + \\
& \qquad u^{(2)} \left(- \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(3)} & y^{(3)} \end{vmatrix} + x \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(3)} \end{vmatrix} - y \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(3)} \end{vmatrix} \right) + u^{(3)} \left(\begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(2)} & y^{(2)} \end{vmatrix} - x \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(2)} \end{vmatrix} + y \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix} \right)
\end{aligned}
\tag{6.161}
$$

Note also that

$$
u^{(1)} \left(1 \begin{vmatrix} x^{(2)} & y^{(2)} \\ x^{(3)} & y^{(3)} \end{vmatrix} - x \begin{vmatrix} 1 & y^{(2)} \\ 1 & y^{(3)} \end{vmatrix} + y \begin{vmatrix} 1 & x^{(2)} \\ 1 & x^{(3)} \end{vmatrix} \right) = u^{(1)} \begin{vmatrix} 1 & x & y \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}
$$

$$
u^{(2)} \left(-1 \begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(3)} & y^{(3)} \end{vmatrix} + x \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(3)} \end{vmatrix} - y \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(3)} \end{vmatrix} \right) = u^{(2)} \begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x & y \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}
$$

$$
u^{(3)} \left(\begin{vmatrix} x^{(1)} & y^{(1)} \\ x^{(2)} & y^{(2)} \end{vmatrix} - x \begin{vmatrix} 1 & y^{(1)} \\ 1 & y^{(2)} \end{vmatrix} + y \begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix} \right) = u^{(3)} \begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x & y \end{vmatrix}
$$

Then, the displacement field (6.161) can also be expressed as follows:

$$
\Rightarrow u(x,y) = u^{(1)} \frac{\begin{vmatrix} 1 & x & y \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}} + u^{(2)} \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x & y \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}} + u^{(3)} \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}
$$

with that we can define the shape functions as follows:

$$
N_1(\vec{\boldsymbol{x}}) = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}, \quad N_2(\vec{\boldsymbol{x}}) = \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x & y \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}, \quad N_3(\vec{\boldsymbol{x}}) = \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{vmatrix}}
\tag{6.162}
$$

Note that the following is also true:

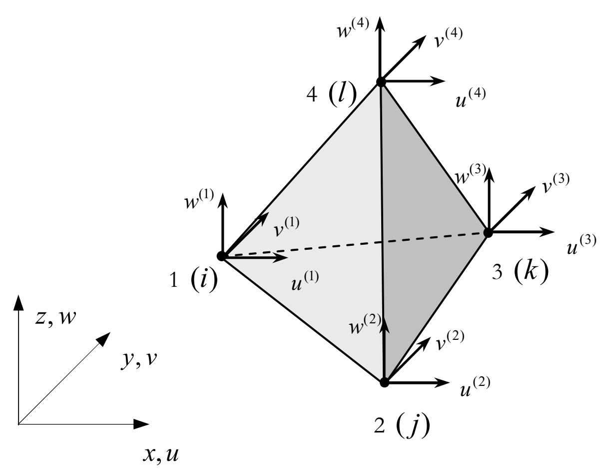
$$\frac{\partial N_1}{\partial x}=\frac{\dfrac{\partial}{\partial x}\begin{vmatrix}1 & x & y\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}=\frac{\begin{vmatrix}0 & 1 & 0\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}=\frac{y^{(2)}-y^{(3)}}{2A}$$

$$\frac{\partial N_1}{\partial y}=\frac{\dfrac{\partial}{\partial y}\begin{vmatrix}1 & x & y\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}=\frac{\begin{vmatrix}0 & 0 & 1\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}=\frac{x^{(3)}-x^{(2)}}{2A}$$

and so on. Note that it is also true that

$$\int N_1 dA=\frac{\displaystyle\int\begin{vmatrix}1 & x & y\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}dA}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}=\frac{\begin{vmatrix}\int dA & \int xdA & \int ydA\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\ 1 & x^{(2)} & y^{(2)}\\ 1 & x^{(3)} & y^{(3)}\end{vmatrix}}$$

NOTE 3.1: The procedure used to obtain the shape functions given by equations in (6.162) can be extrapolated in order to obtain the shape functions for other elements. For example, let us consider the tetrahedron element with 4 nodes, (see Figure 6.52), in which $u(\vec{x})\equiv u(x,y,z)=a_1+xa_2+ya_3+za_4$ (linear function).



$$\begin{Bmatrix}u^{(1)}\\ u^{(2)}\\ u^{(3)}\\ u^{(4)}\end{Bmatrix}=\begin{bmatrix}1 & x^{(1)} & y^{(1)} & z^{(1)}\\ 1 & x^{(2)} & y^{(2)} & z^{(2)}\\ 1 & x^{(3)} & y^{(3)} & z^{(3)}\\ 1 & x^{(4)} & y^{(4)} & z^{(4)}\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\\ a_3\\ a_4\end{Bmatrix}$$

Figure 6.52: Tetrahedron with 4 nodes.

By analogy with the shape functions (6.162) and by considering the relationship between the nodal displacements and the coefficients a_i, (see Figure 6.52), the shape functions can be obtained as follows:

$$N_1 = \frac{\begin{vmatrix} 1 & x & y & z \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}, \quad N_2 = \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x & y & z \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}, \quad N_3 = \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x & y & z \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}},$$

$$N_4 = \frac{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x & y & z \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}} \qquad \textit{where} \quad \begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix} = 3!V = 6V \tag{6.163}$$

where V stands for the tetrahedron volume. Then, the displacement field becomes:

$$u(x,y,z) = N_1 u^{(1)} + N_2 u^{(2)} + N_3 u^{(3)} + N_4 u^{(4)}$$

By considering the same approximation for the fields $v(\vec{x})$ and $w(\vec{x})$, we can obtain:

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \left[\begin{array}{ccc:ccc:ccc:ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array}\right] \left\{\begin{array}{c} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ \hdashline u^{(2)} \\ v^{(2)} \\ w^{(2)} \\ \hdashline u^{(3)} \\ v^{(3)} \\ w^{(3)} \\ \hdashline u^{(4)} \\ v^{(4)} \\ w^{(4)} \end{array}\right\} \tag{6.164}$$

$$\{\boldsymbol{u}(\vec{\boldsymbol{x}})\} = [\boldsymbol{N}(\vec{\boldsymbol{x}})]\{\boldsymbol{u}^{(e)}\} \tag{6.165}$$

In order to construct the polynomial we can resort to the Pascal's polynomial in 3D, (see Figure 6.53).

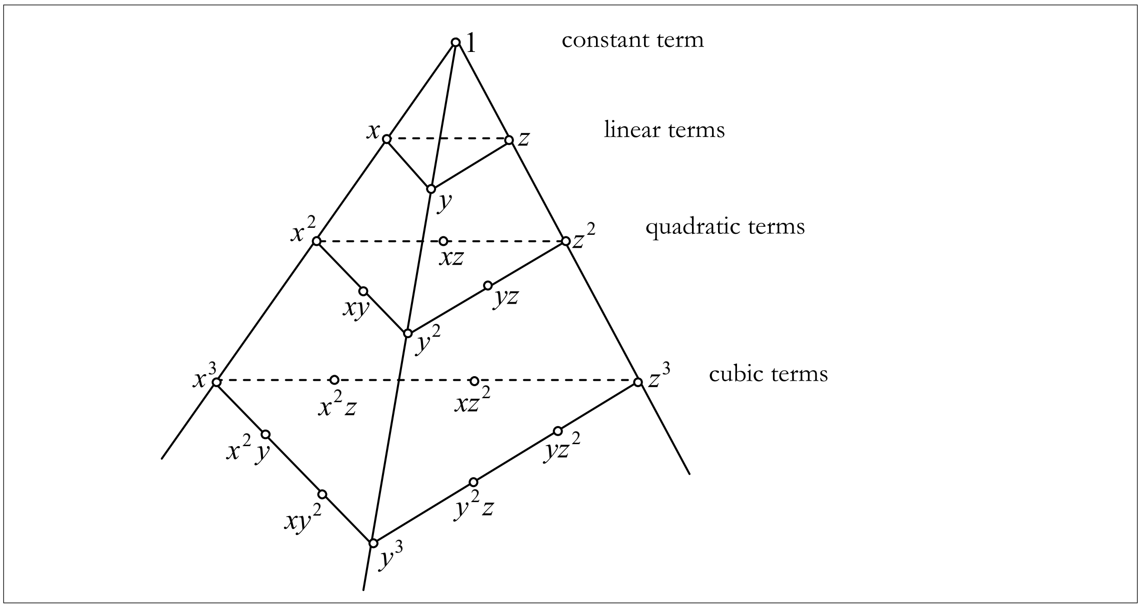


Figure 6.53: Pascal's polynomial for 3D.

NOTE 3.2: In NOTE 3 we have obtained the shape functions in the Cartesian system $(\vec{x})$ for the triangle using linear function. We can also obtain the shape functions in another system. For example, let us obtain the shape functions for the triangle described in Figure 6.54. Then, by using the definition in equation (6.162) we can obtain:

$$N_1(\vec{\xi}) = \frac{\begin{vmatrix} 1 & \xi & \eta \\ 1 & \xi^{(2)} & \eta^{(2)} \\ 1 & \xi^{(3)} & \eta^{(3)} \end{vmatrix}}{\begin{vmatrix} 1 & \xi^{(1)} & \eta^{(1)} \\ 1 & \xi^{(2)} & \eta^{(2)} \\ 1 & \xi^{(3)} & \eta^{(3)} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \xi & \eta \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = 1 - \xi - \eta \; ; \; N_2(\vec{\xi}) = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & \xi & \eta \\ 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = \xi \; ; \; N_3(\vec{\xi}) = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \xi & \eta \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = \eta$$

Let us adopt another nomenclature such as $L_1 = 1 - \xi - \eta$, $L_2 = \xi$, $L_3 = \eta$, which is known as *Area Coordinates*, (see NOTE 5 – Area Coordinates).

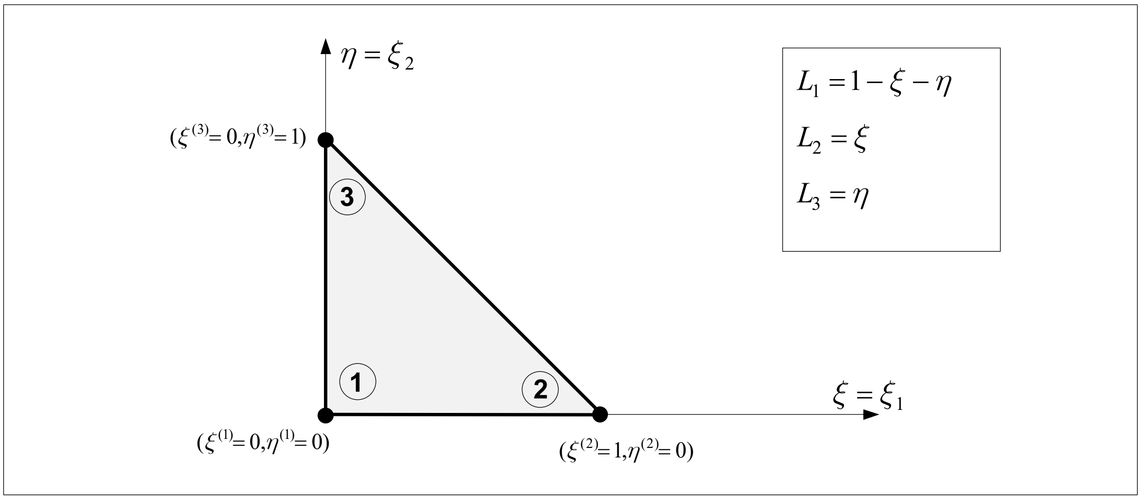


Figure 6.54: Triangle element (3 nodes) – Normalized space.

Now, we will consider the displacement field $u(\vec{x})$ which is approached by a quadratic function, (see Figure 6.55):

$$u(\vec{x}) \equiv u(x,y,z) = a_1 + xa_2 + ya_3 + x^2 a_4 + xya_5 + y^2 a_6$$

The polynomial can be easily obtained by means of Pascal's polynomial, (see Figure 6.42). Note also that to define the coefficients $a_i (i = 1,2,...,6)$ we will need to define 6 nodes. The nodal values by using the above polynomial become:

$$\begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \\ u^{(5)} \\ u^{(6)} \end{Bmatrix} = \begin{bmatrix} 1 & x^{(1)} & y^{(1)} & x^{(1)2} & x^{(1)}y^{(1)} & y^{(1)2} \\ 1 & x^{(2)} & y^{(2)} & x^{(2)2} & x^{(2)}y^{(2)} & y^{(2)2} \\ 1 & x^{(3)} & y^{(3)} & x^{(3)2} & x^{(3)}y^{(3)} & y^{(3)2} \\ 1 & x^{(4)} & y^{(4)} & x^{(4)2} & x^{(4)}y^{(4)} & y^{(4)2} \\ 1 & x^{(5)} & y^{(5)} & x^{(5)2} & x^{(5)}y^{(5)} & y^{(5)2} \\ 1 & x^{(6)} & y^{(6)} & x^{(6)2} & x^{(6)}y^{(6)} & y^{(6)2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \Leftrightarrow \{\boldsymbol{u}\} = [\boldsymbol{H}]\{\boldsymbol{a}\}$$

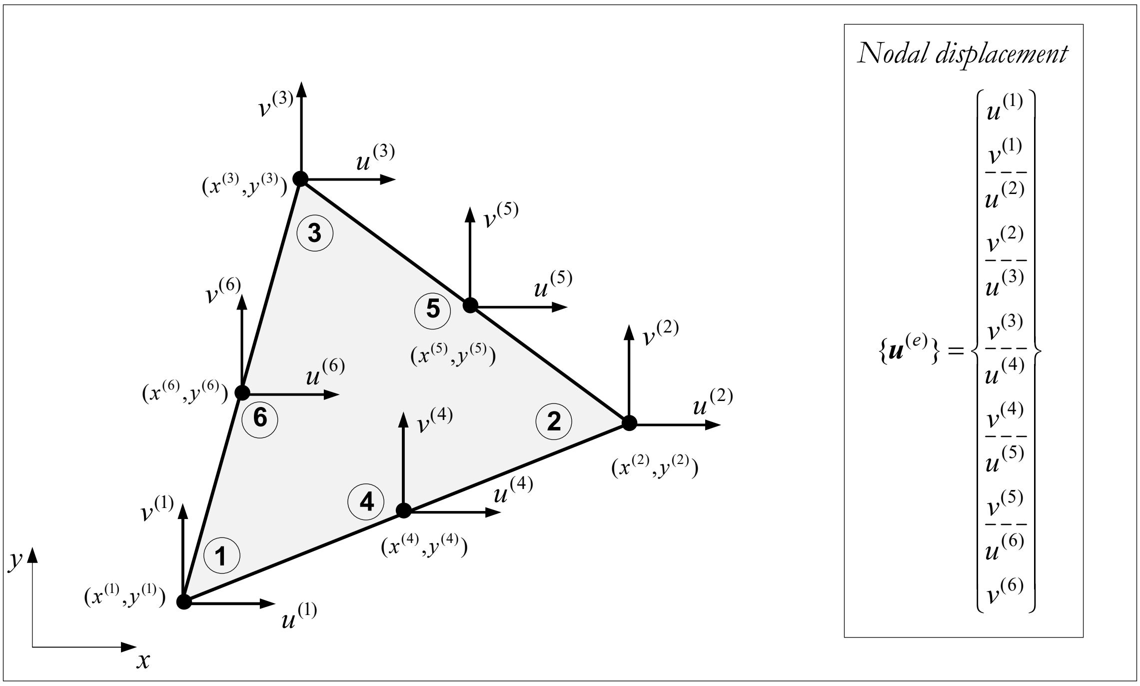


Figure 6.55: Triangle domain – quadratic function.

Then, the shape function $N_1(\vec{x})$ can be obtained as follows:

$$N_1(\vec{x}) = \frac{\begin{vmatrix} 1 & x & y & x^2 & xy & y^2 \\ 1 & x^{(2)} & y^{(2)} & x^{(2)2} & x^{(2)}y^{(2)} & y^{(2)2} \\ 1 & x^{(3)} & y^{(3)} & x^{(3)2} & x^{(3)}y^{(3)} & y^{(3)2} \\ 1 & x^{(4)} & y^{(4)} & x^{(4)2} & x^{(4)}y^{(4)} & y^{(4)2} \\ 1 & x^{(5)} & y^{(5)} & x^{(5)2} & x^{(5)}y^{(5)} & y^{(5)2} \\ 1 & x^{(6)} & y^{(6)} & x^{(6)2} & x^{(6)}y^{(6)} & y^{(6)2} \end{vmatrix}}{|\boldsymbol{H}|}$$

Now, let us consider the normalized space, (see Figure 6.54).

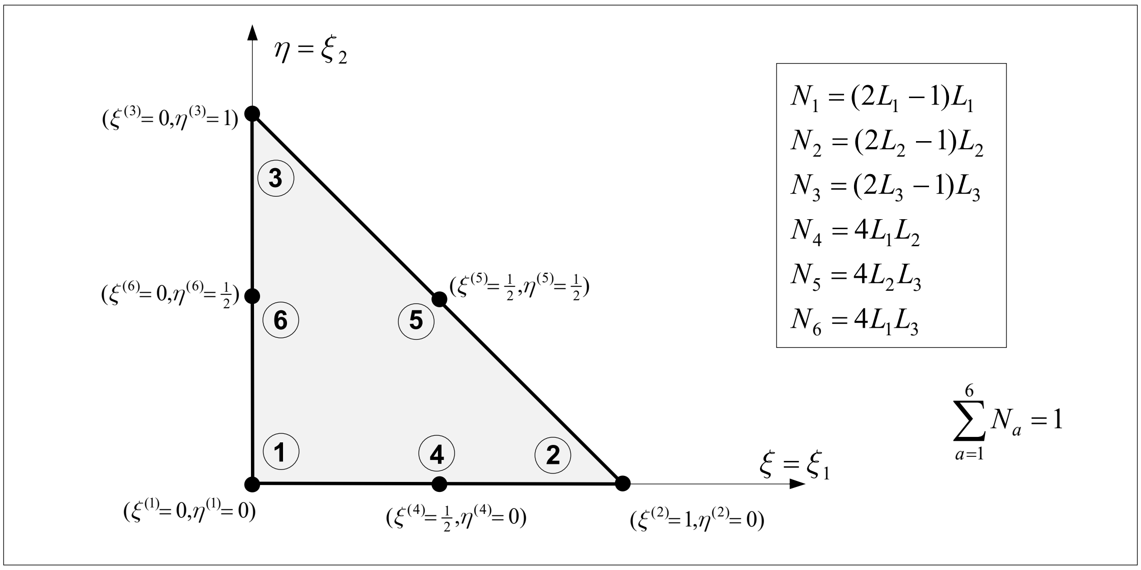


Figure 6.56: Triangle element (6 nodes) – Normalized space.

For this case the matrix $[\boldsymbol{H}]$ becomes

$$[\boldsymbol{H}(\vec{\xi})] = \begin{bmatrix} 1 & \xi^{(1)} & \eta^{(1)} & \xi^{(1)2} & \xi^{(1)}\eta^{(1)} & \eta^{(1)2} \\ 1 & \xi^{(2)} & \eta^{(2)} & \xi^{(2)2} & \xi^{(2)}\eta^{(2)} & \eta^{(2)2} \\ 1 & \xi^{(3)} & \eta^{(3)} & \xi^{(3)2} & \xi^{(3)}\eta^{(3)} & \eta^{(3)2} \\ 1 & \xi^{(4)} & \eta^{(4)} & \xi^{(4)2} & \xi^{(4)}\eta^{(4)} & \eta^{(4)2} \\ 1 & \xi^{(5)} & \eta^{(5)} & \xi^{(5)2} & \xi^{(5)}\eta^{(5)} & \eta^{(5)2} \\ 1 & \xi^{(6)} & \eta^{(6)} & \xi^{(6)2} & \xi^{(6)}\eta^{(6)} & \eta^{(6)2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \left(\frac{1}{2}\right)^2 & \frac{1}{2}\frac{1}{2} & \left(\frac{1}{2}\right)^2 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \left(\frac{1}{2}\right)^2 \end{bmatrix}$$

$$\Rightarrow \quad \det([\boldsymbol{H}]) \equiv |\boldsymbol{H}| = \frac{1}{64}$$

then, the shape function $N_1(\vec{\xi})$ can be obtained as follows:

$$N_1(\vec{\xi}) = 64 \begin{vmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \left(\frac{1}{2}\right)^2 & \frac{1}{2}\frac{1}{2} & \left(\frac{1}{2}\right)^2 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \left(\frac{1}{2}\right)^2 \end{vmatrix} = \underbrace{(2\xi + 2\eta - 1)}_{=-(2L_1-1)}\underbrace{(\xi + \eta - 1)}_{=-L_1} = (2L_1 - 1)L_1$$

And is easy to show that $N_2 = (2L_2 - 1)L_2$ and $N_3 = (2L_3 - 1)L_3$.

The shape function N_4 becomes:

$$N_4 = 64\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 \\ 1 & \frac{1}{2} & \frac{1}{2} & \left(\frac{1}{2}\right)^2 & \frac{1}{2}\frac{1}{2} & \left(\frac{1}{2}\right)^2 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \left(\frac{1}{2}\right)^2 \end{vmatrix} = 4\underbrace{\xi}_{=L_2}\underbrace{(1-\xi-\eta)}_{=L_1} = 4L_1L_2$$

And is easy to show that $N_5 = 4L_2L_3$ and $N_6 = 4L_1L_3$.

Summarizing

$$\begin{aligned} &N_1 = (2L_1-1)L_1 \quad ; \quad N_2 = (2L_2-1)L_2 \quad ; \quad N_3 = (2L_3-1)L_3 \\ &N_4 = 4L_1L_2 \quad ; \quad N_5 = 4L_2L_3 \quad ; \quad N_6 = 4L_1L_3 \end{aligned} \tag{6.166}$$

NOTE 3.3: Another example: let us consider the quadrangular element, (see Figure 6.57), in which the displacement field can be approached by the function:

$$\begin{cases} u(x,y) = a_1 + a_2x + a_3y + a_4xy \\ v(x,y) = a_5 + a_6x + a_7y + a_8xy \end{cases}$$

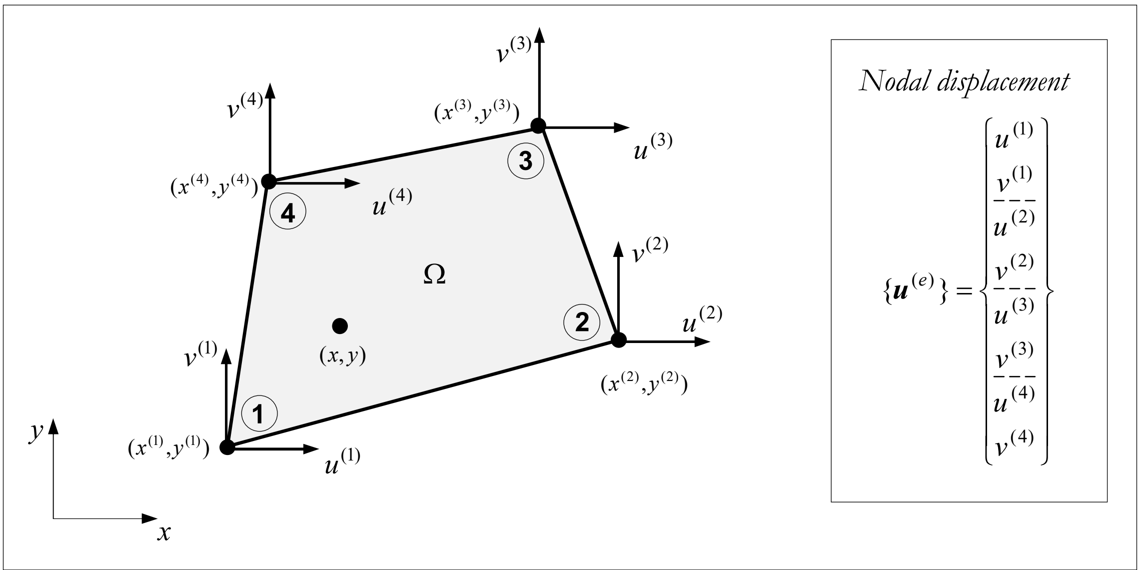


Figure 6.57: Quadrilateral element.

For the displacement field $u(x,y) = a_1 + a_2x + a_3y + a_4xy$ we can obtain the nodal displacements:

$$\begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \end{Bmatrix} = \begin{bmatrix} 1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)} \\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)} \\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)} \\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

Then, the shape functions can be obtained as follows:

$$N_1=\frac{\begin{vmatrix}1 & x & y & xy\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}};\quad N_2=\frac{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x & y & xy\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}};$$

$$N_3=\frac{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x & y & xy\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}};\quad N_4=\frac{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x & y & xy\end{vmatrix}}{\begin{vmatrix}1 & x^{(1)} & y^{(1)} & x^{(1)}y^{(1)}\\ 1 & x^{(2)} & y^{(2)} & x^{(2)}y^{(2)}\\ 1 & x^{(3)} & y^{(3)} & x^{(3)}y^{(3)}\\ 1 & x^{(4)} & y^{(4)} & x^{(4)}y^{(4)}\end{vmatrix}} \tag{6.167}$$

Then

$$\begin{Bmatrix}u(x,y)\\ v(x,y)\end{Bmatrix}=\left[\begin{array}{cc:cc:cc:cc}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4\end{array}\right]\left\{\begin{array}{c}u^{(1)}\\ v^{(1)}\\ \hdashline u^{(2)}\\ v^{(2)}\\ \hdashline u^{(3)}\\ v^{(3)}\\ \hdashline u^{(4)}\\ v^{(4)}\end{array}\right\}\quad\Leftrightarrow\quad \{\boldsymbol{u}(x,y)\}=[\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\}$$

Let us consider a particular case, the regular quadrilateral (rectangle), (see Figure 6.58). In this case the shape functions become:

$$N_1=\frac{\begin{vmatrix}1 & x & y & xy\\ 1 & a & 0 & 0\\ 1 & a & b & ab\\ 1 & 0 & b & 0\end{vmatrix}}{\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & a & 0 & 0\\ 1 & a & b & ab\\ 1 & 0 & b & 0\end{vmatrix}}=\frac{1}{ab}(ab-bx-ay+xy);\quad N_2=\frac{\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & x & y & xy\\ 1 & a & b & ab\\ 1 & 0 & b & 0\end{vmatrix}}{\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & a & 0 & 0\\ 1 & a & b & ab\\ 1 & 0 & b & 0\end{vmatrix}}=\frac{1}{ab}(bx-xy);$$

$$N_3 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & x & y & xy \\ 1 & 0 & b & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \\ 1 & 0 & b & 0 \end{vmatrix}} = \frac{1}{ab} xy \,; \quad N_4 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \\ 1 & x & y & xy \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \\ 1 & 0 & b & 0 \end{vmatrix}} = \frac{1}{ab}(ay - xy)$$

where

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \\ 1 & 0 & b & 0 \end{vmatrix} = -(ab)^2$$

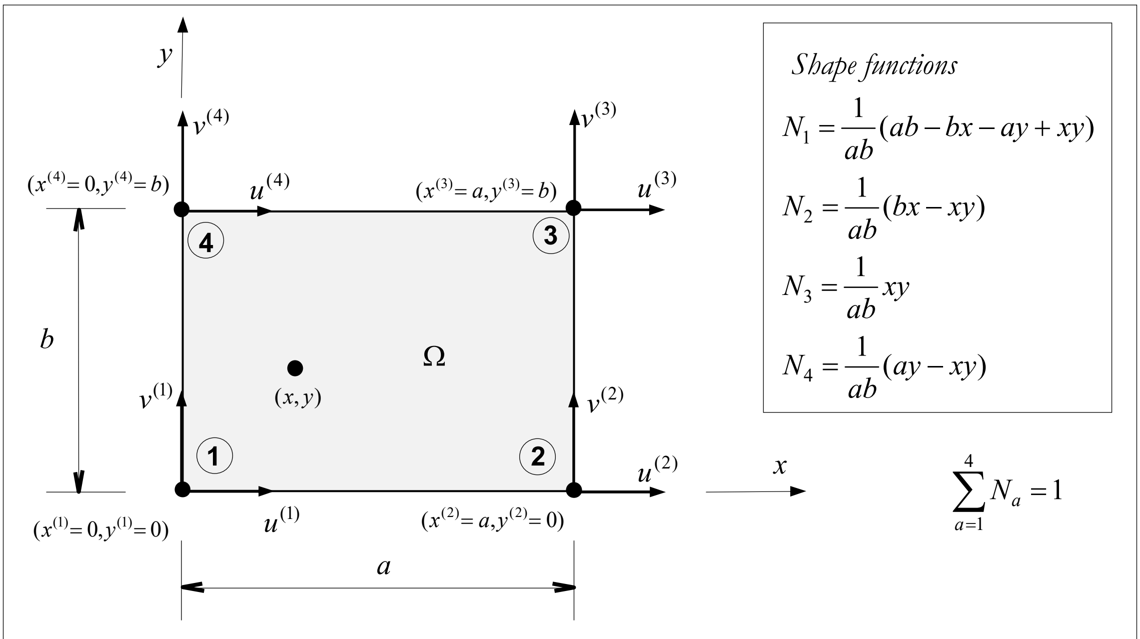


Figure 6.58: Rectangle element – shape functions.

We can apply the same procedure to obtain the shape functions in the normalized space, (see Figure 6.59). Then, if we replace (x, y) by (ξ, η) in the shape functions (6.167) we can obtain:

$$
N_1 = \frac{\begin{vmatrix} 1 & \xi & \eta & \xi\eta \\ 1 & \xi^{(2)} & \eta^{(2)} & \xi^{(2)}\eta^{(2)} \\ 1 & \xi^{(3)} & \eta^{(3)} & \xi^{(3)}\eta^{(3)} \\ 1 & \xi^{(4)} & \eta^{(4)} & x^{(4)}\eta^{(4)} \end{vmatrix}}{\begin{vmatrix} 1 & \xi^{(1)} & \eta^{(1)} & \xi^{(1)}\eta^{(1)} \\ 1 & \xi^{(2)} & \eta^{(2)} & \xi^{(2)}\eta^{(2)} \\ 1 & \xi^{(3)} & \eta^{(3)} & \xi^{(3)}\eta^{(3)} \\ 1 & \xi^{(4)} & \eta^{(4)} & \xi^{(4)}\eta^{(4)} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \xi & \eta & \xi\eta \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}} = \frac{1}{4}(1-\xi)(1-\eta) \,;
$$

$$
N_2 = \frac{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & \xi & \eta & \xi\eta \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}} = \frac{1}{4}(1+\xi)(1-\eta) \,; \quad N_3 = \frac{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & \xi & \eta & \xi\eta \\ 1 & -1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}} = \frac{1}{4}(1+\xi)(1+\eta) \,;
$$

$$
N_4 = \frac{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & \xi & \eta & \xi\eta \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}} = \frac{1}{4}(1-\xi)(1+\eta) \quad \text{where} \quad \begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = -16 \tag{6.168}
$$

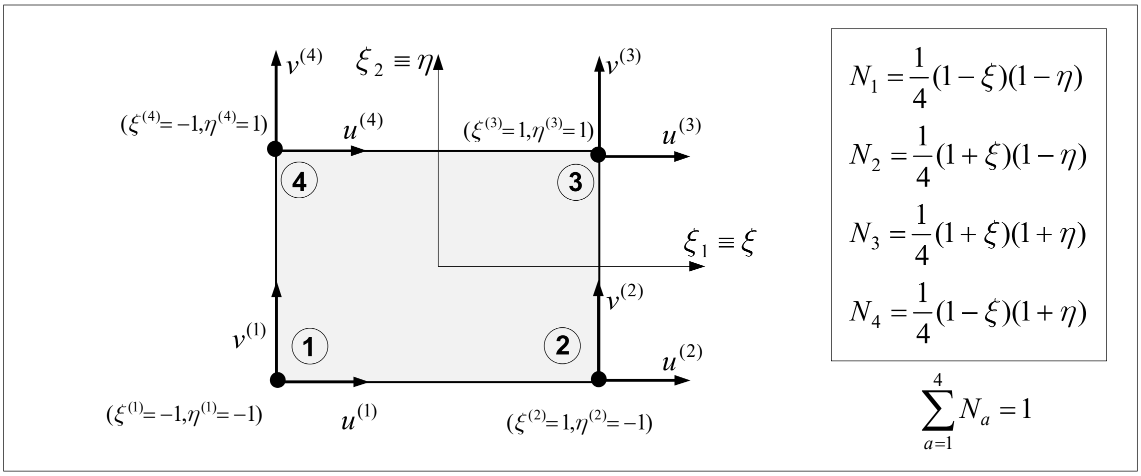


Figure 6.59: Rectangle element – normalized space.

NOTE 3.4: Another example: let us consider a one-dimensional case in which the displacement field is approach by a quadratic function ($u(x)=a_1+a_2x+a_3x^2$), so we will need three points in order to define the quadratic function, (see Figure 6.60). Next, we will obtain the shape functions $N_i(x)$ in order to express the displacement field:

$$u(x)=N_1u^{(1)}+N_2u^{(2)}+N_3u^{(3)}$$

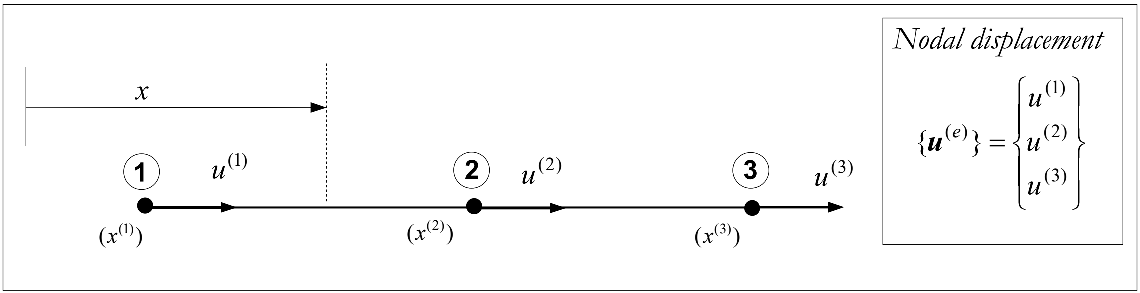


Figure 6.60: 1D element – quadratic function (generic system).

$$u(x)=a_1+a_2x+a_3x^2 \xrightarrow{\text{Nodal values}} \begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{Bmatrix}=\begin{bmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x^{(3)} & x^{(3)2} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

And the shape functions become:

$$N_1(x)=\frac{\begin{vmatrix} 1 & x & x^2 \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x^{(3)} & x^{(3)2} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x^{(3)} & x^{(3)2} \end{vmatrix}};\quad N_2(x)=\frac{\begin{vmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x & x^2 \\ 1 & x^{(3)} & x^{(3)2} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x^{(3)} & x^{(3)2} \end{vmatrix}};\quad N_3(x)=\frac{\begin{vmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & x^{(1)2} \\ 1 & x^{(2)} & x^{(2)2} \\ 1 & x^{(3)} & x^{(3)2} \end{vmatrix}} \tag{6.169}$$

Let us consider a particular case which is described in Figure 6.61. For this particular case the shape functions $N_i(x)$ are:

$$N_1=\frac{\begin{vmatrix} 1 & x & x^2 \\ 1 & \frac{L}{2} & (\frac{L}{2})^2 \\ 1 & L & L^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0^2 \\ 1 & \frac{L}{2} & (\frac{L}{2})^2 \\ 1 & L & L^2 \end{vmatrix}}=1-\frac{3x}{L}+\frac{2x^2}{L^2};\quad N_2=\frac{\begin{vmatrix} 1 & 0 & 0^2 \\ 1 & x & x^2 \\ 1 & L & L^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0^2 \\ 1 & \frac{L}{2} & (\frac{L}{2})^2 \\ 1 & L & L^2 \end{vmatrix}}=\frac{4x}{L}-\frac{4x^2}{L^2};\quad N_3=\frac{\begin{vmatrix} 1 & 0 & 0^2 \\ 1 & \frac{L}{2} & (\frac{L}{2})^2 \\ 1 & x & x^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0^2 \\ 1 & \frac{L}{2} & (\frac{L}{2})^2 \\ 1 & L & L^2 \end{vmatrix}}=\frac{-x}{L}+\frac{2x^2}{L^2}$$

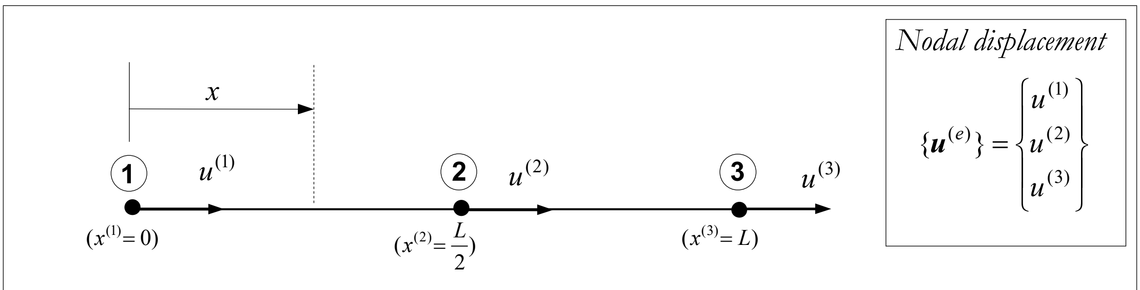


Figure 6.61: 1D element – quadratic function.

Note that the above shape functions are the same as the one obtained in **Problem 5.26** NOTE 1. We can also apply the equations in (6.169) for another system which is described in Figure 6.62.

Then, by considering the nodal values given by Figure 6.62, the equation (6.169) becomes:

$$N_1 = \frac{\begin{vmatrix} 1 & \xi & \xi^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \end{vmatrix}} = \frac{\xi}{2}(\xi - 1); \quad N_2 = \frac{\begin{vmatrix} 1 & -1 & (-1)^2 \\ 1 & \xi & \xi^2 \\ 1 & 1 & 1^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \end{vmatrix}} = 1 - \xi^2; \quad N_3 = \frac{\begin{vmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & \xi & \xi^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \end{vmatrix}} = \frac{\xi}{2}(1 + \xi) \tag{6.170}$$

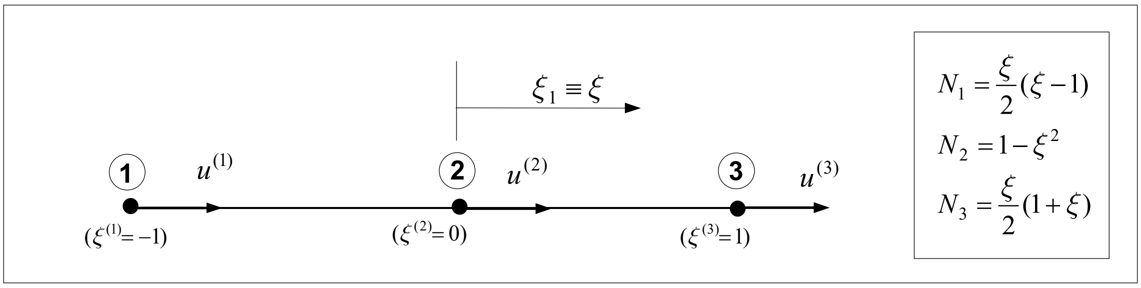


Figure 6.62: Quadratic element (normalized space).

NOTE 3.5: Another example: let us consider a one dimensional case in which the displacement field (according to z-direction) is approached by a cubic function ($w(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$), (see Figure 6.63). The nodal "displacement" vector is represented by:

$$\{\boldsymbol{u}^{(e)}\} = \begin{Bmatrix} w^{(1)} \\ w'^{(1)} \\ w^{(2)} \\ w'^{(2)} \end{Bmatrix}$$

where w' is the derivative of $w(x)$ with respect to x. Next, we will obtain the shape functions in order to obtain the displacement field:

$$w(x) = N_1 w^{(1)} + N_2 w'^{(1)} + N_3 w^{(2)} + N_4 w'^{(2)}$$

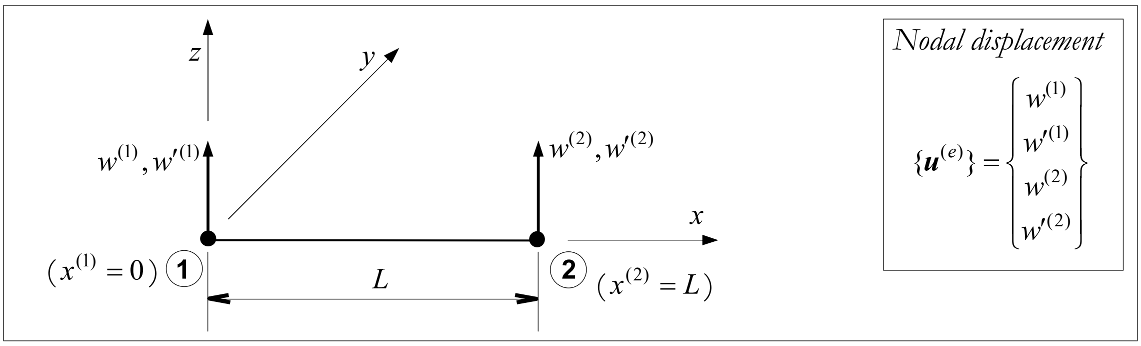


Figure 6.63: 1D element.

The derivative of $w(x)=a_1x^3+a_2x^2+a_3x+a_4$ with respect to x is given by:

$$w(x)=a_1x^3+a_2x^2+a_3x+a_4 \xrightarrow{\text{derivative}} \frac{\partial w(x)}{\partial x}\equiv w'(x)=3a_1x^2+2a_2x+a_3$$

Then, the relationship between the nodal values and the coefficients a_i is given by:

$$\left.\begin{aligned} &w(x=0)=w^{(1)}=a_4\\ &w'(x=0)=w'^{(1)}=a_3\\ &w(x=L)=w^{(2)}=a_1L^3+a_2L^2+a_3L+a_4\\ &w'(x=L)=w'^{(2)}=3a_1L^2+2a_2L+a_3 \end{aligned}\right\} \xrightarrow{\text{Matricial}} \begin{Bmatrix} w^{(1)}\\ w'^{(1)}\\ w^{(2)}\\ w'^{(2)} \end{Bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{bmatrix}\begin{Bmatrix} a_1\\ a_2\\ a_3\\ a_4 \end{Bmatrix}$$

Then, the shape functions can be obtained as follows

$$N_1=\frac{\begin{vmatrix} x^3 & x^2 & x & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}=\left[2\left(\frac{x}{L}\right)^3-3\left(\frac{x}{L}\right)^2+1\right]; \quad N_2=\frac{\begin{vmatrix} 0 & 0 & 0 & 1\\ x^3 & x^2 & x & 1\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}=\left[\frac{x^3}{L^2}-\frac{2x^2}{L}+x\right];$$

$$N_3=\frac{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ x^3 & x^2 & x & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}=\left[-2\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2\right]; \quad N_4=\frac{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ x^3 & x^2 & x & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ L^3 & L^2 & L & 1\\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}=\left[\frac{x^3}{L^2}-\frac{x^2}{L}\right]$$

Then,

$$w=N_1w^{(1)}+N_2w'^{(1)}+N_3w^{(2)}+N_4w'^{(2)}$$

$$w=\left[2\left(\frac{x}{L}\right)^3-3\left(\frac{x}{L}\right)^2+1\right]w^{(1)}+\left[\frac{x^3}{L^2}-\frac{2x^2}{L}+x\right]w'^{(1)}+\left[-2\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2\right]w^{(2)}+\left[\frac{x^3}{L^2}-\frac{x^2}{L}\right]w'^{(2)}$$

If we want to obtain the function $w'=\overline{N}_1w^{(1)}+\overline{N}_2w'^{(1)}+\overline{N}_3w^{(2)}+\overline{N}_4w'^{(2)}$, we can obtain similarly, i.e., now instead of replacing the terms $(x^3,x^2,x,1)$ we will replace the terms related to the derivative function $w'(x)=3a_1x^2+2a_2x+a_3$, $(3x^2,2x,1,0)$, i.e.:

$$
\overline{N}_1 = \frac{\begin{vmatrix} 3x^2 & 2x & 1 & 0 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}} = \frac{6x^2}{L^3} - \frac{6x}{L^2} = \frac{\partial N_1}{\partial x}; \quad \overline{N}_2 = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3x^2 & 2x & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}} = \frac{3x^2}{L^2} - \frac{4x}{L} + 1 = \frac{\partial N_2}{\partial x}
$$

$$
\overline{N}_3 = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 3x^2 & 2x & 1 & 0 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}} = -\frac{6x^2}{L^3} + \frac{6x}{L^2} = \frac{\partial N_3}{\partial x}; \quad \overline{N}_4 = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3x^2 & 2x & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{vmatrix}} = \frac{3x^2}{L^2} - \frac{2x}{L} = \frac{\partial N_4}{\partial x}
$$

then

$$
\begin{aligned}
w' &= \frac{\partial N_1}{\partial x} w^{(1)} + \frac{\partial N_2}{\partial x} w'^{(1)} + \frac{\partial N_3}{\partial x} w^{(2)} + \frac{\partial N_4}{\partial x} w'^{(2)} \\
w' &= \overline{N}_1 w^{(1)} + \overline{N}_2 w'^{(1)} + \overline{N}_3 w^{(2)} + \overline{N}_4 w'^{(2)} \\
w' &= \left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] w^{(1)} + \left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] w'^{(1)} + \left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] w^{(2)} + \left[\frac{3x^2}{L^2} - \frac{2x}{L}\right] w'^{(2)}
\end{aligned}
$$

Next, let us consider a very simple problem which was already discussed in **Problem 5.25**-NOTE 4. In this case the function is linear, (see Figure 6.64), so, $u(x) = a_1 + a_2 x$, and

$$
\begin{Bmatrix} u^{(1)} \\ u^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}
$$

Then, the shape functions for this problem can be obtained as follows:

$$
N_1(x) = \frac{\begin{vmatrix} 1 & x \\ 1 & x^{(2)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix}} = \frac{(x^{(2)} - x)}{(x^{(2)} - x^{(1)})} \quad ; \quad N_2(x) = \frac{\begin{vmatrix} 1 & x^{(1)} \\ 1 & x \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} \\ 1 & x^{(2)} \end{vmatrix}} = \frac{(x - x^{(1)})}{(x^{(2)} - x^{(1)})} \tag{6.171}
$$

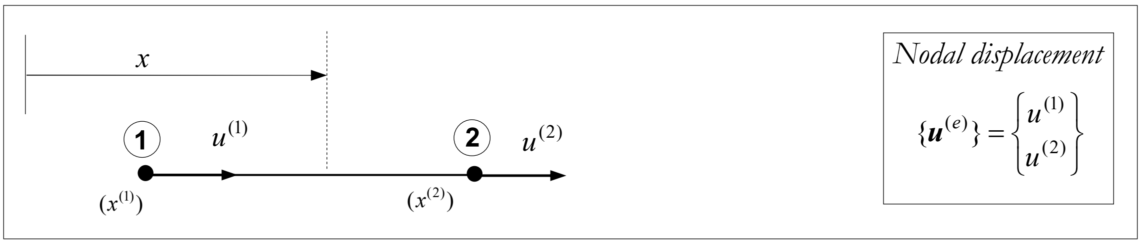


Figure 6.64: 1D element – linear function.

Now if we consider the normalized space, (see Figure 6.65), the shape functions become:

$$N_1(\xi)=\frac{\begin{vmatrix}1 & \xi\\ 1 & \xi^{(2)}\end{vmatrix}}{\begin{vmatrix}1 & \xi^{(1)}\\ 1 & \xi^{(2)}\end{vmatrix}}=\frac{\begin{vmatrix}1 & \xi\\ 1 & 1\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}}=\frac{(1-\xi)}{2} \quad ; \quad N_2(\xi)=\frac{\begin{vmatrix}1 & -1\\ 1 & \xi\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}}=\frac{(1+\xi)}{2}$$

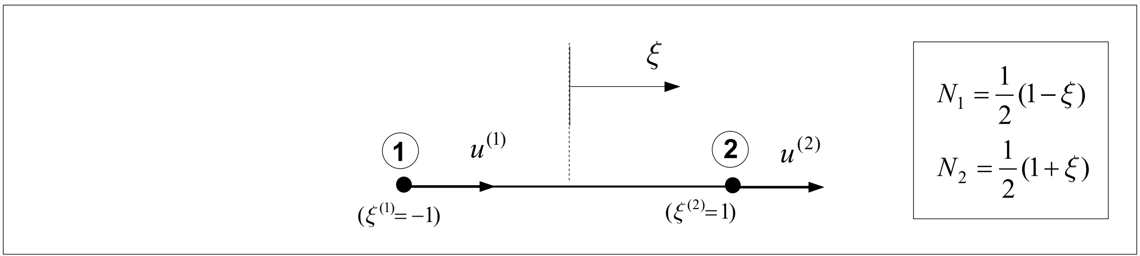


Figure 6.65: 1D element – linear function (normalized space).

Note that the shape functions can be used to approach any function even the geometry, so, if we want to represent the geometry x by using the normalized space, (see Figure 6.66), it is enough to do:

$$x=N_1(\xi)x^{(i)}+N_2(\xi)x^{(f)}=\frac{(1-\xi)}{2}x^{(i)}+\frac{(1+\xi)}{2}x^{(f)}=\frac{(x^{(i)}+x^{(f)})}{2}+\frac{(x^{(f)}-x^{(i)})}{2}\xi \qquad (6.172)$$

and the differential dx can be obtained as follows:

$$dx=d\left(\frac{(x^{(i)}+x^{(f)})}{2}+\frac{(x^{(f)}-x^{(i)})}{2}\xi\right)=\frac{(x^{(f)}-x^{(i)})}{2}d\xi \qquad (6.173)$$

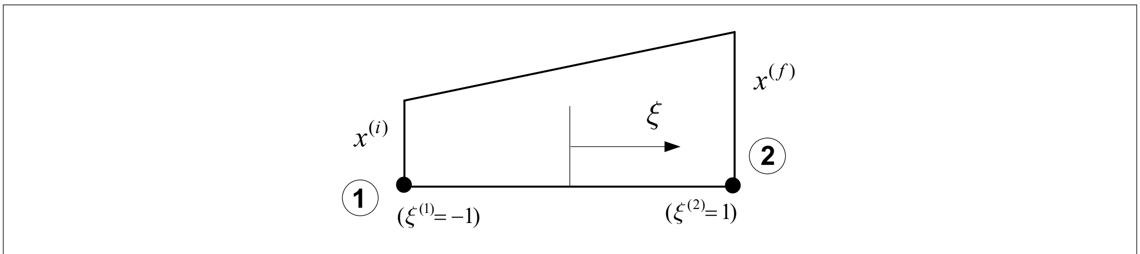


Figure 6.66: 1D element – linear function (normalized space).

Note that the equation in (6.172) is the transformation between the normalized system (Figure 6.65) and the Cartesian system (Figure 6.64).

When the displacement and the geometry are approached by the same shape functions the element is called *Isoparametric Element*.

NOTE 4: Shape Functions in the Normalized Space.

NOTE 4.1: Shape Functions for 1D in the Normalized Space.

We can generalize the shape functions for one-dimensional (1D) in the Normalized space by using the Lagrange's Polynomial of degree $(n-1)$, namely:

$$
\begin{aligned}
N_a^{(n-1)}(\xi) &= \frac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})\cdots(\xi-\xi^{(a-1)})(\xi-\xi^{(a+1)})\cdots(\xi-\xi^{(n)})}{(\xi^{(a)}-\xi^{(1)})(\xi^{(a)}-\xi^{(2)})\cdots(\xi^{(a)}-\xi^{(a-1)})(\xi^{(a)}-\xi^{(a+1)})\cdots(\xi^{(a)}-\xi^{(n)})} \\
&= \prod_{a=1(a\neq j)}^{n}\left(\frac{(\xi-\xi^{(a)})}{(\xi^{(a)}-\xi^{(j)})}\right) \qquad (n - \text{number of nodes})
\end{aligned}
\tag{6.174}
$$

For example, for the element with 3-nodes, described in Figure 6.62, we have:

$$
N_a^{(2)}(\xi) = \frac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})}{(\xi^{(a)}-\xi^{(1)})(\xi^{(a)}-\xi^{(2)})(\xi^{(a)}-\xi^{(3)})}
$$

$$
\Rightarrow \begin{cases}
N_1^{(2)}(\xi) = \dfrac{(\xi-\xi^{(2)})(\xi-\xi^{(3)})}{(\xi^{(1)}-\xi^{(2)})(\xi^{(1)}-\xi^{(3)})} = \dfrac{(\xi-0)(\xi-1)}{((-1)-0)((-1)-1)} = \dfrac{1}{2}\xi(\xi-1) \\[2ex]
N_2^{(2)}(\xi) = \dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(3)})}{(\xi^{(2)}-\xi^{(1)})(\xi^{(2)}-\xi^{(3)})} = \dfrac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = (1-\xi^2) \\[2ex]
N_3^{(2)}(\xi) = \dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})}{(\xi^{(3)}-\xi^{(1)})(\xi^{(3)}-\xi^{(2)})} = \dfrac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)} = \dfrac{1}{2}\xi(\xi+1)
\end{cases}
$$

which results match with the one in equation (6.170).

The shape functions for the 1D element with 4-nodes, described in Figure 6.67, can be obtained as follows:

$$
N_a^{(3)}(\xi) = \frac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})}{(\xi^{(a)}-\xi^{(1)})(\xi^{(a)}-\xi^{(2)})(\xi^{(a)}-\xi^{(3)})(\xi^{(a)}-\xi^{(4)})}
$$

Then

$$
\begin{cases}
N_1^{(3)} = \dfrac{(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})}{(\xi^{(1)}-\xi^{(2)})(\xi^{(1)}-\xi^{(3)})(\xi^{(1)}-\xi^{(4)})} = \dfrac{[\xi-(\frac{-1}{3})][\xi-(\frac{1}{3})](\xi-1)}{[(-1)-(\frac{-1}{3})][(-1)-(\frac{1}{3})][(-1)-1]} = \dfrac{1}{16}(9\xi^2-1)(1-\xi) \\[2ex]
N_2^{(3)} = \dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})}{(\xi^{(2)}-\xi^{(1)})(\xi^{(2)}-\xi^{(3)})(\xi^{(2)}-\xi^{(4)})} = \dfrac{[\xi-(-1)][\xi-(\frac{1}{3})](\xi-1)}{[(\frac{-1}{3})-(-1)][(\frac{-1}{3})-(\frac{1}{3})][(\frac{-1}{3})-1]} = \dfrac{9}{16}(1-\xi^2)(1-3\xi) \\[2ex]
N_3^{(3)} = \dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(4)})}{(\xi^{(3)}-\xi^{(1)})(\xi^{(3)}-\xi^{(2)})(\xi^{(3)}-\xi^{(4)})} = \dfrac{[\xi-(-1)][\xi-(\frac{-1}{3})](\xi-1)}{[(\frac{1}{3})-(-1)][(\frac{1}{3})-(\frac{-1}{3})][(\frac{1}{3})-1]} = \dfrac{9}{16}(1-\xi^2)(1+3\xi) \\[2ex]
N_4^{(3)} = \dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})}{(\xi^{(4)}-\xi^{(1)})(\xi^{(4)}-\xi^{(2)})(\xi^{(4)}-\xi^{(3)})} = \dfrac{[\xi-(-1)][\xi-(\frac{-1}{3})][\xi-(\frac{1}{3})]}{[(1)-(-1)][(1)-(\frac{-1}{3})][(1)-(\frac{1}{3})]} = \dfrac{1}{16}(9\xi^2-1)(1+\xi)
\end{cases}
$$

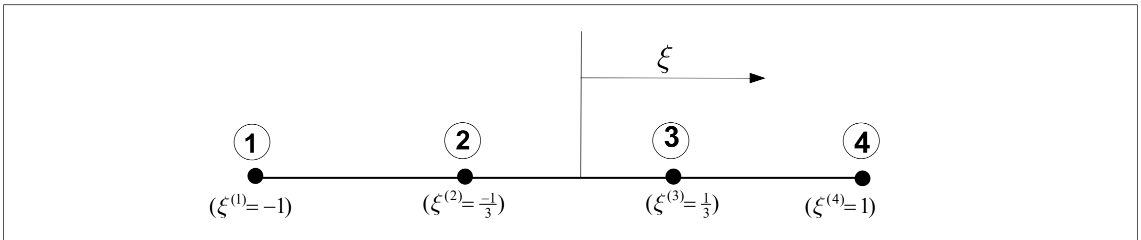


Figure 6.67: Cubic element (normalized space).

The shape functions for the element with 5-nodes described in Figure 6.68 can be obtained as follows:

$$N_a^{(4)}(\xi)=\frac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})(\xi^{(a)}-\xi^{(5)})}{(\xi^{(a)}-\xi^{(1)})(\xi^{(a)}-\xi^{(2)})(\xi^{(a)}-\xi^{(3)})(\xi^{(a)}-\xi^{(4)})(\xi^{(a)}-\xi^{(5)})}$$

Then

$$\begin{cases}
N_1^{(4)}(\xi)=\dfrac{(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})(\xi^{(a)}-\xi^{(5)})}{(\xi^{(1)}-\xi^{(2)})(\xi^{(1)}-\xi^{(3)})(\xi^{(1)}-\xi^{(4)})(\xi^{(1)}-\xi^{(5)})}=\dfrac{1}{6}\xi(4\xi^2-1)(\xi-1)\\
N_2^{(4)}(\xi)=\dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})(\xi^{(a)}-\xi^{(5)})}{(\xi^{(2)}-\xi^{(1)})(\xi^{(2)}-\xi^{(3)})(\xi^{(2)}-\xi^{(4)})(\xi^{(2)}-\xi^{(5)})}=\dfrac{4}{3}\xi(\xi^2-1)(1-2\xi)\\
N_3^{(4)}(\xi)=\dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(4)})(\xi^{(a)}-\xi^{(5)})}{(\xi^{(3)}-\xi^{(1)})(\xi^{(3)}-\xi^{(2)})(\xi^{(3)}-\xi^{(4)})(\xi^{(3)}-\xi^{(5)})}=\xi(1-\xi^2)(1-4\xi^2)\\
N_4^{(4)}(\xi)=\dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi^{(a)}-\xi^{(5)})}{(\xi^{(4)}-\xi^{(1)})(\xi^{(4)}-\xi^{(2)})(\xi^{(4)}-\xi^{(3)})(\xi^{(4)}-\xi^{(5)})}=\dfrac{4}{3}\xi(1-\xi^2)(1+2\xi)\\
N_5^{(4)}(\xi)=\dfrac{(\xi-\xi^{(1)})(\xi-\xi^{(2)})(\xi-\xi^{(3)})(\xi-\xi^{(4)})}{(\xi^{(5)}-\xi^{(1)})(\xi^{(5)}-\xi^{(2)})(\xi^{(5)}-\xi^{(3)})(\xi^{(5)}-\xi^{(4)})}=\dfrac{1}{6}\xi(4\xi^2-1)(1+\xi)
\end{cases}$$

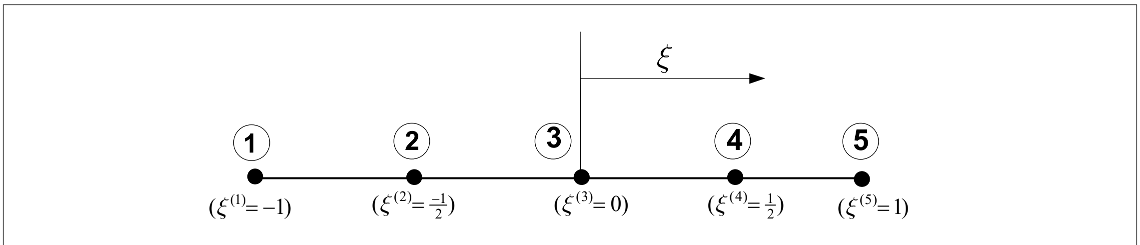
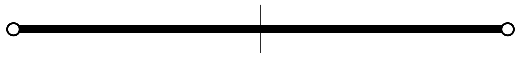


Figure 6.68: Quartic element (normalized space).

Table 6.1: Shape functions for 1D in the Normalized Space $[-1,1]$.

Linear Element  $\xi^{(1)}=-1$ $\quad$ $\xi^{(2)}=1$	$\begin{cases} N_1^{(1)}(\xi)=\frac{1}{2}(1-\xi) \\ N_2^{(1)}(\xi)=\frac{1}{2}(1+\xi) \end{cases}$
Quadratic Element  $\xi^{(1)}=-1$ $\quad$ $\xi^{(2)}=0$ $\quad$ $\xi^{(3)}=1$	$\begin{cases} N_1^{(2)}(\xi)=\frac{1}{2}\xi(\xi-1) \\ N_2^{(2)}(\xi)=(1-\xi^2) \\ N_3^{(2)}(\xi)=\frac{1}{2}\xi(\xi+1) \end{cases}$
Cubic Element  $\xi^{(1)}=-1$ $\quad$ $\xi^{(2)}=\frac{-1}{3}$ $\quad$ $\xi^{(3)}=\frac{1}{3}$ $\quad$ $\xi^{(4)}=1$	$\begin{cases} N_1^{(3)}(\xi)=\frac{1}{16}(9\xi^2-1)(1-\xi) \\ N_2^{(3)}(\xi)=\frac{9}{16}(1-\xi^2)(1-3\xi) \\ N_3^{(3)}(\xi)=\frac{9}{16}(1-\xi^2)(1+3\xi) \\ N_4^{(3)}(\xi)=\frac{1}{16}(9\xi^2-1)(1+\xi) \end{cases}$
Quartic Element  $\xi^{(1)}=-1$ $\quad$ $\xi^{(2)}=-\frac{1}{2}$ $\quad$ $\xi^{(3)}=0$ $\quad$ $\xi^{(4)}=\frac{1}{2}$ $\quad$ $\xi^{(5)}=1$	$\begin{cases} N_1^{(4)}(\xi)=\frac{1}{6}\xi(4\xi^2-1)(\xi-1) \\ N_2^{(4)}(\xi)=\frac{4}{3}\xi(\xi^2-1)(1-2\xi) \\ N_3^{(4)}(\xi)=\xi(1-\xi^2)(1-4\xi^2) \\ N_4^{(4)}(\xi)=\frac{4}{3}\xi(1-\xi^2)(1+2\xi) \\ N_5^{(4)}(\xi)=\frac{1}{6}\xi(4\xi^2-1)(1+\xi) \end{cases}$

NOTE 4.2: Shape Functions for 2D in the Normalized Space.

The shape functions for two-dimensional (2D) elements in the normalized space can be obtained by combining the shape functions for 1D according to the directions ξ and η. For example, for the case presented in Figure 6.59 and by taking into account the Table 6.1 we can conclude that:

$$
\begin{cases}
N_1(\xi,\eta)=N_1^{(1)}(\xi)N_1^{(1)}(\eta)=\frac{1}{2}(1-\xi)\frac{1}{2}(1-\eta)=\frac{1}{4}(1-\xi)(1-\eta) \\
N_2(\xi,\eta)=N_2^{(1)}(\xi)N_1^{(1)}(\eta)=\frac{1}{2}(1+\xi)\frac{1}{2}(1-\eta)=\frac{1}{4}(1+\xi)(1-\eta) \\
N_3(\xi,\eta)=N_2^{(1)}(\xi)N_2^{(1)}(\eta)=\frac{1}{2}(1+\xi)\frac{1}{2}(1+\eta)=\frac{1}{4}(1+\xi)(1+\eta) \\
N_4(\xi,\eta)=N_1^{(1)}(\xi)N_2^{(1)}(\eta)=\frac{1}{2}(1-\xi)\frac{1}{2}(1+\eta)=\frac{1}{4}(1-\xi)(1+\eta)
\end{cases}
\tag{6.175}
$$

which results match the shape functions given in Figure 6.59.

For the rectangle with 9 nodes (see Figure 6.69), the shape functions can be obtained by combining the shape functions given by 1D element with 3 nodes (see Table 6.1), i.e.:

$$
\begin{cases}
N_1(\xi,\eta) = N_1^{(2)}(\xi)N_1^{(2)}(\eta) = \frac{1}{2}\xi(\xi-1)\frac{1}{2}\eta(\eta-1) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1) \\
N_2(\xi,\eta) = N_1^{(2)}(\xi)N_1^{(2)}(\eta) = (1-\xi^2)\frac{1}{2}\eta(\eta-1) = \frac{1}{2}\eta(1-\xi^2)(\eta-1) \\
N_3(\xi,\eta) = N_1^{(2)}(\xi)N_1^{(2)}(\eta) = \frac{1}{2}\xi(\xi+1)\frac{1}{2}\eta(\eta-1) = \frac{1}{4}\xi\eta(\xi+1)(\eta-1) \\
N_4(\xi,\eta) = N_3^{(2)}(\xi)N_2^{(2)}(\eta) = \frac{1}{2}\xi(\xi+1)(1-\eta^2) = \frac{1}{2}\xi(\xi+1)(1-\eta^2) \\
N_5(\xi,\eta) = N_3^{(2)}(\xi)N_3^{(2)}(\eta) = \frac{1}{2}\xi(\xi+1)\frac{1}{2}\eta(\eta+1) = \frac{1}{4}\xi\eta(\xi+1)(\eta+1) \\
N_6(\xi,\eta) = N_2^{(2)}(\xi)N_3^{(2)}(\eta) = (1-\xi^2)\frac{1}{2}\eta(\eta+1) = \frac{1}{2}\eta(1-\xi^2)(\eta+1) \\
N_7(\xi,\eta) = N_1^{(2)}(\xi)N_3^{(2)}(\eta) = \frac{1}{2}\xi(\xi-1)\frac{1}{2}\eta(\eta+1) = \frac{1}{4}\xi\eta(\xi-1)(\eta+1) \\
N_8(\xi,\eta) = N_1^{(2)}(\xi)N_2^{(2)}(\eta) = \frac{1}{2}\xi(\xi-1)(1-\eta^2) = \frac{1}{2}\xi(\xi-1)(1-\eta^2) \\
N_9(\xi,\eta) = N_2^{(2)}(\xi)N_2^{(2)}(\eta) = (1-\xi^2)(1-\eta^2) = (1-\xi^2)(1-\eta^2)
\end{cases}
\tag{6.176}
$$

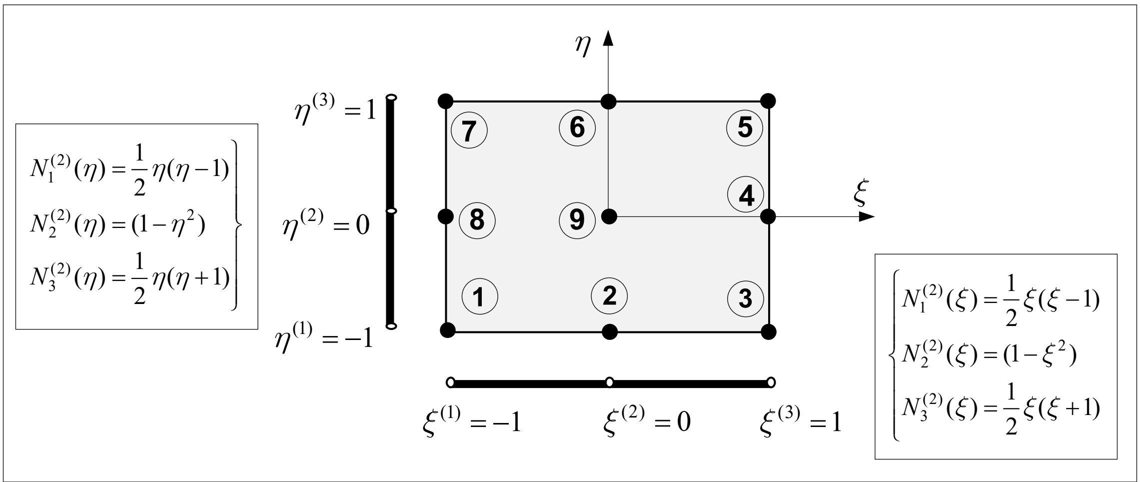


Figure 6.69: Quadrilateral element – quadratic function – normalized space.

For the rectangle with 16 nodes, (see Figure 6.70), the shape functions are obtained by combining the 1D shape functions for 1D element with 4 nodes $N_i^{(3)}$, i.e.:

$$
\begin{aligned}
&N_1(\xi,\eta) = N_1^{(3)}(\xi)N_1^{(3)}(\eta); \quad N_2(\xi,\eta) = N_2^{(3)}(\xi)N_1^{(3)}(\eta); \quad N_3(\xi,\eta) = N_3^{(3)}(\xi)N_1^{(3)}(\eta); \\
&N_4(\xi,\eta) = N_4^{(3)}(\xi)N_1^{(3)}(\eta); \quad N_5(\xi,\eta) = N_4^{(3)}(\xi)N_2^{(3)}(\eta); \quad N_6(\xi,\eta) = N_4^{(3)}(\xi)N_3^{(3)}(\eta); \\
&N_7(\xi,\eta) = N_4^{(3)}(\xi)N_4^{(3)}(\eta); \quad N_8(\xi,\eta) = N_3^{(3)}(\xi)N_4^{(3)}(\eta); \quad N_9(\xi,\eta) = N_2^{(3)}(\xi)N_4^{(3)}(\eta); \\
&N_{10}(\xi,\eta) = N_1^{(3)}(\xi)N_4^{(3)}(\eta); \quad N_{11}(\xi,\eta) = N_4^{(3)}(\xi)N_3^{(3)}(\eta); \quad N_{12}(\xi,\eta) = N_4^{(3)}(\xi)N_2^{(3)}(\eta); \\
&N_{13}(\xi,\eta) = N_2^{(3)}(\xi)N_2^{(3)}(\eta); \quad N_{14}(\xi,\eta) = N_3^{(3)}(\xi)N_2^{(3)}(\eta); \quad N_{15}(\xi,\eta) = N_3^{(3)}(\xi)N_3^{(3)}(\eta); \\
&N_{16}(\xi,\eta) = N_2^{(3)}(\xi)N_3^{(3)}(\eta)
\end{aligned}
\tag{6.177}
$$

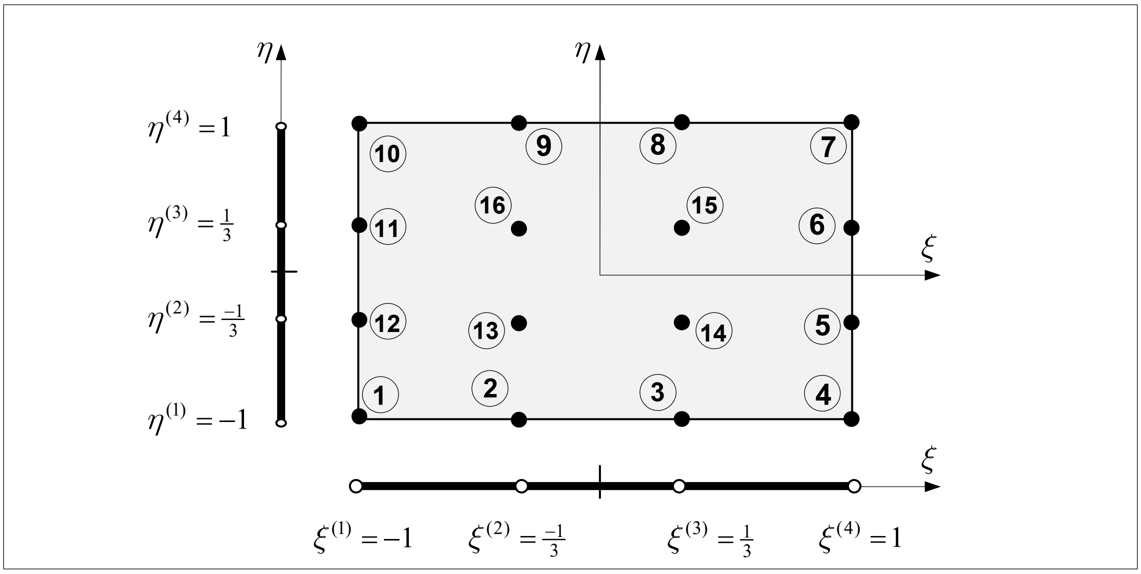


Figure 6.70: Quadrilateral element – cubic function – normalized space.

NOTE 4.3: Shape Functions for 3D in the Normalized Space.

The shape functions for the hexahedron element with 8 nodes in the normalized space, (see Figure 6.71), can be obtained by combining the shape functions for 1D according to the directions ξ, η and ζ, i.e.:

$$
\begin{cases}
N_1(\vec{\xi}) = N_1^{(1)}(\xi)N_1^{(1)}(\eta)N_1^{(1)}(\zeta) = \dfrac{1}{2}(1-\xi)\dfrac{1}{2}(1-\eta)\dfrac{1}{2}(1-\zeta) = \dfrac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \\
N_2(\vec{\xi}) = N_2^{(1)}(\xi)N_1^{(1)}(\eta)N_1^{(1)}(\zeta) = \dfrac{1}{2}(1+\xi)\dfrac{1}{2}(1-\eta)\dfrac{1}{2}(1-\zeta) = \dfrac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \\
N_3(\vec{\xi}) = N_2^{(1)}(\xi)N_1^{(1)}(\eta)N_1^{(1)}(\zeta) = \dfrac{1}{2}(1+\xi)\dfrac{1}{2}(1+\eta)\dfrac{1}{2}(1-\zeta) = \dfrac{1}{8}(1+\xi)(1+\eta)(1-\zeta) \\
N_4(\vec{\xi}) = N_1^{(1)}(\xi)N_2^{(1)}(\eta)N_1^{(1)}(\zeta) = \dfrac{1}{2}(1-\xi)\dfrac{1}{2}(1+\eta)\dfrac{1}{2}(1-\zeta) = \dfrac{1}{8}(1-\xi)(1+\eta)(1-\zeta) \\
N_5(\vec{\xi}) = N_1^{(1)}(\xi)N_1^{(1)}(\eta)N_2^{(1)}(\zeta) = \dfrac{1}{2}(1-\xi)\dfrac{1}{2}(1-\eta)\dfrac{1}{2}(1+\zeta) = \dfrac{1}{8}(1-\xi)(1-\eta)(1+\zeta) \\
N_6(\vec{\xi}) = N_2^{(1)}(\xi)N_1^{(1)}(\eta)N_2^{(1)}(\zeta) = \dfrac{1}{2}(1+\xi)\dfrac{1}{2}(1-\eta)\dfrac{1}{2}(1+\zeta) = \dfrac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \\
N_7(\vec{\xi}) = N_2^{(1)}(\xi)N_2^{(1)}(\eta)N_2^{(1)}(\zeta) = \dfrac{1}{2}(1+\xi)\dfrac{1}{2}(1+\eta)\dfrac{1}{2}(1+\zeta) = \dfrac{1}{8}(1+\xi)(1+\eta)(1+\zeta) \\
N_8(\vec{\xi}) = N_1^{(1)}(\xi)N_2^{(1)}(\eta)N_2^{(1)}(\zeta) = \dfrac{1}{2}(1-\xi)\dfrac{1}{2}(1+\eta)\dfrac{1}{2}(1+\zeta) = \dfrac{1}{8}(1-\xi)(1+\eta)(1+\zeta)
\end{cases}
\tag{6.178}
$$

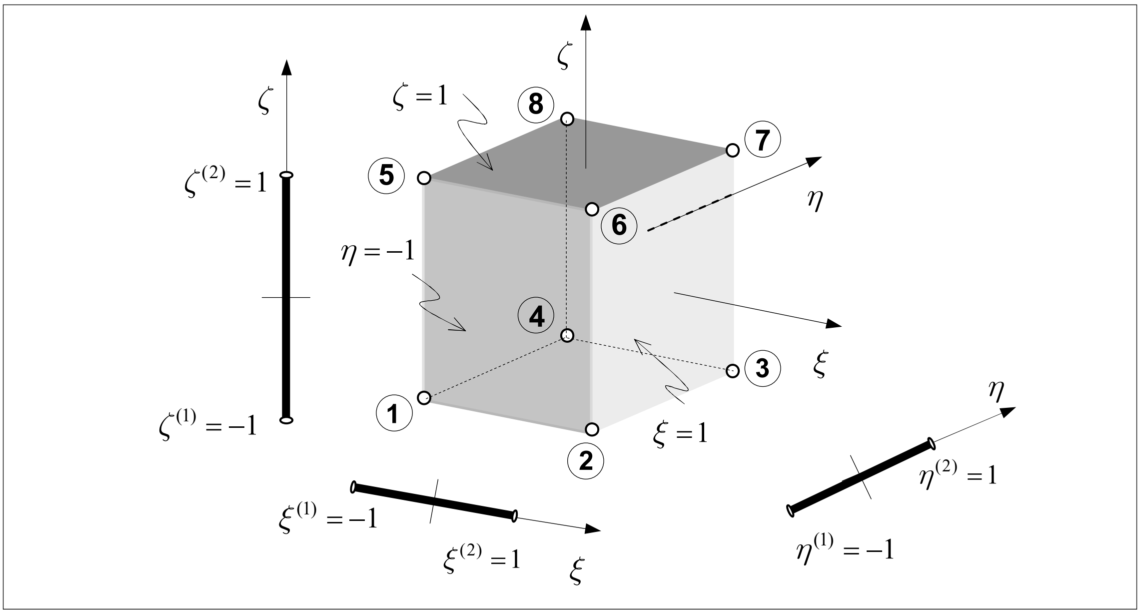


Figure 6.71: Hexahedron element with 8 nodes – linear function – normalized space.

The shape functions for the hexahedron element with 27 nodes in the normalized space, (Oñate (1992)), can be obtained by combining the shape functions for 1D according to the directions ξ, η and ζ, (see Figure 6.72).

For the nodes $i = 1,2,3,4,5,6,7,8$:

$$N_i(\vec{\xi}) = \frac{1}{8}(\xi^2 + \xi\xi^{(i)})(\eta^2 + \eta\eta^{(i)})(\zeta^2 + \zeta\zeta^{(i)}) \tag{6.179}$$

For the nodes $i = 9,10,11,12,13,14,15,16,17,18,19,20$:

$$\begin{aligned} N_i(\vec{\xi}) = \frac{1}{4}\eta^{(i)^2}(\eta^2 - \eta\eta^{(i)})\zeta^{(i)^2}(\zeta^2 - \zeta\zeta^{(i)})(1-\xi^2) + \frac{1}{4}\zeta^{(i)^2}(\zeta^2 - \zeta\zeta^{(i)}) + \\ \xi^{(i)^2}(\xi^2 - \xi\xi^{(i)})(1-\eta^2) + \frac{1}{4}\xi^{(i)^2}(\xi^2 - \xi\xi^{(i)})\eta^{(i)^2}(\eta^2 - \eta\eta^{(i)})(1-\zeta^2) \end{aligned} \tag{6.180}$$

For the nodes $i = 21,22,23,24,25,26$:

$$\begin{aligned} N_i(\vec{\xi}) = \frac{1}{2}(1-\xi^2)(1-\eta^2)(\zeta^2 - \zeta^2\zeta^{(i)}) + \frac{1}{2}(1-\eta^2)(1-\zeta^2)(\xi^2 - \xi^2\xi^{(i)^2}) + \\ \frac{1}{2}(1-\xi^2)(1-\zeta^2)(\eta^2 - \eta^2\eta^{(i)}) \end{aligned} \tag{6.181}$$

For the node $i = 27$:

$$N_{27}(\vec{\xi}) = N_2^{(2)}(\xi)N_2^{(2)}(\eta)N_2^{(2)}(\zeta) = (1-\xi^2)(1-\eta^2)(1-\zeta^2) \tag{6.182}$$

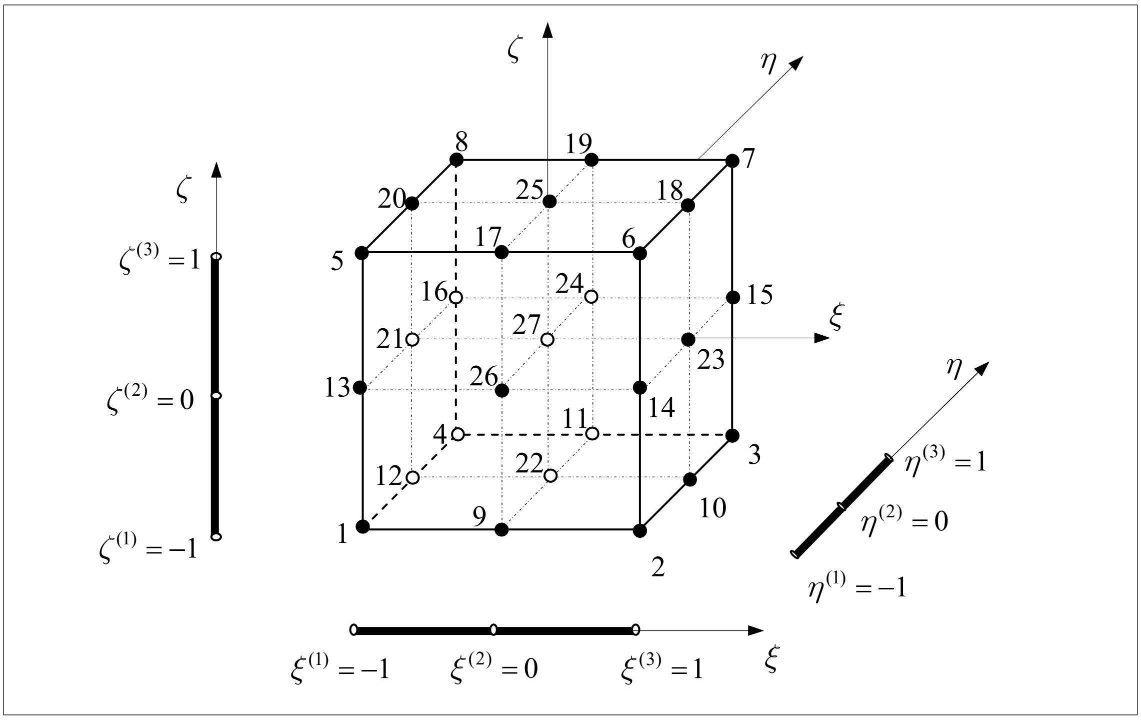


Figure 6.72: Hexahedron element with 27 nodes – quadratic function – normalized space.

NOTE 5: Area Coordinates

Let us consider the triangle and the plane defined in Figure 6.73(a), and by using triangle properties the following is true:

$$\frac{1}{h}=\frac{L_1}{h_1} \quad \Rightarrow \quad L_1=\frac{h_1}{h} \tag{6.183}$$

Note that the areas A and A_1, (see Figure 6.73(b)), can be expressed as follows:

$$A=\frac{bh}{2} \quad \Rightarrow \quad h=\frac{2A}{b} \tag{6.184}$$

and

$$A_1=\frac{bh_1}{2} \quad \Rightarrow \quad h_1=\frac{2A_1}{b} \tag{6.185}$$

Then

$$L_1=\frac{h_1}{h}=\frac{\frac{2A_1}{b}}{\frac{2A}{b}}=\frac{A_1}{A} \tag{6.186}$$

In Figure 6.73(b) we can appreciate how L_1 changes into the triangle.

In the same fashion we can define that $L_2=\frac{A_2}{A}$ and $L_3=\frac{A_3}{A}$, then defining the Area Coordinates:

$$L_1=\frac{A_1}{A} \quad ; \quad L_2=\frac{A_2}{A} \quad ; \quad L_3=\frac{A_3}{A} \tag{6.187}$$

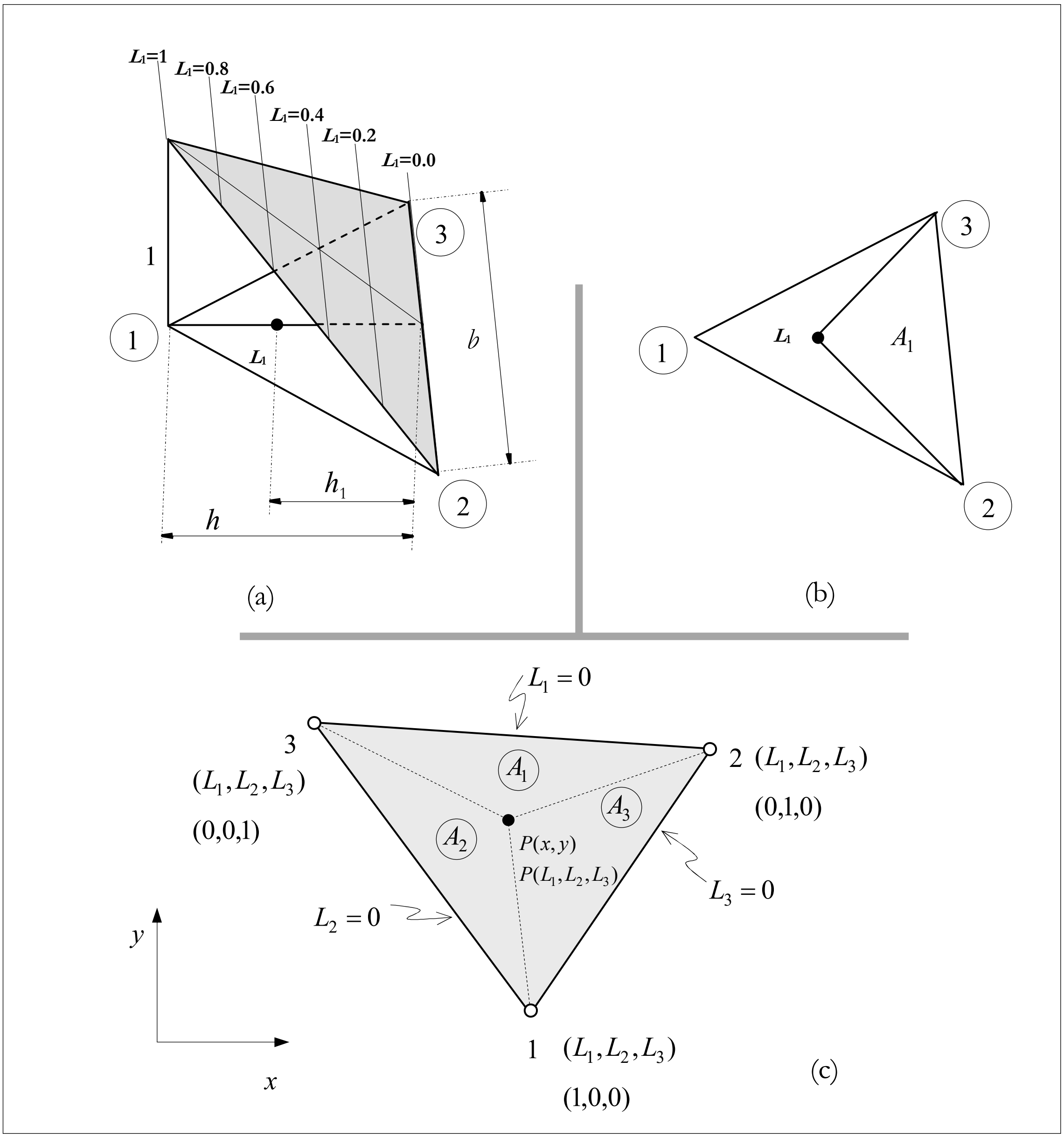


Figure 6.73: Area coordinates.

Note that since $A = A_1 + A_2 + A_3$ the following is true:

$$L_1 + L_2 + L_3 = \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1 \tag{6.188}$$

The geometry can be approached by using area coordinates as follows:

$$\begin{cases} x = L_1 x^{(1)} + L_2 x^{(2)} + L_3 x^{(3)} \\ y = L_1 y^{(1)} + L_2 y^{(2)} + L_3 y^{(3)} \end{cases} \xrightarrow{\text{Matrix form}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x^{(1)} & x^{(2)} & x^{(3)} \\ y^{(1)} & y^{(2)} & y^{(3)} \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \tag{6.189}$$

If the function is in terms of area coordinates $f = f(L_1, L_2, L_3)$, the derivative with respect to x and y can be obtained as follows:

$$\frac{\partial f}{\partial x} = \sum_{a=1}^{3} \frac{\partial f}{\partial L_a} \frac{\partial L_a}{\partial x} \quad ; \quad \frac{\partial f}{\partial y} = \sum_{a=1}^{3} \frac{\partial f}{\partial L_a} \frac{\partial L_a}{\partial y} \tag{6.190}$$

The integral can easily be obtained:

$$\int_L L_1^a L_2^b dL = \frac{a!b!}{(a+b+1)!} L$$
$$\int_A L_1^a L_2^b L_3^c dA = \frac{a!b!c!}{(a+b+c+2)!} 2A \tag{6.191}$$
$$\int_V L_1^a L_2^b L_3^c L_4^d dV = \frac{a!b!c!d!}{(a+b+c+d+3)!} 3!V$$

where $n!$ stands for the factorial of n, i.e. $n! = 1 \times 2 \times 3 \cdots \times (n-1) \times n$

Triangle element with 3-nodes.

We can use the Lagrange's Polynomial, (see equation (6.174)), in order to derive the shape functions for triangles and tetrahedrons. In this case the coordinates is constituted by the coordinates L_i. For example, for the triangle with 3-nodes, (see Figure 6.74) we have:

$$N_1(L_i) = \frac{(L_1 - L_1^{(2-3)})}{(L_1^{(1)} - L_1^{(2-3)})} = \frac{(L_1 - 0)}{(1-0)} = L_1$$
$$N_2(L_i) = \frac{(L_2 - L_2^{(1-3)})}{(L_2^{(2)} - L_2^{(1-3)})} = \frac{(L_2 - 0)}{(1-0)} = L_2 \tag{6.192}$$
$$N_3(L_i) = \frac{(L_3 - L_3^{(1-2)})}{(L_3^{(3)} - L_3^{(1-2)})} = \frac{(L_3 - 0)}{(1-0)} = L_3$$

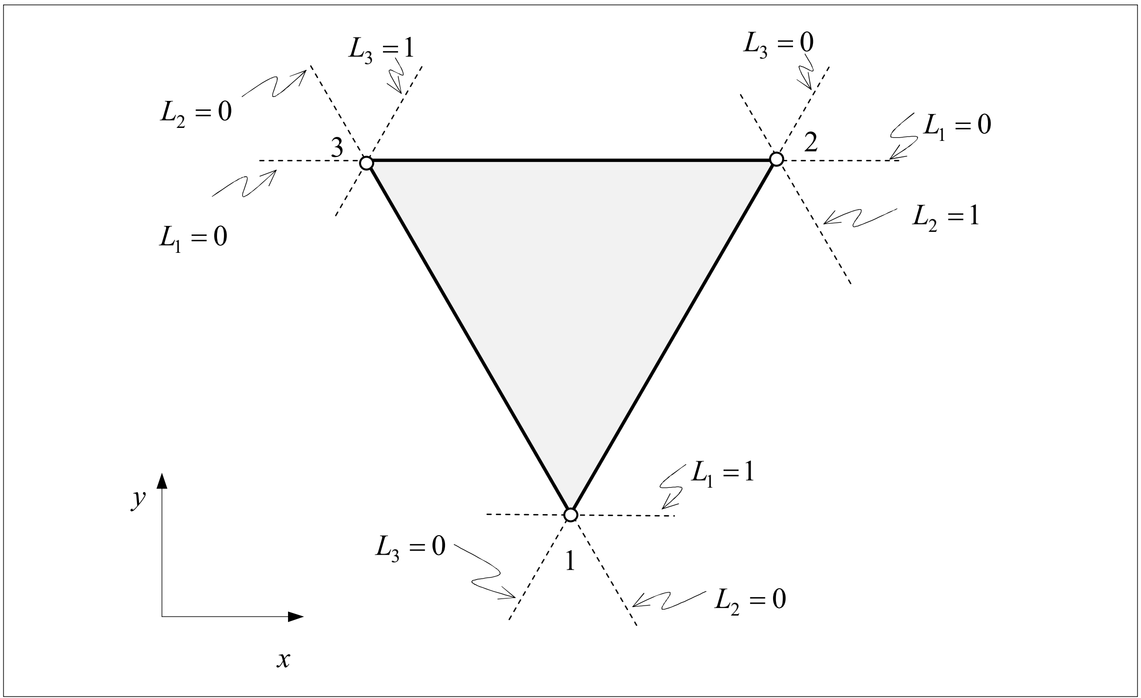


Figure 6.74: Area coordinates – Triangle with 3-nodes.

Triangle with 6-nodes, (see Figure 6.75). For this case we have:

$$N_1 = \frac{(L_1 - L_1^{(4-6)})(L_1 - L_1^{(2-3)})}{(L_1^{(1)} - L_1^{(4-6)})(L_1^{(1)} - L_1^{(2-3)})} = \frac{(L_1 - \frac{1}{2})(L_1 - 0)}{(1 - \frac{1}{2})(1 - 0)} = L_1(2L_1 - 1)$$

$$N_2 = \frac{(L_2 - L_2^{(4-5)})(L_2 - L_2^{(1-3)})}{(L_2^{(2)} - L_2^{(4-5)})(L_2^{(2)} - L_2^{(1-3)})} = \frac{(L_2 - \frac{1}{2})(L_2 - 0)}{(L_2^{(2)} - \frac{1}{2})(L_2^{(2)} - 0)} = L_2(2L_2 - 1)$$

$$N_3 = \frac{(L_3 - L_3^{(6-5)})(L_3 - L_3^{(1-2)})}{(L_3^{(3)} - L_3^{(6-5)})(L_3^{(3)} - L_3^{(1-2)})} = \frac{(L_3 - \frac{1}{2})(L_3 - 0)}{(L_3^{(3)} - \frac{1}{2})(L_3^{(3)} - 0)} = L_3(2L_3 - 1)$$

$$N_4 = N_4(L_1)N_4(L_2) = 4L_1L_2$$
$$N_5 = N_5(L_2)N_5(L_3) = 4L_2L_3$$
$$N_6 = N_6(L_1)N_6(L_3) = 4L_1L_3$$

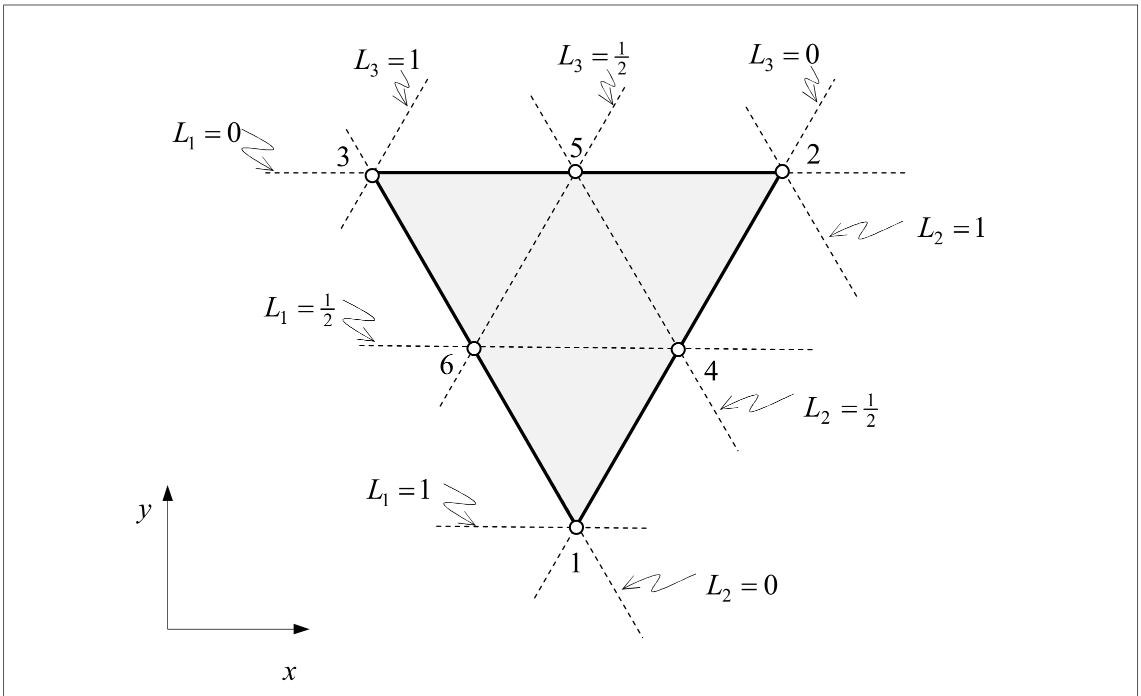


Figure 6.75: Area coordinates – Triangle with 6-nodes.

Triangle with 10-nodes, (see Figure 6.76)

For this case the shape function N_1 can be obtained as follows:

$$N_1 = \frac{(L_1 - L_1^{(4-9)})(L_1 - L_1^{(5-8)})(L_1 - L_1^{(2-3)})}{(L_1^{(1)} - L_1^{(4-9)})(L_1^{(1)} - L_1^{(5-8)})(L_1^{(1)} - L_1^{(2-3)})} = \frac{(L_1 - \frac{2}{3})(L_1 - \frac{1}{3})(L_1 - 0)}{(1 - \frac{2}{3})(1 - \frac{1}{3})(1 - 0)} = \frac{1}{2}L_1(3L_1 - 2)(3L_1 - 1)$$

$$N_2 = \frac{(L_2 - L_2^{(5-6)})(L_2 - L_2^{(4-7)})(L_2 - L_2^{(1-3)})}{(L_2^{(2)} - L_2^{(5-6)})(L_2^{(2)} - L_2^{(4-7)})(L_2^{(2)} - L_2^{(1-3)})} = \frac{(L_2 - \frac{2}{3})(L_2 - \frac{1}{3})(L_2 - 0)}{(1 - \frac{2}{3})(1 - \frac{1}{3})(1 - 0)} = \frac{1}{2}L_2(3L_2 - 2)(3L_2 - 1)$$

$$N_3 = \frac{(L_3 - L_3^{(7-8)})(L_3 - L_3^{(6-9)})(L_3 - L_3^{(1-2)})}{(L_3^{(3)} - L_3^{(7-8)})(L_3^{(3)} - L_3^{(6-9)})(L_3^{(3)} - L_3^{(1-2)})} = \frac{(L_3 - \frac{2}{3})(L_3 - \frac{1}{3})(L_3 - 0)}{(1 - \frac{2}{3})(1 - \frac{1}{3})(1 - 0)} = \frac{1}{2}L_3(3L_3 - 2)(3L_3 - 1)$$

Note that the node 4, (see Figure 6.76), depends on coordinates L_1 and L_2:

$$N_4(L_1)=\frac{(L_1-L_1^{(5-8)})(L_1-L_1^{(2-3)})}{(L_1^{(4)}-L_1^{(5-8)})(L_1^{(4)}-L_1^{(2-3)})}=\frac{(L_1-\frac{2}{3})(L_1-0)}{(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-0)}=\frac{3}{2}L_1(3L_1-1)$$

$$N_4(L_2)=\frac{(L_2-L_2^{(1-3)})}{(L_2^{(4)}-L_2^{(1-3)})}=\frac{(L_2-0)}{(\frac{1}{3}-0)}=3L_2$$

$$\Rightarrow N_4(L_1,L_2)=N_4(L_1)N_4(L_2)=\frac{3}{2}L_1(3L_1-1)3L_2=\frac{9}{2}L_1L_2(3L_1-1)$$

Similarly we can obtain:

$$N_5(L_1,L_2)=N_5(L_1)N_5(L_2)=\frac{9}{2}L_1L_2(3L_2-1)$$

$$N_6(L_2,L_3)=N_5(L_2)N_5(L_3)=\frac{9}{2}L_2L_3(3L_2-1)$$

$$N_7(L_2,L_3)=N_5(L_2)N_5(L_3)=\frac{9}{2}L_2L_3(3L_3-1)$$

$$N_8(L_1,L_3)=N_5(L_1)N_5(L_3)=\frac{9}{2}L_1L_3(3L_3-1)$$

$$N_9(L_1,L_3)=N_5(L_1)N_5(L_3)=\frac{9}{2}L_1L_3(3L_1-1)$$

And the node 10:

$$N_{10}(L_1,L_2,L_3)=27L_1L_2L_3$$

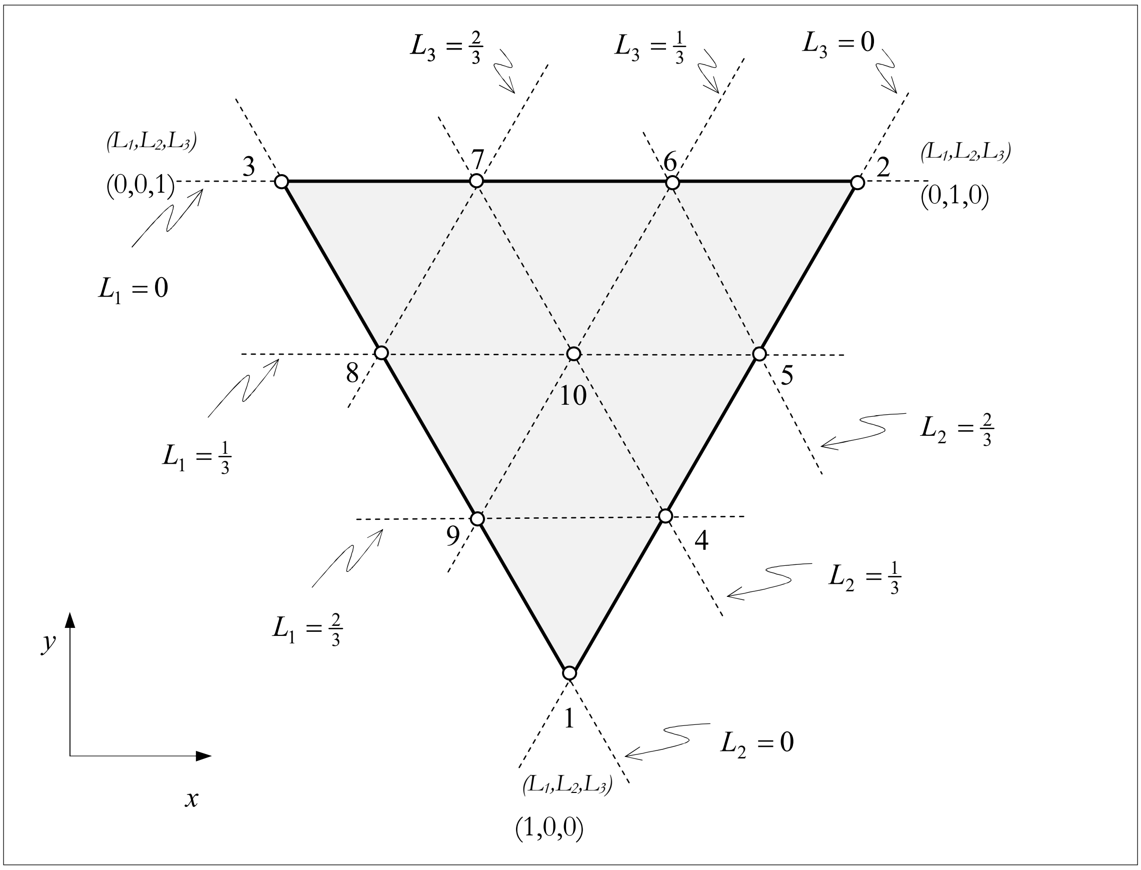


Figure 6.76: Area coordinates – Triangle with 10-nodes.

NOTE 6: Transformation between the Original System and the Normalized Space.

To obtain the transformation between the original system and the normalized space we will make an analogy with the Continuum Kinematics discussed in Chapter 2 in Chaves (2013). Let us suppose that the Reference Configuration represents the Normalized space and the Current Configuration represents the Original system, (see Figure 6.77). In Figure 6.77, $\boldsymbol{F}$ is the Jacobian ("Deformation Gradient"), $J = \det(\boldsymbol{F}) \equiv |\boldsymbol{F}|$ represents the Jacobian determinant, $\boldsymbol{\sigma}(\vec{x})$ is the Cauchy Stress tensor, $\mathbf{S}(\vec{X})$ is the Second Piola-Kirchhoff stress tensor, $\boldsymbol{E}(\vec{X})$ is the Green-Lagrange strain tensor.

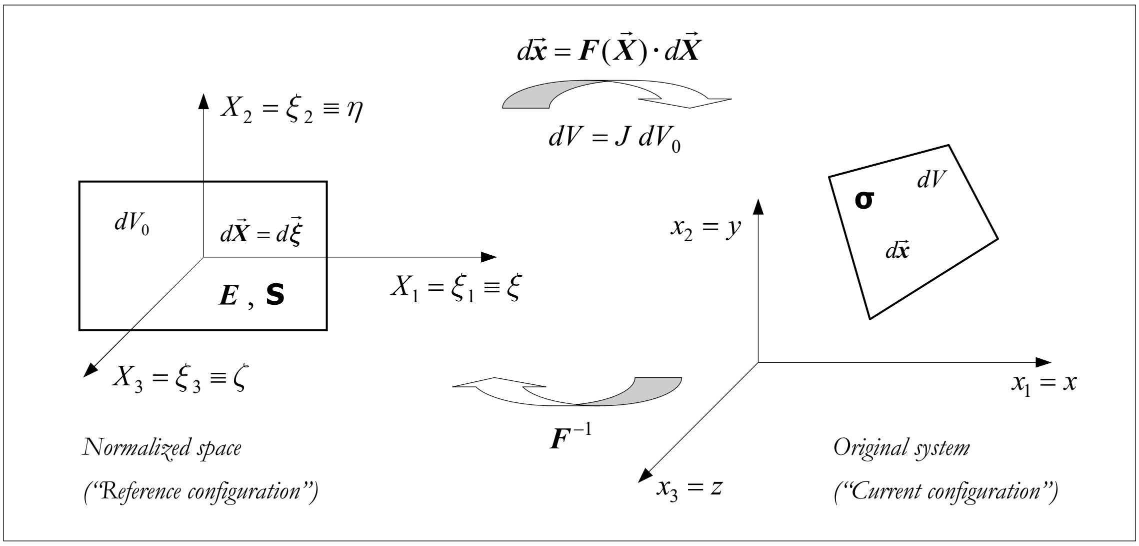


Figure 6.77: Normalized space and the original system.

Here the Jacobian $\boldsymbol{F}$ ("Deformation Gradient") is given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \tag{6.193}$$

The inverse can easily be obtained by using the definition $\boldsymbol{F}^{-1} = \frac{1}{J}[\mathrm{cof}(\boldsymbol{F})]^T$, i.e.:

$$F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j} = \frac{\partial \xi_i}{\partial x_j} = \frac{1}{J} \begin{bmatrix} \begin{vmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{vmatrix} & -\begin{vmatrix} F_{21} & F_{23} \\ F_{31} & F_{33} \end{vmatrix} & \begin{vmatrix} F_{21} & F_{22} \\ F_{31} & F_{32} \end{vmatrix} \\ -\begin{vmatrix} F_{12} & F_{13} \\ F_{32} & F_{33} \end{vmatrix} & \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} & -\begin{vmatrix} F_{11} & F_{12} \\ F_{31} & F_{32} \end{vmatrix} \\ \begin{vmatrix} F_{12} & F_{13} \\ F_{22} & F_{23} \end{vmatrix} & -\begin{vmatrix} F_{11} & F_{13} \\ F_{21} & F_{23} \end{vmatrix} & \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \end{bmatrix}^T \tag{6.194}$$

Note that for the two-dimensional case the above equations become:

$$F_{ij}^{2D} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{6.195}$$

and $\boldsymbol{F}^{(2D)^{-1}} = \frac{1}{J^{(2D)}}[\mathsf{cof}(\boldsymbol{F}^{(2D)})]^T$:

$$(F_{ij}^{2D})^{-1} = \frac{\partial X_i}{\partial x_j} = \frac{\partial \xi_i}{\partial x_j} = \frac{1}{J}\begin{bmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{bmatrix}^T = \frac{1}{J}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}^T \tag{6.196}$$

The stiffness matrix is defined in the original space (current configuration - $\vec{\boldsymbol{x}}$) and by using the transformation $dV = J\,dV_0$ we can obtain:

$$[\boldsymbol{k}^{(e)}] = \int_V [\mathbf{B}(\vec{x})]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\,dV = \int_{V_0} [\mathbf{B}(\vec{x})]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\,J\,dV_0 \tag{6.197}$$

Note that the integrand is defined in the original space. Let us adopt that the geometry can be approached by the same shape functions as those used to approach the displacement *(Isoparametric Element)*, i.e.:

$$x_1 = x = \sum_{a=1}^{n} N_a(\vec{\xi})x^{(a)} \quad ; \quad x_2 = y = \sum_{a=1}^{n} N_a(\vec{\xi})y^{(a)} \quad ; \quad x_3 = z = \sum_{a=1}^{n} N_a(\vec{\xi})z^{(a)} \tag{6.198}$$

where n is the number of nodes. With that the Jacobian can be obtained as follows:

$$F_{ij} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})x^{(a)}\right)}{\partial \xi} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})x^{(a)}\right)}{\partial \eta} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})x^{(a)}\right)}{\partial \zeta} \\ \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})y^{(a)}\right)}{\partial \xi} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})y^{(a)}\right)}{\partial \eta} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})y^{(a)}\right)}{\partial \zeta} \\ \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})z^{(a)}\right)}{\partial \xi} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})z^{(a)}\right)}{\partial \eta} & \frac{\partial\left(\sum_{a=1}^{n} N_a(\vec{\xi})z^{(a)}\right)}{\partial \zeta} \end{bmatrix} \tag{6.199}$$

which can also be expressed as follows:

$$F_{ij}(\vec{\xi}) = \sum_{a=1}^{n} \begin{bmatrix} \frac{\partial N_a(\vec{\xi})}{\partial \xi}x^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \eta}x^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \zeta}x^{(a)} \\ \frac{\partial N_a(\vec{\xi})}{\partial \xi}y^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \eta}y^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \zeta}y^{(a)} \\ \frac{\partial N_a(\vec{\xi})}{\partial \xi}z^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \eta}z^{(a)} & \frac{\partial N_a(\vec{\xi})}{\partial \zeta}z^{(a)} \end{bmatrix} \tag{6.200}$$

To obtain the matrix $[\mathbf{B}(\vec{x})]$ we have to calculate the derivative of the shape functions $\boldsymbol{N}(\vec{x})$ with respect to $(\vec{x})$. Recall that given a scalar field in the Reference configuration

$\phi(\vec{X})$ and the same scalar field defined in the Current configuration $\phi(\vec{x})$ the follow is true:

$$\nabla_{\vec{X}}\phi(\vec{X}) \equiv \frac{\partial\phi(\vec{X})}{\partial\vec{X}} = \frac{\partial\phi(\vec{X}(\vec{x}))}{\partial\vec{x}} \cdot \frac{\partial\vec{x}}{\partial\vec{X}} = \frac{\partial\phi(\vec{x})}{\partial\vec{x}} \cdot \boldsymbol{F} = \nabla_{\vec{x}}\phi(\vec{x}) \cdot \boldsymbol{F}$$

and

$$\nabla_{\vec{x}}\phi(\vec{x}) \equiv \frac{\partial\phi(\vec{x})}{\partial\vec{x}} = \frac{\partial\phi(\vec{x}(\vec{X}))}{\partial\vec{X}} \cdot \frac{\partial\vec{X}}{\partial\vec{x}} = \frac{\partial\phi(\vec{X})}{\partial\vec{X}} \cdot \boldsymbol{F}^{-1} = \{\nabla_{\vec{X}}\phi(\vec{X})\} \cdot \boldsymbol{F}^{-1} = \boldsymbol{F}^{-T} \cdot \{\nabla_{\vec{X}}\phi(\vec{X})\}$$

So, if we consider that $\phi(\vec{x}) = N_a(\vec{x})$ and $\vec{X} = \vec{\xi}$ we can conclude that:

$$\nabla_{\vec{x}}(N_a(\vec{x})) = \frac{\partial(N_a(\vec{x}))}{\partial\vec{x}} = \frac{\partial(N_a(\vec{x}(\vec{\xi})))}{\partial\vec{\xi}} \cdot \frac{\partial\vec{\xi}}{\partial\vec{x}} = \frac{\partial(N_a(\vec{\xi}))}{\partial\vec{\xi}} \cdot \boldsymbol{F}^{-1} = \boldsymbol{F}^{-T} \cdot \frac{\partial(N_a(\vec{\xi}))}{\partial\vec{\xi}} \tag{6.201}$$

which in indicial notation becomes:

$$(N_a(\vec{x}))_{,i} = \frac{\partial(N_a(\vec{x}))}{\partial x_i} = \frac{\partial(N_a(\vec{x}(\vec{\xi})))}{\partial\xi_k}\frac{\partial\xi_k}{\partial x_i} = \frac{\partial(N_a(\vec{\xi}))}{\partial\xi_k} F^{-1}_{ki} = F^{-1}_{ki}\frac{\partial(N_a(\vec{\xi}))}{\partial\xi_k} \tag{6.202}$$

More explicitly we have:

$$\begin{aligned}(N_a(\vec{x}))_{,i} &= \frac{\partial(N_a(\vec{\xi}))}{\partial\xi_k} F^{-1}_{ki} = \frac{\partial(N_a(\vec{\xi}))}{\partial\xi_1} F^{-1}_{1i} + \frac{\partial(N_a(\vec{\xi}))}{\partial\xi_2} F^{-1}_{2i} + \frac{\partial(N_a(\vec{\xi}))}{\partial\xi_3} F^{-1}_{3i} \\ &= \frac{\partial(N_a(\vec{\xi}))}{\partial\xi} F^{-1}_{1i} + \frac{\partial(N_a(\vec{\xi}))}{\partial\eta} F^{-1}_{2i} + \frac{\partial(N_a(\vec{\xi}))}{\partial\zeta} F^{-1}_{3i}\end{aligned} \tag{6.203}$$

The equation (6.202) can also be expressed in matrix form as follows:

$$\begin{Bmatrix} \dfrac{\partial(N_a(\vec{x}))}{\partial x_1} \\ \dfrac{\partial(N_a(\vec{x}))}{\partial x_2} \\ \dfrac{\partial(N_a(\vec{x}))}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_1} & \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_2} & \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_3} \end{Bmatrix} [F^{-1}_{ki}] = [F^{-1}_{ki}]^T \begin{Bmatrix} \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_1} \\ \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_2} \\ \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_3} \end{Bmatrix} \tag{6.204}$$

For two-dimensional case we have:

$$\begin{aligned} &\begin{Bmatrix} \dfrac{\partial(N_a(\vec{x}))}{\partial x_1} \\ \dfrac{\partial(N_a(\vec{x}))}{\partial x_2} \end{Bmatrix} = [F^{-1}_{ki}]^T \begin{Bmatrix} \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_1} \\ \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi_2} \end{Bmatrix} \\ \Rightarrow &\begin{Bmatrix} \dfrac{\partial(N_a(\vec{x}))}{\partial x} \\ \dfrac{\partial(N_a(\vec{x}))}{\partial y} \end{Bmatrix} = \frac{1}{J}\begin{bmatrix} \dfrac{\partial y}{\partial\eta} & -\dfrac{\partial y}{\partial\xi} \\ -\dfrac{\partial x}{\partial\eta} & \dfrac{\partial x}{\partial\xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial(N_a(\vec{\xi}))}{\partial\xi} \\ \dfrac{\partial(N_a(\vec{\xi}))}{\partial\eta} \end{Bmatrix} \end{aligned} \tag{6.205}$$

where we have used the equation in (6.196) in order to express $[F^{-1}_{ki}]^T$ for 2D case.

NOTE 7: Introduction to Numerical Integration (or Quadrature)

Let us consider the function $f(x)$ and we want to obtain the integral $\int\limits_{x^{(i)}}^{x^{(f)}} f(x)dx$. If the function varies linearly into the interval we can use the shape functions, given by (6.171), to approach the function, i.e. $f(x) = f_1 N_1(x) + f_2 N_2(x) = \sum_{a=1}^{2} f_a N_a(x)$, so, the integral can be represented as follows:

$$\int\limits_{x^{(i)}}^{x^{(f)}} f(x)dx = \int\limits_{x^{(i)}}^{x^{(f)}} \sum_{a=1}^{2} f_a N_a(x)dx = \sum_{a=1}^{2}\left(f_a \int\limits_{x^{(i)}}^{x^{(f)}} N_a(x)dx \right) = \sum_{a=1}^{2}(f_a P_a)$$

where we have considered that $P_a = \int\limits_{x^{(i)}}^{x^{(f)}} N_a(x)dx$. Note that we can use the normalized space to solve P_a. Then, by considering the equations (6.172) and (6.173) we can obtain that:

$$P_a = \int\limits_{x^{(i)}}^{x^{(f)}} N_a(x)dx = \int\limits_{-1}^{1} N_a(\xi)\frac{(x^{(f)} - x^{(i)})}{2}d\xi = \frac{(x^{(f)} - x^{(i)})}{2}\int\limits_{-1}^{1} N_a(\xi)d\xi = \frac{(x^{(f)} - x^{(i)})}{2}W_a$$

Note also that

$$W_1 = \int\limits_{-1}^{1} N_1(\xi)d\xi = \int\limits_{-1}^{1}\frac{(1-\xi)}{2}d\xi = 1 \quad ; \quad W_2 = \int\limits_{-1}^{1} N_2(\xi)d\xi = \int\limits_{-1}^{1}\frac{(1+\xi)}{2}d\xi = 1$$

Then, we can conclude that:

$$\int\limits_{x^{(i)}}^{x^{(f)}} f(x)dx = \sum_{a=1}^{2}(f_a P_a) = \sum_{a=1}^{2}\left(f_a \frac{(x^{(f)} - x^{(i)})}{2}W_a \right) = \frac{(x^{(f)} - x^{(i)})}{2}\sum_{a=1}^{2}(f_a W_a)$$

If we are using a quadratic function, (see equations in (6.170)), to approach the function $f(x)$, the parameters W_a $(a = 1,2,3)$ can be obtained as follows:

$$W_1 = \int\limits_{-1}^{1} N_1(\xi)d\xi = \int\limits_{-1}^{1}\left(\frac{\xi}{2}(\xi - 1)\right)d\xi = \frac{1}{3} \quad ; \quad W_2 = \int\limits_{-1}^{1} N_2(\xi)d\xi = \int\limits_{-1}^{1}(1-\xi^2)d\xi = \frac{4}{3}$$

$$W_3 = \int\limits_{-1}^{1} N_3(\xi)d\xi = \int\limits_{-1}^{1}\left(\frac{\xi}{2}(1+\xi)\right)d\xi = \frac{1}{3}$$

For example, let us obtain numerically the integral $\int\limits_{2}^{3} f(x)dx = \int\limits_{2}^{3}(4x^2 + 5x)dx = 37.833$. Let us consider that

$$\int\limits_{x^{(i)}}^{x^{(f)}} f(x)dx = \frac{(x^{(f)} - x^{(i)})}{2}\sum_{a=1}^{3}(f_a W_a) = \frac{(x^{(f)} - x^{(i)})}{2}(f_1 W_1 + f_2 W_2 + f_3 W_3)$$

where

$f_1 = f(x=2) = 4\times(2)^2 + 5\times(2) = 26$; $f_2 = f(x=2.5) = 4\times(2.5)^2 + 5\times(2.5) = 37.5$

$f_3 = f(x=3) = 4\times(3)^2 + 5\times(3) = 51$

Then

$$\int_{x^{(i)}}^{x^{(f)}} f(x)dx = \frac{(x^{(f)} - x^{(i)})}{2}(f_1 W_1 + f_2 W_2 + f_3 W_3) = \frac{(3-2)}{2}\left(26\frac{1}{3} + 37.5\frac{4}{3} + 51\frac{1}{3}\right) = 37.833$$

which matches the exact solution. This particular numerical integration is known as *Newton-Cotes formula*. There are other quadratures more suitable, (Bathe (1996)), e.g. *Gauss-Legendre Quadrature* which is widely used in the Finite Element Technique in order to obtain numerically the integrals, for example, the stiffness matrix $[\boldsymbol{k}^{(e)}] = \int_V [\mathbf{B}(\vec{x})]^T [\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{x})]\, dV$.

In general, the numerical integration for 1D, 2D and 3D, in the Normalized Space, are represented respectively by:

$$\begin{aligned}
&\int_{L_0} f(\xi)d\xi = \sum_{p=1}^{n_p} W_p f(\xi_p) \\
&\int_{A_0} f(\xi,\eta,\zeta)d\xi d\eta = \sum_{q=1}^{n_q}\sum_{p=1}^{n_p} W_q W_p f(\xi_p,\eta_q) \\
&\int_{V_0} f(\xi,\eta,\zeta)d\xi d\eta d\zeta = \sum_{r=1}^{n_r}\sum_{q=1}^{n_q}\sum_{p=1}^{n_p} W_r W_q W_p f(\xi_p,\eta_q,\zeta_r)
\end{aligned} \tag{6.206}$$

where n_p, n_q and n_r are the number of integration points according to the directions ξ, η and ζ, respectively. The weight W_i and the point (ξ_p,η_q,ζ_r) in which the function is evaluated $f(\xi_p,\eta_q,\zeta_r)$ will depend on the numerical technique employed. For example, by using the Gauss-Legendre interpolation the integration points and the weights are given by Table 6.2, (Bathe (1996), Oñate (1992), Chaves&Mínguez(2009)).

Table 6.2: Integration points and weights by using Gauss-Legendre Quadrature.

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{p=1}^{n_p} W_p f(\xi_p)$$

Integration points - ξ_p		*Weights* - W_p
	$n_p = 1$	
0.00000 00000 00000		2.00000 00000 00000
	$n_p = 2$	
$\pm$0.57735 02691 89626		1.00000 00000 00000
	$n_p = 3$	
0.00000 00000 00000		0.88888 88888 88888
$\pm$0.77459 66692 41483		0.55555 55555 55555
	$n_p = 4$	
$\pm$0.33998 10435 84856		0.65214 51548 61630
$\pm$0.86113 63115 94053		0.34785 48451 37448
	$n_p = 5$	
0.00000 00000 00000		0.56888 88888 88889
$\pm$0.53846 93101 05683		0.47862 86704 86297
$\pm$0.90617 98459 38664		0.23692 68850 56182
	$n_p = 6$	
$\pm$0.23861 91860 83197		0.46791 39345 72689
$\pm$0.66120 93864 66264		0.36076 15730 13980
$\pm$0.93246 95142 03152		0.17132 44923 79162
	$n_p = 7$	
0.00000 00000 00000		0.41795 91836 73469
$\pm$0.40584 51513 77397		0.38183 00505 05069
$\pm$0.74153 11855 99394		0.27970 53914 37510
$\pm$0.94910 79123 42758		0.12948 49661 68862
	$n_p = 8$	
$\pm$0.18343 46424 95650		0.36268 37833 78362
$\pm$0.52553 24099 16329		0.31370 66458 77676
$\pm$0.79666 64774 13627		0.22238 10344 53374
$\pm$0.96028 98564 97536		0.10122 85362 90370

Problem 6.41

Returning to three-dimensional problem, obtain the explicit form for the stiffness matrix, from the relationship $\{\boldsymbol{f}^{(e)}\}=[\boldsymbol{k}^{(e)}]\{\boldsymbol{u}^{(e)}\}$, by considering the tetrahedron as sub-domain, (see Figure 6.52). Consider that the displacement fields are approached by considering a linear function $\{\boldsymbol{u}(\vec{\boldsymbol{x}})\}=[\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\}$, (see equation (6.164)). Consider also the Orthotropic Symmetry for the elasticity tensor.

Hint: Use the information given by NOTE 3.1 in **Problem 6.40**.

Solution:

We have to obtain the explicit form of $[\boldsymbol{k}^{(e)}]=\int\limits_V [\mathbf{B}(\vec{\boldsymbol{x}})]^T[\boldsymbol{\mathcal{C}}][\mathbf{B}(\vec{\boldsymbol{x}})]\,dV$.

The relationship between the infinitesimal strain field $\{\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}})\}$ and the nodal displacement $\{\boldsymbol{u}^{(e)}\}$ is given by $\{\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}})\}=[\boldsymbol{L}^{(1)}]\{\boldsymbol{u}(\vec{\boldsymbol{x}})\}=[\boldsymbol{L}^{(1)}][\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\}=[\mathbf{B}(\vec{\boldsymbol{x}})]\{\boldsymbol{u}^{(e)}\}$, (see **Problem 5.8**):

$$\{\boldsymbol{\varepsilon}\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial z}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0 & 0\\ 0 & \dfrac{\partial}{\partial y} & 0\\ 0 & 0 & \dfrac{\partial}{\partial z}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u(\vec{\boldsymbol{x}})\\ v(\vec{\boldsymbol{x}})\\ w(\vec{\boldsymbol{x}})\end{Bmatrix}$$

and by considering the displacement field $\{\boldsymbol{u}(\vec{\boldsymbol{x}})\}=[\boldsymbol{N}(\vec{\boldsymbol{x}})]\{\boldsymbol{u}^{(e)}\}$, (see equation (6.164)), we can obtain:

$$\{\boldsymbol{\varepsilon}\}=\begin{bmatrix}\dfrac{\partial}{\partial x} & 0 & 0\\ 0 & \dfrac{\partial}{\partial y} & 0\\ 0 & 0 & \dfrac{\partial}{\partial z}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x}\end{bmatrix}\left[\begin{array}{ccc:ccc:ccc:ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0\\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0\\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4\end{array}\right]\left\{\begin{array}{c}u^{(1)}\\ v^{(1)}\\ w^{(1)}\\ \hdashline u^{(2)}\\ v^{(2)}\\ w^{(2)}\\ \hdashline u^{(3)}\\ v^{(3)}\\ w^{(3)}\\ \hdashline u^{(4)}\\ v^{(4)}\\ w^{(4)}\end{array}\right\}$$

Then, the $[\mathbf{B}(\vec{\boldsymbol{x}})]$ matrix becomes

$$
[\mathbf{B}(\vec{x})] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc|ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array}\right]
$$

$$
[\mathbf{B}] = \left[\begin{array}{ccc|ccc|ccc|ccc}
\frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & 0 & \frac{\partial N_3}{\partial x} & 0 & 0 & \frac{\partial N_4}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & 0 & 0 & \frac{\partial N_3}{\partial y} & 0 & 0 & \frac{\partial N_4}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_1}{\partial z} & 0 & 0 & \frac{\partial N_2}{\partial z} & 0 & 0 & \frac{\partial N_3}{\partial z} & 0 & 0 & \frac{\partial N_4}{\partial z} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & 0 \\
0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial y} \\
\frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial z} & 0 & \frac{\partial N_4}{\partial x}
\end{array}\right]
$$

$$
\Rightarrow [\mathbf{B}] = [[\overline{\mathbf{B}}_1]_{6\times 3} \mid [\overline{\mathbf{B}}_2]_{6\times 3} \mid [\overline{\mathbf{B}}_3]_{6\times 3} \mid [\overline{\mathbf{B}}_4]_{6\times 3}] \tag{6.207}
$$

The matrix $[\overline{\mathbf{B}}_i]_{6\times 3}$ from the equation (6.207) can be rewritten as follows:

$$
[\overline{\mathbf{B}}_i]_{6\times 3} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \\ b_i & a_i & 0 \\ 0 & c_i & b_i \\ c_i & 0 & a_i \end{bmatrix}
$$

As we will see later, the matrices $[\overline{\mathbf{B}}_i]$ are constant, i.e. it is independent of $\vec{x}$, then

$$[\boldsymbol{k}^{(e)}] = \int_V [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}]\, dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T \\ [\overline{\mathbf{B}}_2]^T \\ [\overline{\mathbf{B}}_3]^T \\ [\overline{\mathbf{B}}_4]^T \end{bmatrix} [\mathcal{C}][[\overline{\mathbf{B}}_1] \mid [\overline{\mathbf{B}}_2] \mid [\overline{\mathbf{B}}_3] \mid [\overline{\mathbf{B}}_4]] \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_2]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_3]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_4]^T[\mathcal{C}] \end{bmatrix} [[\overline{\mathbf{B}}_1] \mid [\overline{\mathbf{B}}_2] \mid [\overline{\mathbf{B}}_3] \mid [\overline{\mathbf{B}}_4]] \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \end{bmatrix} \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V [[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j]]\, dV \qquad (i,j = 1,2,3,4)$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = [[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j]]\int_V dV = V[[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j]] \qquad (i,j = 1,2,3,4)$$

By considering the Orthotropic Symmetry for the elasticity tensor:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55} \end{bmatrix}$$

we can obtain:

$$[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j] = \frac{1}{36V^2} \begin{bmatrix} a_i & 0 & 0 & b_i & 0 & c_i \\ 0 & b_i & 0 & a_i & c_i & 0 \\ 0 & 0 & c_i & 0 & b_i & a_i \end{bmatrix} \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55} \end{bmatrix} \begin{bmatrix} a_j & 0 & 0 \\ 0 & b_j & 0 \\ 0 & 0 & c_j \\ b_j & a_j & 0 \\ 0 & c_j & b_j \\ c_j & 0 & a_j \end{bmatrix}$$

$$[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j] = \frac{1}{36V^2}[\overline{\mathbf{k}}_{ij}] \qquad (i,j = 1,2,3,4)$$

where

$$[\overline{\mathbf{k}}_{ij}]_{3\times 3} = \begin{bmatrix} a_i\mathcal{C}_{11}a_j + b_i\mathcal{C}_{44}b_j + c_i\mathcal{C}_{66}c_j & a_i\mathcal{C}_{12}b_j + b_i\mathcal{C}_{44}a_j & a_i\mathcal{C}_{13}c_j + c_i\mathcal{C}_{66}a_j \\ b_i\mathcal{C}_{12}a_j + a_i\mathcal{C}_{44}b_j & b_i\mathcal{C}_{22}b_j + a_i\mathcal{C}_{44}a_j + c_i\mathcal{C}_{55}c_j & b_i\mathcal{C}_{23}c_j + c_i\mathcal{C}_{55}b_j \\ c_i\mathcal{C}_{13}a_j + a_i\mathcal{C}_{66}c_j & c_i\mathcal{C}_{23}b_j + b_i\mathcal{C}_{55}c_j & c_i\mathcal{C}_{33}c_j + b_i\mathcal{C}_{55}b_j + a_i\mathcal{C}_{66}a_j \end{bmatrix}$$

Then, the stiffness matrix becomes

$$[\boldsymbol{k}^{(e)}]_{12\times 12} = V[[\overline{\mathbf{B}}_i]^T[\boldsymbol{\mathcal{C}}][\overline{\mathbf{B}}_j]] = \frac{V}{36V^2}[\overline{\mathbf{k}}_{ij}] = \frac{1}{36V}\begin{bmatrix} [\overline{\mathbf{k}}_{11}] & [\overline{\mathbf{k}}_{12}] & [\overline{\mathbf{k}}_{13}] & [\overline{\mathbf{k}}_{14}] \\ [\overline{\mathbf{k}}_{21}] & [\overline{\mathbf{k}}_{22}] & [\overline{\mathbf{k}}_{23}] & [\overline{\mathbf{k}}_{24}] \\ [\overline{\mathbf{k}}_{31}] & [\overline{\mathbf{k}}_{32}] & [\overline{\mathbf{k}}_{33}] & [\overline{\mathbf{k}}_{34}] \\ [\overline{\mathbf{k}}_{41}] & [\overline{\mathbf{k}}_{42}] & [\overline{\mathbf{k}}_{43}] & [\overline{\mathbf{k}}_{44}] \end{bmatrix}$$

Now we have to obtain the derivatives of $[\boldsymbol{N}]$ in order to derive the coefficients a_i, b_i and c_i. The shape functions for the linear tetrahedron were obtained in equation (6.163). Note that the shape function N_1 can be rewritten as follows:

$$N_1 = \frac{\begin{vmatrix} 1 & x & y & z \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}{\begin{vmatrix} 1 & x^{(1)} & y^{(1)} & z^{(1)} \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & x & y & z \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}}{6V} \quad \Rightarrow \quad 6VN_1 = \begin{vmatrix} 1 & x & y & z \\ 1 & x^{(2)} & y^{(2)} & z^{(2)} \\ 1 & x^{(3)} & y^{(3)} & z^{(3)} \\ 1 & x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix}$$

$$\Rightarrow 6VN_1 = \begin{vmatrix} x^{(2)} & y^{(2)} & z^{(2)} \\ x^{(3)} & y^{(3)} & z^{(3)} \\ x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix} - x\begin{vmatrix} 1 & y^{(2)} & z^{(2)} \\ 1 & y^{(3)} & z^{(3)} \\ 1 & y^{(4)} & z^{(4)} \end{vmatrix} + y\begin{vmatrix} 1 & x^{(2)} & z^{(2)} \\ 1 & x^{(3)} & z^{(3)} \\ 1 & x^{(4)} & z^{(4)} \end{vmatrix} - z\begin{vmatrix} 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \\ 1 & x^{(4)} & y^{(4)} \end{vmatrix}$$

Then, the derivatives are given by:

$$6V\frac{\partial N_1}{\partial x} = -\begin{vmatrix} 1 & y^{(2)} & z^{(2)} \\ 1 & y^{(3)} & z^{(3)} \\ 1 & y^{(4)} & z^{(4)} \end{vmatrix} = a_1 \;;\; 6V\frac{\partial N_1}{\partial y} = \begin{vmatrix} 1 & x^{(2)} & z^{(2)} \\ 1 & x^{(3)} & z^{(3)} \\ 1 & x^{(4)} & z^{(4)} \end{vmatrix} = b_1 \;;\; 6V\frac{\partial N_1}{\partial z} = -\begin{vmatrix} 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \\ 1 & x^{(4)} & y^{(4)} \end{vmatrix} = c_1$$

The shape function N_2, (see equation (6.163)), can be rewritten as follows:

$$6VN_2 = -\begin{vmatrix} x^{(1)} & y^{(1)} & z^{(1)} \\ x^{(3)} & y^{(3)} & z^{(3)} \\ x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix} + x\begin{vmatrix} 1 & y^{(1)} & z^{(1)} \\ 1 & y^{(3)} & z^{(3)} \\ 1 & y^{(4)} & z^{(4)} \end{vmatrix} - y\begin{vmatrix} 1 & x^{(1)} & z^{(1)} \\ 1 & x^{(3)} & z^{(3)} \\ 1 & x^{(4)} & z^{(4)} \end{vmatrix} + z\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(3)} & y^{(3)} \\ 1 & x^{(4)} & y^{(4)} \end{vmatrix}$$

and the derivatives:

$$6V\frac{\partial N_2}{\partial x} = \begin{vmatrix} 1 & y^{(1)} & z^{(1)} \\ 1 & y^{(3)} & z^{(3)} \\ 1 & y^{(4)} & z^{(4)} \end{vmatrix} = a_2 \;;\; 6V\frac{\partial N_2}{\partial y} = -\begin{vmatrix} 1 & x^{(1)} & z^{(1)} \\ 1 & x^{(3)} & z^{(3)} \\ 1 & x^{(4)} & z^{(4)} \end{vmatrix} = b_2 \;;\; 6V\frac{\partial N_2}{\partial z} = \begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(3)} & y^{(3)} \\ 1 & x^{(4)} & y^{(4)} \end{vmatrix} = c_2$$

The shape function N_3, (see equation (6.163)), can be rewritten as follows:

$$6VN_3 = \begin{vmatrix} x^{(1)} & y^{(1)} & z^{(1)} \\ x^{(3)} & y^{(3)} & z^{(3)} \\ x^{(4)} & y^{(4)} & z^{(4)} \end{vmatrix} - x\begin{vmatrix} 1 & y^{(1)} & z^{(1)} \\ 1 & y^{(2)} & z^{(2)} \\ 1 & y^{(4)} & z^{(4)} \end{vmatrix} + y\begin{vmatrix} 1 & x^{(1)} & z^{(1)} \\ 1 & x^{(2)} & z^{(2)} \\ 1 & x^{(4)} & z^{(4)} \end{vmatrix} - z\begin{vmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(4)} & y^{(4)} \end{vmatrix}$$

and the derivatives:

$$6V\frac{\partial N_3}{\partial x}=-\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(2)} & z^{(2)}\\1 & y^{(4)} & z^{(4)}\end{vmatrix}=a_3\ ;\ 6V\frac{\partial N_3}{\partial y}=\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(2)} & z^{(2)}\\1 & x^{(4)} & z^{(4)}\end{vmatrix}=b_3\ ;\ 6V\frac{\partial N_3}{\partial z}=-\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(2)} & y^{(2)}\\1 & x^{(4)} & y^{(4)}\end{vmatrix}=c_3$$

The shape function N_4, (see equation (6.163)), can be rewritten as follows:

$$6VN_4=-\begin{vmatrix}x^{(1)} & y^{(1)} & z^{(1)}\\x^{(3)} & y^{(3)} & z^{(3)}\\x^{(4)} & y^{(4)} & z^{(4)}\end{vmatrix}+x\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(2)} & z^{(2)}\\1 & y^{(3)} & z^{(3)}\end{vmatrix}-y\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(2)} & z^{(2)}\\1 & x^{(3)} & z^{(3)}\end{vmatrix}+z\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(2)} & y^{(2)}\\1 & x^{(3)} & y^{(3)}\end{vmatrix}$$

and the derivatives:

$$6V\frac{\partial N_4}{\partial x}=\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(2)} & z^{(2)}\\1 & y^{(3)} & z^{(3)}\end{vmatrix}=a_4\ ;\ 6V\frac{\partial N_4}{\partial y}=-\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(2)} & z^{(2)}\\1 & x^{(3)} & z^{(3)}\end{vmatrix}=b_4\ ;\ 6V\frac{\partial N_4}{\partial z}=\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(2)} & y^{(2)}\\1 & x^{(3)} & y^{(3)}\end{vmatrix}=c_4$$

Then, we summarize the coefficients as follows:

$$a_1=-\begin{vmatrix}1 & y^{(2)} & z^{(2)}\\1 & y^{(3)} & z^{(3)}\\1 & y^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad b_1=\begin{vmatrix}1 & x^{(2)} & z^{(2)}\\1 & x^{(3)} & z^{(3)}\\1 & x^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad c_1=-\begin{vmatrix}1 & x^{(2)} & y^{(2)}\\1 & x^{(3)} & y^{(3)}\\1 & x^{(4)} & y^{(4)}\end{vmatrix}$$

$$a_2=\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(3)} & z^{(3)}\\1 & y^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad b_2=-\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(3)} & z^{(3)}\\1 & x^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad c_2=\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(3)} & y^{(3)}\\1 & x^{(4)} & y^{(4)}\end{vmatrix}$$

$$a_3=-\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(2)} & z^{(2)}\\1 & y^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad b_3=\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(2)} & z^{(2)}\\1 & x^{(4)} & z^{(4)}\end{vmatrix}\quad ;\quad c_3=-\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(2)} & y^{(2)}\\1 & x^{(4)} & y^{(4)}\end{vmatrix}$$

$$a_4=\begin{vmatrix}1 & y^{(1)} & z^{(1)}\\1 & y^{(2)} & z^{(2)}\\1 & y^{(3)} & z^{(3)}\end{vmatrix}\quad ;\quad b_4=-\begin{vmatrix}1 & x^{(1)} & z^{(1)}\\1 & x^{(2)} & z^{(2)}\\1 & x^{(3)} & z^{(3)}\end{vmatrix}\quad ;\quad c_4=\begin{vmatrix}1 & x^{(1)} & y^{(1)}\\1 & x^{(2)} & y^{(2)}\\1 & x^{(3)} & y^{(3)}\end{vmatrix}$$

Problem 6.42

Obtain the displacements for the structure described in Figure 6.78 which can be approached by the state of plane stress. As mechanical properties consider that $E=10$ (Young's modulus) and $\nu=0.0$ (Poisson's ratio). Consider that the domain is discretized by two triangles with 3-nodes (CST), (see Figure 6.78).

NOTE: Note that this an academic example, since due to the fact that the thickness is not very small so that in a real practical case we cannot adopt the state of plane stress.

Solution:

For the problem presented here we have 2 degrees-of-freedom per node, and a total of 8 degrees-of-freedom, (see Figure 6.79), so, the discrete system which represents this problem is given by:

$$\{\boldsymbol{F}\}_{8\times1}=[\boldsymbol{K}]_{8\times8}\{\boldsymbol{U}\}_{8\times1}\tag{6.208}$$

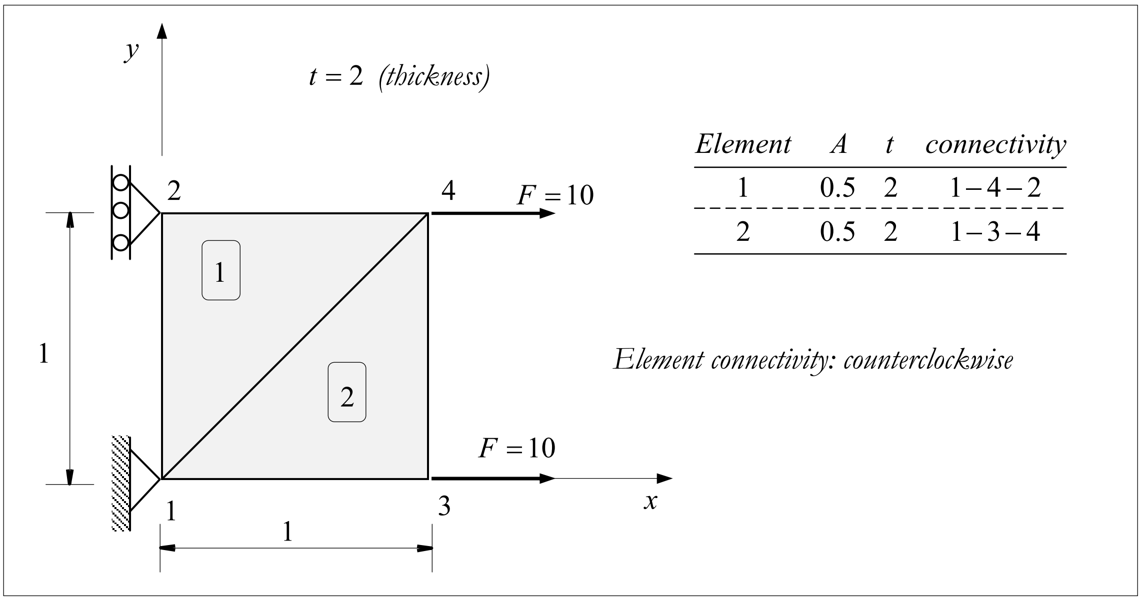


Figure 6.78: Dicretization by using two-CST elements.

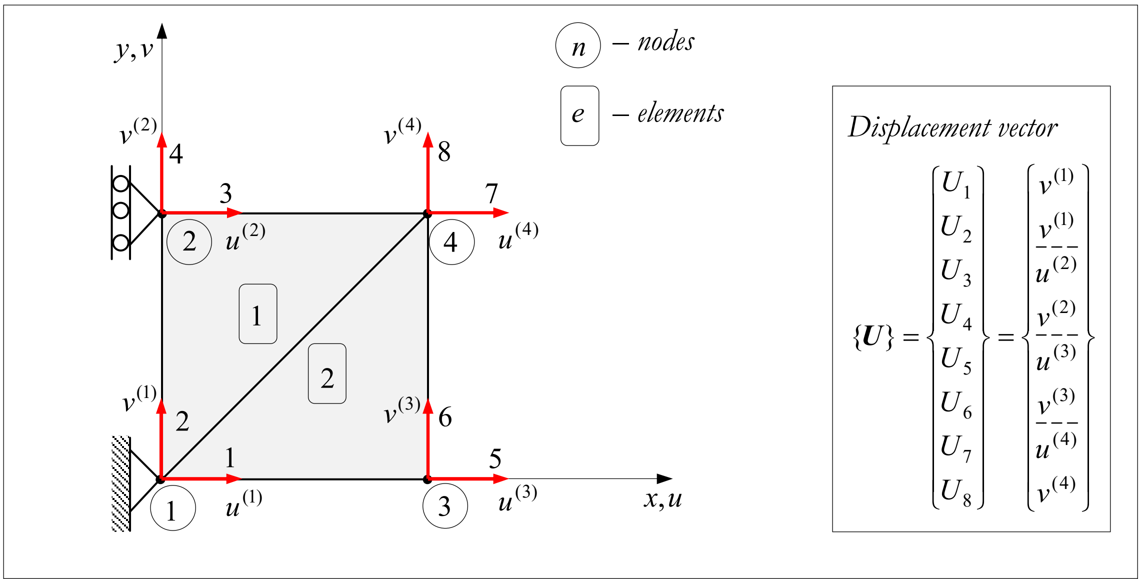


Figure 6.79: Degrees-of-freedom – Global system.

The global stiffness matrix and the global force vector are obtained by the contribution of each finite element and they are represented, respectively, by:

$$[\boldsymbol{K}] = \mathop{\mathbf{A}}_{e=1}^{N_{elem}} [\boldsymbol{k}^{(e)}] \quad ; \quad \{\boldsymbol{F}\} = \mathop{\mathbf{A}}_{e=1}^{N_{elem}} [\boldsymbol{f}^{(e)}] \tag{6.209}$$

where $\mathbf{A}$ stands for assemble operator. For this problem, we only have nodal forces, so, the global force vector is given by:

$$\{\boldsymbol{F}\} = \begin{array}{c} \overbrace{}^{global\ degrees\text{-}of\text{-}freedom} \\ \Downarrow \end{array}$$

$$\{\boldsymbol{F}\} = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \end{matrix} \right\} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \qquad (6.210)$$

The elasticity matrix for the state of plane stress is given by:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad (6.211)$$

The stiffness matrix for CST-element can be obtained by means of equation (6.157), i.e.:

$$[\boldsymbol{k}^{(e)}]_{6\times 6} = \frac{t^{(e)}}{4A^{(e)}} \begin{bmatrix} [\overline{\mathbf{k}}_{11}] & [\overline{\mathbf{k}}_{12}] & [\overline{\mathbf{k}}_{13}] \\ [\overline{\mathbf{k}}_{21}] & [\overline{\mathbf{k}}_{22}] & [\overline{\mathbf{k}}_{23}] \\ [\overline{\mathbf{k}}_{31}] & [\overline{\mathbf{k}}_{32}] & [\overline{\mathbf{k}}_{33}] \end{bmatrix} = \frac{t^{(e)}}{4A^{(e)}} \begin{bmatrix} \bar{k}_{11}^{(e)} & \bar{k}_{12}^{(e)} & \bar{k}_{13}^{(e)} & \bar{k}_{14}^{(e)} & \bar{k}_{15}^{(e)} & \bar{k}_{16}^{(e)} \\ & \bar{k}_{22}^{(e)} & \bar{k}_{23}^{(e)} & \bar{k}_{24}^{(e)} & \bar{k}_{25}^{(e)} & \bar{k}_{26}^{(e)} \\ & & \bar{k}_{33}^{(e)} & \bar{k}_{34}^{(e)} & \bar{k}_{35}^{(e)} & \bar{k}_{36}^{(e)} \\ & & & \bar{k}_{44}^{(e)} & \bar{k}_{45}^{(e)} & \bar{k}_{46}^{(e)} \\ & symmetric & & & \bar{k}_{55}^{(e)} & \bar{k}_{56}^{(e)} \\ & & & & & \bar{k}_{66}^{(e)} \end{bmatrix} \qquad (6.212)$$

where

$$\begin{aligned}
&\bar{k}_{11}^{(e)} = a_1^2\mathcal{C}_{11} + b_1^2\mathcal{C}_{33}; && \bar{k}_{12}^{(e)} = a_1\mathcal{C}_{12}b_1 + b_1\mathcal{C}_{33}a_1; && \bar{k}_{13}^{(e)} = a_1\mathcal{C}_{11}a_2 + b_1\mathcal{C}_{33}b_2; \\
&\bar{k}_{14}^{(e)} = a_1\mathcal{C}_{12}b_2 + b_1\mathcal{C}_{33}a_2; && \bar{k}_{15}^{(e)} = a_1\mathcal{C}_{11}a_3 + b_1\mathcal{C}_{33}b_3; && \bar{k}_{16}^{(e)} = a_1\mathcal{C}_{12}b_3 + b_1\mathcal{C}_{33}a_3; \\
&\bar{k}_{22}^{(e)} = b_1^2\mathcal{C}_{22} + a_1^2\mathcal{C}_{33}; && \bar{k}_{23}^{(e)} = b_1\mathcal{C}_{12}a_2 + a_1\mathcal{C}_{33}b_2; && \bar{k}_{24}^{(e)} = b_1\mathcal{C}_{22}b_2 + a_1\mathcal{C}_{33}a_2; \\
&\bar{k}_{25}^{(e)} = b_1\mathcal{C}_{12}a_3 + a_1\mathcal{C}_{33}b_3; && \bar{k}_{26}^{(e)} = b_1\mathcal{C}_{22}b_3 + a_1\mathcal{C}_{33}a_3; && \bar{k}_{33}^{(e)} = a_2^2\mathcal{C}_{11} + b_2^2\mathcal{C}_{33}; \\
&\bar{k}_{34}^{(e)} = a_2\mathcal{C}_{12}b_2 + b_2\mathcal{C}_{33}a_2; && \bar{k}_{35}^{(e)} = a_2\mathcal{C}_{11}a_3 + b_2\mathcal{C}_{33}b_3; && \bar{k}_{36}^{(e)} = a_2\mathcal{C}_{12}b_3 + b_2\mathcal{C}_{33}a_3; \\
&\bar{k}_{44}^{(e)} = b_2^2\mathcal{C}_{22} + a_2^2\mathcal{C}_{33}; && \bar{k}_{45}^{(e)} = b_2\mathcal{C}_{12}a_3 + a_2\mathcal{C}_{33}b_3; && \bar{k}_{46}^{(e)} = b_2\mathcal{C}_{22}b_3 + a_2\mathcal{C}_{33}a_3; \\
&\bar{k}_{55}^{(e)} = a_3^2\mathcal{C}_{11} + b_3^2\mathcal{C}_{33}; && \bar{k}_{56}^{(e)} = a_3\mathcal{C}_{12}b_3 + b_3\mathcal{C}_{33}a_3; && \bar{k}_{66}^{(e)} = b_3^2\mathcal{C}_{22} + a_3^2\mathcal{C}_{33};
\end{aligned} \qquad (6.213)$$

with

$$\begin{aligned}
&a_1 = y^{(2)} - y^{(3)}; \quad a_2 = y^{(3)} - y^{(1)}; \quad a_3 = y^{(1)} - y^{(2)}; \\
&b_1 = x^{(3)} - x^{(2)}; \quad b_2 = x^{(1)} - x^{(3)}; \quad b_3 = x^{(2)} - x^{(1)}.
\end{aligned} \qquad (6.214)$$

Element – 1

$$\frac{t^{(e=1)}}{4A^{(e=1)}} = \frac{2}{4\times 0.5} = 1$$

$$\left.\begin{aligned} x^{(1)} &= 0; \quad y^{(1)} = 0; \\ x^{(2)} &= 1; \quad y^{(2)} = 1; \\ x^{(3)} &= 0; \quad y^{(3)} = 1; \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &a_1 = 0; \quad a_2 = 1; \quad a_3 = -1; \\ &b_1 = -1; \quad b_2 = 0; \quad b_3 = 1. \end{aligned}$$

$$
\begin{array}{c}
\begin{array}{cccccc c}
1 & 2 & 7 & 8 & 3 & 4 & \textit{Global}
\end{array}\\
[\boldsymbol{k}^{(1)}]_{6\times 6} = \left[\begin{array}{cccccc}
k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} \\
 & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} \\
 & & k_{33}^{(1)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} \\
 & & & k_{44}^{(1)} & k_{45}^{(1)} & k_{46}^{(1)} \\
 & \textit{symmetric} & & & k_{55}^{(1)} & k_{56}^{(1)} \\
 & & & & & k_{66}^{(1)}
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \\ 3 \\ 4 \end{array}\\
= \left[\begin{array}{cccccc}
5 & 0 & 0 & -5 & -5 & 5 \\
 & 10 & 0 & 0 & 0 & -10 \\
 & & 10 & 0 & -10 & 0 \\
 & & & 5 & 5 & -5 \\
 & \textit{symmetric} & & & 15 & -5 \\
 & & & & & 15
\end{array}\right]
\end{array}
\tag{6.215}
$$

Element – 2

$$
\frac{t^{(e=2)}}{4A^{(e=2)}} = \frac{2}{4\times 0.5} = 1
$$

$$
\left.\begin{array}{ll}
x^{(1)} = 0; & y^{(1)} = 0; \\
x^{(2)} = 1; & y^{(2)} = 0; \\
x^{(3)} = 1; & y^{(3)} = 1;
\end{array}\right\}
\quad \Rightarrow \quad
\begin{array}{lll}
a_1 = -1; & a_2 = 1; & a_3 = 0; \\
b_1 = 0; & b_2 = -1; & b_3 = 1.
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc c}
1 & 2 & 5 & 6 & 7 & 8 & \textit{Global}
\end{array}\\
[\boldsymbol{k}^{(2)}]_{6\times 6} = \left[\begin{array}{cccccc}
k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\
 & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\
 & & k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\
 & & & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\
 & \textit{symmetric} & & & k_{55}^{(2)} & k_{56}^{(2)} \\
 & & & & & k_{66}^{(2)}
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\\
= \left[\begin{array}{cccccc}
10 & 0 & -10 & 0 & 0 & 0 \\
 & 5 & 5 & -5 & 5 & 0 \\
 & & 15 & -5 & -5 & 0 \\
 & & & 15 & 5 & -10 \\
 & \textit{symmetric} & & & 5 & 0 \\
 & & & & & 10
\end{array}\right]
\end{array}
\tag{6.216}
$$

Global Stiffness matrix

$$
[\boldsymbol{K}] = \begin{array}{cc}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \mathit{Global} \\
\left[\begin{array}{cccccccc}
k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(1)}+k_{12}^{(2)} & k_{15}^{(1)} & k_{16}^{(1)} & k_{13}^{(2)} & k_{14}^{(2)} & k_{13}^{(1)}+k_{15}^{(2)} & k_{14}^{(1)}+k_{16}^{(2)} \\
 & k_{22}^{(1)}+k_{22}^{(2)} & k_{25}^{(1)} & k_{26}^{(1)} & k_{23}^{(2)} & k_{24}^{(2)} & k_{23}^{(1)}+k_{25}^{(2)} & k_{24}^{(1)}+k_{26}^{(2)} \\
 & & k_{55}^{(1)} & k_{56}^{(1)} & 0 & 0 & k_{35}^{(1)} & k_{45}^{(1)}+k_{36}^{(1)} \\
 & & & k_{66}^{(1)} & 0 & 0 & k_{36}^{(1)} & k_{46}^{(1)} \\
 & \mathit{symmetric} & & & k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\
 & & & & & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\
 & & & & & & k_{33}^{(1)}+k_{55}^{(2)} & k_{34}^{(1)}+k_{56}^{(2)} \\
 & & & & & & & k_{44}^{(1)}+k_{66}^{(2)}
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
\tag{6.217}
$$

$$
\{\boldsymbol{F}\}_{8\times1} = [\boldsymbol{K}]_{8\times8}\{\boldsymbol{U}\}_{8\times1} \quad \Rightarrow \quad
\begin{bmatrix}
15 & 0 & -5 & 5 & -10 & 0 & 0 & -5 \\
 & 15 & 0 & -10 & 5 & -5 & 5 & 0 \\
 & & 15 & -5 & 0 & 0 & -10 & 5 \\
 & & & 15 & 0 & 0 & 0 & -5 \\
 & & & & 15 & -5 & -5 & 0 \\
 & & & & & 15 & 5 & -10 \\
 & \mathit{symmetric} & & & & & 15 & 0 \\
 & & & & & & & 15
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \end{Bmatrix}
$$

By applying the boundary conditions we have

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 15 & 0 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & 15 & -5 & -5 & 0 \\
0 & 0 & 0 & 0 & -5 & 15 & 5 & -10 \\
0 & 0 & 0 & 0 & -5 & 5 & 15 & 0 \\
0 & 0 & 0 & -5 & 0 & -10 & 0 & 15
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \end{Bmatrix}
\xrightarrow{\mathit{Solve}}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}
=
\begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ u^{(2)} \\ v^{(2)} \\ u^{(3)} \\ v^{(3)} \\ u^{(4)} \\ v^{(4)} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}
$$

It is interesting to compare with the result presented in equation (6.113) - **Problem 6.34**:

$$
\begin{cases}
u(x=1) = u^{(3)} = u^{(4)} = \dfrac{F}{EA}x = \dfrac{20}{10\times 2}\times 1 = 1 \\[2ex]
v(x=1) = v^{(3)} = v^{(4)} = \dfrac{-\nu F}{EA}y = 0
\end{cases}
$$

here $A = t\times 1 = 2$ is cross section area and $F = 2\times 10 = 20$ is the total force.

Problem 6.43

Obtain the explicit formation for the stiffness matrix for the element presented in Figure 6.58.

Solution:

The displacement field is given by:

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \left[\begin{array}{cc:cc:cc:cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hdashline u^{(2)} \\ v^{(2)} \\ \hdashline u^{(3)} \\ v^{(3)} \\ \hdashline u^{(4)} \\ v^{(4)} \end{Bmatrix} \qquad \Leftrightarrow \qquad \{\boldsymbol{u}(x,y)\} = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\}$$

And the shape functions are:

$$N_1 = \frac{1}{ab}(ab - bx - ay + xy); \quad N_2 = \frac{1}{ab}(bx - xy);$$
$$N_3 = \frac{1}{ab}xy; \qquad N_4 = \frac{1}{ab}(ay - xy).$$

The $[\mathbf{B}(\vec{x})]$ can be obtained as follows:

$$[\mathbf{B}(\vec{x})] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left[\begin{array}{cc:cc:cc:cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right]$$

$$= \left[\begin{array}{cc:cc:cc:cc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{array}\right]$$

$$= \frac{1}{ab}\left[\begin{array}{cc:cc:cc:cc} y-b & 0 & b-y & 0 & y & 0 & -y & 0 \\ 0 & x-a & 0 & -x & 0 & x & 0 & a-x \\ x-a & y-b & -x & b-y & x & y & a-x & -y \end{array}\right]$$

$$= \frac{1}{ab}\left[\begin{array}{cc:cc:cc:cc} g_1 & 0 & g_2 & 0 & g_3 & 0 & g_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \\ h_1 & g_1 & h_2 & g_2 & h_3 & g_3 & h_4 & g_4 \end{array}\right]$$

$$\Rightarrow [\mathbf{B}] = [[\overline{\mathbf{B}}_1]_{3\times 2} \,\vdots\, [\overline{\mathbf{B}}_2]_{3\times 2} \,\vdots\, [\overline{\mathbf{B}}_3]_{3\times 2} \,\vdots\, [\overline{\mathbf{B}}_4]_{3\times 2}] \tag{6.218}$$

The matrix $[\overline{\mathbf{B}}_i]_{3\times 2}$ can be rewritten as follows:

$$[\overline{\mathbf{B}}_i]_{3\times 2} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{ab}\begin{bmatrix} g_i & 0 \\ 0 & h_i \\ h_i & g_i \end{bmatrix}$$

The stiffness matrix can be obtained as follows

$$[\boldsymbol{k}^{(e)}] = \int_V [\mathbf{B}]^T [\mathcal{C}][\mathbf{B}]\, dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T \\ [\overline{\mathbf{B}}_2]^T \\ [\overline{\mathbf{B}}_3]^T \\ [\overline{\mathbf{B}}_4]^T \end{bmatrix} [\mathcal{C}][[\overline{\mathbf{B}}_1] \mid [\overline{\mathbf{B}}_2] \mid [\overline{\mathbf{B}}_3] \mid [\overline{\mathbf{B}}_4]] \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_2]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_3]^T[\mathcal{C}] \\ [\overline{\mathbf{B}}_4]^T[\mathcal{C}] \end{bmatrix} [[\overline{\mathbf{B}}_1] \mid [\overline{\mathbf{B}}_2] \mid [\overline{\mathbf{B}}_3] \mid [\overline{\mathbf{B}}_4]] \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V \left(\begin{bmatrix} [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_1]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_2]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_3]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \\ [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_1] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_2] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_3] & [\overline{\mathbf{B}}_4]^T[\mathcal{C}][\overline{\mathbf{B}}_4] \end{bmatrix} \right) dV$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = \int_V [[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j]]\, dV \qquad (i,j = 1,2,3,4)$$

$$\Rightarrow [\boldsymbol{k}^{(e)}] = t\int_A [[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j]]\, dA$$

The elasticity matrix is given by:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{bmatrix}$$

Then, we can obtain:

$$[\overline{\mathbf{B}}_i]^T[\mathcal{C}][\overline{\mathbf{B}}_j] = \frac{1}{(ab)^2}\begin{bmatrix} g_i & 0 & h_i \\ 0 & h_i & g_i \end{bmatrix}\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{bmatrix}\begin{bmatrix} g_j & 0 \\ 0 & h_j \\ h_j & g_j \end{bmatrix} = \frac{1}{(ab)^2}[\overline{\mathbf{k}}_{ij}]$$

where

$$[\overline{\mathbf{k}}_{ij}]_{2\times 2} = \begin{bmatrix} g_i\mathcal{C}_{11}g_j + h_i\mathcal{C}_{33}h_j & g_i\mathcal{C}_{12}h_j + h_i\mathcal{C}_{33}g_j \\ h_i\mathcal{C}_{12}g_j + g_i\mathcal{C}_{33}h_j & h_i\mathcal{C}_{22}h_j + g_i\mathcal{C}_{33}g_j \end{bmatrix}$$

$$[\boldsymbol{k}^{(e)}] = \frac{t}{(ab)^2}\int_A [\overline{\mathbf{k}}_{ij}]_{2\times 2}\, dA = \frac{t}{(ab)^2}\int_{y=0}^{y=b}\int_{x=0}^{x=a} [\overline{\mathbf{k}}_{ij}]_{2\times 2}\, dxdy \qquad (i,j = 1,2,3,4)$$

Then, the stiffness matrix becomes

$$[\boldsymbol{k}^{(e)}]_{8\times 8}=\frac{t}{(ab)^2}\begin{bmatrix}[\overline{\mathbf{k}}_{11}] & [\overline{\mathbf{k}}_{12}] & [\overline{\mathbf{k}}_{13}] & [\overline{\mathbf{k}}_{14}]\\ [\overline{\mathbf{k}}_{21}] & [\overline{\mathbf{k}}_{22}] & [\overline{\mathbf{k}}_{23}] & [\overline{\mathbf{k}}_{24}]\\ [\overline{\mathbf{k}}_{31}] & [\overline{\mathbf{k}}_{32}] & [\overline{\mathbf{k}}_{33}] & [\overline{\mathbf{k}}_{34}]\\ [\overline{\mathbf{k}}_{41}] & [\overline{\mathbf{k}}_{42}] & [\overline{\mathbf{k}}_{43}] & [\overline{\mathbf{k}}_{44}]\end{bmatrix}$$

After the integration is taken place we can obtain:

$$[\boldsymbol{k}^{(e)}]=t\begin{bmatrix}k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} & k_{14}^{(e)} & k_{15}^{(e)} & k_{16}^{(e)} & k_{17}^{(e)} & k_{18}^{(e)}\\ & k_{22}^{(e)} & k_{23}^{(e)} & k_{24}^{(e)} & k_{25}^{(e)} & k_{26}^{(e)} & k_{27}^{(e)} & k_{28}^{(e)}\\ & & k_{33}^{(e)} & k_{34}^{(e)} & k_{35}^{(e)} & k_{36}^{(e)} & k_{37}^{(e)} & k_{38}^{(e)}\\ & & & k_{44}^{(e)} & k_{45}^{(e)} & k_{46}^{(e)} & k_{47}^{(e)} & k_{48}^{(e)}\\ & & & & k_{55}^{(e)} & k_{56}^{(e)} & k_{57}^{(e)} & k_{58}^{(e)}\\ & \textit{symmetric} & & & & k_{66}^{(e)} & k_{67}^{(e)} & k_{68}^{(e)}\\ & & & & & & k_{77}^{(e)} & k_{78}^{(e)}\\ & & & & & & & k_{88}^{(e)}\end{bmatrix} \tag{6.219}$$

where

$$\begin{array}{lll}
k_{11}^{(e)}=\dfrac{b\mathcal{C}_{11}}{3a}+\dfrac{a\mathcal{C}_{33}}{3b}; & k_{12}^{(e)}=\dfrac{\mathcal{C}_{12}+\mathcal{C}_{33}}{4}; & k_{13}^{(e)}=\dfrac{-b\mathcal{C}_{11}}{3a}+\dfrac{a\mathcal{C}_{33}}{6b};\\
k_{14}^{(e)}=\dfrac{\mathcal{C}_{12}-\mathcal{C}_{33}}{4}; & k_{15}^{(e)}=\dfrac{-b\mathcal{C}_{11}}{6a}-\dfrac{a\mathcal{C}_{33}}{6b}; & k_{16}^{(e)}=-k_{12}^{(e)};\\
k_{17}^{(e)}=\dfrac{b\mathcal{C}_{11}}{6a}-\dfrac{a\mathcal{C}_{33}}{3b}; & k_{18}^{(e)}=-k_{14}^{(e)}; & k_{22}^{(e)}=\dfrac{a\mathcal{C}_{11}}{3b}+\dfrac{b\mathcal{C}_{33}}{3a};\\
k_{23}^{(e)}=-k_{14}^{(e)}; & k_{24}^{(e)}=\dfrac{a\mathcal{C}_{11}}{6b}-\dfrac{b\mathcal{C}_{33}}{3a}; & k_{25}^{(e)}=-k_{12}^{(e)};\\
k_{26}^{(e)}=\dfrac{-a\mathcal{C}_{11}}{6b}-\dfrac{b\mathcal{C}_{33}}{6a}; & k_{27}^{(e)}=-k_{14}^{(e)}; & k_{28}^{(e)}=\dfrac{-a\mathcal{C}_{11}}{3b}+\dfrac{b\mathcal{C}_{33}}{6a};\\
k_{33}^{(e)}=k_{11}^{(e)}; & k_{34}^{(e)}=-k_{12}^{(e)}; & k_{35}^{(e)}=k_{17}^{(e)};\\
k_{36}^{(e)}=k_{14}^{(e)}; & k_{37}^{(e)}=k_{15}^{(e)}; & k_{38}^{(e)}=k_{12}^{(e)};\\
k_{44}^{(e)}=k_{22}^{(e)}; & k_{45}^{(e)}=-k_{14}^{(e)}; & k_{46}^{(e)}=k_{28}^{(e)};\\
k_{47}^{(e)}=k_{12}^{(e)}; & k_{48}^{(e)}=k_{26}^{(e)}; & k_{55}^{(e)}=k_{11}^{(e)};\\
k_{56}^{(e)}=k_{12}^{(e)}; & k_{57}^{(e)}=k_{13}^{(e)}; & k_{58}^{(e)}=k_{14}^{(e)};\\
k_{66}^{(e)}=k_{22}^{(e)}; & k_{67}^{(e)}=-k_{14}^{(e)}; & k_{68}^{(e)}=k_{24}^{(e)};\\
k_{77}^{(e)}=k_{11}^{(e)}; & k_{78}^{(e)}=-k_{12}^{(e)}; & k_{88}^{(e)}=k_{22}^{(e)};
\end{array} \tag{6.220}$$

Example: If we consider the structure and data given in **Problem 6.42**, and if we only consider one rectangle element, (see Figure 6.80), the stiffness matrix becomes:

$$[\boldsymbol{k}^{(1)}]=t\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} & k_{17}^{(1)} & k_{18}^{(1)} \\ & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} & k_{27}^{(1)} & k_{28}^{(1)} \\ & & k_{33}^{(1)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} & k_{37}^{(1)} & k_{38}^{(1)} \\ & & & k_{44}^{(1)} & k_{45}^{(1)} & k_{46}^{(1)} & k_{47}^{(1)} & k_{48}^{(1)} \\ & & & & k_{55}^{(1)} & k_{56}^{(1)} & k_{57}^{(1)} & k_{58}^{(1)} \\ & \mathit{symmetric} & & & & k_{66}^{(1)} & k_{67}^{(1)} & k_{68}^{(1)} \\ & & & & & & k_{77}^{(1)} & k_{78}^{(1)} \\ & & & & & & & k_{88}^{(1)} \end{bmatrix}$$

$$=2\left(\frac{1}{4}\begin{bmatrix} 20 & 5 & -5 & -5 & -5 & -5 & 0 & 5 \\ 5 & 20 & 5 & 0 & -5 & -5 & 5 & -5 \\ -5 & 5 & 20 & -5 & 0 & -5 & 0 & 5 \\ -5 & 0 & -5 & 20 & 5 & -5 & 5 & -5 \\ -5 & -5 & 0 & 5 & 20 & 5 & -5 & -5 \\ -5 & -5 & -5 & -5 & 5 & 20 & 5 & 0 \\ 0 & 5 & 0 & 5 & -5 & 5 & 20 & -5 \\ 5 & -5 & 5 & -5 & -5 & 0 & -5 & 20 \end{bmatrix}\right)$$

The system to be solved is

$$\{\boldsymbol{F}\}_{8\times1}=[\boldsymbol{K}]_{8\times8}\{\boldsymbol{U}\}_{8\times1} \quad\Rightarrow\quad \frac{1}{2}\begin{bmatrix} 20 & 5 & -5 & -5 & -5 & -5 & 0 & 5 \\ 5 & 20 & 5 & 0 & -5 & -5 & 5 & -5 \\ -5 & 5 & 20 & -5 & 0 & -5 & 0 & 5 \\ -5 & 0 & -5 & 20 & 5 & -5 & 5 & -5 \\ -5 & -5 & 0 & 5 & 20 & 5 & -5 & -5 \\ -5 & -5 & -5 & -5 & 5 & 20 & 5 & 0 \\ 0 & 5 & 0 & 5 & -5 & 5 & 20 & -5 \\ 5 & -5 & 5 & -5 & -5 & 0 & -5 & 20 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

By applying the boundary conditions we have

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & -5 & 0 & -5 & 0 & 5 \\ 0 & 0 & -5 & 20 & 5 & -5 & 0 & -5 \\ 0 & 0 & 0 & 5 & 20 & 5 & 0 & -5 \\ 0 & 0 & -5 & -5 & 5 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & -5 & -5 & 0 & 0 & 20 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\xrightarrow{\mathit{Solve}}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}=\begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ \hline u^{(2)} \\ v^{(2)} \\ \hline u^{(3)} \\ v^{(3)} \\ \hline u^{(4)} \\ v^{(4)} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that the node numeration is different from the one presented in **Problem 6.42**.

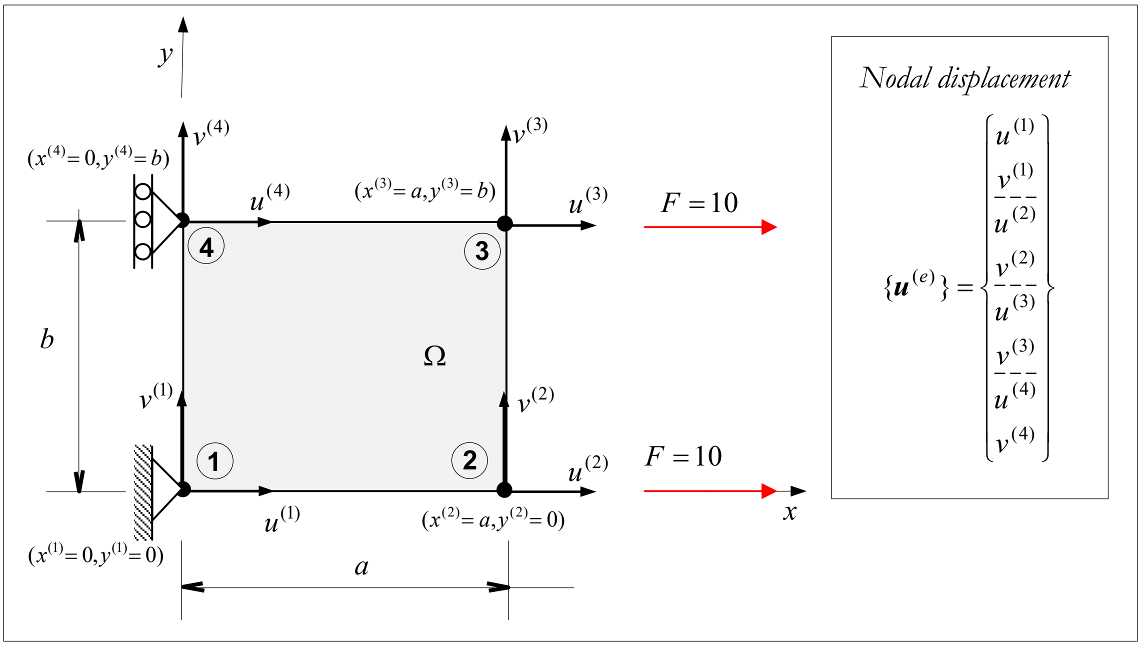


Figure 6.80: Rectangle element.

Finite Element Technique References

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6.4 Introduction to Torsion

Problem 6.44

Consider the hypothesis (approximations) for the Saint-Venant torsion problem, (see Figure 6.81):

- The body is prismatic (consider as prismatic axis the x_1-axis);
- The unit torsion angle θ (*angle of twist per unit length*) is constant according to the x_1-axis;
- The projection of transversal cross section on $(x_2 - x_3)$-plane has rigid body motion (rotation about the x_1-axis).

Show that this problem can be governed by the equation:

$$\boxed{\nabla^2 \mathsf{u}_1 = 0 \qquad with \qquad \mathsf{u}_1 = \mathsf{u}_1(x_2, x_3)} \tag{6.221}$$

or

$$G\left(\frac{\partial^2 \mathsf{u}_1}{\partial x_2^2} + \frac{\partial^2 \mathsf{u}_1}{\partial x_3^2}\right) = 0, \text{ where } G \text{ is the shear modulus} \tag{6.222}$$

or

$$G\frac{\partial}{\partial x_2}\left(\frac{\partial \mathsf{u}_1}{\partial x_2} - x_3\,\theta\right) + G\frac{\partial}{\partial x_3}\left(\frac{\partial \mathsf{u}_1}{\partial x_3} + x_2\,\theta\right) = 0 \tag{6.223}$$

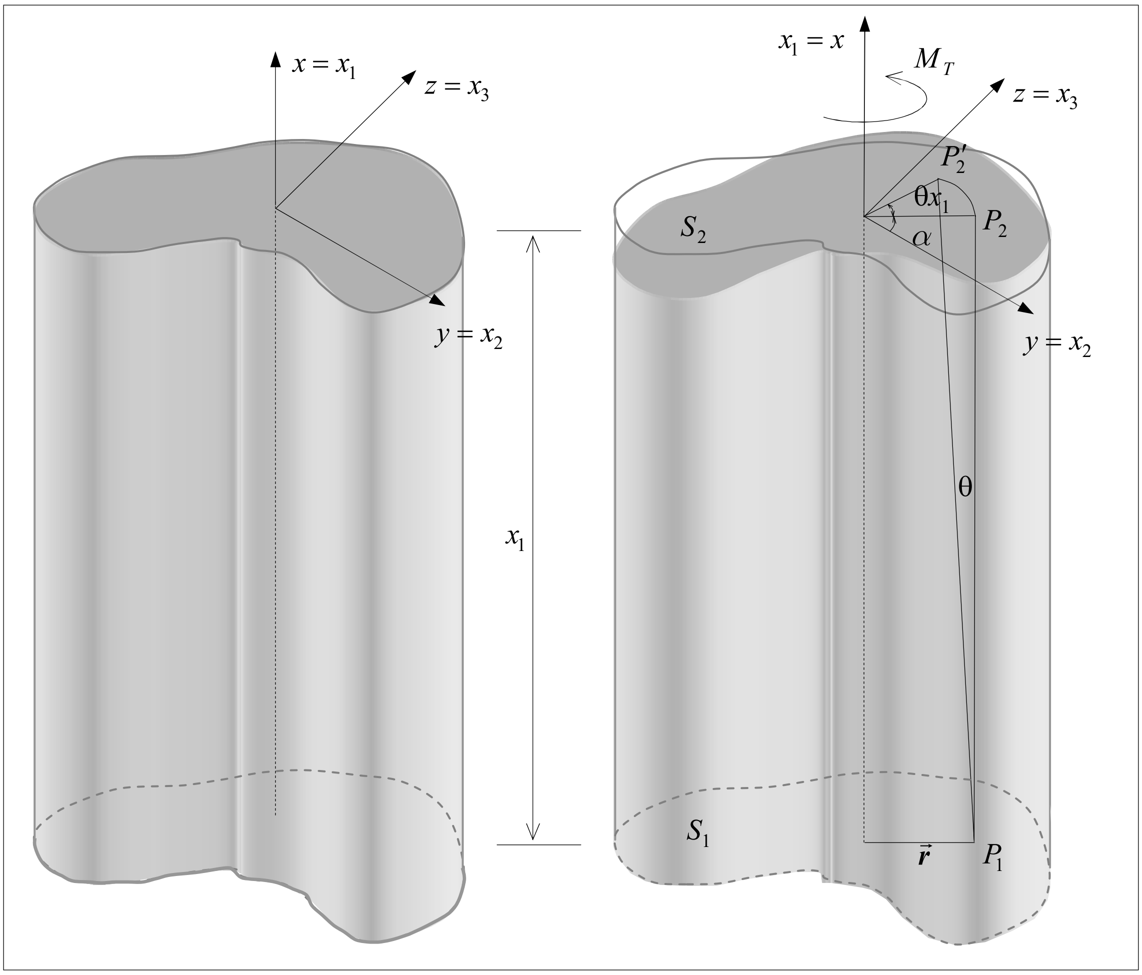


Figure 6.81: Torque applied to a prismatic body.

Obs.:

With these hypotheses, in general, the transversal cross section does not remain plane after deformation.

Considering the prismatic body and by applying the torque at the free end, the body displaces as indicated in Figure 6.81.

Solution:

Let us consider the point P_1 located on the fixed section S_1 whose position vector is $\vec{\boldsymbol{r}}$, (see Figure 6.81). Also consider another cross section S_2 (free to rotate and warping) which distance from the section S_1 is x_1 and by projecting the point P_1 on the cross section S_2 we obtain the point P_2. After the torque is applied the point P_2 moves to P_2' as indicated in Figure 6.82.

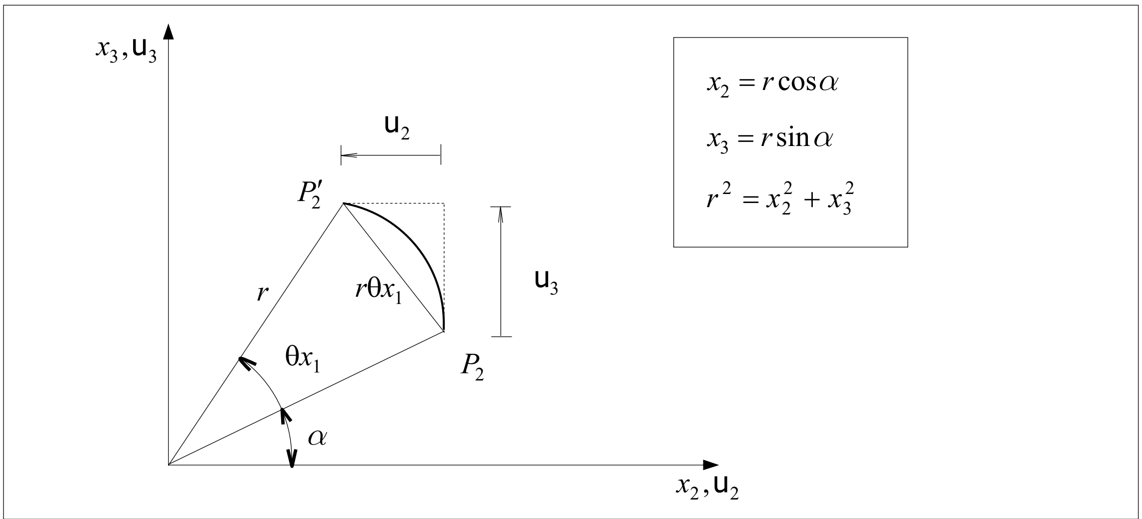


Figure 6.82: Motion of the prismatic body cross section.

Geometrically, (see Figure 6.82), we can obtain the displacements:

$$\begin{cases} \mathrm{u}_2 = -r\,\theta\,x_1 \sin\alpha = -x_3\,\theta\,x_1 \\ \mathrm{u}_3 = r\,\theta\,x_1 \cos\alpha = x_2\,\theta\,x_1 \end{cases} \tag{6.224}$$

where $x_2 = r\cos\alpha$, $x_3 = r\sin\alpha$, and u_2 stands for the displacement according x_2-direction, and u_3 is the displacement according to the x_3-direction. The displacement of the point P_2 according to the x_1-direction can be any, thus we summarize:

$$\mathrm{u}_1 = \mathrm{u}_1(x_2, x_3) \quad ; \quad \mathrm{u}_2 = -x_3\,\theta\,x_1 \quad ; \quad \mathrm{u}_3 = x_2\,\theta\,x_1 \tag{6.225}$$

The displacement u_1 (warping function) is the warping of the cross section, which is independent of x_1.

(Kinematic equations)

The strain-displacement relationships become:

$$\varepsilon_{11} = \frac{\partial \mathrm{u}_1}{\partial x_1} = 0\,; \qquad \varepsilon_{22} = \frac{\partial \mathrm{u}_2}{\partial x_2} = 0\,; \qquad \varepsilon_{33} = \frac{\partial \mathrm{u}_3}{\partial x_3} = 0\,;$$

$$\begin{aligned}\varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_1}{\partial x_2} + \frac{\partial \mathrm{u}_2}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right)\;;\end{aligned} \qquad \begin{aligned}\varepsilon_{23} &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_2}{\partial x_3} + \frac{\partial \mathrm{u}_3}{\partial x_2}\right) \\ &= \frac{1}{2}\left(-\theta\,x_1 + \theta\,x_1\right); \\ &= 0\end{aligned} \qquad \begin{aligned}\varepsilon_{13} &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + \frac{\partial \mathrm{u}_3}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right)\end{aligned}$$

Thus

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) & \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) & 0 & 0 \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) & 0 & 0 \end{bmatrix} \tag{6.226}$$

Note that the compatibility equations, (see **Problem 5.11**), are automatically satisfied, since the displacement field is continuous, a fact verified by the fulfillment of the compatibility equations, i.e.:

$$\begin{cases} S_{11} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 & \text{(a)} \\ S_{22} = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\dfrac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0 & \text{(b)} \\ S_{33} = \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 & \text{(c)} \\ S_{12} = \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0 & \text{(d)} \\ S_{23} = \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = 0 & \text{(e)} \\ S_{13} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{13}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) - \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = 0 & \text{(f)} \end{cases} \tag{6.227}$$

From equations (6.227) (d) and (f) we can conclude that:

$$\frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0 \quad ; \quad \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0 \quad \Rightarrow \quad \left(\frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2}\right) = \text{constant}$$

In fact, and is equal to:

$$\frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2} = \frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) - \frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) = \frac{1}{2}(-\theta - \theta) = -\theta$$

In other words, the compatibility equations are satisfied if the following is true:

$$\frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2} = -\theta \xrightarrow{\text{Engineering notation}} \frac{\partial \gamma_{xy}}{\partial x_3} - \frac{\partial \gamma_{xz}}{\partial x_2} = -2\theta$$

(Constitutive equations)

The constitutive equations for stress are given by:

$$\sigma_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij} + \frac{E}{(1+\nu)} \varepsilon_{ij} = \frac{E}{(1+\nu)} \varepsilon_{ij} \tag{6.228}$$

Note that $\varepsilon_{kk} = 0$ (trace of $\boldsymbol{\varepsilon}$ is zero). Then, by substituting the strain field into the above equation we can obtain:

$$\begin{aligned}
\sigma_{ij} = \frac{E}{(1+\nu)} \varepsilon_{ij} &= \frac{1}{2} \frac{E}{(1+\nu)} \begin{bmatrix} 0 & \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3 \theta \right) & \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2 \theta \right) \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3 \theta \right) & 0 & 0 \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2 \theta \right) & 0 & 0 \end{bmatrix} \\
= \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} &= G \begin{bmatrix} 0 & \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3 \theta \right) & \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2 \theta \right) \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3 \theta \right) & 0 & 0 \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2 \theta \right) & 0 & 0 \end{bmatrix}
\end{aligned} \tag{6.229}$$

Let us suppose that we need to obtain the stress components in a new system $x'_1 - x'_2 - x'_3$, which is formed by a rotation around the x_1-axis, (see Figure 6.83). Then the transformation matrix from $x_1 - x_2 - x_3$ to $x'_1 - x'_2 - x'_3$ is given as follows:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{\mathrm{n}}_2 & \hat{\mathrm{n}}_3 \\ 0 & \hat{\mathrm{s}}_2 & \hat{\mathrm{s}}_3 \end{bmatrix} \tag{6.230}$$

Recall that the transformation law, (see **Problem 1.99**), for a second-order tensor components is given by

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl} \xrightarrow{\text{Matrix form}} \boldsymbol{\sigma}' = \boldsymbol{\mathcal{A}} \boldsymbol{\sigma} \boldsymbol{\mathcal{A}}^T \xrightarrow{Voigt} \{\boldsymbol{\sigma}'\} = [\boldsymbol{\mathcal{M}}]\{\boldsymbol{\sigma}\}$$

where

$$[\mathcal{M}] = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\ a_{21}^2 & a_{22}^{22} & a_{23}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\ a_{21}a_{11} & a_{22}a_{12} & a_{13}a_{23} & (a_{11}a_{22} + a_{12}a_{21}) & (a_{13}a_{22} + a_{12}a_{23}) & (a_{13}a_{21} + a_{11}a_{23}) \\ a_{31}a_{21} & a_{32}a_{22} & a_{33}a_{23} & (a_{31}a_{22} + a_{32}a_{21}) & (a_{33}a_{22} + a_{32}a_{23}) & (a_{33}a_{21} + a_{31}a_{23}) \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12} + a_{32}a_{11}) & (a_{33}a_{12} + a_{32}a_{13}) & (a_{33}a_{11} + a_{31}a_{13}) \end{bmatrix}$$

Then, for this particular transformation, (see equation (6.230)), we can obtain:

$$\begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{n}_2^2 & \hat{n}_3^2 & 0 & 2\hat{n}_2\hat{n}_3 & 0 \\ 0 & \hat{s}_2^2 & \hat{s}_3^2 & 0 & 2\hat{s}_2\hat{s}_3 & 0 \\ 0 & 0 & 0 & \hat{n}_2 & 0 & \hat{n}_3 \\ 0 & \hat{s}_2\hat{n}_2 & \hat{s}_3\hat{n}_3 & 0 & (\hat{s}_3\hat{n}_2 + \hat{s}_2\hat{n}_3) & 0 \\ 0 & 0 & 0 & \hat{s}_2 & 0 & \hat{s}_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} = 0 \\ \sigma_{22} = 0 \\ \sigma_{33} = 0 \\ \sigma_{12} \\ \sigma_{23} = 0 \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hat{n}_2\sigma_{12} + \hat{n}_3\sigma_{13} \\ 0 \\ \hat{s}_2\sigma_{12} + \hat{s}_3\sigma_{13} \end{bmatrix} \tag{6.231}$$

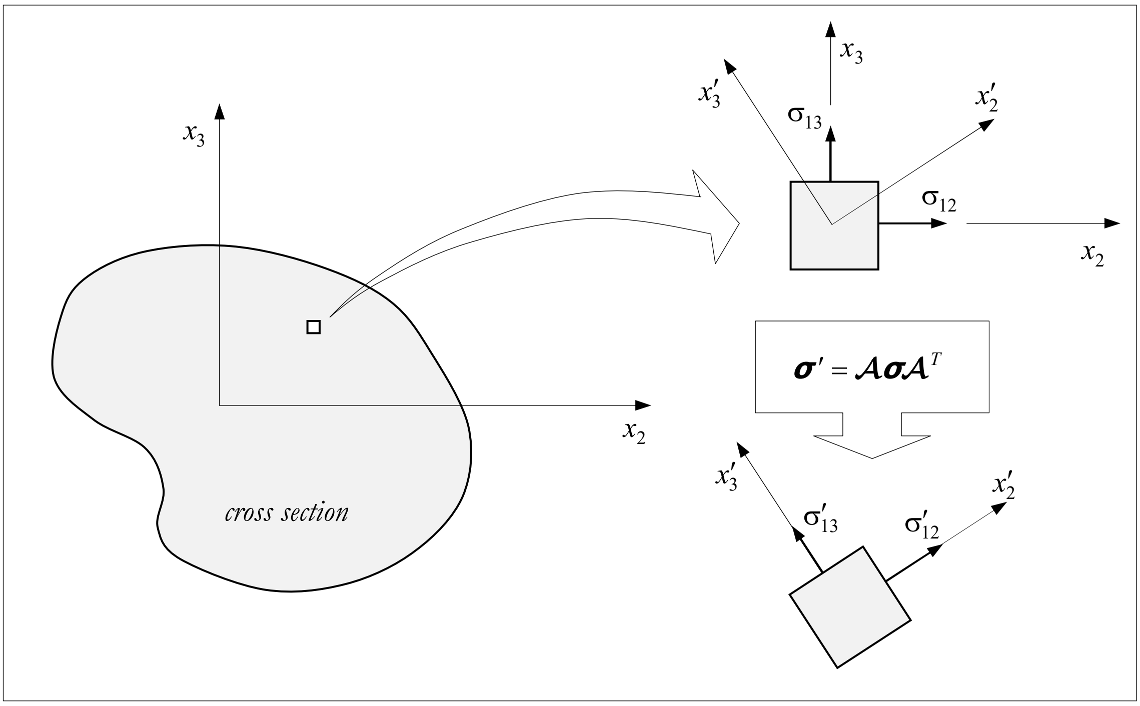


Figure 6.83

(Equilibrium equations)

Using the equilibrium equations without considering the body forces we can obtain:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{13}}{\partial x_1} + \dfrac{\partial \sigma_{23}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0 \\ \dfrac{\partial \sigma_{12}}{\partial x_1} = 0 \Rightarrow \sigma_{12} = \sigma_{12}(x_2, x_3) \\ \dfrac{\partial \sigma_{13}}{\partial x_1} = 0 \Rightarrow \sigma_{13} = \sigma_{13}(x_2, x_3) \end{cases} \tag{6.232}$$

Note that the stresses σ_{12} and σ_{13} are not function of x_1, i.e. they do not vary with x_1. By substituting the values of σ_{12} and σ_{13} into the first equation of the equilibrium equations we can obtain:

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad \Rightarrow \quad G\frac{\partial}{\partial x_2}\left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) + G\frac{\partial}{\partial x_3}\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) = 0 \tag{6.233}$$

$$G\left(\frac{\partial^2 \mathrm{u}_1}{\partial x_2^2} + \frac{\partial^2 \mathrm{u}_1}{\partial x_3^2}\right) = 0 \tag{6.234}$$

$$\boxed{\nabla^2 \mathrm{u}_1 = 0 \qquad \text{where} \qquad \mathrm{u}_1 = \mathrm{u}_1(x_2, x_3)} \tag{6.235}$$

which is the *differential equation of Saint-Venant torsion*. The displacement according to x_1 – direction is known as warping, as we will see later for the circular cross section there is no warping phenomenon, i.e. $\mathsf{u}_1 = 0$, (see Figure 6.84).

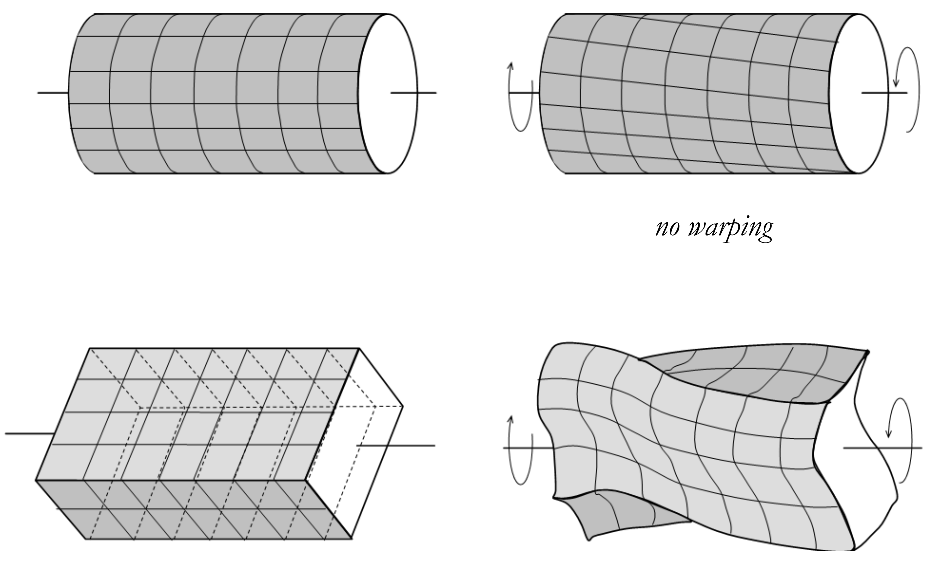


Figure 6.84: Torsion.

External Equilibrium

According to the problem statement, (see Figure 6.81), and by considering the external force equilibrium the following must be true:

External moment (Torque), (see **Problem 4.22**):

$$M_x \equiv M_T = \int_A (\sigma_{13} x_2 - \sigma_{12} x_3) dA \tag{6.236}$$

By using the stress components σ_{12} and σ_{13} given by the equations in (6.229), the above equation can also be written as follows:

$$\begin{aligned}
&M_T = \int_A (\sigma_{13} x_2 - \sigma_{12} x_3) dA = G \int_A \left[\left(\frac{\partial \mathsf{u}_1}{\partial x_3} + x_2 \,\theta \right) x_2 - \left(\frac{\partial \mathsf{u}_1}{\partial x_2} - x_3 \,\theta \right) x_3 \right] dA \\
&\Rightarrow M_T = G \int_A \left[\frac{\partial \mathsf{u}_1}{\partial x_3} x_2 + x_2^2 \,\theta - \frac{\partial \mathsf{u}_1}{\partial x_2} x_3 + x_3^2 \,\theta \right] dA = G\theta \int_A \left[\frac{\partial \mathsf{u}_1}{\partial x_3} \frac{x_2}{\theta} - \frac{\partial \mathsf{u}_1}{\partial x_2} \frac{x_3}{\theta} + x_2^2 + x_3^2 \right] dA \\
&\Rightarrow M_T = G\theta \mathsf{J}_T
\end{aligned} \tag{6.237}$$

where we have introduced the polar moment of inertia:

$$\mathsf{J}_T = \int_A \left[\frac{\partial \mathsf{u}_1}{\partial x_3} \frac{x_2}{\theta} - \frac{\partial \mathsf{u}_1}{\partial x_2} \frac{x_3}{\theta} + x_2^2 + x_3^2 \right] dA \tag{6.238}$$

where J_T is the Saint-Venant torsional stiffness of the section. The *shearing forces* according to the x_2-direction and x_3-direction are equal to zero, so:

$$Q_{x_2} = \int_A \sigma_{12} dA = \int_A G\left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) dA = 0 \quad ; \quad Q_{x_3} = \int_A \sigma_{13} dA = \int_A G\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) dA = 0 \tag{6.239}$$

where we have used the stress components σ_{12} and σ_{13} given by the equations in (6.229).

To complete the boundary conditions for the torsion problem. The boundary condition is defined by the absence of normal stress component on the external surface of the prismatic body $\sigma'_{13} = 0$ on the boundary surface, (see Figure 6.85). By means of equation (6.231) we can conclude that:

$$\begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hat{\mathrm{n}}_2\sigma_{12} + \hat{\mathrm{n}}_3\sigma_{13} \\ 0 \\ \hat{\mathrm{s}}_2\sigma_{12} + \hat{\mathrm{s}}_3\sigma_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tau \end{bmatrix} \tag{6.240}$$

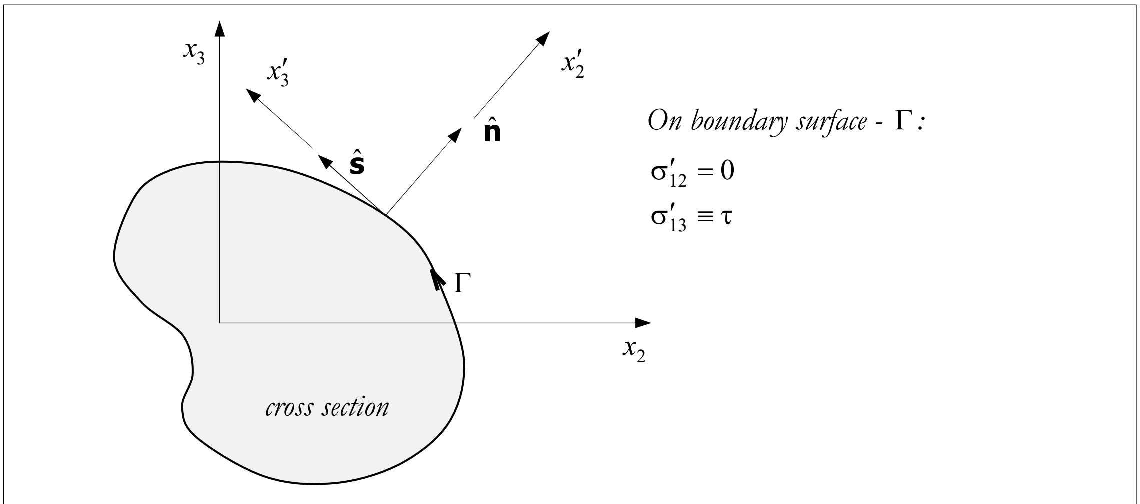


Figure 6.85

NOTE 1: Prismatic circular cross-sectional rod

Note that, when the cross section is circular there is no warping, since $\mathrm{u}_1 = 0$.

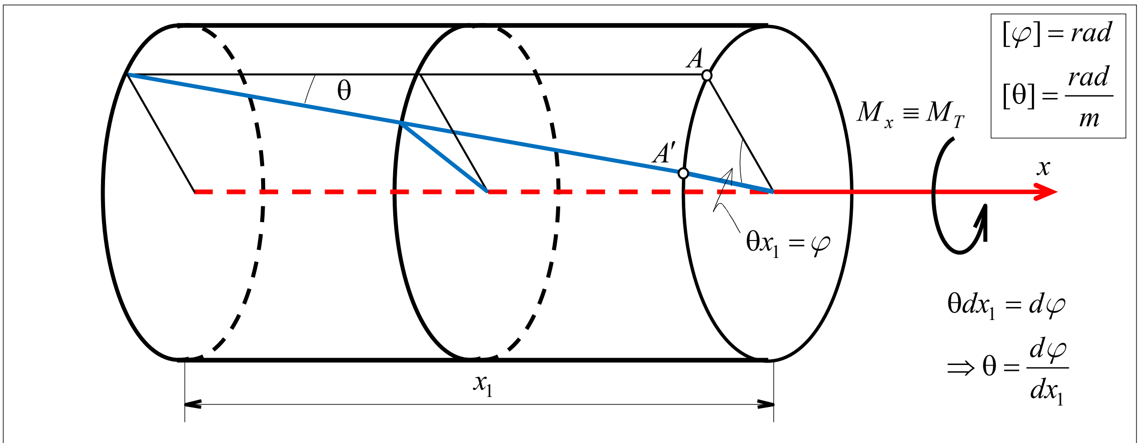


Figure 6.86: The rod (circular cross section) subjected to torque.

Then, in this case the strain field on the cross section is given by:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -x_3\theta & x_2\theta \\ -x_3\theta & 0 & 0 \\ x_2\theta & 0 & 0 \end{bmatrix} = \frac{\theta}{2}\begin{bmatrix} 0 & -x_3 & x_2 \\ -x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}$$

The stress field on the cross section is given by:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0 & -Gx_3\theta & Gx_2\theta \\ -Gx_3\theta & 0 & 0 \\ Gx_2\theta & 0 & 0 \end{bmatrix} = G\theta\begin{bmatrix} 0 & -x_3 & x_2 \\ -x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}$$

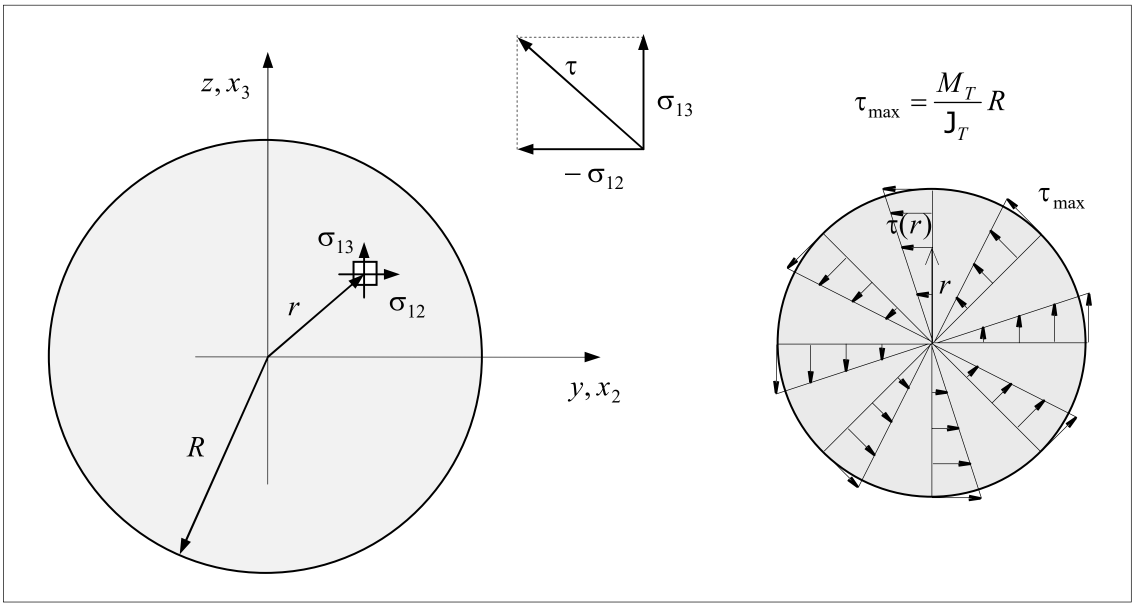


Figure 6.87: Tangential stresses – Moment of torsion.

The moment of torsion is given by:

$$\begin{aligned} M_T &= \int_A (\sigma_{13}x_2 - \sigma_{12}x_3)dA = \int_A ((G\theta x_2)x_2 - (-G\theta x_3)x_3)dA = G\theta\int_A (x_2^2 + x_3^2)dA \\ &= G\theta\int_A r^2 dA = G\theta J_T \end{aligned} \tag{6.241}$$

where $J_T = \int_A r^2 dA = \frac{\pi R^4}{2}$ is the polar inertia moment.

According to Figure 6.87 we can conclude that:

$$\begin{aligned} &(-\sigma_{12})^2 + (\sigma_{13})^2 = \tau^2 \quad \Rightarrow \quad (G\theta)^2(x_2^2 + x_3^2) = \tau^2 \\ &\Rightarrow (G\theta)^2 r^2 = \tau^2 \quad \Rightarrow \quad \tau(r) = G\theta r \end{aligned}$$

And according to the equation in (6.241) the relationship $G\theta = \frac{M_T}{J_T}$ holds, with which the above equation can be rewritten as follows:

$$\boxed{\tau(r) = G\theta r \quad \Rightarrow \quad \tau(r) = \frac{M_T}{J_T} r} \tag{6.242}$$

and the unit torsion angle *(angle of twist per unit length)* can be obtained as follows:

$$\theta = \frac{M_T}{G\mathrm{J}_T} \quad ; \quad \theta = \frac{\tau(r)}{Gr} \quad ; \quad \frac{M_T}{\mathrm{J}_T} = \frac{\tau(r)}{r} = G\theta \tag{6.243}$$

where the term $G\mathrm{J}_T$ stands for the torsional rigidity (or torsional stiffness).

The maximum value of $\tau(r)$ occurs in $r = R$, (see Figure 6.87):

$$\tau(r = R) = \tau_{\max} = G\theta R \quad \Rightarrow \quad \tau(r = R) = \tau_{\max} = \frac{M_T}{\mathrm{J}_T} R$$

For the hollow circular section the expressions are the same in which the polar inertia moment is given by $\mathrm{J}_T = \int_A r^2 dA = \frac{\pi}{2}(R_2^4 - R_1^4)$, (see Figure 6.88).

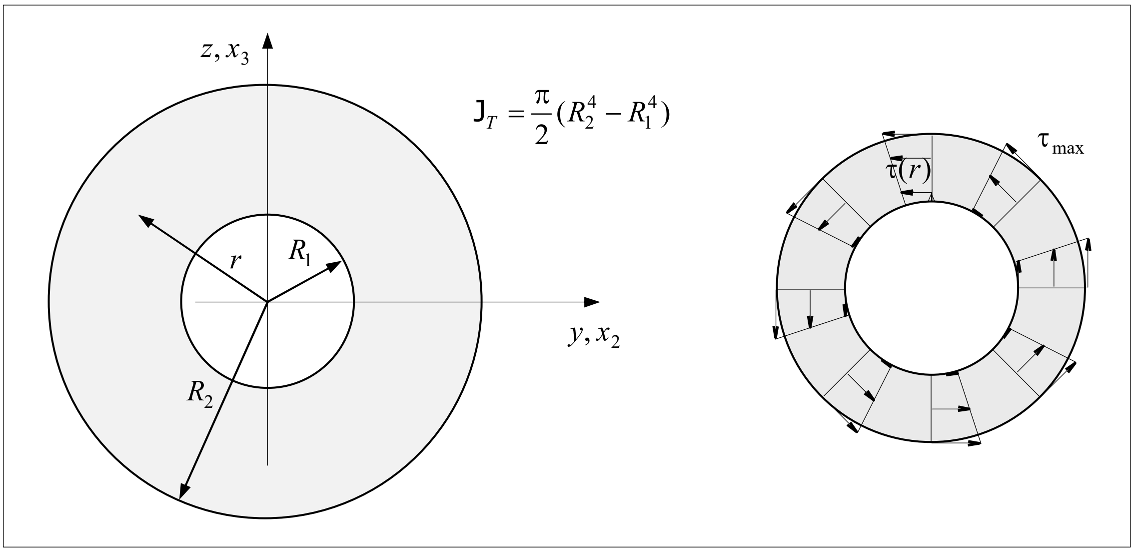


Figure 6.88: Tangential stresses – hollow circular cross section.

NOTE 2: Prandtl's Stress Function

Let us adopt the Prandtl's stress function ϕ such as:

$$\boxed{\sigma_{12} = \frac{\partial \phi}{\partial x_3} \quad ; \quad \sigma_{13} = -\frac{\partial \phi}{\partial x_2}} \tag{6.244}$$

Notice that this function satisfies the equilibrium equation (6.232):

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial x_3}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial \phi}{\partial x_2}\right) = 0$$

Recall that the governing equation for the linear elastic problem can be represented by the 6 stress components, which is known as stress formulation, (see **Problem 5.16** and **Problem 5.19**-NOTE 2). In addition, if we are considering a static problem in which the body force does not vary with $\vec{x}$ we fall back in Beltrami's equations, (see **Problem 5.16**-NOTE 1):

$$\boxed{\sigma_{ij,kk} + \frac{1}{(1+\nu)} \sigma_{kk,ij} = 0_{ij}}$$
$$\boxed{\nabla_{\vec{x}}^2 \boldsymbol{\sigma} + \frac{1}{(1+\nu)} \nabla_{\vec{x}} \left[\nabla_{\vec{x}} \left[\mathrm{Tr}(\boldsymbol{\sigma})\right]\right] = \mathbf{0}} \quad \textit{Beltrami's equations} \tag{6.245}$$

Taking into account the stress components, (see equation (6.229)):

$$\sigma_{ij} = G\begin{bmatrix} 0 & \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) & \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) & 0 & 0 \\ \left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix}$$

where the trace of stress tensor is zero $\sigma_{kk} = 0$, the Beltrami's equations become:

$$\sigma_{ij,kk} = \sigma_{ij,11} + \sigma_{ij,22} + \sigma_{ij,22} = 0_{ij}$$

$$\Rightarrow \sigma_{ij,kk} = \begin{bmatrix} \sigma_{11,11} & \sigma_{12,11} & \sigma_{13,11} \\ \sigma_{12,11} & \sigma_{22,11} & \sigma_{23,11} \\ \sigma_{13,11} & \sigma_{23,11} & \sigma_{33,11} \end{bmatrix} + \begin{bmatrix} \sigma_{11,22} & \sigma_{12,22} & \sigma_{13,22} \\ \sigma_{12,22} & \sigma_{22,22} & \sigma_{23,22} \\ \sigma_{13,22} & \sigma_{23,22} & \sigma_{33,22} \end{bmatrix} + \begin{bmatrix} \sigma_{11,33} & \sigma_{12,33} & \sigma_{13,33} \\ \sigma_{12,33} & \sigma_{22,33} & \sigma_{23,33} \\ \sigma_{13,33} & \sigma_{23,33} & \sigma_{33,33} \end{bmatrix} = 0_{ij}$$

$$\Rightarrow \sigma_{ij,kk} = \begin{bmatrix} 0 & (\sigma_{12,11} + \sigma_{12,22} + \sigma_{12,33}) & (\sigma_{13,11} + \sigma_{13,22} + \sigma_{13,33}) \\ (\sigma_{12,11} + \sigma_{12,22} + \sigma_{12,33}) & 0 & 0 \\ (\sigma_{13,11} + \sigma_{13,22} + \sigma_{13,33}) & 0 & 0 \end{bmatrix} = 0_{ij}$$

With that we can obtain:

$$\begin{cases} \sigma_{12,11} + \sigma_{12,22} + \sigma_{12,33} = 0 \\ \sigma_{13,11} + \sigma_{13,22} + \sigma_{13,33} = 0 \end{cases}$$

Note that $\sigma_{12} = \sigma_{12}(x_2, x_3)$ and $\sigma_{13} = \sigma_{13}(x_2, x_3)$, (see equation (6.232)), so, the above equations become:

$$\begin{cases} \sigma_{12,22} + \sigma_{12,33} = 0 \\ \sigma_{13,22} + \sigma_{13,33} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial^2 \sigma_{12}}{\partial x_2^2} + \dfrac{\partial^2 \sigma_{12}}{\partial x_3^2} = 0 \\ \dfrac{\partial^2 \sigma_{13}}{\partial x_2^2} + \dfrac{\partial^2 \sigma_{13}}{\partial x_3^2} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \nabla^2_{\vec{x}} \sigma_{12} = 0 \\ \nabla^2_{\vec{x}} \sigma_{13} = 0 \end{cases} \tag{6.246}$$

If we consider the stresses defined in (6.244) into the equations (6.246) we can obtain:

$$\begin{cases} \nabla^2_{\vec{x}} \sigma_{12} = 0 \\ \nabla^2_{\vec{x}} \sigma_{13} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \nabla^2_{\vec{x}}\left(\dfrac{\partial \phi}{\partial x_3}\right) = \dfrac{\partial}{\partial x_3}\left(\nabla^2_{\vec{x}}\phi\right) = 0 \Rightarrow \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}\right) = 0 \\ \nabla^2_{\vec{x}}\left(\dfrac{-\partial \phi}{\partial x_2}\right) = \dfrac{-\partial}{\partial x_2}\left(\nabla^2_{\vec{x}}\phi\right) = 0 \Rightarrow \dfrac{\partial}{\partial x_2}\left(\nabla^2_{\vec{x}}\phi\right) = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}\right) = 0 \end{cases} \tag{6.247}$$

As $\nabla^2_{\vec{x}}\phi$ does not vary with x_2 and x_3 we can conclude that:

$$\boxed{\nabla^2_{\vec{x}}\phi = \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \mathrm{C} = \text{constant}} \tag{6.248}$$

Then, any function ϕ which satisfies the above equation will fulfill the equilibrium and compatibility equations.

Starting from the stress components, (see equation (6.229)):

$$\sigma_{12} = G\left(\frac{\partial \mathrm{u}_1}{\partial x_2} - x_3\,\theta\right) \quad \Rightarrow \quad \frac{\partial \mathrm{u}_1}{\partial x_2} = \frac{\sigma_{12}}{G} + x_3\,\theta$$
$$\sigma_{13} = G\left(\frac{\partial \mathrm{u}_1}{\partial x_3} + x_2\,\theta\right) \quad \Rightarrow \quad \frac{\partial \mathrm{u}_1}{\partial x_3} = \frac{\sigma_{13}}{G} - x_2\,\theta \tag{6.249}$$

and taking the derivative σ_{12} with respect to x_3, and taking the derivative of σ_{13} with respect to x_2 we can obtain

$$\frac{\partial}{\partial x_3}\frac{\partial \mathrm{u}_1}{\partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\sigma_{12}}{G} + x_3\,\theta\right) = \frac{1}{G}\frac{\partial \sigma_{12}}{\partial x_3} + \theta$$
$$\frac{\partial}{\partial x_2}\frac{\partial \mathrm{u}_1}{\partial x_3} = \frac{\partial}{\partial x_2}\left(\frac{\sigma_{13}}{G} - x_2\,\theta\right) = \frac{1}{G}\frac{\partial \sigma_{13}}{\partial x_2} - \theta$$

and by subtracting the two expressions we can obtain that

$$\frac{\partial}{\partial x_3}\frac{\partial \mathrm{u}_1}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial \mathrm{u}_1}{\partial x_3} = \frac{1}{G}\frac{\partial \sigma_{12}}{\partial x_3} + \theta - \frac{1}{G}\frac{\partial \sigma_{13}}{\partial x_2} + \theta$$
$$\Rightarrow 0 = \frac{1}{G}\frac{\partial \sigma_{12}}{\partial x_3} + 2\theta - \frac{1}{G}\frac{\partial \sigma_{13}}{\partial x_2}$$
$$\Rightarrow \frac{\partial \sigma_{12}}{\partial x_3} - \frac{\partial \sigma_{13}}{\partial x_2} = -2G\theta$$

and by substituting the stress values $\sigma_{12} = \frac{\partial \phi}{\partial x_3}$ and $\sigma_{13} = -\frac{\partial \phi}{\partial x_2}$ into the above equation we can obtain:

$$\frac{\partial \sigma_{12}}{\partial x_3} - \frac{\partial \sigma_{13}}{\partial x_2} = -2G\theta \quad \Rightarrow \quad \frac{\partial}{\partial x_3}\frac{\partial \phi}{\partial x_3} + \frac{\partial}{\partial x_2}\frac{\partial \phi}{\partial x_2} = \left(\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}\right) = -2G\theta$$

and if we take into account the equation in (6.248) we can conclude that:

$$\boxed{\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\theta = \text{constant} = \mathrm{C}} \quad \Bigg| \quad \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \phi) \equiv \nabla^2_{\vec{x}} \phi = -2G\theta = \text{constant} \tag{6.250}$$

where $\nabla^2_{\vec{x}}$ stands for the Laplacian.

To complete the problem statement we have to define the boundary conditions. To the torsion problem the boundary condition is defined by the absence of normal traction vector on the external surface of the prismatic body, (see Figure 6.89).

The traction vector in terms of the Cauchy stress tensor becomes:

$$\mathrm{t}_i^{(\hat{\mathbf{n}})} = \sigma_{ij}\hat{\mathrm{n}}_j \quad \Rightarrow \quad \begin{Bmatrix} \mathrm{t}_1^{(\hat{\mathbf{n}})} \\ \mathrm{t}_2^{(\hat{\mathbf{n}})} \\ \mathrm{t}_3^{(\hat{\mathbf{n}})} \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} \hat{\mathrm{n}}_1 \\ \hat{\mathrm{n}}_2 \\ \hat{\mathrm{n}}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$
$$\Rightarrow \begin{Bmatrix} \mathrm{t}_1^{(\hat{\mathbf{n}})} \\ \mathrm{t}_2^{(\hat{\mathbf{n}})} \\ \mathrm{t}_3^{(\hat{\mathbf{n}})} \end{Bmatrix} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathrm{n}}_1 = 0 \\ \hat{\mathrm{n}}_2 \\ \hat{\mathrm{n}}_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{12}\hat{\mathrm{n}}_2 + \sigma_{13}\hat{\mathrm{n}}_3 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

with that the boundary condition is:

$$\sigma_{12}\hat{\mathrm{n}}_2 + \sigma_{13}\hat{\mathrm{n}}_3 = 0$$

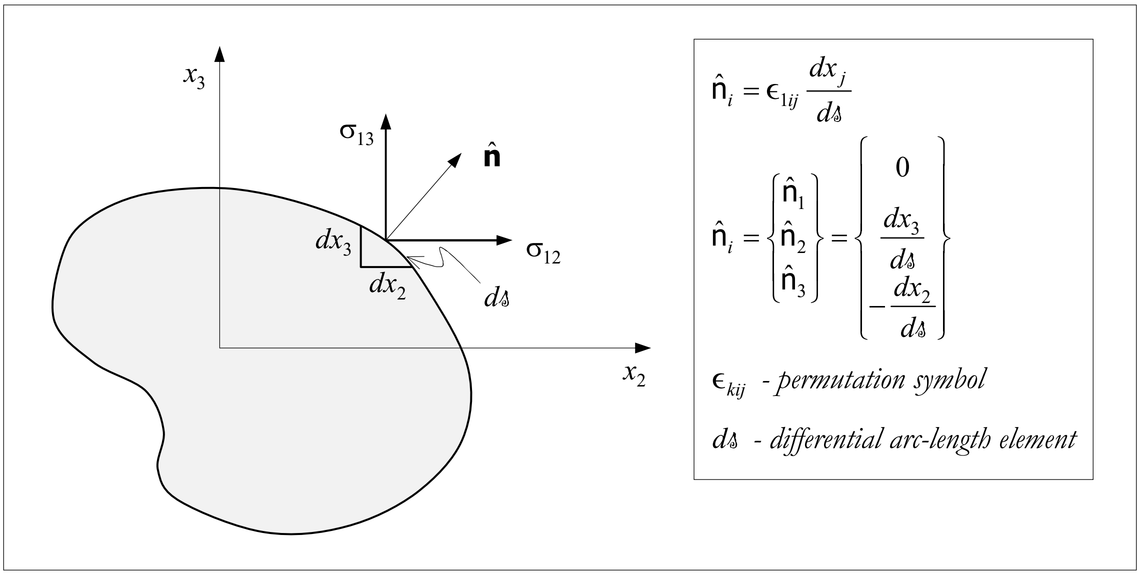


Figure 6.89

Taking into account the stresses given by (6.244), the above condition can be rewritten as follows:

$$\sigma_{12}\hat{\mathsf{n}}_2 + \sigma_{13}\hat{\mathsf{n}}_3 = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial x_3}\hat{\mathsf{n}}_2 - \frac{\partial \phi}{\partial x_2}\hat{\mathsf{n}}_3 = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial x_3}\frac{dx_3}{d\mathcal{s}} - \frac{\partial \phi}{\partial x_2}\left(-\frac{dx_2}{d\mathcal{s}}\right) = 0$$
$$\Rightarrow \frac{\partial \phi}{\partial x_3}\frac{dx_3}{d\mathcal{s}} + \frac{\partial \phi}{\partial x_2}\frac{dx_2}{d\mathcal{s}} = 0 \quad \Rightarrow \quad \frac{d\phi}{d\mathcal{s}} = 0 \tag{6.251}$$

With that we can conclude that ϕ is constant on the boundary and can be assume any value, with which we adopt zero.

Let us consider that:

$$\mathsf{F}_3(x_2, x_3) = \sigma_{12} = \frac{\partial \phi}{\partial x_3} \quad ; \quad \mathsf{F}_2(x_2, x_3) = -\sigma_{13} = \frac{\partial \phi}{\partial x_2} \tag{6.252}$$

The function ϕ is a compatible field if and only if:

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x_2} &= \mathsf{F}_2(x_2, x_3) \\ \frac{\partial \phi}{\partial x_3} &= \mathsf{F}_3(x_2, x_3) \end{aligned}\right\} \xrightarrow{\text{compatible iff}} \frac{\partial \mathsf{F}_2}{\partial x_3} = \frac{\partial \mathsf{F}_3}{\partial x_2} \tag{6.253}$$

If we consider the Green's theorem, (see Chapter 1 of the textbook), we can establish that:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\Gamma} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{e}}_1 dA \xrightarrow{\textit{components}} \oint_{\Gamma} \mathsf{F}_2 dx_2 + \mathsf{F}_3 dx_3 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_2}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_2} \right) dA_1$$

With which we can conclude:

$$\oint_{\Gamma} \mathsf{F}_2 dx_2 + \mathsf{F}_3 dx_3 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_2}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_2} \right) dA_1 \quad \Rightarrow \quad \oint_{\Gamma} \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = \int_{\Omega} \left(\frac{\partial}{\partial x_3}\frac{\partial \phi}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial \phi}{\partial x_3} \right) dA_1 = 0$$
$$\Rightarrow \oint_{\Gamma} \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = \oint_{\Gamma} \nabla_{\vec{x}} \phi \cdot d\vec{\boldsymbol{x}} = \oint_{\Gamma} d\phi = 0$$

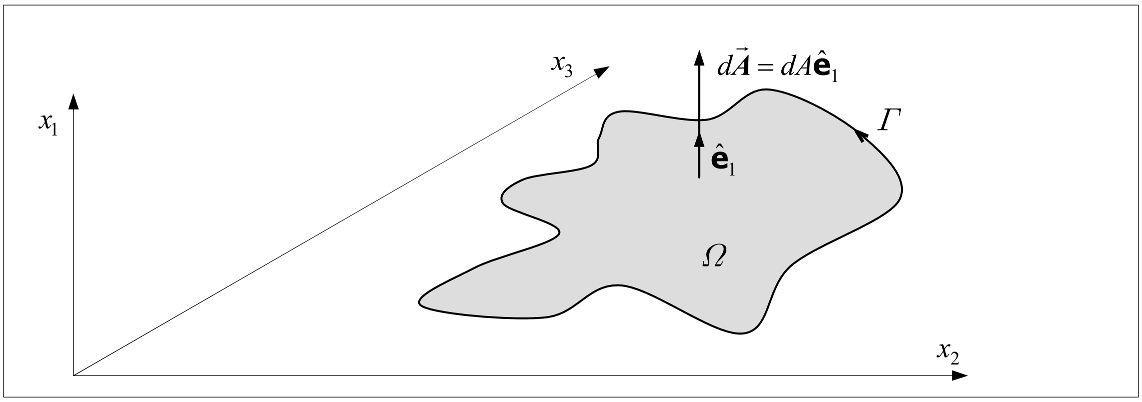


Figure 6.90: Green's theorem.

Now let us obtain the expression for the moment of torsion. Recalling the moment of torsion equation we can obtain:

$$M_T = \int_A (\sigma_{13}x_2 - \sigma_{12}x_3)dA = -\int_A (\phi_{,2}x_2 + \phi_{,3}x_3)dA = -\int_A (\phi_{,i}x_i)dA \qquad (i = 2,3)$$

note that $(\phi x_i)_{,i} = \phi_{,i}x_i + \phi x_{i,i} = \phi_{,i}x_i + \phi\delta_{ii}^{(2D)} = \phi_{,i}x_i + 2\phi \Rightarrow \phi_{,i}x_i = (\phi x_i)_{,i} - 2\phi$, where we have applied the trace of the Kronecker delta for 2D, i.e. $x_{i,i} = \delta_{ii}^{(2D)} = \delta_{22} + \delta_{33} = 2$, thus

$$\begin{aligned} M_T &= -\int_A (\phi_{,i}x_i)dA = -\int_A ((\phi x_i)_{,i} - 2\phi)dA = -\int_A (\phi x_i)_{,i}\,dA + \int_A (2\phi)dA \\ &= -\int_\Gamma (\phi x_i)\hat{\mathsf{n}}_i d\Gamma + \int_A (2\phi)dA \\ &= -2A\phi_\Gamma + \int_A (2\phi)dA \end{aligned}$$

If we consider that on the boundary the value of ϕ is zero, *i.e.* $\phi_\Gamma = 0$, we can obtain that:

$$\boxed{M_T = 2\int_A \phi\, dA} \tag{6.254}$$

The same can be shown by means of tensorial notation (in 2D):

$$\begin{aligned} M_T &= -\int_A (\phi_{,i}x_i)dA = -\int_A [(\nabla_{\vec{x}}\phi)\cdot\vec{x}]dA = -\int_A [\nabla_{\vec{x}}\cdot(\phi\vec{x})]dA + \int_A [\phi(\nabla_{\vec{x}}\cdot\vec{x})]dA \\ &= -\int_\Gamma \phi\vec{x}\cdot\hat{\mathbf{n}}d\Gamma + \int_A 2\phi\, dA = -2A\phi_\Gamma + \int_A 2\phi\, dA \\ &= 2\int_A \phi\, dA \end{aligned} \tag{6.255}$$

where we have used the property $2A = \int_\Gamma \vec{x}\cdot\hat{\mathbf{n}}d\Gamma$, (see **Problem 1.128**).

It is easy to show that there exist iso-curves of ϕ, (see Figure 6.91). Note that for the system $(x_2' - x_3')$ on iso-curve of ϕ, we have:

$$\sigma_{12}' = \frac{\partial\phi}{\partial x_3'} = 0 \quad ; \quad -\sigma_{13}' = \tau = \frac{\partial\phi}{\partial x_2'} \tag{6.256}$$

That is, τ is equal to the slope of ϕ according to the x_2'-direction, which has the same direction as $\nabla\phi$, (see Figure 6.91).

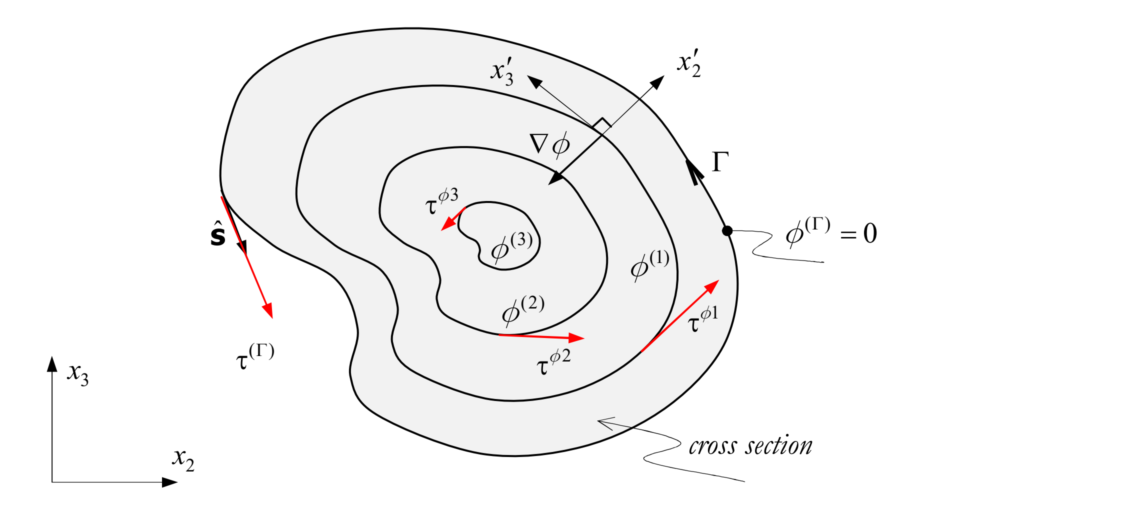


Figure 6.91: Iso-curves of ϕ.

As additional note, next we will show in a different way that the shearing force on the cross section is zero, and that the net moment is equal twice the volume formed by $\phi(x_2,x_3)$. We start from the shearing force (Q_{x_2}) according to the x_2-direction:

$$
\begin{aligned}
Q_{x_2} &= \int_A \sigma_{12}\,dA = \int_A \frac{\partial\phi}{\partial x_3}dA = \int_{x_2^{(1)}}^{x_2^{(2)}}\int_{x_3^{(1)}}^{x_3^{(2)}} \frac{\partial\phi}{\partial x_3}dx_3dx_2 = \int_{x_2^{(1)}}^{x_2^{(2)}}\left[\int_{x_3^{(1)}}^{x_3^{(2)}} \frac{\partial\phi}{\partial x_3}\times 1dx_3\right]dx_2 \\
\Rightarrow Q_{x_2} &= \int_{x_2^{(1)}}^{x_2^{(2)}}\left[\phi\Big|_{x_3^{(1)}}^{x_3^{(2)}} - \int_{x_3^{(1)}}^{x_3^{(2)}} \underbrace{\phi\times\frac{\partial(1)}{\partial x_3}dx_3}_{=0}\right]dx_2 = \int_{x_2^{(1)}}^{x_2^{(2)}}\left[\phi(x_3^{(2)})-\phi(x_3^{(1)})\right]dx_2 = 0
\end{aligned}
\tag{6.257}
$$

where we have applied the integration by part. Note that $\phi(x_3^{(2)})$ and $\phi(x_3^{(1)})$ are values of the membrane on the bar surface, as we have shown previously, ϕ is constant on the boundary so, $\phi(x_3^{(2)})-\phi(x_3^{(1)})=0$. The same procedure can be used to show that $Q_{x_3}=0$.

As we have seen previously, the net moment is obtained as follows:

$$
M_T = \int_A (\sigma_{13}x_2 - \sigma_{12}x_3)dA = -\int_A\left[\frac{\partial\phi}{\partial x_2}x_2 + \frac{\partial\phi}{\partial x_3}x_3\right]dA
\tag{6.258}
$$

Also note that:

$$
\begin{aligned}
\int_A \frac{\partial\phi}{\partial x_3}x_3dA &= \int_{x_2^{(1)}}^{x_2^{(2)}}\left[\int_{x_3^{(1)}}^{x_3^{(2)}}\frac{\partial\phi}{\partial x_3}x_3dx_3\right]dx_2 = \int_{x_2^{(1)}}^{x_2^{(2)}}\left[\phi x_3\Big|_{x_3^{(1)}}^{x_3^{(2)}} - \int_{x_3^{(1)}}^{x_3^{(2)}}\phi\times\frac{\partial(x_3)}{\partial x_3}dx_3\right]dx_2 = -\int_{x_2^{(1)}}^{x_2^{(2)}}\left[\int_{x_3^{(1)}}^{x_3^{(2)}}\phi dx_3\right]dx_2 \\
&= -\int \phi dA
\end{aligned}
$$

where we have used that ϕ at any point of the boundary is zero, i.e. $\phi^{(\Gamma)}=0$, so the term $\underbrace{\phi(x_3^{(2)})}_{=0}x_3^{(2)} - \underbrace{\phi(x_3^{(1)})}_{=0}x_3^{(1)}=0$. Similarly, we can obtain that $\int_A \frac{\partial\phi}{\partial x_2}x_2dA = -\int\phi dA$. With that the equation in (6.258) becomes $M_T = 2\int_A \phi dA$, (see equation (6.254)).

NOTE 3: If we consider that $\tau_1 = 0$, $\tau_2 = \sigma_{12} = \dfrac{\partial \phi}{\partial x_3}$, and $\tau_3 = \sigma_{13} = \dfrac{-\partial \phi}{\partial x_2}$, the following is true $(\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 = -\nabla^2_{\vec{x}} \phi$, (see **Problem 1.107**). And by taking into account the equation (6.250) we can conclude that $(\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 = -\nabla^2_{\vec{x}} \phi = 2G\theta$, then if we take the integral over the cross section area we can obtain:

$$\int_A (\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 dA = \int_A (2G\theta) dA = 2\theta \int_A G dA = \underbrace{2G\theta A}_{G-\text{constant}} \tag{6.259}$$

By using the Stoke's theorem we can say that $\int_A (\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 dA = \oint_\Gamma \vec{\tau} \cdot \hat{\mathbf{s}} d\Gamma = \oint_\Gamma \tau^{(\Gamma)} \, d\Gamma$, then the above equation can be rewritten as follows:

$$\boxed{\int_A (\nabla_{\vec{x}} \wedge \vec{\tau}) \cdot \hat{\mathbf{e}}_1 dA = \oint_\Gamma \tau^{(\Gamma)} \, d\Gamma = 2G\theta A} \tag{6.260}$$

where $\tau^{(\Gamma)}$ is the tangential stress on the boundary, (see Figure 6.91).

NOTE 4: Prandtl's membrane analogy (Soap-film analogy)

Consider a homogenous membrane fixed at its extremities, which is subjected to a uniform lateral pressure p, $[p]=\frac{N}{m^2}$, (see Figure 6.92). Due to the pressure in the membrane a stress state S appears. Next, we will define which equation governs such problem.

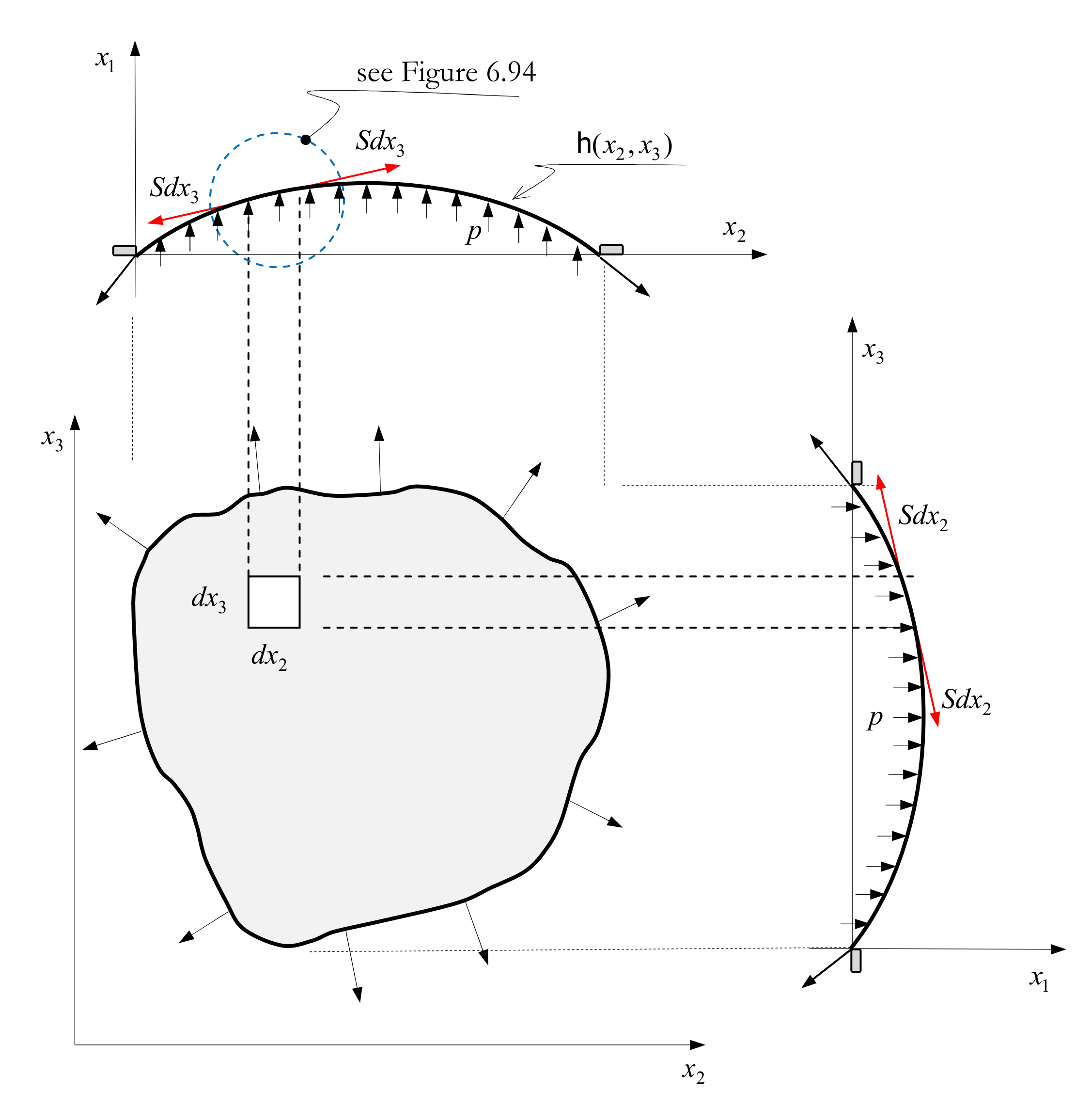


Figure 6.92: Membrane under pressure.

Note that the slope of the membrane at the point (x_2,x_3) is given by the derivative of the function $\mathsf{h}(x_2,x_3)$, i.e. $\frac{\partial \mathsf{h}}{\partial x_2}$ and $\frac{\partial \mathsf{h}}{\partial x_3}$, which are tangents to the curve at the point. We denoted by $\tan(\alpha_2)=\frac{\partial \mathsf{h}}{\partial x_2}$ and $\tan(\alpha_3)=\frac{\partial \mathsf{h}}{\partial x_3}$, and if we are considering small angles the relationships $\tan(\alpha)\approx\sin(\alpha)\approx\alpha$ hold. In the differential element $dx_2 - dx_3$ the variation of the tangents are indicated in Figure 6.93.

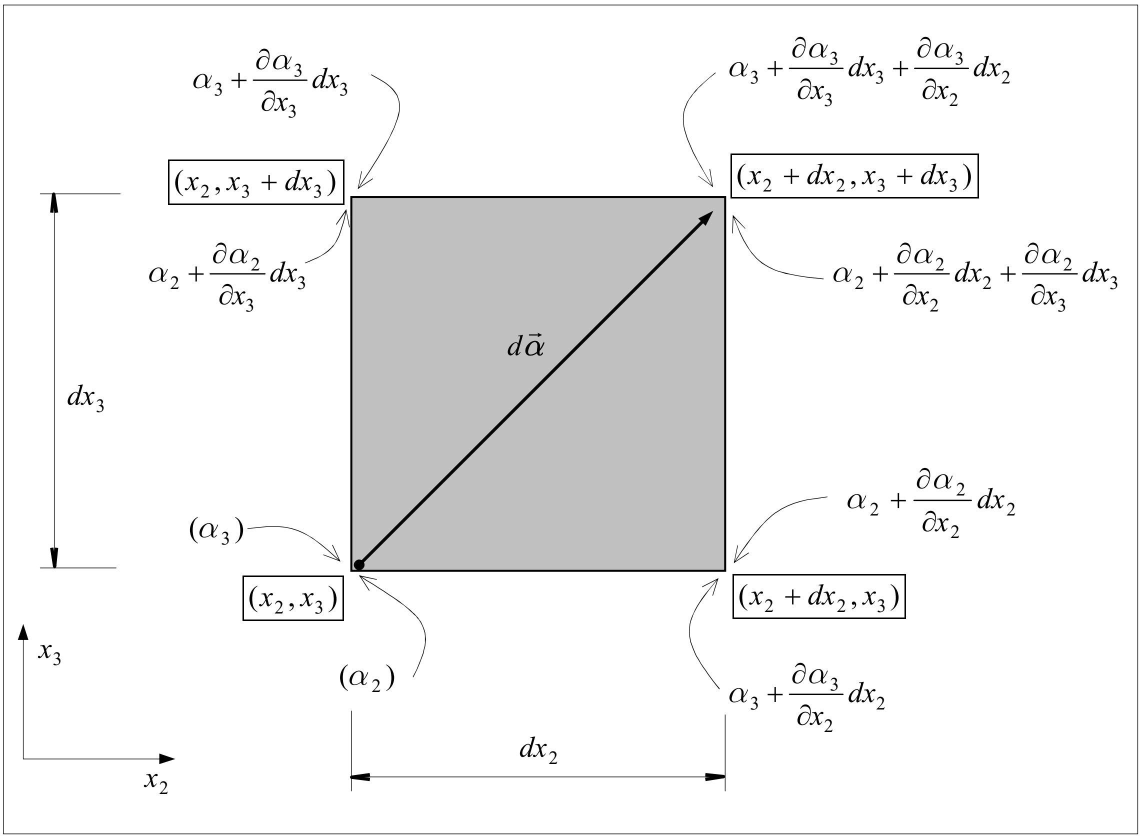


Figure 6.93: Variation of the tangents in the differential element.

If we consider that there is no distortional of the tangents the terms $\frac{\partial \alpha_3}{\partial x_2} dx_2 = 0$ and $\frac{\partial \alpha_2}{\partial x_3} dx_3 = 0$ hold, (see Figure 6.94).

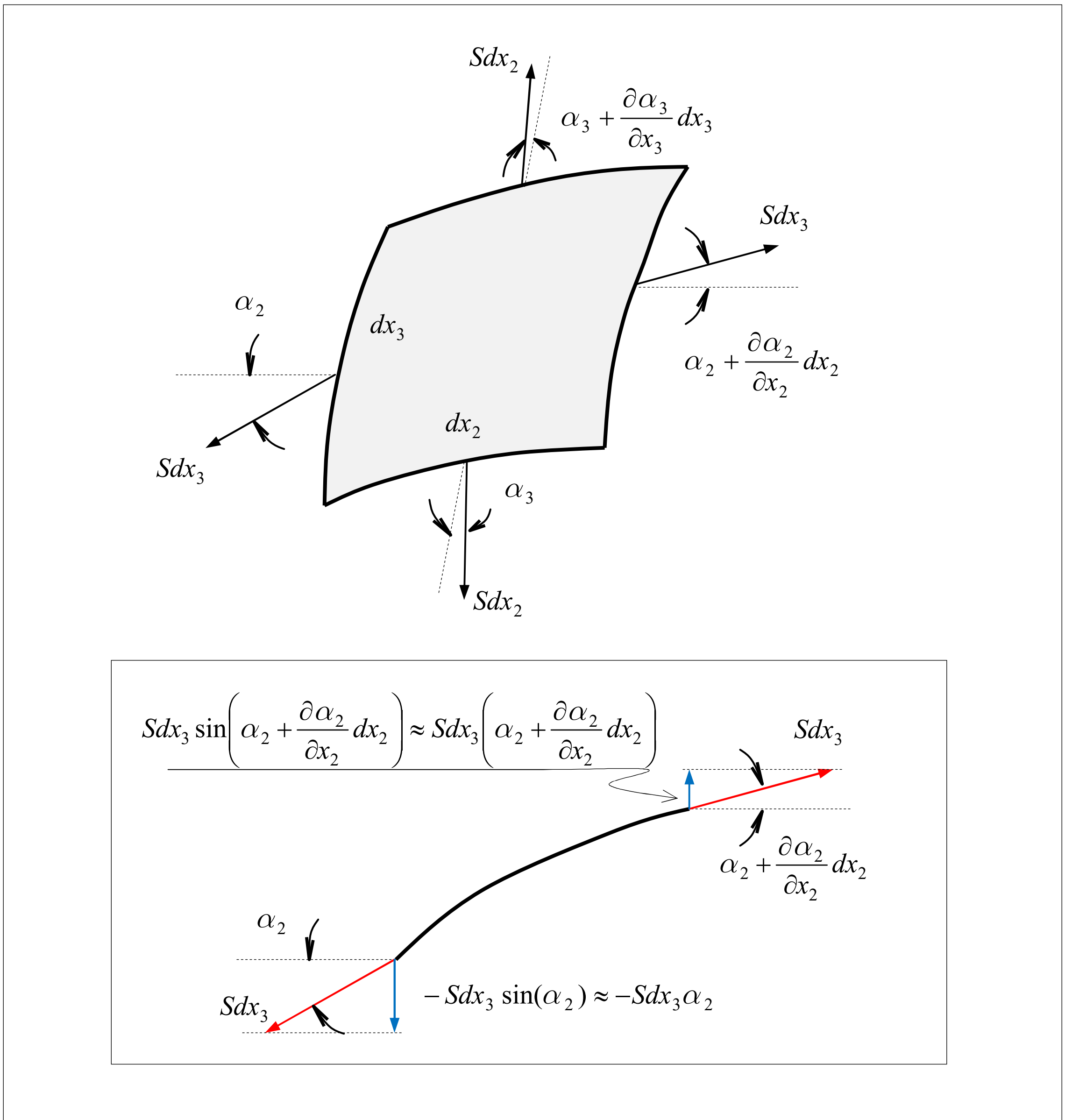


Figure 6.94

By apply the force equilibrium condition according to x_1 -direction, we can obtain:

$$\sum F_{x1} = 0$$

$$pdx_2dx_3 + Sdx_3 \sin\left(\frac{\partial h}{\partial x_2} + \frac{\partial^2 h}{\partial x_2^2} dx_2\right) - Sdx_3 \sin\left(\frac{\partial h}{\partial x_2}\right) + Sdx_2 \sin\left(\frac{\partial h}{\partial x_3} + \frac{\partial^2 h}{\partial x_3^2} dx_3\right) - Sdx_2 \sin\left(\frac{\partial h}{\partial x_3}\right) = 0$$

$$\Rightarrow pdx_2dx_3 + Sdx_3\left(\frac{\partial h}{\partial x_2} + \frac{\partial^2 h}{\partial x_2^2} dx_2\right) - Sdx_3\left(\frac{\partial h}{\partial x_2}\right) + Sdx_2\left(\frac{\partial h}{\partial x_3} + \frac{\partial^2 h}{\partial x_3^2} dx_3\right) - Sdx_2\left(\frac{\partial h}{\partial x_3}\right) = 0$$

$$\Rightarrow pdx_2dx_3 + Sdx_3\frac{\partial h}{\partial x_2} + \frac{\partial^2 h}{\partial x_2^2} Sdx_3dx_2 - Sdx_3\frac{\partial h}{\partial x_2} + Sdx_2\frac{\partial h}{\partial x_3} + \frac{\partial^2 h}{\partial x_3^2} Sdx_2dx_3 - Sdx_2\frac{\partial h}{\partial x_3} = 0$$

$$\Rightarrow pdx_2dx_3 + \frac{\partial^2 h}{\partial x_2^2} Sdx_3dx_2 + \frac{\partial^2 h}{\partial x_3^2} Sdx_2dx_3 = 0$$

$$\Rightarrow pdx_2dx_3 + Sdx_2dx_3\left(\frac{\partial^2 h}{\partial x_2^2} + \frac{\partial^2 h}{\partial x_3^2}\right) = 0$$

with which we can conclude that the governing equation for the membrane under pressure is given by:

$$\boxed{\frac{\partial^2 \mathsf{h}}{\partial x_2^2}+\frac{\partial^2 \mathsf{h}}{\partial x_3^2}=-\frac{p}{S}} \qquad \nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\mathsf{h})\equiv\nabla^2_{\vec{x}}\mathsf{h}=-\frac{p}{S} \tag{6.261}$$

Making an analogy between the above equation and the torsion problem equation (6.250), we can conclude that

$$\mathsf{h}=\phi \qquad \text{and} \qquad 2G\theta=\frac{p}{S}=\text{constant} \tag{6.262}$$

with which we can say that the moment of torsion, (see equation (6.254)), is equal to two times the volume defined by the membrane:

$$M_T=2\int_A \phi\, dA=2\int_A \mathsf{h}(x_2,x_3)\, dA=2V_{memb} \tag{6.263}$$

Problem 6.45

Consider a prismatic bar in which there is a longitudinal cavity, and the cross section of the bar is shown in Figure 6.95. Obtain the equation for M_T in terms of the areas A_0 and A_i.

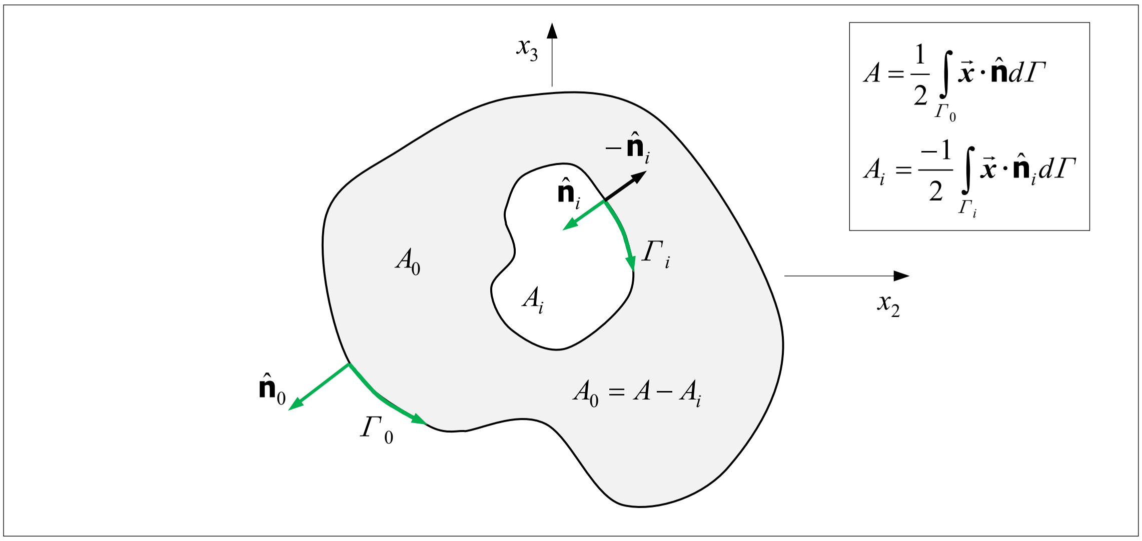


Figure 6.95: Cross section of the bar with longitudinal cavity.

Solution:

The equation in (6.255) states that

$$M_T=-\int_\Gamma \phi\vec{x}\cdot\hat{\mathbf{n}}\,d\Gamma+\int_A 2\phi\, dA \tag{6.264}$$

In order to apply the above equation we will decompose the area as shown in Figure 6.96, then, the torque can be expressed as follows:

$$M_T=M_T^{\overline{A}_0}+M_T^{\overline{\overline{A}}_0}=\left(-\int_{\Gamma_{\overline{A}_0}}\phi\vec{x}\cdot\hat{\mathbf{n}}\,d\Gamma+\int_{\overline{A}_0}2\phi\, dA\right)+\left(-\int_{\Gamma_{\overline{\overline{A}}_0}}\phi\vec{x}\cdot\hat{\mathbf{n}}\,d\Gamma+\int_{\overline{\overline{A}}_0}2\phi\, dA\right) \tag{6.265}$$

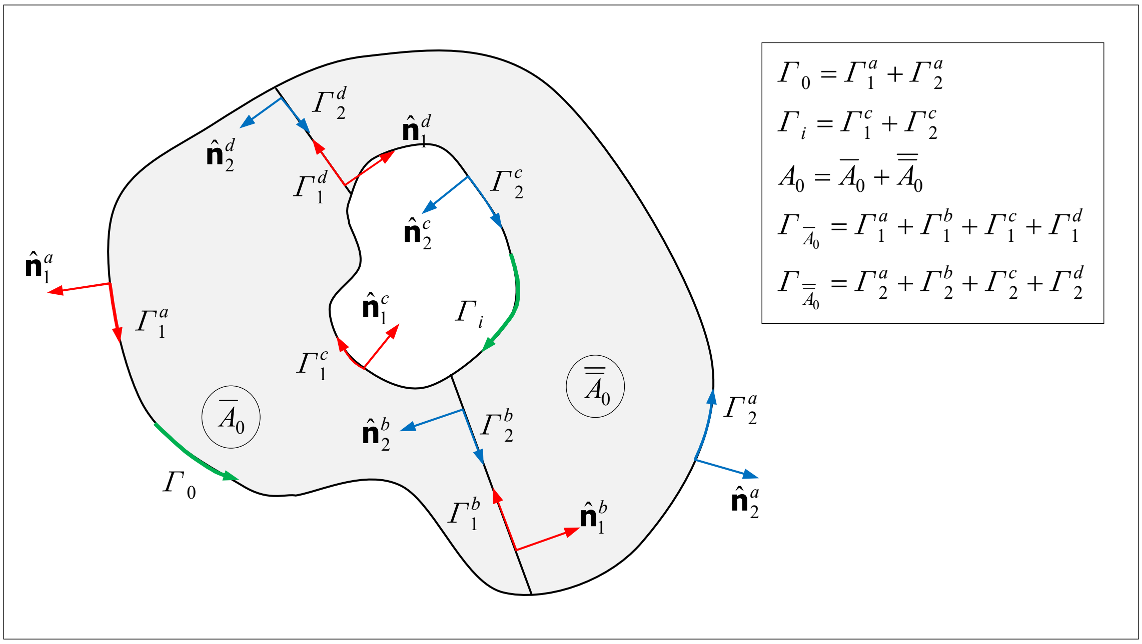


Figure 6.96: Decomposition of the cross.

By considering that:

1) The boundary of the area $\overline{A}_0$ is $\Gamma_{\overline{A}_0} = \Gamma_1^a + \Gamma_1^b + \Gamma_1^c + \Gamma_1^d$;

2) The boundary of the area $\overline{\overline{A}}_0$ is $\Gamma_{\overline{\overline{A}}_0} = \Gamma_2^a + \Gamma_2^b + \Gamma_2^c + \Gamma_2^d$;

3) There is continuity of the function ϕ on the boundaries Γ^b and Γ^d, then we can conclude that $\phi^{(\Gamma_1^b)} = \phi^{(\Gamma_2^b)}$ and $\phi^{(\Gamma_1^d)} = \phi^{(\Gamma_2^d)}$. Note also that $\hat{\mathbf{n}}_2^b = -\hat{\mathbf{n}}_1^b$, $\hat{\mathbf{n}}_2^d = -\hat{\mathbf{n}}_1^d$.

Then, the equation (6.265) can be rewritten as follows

$$
\begin{aligned}
M_T &= \left(-\int_{\Gamma_{\overline{A}_0}} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\overline{A}_0} 2\phi\, dA \right) + \left(-\int_{\Gamma_{\overline{\overline{A}}_0}} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\overline{\overline{A}}_0} 2\phi\, dA \right) \\
&= -\int_{\Gamma_1^a} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_1^b} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_1^c} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_1^d} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma \\
&\quad - \int_{\Gamma_2^a} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_2^b} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_2^c} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma - \int_{\Gamma_2^d} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \left(\int_{\overline{\overline{A}}_0} 2\phi\, dA + \int_{\overline{A}_0} 2\phi\, dA \right)
\end{aligned}
$$

$$
\begin{aligned}
M_T = &-\left(\int_{\Gamma_1^a} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\Gamma_2^a} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma \right) - \underbrace{\left(\int_{\Gamma_1^b} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\Gamma_2^b} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma \right)}_{=0} - \underbrace{\left(\int_{\Gamma_1^d} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\Gamma_2^d} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma \right)}_{=0} \\
&- \left(\int_{\Gamma_2^c} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\Gamma_1^c} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma \right) + \left(\int_{A_0} 2\phi\, dA \right)
\end{aligned}
$$

$$
M_T = -\int_{\Gamma_0} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{\Gamma_i} \phi\vec{\mathbf{x}}\cdot\hat{\mathbf{n}}\,d\Gamma + \int_{A_0} 2\phi\, dA \tag{6.266}
$$

And by means of the equation in (6.255) we can conclude that

$$
\begin{aligned}
M_T &= -\int_{\Gamma_0} \phi \vec{\mathbf{x}} \cdot \hat{\mathbf{n}} d\Gamma + \int_{\Gamma_i} \phi \vec{\mathbf{x}} \cdot \hat{\mathbf{n}} d\Gamma + \int_{A_0} 2\phi \, dA \\
&= -2A\phi_{\Gamma_0} + 2A_i \phi_{\Gamma_i} + \int_{A_0} 2\phi \, dA = 2A_i \phi_{\Gamma_i} + \int_{A_0} 2\phi \, dA
\end{aligned}
\tag{6.267}
$$

where $A = A_0 + A_i$, and we have considered that ϕ on the boundary Γ_0 is zero, i.e. $\phi_{\Gamma_0} = 0$.

The equation (6.267) could also have obtained by considering the equation in (6.255), in which $M_T = M_T^A - M_T^{A_i}$, $A = A_0 + A_i$, $\phi_{\Gamma_0} = \phi_\Gamma$, then

$$
\begin{aligned}
M_T = M_T^A - M_T^{A_i} &= \left(-2A\phi_\Gamma + \int_{A = A_0 + A_i} 2\phi \, dA \right) - \left(-2A_i \phi_{\Gamma_i} + \int_{A_i} 2\phi \, dA \right) \\
&= -2A\phi_\Gamma + 2A_i \phi_{\Gamma_i} + \int_{A_0} 2\phi \, dA + \int_{A_i} 2\phi \, dA - \int_{A_i} 2\phi \, dA = -2A \underbrace{\phi_{\Gamma_0}}_{=0} + 2A_i \phi_{\Gamma_i} + \int_{A_0} 2\phi \, dA \\
&= 2A_i \phi_{\Gamma_i} + \int_{A_0} 2\phi \, dA
\end{aligned}
$$

NOTE 1: Note that the above equation is also true if the bar contains n longitudinal cavities, so in this case the torque is given by

$$
M_T = \sum_{i=1}^{n} (2A_i \phi_{\Gamma_i}) + \int_{A_0} 2\phi \, dA
\tag{6.268}
$$

NOTE 2: Note that if we are dealing with open cross section the torque is smaller than the closed cross section, (see Figure 6.97).

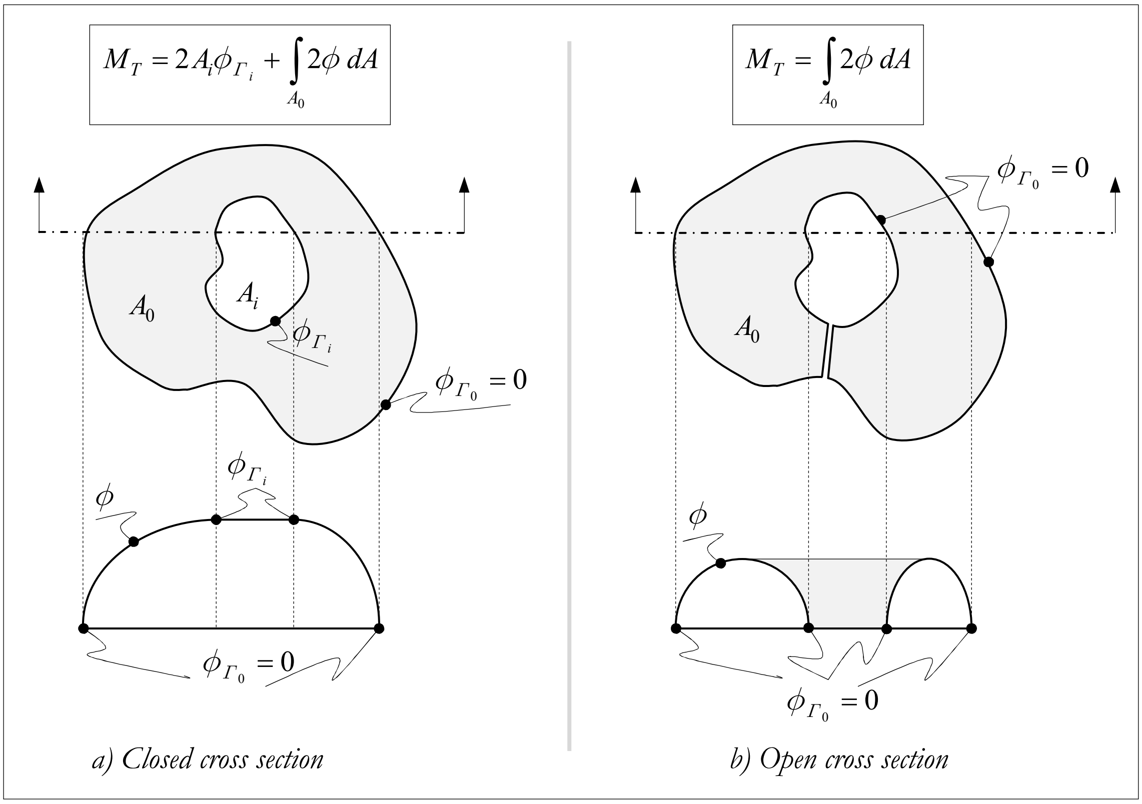


Figure 6.97

Problem 6.46

Using the Prandtl's stress function, a) show that for an elliptical cross section which is subjected to the torque M_T the tangential stresses are:

$$\sigma_{12} = -\frac{2M_T}{\pi a b^3} x_3 \quad ; \quad \sigma_{13} = \frac{2M_T}{\pi a^3 b} x_2 \tag{6.269}$$

b) Draw the tangential stress distribution on the cross section; c) Obtain the function $\mathsf{u}_1(x_2, x_3)$.

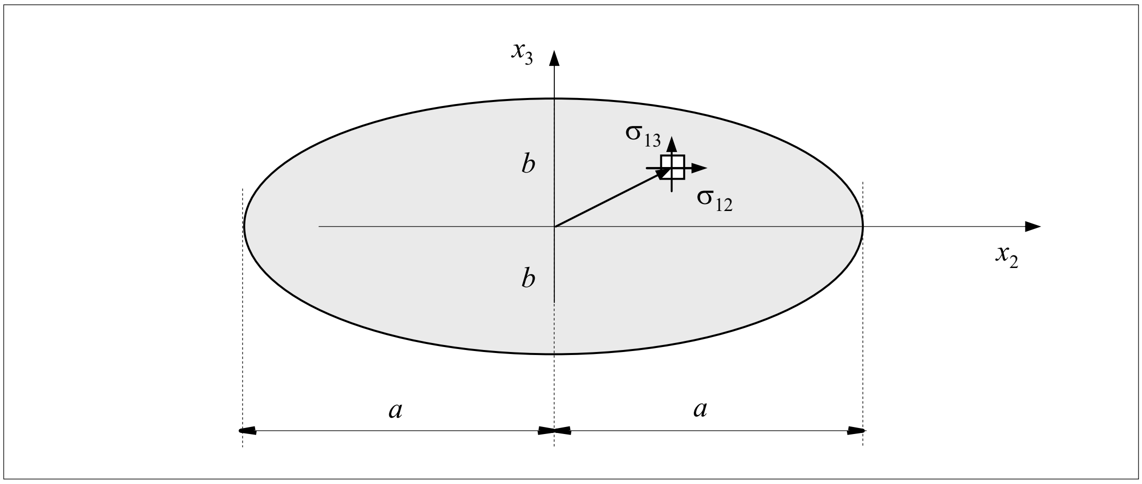


Figure 6.98: Elliptical cross section.

Solution:

a) The ellipse equation is given by:

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1 = 0$$

Since the value of the stress function ϕ on the boundary is constant, we can assume that:

$$\phi = m\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) \tag{6.270}$$

where m is a constant to be determined. From the above equation we can obtain:

$$\frac{\partial \phi}{\partial x_2} = m\left(\frac{2x_2}{a^2}\right) \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x_2^2} = \frac{2m}{a^2}$$

$$\frac{\partial \phi}{\partial x_3} = m\left(\frac{2x_3}{b^2}\right) \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x_3^2} = \frac{2m}{b^2}$$

and by substituting the above equations into the equation (6.250) we obtain that:

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\theta = \text{constant} = \mathrm{C} \quad \Rightarrow \quad \frac{2m}{a^2} + \frac{2m}{b^2} = \mathrm{C}$$

$$\Rightarrow m\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = \mathrm{C} \quad \Rightarrow \quad m\left(\frac{2(a^2 + b^2)}{a^2 b^2}\right) = \mathrm{C} \tag{6.271}$$

$$\Rightarrow m = \frac{a^2 b^2}{2(a^2 + b^2)} \mathrm{C}$$

Substituting the value of m into the equation (6.270) we can obtain:

$$\phi = m\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right) \quad \Rightarrow \quad \phi = \frac{a^2b^2}{2(a^2+b^2)}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\mathrm{C} \tag{6.272}$$

Next step: Determine C

By substituting the value of ϕ into the moment of torsion given by (6.254) we can obtain:

$$M_T = 2\int_A \phi \, dA = 2\int_A \left\{\frac{a^2b^2}{2(a^2+b^2)}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\mathrm{C}\right\} dA = \frac{a^2b^2}{(a^2+b^2)}\mathrm{C}\int_A \left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right) dA$$
$$= \frac{a^2b^2}{(a^2+b^2)}\left[\frac{1}{a^2}\int_A x_2^2 \, dA + \frac{1}{b^2}\int_A x_3^2 \, dA - \int_A dA\right]\mathrm{C} = \frac{a^2b^2}{(a^2+b^2)}\left[\frac{1}{a^2}I_{x3}+\frac{1}{b^2}I_{x2}-A\right]\mathrm{C}$$

where:

$I_{x3} = \int_A x_2^2 \, dA$ - moment of inertia of the cross-sectional area about the x_3-axis;

$I_{x2} = \int_A x_3^2 \, dA$ - moment of inertia cross-sectional area about x_2-axis;

$A = \int_A dA$ - cross-sectional area.

For an elliptical cross section it fulfils that $I_{x3} = \dfrac{\pi b a^3}{4}$, $I_{x2} = \dfrac{\pi b^3 a}{4}$ and $A = \pi ab$. Then, the expression for the moment of torsion becomes:

$$M_T = \frac{a^2b^2}{(a^2+b^2)}\left[\frac{1}{a^2}I_{x3}+\frac{1}{b^2}I_{x2}-A\right]\mathrm{C} = \frac{a^2b^2}{(a^2+b^2)}\left[\frac{1}{a^2}\frac{\pi b a^3}{4}+\frac{1}{b^2}\frac{\pi b^3 a}{4}-\pi ab\right]\mathrm{C}$$
$$= -\frac{\pi a^3 b^3}{2(a^2+b^2)}\mathrm{C}$$

And the value of C can be determined by:

$$\mathrm{C} = \frac{-2M_T(a^2+b^2)}{\pi a^3 b^3}$$

Finally, the stress function (6.272) becomes:

$$\phi = \frac{a^2b^2}{2(a^2+b^2)}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\mathrm{C} \quad \Rightarrow \quad \phi = \frac{-a^2b^2}{2(a^2+b^2)}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\frac{2M_T(a^2+b^2)}{\pi a^3 b^3}$$
$$\Rightarrow \phi = \frac{-M_T}{\pi ab}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)$$

The stresses defined in (6.244) can be expressed by

$$\begin{aligned}
\sigma_{12} &= \frac{\partial \phi}{\partial x_3} = \frac{\partial}{\partial x_3}\left[\frac{-M_T}{\pi ab}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\right] = \frac{-2M_T}{\pi ab^3}x_3 \\
\sigma_{13} &= -\frac{\partial \phi}{\partial x_2} = -\frac{\partial}{\partial x_2}\left[\frac{-M_T}{\pi ab}\left(\frac{x_2^2}{a^2}+\frac{x_3^2}{b^2}-1\right)\right] = \frac{2M_T}{\pi a^3 b}x_2
\end{aligned} \tag{6.273}$$

b) By means of the above equations we can obtain that:

$$x_3 = 0 \quad \Rightarrow \quad \begin{cases} \sigma_{12} = 0 \\ \sigma_{13} = \dfrac{2M_T}{\pi a^3 b} x_2 \xrightarrow{(x_2 = a)} \sigma_{13\max} = \dfrac{2M_T}{\pi a^2 b} \end{cases}$$

$$x_2 = 0 \quad \Rightarrow \quad \begin{cases} \sigma_{12} = \dfrac{-2M_T}{\pi a b^3} x_3 \xrightarrow{(x_3 = b)} \sigma_{12\max} = \dfrac{-2M_T}{\pi a b^2} \\ \sigma_{13} = 0 \end{cases}$$

whose components can be appreciated in Figure 6.99. By means of the Pythagorean Theorem the resultant tangential stress can be obtained:

$$\tau^2 = (\sigma_{12})^2 + (\sigma_{13})^2 = \left(\frac{-2M_T}{\pi a b^3} x_3\right)^2 + \left(\frac{2M_T}{\pi a^3 b} x_2\right)^2 \quad \Rightarrow \quad \tau = \frac{2M_T}{\pi a b}\sqrt{\frac{x_2^2}{b^4} + \frac{x_3^2}{a^4}}$$

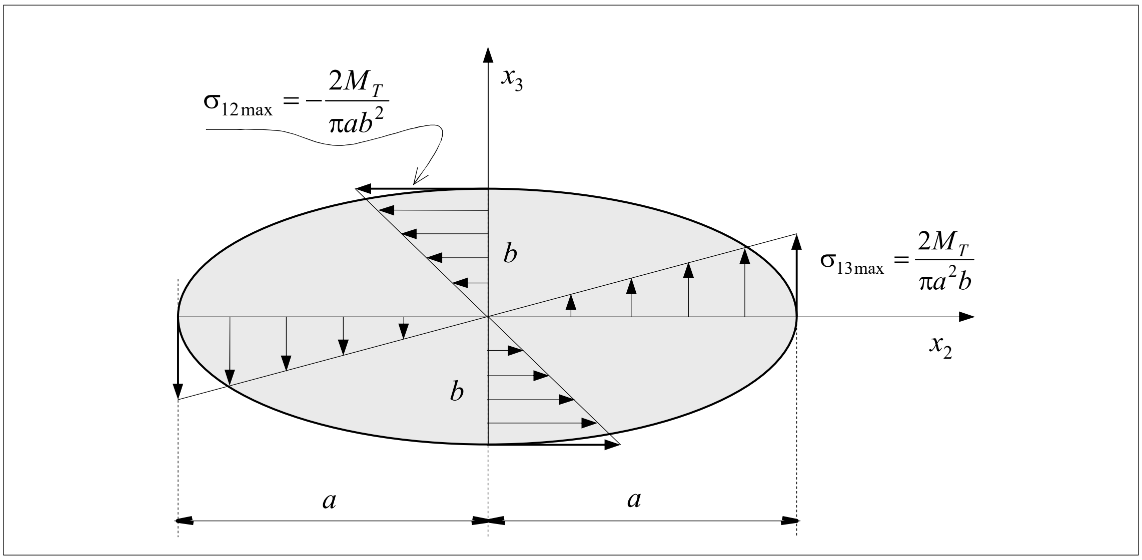


Figure 6.99: Tangential stress distribution in the elliptical cross section.

c) We can obtain the angle of twist per unit length by means of equation (6.271):

$$\mathrm{C} = -2G\theta \quad \Rightarrow \quad \theta = \frac{-\mathrm{C}}{2G} = \frac{-\left(\dfrac{-2M_T(a^2+b^2)}{\pi a^3 b^3}\right)}{2G} = \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G}$$

Taking into account the displacement field given by (6.224), we can obtain:

$$\begin{cases} \mathrm{u}_2 = -x_3\,\theta\,x_1 = \dfrac{-M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3\,x_1 \\ \mathrm{u}_3 = x_2\,\theta\,x_1 = \dfrac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_2\,x_1 \end{cases} \tag{6.274}$$

By considering the above equations and the one in (6.249) we can obtain:

$$\begin{aligned} \frac{\partial \mathrm{u}_1}{\partial x_2} &= \frac{\sigma_{12}}{G} + x_3\,\theta \quad \Rightarrow \quad \frac{\partial \mathrm{u}_1}{\partial x_2} = \frac{\sigma_{12}}{G} + \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3 \\ \frac{\partial \mathrm{u}_1}{\partial x_3} &= \frac{\sigma_{13}}{G} - x_2\,\theta \quad \Rightarrow \quad \frac{\partial \mathrm{u}_1}{\partial x_3} = \frac{\sigma_{13}}{G} - \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_2 \end{aligned} \tag{6.275}$$

Integrating the above equations we can obtain that:

$$\int \partial \mathrm{u}_1 = \int \left(\frac{\sigma_{12}}{G} + \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3 \right) \partial x_2 \quad \Rightarrow \quad \mathrm{u}_1 = \frac{\sigma_{12}}{G} x_2 + \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3\, x_2 + f(x_3)$$

$$\int \partial \mathrm{u}_1 = \int \left(\frac{\sigma_{13}}{G} - \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_2 \right) \partial x_3 \quad \Rightarrow \quad \mathrm{u}_1 = \frac{\sigma_{13}}{G} x_3 - \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3\, x_2 + f(x_2)$$

By substituting the values of σ_{12} and σ_{13}, (see equations (6.273)), into the above equations we can obtain:

$$\mathrm{u}_1 = \left(-\frac{2M_T}{\pi a b^3} x_3 \right) \frac{1}{G} x_2 + \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3\, x_2 + f(x_3) = \frac{M_T x_3\, x_2}{G\pi a^3 b^3}(b^2 - a^2) + f(x_3)$$

$$\mathrm{u}_1 = \left(\frac{2M_T}{\pi a^3 b} x_2 \right) \frac{1}{G} x_3 - \frac{M_T(a^2+b^2)}{\pi a^3 b^3 G} x_3\, x_2 + f(x_2) = \frac{M_T x_3\, x_2}{G\pi a^3 b^3}(b^2 - a^2) + f(x_2)$$

Note that the two above equations must be the same at the same point (x_2, x_3), hence $f(x_2) = f(x_3) = 0$, thus the warping function is given by:

$$\mathrm{u}_1(x_2, x_3) = \frac{M_T}{G} \frac{(b^2 - a^2)}{\pi a^3 b^3} x_2\, x_3$$

The above function in the cross-section can be appreciated in Figure 6.100.

For the particular case when $a = b$, we recover the expressions for the circular cross section, (see **Problem 6.44** - NOTE 1), and in this case there is no warping since $\mathrm{u}_1(x_2, x_3) = 0$.

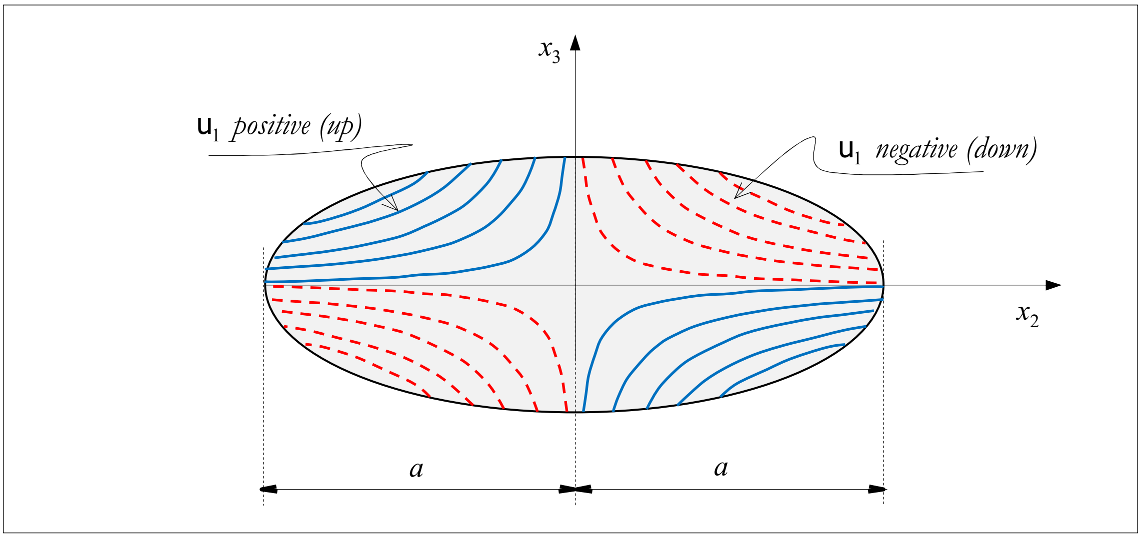


Figure 6.100: Function $\mathrm{u}_1(x_2, x_3)$.

By means of stress components given by (6.273) we can show that, and as expected, the shearing forces are equal to zero:

$$\begin{aligned} Q_{x_2} &= \int_A \sigma_{12}\, dA = -\int_A \frac{2M_T}{\pi a b^3} x_3\, dA = -\frac{2M_T}{\pi a b^3} \underbrace{\int_A x_3\, dA}_{=0} = 0 \\ Q_{x_3} &= \int_A \sigma_{13}\, dA = \int_A \frac{2M_T}{\pi a^3 b} x_2\, dA = \frac{2M_T}{\pi a^3 b} \underbrace{\int_A x_2\, dA}_{=0} = 0 \end{aligned} \qquad (6.276)$$

where $\int_A x_3 dA$ is the first moment of area about x_3-axis, and is equal to zero, since the reference system is located at the geometrical center, (see Complementary Note 2 at the end of Chapter 1).

Problem 6.47

Consider a circular cross section with radius R, and that the Prandtl's stress function is given by:

$$\phi = K(x_2^2 + x_3^2 - R^2) \tag{6.277}$$

Obtain the stress and displacement fields.

Solution:

By means of equation (6.250) we can obtain:

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\theta \quad \Rightarrow \quad 4K = -2G\theta \quad \Rightarrow \quad K = -\frac{G\theta}{2} \tag{6.278}$$

thus,

$$\phi = -\frac{G\theta}{2}(x_2^2 + x_3^2 - R^2) \tag{6.279}$$

And by applying the net moment defined in (6.254) we can obtain:

$$\begin{aligned} M_T &= 2\int_A \phi \, dA = -G\theta \int_A (x_2^2 + x_3^2 - R^2)\, dA = -G\theta\left[\int_A (x_2^2 + x_3^2)\, dA - R^2 \int_A dA\right] \\ &= -G\theta\left[\int_A (r^2)\, dA - R^2 A\right] = -G\theta\left[J_T - R^2 A\right] = -G\theta\left[\frac{\pi R^4}{2} - R^2 \pi R^2\right] \\ &= \frac{\pi R^4}{2} G\theta \end{aligned} \tag{6.280}$$

where $J_T = \int_A r^2 dA = \frac{\pi R^4}{2}$ is the polar inertia moment, and the circle area is $A = \pi R^2$, then $\theta = \frac{2M_T}{\pi G R^4}$.

The stress field, (see equations in (6.244)), becomes:

$$\begin{cases} \sigma_{12} = \dfrac{\partial \phi}{\partial x_3} = -\dfrac{G\theta}{2}\dfrac{\partial}{\partial x_3}(x_2^2 + x_3^2 - R^2) = -G\theta x_3 = -\dfrac{2M_T}{\pi R^4} x_3 \\ \sigma_{13} = -\dfrac{\partial \phi}{\partial x_2} = \dfrac{G\theta}{2}\dfrac{\partial}{\partial x_2}(x_2^2 + x_3^2 - R^2) = G\theta x_2 = \dfrac{2M_T}{\pi R^4} x_2 \end{cases} \tag{6.281}$$

By using the stress definition (6.229) we can obtain:

$$\left.\begin{aligned} \sigma_{12} &= \left(\frac{\partial u_1}{\partial x_2} - x_3\,\theta\right) \quad \Rightarrow \quad \frac{\partial u_1}{\partial x_2} = \frac{\sigma_{12}}{G} + x_3\,\theta = \frac{-G\theta x_3}{G} + x_3\,\theta = 0 \\ \sigma_{13} &= \left(\frac{\partial u_1}{\partial x_3} + x_2\,\theta\right) \quad \Rightarrow \quad \frac{\partial u_1}{\partial x_3} = \frac{\sigma_{13}}{G} - x_2\,\theta = \frac{G\theta x_2}{G} - x_2\,\theta = 0 \end{aligned}\right\} \Rightarrow u_1 = 0$$

As expected $u_1 = 0$, since for circular cross section there is no warping. This problem was already discussed in **Problem 6.44** - **NOTE 1**.

6.4.1 Torsion of Thin-Walled Cross Section

6.4.1.1 Open Thin-Walled Section

Problem 6.48

Apply the torsion theory to obtain the maximum shearing stress (τ_{max}) in a thin rectangular section described in Figure 6.101. Express the result in terms of (M_T, t, b). Consider that the Prandtl's stress function, (Ugural&Fenster (1984)), is given by:

$$\phi = K\left[x_3^2 - \left(\frac{t}{2}\right)^2\right] \qquad (6.282)$$

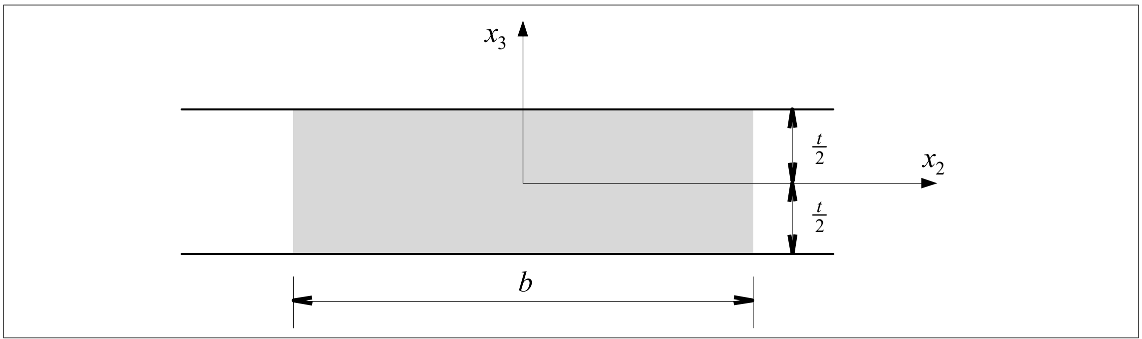


Figure 6.101: Thin rectangular cross section.

Solution:

By means of equation (6.250) we can obtain:

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\theta \quad \Rightarrow \quad 2K = -2G\theta \quad \Rightarrow \quad K = -G\theta \qquad (6.283)$$

thus,

$$\phi = -G\theta\left[x_3^2 - \left(\frac{t}{2}\right)^2\right] \qquad (6.284)$$

And by applying the net moment defined in equation (6.254) we can obtain:

$$M_T = 2\int_A \phi \, dA = -2G\theta \int_A \left[x_3^2 - \left(\frac{t}{2}\right)^2\right] dA = -2G\theta \int_A \left[x_3^2 - \left(\frac{t}{2}\right)^2\right] dA$$

$$= -2G\theta \int_{\frac{-b}{2}}^{\frac{b}{2}} \left[\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[x_3^2 - \left(\frac{t}{2}\right)^2\right] dx_3\right] dx_2 = -2G\theta \int_{\frac{-b}{2}}^{\frac{b}{2}} \left[\left[\frac{x_3^3}{3} - \left(\frac{t}{2}\right)^2 x_3\right]\Bigg|_{\frac{-t}{2}}^{\frac{t}{2}}\right] dx_2$$

$$= -2G\theta \int_{\frac{-b}{2}}^{\frac{b}{2}} \left[\frac{t^3}{6}\right] dx_2 = -2G\theta \left[\frac{t^3}{6} x_2\right]\Bigg|_{\frac{-b}{2}}^{\frac{b}{2}}$$

Then, we can obtain

$$M_T = G\theta \frac{t^3 b}{3} \tag{6.285}$$

By considering that $M_T = G\theta J_T$, (see equation (6.237)), we can conclude that $J_T = \frac{t^3 b}{3}$, then, $G\theta = \frac{3M_T}{t^3 b}$.

The tangential stress field becomes:

$$\begin{cases} \sigma_{12} = \dfrac{\partial \phi}{\partial x_3} = -G\theta \dfrac{\partial}{\partial x_3}\left[x_3^2 - \left(\dfrac{t}{2}\right)^2\right] = -2G\theta x_3 = -\dfrac{6M_T}{t^3 b} x_3 \\ \sigma_{13} = -\dfrac{\partial \phi}{\partial x_2} = G\theta \dfrac{\partial}{\partial x_2}\left[x_3^2 - \left(\dfrac{t}{2}\right)^2\right] = 0 \end{cases}$$

The maximum shearing stress occurs at $x_3 = \pm \frac{t}{2}$, thus:

$$\sigma_{12}(x_3 = \pm \frac{t}{2}) = \tau_{\max} = \mp \frac{6M_T}{t^3 b} \frac{t}{2} = \mp \frac{3M_T}{t^2 b}$$

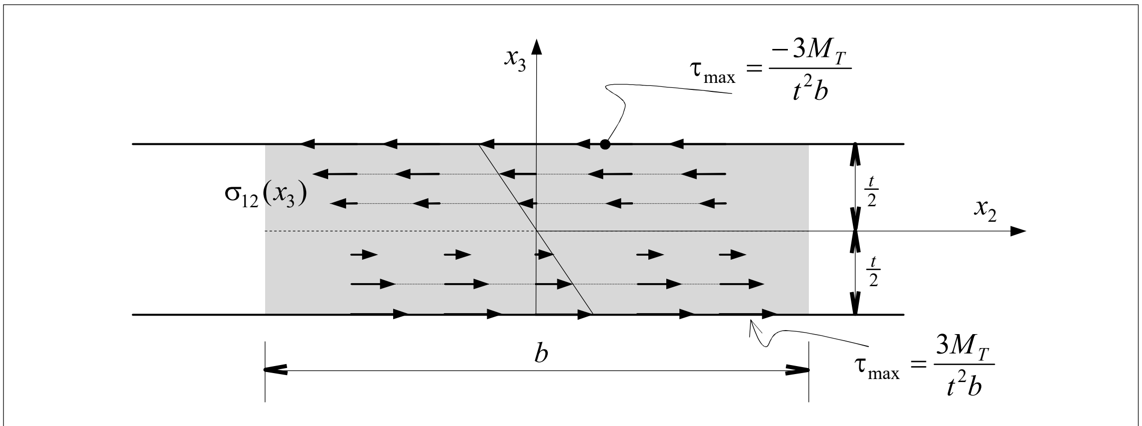


Figure 6.102: Stress distribution.

NOTE 1: Let us obtain the same result by means of Prandtl's membrane analogy, (see **Problem 6.44** – NOTE 4). The membrane deflection (h) for thin rectangular cross section, (see Figure 6.101), can be appreciated in Figure 6.103.

Note that the membrane deflection does not depend on x_2, then $\frac{\partial h}{\partial x_2} = 0$. With that the equation in (6.261) becomes:

$$\frac{\partial^2 h}{\partial x_2^2} + \frac{\partial^2 h}{\partial x_3^2} = -\frac{p}{S} \quad \Rightarrow \quad \frac{\partial^2 h}{\partial x_3^2} = -\frac{p}{S} \tag{6.286}$$

Then by integrate the above equation over x_3 we can obtain:

$$\frac{\partial^2 h}{\partial x_2^2} + \frac{\partial^2 h}{\partial x_3^2} = -\frac{p}{S} \quad \Rightarrow \quad \frac{d^2 h}{dx_3^2} = -\frac{p}{S} \xrightarrow{\text{integrating over } x_3} \frac{dh}{dx_3} = -\frac{p}{S} x_3 + C_1 \tag{6.287}$$

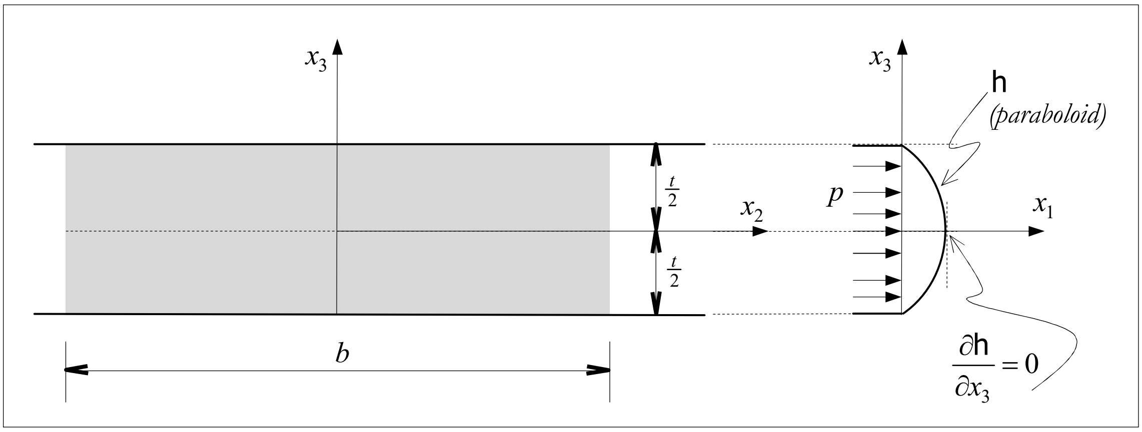


Figure 6.103: Membrane deflection for thin rectangular cross section.

Note that at $x_3=0$ the membrane slope is equal to zero, i.e. $\frac{\partial h}{\partial x_3}=0$, (see Figure 6.103), then the constant of integration equals zero, $C_1=0$. Then,

$$\frac{dh}{dx_3}=-\frac{p}{S}x_3 \xrightarrow{\text{integrating over } x_3} h=-\frac{p}{S}\frac{x_3^2}{2}+C_2 \tag{6.288}$$

At $x_3=\pm\frac{t}{2}$ the membrane deflection is equal to zero, thus

$$h(x_3=\pm\tfrac{t}{2})=-\frac{p}{S}\frac{\left(\frac{t}{2}\right)^2}{2}+C_2=0 \quad\Rightarrow\quad C_2=\frac{pt^2}{8S}$$

With that the membrane deflection equation becomes:

$$h=-\frac{p}{S}\frac{x_3^2}{2}+\frac{pt^2}{8S}=\frac{p}{2S}\left[\left(\frac{t}{2}\right)^2-x_3^2\right] \tag{6.289}$$

The maximum deflection of the membrane occurs at $x_3=0$, and is equal to $h_{\max}=\frac{pt^2}{8S}$. The volume of the paraboloid defined by the membrane is equal to:

$$V_{memb}=\frac{2}{3}bh_{\max}t=\frac{2}{3}b\frac{pt^2}{8S}t=\frac{bt^3p}{12S}=2G\theta\frac{bt^3}{12}=\frac{G\theta bt^3}{6} \tag{6.290}$$

where we have used the definition $\frac{p}{S}=2G\theta$, (see equation (6.262)). And by using equation (6.263) the moment of torsion can be expressed as follows:

$$M_T=2\int_A h(x_2,x_3)\,dA=2V_{memb}=\frac{G\theta bt^3}{3} \tag{6.291}$$

By considering that $M_T=G\theta J_T$, (see equation (6.237)), we can conclude that

$$J_{T_{eff}}=\frac{M_T}{G\theta}=\frac{bt^3}{3} \tag{6.292}$$

The maximum tangential stress, (see Figure 6.102), can be expressed in terms of $J_{T_{eff}}$ as follows:

$$\tau_{max} = \frac{3M_T}{t^2 b} = \frac{M_T t}{\left(\frac{t^3 b}{3}\right)} = \frac{M_T t}{J_{T_{eff}}} \tag{6.293}$$

This solution is the base to solve other cases in which the cross section is made up by several elongated rectangles (thin open wall cross section), (see Figure 6.104), with the condition $\frac{b_i}{t_i} > 10$.

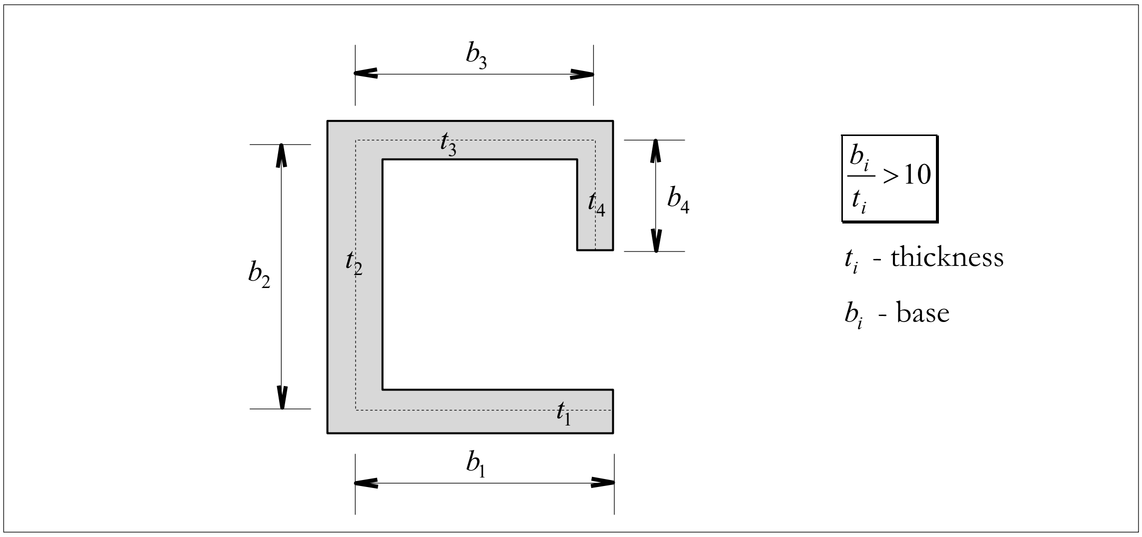


Figure 6.104: Thin open wall cross section.

In this case we have:

$$\boxed{J_{T_{eff}} = \sum_{i=1}^{n} \frac{b_i t_i^3}{3}} \tag{6.294}$$

and

$$\boxed{\theta = \frac{M_T}{GJ_{T_{eff}}} = \frac{M_T}{G\sum_{i=1}^{n} \frac{b_i t_i^3}{3}}} \tag{6.295}$$

And the tangential stress can be evaluated as follows

$$\boxed{\tau = \frac{M_T t}{J_{T_{eff}}}} \quad \text{or} \quad \boxed{\tau = tG\theta} \tag{6.296}$$

For example, let us consider Figure 6.104 in which $t_1 = t_2 = t_3 = t_4 = t$, $b_1 = b_3 = a$ and $b_2 = b_4 = \frac{a}{2}$. Then, the equation (6.294) becomes:

$$J_{T_{eff}} = \sum_{i=1}^{4} \frac{b_i t_i^3}{3} = \frac{b_1 t_1^3 + b_2 t_2^3 + b_3 t_3^3 + b_4 t_4^3}{3} = \frac{at^3 + \frac{a}{2}t^3 + at^3 + \frac{a}{2}t^3}{3} = at^3 \tag{6.297}$$

And the angle of twist per unit length, (see equation (6.295)), becomes:

$$\theta = \frac{M_T}{GJ_{T_{eff}}} = \frac{M_T}{Gat^3} \tag{6.298}$$

The maximum tangential stress, (see equation (6.293)), becomes

$$\tau_{\max} = \frac{M_T t}{J_{T_{eff}}} = \frac{M_T t}{a t^3} = \frac{M_T}{a t^2} \tag{6.299}$$

6.4.1.2 Closed Thin-Walled Section

Problem 6.49

Obtain the equations for torsion problem by means of membrane analogy and by considering a thin-walled closed section.

Hypothesis: Consider the membrane deflection as the one described in Figure 6.105.

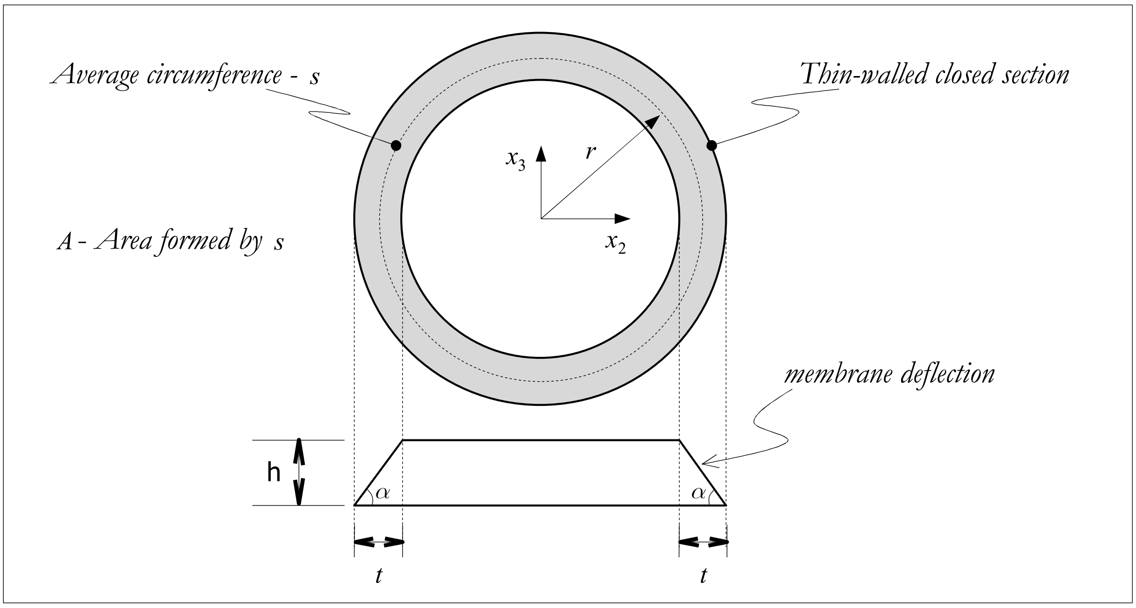


Figure 6.105: Thin-walled closed section.

Solution:

According to the membrane analogy, (see equation (6.262)), it is true that:

$$\mathsf{h} = \phi \quad \text{and} \quad 2G\theta = \frac{p}{S} \tag{6.300}$$

And according to Figure 6.105 the membrane slope is constant, and then the tangential stress is also constant, (see equation (6.256)). The tangential stress (τ) can be obtained by means of membrane slope:

$$\tan\alpha = \tau = \frac{\mathsf{h}}{t} \tag{6.301}$$

The moment of torsion is related to the membrane volume by $M_T = 2V_{memb}$, and the membrane volume can be obtained as follows $V_{memb} = A\mathsf{h}$, where A is the area formed by the average circumference s, (see Figure 6.105). Then $M_T = 2A\mathsf{h}$, and the stress (6.301) becomes:

$$\tau = \frac{\mathsf{h}}{t} = \frac{M_T}{2At} \tag{6.302}$$

which is the same as the one presented in NOTE 2-**Problem 4.27**.

We have obtained that $\oint_{\Gamma} \tau \, d\Gamma = 2G\theta A$, (see equation (6.260)), thus:

$$\theta = \frac{\oint_{\Gamma} \tau \, d\Gamma}{2GA} = \frac{\oint_{\Gamma} \left(\frac{M_T}{2At} \right) d\Gamma}{2GA} \tag{6.303}$$

The above equation is known as Bredt's formula (Ugural&Fenster (1984)). When the thickness is constant we can conclude that:

$$\theta = \frac{\oint_{\Gamma} \left(\frac{M_T}{2At} \right) d\Gamma}{2GA} = \frac{M_T s}{4GA^2 t} \tag{6.304}$$

where s is the perimeter of the cross section. For the case presented in Figure 6.105 we have

$$\tau = \frac{M_T}{2At} = \frac{M_T}{2(\pi r^2)t} = \frac{M_T}{2\pi r^2 t} \tag{6.305}$$

and $\theta = \frac{M_T s}{4GA^2 t} = \frac{M_T (2\pi r)}{4G(\pi r^2)^2 t} = \frac{M_T}{2G\pi r^3 t}$.

Once the angle of twist per unit length is obtained we can obtain the J_T by means of the equation $M_T = G\mathrm{J}_T \theta$, (see equation (6.237)):

$$M_T = G\mathrm{J}_T \theta = \frac{G\mathrm{J}_T M_T}{2G\pi r^3 t} \quad \Rightarrow \quad \mathrm{J}_{T_{eff}} = 2\pi r^3 t \tag{6.306}$$

NOTE 1: Multiply Connected Thin-Walled Section, (Sechler (1952))

We can also use the previous study to solve problem with multiply connected thin-walled section, (see Figure 6.106).

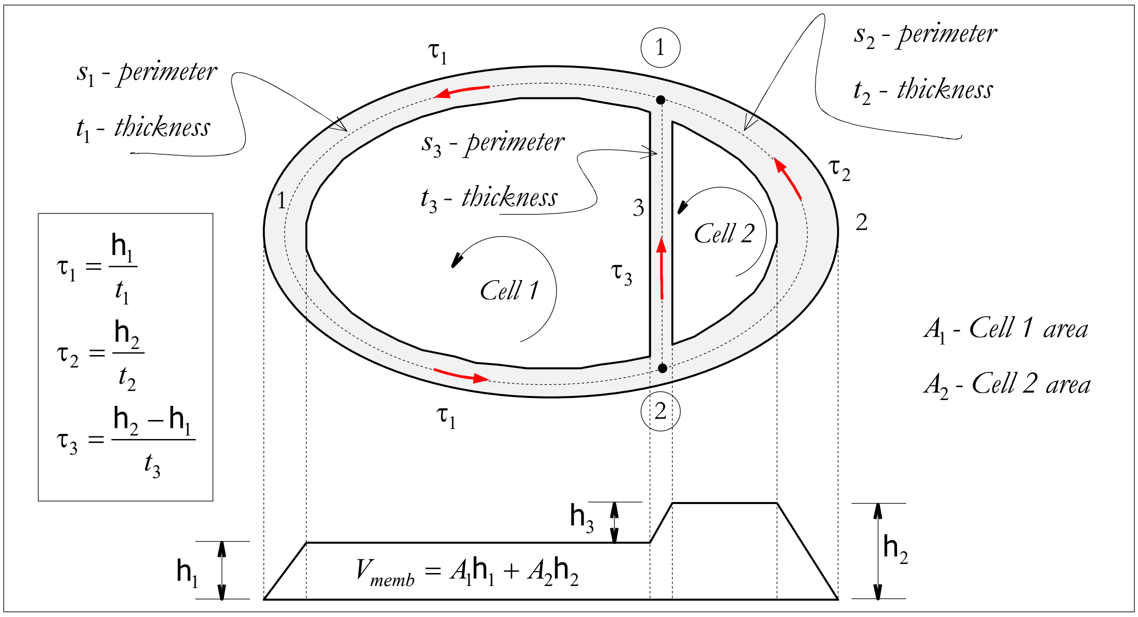


Figure 6.106: Multiply connected thin-walled section.

In this case we have two cells. For each cell we apply the definition $\oint_{\Gamma} \tau \, d\Gamma = 2G\theta A$, (see equation (6.260)), thus

Cell 1: $$\oint_{\Gamma} \tau \, d\Gamma = 2G\theta A_1 \quad \Rightarrow \quad \tau_1 s_1 + \tau_3 s_3 = 2GA_1\theta \quad \Rightarrow \quad \tau_1 s_1 + \tau_3 s_3 - 2GA_1\theta = 0 \tag{6.307}$$

Cell 2: $$\oint_{\Gamma} \tau \, d\Gamma = 2G\theta A_2 \quad \Rightarrow \quad \tau_2 s_2 - \tau_3 s_3 = 2GA_2\theta \quad \Rightarrow \quad \tau_2 s_2 - \tau_3 s_3 - 2GA_2\theta = 0 \tag{6.308}$$

We apply the equation $M_T = 2V_{memb}$, where the membrane volume is $V_{memb} = A_1\mathsf{h}_1 + A_2\mathsf{h}_2$, thus:

$$M_T = 2V_{memb} = 2(A_1 t_1 \tau_1 + A_2 t_2 \tau_2) \quad \Rightarrow \quad A_1 t_1 \tau_1 + A_2 t_2 \tau_2 = \frac{M_T}{2} \tag{6.309}$$

Next we apply the flux continuity at each node. In this case we have two nodes:

Node 1: $$t_1\tau_1 = t_2\tau_2 + t_3\tau_3 \quad \Rightarrow \quad -t_1\tau_1 + t_2\tau_2 + t_3\tau_3 = 0 \tag{6.310}$$

and

Node 2: $$t_2\tau_2 + t_3\tau_3 = t_1\tau_1 \quad \Rightarrow \quad t_1\tau_1 - t_2\tau_2 - t_3\tau_3 = 0 \tag{6.311}$$

Note that the last equation is redundant one.

Then, by considering the equations (6.307), (6.308), (6.309) and (6.310), we can construct the following set of equations:

$$\begin{array}{r} \textit{Cell 1} \\ \textit{Cell 2} \\ \textit{moment} \\ \textit{flux at node 1} \end{array} \begin{bmatrix} s_1 & 0 & s_3 & -2GA_1 \\ 0 & s_2 & -s_3 & -2GA_2 \\ A_1 t_1 & A_2 t_2 & 0 & 0 \\ -t_1 & t_2 & t_3 & 0 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{M_T}{2} \\ 0 \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{F}]\{\tau\} = \{\boldsymbol{g}\} \tag{6.312}$$

There is solver that does not accept that the terms of principal diagonal are zero, if it is the case we can overcome this drawback by sum the third and fourth rows by the first row. Recall that if we sum rows the result is not affected. Then:

$$\begin{bmatrix} s_1 & 0 & s_3 & -2GA_1 \\ 0 & s_2 & -s_3 & -2GA_2 \\ A_1 t_1 + s_1 & A_2 t_2 & s_3 & -2GA_1 \\ -t_1 + s_1 & t_2 & t_3 + s_3 & -2GA_1 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{M_T}{2} + (g_1 = 0) \\ 0 + (g_1 = 0) \end{Bmatrix} \quad \Leftrightarrow \quad [\overline{\boldsymbol{F}}]\{\tau\} = \{\overline{\boldsymbol{g}}\} \tag{6.313}$$

The solution for the above set of equations is:

$$\{\tau\} = \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \theta \end{Bmatrix} = \frac{1}{\det[\overline{\boldsymbol{F}}]} \begin{Bmatrix} -GM_T(s_2 t_3 A_1 + s_3 t_2 A_2 + s_3 t_2 A_1) \\ -GM_T(s_1 t_3 A_2 + s_3 t_1 A_2 + s_3 t_1 A_1) \\ GM_T(s_1 t_2 A_2 - s_2 t_1 A_1) \\ \dfrac{-1}{2} M_T(s_1 s_2 t_3 + s_1 s_3 t_2 + s_2 s_3 t_1) \end{Bmatrix} \tag{6.314}$$

where $\det[\bar{F}] = -2G[t_1 t_2 s_3 (A_1 + A_2)^2 + t_3(s_1 t_2 A_2^2 + s_2 t_1 A_1^2)]$. Note that, when $t = t_1 = t_2$, $s = s_1 = s_2$, $A = A_1 = A_2$, the solution becomes:

$$\{\tau\} = \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \theta \end{Bmatrix} = \begin{Bmatrix} \dfrac{M_T}{4At} \\ \dfrac{M_T}{4At} \\ 0 \\ \dfrac{sM_T}{8GA^2 t} \end{Bmatrix} \tag{6.315}$$

Once the angle of twist per unit length is obtained we can obtain the J_T by means of the equation $M_T = GJ_T\theta$, (see equation (6.237)):

$$M_T = GJ_T\theta = \frac{GJ_T s M_T}{8GA^2 t} \quad \Rightarrow \quad J_{T_{eff}} = \frac{8A^3 t}{s} \tag{6.316}$$

Problem 6.50

Obtain the tangential stress in each segment and the angle of twist θ for the multiply connected thin-walled section described in Figure 6.107, Cervera&Blanco (2004), in which the moment of torsion is $M_T = 5\times 10^3\, Nm$ and the shear modulus is $G = 1\times 10^{11}\, Pa$.

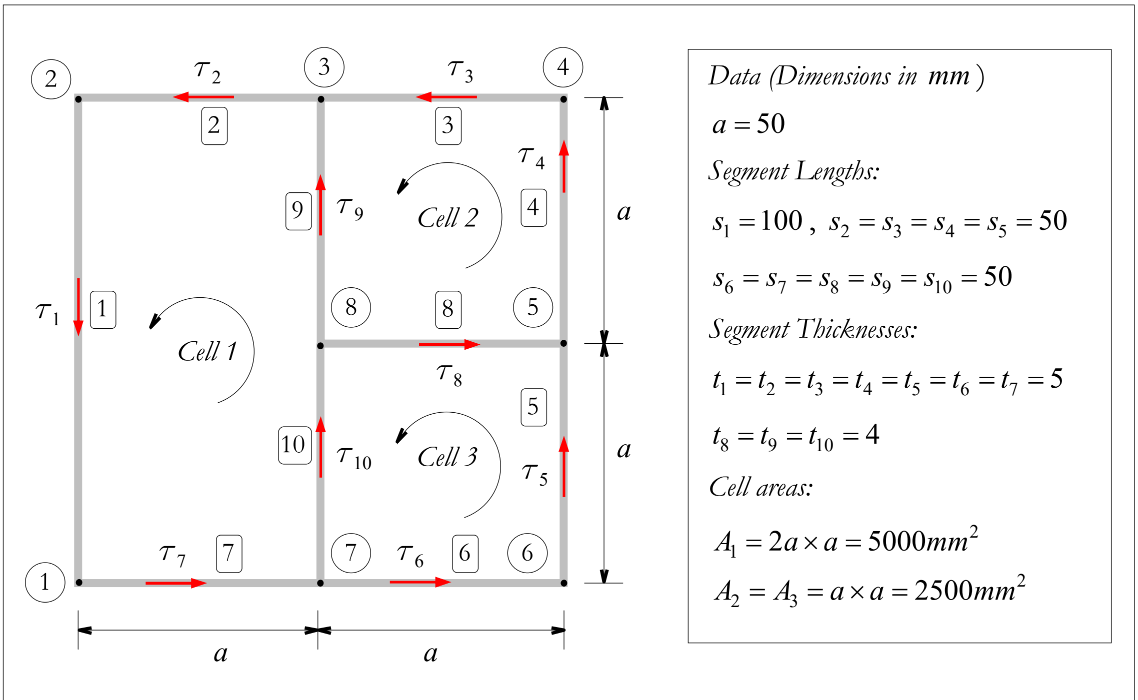


Figure 6.107: Multiply connected thin-walled section.

Solution:

For the three cells we apply the equation (6.260), i.e.:

Cell 1: $\displaystyle\oint_{\Gamma_{C1}} \tau\, d\Gamma = 2G\theta A_1 \quad \Rightarrow \quad \tau_1 s_1 + \tau_2 s_2 + \tau_7 s_7 + \tau_9 s_9 + \tau_{10} s_{10} = 2GA_1\theta$

Cell 2: $\displaystyle\oint_{\Gamma_{C2}} \tau \, d\Gamma = 2G\theta A_2 \qquad \Rightarrow \qquad \tau_3 s_3 + \tau_4 s_4 + \tau_8 s_8 - \tau_9 s_9 = 2GA_2\theta$

Cell 3: $\displaystyle\oint_{\Gamma_{C3}} \tau \, d\Gamma = 2G\theta A_3 \qquad \Rightarrow \qquad \tau_5 s_5 + \tau_6 s_6 - \tau_8 s_8 - \tau_{10} s_{10} = 2GA_3\theta$

Next the following must fulfill:

$M_T = 2V_{memb} = 2(A_1 t_1 \tau_1 + A_2 t_4 \tau_4 + A_3 t_6 \tau_6)$ (moment equation)

We have 11 unknowns and 4 equations, the 7 missing equations can be obtained by tangential stress flux compatibility at each node. Note that we have 8 nodes, but we only need 7 of them:

Node 1: $-t_1\tau_1 + t_7\tau_7 = 0$

Node 2: $-t_1\tau_1 + t_2\tau_2 = 0$

Node 3: $-t_2\tau_2 + t_3\tau_3 + t_9\tau_9 = 0$

Node 4: $-t_3\tau_3 + t_4\tau_4 = 0$

Node 5: $-t_4\tau_4 + t_5\tau_5 + t_8\tau_8 = 0$

Node 6: $-t_5\tau_5 + t_6\tau_6 = 0$

Node 7: $-t_6\tau_6 + t_7\tau_7 - t_{10}\tau_{10} = 0$

Then, by restructuring the above 11 equations in matrix form we can obtain:

$$
\begin{array}{r}
\textit{Cell 1}\\ \textit{Cell 2}\\ \textit{Cell 3}\\ \textit{moment}\\ \textit{node 1}\\ \textit{node 2}\\ \textit{node 3}\\ \textit{node 4}\\ \textit{node 5}\\ \textit{node 6}\\ \textit{node 7}
\end{array}
\begin{bmatrix}
s_1 & s_2 & 0 & 0 & 0 & 0 & s_7 & 0 & s_9 & s_{10} & -2GA_1\\
0 & 0 & s_3 & s_4 & 0 & 0 & 0 & s_8 & -s_9 & 0 & -2GA_2\\
0 & 0 & 0 & 0 & s_5 & s_6 & 0 & -s_8 & 0 & -s_{10} & -2GA_3\\
A_1 t_1 & 0 & 0 & A_2 t_4 & 0 & A_3 t_6 & 0 & 0 & 0 & 0 & 0\\
t_1 & 0 & 0 & 0 & 0 & 0 & -t_7 & 0 & 0 & 0 & 0\\
-t_1 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -t_2 & t_3 & 0 & 0 & 0 & 0 & 0 & t_9 & 0 & 0\\
0 & 0 & -t_3 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -t_4 & t_5 & 0 & 0 & t_8 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -t_5 & t_6 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -t_6 & t_7 & 0 & 0 & -t_{10} & 0
\end{bmatrix}
\begin{Bmatrix}
\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\\ \tau_5\\ \tau_6\\ \tau_7\\ \tau_8\\ \tau_9\\ \tau_{10}\\ \theta
\end{Bmatrix}
=
\begin{Bmatrix}
0\\ 0\\ 0\\ \dfrac{M_T}{2}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0
\end{Bmatrix}
$$

$$[\boldsymbol{F}]\{\tau\} = \{\boldsymbol{g}\} \tag{6.317}$$

The above matrix $[\boldsymbol{F}]$ has zeros in the principal diagonal, we will sum rows to overcome this drawback, and then after summation is taken place we can obtain:

$$
\left[\begin{array}{ccccccccccc}
s_1 & s_2 & 0 & 0 & 0 & 0 & s_7 & 0 & s_9 & s_{10} & -2GA_1 \\
s_1 & s_2 & s_3 & s_4 & 0 & 0 & s_7 & s_8 & 0 & s_{10} & -2G(A_2+A_1) \\
0 & 0 & -t_3 & t_4 & s_5 & s_6 & 0 & -s_8 & 0 & -s_{10} & -2GA_3 \\
\hdashline
A_1t_1 & 0 & 0 & A_2t_4 & 0 & A_3t_6 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
t_1 & 0 & 0 & 0 & -t_5 & t_6 & t_7 & 0 & 0 & 0 & 0 \\
-t_1 & t_2 & 0 & 0 & -t_5 & t_6 & 0 & 0 & 0 & 0 & 0 \\
0 & -t_2 & t_3 & 0 & 0 & -t_6 & t_7 & 0 & t_9 & -t_{10} & 0 \\
0 & 0 & -t_3 & 0 & t_5 & 0 & 0 & t_8 & 0 & 0 & 0 \\
0 & -t_2 & t_3 & -t_4 & t_5 & -t_6 & t_7 & t_8 & t_9 & -t_{10} & 0 \\
0 & 0 & 0 & 0 & -t_5 & 0 & t_7 & 0 & 0 & -t_{10} & 0 \\
s_1 & s_2 & 0 & 0 & 0 & -t_6 & t_7+s_7 & 0 & s_9 & s_{10}-t_{10} & -2GA_1
\end{array}\right]
\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \\ \tau_{10} \\ \theta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{M_T}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

After substituting the data and solving the above system we can obtain:

$$
\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = \tau_7 = 50 \times 10^6 Pa \quad ; \quad \tau_8 = \tau_9 = \tau_{10} = 0 \quad ; \quad \theta = 0.01 \frac{rad}{m}
$$

Note that, this problem is the same as the result presented in equation (6.315), in which:

$$
\tau = \frac{M_T}{4At} = \frac{5 \times 10^3}{4 \times (5000 \times 10^{-6}) \times (5 \times 10^{-3})} = 50 \times 10^6 Pa \,,
$$

$$
\theta = \frac{sM_T}{8GA^2 t} = \frac{s\tau}{2GA} = \frac{(200 \times 10^{-3}) \times (50 \times 10^6)}{2(10^{11})(5000 \times 10^{-6})} = 0.01 \frac{rad}{m} \,,
$$

where $s = s_1 + s_2 + s_7 + s_3 + s_4 + s_5 + s_6$ is the total perimeter.

NOTE 1: We can also use this methodology for a cross section formed by only one cell, (see Figure 6.108).

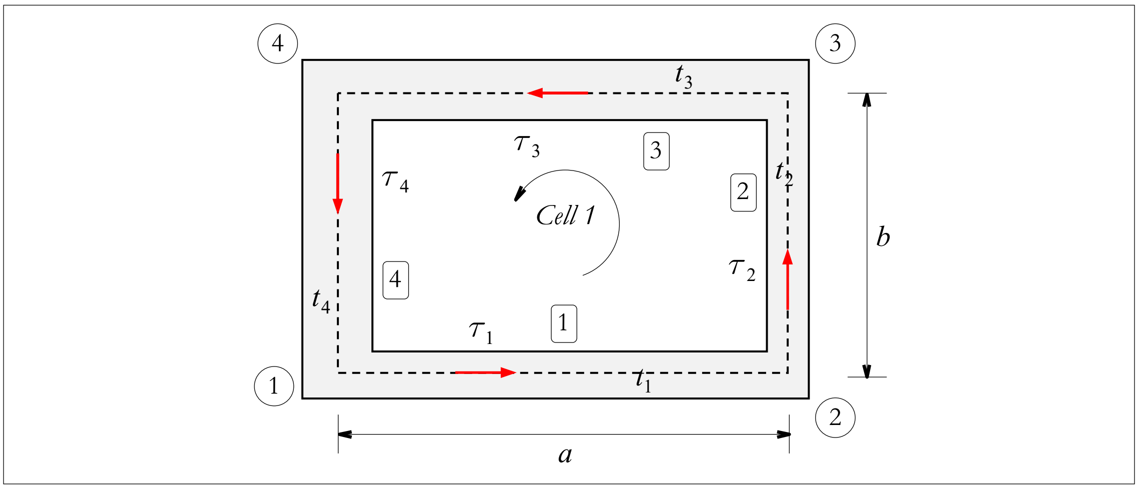


Figure 6.108: Thin-walled section.

Solution for the problem described in Figure 6.108*:*

Cell 1: $\displaystyle\oint_{\Gamma_{C1}} \tau \, d\Gamma = 2G\theta A \quad \Rightarrow \quad \tau_1 s_1 + \tau_2 s_2 + \tau_3 s_3 + \tau_4 s_4 = 2GA\theta$

The moment equation is:

$$
M_T = 2V_{memb} = 2At_2\tau_2
$$

We have 5 unknowns and 2 equations, the 3 missing equations can be obtained by tangential stress flux compatibility at each node. Note that we have 4 nodes, but we only need 3 of them:

Node 1: $-t_4\tau_4 + t_1\tau_1 = 0$

Node 2: $-t_1\tau_1 + t_2\tau_2 = 0$

Node 3: $-t_2\tau_2 + t_3\tau_3 = 0$

Then, the set of equations to be solved is:

$$\begin{array}{c} \textit{Cell 1} \\ \textit{moment} \\ \textit{node 1} \\ \textit{node 2} \\ \textit{node 3} \end{array}\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & -2GA \\ \hline 0 & At_2 & 0 & 0 & 0 \\ \hline t_1 & 0 & 0 & -t_4 & 0 \\ -t_1 & t_2 & 0 & 0 & 0 \\ 0 & -t_2 & t_3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{M_T}{2} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

And the solution is:

$$\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \theta \end{Bmatrix} = M_T \begin{Bmatrix} \frac{1}{2At_1} \\ \frac{1}{2At_2} \\ \frac{1}{2At_3} \\ \frac{1}{2At_4} \\ \frac{1}{4A^2G}\left(\frac{s_1}{t_1}+\frac{s_2}{t_2}+\frac{s_3}{t_3}+\frac{s_4}{t_4}\right) \end{Bmatrix} \tag{6.318}$$

For the particular case when $t_1 = t_2 = t_3 = t_4 = t$, $s_1 = s_3 = a$, $s_2 = s_4 = b$, $A = ab$ and the total perimeter $s_T = s_1 + s_2 + s_3 + s_4 = 2(a+b)$, we can obtain

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau = \frac{M_T}{2At} = \frac{M_T}{2abt}$$

$$\theta = \frac{M_T}{4A^2G}\left(\frac{s_1}{t_1}+\frac{s_2}{t_2}+\frac{s_3}{t_3}+\frac{s_4}{t_4}\right) = \frac{M_T s_T}{4A^2Gt} = \frac{M_T(2(a+b))}{4(ab)^2 Gt} = \frac{M_T(a+b)}{2a^2b^2Gt}$$

By considering that $M_T = G\theta \mathrm{J}_T$, (see equation (6.237)), we can conclude that

$$\theta = \frac{M_T(a+b)}{2a^2b^2Gt} = \frac{M_T}{G\mathrm{J}_T} \quad \Rightarrow \quad \mathrm{J}_T = \frac{2a^2b^2t}{(a+b)}$$

For the particular case when $t_1 = t_2 = t_3 = t_4 = t$, $s_1 = s_3 = a$, $s_2 = s_4 = b = \frac{a}{2}$, $A = \frac{a^2}{2}$ and the total perimeter $s_T = s_1 + s_2 + s_3 + s_4 = 3a$, the above equations become

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau = \frac{M_T}{2At} = \frac{M_T}{a^2t}$$

$$\theta = \frac{M_T}{4A^2G}\left(\frac{s_1}{t_1}+\frac{s_2}{t_2}+\frac{s_3}{t_3}+\frac{s_4}{t_4}\right) = \frac{M_T(a+b)}{2a^2b^2Gt} = \frac{3M_T}{a^3Gt}$$

By considering that $M_T = G\theta J_T$, (see equation (6.237)), we can conclude that

$$\theta = \frac{M_T s_T}{4A^2 Gt} = \frac{3M_T}{a^3 Gt} = \frac{M_T}{GJ_T} \quad \Rightarrow \quad J_T = \frac{4A^2 t}{s_T} = \frac{a^3 t}{3}$$

Note that this particular case has the same geometry as the one given by the results in equations (6.297)-(6.299), in which we were dealing with an open cross section. As we can see, since $t << a$, the close cross section is more rigid than the open close, a fact that can be checked by the angle of twist per unit length, (see equation (6.298)):

$$\theta^{(close)} = \frac{3M_T}{a^3 Gt} = \left[3\left(\frac{t}{a}\right)^2\right]\frac{M_T}{Gat^3} = \underbrace{\left[3\left(\frac{t}{a}\right)^2\right]}_{<<1}\theta^{(open)} \quad \Rightarrow \quad \theta^{(close)} < \theta^{(open)}$$

And according to equation (6.297) we can also conclude

$$J_T^{(Open)} = at^3 << \frac{a^3 t}{3} = J_T^{(Close)}$$

We can also apply the same methodology to cross section formed by curved segment, (see Figure 6.109), in which the curved segment is discretized by linear segments.

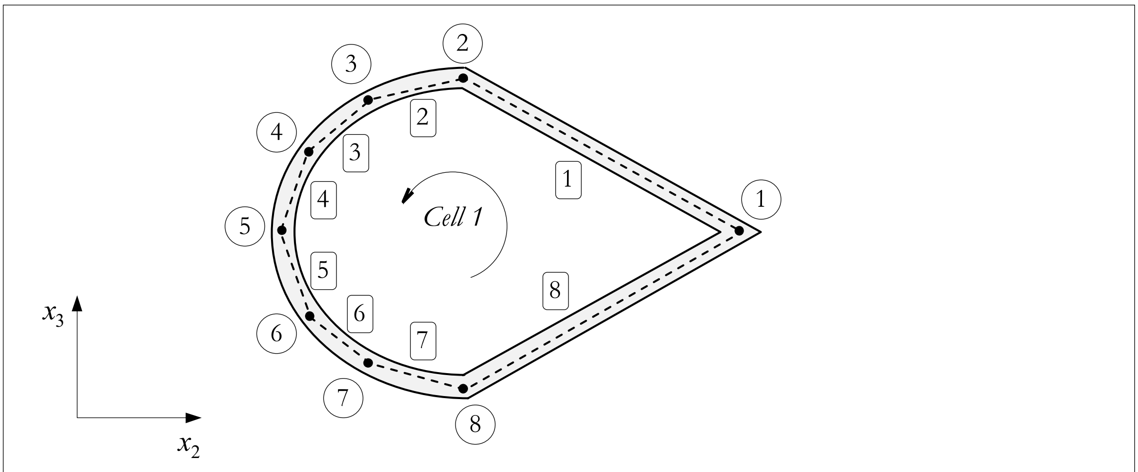


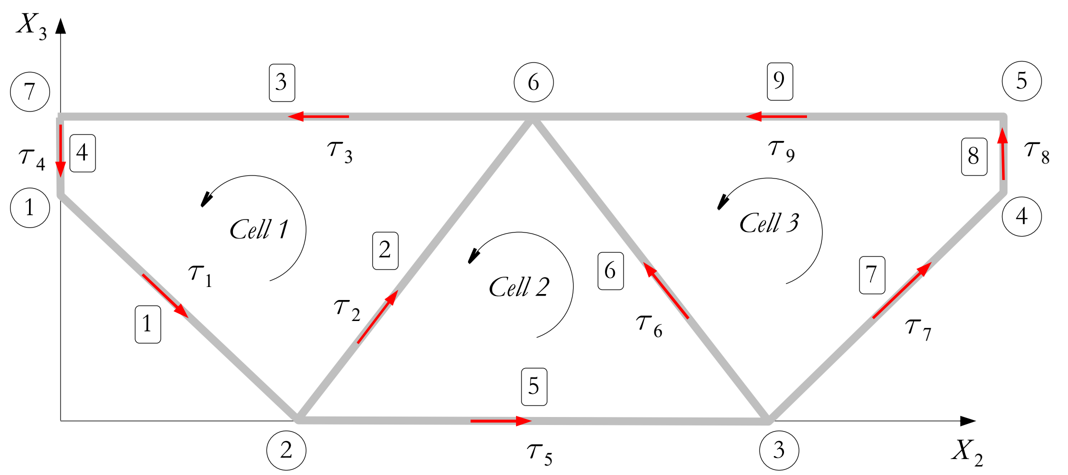
Figure 6.109: Thin-walled section formed by curved segment.

If we consider the e-element connectivity as Node $i(x_2^{(i)}, x_3^{(i)}) \rightarrow$ Node $j(x_2^{(j)}, x_3^{(j)})$, the area of the cell can be obtained as follows:

$$A_{Cell} = \sum_{e=1}^{N_{elem}}\left[(x_3^{(i)} + x_3^{(j)})\frac{(x_2^{(i)} - x_2^{(j)})}{2}\right]$$

Problem 6.51

Obtain the tangential stress in each segment and the angle of twist per unit length θ for the multiply connected thin-walled section described in Figure 6.110, in which the moment of torsion is $M_T = 360$ and the mechanical properties are $E = 2.1\times10^7$ (Young's modulus), $\nu = 0.3$ (Poisson's ratio).



Nodes	Coordinates X_2	Coordinates X_3
1	0	1.5
2	3	0
3	9	0
4	12	1.5
5	12	2
6	6	2
7	0	2

Element e	Connectivity $i \rightarrow j$	Thickness
1	$1\rightarrow2$	$t_1 = 0.03$
2	$2\rightarrow6$	$t_2 = 0.01$
3	$6\rightarrow7$	$t_3 = 0.03$
4	$7\rightarrow1$	$t_4 = 0.03$
5	$2\rightarrow3$	$t_5 = 0.03$
6	$3\rightarrow6$	$t_6 = 0.01$
7	$3\rightarrow4$	$t_7 = 0.03$
8	$4\rightarrow5$	$t_8 = 0.03$
9	$5\rightarrow6$	$t_9 = 0.03$

Figure 6.110: Multiply connected thin-walled section.

Solution:

Loop in Elements in order to calculate the element lengths:

$$s_e = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$$

$$e = 1 (i = 1; j = 2) \Rightarrow s_1 = \sqrt{(3-0)^2 + (0-1.5)^2} = \sqrt{11.25}$$

$s_2 = \sqrt{13}$, $s_3 = 6$, $s_4 = 0.5$, $s_5 = 6$, $s_6 = \sqrt{13}$, $s_7 = \sqrt{11.25}$, $s_8 = 0.5$, $s_9 = 6$

Loop in Cells in order to calculate the cell areas:

Cell 1: Elements 1,2,3,4: $$A_1 = \sum_{e=1}^{N_{elem}} \left[(X_3^{(i)} + X_3^{(j)}) \frac{(X_2^{(i)} - X_2^{(j)})}{2} \right] = \sum_{e=1}^{4} \left[(X_3^{(i)} + X_3^{(j)}) \frac{(X_2^{(i)} - X_2^{(j)})}{2} \right]$$

$$A_1 = (0+1.5)\frac{(0-3)}{2} + (0+2)\frac{(3-6)}{2} + (2+2)\frac{(6-0)}{2} + (2+1.5)\frac{(0-0)}{2} = 6.75$$

Cell 2: Elements 5,6,2: Area: $A_2 = 6$

Cell 3: Elements $7,8,9,6$: Area: $A_3 = 6.75$

For the three cells we apply the equation (6.260), i.e.:

Cell 1: $\oint_{\Gamma_{C1}} \tau \, d\Gamma = 2G\theta A_1 \quad \Rightarrow \quad \tau_1 s_1 + \tau_2 s_2 + \tau_3 s_3 + \tau_4 s_4 - 2GA_1\theta = 0$

Cell 2: $\oint_{\Gamma_{C2}} \tau \, d\Gamma = 2G\theta A_2 \quad \Rightarrow \quad \tau_5 s_5 + \tau_6 s_6 - \tau_2 s_2 - 2GA_2\theta = 0$

Cell 3: $\oint_{\Gamma_{C3}} \tau \, d\Gamma = 2G\theta A_3 \quad \Rightarrow \quad \tau_7 s_7 + \tau_8 s_8 + \tau_9 s_9 - \tau_6 s_6 - 2GA_3\theta = 0$

The moment equation:

$$M_T = 2V_{memb} = 2(A_1 t_1 \tau_1 + A_2 t_5 \tau_5 + A_3 t_7 \tau_7)$$

We have 10 unknowns and 4 equations, the 6 missing equations can be obtained by tangential stress flux compatibility at each node. Note that we have 7 nodes, but we only need 6 of them:

Node 1: $t_4\tau_4 - t_1\tau_1 = 0$

Node 2: $t_1\tau_1 - t_2\tau_2 - t_5\tau_5 = 0$

Node 3: $t_5\tau_5 - t_6\tau_6 - t_7\tau_7 = 0$

Node 4: $t_7\tau_7 - t_8\tau_8 = 0$

Node 5: $t_8\tau_8 - t_9\tau_9 = 0$

Node 6: $t_9\tau_9 + t_6\tau_6 + t_2\tau_2 - t_3\tau_3 = 0$

Then, by restructuring the above 10 equations in matrix form we can obtain:

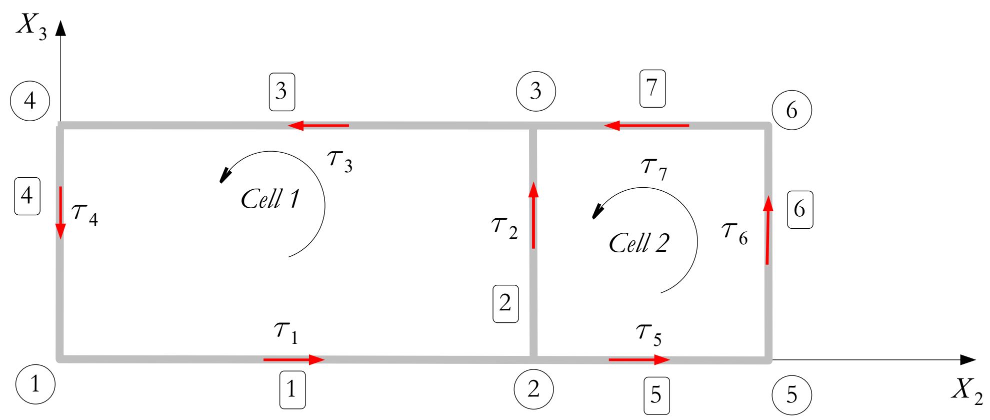
$$\begin{array}{l} \textit{Cell 1} \\ \textit{Cell 2} \\ \textit{Cell 3} \\ \textit{moment} \\ \textit{node 1} \\ \textit{node 2} \\ \textit{node 3} \\ \textit{node 4} \\ \textit{node 5} \\ \textit{node 6} \end{array} \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & 0 & 0 & 0 & 0 & 0 & -2GA_1 \\ 0 & -s_2 & 0 & 0 & s_5 & s_6 & 0 & 0 & 0 & -2GA_2 \\ 0 & 0 & 0 & 0 & 0 & -s_6 & s_7 & s_8 & s_9 & -2GA_3 \\ \hdashline A_1 t_1 & 0 & 0 & 0 & A_2 t_5 & 0 & A_3 t_7 & 0 & 0 & 0 \\ \hdashline -t_1 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & 0 & 0 & -t_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_5 & -t_6 & -t_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_7 & -t_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_8 & -t_9 & 0 \\ 0 & t_2 & -t_3 & 0 & 0 & t_6 & 0 & 0 & t_9 & 0 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{M_T}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.319}$$

where $G = \dfrac{E}{2(1+\nu)}$ (*shear modulus*). After substituting the data and solving the above system we can obtain:

$\tau_1 = \tau_3 = \tau_4 = \tau_7 = \tau_8 = \tau_9 = 300.8354544$, $\tau_2 = -\tau_6 = -66.85432$, $\tau_5 = 323.120227$, $\theta = 2.497666 \times 10^{-5}$.

Problem 6.52

Obtain the shear flux in each segment and the angle of twist θ for the multiply connected thin-walled section described in Figure 6.111, in which the moment of torsion is M_T and consider the mechanical property G (shear modulus).



Nodes	Coordinates X_2	Coordinates X_3
1	0	0
2	$2a$	0
3	$2a$	a
4	0	a
5	$3a$	0
6	$3a$	a

Element e	Connectivity $i \to j$	Thickness
1	$1 \to 2$	$t_1 = t$
2	$2 \to 3$	$t_2 = t$
3	$3 \to 4$	$t_3 = t$
4	$4 \to 1$	$t_4 = t$
5	$2 \to 5$	$t_5 = t$
6	$5 \to 6$	$t_6 = t$
7	$6 \to 3$	$t_7 = t$

Figure 6.111: Multiply connected thin-walled section.

Solution:

Loop in Elements in order to calculate the element lengths:

$$s_e = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$$

$$e = 1 (i = 1; j = 2) \Rightarrow s_1 = \sqrt{(2a - 0)^2 + (0 - 0)^2} = 2a$$

$$s_2 = a \,,\; s_3 = 2a \,,\; s_4 = a \,,\; s_5 = a \,,\; s_6 = a \,,\; s_7 = a$$

Loop in Cells in order to calculate the cell areas:

Cell 1: Elements $1,2,3,4$:

$$A_1 = \sum_{e=1}^{N_{elem}} \left[(X_3^{(i)} + X_3^{(j)}) \frac{(X_2^{(i)} - X_2^{(j)})}{2} \right] = \sum_{e=1}^{4} \left[(X_3^{(i)} + X_3^{(j)}) \frac{(X_2^{(i)} - X_2^{(j)})}{2} \right]$$

$$\Rightarrow A_1 = 2a^2$$

Cell 2: Elements $5,6,7,2$: Area: $A_2 = 2a^2$

For the three cells we apply the equation (6.260), i.e.:

Cell 1: $\displaystyle\oint_{\Gamma_{C1}} \tau \, d\Gamma = 2G\theta A_1 \quad \Rightarrow \quad \tau_1 s_1 + \tau_2 s_2 + \tau_3 s_3 + \tau_4 s_4 - 2GA_1\theta = 0$

Cell 2: $\displaystyle\oint_{\Gamma_{C2}} \tau \, d\Gamma = 2G\theta A_2 \quad \Rightarrow \quad \tau_5 s_5 + \tau_6 s_6 + \tau_7 s_7 - \tau_2 s_2 - 2GA_2\theta = 0$

The moment equation:

$$M_T = 2V_{memb} = 2(A_1 t_1 \tau_1 + A_2 t_5 \tau_5)$$

We have 8 unknowns and 3 equations, the 5 missing equations can be obtained by tangential stress flux compatibility at each node. Note that we have 6 nodes, but we only need 5 of them:

Node 1: $t_4\tau_4 - t_1\tau_1 = 0$

Node 2: $t_1\tau_1 - t_2\tau_2 - t_5\tau_5 = 0$

Node 3: $t_2\tau_2 - t_3\tau_3 + t_7\tau_7 = 0$

Node 4: $t_3\tau_3 - t_4\tau_4 = 0$

Node 5: $t_5\tau_5 - t_6\tau_6 = 0$

Then, by restructuring the above 8 equations in matrix form we can obtain:

$$\begin{array}{l} \textit{Cell 1} \\ \textit{Cell 2} \\ \textit{moment} \\ \textit{node 1} \\ \textit{node 2} \\ \textit{node 3} \\ \textit{node 4} \\ \textit{node 5} \end{array}
\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & 0 & 0 & 0 & -2GA_1 \\ 0 & -s_2 & 0 & 0 & s_5 & s_6 & s_7 & -2GA_2 \\ \hline A_1 t_1 & 0 & 0 & 0 & A_2 t_5 & 0 & 0 & 0 \\ \hline -t_1 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & 0 & 0 & -t_5 & 0 & 0 & 0 \\ 0 & t_2 & -t_3 & 0 & 0 & 0 & t_7 & 0 \\ 0 & 0 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_5 & -t_6 & 0 & 0 \end{bmatrix}
\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{M_T}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.320}$$

After substituting the data and solving the above system we can obtain:

$$\tau_1 = \tau_3 = \tau_4 = \frac{9M_T}{52a^2t}, \; \tau_2 = \frac{M_T}{52a^2t}, \; \tau_5 = \tau_6 = \tau_7 = \frac{2M_T}{13a^2t}, \; \theta = \frac{23M_T}{104a^3tG}.$$

And the shear flux ($q = t\tau$) can be obtained as follows

$$q^{(1)} = t_{1,3,4}\tau_{1,3,4} \quad \Rightarrow \quad q^{(1,3,4)} = \frac{9M_T}{52a^2}, \; q^{(2)} = \frac{M_T}{52a^2}, \; q^{(5,6,7)} = \frac{2M_T}{13a^2}$$

6.4.1.3 Combined Open and Closed Thin-Walled Section

Problem 6.53

Consider a hybrid cross section as the one indicated in Figure 6.112 which is made up by a hollow tube with radial fins. Obtain the equations for the tangential stress in the tube and in the fins when the cross section is subjected to a torsion moment M_T.

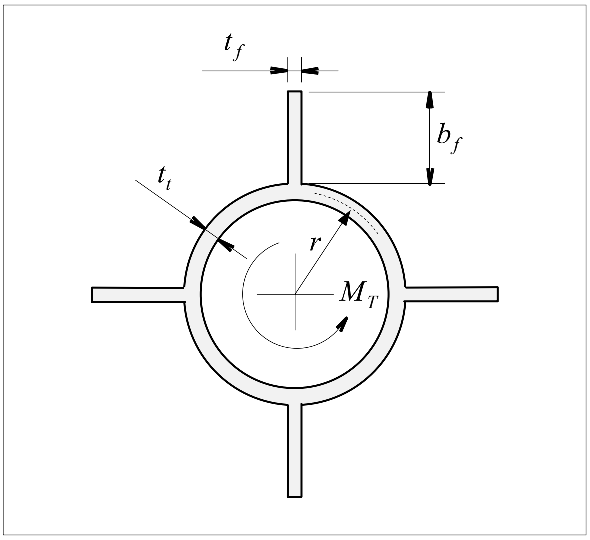


Figure 6.112: Hybrid thin wall cross section.

Solution:

Since we are dealing with a monolithic bar, the angle of twist per unit length is the same for the tube and for the fins, i.e. $\theta = \theta^{(t)} = \theta^{(f)}$. Moreover, we can apply the decomposition of the cross section into the tube and the fins, (see Figure 6.113).

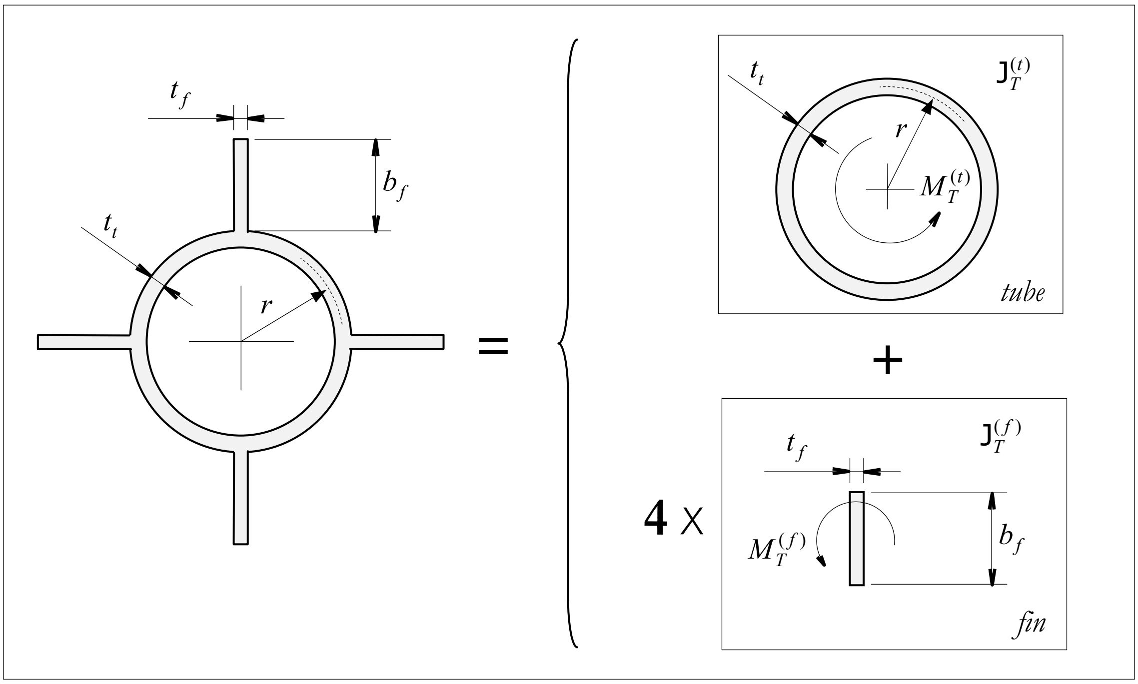


Figure 6.113: Decomposition of the hybrid cross section.

Then, the following is true

$$M_T = M_T^{(t)} + 4M_T^{(f)} \quad ; \quad \mathsf{J}_T = \mathsf{J}_T^{(t)} + 4\mathsf{J}_T^{(f)}$$

By means of the equation $M_T = G\theta \mathsf{J}_T$ we can also conclude that

$$\theta = \theta^{(t)} = \theta^{(f)} \quad \Rightarrow \quad \frac{M_T}{G\mathrm{J}_T} = \frac{M_T^{(t)}}{G\mathrm{J}_T^{(t)}} = \frac{M_T^{(f)}}{G\mathrm{J}_T^{(f)}} \quad \Rightarrow \quad \frac{M_T}{\mathrm{J}_T} = \frac{M_T^{(t)}}{\mathrm{J}_T^{(t)}} = \frac{M_T^{(f)}}{\mathrm{J}_T^{(f)}}$$

$$\Rightarrow \qquad M_T^{(t)} = \frac{M_T \mathrm{J}_T^{(t)}}{\mathrm{J}_T} = \frac{M_T \mathrm{J}_T^{(t)}}{(\mathrm{J}_T^{(t)} + 4\mathrm{J}_T^{(f)})} \quad ; \quad M_T^{(f)} = \frac{M_T \mathrm{J}_T^{(f)}}{\mathrm{J}_T} = \frac{M_T \mathrm{J}_T^{(f)}}{(\mathrm{J}_T^{(t)} + 4\mathrm{J}_T^{(f)})}$$

For the tube we have $\mathrm{J}_T^{(t)} = 2\pi r^3 t$, (see equation (6.306)), and for the fin we have $\mathrm{J}_T^{(f)} = \frac{b_f t_f^3}{3}$, (see equation (6.294)), then

$$\mathrm{J}_T = \mathrm{J}_T^{(t)} + 4\mathrm{J}_T^{(f)} = 2\pi r^3 t_t + \frac{4 b_f t_f^3}{3} = \frac{2}{3}(3\pi r^3 t_t + 2 b_f t_f^3) \quad ; \quad \theta = \frac{M_T}{G\mathrm{J}_T} = \frac{3M_T}{2G(3\pi r^3 t_t + 2 b_f t_f^3)}$$

The tangential stress in the tube can be obtained by means of the equation in (6.305):

$$\tau_{\max}^{(t)} = \frac{M_T^{(t)}}{2A t_t} = \frac{M_T^{(t)}}{2(\pi r^2) t_t} = \frac{M_T^{(t)}}{2\pi r^2 t_t} = \frac{3 M_T r}{2(3\pi r^3 t_t + 2 b_f t_f^3)}$$

And the tangential stress in the fin can be obtained by means of the equation in (6.296):

$$\tau_{\max}^{(f)} = \frac{M_T^{(f)} t_f}{\mathrm{J}_T^{(f)}} = \frac{3 M_T^{(f)} t_f}{b_f t_f^3} = \frac{3 M_T t_f}{2(3\pi r^3 t_t + 2 b_f t_f^3)}$$

NOTE 1: Next we will automatize the above procedure in order to obtain the solution of Hybrid Cross Section. Let us consider the example described in Figure 6.112 which was discretize as indicated in Figure 6.114.

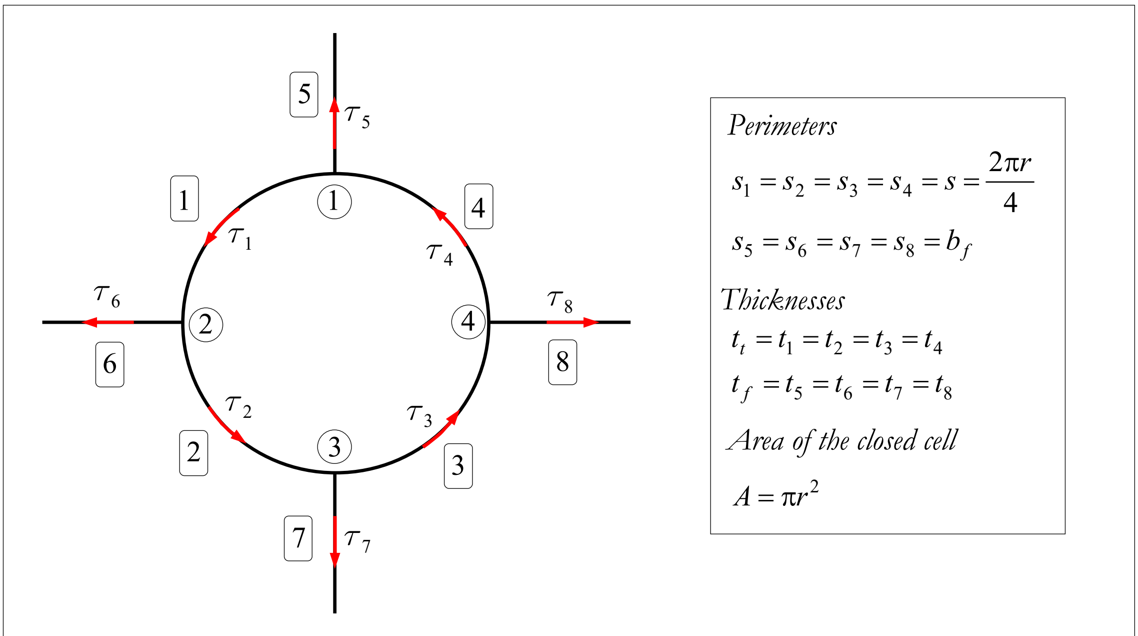


Figure 6.114: Hybrid cross section.

Here we will consider that the fin is an Open Cell in which is valid $\tau = tG\theta$, (see equation (6.296)). For the closed cell we can state that

Closed Cell 1:

$$\oint_{\Gamma_{C1}} \tau \, d\Gamma = 2G\theta A \quad \Rightarrow \quad \tau_1 s_1 + \tau_2 s_2 + \tau_3 s_3 + \tau_4 s_4 = 2GA\theta \quad \Rightarrow \quad \tau_1 s + \tau_2 s + \tau_3 s + \tau_4 s = 2GA\theta$$

And for the fins (open cell):

Open Cell 2 ($e=5$): $\tau_5 = t_5 G\theta \quad \Rightarrow \quad \tau_5 - t_f G\theta = 0$

Open Cell 3 ($e=6$): $\tau_6 = t_6 G\theta \quad \Rightarrow \quad \tau_6 - t_f G\theta = 0$

Open Cell 4 ($e=7$): $\tau_7 = t_7 G\theta \quad \Rightarrow \quad \tau_7 - t_f G\theta = 0$

Open Cell 5 ($e=8$): $\tau_8 = t_8 G\theta \quad \Rightarrow \quad \tau_8 - t_f G\theta = 0$

The moment equation is now given by:

$$M_T = M_T^{(Closed)} + M_T^{(Open)} = 2V_{memb}^{(Closed)} + 4M_T^{(f)} = 2At_1\tau_1 + 4(\mathrm{J}_T^{(f)} G\theta) = 2At_t\tau_1 + 4\left(\frac{b_f t_f^3}{3} G\theta\right)$$

$$\Rightarrow \frac{M_T}{2} = At_t\tau_1 + 4\left(\frac{b_f t_f^3}{6} G\right)\theta$$

We have 9 unknowns and 6 equations, the 3 missing equations can be obtained by tangential stress flux compatibility at each node. Note that we have 4 nodes, but we only need 3 of them, namely:

Node 1: $t_4\tau_4 - t_1\tau_1 = 0 \quad \Rightarrow \quad t_t\tau_4 - t_t\tau_1 = 0$

Node 2: $t_1\tau_1 - t_2\tau_2 = 0 \quad \Rightarrow \quad t_t\tau_1 - t_t\tau_2 = 0$

Node 3: $t_2\tau_2 - t_3\tau_3 = 0 \quad \Rightarrow \quad t_t\tau_2 - t_t\tau_3 = 0$

Then, the set of equations to be solved is:

$$\begin{array}{l} \textit{Cell 1} \\ \textit{Cell 2} \\ \textit{Cell 3} \\ \textit{Cell 4} \\ \textit{Cell 5} \\ \textit{moment} \\ \textit{node 1} \\ \textit{node 2} \\ \textit{node 3} \end{array}
\left[\begin{array}{cccccccc|c}
s & s & s & s & 0 & 0 & 0 & 0 & -2GA \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -t_f G \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -t_f G \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -t_f G \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t_f G \\
\hline
At_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\left(\frac{b_f t_f^3}{6} G\right) \\
\hline
-t_t & 0 & 0 & t_t & 0 & 0 & 0 & 0 & 0 \\
t_t & -t_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_t & -t_t & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \theta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{M_T}{2} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

And the solution is:

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \frac{3AM_T}{2(3A^2 t_t + 4sb_f t_f^3)} = \frac{3M_T r}{2(3\pi r^3 t_t + 2b_f t_f^3)}$$

$$\tau_5 = \tau_6 = \tau_7 = \tau_8 = \frac{3st_f M_T}{(3A^2 t_t + 4sb_f t_f^3)} = \frac{3M_T t_f}{2(3\pi r^3 t_t + 2b_f t_f^3)}$$

$$\theta = \frac{3sM_T}{G(3A^2 t_t + 4sb_f t_f^3)} = \frac{3M_T}{2G(3\pi r^3 t_t + 2b_f t_f^3)}$$

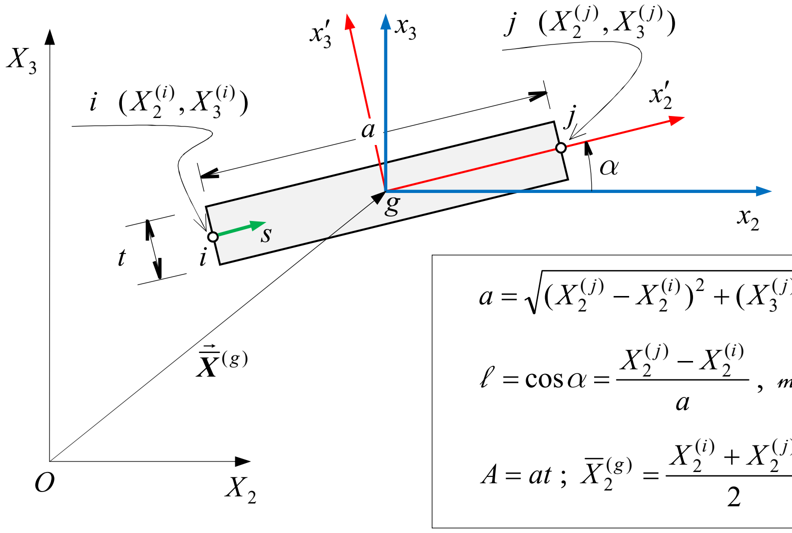
and

$$M_T = G\theta \mathrm{J}_T \quad \Rightarrow \quad \mathrm{J}_T = \frac{M_T}{G\left[\dfrac{3sM_T}{G(3A^2 t_t + 4sb_f t_f^3)}\right]} = \frac{3A^2 t_t + 4sb_f t_f^3}{3s} = \frac{2}{3}(3\pi r^3 t_t + 2b_f t_f^3)$$

6.4.2 Torsion and Bending of Thin-Walled Cross Section

6.4.2.1 Introduction

The shear flux problem for bending was established in Chapter 4 - "*4.2 Some useful concepts for the classical "mechanics of materials*"", and we summarize as indicated in Figure 6.115.



Transformation matrix from $g\vec{x}$ to $g\vec{x}'$

$$\mathcal{A} = \begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix}$$

$$a = \sqrt{(X_2^{(j)} - X_2^{(i)})^2 + (X_3^{(j)} - X_3^{(i)})^2}$$

$$\ell = \cos\alpha = \frac{X_2^{(j)} - X_2^{(i)}}{a} \;,\quad m = \sin\alpha = \frac{X_3^{(j)} - X_3^{(i)}}{a}$$

$$A = at \;;\; \overline{X}_2^{(g)} = \frac{X_2^{(i)} + X_2^{(j)}}{2} \;;\; \overline{X}_3^{(g)} = \frac{X_3^{(i)} + X_3^{(j)}}{2}$$

$$\boxed{\mathrm{I}_{O\vec{X}\,ij} \approx \frac{at}{12}\begin{bmatrix} a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix}}$$

Inertia Tensor of Area related to the system $O\vec{X}$, $(t << a)$

$$\boxed{q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y} s + t\,d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z} s + t\,d_2^{F_z} s^2)F_z}$$ *(Shear Flux)*

$$d_1^{F_y} = -\mathcal{X}^{-1}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] \quad ; \quad d_1^{F_z} = -\mathcal{X}^{-1}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right]$$

$$d_2^{F_y} = -(2\mathcal{X})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] \quad ; \quad d_2^{F_z} = -(2\mathcal{X})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right]$$

$$p_0^{(1)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{23} & \mathrm{I}_{22} \end{vmatrix} \quad ; \quad p_0^{(2)} = -\begin{vmatrix} A\overline{X}_2 & A\overline{X}_3 \\ -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}$$

$$p_2^{(1)} = \begin{vmatrix} A & A\overline{X}_3 \\ A\overline{X}_3 & \mathrm{I}_{22} \end{vmatrix} \quad ; \quad p_2^{(2)} = \begin{vmatrix} A & A\overline{X}_3 \\ -A\overline{X}_2 & \mathrm{I}_{23} \end{vmatrix}$$

$$p_3^{(1)} = -\begin{vmatrix} A & A\overline{X}_2 \\ A\overline{X}_3 & -\mathrm{I}_{23} \end{vmatrix} \quad ; \quad p_3^{(2)} = -\begin{vmatrix} A & A\overline{X}_2 \\ -A\overline{X}_2 & -\mathrm{I}_{33} \end{vmatrix}, \quad \mathcal{X} = \begin{vmatrix} A & A\overline{X}_2 & A\overline{X}_3 \\ A\overline{X}_3 & -\mathrm{I}_{23} & \mathrm{I}_{22} \\ -A\overline{X}_2 & -\mathrm{I}_{33} & \mathrm{I}_{23} \end{vmatrix}$$

$$\boxed{M_O = M_O^{F_y} F_y + M_O^{F_z} F_z = -(X_3^{(S.C.)})F_y + (X_2^{(S.C.)})F_z}$$ *(Torsion moment)*

$$M_O^{F_y} = (6)^{-1}[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)})$$

$$M_O^{F_z} = (6)^{-1}[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)})$$

$$\boxed{f = \left[q_{F_y}^{(i)} a + t\,d_1^{F_y}\frac{a}{2} + t\,d_2^{F_y}\frac{a^3}{3}\right]F_y + \left[q_{F_z}^{(i)} a + t\,d_1^{F_z}\frac{a}{2} + t\,d_2^{F_z}\frac{a^3}{3}\right]F_z}$$ *(Force)*

Figure 6.115

6.4.2.2 Open Cross Section

The problem characterized by bending of open thin-walled section was already discussed in Chapter 4 - "*4.2 Some useful concepts for the classical "mechanics of materials*". And, since we are dealing with linear elasticity, we can use the superposition principle in order to obtain the response with an additional effect due to torsion problem only, (see Figure 6.116).

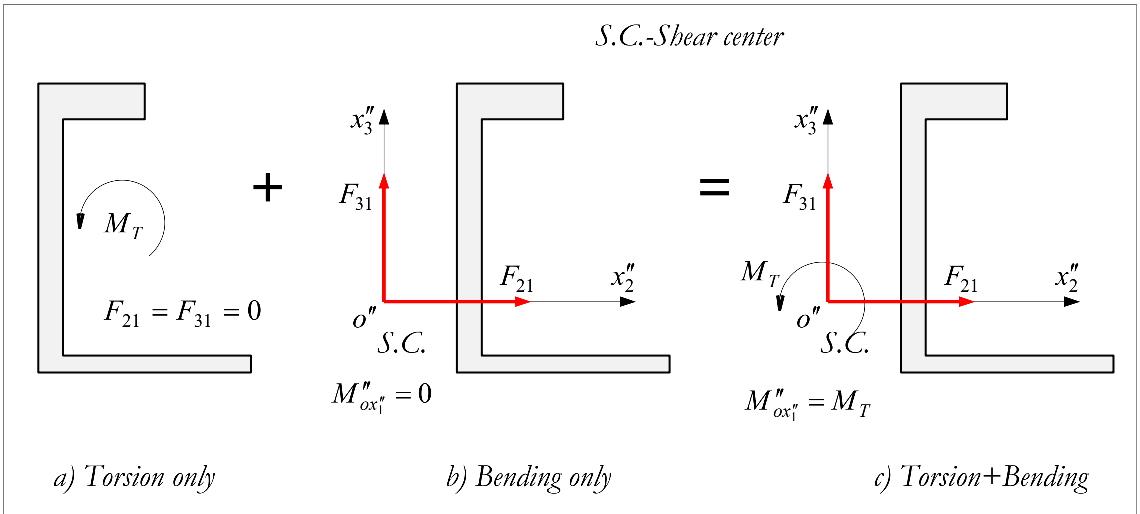


Figure 6.116: Torsion and bending of open section.

The tangential stress distributions, for "torsion only" and for "bending only", can be appreciated in Figure 6.117. In **Problem 4.31** we have obtained the shear flux for the problem described in Figure 6.117(b).

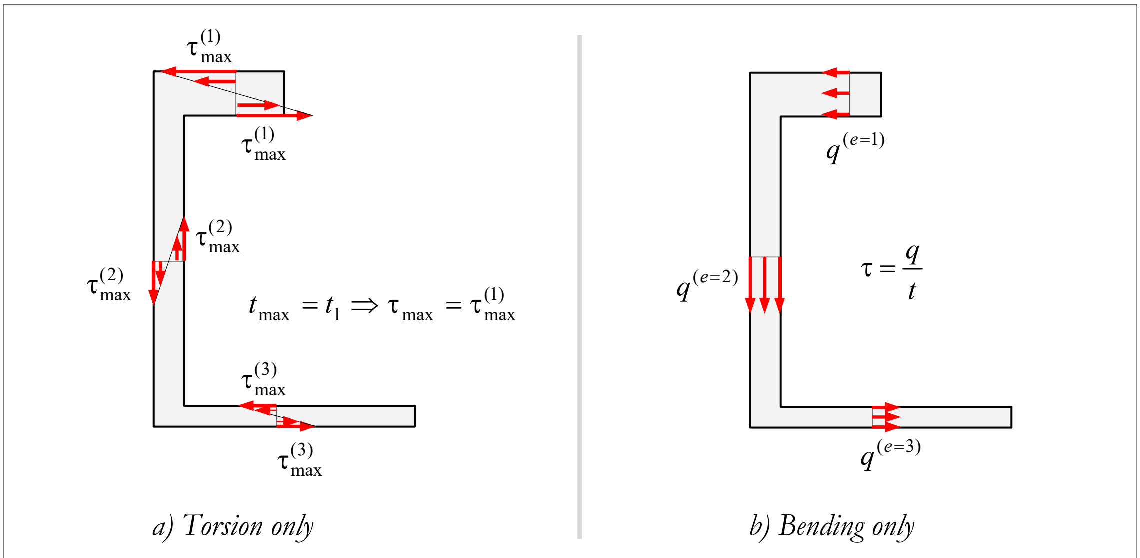
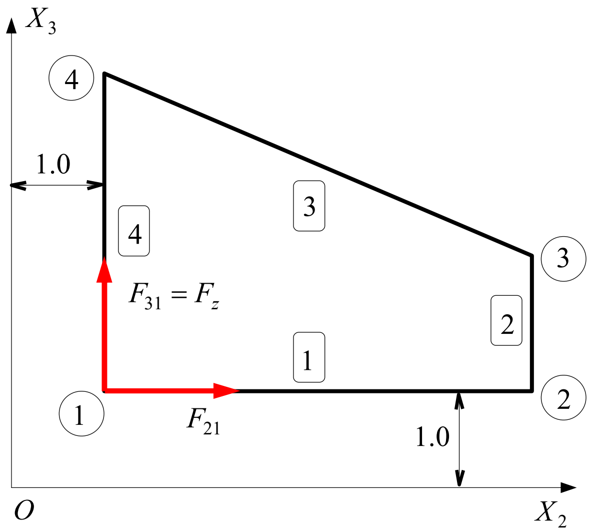


Figure 6.117: Tangential stress distribution for torsion and bending of open section.

6.4.2.3 Closed Cross Section

As example, let us consider the closed cross section as the one indicated in Figure 6.118, where the shearing forces are $F_{21} = F_y = 0$ and $F_{31} = F_z = 1000$, and are applied at node 1.



Node Coordinates

Node	$X_2^{(i)}$	$X_3^{(i)}$
$i=1$	1	1
$i=2$	13	1
$i=3$	14	5
$i=4$	1	10

Flange	*Path* $i \rightarrow j$	*Thickness*(t)
$e=1$	$1 \rightarrow 2$	0.1
$e=2$	$2 \rightarrow 3$	0.1
$e=3$	$3 \rightarrow 4$	0.1
$e=4$	$4 \rightarrow 1$	0.1

Figure 6.118: Closed thin-walled cross section.

The flange geometric characteristics are described in Table 6.1.

Table 6.1

Flange	$(X_2^{(i)};X_3^{(i)})$	$(X_2^{(j)};X_3^{(j)})$	t	a	*Area* $A^{(e)}$	ℓ	m	$\overline{X}_2^{(ge)}$	$\overline{X}_3^{(ge)}$
$e=1$	(1;1)	(13;1)	0.1	12	1.2	1	0	7	1
$e=2$	(13;1)	(13;5)	0.1	4	0.4	0	1	13	3
$e=3$	(13;5)	(1;10)	0.1	13	1.3	$\frac{-12}{13}$	$\frac{5}{13}$	7	7.5
$e=4$	(1;10)	(1;1)	0.1	9	0.9	0	-1	1	5.5

Area Centroid of the Compound and Total Area

The total area is $A = A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} = 1.2 + 0.4 + 1.3 + 0.9 = 3.8$

And the area centroid is given by

$$\overline{X}_2 = \frac{A^{(1)}\overline{X}_2^{(g1)} + A^{(2)}\overline{X}_2^{(g2)} + A^{(3)}\overline{X}_2^{(g3)} + A^{(4)}\overline{X}_2^{(g4)}}{A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)}} = 6.2105$$

$$\overline{X}_3 = \frac{A^{(1)}\overline{X}_3^{(g1)} + A^{(2)}\overline{X}_3^{(g2)} + A^{(3)}\overline{X}_3^{(g3)} + A^{(4)}\overline{X}_3^{(g4)}}{A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)}} = 4.5$$

The Inertia Tensor of Area (for the flange e)

$$\mathrm{I}_{O\vec{X}ij}^{(e)} \approx \frac{at}{12}\begin{bmatrix} a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix} \tag{6.321}$$

The inertia tensors for the flanges related to the system $O\vec{X}$ are

$$\mathrm{I}^{(1)}_{O\vec{X}ij} \approx \begin{bmatrix} 1.2 & -8.4 \\ -8.4 & 73.2 \end{bmatrix};\ \mathrm{I}^{(2)}_{O\vec{X}ij} \approx \begin{bmatrix} 4.1333 & -15.6 \\ -15.6 & 67.6 \end{bmatrix};\ \mathrm{I}^{(3)}_{O\vec{X}ij} \approx \begin{bmatrix} 75.8333 & -61.75 \\ -61.75 & 79.3 \end{bmatrix} \text{ and}$$

$$\mathrm{I}^{(4)}_{O\vec{X}ij} \approx \begin{bmatrix} 33.3 & -4.95 \\ -4.95 & 0.9 \end{bmatrix}$$

Then, the inertia tensor for the compound is given by

$$\left(\mathbf{I}^{(Sys)}_{O\vec{X}}\right)_{ij} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix} = \sum_{e=1}^{3} (\mathbf{I}^{(e)}_{O\vec{X}})_{ij} = \mathrm{I}^{(1)}_{O\vec{X}ij} + \mathrm{I}^{(2)}_{O\vec{X}ij} + \mathrm{I}^{(3)}_{O\vec{X}ij} + \mathrm{I}^{(4)}_{O\vec{X}ij} = \begin{bmatrix} 114.46667 & -90.7 \\ -90.7 & 221 \end{bmatrix}$$

Just to exercise let us calculate the inertia tensor at the Area Centroid, (see Chapter 4), which can be obtained by means of the Steiner's theorem:

$$\mathbf{I}^{(Sys)}_{O\vec{X}} = \overline{\mathbf{I}}^{(Sys)}_{G\vec{x}} - A[(\vec{\overline{X}} \otimes \vec{\overline{X}}) - (\vec{\overline{X}} \cdot \vec{\overline{X}})\mathbf{1}] \quad \Rightarrow \quad \overline{\mathbf{I}}^{(Sys)}_{G\vec{x}} = \mathbf{I}^{(Sys)}_{O\vec{X}} + A[(\vec{\overline{X}} \otimes \vec{\overline{X}}) - (\vec{\overline{X}} \cdot \vec{\overline{X}})\mathbf{1}]$$

whose components are:

$$(\overline{\mathbf{I}}^{(Sys)}_{G\vec{x}})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix} = \mathrm{I}^{(Sys)}_{O\vec{X}\ ij} - A\begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix}$$

$$(\overline{\mathbf{I}}^{(Sys)}_{G\vec{x}})_{ij} = \begin{bmatrix} 114.46667 & -90.7 \\ -90.7 & 221 \end{bmatrix} - (3.8)\begin{bmatrix} (4.5)^2 & -(6.21)(6.21) \\ -(6.21)(6.21) & (6.21)^2 \end{bmatrix} = \begin{bmatrix} 37.5167 & 15.5 \\ 15.5 & 74.432 \end{bmatrix}$$

Shear Flux

By means of Figure 6.115 we can calculate the shear flux in the flanges. Taking into account the geometric properties calculated previously we can calculate the coefficients related to the cross section:

$p_0^{(1)} = -1.15044\times 10^3$; $p_0^{(2)} = -1.63858\times 10^3$; $p_2^{(1)} = 142.56333$; $p_2^{(2)} = 58.9$; $p_3^{(1)} = 58.9$; $p_3^{(2)} = 282.84$, $\boldsymbol{\mathcal{X}} = 9.69826\times 10^3$

Flange 1: (Flange data $(X_2^{(i)}, X_3^{(i)}, t, \ell, m)$ in Table 6.1)

$q_{F_y}^{(i)} = q_{F_y}^{(1)}$, $q_{F_z}^{(i)} = q_{F_z}^{(1)}$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = 0.09785 \quad \Rightarrow \quad d_1^{F_y}t = 0.009785$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 0.13372 \quad \Rightarrow \quad d_1^{F_z}t = 0.013372$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\ \ell + p_3^{(1)}\ m\right] = -7.34994\times 10^{-3} \quad \Rightarrow \quad d_2^{F_y}t = -7.34994\times 10^{-4}$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\ \ell + p_3^{(2)}\ m\right] = -3.03663\times 10^{-3} \quad \Rightarrow \quad d_2^{F_z}t = -3.03663\times 10^{-4}$$

Shear Flux in the Flange 1

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=1)}F_y + q_{F_z}^{(e=1)}F_z$$

The terms $q_{F_y}^{(e=1)}$ and $q_{F_z}^{(e=1)}$ can be evaluated as follows:

$$q_{F_y}^{(e=1)}(s) = q_{F_y}^{(1)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(1)} + (0.009785)s + (-7.34994\times 10^{-4})s^2$$

$$q_{F_z}^{(e=1)}(s) = q_{F_z}^{(1)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(1)} + (0.013372)s + (-3.03663\times 10^{-4})s^2$$

at the end $s = 12$, (node 2)

$$q_{F_y}^{(e=1)}(s=12) = q_{F_y}^{(1)} + (0.009785)s + (-7.34994\times 10^{-4})s^2 = q_{F_y}^{(1)} + 0.0115815 = q_{F_y}^{(2)}$$

$$q_{F_z}^{(e=1)}(s=12) = q_{F_z}^{(1)} + (0.013372)s + (-3.03663\times10^{-4})s^2 = q_{F_z}^{(1)} + 0.1167351 = q_{F_z}^{(2)}$$

The total force in the flange 1 (see **Problem 4.31**-NOTE 1)

$$f^{(e=1)} = \left[q_{F_y}^{(i)} a + t\,d_1^{F_y}\frac{a}{2} + t\,d_2^{F_y}\frac{a^3}{3}\right]F_y + \left[q_{F_z}^{(i)} a + t\,d_1^{F_z}\frac{a}{2} + t\,d_2^{F_z}\frac{a^3}{3}\right]F_z$$

$$\Rightarrow f^{(e=1)} = (12 q_{F_y}^{(1)} + 0.281167)F_y + (12 q_{F_z}^{(1)} + 0.787866)F_z$$

Torsion Moment at O due to Flange 1, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 t d_2^{F_y} + 6q_{F_y}^{(i)}) = -0.281167 - 12 q_{F_y}^{(1)}$$

$$\left(M_O^{F_z}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 t d_2^{F_z} + 6q_{F_z}^{(i)}) = -0.7878655 - 12 q_{F_z}^{(1)}$$

Update Variables

$$q_{F_y}^{(i)} \leftarrow q_{F_y}^{(j)} = q_{F_y}^{(1)} + 0.0115815 \quad ; \quad q_{F_z}^{(i)} \leftarrow q_{F_z}^{(j)} = q_{F_z}^{(1)} + 0.1167351$$

Flange 2: (Flange data $(X_2^{(i)}, X_3^{(i)}, t, \ell, m)$ in Table 6.1)

$$q_{F_y}^{(i)} = q_{F_y}^{(1)} + 0.0115815\,,\; q_{F_z}^{(i)} = q_{F_z}^{(1)} + 0.1167351$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = -0.078548 \quad \Rightarrow \quad d_1^{F_y}t = -0.0078548$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 0.0608398 \quad \Rightarrow \quad d_1^{F_z}t = 0.00608398$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -3.0366259\times10^{-3} \quad \Rightarrow \quad d_2^{F_y}t = -3.0366259\times10^{-4}$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = -0.014582 \quad \Rightarrow \quad d_2^{F_z}t = -0.0014582$$

Shear Flux in the Flange 2

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=2)}F_y + q_{F_z}^{(e=2)}F_z$$

The terms $q_{F_y}^{(e=2)}$ and $q_{F_z}^{(e=2)}$ can be evaluated as follows:

$$q_{F_y}^{(e=2)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(2)} + (-0.0078548)s + (-3.0366259\times10^{-4})s^2$$
$$= (q_{F_y}^{(1)} + 0.0115815) + (-0.0078548)s + (-3.0366259\times10^{-4})s^2$$

$$q_{F_z}^{(e=2)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(2)} + (0.00608398)s + (-0.0014582)s^2$$
$$= (q_{F_z}^{(1)} + 0.1167351) + (0.00608398)s + (-0.0014582)s^2$$

at the end $s=4$, (node 3)

$$q_{F_y}^{(e=2)}(s=4) = q_{F_y}^{(2)} - 0.03627783 = q_{F_y}^{(3)}$$
$$= (q_{F_y}^{(1)} + 0.0115815) - 0.03627783 = q_{F_y}^{(1)} - 0.0246964 = q_{F_y}^{(3)}$$

$$q_{F_z}^{(e=2)}(s=4) = q_{F_z}^{(2)} + 0.00100472 = q_{F_z}^{(3)}$$
$$= (q_{F_z}^{(1)} + 0.1167351) + 0.00100472 = q_{F_z}^{(1)} + 0.1177398 = q_{F_z}^{(3)}$$

The total force in the flange 2 (see **Problem 4.31**-NOTE 1)

$$f^{(e=2)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z$$
$$= (4q_{F_y}^{(2)} - 0.069317) F_y + (4q_{F_z}^{(2)} + 0.017564) F_z$$

by considering $q_{F_y}^{(2)} = q_{F_y}^{(1)} + 0.0115815$, $q_{F_z}^{(2)} = q_{F_z}^{(1)} + 0.1167351$ we can obtain

$$\Rightarrow f^{(e=2)} = (4q_{F_y}^{(1)} - 0.02299078) F_y + (4q_{F_z}^{(1)} + 0.484504) F_z$$

Torsion Moment at O due to Flange 2, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=2)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}] a}{6} (3atd_1^{F_y} + 2a^2 t d_2^{F_y} + 6q_{F_y}^{(i)}) = -0.90111577 + 52 q_{F_y}^{(2)}$$
$$= -0.2988802 + 52 q_{F_y}^{(1)}$$

$$\left(M_O^{F_z}\right)^{(e=2)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}] a}{6} (3atd_1^{F_z} + 2a^2 t d_2^{F_z} + 6q_{F_z}^{(i)}) = 0.228326 + 52 q_{F_z}^{(2)}$$
$$= 6.2985524 + 52 q_{F_z}^{(1)}$$

Update Variables

$$q_{F_y}^{(i)} \leftarrow q_{F_y}^{(j)} = q_{F_y}^{(1)} - 0.0246964 \quad ; \quad q_{F_z}^{(i)} \leftarrow q_{F_z}^{(j)} = q_{F_z}^{(1)} + 0.1177398$$

Flange 3: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.1)

$$q_{F_y}^{(i)} = q_{F_y}^{(3)} = q_{F_y}^{(1)} - 0.0246964,\; q_{F_z}^{(i)} = q_{F_z}^{(3)} = q_{F_z}^{(1)} + 0.1177398$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = -0.10284 \quad \Rightarrow \quad d_1^{F_y} t = -0.010284$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -0.05582 \quad \Rightarrow \quad d_1^{F_z} t = -0.005582$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\, \ell + p_3^{(1)}\, m\right] = 5.61663 \times 10^{-3} \quad \Rightarrow \quad d_2^{F_y} t = 5.61663 \times 10^{-4}$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\, \ell + p_3^{(2)}\, m\right] = -2.80542 \times 10^{-3} \quad \Rightarrow \quad d_2^{F_z} t = -2.80542 \times 10^{-4}$$

Shear Flux in the Flange 3

$$q(s) = (q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2) F_y + (q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2) F_z = q_{F_y}^{(e=3)} F_y + q_{F_z}^{(e=3)} F_z$$

The terms $q_{F_y}^{(e=3)}$ and $q_{F_z}^{(e=3)}$ can be evaluated as follows:

$$q_{F_y}^{(e=3)}(s) = q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2 = q_{F_y}^{(3)} + (-0.010284) s + (5.61663 \times 10^{-4}) s^2$$
$$= (q_{F_y}^{(1)} - 0.0246964) + (-0.010284) s + (5.61663 \times 10^{-4}) s^2$$

$$q_{F_z}^{(e=3)}(s) = q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2 = q_{F_z}^{(3)} + (-0.005582) s + (-2.80542 \times 10^{-4}) s^2$$
$$= (q_{F_z}^{(1)} + 0.1177398) + (-0.005582) s + (-2.80542 \times 10^{-4}) s^2$$

at the end $s = 13$, (node 4)

$$q_{F_y}^{(e=3)}(s = 13) = (q_{F_y}^{(1)} - 0.0246964) - 0.0387724 = q_{F_y}^{(3)} - 0.0387724 = q_{F_y}^{(4)}$$
$$= q_{F_y}^{(1)} - 0.0634688 = q_{F_y}^{(4)}$$

$$q_{F_z}^{(e=3)}(s=13)=(q_{F_z}^{(1)}+0.1177398)-0.1199726=q_{F_z}^{(3)}-0.1199726=q_{F_z}^{(4)}$$
$$=q_{F_z}^{(1)}-0.002232771=q_{F_z}^{(4)}$$

The total force in the flange 3 (see **Problem 4.31**-NOTE 1)

$$f^{(e=3)}=\left[q_{F_y}^{(i)}a+t\,d_1^{F_y}\frac{a}{2}+t\,d_2^{F_y}\frac{a^3}{3}\right]F_y+\left[q_{F_z}^{(i)}a+t\,d_1^{F_z}\frac{a}{2}+t\,d_2^{F_z}\frac{a^3}{3}\right]F_z$$
$$=(13q_{F_y}^{(3)}-0.4576828)F_y+(13q_{F_z}^{(3)}-0.6770968)F_z$$

by considering $q_{F_y}^{(i)}=q_{F_y}^{(1)}-0.0246964$, $q_{F_z}^{(i)}=q_{F_z}^{(1)}+0.1177398$ we can obtain

$$\Rightarrow f^{(e=3)}=(13q_{F_y}^{(1)}-0.7787357)F_y+(13q_{F_z}^{(1)}+0.853521)F_z$$

Torsion Moment at O due to Flange 3, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=3)}=\frac{[m\,X_2^{(i)}-\ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y}+2a^2td_2^{F_y}+6q_{F_y}^{(i)})=-4.4007962+125q_{F_y}^{(3)}$$
$$=-7.48784+125q_{F_y}^{(1)}$$

$$\left(M_O^{F_z}\right)^{(e=3)}=\frac{[m\,X_2^{(i)}-\ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z}+2a^2td_2^{F_z}+6q_{F_z}^{(i)})=-6.510547+125q_{F_z}^{(3)}$$
$$=8.206993+125q_{F_z}^{(1)}$$

Update Variables

$$q_{F_y}^{(i)}\leftarrow q_{F_y}^{(j)}=q_{F_y}^{(1)}-0.0634688\quad;\quad q_{F_z}^{(i)}\leftarrow q_{F_z}^{(j)}=q_{F_z}^{(1)}-0.002232771$$

Flange 4: (Flange data ($X_2^{(i)},X_3^{(i)},t,\ell,m$) in Table 6.1)

$$q_{F_y}^{(i)}=q_{F_y}^{(4)}=q_{F_y}^{(1)}-0.0634688,\ q_{F_z}^{(i)}=q_{F_z}^{(4)}=q_{F_z}^{(1)}-0.002232771$$

$$d_1^{F_y}=-\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)}+p_2^{(1)}X_2^{(i)}+p_3^{(1)}X_3^{(i)}\right]=0.04319\quad\Rightarrow\quad d_1^{F_y}t=0.004319$$
$$d_1^{F_z}=-\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)}+p_2^{(2)}X_2^{(i)}+p_3^{(2)}X_3^{(i)}\right]=-0.12876\quad\Rightarrow\quad d_1^{F_z}t=-0.012876$$
$$d_2^{F_y}=-(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell+p_3^{(1)}\,m\right]=3.03663\times10^{-3}\quad\Rightarrow\quad d_2^{F_y}t=3.03663\times10^{-4}$$
$$d_2^{F_z}=-(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell+p_3^{(2)}\,m\right]=0.01458\quad\Rightarrow\quad d_2^{F_z}t=0.001458$$

Shear Flux in the Flange 4

$$q(s)=(q_{F_y}^{(i)}+t\,d_1^{F_y}s+t\,d_2^{F_y}s^2)F_y+(q_{F_z}^{(i)}+t\,d_1^{F_z}s+t\,d_2^{F_z}s^2)F_z=q_{F_y}^{(e=4)}F_y+q_{F_z}^{(e=4)}F_z$$

The terms $q_{F_y}^{(e=4)}$ and $q_{F_z}^{(e=4)}$ can be evaluated as follows:

$$q_{F_y}^{(e=4)}(s)=q_{F_y}^{(i)}+t\,d_1^{F_y}s+t\,d_2^{F_y}s^2=q_{F_y}^{(4)}+(0.004319)s+(3.03663\times10^{-4})s^2$$
$$=(q_{F_y}^{(1)}-0.0634688)+(0.004319)s+(3.03663\times10^{-4})s^2$$

$$q_{F_z}^{(e=4)}(s)=q_{F_z}^{(i)}+t\,d_1^{F_z}s+t\,d_2^{F_z}s^2=q_{F_z}^{(4)}+(-0.012876)s+(0.001458)s^2$$
$$=(q_{F_z}^{(1)}-0.002232771)+(-0.012876)s+(0.001458)s^2$$

at the end $s=9$, (node 1)

$$q_{F_y}^{(e=4)}(s=9)=(q_{F_y}^{(1)}-0.0634688)+0.0634688=q_{F_y}^{(1)}$$

$$q_{F_z}^{(e=4)}(s=9) = (q_{F_z}^{(1)} - 0.002232771) + 0.002232771 = q_{F_z}^{(1)}$$

The total force in the flange 4 (see **Problem 4.31**-NOTE 1)

$$f^{(e=4)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z$$

$$= (9 q_{F_y}^{(4)} + 0.248715) F_y + (9 q_{F_z}^{(4)} - 0.167124) F_z$$

by considering $q_{F_y}^{(i)} = q_{F_y}^{(4)} = q_{F_y}^{(1)} - 0.0634688$, $q_{F_z}^{(i)} = q_{F_z}^{(4)} = q_{F_z}^{(1)} - 0.002232771$ we get

$$\Rightarrow f^{(e=4)} = (9 q_{F_y}^{(1)} - 0.3225045) F_y + (9 q_{F_z}^{(1)} - 0.1872187) F_z$$

Torsion Moment at O due to Flange 4, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=4)} = \frac{[m X_2^{(i)} - \ell X_3^{(i)}] a}{6} (3 a t d_1^{F_y} + 2 a^2 t d_2^{F_y} + 6 q_{F_y}^{(i)}) = -0.24871451 - 9 q_{F_y}^{(4)} = 0.3225 - 9 q_{F_y}^{(1)}$$

$$\left(M_O^{F_z}\right)^{(e=4)} = \frac{[m X_2^{(i)} - \ell X_3^{(i)}] a}{6} (3 a t d_1^{F_z} + 2 a^2 t d_2^{F_z} + 6 q_{F_z}^{(i)}) = 0.167124 - 9 q_{F_z}^{(4)} = 0.18722 - 9 q_{F_z}^{(1)}$$

The Total Torsion Moment at O

$$\left(M_O^{F_y}\right)^{(Sys)} = \left(M_O^{F_y}\right)^{(e=1)} + \left(M_O^{F_y}\right)^{(e=2)} + \left(M_O^{F_y}\right)^{(e=3)} + \left(M_O^{F_y}\right)^{(e=4)} = -7.7453862 + 156 q_{F_y}^{(1)}$$

$$\left(M_O^{F_z}\right)^{(Sys)} = \left(M_O^{F_z}\right)^{(e=1)} + \left(M_O^{F_z}\right)^{(e=2)} + \left(M_O^{F_z}\right)^{(e=3)} + \left(M_O^{F_z}\right)^{(e=4)} = 13.9048382 + 156 q_{F_z}^{(1)}$$

$$M_O = M_O^{F_y} F_y + M_O^{F_z} F_z = -(X_3^{(S.C.)}) F_y + (X_2^{(S.C.)}) F_z \tag{6.322}$$

$$\Rightarrow M_O = (-7.7453862 + 156 q_{F_y}^{(1)}) F_y + (13.9048382 + 156 q_{F_z}^{(1)}) F_z = -(X_3^{(S.C.)}) F_y + (X_2^{(S.C.)}) F_z \tag{6.323}$$

The Total Force due to shear flux

$$(F_T)^{(Sys)} = f^{(e=3)} + f^{(e=3)} + f^{(e=3)} + f^{(e=3)}$$

$$= (-0.843064 + 38 q_{F_y}^{(1)}) F_y + (1.9386719 + 38 q_{F_z}^{(1)}) F_z$$

The Shear Center

Note that when the total force is at the shear center the torsion moment is zero, so

$$(-0.843064 + 38 q_{F_y}^{(1)}) F_y \times (-X_3^{(S.C.)}) = 0 \quad \Rightarrow \quad q_{F_y}^{(1)} = 0.02218589$$

$$(1.9386719 + 38 q_{F_z}^{(1)}) F_z \times (X_2^{(S.C.)}) = 0 \quad \Rightarrow \quad q_{F_z}^{(1)} = -0.05101768$$

then, the total moment becomes

$$M_O = (-7.7453862 + 156 q_{F_y}^{(1)}) F_y + (13.9048382 + 156 q_{F_z}^{(1)}) F_z = -(X_3^{(S.C.)}) F_y + (X_2^{(S.C.)}) F_z$$

$$\Rightarrow M_O = (-7.7453862 + 156 \times (0.02218589)) F_y + (13.9048382 + 156 \times (-0.05101768)) F_z$$

$$\Rightarrow M_O = -(4.2843866) F_y + (5.9460799) F_z = -(X_3^{(S.C.)}) F_y + (X_2^{(S.C.)}) F_z$$

with that we can conclude that the shear center is located at:

$$X_2^{(S.C.)} = 5.9460799 \quad ; \quad X_3^{(S.C.)} = 4.2843866$$

Shear Flux due to Bending Only ($q_{F_y}^{(1)} = 0.02218589; q_{F_z}^{(1)} = -0.05101768$ **)**

Flange 1

$$q_{F_y}^{(e=1)} = q_{F_y}^{(1)} + 0.009785s - 7.34994\times10^{-4}s^2 = 0.02218589 + 0.009785s - 7.34994\times10^{-4}s^2$$

$$q_{F_z}^{(e=1)} = q_{F_z}^{(1)} + 0.013372s - 3.03663\times10^{-4}s^2 = -0.05101768 + 0.013372s - 3.03663\times10^{-4}s^2$$

$$q^{(e=1)}(s) = q_{F_y}^{(e=1)}\underbrace{F_y}_{=0} + q_{F_z}^{(e=1)}F_z = -51.01768 + 13.372s - 0.303663s^2$$

at the end $s = 12$, (node 2)

$$q_{F_y}^{(e=1)} = q_{F_y}^{(1)} + 0.0115815 = 0.02218589 + 0.0115815 = 0.0337673 = q_{F_y}^{(2)}$$

$$q_{F_z}^{(e=1)} = q_{F_z}^{(1)} + 0.1167351 = -0.05101768 + 0.1167351 = 0.0657174 = q_{F_z}^{(2)}$$

The total force at the flange 1

$$f^{(e=1)} = (12q_{F_y}^{(1)} + 0.281167)F_y + (12q_{F_z}^{(1)} + 0.787866)F_z = (0.5473978)F_y + (0.1756534)F_z$$

Flange 2

$$q_{F_y}^{(e=2)}(s) = q_{F_y}^{(1)} + 0.0115815 - 0.0078548s - 3.0366259\times10^{-4}s^2$$
$$= 0.0337673 - 0.0078548s - 3.0366259\times10^{-4}s^2$$

$$q_{F_z}^{(e=2)}(s) = q_{F_z}^{(1)} + 0.1167351 + 0.00608398s - 0.0014582s^2$$
$$= 0.0657174 + 0.00608398s - 0.0014582s^2$$

$$q^{(e=2)}(s) = q_{F_y}^{(e=2)}\underbrace{F_y}_{=0} + q_{F_z}^{(e=2)}F_z = 65.7174 + 6.08398s - 1.4582s^2$$

at the end $s = 4$, (node 3)

$$q_{F_y}^{(e=2)}(s=4) = q_{F_y}^{(1)} - 0.0246964 = 0.02218589 - 0.0246964 = -2.5104\times10^{-3} = q_{F_y}^{(3)}$$

$$q_{F_z}^{(e=2)}(s=4) = q_{F_z}^{(1)} + 0.1177398 = (-0.05101768) + 0.1177398 = 0.0667222 = q_{F_z}^{(3)}$$

The total force at the flange 2

$$f^{(e=2)} = (4q_{F_y}^{(1)} - 0.02299078)F_y + (4q_{F_z}^{(1)} + 0.484504)F_z = (0.0657528)F_y + (0.2804333)F_z$$

Flange 3

$$q_{F_y}^{(e=3)}(s) = q_{F_y}^{(1)} - 0.0246964 - 0.010284s + 5.61663\times10^{-4}s^2$$
$$= -2.51048\times10^{-3} - 0.010284s + 5.61663\times10^{-4}s^2$$

$$q_{F_z}^{(e=3)}(s) = q_{F_z}^{(1)} + 0.1177398 - 0.005582s - 2.80542\times10^{-4}s^2$$
$$= 0.0667222 - 0.005582s - 2.80542\times10^{-4}s^2$$

at the end $s = 13$, (node 4)

$$q_{F_y}^{(e=3)}(s=13) = q_{F_y}^{(1)} - 0.0634688 = -0.0412829 = q_{F_y}^{(4)}$$

$$q_{F_z}^{(e=3)}(s=13) = q_{F_z}^{(1)} - 0.002232771 = -0.0532505 = q_{F_z}^{(4)}$$

The total force in the flange 3 (see **Problem 4.31**-NOTE 1)

$$f^{(e=3)} = (13q_{F_y}^{(1)} - 0.7787357)F_y + (13q_{F_z}^{(1)} + 0.853521)F_z = -0.4903191F_y + 0.1902911F_z$$

Flange 4

$$q_{F_y}^{(e=4)}(s) = q_{F_y}^{(1)} - 0.0634688 + 0.004319s + 3.03663\times10^{-4}s^2$$
$$= -0.0412829 + 0.004319s + 3.03663\times10^{-4}s^2$$
$$q_{F_z}^{(e=4)}(s) = q_{F_z}^{(1)} - 0.002232771 - 0.012876s + 0.001458s^2$$
$$= -0.0532505 - 0.012876s + 0.001458s^2$$

at the end $s = 9$, (node 1)

$$q_{F_y}^{(e=4)}(s=9) = (q_{F_y}^{(1)} - 0.0634688) + 0.0634688 = 0.0221859 = q_{F_y}^{(1)}$$

$$q_{F_z}^{(e=4)}(s=9) = (q_{F_z}^{(1)} - 0.002232771) + 0.002232771 = -0.0510177 = q_{F_z}^{(1)}$$

The total force in the flange 4 (see **Problem 4.31**-NOTE 1)

$$f^{(e=4)} = (9q_{F_y}^{(1)} - 0.3225045)F_y + (9q_{F_z}^{(1)} - 0.1872187)F_z = -0.1228315F_y - 0.6463778F_z$$

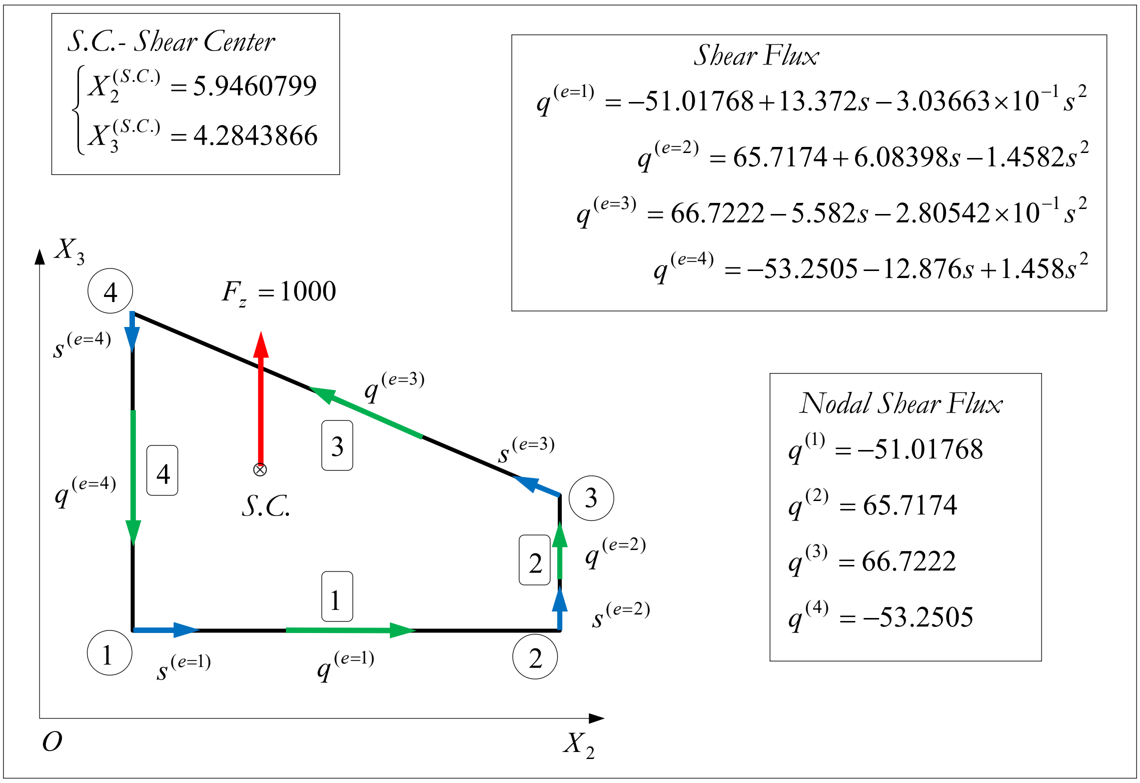


Figure 6.119: Shear flux due to F_z - Bending only.

Torsion only

The solution for this problem was already done, (see equation (6.318)):

$$\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \theta \end{Bmatrix} = M_T \begin{Bmatrix} \dfrac{1}{2At_1} \\ \dfrac{1}{2At_2} \\ \dfrac{1}{2At_3} \\ \dfrac{1}{2At_4} \\ \dfrac{1}{4A^2G}\left(\dfrac{s_1}{t_1}+\dfrac{s_2}{t_2}+\dfrac{s_3}{t_3}+\dfrac{s_4}{t_4}\right) \end{Bmatrix} \Rightarrow \begin{cases} \tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau = \dfrac{M_T}{2At} \\ \theta = \dfrac{s_T}{4A^2Gt} \end{cases}$$

where $M_T = -4946.08$, $A_{Cell} = \left(12\times 9 - \dfrac{12\times 5}{2}\right) = 78$, $t = 0.1$

$$\tau = \frac{M_T}{2A_{Cell}t} = -317.05641 \quad \Rightarrow \quad \tau t = q = \frac{M_T}{2A_{Cell}} = -31.705641$$

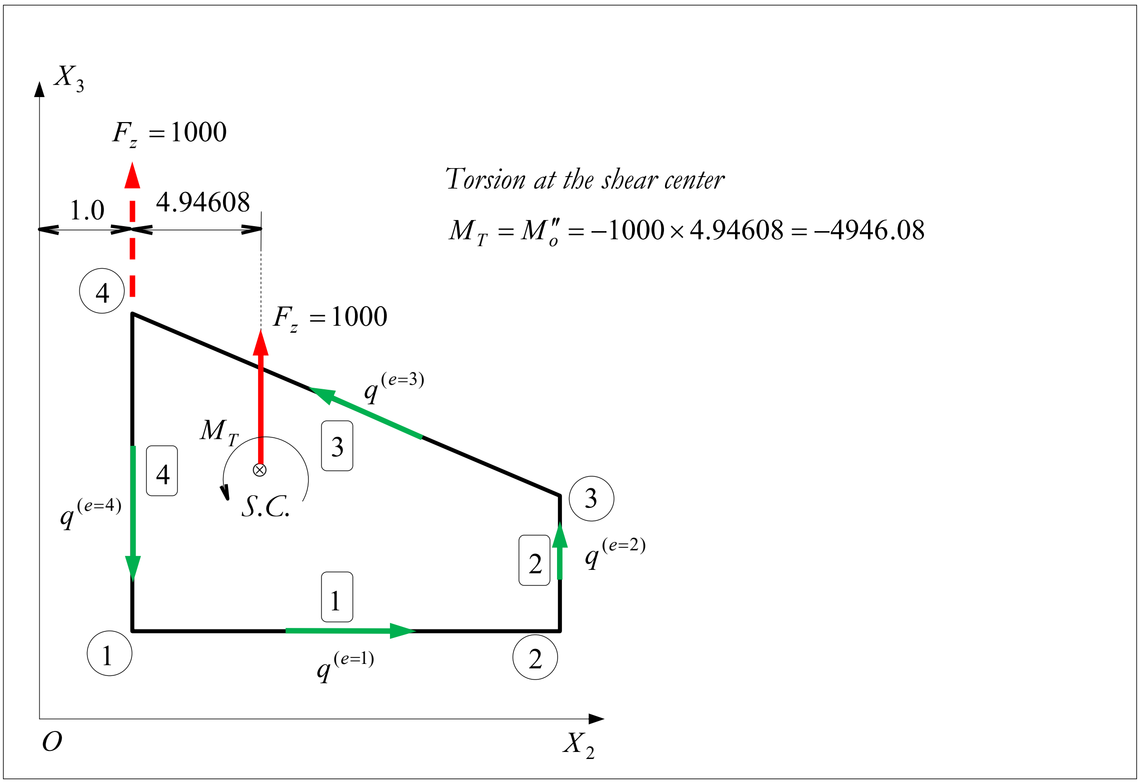


Figure 6.120

NOTE 1: Note that the total moment at the point O obtained in (6.323), $M_O^{F_z} F_z$, must be equal to the moment produced by the force $F_z = 1000$ at the same point, so

$$\begin{aligned} &M_O^{F_z} F_z = F_z \times 1 \quad \Rightarrow \quad (13.9048382 + 156 q_{F_z}^{(1)})F_z = F_z \times 1 \\ &\Rightarrow 13.9048382 + 156 q_{F_z}^{(1)} - 1 = 0 \quad \Rightarrow \quad q_{F_z}^{(1)} = -0.08272332 \end{aligned} \tag{6.324}$$

Note that we have found for bending only problem that $q_{F_z}^{(1)\_bending} = -0.05101768$ (bending), and for torsion only $q_{F_z}^{(1)\_torsion} = \frac{q}{F_z} = \frac{-31.705641}{1000} = -0.031705641$ (torsion), then

$$q_{F_z}^{(1)} = q_{F_z}^{(1)\_bending} + q_{F_z}^{(1)\_torsion} = -0.05101768 - 0.031705641 = -0.08272332$$

which matches the result in (6.324).

NOTE 2:

Just by considering the effect due to F_z we have obtained:

Flange 1

$$\boxed{q_{F_z}^{(1)} + 0.1167351 = q_{F_z}^{(2)}},\; f^{(e=1)} = (12q_{F_z}^{(1)} + 0.787866)F_z\,,\; \left(M_O^{F_z}\right)^{(e=1)} = -0.7878655 - 12q_{F_z}^{(1)}$$

Flange 2

$$\boxed{q_{F_z}^{(2)} + 0.00100472 = q_{F_z}^{(3)}},\; f^{(e=2)} = (4q_{F_z}^{(2)} + 0.017564)F_z\,,\; \left(M_O^{F_z}\right)^{(e=2)} = 0.228326 + 52q_{F_z}^{(2)}$$

Flange 3

$$\boxed{q_{F_z}^{(3)} - 0.1199726 = q_{F_z}^{(4)}},\; f^{(e=3)} = (13q_{F_z}^{(3)} - 0.6770968)F_z\,,\; \left(M_O^{F_z}\right)^{(e=3)} = -6.510547 + 125q_{F_z}^{(3)}$$

Flange 4

$q_{F_z}^{(1)} = q_{F_z}^{(1)}$ (redundant equation), $f^{(e=4)} = (9q_{F_z}^{(4)} - 0.167124)F_z$, $\left(M_O^{F_z}\right)^{(e=4)} = 0.167124 - 9q_{F_z}^{(4)}$

The total moment can also be represented as follows

$$\begin{aligned}
M_O^{F_z} F_z &= [(-0.7878655 - 12q_{F_z}^{(1)}) + (0.228326 + 52q_{F_z}^{(2)}) + (-6.510547 + 125q_{F_z}^{(3)}) \\
&\qquad + (0.167124 - 9q_{F_z}^{(4)})]F_z \\
&= [-12q_{F_z}^{(1)} + 52q_{F_z}^{(2)} + 125q_{F_z}^{(3)} - 9q_{F_z}^{(4)} - 6.90296222]F_z
\end{aligned}$$

which must be equal to $1 \times F_z$, (see Figure 6.118).

$$\boxed{[-12q_{F_z}^{(1)} + 52q_{F_z}^{(2)} + 125q_{F_z}^{(3)} - 9q_{F_z}^{(4)} - 6.90296222]F_z = 1 \times F_z}$$

Then, we can construct the following system:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -12 & 52 & 125 & -9 \end{bmatrix} \begin{Bmatrix} q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \end{Bmatrix} = \begin{Bmatrix} -0.1167351 \\ -0.00100472 \\ 0.1199726 \\ 6.90296222 + 1 \end{Bmatrix} \xrightarrow{Solve} \begin{Bmatrix} q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \end{Bmatrix} = \begin{Bmatrix} -0.082723322 \\ 0.034011797 \\ 0.035016513 \\ -0.084956093 \end{Bmatrix}$$

Bending Only

If the force F_z is applied at the shear center the total moment is zero, so

$$\begin{aligned}
& M_O^{F_z} F_z - X_2^{(S.C.)} F_z = 0 \\
& \Rightarrow -12q_{F_z}^{(1)} + 52q_{F_z}^{(2)} + 125q_{F_z}^{(3)} - 9q_{F_z}^{(4)} - X_2^{(S.C.)} = 6.90296222
\end{aligned}$$

The total force can be represented as follows

$$
\begin{aligned}
F_T^{F_z} &= [(12q_{F_z}^{(1)} + 0.787866) + (4q_{F_z}^{(2)} + 0.017564) + (13q_{F_z}^{(3)} - 0.6770968) + (9q_{F_z}^{(4)} - 0.167124)]F_z \\
&= [12q_{F_z}^{(1)} + 4q_{F_z}^{(2)} + 13q_{F_z}^{(3)} + 9q_{F_z}^{(4)} - 0.03879148]F_z
\end{aligned}
$$

And, it is also true that

$$
F_T^{F_z} \times X_2^{(S.C.)} = 0 \quad \Rightarrow \quad 12q_{F_z}^{(1)} + 4q_{F_z}^{(2)} + 13q_{F_z}^{(3)} + 9q_{F_z}^{(4)} = 0.03879148
$$

Note that condition is the same as the equation in (6.260) in which:

$$
\oint_\Gamma \tau^{(\Gamma)} \, d\Gamma = 2G\theta A \quad \Rightarrow \quad \oint_\Gamma \frac{q}{t} \, d\Gamma = 2G\theta A \quad \xrightarrow{t\text{ - constant}} \quad \oint_\Gamma q \, d\Gamma = 2G\theta A t
$$

And if the forces are applied at the shear center the angle of twist per unit length (or the torsion) is equal to zero, so

$$
\oint_\Gamma q \, d\Gamma = 2G\theta A t = \frac{M_T s}{\underbrace{2A}_{=0}} \quad \xrightarrow{\text{Shear Center } \theta=0} \quad \oint_\Gamma q \, d\Gamma = 0
$$

(see equation (6.304))

And the system to be solved is:

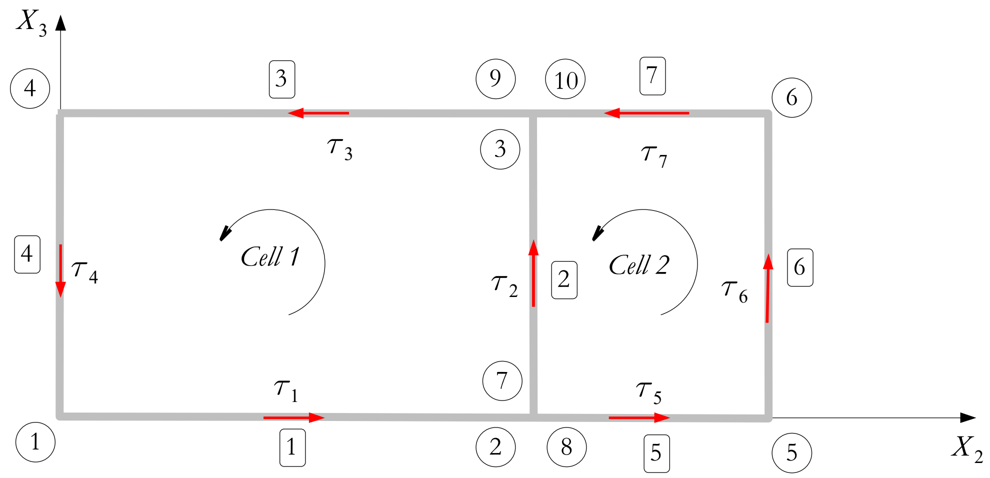
$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
-12 & 52 & 125 & -9 & -1 \\
12 & 4 & 13 & 9 & 0
\end{bmatrix}
\begin{Bmatrix}
q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \\ X_2^{(S.C.)}
\end{Bmatrix}
=
\begin{Bmatrix}
-0.1167351 \\ -0.00100472 \\ 0.1199726 \\ 6.90296222 \\ 0.03879148
\end{Bmatrix}
\xrightarrow{Solve}
\begin{Bmatrix}
q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \\ X_2^{(S.C.)}
\end{Bmatrix}
=
\begin{Bmatrix}
-0.051017682 \\ 0.065717437 \\ 0.066722153 \\ -0.053250453 \\ 5.946079856
\end{Bmatrix}
$$

Problem 6.54

Consider the cross section described in Figure 6.111 in which $a=2$ and $t=0.1$. Obtain the shear flux in each segment and the shear center of the cross section.

Solution:

In order to automatize the procedure we will create additional nodes at the point in which bifurcation occurs, (see Figure 6.121).



Nodes	Coordinates X_2	Coordinates X_3
1	0	0
2,7,8	$2a$	0
3,9,10	$2a$	a
4	0	a
5	$3a$	0
6	$3a$	a

Element e	*Connectivity* $i \to j$	*Thickness*
1	$1 \to 2$	$t_1 = t$
2	$7 \to 3$	$t_2 = t$
3	$9 \to 4$	$t_3 = t$
4	$4 \to 1$	$t_4 = t$
5	$8 \to 5$	$t_5 = t$
6	$5 \to 6$	$t_6 = t$
7	$6 \to 10$	$t_7 = t$

Figure 6.121

The flange geometric characteristics are described in Table 6.2.

Table 6.2

Flange	$(X_2^{(i)};X_3^{(i)})$	$(X_2^{(j)};X_3^{(j)})$	t	a	*Area* $A^{(e)}$	ℓ	m	$\overline{X}_2^{(ge)}$	$\overline{X}_3^{(ge)}$
$e=1$	$(0;0)$	$(4;0)$	0.1	4	0.4	1	0	2	0
$e=2$	$(4;0)$	$(4;2)$	0.1	2	0.2	1	0	4	1
$e=3$	$(4;2)$	$(2;2)$	0.1	4	0.4	-1	0	2	2
$e=4$	$(0;2)$	$(0;0)$	0.1	2	0.2	0	-1	0	1
$e=5$	$(4;0)$	$(6;0)$	0.1	2	0.2	1	0	5	0
$e=6$	$(6;0)$	$(6;2)$	0.1	2	0.2	0	1	6	1
$e=7$	$(6;2)$	$(4;2)$	0.1	2	0.2	-1	0	5	2

Total Area and Area Centroid of the Compound

The total area is $A = A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} + A^{(5)} + A^{(6)} + A^{(7)} = 1.8$

And the area centroid is given by

$$\overline{X}_2 = \frac{\sum_{e=1}^{7} A^{(e)} \overline{X}_2^{(ge)}}{A} = 3.1111111 \quad ; \quad \overline{X}_3 = \frac{\sum_{e=1}^{7} A^{(e)} \overline{X}_3^{(ge)}}{A} = 1.0$$

The Inertia Tensor of Area (for the flange e)

$$\mathrm{I}_{O\vec{X}\,ij}^{(e)} \approx \frac{at}{12}\begin{bmatrix} a^2 m^2 + 12(\overline{X}_3^{(g)})^2 & -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} \\ -\ell m a^2 - 12\overline{X}_2^{(g)}\overline{X}_3^{(g)} & a^2\ell^2 + 12(\overline{X}_2^{(g)})^2 \end{bmatrix} \tag{6.325}$$

The inertia tensors for the flanges related to the system $O\vec{\boldsymbol{X}}$ are

$$\mathrm{I}_{O\vec{X}\,ij}^{(1)} \approx \begin{bmatrix} 0 & 0 \\ 0 & 2.1333333 \end{bmatrix};\ \mathrm{I}_{O\vec{X}\,ij}^{(2)} \approx \begin{bmatrix} 0.2666667 & -0.8 \\ -0.8 & 3.2 \end{bmatrix};\ \mathrm{I}_{O\vec{X}\,ij}^{(3)} \approx \begin{bmatrix} 1.6 & -1.6 \\ -1.6 & 2.133333 \end{bmatrix};$$

$$\mathrm{I}_{O\vec{X}\,ij}^{(4)} \approx \begin{bmatrix} 0.2666667 & 0 \\ 0 & 0 \end{bmatrix};\ \mathrm{I}_{O\vec{X}\,ij}^{(5)} \approx \begin{bmatrix} 0 & 0 \\ 0 & 5.0666667 \end{bmatrix};\ \mathrm{I}_{O\vec{X}\,ij}^{(6)} \approx \begin{bmatrix} 0.2666667 & -1.2 \\ -1.2 & 7.2 \end{bmatrix};$$

$$\mathrm{I}_{O\vec{X}\,ij}^{(7)} \approx \begin{bmatrix} 0.8 & -2 \\ -2 & 5.0666667 \end{bmatrix}$$

Then, the inertia tensor for the compound is given by

$$\left(\mathbf{I}_{O\vec{X}}^{(Sys)}\right)_{ij} = \sum_{e=1}^{7} (\mathbf{I}_{O\vec{X}}^{(e)})_{ij} = \begin{bmatrix} 3.2 & -5.6 \\ -5.6 & 24.8 \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{23} & \mathrm{I}_{33} \end{bmatrix}$$

Just as exercise, let us calculate the inertia tensor at the Area Centroid, (see Chapter 4), which can be obtained by means of the Steiner's theorem:

$$\mathbf{I}_{O\vec{X}}^{(Sys)} = \overline{\mathbf{I}}_{G\vec{x}}^{(Sys)} - A[(\vec{\overline{\boldsymbol{X}}} \otimes \vec{\overline{\boldsymbol{X}}}) - (\vec{\overline{\boldsymbol{X}}} \cdot \vec{\overline{\boldsymbol{X}}})\mathbf{1}] \quad \Rightarrow \quad \overline{\mathbf{I}}_{G\vec{x}}^{(Sys)} = \mathbf{I}_{O\vec{X}}^{(Sys)} + A[(\vec{\overline{\boldsymbol{X}}} \otimes \vec{\overline{\boldsymbol{X}}}) - (\vec{\overline{\boldsymbol{X}}} \cdot \vec{\overline{\boldsymbol{X}}})\mathbf{1}]$$

whose components are:

$$(\overline{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \mathrm{I}_{O\vec{X}\ ij}^{(Sys)} - A\begin{bmatrix} \overline{X}_3^2 & -\overline{X}_2\overline{X}_3 \\ -\overline{X}_2\overline{X}_3 & \overline{X}_2^2 \end{bmatrix} \quad \Rightarrow \quad (\overline{\mathbf{I}}_{G\vec{x}}^{(Sys)})_{ij} = \begin{bmatrix} \overline{\mathrm{I}}_{G22} & \overline{\mathrm{I}}_{G23} \\ \overline{\mathrm{I}}_{G23} & \overline{\mathrm{I}}_{G33} \end{bmatrix} = \begin{bmatrix} 1.4 & 0 \\ 0 & 7.3777778 \end{bmatrix}$$

Shear Flux

By means of Figure 6.115 we can calculate the shear flux in the flanges. Taking into account the geometric properties calculated previously we can calculate the coefficients related to the cross section:

$p_0^{(1)} = -7.84$; $p_0^{(2)} = -13.28$; $p_2^{(1)} = 2.52$; $p_2^{(2)} = 2.487\times10^{-15}$; $p_3^{(1)} = 2.487\times10^{-15}$; $p_3^{(2)} = 13.28$, $\boldsymbol{\mathcal{X}} = 18.592$

Flange 1: (Flange data $(X_2^{(i)}, X_3^{(i)}, t, \ell, m)$ in Table 6.2)

$q_{F_y}^{(i)} = q_{F_y}^{(1)}$, $q_{F_z}^{(i)} = q_{F_z}^{(1)}$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = 0.4216867 \quad \Rightarrow \quad d_1^{F_y}t = 0.04216867$$

$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 0.7142857 \quad \Rightarrow \quad d_1^{F_z}t = 0.07142857$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -0.0677711 \quad \Rightarrow \quad d_2^{F_y}t = -0.00677711$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = -0 \quad \Rightarrow \quad d_2^{F_z}t = 0$$

Shear Flux in the Flange 1

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=1)}F_y + q_{F_z}^{(e=1)}F_z$$

The terms $q_{F_y}^{(e=1)}$ and $q_{F_z}^{(e=1)}$ can be evaluated as follows:

$$q_{F_y}^{(e=1)}(s) = q_{F_y}^{(1)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(1)} + 0.04216867s - 0.00677711s^2$$

$$q_{F_z}^{(e=1)}(s) = q_{F_z}^{(1)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(1)} + 0.07142857s$$

at the end $s = 4$, (node 2)

$$q_{F_y}^{(e=1)}(s=4) = q_{F_y}^{(1)} + 0.04216867s - 0.00677711s^2 = q_{F_y}^{(1)} + 0.060241 = q_{F_y}^{(2)}$$

$$q_{F_z}^{(e=1)}(s=4) = q_{F_z}^{(1)} + 0.07142857s = q_{F_z}^{(1)} + 0.2857143 = q_{F_z}^{(2)}$$

The total force in the flange 1, (see **Problem 4.31**-NOTE 1)

$$f^{(e=1)} = \left[q_{F_y}^{(i)}a + t\,d_1^{F_y}\frac{a}{2} + t\,d_2^{F_y}\frac{a^3}{3}\right]F_y + \left[q_{F_z}^{(i)}a + t\,d_1^{F_z}\frac{a}{2} + t\,d_2^{F_z}\frac{a^3}{3}\right]F_z$$
$$\Rightarrow f^{(e=1)} = (4q_{F_y}^{(1)} + 0.1927711)F_y + (4q_{F_z}^{(1)} + 0.5714286)F_z$$

Torsion Moment at O due to Flange 1, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

$$\left(M_O^{F_z}\right)^{(e=1)} = \frac{[m\,X_2^{(i)} - \ell\,X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Flange 2: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(7)},\; q_{F_z}^{(i)} = q_{F_z}^{(7)}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = -0.1204819 \quad \Rightarrow \quad d_1^{F_y}t = -0.01204819$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 0.7142857 \quad \Rightarrow \quad d_1^{F_z}t = 0.07142857$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -0 \quad \Rightarrow \quad d_2^{F_y}t = 0$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = -0.3571429 \quad \Rightarrow \quad d_2^{F_z}t = -0.03571429$$

Shear Flux in the Flange 2

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=2)}F_y + q_{F_z}^{(e=2)}F_z$$

The terms $q_{F_y}^{(e=2)}$ and $q_{F_z}^{(e=2)}$ can be evaluated as follows:

$$q_{F_y}^{(e=2)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(7)} - 0.01204819s$$

$$q_{F_z}^{(e=2)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(7)} + 0.07142857s - 0.03571429s^2$$

at the end $s = 2$, (node 3)

$$q_{F_y}^{(e=2)}(s=2) = q_{F_y}^{(7)} - 0.0240964 = q_{F_y}^{(3)}$$

$$q_{F_z}^{(e=2)}(s=2) = q_{F_z}^{(7)} + 0 = q_{F_z}^{(3)}$$

The total force in the flange 2, (see **Problem 4.31**-NOTE 1)

$$f^{(e=2)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3}\right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3}\right] F_z$$
$$= (2q_{F_y}^{(7)} - 0.02409639)F_y + (2q_{F_z}^{(7)} + 0.047619)F_z$$

Torsion Moment at O due to Flange 2, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=2)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = 8q_{F_y}^{(7)} - 0.0963855$$

$$\left(M_O^{F_z}\right)^{(e=2)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2 td_2^{F_z} + 6q_{F_z}^{(i)}) = 8q_{F_z}^{(7)} + 0.1904762$$

Flange 3: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.1)

$$q_{F_y}^{(i)} = q_{F_y}^{(9)},\ q_{F_z}^{(i)} = q_{F_z}^{(9)}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = -0.10284 \quad \Rightarrow \quad d_1^{F_y} t = -0.010284$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -0.71429 \quad \Rightarrow \quad d_1^{F_z} t = -0.071429$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 0.06777 \quad \Rightarrow \quad d_2^{F_y} t = 0.006777$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0 \quad \Rightarrow \quad d_2^{F_z} t = 0$$

Shear Flux in the Flange 3

$$q(s) = (q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2)F_y + (q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2)F_z = q_{F_y}^{(e=3)} F_y + q_{F_z}^{(e=3)} F_z$$

The terms $q_{F_y}^{(e=3)}$ and $q_{F_z}^{(e=3)}$ can be evaluated as follows:

$$q_{F_y}^{(e=3)}(s) = q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2 = q_{F_y}^{(9)} - 0.010284s + 0.006777s^2$$

$$q_{F_z}^{(e=3)}(s) = q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2 = q_{F_z}^{(9)} - 0.071429s$$

at the end $s = 4$, (node 4)

$$q_{F_y}^{(e=3)}(s=4) = q_{F_y}^{(9)} - 0.010284s + 0.006777s^2 = q_{F_y}^{(9)} + 0.060241 = q_{F_y}^{(4)}$$

$$q_{F_z}^{(e=3)}(s=4) = q_{F_z}^{(9)} - 0.071429s = q_{F_z}^{(9)} - 0.2857143 = q_{F_z}^{(4)}$$

The total force in the flange 3, (see **Problem 4.31**-NOTE 1)

$$f^{(e=3)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3}\right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3}\right] F_z$$
$$= (4q_{F_y}^{(9)} + 0.0481928)F_y + (4q_{F_z}^{(9)} - 0.5714286)F_z$$

Torsion Moment at O due to Flange 3, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=3)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2 td_2^{F_y} + 6q_{F_y}^{(i)}) = 8q_{F_y}^{(9)} + 0.09639$$

$$\left(M_O^{F_z}\right)^{(e=3)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 8q_{F_z}^{(9)} - 1.14286$$

Flange 4: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(4)},\ q_{F_z}^{(i)} = q_{F_z}^{(4)}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = 0.42169 \quad \Rightarrow \quad d_1^{F_y}t = 0.042169$$

$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = -0.71429 \quad \Rightarrow \quad d_1^{F_z}t = -0.071429$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 0 \quad \Rightarrow \quad d_2^{F_y}t = 0$$

$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0.35714 \quad \Rightarrow \quad d_2^{F_z}t = 0.035714$$

Shear Flux in the Flange 4

$$q(s) = (q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2)F_y + (q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2)F_z = q_{F_y}^{(e=4)}F_y + q_{F_z}^{(e=4)}F_z$$

The terms $q_{F_y}^{(e=4)}$ and $q_{F_z}^{(e=4)}$ can be evaluated as follows:

$$q_{F_y}^{(e=4)}(s) = q_{F_y}^{(i)} + t\,d_1^{F_y}s + t\,d_2^{F_y}s^2 = q_{F_y}^{(4)} + 0.042169s$$

$$q_{F_z}^{(e=4)}(s) = q_{F_z}^{(i)} + t\,d_1^{F_z}s + t\,d_2^{F_z}s^2 = q_{F_z}^{(4)} - 0.071429s + 0.035714s^2$$

at the end $s = 2$, (node 1)

$$q_{F_y}^{(e=4)}(s=2) = q_{F_y}^{(4)} + 0.042169s = q_{F_y}^{(4)} + 0.0843373 = q_{F_y}^{(1)}$$

$$q_{F_z}^{(e=4)}(s=2) = q_{F_z}^{(4)} - 0.071429s + 0.035714s^2 = q_{F_z}^{(4)} + 0 = q_{F_z}^{(1)}$$

The total force in the flange 4, (see **Problem 4.31**-NOTE 1)

$$f^{(e=4)} = \left[q_{F_y}^{(i)}a + t\,d_1^{F_y}\frac{a}{2} + t\,d_2^{F_y}\frac{a^3}{3}\right]F_y + \left[q_{F_z}^{(i)}a + t\,d_1^{F_z}\frac{a}{2} + t\,d_2^{F_z}\frac{a^3}{3}\right]F_z$$
$$= (2q_{F_y}^{(4)} + 0.0843373)F_y + (2q_{F_z}^{(4)} - 0.047619)F_z$$

Torsion Moment at O due to Flange 4, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=4)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

$$\left(M_O^{F_z}\right)^{(e=4)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Flange 5: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(8)},\ q_{F_z}^{(i)} = q_{F_z}^{(8)}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)}X_2^{(i)} + p_3^{(1)}X_3^{(i)}\right] = -0.12048 \quad \Rightarrow \quad d_1^{F_y}t = -0.012048$$

$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)}X_2^{(i)} + p_3^{(2)}X_3^{(i)}\right] = 0.71429 \quad \Rightarrow \quad d_1^{F_z}t = 0.071429$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = -0.0677711 \quad \Rightarrow \quad d_2^{F_y}t = -0.00677711$$

$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0 \quad \Rightarrow \quad d_2^{F_z}t = 0$$

Shear Flux in the Flange 5

$$q(s) = (q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2) F_y + (q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2) F_z = q_{F_y}^{(e=5)} F_y + q_{F_z}^{(e=5)} F_z$$

The terms $q_{F_y}^{(e=5)}$ and $q_{F_z}^{(e=5)}$ can be evaluated as follows:

$$q_{F_y}^{(e=5)}(s) = q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2 = q_{F_y}^{(8)} - 0.012048s - 0.00677711s^2$$

$$q_{F_z}^{(e=5)}(s) = q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2 = q_{F_z}^{(8)} + 0.071429s$$

at the end $s = 2$, (node 5)

$$q_{F_y}^{(e=5)}(s=2) = q_{F_y}^{(8)} - 0.012048s - 0.00677711s^2 = q_{F_y}^{(8)} - 0.0512048 = q_{F_y}^{(5)}$$

$$q_{F_z}^{(e=5)}(s=2) = q_{F_z}^{(8)} + 0.071429s = q_{F_z}^{(8)} + 0.1428571 = q_{F_z}^{(5)}$$

The total force in the flange 5, (see **Problem 4.31**-NOTE 1)

$$f^{(e=5)} = \left[q_{F_y}^{(i)} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q_{F_z}^{(i)} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z$$
$$= (2q_{F_y}^{(8)} - 0.0421687) F_y + (2q_{F_z}^{(8)} + 0.14285771) F_z$$

Torsion Moment at O due to Flange 5, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y} \right)^{(e=5)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6} (3atd_1^{F_y} + 2a^2 t d_2^{F_y} + 6q_{F_y}^{(i)}) = 0$$

$$\left(M_O^{F_z} \right)^{(e=5)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6} (3atd_1^{F_z} + 2a^2 t d_2^{F_z} + 6q_{F_z}^{(i)}) = 0$$

Flange 6: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.2)

$$q_{F_y}^{(i)} = q_{F_y}^{(5)}\,,\; q_{F_z}^{(i)} = q_{F_z}^{(5)}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1} \left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)} \right] = -0.3915663 \quad \Rightarrow \quad d_1^{F_y} t = -0.03915663$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1} \left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)} \right] = 0.7142857 \quad \Rightarrow \quad d_1^{F_z} t = 0.07142857$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1} \left[p_2^{(1)}\, \ell + p_3^{(1)}\, m \right] = 0 \quad \Rightarrow \quad d_2^{F_y} t = 0$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1} \left[p_2^{(2)}\, \ell + p_3^{(2)}\, m \right] = -0.3571429 \quad \Rightarrow \quad d_2^{F_z} t = -0.03571429$$

Shear Flux in the Flange 6

$$q(s) = (q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2) F_y + (q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2) F_z = q_{F_y}^{(e=6)} F_y + q_{F_z}^{(e=6)} F_z$$

The terms $q_{F_y}^{(e=6)}$ and $q_{F_z}^{(e=6)}$ can be evaluated as follows:

$$q_{F_y}^{(e=6)}(s) = q_{F_y}^{(i)} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2 = q_{F_y}^{(5)} - 0.03915663s$$

$$q_{F_z}^{(e=6)}(s) = q_{F_z}^{(i)} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2 = q_{F_z}^{(5)} + 0.07142857s - 0.3571429s^2$$

at the end $s = 2$, (node 6)

$$q_{F_y}^{(e=6)}(s=2) = q_{F_y}^{(5)} - 0.03915663s = q_{F_y}^{(5)} - 0.0783133 = q_{F_y}^{(6)}$$

$$q_{F_z}^{(e=6)}(s=2) = q_{F_z}^{(5)} + 0.07142857s - 0.3571429s^2 = q_{F_z}^{(5)} + 0 = q_{F_z}^{(6)}$$

The total force in the flange 6, (see **Problem 4.31**-NOTE 1)

$$f^{(e=6)} = \left[q^{(i)}_{F_y} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q^{(i)}_{F_z} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z$$
$$= (2q^{(5)}_{F_y} - 0.0783133) F_y + (2q^{(5)}_{F_z} + 0.047619) F_z$$

Torsion Moment at O due to Flange 6, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=6)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q^{(i)}_{F_y}) = 12q^{(5)}_{F_y} - 0.4698795$$

$$\left(M_O^{F_z}\right)^{(e=6)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q^{(i)}_{F_z}) = 12q^{(5)}_{F_z} + 0.2857143$$

Flange 7: (Flange data ($X_2^{(i)}, X_3^{(i)}, t, \ell, m$) in Table 6.2)

$$q^{(i)}_{F_y} = q^{(6)}_{F_y}\,,\; q^{(i)}_{F_z} = q^{(6)}_{F_z}$$

$$d_1^{F_y} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(1)} + p_2^{(1)} X_2^{(i)} + p_3^{(1)} X_3^{(i)}\right] = -0.3915663 \quad \Rightarrow \quad d_1^{F_y} t = -0.03915663$$
$$d_1^{F_z} = -\boldsymbol{\mathcal{X}}^{-1}\left[p_0^{(2)} + p_2^{(2)} X_2^{(i)} + p_3^{(2)} X_3^{(i)}\right] = -0.7142857 \quad \Rightarrow \quad d_1^{F_z} t = -0.07142857$$

$$d_2^{F_y} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(1)}\,\ell + p_3^{(1)}\,m\right] = 0.0677711 \quad \Rightarrow \quad d_2^{F_y} t = 0.00677711$$
$$d_2^{F_z} = -(2\boldsymbol{\mathcal{X}})^{-1}\left[p_2^{(2)}\,\ell + p_3^{(2)}\,m\right] = 0 \quad \Rightarrow \quad d_2^{F_z} t = 0$$

Shear Flux in the Flange 7

$$q(s) = (q^{(i)}_{F_y} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2) F_y + (q^{(i)}_{F_z} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2) F_z = q^{(e=7)}_{F_y} F_y + q^{(e=7)}_{F_z} F_z$$

The terms $q^{(e=7)}_{F_y}$ and $q^{(e=7)}_{F_z}$ can be evaluated as follows:

$$q^{(e=7)}_{F_y}(s) = q^{(i)}_{F_y} + t\, d_1^{F_y} s + t\, d_2^{F_y} s^2 = q^{(6)}_{F_y} - 0.03915663s + 0.00677711s^2$$

$$q^{(e=7)}_{F_z}(s) = q^{(i)}_{F_z} + t\, d_1^{F_z} s + t\, d_2^{F_z} s^2 = q^{(6)}_{F_z} - 0.07142857s$$

at the end $s = 2$, (node 10)

$$q^{(e=7)}_{F_y}(s=2) = q^{(6)}_{F_y} - 0.03915663s + 0.00677711s^2 = q^{(6)}_{F_y} - 0.0512048 = q^{(10)}_{F_y}$$

$$q^{(e=7)}_{F_z}(s=2) = q^{(6)}_{F_z} - 0.07142857s = q^{(6)}_{F_z} - 0.1428571 = q^{(10)}_{F_z}$$

The total force in the flange 7, (see **Problem 4.31**-NOTE 1)

$$f^{(e=7)} = \left[q^{(i)}_{F_y} a + t\, d_1^{F_y} \frac{a}{2} + t\, d_2^{F_y} \frac{a^3}{3} \right] F_y + \left[q^{(i)}_{F_z} a + t\, d_1^{F_z} \frac{a}{2} + t\, d_2^{F_z} \frac{a^3}{3} \right] F_z$$
$$= (2q^{(6)}_{F_y} - 0.060241) F_y + (2q^{(6)}_{F_z} - 0.1428571) F_z$$

Torsion Moment at O due to Flange 7, (see **Problem 4.31**-NOTE 1):

$$\left(M_O^{F_y}\right)^{(e=7)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_y} + 2a^2td_2^{F_y} + 6q^{(i)}_{F_y}) = 4q^{(6)}_{F_y} - 0.1204819$$

$$\left(M_O^{F_z}\right)^{(e=7)} = \frac{[m\, X_2^{(i)} - \ell\, X_3^{(i)}]a}{6}(3atd_1^{F_z} + 2a^2td_2^{F_z} + 6q^{(i)}_{F_z}) = 4q^{(6)}_{F_z} - 0.2857143$$

The Total Torsion Moment at O

$$\left(M_O^{F_y}\right)^{(Sys)} = \sum_{e=1}^{7}\left(M_O^{F_y}\right)^{(e)}$$
$$= (0) + (8q_{F_y}^{(7)} - 0.0963855) + (8q_{F_y}^{(9)} + 0.09639) + (0) + (0) + (12q_{F_y}^{(5)} - 0.4698795) + (4q_{F_y}^{(6)} - 0.1204819)$$

$$\left(M_O^{F_z}\right)^{(Sys)} = \sum_{e=1}^{7}\left(M_O^{F_z}\right)^{(e)}$$
$$= (0) + (8q_{F_z}^{(7)} + 0.1904762) + (8q_{F_z}^{(9)} - 1.14286) + (0) + (0) + (12q_{F_z}^{(5)} + 0.2857143) + (4q_{F_z}^{(6)} - 0.2857143)$$

By restructuring the above two equations we can obtain

$$\left(M_O^{F_y}\right)^{(Sys)} = 12q_{F_y}^{(5)} + 4q_{F_y}^{(6)} + 8q_{F_y}^{(9)} + 8q_{F_y}^{(7)} - 0.5903614$$

$$\left(M_O^{F_z}\right)^{(Sys)} = 12q_{F_z}^{(5)} + 4q_{F_z}^{(6)} + 8q_{F_z}^{(7)} + 8q_{F_z}^{(9)} - 0.952381$$

$$M_O = M_O^{F_y} F_y + M_O^{F_z} F_z = -(X_3^{(S.C.)})F_y + (X_2^{(S.C.)})F_z \tag{6.326}$$

Solution due to F_y

For each flange we have obtained the following relationships between nodal shear fluxes:

$$q_{F_y}^{(1)} + 0.060241 = q_{F_y}^{(2)} \quad \Rightarrow \quad -q_{F_y}^{(1)} + q_{F_y}^{(2)} = 0.060241 \;\; \text{(flange 1)} \tag{6.327}$$

$$q_{F_y}^{(7)} - 0.0240964 = q_{F_y}^{(3)} \quad \Rightarrow \quad q_{F_y}^{(3)} - q_{F_y}^{(7)} = -0.0240964 \;\; \text{(flange 2)} \tag{6.328}$$

$$q_{F_y}^{(9)} + 0.060241 = q_{F_y}^{(4)} \quad \Rightarrow \quad q_{F_y}^{(4)} - q_{F_y}^{(9)} = 0.060241 \;\; \text{(flange 3)} \tag{6.329}$$

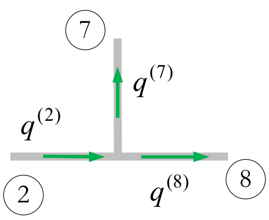
$$q_{F_y}^{(8)} - 0.0512048 = q_{F_y}^{(5)} \quad \Rightarrow \quad q_{F_y}^{(5)} - q_{F_y}^{(8)} = -0.0512048 \;\; \text{(flange 5)} \tag{6.330}$$

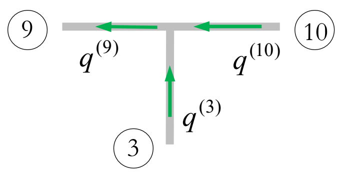
$$q_{F_y}^{(5)} - 0.0783133 = q_{F_y}^{(6)} \quad \Rightarrow \quad q_{F_y}^{(6)} - q_{F_y}^{(5)} = -0.0783133 \;\; \text{(flange 6)} \tag{6.331}$$

$$q_{F_y}^{(6)} - 0.0512048 = q_{F_y}^{(10)} \quad \Rightarrow \quad q_{F_y}^{(10)} - q_{F_y}^{(6)} = -0.0512048 \;\; \text{(flange 7)} \tag{6.332}$$

For the two bifurcation points the following must be true

Shear flux compatibility at nodes 2,7,8 | *Shear flux compatibility at nodes 3,9,10*





$$q^{(7)} + q^{(8)} = q^{(2)} \qquad \Big| \qquad q^{(3)} + q^{(10)} = q^{(9)} \tag{6.333}$$

Another equation we can add to the system is the total torsion moment obtained previously, and since we are searching for the shear center, the following must be true

$$M_O^{F_y} F_y - (-X_3^{(S.C.)})F_y = 0 \quad \Rightarrow \quad 12q_{F_y}^{(5)} + 4q_{F_y}^{(6)} + 8q_{F_y}^{(9)} + 8q_{F_y}^{(7)} - (-X_3^{(S.C.)}) = 0.5903614 \tag{6.334}$$

Up to now we have 9 equations and 11 unknowns (10 for shear fluxes and 1 for $X_3^{(S.C.)}$).

The two missing equations can be added by considering the following equation for each cell:

$$\oint_{\Gamma} q\, d\Gamma = 2G\theta A t = \underbrace{\frac{M_T s}{2A}}_{=0} \xrightarrow{\text{Shear Center } \theta=0} \oint_{\Gamma} q\, d\Gamma = 0$$

Note that we have obtained previously the forces for each flange, (see Figure 6.122).

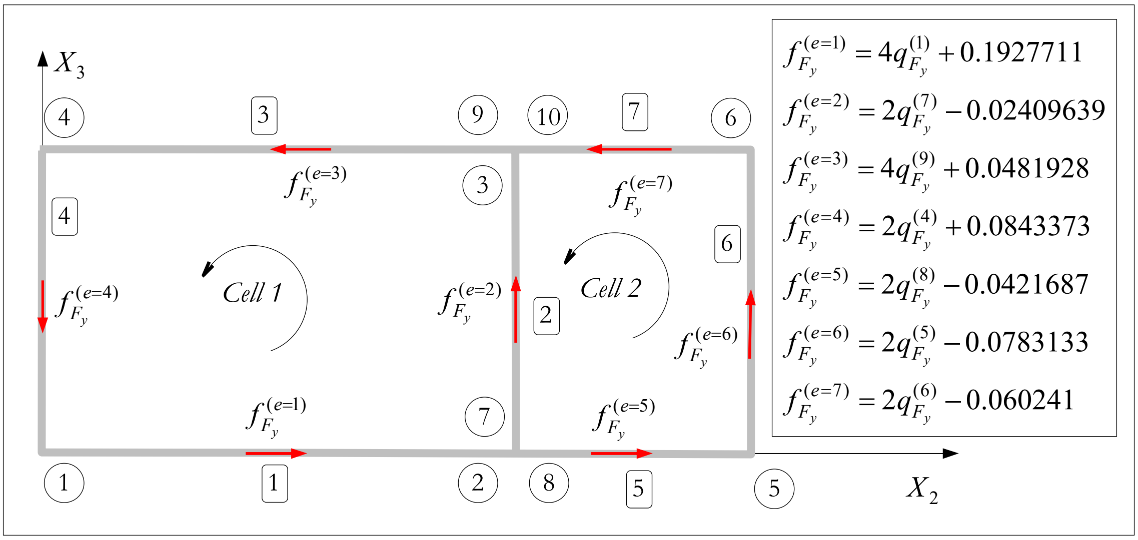


Figure 6.122: Forces on the flanges due to F_y.

Cell 1 ($f_{F_y}^{(e=1)} + f_{F_y}^{(e=2)} + f_{F_y}^{(e=3)} + f_{F_y}^{(e=4)} = 0$)

$$(4q_{F_y}^{(1)} + 0.1927711) + (2q_{F_y}^{(7)} - 0.02409639) + (4q_{F_y}^{(9)} + 0.0481928) + (2q_{F_y}^{(4)} + 0.0843373) = 0$$

$$\Rightarrow 4q_{F_y}^{(1)} + 2q_{F_y}^{(4)} + 2q_{F_y}^{(7)} + 4q_{F_y}^{(9)} = -0.30120482 \qquad (6.335)$$

Cell 2 ($f_{F_y}^{(e=5)} + f_{F_y}^{(e=6)} + f_{F_y}^{(e=7)} - f_{F_y}^{(e=2)} = 0$)

$$(2q_{F_y}^{(8)} - 0.0421687) + (2q_{F_y}^{(5)} - 0.0783133) + (2q_{F_y}^{(6)} - 0.060241) - (2q_{F_y}^{(7)} - 0.02409639) = 0$$

$$\Rightarrow 2q_{F_y}^{(5)} + 2q_{F_y}^{(6)} - 2q_{F_y}^{(7)} + 2q_{F_y}^{(8)} = 0.15662651 \qquad (6.336)$$

Then, by restructuring the equations (6.327)-(6.336) in matrix form we can obtain

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 12 & 4 & 8 & 0 & 8 & 0 & -1 \\ 4 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_{F_y}^{(1)} \\ q_{F_y}^{(2)} \\ q_{F_y}^{(3)} \\ q_{F_y}^{(4)} \\ q_{F_y}^{(5)} \\ q_{F_y}^{(6)} \\ q_{F_y}^{(7)} \\ q_{F_y}^{(8)} \\ q_{F_y}^{(9)} \\ q_{F_y}^{(10)} \\ -X_3^{(S.C.)} \end{Bmatrix} = \begin{Bmatrix} 0.06024096 \\ -0.02409639 \\ 0.06024096 \\ -0.05120482 \\ -0.07831325 \\ -0.05120482 \\ 0 \\ 0 \\ 0.59036145 \\ -0.30120482 \\ 0.15662651 \end{Bmatrix} \xrightarrow{\text{Solve}}$$

And the solution (nodal shear flux + shear center) is

$$\begin{Bmatrix} q_{F_y}^{(1)} \\ q_{F_y}^{(2)} \\ q_{F_y}^{(3)} \\ q_{F_y}^{(4)} \\ q_{F_y}^{(5)} \\ q_{F_y}^{(6)} \\ q_{F_y}^{(7)} \\ q_{F_y}^{(8)} \\ q_{F_y}^{(9)} \\ q_{F_y}^{(10)} \\ -X_3^{(S.C.)} \end{Bmatrix} = \begin{Bmatrix} 0.04216867 \\ 0.10240964 \\ -0.01204819 \\ -0.04216867 \\ 0.03915663 \\ -0.03915663 \\ 0.01204819 \\ 0.09036145 \\ -0.10240964 \\ -0.09036145 \\ -1.0 \end{Bmatrix}$$

Solution due to F_z

For each flange we have obtained the following relationships between nodal shear fluxes:

$$q_{F_z}^{(1)} + 0.2857143 = q_{F_z}^{(2)} \quad \Rightarrow \quad -q_{F_z}^{(1)} + q_{F_z}^{(2)} = 0.2857143 \text{ (flange 1)} \tag{6.337}$$

$$q_{F_z}^{(7)} = q_{F_z}^{(3)} \quad \Rightarrow \quad q_{F_z}^{(3)} - q_{F_z}^{(7)} = 0 \text{ (flange 2)} \tag{6.338}$$

$$q_{F_z}^{(9)} - 0.2857143 = q_{F_z}^{(4)} \quad \Rightarrow \quad q_{F_z}^{(4)} - q_{F_z}^{(9)} = -0.2857143 \text{ (flange 3)} \tag{6.339}$$

$$q_{F_z}^{(8)} + 0.1428571 = q_{F_z}^{(5)} \quad \Rightarrow \quad q_{F_z}^{(5)} - q_{F_z}^{(8)} = 0.1428571 \text{ (flange 5)} \tag{6.340}$$

$$q_{F_z}^{(5)} = q_{F_z}^{(6)} \quad \Rightarrow \quad q_{F_z}^{(6)} - q_{F_z}^{(5)} = 0 \text{ (flange 6)} \tag{6.341}$$

$$q_{F_z}^{(6)} - 0.1428571 = q_{F_z}^{(10)} \quad \Rightarrow \quad q_{F_z}^{(10)} - q_{F_z}^{(6)} = -0.1428571 \text{ (flange 7)} \tag{6.342}$$

For the two bifurcation points the following must be true

$$q_{F_z}^{(7)} + q_{F_z}^{(8)} = q_{F_z}^{(2)} \qquad \Big| \qquad q_{F_z}^{(3)} + q_{F_z}^{(10)} = q_{F_z}^{(9)} \tag{6.343}$$

Another equation due to the total torsion moment obtained previously, (see equation (6.326)), and since we are searching for the shear center, the following must be true

$$M_O^{F_z} F_z - X_2^{(S.C.)} F_z = 0 \quad \Rightarrow \quad 12q_{F_z}^{(5)} + 4q_{F_z}^{(6)} + 8q_{F_z}^{(7)} + 8q_{F_z}^{(9)} - X_2^{(S.C.)} = 0.952381 \tag{6.344}$$

Up to now we have 9 equations and 11 unknowns (10 for shear fluxes and 1 for $X_2^{(S.C.)}$).

The two missing equations can be added by considering the following equation for each cell:

$$\oint_\Gamma q \, d\Gamma = 2G\theta A t = \underbrace{\frac{M_T s}{2A}}_{=0} \xrightarrow{\text{Shear Center } \theta=0} \oint_\Gamma q \, d\Gamma = 0$$

Note that we have obtained previously the forces for each flange, (see Figure 6.123).

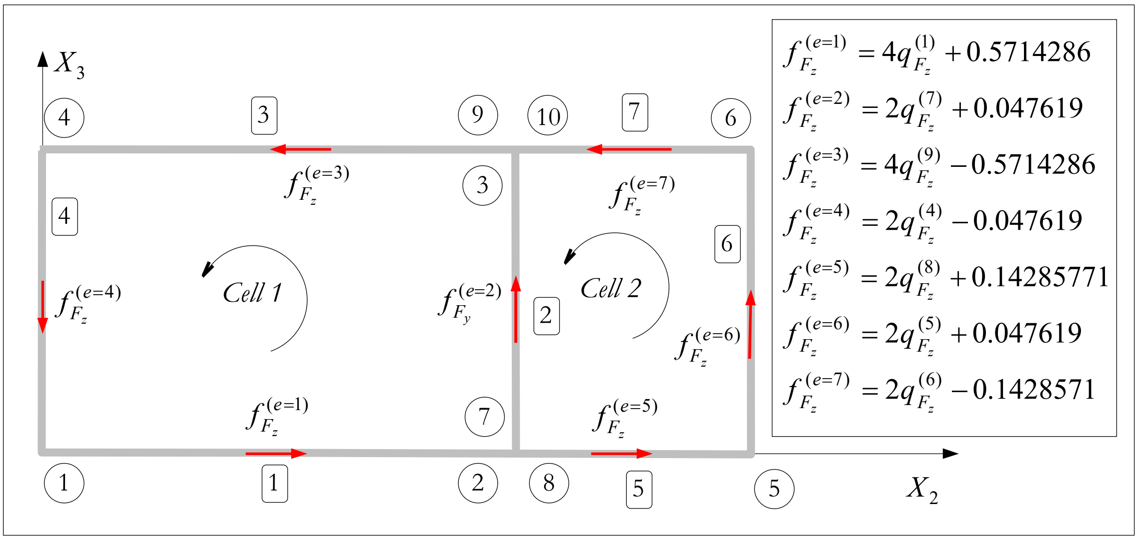


Figure 6.123: Forces on the flanges due to F_z.

Cell 1 ($f_{F_z}^{(e=1)} + f_{F_z}^{(e=2)} + f_{F_z}^{(e=3)} + f_{F_z}^{(e=4)} = 0$)

$$(4q_{F_z}^{(1)} + 0.5714286) + (2q_{F_z}^{(7)} + 0.047619) + (4q_{F_z}^{(9)} - 0.5714286) + (2q_{F_z}^{(4)} - 0.047619) = 0$$

$$\Rightarrow 4q_{F_z}^{(1)} + 2q_{F_z}^{(4)} + 2q_{F_z}^{(7)} + 4q_{F_z}^{(9)} = 0 \tag{6.345}$$

Cell 2 ($f_{F_z}^{(e=5)} + f_{F_z}^{(e=6)} + f_{F_z}^{(e=7)} - f_{F_z}^{(e=2)} = 0$)

$$(2q_{F_z}^{(8)} + 0.14285771) + (2q_{F_z}^{(5)} + 0.047619) + (2q_{F_z}^{(6)} - 0.1428571) - (2q_{F_z}^{(7)} + 0.047619) = 0$$

$$\Rightarrow 2q_{F_z}^{(5)} + 2q_{F_z}^{(6)} - 2q_{F_z}^{(7)} + 2q_{F_z}^{(8)} = 0 \tag{6.346}$$

Then, by restructuring the equations (6.337)-(6.346) in matrix form we can obtain

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 12 & 4 & 8 & 0 & 8 & 0 & -1 \\ 4 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \\ q_{F_z}^{(5)} \\ q_{F_z}^{(6)} \\ q_{F_z}^{(7)} \\ q_{F_z}^{(8)} \\ q_{F_z}^{(9)} \\ q_{F_z}^{(10)} \\ X_2^{(S.C.)} \end{Bmatrix} = \begin{Bmatrix} 0.28571429 \\ 0 \\ -0.28571429 \\ 0.14285714 \\ 0 \\ -0.14285714 \\ 0 \\ 0 \\ 0.95238095 \\ 0 \\ 0 \end{Bmatrix} \xrightarrow{Solve}$$

And the solution (nodal shear flux + shear center) is

$$\begin{Bmatrix} q_{F_z}^{(1)} \\ q_{F_z}^{(2)} \\ q_{F_z}^{(3)} \\ q_{F_z}^{(4)} \\ q_{F_z}^{(5)} \\ q_{F_z}^{(6)} \\ q_{F_z}^{(7)} \\ q_{F_z}^{(8)} \\ q_{F_z}^{(9)} \\ q_{F_z}^{(10)} \\ X_2^{(S.C.)} \end{Bmatrix} = \begin{Bmatrix} -0.14906832 \\ 0.13664596 \\ 0.17391304 \\ -0.14906832 \\ 0.10559006 \\ 0.10559006 \\ 0.17391304 \\ -0.03726708 \\ 0.13664596 \\ -0.03726708 \\ 3.22153209 \end{Bmatrix}$$

NOTE: For a given F_y and F_z located at $(X_2^{(F)}, X_3^{(F)})$, (see Figure 6.124), the torsion moment is given by:

$$M_T = -F_y(X_3^{(F)} - X_3^{(S.C.)}) + F_z(X_2^{(F)} - X_2^{(S.C.)})$$

And we can superimpose the torsion only effect, (see **Problem 6.52**).

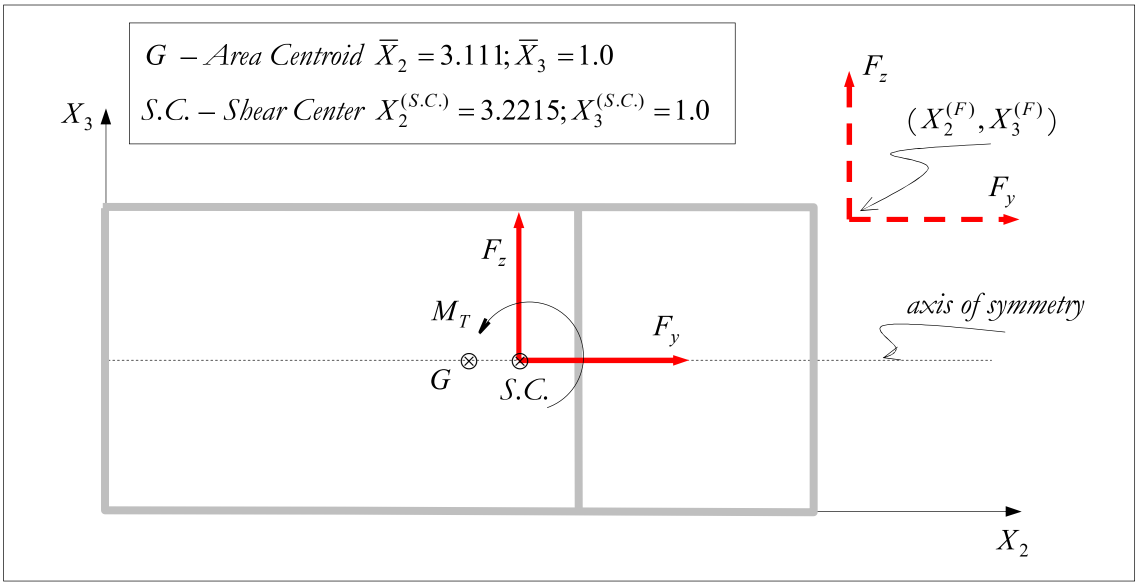


Figure 6.124

6.4.3 The Total Potential Energy for Torsion Problem

In Chapter 5, (see **Problem 5.25**), we have defined the total potential energy Π defined as follows:

$$\boxed{\Pi(\vec{\mathbf{u}}) = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV - \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS - \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \quad \textit{The total potential energy} \tag{6.347}$$

where

$$\boxed{U^{int} = \int_V \Psi^e(\boldsymbol{\varepsilon})\, dV = \int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV} \quad \textit{The internal potential energy} \tag{6.348}$$

and

$$\boxed{U^{ext} = \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS + \int_V (\rho\vec{\mathbf{b}}) \cdot \vec{\mathbf{u}} dV} \quad \textit{The external potential energy} \tag{6.349}$$

For the torsion problem the stress state is only given by σ_{12} and σ_{13}, (see equation (6.229)), then the internal potential energy becomes:

$$U^{int} = \int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV = \int_V \frac{1}{2} \sigma_{ij}\varepsilon_{ij}\, dV = \int_V \frac{1}{2}(2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13})\, dV = \int_V \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2)\, dV \tag{6.350}$$

where we have used the definition $\sigma_{12} = G2\varepsilon_{12} = G\gamma_{xy}$ and $\sigma_{13} = G2\varepsilon_{13} = G\gamma_{xz}$. Recall that

$$\sigma_{12} = \frac{\partial\phi}{\partial x_3} \quad ; \quad \sigma_{13} = -\frac{\partial\phi}{\partial x_2} \tag{6.351}$$

where $\phi = \phi(x_2, x_3)$ is the Prandtl's stress function, (see equation (6.244)). Then the internal potential energy can be rewritten as follows:

$$U^{int} = \int_V \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2)\, dV = \int_V \frac{1}{2G}\left[\left(\frac{\partial\phi}{\partial x_3}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right] dV = \int_V \frac{1}{2G}(\phi_{,k}\phi_{,k})\, dV \tag{6.352}$$

For a position x_1 the above equation becomes:

$$U^{int} = \int_V \frac{1}{2G}\left[\left(\frac{\partial\phi}{\partial x_3}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right] dV = x_1 \int_A \frac{1}{2G}\left[\left(\frac{\partial\phi}{\partial x_3}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right] dA \tag{6.353}$$

The external potential energy without body forces becomes:

$$U^{ext} = \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{\mathbf{u}} dS = \int_{S_\sigma} \mathrm{t}_k^* \mathrm{u}_k\, dS = \int_{S_\sigma} (\mathrm{t}_1^*\mathrm{u}_1 + \mathrm{t}_2^*\mathrm{u}_2 + \mathrm{t}_3^*\mathrm{u}_3) dS = \int_{S_\sigma} (\sigma_{12}^*\mathrm{u}_2 + \sigma_{13}^*\mathrm{u}_3) dS$$

where we have considered that for the torsion problem $\mathrm{t}_1^* = 0$, $\mathrm{t}_2^* = \sigma_{12}^*$ and $\mathrm{t}_3^* = \sigma_{12}^*$. Note that that according to displacement equation (6.224), the above equation becomes:

$$\begin{aligned} U^{ext} &= \int_{S_\sigma} (\sigma_{12}^*\mathrm{u}_2 + \sigma_{13}^*\mathrm{u}_3) dS = \int_{S_\sigma} (-\sigma_{12}^* x_3\, \theta\, x_1 + \sigma_{13}^* x_2\, \theta\, x_1) dS = x_1\theta \int_{S_\sigma} (-\sigma_{12}^* x_3 + \sigma_{13}^* x_2) dS \\ \Rightarrow U^{ext} &= x_1\theta \int_{S_\sigma} (-\sigma_{12}^* x_3 + \sigma_{13}^* x_2) dS = x_1\theta M_T = 2x_1\theta \int_A \phi\, dA = x_1 \int_A 2\theta\phi\, dA \end{aligned} \tag{6.354}$$

where M_T is the torque applied.

Then, the total potential energy becomes:

$$\Pi = U^{int} - U^{ext} \qquad [J]$$

$$\Rightarrow \Pi = x_1 \int_A \frac{1}{2G}\left[\left(\frac{\partial \phi}{\partial x_3}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2\right] dA - x_1 \int_A 2\theta\phi \, dA$$

$$\Rightarrow \frac{\Pi}{x_1} = \overline{\Pi} = \int_A \frac{1}{2G}\left[\left(\frac{\partial \phi}{\partial x_3}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2\right] dA - \int_A 2\theta\phi \, dA \qquad \left[\frac{J}{m}\right] \tag{6.355}$$

Problem 6.55

Considering a torsion problem in a rectangular cross section ($2a \times 2b$), (see Figure 6.125), and by considering that the displacement field u_1 is approached by the function $\mathsf{u}_1 = K\theta x_2 x_3$, obtain the tangential stress field and the J_T (polar moment of inertia).

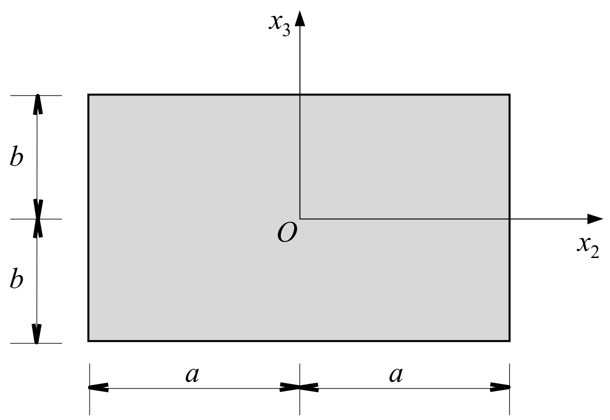


Figure 6.125: Rectangular cross section.

Solution:

The internal potential energy U^{int}, (see equation (6.350)), is:

$$U^{int} = \int_V \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, dV = \int_V \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2) \, dV = x_1 \int_A \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2) \, dA \tag{6.356}$$

The stress field, (see equation (6.229)), can be expressed in terms of the field u_1 as follows:

$$\sigma_{ij} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} = G \begin{bmatrix} 0 & \left(\frac{\partial \mathsf{u}_1}{\partial x_2} - x_3\,\theta\right) & \left(\frac{\partial \mathsf{u}_1}{\partial x_3} + x_2\,\theta\right) \\ \left(\frac{\partial \mathsf{u}_1}{\partial x_2} - x_3\,\theta\right) & 0 & 0 \\ \left(\frac{\partial \mathsf{u}_1}{\partial x_3} + x_2\,\theta\right) & 0 & 0 \end{bmatrix} \tag{6.357}$$

When $\mathsf{u}_1 = K\theta x_2 x_3$ we can obtain:

$$\sigma_{ij} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} = G \begin{bmatrix} 0 & (K\theta x_3 - x_3\,\theta) & (K\theta x_2 + x_2\,\theta) \\ (K\theta x_3 - x_3\,\theta) & 0 & 0 \\ (K\theta x_2 + x_2\,\theta) & 0 & 0 \end{bmatrix} \tag{6.358}$$

Then, the internal potential energy can be rewritten in terms of the constant K :

$$
\begin{aligned}
U^{int}(K) &= x_1 \int_A \frac{1}{2G}(\sigma_{12}^2 + \sigma_{13}^2)\, dA = x_1 \int_A \frac{G^2}{2G}[(K\theta x_3 - x_3\,\theta)^2 + (K\theta x_2 + x_2\,\theta)^2]\, dA \\
&= \frac{G}{2} x_1 \int_A [(\theta x_3)^2 (K-1)^2 + (\theta x_2)^2 (K+1)^2]\, dA \\
&= \frac{G(K-1)^2\theta^2}{2} x_1 \int_A (x_3)^2\, dA + \frac{G\theta^2 (K+1)^2}{2} x_1 \int_A (x_2)^2\, dA \\
&= \frac{G(K-1)^2\theta^2}{2} x_1 \mathrm{I}_{22} + \frac{G\theta^2 (K+1)^2}{2} x_1 \mathrm{I}_{33}
\end{aligned}
\tag{6.359}
$$

where I_{ij} is the inertia tensor of area related to the system $O - x_2 - x_3$:

$$
\begin{aligned}
\mathrm{I}_{O\;ij}^{(A)} &= \begin{bmatrix} \mathrm{I}_{11} & 0 & 0 \\ 0 & \mathrm{I}_{22} & 0 \\ 0 & 0 & \mathrm{I}_{33} \end{bmatrix} = \begin{bmatrix} \int_A (x_2^2 + x_3^2) dA & 0 & 0 \\ 0 & \int_A x_3^2 dA & 0 \\ 0 & 0 & \int_A x_2^2 dA \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{I}_{22} + \mathrm{I}_{33} & 0 & 0 \\ 0 & \dfrac{(2a)(2b)^3}{12} & 0 \\ 0 & 0 & \dfrac{(2b)(2a)^3}{12} \end{bmatrix} = \begin{bmatrix} \mathrm{I}_{22} + \mathrm{I}_{33} & 0 & 0 \\ 0 & \dfrac{4ab^3}{3} & 0 \\ 0 & 0 & \dfrac{4ba^3}{3} \end{bmatrix}
\end{aligned}
\tag{6.360}
$$

Taking into account the external potential energy $U^{ext} = x_1 \theta M_T$, the total potential energy becomes:

$$
\begin{aligned}
&\Pi = U^{int} - U^{ext} \\
&\Rightarrow \Pi(K) = \frac{G(K-1)^2\theta^2}{2} x_1 \mathrm{I}_{22} + \frac{G\theta^2 (K+1)^2}{2} x_1 \mathrm{I}_{33} - x_1 \theta M_T
\end{aligned}
\tag{6.361}
$$

As we are looking for the stationary point, the following must be true:

$$
\begin{aligned}
&\frac{\partial \Pi(K)}{\partial K} = \frac{\partial}{\partial K}\left(\frac{G(K-1)^2\theta^2}{2} x_1 \mathrm{I}_{22} + \frac{G\theta^2 (K+1)^2}{2} x_1 \mathrm{I}_{33} - x_1 \theta M_T \right) = 0 \\
&\Rightarrow G(K-1)\theta^2 x_1 \mathrm{I}_{22} + G\theta^2 (K+1) x_1 \mathrm{I}_{33} = 0 \quad \Rightarrow \quad (K-1)\mathrm{I}_{22} + (K+1)\mathrm{I}_{33} = 0 \\
&\Rightarrow K = \frac{\mathrm{I}_{22} - \mathrm{I}_{33}}{\mathrm{I}_{22} + \mathrm{I}_{33}}
\end{aligned}
\tag{6.362}
$$

Then, the displacement field $\mathrm{u}_1 = K\theta x_2 x_3$ becomes:

$$
\mathrm{u}_1 = K\theta x_2 x_3 = \left(\frac{\mathrm{I}_{22} - \mathrm{I}_{33}}{\mathrm{I}_{22} + \mathrm{I}_{33}} \right) \theta x_2 x_3 = \left(\frac{b^2 - a^2}{a^2 + b^2} \right) \theta x_2 x_3
\tag{6.363}
$$

and the stress field:

$$\sigma_{ij}=\begin{bmatrix}0 & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & 0 & 0\\ \sigma_{13} & 0 & 0\end{bmatrix}=G\begin{bmatrix}0 & \theta x_3\left[\left(\dfrac{\mathrm{I}_{22}-\mathrm{I}_{33}}{\mathrm{I}_{22}+\mathrm{I}_{33}}\right)-1\right] & \theta x_2\left[\left(\dfrac{\mathrm{I}_{22}-\mathrm{I}_{33}}{\mathrm{I}_{22}+\mathrm{I}_{33}}\right)+1\right]\\ \theta x_3\left[\left(\dfrac{\mathrm{I}_{22}-\mathrm{I}_{33}}{\mathrm{I}_{22}+\mathrm{I}_{33}}\right)-1\right] & 0 & 0\\ \theta x_2\left[\left(\dfrac{\mathrm{I}_{22}-\mathrm{I}_{33}}{\mathrm{I}_{22}+\mathrm{I}_{33}}\right)+1\right] & 0 & 0\end{bmatrix} \tag{6.364}$$

$$=\frac{2G\theta}{\mathrm{I}_{22}+\mathrm{I}_{33}}\begin{bmatrix}0 & -x_3\mathrm{I}_{33} & x_2\mathrm{I}_{22}\\ -x_3\mathrm{I}_{33} & 0 & 0\\ x_2\mathrm{I}_{22} & 0 & 0\end{bmatrix}=\frac{2G\theta}{(a^2+b^2)}\begin{bmatrix}0 & -x_3a^2 & x_2b^2\\ -x_3a^2 & 0 & 0\\ x_2b^2 & 0 & 0\end{bmatrix}$$

As we have seen previously, the net moment is obtained as follows:

$$M_T=\int_A(\sigma_{13}x_2-\sigma_{12}x_3)dA=\frac{2G\theta}{\mathrm{I}_{22}+\mathrm{I}_{33}}\int_A(\mathrm{I}_{22}x_2x_2+\mathrm{I}_{33}x_3x_3)dA=\frac{2G\theta}{\mathrm{I}_{22}+\mathrm{I}_{33}}\int_A(\mathrm{I}_{22}x_2^2+\mathrm{I}_{33}x_3^2)dA$$

$$=\frac{2G\theta}{\mathrm{I}_{22}+\mathrm{I}_{33}}\left[\mathrm{I}_{22}\int_A x_2^2dA+\mathrm{I}_{33}\int_A x_3^2dA\right]=\frac{2G\theta}{\mathrm{I}_{22}+\mathrm{I}_{33}}\left[\mathrm{I}_{22}\mathrm{I}_{33}+\mathrm{I}_{33}\mathrm{I}_{22}\right] \tag{6.365}$$

$$=4G\theta\frac{\mathrm{I}_{22}\mathrm{I}_{33}}{\mathrm{I}_{22}+\mathrm{I}_{33}}=4G\theta\frac{4a^3b^3}{3(a^2+b^2)}=\frac{16a^3b^3}{3(a^2+b^2)}G\theta$$

Then, by using the equation $\mathrm{J}_{T_{eff}}=\dfrac{M_T}{G\theta}$, we can obtain:

$$\mathrm{J}_{T_{eff}}=\frac{M_T}{G\theta}=\frac{\left(\dfrac{16a^3b^3}{3(a^2+b^2)}\right)G\theta}{G\theta}=\frac{16a^3b^3}{3(a^2+b^2)} \tag{6.366}$$

When $b=a$ the above equation becomes:

$$\mathrm{J}_{T_{eff}}=\frac{M_T}{G\theta}=\frac{8}{3}a^4=2.66666667a^4 \tag{6.367}$$

and if we compare with the exact solution $\mathrm{J}_{T_{eff}}=2.2496a^4$ we can see that the error is approximately 18.5%.

6.4.4 Introduction to Finite Element for Torsion Problems

Problem 6.56

Starting from the total potential energy obtain an expression equivalent to the torsion problem such as $[\boldsymbol{k}^{(e)}]\{\widetilde{\phi}^{(e)}\} = \{\boldsymbol{f}^{(e)}\}$, where $\{\phi^{(e)}\} = G\theta\{\widetilde{\phi}^{(e)}\}$ is the nodal membrane deflection vector. Consider also that the membrane deflection field is $\phi = \phi(x_2, x_3) = [\boldsymbol{N}(\vec{\boldsymbol{x}})]\{\phi^{(e)}\}$.

Solution:

We can approach the field $\phi = \phi(x_2, x_3)$ by means of shape functions as follows:

$$\phi = \phi(x_2, x_3) = [N_1 \quad N_2 \quad \cdots \quad N_n]\begin{Bmatrix} \phi_1^{(e)} \\ \phi_2^{(e)} \\ \vdots \\ \phi_n^{(e)} \end{Bmatrix} = [\boldsymbol{N}(\vec{\boldsymbol{x}})]_{1\times n}\{\phi^{(e)}\}_{n\times 1} \tag{6.368}$$

where n is the number of nodes. With that the potential $\overline{\Pi}$, (see equation (6.355)), can be written in terms of nodal values $\{\phi^{(e)}\}$ as follows:

$$\begin{aligned} \overline{\Pi} &= \int_A \frac{1}{2G}\left[\left(\frac{\partial \phi}{\partial x_3}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2\right] dA - \int_A 2\theta\phi \, dA \\ \overline{\Pi}\left(\{\phi^{(e)}\}\right) &= \int_A \frac{1}{2G}\left[\left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_3}\right)^2 + \left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_2}\right)^2\right] dA - \int_A 2\theta[\boldsymbol{N}]\{\phi^{(e)}\} \, dA \end{aligned} \tag{6.369}$$

As we are looking for a stationary point, the following must be true:

$$\frac{\partial \overline{\Pi}\left(\{\phi^{(e)}\}\right)}{\partial \{\phi^{(e)}\}} = \frac{\partial}{\partial \{\phi^{(e)}\}}\left(\int_A \frac{1}{2G}\left[\left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_3}\right)^2 + \left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_2}\right)^2\right] dA - \int_A 2\theta[\boldsymbol{N}]\{\phi^{(e)}\} \, dA\right) = \{\boldsymbol{0}\} \tag{6.370}$$

thus

$$\frac{\partial \overline{\Pi}\left(\{\phi^{(e)}\}\right)}{\partial \{\phi^{(e)}\}} = \int_A \frac{1}{2G}\left[2\left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_3}\right)\frac{\partial [\boldsymbol{N}]^T}{\partial x_3} + 2\left(\frac{\partial [\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_2}\right)\frac{\partial [\boldsymbol{N}]^T}{\partial x_2}\right] dA - \int_A 2\theta[\boldsymbol{N}]^T \, dA = \{\boldsymbol{0}\} \tag{6.371}$$

or

$$\left(\int_A \left[\frac{\partial [\boldsymbol{N}]^T}{\partial x_3}\frac{\partial [\boldsymbol{N}]}{\partial x_3} + \frac{\partial [\boldsymbol{N}]^T}{\partial x_2}\frac{\partial [\boldsymbol{N}]}{\partial x_2}\right] dA\right)\frac{\{\phi^{(e)}\}}{G} - 2\theta\int_A [\boldsymbol{N}]^T \, dA = \{\boldsymbol{0}\} \tag{6.372}$$

Note that:

$$\frac{\partial[\boldsymbol{N}]^T}{\partial x_2}\frac{\partial[\boldsymbol{N}]}{\partial x_2}+\frac{\partial[\boldsymbol{N}]^T}{\partial x_3}\frac{\partial[\boldsymbol{N}]}{\partial x_3}=\begin{bmatrix}\frac{\partial[\boldsymbol{N}]^T}{\partial x_2} & \frac{\partial[\boldsymbol{N}]^T}{\partial x_3}\end{bmatrix}\begin{bmatrix}\frac{\partial[\boldsymbol{N}]}{\partial x_2}\\ \frac{\partial[\boldsymbol{N}]}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_3}\\ \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_3}\\ \vdots & \vdots\\ \frac{\partial N_n}{\partial x_2} & \frac{\partial N_n}{\partial x_3}\end{bmatrix}\begin{bmatrix}\frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \cdots & \frac{\partial N_n}{\partial x_2}\\ \frac{\partial N_1}{\partial x_3} & \frac{\partial N_2}{\partial x_3} & \cdots & \frac{\partial N_n}{\partial x_3}\end{bmatrix}$$

$$=[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}] \tag{6.373}$$

where we have considered that:

$$[\overline{\boldsymbol{B}}]=\begin{bmatrix}\frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \cdots & \frac{\partial N_n}{\partial x_2}\\ \frac{\partial N_1}{\partial x_3} & \frac{\partial N_2}{\partial x_3} & \cdots & \frac{\partial N_n}{\partial x_3}\end{bmatrix}_{2\times n} \tag{6.374}$$

Then, the equation (6.372) becomes:

$$\begin{aligned}&\left(\int_A[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]\,dA\right)\frac{\{\phi^{(e)}\}}{G}-2\theta\int_A[\boldsymbol{N}]^T\,dA=\{\boldsymbol{0}\}\\ &\Rightarrow\left(\int_A[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]\,dA\right)\frac{\{\phi^{(e)}\}}{G\theta}=2\int_A[\boldsymbol{N}]^T\,dA\\ &\Rightarrow\left(\int_A[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]\,dA\right)\{\tilde{\phi}^{(e)}\}=2\int_A[\boldsymbol{N}]^T\,dA\\ &\Rightarrow[\boldsymbol{k}^{(e)}]_{n\times n}\{\tilde{\phi}^{(e)}\}_{n\times 1}=\{\boldsymbol{f}^{(e)}\}_{n\times 1}\end{aligned} \tag{6.375}$$

where we have considered that $\{\tilde{\phi}^{(e)}\}=\dfrac{\{\phi^{(e)}\}}{G\theta}$. To solve the above equation we have to introduce the boundary condition which is that on the boundary it must fulfill $\tilde{\phi}^{(e)}_{boundary}=0$. Once the problem is solved we can obtain:

- The angle of twist θ:

$$M_T=2V_{memb}=2\int_A\phi\,dA=2\int_A[\boldsymbol{N}]\{\phi^{(e)}\}\,dA=2\int_A[\boldsymbol{N}]G\theta\{\tilde{\phi}^{(e)}\}\,dA=2G\theta\int_A[\boldsymbol{N}]\{\tilde{\phi}^{(e)}\}\,dA \tag{6.376}$$

Note also that:

$$J_{T_{eff}}=\frac{M_T}{G\theta} \tag{6.377}$$

- The tangential stress can be obtained as follows:

$$\begin{Bmatrix}\sigma_{12}\\ \sigma_{13}\end{Bmatrix}=\begin{Bmatrix}\frac{\partial\phi}{\partial x_3}\\ -\frac{\partial\phi}{\partial x_2}\end{Bmatrix}=\begin{Bmatrix}\frac{\partial[\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_3}\\ -\frac{\partial[\boldsymbol{N}]\{\phi^{(e)}\}}{\partial x_2}\end{Bmatrix}=\begin{bmatrix}\frac{\partial[\boldsymbol{N}]}{\partial x_3}\\ -\frac{\partial[\boldsymbol{N}]}{\partial x_2}\end{bmatrix}\{\phi^{(e)}\}=\begin{bmatrix}\frac{\partial N_1}{\partial x_3} & \frac{\partial N_2}{\partial x_3} & \cdots & \frac{\partial N_n}{\partial x_3}\\ \frac{-\partial N_1}{\partial x_2} & \frac{-\partial N_2}{\partial x_2} & \cdots & \frac{-\partial N_n}{\partial x_2}\end{bmatrix}\{\phi^{(e)}\} \tag{6.378}$$

where $\{\phi^{(e)}\}=G\theta\{\tilde{\phi}^{(e)}\}$.

NOTE 1: Let us make the same development from equation (6.369) to (6.375) by using indicial notation:

$$\overline{\Pi} = \int_A \frac{1}{2G}\left[\left(\frac{\partial \phi}{\partial x_3}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2\right] dA - \int_A 2\theta\phi \, dA = \int_A \frac{1}{2G}(\phi_{,k}\phi_{,k}) \, dA - \int_A 2\theta\phi \, dA \tag{6.379}$$

By considering that $\phi(\vec{x}) = N_i \phi_i^{(e)}$ we have $\phi_{,k} = N_{i,k}\phi_i^{(e)}$. Note that $i = 1,2,\cdots,n$ (n is the number of nodes) and $k = 1,2,3$. Then, the above equation becomes:

$$\begin{aligned}
&\overline{\Pi} = \int_A \frac{1}{2G}(\phi_{,k}\phi_{,k}) \, dA - \int_A 2\theta\phi \, dA = \int_A \frac{1}{2G}(N_{i,k}\phi_i^{(e)} N_{p,k}\phi_p^{(e)}) \, dA - \int_A 2\theta N_i \phi_i^{(e)} \, dA \\
&\Rightarrow \overline{\Pi} = \int_A \frac{1}{2G}(N_{i,k} N_{p,k}\phi_i^{(e)}\phi_p^{(e)}) \, dA - \int_A 2\theta N_i \phi_i^{(e)} \, dA
\end{aligned} \tag{6.380}$$

Taking the derivative with respect to $\phi_j^{(e)}$ $(j = 1,2,\cdots,n)$ we can obtain:

$$\begin{aligned}
&\frac{\partial \overline{\Pi}}{\partial \phi_j^{(e)}} = \frac{\partial}{\partial \phi_j^{(e)}}\left(\int_A \frac{1}{2G}(N_{i,k} N_{p,k}\phi_i^{(e)}\phi_p^{(e)}) \, dA - \int_A 2\theta N_i \phi_i^{(e)} \, dA\right) = 0_j \\
&\Rightarrow \int_A \frac{1}{2G} N_{i,k} N_{p,k}\left(\frac{\partial(\phi_i^{(e)}\phi_p^{(e)})}{\partial \phi_j^{(e)}}\right) dA - \int_A 2\theta N_i \left(\frac{\partial \phi_i^{(e)}}{\partial \phi_j^{(e)}}\right) dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G} N_{i,k} N_{p,k}\left(\frac{\partial(\phi_i^{(e)})}{\partial \phi_j^{(e)}}\phi_p^{(e)} + \phi_i^{(e)}\frac{\partial(\phi_p^{(e)})}{\partial \phi_j^{(e)}}\right) dA - \int_A 2\theta N_i \delta_{ij} \, dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G} N_{i,k} N_{p,k}\left(\delta_{ij}\phi_p^{(e)} + \phi_i^{(e)}\delta_{pj}\right) dA - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G}\left(N_{i,k} N_{p,k}\delta_{ij}\phi_p^{(e)} + N_{i,k} N_{p,k}\phi_i^{(e)}\delta_{pj}\right) dA - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G}\left(N_{j,k} N_{p,k}\phi_p^{(e)} + N_{i,k} N_{j,k}\phi_i^{(e)}\right) dA - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G}\left(N_{j,k} N_{p,k}\phi_p^{(e)} + N_{p,k} N_{j,k}\phi_p^{(e)}\right) dA - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \int_A \frac{1}{2G}\left(N_{j,k} N_{p,k} + N_{p,k} N_{j,k}\right)\phi_p^{(e)} \, dA - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \left(\int_A \frac{1}{G}\left(N_{j,k} N_{p,k}\right) dA\right)\phi_p^{(e)} - \int_A 2\theta N_j \, dA = 0_j \\
&\Rightarrow \left(\int_A \left(N_{j,k} N_{p,k}\right) dA\right)\frac{\phi_p^{(e)}}{G\theta} - \int_A 2N_j \, dA = 0_j \\
&\Rightarrow k_{jp}^{(e)}\frac{\phi_p^{(e)}}{G\theta} = 2\int_A N_j \, dA = f_j^{(e)} \\
&\Rightarrow k_{jp}^{(e)}\widetilde{\phi}_p^{(e)} = f_j^{(e)}
\end{aligned} \tag{6.381}$$

where $i, j, p = 1,2,\cdots,n$ and $k = 1,2,3$. Note also that neither N_i nor ϕ vary with x_1, so $\frac{\partial N_i}{\partial x_1} \equiv N_{i,1} = 0_i$ and $\frac{\partial \phi}{\partial x_1} \equiv \phi_{,1} = 0$.

Problem 6.57

By taking into account **Problem 6.56** obtain the explicit formulation for the equation $[\boldsymbol{k}^{(e)}]\{\widetilde{\phi}^{(e)}\} = \{\boldsymbol{f}^{(e)}\}$, when the sub-domain (finite element) is a triangle with three nodes, (see Figure 6.126).

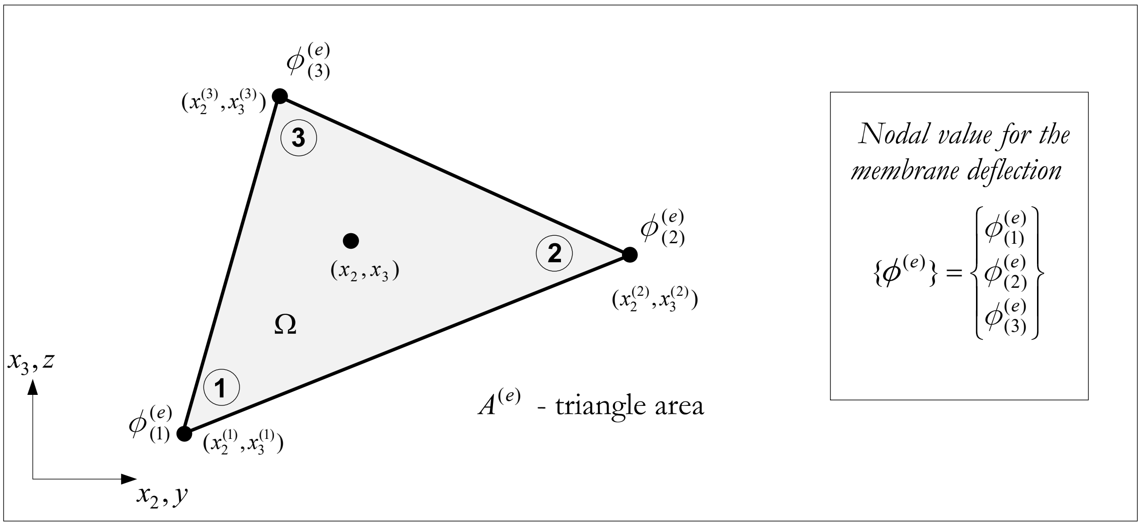


Figure 6.126: Domain Ω.

NOTE: The element connectivity orientation must be counterclockwise.

Solution:

In **Problem 6.40** we have obtained the shape functions for the triangle with three nodes, and by considering the formulation on the plane $x_2 - x_3$ the shape functions can be rewritten as follows:

$$\begin{cases} N_1(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2(x_3^{(2)} - x_3^{(3)}) + x_3(x_2^{(3)} - x_2^{(2)}) + (x_2^{(2)}x_3^{(3)} - x_3^{(2)}x_2^{(3)})] \\ N_2(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2(x_3^{(3)} - x_3^{(1)}) + x_3(x_2^{(1)} - x_2^{(3)}) + (x_2^{(3)}x_3^{(1)} - x_3^{(3)}x_2^{(1)})] \\ N_3(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2(x_3^{(1)} - x_3^{(2)}) + x_3(x_2^{(2)} - x_2^{(1)}) + (x_2^{(1)}x_3^{(2)} - x_3^{(1)}x_2^{(2)})] \end{cases}$$
$$\Rightarrow \begin{cases} N_1(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2 b_1 + x_3 c_1 + d_1] \\ N_2(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2 b_2 + x_3 c_2 + d_2] \\ N_3(x_2,x_3) = \dfrac{1}{2A^{(e)}}[x_2 b_3 + x_3 c_3 + d_3] \end{cases} \tag{6.382}$$

where $A^{(e)}$ is the triangle area and we have considered that:

$$\begin{aligned} b_1 &= (x_3^{(2)} - x_3^{(3)}) \quad ; \quad c_1 = (x_2^{(3)} - x_2^{(2)}) \quad ; \quad d_1 = (x_2^{(2)}x_3^{(3)} - x_3^{(2)}x_2^{(3)}) \\ b_2 &= (x_3^{(3)} - x_3^{(1)}) \quad ; \quad c_2 = (x_2^{(1)} - x_2^{(3)}) \quad ; \quad d_2 = (x_2^{(3)}x_3^{(1)} - x_3^{(3)}x_2^{(1)}) \\ b_3 &= (x_3^{(1)} - x_3^{(2)}) \quad ; \quad c_3 = (x_2^{(2)} - x_2^{(1)}) \quad ; \quad d_3 = (x_2^{(1)}x_3^{(2)} - x_3^{(1)}x_2^{(2)}) \end{aligned} \tag{6.383}$$

For this case the matrix $[\overline{\boldsymbol{B}}]$, (see equation (6.374)), becomes:

$$[\overline{\boldsymbol{B}}]=\begin{bmatrix}\dfrac{\partial N_1}{\partial x_2} & \dfrac{\partial N_2}{\partial x_2} & \dfrac{\partial N_3}{\partial x_2}\\ \dfrac{\partial N_1}{\partial x_3} & \dfrac{\partial N_2}{\partial x_3} & \dfrac{\partial N_3}{\partial x_3}\end{bmatrix}_{2\times 3}=\frac{1}{2A^{(e)}}\begin{bmatrix}b_1 & b_2 & b_2\\ c_1 & c_2 & c_3\end{bmatrix} \tag{6.384}$$

As we can see, $[\overline{\boldsymbol{B}}]$ is constant, then the stiffness matrix, (see equation (6.375)), can be obtained as follows:

$$[\boldsymbol{k}^{(e)}]=\int_A [\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]\,dA=[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]\int_A dA=A^{(e)}[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}] \tag{6.385}$$

or

$$[\boldsymbol{k}^{(e)}]=A^{(e)}[\overline{\boldsymbol{B}}]^T[\overline{\boldsymbol{B}}]=\frac{1}{4A^{(e)}}\begin{bmatrix}b_1^2+c_1^2 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3\\ b_1b_2+c_1c_2 & b_2^2+c_2^2 & b_2b_3+c_2c_3\\ b_1b_3+c_1c_3 & b_2b_3+c_2c_3 & b_3^2+c_3^2\end{bmatrix} \tag{6.386}$$

The nodal "force" vector $\{\boldsymbol{f}^{(e)}\}$, (see equation (6.375)), can be obtained as follows:

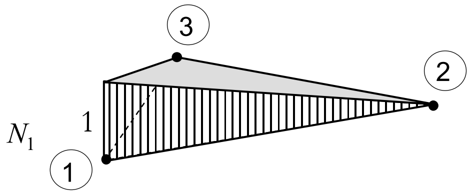
$$\{\boldsymbol{f}^{(e)}\}=2\int_A [\boldsymbol{N}]^T\,dA=2\int_A \begin{Bmatrix}N_1\\ N_2\\ N_3\end{Bmatrix}dA=\frac{2}{2A^{(e)}}\int_A \begin{Bmatrix}[x_2b_1+x_3c_1+d_1]\\ [x_2b_2+x_3c_2+d_2]\\ [x_2b_3+x_3c_3+d_3]\end{Bmatrix}dA=\frac{2A^{(e)}}{3}\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix} \tag{6.387}$$

The tangential stresses, (see equation (6.378)), can be obtained as follows:

$$\begin{Bmatrix}\sigma_{12}^{(e)}\\ \sigma_{13}^{(e)}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\phi}{\partial x_3}\\ -\dfrac{\partial\phi}{\partial x_2}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial N_1}{\partial x_3} & \dfrac{\partial N_2}{\partial x_3} & \dfrac{\partial N_3}{\partial x_3}\\ \dfrac{-\partial N_1}{\partial x_2} & \dfrac{-\partial N_3}{\partial x_2} & \dfrac{-\partial N_3}{\partial x_2}\end{bmatrix}\begin{Bmatrix}\phi_{(1)}^{(e)}\\ \phi_{(2)}^{(e)}\\ \phi_{(3)}^{(e)}\end{Bmatrix}=\frac{1}{2A^{(e)}}\begin{bmatrix}c_1 & c_2 & c_3\\ -b_1 & -b_2 & -b_3\end{bmatrix}\begin{Bmatrix}\phi_{(1)}^{(e)}\\ \phi_{(2)}^{(e)}\\ \phi_{(3)}^{(e)}\end{Bmatrix} \tag{6.388}$$

Note that for the triangle of three-nodes the stress field into the element is constant.

Note that the integral $\int_A N_1 dA$ from (6.387) is the volume formed by the shape function N_1, (see Figure 6.127).

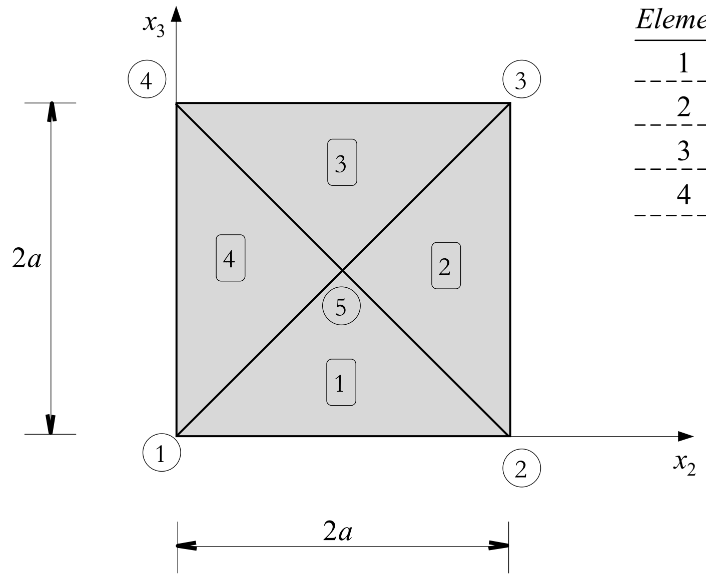


$$\int_A N_1 dA = V = \frac{1}{3}A^{(e)}$$

Figure 6.127: Shape function N_1.

Problem 6.58

Considering the torsion problem in the squared cross section ($2a \times 2a$). Obtain the solution for this problem by considering the discretization by using 4 finite elements, (see Figure 6.128).



Element	*Area*	*Connectivity*
1	a^2	$1-2-5$
2	a^2	$2-3-5$
3	a^2	$3-4-5$
4	a^2	$4-1-5$

Figure 6.128: Squared cross section.

Solution:

In order to construct the global stiffness matrix and the global nodal force vector we will need to define for each finite element the stiffness matrix and the nodal force vector.

Element 1: Connectivity: 1 (local 1)-2(local 2)-5 (local 3),

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = 0 \quad ; \quad c_1 = (x_2^{(3)} - x_2^{(2)}) = -a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = a \quad ; \quad c_2 = (x_2^{(1)} - x_2^{(3)}) = -a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = 0 \quad ; \quad c_3 = (x_2^{(2)} - x_2^{(1)}) = 2a
\end{aligned}
$$

By using the equation in (6.386) we can obtain:

$$
[\boldsymbol{k}^{(1)}] = \frac{1}{4a^2}\begin{bmatrix} a^2 & a^2 & -2a^2 \\ a^2 & 2a^2 & -2a^2 \\ -2a^2 & -2a^2 & 4a^2 \end{bmatrix} = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} \mathit{1} & \mathit{2} & \mathit{5} \end{array} \quad \mathit{Global} \\ \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{5} \end{array} \end{array} \tag{6.389}
$$

Nodal "force" vector

$$
\{\boldsymbol{f}^{(1)}\} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \end{Bmatrix} = \frac{2A^{(1)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{2a^2}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \begin{array}{c} \mathit{Global} \\ \mathit{1} \\ \mathit{2} \\ \mathit{5} \end{array} \tag{6.390}
$$

Element 2: Connectivity: 2 (local 1)-3(local 2)-5 (local 3),

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = a \quad &;\quad c_1 &= (x_2^{(3)} - x_2^{(2)}) = -a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = a \quad &;\quad c_2 &= (x_2^{(1)} - x_2^{(3)}) = a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = -2a \quad &;\quad c_3 &= (x_2^{(2)} - x_2^{(1)}) = 0
\end{aligned}
$$

By using equation (6.386) we can obtain:

$$
[\boldsymbol{k}^{(2)}] = \frac{1}{4a^2}\begin{bmatrix} 2a^2 & 0 & -2a^2 \\ 0 & 2a^2 & -2a^2 \\ -2a^2 & -2a^2 & 4a^2 \end{bmatrix} = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} \mathit{2} & \mathit{3} & \mathit{5} \end{array} \; \mathit{Global} \\ \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{array}{c} \mathit{2} \\ \mathit{3} \\ \mathit{5} \end{array} \end{array} \tag{6.391}
$$

Nodal "force" vector

$$
\{\boldsymbol{f}^{(2)}\} = \begin{Bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \frac{2A^{(2)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{2a^2}{3}\begin{array}{c} \mathit{Global} \\ \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \begin{array}{c} \mathit{2} \\ \mathit{3} \\ \mathit{5} \end{array} \end{array} \tag{6.392}
$$

Element 3: Connectivity: 3 (local 1)-4(local 2)-5 (local 3),

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = a \quad &;\quad c_1 &= (x_2^{(3)} - x_2^{(2)}) = a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = -a \quad &;\quad c_2 &= (x_2^{(1)} - x_2^{(3)}) = a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = 0 \quad &;\quad c_3 &= (x_2^{(2)} - x_2^{(1)}) = -2a
\end{aligned}
$$

By using equation (6.386) we can obtain:

$$
[\boldsymbol{k}^{(3)}] = \frac{1}{4a^2}\begin{bmatrix} 2a^2 & -a^2 & -2a^2 \\ -a^2 & 2a^2 & -2a^2 \\ -2a^2 & -2a^2 & 4a^2 \end{bmatrix} = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} \mathit{3} & \mathit{4} & \mathit{5} \end{array} \; \mathit{Global} \\ \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{array}{c} \mathit{3} \\ \mathit{4} \\ \mathit{5} \end{array} \end{array} \tag{6.393}
$$

Nodal "force" vector

$$
\{\boldsymbol{f}^{(3)}\} = \begin{Bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} = \frac{2A^{(3)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{2a^2}{3}\begin{array}{c} \mathit{Global} \\ \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \begin{array}{c} \mathit{3} \\ \mathit{4} \\ \mathit{5} \end{array} \end{array} \tag{6.394}
$$

Element 4: Connectivity: 4 (local 1)-1(local 2)-5 (local 3),

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = -a \quad &;\quad c_1 &= (x_2^{(3)} - x_2^{(2)}) = a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = -a \quad &;\quad c_2 &= (x_2^{(1)} - x_2^{(3)}) = -a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = 2a \quad &;\quad c_3 &= (x_2^{(2)} - x_2^{(1)}) = 0
\end{aligned}
$$

By using equation (6.386) we can obtain:

$$[\boldsymbol{k}^{(4)}] = \frac{1}{4a^2}\begin{bmatrix} 2a^2 & 0 & -2a^2 \\ 0 & 2a^2 & -2a^2 \\ -2a^2 & -2a^2 & 4a^2 \end{bmatrix} = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} \mathit{4} & \mathit{1} & \mathit{5} \end{array} \\ \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \end{array} \begin{array}{l} \textit{Global} \\ \mathit{4} \\ \mathit{1} \\ \mathit{5} \end{array} \tag{6.395}$$

Nodal "force" vector

$$\{\boldsymbol{f}^{(4)}\} = \begin{Bmatrix} f_1^{(4)} \\ f_2^{(4)} \\ f_3^{(4)} \end{Bmatrix} = \frac{2A^{(4)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{2a^2}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \begin{array}{l} \textit{Global} \\ \mathit{4} \\ \mathit{1} \\ \mathit{5} \end{array} \tag{6.396}$$

Then, the global stiffness matrix and global nodal force vector can be obtained by:

$$[\boldsymbol{K}] = \mathop{\mathbf{A}}_{e=1}^{N_{elem}} [\boldsymbol{k}^{(e)}] \quad ; \quad \{\boldsymbol{F}\} = \mathop{\mathbf{A}}_{e=1}^{N_{elem}} [\boldsymbol{f}^{(e)}] \tag{6.397}$$

where $\mathbf{A}$ stands for assemble operator. Making the contribution to the respective degree-of-freedom we can obtain:

<u>Global Stiffness Matrix:</u>

$$[\boldsymbol{K}] = \begin{bmatrix} (k_{11}^{(1)} + k_{22}^{(4)}) & k_{12}^{(1)} & 0 & k_{21}^{(4)} & (k_{13}^{(1)} + k_{23}^{(4)}) \\ k_{21}^{(1)} & (k_{22}^{(1)} + k_{11}^{(2)}) & k_{12}^{(2)} & 0 & (k_{23}^{(1)} + k_{13}^{(1)}) \\ 0 & k_{21}^{(2)} & (k_{22}^{(2)} + k_{11}^{(3)}) & k_{12}^{(3)} & (k_{23}^{(2)} + k_{13}^{(3)}) \\ k_{12}^{(4)} & 0 & k_{21}^{(3)} & (k_{22}^{(3)} + k_{11}^{(4)}) & (k_{23}^{(3)} + k_{13}^{(4)}) \\ (k_{31}^{(1)} + k_{32}^{(4)}) & (k_{23}^{(1)} + k_{13}^{(1)}) & k_{32}^{(2)} + k_{31}^{(3)} & (k_{32}^{(3)} + k_{31}^{(4)}) & (k_{33}^{(1)} + k_{33}^{(2)} + k_{33}^{(3)} + k_{33}^{(4)}) \end{bmatrix} \tag{6.398}$$

Note that the matrix $[\boldsymbol{K}]$ is symmetric and that $\det[\boldsymbol{K}] = 0$.

<u>Global nodal force vector:</u>

$$\{\boldsymbol{F}\} = \begin{Bmatrix} f_1^{(1)} + f_2^{(4)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + f_1^{(4)} \\ f_3^{(1)} + f_3^{(2)} + f_3^{(3)} + f_3^{(4)} \end{Bmatrix} = \frac{2a^2}{3}\begin{Bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 4 \end{Bmatrix} \tag{6.399}$$

With that the system can be constructed:

$$[\boldsymbol{K}]\{\tilde{\phi}\} = \{\boldsymbol{F}\} \tag{6.400}$$

The boundary condition is applied on the boundary in which $\tilde{\phi}_{boundary} = 0$. Then, for the problem presented here $\tilde{\phi}_1 = \tilde{\phi}_2 = \tilde{\phi}_3 = \tilde{\phi}_4 = 0$, then the system to be solved is:

$$[\overline{\boldsymbol{K}}]\{\tilde{\phi}\} = \{\overline{\boldsymbol{F}}\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}\begin{Bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \end{Bmatrix} = \frac{2a^2}{3}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{Bmatrix} \xrightarrow{solve} \begin{Bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \end{Bmatrix} = \frac{2a^2}{3}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{6.401}$$

where we have considered that $k_{33}^{(1)}+k_{33}^{(2)}+k_{33}^{(3)}+k_{33}^{(4)}=4$. Then $\tilde{\phi}_5=\dfrac{2a^2}{3}$ and $\phi_5=G\theta\tilde{\phi}_5=G\theta\dfrac{2a^2}{3}$.

The moment of torsion is equal to two times the membrane volume:

$$M_T=2V_{memb}=2\sum_{e=1}^{N_{elem}=4}\sum_{j=1}^{N_{nodes}=3}\frac{\phi_j^{(e)}}{3}A^{(e)}=2\left[4\left(\frac{2a^2}{3}G\theta\frac{a^2}{3}\right)\right]=\frac{16}{9}G\theta a^4 \tag{6.402}$$

and

$$\mathrm{J}_{T_{eff}}=\frac{M_T}{G\theta}=\frac{\frac{16}{9}G\theta a^4}{G\theta}=\frac{16}{9}a^4\approx 1.77778a^4 \tag{6.403}$$

and if we compare with the exact solution $\mathrm{J}_{T_{eff}}=2.2496a^4$ we can see that the error is approximately 21%. To improve the result we have to discretize the domain by using more finite elements.

Note that for this case the element slope is constant so is the tangential stress, (see equation(6.388)):

$$\begin{Bmatrix}\sigma_{12}^{(e)}\\ \sigma_{13}^{(e)}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial N_1}{\partial x_3} & \dfrac{\partial N_2}{\partial x_3} & \dfrac{\partial N_3}{\partial x_3}\\ \dfrac{-\partial N_1}{\partial x_2} & \dfrac{-\partial N_2}{\partial x_2} & \dfrac{-\partial N_3}{\partial x_2}\end{bmatrix}\begin{Bmatrix}\phi_1^{(e)}\\ \phi_2^{(e)}\\ \phi_3^{(e)}\end{Bmatrix}=\frac{1}{2A^{(e)}}\begin{bmatrix}c_1 & c_2 & c_3\\ -b_1 & -b_2 & -b_3\end{bmatrix}\begin{Bmatrix}\phi_1^{(e)}\\ \phi_2^{(e)}\\ \phi_3^{(e)}\end{Bmatrix} \tag{6.404}$$

For the **element 1** we have:

$$\begin{aligned}
b_1&=(x_3^{(2)}-x_3^{(3)})=0 \quad ; \quad c_1=(x_2^{(3)}-x_2^{(2)})=-a\\
b_2&=(x_3^{(3)}-x_3^{(1)})=a \quad ; \quad c_2=(x_2^{(1)}-x_2^{(3)})=-a\\
b_3&=(x_3^{(1)}-x_3^{(2)})=0 \quad ; \quad c_3=(x_2^{(2)}-x_2^{(1)})=2a
\end{aligned}$$

and $\phi_1^{(1)}=\phi_1$, $\phi_2^{(1)}=\phi_2$, $\phi_3^{(1)}=\phi_5$, then

$$\begin{Bmatrix}\sigma_{12}^{(1)}\\ \sigma_{13}^{(1)}\end{Bmatrix}=\frac{1}{2a^2}\begin{bmatrix}c_1 & c_2 & c_3\\ -b_1 & -b_2 & -b_3\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_5\end{Bmatrix}=\frac{1}{2a^2}\begin{bmatrix}-a & -a & 2a\\ 0 & -a & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ G\theta\dfrac{2a^2}{3}\end{Bmatrix}=\frac{2}{3}G\theta a\begin{Bmatrix}1\\ 0\end{Bmatrix} \tag{6.405}$$

For the **element 2** we have:

$$\begin{aligned}
b_1&=(x_3^{(2)}-x_3^{(3)})=a \quad ; \quad c_1=(x_2^{(3)}-x_2^{(2)})=-a\\
b_2&=(x_3^{(3)}-x_3^{(1)})=a \quad ; \quad c_2=(x_2^{(1)}-x_2^{(3)})=a\\
b_3&=(x_3^{(1)}-x_3^{(2)})=-2a \quad ; \quad c_3=(x_2^{(2)}-x_2^{(1)})=0
\end{aligned}$$

and $\phi_1^{(2)}=\phi_2$, $\phi_2^{(2)}=\phi_3$, $\phi_3^{(2)}=\phi_5$, then

$$\begin{Bmatrix}\sigma_{12}^{(2)}\\ \sigma_{13}^{(2)}\end{Bmatrix}=\frac{1}{2a^2}\begin{bmatrix}c_1 & c_2 & c_3\\ -b_1 & -b_2 & -b_3\end{bmatrix}\begin{Bmatrix}\phi_2\\ \phi_3\\ \phi_5\end{Bmatrix}=\frac{1}{2a^2}\begin{bmatrix}-a & a & 0\\ -a & -a & 2a\end{bmatrix}\begin{Bmatrix}0\\ 0\\ G\theta\dfrac{2a^2}{3}\end{Bmatrix}=\frac{2}{3}G\theta a\begin{Bmatrix}0\\ 1\end{Bmatrix} \tag{6.406}$$

For the **element 3** we have:

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = a \quad ; \quad c_1 = (x_2^{(3)} - x_2^{(2)}) = a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = -a \quad ; \quad c_2 = (x_2^{(1)} - x_2^{(3)}) = a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = 0 \quad ; \quad c_3 = (x_2^{(2)} - x_2^{(1)}) = -2a
\end{aligned}
$$

and $\phi_1^{(3)} = \phi_3$, $\phi_2^{(3)} = \phi_4$, $\phi_3^{(3)} = \phi_5$, then

$$
\begin{Bmatrix} \sigma_{12}^{(3)} \\ \sigma_{13}^{(3)} \end{Bmatrix} = \frac{1}{2a^2} \begin{bmatrix} c_1 & c_2 & c_3 \\ -b_1 & -b_2 & -b_3 \end{bmatrix} \begin{Bmatrix} \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \frac{1}{2a^2} \begin{bmatrix} a & a & -2a \\ -a & a & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ G\theta \dfrac{2a^2}{3} \end{Bmatrix} = \frac{-2}{3} G\theta a \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{6.407}
$$

For the **element 4** we have:

$$
\begin{aligned}
b_1 &= (x_3^{(2)} - x_3^{(3)}) = -a \quad ; \quad c_1 = (x_2^{(3)} - x_2^{(2)}) = a \\
b_2 &= (x_3^{(3)} - x_3^{(1)}) = -a \quad ; \quad c_2 = (x_2^{(1)} - x_2^{(3)}) = -a \\
b_3 &= (x_3^{(1)} - x_3^{(2)}) = 2a \quad ; \quad c_3 = (x_2^{(2)} - x_2^{(1)}) = 0
\end{aligned}
$$

and $\phi_1^{(4)} = \phi_4$, $\phi_2^{(4)} = \phi_1$, $\phi_3^{(4)} = \phi_5$, then

$$
\begin{Bmatrix} \sigma_{12}^{(4)} \\ \sigma_{13}^{(4)} \end{Bmatrix} = \frac{1}{2a^2} \begin{bmatrix} c_1 & c_2 & c_3 \\ -b_1 & -b_2 & -b_3 \end{bmatrix} \begin{Bmatrix} \phi_4 \\ \phi_1 \\ \phi_5 \end{Bmatrix} = \frac{1}{2a^2} \begin{bmatrix} a & -a & 0 \\ a & a & -2a \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ G\theta \dfrac{2a^2}{3} \end{Bmatrix} = \frac{-2}{3} G\theta a \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{6.408}
$$

In Figure 6.129 we can appreciate the tangential stress distributions, which is a very poor solution. We will need more elements to achieve a better stress distribution.

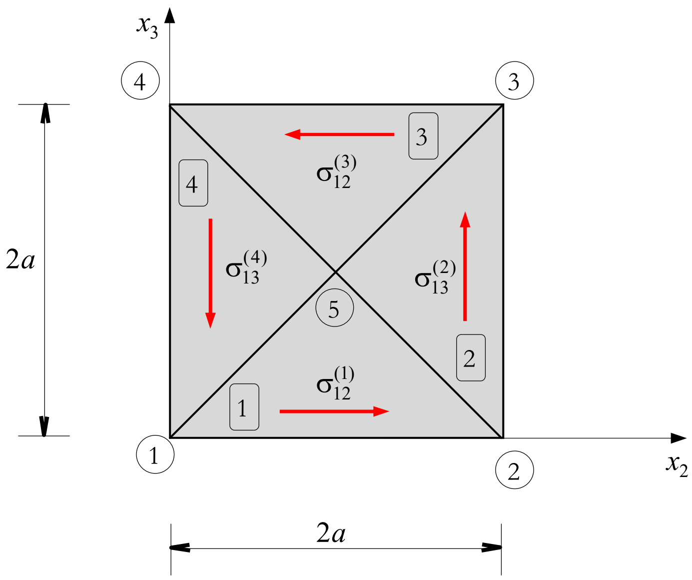


Figure 6.129: Tangential stress distribution.

The membrane deflection can be appreciated in Figure 6.130. Note that for this case the membrane deflection is a pyramid. Recall that the pyramid volume is:

$$V_{pyr}=\frac{1}{3}(A_{base}\times h_{apex})=\frac{1}{3}4a^2G\theta\frac{2a^2}{3}=\frac{8}{9}G\theta a^4=\frac{M_T}{2}$$

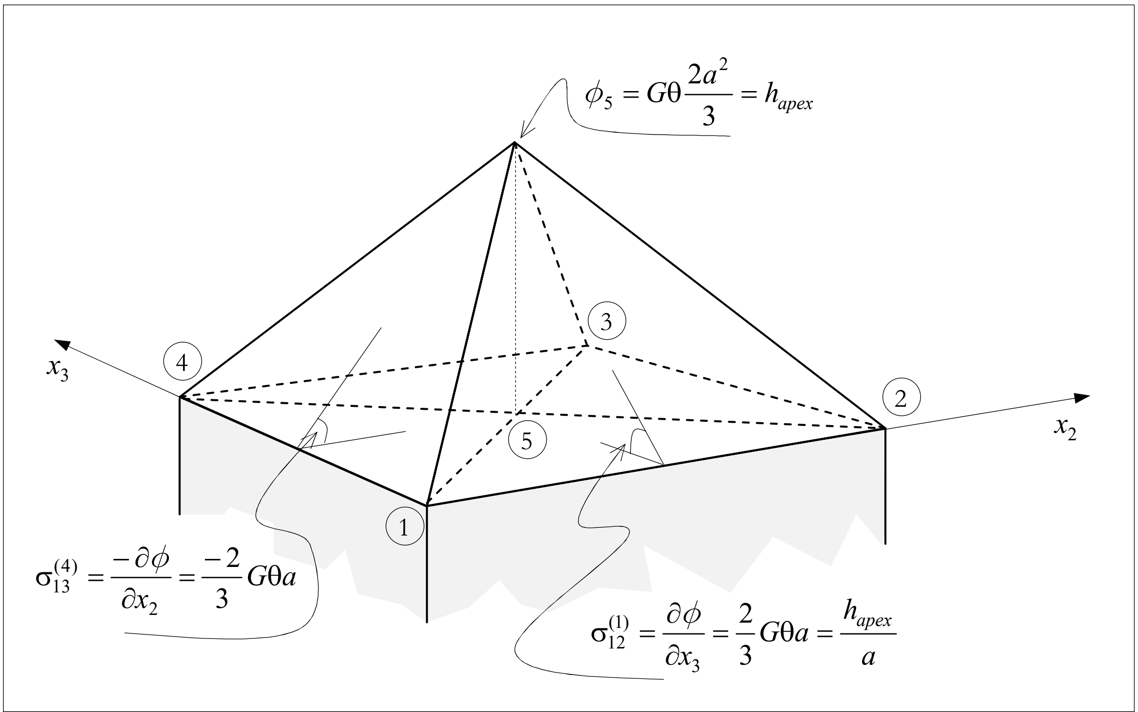


Figure 6.130: Membrane deflection for **Problem 6.58**.

In Figure 6.131 we can appreciate another example for the membrane deflection and in Figure 6.132 the correspondent tangential stress.

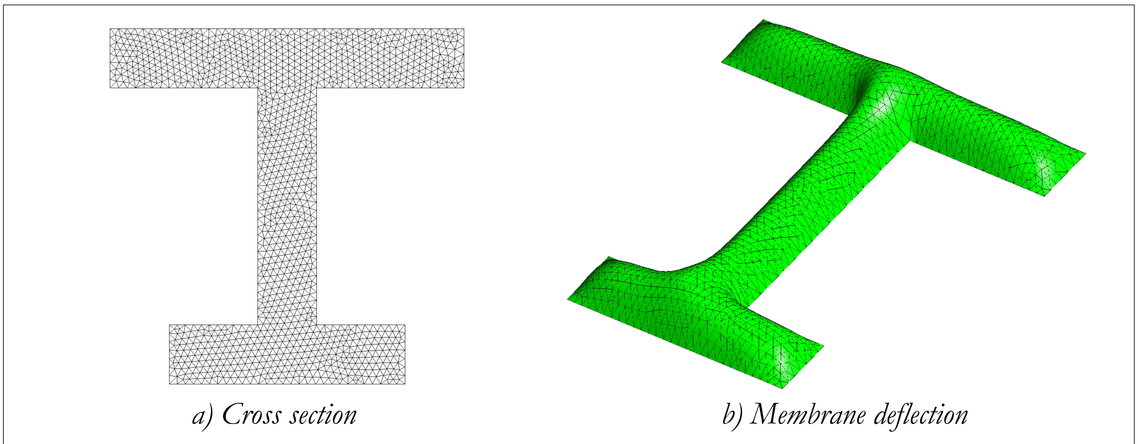


Figure 6.131: Membrane deflection.

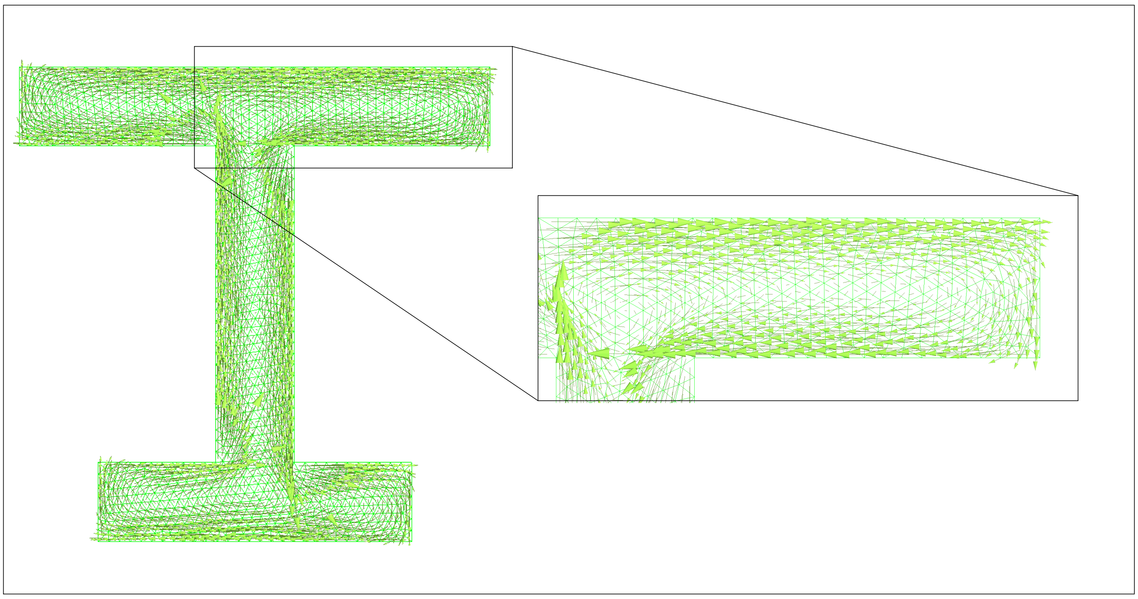


Figure 6.132: Tangential stress distribution.

Torsion References

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6.5 Introduction to One-Dimensional Elements (1D)

Problem 6.59

Consider the bar element which presents one dimension greater than the other two. Obtain the internal forces in the cross-sectional area of the bar. The coordinate system is located at the Area Centroid and is principal axis of inertia, (see Figure 6.133), and use engineering notation.

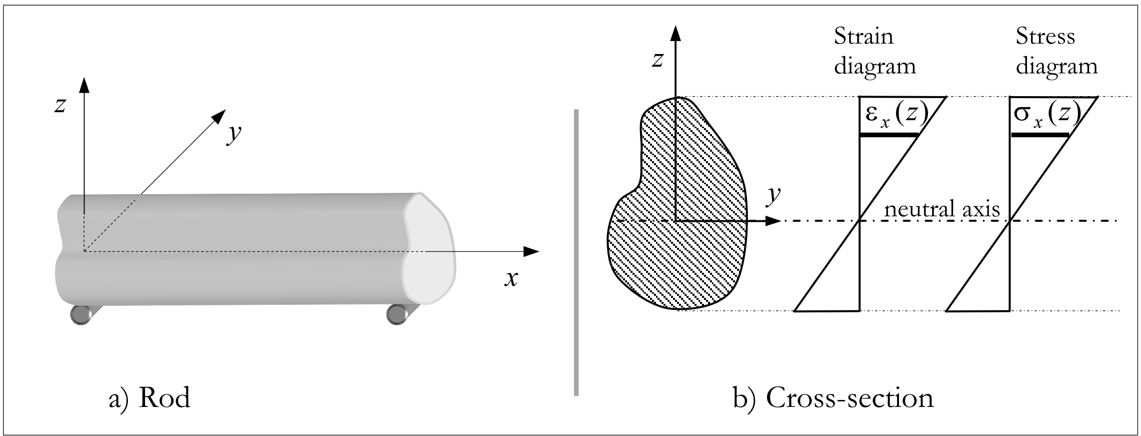


Figure 6.133: The bar.

Hypothesis:

- Small deformation regime and small rotation;
- Homogeneous, elastic, linear and isotropic material;
- To obtain the internal forces due to the stress component σ_x consider that any cross-sectional area defined by a plane remains plane after deformation.

Note that this problem was already discussed in **Problem 4.22**. Here we will present the solution from another point of view.

Solution:

The internal forces are obtained by integrate over the cross-sectional area of the bar. Then, in a generic case, on the face of the cross section (according to the system adopted) we can appear the stresses σ_x, τ_{xy} and τ_{xz}.

If we make a cut in the bar according to the plane defined by Π, the stress state at an arbitrary point is the one as indicated in Figure 6.134.

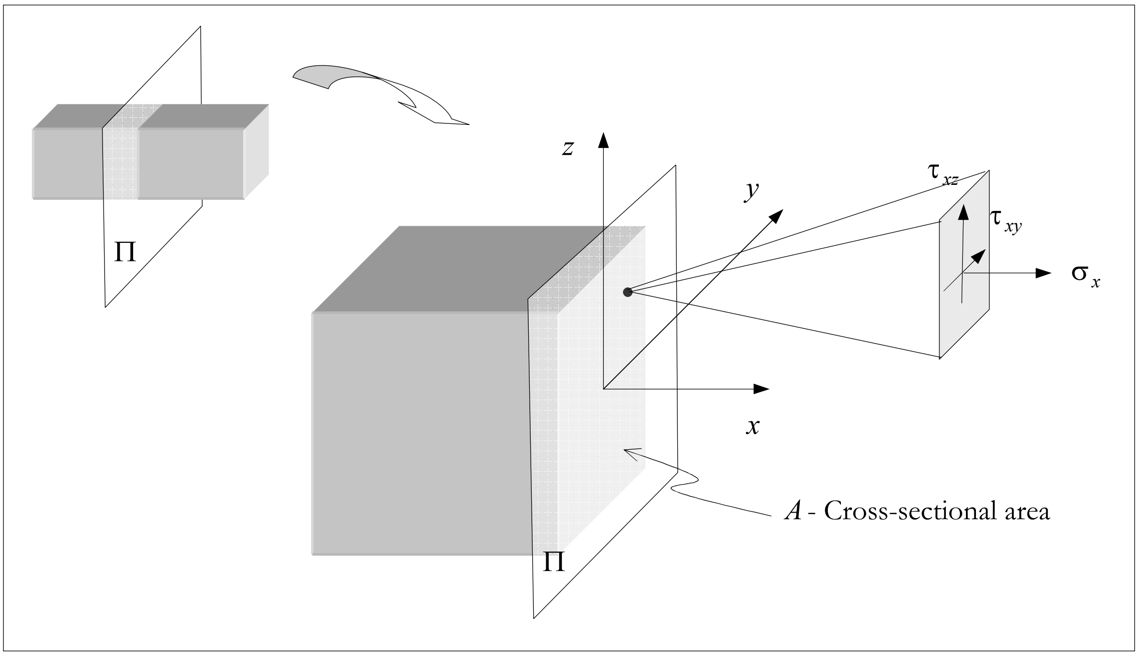


Figure 6.134: Stress field on the cross section.

The next step is to establish how the stress field varies on the cross section.

As the material is elastic and linear, the stress varies linearly with deformation ($\sigma_x = E\varepsilon_x$). In addition, as we are dealing with the small deformation regime the relationship $\varepsilon_x = \dfrac{\partial u}{\partial x}$ holds. Then, if the displacement field $u(y,z)$ on the cross-sectional area defines a plane so the strain and stress do.

a) We can take the following possibilities:

The cross-sectional area displaces according x-direction. In this case the strain is constant on the cross section, as consequence the normal stress field on cross section is also constant. By integrate the normal stress over the area of the cross section we obtain the internal force N (the axial force) which could be positive (tensile) or negative (compression).

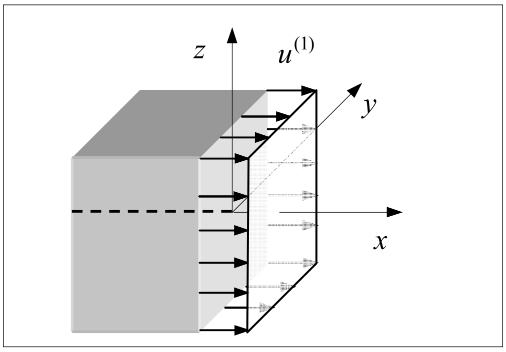


Figure 6.135

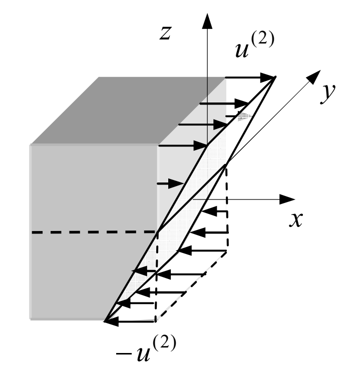


Figure 6.136

b) Another possibility for the displacement field on the cross section is when the cross section rotates about y-axis. In this case the displacement field is the one showed in Figure 6.136. The strain and normal stress also vary according to a plane on the cross section. Note that, if we integrate the normal stress over the area we obtain zero as result, i.e. the resultant force is zero, but there is moment according to the y-direction, so, the bar is subjected to *pure bending*. We denote by M_y the bending moment according to the y-direction. The displacement $u^{(2)}$ can be caused when the bar is subjected to a deflection according to z-direction (displacement $w(x)$).

c) Another possibility is when the cross section rotates about the z-axis, (see Figure 6.137). In this case the resultant force equals zero, and there is a bending moment according the z-direction which is denoted by M_z. The displacement $u^{(3)}$ can be caused when the bar is subjected to a deflection according to the y-direction (displacement $v(x)$).

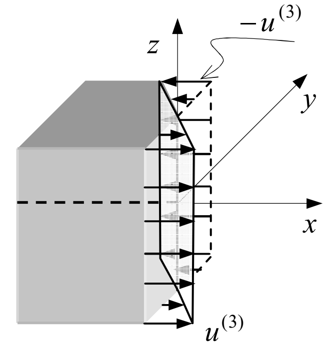


Figure 6.137

The combination of the previous cases is also possible. In general, the normal stress field σ_x on the cross section is illustrated in Figure 6.138.

If we consider Figure 6.139, we can also express the bending moment M_y as follows:

$$M_y = \int_A \sigma_x^{(2)} z dA = \int_A \frac{\sigma_S z}{c} z dA = \frac{\sigma_S}{c} \int_A z^2 dA = \frac{\sigma_S}{c} I_y \tag{6.409}$$

where $I_y = \int_A z^2 dA$ is the inertia moment of area about the y-direction. Taking into account that $\frac{\sigma_S}{c} = \frac{\sigma_x^{(2)}}{z}$ we can also obtain:

$$\sigma_x^{(2)}(z) = \frac{M_y}{I_y} z \tag{6.410}$$

Similarly, we can obtain:

$$\sigma_x^{(3)}(y) = \frac{-M_z}{I_z} y \tag{6.411}$$

Taking into account that $\sigma_x = E\varepsilon_x$, the above equations can be rewritten as follows:

$$\varepsilon_x^{(2)}(z) = \frac{M_y}{EI_y} z \quad ; \quad \varepsilon_x^{(3)}(y) = \frac{-M_z}{EI_z} y \tag{6.412}$$

where EI_{x_i} is the *modulus of flexural rigidity* about x_i of the bar.

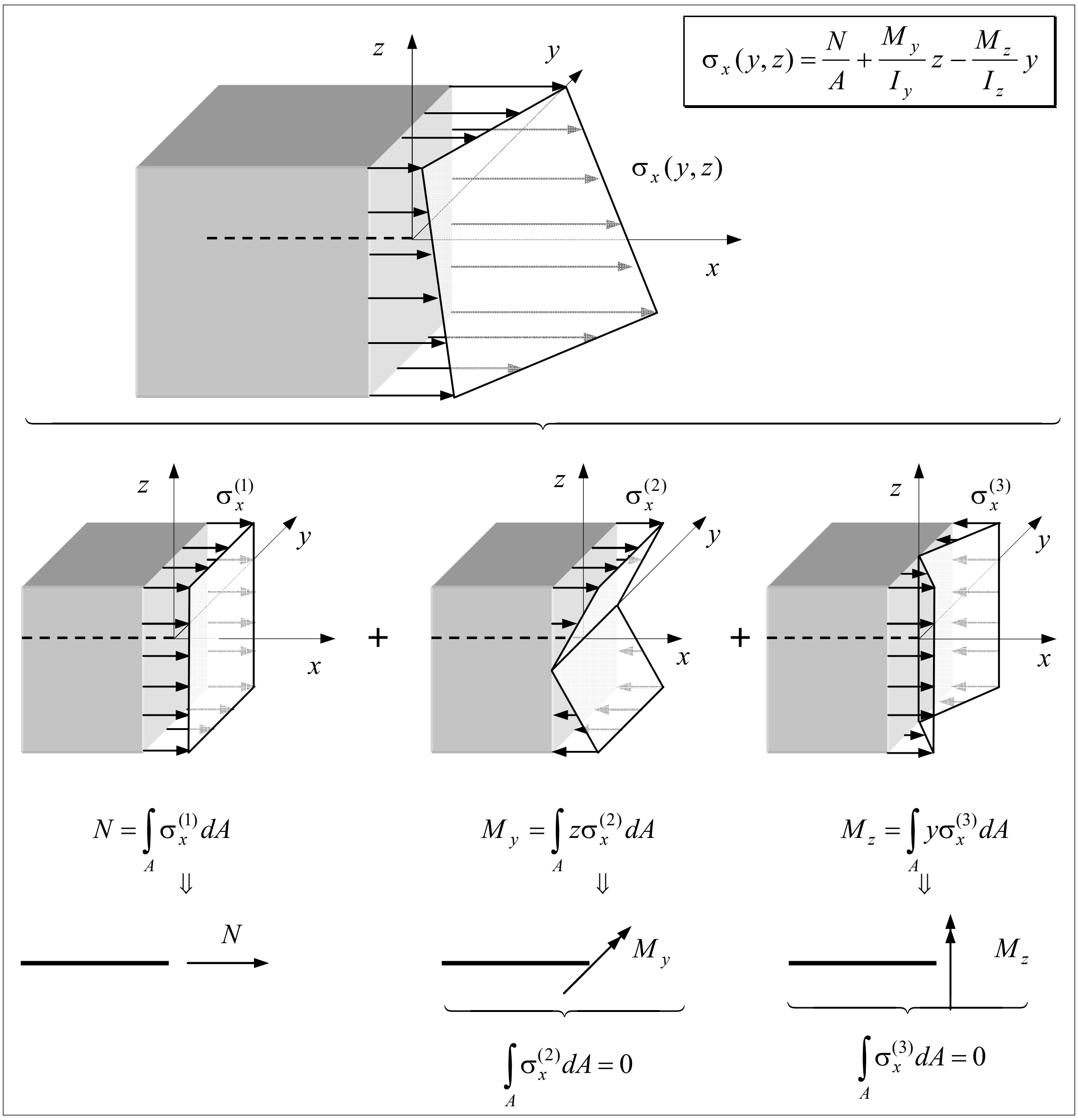


Figure 6.138: The axial force and bending moments.

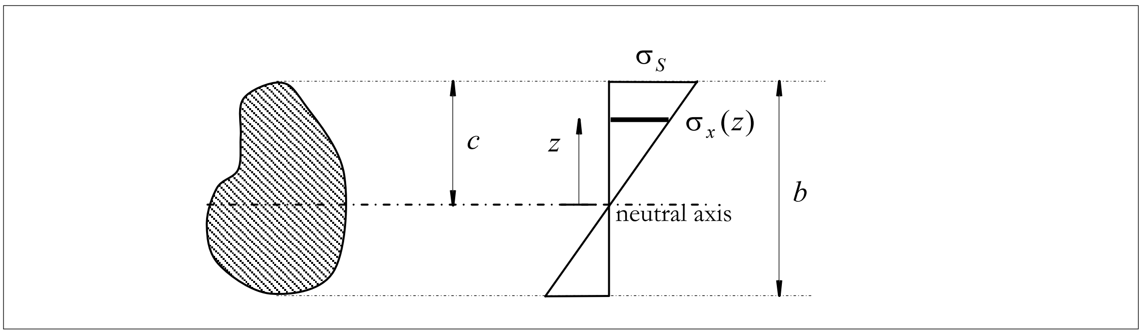


Figure 6.139: Normal stress distribution on the cross section.

Shearing forces and moment of torsion, (see **Problem 4.22**):

By integrate the tangential stresses (shearing stresses) τ_{xy} and τ_{xz} over the cross-sectional area we obtain the shearing forces Q_y and Q_z, respectively, (see Figure 6.140):

$$Q_y = \int_A \tau_{xy} dA \quad ; \quad Q_z = \int_A \tau_{xz} dA \tag{6.413}$$

and the moment of torsion (torque) (M_T):

$$M_T = \int_A (\tau_{xz} y - \tau_{xy} z) dA \tag{6.414}$$

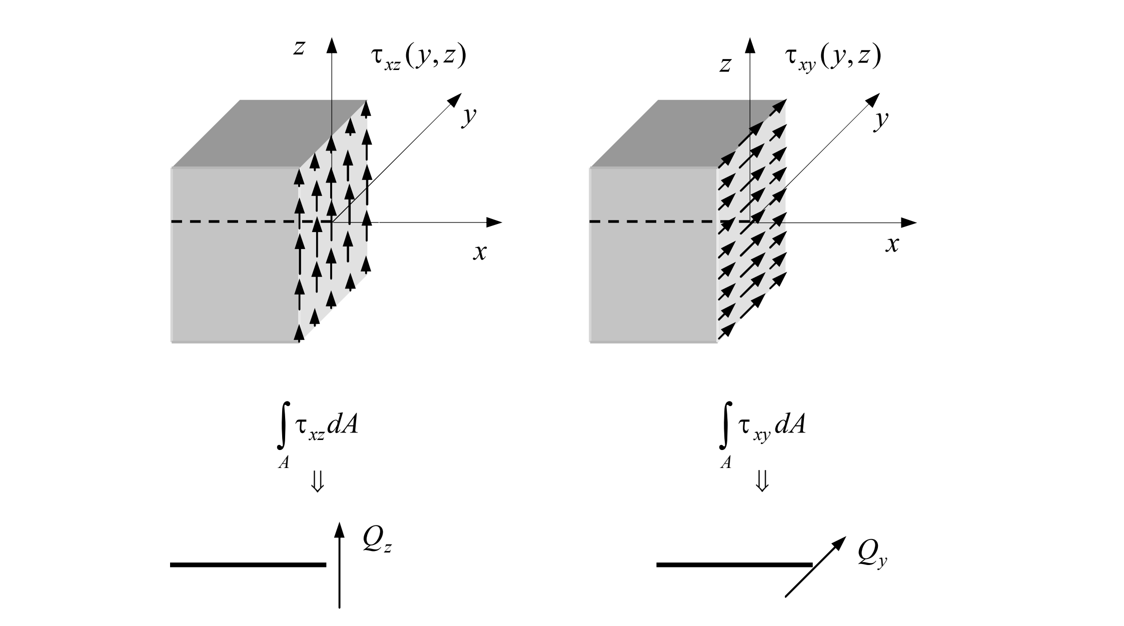


Figure 6.140: Tangential stresses – Shearing forces.

We summarize in Figure 6.141 the internal forces (N, Q_y, Q_z) and internal moments (M_x, M_y, M_z) that could appear on the cross section of the bar.

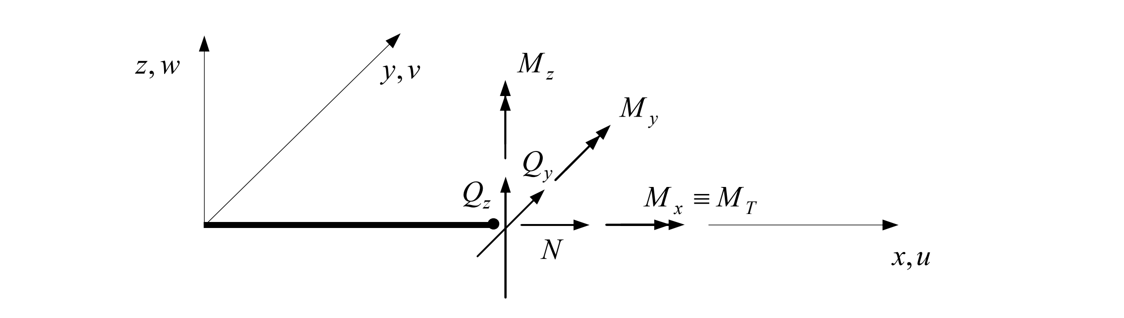


Figure 6.141: Internal forces and internal moments on the cross section of the bar.

NOTE 1: The internal forces and internal moments depend on the external actions (loads) in which the bar is subjected, (see Figure 6.142).

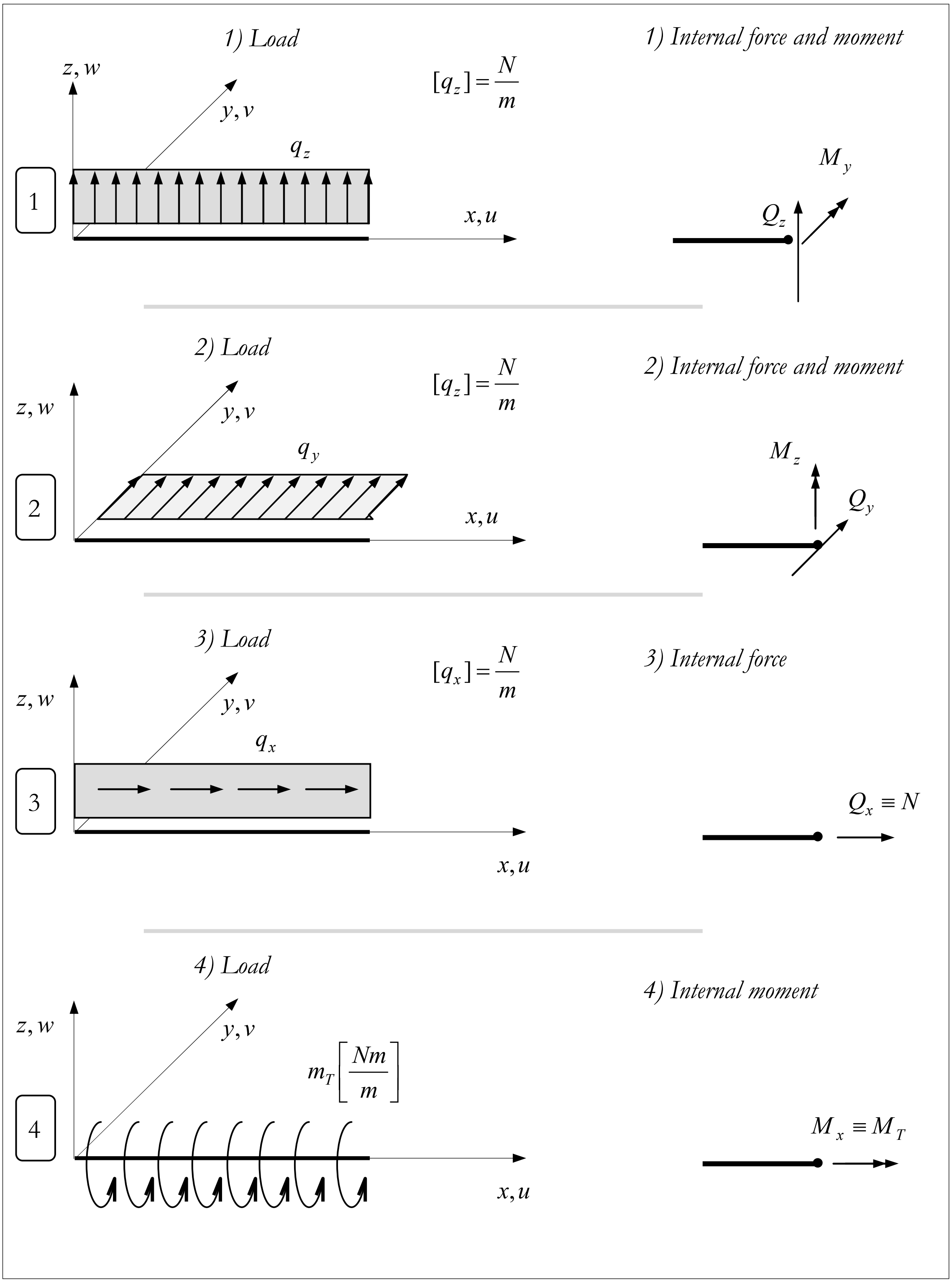


Figure 6.142: Some cases of external loads.

NOTE 2: Warping of the Cross Section

In **Problem 6.44** we have shown that only when we are dealing with a circular cross section the section remains planar after the torsion is applied.

The warping of the cross section appears due to the non-homogeneous tangential stress field on the cross section, (see Figure 6.144(a)). In circular section there is no warping since the tangential stress varies linearly according to the radius as show in Figure 6.144(b).

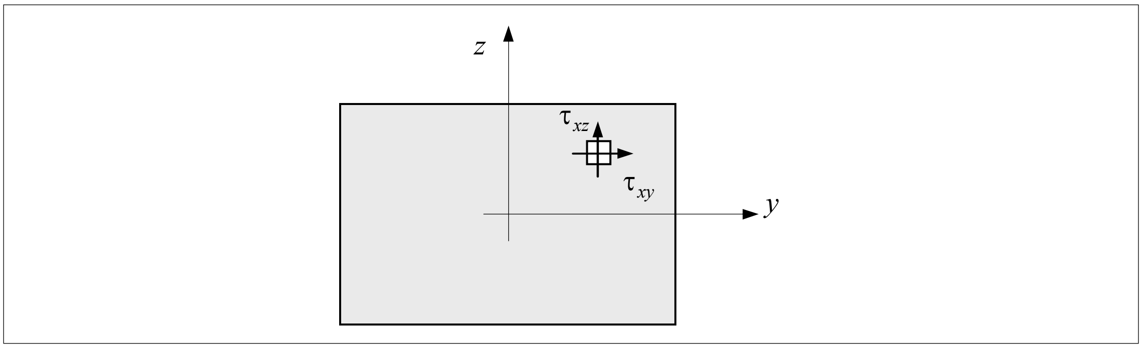


Figure 6.143: Tangential stresses (shearing stresses) – Moment of torsion.

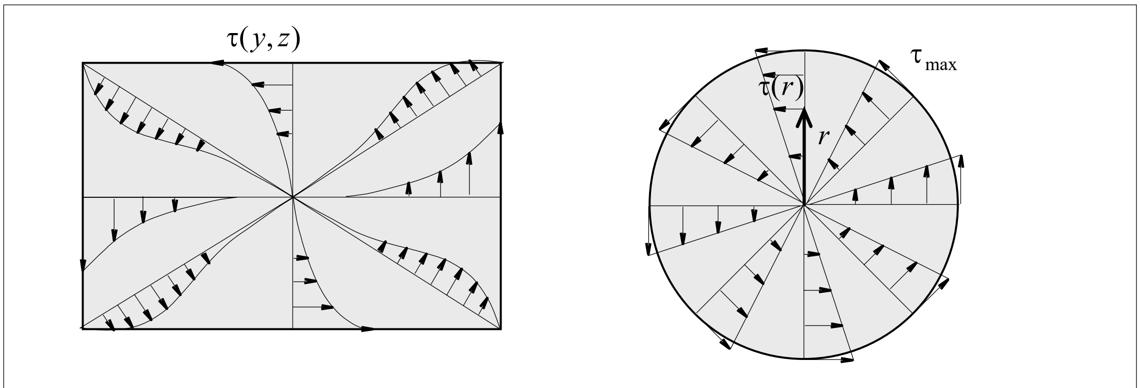


Figure 6.144: Distribution of the tangential stress.

NOTE 3: Deflection of the Bar

Initially, let us consider the defection only according to the z-direction (displacement $w(x)$), (see Figure 6.145). By means of Figure 6.145 we can conclude that:

$$
\begin{aligned}
&u^{(2)} = -w_{,x}\, z \quad \Rightarrow \quad \varepsilon_x^{(2)}(z) = \frac{du^{(2)}}{dx} = -\frac{d^2 w}{dx^2} z \equiv -w_{,xx}\, z \\
&\sigma_x^{(2)}(z) = E\varepsilon_x^{(2)} = -Ew_{,xx}\, z
\end{aligned}
\tag{6.415}
$$

Note that, if we compare the equations (6.416) and (6.415) they have the reversed sign since the bending moment M_z (same direction and sense as z-axis) produces the displacement field contrary as the one presented in Figure 6.145.

Then, if we consider the defection of the bar according to the y-direction (displacement $v(x)$), (see Figure 6.146), we can obtain:

$$
\begin{aligned}
&u^{(3)} = -v_{,x}\, y \qquad \Rightarrow \qquad \varepsilon_x^{(3)}(y) = \frac{du^{(3)}}{dx} = -\frac{d^2 v}{dx^2} y \equiv -v_{,xx}\, y \\
&\sigma_x^{(3)}(y) = E\varepsilon_x^{(3)} = -Ev_{,xx}\, y
\end{aligned}
\tag{6.416}
$$

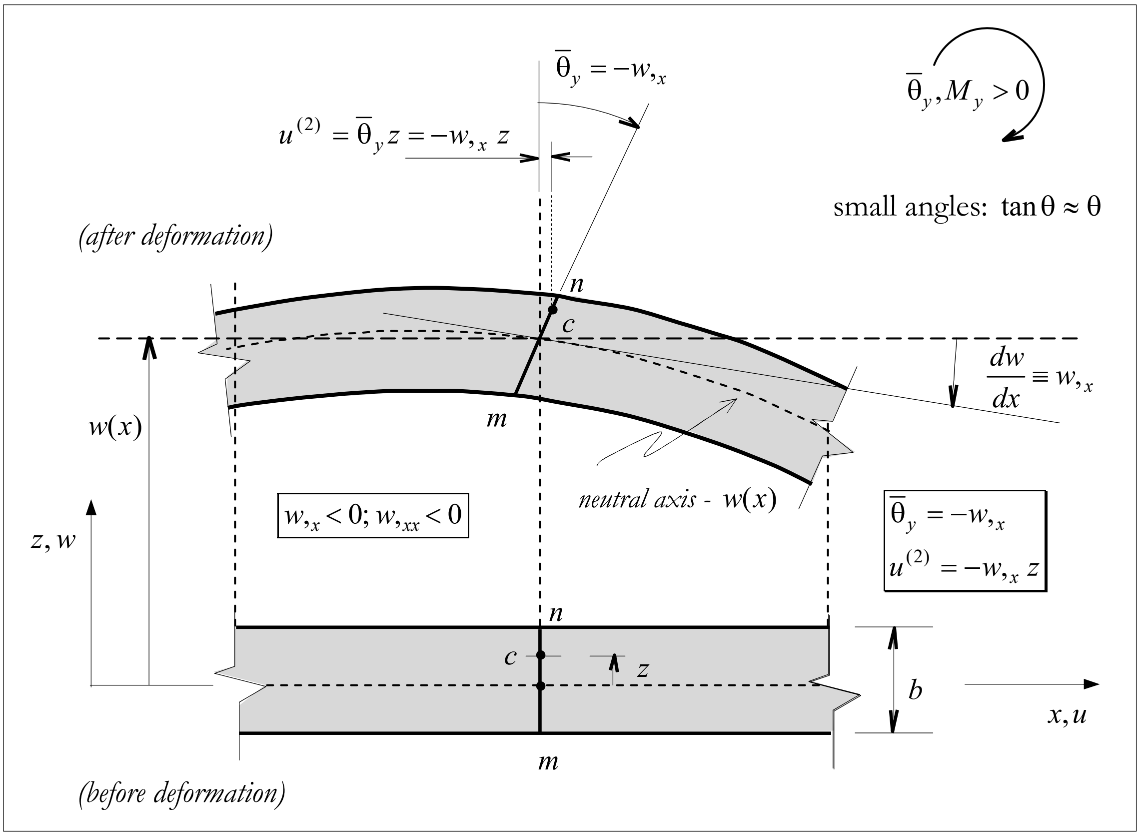


Figure 6.145: Displacement on the transversal cross section due to M_y.

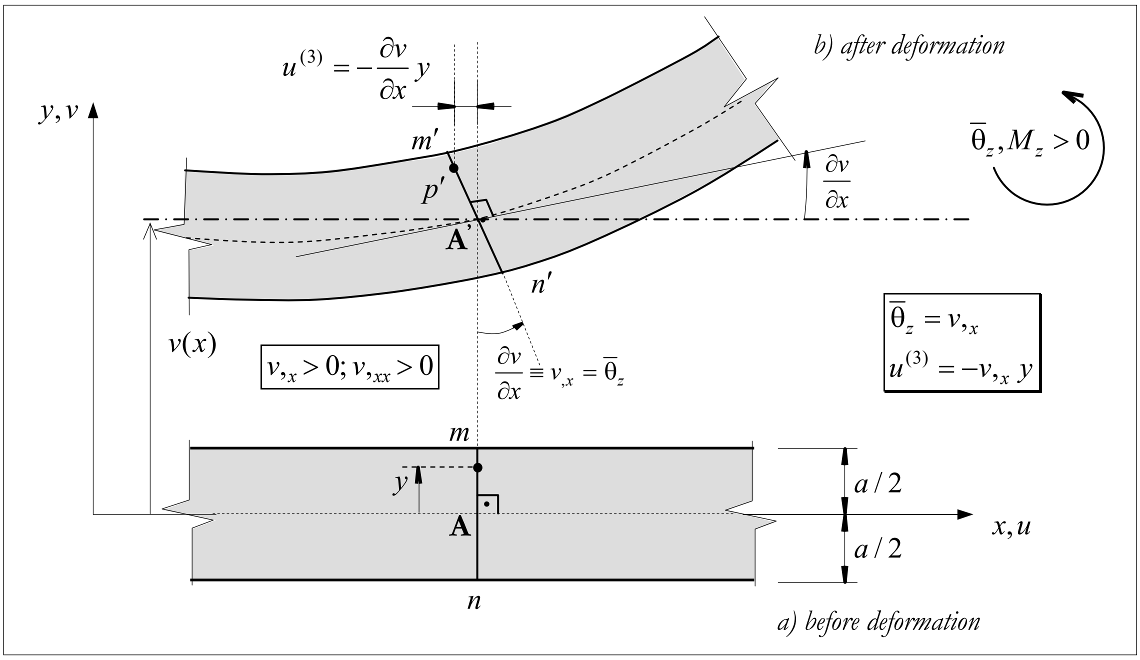


Figure 6.146: Displacement on the transversal cross section due to M_z.

If we compare the equations (6.410) and (6.415) we can conclude that:

$$\left.\begin{aligned}\sigma_x^{(2)}(z) &= \frac{M_y}{I_y} z \\ \sigma_x^{(2)}(z) &= -E w_{,xx} z\end{aligned}\right\} \quad \Rightarrow \quad w_{,xx} = \frac{-M_y}{EI_y} \tag{6.417}$$

Similarly, if we compare the equations (6.411) and (6.416) we can conclude:

$$\left.\begin{aligned}\sigma_x^{(3)}(y) &= \frac{-M_z}{I_z} y \\ \sigma_x^{(3)}(y) &= -E v_{,xx}\, y\end{aligned}\right\} \quad \Rightarrow \quad v_{,xx} = \frac{M_z}{EI_z} \tag{6.418}$$

Note that, to obtain the equations (6.417) and (6.418) we have already considered the *kinematic equations* and the *constitutive equations*. To complete the governing equations of the IBVP we have introduce the *equilibrium equations*. As in the cross section of the bar we have lost the information of the symmetry of the Cauchy stress tensor we have to apply the Principle of Linear Momentum (equilibrium equations or Summation of forces equal zero) and the Principle of the Angular Momentum (summation of the moments equal zero). In other words, we have to apply the summation of forces and moments equal zero in the differential element of bar dx, (see **Problem 6.60**).

NOTE 3.1: As additional information, let us consider now that the beam element is lying on the y-axis as indicated in Figure 6.147. Then, we can conclude that:

$$\begin{gathered} v = -w_{,y} z \quad \Rightarrow \quad \varepsilon_y(z) = \frac{dv}{dy} = -\frac{d^2 w}{dy^2} z \equiv -w_{,yy} z \\ \sigma_y(z) = E\varepsilon_y = -E w_{,yy} z \end{gathered} \tag{6.419}$$

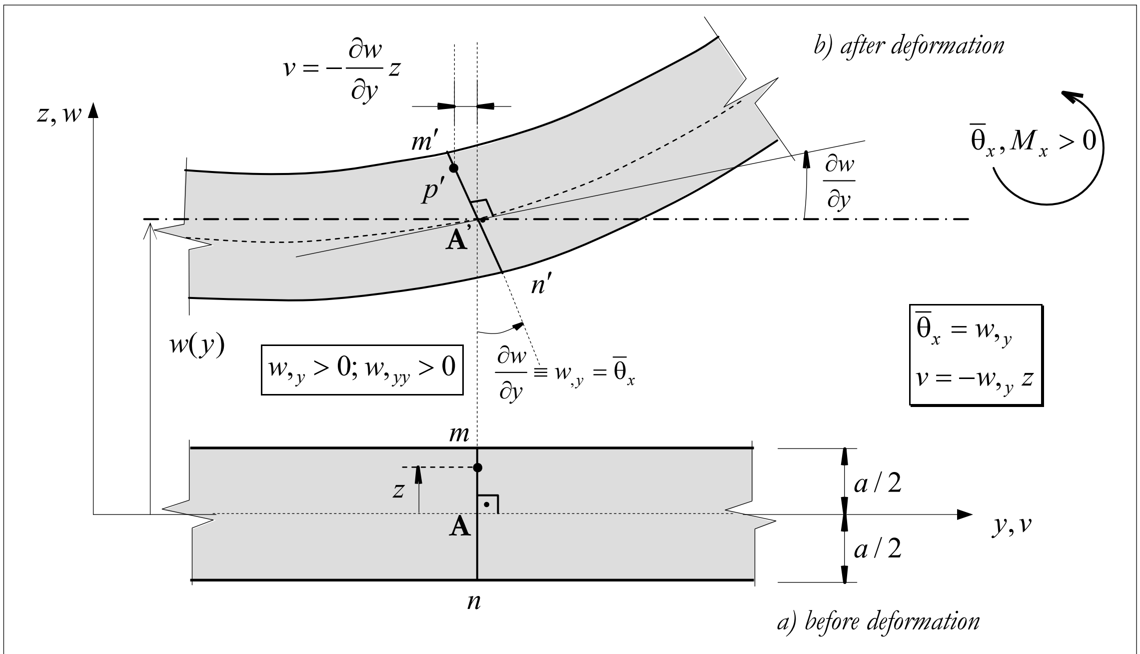


Figure 6.147: Displacement on the transversal cross section $(m\text{-}n)$ due to M_y.

NOTE 4: Tangential stress on the cross section of the bar

Let us suppose that we have several layers of smooth plates as indicated in Figure 6.148 (a). As the plates can displace freely between layers after the load is applied, (see Figure 6.148 (b)), there is no tangential stress between layers. If all plates are united to form a single monolithic piece (Figure 6.148 (c)), the displacement between layers is limited, and so the tangential stress appears. This tangential stress is what cause the appearance of the tangential stress on the cross section, (see Figure 6.148 (d)). Recall from **Problem 4.22**-NOTE 4 that the shearing force is caused by the variation of bending moment.

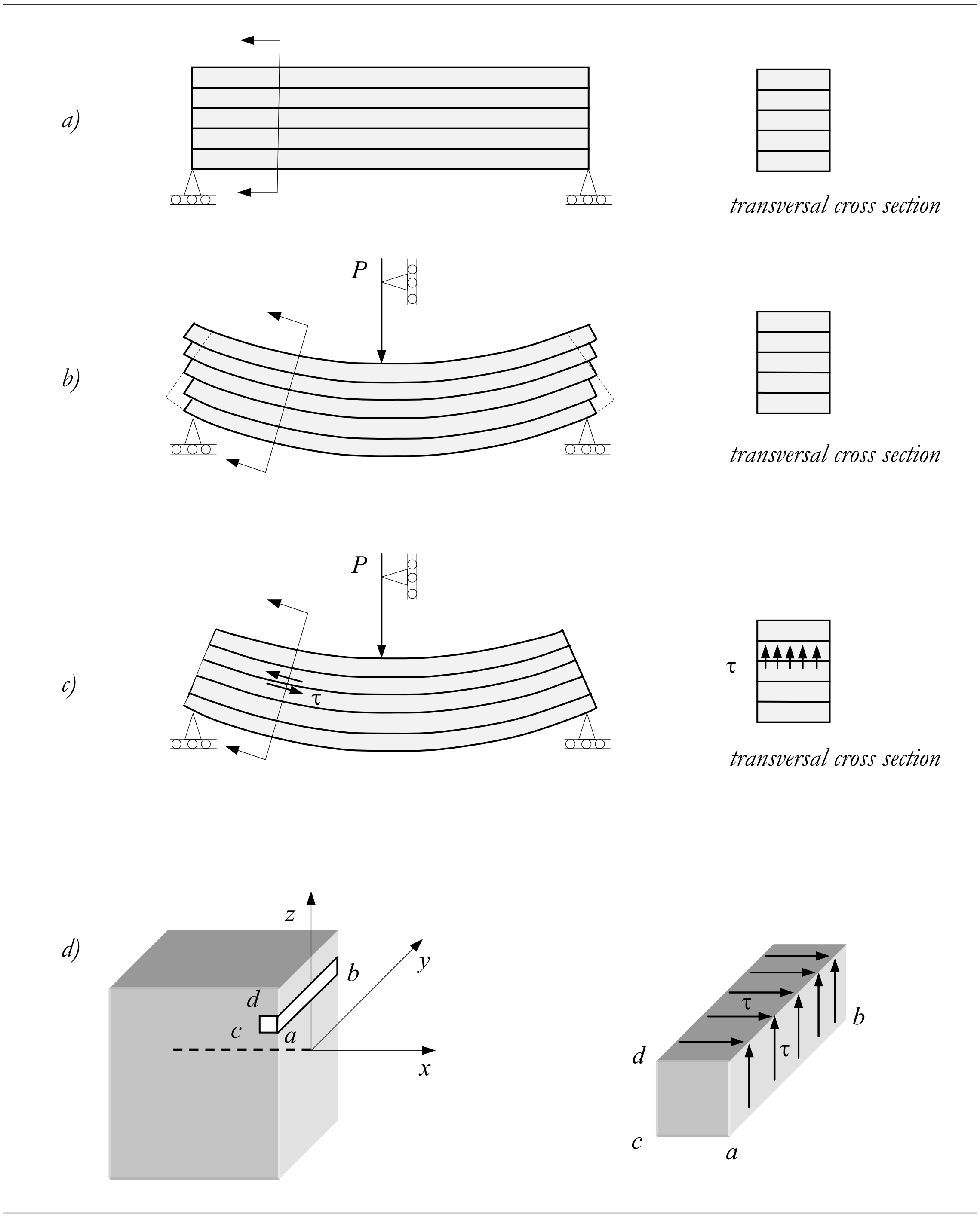


Figure 6.148: Beam subjected to bending.

Problem 6.60

Obtain the governing equation for the beam in which the flexural rigidity (EI_y) is constant. The beam is subjected to an uniformly distributed load per unit length (q_z), (see Figure 6.149).

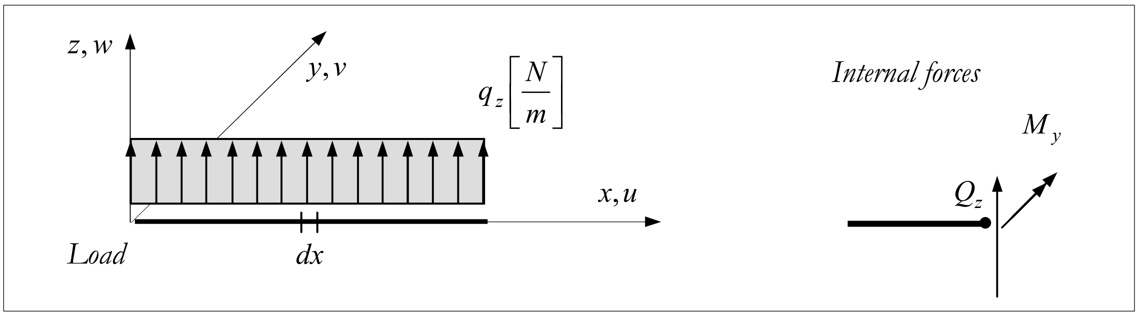


Figure 6.149: Beam subjected to uniformly distributed load.

Solution:

Let us consider the differential beam element dx in which the internal forces are indicated in Figure 6.150.

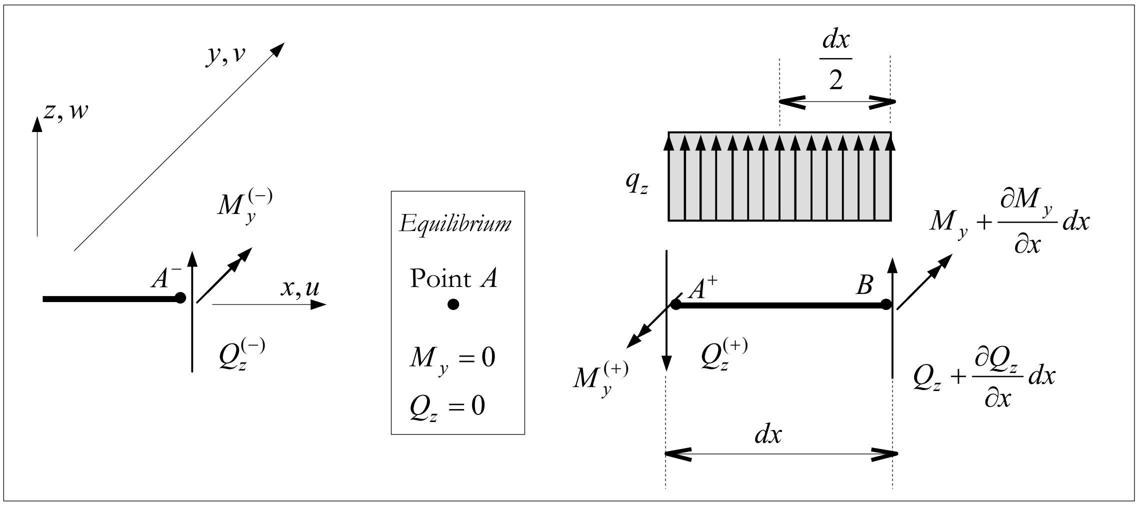


Figure 6.150: Differential beam element.

By applying the equilibrium of force and moment (at point B) in the differential beam element we can obtain:

$$\sum F_z = 0 \quad \Rightarrow \quad -Q_z + \left(Q_z + \frac{\partial Q_z}{\partial x} dx\right) + q_z dx = 0 \quad \Rightarrow \quad \frac{\partial Q_z}{\partial x} = -q_z \tag{6.420}$$

$$\sum M_{yB} = 0 \;\Rightarrow\; -M_y - Q_z dx + \left(M_y + \frac{\partial M_y}{\partial x} dx\right) + q_z dx \frac{dx}{2} = 0 \quad \Rightarrow \quad \frac{\partial M_y}{\partial x} = Q_z \tag{6.421}$$

where we have considered that $dxdx \approx 0$.

To complete the governing equation we have to introduce the kinematic and constitutive equations. Note that the equation in (6.417) is already considering these equations, then

$$w,_{xx} = -\frac{M_y}{EI_y} \quad \Rightarrow \quad M_y = -EI_y w,_{xx}$$

$$\Rightarrow \frac{\partial M_y}{\partial x} = Q_z = -\frac{\partial}{\partial x}\left(EI_y w,_{xx}\right) = -EI_y w,_{xxx} \equiv -EI_y \frac{\partial^3 w}{\partial x^3}$$
$$\Rightarrow \frac{\partial^2 M_y}{\partial x^2} = \frac{\partial Q_z}{\partial x} = -\frac{\partial^2}{\partial x^2}\left(EI_y w,_{xx}\right) = -EI_y \frac{\partial^4 w}{\partial x^4} = -q_z \tag{6.422}$$

with that the beam differential equation can be represented by:

$$\boxed{\frac{\partial Q_z}{\partial x} = -q_z} \quad ; \quad \boxed{\frac{\partial M_y}{\partial x} = Q_z}$$
$$\boxed{\frac{\partial^2 w}{\partial x^2} = \frac{-M_y}{EI_y}} \quad \text{or} \quad \boxed{\frac{\partial^3 w}{\partial x^3} = \frac{-Q_z}{EI_y}} \quad \text{or} \quad \boxed{\frac{\partial^4 w}{\partial x^4} = \frac{q_z}{EI_y}} \tag{6.423}$$

NOTE 1:

Now, if we consider the beam presented in Figure 6.151 and by considering the equilibrium we can obtain:

$$\sum F_y = 0 \quad \Rightarrow \quad -Q_y + \left(Q_y + \frac{\partial Q_y}{\partial x}dx\right) + q_y dx = 0 \quad \Rightarrow \quad \boxed{\frac{\partial Q_y}{\partial x} = -q_y}$$

$$\sum M_{zB} = 0 \;\Rightarrow\; -M_z + Q_y dx + \left(M_z + \frac{\partial M_z}{\partial x}dx\right) - q_y dx \frac{dx}{2} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial M_z}{\partial x} = -Q_y}$$

and

$$\boxed{\frac{\partial^2 v}{\partial x^2} = \frac{M_z}{EI_z}} \quad \text{or} \quad \boxed{\frac{\partial^3 v}{\partial x^3} = \frac{Q_y}{EI_z}} \quad \text{or} \quad \boxed{\frac{\partial^4 v}{\partial x^4} = \frac{-q_y}{EI_z}} \tag{6.424}$$

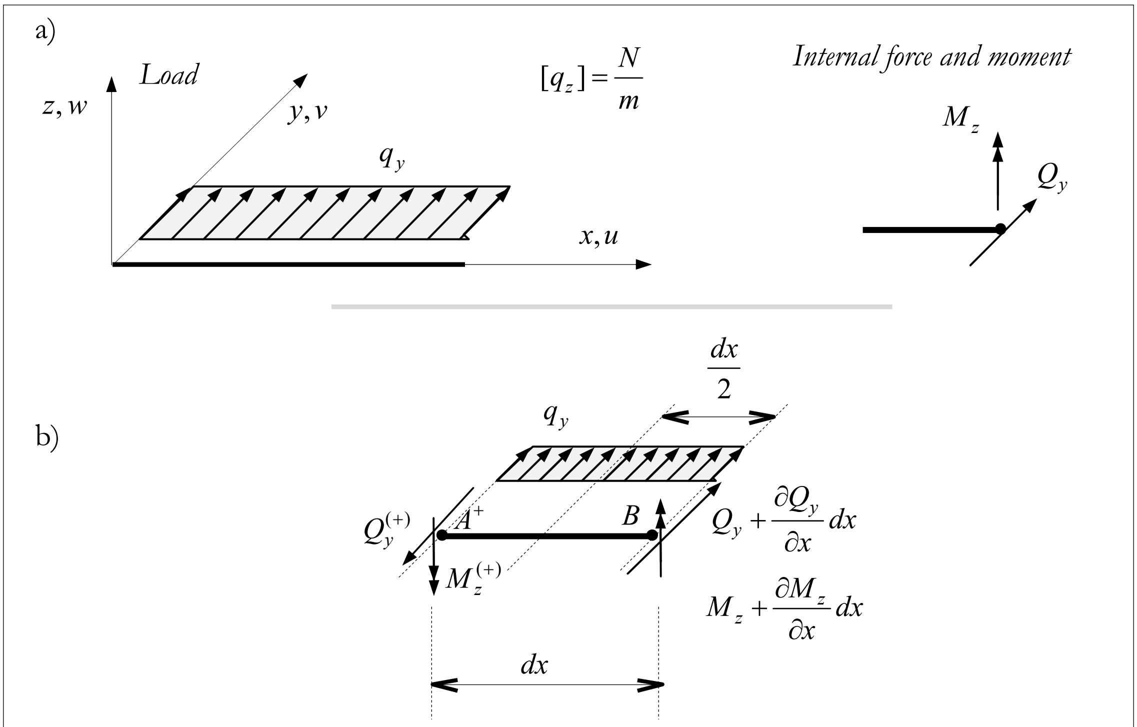


Figure 6.151: Differential beam element.

NOTE 2: Tangential stress on the cross section

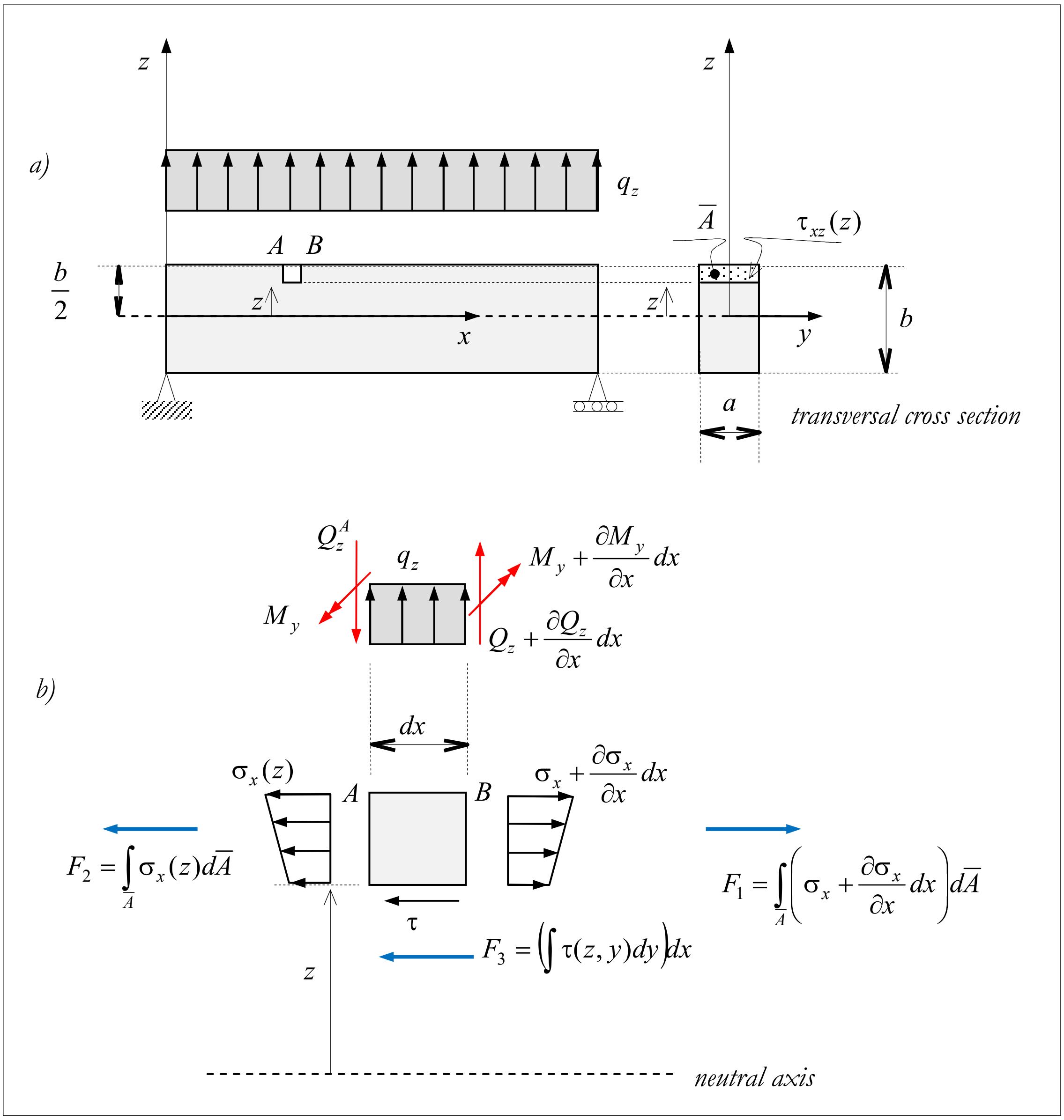


Figure 6.152:

Applying the force equilibrium according the x-direction, (see Figure 6.152 (b)), we can obtain:

$$
\begin{aligned}
&\sum F_x = 0 \;\Rightarrow\; F_1 - F_2 - F_3 = 0 \;\Rightarrow\; \int_{\bar{A}} \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) d\bar{A} - \int_{\bar{A}} \sigma_x d\bar{A} - dx \int \tau(z,y) dy = 0 \\
&\Rightarrow dx \int \tau(z,y) dy = \int_{\bar{A}} \left(\frac{\partial \sigma_x(z)}{\partial x} dx \right) d\bar{A}
\end{aligned}
\tag{6.425}
$$

If we invoke the equations (6.410) and (6.421) we can obtain:

$$
\sigma_x^{(2)}(z) = \frac{M_y}{I_y} z \quad \Rightarrow \quad \frac{\partial \sigma_x^{(2)}(z)}{\partial x} = \frac{\partial}{\partial x} \left(\frac{M_y}{I_y} z \right) = \frac{\partial M_y}{\partial x} \frac{z}{I_y} = \frac{Q_z z}{I_y}
$$

where we are considering that I_y is constant along the beam. Taking into account the above equation into the equation in (6.425) we can obtain:

$$dx\int \tau(z,y)dy = \int_{\overline{A}}\left(\frac{\partial \sigma_x(z)}{\partial x}dx\right)d\overline{A} \quad \Rightarrow \Rightarrow dx\int \tau(z,y)dy = \int_{\overline{A}}\left(\frac{Q_z z}{I_y}dx\right)d\overline{A} = \frac{Q_z}{I_y}dx\int_{\overline{A}} z d\overline{A}$$

$$\Rightarrow \int \tau(z,y)dy = \frac{Q_z}{I_y}\int_{\overline{A}} z d\overline{A}$$

with that we can conclude that:

$$\boxed{\int \tau(z,y)dy = \frac{Q_z}{I_y}\int_{\overline{A}} z d\overline{A} = \frac{Q_z \chi_z}{I_y} \qquad \text{where} \qquad \chi_z = \int_{\overline{A}} z d\overline{A}} \tag{6.426}$$

The above equation was also obtained in **Problem 4.29**. For the cross section $a\times b$ and if we considering as a good approximation that

$$\tau_{ave}(z)a = \int \tau(z,y)dy$$

where $\tau_{ave}(z)$ is the average of the tangential stress at (z).

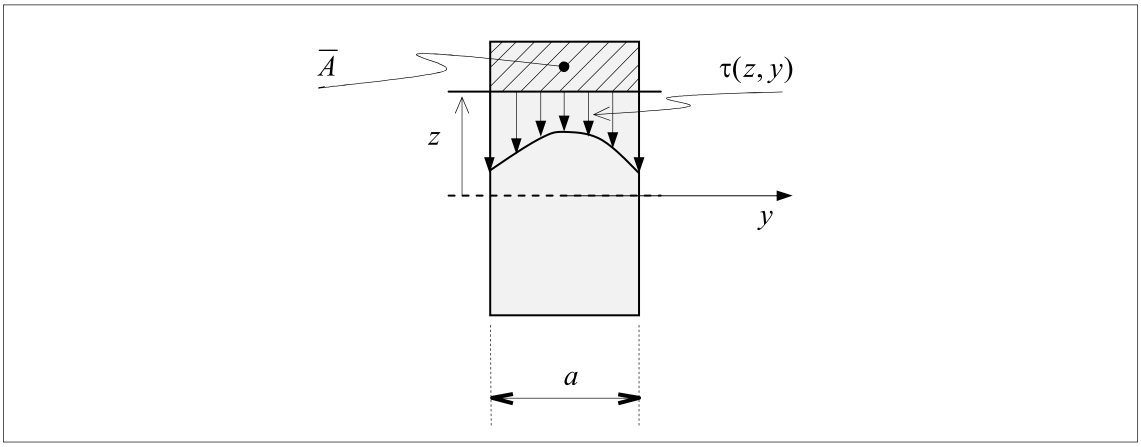


Figure 6.153: Tangential stress on the cross section.

Then, we can obtain:

$$\tau_{ave}(z) = \frac{Q_z \chi_z}{I_y} \qquad \Rightarrow \qquad \tau_{ave}(z) = \frac{Q_z \chi_z}{I_y a}$$

References of Mechanics of Materials

BEER, F.P.; JOHNSTON (JR.), E.R.; DEWOLF, J.T. & MAZUREK, D.F. (2012). *Mechanics of Materials*. 6th Edition. McGraw-Hill, New York-USA.

Problem 6.61

Show:

a) Mohr's First Theorem

"The change in slope of a defection curve between two points is equal to the area diagram of $\frac{M_y}{EI_y}$ between these two points."

b) Mohr's Second Theorem

Let us consider a beam subjected to bending between two point, namely A and B. Let Δd be the distance between the point B and point D, where D is the point intercepted by the tangent line at the point A and the vertical line through the point B, (see Figure 6.154). The Mohr's second theorem states:

"The distance Δd is equal to the first moment of the diagram $\dfrac{M_y}{EI_y}$ about the axis where Δd is measured."

a) Mohr's first theorem

Solution:

We have seen at the end of Chapter 1 (Complementary Note 1 - curvature) that the following relationships are true:

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{dx}\frac{dx}{ds} = \frac{w_{,xx}}{[1+(w_{,x})^2]}\frac{1}{[1+(w_{,x})^2]^{\frac{1}{2}}} = \frac{w_{,xx}}{[1+(w_{,x})^2]^{\frac{3}{2}}} = \frac{1}{r} \tag{6.427}$$

$$\int \kappa ds = \int_A^B d\psi = \psi_B - \psi_A \equiv \Delta\psi_{B\_A} \tag{6.428}$$

where $w_{,x} \equiv \dfrac{\partial w}{\partial x}$, $w_{,xx} \equiv \dfrac{\partial^2 w}{\partial x^2}$, in which w stands for the deflection, κ is the curvature, and ds is the differential arc-length element.

For small curvature it fulfils:

$$\kappa = \frac{1}{r} \approx w'' \equiv \frac{\partial^2 w}{\partial x^2} \equiv w_{,xx} \quad ; \quad ds \approx dx \quad ; \quad \tan\psi \approx \psi \quad ; \quad \cos\psi \approx 1 \quad ; \quad \frac{d\psi}{ds} \approx \frac{d\psi}{dx}$$

Taking into account the equation (6.417) we can conclude that:

$$w_{,xx} = \frac{-M_y}{EI_y} \quad \Rightarrow \quad \kappa = \kappa_y = \frac{-M_y}{EI_y}$$

and if we apply the equation in (6.428) we can obtain:

$$\int_A^B \kappa ds = \int_A^B d\psi = \Delta\psi_{B\_A} \quad \Rightarrow \quad -\int_A^B \frac{M_y}{EI_y} ds \approx -\int_A^B \frac{M_y}{EI_y} dx = \int_A^B d\psi = \Delta\psi_{B\_A} \tag{6.429}$$

where the expression $\displaystyle\int \frac{M_y}{EI_y} ds$ is the area of the diagram defined by $\dfrac{M_y}{EI_y}$, (see Figure 6.154).

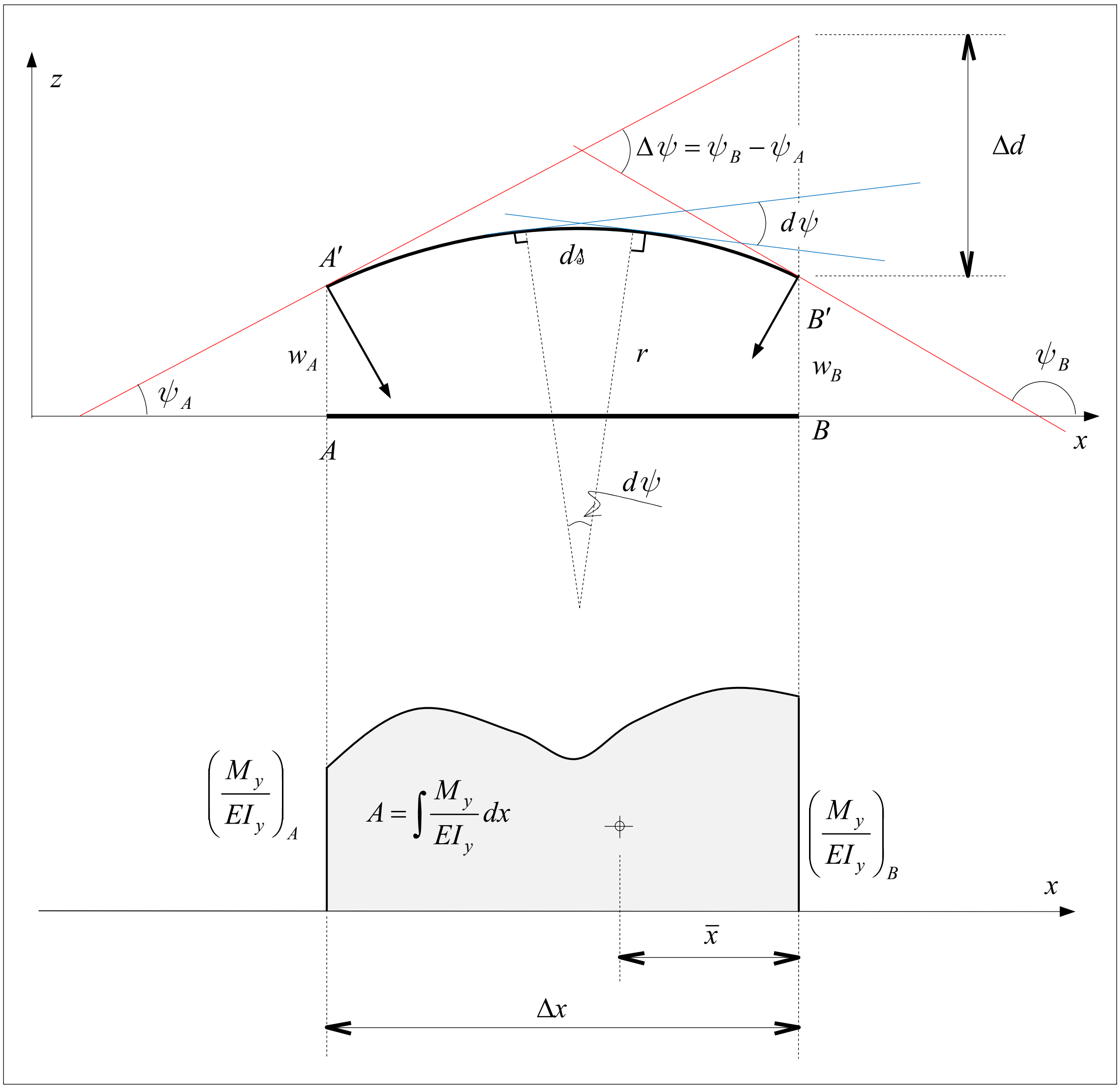


Figure 6.154

b) Mohr's second theorem

By multiply the equation (6.427) by x we can obtain:

$$\kappa x = \frac{d\psi}{ds} x \xrightarrow{\text{integrating}} \int_A^B \kappa x ds = \int_A^B x d\psi \quad \Rightarrow \quad \int_A^B -\frac{M_y}{EI_y} x dx = \int_A^B x d\psi = (x\psi)_B - (x\psi)_A \approx \Delta d$$

$$\Rightarrow -\left(\int_A^B \frac{M_y}{EI_y} dx\right)\bar{x} = \int_A^B x d\psi = (x\psi)_B - (x\psi)_A \approx \Delta d$$

where $\int_A^B \frac{M_y}{EI_y} x dx$ is the first moment of area of the diagram $\frac{M_y}{EI_y}$.

6.5.1 The Potential Energy for 1D Structural Element

Problem 6.62

Obtain the internal potential energy for the bar element of length L in function of forces and moments, (see Figure 6.155).

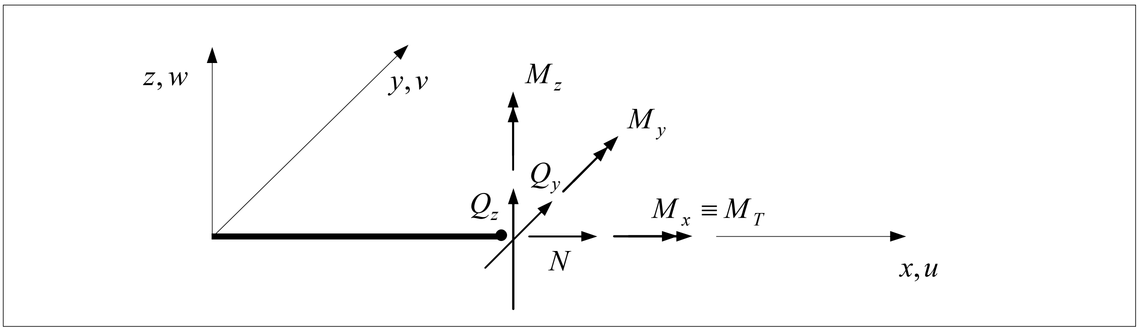


Figure 6.155: Forces and moments in the bar.

Solution:

The internal potential energy is given by:

$$
\begin{aligned}
U^{int} &= \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} dV = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij} dV \\
&= \frac{1}{2}\int_V (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{13}\varepsilon_{13}) dV \\
&= \frac{1}{2}\int_V (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}) dV
\end{aligned}
$$

On the transversal cross section of the bar, the normal stress σ_x can be represented by $\sigma_x = \sigma_x^{(1)} + \sigma_x^{(2)} + \sigma_x^{(3)}$. The internal potential energy associated with the stress $\sigma_x^{(1)} = E\varepsilon_x^{(1)}$, (see Figure 6.138), can be expressed in function of the axial force N:

$$
U^{int} = \frac{1}{2}\int_V \sigma_x^{(1)}\varepsilon_x^{(1)} dV = \frac{1}{2}\int_V \frac{\sigma_x^{(1)2}}{E} dV = \frac{1}{2}\int_0^L \frac{N^2}{EA^2}\int_A dA dx = \frac{1}{2}\int_0^L \frac{N^2}{EA} dx \qquad (6.430)
$$

Similarly, we can obtain the internal potential energy associated with the normal stress $\sigma_x^{(2)} = E\varepsilon_x^{(2)}$ in terms of bending moment M_y, (see equation (6.412)):

$$
U^{int} = \frac{1}{2}\int_V \sigma_x^{(2)}\varepsilon_x^{(2)} dV = \frac{1}{2}\int_0^L \int_A \frac{M_y}{I_y} z \frac{M_y}{EI_y} z dA dx = \frac{1}{2}\int_0^L \frac{M_y^2}{EI_y^2}\int_A z^2 dA dx = \frac{1}{2}\int_0^L \frac{M_y^2}{EI_y} dx \qquad (6.431)
$$

Similarly, if we consider $\sigma_x^{(3)}$, (see equation (6.412)), we can obtain:

$$
U^{int} = \frac{1}{2}\int_0^L \frac{M_z^2}{EI_z} dx \qquad (6.432)
$$

The component $\tau_{xz}(z) = G\gamma_{xz}$ is associated with the shearing force Q_z, (see equation (6.413)), where G is the shear modulus (also called transversal elastic modulus). In equation (6.426) we have obtained that:

$$\int \tau(z,y)dy = \frac{Q_z}{I_y}\int_{\overline{A}} z d\overline{A} = \frac{Q_z \chi_z}{I_y} \qquad \text{where} \qquad \chi_z = \int_{\overline{A}} z d\overline{A} \tag{6.433}$$

Then, the internal potential energy associated with $\tau(z,y) = G\gamma$ is given by:

$$\begin{aligned} U^{int} &= \frac{1}{2}\int_V \tau\gamma dV = \frac{1}{2}\int_V \frac{\tau_{xz}^2}{G} dV = \frac{1}{2}\int_0^L \frac{1}{G}\int_A \tau^2 dA dx = \frac{1}{2}\int_0^L \frac{1}{G}\left(\int\int \tau^2 dy dz\right) dx \\ &= \frac{1}{2}\int_0^L \frac{1}{G}\left(\int \left(\frac{Q_z \chi_z}{I_y}\right)^2 dz\right) dx = \frac{1}{2}\int_0^L \frac{Q_z^2}{G}\left(\int \left(\frac{\chi_z}{I_y}\right)^2 dz\right) dx = \frac{1}{2}\int_0^L \frac{\varsigma_z Q_z^2}{GA} dx \end{aligned} \tag{6.434}$$

where ς_z is the shape factor of the cross section along z-axis, which is given by:

$$\varsigma_z = A\int \left(\frac{\chi_z}{I_y}\right)^2 dz \tag{6.435}$$

Similarly, we can obtain:

$$U^{int} = \frac{1}{2}\int_V \tau_{xy}\gamma_{xy} dV = \frac{1}{2}\int_V \frac{\tau_{xy}^2}{G} dV = \frac{1}{2}\int_0^L \frac{\varsigma_y Q_y^2}{GA} dx \qquad \text{where} \qquad \varsigma_y = A\int \left(\frac{\chi_z}{I_y}\right)^2 dz \tag{6.436}$$

If we consider a circular cross section, the tangential stress field in the cross section, (see **Problem 6.44**-NOTE 1), is given:

$$\tau(r) = \frac{M_T}{\mathtt{J}_T} r$$

Then, the internal `potential energy due to the tangential stress $\tau(r) = G\gamma(r)$ can be obtained by:

$$U^{int} = \frac{1}{2}\int_V \tau(r)\gamma(r) dV = \frac{1}{2}\int_V \frac{\tau^2}{G} dV = \frac{1}{2}\int_V \frac{1}{G}\left(\frac{M_T}{\mathtt{J}_T} r\right)^2 dV = \frac{1}{2}\int_0^L \frac{M_T^2}{G\mathtt{J}_T^2}\int_A r^2 dA dx = \frac{1}{2}\int_0^L \frac{M_T^2}{G\mathtt{J}_T} dx \tag{6.437}$$

where $\mathtt{J}_T = \int_A r^2 dA$ is the polar moment of inertia of the circular cross section. We can obtain an equivalent polar moment of inertia for another shape of the cross section which is denoted by J_{Eq}. Thus, we can write:

$$U^{int} = \frac{1}{2}\int_0^L \frac{M_T^2}{GJ_{Eq}} dx \tag{6.438}$$

Then, the internal potential energy for the bar element can be represented as follows:

$$\boxed{U^{int} = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M_y^2}{EI_y} + \frac{M_z^2}{EI_z} + \frac{\varsigma_y Q_y^2}{GA} + \frac{\varsigma_z Q_z^2}{GA} + \frac{M_T^2}{GJ_{Eq}}\right) dx} \tag{6.439}$$

NOTE 1: The External Potential Energy

Next we will provide some equations related to the external potential energy due to some external loads:

- Concentrated force: $U^{ext} = P w_p$, where P is the concentrated force and w_p is the deflection according to P-direction, (see Figure 6.156).

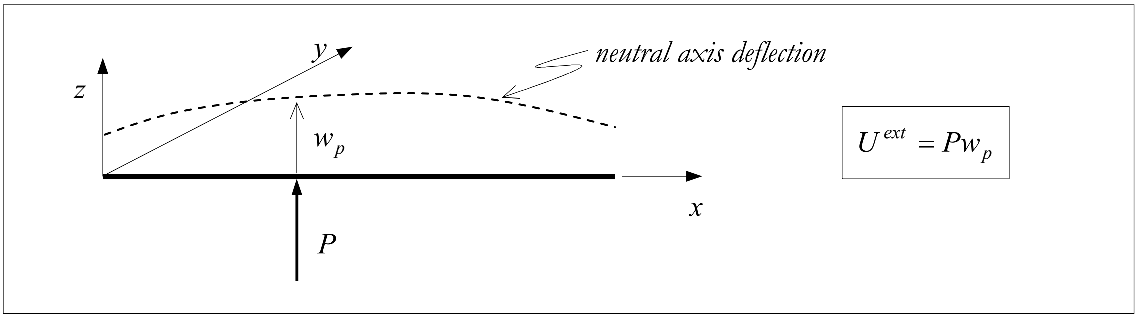


Figure 6.156: Concentrated force.

- Distributed load: $U^{ext} = \int_0^L q_z(x) w(x) dx$, where $q_z(x)$ - distributed load, $w(x)$ - deflection according to z-direction, (see Figure 6.157).

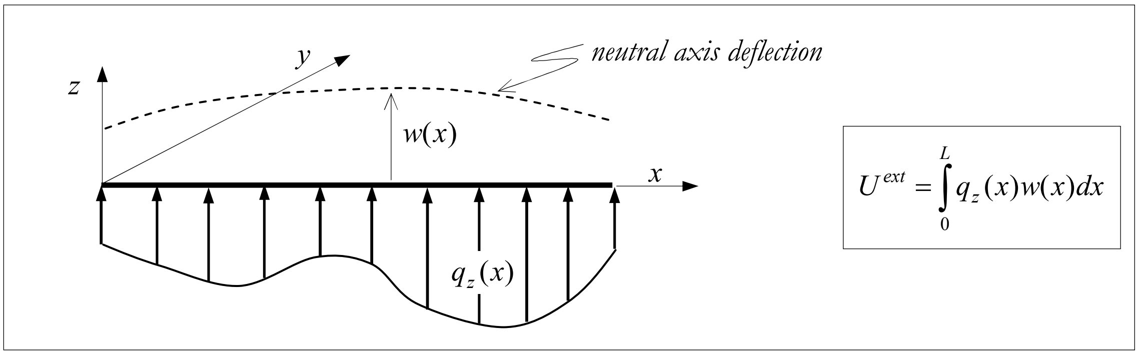


Figure 6.157: Uniformly distributed load.

In the case that $q_z(x)$ is uniformly distributed load we obtain $U^{ext} = q_z \int_0^L w(x) dx$.

- Concentrated moment load: $U^{ext} = M_y^A \overline{\theta}_{yA}$, (see Figure 6.158), where $\overline{\theta}$ stands for rotation.

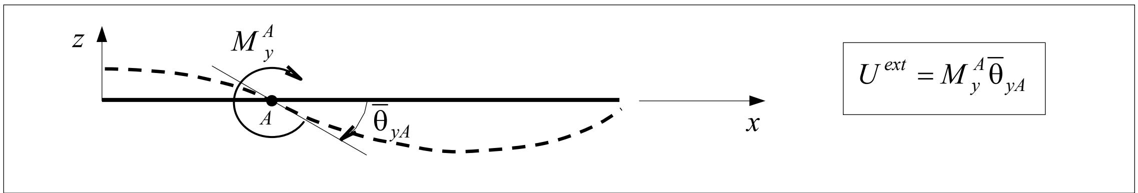


Figure 6.158: Concentrated moment at point A.

NOTE 2: Example of application. Let us consider a beam element in which the nodal forces (shear and moment) and nodal displacement (deflection and rotation) are indicated in Figure 6.159.

Let us consider that each node there is two "displacements" (degrees-of-freedom): deflection (w) and rotation ($\overline{\theta}_y = -\frac{dw}{dx} \equiv -w'$). We will use also the notation $\frac{d^2 w}{dx^2} \equiv w''$.

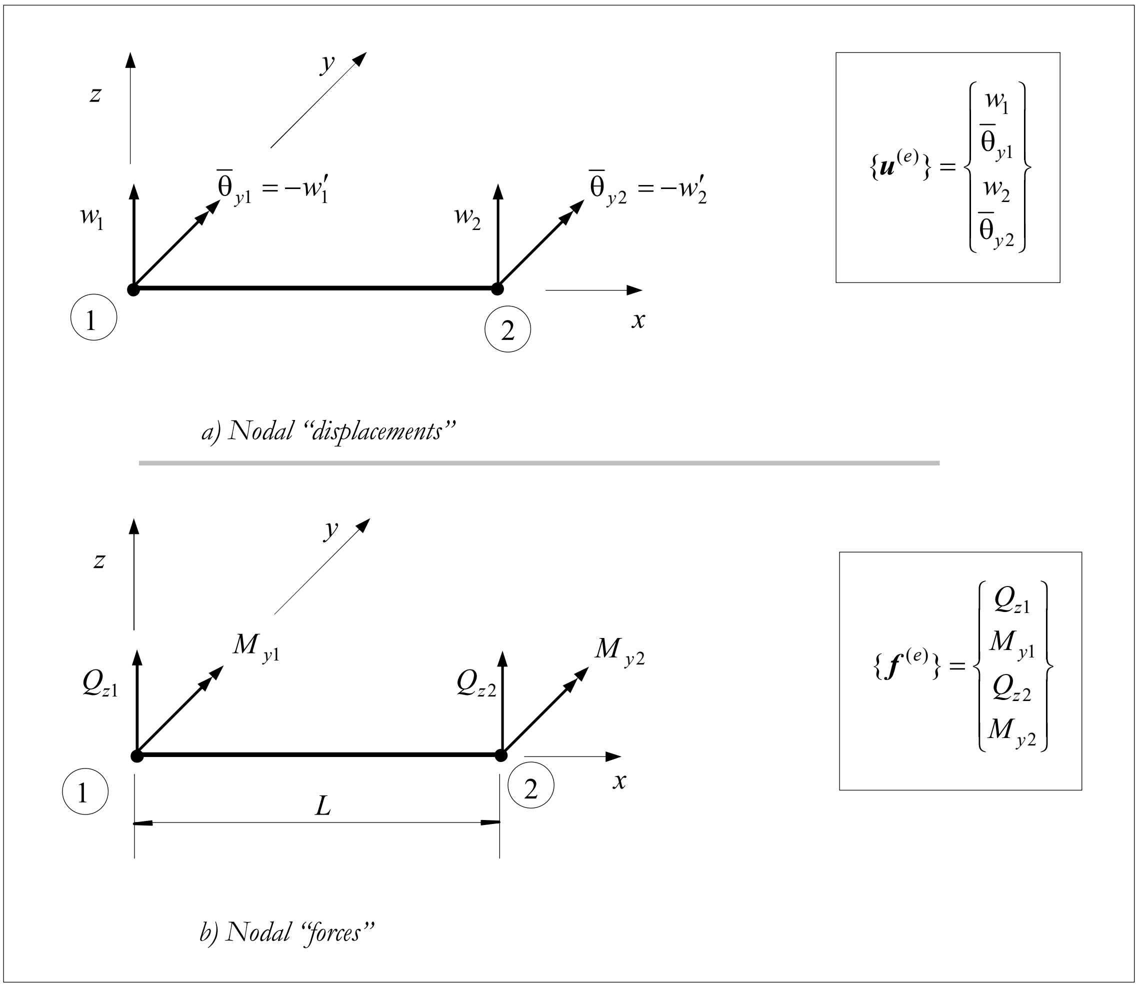


Figure 6.159: Beam element.

Let us adopt a cubic function to approach the deflection w:

$$w = ax^3 + bx^2 + cx + d \tag{6.440}$$

The first derivative of w becomes:

$$\frac{dw}{dx} \equiv w' = 3ax^2 + 2bx + c \tag{6.441}$$

Then, at the ends (nodes) of the beam the following fulfills:

$$\begin{aligned} x = 0 \;\; (w = w_1) \quad &\Rightarrow \quad w_1 = d \\ x = 0 \;\; (w' = w_1') \quad &\Rightarrow \quad \frac{dw}{dx} = w_1' = c \end{aligned} \tag{6.442}$$

$$\begin{aligned} x = L \;\; (w = w_2) \quad &\Rightarrow \quad w_2 = aL^3 + bL^2 + cL + d \\ x = L \;\; (w' = w_2') \quad &\Rightarrow \quad w_2' = 3aL^2 + 2bL + c \end{aligned} \tag{6.443}$$

By restructuring the above equations in matrix form we can obtain:

$$\begin{bmatrix} w_1 \\ w_1' \\ w_2 \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{\text{Reverse}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{L^4} \begin{bmatrix} 2L & L^2 & -2L & L^2 \\ -3L^2 & -2L^3 & 3L^2 & -L^3 \\ 0 & L^4 & 0 & 0 \\ L^4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_1' \\ w_2 \\ w_2' \end{bmatrix} \tag{6.444}$$

where the coefficients are:

$$a = \frac{2}{L^3}(w_1 - w_2) + \frac{1}{L^2}(w_2' + w_1') \tag{6.445}$$

$$b = \frac{3}{L^2}(w_2 - w_1) - \frac{1}{L}(w_2' + 2w_1') \tag{6.446}$$

$$c = w_1' \tag{6.447}$$

$$d = w_1 \tag{6.448}$$

By substituting the values of a, b, c and d into the equation of the deflection (6.440), we can obtain:

$$w = w_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + w_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] + w_1'\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] + w_2'\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \tag{6.449}$$

Recall that we have adopted as degree-of-freedom the rotation $\overline{\theta}_y$ which is in agreement with the coordinate system adopted, (see Figure 6.159), and the above equation is in function of w' (deflection derivative). But, they are related to each other by means of the equation $\overline{\theta}_y = -w'$. With that, the equation of the deflection becomes:

$$w = w_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + w_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] - \overline{\theta}_{y1}\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] - \overline{\theta}_{y2}\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \tag{6.450}$$

and the first derivative becomes:

$$w' = -\overline{\theta}_y = w_1\left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] + w_2\left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] - \overline{\theta}_{y1}\left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] - \overline{\theta}_{y2}\left[\frac{3x^2}{L^2} - \frac{2x}{L}\right] \tag{6.451}$$

The second derivative:

$$w'' = w_1\left[\frac{12x}{L^3} - \frac{6}{L^2}\right] + w_2\left[-\frac{12x}{L^3} + \frac{6}{L^2}\right] - \overline{\theta}_{y1}\left[\frac{6x}{L^2} - \frac{4}{L}\right] - \overline{\theta}_{y2}\left[\frac{6x}{L^2} - \frac{2}{L}\right] \tag{6.452}$$

It will be useful to obtain analytically the following integrals:

$$\int_0^L w(x)dx = \frac{L}{2}w_1 + \frac{L}{2}w_2 - \frac{L^2}{12}\overline{\theta}_{y1} + \frac{L^2}{12}\overline{\theta}_{y2} \tag{6.453}$$

$$\begin{aligned}\int_0^L w^2 dx = {} & \frac{13L}{35}w_1^2 + \frac{13L}{35}w_2^2 + \frac{L^3}{105}\overline{\theta}_{y1}^2 + \frac{L^3}{105}\overline{\theta}_{y2}^2 + \frac{9L}{35}w_1 w_2 - \frac{11L^2}{105}w_1\overline{\theta}_{y1} + \frac{13L^2}{210}w_1\overline{\theta}_{y2} \\ & - \frac{13L^2}{210}w_2\overline{\theta}_{y1} + \frac{11L^2}{105}w_2\overline{\theta}_{y2} - \frac{L^3}{70}\overline{\theta}_{y1}\overline{\theta}_{y2}\end{aligned} \tag{6.454}$$

$$\int_0^L w'^2 dx = \frac{6}{5L}w_1^2 + \frac{6}{5L}w_2^2 + \frac{2L}{15}\overline{\theta}_{y1}^2 + \frac{2L}{15}\overline{\theta}_{y2}^2 - \frac{12}{5L}w_1 w_2 - \frac{1}{5}w_1\overline{\theta}_{y1} - \frac{1}{5}w_1\overline{\theta}_{y2} + \frac{1}{5}w_2\overline{\theta}_{y1} + \frac{1}{5}w_2\overline{\theta}_{y2} - \frac{L}{15}\overline{\theta}_{y1}\overline{\theta}_{y2} \tag{6.455}$$

$$\int_0^L w''^2 dx = \frac{12}{L^3}w_1^2 + \frac{12}{L^3}w_2^2 + \frac{4}{L}\overline{\theta}_{y1}^2 + \frac{4}{L}\overline{\theta}_{y2}^2 - \frac{24}{L^3}w_1 w_2 - \frac{12}{L^2}w_1\overline{\theta}_{y1} - \frac{12}{L^2}w_1\overline{\theta}_{y2} + \frac{12}{L^2}w_2\overline{\theta}_{y1} + \frac{12}{L^2}w_2\overline{\theta}_{y2} + \frac{4}{L}\overline{\theta}_{y1}\overline{\theta}_{y2} \tag{6.456}$$

$$\int_0^L x\, w(x) dx = \frac{3L^2}{20}w_1 + \frac{7L^2}{20}w_2 - \frac{L^3}{30}\overline{\theta}_{y1} + \frac{L^3}{20}\overline{\theta}_{y2} \tag{6.457}$$

Let us consider a beam element with the flexural rigidity EI_y constant into the beam element in which is under the uniformly distributed load q_z, (see Figure 6.160).

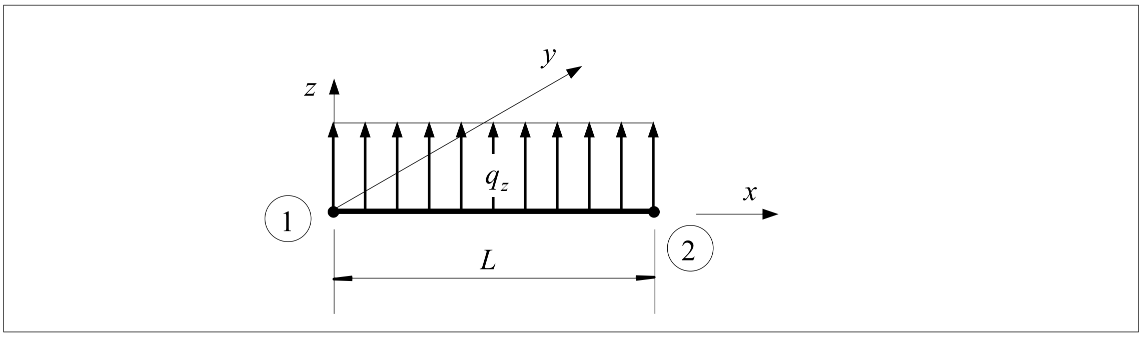


Figure 6.160: Beam under uniformly distributed load.

The total potential energy is given by:

$$\Pi = U^{int} - U^{ext} = \int_0^L \frac{EI_y}{2} w''^2 dx - \int_0^L q_z w(x) dx \tag{6.458}$$

As we are considering that q_z is independent of x, the external potential energy becomes:

$$U^{ext} = \int_0^L q_z w(x) dx = q_z \int_0^L w(x) dx \tag{6.459}$$

By considering the equation in (6.453) into the above equation we can express U^{ext} in terms of the nodal parameters w_1, w_2, $\overline{\theta}_{y1}$ and $\overline{\theta}_{y2}$, i.e.:

$$U^{ext} = q_z \int_0^L w(x) dx = q_z \left(\frac{L}{2}w_1 + \frac{L}{2}w_2 - \frac{L^2}{12}\overline{\theta}_{y1} + \frac{L^2}{12}\overline{\theta}_{y2} \right) \tag{6.460}$$

Considering that EI_y is constant in the beam element, the internal potential energy becomes:

$$U^{int} = \frac{EI_y}{2} \int_0^L w''^2 dx \tag{6.461}$$

Using the equation (6.456) the above equation can be represented as follows:

$$U^{int} = \frac{EI_y}{2}\int_0^L w''^2 dx = \frac{EI_y}{2}\left(\frac{12}{L^3}w_1^2 + \frac{12}{L^3}w_2^2 + \frac{4}{L}\overline{\theta}_{y1}^2 + \frac{4}{L}\overline{\theta}_{y2}^2 - \frac{24}{L^3}w_1 w_2 - \frac{12}{L^2}w_1\overline{\theta}_{y1} \right.$$
$$\left. - \frac{12}{L^2}w_1\overline{\theta}_{y2} + \frac{12}{L^2}w_2\overline{\theta}_{y1} + \frac{12}{L^2}w_2\overline{\theta}_{y2} + \frac{4}{L}\overline{\theta}_{y1}\overline{\theta}_{y2}\right) \tag{6.462}$$

Then, the total potential energy (6.458), $\Pi = U^{int} - U^{ext}$, can be written as follows:

$$\Pi = \frac{EI_y}{2}\left(\frac{12}{L^3}w_1^2 + \frac{12}{L^3}w_2^2 + \frac{4}{L}\overline{\theta}_{y1}^2 + \frac{4}{L}\overline{\theta}_{y2}^2 - \frac{24}{L^3}w_1 w_2 - \frac{12}{L^2}w_1\overline{\theta}_{y1} - \frac{12}{L^2}w_1\overline{\theta}_{y2}\right.$$
$$\left. + \frac{12}{L^2}w_2\overline{\theta}_{y1} + \frac{12}{L^2}w_2\overline{\theta}_{y2} + \frac{4}{L}\overline{\theta}_{y1}\overline{\theta}_{y2}\right) - q_z\left(\frac{L}{2}w_1 + \frac{L}{2}w_2 - \frac{L^2}{12}\overline{\theta}_{y1} + \frac{L^2}{12}\overline{\theta}_{y2}\right) \tag{6.463}$$

As we are looking for the stationary state (equilibrium) the following must hold:

$$\frac{\partial \Pi}{\partial w_1} = 0 \quad \Rightarrow \quad \frac{EI_y}{2}\left\{\frac{24}{L^3}w_1 - \frac{24}{L^3}w_2 - \frac{12}{L^2}\overline{\theta}_{y1} - \frac{12}{L^2}\overline{\theta}_{y2}\right\} - q_z\frac{L}{2} = 0 \tag{6.464}$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y1}} = 0 \quad \Rightarrow \quad \frac{EI_y}{2}\left\{\frac{8}{L}\overline{\theta}_{y1} - \frac{12}{L^2}w_1 + \frac{12}{L^2}w_2 + \frac{4}{L}\overline{\theta}_{y2}\right\} + q_z\frac{L^2}{12} = 0 \tag{6.465}$$

$$\frac{\partial \Pi}{\partial w_2} = 0 \quad \Rightarrow \quad \frac{EI_y}{2}\left\{\frac{24}{L^3}w_2 - \frac{24}{L^3}w_1 + \frac{12}{L^2}\overline{\theta}_{y1} + \frac{12}{L^2}\overline{\theta}_{y2}\right\} - q_z\frac{L}{2} = 0 \tag{6.466}$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y2}} = 0 \quad \Rightarrow \quad \frac{EI_y}{2}\left\{\frac{8}{L}\overline{\theta}_{y2} - \frac{12}{L^2}w_1 + \frac{12}{L^2}w_2 + \frac{4}{L}\overline{\theta}_{y1}\right\} - q_z\frac{L^2}{12} = 0 \tag{6.467}$$

Restructuring the above set of equations in matrix form we can obtain:

$$\begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} \frac{q_z L}{2} \\ \frac{-q_z L^2}{12} \\ \frac{q_z L}{2} \\ \frac{q_z L^2}{12} \end{Bmatrix} \tag{6.468}$$

or:

$$[\boldsymbol{Ke}^{(1)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\} \tag{6.469}$$

in which:

$$\boxed{[\boldsymbol{Ke}^{(1)}] = \begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \quad ; \quad \{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{q_z L}{2} \\ \frac{-q_z L^2}{12} \\ \frac{q_z L}{2} \\ \frac{q_z L^2}{12} \end{Bmatrix}} \tag{6.470}$$

where $\{\boldsymbol{f}_{Eq}^{(e)}\}$ is the consistent load vector and $[\boldsymbol{Ke}^{(1)}]$ is the stiffness matrix for the beam element, and note that $[\boldsymbol{Ke}^{(1)}]$ has no inverse since $\det[\boldsymbol{Ke}^{(1)}]=0$. In order to obtain the unique solution of the set of equations (6.468) we must introduce the boundary conditions. The same equation in (6.468) can be obtained by means of the Principle of Virtual Work.

Example 1: Let us consider that the beam is fixed at one end $(x=0)$, (see Figure 6.161).

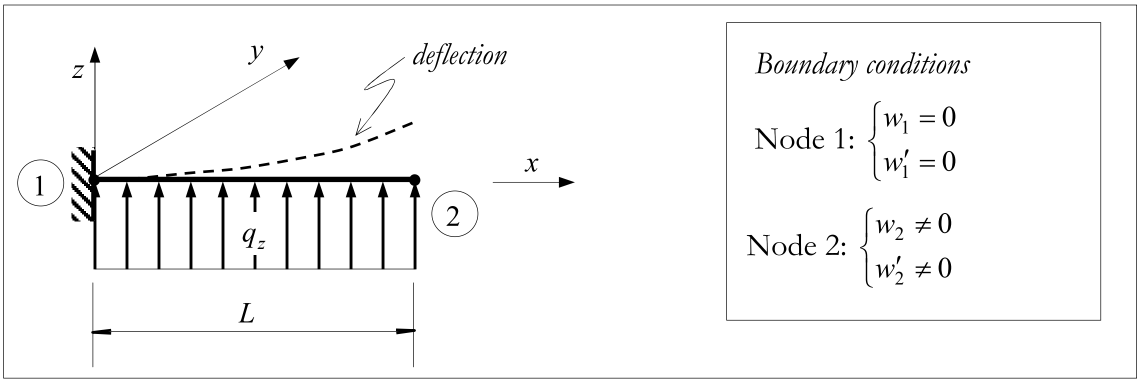


Figure 6.161: Cantilever under uniformly distributed load.

The force vector $\{\boldsymbol{f}^{(e)}\}$ is constructed by the consistent load vector ($\{\boldsymbol{f}_{Eq}^{(e)}\}$) and by the concentrated (nodal) force vector $\{\boldsymbol{f}_0^{(e)}\}$ which is zero for the problem presented in Figure 6.161.

By applying the boundary conditions to the equation in (6.468) we can obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ 0 & 0 & \dfrac{6EI_y}{L^2} & \dfrac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{q_z L}{2} \\ \dfrac{q_z L^2}{12} \end{Bmatrix} \quad \Leftrightarrow \quad [\overline{\boldsymbol{Ke}}^{(1)}]\ \{\boldsymbol{u}^{(e)}\} = \{\bar{\boldsymbol{f}}_{\boldsymbol{Eq}}^{(e)}\} \tag{6.471}$$

By solving the above equation we can obtain:

$$\begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{q_z L^4}{8EI_y} \\ \dfrac{-q_z L^3}{6EI_y} \end{Bmatrix} \tag{6.472}$$

The above solution (deflection and rotation) matches the exact solution. The moment at node 1 is given by $M_{y1} = -EI_y w_1''$. And by means of the equation in (6.452) we can obtain:

$$w'' = w_1\left[\frac{12x}{L^3} - \frac{6}{L^2}\right] + w_2\left[-\frac{12x}{L^3} + \frac{6}{L^2}\right] - \overline{\theta}_{y1}\left[\frac{6x}{L^2} - \frac{4}{L}\right] - \overline{\theta}_{y2}\left[\frac{6x}{L^2} - \frac{2}{L}\right]$$

$$\Rightarrow w''(x=0) = w_1'' = w_2\left[\frac{6}{L^2}\right] - \overline{\theta}_{y2}\left[-\frac{2}{L}\right]$$

$$\Rightarrow w_1'' = \frac{q_z L^4}{8EI_y}\left[\frac{6}{L^2}\right] - \frac{q_z L^3}{6EI_y}\left[\frac{2}{L}\right] \quad \Rightarrow \quad \Rightarrow w_1'' = \left(\frac{5}{6}\right)\frac{q_z L^2}{2EI_y}$$

Then, the moment becomes:

$$M_{y1} = -EI_y w_1'' = -EI_y\left(\frac{5}{6}\right)\frac{q_z L^2}{2EI_y} = -\left(\frac{5}{6}\right)\frac{q_z L^2}{2} \tag{6.473}$$

And if we compare with the exact solution $M_{y1}^{exact} = \dfrac{-q_z L^2}{2}$ the error is 16.6%.

NOTE 3: Note that the differential equation for the beam problem, given by equation (6.225), $\dfrac{\partial^4 w}{\partial x^4} = \dfrac{q_z}{EI_y}$, requires a fourth-order function if $\left(\dfrac{q_z}{EI_y}\right)$ is a constant into the beam. For the problem established here we have adopted a third-order function, (see equation (6.449)). Due to this fact we have errors associated with M_y and Q_z. To overcome this drawback we will establish a procedure in order to achieve accuracy for internal forces at the beam element nodes.

Loading vector for the beam element

Once the displacements are obtained (6.472), the internal forces can be obtained by means of the equation (6.469), i.e.:

$$\{\bar{\boldsymbol{f}}^{(e)}\} = [\boldsymbol{Ke}^{(1)}]\ \{\boldsymbol{u}^{(e)}\} \tag{6.474}$$

with which we can obtain:

$$\{\bar{\boldsymbol{f}}^{(e)}\} = \begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{q_z L^4}{8EI_y} \\ \frac{-q_z L^3}{6EI_y} \end{Bmatrix} = \begin{Bmatrix} \frac{-q_z L}{2} \\ \frac{5q_z L^2}{12} \\ \frac{q_z L}{2} \\ \frac{q_z L^2}{12} \end{Bmatrix} \tag{6.475}$$

When we are dealing with the traditional finite element technique, the internal forces of the beam element are given by $\{\bar{\boldsymbol{f}}^{(e)}\}$, (see Figure 6.162(a)). The error can be minimized by dividing the beam element in more elements. From structural analysis, the exact solution is given by Figure 6.162(c), and we can verify that the reactions at the ends of the beam element can be obtained as follows:

$$\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} + \{\tilde{\bar{\boldsymbol{f}}}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}_{Eq}^{(e)}\} \tag{6.476}$$

where $\{\boldsymbol{f}_{Eq}^{(e)}\}$ is given by equation (6.470). Then, the internal forces acting at the ends of the beam can be obtained as follows:

$$\{\boldsymbol{f}_{int}^{(e)}\} = -\{\boldsymbol{R}^{(e)}\} \tag{6.477}$$

The vector $\{\tilde{\bar{\boldsymbol{f}}}^{(e)}\} = -\{\boldsymbol{f}_{Eq}^{(e)}\}$, (see equation (6.470)), is known as loading vector. Note that the vector given in Figure 6.162(b) represents the reactions that appear when the beam is fixed at both ends, (see Figure 6.163), (Gere&Weaver (1965)). And these reactions are the same as the one obtained by means of the equation (6.470).

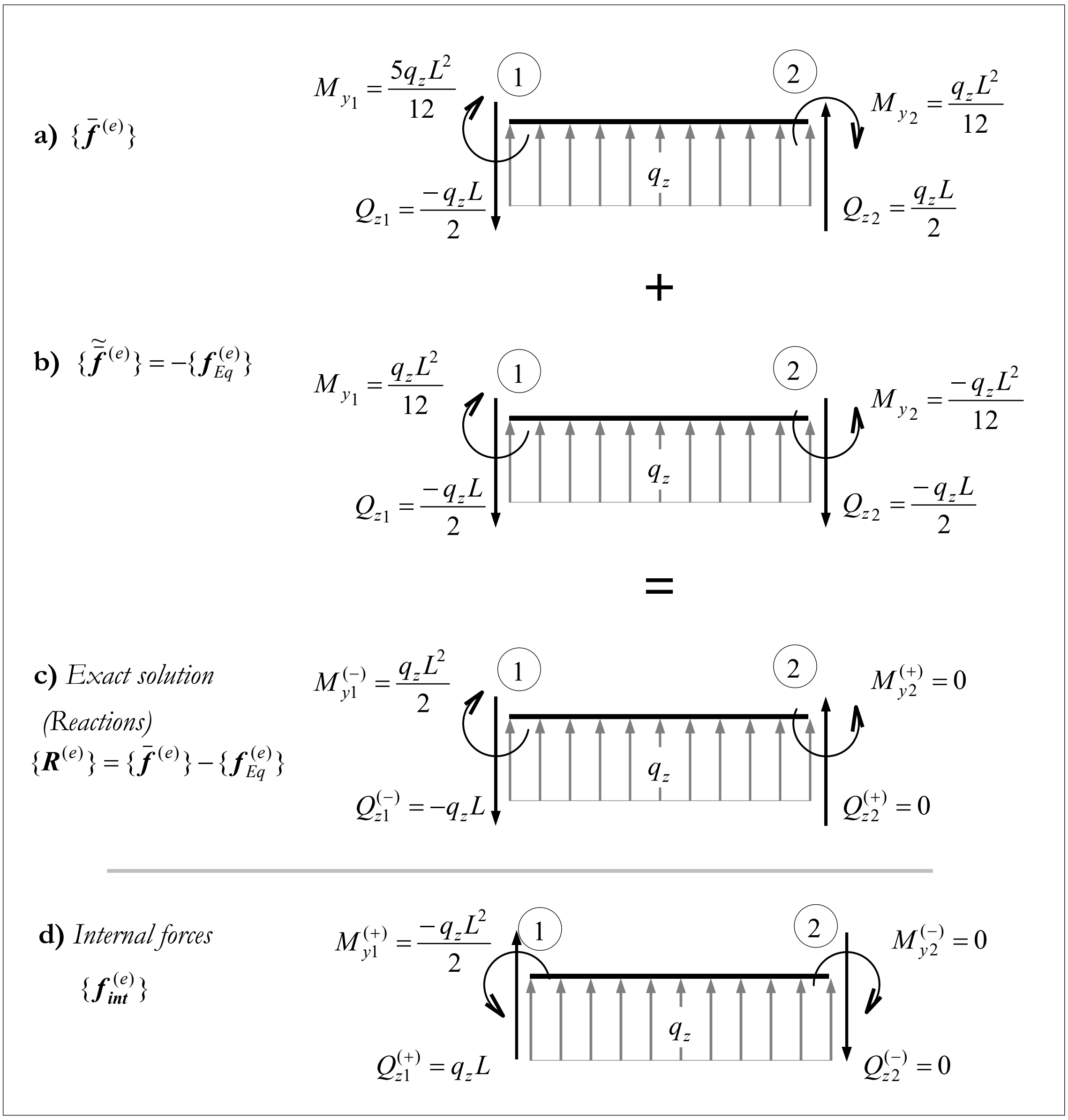


Figure 6.162: Reaction in the beam element.

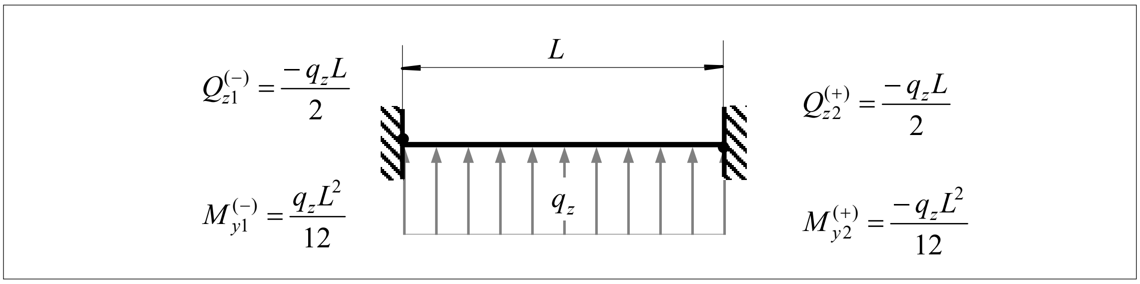


Figure 6.163: Reaction forces of the beam with fixed ends.

NOTE 4: Analytical solution using direct integration

In this sub-section we will obtain the analytical solution (the exact one) for the problem described in Figure 6.164. To obtain the analytical solution we will use the direct integration, and we will start from the moment function of the beam, (see Figure 6.164):

$$M_y(x) = -q_z(L-x)\frac{(L-x)}{2} = \frac{-q_z}{2}(x^2 - 2Lx + L^2)$$

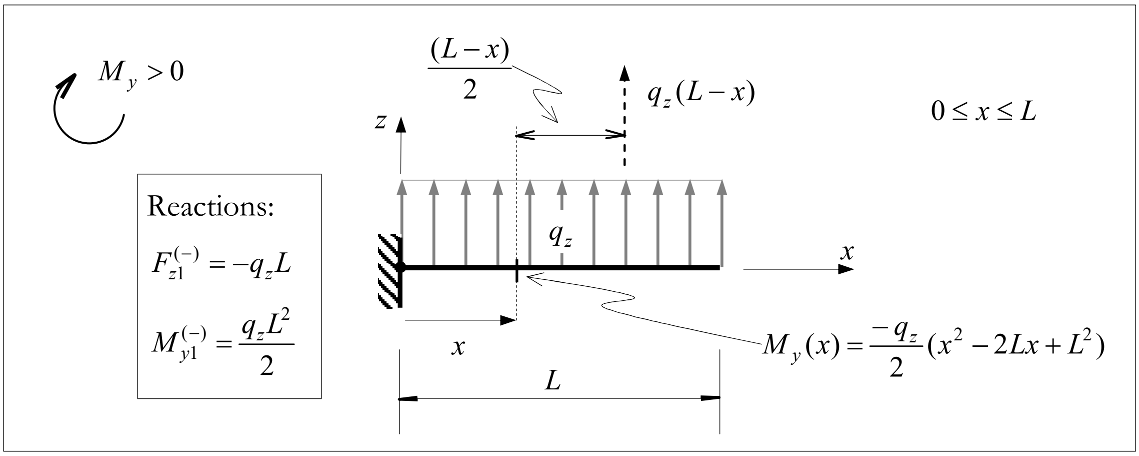


Figure 6.164: Beam fixed at one end under uniformly distributed load.

Recall that the differential equation of the beam in function of the deflection (w), (see equation (6.417)), is given by:

$$M_y = -EI_y w'' \equiv -EI_y \frac{d^2 w}{dx^2} \quad \Rightarrow \quad -EI_y \frac{d^2 w}{dx^2} = \frac{-q_z}{2}(x^2 - 2Lx + L^2)$$

By means of direct integration we can obtain:

$$EI_y \frac{d^2 w}{dx^2} = \frac{q_z}{2}(x^2 - 2Lx + L^2) \xrightarrow{\text{integrating}} EI_y \frac{dw}{dx} = \frac{q_z}{6}(x^3 - 3Lx^2 + 3L^2 x) + C_1$$

The constant of integration C_1 can be obtained by means of the boundary condition, i.e. the rotation is zero at $(x = 0)$, so, $\frac{dw}{dx} \equiv w'(x = 0) = 0$ with which we can obtain $C_1 = 0$. Then, the first derivative of the deflection becomes:

$$\frac{dw}{dx} \equiv w' = \frac{q_z}{6EI_y}\left(3L^2 x - 3Lx^2 + x^3\right)$$

By integrating once more we can obtain the deflection of the beam $w(x)$:

$$\frac{dw}{dx} = \frac{q_z}{6EI_y}\left(3L^2 x - 3Lx^2 + x^3\right) \xrightarrow{\text{integrating}} w = \frac{q_z}{6EI_y}\left(\frac{3L^2 x^2}{2} - \frac{3Lx^3}{3} + \frac{x^4}{4}\right) + C_2$$

where the constant of integration C_2 can be obtained as follows $w(x = 0) = 0 \Rightarrow C_2 = 0$. Then, the deflection of the beam becomes:

$$w(x) = \frac{q_z x^2}{24EI_y}\left(6L^2 - 4Lx + x^2\right)$$

When $x = L$ we can obtain:

deflection: $w(x = L) = \frac{q_z x^2}{24EI_y}\left(6L^2 - 4Lx + x^2\right) = \frac{q_z L^2}{24EI_y}\left(6L^2 - 4L^2 + L^2\right) = \frac{q_z L^4}{8EI_y}$

rotation: $\frac{dw}{dx} \equiv w'(x = L) = \frac{q_z}{6EI_y}\left(3L^2 x - 3Lx^2 + x^3\right) = \frac{q_z}{6EI_y}\left(3L^3 - 3L^3 + L^3\right) = \frac{q_z L^3}{6EI_y} = -\overline{\theta}_y$

which matches the result in (6.472).

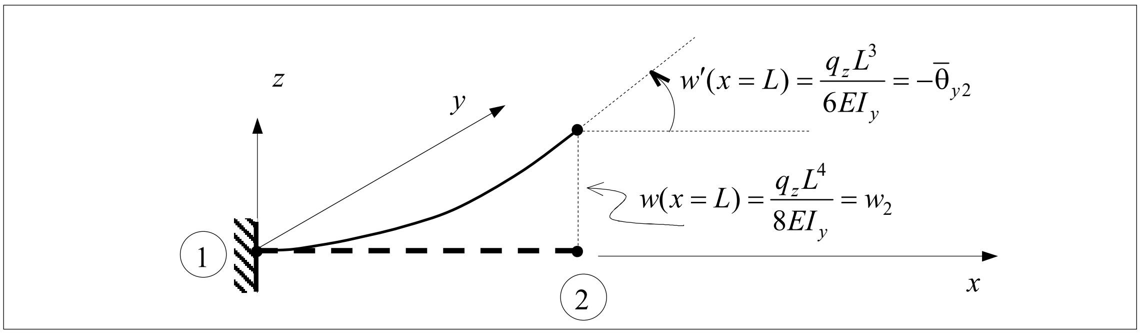


Figure 6.165: Displacements at the free-end of the beam.

Example 2: Let us consider that the beam is fixed at one end $(x=0)$, (see Figure 6.166).

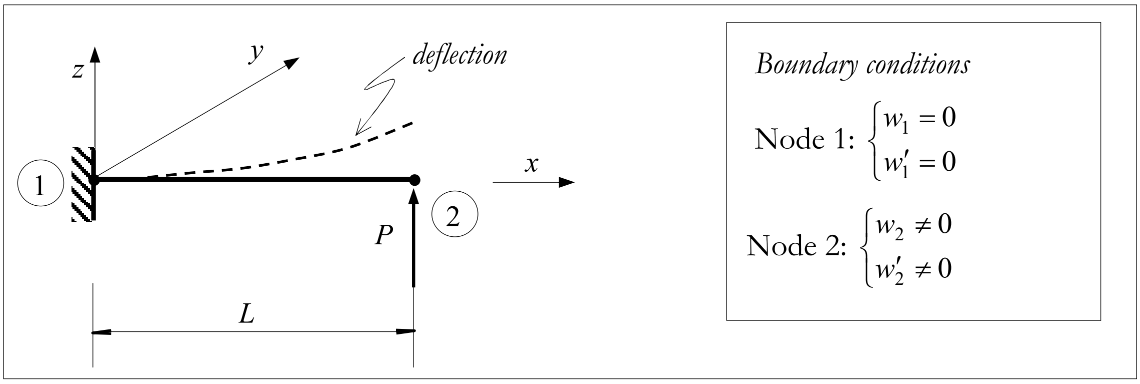


Figure 6.166: Cantilever under concentrated force.

This example is very similar to the previous Example 1. The only difference is the force vector in which we must add the concentrated force vector $\{\boldsymbol{f}_0^{(e)}\}$:

$$\{\boldsymbol{f}_0^{(e)}\} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix} \tag{6.478}$$

By applying the boundary conditions to the equation in (6.468) we can obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ 0 & 0 & \dfrac{6EI_y}{L^2} & \dfrac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \underbrace{\begin{Bmatrix} \dfrac{q_z L}{2} \\ \dfrac{-q_z L^2}{12} \\ \dfrac{q_z L}{2} \\ \dfrac{q_z L^2}{12} \end{Bmatrix}}_{\substack{\text{Consistent} \\ \text{load vector} \\ =\{\boldsymbol{f}_{Eq}^{(e)}\}}} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix}}_{\substack{\text{Concentrated} \\ \text{force vector} \\ =\{\boldsymbol{f}_0^{(e)}\}}} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{\substack{\text{Consistent} \\ \text{load vector}}} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix}}_{\substack{\text{Concentrated} \\ \text{force vector}}} \tag{6.479}$$

Where we have considered $\{\boldsymbol{f}_{Eq}^{(e)}\} = \{\boldsymbol{0}\}$. Solving the above equation we can obtain:

$$\begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{PL^3}{3EI_y} \\ \dfrac{-PL^2}{2EI_y} \end{Bmatrix} \tag{6.480}$$

Reaction Calculation

$$\{\bar{\boldsymbol{f}}^{(e)}\} = \begin{bmatrix} \dfrac{12EI_y}{L^3} & \dfrac{-6EI_y}{L^2} & \dfrac{-12EI_y}{L^3} & \dfrac{-6EI_y}{L^2} \\ \dfrac{-6EI_y}{L^2} & \dfrac{4EI_y}{L} & \dfrac{6EI_y}{L^2} & \dfrac{2EI_y}{L} \\ \dfrac{-12EI_y}{L^3} & \dfrac{6EI_y}{L^2} & \dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ \dfrac{-6EI_y}{L^2} & \dfrac{2EI_y}{L} & \dfrac{6EI_y}{L^2} & \dfrac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dfrac{PL^3}{3EI_y} \\ \dfrac{-PL^2}{2EI_y} \end{Bmatrix} = \begin{Bmatrix} -P \\ PL \\ P \\ 0 \end{Bmatrix} \tag{6.481}$$

Then, by apply the equation (6.476), $\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}_{Eq}^{(e)}\}$, we can obtain:

$$\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} -P \\ PL \\ P \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P \\ PL \\ P \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{z1}^{(-)} \\ M_{y1}^{(-)} \\ F_{z2}^{(+)} \\ M_{y1}^{(+)} \end{Bmatrix} \tag{6.482}$$

And the internal forces at the extremities of the beam are:

$$\{\boldsymbol{f}_{int}^{(e)}\} = -\{\boldsymbol{R}^{(e)}\} = \begin{Bmatrix} F_{z1}^{(+)} \\ M_{y1}^{(+)} \\ F_{z2}^{(-)} \\ M_{y2}^{(-)} \end{Bmatrix} = \begin{Bmatrix} P \\ -PL \\ -P \\ 0 \end{Bmatrix} \tag{6.483}$$

NOTE 5: Analytical solution using direct integration

In this sub-section we will obtain the analytical solution (the exact one) for the problem described in Figure 6.167. To obtain the analytical solution we will use the direct integration, and we will start from the moment function of the beam, (see Figure 6.167):

$$M_y(x) = -P(L-x)$$

Recall that the differential equation of the beam in function of the deflection (w), (see equation (6.417)), is given by:

$$M_y = -EI_y w'' \equiv -EI_y \frac{d^2 w}{dx^2} \quad \Rightarrow \quad -EI_y \frac{d^2 w}{dx^2} = -P(L-x)$$

By means of direct integration we can obtain:

$$EI_y \frac{d^2 w}{dx^2} = P(L-x) \xrightarrow{\text{integrating}} EI_y \frac{dw}{dx} = P\left(Lx - \frac{x^2}{2}\right) + C_1$$

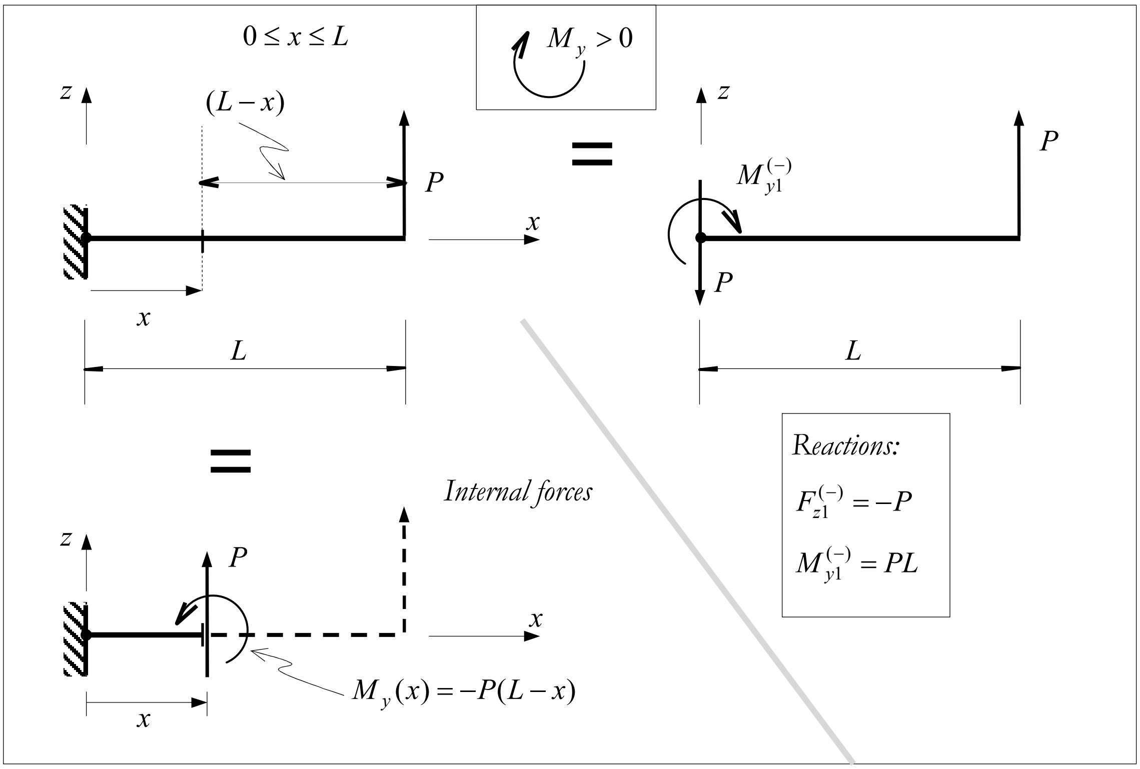


Figure 6.167: Beam fixed at one end under concentrated force.

The constant of integration C_1 can be obtained by means of the boundary condition $\frac{dw}{dx} \equiv w'(x=0) = 0$ with which we can obtain $C_1 = 0$. Then, the first derivative of the deflection becomes:

$$\frac{dw}{dx} \equiv w' = \frac{P}{EI_y}\left(Lx - \frac{x^2}{2}\right) = \frac{P}{2EI_y}\left(2Lx - x^2\right)$$

By integrating once more we can obtain the deflection of the beam $w(x)$:

$$\frac{dw}{dx} = \frac{P}{EI_y}\left(Lx - \frac{x^2}{2}\right) \xrightarrow{\text{integrating}} w = \frac{P}{EI_y}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

where the constant of integration C_2 can be obtained as follows $w(x=0) = 0 \Rightarrow C_2 = 0$. Then, the deflection of the beam becomes:

$$w(x) = \frac{P}{6EI_y}(3Lx^2 - x^3)$$

When $x = L$ we can obtain:

- deflection: $w(x=L) = w_2 = \frac{P}{6EI_y}(3Lx^2 - x^3) = \frac{P}{6EI_y}(3LL^2 - L^3) = \frac{PL^3}{3EI_y}$
- rotation: $\frac{dw}{dx} \equiv w'(x=L) = -\overline{\theta}_{y2} = \frac{P}{2EI_y}\left(2Lx - x^2\right) = \frac{P}{2EI_y}\left(2LL - L^2\right) = \frac{PL^2}{2EI_y}$

which matches the solution in (6.480).

Shearing Force

Taking into account that

$$EI_y \frac{d^2 w}{dx^2} = P(L-x) \xrightarrow{\text{derivative}} \frac{d^3 w}{dx^3} = \frac{-P}{EI_y}$$

And according to equation in (6.423) we can conclude that:

$$\frac{\partial^3 w}{\partial x^3} = \frac{-Q_z}{EI_y} \quad \Rightarrow \quad \frac{-Q_z}{EI_y} = \frac{-P}{EI_y} \quad \Rightarrow \quad Q_z(x) = P \quad \text{(constant)}$$

NOTE 6: When are dealing with steel structure the sign of bending moment plays no role at the time of the design of the structure, since the steel has the same behavior when is under either traction or compression. This scenario changes when we are dealing with reinforced concrete structure, since the concrete has little capacity to support tensile action. The bending moment sign convention for concrete structure is defined positive if the lower fibers are under traction and negative if the upper fibers are in traction, (see Figure 6.168). Nevertheless, in this book we are adopting the positive values (displacements, rotations, forces and moments) if they are in accordance with the axes orientation.

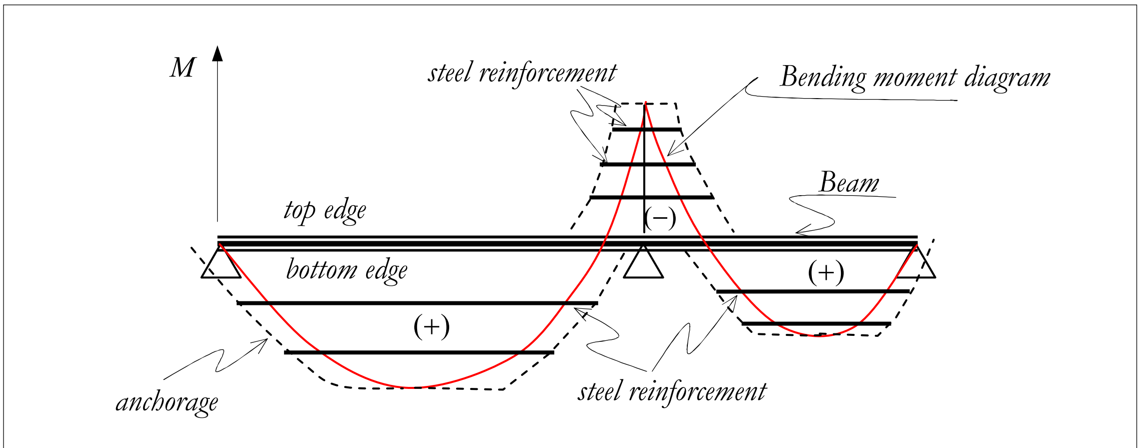


Figure 6.168: Bending moment sign convention for reinforced concrete design.

Problem 6.63

Obtain the explicit expression for the stiffness matrix from the equation $[\boldsymbol{Ke}^{(3)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}\}$ for the problem described in **Problem 6.62**-NOTE 2 by using the Principle of Virtual Work, (see **Problem 5.22**). Obtain also the consistent mass matrix given in **Problem 5.24**.

Solution:

According to **Problem 6.62**-NOTE 2 we have found out the deflection function, which can also be written as follows:

$$w = w_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + w_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] - \overline{\theta}_{y1}\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] - \overline{\theta}_{y2}\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right]$$

$$\Rightarrow w(x) = w_1 N_1 + \overline{\theta}_{y1} N_2 + w_2 N_3 + \overline{\theta}_{y2} N_4$$

$$\Rightarrow w(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ -- \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} \quad \Rightarrow \quad w(x) = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\} \tag{6.484}$$

where N_i are the shape functions. Note that the displacement vector is made up by displacements and rotations, and

$$\begin{aligned} [\boldsymbol{N}] &= \left[\left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \right) \quad \left(\frac{-x^3}{L^2} + \frac{2x^2}{L} - x \right) \quad \left(-\frac{2x^3}{L^3} + \frac{3x^2}{L^2} \right) \quad \left(\frac{-x^3}{L^2} + \frac{x^2}{L} \right) \right] \\ \Rightarrow \frac{\partial [\boldsymbol{N}]}{\partial x} &= \left[\left(\frac{6x^2}{L^3} - \frac{6x}{L^2} \right) \quad \left(\frac{-3x^2}{L^2} + \frac{4x}{L} - 1 \right) \quad \left(-\frac{6x^2}{L^3} + \frac{6x}{L^2} \right) \quad \left(\frac{-3x^2}{L^2} + \frac{2x}{L} \right) \right] \\ \Rightarrow \frac{\partial^2 [\boldsymbol{N}]}{\partial x^2} &\equiv [\boldsymbol{B}_V] = \left[\left(\frac{12x}{L^3} - \frac{6}{L^2} \right) \quad \left(\frac{-6x}{L^2} + \frac{4}{L} \right) \quad \left(-\frac{12x}{L^3} + \frac{6}{L^2} \right) \quad \left(\frac{-6x}{L^2} + \frac{2}{L} \right) \right] \end{aligned} \tag{6.485}$$

Next we will try to relate strain and stress to the displacement vector $\{\boldsymbol{u}^{(e)}\}$. For this beam problem the strain is given by the equation $\varepsilon_x^{(2)}(z) = -\frac{d^2 w}{dx^2} z \equiv -w,_{xx} z$, (see **Problem 6.59**-NOTE 3). The term $w,_{xx}$ can be obtained as follows:

$$\begin{aligned} & w(x) = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\} \quad \Rightarrow \quad \frac{\partial w(x)}{\partial x} = \frac{\partial}{\partial x}\left([\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\}\right) = \frac{\partial [\boldsymbol{N}]}{\partial x}\{\boldsymbol{u}^{(e)}\} \\ & \Rightarrow \frac{\partial^2 w(x)}{\partial x^2} \equiv w,_{xx} = \frac{\partial^2 [\boldsymbol{N}]}{\partial x^2}\{\boldsymbol{u}^{(e)}\} \end{aligned} \tag{6.486}$$

where $\frac{\partial^2 [\boldsymbol{N}]}{\partial x^2}$ is given by equation in (6.485). Then:

$$\begin{aligned} & \varepsilon_x^{(2)} = \{\boldsymbol{\varepsilon}\} = -\frac{d^2 w}{dx^2} z \equiv -z \frac{\partial^2 [\boldsymbol{N}]}{\partial x^2}\{\boldsymbol{u}^{(e)}\} = -z[\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\} \\ & \sigma_x^{(2)} = E\varepsilon_x^{(2)} \quad or \quad \{\boldsymbol{\sigma}\} = [E]\{\boldsymbol{\varepsilon}\} = -z[E][\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\} \end{aligned} \tag{6.487}$$

We will adopt the same approximation for the virtual field ($\overline{\varepsilon}_x^{(2)}$), i.e.:

$$\overline{\varepsilon}_x^{(2)} = \{\overline{\boldsymbol{\varepsilon}}\} = -z[\boldsymbol{B}_V]\{\overline{\boldsymbol{u}}^{(e)}\} \quad \text{(Virtual field)} \tag{6.488}$$

And according to the problem established in **Problem 5.22** we can conclude that:

$$\{\boldsymbol{F}^{(e)}\}^T\{\overline{\boldsymbol{u}}^{(e)}\} = \int_V \{\boldsymbol{\sigma}\}^T\{\overline{\boldsymbol{\varepsilon}}\}\, dV \quad or \quad \{\overline{\boldsymbol{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \int_V \{\overline{\boldsymbol{\varepsilon}}\}^T\{\boldsymbol{\sigma}\}\, dV$$

$$\Rightarrow \{\overline{\boldsymbol{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \int_V \{-z[\boldsymbol{B}_V]\{\overline{\boldsymbol{u}}^{(e)}\}\}^T\{-z[E][\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\}\}\, dV$$

$$\Rightarrow \{\overline{\boldsymbol{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \int_V z^2\{\overline{\boldsymbol{u}}^{(e)}\}^T[\boldsymbol{B}_V]^T[E][\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\}\, dV$$

$$\Rightarrow \{\overline{\boldsymbol{u}}^{(e)}\}^T\{\boldsymbol{F}^{(e)}\} = \{\overline{\boldsymbol{u}}^{(e)}\}^T\left(\int_V z^2[\boldsymbol{B}_V]^T[E][\boldsymbol{B}_V]\, dV\right)\{\boldsymbol{u}^{(e)}\}$$

$$\Rightarrow \{\boldsymbol{F}^{(e)}\} = \left(\int_V z^2[\boldsymbol{B}_V]^T[E][\boldsymbol{B}_V]\, dV\right)\{\boldsymbol{u}^{(e)}\}$$

$$\Rightarrow \{\boldsymbol{F}^{(e)}\} = [\boldsymbol{Ke}]\{\boldsymbol{u}^{(e)}\} \tag{6.489}$$

Taking into account that $[E] = E$ and I_y are constant into the element, the explicit form for the stiffness matrix is given by:

$$\begin{aligned}
&[\boldsymbol{Ke}] = \int_V z^2 [\boldsymbol{B}_V(x)]^T [E][\boldsymbol{B}_V(x)]\, dV = E\int_V z^2 [\boldsymbol{B}_V(x)]^T [\boldsymbol{B}_V(x)]\, dV \\
&\Rightarrow [\boldsymbol{Ke}] = E\int_0^L [\boldsymbol{B}_V(x)]^T [\boldsymbol{B}_V(x)] \underbrace{\int_A z^2 \, dA}_{=I_y} dx = E\int_0^L [\boldsymbol{B}_V(x)]^T [\boldsymbol{B}_V(x)]\, I_y dx \\
&\Rightarrow [\boldsymbol{Ke}] = EI_y \int_0^L [\boldsymbol{B}_V(x)]^T [\boldsymbol{B}_V(x)] dx \equiv EI_y \int_0^L [\boldsymbol{Bb}_V] dx
\end{aligned} \tag{6.490}$$

where the multiplication of matrices $[\boldsymbol{B}_V(x)]^T [\boldsymbol{B}_V(x)] \equiv [\boldsymbol{Bb}_V]$ is given by

$$[\boldsymbol{Bb}_V] = \begin{bmatrix}
\dfrac{36(2x-L)^2}{L^6} & \dfrac{-12(2x-L)(3x-2L)}{L^5} & \dfrac{-36(2x-L)^2}{L^6} & \dfrac{-12(2x-L)(3x-L)}{L^5} \\
 & \dfrac{4(3x-2L)^2}{L^4} & \dfrac{12(2x-L)(3x-2L)}{L^5} & \dfrac{4(3x-2L)(3x-L)}{L^4} \\
 & & \dfrac{36(2x-L)^2}{L^6} & \dfrac{12(2x-L)(3x-L)}{L^5} \\
 & \textit{symmetric} & & \dfrac{4(3x-L)^2}{L^4}
\end{bmatrix}$$

Note that the component $[\boldsymbol{Ke}]_{11} = EI_y \int_0^L [\boldsymbol{Bb}_V]_{11} dx = EI_y \int_0^L \dfrac{36(2x-L)^2}{L^6} dx = \dfrac{12EI_y}{L^3}$, and in the same fashion we can obtain the remaining components of the matrix $[\boldsymbol{Ke}]$. With that we can obtain the same stiffness matrix as the one in **Problem 6.62**-NOTE 2.

In the same way we can obtain the mass matrix given in **Problem 5.24** by:

$$[\boldsymbol{M}^{(e)}] = \int_V \rho [\boldsymbol{N}(x)]^T [\boldsymbol{N}(x)]\, dV \tag{6.491}$$

where ρ is the mass density. For this problem the matrix $[\boldsymbol{N}]$ is given by equation in (6.485), and:

$$[\boldsymbol{M}^{(e)}] = \rho\int_0^L [\boldsymbol{N}(x)]^T [\boldsymbol{N}(x)] \underbrace{\int_A dA}_{=A} dx = \rho A\int_0^L [\boldsymbol{N}(x)]^T [\boldsymbol{N}(x)] dx \equiv \rho A \int_0^L [\boldsymbol{Nn}] dx \tag{6.492}$$

where

$$[\boldsymbol{N}]^T[\boldsymbol{N}] \equiv [\boldsymbol{Nn}] = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} = \begin{bmatrix}
(N_1)^2 & N_1N_2 & N_1N_3 & N_1N_4 \\
N_1N_2 & (N_2)^2 & N_2N_3 & N_2N_4 \\
N_1N_3 & N_2N_3 & (N_3)^2 & N_3N_4 \\
N_1N_4 & N_2N_4 & N_3N_4 & (N_4)^2
\end{bmatrix} \tag{6.493}$$

The first term of $[\boldsymbol{M}^{(e)}]$ can be obtained as follows:

$$[\boldsymbol{M}^{(e)}]_{11} = \rho A \int_0^L [\boldsymbol{Nn}]_{11} dx = \rho A \int_0^L (N_1)^2 dx = \rho A \int_0^L \left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \right)^2 dx = \rho A \frac{13L}{35}$$

Then, after the integration (6.492) is carried out we can obtain:

$$[\boldsymbol{M}^{(e)}] = \rho A \int_0^L [\boldsymbol{N}(x)]^T [\boldsymbol{N}(x)] dx = \rho A \begin{bmatrix} \frac{13L}{35} & \frac{-11L^2}{210} & \frac{9L}{70} & \frac{13L^2}{420} \\ \frac{-11L^2}{210} & \frac{L^3}{105} & \frac{-13L^2}{420} & \frac{-L^3}{140} \\ \frac{9L}{70} & \frac{-13L^2}{420} & \frac{13L}{35} & \frac{11L^2}{210} \\ \frac{13L^2}{420} & \frac{-L^3}{140} & \frac{11L^2}{210} & \frac{L^3}{105} \end{bmatrix} \tag{6.494}$$

The matrix in (6.494) is known as the *Consistent Mass Matrix*.

NOTE 1: Note that the internal potential energy due to bending moment M_y can be written as follows:

$$U^{int} = \frac{1}{2} \int_V \sigma_x^{(2)} \varepsilon_x^{(2)} dV = \frac{1}{2} \int_0^L \frac{M_y^2}{EI_y} dx = \frac{1}{2} \int_0^L EI_y (w_{,xx})^2 dx \tag{6.495}$$

where we have considered $M_y = -EI_y w_{,xx}$. By considering that the deflection can be approached by the shape functions:

$$w(x) = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\} \quad ; \quad w_{,xx} \equiv w'' = [\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\}$$

The total potential energy can be represented as follows

$$\Pi = U^{int} - U^{ext} = \frac{1}{2} \int_0^L EI_y (w_{,xx})^2 dx - U^{ext} = \frac{1}{2} \int_0^L EI_y ([\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\})([\boldsymbol{B}_V]\{\boldsymbol{u}^{(e)}\}) dx - U^{ext}$$

$$\Rightarrow \Pi = \{\boldsymbol{u}^{(e)}\}^T \left(\frac{1}{2} \int_0^L EI_y [\boldsymbol{B}_V]^T [\boldsymbol{B}_V] dx \right) \{\boldsymbol{u}^{(e)}\} - U^{ext}$$

As we are looking for the stationary state (equilibrium) the following must hold:

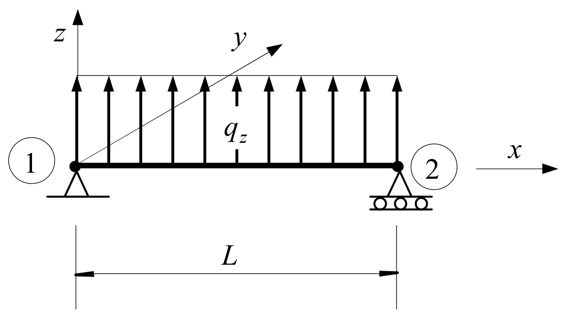
$$\frac{\partial \Pi}{\partial \{\boldsymbol{u}^{(e)}\}} = \left(\int_0^L EI_y [\boldsymbol{B}_V]^T [\boldsymbol{B}_V] dx \right) \{\boldsymbol{u}^{(e)}\} - \frac{\partial U^{ext}}{\partial \{\boldsymbol{u}^{(e)}\}} = \{\boldsymbol{0}\}$$

$$\Rightarrow \left(\int_0^L EI_y [\boldsymbol{B}_V]^T [\boldsymbol{B}_V] dx \right) \{\boldsymbol{u}^{(e)}\} = \frac{\partial U^{ext}}{\partial \{\boldsymbol{u}^{(e)}\}} = \{\boldsymbol{f}_{Eq}^{(e)}\}$$

$$\Rightarrow [\boldsymbol{Ke}^{(1)}] \; \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\}$$

Problem 6.64

Obtain the rotations and reaction forces at the nodes 1 and 2 for the beam described in Figure 6.169.



Boundary conditions

Node 1: $\begin{cases} w_1 = 0 \\ w_1' \neq 0 \end{cases}$

Node 2: $\begin{cases} w_2 = 0 \\ w_2' \neq 0 \end{cases}$

Figure 6.169: Beam bi-supported under uniformly distributed load.

Solution:

We will adopt the same procedure described in the previous problem, (see **Problem 6.62**-NOTE 2). So, for this problem is also valid that:

$$\begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} \frac{q_z L}{2} \\ \frac{-q_z L^2}{12} \\ \frac{q_z L}{2} \\ \frac{q_z L^2}{12} \end{Bmatrix} = \{\boldsymbol{f}^{(e)}\} \tag{6.496}$$

and by applying the boundary conditions, (see Figure 6.169), we can obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4EI_y}{L} & 0 & \frac{2EI_y}{L} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2EI_y}{L} & 0 & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 = 0 \\ \overline{\theta}_{y1} \\ w_2 = 0 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{-q_z L^2}{12} \\ 0 \\ \frac{q_z L^2}{12} \end{Bmatrix} \tag{6.497}$$

which is the same as

$$\begin{bmatrix} \frac{4EI_y}{L} & \frac{2EI_y}{L} \\ \frac{2EI_y}{L} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} \overline{\theta}_{y1} \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} \frac{-q_z L^2}{12} \\ \frac{q_z L^2}{12} \end{Bmatrix} \tag{6.498}$$

By solving the above system we can obtain:

$$\begin{Bmatrix} \overline{\theta}_{y1} \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{bmatrix} \frac{4EI_y}{L} & \frac{2EI_y}{L} \\ \frac{2EI_y}{L} & \frac{4EI_y}{L} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{-q_z L^2}{12} \\ \frac{q_z L^2}{12} \end{Bmatrix} = \frac{q_z L^3}{24EI_y} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{6.499}$$

which matches the exact solution.

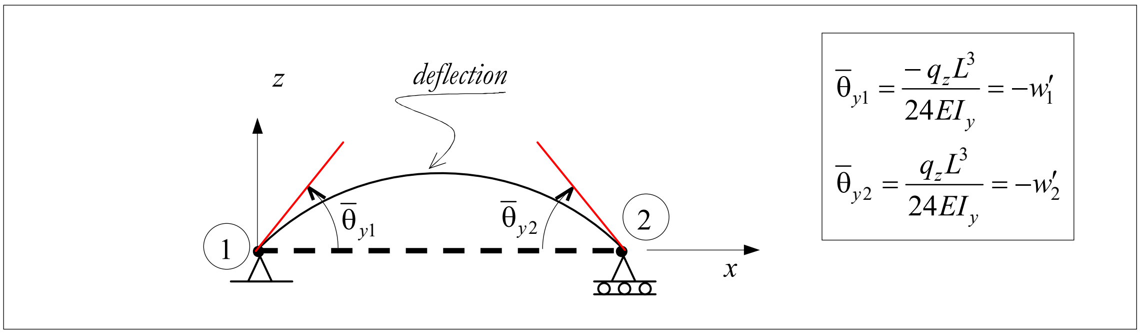


Figure 6.170: Rotations at the ends of the beam.

Reaction Calculation

$$\{\bar{\boldsymbol{f}}^{(e)}\} = \begin{bmatrix} \dfrac{12EI_y}{L^3} & \dfrac{-6EI_y}{L^2} & \dfrac{-12EI_y}{L^3} & \dfrac{-6EI_y}{L^2} \\ \dfrac{-6EI_y}{L^2} & \dfrac{4EI_y}{L} & \dfrac{6EI_y}{L^2} & \dfrac{2EI_y}{L} \\ \dfrac{-12EI_y}{L^3} & \dfrac{6EI_y}{L^2} & \dfrac{12EI_y}{L^3} & \dfrac{6EI_y}{L^2} \\ \dfrac{-6EI_y}{L^2} & \dfrac{2EI_y}{L} & \dfrac{6EI_y}{L^2} & \dfrac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ \dfrac{-q_z L^3}{24EI_y} \\ 0 \\ \dfrac{q_z L^3}{24EI_y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{-q_z L^2}{12} \\ 0 \\ \dfrac{q_z L^2}{12} \end{Bmatrix} \tag{6.500}$$

Then, by apply the equation (6.476), $\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}^{(e)}_{Eq}\}$, we can obtain:

$$\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}^{(e)}\} = \begin{Bmatrix} 0 \\ \dfrac{-q_z L^2}{12} \\ 0 \\ \dfrac{q_z L^2}{12} \end{Bmatrix} - \begin{Bmatrix} \dfrac{q_z L}{2} \\ \dfrac{-q_z L^2}{12} \\ \dfrac{q_z L}{2} \\ \dfrac{q_z L^2}{12} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-q_z L}{2} \\ 0 \\ \dfrac{-q_z L}{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{z1}^{(-)} \\ M_{y1}^{(-)} \\ F_{z2}^{(+)} \\ M_{y1}^{(+)} \end{Bmatrix} \tag{6.501}$$

And the internal forces at the ends of the beam are:

$$\{\boldsymbol{f}^{(e)}_{int}\} = -\{\boldsymbol{R}^{(e)}\} = \begin{Bmatrix} F_{z1}^{(+)} \\ M_{y1}^{(+)} \\ F_{z2}^{(-)} \\ M_{y2}^{(-)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{q_z L}{2} \\ 0 \\ \dfrac{q_z L}{2} \\ 0 \end{Bmatrix} \tag{6.502}$$

NOTE 1: Analytical solution by using the direct integration

In this sub-section we will obtain the analytical solution (the exact one) for the problem described in Figure 6.169. To obtain the analytical solution we will use the direct integration, and we will start from the moment function of the beam, (see Figure 6.171):

$$M_y(x) = -F_{z2}^{(-)}(L-x) - q_z(L-x)\frac{(L-x)}{2} = \frac{q_z L}{2}(L-x) - q_z(L-x)\frac{(L-x)}{2} = \frac{q_z}{2}(Lx - x^2) \tag{6.503}$$

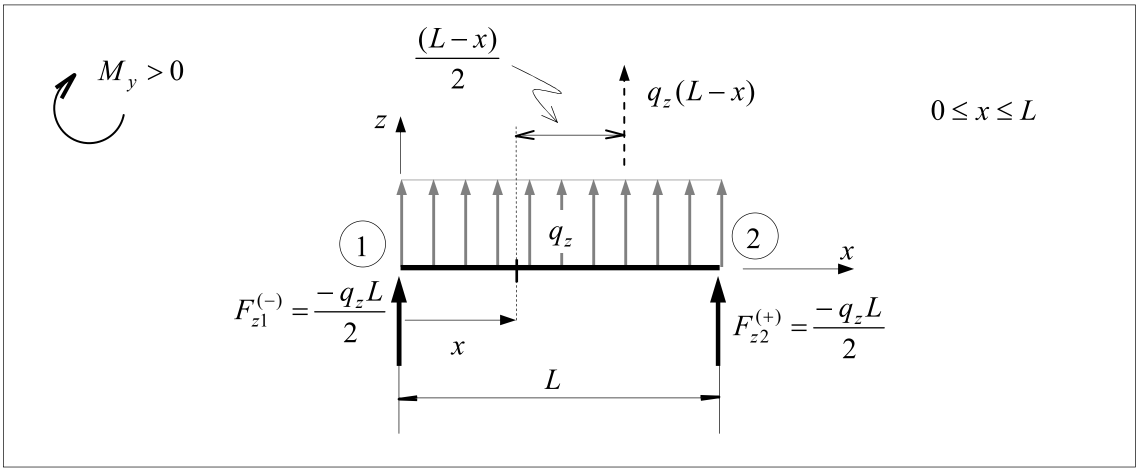


Figure 6.171: Beam bi-supported under uniformly distributed load.

Recall that the differential equation for the beam in function of the deflection (w), (see equation (6.417)), is given by:

$$M_y = -EI_y w'' \equiv -EI_y \frac{d^2w}{dx^2} \quad \Rightarrow \quad -EI_y \frac{d^2w}{dx^2} = \frac{q_z}{2}(Lx - x^2)$$

By means of direct integration we can obtain:

$$EI_y \frac{d^2w}{dx^2} = \frac{-q_z}{2}(Lx - x^2) \xrightarrow{\text{integrating}} EI_y \frac{dw}{dx} = \frac{-q_z}{12}(3Lx^2 - 2x^3) + C_1$$

Due to the symmetry, at the middle of the beam span the condition $\frac{dw}{dx} \equiv w'(x = \frac{L}{2}) = 0$ holds. Then, the constant of integration C_1 can be obtained as follows:

$$EI_y w'(x = \tfrac{L}{2}) = \frac{-q_z}{12}\left[3L\left(\frac{L}{2}\right)^2 - 2\left(\frac{L}{2}\right)^3\right] + C_1 = 0 \quad \Rightarrow \quad C_1 = \frac{q_z L^3}{24}$$

Then

$$EI_y \frac{dw}{dx} = \frac{-q_z}{12}(3Lx^2 - 2x^3) + \frac{q_z L^3}{24} \quad \Rightarrow \quad \frac{dw}{dx} \equiv w' = \frac{-q_z}{24EI_y}(6Lx^2 - 4x^3 - L^3) \tag{6.504}$$

By integrating once more we obtain the deflection of the beam $w(x)$:

$$\frac{dw}{dx} = \frac{-q_z}{24EI_y}(6Lx^2 - 4x^3 - L^3) \xrightarrow{\text{integrating}} w = \frac{-q_z}{24EI_y}(2Lx^3 - x^4 - L^3 x) + C_2$$

where the constant of integration C_2 can be obtained as follows $w(x=0) = 0 \Rightarrow C_2 = 0$. Then, the deflection of the beam becomes:

$$w(x) = \frac{-q_z}{24EI_y}(2Lx^3 - x^4 - L^3 x) \tag{6.505}$$

Then, we can obtain:

Deflection at $x = \frac{L}{2}$:

$$w(x=\tfrac{L}{2})=\frac{-q_z}{24EI_y}\left[2L\left(\frac{L}{2}\right)^3-\left(\frac{L}{2}\right)^4-L^3\left(\frac{L}{2}\right)\right]=\frac{5q_zL^4}{384EI_y} \tag{6.506}$$

First derivative of the deflection, (see equation (6.504)):

$$(x=0)\quad\Rightarrow\quad w'(x=0)=w_1'=\frac{-q_z}{24EI_y}(6L(0)^2-4(0)^3-L^3)=\frac{q_zL^3}{24EI_y}=-\overline{\theta}_{y1}$$

$$(x=L)\quad\Rightarrow\quad w'(x=L)=w_2'=\frac{-q_z}{24EI_y}(6L(L)^2-4(L)^3-L^3)=\frac{-q_zL^3}{24EI_y}=-\overline{\theta}_{y2}$$

which matches the equation in (6.499).

NOTE 2: Checking the results

According to equations in (6.423) the following is true:

$$\frac{\partial^2 w}{\partial x^2}=\frac{-M_y}{EI_y}\quad\text{and}\quad\frac{\partial^3 w}{\partial x^3}=\frac{-Q_z}{EI_y} \tag{6.507}$$

By means of equation (6.505) we can obtain:

$$w'=\frac{\partial w(x)}{\partial x}=\frac{\partial}{\partial x}\left[\frac{-q_z}{24EI_y}(2Lx^3-x^4-L^3x)\right]=\frac{-q_z}{24EI_y}(6Lx^2-4x^3-L^3)$$

$$w''=\frac{\partial^2 w(x)}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{-q_z}{24EI_y}(6Lx^2-4x^3-L^3)\right]=\frac{-q_z}{2EI_y}(Lx-x^2)$$

$$w'''=\frac{\partial^3 w(x)}{\partial x^3}=\frac{\partial}{\partial x}\left[\frac{-q_z}{2EI_y}(Lx-x^2)\right]=\frac{-q_z}{2EI_y}(L-2x)$$

With that the equations (6.507) become:

$$\frac{\partial^2 w}{\partial x^2}=\frac{-M_y}{EI_y}\quad\Rightarrow\quad\frac{-q_z}{2EI_y}(Lx-x^2)=\frac{-M_y}{EI_y}\quad\Rightarrow\quad M_y(x)=\frac{q_z}{2}(Lx-x^2) \tag{6.508}$$

which result matches the equation in (6.503). And

$$\frac{\partial^3 w}{\partial x^3}=\frac{-Q_z}{EI_y}\quad\Rightarrow\quad\frac{-Q_z}{EI_y}=\frac{-q_z}{2EI_y}(L-2x)\quad\Rightarrow\quad Q_z(x)=\frac{q_z}{2}(L-2x)\ \Rightarrow\begin{cases}Q_z(x=0)=\dfrac{q_zL}{2}\\[2ex] Q_z(x=L)=\dfrac{-q_zL}{2}\end{cases}$$

The functions $M_y(x)$ and $Q_z(x)$ can be appreciated in Figure 6.172.

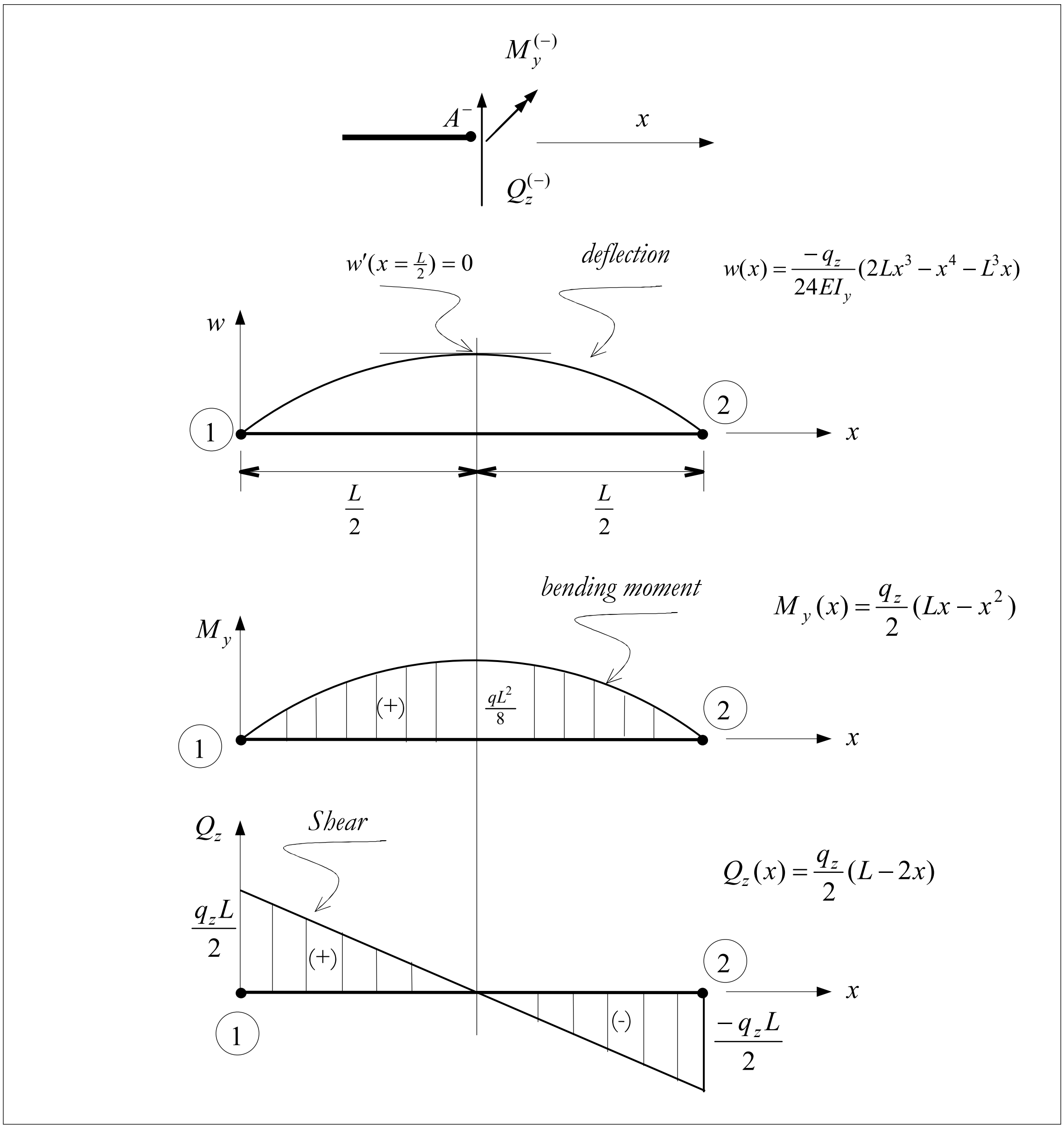


Figure 6.172: Deflection and internal "forces".

NOTE 3: Note that the function adopted to approach the displacement $w(x)$ is given by, (see equation (6.450)):

$$w=w_1\left[2\left(\frac{x}{L}\right)^3-3\left(\frac{x}{L}\right)^2+1\right]+w_2\left[-2\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2\right]-\overline{\theta}_{y1}\left[\frac{x^3}{L^2}-\frac{2x^2}{L}+x\right]-\overline{\theta}_{y2}\left[\frac{x^3}{L^2}-\frac{x^2}{L}\right]$$

For this example we have that $w_1=w_2=0$, and $\begin{Bmatrix}\overline{\theta}_{y1}\\ \overline{\theta}_{y2}\end{Bmatrix}=\frac{q_zL^3}{24EI_y}\begin{Bmatrix}-1\\ 1\end{Bmatrix}$, thus the above equation becomes:

$$w(x)=\frac{q_zL^3}{24EI_y}\left[\frac{x^3}{L^2}-\frac{2x^2}{L}+x\right]-\frac{q_zL^3}{24EI_y}\left[\frac{x^3}{L^2}-\frac{x^2}{L}\right]$$

At the middle of the beam the displacement is:

$$w(x=\tfrac{L}{2})=\frac{q_z L^3}{24EI_y}\left[\frac{\left(\frac{L}{2}\right)^3}{L^2}-\frac{2\left(\frac{L}{2}\right)^2}{L}+\left(\frac{L}{2}\right)\right]-\frac{q_z L^3}{24EI_y}\left[\frac{\left(\frac{L}{2}\right)^3}{L^2}-\frac{\left(\frac{L}{2}\right)^2}{L}\right]=\frac{1}{96}\frac{q_z L^4}{EI_y}$$

$$\Rightarrow w(x=\tfrac{L}{2})=\frac{1}{96}\frac{q_z L^4}{EI_y}=\left(\frac{4}{5}\right)\frac{5q_z L^4}{384EI_y}$$

And if we compare with the exact solution, (see equation (6.506)), the error is 20%.

NOTE 4: Let us consider the problem described in Figure 6.169 in which additionally we have concentrated moments at the nodes 1 and 2, (see Figure 6.173). Then, by adding the concentrated force vector to the vector $\{\boldsymbol{f}^{(e)}\}$ and by applying the boundary conditions we can obtain:

$$\begin{bmatrix}\frac{4EI_y}{L} & \frac{2EI_y}{L}\\ \frac{2EI_y}{L} & \frac{4EI_y}{L}\end{bmatrix}\begin{Bmatrix}\theta_{y1}\\ \theta_{y2}\end{Bmatrix}=\underbrace{\begin{Bmatrix}\frac{-q_z L^2}{12}\\ \frac{q_z L^2}{12}\end{Bmatrix}}_{=\{\boldsymbol{f}_{Eq}^{(e)}\}}+\underbrace{\begin{Bmatrix}\overline{M}_y^{(1)}\\ \overline{M}_y^{(2)}\end{Bmatrix}}_{=\{\boldsymbol{f}_0^{(e)}\}} \tag{6.509}$$

Solving the above equation we can obtain:

$$\begin{Bmatrix}\theta_{y1}\\ \theta_{y2}\end{Bmatrix}=\begin{bmatrix}\frac{4EI_y}{L} & \frac{2EI_y}{L}\\ \frac{2EI_y}{L} & \frac{4EI_y}{L}\end{bmatrix}^{-1}\left(\begin{Bmatrix}\frac{-q_z L^2}{12}\\ \frac{q_z L^2}{12}\end{Bmatrix}+\begin{Bmatrix}\overline{M}_y^{(1)}\\ \overline{M}_y^{(2)}\end{Bmatrix}\right)=\begin{Bmatrix}\frac{-q_z L^3}{24EI_y}+\frac{\overline{M}_y^{(1)}L}{3EI_y}-\frac{\overline{M}_y^{(2)}L}{6EI_y}\\ \frac{q_z L^3}{24EI_y}-\frac{\overline{M}_y^{(1)}L}{6EI_y}+\frac{\overline{M}_y^{(2)}L}{3EI_y}\end{Bmatrix} \tag{6.510}$$

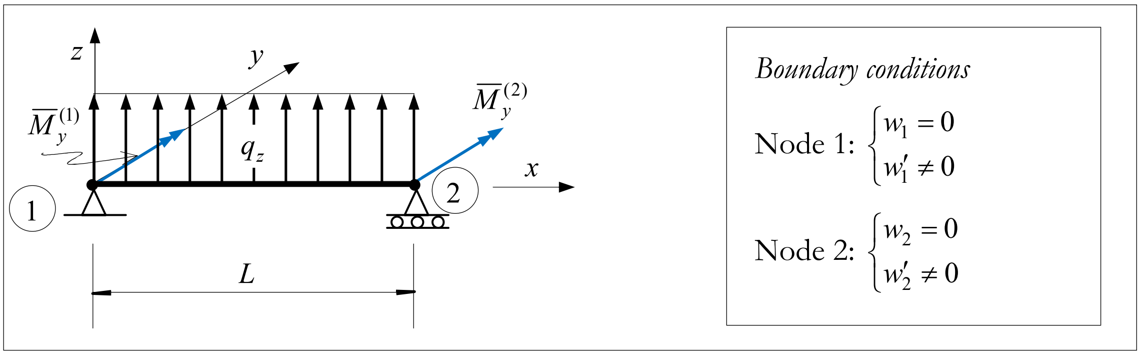


Figure 6.173: Beam bi-supported under uniformly distributed load and concentrated moments at the nodes.

Then the reaction vector can be obtained as follows:

$$\{\bar{\boldsymbol{f}}^{(e)}\}=\begin{bmatrix}\frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2}\\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L}\\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L}\end{bmatrix}\begin{Bmatrix}0\\ \frac{-q_z L^3}{24EI_y}+\frac{\overline{M}_y^{(1)}L}{3EI_y}-\frac{\overline{M}_y^{(2)}L}{6EI_y}\\ 0\\ \frac{q_z L^3}{24EI_y}-\frac{\overline{M}_y^{(1)}L}{6EI_y}+\frac{\overline{M}_y^{(2)}L}{3EI_y}\end{Bmatrix}=\begin{Bmatrix}\frac{-(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L}\\ \overline{M}_y^{(1)}-\frac{q_z L^2}{12}\\ \frac{(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L}\\ \overline{M}_y^{(2)}+\frac{q_z L^2}{12}\end{Bmatrix} \tag{6.511}$$

Then, by applying the equation (6.476), i.e. $\{\boldsymbol{R}^{(e)}\}=\{\bar{\boldsymbol{f}}^{(e)}\}-\{\boldsymbol{f}_{Eq}^{(e)}\}$, we can obtain:

$$\{\boldsymbol{R}^{(e)}\}=\{\bar{\boldsymbol{f}}^{(e)}\}-\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \dfrac{-(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L} \\ \overline{M}_y^{(1)}-\dfrac{q_z L^2}{12} \\ \hdashline \dfrac{(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L} \\ \overline{M}_y^{(2)}+\dfrac{q_z L^2}{12} \end{Bmatrix} - \begin{Bmatrix} \dfrac{q_z L}{2} \\ \dfrac{-q_z L^2}{12} \\ \dfrac{q_z L}{2} \\ \dfrac{q_z L^2}{12} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L}-\dfrac{q_z L}{2} \\ \overline{M}_y^{(1)} \\ \hdashline \dfrac{(\overline{M}_y^{(1)}+\overline{M}_y^{(2)})}{L}-\dfrac{q_z L}{2} \\ \overline{M}_y^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{z1}^{(-)} \\ M_{y1}^{(-)} \\ F_{z2}^{(+)} \\ M_{y1}^{(+)} \end{Bmatrix}$$

(6.512)

Problem 6.65

Obtain the rotations at the extremities of the beam described in Figure 6.174, and also obtain the deflection at the middle of the beam length. Use a fourth-order function to approach the displacement $w(x)$.

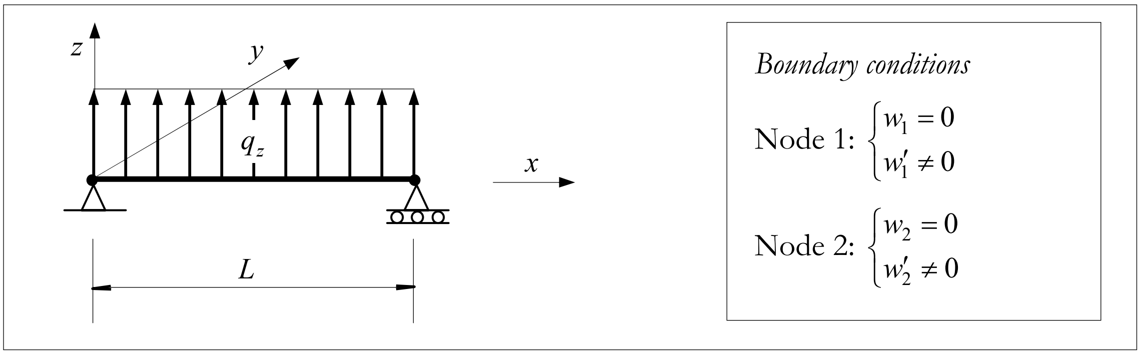


Figure 6.174: Beam bi-supported under uniformly distributed load.

Solution:

We adopt the function $w(x)=a_1+a_2x+a_3x^2+a_4x^3+a_5x^4$.

Let us apply the boundary conditions to $w(x)=a_1+a_2x+a_3x^2+a_4x^3+a_5x^4$, i.e.:

$$w(x=0)=a_1+a_2 0+a_3 0^2+a_4 0^3+a_5 0^4=0 \quad \Rightarrow \quad a_1=0$$
$$w(x=L)=\underbrace{a_1}_{=0}+a_2L+a_3L^2+a_4L^3+a_5L^4=0 \quad \Rightarrow \quad a_2+a_3L+a_4L^2+a_5L^3=0$$
$$\Rightarrow a_2=-(a_3L+a_4L^2+a_5L^3)$$

Then the displacement function can be rewritten as follows:

$$w(x)=a_2x+a_3x^2+a_4x^3+a_5x^4=-(a_3L+a_4L^2+a_5L^3)x+a_3x^2+a_4x^3+a_5x^4$$

or

$$w(x)=a_3(x^2-Lx)+a_4(x^3-L^2x)+a_5(x^4-L^2x) \tag{6.513}$$

The total potential energy is given by $\Pi=U^{int}-U^{ext}$, where:

Internal Potential Energy

$$U^{int}=\int_0^L \frac{EI_y}{2}w''^2dx=\frac{EI_y}{2}\int_0^L w''^2dx \tag{6.514}$$

where

$$
\begin{aligned}
&w'(x)=\frac{\partial w}{\partial x}=a_3(2x-L)+a_4(3x^2-L^2)+a_5(4x^3-L^2)\\
&\Rightarrow w''(x)=\frac{\partial^2 w}{\partial x^2}=2a_3+6a_4x+12a_5x^2\\
&\Rightarrow (w''(x))^2=4a_3^2+24a_3a_4x+48a_3a_5x^2+36a_4^2x^2+144a_4a_5x^3+144a_5^2x^4\\
&\Rightarrow \int_0^L [w''(x)]^2\,dx=4a_3^2L+12a_3a_4L^2+16a_3a_5L^3+12a_4^2L^3+36a_4a_5L^4+\frac{144}{5}a_5^2L^5
\end{aligned}
\tag{6.515}
$$

thus

$$
U^{int}=\int_0^L \frac{EI_y}{2}w''^2dx=\frac{EI_y}{2}\left[4a_3^2L+12a_3a_4L^2+16a_3a_5L^3+12a_4^2L^3+36a_4a_5L^4+\frac{144}{5}a_5^2L^5\right]
$$

External Potential Energy

As we are considering that q_z is independent of x, the external potential energy becomes:

$$
\begin{aligned}
U^{ext}&=\int_0^L q_z w(x)dx=q_z\int_0^L w(x)dx=q_z\int_0^L[a_3(x^2-Lx)+a_4(x^3-L^2x)+a_5(x^4-L^2x)]dx\\
&=-\left(\frac{q_zL^3}{6}a_3+\frac{q_zL^4}{4}a_4+\frac{3q_zL^5}{10}a_5\right)
\end{aligned}
\tag{6.516}
$$

Then, the Total Potential Energy, $\Pi(a_3,a_4,a_5)=U^{int}-U^{ext}$, becomes:

$$
\begin{aligned}
\Pi(a_3,a_4,a_5)=\frac{EI_y}{2}&\left[4a_3^2L+12a_3a_4L^2+16a_3a_5L^3+12a_4^2L^3+36a_4a_5L^4+\frac{144}{5}a_5^2L^5\right]\\
&+\left(\frac{q_zL^3}{6}a_3+\frac{q_zL^4}{4}a_4+\frac{3q_zL^5}{10}a_5\right)
\end{aligned}
\tag{6.517}
$$

As we are looking for the stationary state the following must hold:

$$
\frac{\partial\Pi}{\partial a_3}=0 \quad\Rightarrow\quad \frac{EI_y}{2}\left\{8La_3+12L^2a_4+16L^3a_5\right\}+\frac{q_zL^3}{6}=0 \tag{6.518}
$$

$$
\frac{\partial\Pi}{\partial a_4}=0 \quad\Rightarrow\quad \frac{EI_y}{2}\left\{12L^2a_3+24L^3a_4+36L^4a_5\right\}+\frac{q_zL^4}{4}=0 \tag{6.519}
$$

$$
\frac{\partial\Pi}{\partial a_5}=0 \quad\Rightarrow\quad \frac{EI_y}{2}\left\{16L^3a_3+36L^4a_4+\frac{288}{5}L^5a_5\right\}+\frac{3q_zL^5}{10}=0 \tag{6.520}
$$

Restructuring the above set of equations in matrix form we can obtain:

$$
EI_y\begin{bmatrix}4L & 6L^2 & 8L^3\\ 6L^2 & 12L^3 & 18L^4\\ 8L^3 & 18L^4 & \frac{144}{5}L^5\end{bmatrix}\begin{Bmatrix}a_3\\a_4\\a_5\end{Bmatrix}=\begin{Bmatrix}\dfrac{-q_zL^3}{6}\\ \dfrac{-q_zL^4}{4}\\ \dfrac{-3q_zL^5}{10}\end{Bmatrix}\xrightarrow{Solve}\begin{Bmatrix}a_3\\a_4\\a_5\end{Bmatrix}=\begin{Bmatrix}0\\ \dfrac{-q_zL}{12EI_y}\\ \dfrac{q_z}{24EI_y}\end{Bmatrix} \tag{6.521}
$$

Then, by substituting the coefficients (a_3,a_4,a_5) into the displacement function (6.513) we can obtain:

$$
\begin{aligned}
& w(x) = a_3(x^2 - Lx) + a_4(x^3 - L^2x) + a_5(x^4 - L^2x) \\
& \Rightarrow w(x) = 0(x^2 - Lx) + \left(\frac{-q_z L}{12EI_y}\right)(x^3 - L^2x) + \left(\frac{q_z}{24EI_y}\right)(x^4 - L^2x) \\
& \Rightarrow w(x) = \frac{q_z}{24EI_y}(x^4 - 2Lx^3 + L^3x)
\end{aligned}
\tag{6.522}
$$

which matches the exact solution, (see equation (6.505)). Then,

$$
w(x) = \frac{q_z}{24EI_y}(x^4 - 2Lx^3 + L^3x) \quad \Rightarrow \quad w'(x) = \frac{q_z}{24EI_y}(4x^3 - 6Lx^2 + L^3) = -\overline{\theta}_y(x)
$$

$$
\overline{\theta}_y(x) = \frac{-q_z}{24EI_y}(4x^3 - 6Lx^2 + L^3) \quad \Rightarrow \quad \begin{cases} \overline{\theta}_y(x=0) \equiv \overline{\theta}_{y1} = \dfrac{-q_z L^3}{24EI_y} \\ \overline{\theta}_y(x=L) \equiv \overline{\theta}_{y2} = \dfrac{q_z L^3}{24EI_y} \end{cases}
$$

NOTE 1: Let us obtain the stiffness matrix correspondent to this case by using the function $w(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4$. To determine the coefficients (a_i) we will need to define 5 points (nodes), (see Figure 6.175).

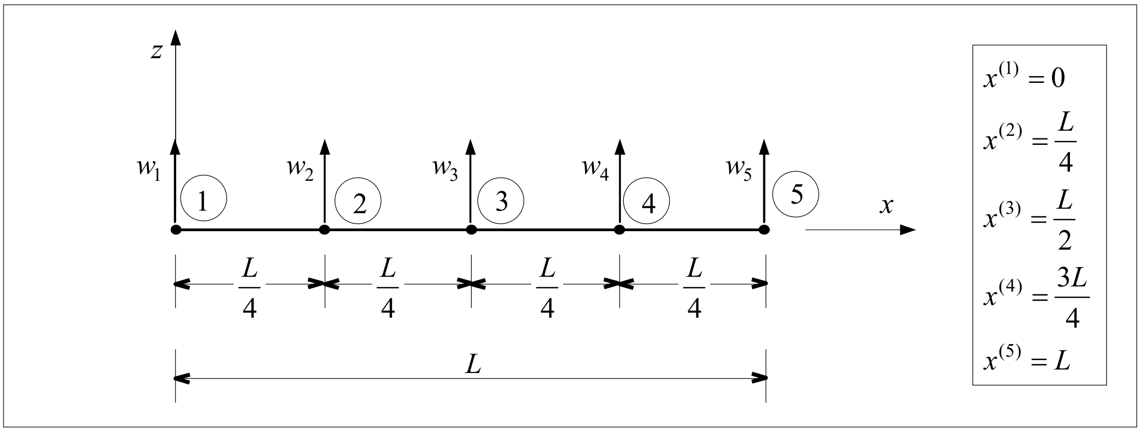


Figure 6.175: Discretization of the beam (5 degrees-of-freedom).

Applying the function $w(x)$ for each node we can obtain:

$$
\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{L}{4} & \frac{L^2}{16} & \frac{L^3}{64} & \frac{L^4}{256} \\ 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} & \frac{L^4}{16} \\ 1 & \frac{3L}{4} & \frac{9L^2}{16} & \frac{27L^3}{64} & \frac{81L^4}{256} \\ 1 & L & L^2 & L^3 & L^4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}
\tag{6.523}
$$

and its inverse is given by:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-25}{3L} & \frac{16}{L} & \frac{-12}{L} & \frac{16}{3L} & \frac{-1}{L} \\ \frac{70}{3L^2} & \frac{-208}{3L^2} & \frac{76}{L^2} & \frac{-112}{3L^2} & \frac{22}{3L^2} \\ \frac{-80}{3L^3} & \frac{96}{L^3} & \frac{-128}{L^3} & \frac{224}{3L^3} & \frac{-16}{3L^3} \\ \frac{32}{3L^4} & \frac{-128}{3L^4} & \frac{64}{L^4} & \frac{-128}{3L^4} & \frac{32}{3L^4} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{Bmatrix} \tag{6.524}$$

Then, the function $w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4$ can be written as follows:

$$w(x) = N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4 + N_5 w_5$$

or

$$\begin{aligned} w(x) = {} & \left(1 - \frac{25x}{3L} + \frac{70x^2}{3L^2} - \frac{80x^3}{3L^3} + \frac{32x^4}{3L^4}\right) w_1 + \left(\frac{16x}{L} - \frac{208x^2}{3L^2} + \frac{96x^3}{L^3} - \frac{128x^4}{3L^4}\right) w_2 + \\ & \left(\frac{-12x}{3L} + \frac{76x^2}{L^2} - \frac{128x^3}{L^3} + \frac{64x^4}{L^4}\right) w_3 + \left(\frac{16x}{3L} - \frac{112x^2}{3L^2} + \frac{224x^3}{3L^3} - \frac{128x^4}{3L^4}\right) w_4 + \\ & \left(\frac{-x}{L} + \frac{22x^2}{3L^2} - \frac{16x^3}{L^3} + \frac{32x^4}{3L^4}\right) w_5 \end{aligned}$$

Following the same procedure used from equation (6.458) to (6.468) we can finally obtain:

$$\underbrace{\frac{8EI_y}{15L^3} \begin{bmatrix} 494 & -1376 & 1444 & -736 & 174 \\ -1376 & 4224 & -5056 & 2944 & -736 \\ 1444 & -5056 & 7224 & -5056 & 1444 \\ -736 & 2944 & -5056 & 4224 & -1376 \\ 174 & -736 & 1444 & -1376 & 494 \end{bmatrix}}_{=[\boldsymbol{Ke}^{(e)}]} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{Bmatrix} = \underbrace{\begin{Bmatrix} \frac{7q_z L}{90} \\ \frac{16q_z L}{45} \\ \frac{2q_z L}{15} \\ \frac{16q_z L}{45} \\ \frac{7q_z L}{90} \end{Bmatrix}}_{=\{\boldsymbol{f}_{Eq}^{(e)}\}} \tag{6.525}$$

Note that for this case we do not have the continuity rotation between elements, and we cannot apply rotation equal to zero, (see Figure 6.176).

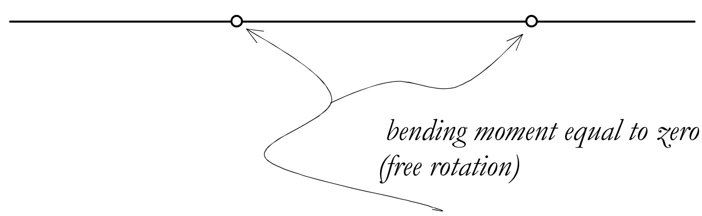


Figure 6.176: Free rotation.

NOTE 2: Note that the shape function N_1 could have been obtained by means of the procedure used in **Problem 6.40** – NOTE 3, i.e.:

$$
N_1 = \frac{\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & \frac{L}{4} & \frac{L^2}{16} & \frac{L^3}{64} & \frac{L^4}{256} \\ 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} & \frac{L^4}{16} \\ 1 & \frac{3L}{4} & \frac{9L^2}{16} & \frac{27L^3}{64} & \frac{81L^4}{256} \\ 1 & L & L^2 & L^3 & L^4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{L}{4} & \frac{L^2}{16} & \frac{L^3}{64} & \frac{L^4}{256} \\ 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} & \frac{L^4}{16} \\ 1 & \frac{3L}{4} & \frac{9L^2}{16} & \frac{27L^3}{64} & \frac{81L^4}{256} \\ 1 & L & L^2 & L^3 & L^4 \end{vmatrix}} = 1 - \frac{25x}{3L} + \frac{70x^2}{3L^2} - \frac{80x^3}{3L^3} + \frac{32x^4}{3L^4}
$$

Problem 6.66

Obtain the consistent load vectors for the cases: a) beam presented in Figure 6.177 and b) Figure 6.178. And the boundary conditions are: Node 1 - $w_1 = 0$, $w_1' \neq 0$; Node 2 - $w_2 = 0$, $w_2' \neq 0$.

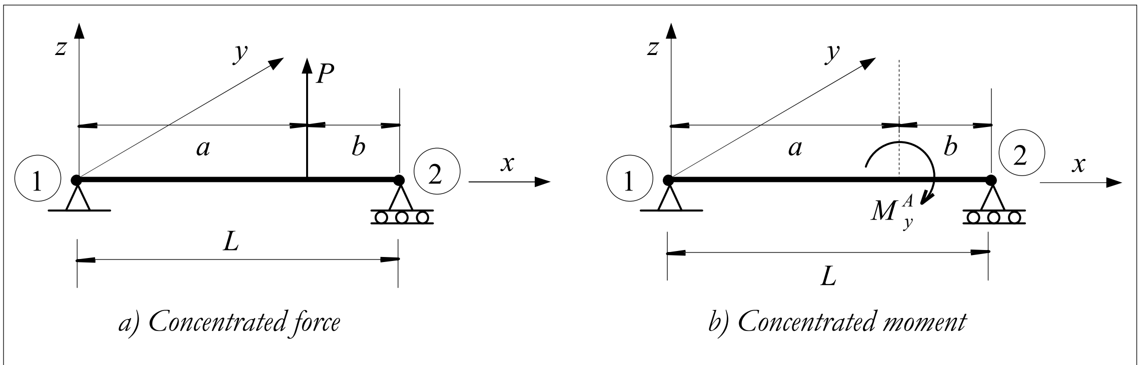


Figure 6.177: Concentrated load and moment.

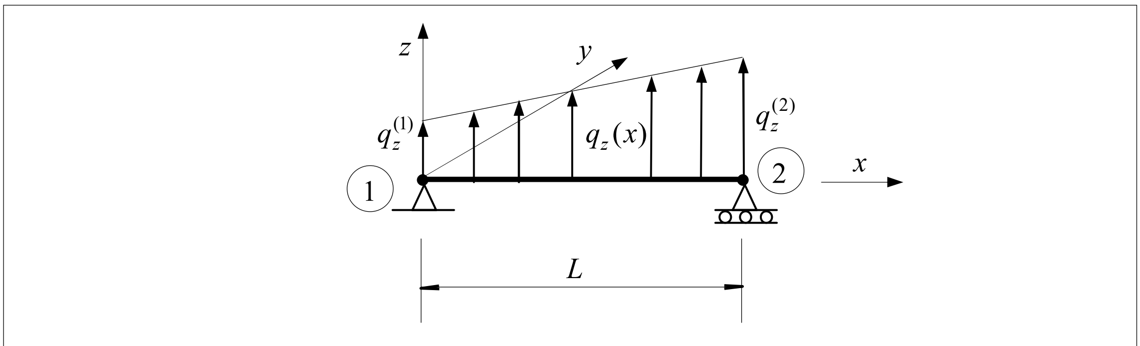


Figure 6.178: Linearly distributed load.

Solution:

Case a (Figure 6.177(a))**:** For this case the stiffness matrix is the same as the one presented in equation (6.470). And the External Potential Energy, (see Figure 6.156), for this case becomes:

$$U^{ext} = Pw_p = Pw(x=a) \tag{6.526}$$

Taking into account the deflection function w, (see equation (6.450)), when $(x=a)$:

$$w = w_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + w_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] - \bar{\theta}_{y1}\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] - \bar{\theta}_{y2}\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right]$$

$$\Rightarrow w_P = w_1\left[2\left(\frac{a}{L}\right)^3 - 3\left(\frac{a}{L}\right)^2 + 1\right] + w_2\left[-2\left(\frac{a}{L}\right)^3 + 3\left(\frac{a}{L}\right)^2\right] - \bar{\theta}_{y1}\left[\frac{a^3}{L^2} - \frac{2a^2}{L} + a\right] - \bar{\theta}_{y2}\left[\frac{a^3}{L^2} - \frac{a^2}{L}\right] \tag{6.527}$$

Then,

$$U^{ext} = P\left(w_1\left[\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right] + \bar{\theta}_{y1}\left[\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right] + w_2\left[-\frac{2a^3}{L^3} + \frac{3a^2}{L^2}\right] + \bar{\theta}_{y2}\left[\frac{-a^3}{L^2} + \frac{a^2}{L}\right]\right)$$

To achieve the equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial w_1} = \frac{\partial U^{int}}{\partial w_1} - \frac{\partial U^{ext}}{\partial w_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_1} - \left[\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right]P = 0 \tag{6.528}$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{y1}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{y1}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{y1}} - \left[\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right]P = 0 \tag{6.529}$$

$$\frac{\partial \Pi}{\partial w_2} = \frac{\partial U^{int}}{\partial w_2} - \frac{\partial U^{ext}}{\partial w_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_2} - \left[-\frac{2a^3}{L^3} + \frac{3a^2}{L^2}\right]P = 0 \tag{6.530}$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{y2}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{y2}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{y2}} - \left[\frac{-a^3}{L^2} + \frac{a^2}{L}\right]P = 0 \tag{6.531}$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \left(\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right)P \\ \left(\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right)P \\ \left(\frac{-2a^3}{L^3} + \frac{3a^2}{L^2}\right)P \\ \left(\frac{-a^3}{L^2} + \frac{a^2}{L}\right)P \end{Bmatrix}} \quad \text{or} \quad \boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{Pb^2}{L^3}(3a+b) \\ \frac{-Pab^2}{L^2} \\ \frac{Pa^2}{L^3}(a+3b) \\ \frac{Pa^2 b}{L^2} \end{Bmatrix}} \tag{6.532}$$

and the system $[\boldsymbol{Ke}^{(1)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}\}$ becomes:

$$\begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} \left(\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right)P \\ \left(\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right)P \\ \left(\frac{-2a^3}{L^3} + \frac{3a^2}{L^2}\right)P \\ \left(\frac{-a^3}{L^2} + \frac{a^2}{L}\right)P \end{Bmatrix} \tag{6.533}$$

To solve the problem we must apply the boundary conditions to the above system, so:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4EI_y}{L} & 0 & \frac{2EI_y}{L} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2EI_y}{L} & 0 & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \left(\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right)P \\ 0 \\ \left(\frac{-a^3}{L^2} + \frac{a^2}{L}\right)P \end{Bmatrix}$$

Solving we can obtain:

$$\begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \frac{-Pa}{6LEI_y} \begin{Bmatrix} 0 \\ (a^2 - 3aL + 2L^2) \\ 0 \\ (a^2 - L^2) \end{Bmatrix} \tag{6.534}$$

which matches the exact solution. The vector $\{\bar{\boldsymbol{f}}^{(e)}\}$ can be obtained as follows:

$$\begin{bmatrix} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ \left(\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right)P \\ 0 \\ \left(\frac{-a^3}{L^2} + \frac{a^2}{L}\right)P \end{Bmatrix} = \frac{Pa}{L^3} \begin{Bmatrix} (2a^2 - 3aL + L^2) \\ -L(a^2 - 2aL + L^2) \\ -(2a^2 - 3aL + L^2) \\ -L(a-L) \end{Bmatrix}$$

Then, by applying the equation (6.476), $\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}^{(e)}_{Eq}\}$, we can obtain the reaction forces:

$$\{\boldsymbol{R}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} - \{\boldsymbol{f}^{(e)}_{Eq}\} = \frac{Pa}{L^3} \begin{Bmatrix} (2a^2 - 3aL + L^2) \\ -L(a^2 - 2aL + L^2) \\ -(2a^2 - 3aL + L^2) \\ -L(a-L) \end{Bmatrix} - \begin{Bmatrix} \left(\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right)P \\ \left(\frac{-a^3}{L^2} + \frac{2a^2}{L} - a\right)P \\ \left(\frac{-2a^3}{L^3} + \frac{3a^2}{L^2}\right)P \\ \left(\frac{-a^3}{L^2} + \frac{a^2}{L}\right)P \end{Bmatrix} = \begin{Bmatrix} \frac{P}{L}(a-L) \\ 0 \\ \frac{-Pa}{L} \\ 0 \end{Bmatrix}$$

Case a (Figure 6.177(b))**:** For this case we will only obtain the consistent load vector. The External Potential Energy, (see Figure 6.158), due to the concentrated moment is given by:

$$U^{ext} = M_y^A \overline{\theta}_{yA} \tag{6.535}$$

Taking into account the derivative of the deflection function w, (see equation (6.451)), when $(x = a)$:

$$\overline{\theta}_y(x) = -w_1\left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] - w_2\left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] + \overline{\theta}_{y1}\left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] + \overline{\theta}_{y2}\left[\frac{3x^2}{L^2} - \frac{2x}{L}\right]$$

$$\Rightarrow \overline{\theta}_y(x = a) = -w_1\left[\frac{6a^2}{L^3} - \frac{6a}{L^2}\right] - w_2\left[-\frac{6a^2}{L^3} + \frac{6a}{L^2}\right] + \overline{\theta}_{y1}\left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1\right] + \overline{\theta}_{y2}\left[\frac{3a^2}{L^2} - \frac{2a}{L}\right]$$

Then,

$$U^{ext} = M_y^A\left(-w_1\left[\frac{6a^2}{L^3} - \frac{6a}{L^2}\right] - w_2\left[-\frac{6a^2}{L^3} + \frac{6a}{L^2}\right] + \overline{\theta}_{y1}\left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1\right] + \overline{\theta}_{y2}\left[\frac{3a^2}{L^2} - \frac{2a}{L}\right]\right)$$

To achieve the equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial w_1} = \frac{\partial U^{int}}{\partial w_1} - \frac{\partial U^{ext}}{\partial w_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_1} + M_y^A\left[\frac{6a^2}{L^3} - \frac{6a}{L^2}\right] = 0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y1}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{y1}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{y1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{y1}} - M_y^A\left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1\right] = 0$$

$$\frac{\partial \Pi}{\partial w_2} = \frac{\partial U^{int}}{\partial w_2} - \frac{\partial U^{ext}}{\partial w_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_2} + M_y^A\left[-\frac{6a^2}{L^3} + \frac{6a}{L^2}\right] = 0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y2}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{y2}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{y2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{y2}} - M_y^A\left[\frac{3a^2}{L^2} - \frac{2a}{L}\right] = 0$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = M_y^A\begin{Bmatrix} \dfrac{-6a^2}{L^3} + \dfrac{6a}{L^2} \\ \dfrac{3a^2}{L^2} - \dfrac{4a}{L} + 1 \\ \dfrac{6a^2}{L^3} - \dfrac{6a}{L^2} \\ \dfrac{3a^2}{L^2} - \dfrac{2a}{L} \end{Bmatrix}} \quad \text{or} \quad \boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \dfrac{6M_y^A ab}{L^3} \\ \dfrac{M_y^A b}{L^2}(b - 2a) \\ -\dfrac{6M_y^A ab}{L^3} \\ \dfrac{M_y^A a}{L^2}(a - 2b) \end{Bmatrix}} \tag{6.536}$$

Case b: For this case the distributed load can be represented by $q_z(x) = q_z^{(1)} + \frac{x}{L}(q_z^{(2)} - q_z^{(1)})$, and the external potential energy becomes:

$$U^{ext} = \int_0^L q(x)w(x)dx = \int_0^L \left[q_z^{(1)} + \frac{x}{L}(q_z^{(2)} - q_z^{(1)})\right]w(x)dx$$

Taking into account the deflection function w, (see equation (6.450)), and after the integral is solved we can obtain:

$$U^{ext} = \left(\frac{3Lq_z^{(2)}}{20} + \frac{7Lq_z^{(1)}}{20} \right) w_1 + \left(\frac{-L^2 q_z^{(2)}}{30} - \frac{L^2 q_z^{(1)}}{20} \right) \overline{\theta}_{y1} + \left(\frac{7Lq_z^{(2)}}{20} + \frac{3Lq_z^{(1)}}{20} \right) w_2 + \left(\frac{L^2 q_z^{(2)}}{20} + \frac{L^2 q_z^{(1)}}{30} \right) \overline{\theta}_{y2}$$

To achieve the equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial w_1} = \frac{\partial U^{int}}{\partial w_1} - \frac{\partial U^{ext}}{\partial w_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_1} - \left[\frac{3Lq_z^{(2)}}{20} + \frac{7Lq_z^{(1)}}{20} \right] = 0 \tag{6.537}$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y1}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{y1}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{y1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{y1}} - \left[\frac{-L^2 q_z^{(2)}}{30} - \frac{L^2 q_z^{(1)}}{20} \right] = 0 \tag{6.538}$$

$$\frac{\partial \Pi}{\partial w_2} = \frac{\partial U^{int}}{\partial w_2} - \frac{\partial U^{ext}}{\partial w_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial w_2} - \left[\frac{7Lq_z^{(2)}}{20} + \frac{3Lq_z^{(1)}}{20} \right] = 0 \tag{6.539}$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{y2}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{y2}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{y2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{y2}} - \left[\frac{L^2 q_z^{(2)}}{20} + \frac{L^2 q_z^{(1)}}{30} \right] = 0 \tag{6.540}$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \dfrac{L}{20}(7q_z^{(1)} + 3q_z^{(2)}) \\ \dfrac{-L^2}{60}(3q_z^{(1)} + 2q_z^{(2)}) \\ \dfrac{L}{20}(3q_z^{(1)} + 7q_z^{(2)}) \\ \dfrac{L^2}{60}(2q_z^{(1)} + 3q_z^{(2)}) \end{Bmatrix}} \tag{6.541}$$

Note that for the particular case when $q_z^{(1)} = q_z^{(2)} = q_z$, the above equation must match the equation for $\{\boldsymbol{f}_{Eq}^{(e)}\}$ given by the equation in (6.470).

Problem 6.67

Obtain the explicit equation $[\boldsymbol{Ke}^{(2)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}_{Eq}\}$ for the beam presented in Figure 6.179.

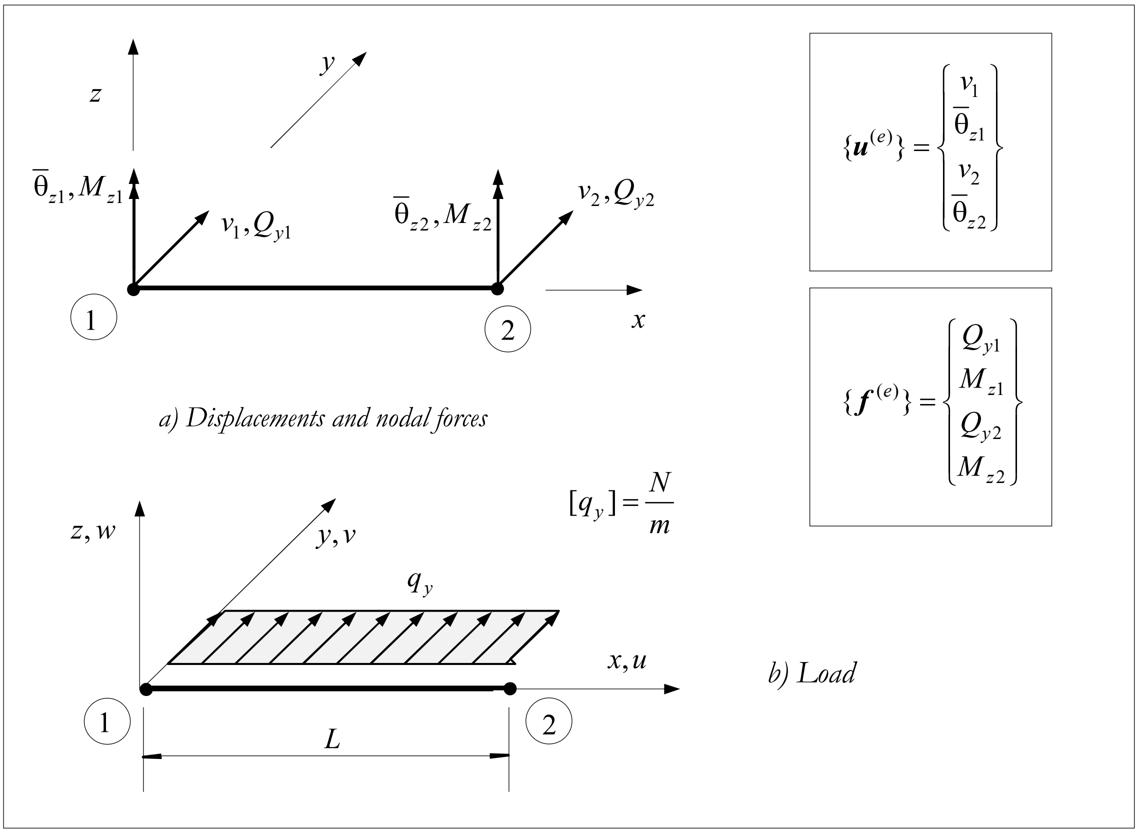


Figure 6.179: Beam element.

Solution:

The problem presented here is similar to the one presented in **Problem 6.62** – NOTE 2. The displacement according to y-direction can be represented by $v = ax^3 + bx^2 + cx + d$, (see equation (6.440)). Then, we can obtain a similar equation presented in (6.449), i.e.:

$$v = v_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + v_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] + v_1'\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] + v_2'\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right] \tag{6.542}$$

For this case we have that $\overline{\theta}_z = v'$, (see Figure 6.146). And the derivative of the above equation with respect to x becomes

$$v' = \overline{\theta}_z = v_1\left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] + v_2\left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] + \overline{\theta}_{z2}\left[\frac{3x^2}{L^2} - \frac{2x}{L}\right] \tag{6.543}$$

and

$$v'' = v_1\left[\frac{12x}{L^3} - \frac{6}{L^2}\right] + v_2\left[-\frac{12x}{L^3} + \frac{6}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{6x}{L^2} - \frac{4}{L}\right] + \overline{\theta}_{z2}\left[\frac{6x}{L^2} - \frac{2}{L}\right] \tag{6.544}$$

It will be useful to obtain analytically the following integrals:

$$\int_0^L v(x)dx = \frac{L}{2}v_1 + \frac{L}{2}v_2 + \frac{L^2}{12}\overline{\theta}_{z1} - \frac{L^2}{12}\overline{\theta}_{z2} \tag{6.545}$$

$$\begin{aligned}\int_0^L v^2 dx = {} & \frac{13L}{35}v_1^2 + \frac{13L}{35}v_2^2 + \frac{L^3}{105}\overline{\theta}_{z1}^2 + \frac{L^3}{105}\overline{\theta}_{z2}^2 + \frac{9L}{35}v_1v_2 + \frac{11L^2}{105}v_1\overline{\theta}_{z1} - \frac{13L^2}{210}v_1\overline{\theta}_{z2} \\ & + \frac{13L^2}{210}v_2\overline{\theta}_{z1} - \frac{11L^2}{105}v_2\overline{\theta}_{z2} - \frac{L^3}{70}\overline{\theta}_{z1}\overline{\theta}_{z2}\end{aligned} \tag{6.546}$$

$$\begin{aligned}\int_0^L v'^2 dx = {} & \frac{6}{5L}v_1^2 + \frac{6}{5L}v_2^2 + \frac{2L}{15}\overline{\theta}_{z1}^2 + \frac{2L}{15}\overline{\theta}_{z2}^2 - \frac{12}{5L}v_1v_2 + \frac{1}{5}v_1\overline{\theta}_{z1} + \frac{1}{5}v_1\overline{\theta}_{z2} \\ & - \frac{1}{5}v_2\overline{\theta}_{z1} - \frac{1}{5}v_2\overline{\theta}_{z2} - \frac{L}{15}\overline{\theta}_{z1}\overline{\theta}_{z2}\end{aligned} \tag{6.547}$$

$$\begin{aligned}\int_0^L v''^2 dx = {} & \frac{12}{L^3}v_1^2 + \frac{12}{L^3}v_2^2 + \frac{4}{L}\overline{\theta}_{z1}^2 + \frac{4}{L}\overline{\theta}_{z2}^2 - \frac{24}{L^3}v_1v_2 + \frac{12}{L^2}v_1\overline{\theta}_{z1} + \frac{12}{L^2}v_1\overline{\theta}_{z2} \\ & - \frac{12}{L^2}v_2\overline{\theta}_{z1} - \frac{12}{L^2}v_2\overline{\theta}_{z2} + \frac{4}{L}\overline{\theta}_{z1}\overline{\theta}_{z2}\end{aligned} \tag{6.548}$$

$$\int_0^L x\, v(x)dx = \frac{3L^2}{20}v_1 + \frac{7L^2}{20}v_2 + \frac{L^3}{30}\overline{\theta}_{z1} - \frac{L^3}{20}\overline{\theta}_{z2} \tag{6.549}$$

We will follow the same procedure adopted from the equation (6.450) to (6.468) in order to obtain:

$$U^{ext} = q_y\int_0^L v(x)dx = q_y\left(\frac{L}{2}v_1 + \frac{L}{2}v_2 + \frac{L^2}{12}\overline{\theta}_{z1} - \frac{L^2}{12}\overline{\theta}_{z2}\right) \tag{6.550}$$

Considering that EI_z is constant in the beam element, the internal potential energy becomes:

$$U^{int} = \frac{EI_z}{2}\int_0^L v''^2 dx \tag{6.551}$$

and:

$$\begin{aligned}U^{int} = \frac{EI_z}{2}\int_0^L v''^2 dx = \frac{EI_z}{2}\Bigg(& \frac{12}{L^3}v_1^2 + \frac{12}{L^3}v_2^2 + \frac{4}{L}\overline{\theta}_{z1}^2 + \frac{4}{L}\overline{\theta}_{z2}^2 - \frac{24}{L^3}v_1v_2 + \frac{12}{L^2}v_1\overline{\theta}_{z1} + \\ & \frac{12}{L^2}v_1\overline{\theta}_{z2} - \frac{12}{L^2}v_2\overline{\theta}_{z1} - \frac{12}{L^2}v_2\overline{\theta}_{z2} + \frac{4}{L}\overline{\theta}_{z1}\overline{\theta}_{z2}\Bigg)\end{aligned} \tag{6.552}$$

Then, the total potential energy (6.458), $\Pi = U^{int} - U^{ext}$, can be written as follows:

$$\begin{aligned}\Pi = \frac{EI_z}{2}\Bigg(& \frac{12}{L^3}v_1^2 + \frac{12}{L^3}v_2^2 + \frac{4}{L}\overline{\theta}_{z1}^2 + \frac{4}{L}\overline{\theta}_{z2}^2 - \frac{24}{L^3}v_1v_2 + \frac{12}{L^2}v_1\overline{\theta}_{z1} + \frac{12}{L^2}v_1\overline{\theta}_{z2} - \frac{12}{L^2}v_2\overline{\theta}_{z1} \\ & - \frac{12}{L^2}v_2\overline{\theta}_{z2} + \frac{4}{L}\overline{\theta}_{z1}\overline{\theta}_{z2}\Bigg) - q_y\left(\frac{L}{2}v_1 + \frac{L}{2}v_2 + \frac{L^2}{12}\overline{\theta}_{z1} - \frac{L^2}{12}\overline{\theta}_{z2}\right)\end{aligned} \tag{6.553}$$

As we are looking for the stationary state the following must hold:

$$\frac{\partial \Pi}{\partial v_1} = 0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{24}{L^3}v_1 - \frac{24}{L^3}v_2 + \frac{12}{L^2}\overline{\theta}_{z1} + \frac{12}{L^2}\overline{\theta}_{z2}\right\} - q_y\frac{L}{2} = 0 \tag{6.554}$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z1}} = 0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{8}{L}\bar{\theta}_{z1} + \frac{12}{L^2}v_1 - \frac{12}{L^2}v_2 + \frac{4}{L}\bar{\theta}_{z2}\right\} - q_y\frac{L^2}{12} = 0 \tag{6.555}$$

$$\frac{\partial \Pi}{\partial v_2} = 0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{24}{L^3}v_2 - \frac{24}{L^3}v_1 - \frac{12}{L^2}\bar{\theta}_{z1} - \frac{12}{L^2}\bar{\theta}_{z2}\right\} - q_y\frac{L}{2} = 0 \tag{6.556}$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z2}} = 0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{8}{L}\bar{\theta}_{z2} + \frac{12}{L^2}v_1 - \frac{12}{L^2}v_2 + \frac{4}{L}\bar{\theta}_{z1}\right\} + q_y\frac{L^2}{12} = 0 \tag{6.557}$$

Restructuring the above set of equations in matrix form we can obtain:

$$\begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\ \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{Bmatrix} v_1 \\ \bar{\theta}_{z1} \\ v_2 \\ \bar{\theta}_{z2} \end{Bmatrix} = \begin{Bmatrix} \frac{q_y L}{2} \\ \frac{q_y L^2}{12} \\ \frac{q_y L}{2} \\ \frac{-q_y L^2}{12} \end{Bmatrix} \tag{6.558}$$

$$[\boldsymbol{Ke}^{(2)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\}$$

where

$$\boxed{[\boldsymbol{Ke}^{(2)}] = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\ \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \quad ; \quad \{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{q_y L}{2} \\ \frac{q_y L^2}{12} \\ \frac{q_y L}{2} \\ \frac{-q_y L^2}{12} \end{Bmatrix}} \tag{6.559}$$

NOTE: The consistent load vector related to the load described in Figure 6.180 (a) and (b).

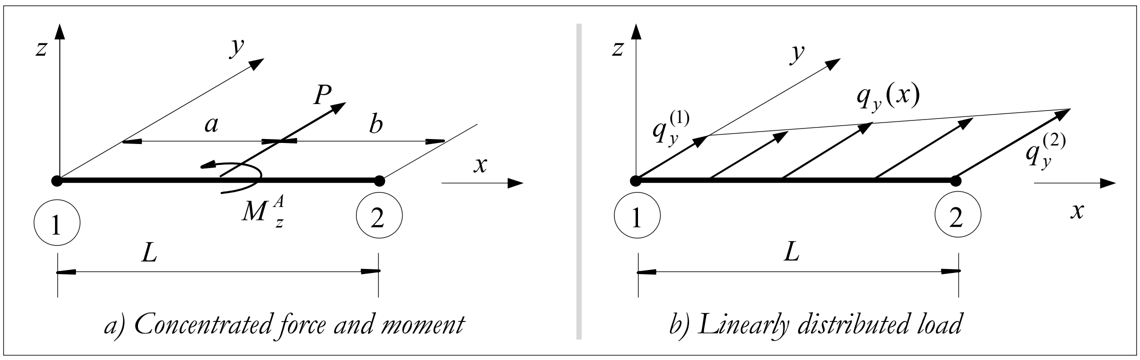


Figure 6.180: Load lies on plane $x-y$.

For the case a) the External Potential Energy due to the concentrated force is given by:

$$U^{ext} = Pv_p = Pv(x=a)$$

Taking into account the deflection function v, (see equation (6.542)), when $(x=a)$:

$$v = v_1\left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1\right] + v_2\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] + v_1'\left[\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right] + v_2'\left[\frac{x^3}{L^2} - \frac{x^2}{L}\right]$$

$$\Rightarrow v_P = v_1\left[\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right] + v_2\left[-\frac{2a^3}{L^3} + \frac{3a^2}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{a^3}{L^2} - \frac{2a^2}{L} + a\right] + \overline{\theta}_{z2}\left[\frac{a^3}{L^2} - \frac{a^2}{L}\right]$$

where we have also considered the definition $\overline{\theta}_z = v'$. Then,

$$U^{ext} = P\left(v_1\left[\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right] + v_2\left[-\frac{2a^3}{L^3} + \frac{3a^2}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{a^3}{L^2} - \frac{2a^2}{L} + a\right] + \overline{\theta}_{z2}\left[\frac{a^3}{L^2} - \frac{a^2}{L}\right]\right)$$

To achieve equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial v_1} = \frac{\partial U^{int}}{\partial v_1} - \frac{\partial U^{ext}}{\partial v_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_1} - \left[\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right]P = 0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{z1}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{z1}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{z1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{z1}} - \left[\frac{a^3}{L^2} - \frac{2a^2}{L} + a\right]P = 0$$

$$\frac{\partial \Pi}{\partial v_2} = \frac{\partial U^{int}}{\partial v_2} - \frac{\partial U^{ext}}{\partial v_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_2} - \left[-\frac{2a^3}{L^3} + \frac{3a^2}{L^2}\right]P = 0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{z2}} = \frac{\partial U^{int}}{\partial \overline{\theta}_{z2}} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{z2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \overline{\theta}_{z2}} - \left[\frac{a^3}{L^2} - \frac{a^2}{L}\right]P = 0$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \left(\frac{2a^3}{L^3} - \frac{3a^2}{L^2} + 1\right)P \\ \left(\frac{a^3}{L^2} - \frac{2a^2}{L} + a\right)P \\ \left(\frac{-2a^3}{L^3} + \frac{3a^2}{L^2}\right)P \\ \left(\frac{a^3}{L^2} - \frac{a^2}{L}\right)P \end{Bmatrix}} \quad \text{or} \quad \boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{Pb^2}{L^3}(3a+b) \\ \frac{Pab^2}{L^2} \\ \frac{Pa^2}{L^3}(a+3b) \\ \frac{-Pa^2 b}{L^2} \end{Bmatrix}} \tag{6.560}$$

The External Potential Energy due to the concentrated moment is given by:

$$U^{ext} = M_z^A \overline{\theta}_{zA} \tag{6.561}$$

Taking into account the derivative of the deflection function v, (see equation (6.543)), when $(x=a)$:

$$\overline{\theta}_z(x) = v_1\left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] + v_2\left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] + \overline{\theta}_{z2}\left[\frac{3x^2}{L^2} - \frac{2x}{L}\right]$$

$$\Rightarrow \overline{\theta}_z(x=a) \equiv \overline{\theta}_{zA} = v_1\left[\frac{6a^2}{L^3} - \frac{6a}{L^2}\right] + v_2\left[-\frac{6a^2}{L^3} + \frac{6a}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1\right] + \overline{\theta}_{z2}\left[\frac{3a^2}{L^2} - \frac{2a}{L}\right]$$

Then,

$$U^{ext} = M_z^A \left(v_1 \left[\frac{6a^2}{L^3} - \frac{6a}{L^2} \right] + v_2 \left[-\frac{6a^2}{L^3} + \frac{6a}{L^2} \right] + \bar{\theta}_{z1} \left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1 \right] + \bar{\theta}_{z2} \left[\frac{3a^2}{L^2} - \frac{2a}{L} \right] \right)$$

To achieve the equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial v_1} = \frac{\partial U^{int}}{\partial v_1} - \frac{\partial U^{ext}}{\partial v_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_1} - M_z^A \left[\frac{6a^2}{L^3} - \frac{6a}{L^2} \right] = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z1}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{z1}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{z1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{z1}} - M_z^A \left[\frac{3a^2}{L^2} - \frac{4a}{L} + 1 \right] = 0$$

$$\frac{\partial \Pi}{\partial v_2} = \frac{\partial U^{int}}{\partial v_2} - \frac{\partial U^{ext}}{\partial v_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_2} - M_z^A \left[-\frac{6a^2}{L^3} + \frac{6a}{L^2} \right] = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z2}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{z2}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{z2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{z2}} - M_z^A \left[\frac{3a^2}{L^2} - \frac{2a}{L} \right] = 0$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = M_z^A \begin{Bmatrix} \frac{6a^2}{L^3} - \frac{6a}{L^2} \\ \frac{3a^2}{L^2} - \frac{4a}{L} + 1 \\ \frac{-6a^2}{L^3} + \frac{6a}{L^2} \\ \frac{3a^2}{L^2} - \frac{2a}{L} \end{Bmatrix}} \quad \text{or} \quad \boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{-6M_z^A ab}{L^3} \\ \frac{M_z^A b}{L^2}(b-2a) \\ \frac{6M_z^A ab}{L^3} \\ \frac{M_z^A a}{L^2}(a-2b) \end{Bmatrix}} \tag{6.562}$$

For the case b) the distributed load can be represented by $q_y(x) = q_y^{(1)} + \frac{x}{L}(q_y^{(2)} - q_y^{(1)})$, and the External Potential Energy becomes:

$$U^{ext} = \int_0^L q_y(x) v(x) dx = \int_0^L \left[q_y^{(1)} + \frac{x}{L}(q_y^{(2)} - q_y^{(1)}) \right] v(x) dx$$

Taking into account the deflection function y and after the integral is solved we can obtain:

$$U^{ext} = \frac{L}{20}(3q_y^{(2)} + 7q_y^{(1)})v_1 + \frac{L^2}{60}(2q_y^{(2)} + 3q_y^{(1)})\bar{\theta}_{z1} + \frac{L}{20}(7q_y^{(2)} + 3q_y^{(1)})v_2 + \frac{L^2}{60}(-3q_y^{(2)} - 2q_y^{(1)})\bar{\theta}_{z2}$$

To achieve the equilibrium, the following must be true:

$$\frac{\partial \Pi}{\partial v_1} = \frac{\partial U^{int}}{\partial v_1} - \frac{\partial U^{ext}}{\partial v_1} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_1} - \left[\frac{L}{20}(3q_y^{(2)} + 7q_y^{(1)}) \right] = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z1}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{z1}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{z1}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{z1}} - \left[\frac{L^2}{60}(2q_y^{(2)} + 3q_y^{(1)}) \right] = 0$$

$$\frac{\partial \Pi}{\partial v_2} = \frac{\partial U^{int}}{\partial v_2} - \frac{\partial U^{ext}}{\partial v_2} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial v_2} - \left[\frac{L}{20}(7q_y^{(2)} + 3q_y^{(1)}) \right] = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{z2}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{z2}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{z2}} = 0 \quad \Rightarrow \quad \frac{\partial U^{int}}{\partial \bar{\theta}_{z2}} - \left[\frac{L^2}{60}(-3q_y^{(2)} - 2q_y^{(1)}) \right] = 0$$

Then, the consistent load vector becomes:

$$\boxed{\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \dfrac{L}{20}(7q_y^{(1)} + 3q_y^{(2)}) \\ \dfrac{L^2}{60}(3q_y^{(1)} + 2q_y^{(2)}) \\ \dfrac{L}{20}(3q_y^{(1)} + 7q_y^{(2)}) \\ \dfrac{-L^2}{60}(2q_y^{(1)} + 3q_y^{(2)}) \end{Bmatrix}} \tag{6.563}$$

In **Problem 6.66** we have obtained the consistent load vector for the case when the load is lying on the plane $x-z$.

Problem 6.68

Obtain the explicit equation $[\boldsymbol{Ke}^{(3)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\}$ for the beam presented in Figure 6.181. Consider $u = ax + b$ as the approximation for the displacement according to x-direction.

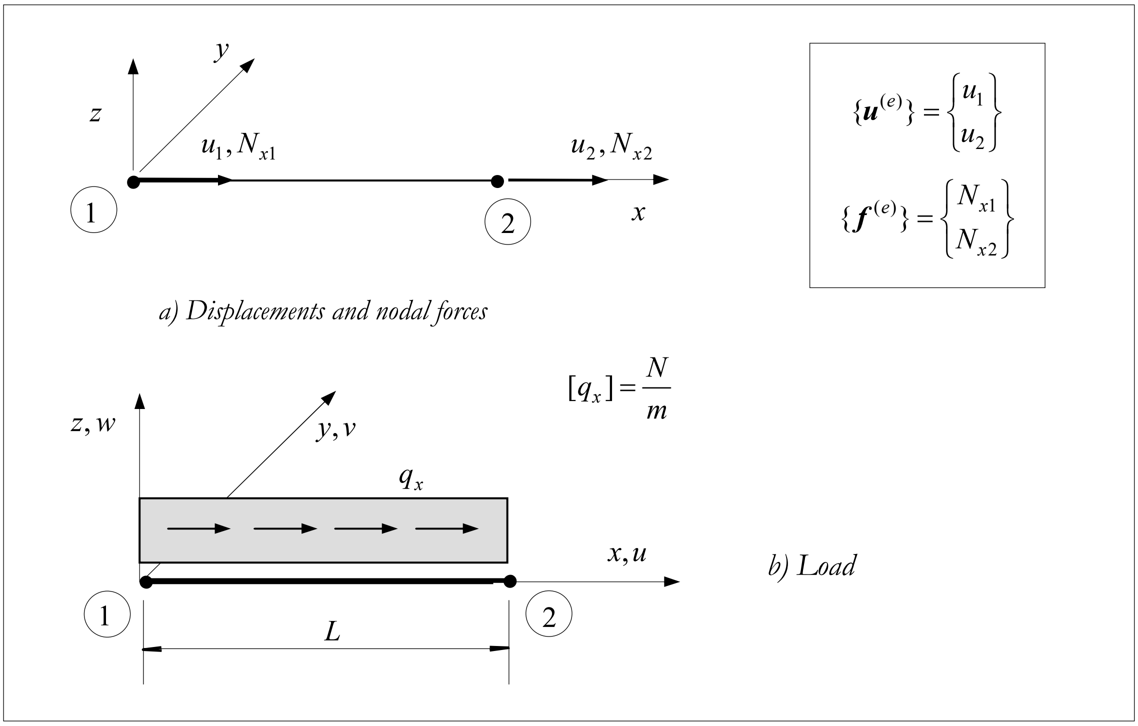


Figure 6.181: Bar element.

Solution:

By applying the displacement ($u = ax + b$) at the beam nodes we can obtain:

$$x = 0 \ \ (u = u_1) \quad \Rightarrow \quad u_1 = b \tag{6.564}$$

$$x = L \ \ (u = u_2) \quad \Rightarrow \quad u_2 = aL + b \tag{6.565}$$

This results in the following set of equations:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ L & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} \xrightarrow{\text{Reverse}} \begin{Bmatrix} a \\ b \end{Bmatrix} = \frac{-1}{L} \begin{bmatrix} 1 & -1 \\ -L & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6.566}$$

where the coefficients are:

$$a = \frac{-1}{L} u_1 + \frac{1}{L} u_2 \tag{6.567}$$

$$b = u_1 \tag{6.568}$$

By substituting the values of a, b into the equation of the displacement $u = ax + b$, we can obtain:

$$\begin{aligned} u(x) &= ax + b = \left(\frac{-1}{L} u_1 + \frac{1}{L} u_2 \right) x + u_1 = \left(1 - \frac{x}{L} \right) u_1 + \frac{x}{L} u_2 \\ u(x) &= N_1 u_1 + N_2 u_2 = [\boldsymbol{N}]\{\boldsymbol{u}^{(e)}\} \end{aligned} \tag{6.569}$$

in which we have considered that

$$[\boldsymbol{N}]=\begin{bmatrix}N_1 & N_2\end{bmatrix}=\begin{bmatrix}\left(1-\dfrac{x}{L}\right) & \left(\dfrac{x}{L}\right)\end{bmatrix} \quad \text{and} \quad \{\boldsymbol{u}^{(e)}\}=\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \tag{6.570}$$

The total potential energy is given by:

$$\Pi=U^{int}-U^{ext}=\frac{1}{2}\int_0^L \frac{N^2}{EA}dx-\int_0^L q_x u(x)dx \tag{6.571}$$

As we are considering that q_x is independent of x, the external potential energy becomes:

$$U^{ext}=\int_0^L q_x u(x)dx=q_x\int_0^L u(x)dx=q_x\int_0^L\left[\left(1-\frac{x}{L}\right)u_1+\frac{x}{L}u_2\right]dx=\frac{q_xL}{2}(u_1+u_2) \tag{6.572}$$

The internal potential energy becomes:

$$\begin{aligned}U^{int}&=\frac{1}{2}\int_0^L\frac{N^2}{EA}dx=\frac{1}{2}\int_0^L\frac{(u'EA)^2}{EA}dx=\frac{EA}{2}\int_0^L u'^2dx=\frac{EA}{2}\int_0^L\left\{\frac{\partial}{\partial x}\left[\left(1-\frac{x}{L}\right)u_1+\frac{x}{L}u_2\right]\right\}^2dx\\&=\frac{EA}{2}\int_0^L\left[\frac{-u_1}{L}+\frac{u_2}{L}\right]^2dx\\&=\frac{EA}{2L}(u_1^2-2u_1u_2+u_2^2)\end{aligned} \tag{6.573}$$

where we have consider that $N=\sigma A=E\varepsilon A=E\dfrac{\partial u}{\partial x}A\equiv Eu'A$ and that EA is constant in the beam element. Note also that the strain and the stress are constant into the element since the displacement field is a linear function, i.e.:

$$\begin{aligned}&\varepsilon(x)=\frac{\partial u(x)}{\partial x}=\frac{\partial([\boldsymbol{N}(x)]\{\boldsymbol{u}^{(e)}\})}{\partial x}=\frac{\partial[\boldsymbol{N}(x)]}{\partial x}\{\boldsymbol{u}^{(e)}\}=[\boldsymbol{B}(x)]\{\boldsymbol{u}^{(e)}\}\\&\Rightarrow\varepsilon(x)=\frac{\partial}{\partial x}\begin{bmatrix}\left(1-\dfrac{x}{L}\right) & \left(\dfrac{x}{L}\right)\end{bmatrix}\{\boldsymbol{u}^{(e)}\}=\begin{bmatrix}\left(\dfrac{-1}{L}\right) & \left(\dfrac{1}{L}\right)\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\frac{1}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\varepsilon\end{aligned} \tag{6.574}$$

as we can observe the strain is independent of x as well as the stress, since

$$\sigma=E\varepsilon=\frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \tag{6.575}$$

Taking into account the equations (6.572) and (6.573), the total potential energy (6.571) can be written as follows:

$$\Pi=U^{int}-U^{ext}=\frac{EA}{2L}(u_1^2-2u_1u_2+u_2^2)-\frac{q_xL}{2}(u_1+u_2) \tag{6.576}$$

As we are looking for the stationary state the following must hold:

$$\frac{\partial\Pi}{\partial u_1}=0 \quad\Rightarrow\quad \frac{EA}{2L}(2u_1-2u_2)-\frac{q_xL}{2}=0 \tag{6.577}$$

$$\frac{\partial\Pi}{\partial u_2}=0 \quad\Rightarrow\quad \frac{EA}{2L}(2u_2-2u_1)-\frac{q_xL}{2}=0 \tag{6.578}$$

Restructuring the above set of equations in matrix form we can obtain:

$$\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}\dfrac{q_xL}{2}\\ \dfrac{q_xL}{2}\end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{Ke}^{(3)}]\ \{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{f}_{Eq}^{(e)}\} \tag{6.579}$$

where

$$\boxed{[\boldsymbol{Ke}^{(3)}]=\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad ; \quad \{\boldsymbol{f}_{Eq}^{(e)}\}=\begin{Bmatrix}\dfrac{q_xL}{2}\\ \dfrac{q_xL}{2}\end{Bmatrix}} \tag{6.580}$$

NOTE 1: The consistent mass matrix

In **Problem 5.24** we have obtain the consistent mass matrix which is given by:

$$[\boldsymbol{M}^{(e)}]=\int_V \rho[\boldsymbol{N}(x)]^T[\boldsymbol{N}(x)]\,dV \tag{6.581}$$

where ρ is the mass density. For this problem the matrix $[\boldsymbol{N}]$ is given by the equation in (6.570), then:

$$[\boldsymbol{M}^{(e)}]=\rho\int_0^L[\boldsymbol{N}(x)]^T[\boldsymbol{N}(x)]\underbrace{\int_A dA}_{=A}dx=\rho A\int_0^L[\boldsymbol{N}(x)]^T[\boldsymbol{N}(x)]dx\equiv\rho A\int_0^L[\boldsymbol{Nn}]dx \tag{6.582}$$

where

$$[\boldsymbol{N}]^T[\boldsymbol{N}]\equiv[\boldsymbol{Nn}]=\begin{bmatrix}N_1\\ N_2\end{bmatrix}\begin{bmatrix}N_1 & N_2\end{bmatrix}=\begin{bmatrix}(N_1)^2 & N_1N_2\\ N_1N_2 & (N_2)^2\end{bmatrix} \tag{6.583}$$

Then, after the integration (6.582) is taken place we can obtain:

$$\begin{aligned}
&[\boldsymbol{M}^{(e)}]=\rho A\int_0^L[\boldsymbol{N}(x)]^T[\boldsymbol{N}(x)]dx=\rho A\begin{bmatrix}\int_0^L(N_1)^2\,dx & \int_0^L N_1N_2dx\\ \int_0^L N_1N_2dx & \int_0^L(N_2)^2\,dx\end{bmatrix}\\
&\Rightarrow[\boldsymbol{M}^{(e)}]=\rho A\begin{bmatrix}\int_0^L\left(1-\dfrac{x}{L}\right)^2dx & \int_0^L\left(1-\dfrac{x}{L}\right)\left(\dfrac{x}{L}\right)dx\\ \int_0^L\left(1-\dfrac{x}{L}\right)\left(\dfrac{x}{L}\right)dx & \int_0^L\left(\dfrac{x}{L}\right)^2dx\end{bmatrix}\\
&\Rightarrow[\boldsymbol{M}^{(e)}]=\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}
\end{aligned} \tag{6.584}$$

which is known as the *Consistent Mass Matrix*.

Problem 6.69

Obtain the explicit equation $[\boldsymbol{Ke}^{(4)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\}$ for the beam presented in Figure 6.182. Consider $\overline{\theta}_x = ax + b$ as the approximation for the rotation about the x-axis.

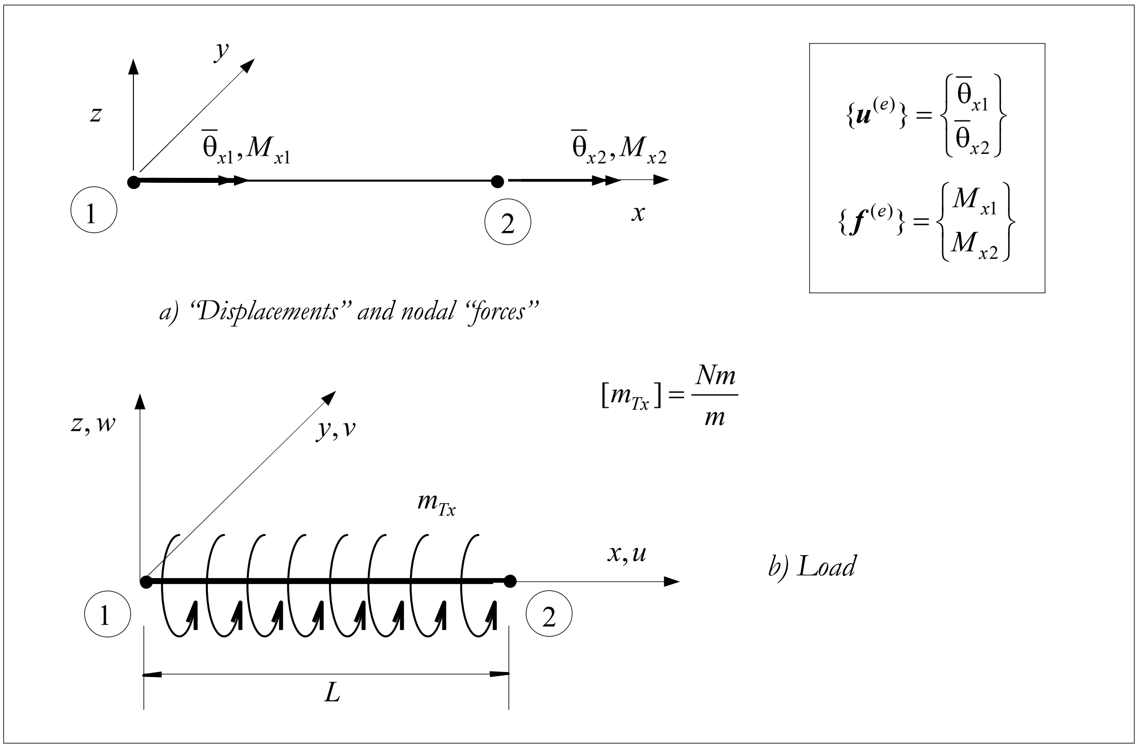


Figure 6.182: Bar element.

Solution:

Note that the approximation for rotation is the same as the one used to approximate the displacement in **Problem 6.68**. By analogy with the previous problem we can obtain that:

$$\overline{\theta}_x(x) = \left(1 - \frac{x}{L}\right)\overline{\theta}_{x1} + \frac{x}{L}\overline{\theta}_{x2} \tag{6.585}$$

As we are considering that q_x is independent of x, the external potential energy becomes:

$$U^{ext} = \int_0^L m_{Tx}\overline{\theta}_x(x)dx = m_{Tx}\int_0^L \overline{\theta}_x(x)dx = \frac{m_{Tx}L}{2}(\overline{\theta}_{x1} + \overline{\theta}_{x2}) \tag{6.586}$$

The internal potential energy due to M_T was obtained in **Problem 6.62**. For torsion problem we have obtained that $M_T = \overline{\theta}_x G\mathrm{J}_T$, then the internal potential energy becomes:

$$U^{int} = \frac{1}{2}\int_0^L \frac{M_T^2}{G\mathrm{J}_T}dx = \frac{1}{2}\int_0^L \frac{(\overline{\theta}_x G\mathrm{J}_T)^2}{G\mathrm{J}_T}dx = \frac{G\mathrm{J}_T}{2}\int_0^L \overline{\theta}_x^2 dx = \frac{G\mathrm{J}_T}{2L}(\overline{\theta}_{x1}^2 - 2\overline{\theta}_{x1}\overline{\theta}_{x2} + \overline{\theta}_{x2}^2) \tag{6.587}$$

The total potential energy is given by:

$$\begin{aligned}\Pi = U^{int} - U^{ext} &= \frac{1}{2}\int_0^L \frac{M_T^2}{G\mathrm{J}_T}dx - \int_0^L m_{Tx}\overline{\theta}_x(x)dx \\ &= \frac{G\mathrm{J}_T}{2L}(\overline{\theta}_{x1}^2 - 2\overline{\theta}_{x1}\overline{\theta}_{x2} + \overline{\theta}_{x2}^2) - \left[\frac{m_{Tx}L}{2}(\overline{\theta}_{x1} + \overline{\theta}_{x2})\right]\end{aligned} \tag{6.588}$$

As we are looking for the stationary state the following must hold:

$$\frac{\partial \Pi}{\partial \bar{\theta}_{x1}} = 0 \quad \Rightarrow \quad \frac{G\mathrm{J}_T}{2L}(2\bar{\theta}_{x1} - 2\bar{\theta}_{x2}) - \frac{m_{Tx}L}{2} = 0 \tag{6.589}$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{x2}} = 0 \quad \Rightarrow \quad \frac{G\mathrm{J}_T}{2L}(2\bar{\theta}_{x2} - 2\bar{\theta}_{x1}) - \frac{m_{Tx}L}{2} = 0 \tag{6.590}$$

Restructuring the above set of equations in matrix form we can obtain:

$$\frac{G\mathrm{J}_T}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{\theta}_{x1} \\ \bar{\theta}_{x2} \end{Bmatrix} = \begin{Bmatrix} \frac{m_{Tx}L}{2} \\ \frac{m_{Tx}L}{2} \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{Ke}^{(4)}]\ \{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}_{Eq}^{(e)}\} \tag{6.591}$$

where

$$[\boldsymbol{Ke}^{(4)}] = \frac{G\mathrm{J}_T}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \quad \{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \frac{m_{Tx}L}{2} \\ \frac{m_{Tx}L}{2} \end{Bmatrix} \tag{6.592}$$

Problem 6.70

Obtain the explicit equation $[\boldsymbol{K}^{(e)}_{Local}]\,\{\boldsymbol{u}^{(e)}_{Local}\}=\{\boldsymbol{F}^{(e)}_{Eq\_L}\}$ for the beam presented in Figure 6.183. Consider as material/geometric characteristic of the transversal cross section as the one described in Figure 6.184 and as external load the one described in Figure 6.189.

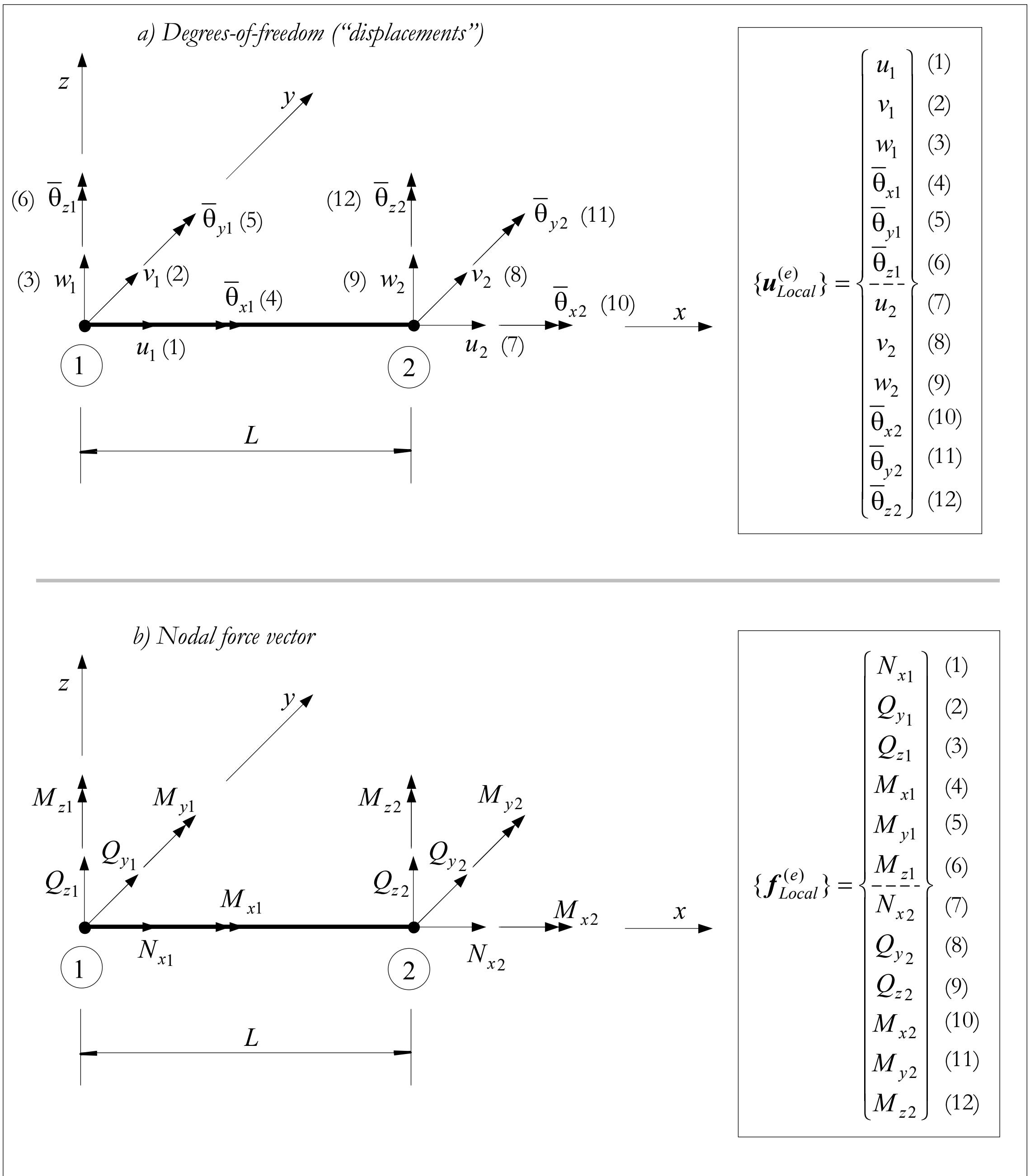
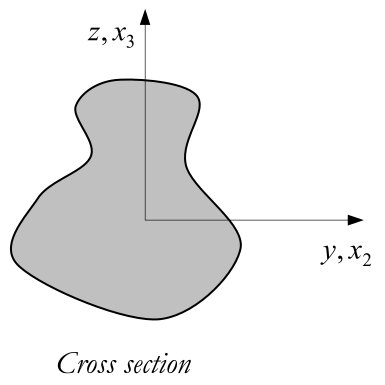


Figure 6.183: Beam element – local system.



$A -$ *Cross-sectional area*

$I_y -$ *Inertia moment of area about the* y*-direction*

$I_z -$ *Inertia moment of area about* z*-direction*

$J_{Eq} -$ *Equivalent polar moment of inertia*

$E -$ *Young's modulus*

$G -$ *Shear modulus*

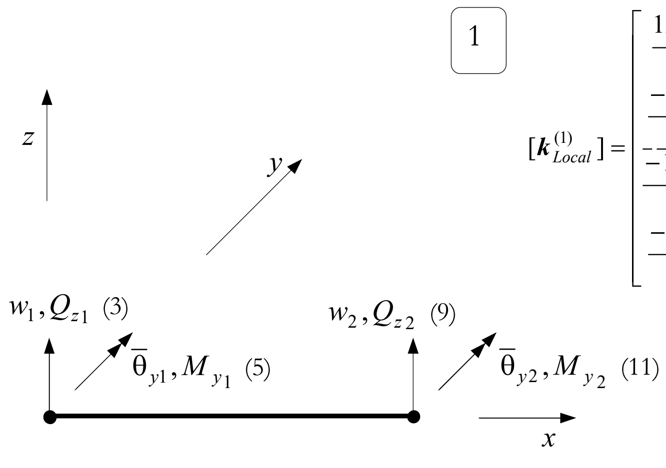
$\nu -$ *Poisson's ratio*

$$G = \frac{E}{2(1+\nu)}$$

Figure 6.184: Material/Geometrical properties of the beam cross section.

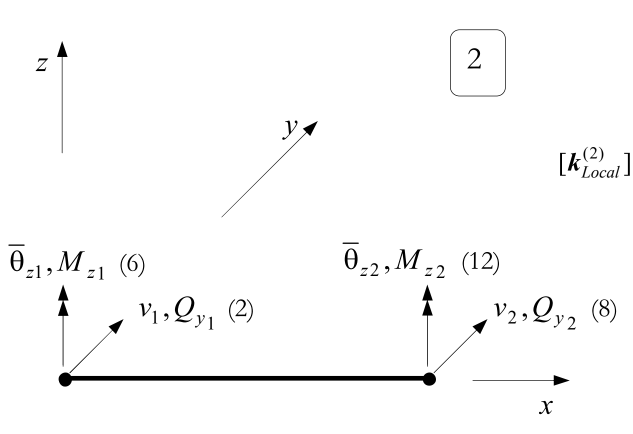
Solution:

As we are dealing with linear elasticity we can apply the superposition principle. Let us consider the stiffness matrices obtained previously, (see Figure 6.185, Figure 6.186, Figure 6.187 and Figure 6.188).



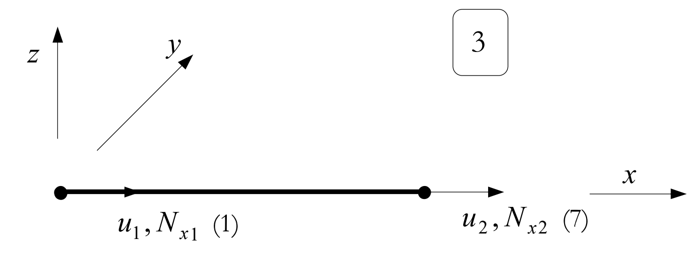
$$[\boldsymbol{k}_{Local}^{(1)}] = \begin{array}{c} \begin{array}{cccc} (3) & (5) & (9) & (11) \end{array} \\ \left[\begin{array}{cc|cc} \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & \frac{-6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline \frac{-12EI_y}{L^3} & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ \frac{-6EI_y}{L^2} & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{array}\right] \end{array} \begin{array}{c} (3) \\ (5) \\ (9) \\ (11) \end{array}$$

Figure 6.185: See **Problem 6.62- NOTE 2**.



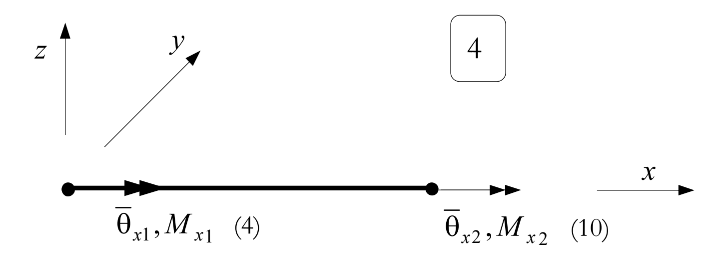
$$[\boldsymbol{k}^{(2)}_{Local}] = \begin{array}{c} \begin{array}{cccc} (2) & (6) & (8) & (12) \end{array} \\ \left[\begin{array}{cc|cc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\ \hline \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \end{array} \begin{array}{c} (2) \\ (6) \\ (8) \\ (12) \end{array}$$

Figure 6.186: See **Problem 6.67**.



$$[\boldsymbol{k}^{(3)}_{Local}] = \begin{array}{c} \begin{array}{cc} (1) & (7) \end{array} \\ \begin{bmatrix} \frac{EA}{L} & \frac{-EA}{L} \\ \frac{-EA}{L} & \frac{EA}{L} \end{bmatrix} \end{array} \begin{array}{c} (1) \\ (7) \end{array}$$

Figure 6.187: See **Problem 6.68**.



$$[\boldsymbol{k}^{(4)}_{Local}] = \begin{array}{c} \begin{array}{cc} (4) & (10) \end{array} \\ \begin{bmatrix} \frac{GJ_{Eq}}{L} & \frac{-GJ_{Eq}}{L} \\ \frac{-GJ_{Eq}}{L} & \frac{GJ_{Eq}}{L} \end{bmatrix} \end{array} \begin{array}{c} (4) \\ (10) \end{array}$$

Figure 6.188: See **Problem 6.69**.

The consistent load vectors can be appreciated in Figure 6.189.

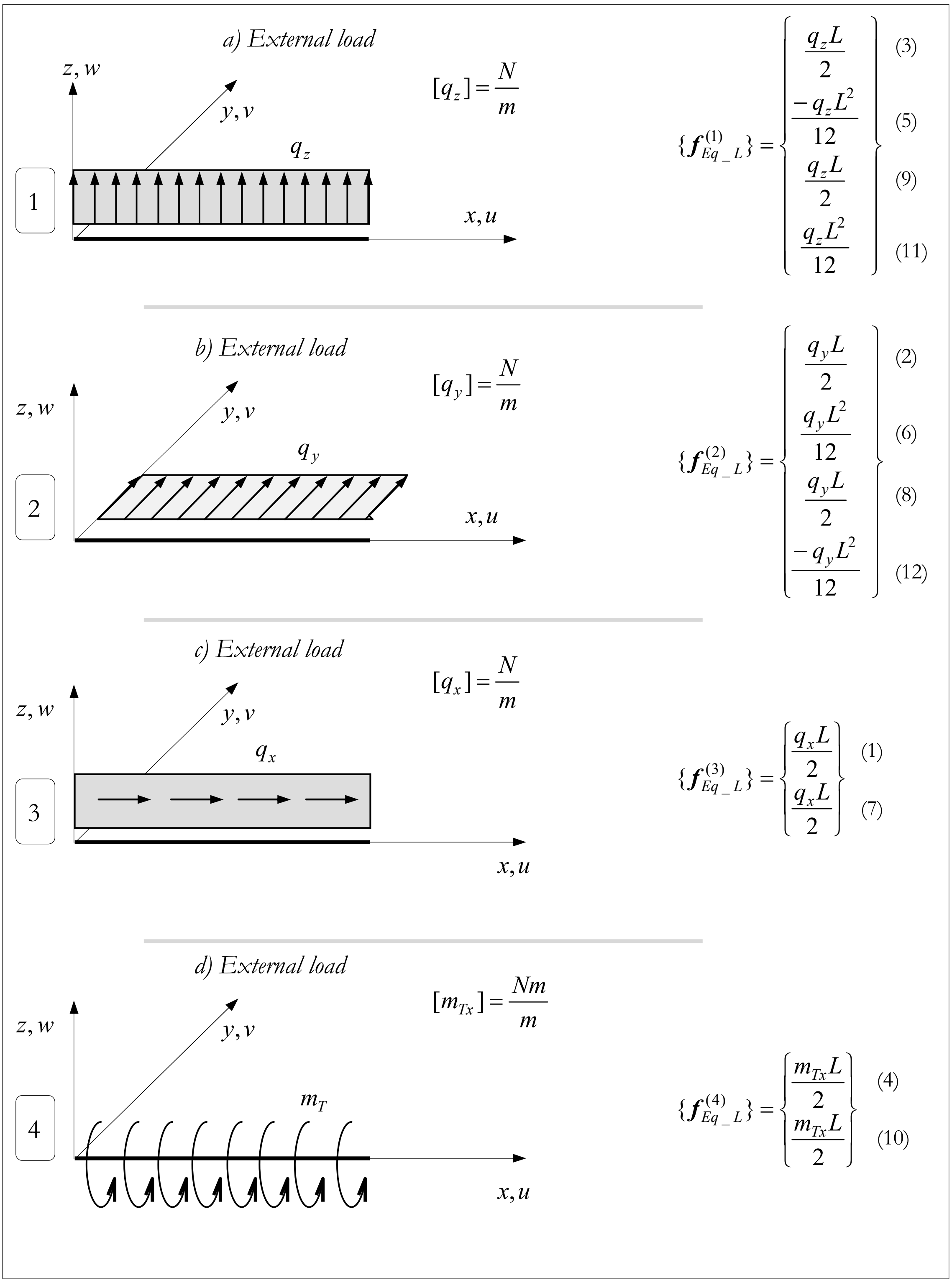


Figure 6.189: The consistent load vectors (Local system).

Then, the stiffness matrix and consistent load vector can be obtained by:

$$[\boldsymbol{K}^{(e)}_{Local}] = \mathop{\mathbf{A}}_{i=1}^{4}[\boldsymbol{k}^{(i)}_{Local}] \quad ; \quad \{\boldsymbol{F}^{(e)}_{Local}\} = \mathop{\mathbf{A}}_{i=1}^{4}\{\boldsymbol{f}^{(i)}_{Eq\_L}\} \tag{6.593}$$

where $\mathbf{A}$ stands for assemble operator. Making the contribution to the respective degree-of-freedom we can obtain the local stiffness matrix:

$$\left[\boldsymbol{K}^{(e)}_{Local}\right]=\left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2}\\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 & 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0\\
0 & 0 & 0 & \frac{GJ_T}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ_T}{L} & 0 & 0\\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0\\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L}\\
\hline
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2}\\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0\\
0 & 0 & 0 & \frac{-GJ_T}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_T}{L} & 0 & 0\\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0\\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]$$

And the consistent load vector by:

$$\{\boldsymbol{F}^{(e)}_{Eq_L}\}=\left\{\begin{array}{c}
\frac{q_x L}{2}\\
\frac{q_y L}{2}\\
\frac{q_z L}{2}\\
\frac{m_{Tx} L}{2}\\
\frac{-q_z L^2}{12}\\
\frac{q_y L^2}{12}\\
\hline
\frac{q_x L}{2}\\
\frac{q_y L}{2}\\
\frac{q_z L}{2}\\
\frac{m_{Tx} L}{2}\\
\frac{q_z L^2}{12}\\
\frac{-q_y L^2}{12}
\end{array}\right\}\begin{array}{c}(1)\\(2)\\(3)\\(4)\\(5)\\(6)\\(7)\\(8)\\(9)\\(10)\\(11)\\(12)\end{array}$$

NOTE 1: Stiffness Matrix related to the Global System

Knowing the vectors $\{\boldsymbol{F}^{(e)}_{Global}\}$ and $\{\boldsymbol{U}^{(e)}_{Global}\}$ in the global system, the transformation law between the global and local systems is represented by the matrix $[\lambda]$, in which fulfils that $\{\boldsymbol{F}^{(e)}_{Local}\}=[\lambda]\{\boldsymbol{F}^{(e)}_{Global}\}$ and $\{\boldsymbol{u}^{(e)}_{Local}\}=[\lambda]\{\boldsymbol{U}^{(e)}_{Global}\}$. With that we can conclude that:

$$\begin{aligned}
&\{\boldsymbol{F}^{(e)}_{Local}\}=[\boldsymbol{K}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}\\
&\Rightarrow[\lambda]\{\boldsymbol{F}^{(e)}_{Global}\}=[\boldsymbol{K}^{(e)}_{Local}][\lambda]\{\boldsymbol{U}^{(e)}_{Global}\}\\
&\Rightarrow[\lambda]^T[\lambda]\{\boldsymbol{F}^{(e)}_{Global}\}=[\lambda]^T[\boldsymbol{K}^{(e)}_{Local}][\lambda]\{\boldsymbol{U}^{(e)}_{Global}\}\\
&\Rightarrow\{\boldsymbol{F}^{(e)}_{Global}\}=[\lambda]^T[\boldsymbol{K}^{(e)}_{Local}][\lambda]\{\boldsymbol{U}^{(e)}_{Global}\}\\
&\Rightarrow\{\boldsymbol{F}^{(e)}_{Global}\}=[\boldsymbol{K}^{(e)}_{Global}]\{\boldsymbol{U}^{(e)}_{Global}\}
\end{aligned}\tag{6.594}$$

where $[\boldsymbol{K}^{(e)}_{Global}]=[\lambda]^T[\boldsymbol{K}^{(e)}_{Local}][\lambda]$.

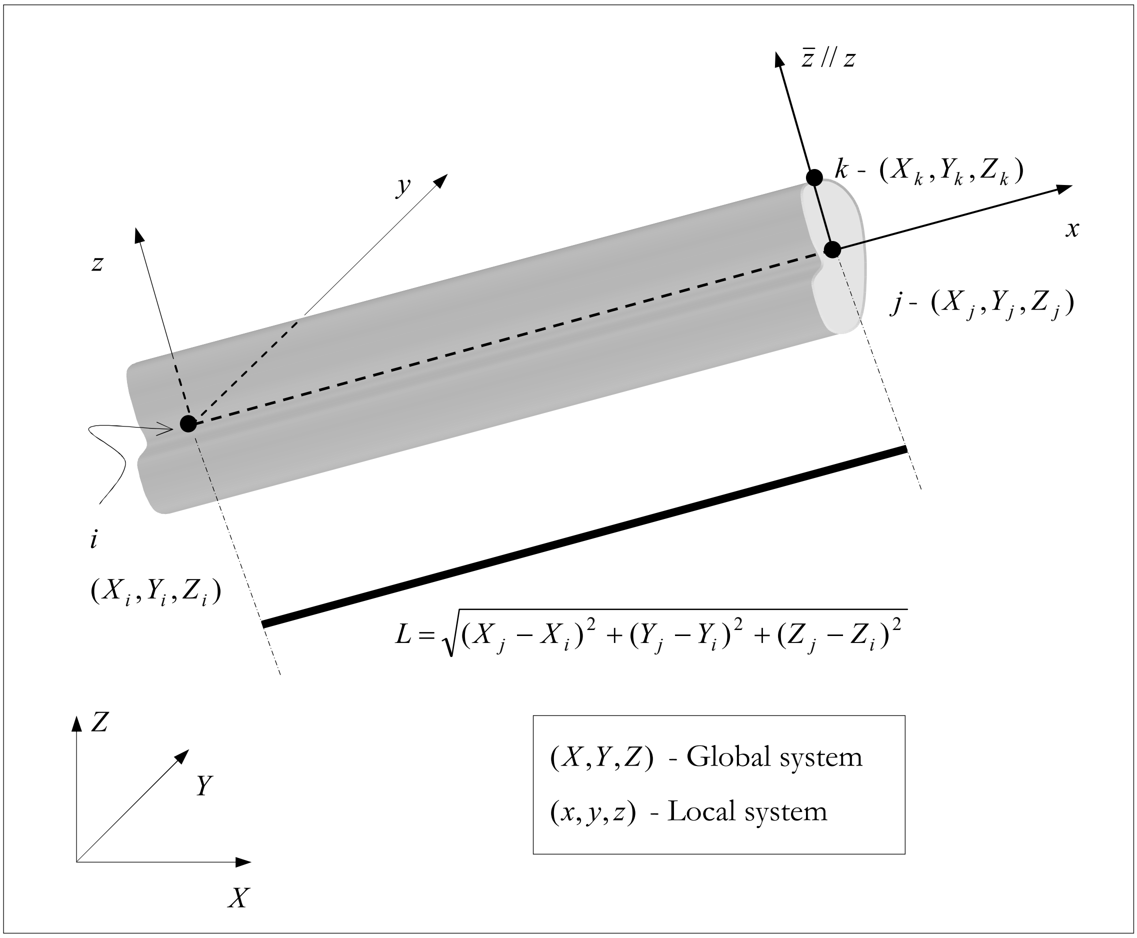


Figure 6.190: Beam element in 3D space.

Next, we will define the transformation matrix $[\lambda]$.

We need to define three points, namely: $i-j-k$

Local system $x-y-z$

- x-direction: according to $\vec{ij}\,//\,x$-direction.

The unit vector according to x'-direction is defined by:

$$a_{11} = \frac{X_j - X_i}{L} \quad ; \quad a_{12} = \frac{Y_j - Y_i}{L} \quad ; \quad a_{13} = \frac{Z_j - Z_i}{L}$$

where

$$L = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}$$

- The unit vector according to $\overrightarrow{ik}$ -direction.

$$p_1 = \frac{X_k - X_i}{d_{(ik)}} \quad ; \quad p_2 = \frac{Y_k - Y_i}{d_{(ik)}} \quad ; \quad p_3 = \frac{Z_k - Z_i}{d_{(ik)}}$$

where

$$d_{(ik)} = \sqrt{(X_k - X_i)^2 + (Y_k - Y_i)^2 + (Z_k - Z_i)^2}$$

- y -direction: perpendicular to the surface defined by $\hat{\mathbf{j}} = \overrightarrow{ik} \wedge \overrightarrow{ij}$

$$\hat{\mathbf{y}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ p_1 & p_2 & p_3 \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = \underbrace{(p_2 a_{13} - p_3 a_{12})}_{a_{21}}\hat{\mathbf{i}} \underbrace{- (p_1 a_{13} - p_3 a_{11})}_{a_{22}}\hat{\mathbf{j}} + \underbrace{(p_1 a_{12} - p_2 a_{11})}_{a_{23}}\hat{\mathbf{k}}$$

- z -direction: according to the convention $\hat{\mathbf{k}} = \hat{\mathbf{i}} \wedge \hat{\mathbf{j}}$

$$\mathbf{z} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = \underbrace{(a_{12}a_{23} - a_{13}a_{22})}_{a_{31}}\hat{\mathbf{i}} \underbrace{- (a_{11}a_{22} - a_{12}a_{21})}_{a_{32}}\hat{\mathbf{j}} + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{a_{33}}\hat{\mathbf{k}} \, .$$

The transformation matrix from the global system $X - Y - Z$ to the local system $x - y - z$ is constituted by the unit vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, i.e.:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

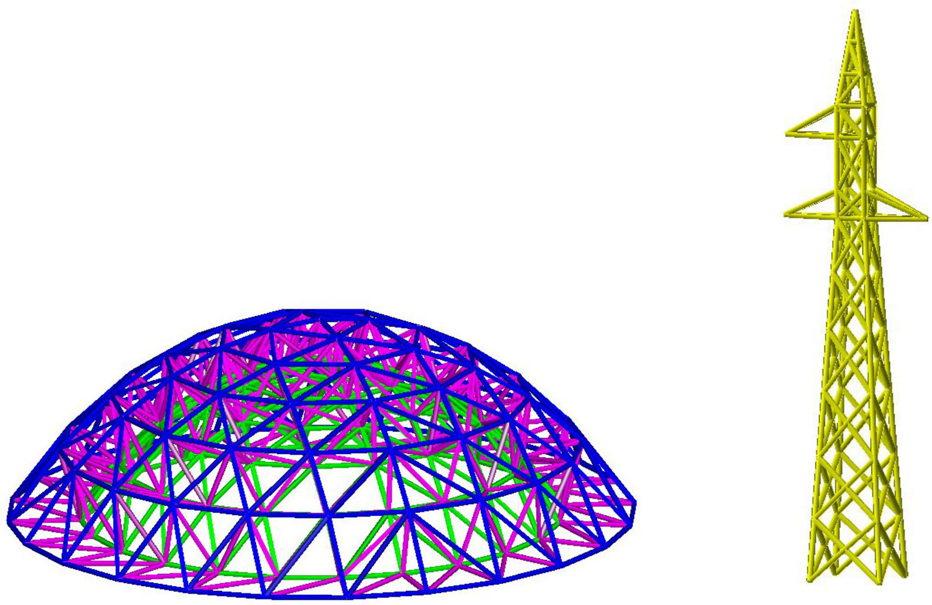
Then, the transformation matrix $[\lambda]$ is represented as follows:

$$[\lambda] = \left[\begin{array}{ccc:ccc:ccc:ccc} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{array}\right] \tag{6.595}$$

NOTE 2: Note that the adopted stiffness matrix will depend on the degree-of-freedom of the structure, which in turns depends on the external load and how the structure was conceived. Some examples follow.

NOTE 2.1: Truss

Truss is a structural element which is subjected only by traction or compression forces and the nodes are free to rotate, i.e. there is no moment at the nodes. The external load (concentrated force) is considered to be applied only at the nodes. Examples for this problem we can quote: an electricity transmission tower (electricity pylon), geodesic dome, among others, (see Figure 6.191).



a) Geodesic dome *b) Electricity transmission tower*

Figure 6.191: Truss examples.

The degrees-of-freedom associated with this type of structure are only translations. For instance, in **Problem 6.68** we have defined this problem by considering one-dimensional space (1D), in case we are dealing with two-dimensional space there will be two degrees-of-freedom per node (u, v), (see Figure 6.193), and for three-dimensional space (3D) there are 3 degrees-of-freedom per node namely: u, v, and w. (see Figure 6.194).

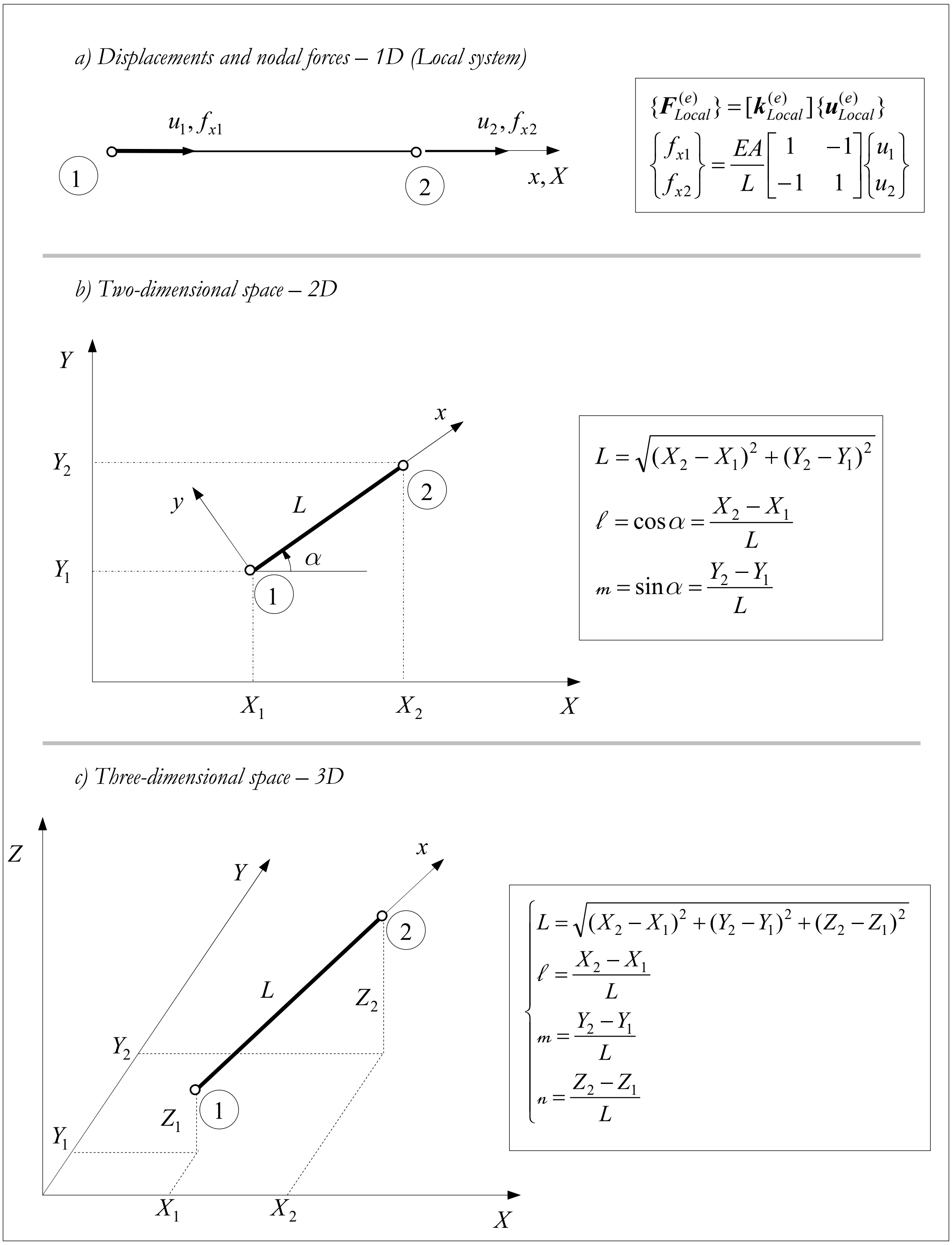


Figure 6.192: Bar element.

Note that the local stiffness matrix for truss element was obtained in **Problem 6.68**. And if we consider that $\{\boldsymbol{u}^{(e)}_{Local}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$ and $\{\boldsymbol{f}^{(e)}_{Local}\} = [\overline{\mathcal{A}}]\{\boldsymbol{f}^{(e)}_{Global}\}$ where $[\overline{\mathcal{A}}]$ is the transformation matrix from the Global system to the Local system we can obtain:

$$\{\boldsymbol{f}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\} \quad \Rightarrow \quad [\overline{\mathcal{A}}]\{\boldsymbol{f}^{(e)}_{Global}\} = [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$$

$$\Rightarrow [\overline{\mathcal{A}}]^T[\overline{\mathcal{A}}]\{\boldsymbol{f}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T[\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$$

$$\Rightarrow \{\boldsymbol{f}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T[\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$$

$$\Rightarrow \boxed{\{\boldsymbol{f}^{(e)}_{Global}\} = [\boldsymbol{k}^{(e)}_{Global}]\{\boldsymbol{u}^{(e)}_{Global}\}} \quad \therefore \quad \boxed{[\boldsymbol{k}^{(e)}_{Global}] = [\overline{\mathcal{A}}]^T [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]} \tag{6.596}$$

Two-Dimensional Space (2D)

Stiffness Matrix for 2D

For the local system the force-displacement relationship, (see Figure 6.193), is given by:

$$\{\boldsymbol{f}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\} \quad \Leftrightarrow \quad \begin{Bmatrix} f_{x1} \\ f_{y1}=0 \\ f_{x2} \\ f_{y2}=0 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{6.597}$$

Transformation Matrix

The transformation matrix from the Global system $X-Y$ to the Local system $x-y$, (see Figure 6.192(b)), is given by:

$$[\mathcal{A}] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix} \tag{6.598}$$

And the transformation matrix for the displacement vector $\{\boldsymbol{u}^{(e)}_{Local}\}$ becomes

$$\xrightarrow{Nodal} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} \ell & m \\ -m & \ell \end{bmatrix}\begin{Bmatrix} u_{Xi} \\ v_{Yi} \end{Bmatrix} \quad \xrightarrow{Element} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{bmatrix}\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ u_{X2} \\ v_{Y2} \end{Bmatrix}}_{\{\boldsymbol{u}^{(e)}_{Local}\}=[\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}} \tag{6.599}$$

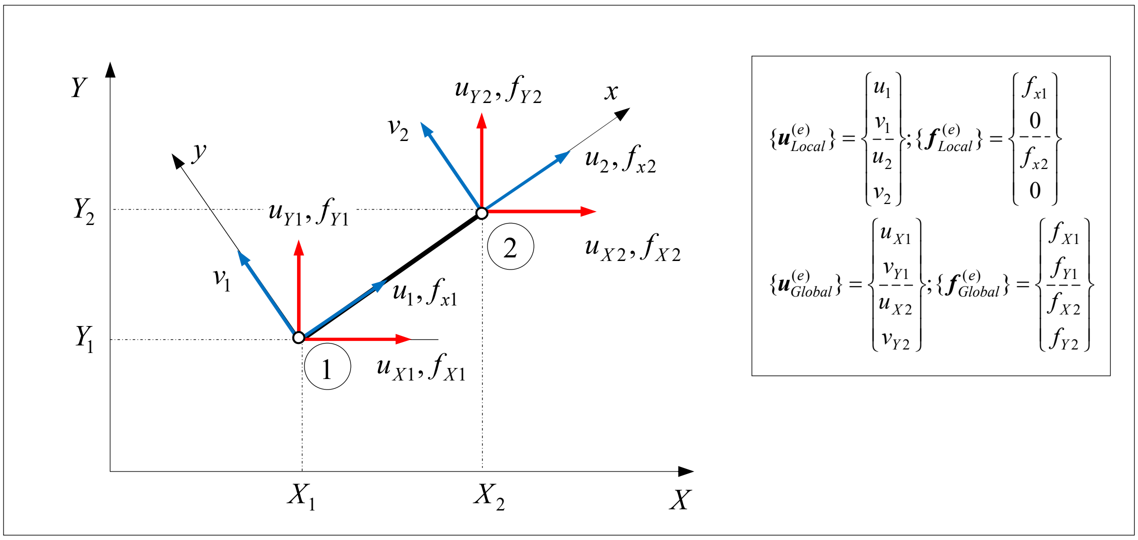


Figure 6.193: Bar element in two-dimensional space – degrees-of-freedom.

Then, the stiffness matrix in the Global system can be expressed as follows:

$$[\boldsymbol{k}^{(e)}_{Global}] = [\overline{\mathcal{A}}]^T [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}] = \frac{EA}{L}\begin{bmatrix} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{bmatrix}$$

$$[\boldsymbol{k}^{(e)}_{Global}] = \frac{EA}{L}\left[\begin{array}{cc|cc} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ \hline -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{array}\right] \tag{6.600}$$

Strain and Stress (2D)

Once the global displacement vector is defined, the strain, (see equation (6.574)), can be obtained as follows:

$$\varepsilon = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \hdashline u_{X2} \\ v_{Y2} \end{Bmatrix} \Rightarrow \varepsilon = \frac{1}{L}(-u_{X1}\ell - v_{Y1}m + u_{X2}\ell + v_{Y2}m) \tag{6.601}$$

where we have considered, (see equation (6.599)), that

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \hdashline u_{X2} \\ v_{Y2} \end{Bmatrix} \tag{6.602}$$

And the stress becomes

$$\sigma = E\varepsilon = \frac{E}{L}[(u_{X2} - u_{X1})\ell + (v_{Y2} - v_{Y1})m] \tag{6.603}$$

Note that $\sigma = \frac{f_{x2}}{A} = \frac{EA}{LA}[(u_{X2} - u_{X1})\ell + (v_{Y2} - v_{Y1})m] = \frac{E}{L}[(u_{X2} - u_{X1})\ell + (v_{Y2} - v_{Y1})m]$.

The Reactions and internal forces can also be obtained in the local system as follows

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$$

$$\{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$$

$$\begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} = 0 \\ \hdashline \bar{f}_{x2} \\ \bar{f}_{y2} = 0 \end{Bmatrix} = \frac{EA}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{Bmatrix} u_1 \\ v_1 \\ \hdashline u_2 \\ v_2 \end{Bmatrix} = \frac{EA}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ \hline 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{array}\right]\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \hdashline u_{X2} \\ v_{Y2} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} = 0 \\ \hdashline \bar{f}_{x2} \\ \bar{f}_{y2} = 0 \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} (u_{X1} - u_{X2})\ell + (v_{Y1} - v_{Y2})m \\ 0 \\ \hdashline (u_{X2} - u_{X1})\ell + (v_{Y2} - v_{Y1})m \\ 0 \end{Bmatrix} \tag{6.604}$$

Note that for the truss problem we do not have the consistent load vector, since the load must be applied at the nodes, so $\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} = \{\bar{\boldsymbol{f}}^{(e)}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$.

It could be interesting to generate an internal global internal vector, related to Global system, i.e.: $\{\boldsymbol{r}^{(e)}_{Global}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_G}\} + [\boldsymbol{k}^{(e)}_{Global}]\{\boldsymbol{u}^{(e)}_{Global}\}$ or we can use $\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T\{\boldsymbol{r}^{(e)}_{Local}\}$:

$$\{\boldsymbol{r}_{Global}^{(e)}\} = \left[\begin{array}{cc|cc} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ \hline 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{array}\right]^T \left\{\begin{array}{c} \bar{f}_{x1} \\ \bar{f}_{y1}=0 \\ \hline \bar{f}_{x2} \\ \bar{f}_{y2}=0 \end{array}\right\} = \left\{\begin{array}{c} \bar{f}_{x1}\ell \\ \bar{f}_{x1}m \\ \hline \bar{f}_{x2}\ell \\ \bar{f}_{x2}m \end{array}\right\}$$

<u>**Three-Dimensional Space (3D)**</u>

Stiffness Matrix for 3D

For the local system the force-displacement relationship is given by:

$$\{\boldsymbol{f}_{Local}^{(e)}\} = [\boldsymbol{k}_{Local}^{(e)}]\{\boldsymbol{u}_{Local}^{(e)}\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} f_{x1} \\ f_{y1}=0 \\ f_{z1}=0 \\ \hline f_{x2} \\ f_{y2}=0 \\ f_{z2}=0 \end{array}\right\} = \frac{EA}{L}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left\{\begin{array}{c} u_1 \\ v_1 \\ w_1 \\ \hline u_2 \\ v_2 \\ w_2 \end{array}\right\} \tag{6.605}$$

Transformation Matrix

The transformation matrix from the Global system $X-Y-Z$ to the Local system $x-y-z$ is given by:

$$[\mathcal{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv \begin{bmatrix} \ell & m & n \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{6.606}$$

And the transformation matrix for the vector $\{\boldsymbol{u}_{Local}^{(e)}\}$ becomes

$$\xrightarrow{Nodal} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \begin{bmatrix} \ell & m & n \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ v_{Y1} \\ w_{Z1} \end{Bmatrix}$$

$$\xrightarrow{Element} \left\{\begin{array}{c} u_1 \\ v_1 \\ w_1 \\ \hline u_2 \\ v_2 \\ w_2 \end{array}\right\} = \left[\begin{array}{ccc|ccc} \ell & m & n & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \ell & m & n \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{array}\right]\left\{\begin{array}{c} u_{X1} \\ v_{Y1} \\ w_{Z1} \\ \hline u_{X2} \\ v_{Y2} \\ w_{Z2} \end{array}\right\} \Rightarrow \{\boldsymbol{u}_{Local}^{(e)}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}_{Global}^{(e)}\} \tag{6.607}$$

Then, the stiffness matrix in the Global system can be expressed as follows:

$$[\boldsymbol{k}_{Global}^{(e)}] = [\overline{\mathcal{A}}]^T[\boldsymbol{k}_{Local}^{(e)}][\overline{\mathcal{A}}] = \frac{EA}{L}[\overline{\mathcal{A}}]^T\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right][\overline{\mathcal{A}}]$$

after the matrix multiplication is taken place we can obtain

$$
[\boldsymbol{k}^{(e)}_{Global}] = \frac{EA}{L}\left[\begin{array}{ccc|ccc}
\ell^2 & \ell m & \ell n & -\ell^2 & -\ell m & -\ell n \\
\ell m & m^2 & mn & -\ell m & -m^2 & -mn \\
\ell n & mn & n^2 & -\ell n & -mn & -n^2 \\
\hline
-\ell^2 & -\ell m & -\ell n & \ell^2 & \ell m & \ell n \\
-\ell m & -m^2 & -mn & \ell m & m^2 & mn \\
-\ell n & -mn & -n^2 & \ell n & mn & n^2
\end{array}\right] \tag{6.608}
$$

Note that we do not need to define the coefficients $a_{21}, a_{22}, \cdots, a_{33}$ in order to obtain the Global elemental stiffness matrix.

Strain and Stress (3D)

Once the global displacement vector is defined the strain, (see equation (6.574)), can be obtained as follows:

$$
\varepsilon = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} \ell & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & n \end{bmatrix}\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ w_{Z1} \\ \hline u_{X2} \\ v_{Y2} \\ w_{Z2} \end{Bmatrix}
$$

$$
\Rightarrow \varepsilon = \frac{1}{L}(-u_{X1}\ell - v_{Y1}m - w_{Z1}n + u_{X2}\ell + v_{Y2}m + w_{Z2}n) \tag{6.609}
$$

where we have considered that

$$
\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \ell & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & n \end{bmatrix}\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ w_{Z1} \\ \hline u_{X2} \\ v_{Y2} \\ w_{Z2} \end{Bmatrix} \tag{6.610}
$$

And the stress becomes

$$
\sigma = E\varepsilon = \frac{E}{L}(-u_{X1}\ell - v_{Y1}m - w_{Z1}n + u_{X2}\ell + v_{Y2}m + w_{Z2}n) \tag{6.611}
$$

Note that $\sigma = \dfrac{f_{x2}}{A} = \dfrac{EA}{LA}[(u_{X2}-u_{X1})\ell + (v_{Y2}-v_{Y1})m] = \dfrac{E}{L}[(u_{X2}-u_{X1})\ell + (v_{Y2}-v_{Y1})m]$.

The internal forces can also be obtained in the local system as follows

$$
\{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}][\mathcal{A}]\{\boldsymbol{u}^{(e)}_{Global}\}
$$

$$
\begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} = 0 \\ \bar{f}_{z1} = 0 \\ \hline \bar{f}_{x2} \\ \bar{f}_{y2} = 0 \\ \bar{f}_{z2} = 0 \end{Bmatrix} = \frac{EA}{L}\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc|ccc}
\ell & m & n & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \ell & m & n \\
0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\
0 & 0 & 0 & a_{31} & a_{32} & a_{33}
\end{array}\right]\begin{Bmatrix} u_{X1} \\ v_{Y1} \\ w_{Z1} \\ \hline u_{X2} \\ v_{Y2} \\ w_{Z2} \end{Bmatrix}
$$

$$\Rightarrow \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1}=0 \\ \bar{f}_{z1}=0 \\ \hline \bar{f}_{x2} \\ \bar{f}_{y2}=0 \\ \bar{f}_{z2}=0 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} (u_{X1}-u_{X2})\ell + (v_{Y1}-v_{Y2})m + (w_{Z1}-w_{Z2})n \\ 0 \\ 0 \\ \hline -[(u_{X1}-u_{X2})\ell + (v_{Y1}-v_{Y2})m + (w_{Z1}-w_{Z2})n] \\ 0 \\ 0 \end{Bmatrix} \tag{6.612}$$

Note that for the truss problem we do not have the consistent load vector, since the load must be applied at the nodes, so $\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{int}\} = \{\bar{\boldsymbol{f}}^{(e)}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$.

It could be interesting to generate an internal global internal vector, related to Global system, i.e.: $\{\boldsymbol{r}^{(e)}_{Global}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_G}\} + [\boldsymbol{k}^{(e)}_{Global}]\{\boldsymbol{u}^{(e)}_{Global}\}$ or we can use $\{\boldsymbol{r}^{(e)}_{Global}\} = [\mathcal{A}]^T\{\boldsymbol{r}^{(e)}_{Local}\}$:

$$\{\boldsymbol{r}^{(e)}_{Global}\} = \left[\begin{array}{ccc|ccc} \ell & m & n & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \ell & m & n \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{array}\right]^T \begin{Bmatrix} \bar{f}_{x1} \\ 0 \\ 0 \\ \hline \bar{f}_{x2} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{x1}\ell \\ \bar{f}_{x1}m \\ \bar{f}_{x1}n \\ \hline \bar{f}_{x2}\ell \\ \bar{f}_{x2}m \\ \bar{f}_{x2}n \end{Bmatrix}$$

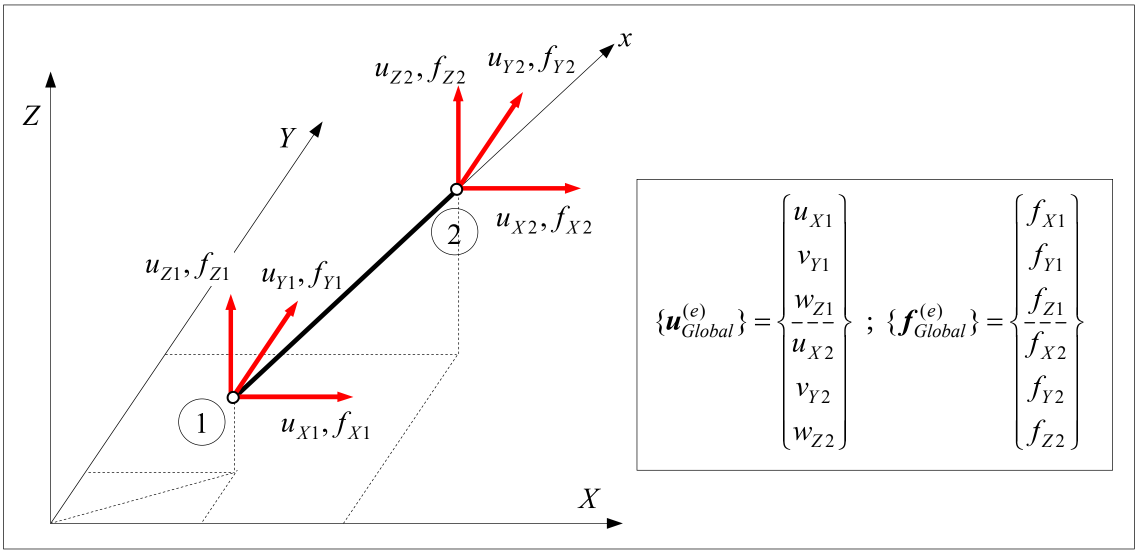


Figure 6.194: Bar element in three-dimensional space.

NOTE 2.2: Slab Floor compound by beams only

If we consider a Slab Floor in which only the beams are taken into account, (see Figure 6.195), the degrees-of-freedom associated with a node are characterized by 1 translation and 2 rotations, i.e. w, $\bar{\theta}_x$, $\bar{\theta}_y$, (Figure 6.196).

For this case the stiffness matrix is made up by combining the problems described in Figure 6.185 and Figure 6.188. And as external loads we will consider those described in Figure 6.189 (a)+(d). Then, if we consider the structure characterized by the degrees-of-freedom described in Figure 6.196 we can obtain:

$$
\left[\begin{array}{ccc|ccc}
\frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} \\
0 & \frac{GJ_{Eq}}{L} & 0 & 0 & \frac{-GJ_{Eq}}{L} & 0 \\
\frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} \\
\hline
\frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} \\
0 & \frac{-GJ_{Eq}}{L} & 0 & 0 & \frac{GJ_{Eq}}{L} & 0 \\
\frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L}
\end{array}\right]
\left\{\begin{array}{c} w_1 \\ \overline{\theta}_{x1} \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{x2} \\ \overline{\theta}_{y2} \end{array}\right\}
=
\left\{\begin{array}{c} \frac{q_z L}{2} \\ \frac{m_{Tx} L}{2} \\ \frac{-q_z L^2}{12} \\ \hline \frac{q_z L}{2} \\ \frac{m_{Tx} L}{2} \\ \frac{q_z L^2}{12} \end{array}\right\}
\tag{6.613}
$$

$$
[\boldsymbol{k}^{(e)}_{Local}]\ \{\boldsymbol{u}^{(e)}_{Local}\} = \{\boldsymbol{f}^{(e)}_{Eq\_L}\}
$$

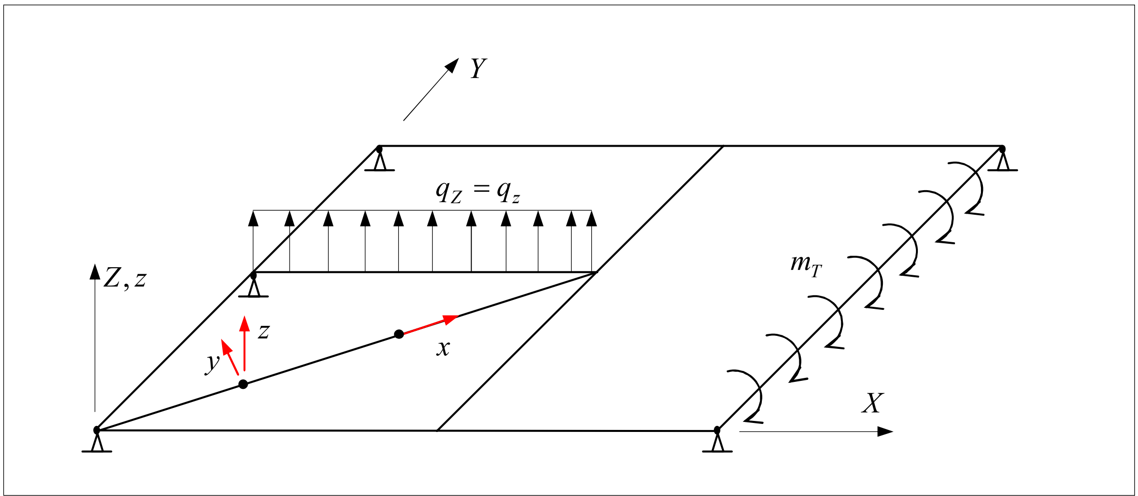


Figure 6.195: Slab floor by considering beams only.

The matrix transformation from the Global system $X-Y-Z$ to the local system $x-y-z$, (see Figure 6.197), is given by:

$$
[\mathcal{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell & m & 0 \\ -m & \ell & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.614}
$$

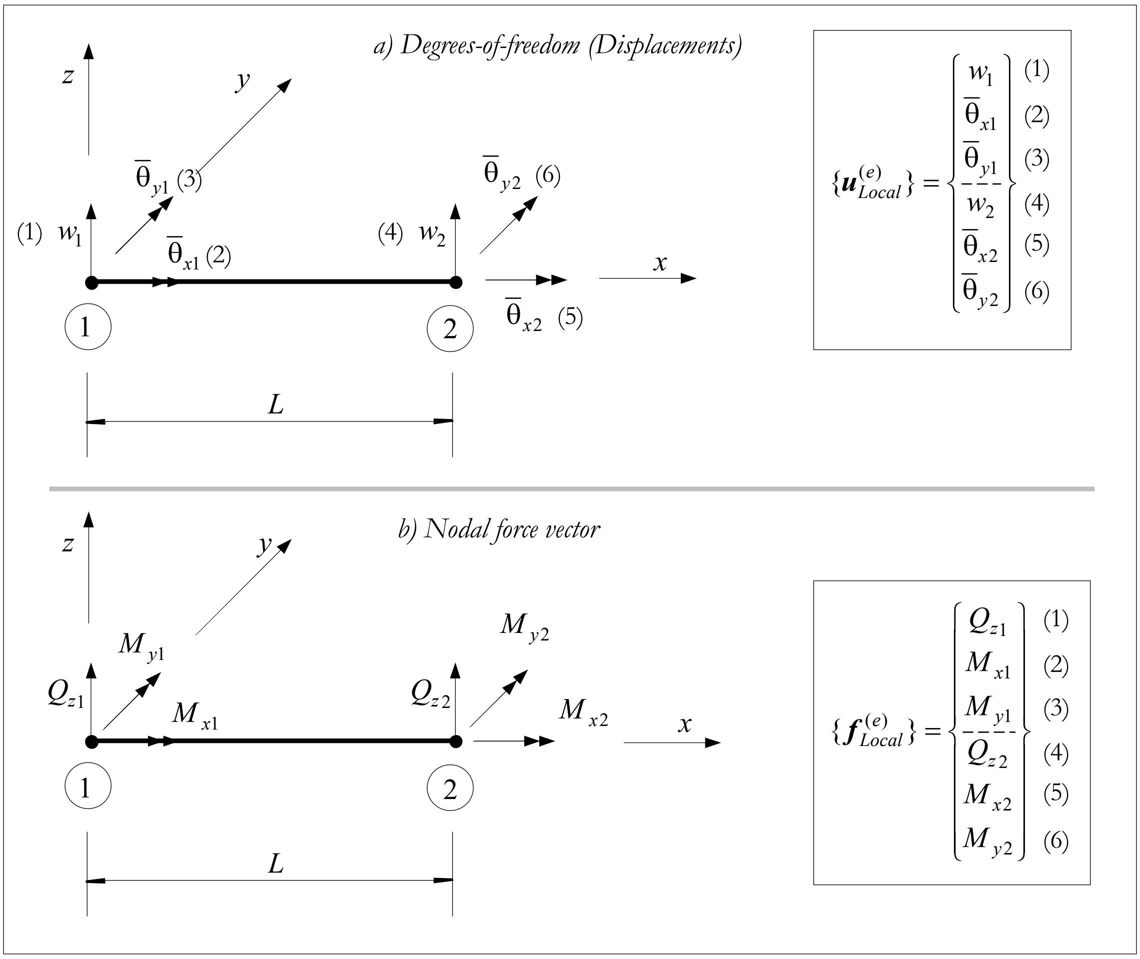


Figure 6.196: Slab floor beam – local system.

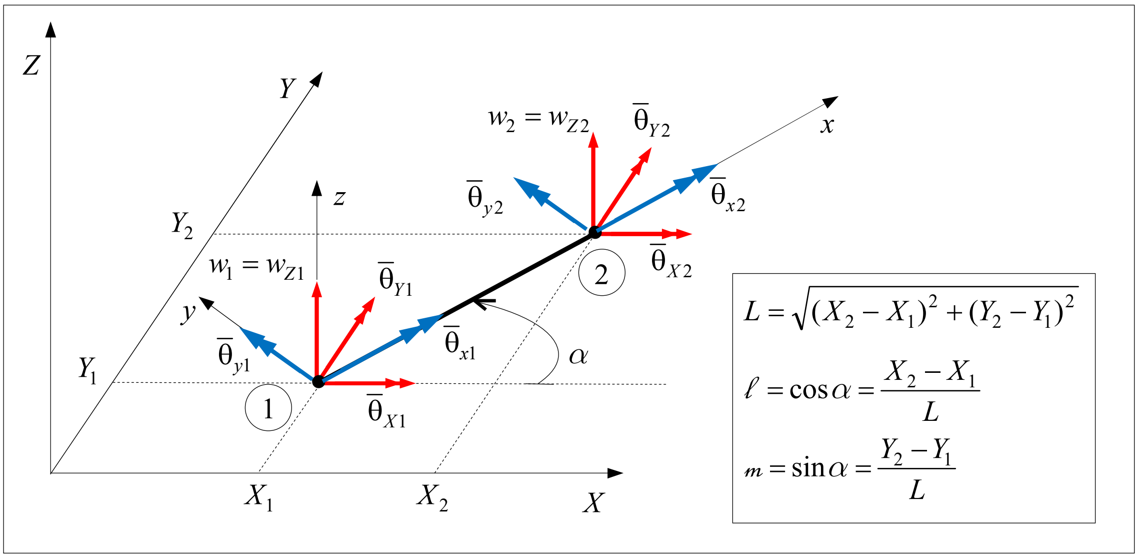


Figure 6.197: Bar element in three-dimensional space.

The transformation matrix for the nodal displacement vector is given by:

$$\xrightarrow{Nodal}\begin{Bmatrix} w_1 \\ \overline{\theta}_{x1} \\ \overline{\theta}_{y1} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ell & m \\ 0 & -m & \ell \end{bmatrix} \begin{Bmatrix} w_{Z1} = w_1 \\ \overline{\theta}_{X1} \\ \overline{\theta}_{Y1} \end{Bmatrix}$$

$$\xrightarrow{Element}\begin{Bmatrix} w_1 \\ \overline{\theta}_{x1} \\ \overline{\theta}_{y1} \\ \hline w_2 \\ \overline{\theta}_{x2} \\ \overline{\theta}_{y2} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell & m & 0 & 0 & 0 \\ 0 & -m & \ell & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ell & m \\ 0 & 0 & 0 & 0 & -m & \ell \end{array}\right] \begin{Bmatrix} w_1 \\ \overline{\theta}_{X1} \\ \overline{\theta}_{Y1} \\ \hline w_2 \\ \overline{\theta}_{X2} \\ \overline{\theta}_{Y2} \end{Bmatrix} \Rightarrow \{\boldsymbol{u}^{(e)}_{Local}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\} \tag{6.615}$$

By considering that:

$$[\boldsymbol{k}^{(e)}_{Local}] = \left[\begin{array}{ccc|ccc} f & 0 & -a & -f & 0 & -a \\ 0 & b & 0 & 0 & -b & 0 \\ -a & 0 & 2d & a & 0 & d \\ \hline -f & 0 & a & f & 0 & a \\ 0 & -b & 0 & 0 & b & 0 \\ -a & 0 & d & a & 0 & 2d \end{array}\right] \quad with \quad \begin{cases} a = \dfrac{6EI_y}{L^2} \\ b = \dfrac{GJ_{Eq}}{L} \\ f = \dfrac{12EI_y}{L^3} \\ d = \dfrac{2EI_y}{L} \end{cases} \tag{6.616}$$

the stiffness matrix in the Global system, (Chaves&Mínguez(2010)), can be expressed as follows:

$$[\boldsymbol{k}^{(e)}_{Global}] = [\overline{\mathcal{A}}]^T[\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}] = [\overline{\mathcal{A}}]^T \left[\begin{array}{ccc|ccc} f & 0 & -a & -f & 0 & -a \\ 0 & b & 0 & 0 & -b & 0 \\ -a & 0 & 2d & a & 0 & d \\ \hline -f & 0 & a & f & 0 & a \\ 0 & -b & 0 & 0 & b & 0 \\ -a & 0 & d & a & 0 & 2d \end{array}\right] [\overline{\mathcal{A}}]$$

thus

$$[\boldsymbol{k}^{(e)}_{Global}] = \left[\begin{array}{ccc|ccc} f & am & -a\ell & -f & am & -a\ell \\ am & b\ell^2 + 2dm^2 & b\ell m - 2d\ell m & -am & -b\ell^2 + dm^2 & -b\ell m - d\ell m \\ -a\ell & b\ell m - 2d\ell m & bm^2 + 2d\ell^2 & a\ell & -b\ell m - d\ell m & -bm^2 + d\ell^2 \\ \hline -f & -am & a\ell & f & -am & a\ell \\ am & -b\ell^2 + dm^2 & -b\ell m - d\ell m & -am & b\ell^2 + 2dm^2 & b\ell m - 2d\ell m \\ -a\ell & -b\ell m - d\ell m & -bm^2 + d\ell^2 & a\ell & b\ell m - 2d\ell m & bm^2 + 2d\ell^2 \end{array}\right] \tag{6.617}$$

The Consistent Load Vector

As external load, (see Figure 6.198), we will only consider:

$$\{\boldsymbol{P}^{(e)}_{Local}\} = \begin{Bmatrix} q_z \\ m_{Tx} \\ 0 \end{Bmatrix} \tag{6.618}$$

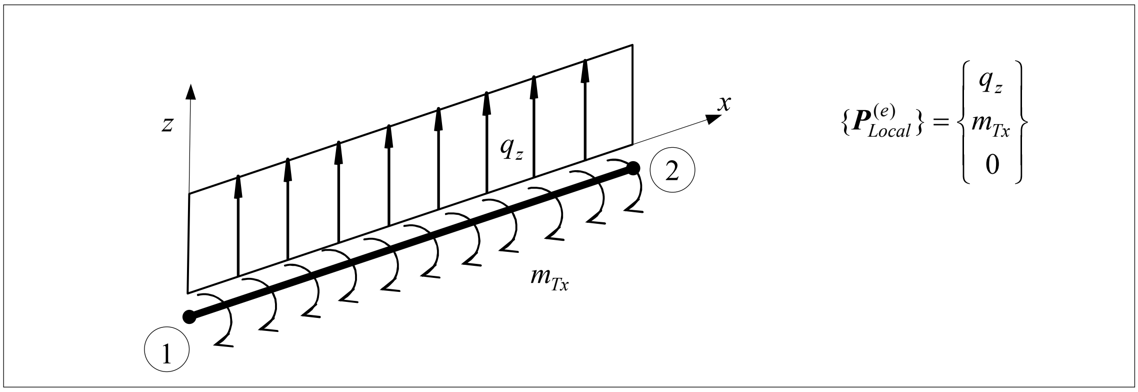


Figure 6.198: Beam element – external load (local system).

And the consistent load vector in the Local system $(x-z)$ is given by:

$$\{\boldsymbol{f}^{(e)}_{Eq\_L}\} = \left\{ \begin{array}{c} \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{-q_z L^2}{12} \\ \hdashline \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{q_z L^2}{12} \end{array} \right\} \tag{6.619}$$

Then, we can obtain the consistent load vector in the Global system as follows:

$$\{\boldsymbol{f}^{(e)}_{Eq\_L}\} = [\overline{\mathcal{A}}]\{\boldsymbol{f}^{(e)}_{Eq\_G}\} \Rightarrow \{\boldsymbol{f}^{(e)}_{Eq\_G}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{f}^{(e)}_{Eq\_L}\}$$

$$\Rightarrow \{\boldsymbol{f}^{(e)}_{Eq\_G}\} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell & m & 0 & 0 & 0 \\ 0 & -m & \ell & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ell & m \\ 0 & 0 & 0 & 0 & -m & \ell \end{array} \right]^T \left\{ \begin{array}{c} \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{-q_z L^2}{12} \\ \hdashline \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{q_z L^2}{12} \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{q_z L}{2} \\ \left(\dfrac{m_{Tx} L}{2}\right)\ell - \left(\dfrac{-q_z L^2}{12}\right)m \\ \left(\dfrac{m_{Tx} L}{2}\right)m + \left(\dfrac{-q_z L^2}{12}\right)\ell \\ \hdashline \dfrac{q_z L}{2} \\ \left(\dfrac{m_{Tx} L}{2}\right)\ell - \left(\dfrac{q_z L^2}{12}\right)m \\ \left(\dfrac{m_{Tx} L}{2}\right)m + \left(\dfrac{q_z L^2}{12}\right)\ell \end{array} \right\} \tag{6.620}$$

Internal Force in the Beam Element

Once the nodal displacements in the Global system are obtained the internal force in the beam element can be obtained as follows:

1) Obtain the nodal displacement in the local system: $\{\boldsymbol{u}^{(e)}_{Local}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$

2) Obtain the force due to these displacements: $\{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$

3) Then, the internal force, (see **Problem 6.62**- NOTE 3), can be obtained as follows:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} \tag{6.621}$$

Explicitly we can obtain:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\left\{\begin{array}{c} \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{-q_z L^2}{12} \\ \hline \dfrac{q_z L}{2} \\ \dfrac{m_{Tx} L}{2} \\ \dfrac{q_z L^2}{12} \end{array}\right\} + \left[\begin{array}{ccc|ccc} f & 0 & -a & -f & 0 & -a \\ 0 & b & 0 & 0 & -b & 0 \\ -a & 0 & 2d & a & 0 & d \\ \hline -f & 0 & a & f & 0 & a \\ 0 & -b & 0 & 0 & b & 0 \\ -a & 0 & d & a & 0 & 2d \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell & m & 0 & 0 & 0 \\ 0 & -m & \ell & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ell & m \\ 0 & 0 & 0 & 0 & -m & \ell \end{array}\right] \left\{\begin{array}{c} w_1 \\ \overline{\theta}_{X1} \\ \overline{\theta}_{Y1} \\ w_2 \\ \overline{\theta}_{X2} \\ \overline{\theta}_{Y2} \end{array}\right\}$$

After the multiplication of matrices, (Chaves&Mínguez(2019)), we can obtain:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = \left\{\begin{array}{c} \dfrac{-q_z L}{2} \\ \dfrac{-m_{Tx} L}{2} \\ \dfrac{q_z L^2}{12} \\ \hline \dfrac{-q_z L}{2} \\ \dfrac{-m_{Tx} L}{2} \\ \dfrac{q_z L^2}{12} \end{array}\right\} + \left\{\begin{array}{c} fw_1 - fw_2 - a\ell(\overline{\theta}_{Y1} + \overline{\theta}_{Y2}) + am(\overline{\theta}_{X1} + \overline{\theta}_{X2}) \\ b\ell(\overline{\theta}_{X1} - \overline{\theta}_{X2}) + bm(\overline{\theta}_{Y1} - \overline{\theta}_{Y2}) \\ -aw_1 + aw_2 + d\ell(2\overline{\theta}_{Y1} + \overline{\theta}_{Y2}) - dm(2\overline{\theta}_{X1} + \overline{\theta}_{X2}) \\ \hline -fw_1 - + fw_2 + a\ell(\overline{\theta}_{Y1} + \overline{\theta}_{Y2}) - am(\overline{\theta}_{X1} + \overline{\theta}_{X2}) \\ -b\ell(\overline{\theta}_{X1} - \overline{\theta}_{X2}) - bm(\overline{\theta}_{Y1} - \overline{\theta}_{Y2}) \\ -aw_1 + aw_2 + d\ell(\overline{\theta}_{Y1} + 2\overline{\theta}_{Y2}) - dm(\overline{\theta}_{X1} + 2\overline{\theta}_{X2}) \end{array}\right\} \tag{6.622}$$

and the reaction in the global system can be obtained as follows

$$\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}^{(e)}_{Local}\} \tag{6.623}$$

NOTE 2.3: Frame Structures

3D Frame Structures. For this type of structure at each node is associated with 6 degrees-of-freedom, (see Figure 6.183).

2D Frame Structures. For this type of structure we will have, per node, 3 degrees-of-freedom associated with 2 translations (u, v) and 1 rotation $(\overline{\theta}_z)$. Locally the internal forces: normal force (N_x), shear force (Q_y) and bending moment (M_z), (see Figure 6.200).

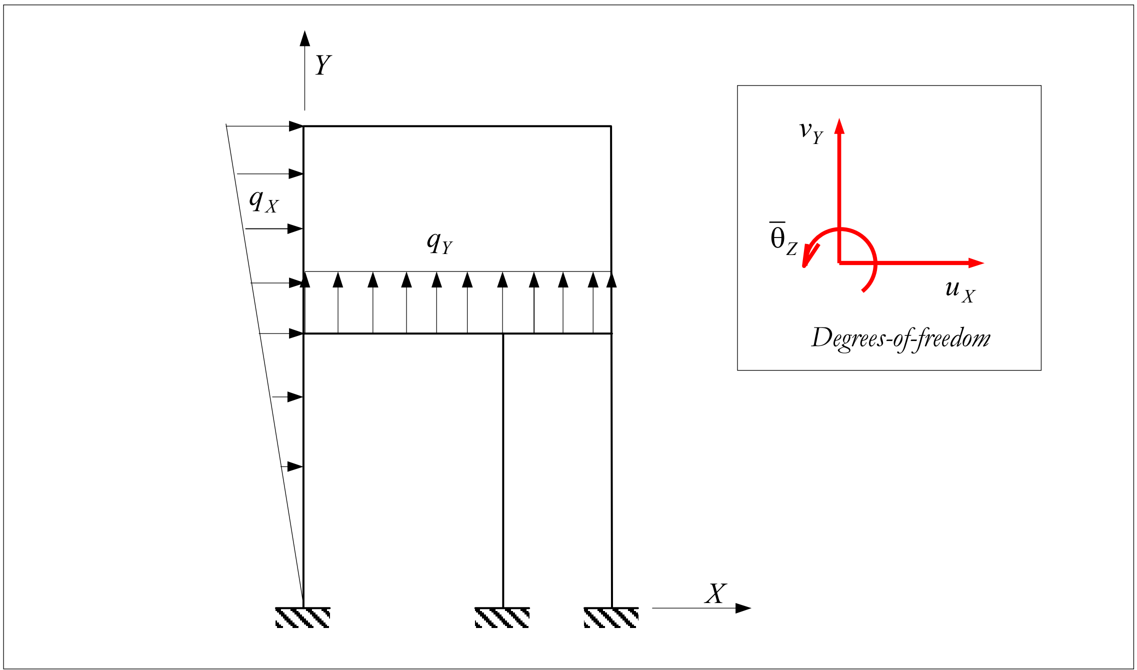


Figure 6.199: 2D Frame Structure.

If we consider the structure of the nodal displacement described in Figure 6.200, and by considering the stiffness matrices given by Figure 6.186 and Figure 6.187 we can obtain:

$$
\left[\begin{array}{ccc|ccc}
\frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\
\hline
\frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right]
\left\{\begin{array}{c} u_1 \\ v_1 \\ \overline{\theta}_{z1} \\ \hline u_2 \\ v_2 \\ \overline{\theta}_{z2} \end{array}\right\}
=
\left\{\begin{array}{c} \frac{q_x L}{2} \\ \frac{q_y L}{2} \\ \frac{q_y L^2}{12} \\ \hline \frac{q_x L}{2} \\ \frac{q_y L}{2} \\ \frac{-q_y L^2}{12} \end{array}\right\}
\tag{6.624}
$$

$$[\boldsymbol{k}^{(e)}_{Local}]\ \{\boldsymbol{u}^{(e)}_{Local}\} = \{\boldsymbol{f}^{(e)}_{Eq\_L}\}$$

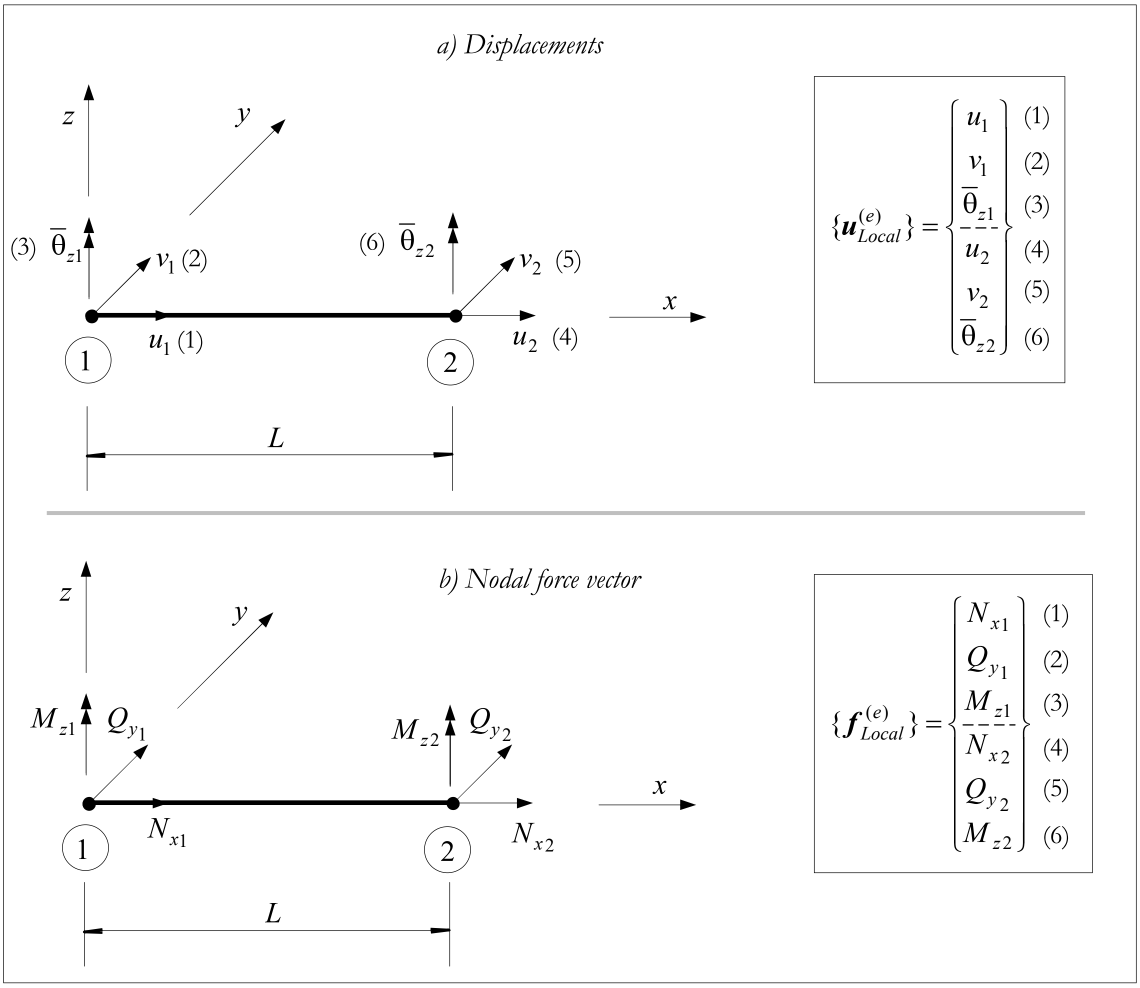


Figure 6.200: 2D frame structure – local system.

<u>Two-Dimensional Space (2D)</u>

Stiffness Matrix for 2D

For the local system the force-displacement relationship is given by the equation in (6.624).

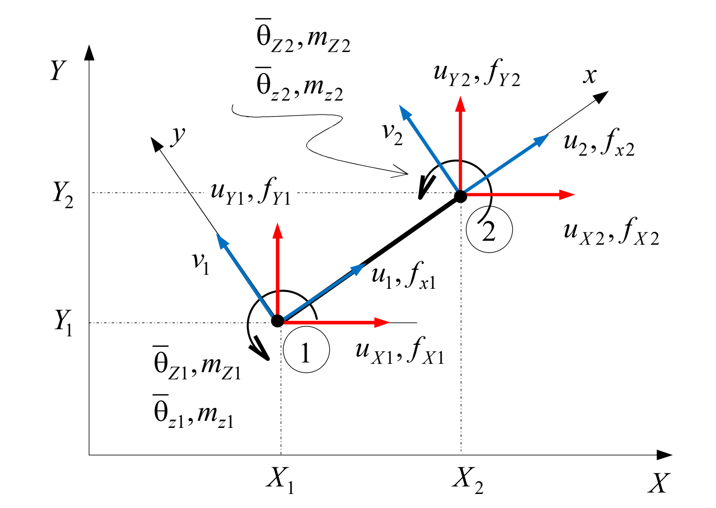
Transformation Matrix

The transformation matrix from the Global system $X-Y$ to the Local system $x-y$, (see Figure 6.192(b)), is given by:

$$[\mathcal{A}] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \ell & m & 0 \\ -m & \ell & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{Nodal} \begin{Bmatrix} u_i \\ v_i \\ \overline{\theta}_{zi} \end{Bmatrix} = \begin{bmatrix} \ell & m & 0 \\ -m & \ell & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{Xi} \\ v_{Yi} \\ \overline{\theta}_{Zi} \end{Bmatrix} \qquad (6.625)$$

And the transformation matrix for the displacement vector $\{\boldsymbol{u}_{Local}^{(e)}\}$ becomes

$$\xrightarrow{Element} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ \overline{\theta}_{z1} \\ u_2 \\ v_2 \\ \overline{\theta}_{z1} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \overline{\theta}_{Z1} \\ u_{X2} \\ v_{Y2} \\ \overline{\theta}_{Z1} \end{Bmatrix}}_{\{\boldsymbol{u}_{Local}^{(e)}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}_{Global}^{(e)}\}} \qquad (6.626)$$



$$
\{\boldsymbol{u}^{(e)}_{Local}\} = \begin{Bmatrix} u_1 \\ v_1 \\ \overline{\theta}_{z1} \\ \hdashline u_2 \\ v_2 \\ \overline{\theta}_{z2} \end{Bmatrix}; \{\boldsymbol{f}^{(e)}_{Local}\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix}
$$

$$
\{\boldsymbol{u}^{(e)}_{Global}\} = \begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \overline{\theta}_{Z1} \\ \hdashline u_{X2} \\ v_{Y2} \\ \overline{\theta}_{Z2} \end{Bmatrix}; \{\boldsymbol{f}^{(e)}_{Global}\} = \begin{Bmatrix} f_{X1} \\ f_{Y1} \\ m_{Z1} \\ \hdashline f_{X2} \\ f_{Y2} \\ m_{Z2} \end{Bmatrix}
$$

Figure 6.201: Beam element in two-dimensional space – 3 DOF per node.

By considering that:

$$
[\boldsymbol{k}^{(e)}_{Local}] = \left[\begin{array}{ccc:ccc} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & c & 0 & -b & c \\ 0 & c & 2d & 0 & -c & d \\ \hdashline -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -c & 0 & b & -c \\ 0 & c & d & 0 & -c & 2d \end{array}\right] \quad with \quad \begin{cases} a = \dfrac{EA}{L} \\ b = \dfrac{12EI_z}{L^3} \\ c = \dfrac{6EI_z}{L^2} \\ d = \dfrac{2EI_z}{L} \end{cases} \tag{6.627}
$$

the stiffness matrix in the Global system can be expressed as follows:

$$
[\boldsymbol{k}^{(e)}_{Global}] = [\overline{\mathcal{A}}]^T [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}] = [\overline{\mathcal{A}}]^T \left[\begin{array}{ccc:ccc} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & c & 0 & -b & c \\ 0 & c & 2d & 0 & -c & d \\ \hdashline -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -c & 0 & b & -c \\ 0 & c & d & 0 & -c & 2d \end{array}\right] [\overline{\mathcal{A}}]
$$

thus

$$
[\boldsymbol{k}^{(e)}_{Global}] = \left[\begin{array}{ccc:ccc} a\ell^2 + bm^2 & a\ell m - b\ell m & -cm & -a\ell^2 - bm^2 & -a\ell m + b\ell m & -cm \\ a\ell m - b\ell m & b\ell^2 + am^2 & c\ell & -a\ell m + b\ell m & -b\ell^2 - am^2 & c\ell \\ -cm & c\ell & 2d & cm & -c\ell & d \\ \hdashline -a\ell^2 - bm^2 & -a\ell m + b\ell m & cm & a\ell^2 + bm^2 & a\ell m - b\ell m & cm \\ -a\ell m + b\ell m & -b\ell^2 - am^2 & -c\ell & a\ell m - b\ell m & b\ell^2 + am^2 & -c\ell \\ -cm & c\ell & d & cm & -c\ell & 2d \end{array}\right] \tag{6.628}
$$

The Consistent Load Vector

As external load, (see Figure 6.202), we will only consider:

$$\{\boldsymbol{P}^{(e)}_{Local}\} = \begin{Bmatrix} q_x \\ q_y \\ 0 \end{Bmatrix} \tag{6.629}$$

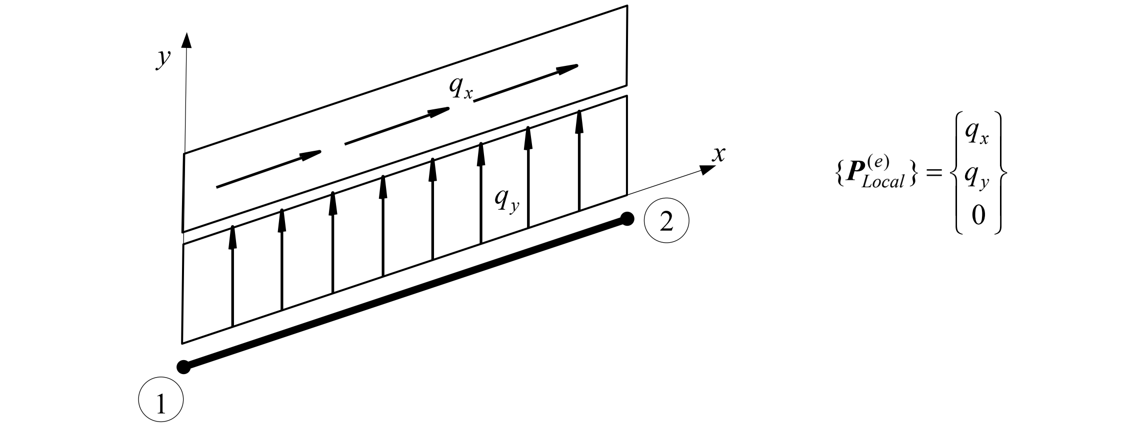


Figure 6.202: Beam element – external load (local system).

And the consistent load vector in the local system, (see equation (6.624)), is given by:

$$\{\boldsymbol{f}^{(e)}_{Eq\_L}\} = \begin{Bmatrix} \dfrac{q_x L}{2} \\ \dfrac{q_y L}{2} \\ \dfrac{q_y L^2}{12} \\ \hdashline \dfrac{q_x L}{2} \\ \dfrac{q_y L}{2} \\ \dfrac{-q_y L^2}{12} \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix} \tag{6.630}$$

Then, we can obtain the consistent load vector in the global system as follows:

$$\{\boldsymbol{f}^{(e)}_{Eq\_L}\} = [\overline{\mathcal{A}}]\{\boldsymbol{f}^{(e)}_{Eq\_G}\} \Rightarrow \{\boldsymbol{f}^{(e)}_{Eq\_G}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{f}^{(e)}_{Eq\_L}\}$$

$$\Rightarrow \{\boldsymbol{f}^{(e)}_{Eq\_G}\} = \left[\begin{array}{ccc:ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^T \begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)}\ell - f_2^{(Eq\_L)} m \\ f_1^{(Eq\_L)} m + f_2^{(Eq\_L)}\ell \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)}\ell - f_5^{(Eq\_L)} m \\ f_4^{(Eq\_L)} m + f_5^{(Eq\_L)}\ell \\ f_6^{(Eq\_L)} \end{Bmatrix} \tag{6.631}$$

or more explicitly

$$\Rightarrow \{\boldsymbol{f}^{(e)}_{Eq\_G}\} = \begin{Bmatrix} f_1^{(Eq\_L)}\ell - f_2^{(Eq\_L)}m \\ f_1^{(Eq\_L)}m + f_2^{(Eq\_L)}\ell \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)}\ell - f_5^{(Eq\_L)}m \\ f_4^{(Eq\_L)}m + f_5^{(Eq\_L)}\ell \\ f_6^{(Eq\_L)} \end{Bmatrix} = \begin{Bmatrix} \left(\frac{q_x L}{2}\right)\ell - \left(\frac{q_y L}{2}\right)m \\ \left(\frac{q_x L}{2}\right)m + \left(\frac{q_y L}{2}\right)\ell \\ \frac{q_y L^2}{12} \\ \hdashline \left(\frac{q_x L}{2}\right)\ell - \left(\frac{q_y L}{2}\right)m \\ \left(\frac{q_x L}{2}\right)m + \left(\frac{q_y L}{2}\right)\ell \\ \frac{-q_y L^2}{12} \end{Bmatrix} \tag{6.632}$$

Internal Force in the Beam Element

Once the nodal displacements in the Global system are obtained the internal force in the beam element can be obtained as follows:

1) Obtain the nodal displacement in the local system: $\{\boldsymbol{u}^{(e)}_{Local}\} = [\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\}$

2) Obtain the force due to these displacements: $\{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = [\boldsymbol{k}^{(e)}_{Local}]\{\boldsymbol{u}^{(e)}_{Local}\}$

3) Then, the internal force related to local system, (see **Problem 6.62**-NOTE 3), can be obtained as follows:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} \tag{6.633}$$

Explicitly we can obtain:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix} + \left[\begin{array}{ccc:ccc} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & c & 0 & -b & c \\ 0 & c & 2d & 0 & -c & d \\ \hdashline -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -c & 0 & b & -c \\ 0 & c & d & 0 & -c & 2d \end{array}\right] \left[\begin{array}{ccc:ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_{X1} \\ v_{Y1} \\ \overline{\theta}_{Z1} \\ \hdashline u_{X2} \\ v_{Y2} \\ \overline{\theta}_{Z1} \end{Bmatrix}$$

After the multiplication of matrices we can obtain:

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix} = -\begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix} + \begin{Bmatrix} a\ell(u_{X1}-u_{X2}) + am(v_{Y1}-v_{Y2}) \\ b\ell(v_{Y1}-v_{Y2}) + bm(u_{X2}-u_{X1}) + c(\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ c\ell(v_{Y1}-v_{Y2}) + cm(u_{X2}-u_{X1}) + d(2\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ \hdashline -a\ell(u_{X1}-u_{X2}) - am(v_{Y1}-v_{Y2}) \\ -b\ell(v_{Y1}-v_{Y2}) - bm(u_{X2}-u_{X1}) - c(\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ c\ell(v_{Y1}-v_{Y2}) + cm(u_{X2}-u_{X1}) + d(\overline{\theta}_{Z1}+2\overline{\theta}_{Z2}) \end{Bmatrix} \tag{6.634}$$

More explicitly

$$
\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{int}\} = \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = -\left\{\begin{array}{c} \dfrac{q_x L}{2} \\ \dfrac{q_y L}{2} \\ \dfrac{q_y L^2}{12} \\ \hdashline \dfrac{q_x L}{2} \\ \dfrac{q_y L}{2} \\ \dfrac{-q_y L^2}{12} \end{array}\right\} + \left\{\begin{array}{c} a\ell(u_{X1}-u_{X2}) + am(v_{Y1}-v_{Y2}) \\ b\ell(v_{Y1}-v_{Y2}) + bm(u_{X2}-u_{X1}) + c(\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ c\ell(v_{Y1}-v_{Y2}) + cm(u_{X2}-u_{X1}) + d(2\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ \hdashline -a\ell(u_{X1}-u_{X2}) - am(v_{Y1}-v_{Y2}) \\ -b\ell(v_{Y1}-v_{Y2}) - bm(u_{X2}-u_{X1}) - c(\overline{\theta}_{Z1}+\overline{\theta}_{Z2}) \\ c\ell(v_{Y1}-v_{Y2}) + cm(u_{X2}-u_{X1}) + d(\overline{\theta}_{Z1}+2\overline{\theta}_{Z2}) \end{array}\right\}
\tag{6.635}
$$

The reaction forces in the Global system can be obtained by $\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}^{(e)}_{Local}\}$:

$$
\{\boldsymbol{r}^{(e)}_{Global}\} = \left[\begin{array}{ccc:ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^T \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hdashline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m^{(int)}_{z2} \end{array}\right\}
\tag{6.636}
$$

Problem 6.71

Consider the structure described in Figure 6.203, (Timoshenko (1940)). Obtain the displacement of the node 1 according to Y-direction.

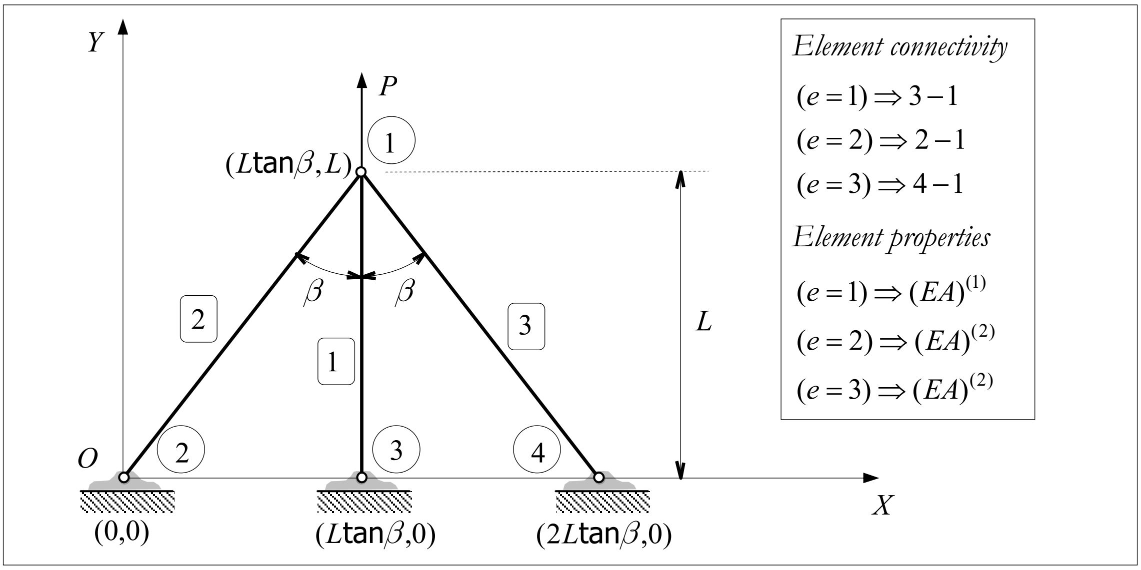


Figure 6.203: 2D truss structure.

Solution:

By considering 2D truss structure we have two degrees-of freedom per node, so the Global displacement vector for the structure has 8 degrees-of-freedom (Number of nodes (4) times Number of degrees-of-freedom per node (2)), (see Figure 6.204).

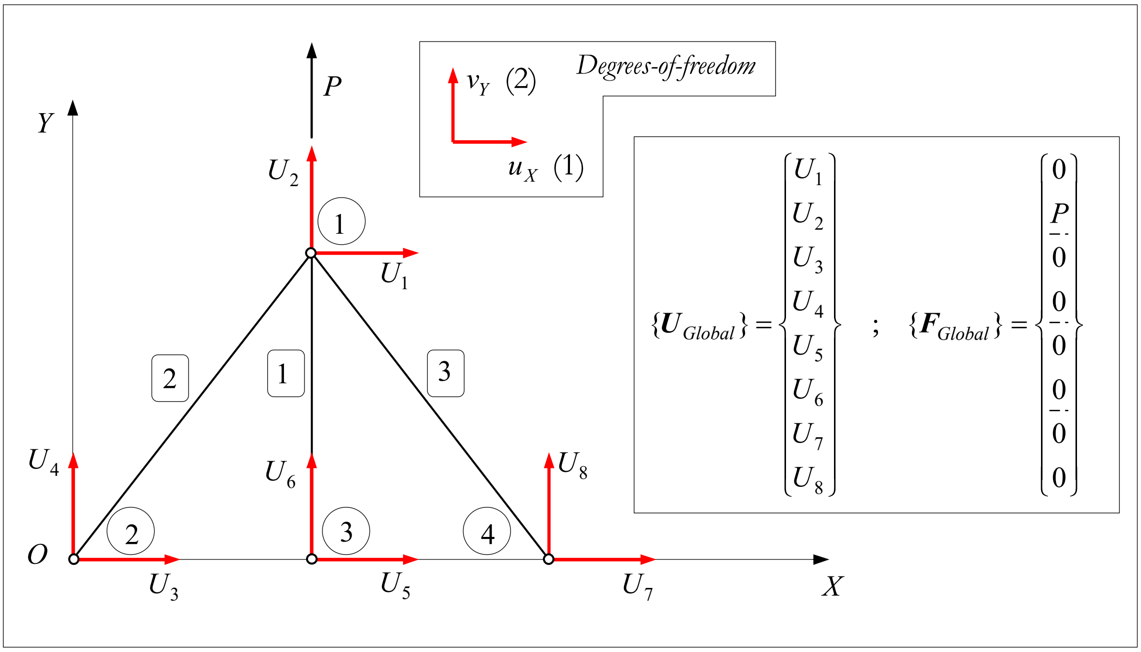


Figure 6.204: Degrees-of-freedom – Global system.

Next we will construct the system $\{\boldsymbol{F}_{Global}\} = [\boldsymbol{K}_{Global}]\{\boldsymbol{U}_{Global}\}$ where $[\boldsymbol{K}_{Global}]$ is the Global stiffness matrix of the structure, and $\{\boldsymbol{F}_{Global}\}$ is the Global nodal force vector.

Construction of the Global Stiffness Matrix - $[\boldsymbol{K}_{Global}]$

The matrix $[\boldsymbol{K}_{Global}]$ can be constructed by assembling the individual bar elements, i.e.:

$$[\boldsymbol{K}_{Global}] = \mathop{\mathbf{A}}_{e=1}^{3} [\boldsymbol{k}^{(e)}_{Global}] \tag{6.637}$$

The stiffness matrix for the 2D truss element can be obtained by using the equation in (6.600), i.e.:

$$[\boldsymbol{k}^{(e)}_{Global}] = \frac{(EA)^{(e)}}{L^{(e)}} \left[\begin{array}{cc|cc} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ \hline -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{array}\right] \tag{6.638}$$

For each element we have

Element	Connectivity : N1 → N2 $N1 \to N2 \begin{cases} \text{Node1}: (X_1, Y_1) \\ \text{Node2}: (X_2, Y_2) \end{cases}$	$L^{(e)}$	$\ell^{(e)} = \frac{X_2 - X_1}{L^{(e)}}$	$m^{(e)} = \frac{Y_2 - Y_1}{L^{(e)}}$	$(EA)^{(e)}$
1	$3 \to 1 \begin{cases} 3: (L\tan\beta, 0) \\ 1: (L\tan\beta, L) \end{cases}$	L	$\ell^{(1)} = 0$	$m^{(1)} = 1$	$(EA)^{(1)}$
2	$2 \to 1 \begin{cases} 2: (0,0) \\ 1: (L\tan\beta, L) \end{cases}$	$\frac{L}{\cos\beta}$	$\ell^{(2)} = \sin\beta$ $\ell^{(2)} = s$	$m^{(2)} = \cos\beta$ $m^{(2)} = c$	$(EA)^{(2)}$
3	$4 \to 1 \begin{cases} 2: (2L\tan\beta, 0) \\ 1: (L\tan\beta, L) \end{cases}$	$\frac{L}{\cos\beta}$	$\ell^{(3)} = -\sin\beta$ $\ell^{(3)} = -s$	$m^{(3)} = \cos\beta$ $m^{(3)} = c$	$(EA)^{(2)}$

And we will consider that $\ell^{(2)} = \sin\beta = s = -\ell^{(3)}$, $m^{(2)} = m^{(2)} = \cos\beta = c$

Element 1: ($\ell^{(1)} = 0$, $m^{(1)} = 1$)

$$[\boldsymbol{k}^{(1)}_{Global}] = \frac{(EA)^{(1)}}{L^{(1)}} \left[\begin{array}{cc|cc} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ \hline -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{array}\right] = \frac{(EA)^{(1)}}{L} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \; Global \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \end{array}$$

Element 2: ($\ell^{(2)} = s$, $m^{(2)} = c$)

$$[\boldsymbol{k}^{(2)}_{Global}] = \frac{(EA)^{(2)}}{L^{(2)}} \left[\begin{array}{cc|cc} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ \hline -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{array}\right] = \frac{c(EA)^{(2)}}{L} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \; Global \\ \left[\begin{array}{cc|cc} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ \hline -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{array}\right] \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} \end{array}$$

Element 3 ($\ell^{(3)} = -s$, $m^{(3)} = c$)

$$[\boldsymbol{k}^{(3)}_{Global}] = \frac{(EA)^{(2)}}{L^{(3)}} \left[\begin{array}{cc|cc} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ \hline -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{array}\right] = \frac{c(EA)^{(2)}}{L} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \; Global \\ \left[\begin{array}{cc|cc} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ \hline -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{array}\right] \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array} \end{array}$$

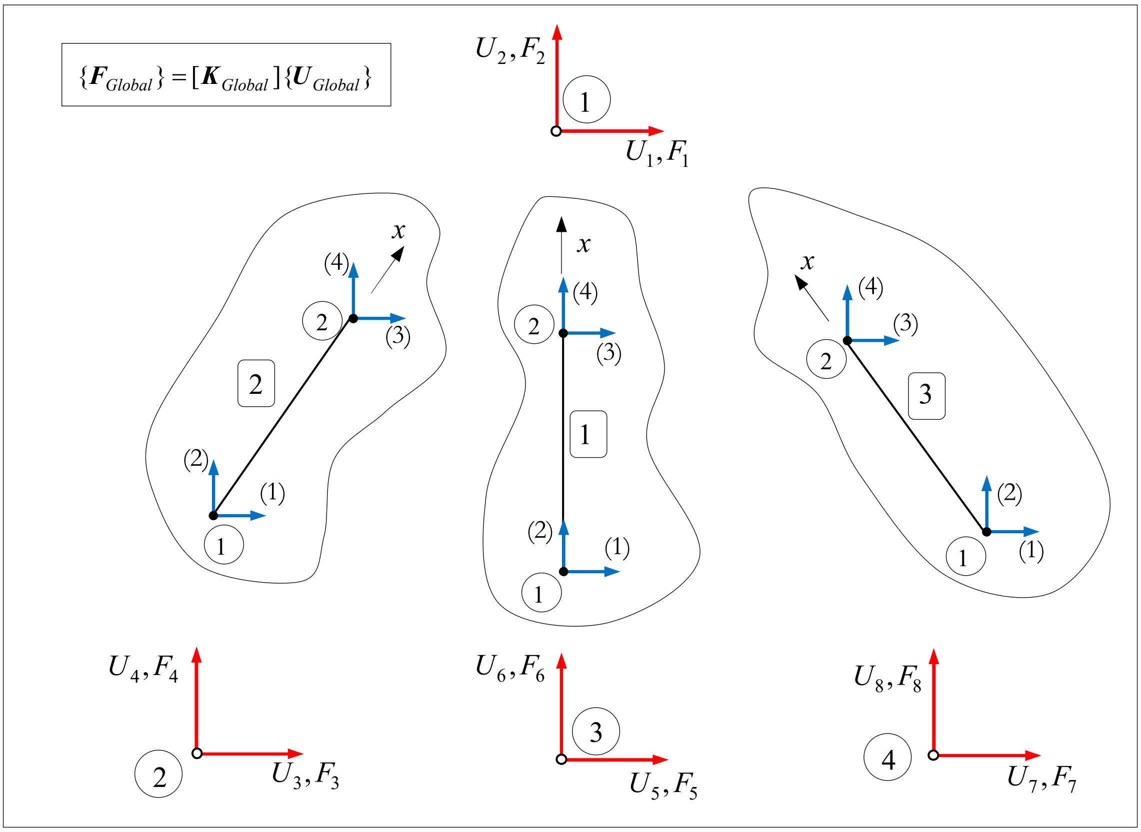


Figure 6.205: Discrete system.

$$
[\boldsymbol{K}_{Global}] = \mathop{\mathbf{A}}_{e=1}^{3} [\boldsymbol{k}^{(e)}_{Global}] = \left[\begin{array}{cc:cc:cc:cc}
k^{(1)}_{33}+k^{(2)}_{33}+k^{(3)}_{33} & k^{(1)}_{34}+k^{(2)}_{34}+k^{(3)}_{34} & k^{(2)}_{31} & k^{(2)}_{32} & k^{(1)}_{31} & k^{(1)}_{32} & k^{(3)}_{31} & k^{(3)}_{32} \\
k^{(1)}_{43}+k^{(2)}_{43}+k^{(3)}_{43} & k^{(1)}_{44}+k^{(2)}_{44}+k^{(3)}_{44} & k^{(2)}_{41} & k^{(2)}_{42} & k^{(1)}_{41} & k^{(1)}_{42} & k^{(3)}_{41} & k^{(3)}_{42} \\
\hdashline
k^{(2)}_{13} & k^{(2)}_{14} & k^{(2)}_{11} & k^{(2)}_{12} & 0 & 0 & 0 & 0 \\
k^{(2)}_{23} & k^{(2)}_{24} & k^{(2)}_{21} & k^{(2)}_{22} & 0 & 0 & 0 & 0 \\
\hdashline
k^{(1)}_{13} & k^{(1)}_{14} & 0 & 0 & k^{(1)}_{11} & k^{(1)}_{12} & 0 & 0 \\
k^{(1)}_{23} & k^{(1)}_{24} & 0 & 0 & k^{(1)}_{21} & k^{(1)}_{22} & 0 & 0 \\
\hdashline
k^{(3)}_{13} & k^{(3)}_{14} & 0 & 0 & 0 & 0 & k^{(3)}_{11} & k^{(3)}_{12} \\
k^{(3)}_{23} & k^{(3)}_{24} & 0 & 0 & 0 & 0 & k^{(3)}_{21} & k^{(3)}_{22}
\end{array}\right]
$$

Construction of the Global Nodal Force Vector - $\{F_{Global}\}$

$$
\{\boldsymbol{F}_{Global}\} = \underbrace{\left(\mathop{\mathbf{A}}_{e=1}^{3} \{\boldsymbol{f}^{(e)}_{Eq\_L}\}\right)}_{=\{\boldsymbol{0}\}} + \{\boldsymbol{F}_0\} = \left\{\begin{array}{c} 0 \\ P \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right\}
$$

Applying the Boundary Conditions

Note that the degrees-of-freedom related to the nodes 2, 3, and 4 cannot move due to the boundary conditions, then the system $\{\boldsymbol{F}_{Global}\} = [\boldsymbol{K}_{Global}]\{\boldsymbol{U}_{Global}\}$ after impose the boundary conditions becomes:

$$\left[\begin{array}{cc|cc|cc|cc} k_{33}^{(1)} + k_{33}^{(2)} + k_{33}^{(3)} & k_{34}^{(1)} + k_{34}^{(2)} + k_{34}^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{43}^{(1)} + k_{43}^{(2)} + k_{43}^{(3)} & k_{44}^{(1)} + k_{44}^{(2)} + k_{44}^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{Bmatrix}$$

By substituting the values we can obtain:

$$\frac{1}{L}\left[\begin{array}{cc|cc|cc|cc} 2c(EA)^{(2)}s^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (EA)^{(1)} + 2(EA)^{(2)}c^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{Bmatrix}$$

Solving the System

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \left\{\begin{array}{c} 0 \\ \left(\dfrac{LP}{(EA)^{(1)} + 2(EA)^{(2)}c^3}\right) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right\}$$

Internal and Reaction Forces

We calculate the vector $\{\bar{\boldsymbol{f}}_{Local}^{(e)}\} = [\boldsymbol{k}_{Local}^{(e)}]\{\boldsymbol{u}_{Local}^{(e)}\}$, (see equation (6.604)):

$$\begin{Bmatrix} \bar{f}_{x1}^{(e)} \\ \bar{f}_{y1}^{(e)} \\ \hline \bar{f}_{x2}^{(e)} \\ \bar{f}_{y2}^{(e)} \end{Bmatrix} = \frac{(EA)^{(e)}}{L^{(e)}} \left\{\begin{array}{c} (u_{X1}^{(e)} - u_{X2}^{(e)})\ell^{(e)} + (v_{Y1}^{(e)} - v_{Y2}^{(e)})m^{(e)} \\ 0 \\ \hline (u_{X2}^{(e)} - u_{X1}^{(e)})\ell^{(e)} + (v_{Y2}^{(e)} - v_{Y1}^{(e)})m^{(e)} \\ 0 \end{array}\right\} \tag{6.639}$$

and the reaction vector in the Local and Global system can be obtained as follows

$$\{\boldsymbol{r}_{Local}^{(e)}\} = -\{\boldsymbol{f}_{int}^{(e)}\} = \{\bar{\boldsymbol{f}}^{(e)}\} = [\boldsymbol{k}_{Local}^{(e)}]\{\boldsymbol{u}_{Local}^{(e)}\} \quad \Rightarrow \quad \{\boldsymbol{r}_{Global}^{(e)}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}_{Local}^{(e)}\}$$

where

$$\{\boldsymbol{r}_{Global}^{(e)}\} = \begin{bmatrix} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{bmatrix}^T \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1}=0 \\ \bar{f}_{x2} \\ \bar{f}_{y2}=0 \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{x1}\ell \\ \bar{f}_{x1}m \\ \bar{f}_{x2}\ell \\ \bar{f}_{x2}m \end{Bmatrix}$$

Element 1: $\ell^{(1)}=0$, $m^{(1)}=1$, $(EA)^{(1)}$, $L^{(1)}=L$, $(u_{X1}^{(1)}=U_5=0)$, $(v_{Y1}^{(1)}=U_6=0)$, $(u_{X2}^{(1)}=U_1=0)$, $\left(v_{Y2}^{(1)}=U_2=\dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right)$, then

$$\{\bar{\boldsymbol{f}}^{(1)}\} = \begin{Bmatrix} \bar{f}_{x1}^{(1)} \\ \bar{f}_{y1}^{(1)} \\ \bar{f}_{x2}^{(1)} \\ \bar{f}_{y2}^{(1)} \end{Bmatrix} = \frac{(EA)^{(1)}}{L^{(1)}} \begin{Bmatrix} (u_{X1}^{(1)}-u_{X2}^{(1)})\ell^{(1)} + (v_{Y1}^{(1)}-v_{Y2}^{(1)})m^{(1)} \\ 0 \\ (u_{X2}^{(1)}-u_{X1}^{(1)})\ell^{(1)} + (v_{Y2}^{(1)}-v_{Y1}^{(1)})m^{(1)} \\ 0 \end{Bmatrix} = \frac{(EA)^{(1)}}{L} \begin{Bmatrix} \left(\dfrac{-LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right) \\ 0 \\ \left(\dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right) \\ 0 \end{Bmatrix}$$

$$= -\{\boldsymbol{f}_{int}^{(1)}\}$$

and

$$\{\boldsymbol{r}_{Global}^{(1)}\} = \begin{Bmatrix} \bar{f}_{x1}\ell \\ \bar{f}_{x1}m \\ \bar{f}_{x2}\ell \\ \bar{f}_{x2}m \end{Bmatrix} = \frac{(EA)^{(1)}}{L} \begin{Bmatrix} \dfrac{-LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\ell \\ \dfrac{-LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}m \\ \dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\ell \\ \dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}m \end{Bmatrix} = \begin{array}{l} \\ \begin{Bmatrix} 0 \\ \dfrac{-(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3} \\ 0 \\ \dfrac{(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3} \end{Bmatrix} \end{array} \begin{array}{l} \textit{Global} \\ 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Element 2: $\ell^{(2)}=s$, $m^{(2)}=c$, $(EA)^{(2)}$, $L^{(2)}=\dfrac{L}{\cos\beta}=\dfrac{L}{c}$, $(u_{X1}^{(2)}=U_3=0)$, $(v_{Y1}^{(2)}=U_4=0)$, $(u_{X2}^{(2)}=U_1=0)$, $\left(v_{Y2}^{(2)}=U_2=\dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right)$, then

$$\{\bar{\boldsymbol{f}}^{(2)}\} = \begin{Bmatrix} \bar{f}_{x1}^{(2)} \\ \bar{f}_{y1}^{(2)} \\ \bar{f}_{x2}^{(2)} \\ \bar{f}_{y2}^{(2)} \end{Bmatrix} = \frac{c(EA)^{(2)}}{L} \begin{Bmatrix} -\left(\dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right)c \\ 0 \\ \left(\dfrac{LP}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right)c \\ 0 \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right) \\ 0 \\ \left(\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}\right) \\ 0 \end{Bmatrix}$$

$$= -\{\boldsymbol{f}_{int}^{(2)}\}$$

and

$$
\{\boldsymbol{r}_{Global}^{(2)}\} = \left\{ \begin{array}{c} \bar{f}_{x1}\ell \\ \bar{f}_{x1}m \\ \hline \bar{f}_{x2}\ell \\ \bar{f}_{x2}m \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}\ell \\ \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}m \\ \hline \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}\ell \\ \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}m \end{array} \right\} = \begin{array}{c} \\ \left\{ \begin{array}{c} \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}s \\ \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}c \\ \hline \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}s \\ \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}c \end{array} \right\} \end{array} \begin{array}{l} \mathit{Global} \\ 3 \\ 4 \\ 1 \\ 2 \end{array}
$$

Element 3: $\ell^{(3)} = -s$, $m^{(2)} = c$, $(EA)^{(2)}$, $L^{(3)} = \dfrac{L}{\cos\beta} = \dfrac{L}{c}$, $(u_{X1}^{(3)} = U_3 = 0)$, $(v_{Y1}^{(3)} = U_4 = 0)$, $(u_{X2}^{(3)} = U_1 = 0)$, $\left(v_{Y2}^{(3)} = U_2 = \dfrac{LP}{(EA)^{(1)} + 2(EA)^{(2)} c^3} \right)$, then

$$
\{\bar{\boldsymbol{f}}^{(3)}\} = \left\{ \begin{array}{c} \bar{f}_{x1}^{(3)} \\ \bar{f}_{y1}^{(3)} \\ \hline \bar{f}_{x2}^{(3)} \\ \bar{f}_{y2}^{(3)} \end{array} \right\} = \frac{c(EA)^{(2)}}{L} \left\{ \begin{array}{c} -\left(\dfrac{LP}{(EA)^{(1)} + 2(EA)^{(2)} c^3} \right) c \\ 0 \\ \hline \left(\dfrac{LP}{(EA)^{(1)} + 2(EA)^{(2)} c^3} \right) c \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} \left(\dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3} \right) \\ 0 \\ \hline \left(\dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3} \right) \\ 0 \end{array} \right\}
$$

$$
= -\{\boldsymbol{f}_{\boldsymbol{int}}^{(3)}\}
$$

and

$$
\{\boldsymbol{r}_{Global}^{(3)}\} = \left\{ \begin{array}{c} f_{x1}\ell \\ f_{x1}m \\ \hline f_{x2}\ell \\ f_{x2}m \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}\ell \\ \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}m \\ \hline \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}\ell \\ \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}m \end{array} \right\} = \left\{ \begin{array}{c} \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}s \\ \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}c \\ \hline \dfrac{-c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}s \\ \dfrac{c^2 (EA)^{(2)} P}{(EA)^{(1)} + 2(EA)^{(2)} c^3}c \end{array} \right\} \begin{array}{l} \mathit{Global} \\ 7 \\ 8 \\ 1 \\ 2 \end{array}
$$

The Global reaction vector can be obtained by the contribution of each element:

$$
\mathop{\mathbf{A}}_{e=1}^{3}\{\boldsymbol{r}_{Global}^{(e)}\} = \begin{array}{c} \\ \left\{ \begin{array}{c}
0+\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s+\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}+\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c+\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c \\ \hdashline
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c \\ \hdashline
0 \\
\dfrac{-(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3} \\ \hdashline
\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c
\end{array} \right\} \end{array}
\begin{array}{c} \mathit{Global} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \end{array}
$$

which results

$$
\{\boldsymbol{R}\} = \mathop{\mathbf{A}}_{e=1}^{3}\{\boldsymbol{r}_{Global}^{(e)}\} - \{\boldsymbol{F}_0\} = \left\{ \begin{array}{c}
0 \\
P \\ \hdashline
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c \\ \hdashline
0 \\
\dfrac{-(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}+ \\ \hdashline
\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c
\end{array} \right\} - \left\{ \begin{array}{c} 0 \\ P \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c}
0 \\
0 \\ \hdashline
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c \\ \hdashline
0 \\
\dfrac{-(EA)^{(1)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}+ \\ \hdashline
\dfrac{c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}s \\
\dfrac{-c^2(EA)^{(2)}P}{(EA)^{(1)}+2(EA)^{(2)}c^3}c
\end{array} \right\}
$$

Problem 6.72

Consider the 2D Framed structure described in Figure 6.206, where $L = 5m$ and the mechanical - geometrical properties for the beams are $EI_z = 2\times10^5 kNm^2$ and $EA = 10^7 kN$. Obtain the displacements of the node 1 and 2 according to Y-direction when $q_Y = -20kN/m$.

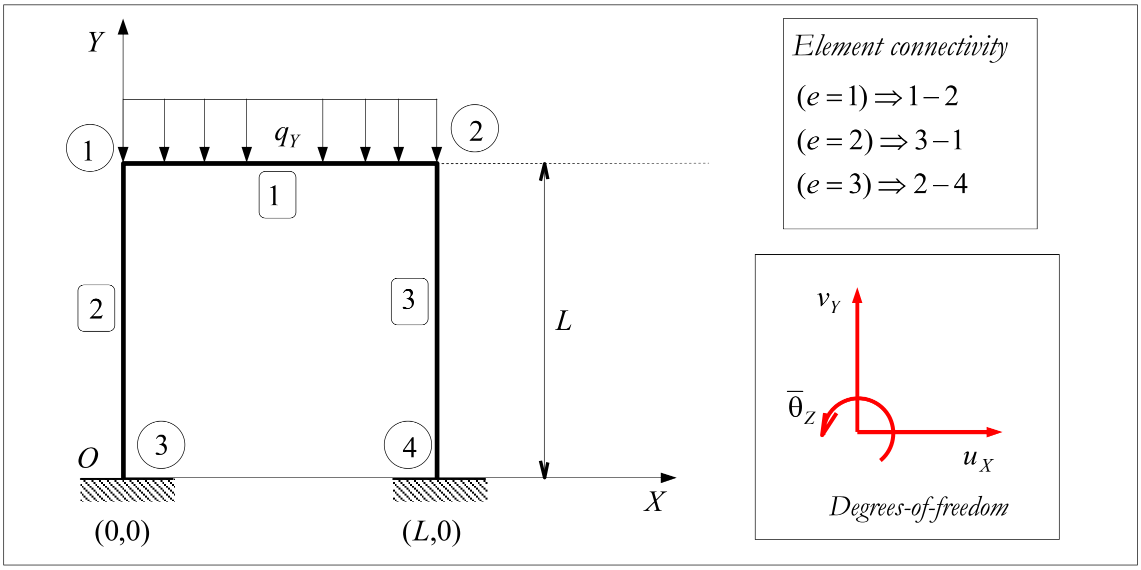


Figure 6.206: 2D Framed structure.

Solution:

By considering 2D framed we have three degrees-of-freedom per node, so the Global displacement vector for the structure has 12 degrees-of-freedom (Number of nodes (4) times Number of degrees-of-freedom per node (3)), (see Figure 6.207).

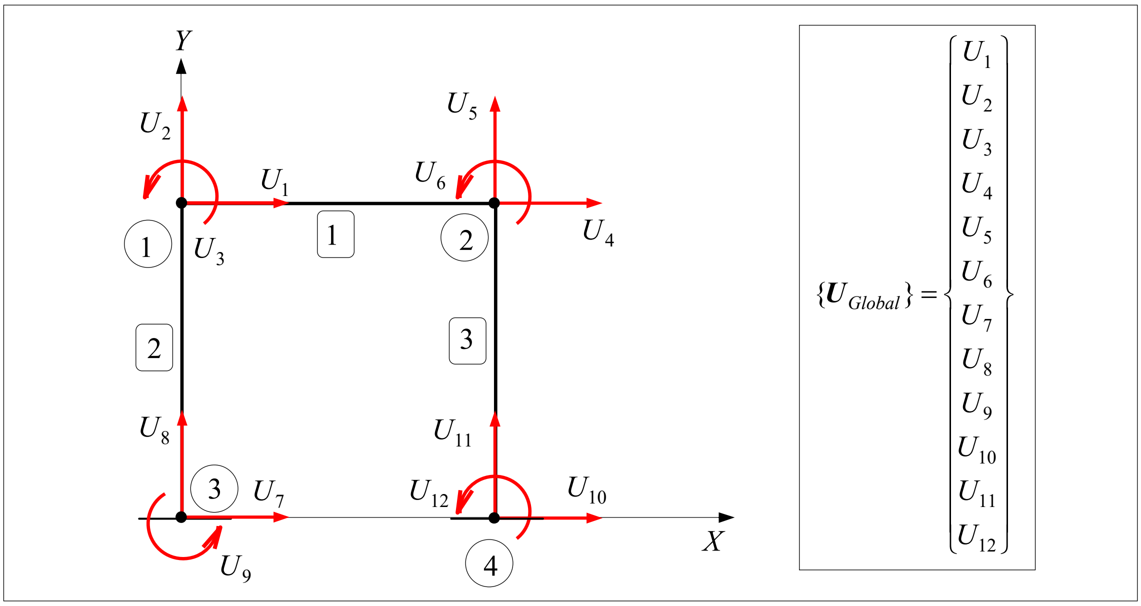


Figure 6.207: Degrees-of-freedom – Global system.

Next we will construct the system $\{\boldsymbol{F}_{Global}\} = [\boldsymbol{K}_{Global}]\{\boldsymbol{U}_{Global}\}$ where $[\boldsymbol{K}_{Global}]$ is the Global stiffness matrix of the structure, and $\{\boldsymbol{F}_{Global}\}$ is the Global nodal force vector.

Construction of the Global Stiffness Matrix - $[\boldsymbol{K}_{Global}]$

The matrix $[\boldsymbol{K}_{Global}]$ can be constructed by assembling the individual beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{12\times 12} = \mathop{\mathbf{A}}_{e=1}^{3} [\boldsymbol{k}^{(e)}_{Global}] \tag{6.640}$$

The stiffness matrix for the 2D frame element can be obtained by using the equation in (6.628), i.e.:

$$[\boldsymbol{k}^{(e)}_{Global}] = \left[\begin{array}{ccc|ccc} a\ell^2 + bm^2 & a\ell m - b\ell m & -cm & -a\ell^2 - bm^2 & -a\ell m + b\ell m & -cm \\ a\ell m - b\ell m & b\ell^2 + am^2 & c\ell & -a\ell m + b\ell m & -b\ell^2 - am^2 & c\ell \\ -cm & c\ell & 2d & cm & -c\ell & d \\ \hline -a\ell^2 - bm^2 & -a\ell m + b\ell m & cm & a\ell^2 + bm^2 & a\ell m - b\ell m & cm \\ -a\ell m + b\ell m & -b\ell^2 - am^2 & -c\ell & a\ell m - b\ell m & b\ell^2 + am^2 & -c\ell \\ -cm & c\ell & d & cm & -c\ell & 2d \end{array}\right] \tag{6.641}$$

For each element we have

Element	Connectivity : N1 $\rightarrow$ N2 $\text{N1} \rightarrow \text{N2}\begin{cases}\text{Node1}:(X_1,Y_1)\\ \text{Node2}:(X_2,Y_2)\end{cases}$	$L^{(e)}$	$\ell^{(e)} = \dfrac{X_2 - X_1}{L^{(e)}}$	$m^{(e)} = \dfrac{Y_2 - Y_1}{L^{(e)}}$
1	$1 \rightarrow 2\begin{cases}1:(0,L)\\ 2:(L,L)\end{cases}$	L	$\ell^{(1)} = 1$	$m^{(1)} = 0$
2	$3 \rightarrow 1\begin{cases}3:(0,0)\\ 1:(0,L)\end{cases}$	L	$\ell^{(2)} = 0$	$m^{(2)} = 1$
3	$2 \rightarrow 4\begin{cases}2:(L,L)\\ 4:(L,0)\end{cases}$	L	$\ell^{(3)} = 0$	$m^{(3)} = -1$

Element 1:

$$[\boldsymbol{k}^{(1)}_{Global}] = 10^6 \times \begin{array}{c} \begin{array}{cccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} \end{array} \; \mathit{Global} \\ \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.0192 & 0.048 & 0 & -0.0192 & 0.048 \\ 0 & 0.048 & 0.16 & 0 & -0.048 & 0.08 \\ \hline -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.0192 & -0.048 & 0 & 0.0192 & -0.048 \\ 0 & 0.048 & 0.08 & 0 & -0.048 & 0.16 \end{array}\right] \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array} \end{array}$$

Element 2:

$$[\boldsymbol{k}^{(2)}_{Global}] = 10^6 \times \begin{array}{c} \begin{array}{cccccc} \mathit{7} & \mathit{8} & \mathit{9} & \mathit{1} & \mathit{2} & \mathit{3} \end{array} \; \mathit{Global} \\ \left[\begin{array}{ccc|ccc} 0.0192 & 0 & -0.048 & -0.0192 & 0 & -0.048 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ -0.048 & 0 & 0.16 & 0.048 & 0 & 0.08 \\ \hline -0.0192 & 0 & 0.048 & 0.0192 & 0 & 0.048 \\ 0 & -2 & -0.048 & 0 & 2 & 0 \\ -0.048 & 0 & 0.08 & 0.048 & 0 & 0.16 \end{array}\right] \begin{array}{c} \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{array} \end{array}$$

Element 3

$$
\begin{array}{c}
\begin{array}{cccccc c} 4 & 5 & 6 & 10 & 11 & 12 & Global \end{array}\\
[\boldsymbol{k}^{(3)}_{Global}] = 10^6 \times \left[\begin{array}{ccc|ccc} 0.0192 & 0 & 0.048 & -0.0192 & 0 & 0.048 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0.048 & 0 & 0.16 & -0.048 & 0 & 0.08 \\ \hline -0.0192 & 0 & -0.048 & 0.0192 & 0 & -0.048 \\ 0 & -2 & -0.048 & 0 & 2 & 0 \\ 0.048 & 0 & 0.08 & -0.048 & 0 & 0.16 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
$$

Once we have assembled the contribution of all elements the stiffness matrix

$$
[\boldsymbol{K}_{Global}] = \mathop{\mathbf{A}}_{e=1}^{3} [\boldsymbol{k}^{(e)}_{Global}]
$$

we will obtain the stiffness of the structure.

Construction of the Global Nodal Force Vector - $\{\boldsymbol{F}_{Global}\}$, (see equation (6.632))

$$
\{\boldsymbol{F}_{Global}\} = \left(\mathop{\mathbf{A}}_{e=1}^{3} \{\boldsymbol{f}^{(e)}_{Eq\_G}\} \right) + \underbrace{\{\boldsymbol{F}_0\}}_{=\{\boldsymbol{0}\}}
$$

Element 1:

$$
\{\boldsymbol{f}^{(1)}_{Eq\_G}\} = \frac{1}{12} \left\{ \begin{array}{c} 6L[q_x \ell + q_y m] \\ 6L[q_x m + q_y \ell] \\ q_y L^2 \\ \hline 6L[q_x \ell + q_y m] \\ 6L[q_x m + q_y \ell] \\ -q_y L^2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -50 \\ -41.666667 \\ \hline 0 \\ -50 \\ 41.666667 \end{array} \right\} \begin{array}{c} Global \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

Element 2 and Element 3 $\{\boldsymbol{f}^{(2)}_{Eq\_G}\} = \{\boldsymbol{f}^{(3)}_{Eq\_G}\} = \{\boldsymbol{0}\}$.

Then,

$$
\{\boldsymbol{F}_{Global}\}_{12\times 1} = \left\{ \begin{array}{c} 0 \\ -50 \\ -41.666667 \\ \hline 0 \\ -50 \\ 41.666667 \\ \hline \{\boldsymbol{0}\}_{6\times 1} \end{array} \right\}
$$

Applying the Boundary Conditions

Note that the nodes 3 and 4 cannot move, so $U_7 = U_8 = U_9 = 0$, $U_{10} = U_{11} = U_{12} = 0$ and after applying these boundary conditions the global stiffness matrix will have the following aspect:

$$[\overline{\boldsymbol{K}}_{Global}]_{12\times12} = \left[\begin{array}{ccc:ccc:ccc:ccc} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{13} & K_{23} & K_{33} & K_{34} & K_{35} & K_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline K_{14} & K_{24} & K_{34} & K_{44} & K_{45} & K_{46} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{15} & K_{25} & K_{35} & K_{45} & K_{55} & K_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{16} & K_{26} & K_{36} & K_{46} & K_{56} & K_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

Then, for the sake of simplicity we will only express here the sub-matrix 6×6, which is

$$[\overline{\overline{\boldsymbol{K}}}_{Global}]_{6\times6} = \begin{bmatrix} k_{11}^{(1)}+k_{44}^{(2)} & k_{12}^{(1)}+k_{45}^{(2)} & k_{13}^{(1)}+k_{46}^{(2)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} \\ k_{21}^{(1)}+k_{54}^{(2)} & k_{22}^{(1)}+k_{55}^{(2)} & k_{23}^{(1)}+k_{56}^{(2)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} \\ k_{31}^{(1)}+k_{64}^{(2)} & k_{32}^{(1)}+k_{65}^{(2)} & k_{33}^{(1)}+k_{66}^{(2)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)}+k_{11}^{(3)} & k_{45}^{(1)}+k_{12}^{(3)} & k_{46}^{(1)}+k_{13}^{(3)} \\ k_{51}^{(1)} & k_{52}^{(1)} & k_{53}^{(1)} & k_{54}^{(1)}+k_{21}^{(3)} & k_{55}^{(1)}+k_{22}^{(3)} & k_{56}^{(1)}+k_{23}^{(3)} \\ k_{61}^{(1)} & k_{62}^{(1)} & k_{63}^{(1)} & k_{64}^{(1)}+k_{31}^{(3)} & k_{65}^{(1)}+k_{32}^{(3)} & k_{66}^{(1)}+k_{33}^{(3)} \end{bmatrix}$$

which results in

$$[\overline{\overline{\boldsymbol{K}}}_{Global}]_{6\times6} = \begin{bmatrix} 2.0192 & 0 & 0.048 & -2 & 0 & 0 \\ 0 & 2.0192 & 0.048 & 0 & -0.0192 & 0.048 \\ 0.048 & 0.048 & 0.32 & 0 & -0.048 & 0.08 \\ -2 & 0 & 0 & 2.0192 & 0 & 0.048 \\ 0 & -0.0192 & -0.048 & 0 & 2.0192 & -0.048 \\ 0 & 0.048 & 0.08 & 0.048 & -0.048 & 0.32 \end{bmatrix} \times 10^6$$

Solving the System

Then, we have to solve the system

$$\begin{bmatrix} 2.0192 & 0 & 0.048 & -2 & 0 & 0 \\ 0 & 2.0192 & 0.048 & 0 & -0.0192 & 0.048 \\ 0.048 & 0.048 & 0.32 & 0 & -0.048 & 0.08 \\ -2 & 0 & 0 & 2.0192 & 0 & 0.048 \\ 0 & -0.0192 & -0.048 & 0 & 2.0192 & -0.048 \\ 0 & 0.048 & 0.08 & 0.048 & -0.048 & 0.32 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \left\{\begin{array}{c} 0 \\ -50 \\ -41.666667 \\ \hdashline 0 \\ -50 \\ 41.666667 \end{array}\right\}$$

and the solution is

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 2.078345 \\ -25 \\ -174.02678 \\ \hdashline -2.078345 \\ -25 \\ 174.02678 \end{Bmatrix} \times 10^{-6}$$

Internal and Reaction Forces

The internal and reaction forces can be obtained by $\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\}$, (see equation (6.635)):

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix} = -\begin{Bmatrix} \frac{q_x L}{2} \\ \frac{q_y L}{2} \\ \frac{q_y L^2}{12} \\ \hdashline \frac{q_x L}{2} \\ \frac{q_y L}{2} \\ \frac{-q_y L^2}{12} \end{Bmatrix} + \begin{Bmatrix} a\ell(u_{X1} - u_{X2}) + am(v_{Y1} - v_{Y2}) \\ b\ell(v_{Y1} - v_{Y2}) + bm(u_{X2} - u_{X1}) + c(\overline{\theta}_{Z1} + \overline{\theta}_{Z2}) \\ c\ell(v_{Y1} - v_{Y2}) + cm(u_{X2} - u_{X1}) + d(2\overline{\theta}_{Z1} + \overline{\theta}_{Z2}) \\ \hdashline -a\ell(u_{X1} - u_{X2}) - am(v_{Y1} - v_{Y2}) \\ -b\ell(v_{Y1} - v_{Y2}) - bm(u_{X2} - u_{X1}) - c(\overline{\theta}_{Z1} + \overline{\theta}_{Z2}) \\ c\ell(v_{Y1} - v_{Y2}) + cm(u_{X2} - u_{X1}) + d(\overline{\theta}_{Z1} + 2\overline{\theta}_{Z2}) \end{Bmatrix}$$

and the reaction vector in the Global system $\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}^{(e)}_{Local}\}$ becomes:

$$\{\boldsymbol{r}^{(e)}_{Global}\} = \left[\begin{array}{ccc:ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^T \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix} = \begin{Bmatrix} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m^{(int)}_{z1} \\ \hdashline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{Bmatrix}$$

Element 1:

Internal force (Local system)

$$\{\boldsymbol{r}^{(1)}_{Local}\} = -\{\boldsymbol{f}^{(1)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix} = -\begin{Bmatrix} 0 \\ -50 \\ -41.666667 \\ \hdashline 0 \\ -50 \\ 41.666667 \end{Bmatrix} + \begin{Bmatrix} 8.31338122 \\ 0 \\ -13.922142 \\ \hdashline -8.31338122 \\ 0 \\ 13.922142 \end{Bmatrix} = \begin{Bmatrix} 8.31338122 \\ 50 \\ 27.74452425 \\ \hdashline -8.31338122 \\ 50 \\ -27.74452425 \end{Bmatrix}$$

and in the global system becomes:

$$
\{\boldsymbol{r}^{(1)}_{Global}\} = \left[\begin{array}{ccc:ccc} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^T \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hdashline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} 8.31338122 \\ 50 \\ 27.74452425 \\ \hdashline -8.31338122 \\ 50 \\ -27.74452425 \end{array}\right\} \begin{array}{l} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

Element 2:

Internal force (Local system)

$$
\{\boldsymbol{r}^{(2)}_{Local}\} = -\{\boldsymbol{f}^{(2)}_{int}\} = \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = -\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \end{array}\right\} + \left\{\begin{array}{c} 50 \\ -8.313812 \\ -13.8223818 \\ \hdashline -50 \\ 8.3133812 \\ -27.744524 \end{array}\right\} = \left\{\begin{array}{c} 50 \\ -8.313812 \\ -13.8223818 \\ \hdashline -50 \\ 8.3133812 \\ -27.744524 \end{array}\right\}
$$

and in the global system becomes:

$$
\{\boldsymbol{r}^{(2)}_{Global}\} = \left\{\begin{array}{c} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hdashline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} 8.31338122 \\ 50 \\ -13.8223818 \\ \hdashline -8.31338122 \\ -50 \\ -27.74452425 \end{array}\right\} \begin{array}{l} \textit{Global} \\ 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array}
$$

Element 3:

Internal force (Local system)

$$
\{\boldsymbol{r}^{(3)}_{Local}\} = -\{\boldsymbol{f}^{(3)}_{int}\} = \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hdashline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = -\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \end{array}\right\} + \left\{\begin{array}{c} 50 \\ 8.313812 \\ 27.744524 \\ \hdashline -50 \\ -8.3133812 \\ 13.8223818 \end{array}\right\} = \left\{\begin{array}{c} 50 \\ 8.313812 \\ 27.744524 \\ \hdashline -50 \\ -8.3133812 \\ 13.8223818 \end{array}\right\}
$$

and in the global system becomes:

$$
\{\boldsymbol{r}^{(3)}_{Global}\} = \left\{\begin{array}{c} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hdashline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} 8.31338122 \\ -50 \\ 27.74452425 \\ \hdashline -8.31338122 \\ 50 \\ 13.8223818 \end{array}\right\} \begin{array}{l} \textit{Global} \\ 4 \\ 5 \\ 6 \\ 10 \\ 11 \\ 12 \end{array}
$$

The Global reaction vector, (see Figure 6.208), becomes:

$$
\{\boldsymbol{R}\} = \mathop{\mathbf{A}}_{e=1}^{3} \{\boldsymbol{r}^{(e)}_{Global}\} = \left\{\begin{array}{c}
8.31338122 + (-8.31338122) \\
50 + (-50) \\
27.74452425 + (-27.74452425) \\ \hdashline
-8.31338122 + 8.31338122 \\
50 + (-50) \\
-27.74452425 + 27.74452425 \\ \hdashline
8.31338122 \\
50 \\
-13.8223818 \\ \hdashline
-8.31338122 \\
50 \\
13.8223818
\end{array}\right\} = \left\{\begin{array}{c}
0 \\
0 \\
0 \\ \hdashline
0 \\
0 \\
0 \\ \hdashline
8.31338122 \\
50 \\
-13.8223818 \\ \hdashline
-8.31338122 \\
50 \\
13.8223818
\end{array}\right\}\begin{array}{l}
\scriptstyle Global \\
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12
\end{array}
$$

and

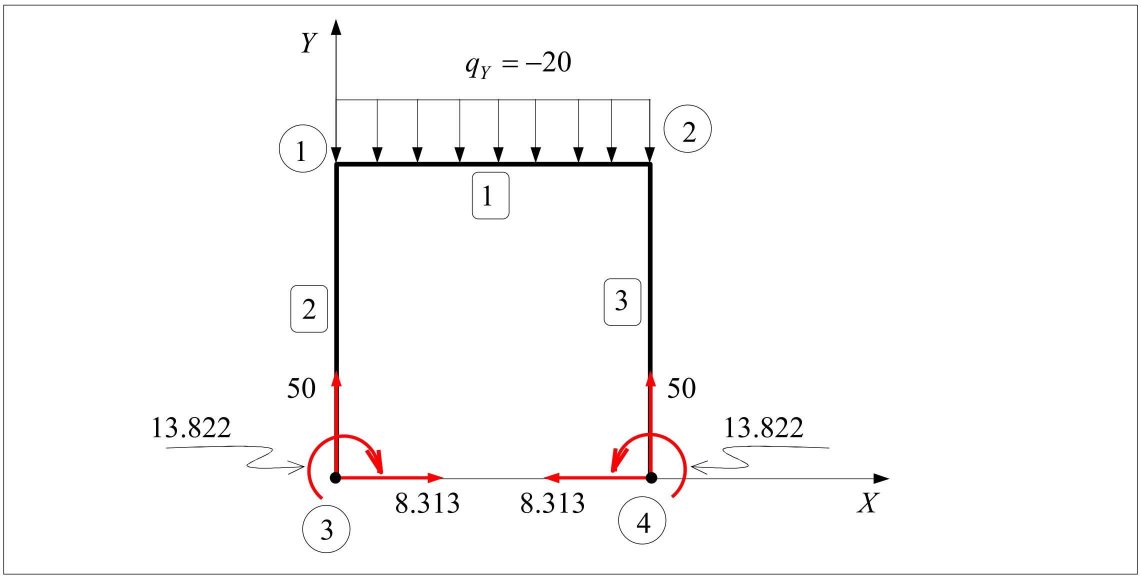


Figure 6.208: Reactions.

Problem 6.73

Consider the 2D Framed structure described in Figure 6.209, where the mechanical - geometrical properties for the beams are $EI_z = 10^4$ and $EA = 3\times10^4$. Obtain the displacements of the node 1 (rotation) and 2.

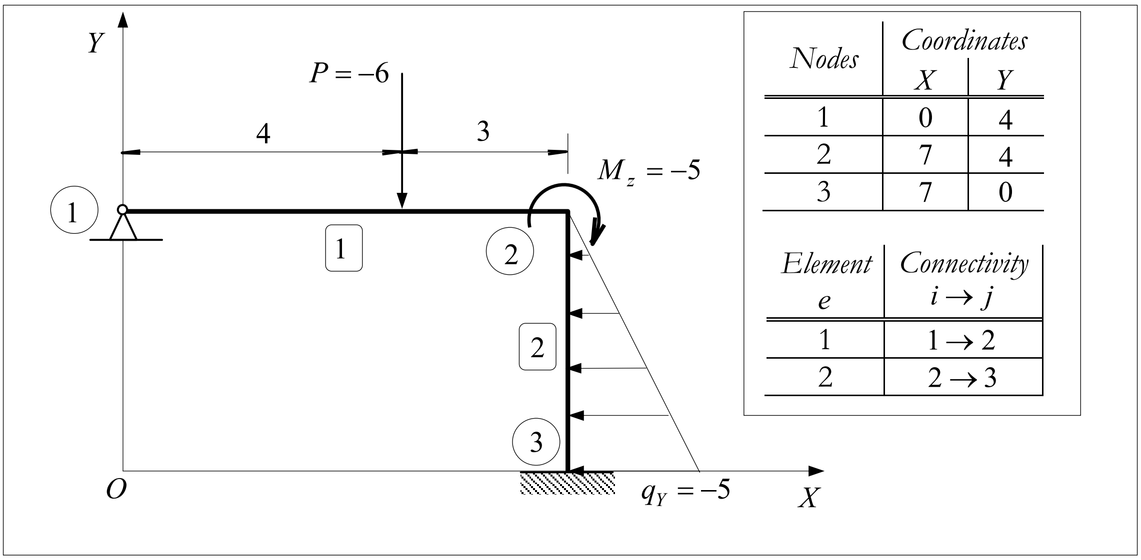


Figure 6.209: 2D Framed structure.

Solution:

By considering 2D framed we have three degrees-of freedom per node, so the Global displacement vector for the structure has 12 degrees-of-freedom (Number of nodes (4) times Number of degrees-of-freedom per node (3)), (see Figure 6.210).

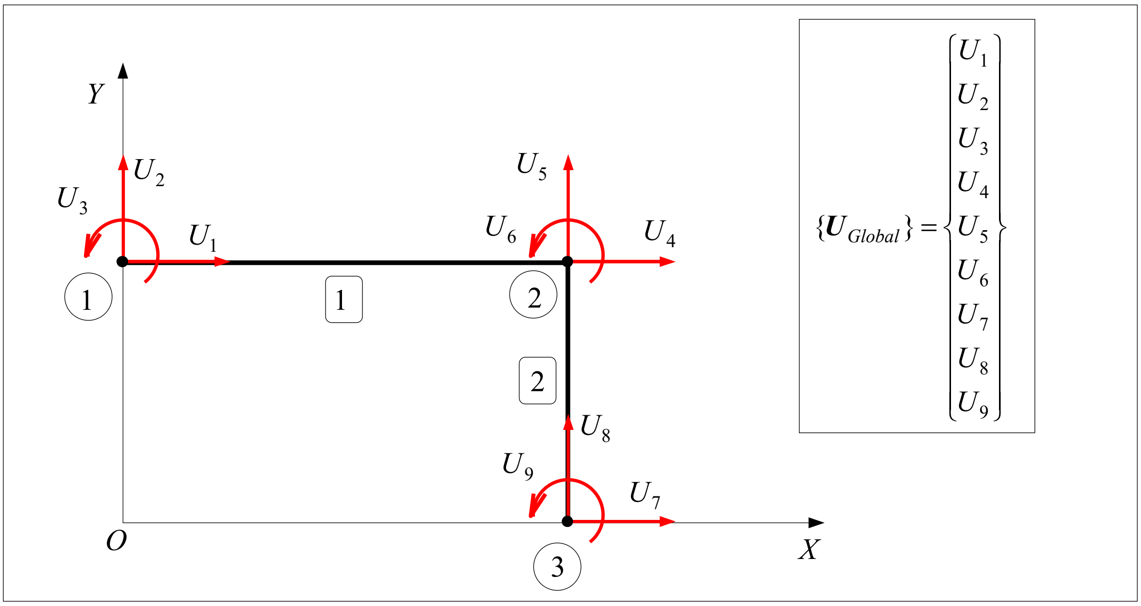


Figure 6.210: Degrees-of-freedom – Global system.

Next we will construct the system $\{\boldsymbol{F}_{Global}\} = [\boldsymbol{K}_{Global}]\{\boldsymbol{U}_{Global}\}$ where $[\boldsymbol{K}_{Global}]$ is the Global stiffness matrix of the structure, and $\{\boldsymbol{F}_{Global}\}$ is the Global nodal force vector.

Construction of the Global Stiffness Matrix - $[\boldsymbol{K}_{Global}]$

The matrix $[\boldsymbol{K}_{Global}]$ can be constructed by assembling the beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{9\times9} = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}] \tag{6.642}$$

The stiffness matrix for the 2D framed element can be obtained by using the equation in (6.628). For each element we have

Element	Connectivity : N1 → N2 $N1 \to N2\begin{cases}\text{Node1}:(X_1,Y_1)\\ \text{Node2}:(X_2,Y_2)\end{cases}$	$L^{(e)}$	$\ell^{(e)} = \dfrac{X_2-X_1}{L^{(e)}}$	$m^{(e)} = \dfrac{Y_2-Y_1}{L^{(e)}}$
1	$1\to 2\begin{cases}1:(0,4)\\ 2:(7,4)\end{cases}$	7	$\ell^{(1)}=1$	$m^{(1)}=0$
2	$3\to 12\begin{cases}3:(7,4)\\ 1:(7,0)\end{cases}$	4	$\ell^{(2)}=0$	$m^{(2)}=-1$

Element 1:

$$[\boldsymbol{k}^{(1)}_{Global}] = 10^3\times \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} \end{matrix} \\ \left[\begin{array}{ccc|ccc} 4.28571 & 0 & 0 & -4.28571 & 0 & 0 \\ 0 & 0.34985 & 1.22449 & 0 & -0.34985 & 1.22449 \\ 0 & 1.22449 & 5.71429 & 0 & -1.22449 & 2.85714 \\ \hline -4.28571 & 0 & 0 & 4.28571 & 0 & 0 \\ 0 & -0.34985 & -1.22449 & 0 & 0.34985 & -1.22449 \\ 0 & 1.22449 & 2.85714 & 0 & -1.22449 & 5.71429 \end{array}\right] \end{array} \begin{array}{c} \mathit{Global} \\ \mathit{1}\\ \mathit{2}\\ \mathit{3}\\ \mathit{4}\\ \mathit{5}\\ \mathit{6} \end{array}$$

Element 2:

$$[\boldsymbol{k}^{(2)}_{Global}] = k^{(2)}_{ij} = 10^3\times \begin{array}{c} \begin{matrix} \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} \end{matrix} \\ \left[\begin{array}{ccc|ccc} 1.875 & 0 & 3.75 & -1.875 & 0 & 3.75 \\ 0 & 7.5 & 0 & 0 & -7.5 & 0 \\ 3.75 & 0 & 10 & -3.75 & 0 & 5 \\ \hline -1.875 & 0 & -3.75 & 1.875 & 0 & -3.75 \\ 0 & -7.5 & 0 & 0 & 7.5 & 0 \\ 3.75 & 0 & 5 & -3.75 & 0 & 10 \end{array}\right] \end{array} \begin{array}{c} \mathit{Global} \\ \mathit{4}\\ \mathit{5}\\ \mathit{6}\\ \mathit{7}\\ \mathit{8}\\ \mathit{9} \end{array}$$

After the contribution of all elements we have assembled we can obtain

$$[\boldsymbol{K}_{Global}] = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}]$$

$$[\boldsymbol{K}_{Global}] = \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} \end{matrix} \\ \begin{bmatrix} k^{(1)}_{11} & k^{(1)}_{12} & k^{(1)}_{13} & k^{(1)}_{14} & k^{(1)}_{15} & k^{(1)}_{16} & 0 & 0 & 0 \\ k^{(1)}_{21} & k^{(1)}_{22} & k^{(1)}_{23} & k^{(1)}_{24} & k^{(1)}_{25} & k^{(1)}_{26} & 0 & 0 & 0 \\ k^{(1)}_{31} & k^{(1)}_{32} & k^{(1)}_{33} & k^{(1)}_{34} & k^{(1)}_{35} & k^{(1)}_{36} & 0 & 0 & 0 \\ k^{(1)}_{41} & k^{(1)}_{42} & k^{(1)}_{43} & k^{(1)}_{44}+k^{(2)}_{11} & k^{(1)}_{45}+k^{(2)}_{12} & k^{(1)}_{46}+k^{(2)}_{13} & k^{(2)}_{14} & k^{(2)}_{15} & k^{(2)}_{16} \\ k^{(1)}_{51} & k^{(1)}_{52} & k^{(1)}_{53} & k^{(1)}_{54}+k^{(2)}_{21} & k^{(1)}_{55}+k^{(2)}_{22} & k^{(1)}_{56}+k^{(2)}_{23} & k^{(2)}_{24} & k^{(2)}_{25} & k^{(2)}_{26} \\ k^{(1)}_{61} & k^{(1)}_{62} & k^{(1)}_{63} & k^{(1)}_{64}+k^{(2)}_{31} & k^{(1)}_{65}+k^{(2)}_{32} & k^{(1)}_{66}+k^{(2)}_{33} & k^{(2)}_{34} & k^{(2)}_{35} & k^{(2)}_{36} \\ 0 & 0 & 0 & k^{(2)}_{41} & k^{(2)}_{42} & k^{(2)}_{43} & k^{(2)}_{44} & k^{(2)}_{45} & k^{(2)}_{46} \\ 0 & 0 & 0 & k^{(2)}_{51} & k^{(2)}_{52} & k^{(2)}_{53} & k^{(2)}_{54} & k^{(2)}_{55} & k^{(2)}_{56} \\ 0 & 0 & 0 & k^{(2)}_{61} & k^{(2)}_{62} & k^{(2)}_{63} & k^{(2)}_{64} & k^{(2)}_{65} & k^{(2)}_{66} \end{bmatrix} \end{array} \begin{array}{c} \mathit{Global} \\ \mathit{1}\\ \mathit{2}\\ \mathit{3}\\ \mathit{4}\\ \mathit{5}\\ \mathit{6}\\ \mathit{7}\\ \mathit{8}\\ \mathit{9} \end{array}$$

By substituting the numerical values we can obtain:

$$[\boldsymbol{K}_{Global}] = \begin{bmatrix} 4.286 & 0 & 0 & -4.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.35 & 1.224 & 0 & -0.35 & 1.224 & 0 & 0 & 0 \\ 0 & 1.224 & 5.714 & 0 & -1.224 & 2.875 & 0 & 0 & 0 \\ -4.286 & 0 & 0 & 6.161 & 0 & 3.75 & -1.875 & 0 & 3.75 \\ 0 & -0.35 & -1.224 & 0 & 7.85 & -1.224 & 0 & -7.5 & 0 \\ 0 & 1.224 & 2.875 & 3.75 & -1.224 & 15.714 & -3.75 & 0 & 5 \\ 0 & 0 & 0 & -1.875 & 0 & -3.75 & 1.875 & 0 & -3.75 \\ 0 & 0 & 0 & 0 & -7.5 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 3.75 & 0 & 5 & -3.75 & 0 & 10 \end{bmatrix} \times 10^3$$

Construction of the Global Nodal Force Vector - $\{\boldsymbol{F}_{Global}\}$, (see equation (6.632))

$$\{\boldsymbol{F}_{Global}\} = \left(\mathop{\mathbf{A}}_{e=1}^{2} \{\boldsymbol{f}^{(e)}_{Eq\_G}\} \right) + \{\boldsymbol{F}_0\}$$

Element 1: Concentrated force, (see **Problem 6.67** –NOTE), ($P=-6$, $a_P=4$, $L=7$):

$$\{\boldsymbol{f}^{(1)}_{Eq\_L}\} = \begin{Bmatrix} 0 \\ \left(\dfrac{2a_P^3}{L^3} - \dfrac{3a_P^2}{L^2} + 1\right)P \\ \left(\dfrac{a_P^3}{L^2} - \dfrac{2a_P^2}{L} + a_P\right)P \\ \hdashline 0 \\ \left(\dfrac{-2a_P^3}{L^3} + \dfrac{3a_P^2}{L^2}\right)P \\ \left(\dfrac{a_P^3}{L^2} - \dfrac{a_P^2}{L}\right)P \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.361516 \\ -4.408163 \\ \hdashline 0 \\ -3.638484 \\ 5.877551 \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix}$$

And by means of the equation $\{\boldsymbol{f}^{(1)}_{Eq\_G}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{f}^{(1)}_{Eq\_L}\}$ we can obtain the vector in the Global system, (see equation (6.631)):

$$\{\boldsymbol{f}^{(1)}_{Eq\_G}\} = \begin{Bmatrix} f_1^{(Eq\_G)\_(1)} \\ f_2^{(Eq\_G)\_(1)} \\ f_3^{(Eq\_G)\_(1)} \\ \hdashline f_4^{(Eq\_G)\_(1)} \\ f_5^{(Eq\_G)\_(1)} \\ f_6^{(Eq\_G)\_(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)}\ell - f_2^{(Eq\_L)}m \\ f_1^{(Eq\_L)}m + f_2^{(Eq\_L)}\ell \\ f_3^{(Eq\_L)} \\ \hdashline f_4^{(Eq\_L)}\ell - f_5^{(Eq\_L)}m \\ f_4^{(Eq\_L)}m + f_5^{(Eq\_L)}\ell \\ f_6^{(Eq\_L)} \end{Bmatrix} \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{l} \begin{Bmatrix} 0 \\ -2.361516 \\ -4.408163 \\ \hdashline 0 \\ -3.638484 \\ 5.877551 \end{Bmatrix} \end{array} \begin{array}{c} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Element 2: Linearly distributed load, (see **Problem 6.67** –NOTE), ($q_y^{(1)}=0$, $q_y^{(2)}=-5$, $L=4$):

$$\{\boldsymbol{f}^{(2)}_{Eq\_L}\} = \begin{Bmatrix} \dfrac{L}{20}(7q_y^{(1)}+3q_y^{(2)}) \\ \dfrac{L^2}{60}(3q_y^{(1)}+2q_y^{(2)}) \\ \dfrac{L}{20}(3q_y^{(1)}+7q_y^{(2)}) \\ \dfrac{-L^2}{60}(2q_y^{(1)}+3q_y^{(2)}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3 \\ -2.666667 \\ 0 \\ -7 \\ 4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)} \\ f_2^{(Eq\_L)} \\ f_3^{(Eq\_L)} \\ f_4^{(Eq\_L)} \\ f_5^{(Eq\_L)} \\ f_6^{(Eq\_L)} \end{Bmatrix}$$

And by means of the equation $\{\boldsymbol{f}^{(2)}_{Eq\_G}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{f}^{(2)}_{Eq\_L}\}$ we can obtain the vector in the Global system, (see equation (6.631)):

$$\{\boldsymbol{f}^{(2)}_{Eq\_G}\} = \begin{Bmatrix} f_1^{(Eq\_G)\_(2)} \\ f_2^{(Eq\_G)\_(2)} \\ f_3^{(Eq\_G)\_(2)} \\ f_4^{(Eq\_G)\_(2)} \\ f_5^{(Eq\_G)\_(2)} \\ f_6^{(Eq\_G)\_(2)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(Eq\_L)}\ell - f_2^{(Eq\_L)}m \\ f_1^{(Eq\_L)}m + f_2^{(Eq\_L)}\ell \\ f_3^{(Eq\_L)} \\ f_4^{(Eq\_L)}\ell - f_5^{(Eq\_L)}m \\ f_4^{(Eq\_L)}m + f_5^{(Eq\_L)}\ell \\ f_6^{(Eq\_L)} \end{Bmatrix} = \begin{array}{c} \\ \begin{Bmatrix} -3 \\ 0 \\ -2.666667 \\ -7 \\ 0 \\ 4 \end{Bmatrix} \end{array} \begin{array}{c} \textit{Global} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Then,

$$\mathop{\mathbf{A}}_{e=1}^{2}\{\boldsymbol{f}^{(e)}_{Eq\_G}\} = \begin{Bmatrix} f_1^{(Eq\_G)\_(1)} \\ f_2^{(Eq\_G)\_(1)} \\ f_3^{(Eq\_G)\_(1)} \\ f_4^{(Eq\_G)\_(1)} + f_1^{(Eq\_G)\_(2)} \\ f_5^{(Eq\_G)\_(1)} + f_2^{(Eq\_G)\_(2)} \\ f_6^{(Eq\_G)\_(1)} + f_3^{(Eq\_G)\_(2)} \\ f_4^{(Eq\_G)\_(2)} \\ f_5^{(Eq\_G)\_(2)} \\ f_6^{(Eq\_G)\_(2)} \end{Bmatrix} = \begin{array}{c} \\ \begin{Bmatrix} 0 \\ -2.361516 \\ -4.4081633 \\ -3 \\ -3.638484 \\ 3.210884 \\ -7 \\ 0 \\ 4 \end{Bmatrix} \end{array} \begin{array}{c} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

And

$$\{\boldsymbol{F}_{Global}\} = \left(\mathop{\mathbf{A}}_{e=1}^{2}\{\boldsymbol{f}^{(e)}_{Eq\_G}\} \right) + \{\boldsymbol{F}_0\} = \begin{Bmatrix} 0 \\ -2.361516 \\ -4.4081633 \\ -3 \\ -3.638484 \\ 3.210884 \\ -7 \\ 0 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.361516 \\ -4.4081633 \\ -3 \\ -3.638484 \\ -1.789116 \\ -7 \\ 0 \\ 4 \end{Bmatrix}$$

Applying the Boundary Conditions

Note that there are restrictions in motion for the following degrees-of-freedom: $U_1 = 0$, $U_2 = 0$, $U_7 = 0$, $U_8 = 0$, $U_9 = 0$ and after applying these boundary conditions the set of equation $\{\boldsymbol{F}_{Global}\} = [\boldsymbol{K}_{Global}]\{\boldsymbol{U}_{Global}\}$ will have the following aspect:

$$
10^3 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.714 & 0 & -1.224 & 2.875 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.161 & 0 & 3.75 & 0 & 0 & 0 \\ 0 & 0 & -1.224 & 0 & 7.85 & -1.224 & 0 & 0 & 0 \\ 0 & 0 & 2.875 & 3.75 & -1.224 & 15.714 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4.4081633 \\ -3 \\ -3.638484 \\ -1.789116 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Solving the System

$$
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -9.7643 \\ -5.8224 \\ -5.914 \\ 1.5654 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-4}
$$

Internal and Reaction Forces

The internal and reaction forces can be obtained by $\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\}$, (see equation (6.634)). And the reaction vector in the Global system $\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}^{(e)}_{Local}\}$, (see equation (6.636)).

Element 1:

Internal force (Local system)

$$
\{\boldsymbol{r}^{(1)}_{Local}\} = -\{\boldsymbol{f}^{(1)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hline f_{x2} \\ f_{y2} \\ m_{z2} \end{Bmatrix} = -\underbrace{\begin{Bmatrix} 0 \\ -2.361516 \\ -4.408163 \\ \hline 0 \\ -3.638484 \\ 5.877551 \end{Bmatrix}}_{=\{\boldsymbol{f}^{(1)}_{Eq\_L}\}} + \underbrace{\begin{Bmatrix} 2.495322 \\ -0.797039 \\ -4.408163 \\ \hline -2.495355 \\ 0.797039 \\ -1.1711112 \end{Bmatrix}}_{=\{\bar{\boldsymbol{f}}^{(1)}_{Local}\}} = \begin{Bmatrix} 2.495322 \\ 1.564477 \\ 0 \\ \hline -2.495355 \\ 4.435523 \\ -7.048662 \end{Bmatrix}
$$

and in the global system becomes:

$$
\{\boldsymbol{r}^{(1)}_{Global}\} = \begin{Bmatrix} r^{(1)}_{G\ 1} \\ r^{(1)}_{G\ 2} \\ r^{(1)}_{G\ 3} \\ \hline r^{(1)}_{G\ 4} \\ r^{(1)}_{G\ 5} \\ r^{(1)}_{G\ 6} \end{Bmatrix} = \begin{Bmatrix} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{Bmatrix} = \begin{Bmatrix} 2.495322 \\ 1.564477 \\ 0 \\ \hline -2.495355 \\ 4.435523 \\ -7.048662 \end{Bmatrix} \begin{array}{c} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

Element 2:

Internal force (Local system)

$$
\{\boldsymbol{r}^{(2)}_{Local}\} = -\{\boldsymbol{f}^{(2)}_{int}\} = \left\{\begin{array}{c} f_{x1} \\ f_{y1} \\ m_{z1} \\ \hline f_{x2} \\ f_{y2} \\ m_{z2} \end{array}\right\} = -\underbrace{\left\{\begin{array}{c} 0 \\ -3 \\ -2.666667 \\ \hline 0 \\ -7 \\ 4 \end{array}\right\}}_{=\{\boldsymbol{f}^{(2)}_{Eq\_L}\}} + \underbrace{\left\{\begin{array}{c} 4.4355232 \\ -0.5046776 \\ -0.6180044 \\ \hline -4.4355232 \\ 0.5046776 \\ -1.4007058 \end{array}\right\}}_{=\{\bar{\boldsymbol{f}}^{(2)}_{Local}\}} = \left\{\begin{array}{c} 4.4355232 \\ 2.495322 \\ 2.048662 \\ \hline -4.4355232 \\ 7.504678 \\ -5.400706 \end{array}\right\}
$$

and in the global system becomes:

$$
\{\boldsymbol{r}^{(2)}_{Global}\} = \left\{\begin{array}{c} r^{(2)}_{G\ 1} \\ r^{(2)}_{G\ 2} \\ r^{(2)}_{G\ 3} \\ \hline r^{(2)}_{G\ 4} \\ r^{(2)}_{G\ 5} \\ r^{(2)}_{G\ 6} \end{array}\right\} = \left\{\begin{array}{c} f_{x1}\ell - f_{y1}m \\ f_{x1}m + f_{y1}\ell \\ m_{z1} \\ \hline f_{x2}\ell - f_{y2}m \\ f_{x2}m + f_{y2}\ell \\ m_{z2} \end{array}\right\} = \left\{\begin{array}{c} 2.49532245 \\ -4.43552318 \\ 2.04866225 \\ \hline 7.50467755 \\ 4.43552318 \\ -5.40070578 \end{array}\right\} \begin{array}{c} \textit{Global} \\ 7 \\ 8 \\ 9 \\ 1 \\ 2 \\ 3 \end{array}
$$

The Global reaction vector becomes:

$$
\mathop{\mathbf{A}}_{e=1}^{2}\{\boldsymbol{r}^{(e)}_{Global}\} = \left\{\begin{array}{c} r^{(1)}_{G\ 1} \\ r^{(1)}_{G\ 2} \\ r^{(1)}_{G\ 3} \\ r^{(1)}_{G\ 4} + r^{(2)}_{G\ 1} \\ r^{(1)}_{G\ 5} + r^{(2)}_{G\ 2} \\ r^{(1)}_{G\ 6} + r^{(2)}_{G\ 3} \\ r^{(2)}_{G\ 4} \\ r^{(2)}_{G\ 5} \\ r^{(2)}_{G\ 6} \end{array}\right\} = \left\{\begin{array}{c} 2.4953224 \\ 1.5644768 \\ 0 \\ \hline 0 \\ 0 \\ -5 \\ \hline 7.5046776 \\ 4.4355232 \\ -5.4007058 \end{array}\right\}
$$

And

$$
\{\boldsymbol{R}\} = \mathop{\mathbf{A}}_{e=1}^{2}\{\boldsymbol{r}^{(e)}_{Global}\} - \{\boldsymbol{F}_0\} = \left\{\begin{array}{c} 2.4953224 \\ 1.5644768 \\ 0 \\ \hline 0 \\ 0 \\ -5 \\ \hline 7.5046776 \\ 4.4355232 \\ -5.4007058 \end{array}\right\} - \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ -5 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 2.4953224 \\ 1.5644768 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 7.5046776 \\ 4.4355232 \\ -5.4007058 \end{array}\right\} \begin{array}{c} \textit{Global} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

Problem 6.74

Obtain the explicit equation $[\boldsymbol{K}^{(e)}]\ \{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{F}^{(e)}\}$ for the beam presented in Figure 6.211, in which the beam is supported by an elastic foundation.

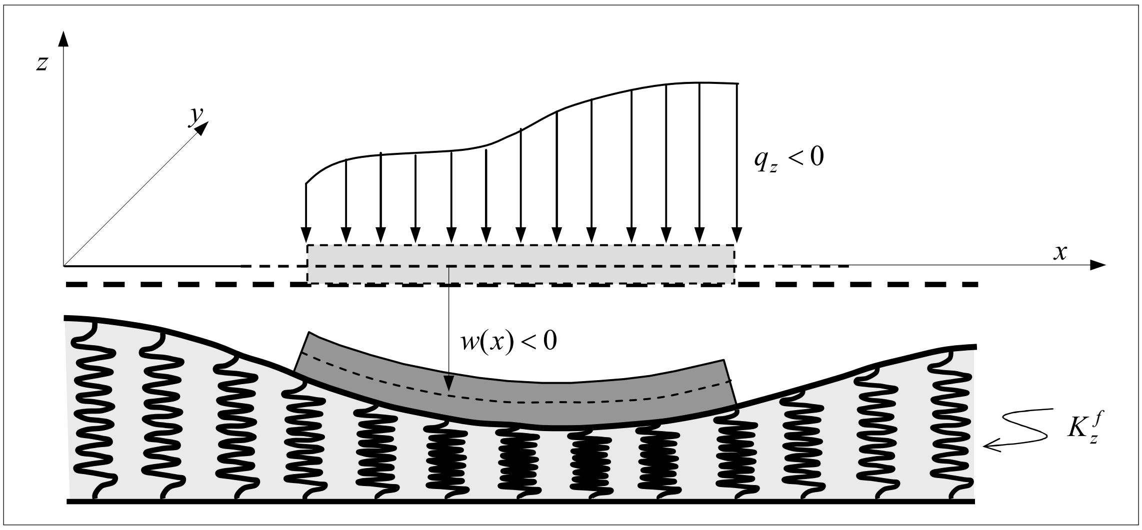


Figure 6.211: Beam element on elastic foundation.

Solution:

In this situation, the total potential energy is given by:

$$\Pi = U^{int} - U^{ext} = (U_{beam} + U_{spring}) - U^{ext} \tag{6.643}$$

According to the stationary principle of the total potential energy we can conclude:

$$\frac{d\Pi}{d\{\boldsymbol{u}^{(e)}\}} = 0 \quad \Rightarrow \quad \frac{d}{d\{\boldsymbol{u}^{(e)}\}}\left(U_{beam} + U_{spring} - U^{ext}\right) = 0 \Rightarrow \frac{dU_{beam}}{d\{\boldsymbol{u}^{(e)}\}} + \frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} - \frac{dU^{ext}}{d\{\boldsymbol{u}^{(e)}\}} = 0 \tag{6.644}$$

or

$$\frac{dU_{beam}}{d\{\boldsymbol{u}^{(e)}\}} + \frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} - \frac{dU^{ext}}{d\{\boldsymbol{u}^{(e)}\}} = 0 \quad \Rightarrow \quad [\boldsymbol{ke}^{(1)}]\{\boldsymbol{u}^{(e)}\} + \frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} = \{\boldsymbol{f}^{(e)}\} \tag{6.645}$$

where $[\boldsymbol{ke}^{(1)}]$ is the same matrix showed in **Problem 6.62**. Then, the term $\frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}}$ can be obtained as follows. The internal energy for the spring, (see Figure 6.212), is given by:

$$U_{spring} = \int_0^L \frac{1}{2} K_z^f w^2 dx \tag{6.646}$$

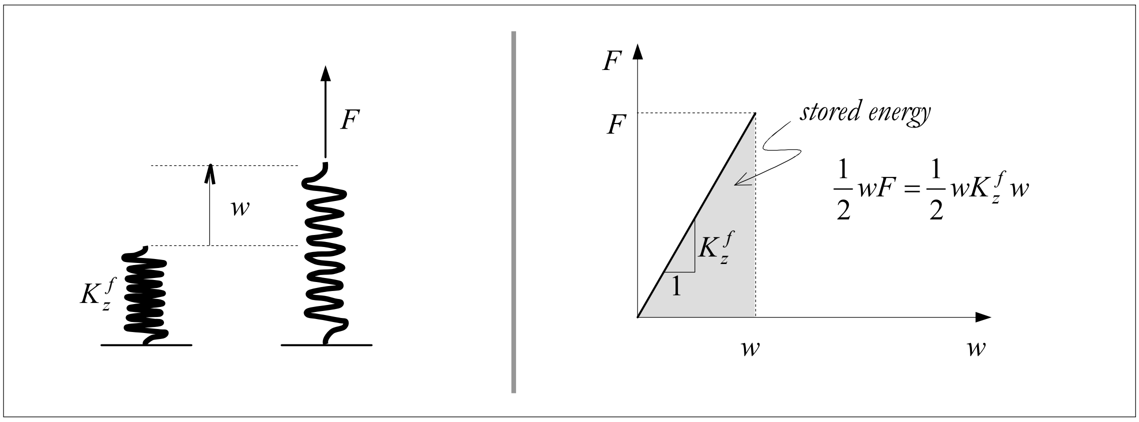


Figure 6.212: Spring element.

Considering that the spring coefficient K_z^f is constant in the beam element the equation (6.646) becomes:

$$U_{spring}=\frac{K_z^f}{2}\int_0^L w^2 dx \tag{6.647}$$

Moreover, by means of equation, (see **Problem 6.62** – NOTE 2):

$$\int_0^L w^2 dx=\frac{13L}{35}w_1^2+\frac{13L}{35}w_2^2+\frac{L^3}{105}\overline{\theta}_{y1}^2+\frac{L^3}{105}\overline{\theta}_{y2}^2+\frac{9L}{35}w_1w_2-\frac{11L^2}{105}w_1\overline{\theta}_{y1}+\frac{13L^2}{210}w_1\overline{\theta}_{y2}$$
$$-\frac{13L^2}{210}w_2\overline{\theta}_{y1}+\frac{11L^2}{105}w_2\overline{\theta}_{y2}-\frac{L^3}{70}\overline{\theta}_{y1}\overline{\theta}_{y2}$$

we can obtain:

$$U_{spring}=\frac{K_f}{2}\left(\frac{13L}{35}w_1^2+\frac{13L}{35}w_2^2+\frac{L^3}{105}\overline{\theta}_{y1}^2+\frac{L^3}{105}\overline{\theta}_{y2}^2+\frac{9L}{35}w_1w_2-\frac{11L^2}{105}w_1\overline{\theta}_{y1}+\right.$$
$$\left.\frac{13L^2}{210}w_1\overline{\theta}_{y2}-\frac{13L^2}{210}w_2\overline{\theta}_{y1}+\frac{11L^2}{105}w_2\overline{\theta}_{y2}-\frac{L^3}{70}\overline{\theta}_{y1}\overline{\theta}_{y2}\right) \tag{6.648}$$

Then

$$\frac{\partial U_{spring}}{\partial w_1}=\frac{K_z^f}{2}\left\{\frac{26L}{35}w_1+\frac{9L}{35}w_2-\frac{11L^2}{105}\overline{\theta}_{y1}+\frac{13L^2}{210}\overline{\theta}_{y2}\right\} \tag{6.649}$$

$$\frac{\partial U_{spring}}{\partial \overline{\theta}_{y1}}=\frac{K_z^f}{2}\left\{\frac{2L^3}{105}\overline{\theta}_{y1}-\frac{11L^2}{105}w_1-\frac{13L^2}{210}w_2-\frac{L^3}{70}\overline{\theta}_{y2}\right\} \tag{6.650}$$

$$\frac{\partial U_{spring}}{\partial w_2}=\frac{K_z^f}{2}\left\{\frac{26L}{35}w_2+\frac{9L}{35}w_1-\frac{13L^2}{210}\overline{\theta}_{y1}+\frac{11L^2}{105}\overline{\theta}_{y2}\right\} \tag{6.651}$$

$$\frac{\partial U_{spring}}{\partial \overline{\theta}_{y2}}=\frac{K_z^f}{2}\left\{\frac{2L^3}{105}\overline{\theta}_{y2}+\frac{13L^2}{210}w_1+\frac{11L^2}{105}w_2-\frac{L^3}{70}\overline{\theta}_{y1}\right\} \tag{6.652}$$

Restructuring the above set of equations in matrix form we can obtain:

$$
\frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} = K_z^f L \left[\begin{array}{cc|cc} \frac{13}{35} & \frac{-11L}{210} & \frac{9}{70} & \frac{13L}{420} \\ \frac{-11L}{210} & \frac{L^2}{105} & \frac{-13L}{420} & \frac{-L^2}{140} \\ \hline \frac{9}{70} & \frac{-13L}{420} & \frac{13}{35} & \frac{11L}{210} \\ \frac{13L}{420} & \frac{-L^2}{140} & \frac{11L}{210} & \frac{L^2}{105} \end{array}\right] \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} \tag{6.653}
$$

$$
\frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} = [\boldsymbol{ke}^{(Spring\_z)}]\{\boldsymbol{u}^{(e)}\}
$$

where

$$
\boxed{[\boldsymbol{ke}^{(Spring\_z)}] = K_z^f \left[\begin{array}{cc|cc} \frac{13L}{35} & \frac{-11L^2}{210} & \frac{9L}{70} & \frac{13L^2}{420} \\ \frac{-11L^2}{210} & \frac{L^3}{105} & \frac{-13L^2}{420} & \frac{-L^3}{140} \\ \hline \frac{9L}{70} & \frac{-13L^2}{420} & \frac{13l}{35} & \frac{11L^2}{210} \\ \frac{13L^2}{420} & \frac{-L^3}{140} & \frac{11L^2}{210} & \frac{L^3}{105} \end{array}\right]} \tag{6.654}
$$

Then, the equation (6.645) becomes

$$
\begin{aligned}
&[\boldsymbol{ke}^{(1)}]\{\boldsymbol{u}^{(e)}\} + \frac{dU_{spring}}{d\{\boldsymbol{u}^{(e)}\}} = \{\boldsymbol{f}^{(e)}\} \quad \Rightarrow \quad [\boldsymbol{ke}^{(1)}]\{\boldsymbol{u}^{(e)}\} + [\boldsymbol{ke}^{(Spring\_z)}]\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}\} \\
&\Rightarrow \Big[\, [\boldsymbol{ke}^{(1)}] + [\boldsymbol{ke}^{(Spring\_z)}] \,\Big]\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}\}
\end{aligned} \tag{6.655}
$$

NOTE 1: Note that if the elastic foundation is orientated according to the plane $x-y$, (see **Problem 6.67**), the internal energy becomes:

$$
\begin{aligned}
U_{spring} = \frac{K_y^f}{2}\int_0^L v^2 dx = \frac{K_y^f}{2}\Bigg(& \frac{13L}{35}v_1^2 + \frac{13L}{35}v_2^2 + \frac{L^3}{105}\overline{\theta}_{z1}^2 + \frac{L^3}{105}\overline{\theta}_{z2}^2 + \frac{9L}{35}v_1 v_2 + \\
& \frac{11L^2}{105}v_1\overline{\theta}_{z1} - \frac{13L^2}{210}v_1\overline{\theta}_{z2} + \frac{13L^2}{210}v_2\overline{\theta}_{z1} - \frac{11L^2}{105}v_2\overline{\theta}_{z2} - \frac{L^3}{70}\overline{\theta}_{z1}\overline{\theta}_{z2}\Bigg)
\end{aligned} \tag{6.656}
$$

And the stiffness matrix becomes:

$$
\boxed{[\boldsymbol{ke}^{(Spring\_y)}] = K_y^f \left[\begin{array}{cc|cc} \frac{13L}{35} & \frac{11L^2}{210} & \frac{9L}{70} & \frac{-13L^2}{420} \\ \frac{11L^2}{210} & \frac{L^3}{105} & \frac{13L^2}{420} & \frac{-L^3}{140} \\ \hline \frac{9L}{70} & \frac{13L^2}{420} & \frac{13L}{35} & \frac{-11L^2}{210} \\ \frac{-13L^2}{420} & \frac{-L^3}{140} & \frac{-11L^2}{210} & \frac{L^3}{105} \end{array}\right]} \tag{6.657}
$$

Problem 6.75

Consider the beam describe in Figure 6.213. Obtain the displacements at the node 2 and the rotations at the nodes 1 and 3.

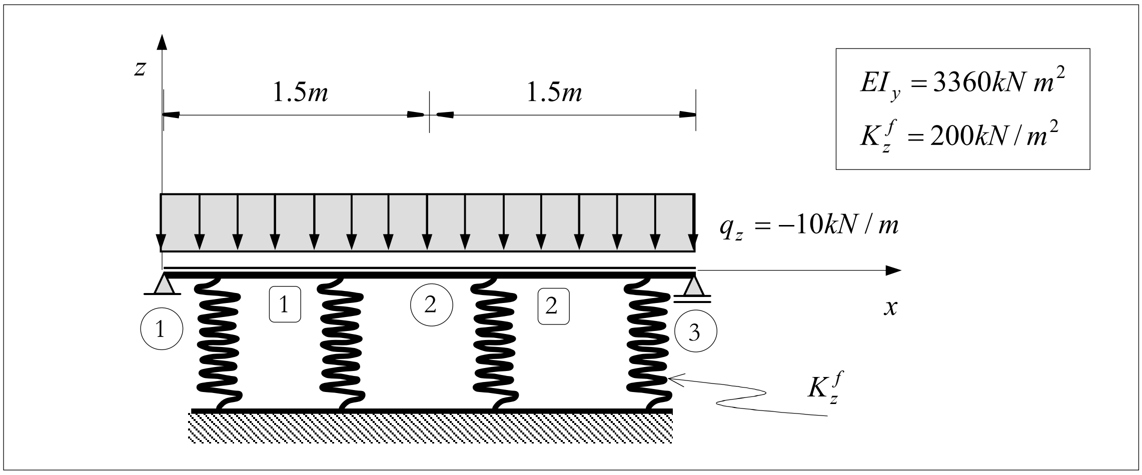


Figure 6.213: Beam and elastic foundation.

Solution:

For this problem the global degrees-of-freedom are described in Figure 6.214.

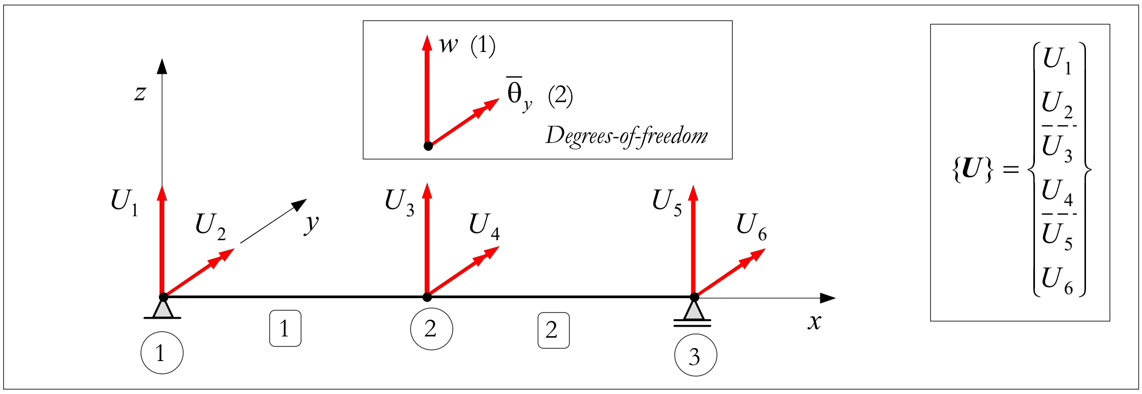


Figure 6.214: Global degrees-of-freedom.

Construction of the Global Stiffness Matrix - $[\boldsymbol{K}_{Global}]$

The matrix $[\boldsymbol{K}_{Global}]$ can be constructed by assembling the individual beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2} [\boldsymbol{k}^{(e)}_{Global}] \tag{6.658}$$

The stiffness matrix for the adopted system was obtained in **Problem 6.62**. And the elastic foundation stiffness matrix was obtained in (6.654).

For each element we have

Element 1:

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \quad Global \\
[\boldsymbol{k1}^{(1)}_{Global}] = [\boldsymbol{k1}^{(1)}_{Local}] = 10^4 \times & \begin{bmatrix} 1.194667 & -0.896 & -1.194667 & -0.896 \\ -0.896 & 0.896 & 0.896 & 0.448 \\ -1.194667 & 0.896 & 1.194667 & 0.896 \\ -0.896 & 0.448 & 0.896 & 0.896 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \quad Global \\
[\boldsymbol{k}^{(Spring\_z)(1)}_{Global}] = [\boldsymbol{k}^{(Spring\_z)(1)}_{Local}] = & \begin{bmatrix} 111.42857 & -23.57143 & 38.57143 & 13.92857 \\ -23.57143 & 6.42857 & -13.92857 & -4.82143 \\ 38.57143 & -13.92857 & 111.42857 & 23.57143 \\ 13.92857 & -4.82143 & 23.57143 & 6.42857 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Then

$$
[\boldsymbol{k}^{(1)}_{Global}] = [\boldsymbol{k1}^{(1)}_{Global}] + [\boldsymbol{k}^{(Spring\_z)(1)}_{Global}]
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \quad Global \\
k^{(1)}_{ij} \equiv [\boldsymbol{k}^{(1)}_{Global}] \approx 10^4 \times & \begin{bmatrix} 1.20581 & -0.89836 & -1.19081 & -0.89461 \\ -0.89836 & 0.89664 & 0.89461 & 0.44752 \\ -1.19081 & 0.89461 & 1.20581 & 0.89836 \\ -0.89461 & 0.44752 & 0.89836 & 6.42857 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

Element 2: Note that $[\boldsymbol{k1}^{(2)}_{Global}] = [\boldsymbol{k1}^{(1)}_{Global}]$, $[\boldsymbol{k}^{(Spring\_z)(2)}_{Global}] = [\boldsymbol{k}^{(Spring\_z)(1)}_{Global}]$, then

$$
\begin{array}{cc}
 & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \quad Global \\
k^{(2)}_{ij} \equiv [\boldsymbol{k}^{(2)}_{Global}] = [\boldsymbol{k}^{(1)}_{Global}] \approx 10^4 \times & \begin{bmatrix} 1.20581 & -0.89836 & -1.19081 & -0.89461 \\ -0.89836 & 0.89664 & 0.89461 & 0.44752 \\ -1.19081 & 0.89461 & 1.20581 & 0.89836 \\ -0.89461 & 0.44752 & 0.89836 & 6.42857 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

After all elements have been assembled we can obtain

$$
[\boldsymbol{K}_{Global}] = \mathop{\mathbf{A}}_{e=1}^{2} [\boldsymbol{k}^{(e)}_{Global}]
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad Global \\
[\boldsymbol{K}_{Global}] = & \begin{bmatrix} k^{(1)}_{11} & k^{(1)}_{12} & k^{(1)}_{13} & k^{(1)}_{14} & 0 & 0 \\ k^{(1)}_{21} & k^{(1)}_{22} & k^{(1)}_{23} & k^{(1)}_{24} & 0 & 0 \\ k^{(1)}_{31} & k^{(1)}_{32} & k^{(1)}_{33} + k^{(2)}_{11} & k^{(1)}_{44} + k^{(2)}_{12} & k^{(2)}_{13} & k^{(2)}_{14} \\ k^{(1)}_{41} & k^{(1)}_{42} & k^{(1)}_{43} + k^{(2)}_{21} & k^{(1)}_{44} + k^{(2)}_{22} & k^{(2)}_{23} & k^{(2)}_{24} \\ 0 & 0 & k^{(2)}_{31} & k^{(2)}_{32} & k^{(2)}_{33} & k^{(2)}_{34} \\ 0 & 0 & k^{(2)}_{41} & k^{(2)}_{42} & k^{(2)}_{43} & k^{(2)}_{44} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

By substituting the numerical values we can obtain:

$$
[\boldsymbol{K}_{Global}] = 10^4 \times
\begin{array}{c}
\begin{array}{cccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} \end{array} \quad \mathit{Global} \\
\begin{bmatrix}
1.20581 & -0.89836 & -1.19081 & -0.89461 & 0 & 0 \\
-0.89836 & 0.89664 & 0.89461 & 0.44752 & 0 & 0 \\
-1.19081 & 0.89461 & 2.41162 & 0 & -1.19081 & -0.89461 \\
-0.89461 & 0.44752 & 0 & 1.79329 & 0.89461 & 0.44752 \\
0 & 0 & -1.19081 & 0.89461 & 1.20581 & 0.89836 \\
0 & 0 & -0.89461 & 0.44752 & 0.89836 & 0.89664
\end{bmatrix}
\begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array}
\end{array}
$$

Construction of the Global Nodal Force Vector - $\{\boldsymbol{F}_{Global}\}$

$$
\{\boldsymbol{F}_{Global}\} = \left(\mathop{\mathbf{A}}_{e=1}^{2} \{\boldsymbol{f}_{Eq\_G}^{(e)}\} \right) + \underbrace{\{\boldsymbol{F}_0\}}_{=\{\boldsymbol{0}\}}
$$

Element 1: uniformly distributed load, ($q_z = -10$, $L = 1.5$):

$$
\{\boldsymbol{f}_{Eq\_G}^{(1)}\} = \{\boldsymbol{f}_{Eq\_L}^{(1)}\} = \begin{Bmatrix} \dfrac{q_z L}{2} \\ \dfrac{-q_z L^2}{12} \\ \dfrac{q_z L}{2} \\ \dfrac{q_z L^2}{12} \end{Bmatrix} = \begin{Bmatrix} -7.5 \\ 1.875 \\ -7.5 \\ -1.875 \end{Bmatrix}
\begin{array}{c} \mathit{Global} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{array}
$$

Element 2: $\{\boldsymbol{f}_{Eq\_G}^{(1)}\} = \{\boldsymbol{f}_{Eq\_G}^{(2)}\}$

Then,

$$
\{\boldsymbol{F}_{Global}\}_{6\times 1} = \left(\mathop{\mathbf{A}}_{e=1}^{2} \{\boldsymbol{f}_{Eq\_G}^{(e)}\} \right) = \begin{Bmatrix} \{\boldsymbol{f}_{Eq\_G}^{(1)}\}_1 \\ \{\boldsymbol{f}_{Eq\_G}^{(1)}\}_2 \\ \{\boldsymbol{f}_{Eq\_G}^{(1)}\}_3 + \{\boldsymbol{f}_{Eq\_G}^{(2)}\}_1 \\ \{\boldsymbol{f}_{Eq\_G}^{(1)}\}_4 + \{\boldsymbol{f}_{Eq\_G}^{(2)}\}_2 \\ \{\boldsymbol{f}_{Eq\_G}^{(2)}\}_3 \\ \{\boldsymbol{f}_{Eq\_G}^{(2)}\}_4 \end{Bmatrix} = \begin{Bmatrix} -7.5 \\ 1.875 \\ -15 \\ 0 \\ -7.5 \\ -1.875 \end{Bmatrix}
\begin{array}{c} \mathit{Global} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array}
$$

Applying the Boundary Conditions

Note that there are restrictions to move for the following degrees-of-freedom: $U_1 = 0$, $U_5 = 0$, and after applying these boundary conditions the set of equation will have the following aspect:

$$
10^4 \times \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.89664 & 0.89461 & 0.44752 & 0 & 0 \\
0 & 0.89461 & 2.41162 & 0 & 0 & -0.89461 \\
0 & 0.44752 & 0 & 1.79329 & 0 & 0.44752 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -0.89461 & 0.44752 & 0 & 0.89664
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.875 \\ -15 \\ 0 \\ 0 \\ -1.875 \end{Bmatrix}
$$

Solving the System

$$\begin{Bmatrix} U_1 \\ U_2 \\ \hline U_3 \\ U_4 \\ \hline U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3.1939374 \\ -2.9916161 \\ 0 \\ 0 \\ -3.1939374 \end{Bmatrix} \times 10^{-3}$$

Internal and Reaction Forces

The internal and reaction forces can be obtained as follows

$$\{\boldsymbol{r}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + \{\bar{\boldsymbol{f}}^{(e)}_{Local}\} = -\{\boldsymbol{f}^{(e)}_{Eq\_L}\} + [\boldsymbol{k}^{(e)}_{Local}][\overline{\mathcal{A}}]\{\boldsymbol{u}^{(e)}_{Global}\} = -\{\boldsymbol{f}^{(e)}_{\boldsymbol{int}}\}$$

And the reaction vector in the Global system $\{\boldsymbol{r}^{(e)}_{Global}\} = [\overline{\mathcal{A}}]^T \{\boldsymbol{r}^{(e)}_{Local}\}$. Note also that for this problem the Global and Local systems have the same orientation, so $[\overline{\mathcal{A}}] = [\boldsymbol{I}]$.

Element 1:

Internal force (Local system=Global system)

$$\{\boldsymbol{r}^{(1)}_{Local}\} = -\{\boldsymbol{f}^{(1)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{z1} \\ m_{y1} \\ f_{z2} \\ m_{y2} \end{Bmatrix} = -\underbrace{\begin{Bmatrix} -7.5 \\ 1.875 \\ \hline -7.5 \\ -1.875 \end{Bmatrix}}_{=\{\boldsymbol{f}^{(1)}_{Eq\_L}\}} + \underbrace{\begin{Bmatrix} 6.931485 \\ 1.875 \\ \hline -7.5 \\ -12.581957 \end{Bmatrix}}_{=\{\bar{\boldsymbol{f}}^{(1)}_{Local}\}} = \begin{Bmatrix} 14.431485 \\ 0 \\ \hline 0 \\ -10.706957 \end{Bmatrix} = \{\boldsymbol{r}^{(1)}_{Global}\}$$

Element 2:

Internal force (Local system=Global system)

$$\{\boldsymbol{r}^{(2)}_{Local}\} = -\{\boldsymbol{f}^{(2)}_{\boldsymbol{int}}\} = \begin{Bmatrix} f_{z1} \\ m_{y1} \\ f_{z2} \\ m_{y2} \end{Bmatrix} = -\underbrace{\begin{Bmatrix} -7.5 \\ 1.875 \\ \hline -7.5 \\ -1.875 \end{Bmatrix}}_{=\{\boldsymbol{f}^{(2)}_{Eq\_L}\}} + \underbrace{\begin{Bmatrix} -7.5 \\ 12.581957 \\ \hline 6.931485 \\ -1.875 \end{Bmatrix}}_{=\{\bar{\boldsymbol{f}}^{(2)}_{Local}\}} = \begin{Bmatrix} 0 \\ 10.706957 \\ \hline 14.431485 \\ 0 \end{Bmatrix} = \{\boldsymbol{r}^{(2)}_{Global}\}$$

The Global reaction vector becomes:

$$\{\boldsymbol{R}\} = \mathop{\mathbf{A}}_{e=1}^{2} \{\boldsymbol{r}^{(e)}_{Global}\} - \underbrace{\{\boldsymbol{F}_0\}}_{=\{\boldsymbol{0}\}} = \begin{Bmatrix} r^{(1)}_{G\ 1} \\ r^{(1)}_{G\ 2} \\ r^{(1)}_{G\ 3} + r^{(2)}_{G\ 1} \\ r^{(1)}_{G\ 4} + r^{(2)}_{G\ 2} \\ r^{(2)}_{G\ 3} \\ r^{(2)}_{G\ 4} \end{Bmatrix} = \begin{array}{cl} & \mathit{Global} \\ \begin{Bmatrix} 14.431485 \\ 0 \\ \hline 0 \\ 0 \\ \hline 14.431485 \\ 0 \end{Bmatrix} & \begin{array}{l} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array} \end{array}$$

References

GERE, J.M. & WEAVER JR., W. (1965). *Analysis of Framed Structures.* Van Nostrand Reinhold, U.S.

SECHLER, E.E. (1952). *Elasticity in engineering.* John Wiley & Sons, Inc., New York.

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UGURAL, A.C.; FENSTER, S.K., (1984). *Advanced strength and applied elasticity - The SI version.* Elsevier Science Publishing Co. Inc., New York.

Problem 6.76

Consider a beam of length L where the internal forces are schematically described in Figure 6.215. Apply the Principle of Complementary Virtual Work to the beam, (see **Problem 5.22 – NOTE 4**).

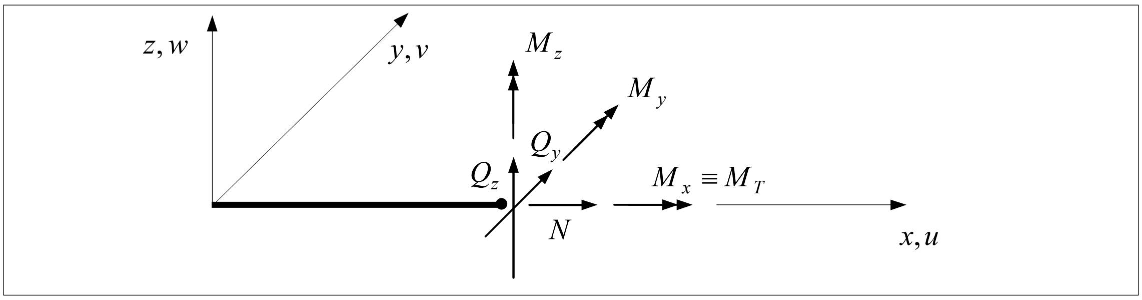

Figure 6.215: Internal forces in the beam.

Obs.: Consider the elastic problem without body forces, and also consider that the beam is only subjected by concentrated loads.

Solution:

In **Problem 5.22 – NOTE 4** we have shown that:

$$\boxed{\underbrace{\vec{\bar{F}}^{loc}\cdot\vec{\bar{\mathbf{u}}}^{loc}}_{\substack{\text{Total external complementary virtual work}\\\text{(due to concentrated forces)}}} = \underbrace{\int_V \bar{\boldsymbol{\sigma}}:\boldsymbol{\varepsilon}\,dV}_{\substack{\text{Total internal}\\\text{complementary virtual work}}}} \tag{6.659}$$

Principle of Complementary Virtual Work (static case without body forces and with concentrated forces)

with $\bar{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}} = \vec{\bar{\mathbf{t}}}^{*}$ on $S_{\boldsymbol{\sigma}}$.

The total internal complementary virtual work:

$$W_{\text{int}} = \int_V \bar{\boldsymbol{\sigma}}:\boldsymbol{\varepsilon}\,dV = \int_V \bar{\sigma}_{ij}\varepsilon_{ij}\,dV = \int_V (\bar{\sigma}_{11}\varepsilon_{11} + \bar{\sigma}_{22}\varepsilon_{22} + \bar{\sigma}_{33}\varepsilon_{33} + 2\bar{\sigma}_{12}\varepsilon_{12} + 2\bar{\sigma}_{23}\varepsilon_{23} + 2\bar{\sigma}_{13}\varepsilon_{13})dV$$
$$= \int_V (\bar{\sigma}_x\varepsilon_x + \bar{\sigma}_y\varepsilon_y + \bar{\sigma}_z\varepsilon_z + \bar{\tau}_{xy}\gamma_{xy} + \bar{\tau}_{yz}\gamma_{yz} + \bar{\tau}_{xz}\gamma_{xz})dV$$

In the previous problem we have expressed the stresses in terms of internal forces, (see **Problem 6.62**):

$$\sigma_x = \sigma_x^{(1)} + \sigma_x^{(2)} + \sigma_x^{(3)}:$$

$$\int_V \bar{\sigma}_x^{(1)}\varepsilon_x^{(1)}dV = \int_V \bar{\sigma}_x^{(1)}\frac{\sigma_x^{(1)}}{E}dV = \int_0^L \frac{\bar{N}}{A}\frac{N}{EA}\int_A dA\,dx = \int_0^L \bar{N}\frac{N}{EA}dx$$

$$\int_V \bar{\sigma}_x^{(2)}\varepsilon_x^{(2)}dV = \int_0^L\int_A \frac{\bar{M}_y}{I_y}z\frac{M_y}{EI_y}z\,dA\,dx = \int_0^L \bar{M}_y\frac{M_y}{EI_y^2}\int_A z^2\,dA\,dx = \int_0^L \bar{M}_y\frac{M_y}{EI_y}dx$$

$$\int_V \bar{\sigma}_x^{(3)}\varepsilon_x^{(3)}dV = \int_0^L \bar{M}_z\frac{M_z}{EI_z}dx$$

The component $\tau_{xz}(z) = G\gamma_{xz}$ is related to the shearing force Q_z, (see equation (6.216)), where G is the shear modulus. In equation (6.228) we have obtained:

$$\int \tau(z,y)dy = \frac{Q_z}{I_y}\int_A z d\overline{A} = \frac{Q_z \chi_z}{I_y} \qquad \text{where} \qquad \chi_z = \int_A z d\overline{A} \tag{6.660}$$

Then, the strain energy associated with $\tau_{xz}(z) = G\gamma_{xz}$ is given by:

$$\begin{aligned}\int_V \overline{\tau}\gamma dV &= \int_V \overline{\tau}_{xz}^2 \frac{\tau_{xz}}{G} dV = \int_0^L \frac{1}{G}\int_A \overline{\tau}\,\tau\, dA dx = \int_0^L \frac{1}{G}\left(\int\int \overline{\tau}\,\tau\, dy dz\right)dx \\ &= \int_0^L \frac{1}{G}\left(\int\left(\frac{\overline{Q}_z \chi_z}{I_y}\right)\left(\frac{Q_z \chi_z}{I_y}\right)dz\right)dx = \int_0^L \overline{Q}_z \frac{Q_z}{G}\left(\int\left(\frac{\chi_z}{I_y}\right)^2 dz\right)dx = \int_0^L \overline{Q}_z \frac{\varsigma_z Q_z}{GA} dx\end{aligned} \tag{6.661}$$

where ς_z is the shape factor which is given by:

$$\varsigma_z = A\int\left(\frac{\chi_z}{I_y}\right)^2 dz \tag{6.662}$$

In the same way we can obtain, (see **Problem 6.62**):

$$\int_V \overline{\tau}_{xy}\gamma_{xy} dV = \int_V \overline{\tau}_{xy}\frac{\tau_{xy}}{G} dV = \int_0^L \overline{Q}_y \frac{\varsigma_y Q_y}{GA} dx \qquad \text{where} \qquad \varsigma_y = A\int_A \left(\frac{\chi_y}{I_z}\right)^2 dA \tag{6.663}$$

The internal complementary virtual work due to the torsion moment becomes:

$$\int_V \overline{\tau}(r)\gamma(r) dV = \int_V \overline{\tau}\frac{\tau}{G} dV = \int_V \frac{1}{G}\left(\frac{\overline{M}_T}{\mathrm{J}} r\right)\left(\frac{M_T}{\mathrm{J}} r\right) dV = \int_0^L \overline{M}_T \frac{M_T}{G\mathrm{J}^2}\int_A r^2 dA dx = \int_0^L \overline{M}_T \frac{M_T}{G\mathrm{J}} dx \tag{6.664}$$

where $\mathrm{J} = \int_A r^2 dA$ is the polar inertia moment. When we are dealing with rectangular cross section we can express the equivalent internal complementary virtual work as follows:

$$\int_0^L \overline{M}_T \frac{M_T}{GJ_{Eq}} dx \tag{6.665}$$

where J_{Eq} is the equivalent inertia moment.

The total internal complementary virtual work:

$$\begin{aligned}W_{\text{int}} &= \int_V (\overline{\sigma}_x \varepsilon_x + \overline{\sigma}_y \varepsilon_y + \overline{\sigma}_z \varepsilon_z + \overline{\tau}_{xy}\gamma_{xy} + \overline{\tau}_{yz}\gamma_{yz} + \overline{\tau}_{xz}\gamma_{xz}) dV \\ &= \int_0^L \left(\overline{N}\frac{N}{EA} + \overline{M}_y \frac{M_y}{EI_y} + \overline{M}_z \frac{M_z}{EI_z} + \overline{Q}_z \frac{\varsigma_z Q_z}{GA} + \overline{Q}_y \frac{\varsigma_y Q_y}{GA} + \overline{M}_T \frac{M_T}{GJ_{Eq}}\right) dx\end{aligned}$$

And the Principle of Complementary Virtual Work becomes:

$$\underbrace{\vec{\bar{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc}}_{\substack{\text{Total external complementary virtual work} \\ \text{(due to concentrated forces)}}} = \underbrace{\int_V \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} \, dV}_{\substack{\text{Total internal} \\ \text{complementary virtual work}}}$$

$$\Rightarrow \vec{\bar{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc} = \int_0^L \left(\bar{N} \frac{N}{EA} + \bar{M}_y \frac{M_y}{EI_y} + \bar{M}_z \frac{M_z}{EI_z} + \bar{Q}_z \frac{\varsigma_z Q_z}{GA} + \bar{Q}_y \frac{\varsigma_y Q_y}{GA} + \bar{M}_T \frac{M_T}{GJ_{Eq}} \right) dx$$

NOTE: For the next example we will apply the principle of complementary virtual work, in which we consider the beam fixed at one end, (see Figure 6.216).

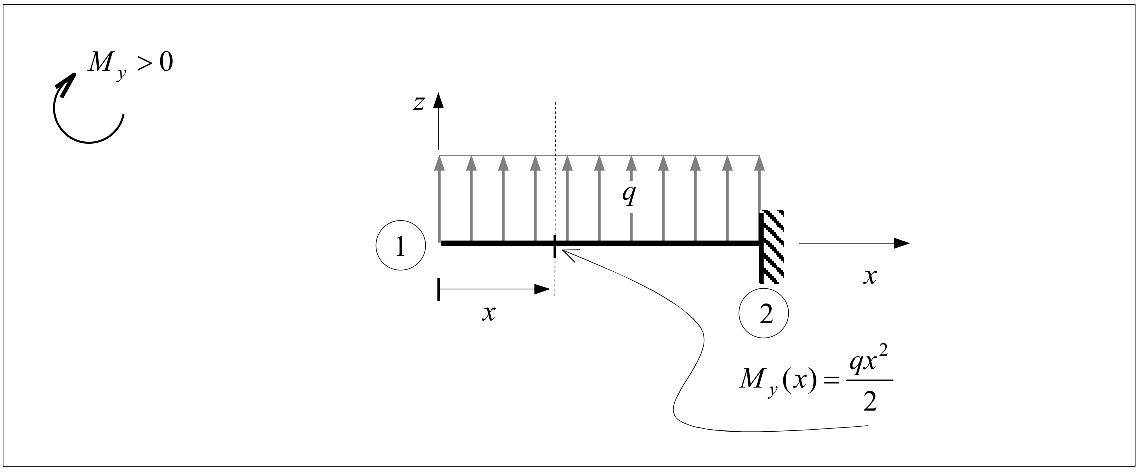

Figure 6.216: Beam fixed at one end under uniformly distributed load.

In this situation the principle of the complementary virtual work becomes:

$$\vec{\bar{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc} = \int_0^L \bar{M}_y \frac{M_y}{EI_y} dx$$

If we want to know the deflection of the beam at $x = 0$ (node 1) we apply a concentrated virtual load $\bar{F}^{(1)} = 1$ at this point. The moment diagram for real and virtual states are presented in Figure 6.217.

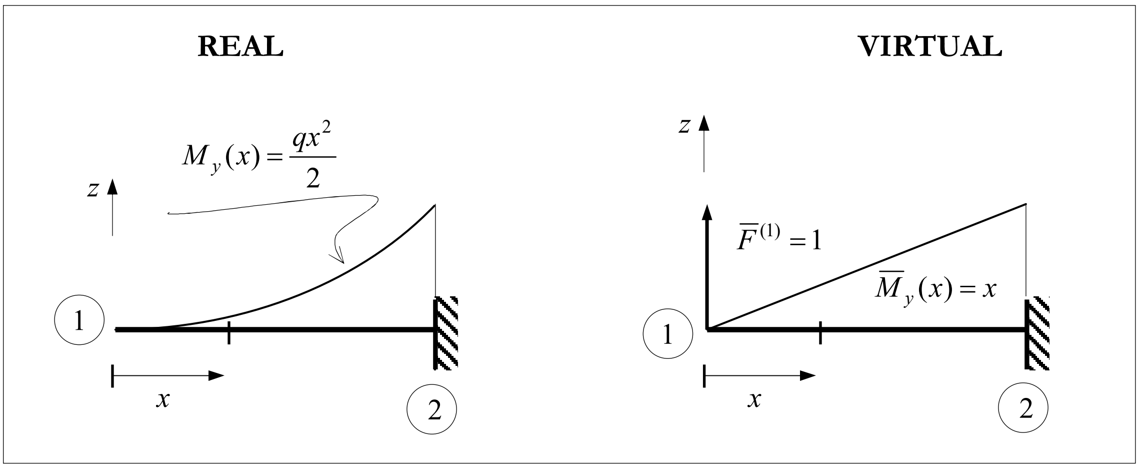

Figure 6.217: Beam fixed at one end under uniformly distributed load.

With that we can state that:

$$\vec{\vec{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc} = \int_0^L \overline{M}_y \frac{M_y}{EI_y} dx \quad \Rightarrow \quad \overline{F}^{(1)} \mathsf{U}^{(1)} = \int_0^L \overline{M}_y \frac{M_y}{EI_y} dx = \int_0^L x \frac{\frac{qx^2}{2}}{EI_y} dx = \frac{q}{2EI_y} \int_0^L x^3 dx$$

Taking into account that $\overline{F}^{(1)} = 1$ and by solving the integral we can obtain:

$$\mathsf{U}^{(1)} = \frac{q}{2EI_y} \int_0^L x^3 dx = \frac{qL^4}{8EI_y}$$

which value matches the result in **Problem 6.62**- NOTE 4.

If we want to know the rotation of the beam at $x = 0$ we apply a concentrated virtual load (moment) $\overline{m}^{(1)} = 1$ at this point. The moment diagram for real and virtual states are presented in Figure 6.218.

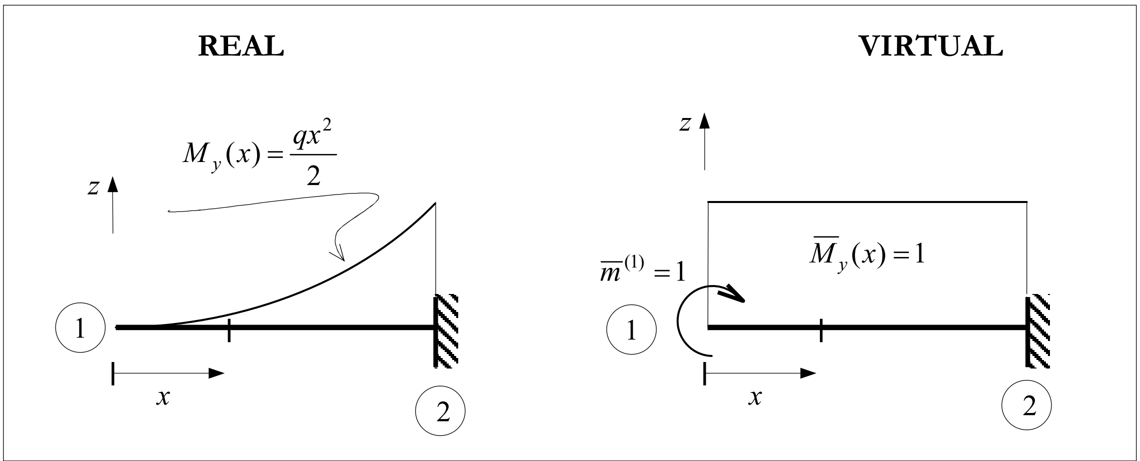

Figure 6.218: Beam fixed at one end under uniformly distributed load.

With that we can state that:

$$\vec{\vec{F}}^{loc} \cdot \vec{\mathbf{u}}^{loc} = \int_0^L \overline{M}_y \frac{M_y}{EI_y} dx \quad \Rightarrow \quad \overline{m}^{(1)} \theta^{(1)} = \int_0^L \overline{M}_y \frac{M_y}{EI_y} dx = \int_0^L 1 \frac{\frac{qx^2}{2}}{EI_y} dx = \frac{q}{2EI_y} \int_0^L x^2 dx$$

Taking into account that $\overline{m}^{(1)} = 1$ and by solving the integral we can obtain:

$$\theta^{(1)} = \frac{q}{2EI_y} \int_0^L x^2 dx = \frac{qL^3}{6EI_y}$$

which value matches the result in **Problem 6.62**- NOTE 4.

Problem 6.77

Consider Figure 6.216 and apply the first and second Mohr Theorem, (see **Problem 6.61**), to obtain the deflection and rotation at the free-end of the beam (node 1 in Figure 6.216).

Solution:

The Mohr's first theorem states: "The change in slope of a deflection curve between two points is equal the area diagram of $\frac{M_y}{EI_y}$ between these two points."

The slope at the point 2, (see Figure 6.219), is zero ($\psi_{(2)} = 0$). Then,

$$\Delta\psi_{(2)\_(1)} = -\int_{(1)}^{(2)} \frac{M_y}{EI_y} dx \quad \Rightarrow \quad \psi_{(2)} - \psi_{(1)} = -\int_{(1)}^{(2)} \frac{M_y}{EI_y} dx \quad \Rightarrow \quad \psi_{(1)} = \int_{(1)}^{(2)} \frac{M_y}{EI_y} dx$$

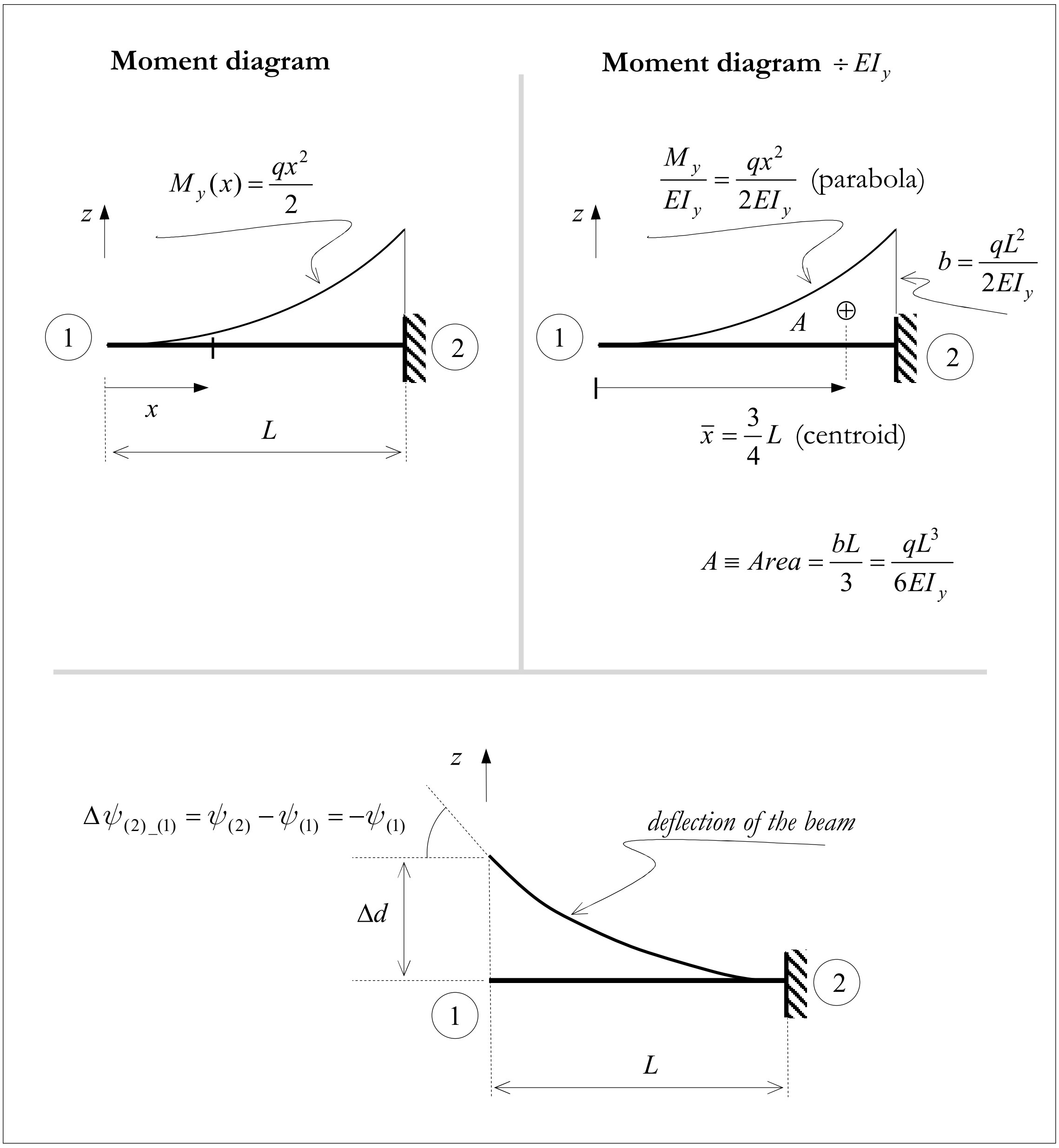

Figure 6.219: Beam fixed at one end under uniformly distributed load.

According to Figure 6.219 we can conclude that:

$$\psi_{(1)} = \int_{(1)}^{(2)} \frac{M_y}{EI_y} dx = A = \frac{qL^3}{6EI_y}$$

The Mohr's second theorem states:

"The distance Δd is equal to the first moment of the diagram $\frac{M_y}{EI_y}$ about the axis where Δd is measured."

According to Figure 6.219 we can conclude that:

$$\Delta d = A\bar{x} = \frac{qL^3}{6EI_y}\frac{3}{4}L = \frac{qL^4}{8EI_y}$$

Problem 6.78

Consider a bar of length L, which has a squared cross section of side a. The elastic constants of the material is assumed to be known (E and $\nu = 0.25$).

a) In the case of Figure 6.220(a), calculate the stored energy (strain energy density) in the bar due to the deformation, and also obtain the total strain energy;
b) Determine the stored energy corresponds to the change of volume and to the change of shape;
c) The same question described in paragraph (a) but considering the case of Figure 6.220(b).

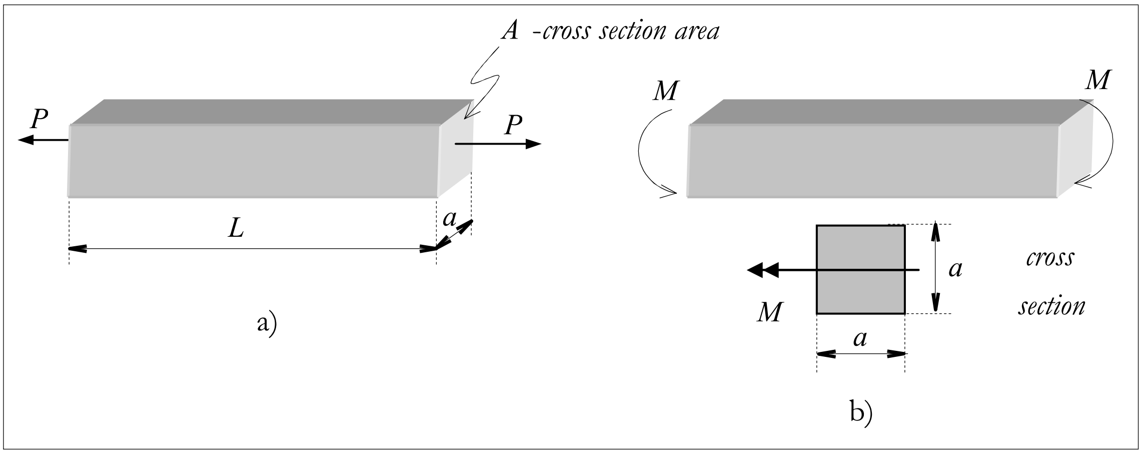

Figure 6.220

Solution:

Considering a one dimensional case:

$$\sigma_x = E\varepsilon_x \Rightarrow \varepsilon_x = \frac{\sigma_x}{E} \qquad with \qquad \sigma_x = \frac{P}{A} \tag{6.666}$$

We know that the strain energy per unit volume is given by:

$$\Psi^e = \frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon} \xrightarrow{\text{one-dimensional}} \Psi^e = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}\sigma_x\frac{\sigma_x}{E} = \frac{1}{2}\frac{P^2}{EA^2} \tag{6.667}$$

Then, the total energy U is given by:

$$\begin{aligned} \Psi^e_x \times (volume) = L \times A \times \frac{P^2}{2EA^2} \Rightarrow \\ \Rightarrow U = \frac{P^2 L}{2EA} \end{aligned} \tag{6.668}$$

The strain energy density (per unit volume) can also be expressed as follows:

$$\Psi^e = \underbrace{\frac{1}{6(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2}_{\Psi^e{}_{vol}} - \underbrace{\frac{1}{2\mu} I\!I_{\boldsymbol{\sigma}^{dev}}}_{\Psi^e{}_{shape}} \tag{6.669}$$

Considering:

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow I_{\boldsymbol{\sigma}} = \sigma_x = \frac{P}{A} \tag{6.670}$$

Calculation of $I\!I_{\boldsymbol{\sigma}^{dev}}$:

$$I\!I_{\boldsymbol{\sigma}^{dev}} = \frac{1}{3}(3 I\!I_{\boldsymbol{\sigma}} - I_{\boldsymbol{\sigma}}^2) = -\frac{I_{\boldsymbol{\sigma}}^2}{3} = -\frac{\sigma_x{}^2}{3} \tag{6.671}$$

Then, the strain energy density associated with the change of volume is:

$$\Psi^e{}_{vol} = \frac{1}{6(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2 = \frac{(1-2\nu)}{6E} I_{\boldsymbol{\sigma}}^2 = \frac{(1-2\nu)}{6E}\sigma_x{}^2 \tag{6.672}$$

$$\Psi^e{}_{vol} = \frac{(1-2\nu)}{6E}\frac{P^2}{A^2} \quad \text{(per unit volume)} \tag{6.673}$$

The strain energy density associated with the change of shape is:

$$\begin{aligned} \Psi^e{}_{shape} &= -\frac{1}{2\mu} I\!I_{\boldsymbol{\sigma}^{dev}} = -\frac{1}{2}\frac{2(1+\nu)}{E} I\!I_{\boldsymbol{\sigma}^{dev}} \\ &= -\frac{(1+\nu)}{E}\left(-\frac{\sigma_x{}^2}{3}\right) \end{aligned} \tag{6.674}$$

$$\Psi^e{}_{shape} = \frac{(1+\nu)}{E}\frac{\sigma_x{}^2}{3} = \frac{(1+\nu)}{3E}\frac{P^2}{A^2} \tag{6.675}$$

Checking:

$$\begin{aligned} \Psi^e{}_{vol} + \Psi^e{}_{shape} &= \frac{(1-2\nu)}{6E}\frac{P^2}{A^2} + \frac{(1+\nu)}{3E}\frac{P^2}{A^2} = \frac{P^2}{6EA^2}\left[(1-2\nu)+2(1+\nu)\right] \\ &= \frac{P^2}{6EA^2}\left[1-2\nu+2+2\nu\right] = \frac{P^2}{2EA^2} = \Psi^e \end{aligned}$$

For the case of bending, we consider the following relationships:

$$\sigma_y = \frac{M\ y}{\mathrm{I}} \qquad \text{where} \qquad \mathrm{I} = \frac{a^4}{12}$$

$$\sigma_y = \frac{12\,M\ y}{a^4}$$

$$\sigma_y = E\varepsilon_y \Rightarrow \varepsilon_y = \frac{\sigma_y}{E}$$

The strain energy density becomes:

$$\Psi^e = \frac{1}{2}\sigma_y\varepsilon_y = \frac{1}{2}\left(\frac{12\,M\ y}{a^4}\frac{\sigma_y}{E}\right) = \frac{1}{2}\left(\frac{12\,M\ y}{a^4}\frac{12\,M\ y}{Ea^4}\right) = \frac{72\,M^2\ y^2}{Ea^8} \tag{6.676}$$

6.5.2 Beam with Varying Cross Section

Problem 6.79

Obtain the Finite Element Formulation for the beam with variable cross section, (see Figure 6.221).

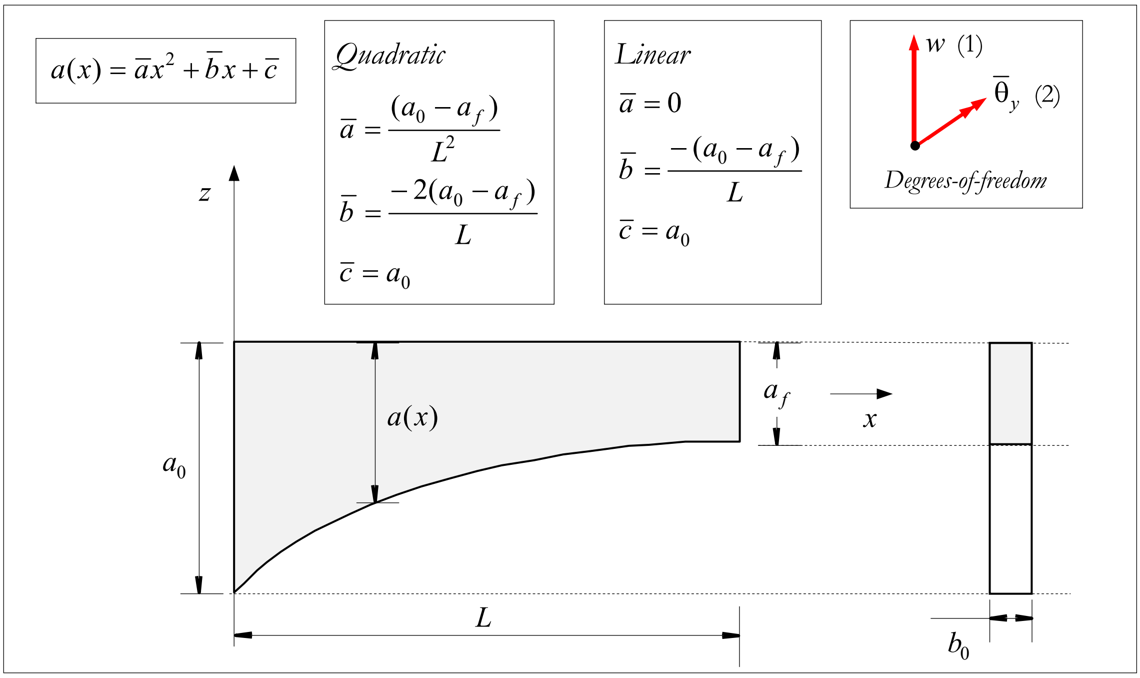

Figure 6.221: Beam with variable cross-section.

Solution:

The internal potential energy is given by

$$U^{int} = \int_0^L \frac{EI_y}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx \equiv \frac{E}{2}\int_0^L I_y (w'')^2 dx \tag{6.677}$$

where the moment of inertia I_y is given by

$$I_y = \frac{b_0[a(x)]^3}{12} = \frac{b_0[\bar{a}x^2 + \bar{b}x + \bar{c}]^3}{12} \tag{6.678}$$

Then, the internal potential energy becomes

$$U^{int} = \frac{E}{2}\int_0^L I_y (w'')^2 dx = \frac{Eb_0}{24}\int_0^L [\bar{a}x^2 + \bar{b}x + \bar{c}]^3 (w'')^2 dx \tag{6.679}$$

In **Problem 6.62** we have obtain an expression for (w'') in terms of the degrees-of-freedom (nodal values of deflection and rotation):

$$w'' = w_1\left[\frac{12x}{L^3} - \frac{6}{L^2}\right] + \bar{\theta}_{y1}\left[\frac{-6x}{L^2} + \frac{4}{L}\right] + w_2\left[-\frac{12x}{L^3} + \frac{6}{L^2}\right] + \bar{\theta}_{y2}\left[\frac{-6x}{L^2} + \frac{2}{L}\right]$$

Then, the internal potential energy becomes

$$U^{int} = \frac{Eb_0}{24}\int_0^L [\bar{a}x^2 + \bar{b}x + \bar{c}]^3 \left\{ w_1\left[\frac{12x}{L^3} - \frac{6}{L^2}\right] + \bar{\theta}_{y1}\left[\frac{-6x}{L^2} + \frac{4}{L}\right] + w_2\left[-\frac{12x}{L^3} + \frac{6}{L^2}\right] + \bar{\theta}_{y2}\left[\frac{-6x}{L^2} + \frac{2}{L}\right]\right\}^2 dx$$

By using the computer software *Mathematica* we can solve the above integral and express it in terms of w_1, $\bar{\theta}_{y1}$, w_2 and $\bar{\theta}_{y2}$, with that the total potential energy can be represented as follows

$$\Pi(w_1, \bar{\theta}_{y1}, w_2, \bar{\theta}_{y2}) = U^{int}(w_1, \bar{\theta}_{y1}, w_2, \bar{\theta}_{y2}) - U^{ext}(w_1, \bar{\theta}_{y1}, w_2, \bar{\theta}_{y2}) =$$

And the stationary point is given by

$$\frac{\partial \Pi}{\partial w_1} = \frac{\partial U^{int}}{\partial w_1} - \frac{\partial U^{ext}}{\partial w_1} = k_{11}w_1 + k_{12}\bar{\theta}_{y1} + k_{13}w_2 + k_{14}\bar{\theta}_{y2} - \frac{\partial U^{ext}}{\partial w_1} = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{y1}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{y1}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y1}} = k_{21}w_1 + k_{22}\bar{\theta}_{y1} + k_{23}w_2 + k_{24}\bar{\theta}_{y2} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y1}} = 0$$

$$\frac{\partial \Pi}{\partial w_2} = \frac{\partial U^{int}}{\partial w_2} - \frac{\partial U^{ext}}{\partial w_2} = k_{31}w_1 + k_{32}\bar{\theta}_{y1} + k_{33}w_2 + k_{34}\bar{\theta}_{y2} - \frac{\partial U^{ext}}{\partial w_2} = 0$$

$$\frac{\partial \Pi}{\partial \bar{\theta}_{y2}} = \frac{\partial U^{int}}{\partial \bar{\theta}_{y2}} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y2}} = k_{41}w_1 + k_{42}\bar{\theta}_{y1} + k_{43}w_2 + k_{44}\bar{\theta}_{y2} - \frac{\partial U^{ext}}{\partial \bar{\theta}_{y2}} = 0$$

By restructuring the above equations in matrix form we can obtain

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U^{ext}}{\partial w_1} \\ \dfrac{\partial U^{ext}}{\partial \bar{\theta}_{y1}} \\ \dfrac{\partial U^{ext}}{\partial w_2} \\ \dfrac{\partial U^{ext}}{\partial \bar{\theta}_{y2}} \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{k}^{(e)}]\{\boldsymbol{u}^{(e)}\} = \frac{\partial U^{ext}}{\partial \{\boldsymbol{u}^{(e)}\}} = \{\boldsymbol{f}_{Eq}^{(e)}\} \tag{6.680}$$

where the components of the stiffness matrix are given by

First row

$$k_{11} = \frac{Eb_0}{840L^3}(792\bar{a}^2\bar{c}L^4 + 792\bar{a}\bar{b}^2L^4 + 294\bar{b}^3L^3 + 1764\bar{a}\bar{b}\bar{c}L^3 + 840\bar{c}^3 + 1008\bar{b}^2\bar{c}L^2 + 720\bar{a}^2\bar{b}L^5$$
$$+1260\bar{b}\bar{c}^2L + 220\bar{a}^3L^6 + 1008\bar{a}\bar{c}^2L^2)$$

$$k_{12} = \frac{-Eb_0}{840L^2}(504\bar{a}\bar{b}\bar{c}L^3 + 228\bar{a}\bar{b}^2L^4 + 420\bar{c}^3 + 210\bar{a}^2\bar{b}L^5 + 228\bar{a}^2\bar{c}L^4 + 65\bar{a}^3L^6$$
$$+294\bar{a}\bar{c}^2L^2 + 420\bar{b}\bar{c}^2L + 84\bar{b}^3L^3 + 294\bar{b}^2\bar{c}L^2)$$

$$k_{13} = \frac{-Eb_0}{840L^3}(792\bar{a}^2\bar{c}L^4 + 792\bar{a}\bar{b}^2L^4 + 294\bar{b}^3L^3 + 1764\bar{a}\bar{b}\bar{c}L^3 + 840\bar{c}^3 + 1008\bar{b}^2\bar{c}L^2 + 720\bar{a}^2\bar{b}L^5$$
$$+1260\bar{b}\bar{c}^2L + 220\bar{a}^3L^6 + 1008\bar{a}\bar{c}^2L^2) = -k_{11}$$

$$k_{14} = \frac{-Eb_0}{840L^2}(1260\bar{a}\bar{b}\bar{c}L^3 + 564\bar{a}\bar{b}^2L^4 + 420\bar{c}^3 + 510\bar{a}^2\bar{b}L^5 + 564\bar{a}^2\bar{c}L^4 + 155\bar{a}^3L^6$$
$$+714\bar{a}\bar{c}^2L^2 + 840\bar{b}\bar{c}^2L + 210\bar{b}^3L^3 + 714\bar{b}^2\bar{c}L^2)$$

Second row

$$k_{21}=k_{12}$$

$$k_{22}=\frac{Eb_0}{840L}(65\bar{a}^2\bar{b}L^5+168\bar{a}\bar{b}\bar{c}L^3+112\bar{b}^2\bar{c}L^2+72\bar{a}\bar{b}^2L^4+72\bar{a}^2\bar{c}L^4+20\bar{a}^3L^6$$
$$+210\bar{b}\bar{c}^2L+112\bar{a}\bar{c}^2L^2+28\bar{b}^3L^3+280\bar{c}^3)$$

$$k_{23}=-k_{12}$$

$$k_{24}=\frac{Eb_0}{840L}(156\bar{a}^2\bar{c}L^4+140\bar{c}^3+45\bar{a}^3L^6+182\bar{b}^2\bar{c}L^2+156\bar{a}\bar{b}^2L^4+336\bar{a}\bar{b}\bar{c}L^3$$
$$+210\bar{b}\bar{c}^2L+182\bar{a}\bar{c}^2L^2+56\bar{b}^3L^3+145\bar{a}^2\bar{b}L^5)$$

Third row

$k_{31}=k_{13}$, $k_{32}=k_{23}$, $k_{33}=k_{11}$, $k_{34}=-k_{14}$

Fourth row

$k_{41}=k_{14}$, $k_{42}=k_{24}$, $k_{43}=k_{34}$,

$$k_{44}=\frac{Eb_0}{840L}(408\bar{a}\bar{b}^2L^4+365\bar{a}^2\bar{b}L^5+110\bar{a}^3L^6+532\bar{b}^2\bar{c}L^2+532\bar{a}\bar{c}^2L^2+630\bar{b}\bar{c}^2L$$
$$+924\bar{a}\bar{b}\bar{c}L^3+408\bar{a}^2\bar{c}L^4+154\bar{b}^3L^3+280\bar{c}^3)$$

And the parameters $\bar{a}$, $\bar{b}$ and $\bar{c}$ are given by Figure 6.221.

NOTE: If we are considering the deflection $v(x)$ instead of $w(x)$, (see **Problem 6.67**), the total potential becomes

$$\Pi(v_1,\bar{\theta}_{z1},v_2,\bar{\theta}_{z2})=U^{int}-U^{ext}=\frac{E}{2}\int_0^L I_z(v'')^2\,dx-U^{ext}$$

where the moment of inertia I_y is given by

$$I_z=\frac{a(x)b_0^3}{12}=\frac{[\bar{a}x^2+\bar{b}x+\bar{c}]b_0^3}{12} \tag{6.681}$$

Then,

$$\Pi(v_1,\bar{\theta}_{z1},v_2,\bar{\theta}_{z2})=\frac{E}{2}\int_0^L I_z(v'')^2\,dx-U^{ext}=\frac{E}{2}\int_0^L\frac{[\bar{a}x^2+\bar{b}x+\bar{c}]b_0^3}{12}(v'')^2\,dx-U^{ext} \tag{6.682}$$

In **Problem 6.67** we have obtained an expression for (v'') in terms of the degrees-of-freedom (nodal values of deflection and rotation):

$$v''=v_1\left[\frac{12x}{L^3}-\frac{6}{L^2}\right]+v_2\left[-\frac{12x}{L^3}+\frac{6}{L^2}\right]+\bar{\theta}_{z1}\left[\frac{6x}{L^2}-\frac{4}{L}\right]+\bar{\theta}_{z2}\left[\frac{6x}{L^2}-\frac{2}{L}\right]$$

And the total potential energy becomes

$$\Pi=\frac{Eb_0^3}{24}\int_0^L[\bar{a}x^2+\bar{b}x+\bar{c}]\left\{v_1\left[\frac{12x}{L^3}-\frac{6}{L^2}\right]+v_2\left[-\frac{12x}{L^3}+\frac{6}{L^2}\right]+\bar{\theta}_{z1}\left[\frac{6x}{L^2}-\frac{4}{L}\right]+\bar{\theta}_{z2}\left[\frac{6x}{L^2}-\frac{2}{L}\right]\right\}^2dx-U^{ext}$$

By using the computer software *Mathematica* we can solve the above integral and express the total potential energy in terms of v_1, $\bar{\theta}_{z1}$, v_2 and $\bar{\theta}_{z2}$. And the stationary point is given by

$$\frac{\partial \Pi}{\partial v_1} = k_{11} v_1 + k_{12} \overline{\theta}_{z1} + k_{13} v_2 + k_{14} \overline{\theta}_{z2} - \frac{\partial U^{ext}}{\partial v_1} = 0$$
$$\frac{\partial \Pi}{\partial \overline{\theta}_{z1}} = k_{21} v_1 + k_{22} \overline{\theta}_{z1} + k_{23} v_2 + k_{24} \overline{\theta}_{z2} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{z1}} = 0$$
$$\frac{\partial \Pi}{\partial v_2} = k_{31} v_1 + k_{32} \overline{\theta}_{z1} + k_{33} v_2 + k_{34} \overline{\theta}_{z2} - \frac{\partial U^{ext}}{\partial v_2} = 0$$
$$\frac{\partial \Pi}{\partial \overline{\theta}_{z2}} = k_{41} v_1 + k_{42} \overline{\theta}_{z1} + k_{43} v_2 + k_{44} \overline{\theta}_{z2} - \frac{\partial U^{ext}}{\partial \overline{\theta}_{z2}} = 0$$

with that we can obtain

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \overline{\theta}_{z1} \\ v_2 \\ \overline{\theta}_{z2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U^{ext}}{\partial v_1} \\ \dfrac{\partial U^{ext}}{\partial \overline{\theta}_{z1}} \\ \dfrac{\partial U^{ext}}{\partial v_2} \\ \dfrac{\partial U^{ext}}{\partial \overline{\theta}_{z2}} \end{Bmatrix} \quad \Leftrightarrow \quad [\boldsymbol{k}^{(e)}]\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{f}^{(e)}_{Eq}\} \tag{6.683}$$

where the components of the stiffness matrix are given by

First row

$$k_{11} = \frac{Eb_0^3}{10L^3}(4\overline{a}L^2 + 5\overline{b}L + 10\overline{c}), \quad k_{12} = \frac{Eb_0^3}{60L^2}(7\overline{a}L^2 + 10\overline{b}L + 30\overline{c})$$

$$k_{13} = \frac{-Eb_0^3}{10L^3}(4\overline{a}L^2 + 5\overline{b}L + 10\overline{c}), \quad k_{14} = \frac{Eb_0^3}{60L^2}(17\overline{a}L^2 + 20\overline{b}L + 30\overline{c})$$

Second row

$$k_{21} = k_{12}, \; k_{22} = \frac{Eb_0^3}{180L}(8\overline{a}L^2 + 15\overline{b}L + 60\overline{c}), \; k_{23} = -k_{12}, \; k_{24} = \frac{Eb_0^3}{180L}(13\overline{a}L^2 + 15\overline{b}L + 30\overline{c})$$

Third row

$$k_{31} = k_{13}, \; k_{32} = k_{23}, \; k_{33} = k_{11}, \; k_{34} = -k_{14}$$

Fourth row

$$k_{41} = k_{14}, \; k_{42} = k_{24}, \; k_{43} = k_{34}, \; k_{44} = \frac{Eb_0^3}{180L}(38\overline{a}L^2 + 45\overline{b}L + 60\overline{c})$$

And the parameters $\overline{a}$, $\overline{b}$ and $\overline{c}$ are given by Figure 6.221.

Problem 6.80

By means of **Problem 6.79** obtain the rotation at node 2 and the moment at node 1 for the problem described in Figure 6.222).

NOTE 0: Although the data are in SI unit, computationally is better to work with the units kN and cm, with that we avoid to deal with large numbers.

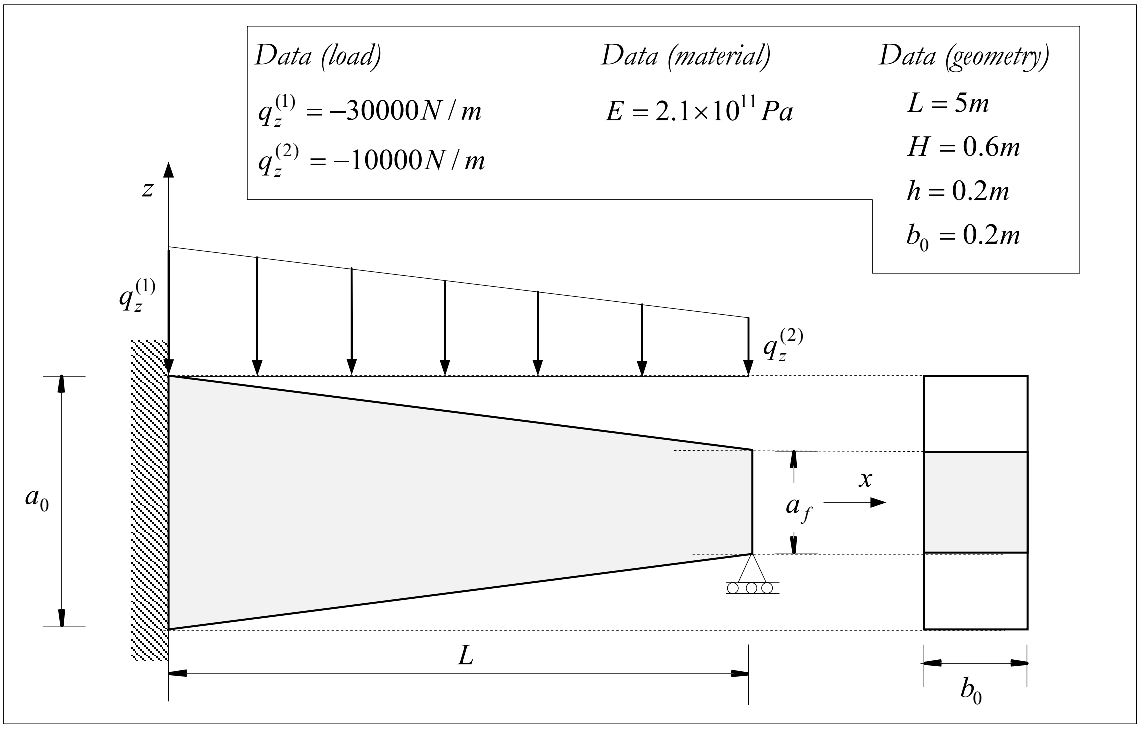

Figure 6.222: Beam with variable cross-section (Linear variation).

Solution:

By using the finite element formulation given by (6.680), where we will adopt 1 finite element, (Figure 6.223).

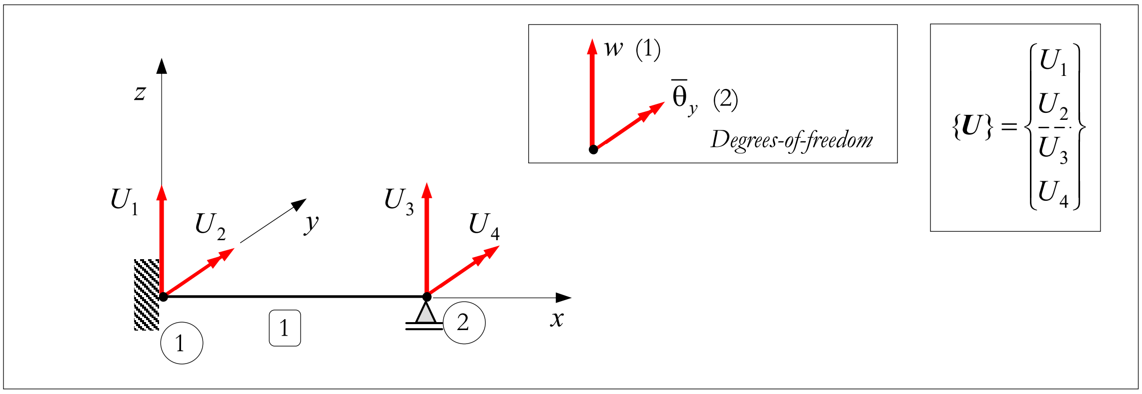

Figure 6.223: Global degrees-of-freedom.

The consistent load vector was obtained in **Problem 6.66**:

$$\{\boldsymbol{f}_{Eq}^{(e)}\} = \begin{Bmatrix} \dfrac{L}{20}(7q_z^{(1)} + 3q_z^{(2)}) \\ \dfrac{-L^2}{60}(3q_z^{(1)} + 2q_z^{(2)}) \\ \dfrac{L}{20}(3q_z^{(1)} + 7q_z^{(2)}) \\ \dfrac{L^2}{60}(2q_z^{(1)} + 3q_z^{(2)}) \end{Bmatrix} = \begin{Bmatrix} -6\times10^4 \\ 4.583333\times10^4 \\ -4\times10^4 \\ -3.75\times10^4 \end{Bmatrix} \tag{6.684}$$

The stiffness matrix given by (6.680), and for linear variation of the height, (see Figure 6.221), we have $a_f = 0.2$, $a_0 = 0.6$, $\bar{a} = 0$, $\bar{b} = \dfrac{(a_f - a_0)}{L} = -0.08$ and $\bar{c} = a_0 = 0.6$, thus

$$[\boldsymbol{k}^{(e)}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} = \begin{bmatrix} 0.311808 & -1.06176 & -0.311808 & -0.49728 \\ -1.06176 & 3.92 & 1.06176 & 1.3888 \\ -0.311808 & 1.06176 & 0.311808 & 0.49728 \\ -0.49728 & 1.3888 & 0.49728 & 1.0976 \end{bmatrix} \times 10^8$$

Then by applying the boundary conditions $U_1 = 0$, $U_2 = 0$ and $U_3 = 0$

$$[\bar{\boldsymbol{k}}^{(e)}]\{\boldsymbol{u}^{(e)}\} = \{\bar{\boldsymbol{f}}_{Eq}^{(e)}\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.0976\times10^8 \end{bmatrix} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.75\times10^4 \end{Bmatrix} \xrightarrow{Solve} \begin{Bmatrix} w_1 \\ \bar{\theta}_{y1} \\ w_2 \\ \bar{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.416545\times10^{-4} \end{Bmatrix}$$

Internal and Reaction Forces

The internal and reaction forces can be obtained as follows

$$\{\boldsymbol{r}_{Local}^{(e)}\} = -\{\boldsymbol{f}_{Eq\_L}^{(e)}\} + \{\bar{\boldsymbol{f}}_{Local}^{(e)}\} = -\{\boldsymbol{f}_{Eq\_L}^{(e)}\} + [\boldsymbol{k}_{Local}^{(e)}][\overline{\mathcal{A}}]\{\boldsymbol{u}_{Global}^{(e)}\} = -\{\boldsymbol{f}_{\boldsymbol{int}}^{(e)}\}$$

And the reaction vector in the Global system $\{\boldsymbol{r}_{Global}^{(e)}\} = [\overline{\mathcal{A}}]^T\{\boldsymbol{r}_{Local}^{(e)}\}$. Note also that for this problem the Global and Local systems have the same orientation, so $[\overline{\mathcal{A}}] = [\boldsymbol{I}]$.

Element 1:

Internal force (Local system=Global system)

$$\{\bar{\boldsymbol{f}}_{Local}^{(e)}\} = [\boldsymbol{k}_{Local}^{(e)}]\{\boldsymbol{u}_{Global}^{(e)}\}$$

$$= 10^8 \times \begin{bmatrix} 0.311808 & -1.06176 & -0.311808 & -0.49728 \\ -1.06176 & 3.92 & 1.06176 & 1.3888 \\ -0.311808 & 1.06176 & 0.311808 & 0.49728 \\ -0.49728 & 1.3888 & 0.49728 & 1.0976 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.416545\times10^{-4} \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.6989796 \\ -4.744898 \\ -1.6989796 \\ -3.75 \end{Bmatrix} \times 10^4$$

$$\{\boldsymbol{r}^{(1)}_{Local}\} = -\{\boldsymbol{f}^{(1)}_{int}\} = \begin{Bmatrix} f_{z1} \\ m_{y1} \\ f_{z2} \\ m_{y2} \end{Bmatrix} = \underbrace{-\begin{Bmatrix} -6 \\ 4.583333 \\ -4 \\ -3.75 \end{Bmatrix}\times 10^4}_{=\{\boldsymbol{f}^{(1)}_{Eq\_L}\}} + \underbrace{\begin{Bmatrix} 1.6989796 \\ -4.744898 \\ -1.6989796 \\ -3.75 \end{Bmatrix}\times 10^4}_{=\{\bar{\boldsymbol{f}}^{(1)}_{Local}\}} = \begin{Bmatrix} 7.6989796 \\ -9.32823129 \\ 2.30102041 \\ 0 \end{Bmatrix}\times 10^4$$

So, the moment at node 1 is $M_y^{(1)} = \{\boldsymbol{r}^{(1)}_{Local}\}_2 = -93282.3129$.

NOTE 1:

For *1 finite element* we have obtained: Rotation at node 2: -0.3416545×10^{-3}; and moment at node 1: $M_y^{(1)} = -93282.3129$

If we discretize into *2 finite elements* we can obtain

Rotation at node 3: $-0.331712829486\times 10^{-3}$

Deflection at node $2(x = 2.5)$: $-0.218494069993\times 10^{-3}$

Moment at node 1: $M_y^{(1)} = -93531.0701$

If we discretize into *4 finite elements* we can obtain

Rotation at node 5: $-0.3094426299\times 10^{-3}$

Deflection at node $3(x = 2.5)$: $-0.2230288945\times 10^{-3}$

Moment at node 1: $M_y^{(1)} = -94106.583974$

6.5.3 Introduction to Linear Buckling Problems

Problem 6.81

Obtain the total potential energy $\Pi = U^{int} - U^{ext}$ in terms of the deflection v for the problem described in Figure 6.224, (Laier&Barreiro (1983)), and for the internal potential energy consider only the effect due to the flexural moment M_z. Consider small deformation and small rotation regime. Consider also that the column has cross section constant.

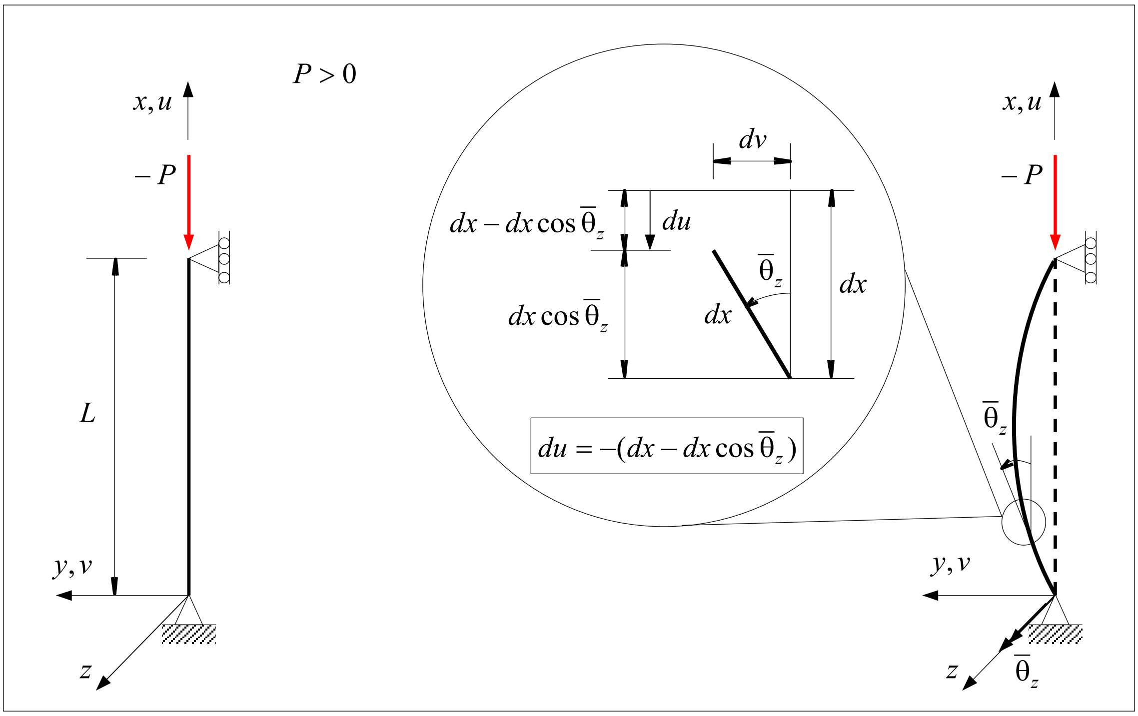

Figure 6.224: Column under compression.

Solution:

The internal potential energy for the deflection problem was established in **Problem 6.67**, which is

$$U^{int} = \frac{EI_z}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx \equiv \frac{EI_z}{2}\int_0^L (v'')^2\, dx \tag{6.685}$$

Note that since we are dealing with small rotations the following is true

$$\frac{dv}{dx} \equiv v' = \tan\overline{\theta}_z \approx \sin\overline{\theta}_z \approx \overline{\theta}_z \quad \textit{(small rotation)} \tag{6.686}$$

The differential of the external potential energy can be represented as follows

$$dU^{ext} = -P du = P(dx - dx\cos\overline{\theta}_z) = P dx(1-\cos\overline{\theta}_z) \tag{6.687}$$

Let us try to express the above equation in terms of sine of the angle. By considering the following trigonometric relation $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, when $\alpha = \beta = \frac{\overline{\theta}_z}{2}$

we can obtain $\cos(\overline{\theta}_z) = \cos^2\left(\frac{\overline{\theta}_z}{2}\right) - \sin^2\left(\frac{\overline{\theta}_z}{2}\right)$, and recall also that $\cos^2\left(\frac{\overline{\theta}_z}{2}\right) + \sin^2\left(\frac{\overline{\theta}_z}{2}\right) = 1$ holds, then the equation in (6.687) can be rewritten as follows

$$dU^{ext} = Pdx(1-\cos\overline{\theta}_z) = Pdx\left\{\cos^2\left(\frac{\overline{\theta}_z}{2}\right) + \sin^2\left(\frac{\overline{\theta}_z}{2}\right) - \left[\cos^2\left(\frac{\overline{\theta}_z}{2}\right) - \sin^2\left(\frac{\overline{\theta}_z}{2}\right)\right]\right\}$$

$$\Rightarrow dU^{ext} = 2P\sin^2\left(\frac{\overline{\theta}_z}{2}\right)dx \tag{6.688}$$

By considering small rotations the following is true

$$\sin^2\left(\frac{\overline{\theta}_z}{2}\right) \approx \left(\frac{\overline{\theta}_z}{2}\right)^2 = \frac{\overline{\theta}_z^{\,2}}{4} = \frac{(v')^2}{4}$$

With that the equation (6.688) can be rewritten as follows

$$dU^{ext} = 2P\sin^2\left(\frac{\overline{\theta}_z}{2}\right)dx \qquad \Rightarrow \qquad dU^{ext} = \frac{P(v')^2}{2}dx \tag{6.689}$$

And the external potential energy becomes:

$$U^{ext} = \int dU^{ext} = \int_0^L \frac{P(v')^2}{2}dx = \frac{P}{2}\int_0^L (v')^2\,dx \tag{6.690}$$

Then, the Total potential Energy can be expressed as follows

$$\boxed{\Pi = U^{int} - U^{ext} = \frac{1}{2}\int_0^L EI_z(v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx} \tag{6.691}$$

NOTE 1: If we are dealing with the deflection $w = w(x)$, the total potential energy is given by:

$$\boxed{\Pi = U^{int} - U^{ext} = \frac{1}{2}\int_0^L EI_y(w'')^2\,dx - \frac{P}{2}\int_0^L (w')^2\,dx} \tag{6.692}$$

NOTE 2: The value for P in which $\Pi = 0$ is called critic load (P_{cr}), then according to equation (6.691) we can conclude that

$$\Pi = \frac{1}{2}\int_0^L EI_z(v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx = 0 \qquad \Rightarrow \qquad P_{cr} = \frac{\int_0^L EI_z(v'')^2\,dx}{\int_0^L (v')^2\,dx}$$

For a exact value for the deflection $v(x)$ we have the exact value for the critical load $P_{cr}^{(exact)}$. And also note that the approximated value for P_{cr} is always greater than $P_{cr}^{(exact)}$.

Problem 6.82

Consider the problem established in **Problem 6.81**, (see Figure 6.224), and also consider the following approximations for the deflection ($v = v(x)$) of the column:

a) $v(x) = C_2(x^2 - Lx)$; b) $v(x) = C_2(x^2 - Lx) + C_3(x^3 - L^2x)$; c) $v(x) = C_4(x^4 - 2Lx^3 + L^3x)$

Obtain the critical load P_{cr} for each case.

NOTE: The exact value for the critical load for the problem describe in Figure 6.224, (Sechler (1952)), is

$$P_{cr}^{(exact)} = \frac{\pi^2 EI_z}{L^2} \approx 9.8696044 \frac{EI_z}{L^2}$$

Solution:

Case a): Taking the derivatives of the deflection $v(x) = C_2(x^2 - Lx)$ we can obtain:

$$v(x) = C_2(x^2 - Lx) \xrightarrow{\frac{d}{dx}} v'(x) = C_2(2x - L) \xrightarrow{\frac{d}{dx}} v'' = 2C_2$$

The total potential energy becomes

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx$$

$$\Rightarrow \Pi = \frac{EI_z}{2}\int_0^L (2C_2)^2\,dx - \frac{P}{2}\int_0^L [C_2(2x-L)]^2\,dx = 2EI_z L C_2^2 - \frac{P}{6}L^3 C_2^2 = \left(2EI_z L - \frac{P}{6}L^3\right)C_2^2$$

The inflection point is given by

$$\frac{\partial \Pi}{\partial C_2} = \frac{\partial}{\partial C_2}\left[\left(2EI_z L - \frac{P}{6}L^3\right)C_2^2\right] = 2\left(2EI_z L - \frac{P}{6}L^3\right)C_2 = 0 \quad \xrightarrow{C_2 \neq 0} \quad P_{cr} = \frac{12 EI_z}{L^2}$$

Note that $P_{cr} > P_{cr}^{(exact)}$ and if we compare with the exact value the error is 21.59%.

Case b): Taking the derivatives of the deflection $v(x) = C_2(x^2 - Lx) + C_3(x^3 - L^2 x)$ we can obtain:

$$v = C_2(x^2 - Lx) + C_3(x^3 - L^2 x) \xrightarrow{\frac{d}{dx}} v'(x) = C_2(2x - L) + C_3(3x^2 - L^2) \xrightarrow{\frac{d}{dx}} v'' = 2C_2 + 6C_3 x$$

The total potential energy becomes

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx$$

$$\Rightarrow \Pi = \frac{EI_z}{2}\int_0^L (2C_2 + 6C_3 x)^2\,dx - \frac{P}{2}\int_0^L [C_2(2x-L) + C_3(3x^2 - L^2)]^2\,dx$$

$$\Rightarrow \Pi(C_2, C_3) = 2EI_z L(C_2^2 + 3LC_2C_3 + 3L^2C_3^2) - \frac{PL^3}{30}(5C_2^2 + 15LC_2C_3 + 12L^2C_3^2)$$

Then

$$\frac{\partial \Pi}{\partial C_2} = \frac{\partial}{\partial K_2}\left[2EI_z L(C_2^2 + 3LC_2C_3 + 3L^2C_3^2) - \frac{PL^3}{30}(5C_2^2 + 15LC_2C_3 + 12L^2C_3^2)\right] = 0$$

$$= 2EI_z L(2C_2 + 3LC_3) - \frac{PL^3}{30}(10C_2 + 15LC_3) = 0$$

$$\frac{\partial \Pi}{\partial C_3} = \frac{\partial}{\partial C_3}\left[2EI_z L(C_2^2 + 3LC_2C_3 + 3L^2C_3^2) - \frac{PL^3}{30}(5C_2^2 + 15LC_2C_3 + 12L^2C_3^2)\right] = 0$$

$$= 2EI_z L(3LC_2 + 6L^2C_3) - \frac{PL^3}{30}(15LC_2 + 24L^2C_3) = 0$$

in matrix form becomes

$$\begin{bmatrix} 4EI_z L & 6EI_z L^2 \\ 6EI_z L^2 & 12EI_z L^3 \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} - P\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{2} \\ \frac{L^4}{2} & \frac{4L^5}{5} \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.693}$$

Note that

$$\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{2} \\ \frac{L^4}{2} & \frac{4L^5}{5} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{48}{L^3} & \frac{-30}{L^4} \\ \frac{-30}{L^4} & \frac{20}{L^5} \end{bmatrix}$$

And the equation in (6.693) can also be expressed as follows:

$$\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{2} \\ \frac{L^4}{2} & \frac{4L^5}{5} \end{bmatrix}^{-1}\begin{bmatrix} 4EI_z L & 6EI_z L^2 \\ 6EI_z L^2 & 12EI_z L^3 \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} - P\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{2} \\ \frac{L^4}{2} & \frac{4L^5}{5} \end{bmatrix}^{-1}\begin{bmatrix} \frac{L^3}{3} & \frac{L^4}{2} \\ \frac{L^4}{2} & \frac{4L^5}{5} \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{12EI_z}{L^2} & \frac{-72EI_z}{L} \\ 0 & \frac{60EI_z}{L^2} \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} - P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} \frac{12EI_z}{L^2} & \frac{-72EI_z}{L} \\ 0 & \frac{60EI_z}{L^2} \end{bmatrix} - P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \textit{with} \quad \begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow ([\boldsymbol{A}] - P[\boldsymbol{1}])\{\boldsymbol{C}\} = \{\boldsymbol{0}\}$$

Note that the above system has non-trivial solution, i.e. $\{\boldsymbol{C}\} \neq \{\boldsymbol{0}\}$, if and only if the determinant of $([\boldsymbol{A}] - P[\boldsymbol{1}])$ is zero, in other words, we are dealing with the eigenvalue-eigenvector problem, where the eigenvalues are P_i and the eigenvectors are $\{\boldsymbol{C}\}$, and the critical value is the smallest value of P_i. After solving the above problem we obtain the following eigenvalues

$$\det([\boldsymbol{A}] - P[\boldsymbol{1}]) = 0 \quad \Rightarrow \quad \left(\frac{12EI_z}{L^2} - P\right)\left(\frac{60EI_z}{L^2} - P\right) = 0 \quad \Rightarrow \quad \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} \frac{12EI_z}{L^2} \\ \frac{60EI_z}{L^2} \end{Bmatrix}$$

$$\Rightarrow P_{cr} = P_1 = \frac{12EI_z}{L^2}$$

which matches the solution for the case (a) in which we have considered a quadratic function for the deflection. Note that at $x = \frac{L}{2}$ the derivative of the deflection is zero, so

$$v'(x = \frac{L}{2}) = C_2(2x - L) + C_3(3x^2 - L^2) = C_2\left(2\frac{L}{2} - L\right) + C_3\left(3\frac{L^2}{4} - L^2\right) = 0 \Rightarrow C_3 = 0$$

In other words, the total potential does not depend on C_3.

Case c): Taking the derivatives of the deflection $v(x) = C_4(x^4 - 2Lx^3 + L^3 x)$ we can obtain:

$$v(x) = C_4(x^4 - 2Lx^3 + L^3 x) \xrightarrow{\frac{d}{dx}} v'(x) = C_4(4x^3 - 6Lx^2 + L^3) \xrightarrow{\frac{d}{dx}} v'' = C_4(12x^2 - 12Lx)$$

The total potential energy becomes

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx$$

$$\Rightarrow \Pi = \frac{EI_z}{2}\int_0^L [C_4(12x^2 - 12Lx)]^2\,dx - \frac{P}{2}\int_0^L [C_4(4x^3 - 6Lx^2 + L^3)]^2\,dx$$

$$\Rightarrow \Pi(C_4) = \frac{12EI_z L^5}{5}C_4^2 - \frac{17PL^7}{70}C_4^2$$

The inflection point is given by

$$\Rightarrow \frac{\partial \Pi}{\partial C_4} = \frac{\partial}{\partial C_4}\left[\frac{12EI_z L^5}{5}C_4^2 - \frac{17PL^7}{70}C_4^2\right] = \left(\frac{24EI_z L^5}{5} - \frac{17PL^7}{35}\right)C_4 = 0$$

$$\xrightarrow{C_4 \neq 0} \quad P_{cr} = \frac{168EI_z}{17L^2} \approx 9.88235\frac{EI_z}{L^2} > P_{cr}^{(exact)}$$

which is a good solution since the error is 0.1292%.

6.5.3.1 Introduction to Finite Element for Column Problems

Problem 6.83

a) Obtain Finite Element Formulation for the problem established in **Problem 6.81**. Consider the degree-of-freedom (beam element) as the one described in Figure 6.225. b) Use this required formulation to solve the problem described in Figure 6.225(a).

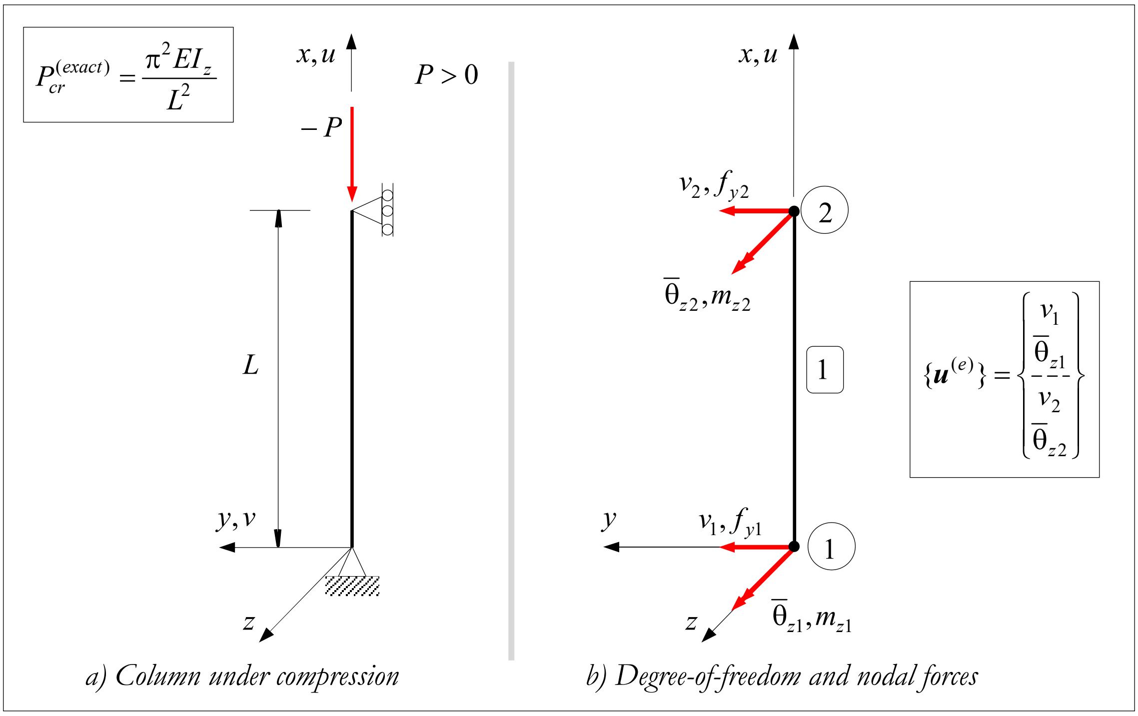

Figure 6.225: Column under compression.

Solution:

a) Let us express the total potential energy in terms of the degrees-of-freedom $\{\boldsymbol{u}^{(e)}\}$. In **Problem 6.67** we have obtained the integrals $\int_0^L v'^2 dx$ and $\int_0^L v''^2 dx$ in terms of $\{\boldsymbol{u}^{(e)}\}$. Then, by considering the total potential energy:

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{P}{2}\int_0^L (v')^2\,dx \tag{6.694}$$

we can obtain

$$\begin{aligned}\Pi = \frac{EI_z}{2}\Bigg[&\frac{12}{L^3}v_1^2 + \frac{12}{L^3}v_2^2 + \frac{4}{L}\overline{\theta}_{z1}^2 + \frac{4}{L}\overline{\theta}_{z2}^2 - \frac{24}{L^3}v_1 v_2 + \frac{12}{L^2}v_1\overline{\theta}_{z1} + \frac{12}{L^2}v_1\overline{\theta}_{z2} - \frac{12}{L^2}v_2\overline{\theta}_{z1} - \frac{12}{L^2}v_2\overline{\theta}_{z2}\\ &+\frac{4}{L}\overline{\theta}_{z1}\overline{\theta}_{z2}\Bigg] - \frac{P}{2}\Bigg[\frac{6}{5L}v_1^2 + \frac{6}{5L}v_2^2 + \frac{2L}{15}\overline{\theta}_{z1}^2 + \frac{2L}{15}\overline{\theta}_{z2}^2 - \frac{12}{5L}v_1v_2 + \frac{1}{5}v_1\overline{\theta}_{z1} + \frac{1}{5}v_1\overline{\theta}_{z2}\\ &-\frac{1}{5}v_2\overline{\theta}_{z1} - \frac{1}{5}v_2\overline{\theta}_{z2} - \frac{L}{15}\overline{\theta}_{z1}\overline{\theta}_{z2}\Bigg]\end{aligned}$$

As we are looking for the stationary state the following must hold:

$$\frac{\partial\Pi}{\partial v_1} = 0 \quad\Rightarrow\quad \frac{EI_z}{2}\left\{\frac{24}{L^3}v_1 + \frac{12}{L^2}\overline{\theta}_{z1} - \frac{24}{L^3}v_2 + \frac{12}{L^2}\overline{\theta}_{z2}\right\} - \frac{P}{2}\left\{\frac{12}{5L}v_1 + \frac{1}{5}\overline{\theta}_{z1} - \frac{12}{5L}v_2 + \frac{1}{5}\overline{\theta}_{z2}\right\} = 0$$

$$\frac{\partial\Pi}{\partial \overline{\theta}_{z1}} = 0 \quad\Rightarrow\quad \frac{EI_z}{2}\left\{\frac{12}{L^2}v_1 + \frac{8}{L}\overline{\theta}_{z1} - \frac{12}{L^2}v_2 + \frac{4}{L}\overline{\theta}_{z2}\right\} - \frac{P}{2}\left\{\frac{1}{5}v_1 + \frac{4L}{15}\overline{\theta}_{z1} - \frac{1}{5}v_2 - \frac{L}{15}\overline{\theta}_{z2}\right\} = 0$$

$$\frac{\partial\Pi}{\partial v_2} = 0 \quad\Rightarrow\quad \frac{EI_z}{2}\left\{\frac{-24}{L^3}v_1 - \frac{12}{L^2}\overline{\theta}_{z1} + \frac{24}{L^3}v_2 - \frac{12}{L^2}\overline{\theta}_{z2}\right\} - \frac{P}{2}\left\{\frac{-12}{5L}v_1 - \frac{1}{5}\overline{\theta}_{z1} + \frac{12}{5L}v_2 - \frac{1}{5}\overline{\theta}_{z2}\right\} = 0$$

$$\frac{\partial\Pi}{\partial \overline{\theta}_{z2}} = 0 \quad\Rightarrow\quad \frac{EI_z}{2}\left\{\frac{12}{L^2}v_1 + \frac{4}{L}\overline{\theta}_{z1} - \frac{12}{L^2}v_2 + \frac{8}{L}\overline{\theta}_{z2}\right\} - \frac{P}{2}\left\{\frac{1}{5}v_1 - \frac{L}{15}\overline{\theta}_{z1} - \frac{1}{5}v_2 + \frac{4L}{15}\overline{\theta}_{z2}\right\} = 0$$

Restructuring the above set of equations in matrix form we can obtain:

$$\left(EI_z\begin{bmatrix}\frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2}\\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L}\\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2}\\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L}\end{bmatrix} - P\begin{bmatrix}\frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10}\\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30}\\ \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10}\\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15}\end{bmatrix}\right)\begin{Bmatrix}v_1\\ \overline{\theta}_{z1}\\ v_2\\ \overline{\theta}_{z2}\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{6.695}$$

$$\Rightarrow ([\boldsymbol{k}^{(e)}] - P[\boldsymbol{kp}_y^{(e)}])\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{0}\}$$

where $[\boldsymbol{kp}_y^{(e)}]$ is called Geometric Matrix.

b) As we are using 1 finite element we can apply directly the above equation and apply the boundary conditions: $v_1 = 0$, $v_2 = 0$, then the above equation becomes

$$\left(\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{4EI_z}{L} & 0 & \frac{2EI_z}{L}\\ 0 & 0 & 1 & 0\\ 0 & \frac{2EI_z}{L} & 0 & \frac{4EI_z}{L}\end{bmatrix} - P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{2L}{15} & 0 & \frac{-L}{30}\\ 0 & 0 & 1 & 0\\ 0 & \frac{-L}{30} & 0 & \frac{2L}{15}\end{bmatrix}\right)\begin{Bmatrix}v_1\\ \overline{\theta}_{z1}\\ v_2\\ \overline{\theta}_{z2}\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \Leftrightarrow ([\overline{\boldsymbol{k}}^{(e)}] - P[\overline{\boldsymbol{kp}_y}^{(e)}])\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{0}\}$$

By applying the solution for eigenvalue-eigenvector problem we can obtain

$$\det([\overline{\boldsymbol{k}}^{(e)}] - P[\overline{\boldsymbol{kp}_y}^{(e)}]) = 0 \quad \Rightarrow \quad \frac{1}{60}(1-P)^2 \frac{(720(EI_z)^2 - 72 EI_z L^2 P + L^4 P^2)}{L^2} = 0$$

We have 4 values for P whose values are

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \dfrac{12 EI_z}{L^2} \\ \dfrac{60 EI_z}{L^2} \end{Bmatrix}$$

The solutions $P_1 = P_2 = 1$ are related to the boundary conditions and can be discarded. The critical value is the smallest one among the other two solutions, i.e. $P_{cr} = \dfrac{12 EI_z}{L^2}$. The solution can be improved by discretizing the column into more finite elements.

NOTE: If we are dealing with the deflection $w = w(x)$, (see **Problem 6.62**-NOTE 1), the following is true

$$U^{ext} = \frac{P}{2}\int_0^L w'^2 dx = \frac{P}{2}\left[\frac{6}{5L}w_1^2 + \frac{6}{5L}w_2^2 + \frac{2L}{15}\overline{\theta}_{y1}^2 + \frac{2L}{15}\overline{\theta}_{y2}^2 - \frac{12}{5L}w_1 w_2 - \frac{1}{5}w_1\overline{\theta}_{y1} - \frac{1}{5}w_1\overline{\theta}_{y2} \right.$$
$$\left. + \frac{1}{5}w_2\overline{\theta}_{y1} + \frac{1}{5}w_2\overline{\theta}_{y2} - \frac{L}{15}\overline{\theta}_{y1}\overline{\theta}_{y2}\right]$$

then

$$\frac{\partial \Pi}{\partial w_1} = 0 \Rightarrow \frac{EI_y}{2}\left\{\frac{24}{L^3}w_1 - \frac{12}{L^2}\overline{\theta}_{y1} - \frac{24}{L^3}w_2 - \frac{12}{L^2}\overline{\theta}_{y2}\right\} - \frac{P}{2}\left\{\frac{12}{5L}w_1 - \frac{1}{5}\overline{\theta}_{y1} - \frac{12}{5L}w_2 - \frac{1}{5}\overline{\theta}_{y2}\right\} = 0$$
$$\frac{\partial \Pi}{\partial \overline{\theta}_{y1}} = 0 \Rightarrow \frac{EI_y}{2}\left\{\frac{-12}{L^2}w_1 + \frac{8}{L}\overline{\theta}_{y1} + \frac{12}{L^2}w_2 + \frac{4}{L}\overline{\theta}_{y2}\right\} - \frac{P}{2}\left\{\frac{-1}{5}w_1 + \frac{4L}{15}\overline{\theta}_{y1} + \frac{1}{5}w_2 - \frac{L}{15}\overline{\theta}_{y2}\right\} = 0$$
$$\frac{\partial \Pi}{\partial w_2} = 0 \Rightarrow \frac{EI_y}{2}\left\{\frac{-24}{L^3}w_1 + \frac{12}{L^2}\overline{\theta}_{y1} + \frac{24}{L^3}w_2 + \frac{12}{L^2}\overline{\theta}_{y2}\right\} - \frac{P}{2}\left\{\frac{-12}{5L}w_1 + \frac{1}{5}\overline{\theta}_{y1} + \frac{12}{5L}w_2 + \frac{1}{5}\overline{\theta}_{y2}\right\} = 0$$
$$\frac{\partial \Pi}{\partial \overline{\theta}_{y2}} = 0 \Rightarrow \frac{EI_y}{2}\left\{\frac{-12}{L^2}w_1 + \frac{4}{L}\overline{\theta}_{y1} + \frac{12}{L^2}w_2 + \frac{8}{L}\overline{\theta}_{y2}\right\} - \frac{P}{2}\left\{\frac{-1}{5}w_1 - \frac{L}{15}\overline{\theta}_{y1} + \frac{1}{5}w_2 + \frac{4L}{15}\overline{\theta}_{y2}\right\} = 0$$

Restructuring the above set of equations in matrix form we can obtain:

$$\left(EI_y \begin{bmatrix} \frac{12}{L^3} & \frac{-6}{L^2} & \frac{-12}{L^3} & \frac{-6}{L^2} \\ \frac{-6}{L^2} & \frac{4}{L} & \frac{6}{L^2} & \frac{2}{L} \\ \frac{-12}{L^3} & \frac{6}{L^2} & \frac{12}{L^3} & \frac{6}{L^2} \\ \frac{-6}{L^2} & \frac{2}{L} & \frac{6}{L^2} & \frac{4}{L} \end{bmatrix} - P \begin{bmatrix} \frac{6}{5L} & \frac{-1}{10} & \frac{-6}{5L} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{2L}{15} & \frac{1}{10} & \frac{-L}{30} \\ \frac{-6}{5L} & \frac{1}{10} & \frac{6}{5L} & \frac{1}{10} \\ \frac{-1}{10} & \frac{-L}{30} & \frac{1}{10} & \frac{2L}{15} \end{bmatrix} \right) \begin{Bmatrix} w_1 \\ \overline{\theta}_{y1} \\ w_2 \\ \overline{\theta}_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.696}$$

$$\Rightarrow ([\boldsymbol{k}^{(e)}] - P[\boldsymbol{kp}_z^{(e)}])\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{0}\}$$

Problem 6.84

Obtain the critical load for the cases described in Figure 6.226(a), (b) and (c), by using the finite element formulation, (see **Problem 6.83**).

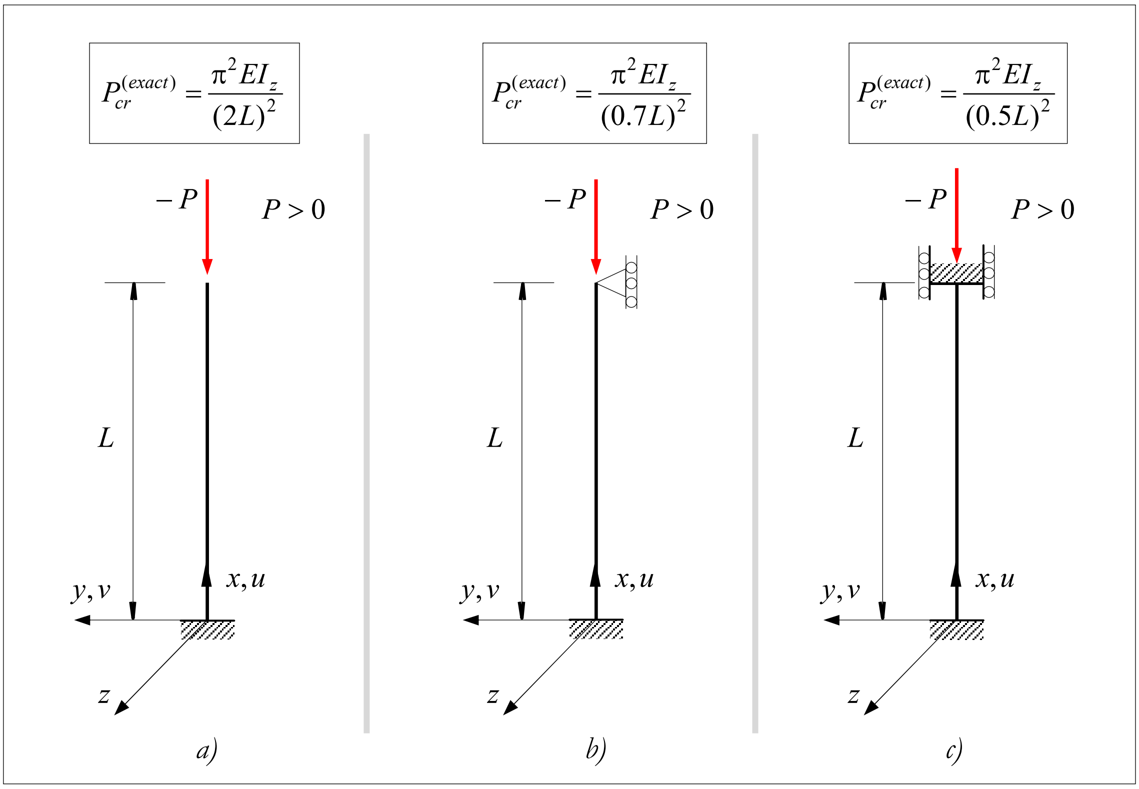

Figure 6.226: Column under compression.

Solution:

This problem is the same one described in **Problem 6.83**, i.e.:

$$\left(EI_z \left[\begin{array}{cc|cc} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \hline \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{array}\right] - P \left[\begin{array}{cc|cc} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30} \\ \hline \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{array}\right] \right) \left\{\begin{array}{c} v_1 \\ \overline{\theta}_{z1} \\ \hline v_2 \\ \overline{\theta}_{z2} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right\}$$

Case a): For case a) the boundary conditions are $v_1 = 0$, $\overline{\theta}_{z1} = 0$, then the above equation becomes:

$$\left(\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] - P \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{6}{5L} & \frac{-1}{10} \\ 0 & 0 & \frac{-1}{10} & \frac{2L}{15} \end{array}\right] \right) \left\{\begin{array}{c} v_1 \\ \overline{\theta}_{z1} \\ \hline v_2 \\ \overline{\theta}_{z2} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right\}$$

$$([\overline{\boldsymbol{k}}^{(e)}] - P[\overline{\boldsymbol{kp}_y}^{(e)}])\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{0}\}$$

the above equation has non-trivial solution if and only if

$$\det([\bar{\boldsymbol{k}}^{(e)}]-P[\overline{\boldsymbol{kp}_y}^{(e)}])=0 \quad \Rightarrow \quad \frac{1}{20}(P-1)^2\frac{(240(EI_z)^2-104EI_zL^2P+3L^4P^2)}{L^4}=0$$

We have 4 values for P whose values are

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1 \\ \hline 32.180705\dfrac{EI_z}{L^2} \\ 2.4859617\dfrac{EI_z}{L^2} \end{Bmatrix}$$

The solutions $P_1=P_2=1$ are related to the boundary conditions and can be discarded. The critical value is the smallest one among the others, i.e.

$$P_{cr}=2.4859617\frac{EI_z}{L^2}>P_{cr}^{(exact)}=2.4674011\frac{EI_z}{L^2}.$$

which error is about 0.752%.

Case b)

For this case the boundary conditions are $v_1=0$, $\bar{\theta}_{z1}=0$ and $v_2=0$, then the system to solved is

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{4EI_z}{L} \end{bmatrix}-P\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{2L}{15} \end{bmatrix}\right)\begin{Bmatrix} v_1 \\ \bar{\theta}_{z1} \\ v_2 \\ \bar{\theta}_{z2} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$([\bar{\boldsymbol{k}}^{(e)}]-P[\overline{\boldsymbol{kp}_y}^{(e)}])\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{0}\}$$

Then

$$\det([\bar{\boldsymbol{k}}^{(e)}]-P[\overline{\boldsymbol{kp}_y}^{(e)}])=0 \quad \Rightarrow \quad \frac{2}{15}(P-1)^3\frac{(L^2P-30EI_z)}{L}=0 \quad \Rightarrow \quad \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1 \\ \hline 1 \\ 30\dfrac{EI_z}{L^2} \end{Bmatrix}$$

$$\Rightarrow P_{cr}=30\frac{EI_z}{L^2}>P_{cr}^{(exact)}=20.1420498\frac{EI_z}{L^2}.$$

which error is about 48.94%. The error can be minimized by considering more finite elements. For instance, if we are adopting two elements we obtain $P_{cr}=20.7088\dfrac{EI_z}{L^2}$, and the error associated with it is about 2.81%.

Case c)

For the case c) we cannot use 1 finite element due to the boundary conditions, so we will adopt two finite elements for the discretization, (see Figure 6.227).

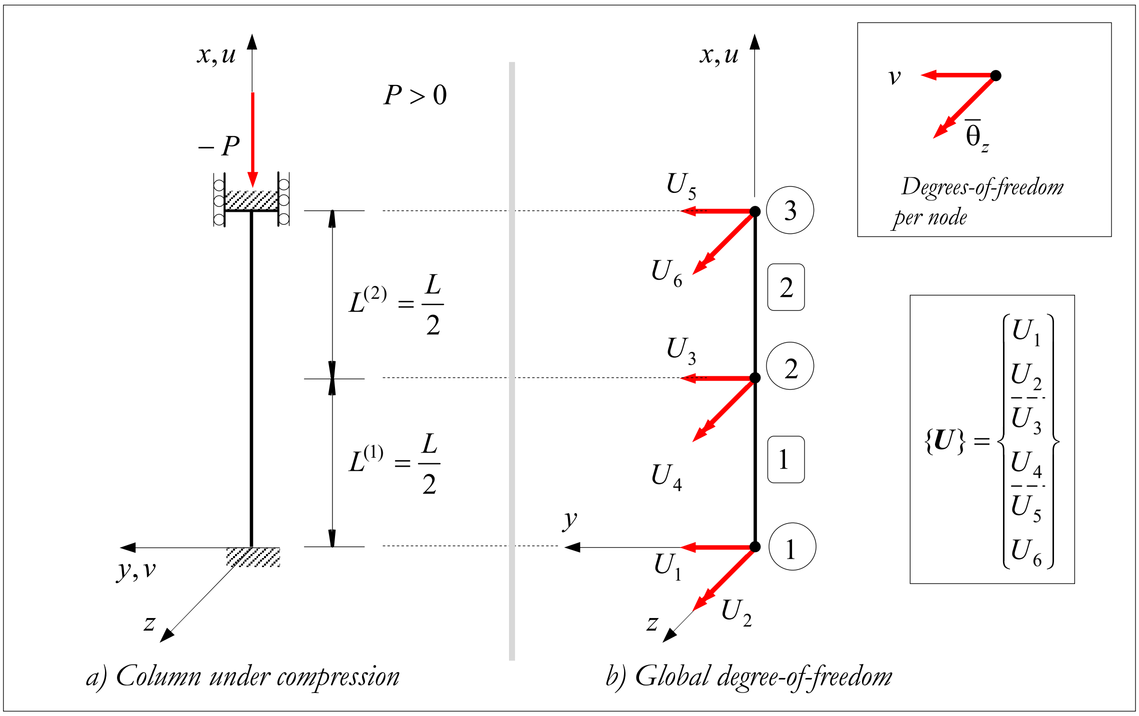

Figure 6.227: Column under compression.

Solution:

Construction of the Global Matrices - $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$

We need to construct the following system

$$[\ [\boldsymbol{K}_{Global}] - P[\boldsymbol{Kp}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\}$$

The matrices $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$ can be constructed by assembling the individual beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2} [\boldsymbol{k}^{(e)}_{Global}] \qquad \Bigg| \qquad [\boldsymbol{Kp}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2} [\boldsymbol{kp}^{(e)}_{Global}] \tag{6.697}$$

where

$$[\boldsymbol{k}^{(e)}_{Global}] = EI_z \left[\begin{array}{cc|cc} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \hline \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{array}\right] \quad ; \quad [\boldsymbol{kp}^{(e)}_{Global}] = \left[\begin{array}{cc|cc} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30} \\ \hline \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{array}\right] \tag{6.698}$$

For this problem the stiffness matrices for both elements are the same.

Element 1 and 2: $(L)^{(1)} = (L)^{(2)} = \frac{L}{2}$, $(EI_z)^{(1)} = (EI_z)^{(2)} = EI_z$ and by substituting these values into the equations in (6.698) we can obtain:

$$
\begin{array}{ccc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \quad \Leftarrow \; Global \; \Rightarrow & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[\boldsymbol{k}^{(1)}_{Global}] = EI_z & \left[\begin{array}{cc|cc}
\frac{96}{L^3} & \frac{24}{L^2} & \frac{-96}{L^3} & \frac{24}{L^2} \\
\frac{24}{L^2} & \frac{8}{L} & \frac{-24}{L^2} & \frac{4}{L} \\ \hline
\frac{-96}{L^3} & \frac{-24}{L^2} & \frac{96}{L^3} & \frac{-24}{L^2} \\
\frac{24}{L^2} & \frac{4}{L} & \frac{-24}{L^2} & \frac{8}{L}
\end{array}\right] \quad ; \quad [\boldsymbol{kp}^{(1)}_{Global}] = & \left[\begin{array}{cc|cc}
\frac{12}{5L} & \frac{1}{10} & \frac{-12}{5L} & \frac{1}{10} \\
\frac{1}{10} & \frac{L}{15} & \frac{-1}{10} & \frac{-L}{60} \\ \hline
\frac{-12}{5L} & \frac{-1}{10} & \frac{12}{5L} & \frac{-1}{10} \\
\frac{1}{10} & \frac{-L}{60} & \frac{-1}{10} & \frac{L}{15}
\end{array}\right]
\end{array}
$$

$$
[\boldsymbol{K}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}] = \begin{bmatrix}
k^{(1)}_{11} & k^{(1)}_{12} & k^{(1)}_{13} & k^{(1)}_{14} & 0 & 0 \\
k^{(1)}_{21} & k^{(1)}_{22} & k^{(1)}_{23} & k^{(1)}_{24} & 0 & 0 \\
k^{(1)}_{31} & k^{(1)}_{32} & k^{(1)}_{33} + k^{(2)}_{11} & k^{(1)}_{34} + k^{(2)}_{12} & k^{(2)}_{13} & k^{(2)}_{14} \\
k^{(1)}_{41} & k^{(1)}_{42} & k^{(1)}_{43} + k^{(2)}_{21} & k^{(1)}_{44} + k^{(2)}_{22} & k^{(2)}_{23} & k^{(2)}_{24} \\
0 & 0 & k^{(2)}_{31} & k^{(2)}_{32} & k^{(2)}_{33} & k^{(2)}_{34} \\
0 & 0 & k^{(2)}_{41} & k^{(2)}_{42} & k^{(2)}_{43} & k^{(2)}_{44}
\end{bmatrix}
$$

After the values are substituted we can obtain

$$
[\boldsymbol{K}_{Global}]_{6\times 6} = EI_z \begin{bmatrix}
\frac{96}{L^3} & \frac{24}{L^2} & \frac{-96}{L^3} & \frac{24}{L^2} & 0 & 0 \\
\frac{24}{L^2} & \frac{8}{L} & \frac{-24}{L^2} & \frac{4}{L} & 0 & 0 \\
\frac{-96}{L^3} & \frac{-24}{L^2} & \frac{192}{L^3} & 0 & \frac{-96}{L^3} & \frac{24}{L^2} \\
\frac{24}{L^2} & \frac{4}{L} & 0 & \frac{16}{L} & \frac{-24}{L^2} & \frac{4}{L} \\
0 & 0 & \frac{-96}{L^3} & \frac{-24}{L^2} & \frac{96}{L^3} & \frac{-24}{L^2} \\
0 & 0 & \frac{24}{L^2} & \frac{4}{L} & \frac{-24}{L^2} & \frac{8}{L}
\end{bmatrix}
$$

And

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad Global \\
[\boldsymbol{Kp}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{kp}^{(e)}_{Global}] = & \begin{bmatrix}
kp^{(1)}_{11} & kp^{(1)}_{12} & kp^{(1)}_{13} & kp^{(1)}_{14} & 0 & 0 \\
kp^{(1)}_{21} & kp^{(1)}_{22} & kp^{(1)}_{23} & kp^{(1)}_{24} & 0 & 0 \\
kp^{(1)}_{31} & kp^{(1)}_{32} & kp^{(1)}_{33} + kp^{(2)}_{11} & kp^{(1)}_{34} + kp^{(2)}_{12} & kp^{(2)}_{13} & kp^{(2)}_{14} \\
kp^{(1)}_{41} & kp^{(1)}_{42} & kp^{(1)}_{43} + kp^{(2)}_{21} & kp^{(1)}_{44} + kp^{(2)}_{22} & kp^{(2)}_{23} & kp^{(2)}_{24} \\
0 & 0 & kp^{(2)}_{31} & kp^{(2)}_{32} & kp^{(2)}_{33} & kp^{(2)}_{34} \\
0 & 0 & kp^{(2)}_{41} & kp^{(2)}_{42} & kp^{(2)}_{43} & kp^{(2)}_{44}
\end{bmatrix}
\end{array}
$$

After the values are substituted we can obtain

$$[\boldsymbol{Kp}_{Global}]_{6\times6} = \begin{bmatrix} \frac{12}{5L} & \frac{1}{10} & \frac{-12}{5L} & \frac{1}{10} & 0 & 0 \\ \frac{1}{10} & \frac{L}{15} & \frac{-1}{10} & \frac{-L}{60} & 0 & 0 \\ \frac{-12}{5L} & \frac{-1}{10} & \frac{24}{5L} & 0 & \frac{-12}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{-L}{60} & 0 & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{60} \\ 0 & 0 & \frac{-12}{5L} & \frac{-1}{10} & \frac{12}{5L} & \frac{-1}{10} \\ 0 & 0 & \frac{1}{10} & \frac{-L}{60} & \frac{-1}{10} & \frac{L}{15} \end{bmatrix}$$

Applying the Boundary Conditions

Note that there are restrictions to move for the following degrees-of-freedom: $U_1 = 0$, $U_2 = 0$, $U_5 = 0$ and $U_6 = 0$, thus

$$[\ [\boldsymbol{K}_{Global}] - P[\boldsymbol{Kp}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\} \xrightarrow{\text{Boundary Conditions}} [\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{192EI_z}{L^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{16EI_z}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - P \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{24}{5L} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2L}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\}$$

The solution for the above system, i.e. the eigenvalues of $\det[\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ] = 0$ are:

$$\frac{16}{25}(P-1)^4 (PL^2 - 40EI_z)\frac{(PL^2 - 120EI_z)}{L^4} = 0$$

$$P_1 = P_2 = P_3 = P_4 = 1; \qquad P_5 = 40\frac{EI_z}{L^2}; \qquad P_5 = 120\frac{EI_z}{L^2}$$

And the critical value is

$$P_{cr} = 40\frac{EI_z}{L^2} > P_{cr}^{(exact)} \approx 39.478\frac{EI_z}{L^2}$$

Problem 6.85

a) Consider the column described in Figure 6.228 in order to obtain the critical load by considering two finite elements, (see Figure 6.228(b)).

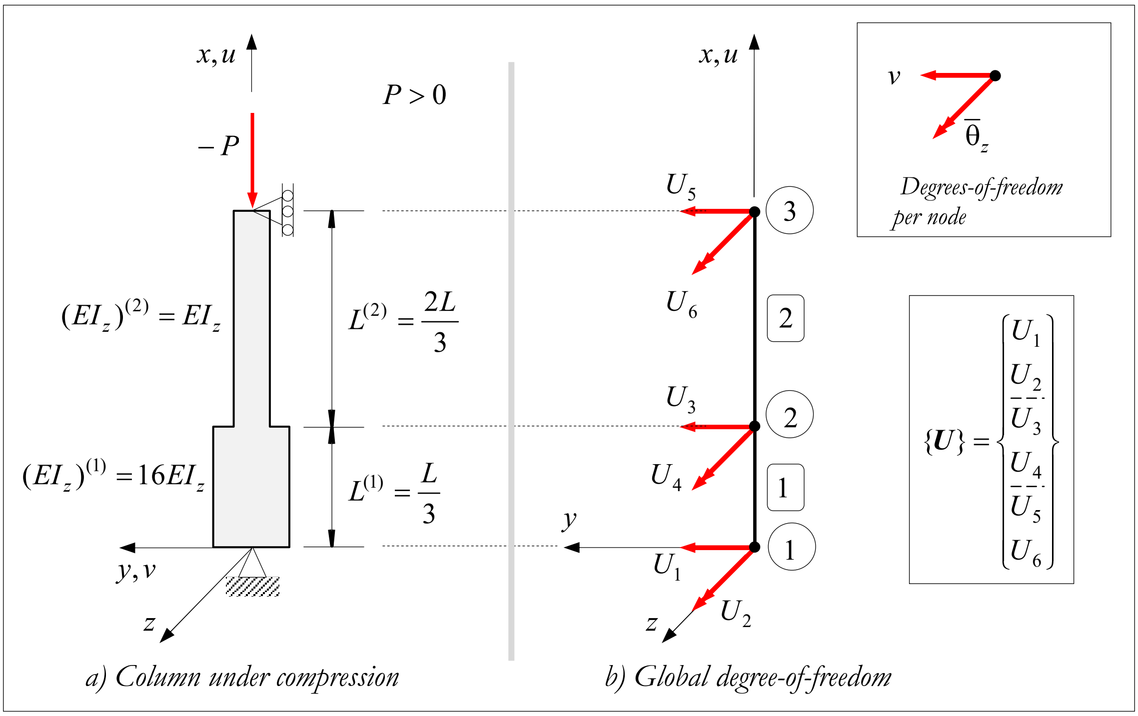

Figure 6.228: Column under compression.

Solution:

Construction of the Global Matrices - $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$

We need to construct the following system

$$[\ [\boldsymbol{K}_{Global}]-P[\boldsymbol{Kp}_{Global}]\ ]\{\boldsymbol{U}\}=\{\boldsymbol{0}\}$$

The matrices $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$ can be constructed by assembling the individual beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{6\times 6}=\mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}] \qquad \Bigg| \qquad [\boldsymbol{Kp}_{Global}]_{6\times 6}=\mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{kp}^{(e)}_{Global}] \tag{6.699}$$

where

$$[\boldsymbol{k}^{(e)}_{Global}]=EI_z\left[\begin{array}{cc:cc} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \hdashline \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{array}\right] \quad ; \quad [\boldsymbol{kp}^{(e)}_{Global}]=\left[\begin{array}{cc:cc} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30} \\ \hdashline \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{array}\right] \tag{6.700}$$

Element 1: $(L)^{(1)}=\dfrac{L}{3}$, $(EI_z)^{(1)}=16EI_z$ and by substituting these values into the equations in (6.700) we can obtain:

$$
\begin{array}{cccccc}
& 1 & 2 & 3 & 4 & \Leftarrow Global \Rightarrow
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}
$$

$$
[\boldsymbol{k}^{(1)}_{Global}] = EI_z \left[\begin{array}{cc|cc}
\frac{5184}{L^3} & \frac{864}{L^2} & \frac{-5184}{L^3} & \frac{864}{L^2} \\
\frac{864}{L^2} & \frac{192}{L} & \frac{-864}{L^2} & \frac{96}{L} \\ \hline
\frac{-5184}{L^3} & \frac{-864}{L^2} & \frac{5184}{L^3} & \frac{-864}{L^2} \\
\frac{864}{L^2} & \frac{96}{L} & \frac{-864}{L^2} & \frac{192}{L}
\end{array}\right] \quad ; \quad
[\boldsymbol{kp}^{(1)}_{Global}] = \left[\begin{array}{cc|cc}
\frac{18}{5L} & \frac{1}{10} & \frac{-18}{5L} & \frac{1}{10} \\
\frac{1}{10} & \frac{2L}{45} & \frac{-1}{10} & \frac{-L}{90} \\ \hline
\frac{-18}{5L} & \frac{-1}{10} & \frac{18}{5L} & \frac{-1}{10} \\
\frac{1}{10} & \frac{-L}{90} & \frac{-1}{10} & \frac{2L}{45}
\end{array}\right]
$$

Element 2: $(L)^{(2)} = \frac{2L}{3}$, $(EI_z)^{(2)} = EI_z$ and by substituting these values into the equations in (6.700) we can obtain:

$$
\begin{array}{cccccc}
& 3 & 4 & 5 & 6 & \Leftarrow Global \Rightarrow
\end{array}
\quad
\begin{array}{cccc}
3 & 4 & 5 & 6
\end{array}
$$

$$
[\boldsymbol{k}^{(2)}_{Global}] = EI_z \left[\begin{array}{cc|cc}
\frac{81}{2L^3} & \frac{27}{2L^2} & \frac{-81}{2L^3} & \frac{27}{2L^2} \\
\frac{27}{2L^2} & \frac{6}{L} & \frac{-27}{2L^2} & \frac{3}{L} \\ \hline
\frac{-81}{2L^3} & \frac{-27}{2L^2} & \frac{81}{2L^3} & \frac{-27}{2L^2} \\
\frac{27}{2L^2} & \frac{3}{L} & \frac{-27}{2L^2} & \frac{6}{L}
\end{array}\right] \quad ; \quad
[\boldsymbol{kp}^{(2)}_{Global}] = \left[\begin{array}{cc|cc}
\frac{9}{5L} & \frac{1}{10} & \frac{-9}{5L} & \frac{1}{10} \\
\frac{1}{10} & \frac{4L}{45} & \frac{-1}{10} & \frac{-L}{45} \\ \hline
\frac{-9}{5L} & \frac{-1}{10} & \frac{9}{5L} & \frac{-1}{10} \\
\frac{1}{10} & \frac{-L}{45} & \frac{-1}{10} & \frac{4L}{45}
\end{array}\right]
$$

Then, the equations in (6.699) become

$$
[\boldsymbol{K}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2} [\boldsymbol{k}^{(e)}_{Global}] =
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & Global \end{array} \\
\begin{bmatrix}
k^{(1)}_{11} & k^{(1)}_{12} & k^{(1)}_{13} & k^{(1)}_{14} & 0 & 0 \\
k^{(1)}_{21} & k^{(1)}_{22} & k^{(1)}_{23} & k^{(1)}_{24} & 0 & 0 \\
k^{(1)}_{31} & k^{(1)}_{32} & k^{(1)}_{33} + k^{(2)}_{11} & k^{(1)}_{34} + k^{(2)}_{12} & k^{(2)}_{13} & k^{(2)}_{14} \\
k^{(1)}_{41} & k^{(1)}_{42} & k^{(1)}_{43} + k^{(2)}_{21} & k^{(1)}_{44} + k^{(2)}_{22} & k^{(2)}_{23} & k^{(2)}_{24} \\
0 & 0 & k^{(2)}_{31} & k^{(2)}_{32} & k^{(2)}_{33} & k^{(2)}_{34} \\
0 & 0 & k^{(2)}_{41} & k^{(2)}_{42} & k^{(2)}_{43} & k^{(2)}_{44}
\end{bmatrix}
\end{array}
$$

After the values are substituted we can obtain

$$
[\boldsymbol{K}_{Global}]_{6\times 6} = EI_z \begin{bmatrix}
\frac{5184}{L^3} & \frac{864}{L^2} & \frac{-5184}{L^3} & \frac{864}{L^2} & 0 & 0 \\
\frac{864}{L^2} & \frac{192}{L} & \frac{-864}{L^2} & \frac{96}{L} & 0 & 0 \\
\frac{-5184}{L^3} & \frac{-864}{L^2} & \frac{10449}{2L^3} & \frac{-1701}{2L^2} & \frac{-81}{2L^3} & \frac{27}{2L^2} \\
\frac{864}{L^2} & \frac{96}{L} & \frac{-1701}{2L^2} & \frac{198}{L} & \frac{-27}{2L^2} & \frac{3}{L} \\
0 & 0 & \frac{-81}{2L^3} & \frac{-27}{2L^2} & \frac{81}{2L^3} & \frac{-27}{2L^2} \\
0 & 0 & \frac{27}{2L^2} & \frac{3}{L} & \frac{-27}{2L^2} & \frac{6}{L}
\end{bmatrix}
$$

And

$$
[\boldsymbol{Kp}_{Global}]_{6\times 6}=\mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{kp}^{(e)}_{Global}]=
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \quad Global \end{array}\\
\begin{bmatrix}
kp^{(1)}_{11} & kp^{(1)}_{12} & kp^{(1)}_{13} & kp^{(1)}_{14} & 0 & 0\\
kp^{(1)}_{21} & kp^{(1)}_{22} & kp^{(1)}_{23} & kp^{(1)}_{24} & 0 & 0\\
kp^{(1)}_{31} & kp^{(1)}_{32} & kp^{(1)}_{33}+kp^{(2)}_{11} & kp^{(1)}_{34}+kp^{(2)}_{12} & kp^{(2)}_{13} & kp^{(2)}_{14}\\
kp^{(1)}_{41} & kp^{(1)}_{42} & kp^{(1)}_{43}+kp^{(2)}_{21} & kp^{(1)}_{44}+kp^{(2)}_{22} & kp^{(2)}_{23} & kp^{(2)}_{24}\\
0 & 0 & kp^{(2)}_{31} & kp^{(2)}_{32} & kp^{(2)}_{33} & kp^{(2)}_{34}\\
0 & 0 & kp^{(2)}_{41} & kp^{(2)}_{42} & kp^{(2)}_{43} & kp^{(2)}_{44}
\end{bmatrix}
\end{array}
$$

After the values are substituted we can obtain

$$
[\boldsymbol{Kp}_{Global}]_{6\times 6}=
\begin{bmatrix}
\frac{18}{5L} & \frac{1}{10} & \frac{-18}{5L} & \frac{1}{10} & 0 & 0\\
\frac{1}{10} & \frac{2L}{45} & \frac{-1}{10} & \frac{-L}{90} & 0 & 0\\
\frac{-18}{5L} & \frac{-1}{10} & \frac{27}{5L} & 0 & \frac{-9}{5L} & \frac{1}{10}\\
\frac{1}{10} & \frac{-L}{90} & 0 & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{45}\\
0 & 0 & \frac{-9}{5L} & \frac{-1}{10} & \frac{9}{5L} & \frac{-1}{10}\\
0 & 0 & \frac{1}{10} & \frac{-L}{45} & \frac{-1}{10} & \frac{4L}{45}
\end{bmatrix}
$$

Applying the Boundary Conditions

Note that there are restrictions to move for the following degrees-of-freedom: $U_1=0$ and $U_5=0$, thus

$$
[\,[\boldsymbol{K}_{Global}]-P[\boldsymbol{Kp}_{Global}]\,]\{\boldsymbol{U}\}=\{\boldsymbol{0}\}\xrightarrow{\text{Boundary Conditions}}[\,[\overline{\boldsymbol{K}}_{Global}]-P[\overline{\boldsymbol{Kp}}_{Global}]\,]\{\boldsymbol{U}\}=\{\boldsymbol{0}\}
$$

$$
\left(EI_z\begin{bmatrix}
\frac{1}{EI_z} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{192}{L} & \frac{-864}{L^2} & \frac{96}{L} & 0 & 0\\
0 & \frac{-864}{L^2} & \frac{10449}{2L^3} & \frac{-1701}{2L^2} & 0 & \frac{27}{2L^2}\\
0 & \frac{96}{L} & \frac{-1701}{2L^2} & \frac{198}{L} & 0 & \frac{3}{L}\\
0 & 0 & 0 & 0 & \frac{1}{EI_z} & 0\\
0 & 0 & \frac{27}{2L^2} & \frac{3}{L} & 0 & \frac{6}{L}
\end{bmatrix}
-P\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2L}{45} & \frac{-1}{10} & \frac{-1}{90} & 0 & 0\\
0 & \frac{-1}{10} & \frac{27}{5L} & 0 & 0 & \frac{1}{10}\\
0 & \frac{-1}{90} & 0 & \frac{2L}{15} & 0 & \frac{-L}{45}\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & \frac{1}{10} & \frac{-L}{45} & 0 & \frac{4L}{45}
\end{bmatrix}\right)
\begin{Bmatrix} U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6 \end{Bmatrix}
=\begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{Bmatrix}
$$

$$
[\,[\overline{\boldsymbol{K}}_{Global}]-P[\overline{\boldsymbol{Kp}}_{Global}]\,]\{\boldsymbol{U}\}=\{\boldsymbol{0}\}
$$

Let us assume that $L=1$ and $EI_z=1$ and the solution for the above system, i.e. the eigenvalues of $\det[\,[\overline{\boldsymbol{K}}_{Global}]-P[\overline{\boldsymbol{Kp}}_{Global}]\,]=0$ are:

$$P_i = \begin{Bmatrix} 5.7331997382\times 10^3 \\ 1.0931697830\times 10^3 \\ 12.13650390 \\ 79.4939748 \\ 1 \\ 1 \end{Bmatrix}$$

Note that the solutions $P=1$ are associated with the boundary conditions and can be discarded. Then, the smallest value is $P_{cr}=12.13650390$. If we compare with the exact solution for this problem which is

$$P_{cr}^{(exact)} = 11.1\frac{EI_z}{L^2}$$

the error is about 9.34%. The eigenvector associated with the eigenvalue $P_{cr}=12.1365$ is:

$$U_i = \begin{Bmatrix} 0 \\ -0.490914 \\ -0.161349 \\ -0.470371 \\ 0 \\ 0.715347 \end{Bmatrix}$$

Problem 6.86

Consider the structure described in Figure 6.229. Obtain the displacements (translation and rotation) at the node 2.

Approach: Do not consider the energy due to the axial force ($EA\to\infty$).

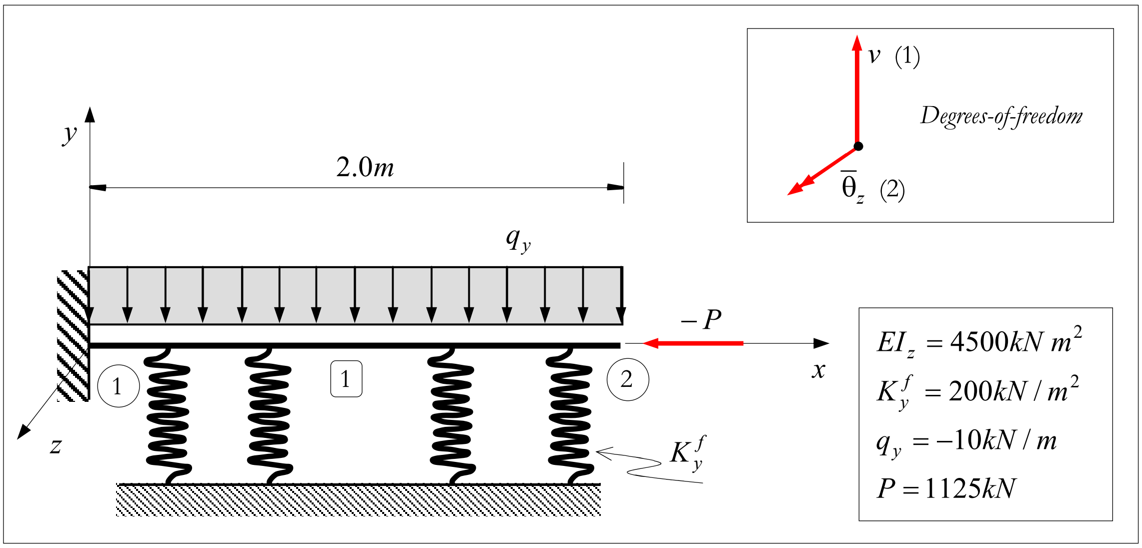

Figure 6.229: Beam and elastic foundation.

Solution:

In **Problem 6.83**, by considering the plane $x-y$, we have shown that the following is true

$$[\boldsymbol{k}^{(e)}]=EI_z\left[\begin{array}{cc:cc}\frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2}\\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L}\\ \hdashline \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2}\\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L}\end{array}\right] \quad ; \quad P[\boldsymbol{kp}_y^{(e)}]=P\left[\begin{array}{cc:cc}\frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10}\\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30}\\ \hdashline \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10}\\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15}\end{array}\right]$$

And in **Problem 6.74** we have shown that the stiffness due to the elastic base is given by

$$[\boldsymbol{ke}^{(Spring\_y)}]=K_y^f L\left[\begin{array}{cc:cc}\frac{13}{35} & \frac{11L}{210} & \frac{9}{70} & \frac{-13L}{420}\\ \frac{11L}{210} & \frac{L^2}{105} & \frac{13L}{420} & \frac{-L^2}{140}\\ \hdashline \frac{9}{70} & \frac{13L}{420} & \frac{13}{35} & \frac{-11L}{210}\\ \frac{-13L}{420} & \frac{-L^2}{140} & \frac{-11L}{210} & \frac{L^2}{105}\end{array}\right]$$

Then, the complete system is represented as follows:

$$[[\boldsymbol{k}^{(e)}]+[\boldsymbol{ke}^{(Spring\_y)}]-P[\boldsymbol{kp}_y^{(e)}]]\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{f}^{(e)}\}$$

The consistent load vector, (see **Problem 6.67**), is given by

$$\{\boldsymbol{f}^{(e)}\}=\left\{\begin{array}{c}\frac{q_yL}{2}\\ \frac{q_yL^2}{12}\\ \hdashline \frac{q_yL}{2}\\ \frac{-q_yL^2}{12}\end{array}\right\}=\left\{\begin{array}{c}-10\\ -3.33333333\\ \hdashline -10\\ 3.33333333\end{array}\right\}$$

After the assemble is complete and the numerical variables are considered we can obtain

$$10^3\times\left[\begin{array}{cc:cc}6.223571 & 6.679405 & -6.023571 & 6.612738\\ 6.679405 & 8.715238 & -6.612738 & 4.563571\\ \hdashline -6.023571 & -6.612738 & 6.223571 & -6.679405\\ 6.612738 & 4.563571 & -6.679405 & 8.715238\end{array}\right]\left\{\begin{array}{c}v_1\\ \overline{\theta}_{z1}\\ \hdashline v_2\\ \overline{\theta}_{z2}\end{array}\right\}=\left\{\begin{array}{c}-10\\ -3.33333333\\ \hdashline -10\\ 3.33333333\end{array}\right\}$$

Applying the boundary conditions, i.e. $v_1=0$, $\overline{\theta}_{z1}=0$:

$$10^3\times\left[\begin{array}{cc:cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hdashline 0 & 0 & 6.223571 & -6.679405\\ 0 & 0 & -6.679405 & 8.715238\end{array}\right]\left\{\begin{array}{c}v_1\\ \overline{\theta}_{z1}\\ \hdashline v_2\\ \overline{\theta}_{z2}\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\\ \hdashline -10\\ 3.3333\end{array}\right\}\xrightarrow{Solve}\left\{\begin{array}{c}v_1\\ \overline{\theta}_{z1}\\ \hdashline v_2\\ \overline{\theta}_{z2}\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\\ \hdashline -6.741\\ -4.784\end{array}\right\}\times 10^{-3}$$

Problem 6.87

Consider the column described in Figure 6.230 in which the load is only due its own weight, i.e. $q_x = \rho g A$, where ρ is the mass density, g is the gravity acceleration and A is the cross-section area, and also consider that all these variables are constant along the column. Note that the unit of q_x is $[q_x] = [\rho g A] = \frac{kg}{m^3}\frac{m}{s^2}m^2 = \frac{kgm}{s^2}\frac{1}{m} = \frac{N}{m}$.

a) By considering the deflection $v = C_3(x^3 - 3Lx^2)$ obtaining the critical value for q_x;

b) Obtain the finite element formulation for this problem.

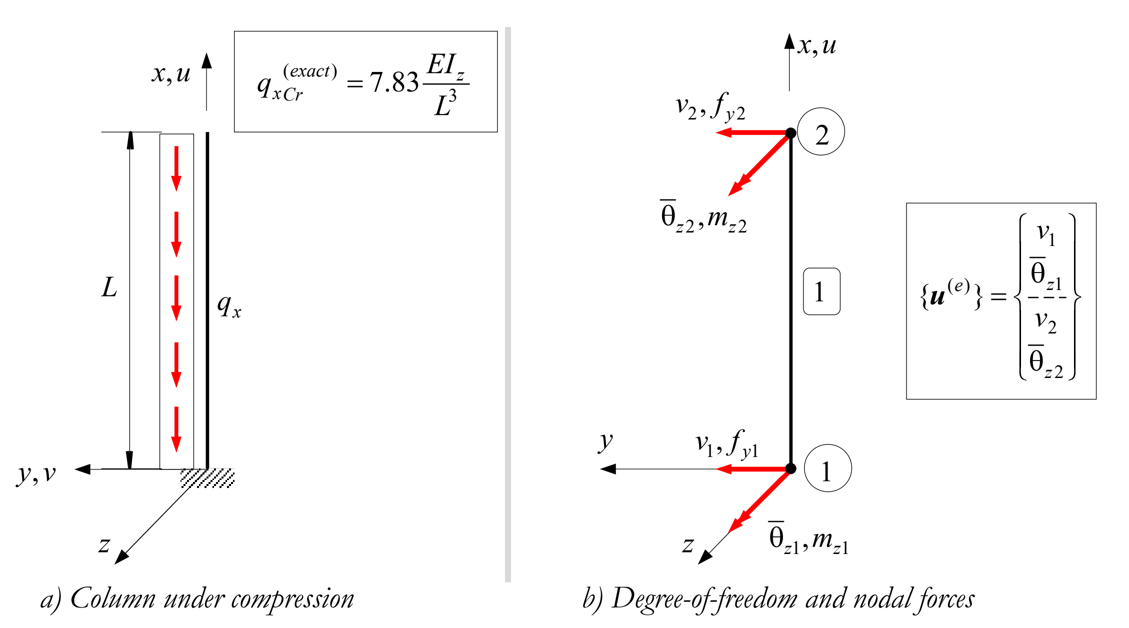

Figure 6.230: Column under compression due to mass density.

Solution:

In **Problem 6.81** we have obtained that $dU^{ext} = \frac{P(v')^2}{2}dx$, now the force is varying along the column, then we can state that $dU^{ext} = \frac{P(x)(v')^2}{2}dx = \frac{q_x(L-x)(v')^2}{2}dx$, then

$$U^{ext} = \int dU^{ext} = \int_0^L \left[\frac{q_x(L-x)}{2}(v')^2\right]dx = \frac{q_x}{2}\int_0^L \left[(L-x)(v')^2\right]dx$$

Note also that we can consider that $dU^{ext} = \frac{dP(v')^2}{2}dx$ and

$$dU^{ext}(x) = \frac{dP}{2}\int_0^x (v')^2\,dx \quad \Rightarrow \quad U^{ext} = \int dU^{ext} = \int_0^L \left[\frac{q_x}{2}\int_0^x (v')^2\,dx\right]dx = \frac{1}{2}\int_0^L \left[q_x \int_0^x (v')^2\,dx\right]dx$$

where we have considered $dP = q_x\,dx$.

Case a) The derivatives of the deflections are:

$$v(x) = C_3(x^3 - 3Lx^2) \xrightarrow{\frac{d}{dx}} v'(x) = C_3(3x^2 - 6Lx) \xrightarrow{\frac{d}{dx}} v'' = C_3(6x - 6L)$$

The total potential energy becomes

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{1}{2}\int_0^L \left[q_x \int_0^x (v')^2\,dx \right] dx$$

$$\Rightarrow \Pi = \frac{EI_z}{2}\int_0^L [C_3(6x-6L)]^2\,dx - \frac{1}{2}\int_0^L \left[q_x \int_0^x [C_3(3x^2-6Lx)]^2\,dx \right] dx$$

$$\Rightarrow \Pi = \frac{EI_z}{2}\int_0^L [C_3(6x-6L)]^2\,dx - \frac{1}{2}\int_0^L \left[q_x\, C_3^2\left(\frac{9}{5}x^5 - 9Lx^4 + 12L^2x^3\right)\right] dx$$

$$\Rightarrow \Pi(C_3) = 6EI_zL^3C_3^2 - \frac{3q_x\,L^6}{4}C_3^2$$

The inflection point is given by

$$\Rightarrow \frac{\partial \Pi}{\partial C_3} = \frac{\partial}{\partial C_3}\left[6EI_zL^3C_3^2 - \frac{3q_x\,L^6}{4}C_3^2\right] = \left(12EI_zL^3 - \frac{3q_x\,L^6}{2}\right)C_3 = 0$$

$$\xrightarrow{C_3 \neq 0} \qquad q_{xCr} = 8\frac{EI_z}{L^3} > q_{xCr}^{(exact)} = 7.83\frac{EI_z}{L^3}$$

The error is 2.17%.

Case b) In **Problem 6.67** we have obtained that

$$v' = \overline{\theta}_z = v_1\left[\frac{6x^2}{L^3} - \frac{6x}{L^2}\right] + v_2\left[-\frac{6x^2}{L^3} + \frac{6x}{L^2}\right] + \overline{\theta}_{z1}\left[\frac{3x^2}{L^2} - \frac{4x}{L} + 1\right] + \overline{\theta}_{z2}\left[\frac{3x^2}{L^2} - \frac{2x}{L}\right]$$

Then, the integral $U^{ext} = \frac{q_x}{2}\int_0^L \left[(L-x)(v')^2\right]dx$ can be expressed in terms of nodal values as follows

$$U^{ext} = \frac{q_x}{2}\int_0^L \left[(L-x)(v')^2\right]dx = q_x\left[\frac{-L}{10}v_2\overline{\theta}_{z2} + \frac{L}{10}v_1\overline{\theta}_{z2} - \frac{L^2}{60}\overline{\theta}_{z1}\overline{\theta}_{z2} + \frac{L^2}{20}\overline{\theta}_{z1}^2 + \frac{3}{10}v_2^2 + \frac{L^2}{60}\overline{\theta}_{z2}^2 + \frac{3}{10}v_1^2 - \frac{3}{5}v_1v_2\right]$$

And by considering the total potential energy:

$$\Pi = U^{int} - U^{ext} = \frac{EI_z}{2}\int_0^L (v'')^2\,dx - \frac{q_x}{2}\int_0^L \left[(L-x)(v')^2\right]dx \tag{6.701}$$

we can obtain

$$\Pi = \frac{EI_z}{2}\left[\frac{12}{L^3}v_1^2 + \frac{12}{L^3}v_2^2 + \frac{4}{L}\overline{\theta}_{z1}^2 + \frac{4}{L}\overline{\theta}_{z2}^2 - \frac{24}{L^3}v_1v_2 + \frac{12}{L^2}v_1\overline{\theta}_{z1} + \frac{12}{L^2}v_1\overline{\theta}_{z2} - \frac{12}{L^2}v_2\overline{\theta}_{z1} - \frac{12}{L^2}v_2\overline{\theta}_{z2} + \frac{4}{L}\overline{\theta}_{z1}\overline{\theta}_{z2}\right] - q_x\left[\frac{-L}{10}v_2\overline{\theta}_{z2} + \frac{L}{10}v_1\overline{\theta}_{z2} - \frac{L^2}{60}\overline{\theta}_{z1}\overline{\theta}_{z2} + \frac{L^2}{20}\overline{\theta}_{z1}^2 + \frac{3}{10}v_2^2 + \frac{L^2}{60}\overline{\theta}_{z2}^2 + \frac{3}{10}v_1^2 - \frac{3}{5}v_1v_2\right]$$

As we are looking for the stationary state the following must hold:

$$\frac{\partial \Pi}{\partial v_1}=0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{24}{L^3}v_1+\frac{12}{L^2}\overline{\theta}_{z1}-\frac{24}{L^3}v_2+\frac{12}{L^2}\overline{\theta}_{z2}\right\}-q_x\left\{\frac{3}{5}v_1-\frac{3}{5}v_2+\frac{L}{10}\overline{\theta}_{z2}\right\}=0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{z1}}=0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{12}{L^2}v_1+\frac{8}{L}\overline{\theta}_{z1}-\frac{12}{L^2}v_2+\frac{4}{L}\overline{\theta}_{z2}\right\}-q_x\left\{\frac{L^2}{10}\overline{\theta}_{z1}-\frac{L^2}{60}\overline{\theta}_{z2}\right\}=0$$

$$\frac{\partial \Pi}{\partial v_2}=0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{-24}{L^3}v_1-\frac{12}{L^2}\overline{\theta}_{z1}+\frac{24}{L^3}v_2-\frac{12}{L^2}\overline{\theta}_{z2}\right\}-q_x\left\{\frac{-3}{5}v_1+\frac{3}{5}v_2-\frac{L}{10}\overline{\theta}_{z2}\right\}=0$$

$$\frac{\partial \Pi}{\partial \overline{\theta}_{z2}}=0 \quad \Rightarrow \quad \frac{EI_z}{2}\left\{\frac{12}{L^2}v_1+\frac{4}{L}\overline{\theta}_{z1}-\frac{12}{L^2}v_2+\frac{8}{L}\overline{\theta}_{z2}\right\}-q_x\left\{\frac{L}{10}v_1-\frac{L^2}{60}\overline{\theta}_{z1}-\frac{L}{10}v_2+\frac{L^2}{30}\overline{\theta}_{z2}\right\}=0$$

Restructuring the above set of equations in matrix form we can obtain:

$$\left(EI_z\begin{bmatrix}\frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2}\\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L}\\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2}\\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L}\end{bmatrix}-q_x\begin{bmatrix}\frac{3}{5} & 0 & \frac{-3}{5} & \frac{L}{10}\\ 0 & \frac{L^2}{10} & 0 & \frac{-L^2}{60}\\ \frac{-3}{5} & 0 & \frac{3}{5} & \frac{-L}{10}\\ \frac{L}{10} & \frac{-L^2}{60} & \frac{-L}{10} & \frac{L^2}{30}\end{bmatrix}\right)\begin{Bmatrix}v_1\\ \overline{\theta}_{z1}\\ v_2\\ \overline{\theta}_{z2}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{6.702}$$

$$\Rightarrow ([\boldsymbol{k}^{(e)}]-q_x[\boldsymbol{kq}^{(e)}])\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{0}\}$$

Note that we are not considering the strain energy due to the axial force, so if the domain is discretized by several elements we have to transfer the concentrated load indirectly, for instance, if the domain is dicretized into 3 finite elements we also have to consider the effect of the concentrated load as the one described in Figure 6.231(b). For each element we have to consider $([\boldsymbol{k}^{(e)}]-q_x[[\boldsymbol{kq}^{(e)}]+\overline{L}_j^{(e)}[\boldsymbol{kp}_y{}^{(e)}]])\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{0}\}$.

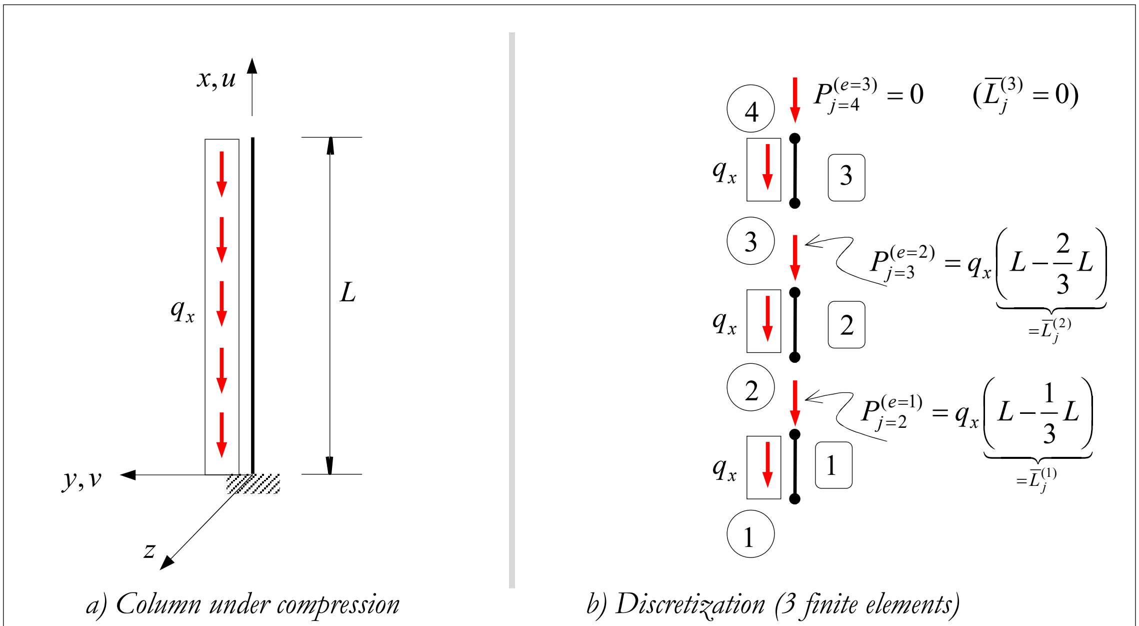

Figure 6.231: Column under compression due to mass density – 3 finite elements.

Let us consider 1 finite element, then by applying the boundary conditions $v_1 = 0$, $\overline{\theta}_{z1} = 0$, the equation in (6.702) becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} - q_x \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & \frac{-L}{10} \\ 0 & 0 & \frac{-L}{10} & \frac{L^2}{30} \end{bmatrix} \right) \begin{Bmatrix} v_1 \\ \overline{\theta}_{z1} \\ v_2 \\ \overline{\theta}_{z2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow ([\overline{\boldsymbol{k}}^{(e)}] - q_x[\overline{\boldsymbol{kq}}^{(e)}])\{\boldsymbol{u}^{(e)}\} = \{\boldsymbol{0}\}$$

and

$$\det([\overline{\boldsymbol{k}}^{(e)}] - q_x[\overline{\boldsymbol{kq}}^{(e)}]) = 0 \quad \Rightarrow \quad q_{xi} = \begin{Bmatrix} 1 \\ 1 \\ 152.111026\frac{EI_z}{L^3} \\ 7.888974\frac{EI_z}{L^3} \end{Bmatrix} \quad \Rightarrow \quad q_{xCr} = 7.888974\frac{EI_z}{L^3}$$

if we are adopting 2 elements the result is $q_{xCr} = 7.857\frac{EI_z}{L^3}$, and by adopting 3 elements the critical value is $q_{xCr} = 7.8421\frac{EI_z}{L^3}$.

Problem 6.88

Obtain the critical load for the problem described in Figure 6.232. As academic problem consider that $L = 1m$, $K_y^f = 19.75kN/m^2$, $EI_z = 10kN/m^2$.

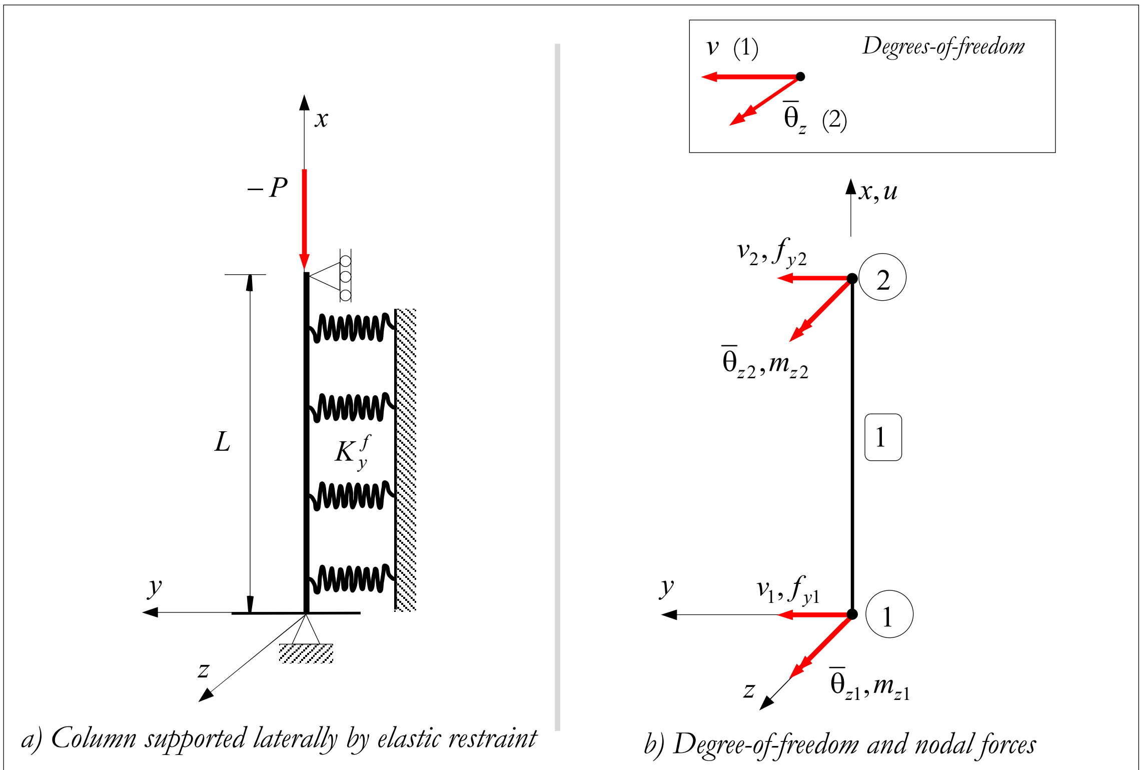

Figure 6.232: Beam and elastic base.

Solution:

In **Problem 6.86** we have established that

$$[[\boldsymbol{k}^{(e)}]+[\boldsymbol{ke}^{(Spring\_y)}]-P[\boldsymbol{kp}_y^{(e)}]]\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{0}\}$$

where

$$[\boldsymbol{k}^{(e)}]=EI_z\begin{bmatrix}\frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2}\\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L}\\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2}\\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L}\end{bmatrix} \quad ; \quad P[\boldsymbol{kp}_y^{(e)}]=P\begin{bmatrix}\frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10}\\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30}\\ \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10}\\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15}\end{bmatrix}$$

and

$$[\boldsymbol{ke}^{(Spring\_y)}]=K_y^f L\begin{bmatrix}\frac{13}{35} & \frac{11L}{210} & \frac{9}{70} & \frac{-13L}{420}\\ \frac{11L}{210} & \frac{L^2}{105} & \frac{13L}{420} & \frac{-L^2}{140}\\ \frac{9}{70} & \frac{13L}{420} & \frac{13}{35} & \frac{-11L}{210}\\ \frac{-13L}{420} & \frac{-L^2}{140} & \frac{-11L}{210} & \frac{L^2}{105}\end{bmatrix}$$

Then, after applying the boundary conditions ($v_1=0$, $v_2=0$) to the system

$$[[\overline{\boldsymbol{k}}^{(e)}]+[\overline{\boldsymbol{ke}}^{(Spring\_y)}]-P[\overline{\boldsymbol{kp}}_y^{(e)}]]\{\boldsymbol{u}^{(e)}\}=\{\boldsymbol{0}\}$$

$$\left(\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{4EI_z}{L}+\frac{L^3K_y^f}{105} & 0 & \frac{2EI_z}{L}-\frac{L^3K_y^f}{140}\\ 0 & 0 & 1 & 0\\ 0 & \frac{2EI_z}{L}-\frac{L^3K_y^f}{140} & 0 & \frac{4EI_z}{L}+\frac{L^3K_y^f}{105}\end{bmatrix}-P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \frac{2L}{15} & 0 & \frac{-L}{30}\\ 0 & 0 & 1 & 0\\ 0 & \frac{-L}{30} & 0 & \frac{2L}{15}\end{bmatrix}\right)\begin{Bmatrix}v_1\\ \theta_{z1}\\ v_2\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

and the solution for $\det[[\overline{\boldsymbol{k}}^{(e)}]+[\overline{\boldsymbol{ke}}^{(Spring\_y)}]-P[\overline{\boldsymbol{kp}}_y^{(e)}]]=0$ is

$$P_i=\begin{Bmatrix}1\\ 1\\ \frac{L^4K_y^f+2520EI_z}{42L^2}\\ \frac{L^4K_y^f+120EI_z}{10L^2}\end{Bmatrix}=\begin{Bmatrix}1\\ 1\\ 1.2\times10^3\\ 241.975\end{Bmatrix}$$

And the critical load is $P_{cr}=241.975$, which is a very poor approximation with an error approximately 50%. In order to obtain a better solution more finite element is needed.

Problem 6.89

By using the finite element formulation, obtain the critical load for the problem described in Figure 6.233.

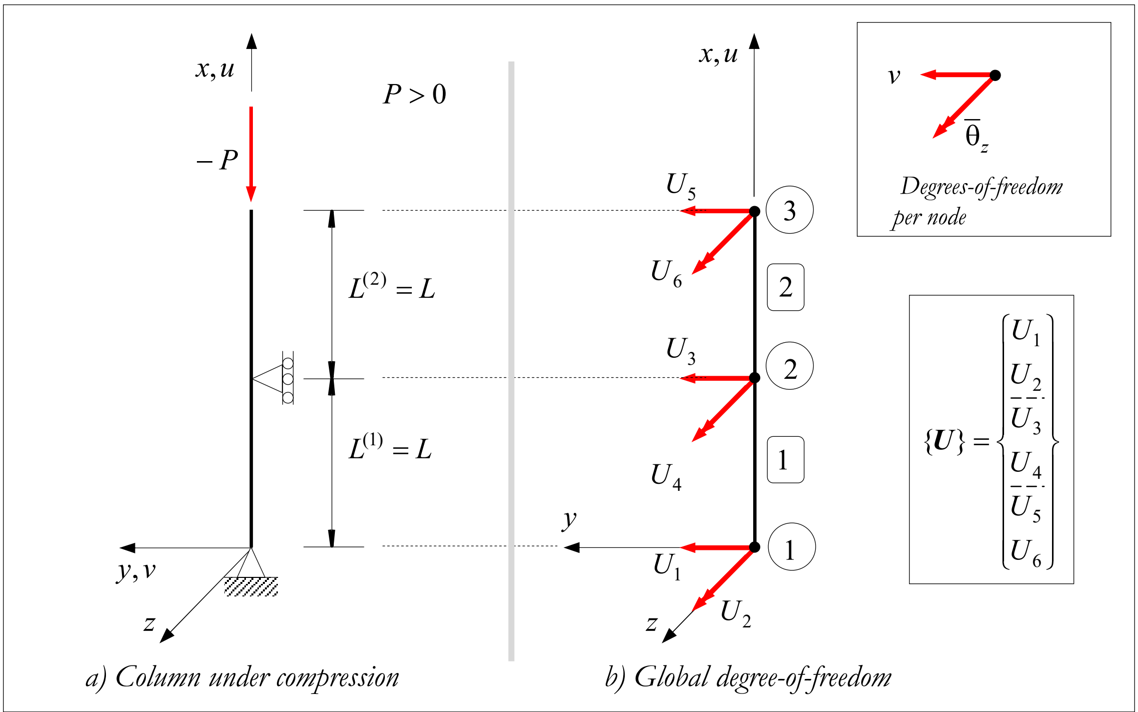

Figure 6.233: Column under compression.

Solution:

Construction of the Global Stiffness Matrices - $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$

We need to construct the following system

$$[\ [\boldsymbol{K}_{Global}]-P[\boldsymbol{Kp}_{Global}]\ ]\{\boldsymbol{U}\}=\{\boldsymbol{0}\}$$

The matrices $[\boldsymbol{K}_{Global}]$ and $[\boldsymbol{Kp}_{Global}]$ can be constructed by assembling the individual beam elements, i.e.:

$$[\boldsymbol{K}_{Global}]_{6\times 6}=\mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}] \qquad \Bigg| \qquad [\boldsymbol{Kp}_{Global}]_{6\times 6}=\mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{kp}^{(e)}_{Global}] \tag{6.703}$$

where

$$[\boldsymbol{k}^{(e)}_{Global}]=EI_z\left[\begin{array}{cc|cc} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \hline \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{array}\right] \quad ; \quad [\boldsymbol{kp}^{(e)}_{Global}]=\left[\begin{array}{cc|cc} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30} \\ \hline \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{array}\right] \tag{6.704}$$

For this problem the stiffness matrices for both elements are the same.

For both elements the stiffness matrices are the same, the

$$[\boldsymbol{K}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{k}^{(e)}_{Global}] = \begin{bmatrix} k^{(1)}_{11} & k^{(1)}_{12} & k^{(1)}_{13} & k^{(1)}_{14} & 0 & 0 \\ k^{(1)}_{21} & k^{(1)}_{22} & k^{(1)}_{23} & k^{(1)}_{24} & 0 & 0 \\ k^{(1)}_{31} & k^{(1)}_{32} & k^{(1)}_{33}+k^{(2)}_{11} & k^{(1)}_{34}+k^{(2)}_{12} & k^{(2)}_{13} & k^{(2)}_{14} \\ k^{(1)}_{41} & k^{(1)}_{42} & k^{(1)}_{43}+k^{(2)}_{21} & k^{(1)}_{44}+k^{(2)}_{22} & k^{(2)}_{23} & k^{(2)}_{24} \\ 0 & 0 & k^{(2)}_{31} & k^{(2)}_{32} & k^{(2)}_{33} & k^{(2)}_{34} \\ 0 & 0 & k^{(2)}_{41} & k^{(2)}_{42} & k^{(2)}_{43} & k^{(2)}_{44} \end{bmatrix}$$

After the values are substituted we can obtain

$$[\boldsymbol{K}_{Global}]_{6\times 6} = EI_z \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} & 0 & 0 \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} & 0 & 0 \\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{24}{L^3} & 0 & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & 0 & \frac{8}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ 0 & 0 & \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ 0 & 0 & \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{bmatrix}$$

And

$$[\boldsymbol{Kp}_{Global}]_{6\times 6} = \mathop{\mathbf{A}}_{e=1}^{2}[\boldsymbol{kp}^{(e)}_{Global}] = \begin{array}{c} \begin{array}{cccccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{Global} \end{array} \\ \begin{bmatrix} kp^{(1)}_{11} & kp^{(1)}_{12} & kp^{(1)}_{13} & kp^{(1)}_{14} & 0 & 0 \\ kp^{(1)}_{21} & kp^{(1)}_{22} & kp^{(1)}_{23} & kp^{(1)}_{24} & 0 & 0 \\ kp^{(1)}_{31} & kp^{(1)}_{32} & kp^{(1)}_{33}+kp^{(2)}_{11} & kp^{(1)}_{34}+kp^{(2)}_{12} & kp^{(2)}_{13} & kp^{(2)}_{14} \\ kp^{(1)}_{41} & kp^{(1)}_{42} & kp^{(1)}_{43}+kp^{(2)}_{21} & kp^{(1)}_{44}+kp^{(2)}_{22} & kp^{(2)}_{23} & kp^{(2)}_{24} \\ 0 & 0 & kp^{(2)}_{31} & kp^{(2)}_{32} & kp^{(2)}_{33} & kp^{(2)}_{34} \\ 0 & 0 & kp^{(2)}_{41} & kp^{(2)}_{42} & kp^{(2)}_{43} & kp^{(2)}_{44} \end{bmatrix} \end{array}$$

After the values are substituted we can obtain

$$[\boldsymbol{Kp}_{Global}]_{6\times 6} = \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & \frac{-6}{5L} & \frac{1}{10} & 0 & 0 \\ \frac{1}{10} & \frac{2L}{15} & \frac{-1}{10} & \frac{-L}{30} & 0 & 0 \\ \frac{-6}{5L} & \frac{-1}{10} & \frac{12}{5L} & 0 & \frac{-6}{5L} & \frac{1}{10} \\ \frac{1}{10} & \frac{-L}{30} & 0 & \frac{4L}{15} & \frac{-1}{10} & \frac{-L}{30} \\ 0 & 0 & \frac{-6}{5L} & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\ 0 & 0 & \frac{1}{10} & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15} \end{bmatrix}$$

Applying the Boundary Conditions

Note that there are restrictions to move for the following degrees-of-freedom: $U_1 = 0$, $U_3 = 0$, thus

$$[\ [\boldsymbol{K}_{Global}] - P[\boldsymbol{Kp}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\} \xrightarrow{\text{Boundary Conditions}} [\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\}$$

$$
\left(
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4EI_z}{L} & 0 & \frac{2EI_z}{L} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{2EI_z}{L} & 0 & \frac{8EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\
0 & 0 & 0 & \frac{-6EI_z}{L^2} & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\
0 & 0 & 0 & \frac{2EI_z}{L} & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L}
\end{bmatrix}
- P
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2L}{15} & 0 & \frac{-L}{30} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{-L}{30} & 0 & \frac{4L}{15} & \frac{-1}{10} & \frac{-L}{30} \\
0 & 0 & 0 & \frac{-1}{10} & \frac{6}{5L} & \frac{-1}{10} \\
0 & 0 & 0 & \frac{-L}{30} & \frac{-1}{10} & \frac{2L}{15}
\end{bmatrix}
\right)
\begin{Bmatrix} U_1 \\ \underline{U_2} \\ U_3 \\ \underline{U_4} \\ U_5 \\ U_6 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ \underline{0} \\ 0 \\ \underline{0} \\ 0 \\ 0 \end{Bmatrix}
$$

$$
[\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ]\{\boldsymbol{U}\} = \{\boldsymbol{0}\}
$$

The solution for the above system, i.e. the eigenvalues of $\det[\ [\overline{\boldsymbol{K}}_{Global}] - P[\overline{\boldsymbol{Kp}}_{Global}]\ ] = 0$ are:

$$
\frac{(P-1)^2(17L^8P^4 - 1776L^6EI_zP^3 + 52704L^4(EI_z)^2P^2 - 449280L^2(EI_z)^3P + 518400(EI_z)^4)}{3600L^6} = 0
$$

And the solution is

$$
P_i = \begin{Bmatrix} 60\frac{EI_z}{L^2} \\ 12\frac{EI_z}{L^2} \\ 31.10915\frac{EI_z}{L^2} \\ 1.36143\frac{EI_z}{L^2} \\ 1 \\ 1 \end{Bmatrix}
\quad \Rightarrow \quad
P_{cr} = 1.36143\frac{EI_z}{L^2} > P_{cr}^{(exact)} \approx 1.359\frac{EI_z}{L^2}
$$

6.6 Introduction to Flexural Plates

Problem 6.90

Obtain expressions for the following fields: displacement, strain and stress for the problem represented in Figure 6.234 in which $w = w(x, y)$ represents the deflection according to z-direction; $\overline{\theta}_x$ and $\overline{\theta}_y$ stand for the rotations according to the directions x and y respectively; and t is the plate thickness.

Hint: In order to obtain the displacement field consider the Euler-Bernoulli beam theory (the classical beam theory), i.e. by combining **Figure 6.145** and **Figure 6.147**.

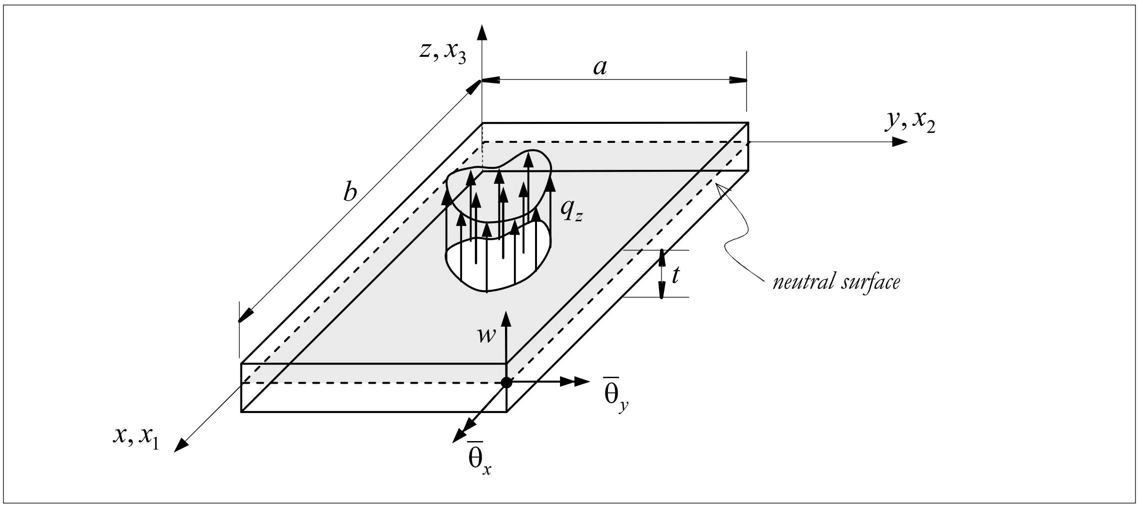

Figure 6.234: Flexural plate.

Solution:

According to **Figure 6.145** and **Figure 6.147** the displacement field, (see Figure 6.235), can be represented as follows:

$$w = w(x, y) \quad ; \quad u = u(x, y, z) = \frac{-\partial w}{\partial x} z \equiv -w_{,x} z \quad ; \quad v = v(x, y, z) = \frac{-\partial w}{\partial y} z \equiv -w_{,y} z \tag{6.705}$$

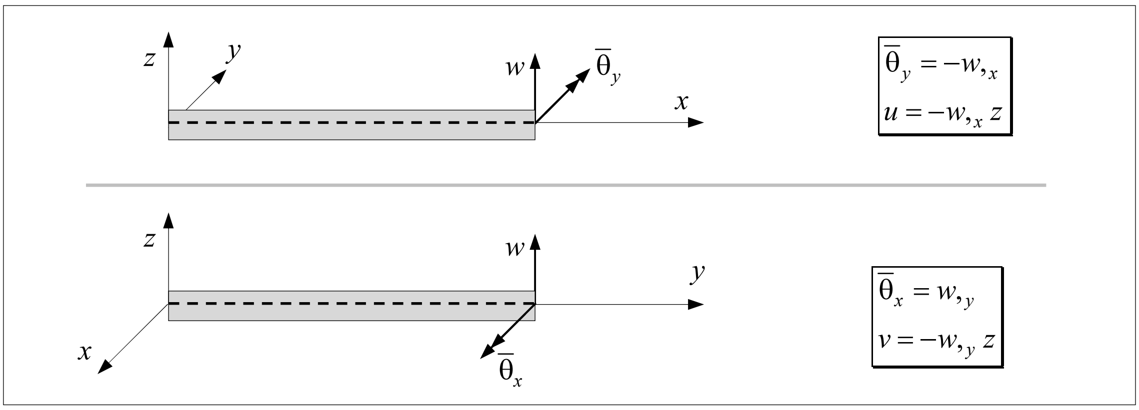

Figure 6.235: Classical beam theory.

The Displacement Field

The displacement field vector, (see Figure 6.236), can be represented as follows:

$$\{\boldsymbol{u}(\vec{\boldsymbol{x}})\} = \begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \mathrm{u}_3 \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} -w,_x z \\ -w,_y z \\ w \end{Bmatrix} = \begin{Bmatrix} \overline{\theta}_y z \\ -\overline{\theta}_x z \\ w \end{Bmatrix} = \begin{Bmatrix} \overline{\beta}_x z \\ \overline{\beta}_y z \\ w \end{Bmatrix} = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{\beta}_x \\ \overline{\beta}_y \\ w \end{Bmatrix} \tag{6.706}$$

where we have introduced the variables $\overline{\beta}_x = \overline{\theta}_y = -w,_x$ and $\overline{\beta}_y = -\overline{\theta}_x = -w,_y$. The above equation in indicial notation becomes

$$\boxed{\mathrm{u}_3 = \mathrm{u}_3(x_1, x_2) \quad ; \quad \mathrm{u}_i = \frac{-\partial \mathrm{u}_3}{\partial x_i} x_3 \equiv -\mathrm{u}_{3,i} x_3 \qquad (i = 1,2)} \tag{6.707}$$

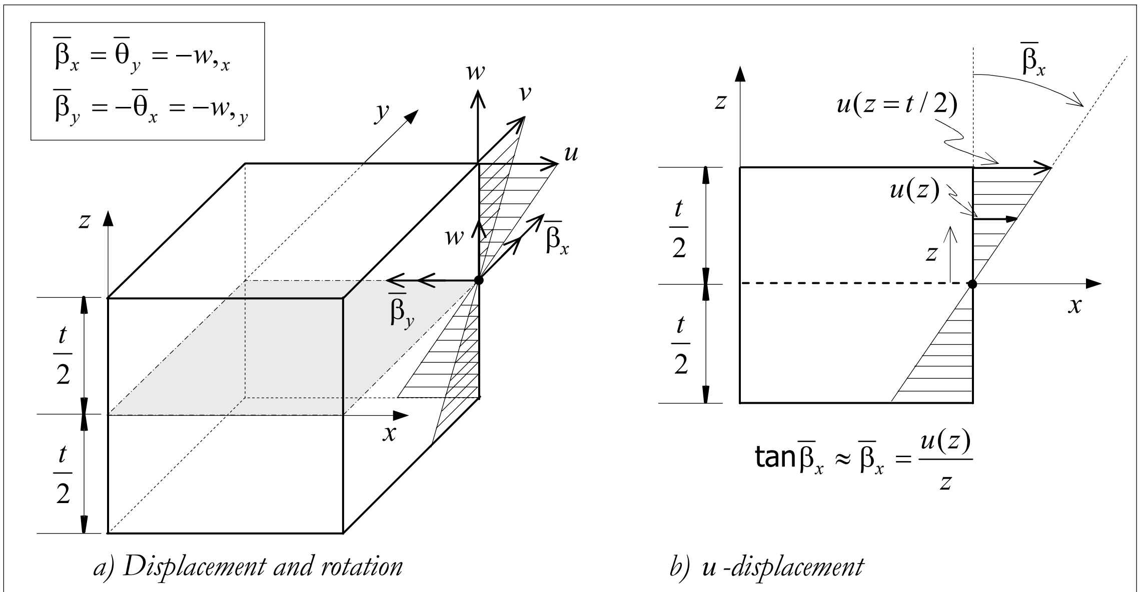

Figure 6.236: Displacement and rotations – Kirchhoff-Love plate theory.

The Strain Field

For small deformation regime the strain and displacement, (see **Problem 5.8**), are related to each other by using Voigt notation as follows:

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \overline{\beta}_x z \\ \overline{\beta}_y z \\ w \end{Bmatrix} = \begin{Bmatrix} \frac{\partial(\overline{\beta}_x z)}{\partial x} \\ \frac{\partial(\overline{\beta}_y z)}{\partial y} \\ \frac{\partial(w(x,y))}{\partial z} \\ \frac{\partial(\overline{\beta}_x z)}{\partial y} + \frac{\partial(\overline{\beta}_y z)}{\partial x} \\ \frac{\partial(\overline{\beta}_y z)}{\partial z} + \frac{\partial(w)}{\partial y} \\ \frac{\partial(\overline{\beta}_x z)}{\partial z} + \frac{\partial(w)}{\partial x} \end{Bmatrix}$$

thus

$$\Rightarrow \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \overline{\beta}_{x,x} z \\ \overline{\beta}_{y,y} z \\ 0 \\ \overline{\beta}_{x,y} z + \overline{\beta}_{y,x} z \\ \overline{\beta}_y + w_{,y} \\ \overline{\beta}_x + w_{,x} \end{Bmatrix} = \begin{Bmatrix} z\overline{\beta}_{x,x} \\ z\overline{\beta}_{y,y} \\ 0 \\ 2z\overline{\beta}_{x,y} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -zw_{,xx} \\ -zw_{,yy} \\ 0 \\ -2zw_{,xy} \\ 0 \\ 0 \end{Bmatrix} \tag{6.708}$$

Note that $\overline{\beta}_x = \overline{\theta}_y = -w_{,x}$ and $\overline{\beta}_y = -\overline{\theta}_x = -w_{,y}$, then the derivatives are $\overline{\beta}_{x,y} = \overline{\theta}_{y,y} = -w_{,xy}$ and $\overline{\beta}_{y,x} = -\overline{\theta}_{x,x} = -w_{,yx}$, so that we can conclude $\overline{\beta}_{x,y} = -w_{,xy} = \overline{\beta}_{y,x}$.

The strain field (6.708) could have been obtained by means of indicial notation:

$$\boxed{\begin{aligned} &\varepsilon_{ij} = [(\nabla \bar{\mathbf{u}})^{sym}]_{ij} = \frac{1}{2}(\mathsf{u}_{i,j} + \mathsf{u}_{j,i}) = \mathsf{u}_{i,j} = \frac{-\partial \mathsf{u}_3}{\partial x_i \partial x_j} x_3 \equiv -\mathsf{u}_{3,ij} x_3 \qquad (i = 1,2) \\ &\varepsilon_{3i} = 0 \qquad (i = 1,2,3) \end{aligned}} \tag{6.709}$$

where we have considered that

$$\mathsf{u}_i = \frac{-\partial \mathsf{u}_3}{\partial x_i} x_3 \equiv -\mathsf{u}_{3,i} x_3 \qquad \Rightarrow \qquad \mathsf{u}_{i,j} = \frac{-\partial \mathsf{u}_3}{\partial x_i \partial x_j} x_3 \equiv -\mathsf{u}_{3,ij} x_3 = \mathsf{u}_{j,i} \qquad (i = 1,2)$$

The Stress Field

To obtain the stress-strain relationship we assume that the stress according to z-direction is zero, i.e. $\sigma_{3i} = \sigma_{i3} = 0$ since we have assume that $\varepsilon_{3i} = 0$, in other words we are dealing with the state of plane stress. And the stress-strain relationship for the state of plane stress in Voigt notation, (see **Problem 6.24**), is given by:

$$\begin{aligned} &\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -zw_{,xx} \\ -zw_{,yy} \\ -2zw_{,xy} \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{-Ez}{1-\nu^2} \begin{bmatrix} w_{,xx} + \nu\, w_{,yy} \\ \nu\, w_{,xx} + w_{,yy} \\ (1-\nu) w_{,xy} \end{bmatrix} \end{aligned} \tag{6.710}$$

where E stands for Young's modulus and ν is the Poisson's ratio. The stress-strain relationship in indicial notation, (see **Problem 6.24**), can be obtained as follows:

$$\begin{cases} \sigma_{ij} = \dfrac{\nu E}{(1-\nu^2)} \mathrm{Tr}(\boldsymbol{\varepsilon}) \delta_{ij} + \dfrac{E}{(1+\nu)} \varepsilon_{ij} \quad ; \quad (i,j = 1,2) \quad \text{with} \quad \mathrm{Tr}(\boldsymbol{\varepsilon}) = \varepsilon_{11} + \varepsilon_{22} \\ \varepsilon_{ij} = \dfrac{-\nu}{E} \mathrm{Tr}(\boldsymbol{\sigma}) \delta_{ij} + \dfrac{(1+\nu)}{E} \sigma_{ij} \qquad (i,j = 1,2,3) \quad (\text{the same as } 3D) \end{cases} \tag{6.711}$$

Taking into account that $\varepsilon_{ij} = -\mathsf{u}_{3,ij} x_3$ and $\mathrm{Tr}(\boldsymbol{\varepsilon}) = \mathrm{Tr}(-\mathsf{u}_{3,ij} x_3) = -\mathsf{u}_{3,kk} x_3$ the above equation for stress becomes

$$\boxed{\sigma_{ij} = \frac{-E\, x_3}{(1-\nu^2)} [\nu\, \mathsf{u}_{3,kk} \delta_{ij} + (1-\nu) \mathsf{u}_{3,ij}] \quad ; \quad (i,j = 1,2)} \tag{6.712}$$

The above equation can also be expressed more explicitly as follows:

$$\sigma_{ij} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,\mathsf{u}_{3,kk}\delta_{ij} + (1-\nu)\mathsf{u}_{3,ij}] \quad ; \quad (i,j=1,2)$$

$$\Rightarrow \sigma_{ij} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,(\mathsf{u}_{3,11} + \mathsf{u}_{3,22})\delta_{ij} + (1-\nu)\mathsf{u}_{3,ij}]$$

$$\Rightarrow \sigma_{ij} = \frac{-E\,x_3}{(1-\nu^2)}\left(\begin{bmatrix} \nu\,(\mathsf{u}_{3,11}+\mathsf{u}_{3,22}) & 0 \\ 0 & \nu\,(\mathsf{u}_{3,11}+\mathsf{u}_{3,22}) \end{bmatrix} + \begin{bmatrix} (1-\nu)\mathsf{u}_{3,11} & (1-\nu)\mathsf{u}_{3,12} \\ (1-\nu)\mathsf{u}_{3,21} & (1-\nu)\mathsf{u}_{3,22} \end{bmatrix}\right)$$

$$\Rightarrow \sigma_{ij} = \frac{-E\,x_3}{1-\nu^2}\begin{bmatrix} \nu\,\mathsf{u}_{3,22}+\mathsf{u}_{3,11} & (1-\nu)\mathsf{u}_{3,12} \\ (1-\nu)\mathsf{u}_{3,12} & \nu\,\mathsf{u}_{3,11}+\mathsf{u}_{3,22} \end{bmatrix} = \frac{-E\,x_3}{1-\nu^2}\begin{bmatrix} \frac{\partial^2\mathsf{u}_3}{\partial x_1^2} + \nu\frac{\partial^2\mathsf{u}_3}{\partial x_2^2} & (1-\nu)\frac{\partial^2\mathsf{u}_3}{\partial x_1\partial x_2} \\ (1-\nu)\frac{\partial^2\mathsf{u}_3}{\partial x_1\partial x_2} & \frac{\partial^2\mathsf{u}_3}{\partial x_2^2} + \frac{\partial^2\mathsf{u}_3}{\partial x_1^2}\nu \end{bmatrix}$$

which matches the equation in (6.710).

NOTE 1: The problem established here is called *Kirchhoff-Love Plate Theory*, and is used to solve flexural plates when the thickness is very small.

NOTE 2: Note that the stresses $\sigma_{3i} = \sigma_{i3} \neq 0$ are not zero, (see Figure 6.237), and we cannot obtain σ_{3i} from the constitutive equations since we have assumed that $\varepsilon_{3i} = 0$. In order to obtain the equations for σ_{3i} we have to consider the equilibrium equations $\sigma_{ij,j} = 0_i$ (without body forces and static case).

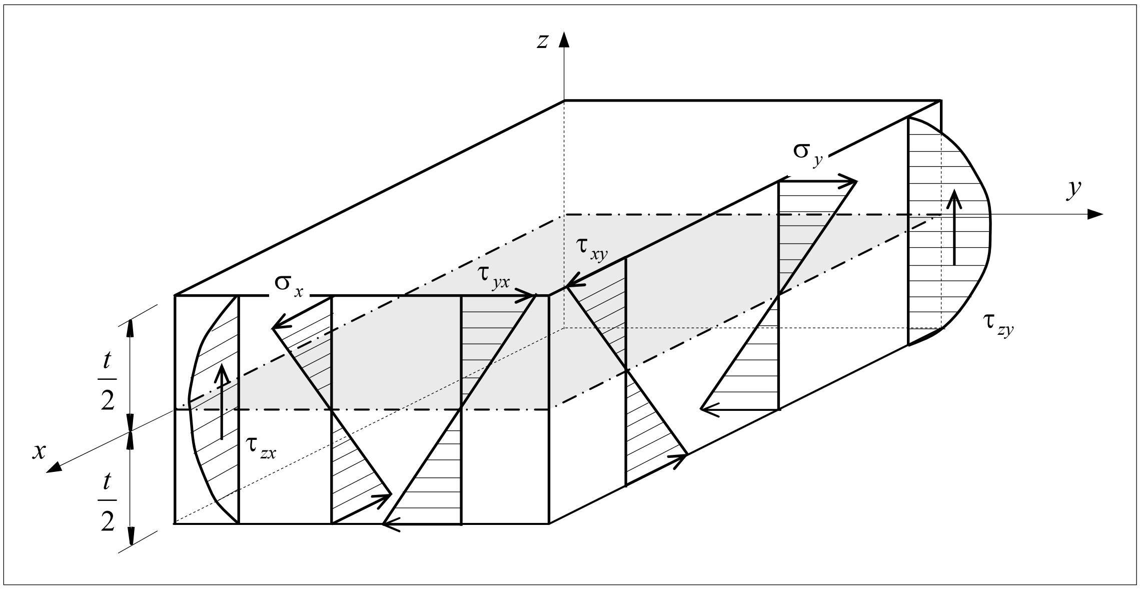

Figure 6.237: Stress distribution – Kirchhoff-Love plate theory.

NOTE 3: The Resultant Moments/Forces

By considering the infinitesimal element described in Figure 6.238 the resultant moment can be expressed as follows

$$m_x = \int_{\frac{-t}{2}}^{\frac{t}{2}} z\sigma_x dz \quad ; \quad m_y = \int_{\frac{-t}{2}}^{\frac{t}{2}} z\sigma_y dz \quad ; \quad m_{xy} = \int_{\frac{-t}{2}}^{\frac{t}{2}} z\tau_{xy} dz \qquad (i,j=1,2) \tag{6.713}$$

note that $\tau_{xy} = \tau_{yx}$ hence $m_{xy} = m_{yx}$. The above equations in indicial notation can be represented as follows:

$$\boxed{m_{ij} = \int_{\frac{-t}{2}}^{\frac{t}{2}} x_3 \sigma_{ij} dx_3 \qquad (i,j=1,2)} \tag{6.714}$$

The resultant forces

$$Q_{31} = Q_x = \int_{\frac{-t}{2}}^{\frac{t}{2}} \tau_{zx} dz \quad ; \quad Q_{32} = Q_y = \int_{\frac{-t}{2}}^{\frac{t}{2}} \tau_{zy} dz \qquad (i,j=1,2) \tag{6.715}$$

or in indicial notation

$$\boxed{Q_{3i} = \int_{\frac{t}{2}}^{\frac{t}{2}} \sigma_{i3} dx_3 \qquad (i,j=1,2)} \tag{6.716}$$

Note that

$$m_{ij,j} = \int_{\frac{-t}{2}}^{\frac{t}{2}} (x_3\sigma_{ij})_{,j} dx_3 = \int_{\frac{-t}{2}}^{\frac{t}{2}} (x_{3,j}\sigma_{ij} + x_3\sigma_{ij,j}) dx_3 = \int_{\frac{-t}{2}}^{\frac{t}{2}} (x_{3,j}\sigma_{ij}) dx_3 + \int_{\frac{-t}{2}}^{\frac{t}{2}} (x_3\sigma_{ij,j}) dx_3$$

$$\Rightarrow m_{ij,j} = \int_{\frac{-t}{2}}^{\frac{t}{2}} (\underbrace{x_{3,1}}_{=0}\sigma_{i1} + \underbrace{x_{3,2}}_{=0}\sigma_{i2} + \underbrace{x_{3,3}}_{=1}\sigma_{i3}) dx_3 + \int_{\frac{-t}{2}}^{\frac{t}{2}} (x_3 \underbrace{\sigma_{ij,j}}_{=0_i}) dx_3 = \int_{\frac{-t}{2}}^{\frac{t}{2}} \sigma_{i3} dx_3 = Q_{3i}$$

$$\boxed{m_{ij,j} - Q_{3i} = 0_i} \quad \textit{Flexural plate differential equation (in terms of moments)} \tag{6.717}$$

where we have applied the equilibrium equations $\sigma_{ij,j} = 0_i$. Taking into account the equation for σ_{ij} given by the equation in (6.712), the moments become

$$m_{ij} = \int_{\frac{-t}{2}}^{\frac{t}{2}} x_3\sigma_{ij} dx_3 = \frac{-E}{(1-\nu^2)}[\nu\, \mathsf{u}_{3,kk}\delta_{ij} + (1-\nu)\mathsf{u}_{3,ij}] \int_{\frac{-t}{2}}^{\frac{t}{2}} x_3^2 dx_3 \qquad (i,j=1,2)$$

$$\Rightarrow m_{ij} = \frac{-Et^3}{12(1-\nu^2)}[\nu\, \mathsf{u}_{3,kk}\delta_{ij} + (1-\nu)\mathsf{u}_{3,ij}]$$

$$\Rightarrow m_{ij} = -D \begin{bmatrix} \frac{\partial^2 \mathsf{u}_3}{\partial x_1^2} + \nu \frac{\partial^2 \mathsf{u}_3}{\partial x_2^2} & (1-\nu)\frac{\partial^2 \mathsf{u}_3}{\partial x_1 \partial x_2} \\ (1-\nu)\frac{\partial^2 \mathsf{u}_3}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathsf{u}_3}{\partial x_2^2} + \frac{\partial^2 \mathsf{u}_3}{\partial x_1^2}\nu \end{bmatrix} \tag{6.718}$$

where we have introduced the parameter $D = \frac{Et^3}{12(1-\nu^2)}$ which is called the *bending stiffness of the plate* or *flexural rigidity of the plate*. The above equation in Voigt notation can be expressed as follows

$$\{\boldsymbol{m}\} = \begin{Bmatrix} m_{11} \\ m_{22} \\ m_{12} \end{Bmatrix} = \begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 \mathsf{u}_3}{\partial x_1^2} \\ \frac{\partial^2 \mathsf{u}_3}{\partial x_2^2} \\ 2\frac{\partial^2 \mathsf{u}_3}{\partial x_1 \partial x_2} \end{Bmatrix} \quad \text{with} \quad D = \frac{Et^3}{12(1-\nu^2)} \tag{6.719}$$

$$\Rightarrow \{\boldsymbol{m}\} = \{\boldsymbol{D}_K\}\{\hat{\boldsymbol{\varepsilon}}\}$$

or in Engineering notation:

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} -D\frac{\partial^2 w}{\partial x^2} - D\nu\frac{\partial^2 w}{\partial y^2} \\ -D\nu\frac{\partial^2 w}{\partial x^2} - D\frac{\partial^2 w}{\partial y^2} \\ -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{6.720}$$

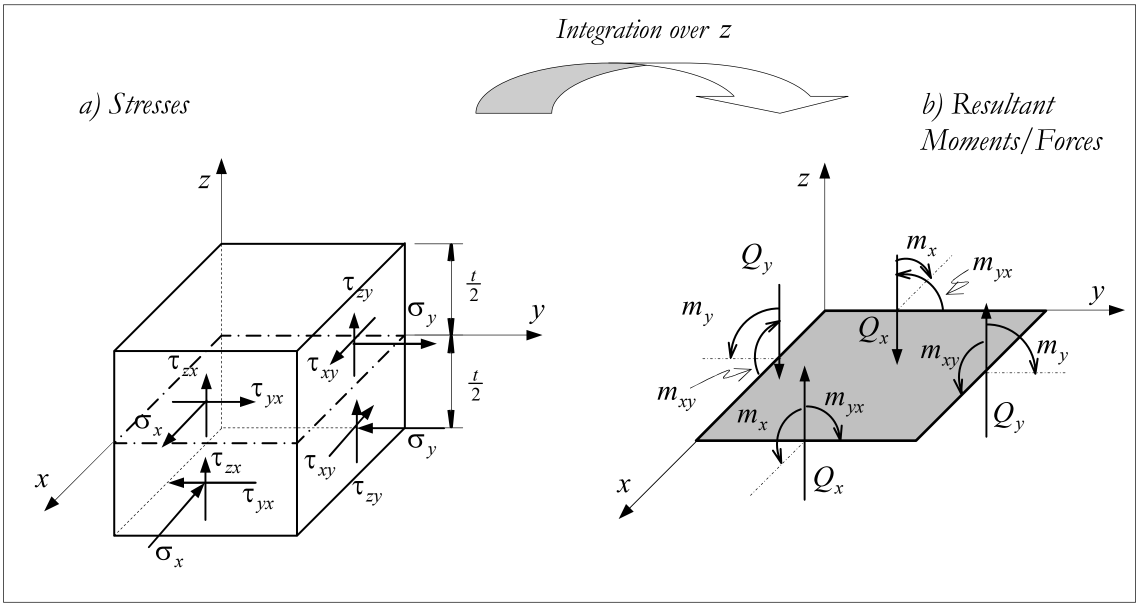

Figure 6.238: Infinitesimal element of plate.

NOTE 4: Plate Differential Equation

Now if we consider the differential element of plate, (see Figure 6.239), and by applying the equilibrium of force according to z-direction we can obtain:

$$\sum F_z = 0$$

$$\Rightarrow q_z dxdy - Q_y dx - Q_x dy + \left(Q_x + \frac{\partial Q_x}{\partial x} dx\right) dy + \left(Q_y + \frac{\partial Q_y}{\partial y} dy\right) dx = 0 \tag{6.721}$$

$$\Rightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z = 0$$

or in indicial notation

$$\boxed{Q_{3i,i} + q_z = 0 \qquad (i,j=1,2)} \tag{6.722}$$

Equilibrium of moments

Total moment about x-direction:

$$\sum M_x = 0$$

$$\Rightarrow m_y dx + m_{yx} dy - \left(m_{yx} + \frac{\partial m_{yx}}{\partial x} dx \right) dy + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy \frac{dy}{2} +$$

$$- \left(m_y + \frac{\partial m_y}{\partial y} dy \right) dx + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dxdy - Q_x dy \frac{dy}{2} + q_z dxdy \frac{dy}{2} = 0$$

$$\Rightarrow \frac{-\partial m_{yx}}{\partial x} dxdy + \frac{\partial Q_x}{\partial x} dxdy \frac{dy}{2} - \frac{\partial m_y}{\partial y} dxdy + Q_y dxdy + \frac{\partial Q_y}{\partial y} dxdydy + q_z dxdy \frac{dy}{2} = 0$$

$$\Rightarrow -\left(\frac{\partial m_{yx}}{\partial x} + \frac{\partial m_y}{\partial y} - Q_y \right) dxdy + \underbrace{\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z \right)}_{=0} dxdy \frac{dy}{2} + \frac{\partial Q_y}{\partial y} dxdy \frac{dy}{2} = 0$$

$$\Rightarrow -\left(\frac{\partial m_{yx}}{\partial x} + \frac{\partial m_y}{\partial y} - Q_y \right) dxdy + \frac{\partial Q_y}{\partial y} dxdy \frac{dy}{2} = 0$$

And by discarding the term related to $dxdydy \approx 0$ we can obtain:

$$\frac{\partial m_{yx}}{\partial x} + \frac{\partial m_y}{\partial y} - Q_y = 0 \qquad (6.723)$$

Total moment about y-direction:

$$\sum M_y = 0$$

$$\Rightarrow -m_x dy - m_{xy} dx + \left(m_{xy} + \frac{\partial m_{xy}}{\partial y} dy \right) dx + \left(m_x + \frac{\partial m_x}{\partial x} dx \right) dy +$$

$$+ Q_y dx \frac{dx}{2} - \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dydx - \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx \frac{dx}{2} - q_z dxdy \frac{dx}{2} = 0$$

$$\Rightarrow \frac{\partial m_{xy}}{\partial y} dydx + \frac{\partial m_x}{\partial x} dxdy - Q_x dydx - \frac{\partial Q_x}{\partial x} dxdydx - \frac{\partial Q_y}{\partial y} dydx \frac{dx}{2} - q_z dxdy \frac{dx}{2} = 0$$

$$\Rightarrow \left(\frac{\partial m_{xy}}{\partial y} + \frac{\partial m_x}{\partial x} - Q_x \right) dxdy - \underbrace{\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z \right)}_{=0} dydx \frac{dx}{2} + \frac{\partial Q_x}{\partial x} dxdy \frac{dx}{2} = 0$$

$$\Rightarrow \left(\frac{\partial m_{xy}}{\partial y} + \frac{\partial m_x}{\partial x} - Q_x \right) dxdy + \frac{\partial Q_x}{\partial x} dxdy \frac{dx}{2} = 0$$

And by discarding the term related to $dxdydx \approx 0$ we can obtain:

$$\frac{\partial m_{xy}}{\partial y} + \frac{\partial m_x}{\partial x} - Q_x = 0 \qquad (6.724)$$

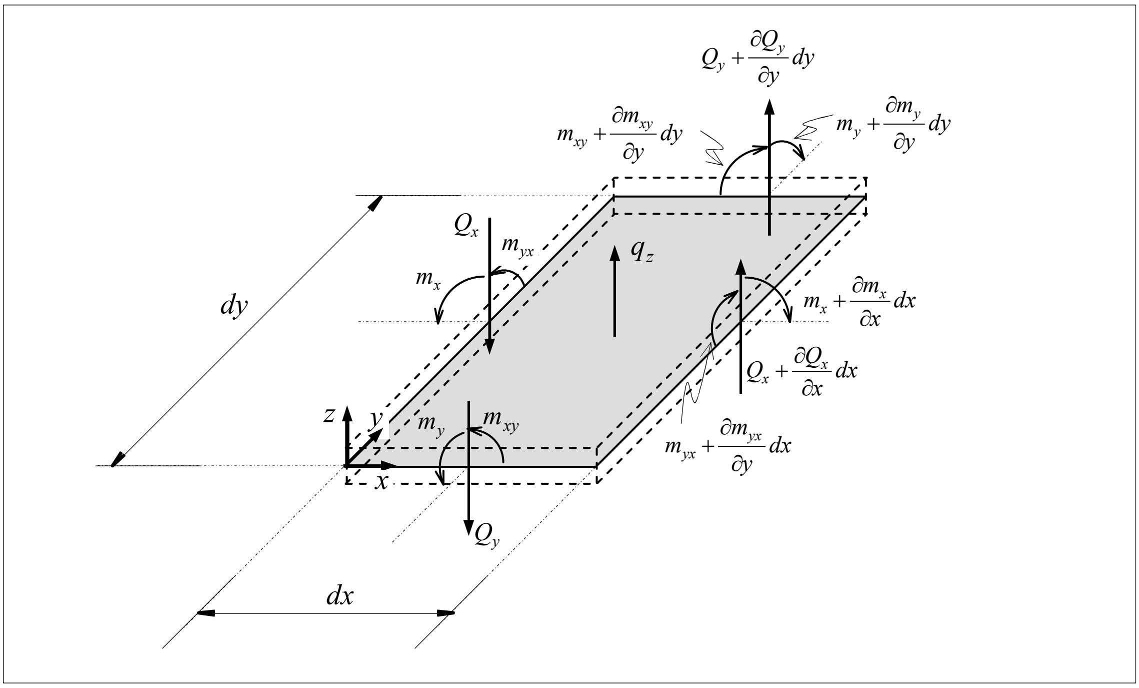

Figure 6.239: Differential element of plate.

According to equation (6.720) we have

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} -D\frac{\partial^2 w}{\partial x^2} - D\nu \frac{\partial^2 w}{\partial y^2} \\ -D\nu\frac{\partial^2 w}{\partial x^2} - D \frac{\partial^2 w}{\partial y^2} \\ -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

Then, the equation (6.724) can be rewritten as follows

$$\begin{aligned} &\frac{\partial m_{xy}}{\partial y} + \frac{\partial m_x}{\partial x} - Q_x = 0 \Rightarrow Q_x = \frac{\partial}{\partial y}\left(-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial}{\partial x}\left(-D\frac{\partial^2 w}{\partial x^2} - D\nu\frac{\partial^2 w}{\partial y^2}\right) \\ &\Rightarrow Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \end{aligned} \tag{6.725}$$

And the equation in (6.723) can be rewritten as follows

$$\begin{aligned} &\frac{\partial m_{yx}}{\partial x} + \frac{\partial m_y}{\partial y} - Q_y = 0 \Rightarrow Q_y = \frac{\partial}{\partial x}\left(-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(-D\nu\frac{\partial^2 w}{\partial x^2} - D\frac{\partial^2 w}{\partial y^2}\right) \\ &\Rightarrow Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \end{aligned} \tag{6.726}$$

Then, by substituting the equations (6.725) and (6.726) into the force equilibrium equation (6.721) we can obtain:

$$\begin{aligned}
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q_z=0\\
&\Rightarrow \frac{\partial}{\partial x}\left(-D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right)+\frac{\partial}{\partial y}\left(-D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right)+q_z=0\\
&\Rightarrow -D\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-D\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+q_z=0\\
&\Rightarrow \frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q_z}{D}
\end{aligned}\tag{6.727}$$

which is the flexural plate differential equation in terms of w-displacement (deflection). The above equation can also be written as follows:

$$\boxed{\begin{aligned}&\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q_z}{D}\\ &or\\ &\nabla^2_{\vec{x}}(\nabla^2_{\vec{x}} w)=\frac{q_z}{D}\quad\Leftrightarrow\quad\nabla^4_{\vec{x}} w=\frac{q_z}{D}\end{aligned}}\quad \textit{Flexural plate differential equation (in terms of deflection } w\textit{)}\tag{6.728}$$

where $\nabla^2_{\vec{x}}\equiv\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}})$ is the Laplacian operator and $\nabla^4_{\vec{x}}$ is the bi-Laplacian operator.

Using indicial notation:

By starting from the equation (6.717) we can obtain:

$$\begin{aligned}
&m_{ij,j}-Q_{3i}=0_i\qquad (i,j=1,2)\\
&\Rightarrow\left(\frac{-Et^3}{12(1-\nu^2)}[\nu\,\mathsf{u}_{3,kk}\delta_{ij}+(1-\nu)\mathsf{u}_{3,ij}]\right)_{,j}-Q_{3i}=0_i\\
&\Rightarrow\frac{-Et^3}{12(1-\nu^2)}[\nu\,\mathsf{u}_{3,kkj}\delta_{ij}+(1-\nu)\mathsf{u}_{3,ijj}]-Q_{3i}=0_i\\
&\Rightarrow -D[\nu\,\mathsf{u}_{3,kki}+(1-\nu)\mathsf{u}_{3,ikk}]-Q_{3i}=0_i\\
&\Rightarrow -D\mathsf{u}_{3,kki}-Q_{3i}=0_i
\end{aligned}\tag{6.729}$$

Taking the derivative with respect to x_i we can obtain

$$\begin{aligned}
&\Rightarrow -D\mathsf{u}_{3,kkii}-Q_{3i,i}=0_{i,i}=0\\
&\Rightarrow -D\mathsf{u}_{3,kkii}+q_z=0\\
&\Rightarrow \mathsf{u}_{3,kkii}=\frac{q_z}{D}\qquad (i,k=1,2)
\end{aligned}\tag{6.730}$$

which is the same as the equation in (6.728).

NOTE 5: The stresses σ_{3i}

Once the problem is solved the stresses σ_{3i}, (see Figure 6.237), can be obtained. We start from the equilibrium equation

$$\sigma_{ij,j}=0_i\quad\Rightarrow\quad\sigma_{i1,1}+\sigma_{i2,2}+\sigma_{i3,3}=0_i\quad\Rightarrow\quad\sigma_{i3,3}=-(\sigma_{i1,1}+\sigma_{i2,2})$$

which is the same as

$$\sigma_{i3,3}=-(\sigma_{i1,1}+\sigma_{i2,2})\quad\Rightarrow\quad\sigma_{i3,3}=-\sigma_{ik,k}\qquad(i=1,2,3;k=1,2)\tag{6.731}$$

According to the equation (6.712) we can obtain

$$
\begin{aligned}
&\sigma_{ik} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,\mathsf{u}_{3,tt}\delta_{ik} + (1-\nu)\mathsf{u}_{3,ik}] \quad ; \quad (i,k,t=1,2) \\
&\Rightarrow \sigma_{ik,k} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,\mathsf{u}_{3,tt}\delta_{ik} + (1-\nu)\mathsf{u}_{3,ik}]_{,k} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,\mathsf{u}_{3,ttk}\delta_{ik} + (1-\nu)\mathsf{u}_{3,ikk}] \\
&\Rightarrow \sigma_{ik,k} = \frac{-E\,x_3}{(1-\nu^2)}[\nu\underbrace{\mathsf{u}_{3,tti}}_{=\mathsf{u}_{3,kki}} + (1-\nu)\mathsf{u}_{3,ikk}] = \frac{-E\,x_3}{(1-\nu^2)}[\nu\,\mathsf{u}_{3,kki} + (1-\nu)\underbrace{\mathsf{u}_{3,ikk}}_{=\mathsf{u}_{3,kki}}] \\
&\Rightarrow \sigma_{ik,k} = \frac{-E\,x_3}{(1-\nu^2)}\mathsf{u}_{3,kki}
\end{aligned}
\tag{6.732}
$$

Taking into account the equation in (6.729) the above equation can be written as follows:

$$
\begin{aligned}
&\sigma_{ik,k} = \frac{-E\,x_3}{(1-\nu^2)}\mathsf{u}_{3,kki} = \frac{-E\,x_3}{(1-\nu^2)}\left(\frac{-Q_{3i}}{D}\right) = \frac{E\,Q_{3i}x_3}{(1-\nu^2)}\frac{12(1-\nu^2)}{Et^3} \\
&\Rightarrow \sigma_{ik,k} = \frac{12Q_{3i}}{t^3}x_3 \qquad (i,k=1,2)
\end{aligned}
\tag{6.733}
$$

where we considered $D = \dfrac{Et^3}{12(1-\nu^2)}$. By substituting the above equation into the equilibrium equation (6.731) we can obtain

$$
\sigma_{i3,3} \equiv \frac{\partial \sigma_{i3}}{\partial x_3} = -\sigma_{ik,k} = \frac{-12Q_{3i}}{t^3}x_3 \qquad (i,k=1,2)
\tag{6.734}
$$

By integrating the above equation over x_3 we can obtain

$$
\sigma_{i3} = \int\left(\frac{-12Q_{3i}}{t^3}x_3\right)\partial x_3 = \frac{-12Q_{3i}}{2t^3}x_3^2 + K
\tag{6.735}
$$

The constant of integration can be obtained by the condition: $x_3 = \pm\dfrac{t}{2} \Rightarrow \sigma_{i3} = 0$:

$$
\sigma_{i3} = \frac{-12Q_{3i}}{2t^3}x_3^2 + K = \frac{-12Q_{3i}}{2t^3}\left(\frac{\pm t}{2}\right)^2 + K = 0 \quad \Rightarrow \quad K = \frac{3Q_{3i}}{2t}
$$

Then

$$
\sigma_{i3} = \frac{-12Q_{3i}}{2t^3}x_3^2 + K = \frac{-6Q_{3i}}{t^3}x_3^2 + \frac{3Q_{3i}}{2t} = \frac{3Q_{3i}}{2t}\left[1-\left(\frac{2x_3}{t}\right)^2\right]
\tag{6.736}
$$

From the equilibrium equation (6.731) we can also obtain $\sigma_{33,3}$

$$
\begin{aligned}
&\sigma_{33,3} \equiv \frac{\partial \sigma_{33}}{\partial x_3} = -\sigma_{3k,k} = -\left(\frac{-6Q_{3k}}{t^3}x_3^2 + \frac{3Q_{3k}}{2t}\right)_{,k} \qquad (k=1,2) \\
&\Rightarrow \frac{\partial \sigma_{33}}{\partial x_3} = \frac{6Q_{3k,k}}{t^3}x_3^2 - \frac{3Q_{3k,k}}{2t} \xrightarrow{\text{integrating over } x_3} \sigma_{33} = \frac{6Q_{3k,k}}{t^3}\frac{x_3^3}{3} - \frac{3Q_{3k,k}}{2t}x_3 + K
\end{aligned}
\tag{6.737}
$$

The constant of integration can be obtained by means $x_3 = \dfrac{-t}{2} \Rightarrow \sigma_{33} = 0$, then

$$\sigma_{33} = \frac{6Q_{3k,k}}{t^3}\frac{\left(\frac{-t}{2}\right)^3}{3} - \frac{3Q_{3k,k}}{2t}\left(\frac{-t}{2}\right) + K = 0 \quad \Rightarrow \quad K = \frac{-Q_{3k,k}}{2}$$

thus

$$\sigma_{33} = \frac{2Q_{3k,k}}{t^3}x_3^3 - \frac{3Q_{3k,k}}{2t}x_3 - \frac{Q_{3k,k}}{2} = -\frac{3}{4}Q_{3k,k}\left[\frac{2}{3} + \frac{2x_3}{t} - \frac{1}{3}\left(\frac{2x_3}{t}\right)^3\right] \tag{6.738}$$

By means of equation in (6.722), the above equation can also be written as follows:

$$\sigma_{33} = \frac{3}{4}q_z\left[\frac{2}{3} + \frac{2x_3}{t} - \frac{1}{3}\left(\frac{2x_3}{t}\right)^3\right] \tag{6.739}$$

And note that when $x_3 = \frac{t}{2}$ we can obtain

$$\sigma_{33} = \frac{3}{4}q_z\left[\frac{2}{3} + \frac{2}{t}\left(\frac{t}{2}\right) - \frac{1}{3}\left(\frac{2}{t}\left(\frac{t}{2}\right)\right)^3\right] = q_z$$

11 Introduction to Fluids

Problem 11.1

Demonstrate whether the following statements are true or false:

a) If the velocity field is steady, then the acceleration field is also;

b) If the velocity field is homogeneous, the acceleration field is always equal to zero;

c) If the velocity field is steady and the medium is incompressible, the acceleration is always zero.

Solution:

a) In a steady velocity field we have $\dfrac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} = \vec{\boldsymbol{0}}$ whereby the acceleration field becomes:

$$a_i = \dot{v}_i = \underbrace{\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial t}}_{=0_i} + v_{i,k} v_k = v_{i,k} v_k$$

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}{\partial t} + \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) = \underbrace{\nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}_{\text{Independent of time}}$$

Then, assumption (a) is TRUE.

b) A homogeneous velocity field implies that $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{v}}(t)$, whereby:

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} + \underbrace{\nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}_{=\boldsymbol{0}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t}$$

Then, assumption (b) is FALSE.

c) A steady velocity field implies that $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$ and an incompressible medium means that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = 0$, so, we can conclude that:

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}{\partial t} + \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) = \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$$

Then, assumption (c) is FALSE.

Problem 11.2

Show the Navier-Stokes-Duhem equations of motion:

$$\boxed{\begin{aligned}&\rho\dot{v}_i = \rho \mathrm{b}_i - p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj}\\ &\rho\dot{\vec{\boldsymbol{v}}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*)[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})] + \mu^* \nabla^2_{\vec{x}} \vec{\boldsymbol{v}}\end{aligned}}$$ *Navier-Stokes-Duhem equations of motion* (11.1)

Solution:

The Navier-Stokes-Duhem equations of motion are a combination of the equations of motion $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{\boldsymbol{v}}}$, $(\sigma_{ij,j} + \rho\,\mathrm{b}_i = \rho\dot{v}_i)$, and the constitutive equations:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda^* \mathrm{Tr}(\mathbf{D})\mathbf{1} + 2\mu^* \mathbf{D} \qquad \Bigg| \qquad \sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} \mathrm{D}_{kk} + 2\mu^* \mathrm{D}_{ij} \tag{11.2}$$

The Cauchy stress tensor divergence ($\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}$) can be evaluated as follows:

$$\begin{aligned}\sigma_{ij,j} &= (-p\delta_{ij} + \lambda^* \delta_{ij} \mathrm{D}_{kk} + 2\mu^* \mathrm{D}_{ij})_{,j} = -p_{,j}\delta_{ij} + \lambda^* (\delta_{ij} \mathrm{D}_{kk})_{,j} + 2\mu^* \mathrm{D}_{ij,j}\\ &= -p_{,i} + \lambda^* \delta_{ij} \mathrm{D}_{kk,j} + 2\mu^* \mathrm{D}_{ij,j} = -p_{,i} + \lambda^* \mathrm{D}_{kk,i} + 2\mu^* \mathrm{D}_{ij,j}\end{aligned} \tag{11.3}$$

Note that, in this formulation, we are considering that the material is homogeneous, i.e. $\lambda^*_{,j} = \mu^*_{,j} = 0_j$.

In addition, by considering $2\mathrm{D}_{ij} = v_{i,j} + v_{j,i}$ and $2\mathrm{D}_{kk} = v_{k,k} + v_{k,k} = 2v_{k,k}$, we can obtain:

$$\Rightarrow 2\mathrm{D}_{ij,j} = v_{i,jj} + v_{j,ij} = v_{i,jj} + v_{j,ji} \qquad \Rightarrow \qquad \mathrm{D}_{kk,j} = v_{k,kj} \tag{11.4}$$

whereby the equation in (11.2) becomes:

$$\begin{aligned}\sigma_{ij,j} &= -p_{,i} + \lambda^* \mathrm{D}_{jj,i} + 2\mu^* \mathrm{D}_{ij,j} = -p_{,i} + \lambda^* v_{j,ji} + \mu^* (v_{i,jj} + v_{j,ji})\\ &= -p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj}\end{aligned} \tag{11.5}$$

Then, by substituting the equation in (11.5) into the equations of motion ($\sigma_{ij,j} + \rho\mathrm{b}_i = \rho\dot{v}_i$), we obtain the Navier-Stokes-Duhem equations of motion for homogeneous materials.

NOTE 1: The explicit form of the equation (11.1) is presented as follows:

$$\begin{aligned}&(\lambda^* + \mu^*) v_{j,ji} + \mu v_{i,jj} + \rho\mathrm{b}_i - p_{,i} = \rho\dot{v}_i\\ &\Rightarrow (\lambda^* + \mu^*)(v_{1,1i} + v_{2,2i} + v_{3,3i}) + \mu^*(v_{i,11} + v_{i,22} + v_{i,33}) + \rho\mathrm{b}_i - p_{,i} = \rho\dot{v}_i\end{aligned}$$

$$\begin{cases}(\lambda^* + \mu^*)(v_{1,11} + v_{2,21} + v_{3,31}) + \mu^*(v_{1,11} + v_{1,22} + v_{1,33}) - p_{,1} + \rho\mathrm{b}_1 = \rho\dot{v}_1\\ (\lambda^* + \mu^*)(v_{1,12} + v_{2,22} + v_{3,32}) + \mu^*(v_{2,11} + \mathrm{u}_{2,22} + v_{2,33}) - p_{,2} + \rho\mathrm{b}_2 = \rho\dot{v}_2\\ (\lambda^* + \mu^*)(v_{1,13} + v_{2,23} + v_{3,33}) + \mu^*(v_{3,11} + v_{3,22} + v_{3,33}) - p_{,3} + \rho\mathrm{b}_3 = \rho\dot{v}_3\end{cases}$$

or:

$$
\begin{cases}
(\lambda^* + \mu^*)\dfrac{\partial}{\partial x_1}\left(\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3}\right) + \mu^*\left(\dfrac{\partial^2 v_1}{\partial x_1^2} + \dfrac{\partial^2 v_1}{\partial x_2^2} + \dfrac{\partial^2 v_1}{\partial x_3^2}\right) + \rho \mathsf{b}_1 - p_{,1} = \rho \dot{v}_1 \\
(\lambda^* + \mu^*)\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3}\right) + \mu^*\left(\dfrac{\partial^2 v_2}{\partial x_1^2} + \dfrac{\partial^2 v_2}{\partial x_2^2} + \dfrac{\partial^2 v_2}{\partial x_3^2}\right) + \rho \mathsf{b}_2 - p_{,2} = \rho \dot{v}_2 \\
(\lambda^* + \mu^*)\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3}\right) + \mu^*\left(\dfrac{\partial^2 v_3}{\partial x_1^2} + \dfrac{\partial^2 v_3}{\partial x_2^2} + \dfrac{\partial^2 v_3}{\partial x_3^2}\right) + \rho \mathsf{b}_3 - p_{,3} = \rho \dot{v}_3
\end{cases}
$$

NOTE 2: We have proven in **Problem 1.106** (Chapter 1) that the following is true:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}}) - \nabla^2_{\vec{x}}\vec{\mathbf{a}} \xrightarrow{\text{indicial}} \epsilon_{ilq}\epsilon_{qjk}\mathsf{a}_{k,jl} = \mathsf{a}_{j,ji} - \mathsf{a}_{i,jj}$$

Then, we can obtain

$$\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}}\vec{v}) \equiv \nabla^2_{\vec{x}}\vec{v} = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \xrightarrow{\text{indicial}} v_{i,jj} = v_{j,ji} - \epsilon_{ilq}\epsilon_{qjk}v_{k,jl}$$

with which the equation (11.1) can also be written as follows:

$$
\begin{aligned}
&(\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj} + \rho \mathsf{b}_i - p_{,i} = \rho \dot{v}_i \\
&\Rightarrow (\lambda^* + \mu^*)v_{j,ji} + \mu^*(v_{j,ji} - \epsilon_{ilq}\epsilon_{qjk}v_{k,jl}) + \rho \mathsf{b}_i - p_{,i} = \rho \dot{v}_i \\
&\Rightarrow (\lambda^* + 2\mu^*)v_{j,ji} - \mu^*\epsilon_{ilq}\epsilon_{qjk}v_{k,jl} + \rho \mathsf{b}_i - p_{,i} = \rho \dot{v}_i
\end{aligned}
$$

and the equivalent in tensorial notation:

$$
\begin{aligned}
&(\lambda^* + \mu^*)[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})] + \mu^*[\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}}\vec{v})] + \rho\vec{\mathbf{b}} - \nabla_{\vec{x}}p = \rho\dot{\vec{v}} \\
&\Rightarrow (\lambda^* + \mu^*)[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})] + \mu^*\left[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] + \rho\vec{\mathbf{b}} - \nabla_{\vec{x}}p = \rho\dot{\vec{v}} \\
&\Rightarrow (\lambda^* + 2\mu^*)[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})] - \mu^*\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] + \rho\vec{\mathbf{b}} - \nabla_{\vec{x}}p = \rho\dot{\vec{v}}
\end{aligned}
$$

In the Cartesian System we have:

$$\vec{v} = v_i\hat{\mathbf{e}}_i = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$$

$$(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \equiv \mathrm{rot}(\vec{v}) = (\mathrm{rot}(\vec{v}))_i\hat{\mathbf{e}}_i = \underbrace{\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_1}\hat{\mathbf{e}}_1 + \underbrace{\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_2}\hat{\mathbf{e}}_2 + \underbrace{\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)}_{=(\mathrm{rot}(\vec{\mathbf{u}}))_3}\hat{\mathbf{e}}_3$$

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{v}) = \left(\frac{\partial(\mathrm{rot}(\vec{v}))_3}{\partial x_2} - \frac{\partial(\mathrm{rot}(\vec{v}))_2}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial(\mathrm{rot}(\vec{v}))_1}{\partial x_3} - \frac{\partial(\mathrm{rot}(\vec{v}))_3}{\partial x_1}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial(\mathrm{rot}(\vec{v}))_2}{\partial x_1} - \frac{\partial(\mathrm{rot}(\vec{v}))_1}{\partial x_2}\right)\hat{\mathbf{e}}_3$$

$$[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v})]_i = \begin{Bmatrix} \dfrac{\partial(\mathrm{rot}(\vec{v}))_3}{\partial x_2} - \dfrac{\partial(\mathrm{rot}(\vec{v}))_2}{\partial x_3} \\ \dfrac{\partial(\mathrm{rot}(\vec{v}))_1}{\partial x_3} - \dfrac{\partial(\mathrm{rot}(\vec{v}))_3}{\partial x_1} \\ \dfrac{\partial(\mathrm{rot}(\vec{v}))_2}{\partial x_1} - \dfrac{\partial(\mathrm{rot}(\vec{v}))_1}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}\right) - \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\right) \\ \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}\right) - \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}\right) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\right) - \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}\right) \end{Bmatrix}$$

NOTE 3: If we are dealing with heterogeneous material, we have:

$$
\begin{aligned}
&\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{ij} \\
&\Rightarrow \sigma_{ij,j} = (-p\delta_{ij} + \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{ij})_{,j} = -p_{,j}\delta_{ij} + (\lambda^*\mathsf{D}_{kk})_{,j}\delta_{ij} + 2(\mu^*\mathsf{D}_{ij})_{,j} \\
&\Rightarrow \sigma_{ij,j} = -p_{,i} + (\lambda^*\mathsf{D}_{kk})_{,i} + 2(\mu^*\mathsf{D}_{ij})_{,j}
\end{aligned}
$$

Taking into account that $2\mathsf{D}_{ij} = v_{i,j} + v_{j,i}$ and $\mathsf{D}_{kk} = v_{k,k}$, the above equation becomes:

$$\sigma_{ij,j} = -p_{,i} + (\lambda^*\mathsf{D}_{kk})_{,i} + 2(\mu^*\mathsf{D}_{ij})_{,j} = -p_{,i} + (\lambda^* v_{k,k})_{,i} + \left[\mu^*(v_{i,j} + v_{j,i})\right]_{,j}$$

whereby

$$\sigma_{ij,j}+\rho \mathsf{b}_i=\rho\dot{v}_i \qquad \Rightarrow \qquad -p_{,i}+(\lambda^* v_{k,k})_{,i}+[\mu^*(v_{i,j}+v_{j,i})]_{,j}+\rho \mathsf{b}_i=\rho\dot{v}_i \tag{11.6}$$

Note that

$v_{k,k}=\mathrm{Tr}(\nabla_{\vec{x}}\vec{\boldsymbol{v}})=(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})$

$\dot{v}_i=\dfrac{D\dot{v}_i}{Dt}=\dfrac{\partial v_i}{\partial t}+\dfrac{\partial v_i}{\partial x_j}v_j=\dfrac{\partial v_i}{\partial t}+\dfrac{\partial v_i}{\partial x_1}v_1+\dfrac{\partial v_i}{\partial x_2}v_2+\dfrac{\partial v_i}{\partial x_3}v_3$, and its explicit components are:

$$a_i=\dot{v}_i=\begin{Bmatrix}\dfrac{\partial v_1}{\partial t}+\dfrac{\partial v_1}{\partial x_1}v_1+\dfrac{\partial v_1}{\partial x_2}v_2+\dfrac{\partial v_1}{\partial x_3}v_3\\[2mm] \dfrac{\partial v_2}{\partial t}+\dfrac{\partial v_2}{\partial x_1}v_1+\dfrac{\partial v_2}{\partial x_2}v_2+\dfrac{\partial v_2}{\partial x_3}v_3\\[2mm] \dfrac{\partial v_3}{\partial t}+\dfrac{\partial v_3}{\partial x_1}v_1+\dfrac{\partial v_3}{\partial x_2}v_2+\dfrac{\partial v_3}{\partial x_3}v_3\end{Bmatrix}$$

$$\begin{aligned}[\mu^*(v_{i,j}+v_{j,i})]_{,j}&=\frac{\partial}{\partial x_j}[\mu^*(v_{i,j}+v_{j,i})]\\&=\frac{\partial}{\partial x_1}[\mu^*(v_{i,1}+v_{1,i})]+\frac{\partial}{\partial x_2}[\mu^*(v_{i,2}+v_{2,i})]+\frac{\partial}{\partial x_3}[\mu^*(v_{i,3}+v_{3,i})]\end{aligned}$$

$$[\mu^*(v_{i,j}+v_{j,i})]_{,j}=\begin{Bmatrix}\dfrac{\partial}{\partial x_1}[2\mu^*(v_{1,1})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{1,2}+v_{2,1})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{1,3}+v_{3,1})]\\[2mm] \dfrac{\partial}{\partial x_1}[\mu^*(v_{2,1}+v_{1,2})]+\dfrac{\partial}{\partial x_2}[2\mu^*(v_{2,2})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{2,3}+v_{3,2})]\\[2mm] \dfrac{\partial}{\partial x_1}[\mu^*(v_{3,1}+v_{1,3})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{3,2}+v_{2,3})]+\dfrac{\partial}{\partial x_3}[2\mu^*(v_{3,3})]\end{Bmatrix}$$

The three equations in (11.6), ($i=1,2,3$), are explicitly given by:

$$\begin{cases}\dfrac{\partial}{\partial x_1}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})]+\dfrac{\partial}{\partial x_1}[2\mu^*(v_{1,1})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{1,2}+v_{2,1})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{1,3}+v_{3,1})]+\rho\mathsf{b}_1-p_{,1}=\rho\dot{v}_1\\[2mm] \dfrac{\partial}{\partial x_2}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})]+\dfrac{\partial}{\partial x_1}[\mu^*(v_{2,1}+v_{1,2})]+\dfrac{\partial}{\partial x_2}[2\mu^*(v_{2,2})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{2,3}+v_{3,2})]+\rho\mathsf{b}_2-p_{,2}=\rho\dot{v}_2\\[2mm] \dfrac{\partial}{\partial x_3}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})]+\dfrac{\partial}{\partial x_1}[\mu^*(v_{3,1}+v_{1,3})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{3,2}+v_{2,3})]+\dfrac{\partial}{\partial x_3}[2\mu^* v_{3,3}]+\rho\mathsf{b}_3-p_{,3}=\rho\dot{v}_3\end{cases}$$

(11.7)

or

$$\boxed{\begin{cases}\dfrac{\partial}{\partial x_1}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})+2\mu^*(v_{1,1})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{1,2}+v_{2,1})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{1,3}+v_{3,1})]+\rho\mathsf{b}_1-p_{,1}=\rho\dot{v}_1\\[2mm] \dfrac{\partial}{\partial x_2}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})+2\mu^*(v_{2,2})]+\dfrac{\partial}{\partial x_1}[\mu^*(v_{2,1}+v_{1,2})]+\dfrac{\partial}{\partial x_3}[\mu^*(v_{2,3}+v_{3,2})]+\rho\mathsf{b}_2-p_{,2}=\rho\dot{v}_2\\[2mm] \dfrac{\partial}{\partial x_3}[\lambda^*(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})+2\mu^*(v_{3,3})]+\dfrac{\partial}{\partial x_1}[\mu^*(v_{3,1}+v_{1,3})]+\dfrac{\partial}{\partial x_2}[\mu^*(v_{3,2}+v_{2,3})]+\rho\mathsf{b}_3-p_{,3}=\rho\dot{v}_3\end{cases}}$$

(11.8)

NOTE 4: To obtain the dynamic equations of motion in a rotating fluid on the sphere, we have to consider the rotation of the Earth and the curvature, and the acceleration for a fixed system was obtained in **Problem 4.38**:

$$\begin{aligned}&\vec{\boldsymbol{a}}_f = \vec{\boldsymbol{a}}_r + 2(\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}}_r) + \boldsymbol{\Omega}^T \cdot \vec{\boldsymbol{v}}_r + \vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{x}}) \\ &\Rightarrow \vec{\boldsymbol{a}}_r = \vec{\boldsymbol{a}}_f - 2(\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}}_r) - \boldsymbol{\Omega}^T \cdot \vec{\boldsymbol{v}}_r - \vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{x}})\end{aligned} \tag{11.9}$$

where $(2(\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}}_r))$ is the Coriolis term, $(\boldsymbol{\Omega}^T \cdot \vec{\boldsymbol{v}}_r)$ is the curvature term, $\vec{\boldsymbol{\omega}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{x}})$ is the centrifugal term, and the components of these terms are:

$$\begin{aligned}&2(\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}}_r)_i = \begin{Bmatrix} 2[\omega_3 v_{r3}\cos(\phi) - \omega_3 v_{r2}\sin(\phi)] \\ 2[\omega_3 v_{r1}\sin(\phi)] \\ -2[\omega_3 v_{r1}\cos(\phi)] \end{Bmatrix} \\ &(\boldsymbol{\Omega}^T \cdot \vec{\boldsymbol{v}}_r)_i = \frac{1}{\|\vec{\boldsymbol{x}}\|}\begin{Bmatrix} -v_{r1}v_{r2}\tan(\phi) + v_{r1}v_{r3} \\ v_{r1}^2\tan(\phi) + v_{r2}v_{r3} \\ -v_{r1}^2 - v_{r2}^2 \end{Bmatrix} \\ &(\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\boldsymbol{r}}))_i = -\|\vec{\omega}\|^2 \vec{\boldsymbol{r}}\end{aligned} \tag{11.10}$$

Then, if we want to consider these terms we replace $\vec{\boldsymbol{a}}_r$ into the Navier-Stokes-Duhem equations.

Problem 11.3

Consider

$$\boxed{\begin{aligned}&\rho\dot{v}_i = \rho \mathrm{b}_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj} \\ &\rho\dot{\vec{\boldsymbol{v}}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*)[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})] + \mu^*\nabla^2_{\vec{x}}\vec{\boldsymbol{v}}\end{aligned}} \quad \textit{Navier-Stokes-Duhem equations of motion} \tag{11.11}$$

Show the equation of vorticity:

$$\boxed{\frac{\partial\vec{\boldsymbol{\omega}}}{\partial t} + 2\nabla_{\vec{x}} \cdot [(\vec{\boldsymbol{\omega}} \otimes \vec{\boldsymbol{v}})^{skew}] - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\,\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}} \quad \textit{The equation of vorticity} \tag{11.12}$$

where $\vec{\boldsymbol{\omega}}$ is vorticity vector and is given by $\vec{\boldsymbol{\omega}} \equiv \mathrm{rot}(\vec{\boldsymbol{v}}) \equiv (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}})$.

Solution:

Taking into account the material time derivative of the Eulerian velocity we obtain:

$$\dot{v}_i \equiv \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}v_j = \frac{\partial v_i}{\partial t} + v_{i,j}v_j \qquad \vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} + \nabla_{\vec{x}}\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \tag{11.13}$$

The resulting components of the algebraic operation $v_{i,j}v_j$ are the components of $(\nabla_{\vec{x}}\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}}$, (see **Problem 1.120** in Chapter 1), and it was shown that:

$$\begin{aligned}(\nabla_{\vec{x}}\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}} &= (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}}) \wedge \vec{\boldsymbol{v}} + \frac{1}{2}\nabla_{\vec{x}}(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}) \\ &= (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}}) \wedge \vec{\boldsymbol{v}} + \frac{1}{2}\nabla_{\vec{x}}(v^2) \\ &= \vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}} + \frac{1}{2}\nabla_{\vec{x}}(v^2)\end{aligned} \tag{11.14}$$

Then, the Eulerian acceleration can also be represented by:

$$\boxed{\vec{a}(\vec{x},t)=\dot{\vec{v}}=\frac{\partial\vec{v}(\vec{x},t)}{\partial t}+\vec{\boldsymbol{\omega}}\wedge\vec{v}+\frac{1}{2}\nabla_{\vec{x}}(v^2)} \tag{11.15}$$

Taking into account (11.13) and (11.14), the equation in (11.11) becomes:

$$\begin{aligned}
&\rho\dot{v}_i=\rho\mathsf{b}_i-p_{,i}+(\lambda^*+\mu^*)v_{j,ji}+\mu^*v_{i,jj}\\
&\rho\dot{\vec{v}}=\rho\vec{\mathbf{b}}-\nabla_{\vec{x}}p+(\lambda^*+\mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})+\mu^*\nabla^2_{\vec{x}}\vec{v}\\
&\Rightarrow\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{\boldsymbol{\omega}}\wedge\vec{v}+\frac{1}{2}\nabla_{\vec{x}}(v^2)\right)=\rho\vec{\mathbf{b}}-\nabla_{\vec{x}}p+(\lambda^*+\mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})+\mu^*\nabla^2_{\vec{x}}\vec{v}\\
&\Rightarrow\frac{\partial\vec{v}}{\partial t}+\vec{\boldsymbol{\omega}}\wedge\vec{v}+\frac{1}{2}\nabla_{\vec{x}}(v^2)-\vec{\mathbf{b}}+\frac{1}{\rho}\nabla_{\vec{x}}p-\frac{(\lambda^*+\mu^*)}{\rho}\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})-\frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\vec{v}=\vec{\mathbf{0}}
\end{aligned} \tag{11.16}$$

Then we take the curl of the above equation:

$$\vec{\nabla}_{\vec{x}}\wedge\left[\frac{\partial\vec{v}}{\partial t}+\vec{\boldsymbol{\omega}}\wedge\vec{v}+\frac{1}{2}\nabla_{\vec{x}}(v^2)-\vec{\mathbf{b}}+\frac{1}{\rho}\nabla_{\vec{x}}p-\frac{(\lambda^*+\mu^*)}{\rho}\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})-\frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\vec{v}\right]=\vec{\mathbf{0}} \tag{11.17}$$

Note that the following relationships hold:

- $\vec{\nabla}_{\vec{x}}\wedge[\nabla_{\vec{x}}(v^2)]=\vec{\mathbf{0}}$, $\vec{\nabla}_{\vec{x}}\wedge[\nabla_{\vec{x}}p]=\vec{\mathbf{0}}$, $\vec{\nabla}_{\vec{x}}\wedge[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})]=\vec{\mathbf{0}}$;
- $$\begin{aligned}&\vec{\nabla}_{\vec{x}}\wedge[(\vec{\nabla}_{\vec{x}}\wedge\vec{v})\wedge\vec{v}]=(\nabla_{\vec{x}}\cdot\vec{v})(\vec{\nabla}_{\vec{x}}\wedge\vec{v})+[\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}}\wedge\vec{v})]\cdot\vec{v}-(\nabla_{\vec{x}}\vec{v})\cdot(\vec{\nabla}_{\vec{x}}\wedge\vec{v})\\ &\Rightarrow\vec{\nabla}_{\vec{x}}\wedge[\vec{\boldsymbol{\omega}}\wedge\vec{v}]=(\nabla_{\vec{x}}\cdot\vec{v})\vec{\boldsymbol{\omega}}+[\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}]\cdot\vec{v}-(\nabla_{\vec{x}}\vec{v})\cdot\vec{\boldsymbol{\omega}}\end{aligned};$$
- $\vec{\nabla}_{\vec{x}}\wedge[\nabla^2_{\vec{x}}\vec{v}]=-\vec{\nabla}_{\vec{x}}\wedge[\vec{\nabla}_{\vec{x}}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})]=\nabla^2_{\vec{x}}[\vec{\nabla}_{\vec{x}}\wedge\vec{v}]=\nabla^2_{\vec{x}}\,\vec{\boldsymbol{\omega}}$;
- $\vec{\nabla}_{\vec{x}}\wedge\left[\dfrac{\partial\vec{v}}{\partial t}\right]=\dfrac{\partial}{\partial t}[\vec{\nabla}_{\vec{x}}\wedge\vec{v}]=\dfrac{\partial\vec{\boldsymbol{\omega}}}{\partial t}$;
- Considering that the field $\vec{\mathbf{b}}$ is conservative, and considering that the curl of any conservative vector field is always zero we obtain $\vec{\nabla}_{\vec{x}}\wedge\vec{\mathbf{b}}=\vec{\mathbf{0}}$.

Considering the previous expression, the equation (11.17) becomes:

$$\frac{\partial\vec{\boldsymbol{\omega}}}{\partial t}+(\nabla_{\vec{x}}\cdot\vec{v})\vec{\boldsymbol{\omega}}+(\nabla_{\vec{x}}\vec{\boldsymbol{\omega}})\cdot\vec{v}-(\nabla_{\vec{x}}\vec{v})\cdot\vec{\boldsymbol{\omega}}-\frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\,\vec{\boldsymbol{\omega}}=\vec{\mathbf{0}} \tag{11.18}$$

Note that the following relationships hold:

$$\begin{aligned}
&(v_i\omega_j)_{,i}=v_{i,i}\,\omega_j+v_i\omega_{j,i} &&\Rightarrow\quad v_{i,i}\,\omega_j=(v_i\omega_j)_{,i}-v_i\omega_{j,i}\\
&(v_i\omega_j)_{,j}=v_{i,j}\,\omega_j+v_i\omega_{j,j} &&\Rightarrow\quad v_{i,j}\,\omega_j=(v_i\omega_j)_{,j}-v_i\omega_{j,j}=(v_i\omega_j)_{,j}
\end{aligned} \tag{11.19}$$

The above equations in tensorial notation are represented by:

$$\begin{aligned}
&(\nabla_{\vec{x}}\cdot\vec{v})\vec{\boldsymbol{\omega}}=\nabla_{\vec{x}}\cdot[\vec{\boldsymbol{\omega}}\otimes\vec{v}]-(\nabla_{\vec{x}}\vec{\boldsymbol{\omega}})\cdot\vec{v}\\
&(\nabla_{\vec{x}}\vec{v})\cdot\vec{\boldsymbol{\omega}}=\nabla_{\vec{x}}\cdot[\vec{v}\otimes\vec{\boldsymbol{\omega}}]-(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{\omega}})\vec{v}=\nabla_{\vec{x}}\cdot[\vec{v}\otimes\vec{\boldsymbol{\omega}}]
\end{aligned} \tag{11.20}$$

where we have applied the definition that the divergence of the curl of a vector is zero, i.e. $\nabla_{\vec{x}}\cdot\vec{\boldsymbol{\omega}}=\nabla_{\vec{x}}\cdot(\vec{\nabla}_{\vec{x}}\wedge\vec{v})=0$. Taking into account (11.20), the equation (11.18) becomes:

$$
\begin{aligned}
&\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}} - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot [\vec{\boldsymbol{\omega}} \otimes \vec{v}] - (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - \nabla_{\vec{x}} \cdot [\vec{v} \otimes \vec{\boldsymbol{\omega}}] - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot [\vec{\boldsymbol{\omega}} \otimes \vec{v}] - \nabla_{\vec{x}} \cdot \left[\vec{v} \otimes \vec{\boldsymbol{\omega}}\right] - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot [\vec{\boldsymbol{\omega}} \otimes \vec{v} - \vec{v} \otimes \vec{\boldsymbol{\omega}}] - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + 2\nabla_{\vec{x}} \cdot [(\vec{\boldsymbol{\omega}} \otimes \vec{v})^{skew}] - \frac{\mu^*}{\rho}\nabla^2_{\vec{x}}\, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}
\end{aligned}
\tag{11.21}
$$

With that we prove the equation of vorticity given by the equation in (11.12).

Problem 11.4

Let us consider a body immersed in a Newtonian fluid. Find the total traction force $\vec{\boldsymbol{E}}$ acting on the closed surface S which delimits the volume V. Consider that the bulk viscosity coefficient to be zero.

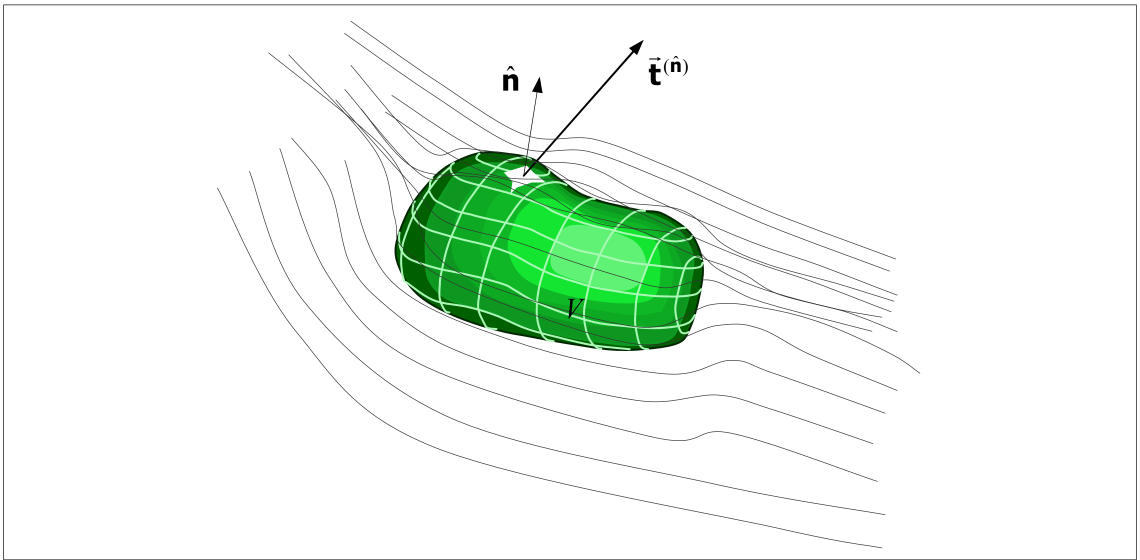

Figure 11.1

Solution: We know that the following holds:

$$dE_i = \mathrm{t}_i^{(\hat{\mathbf{n}})} dS$$

The total traction force is given by the following integral:

$$E_i = \int_S \mathrm{t}_i^{(\hat{\mathbf{n}})} dS = \int_S \sigma_{ij} \hat{\mathrm{n}}_j dS = \int_V \sigma_{ij,j} dV$$

where we have used the relationship $\sigma_{ij}\hat{\mathrm{n}}_j = \mathrm{t}_i^{(\hat{\mathbf{n}})}$.

Then, if the bulk viscosity coefficient is zero, we have: $\kappa^* = 0 \Rightarrow \lambda^* = -\frac{2}{3}\mu^*$ (Stokes' condition).

Next, by considering the stress constitutive equation for Newtonian fluids, we obtain:

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} \mathsf{D}_{kk} + 2\mu^* \mathsf{D}_{ij} = -p\delta_{ij} - \frac{2}{3}\mu^* \delta_{ij} \mathsf{D}_{kk} + 2\mu^* \mathsf{D}_{ij} = -p\delta_{ij} + 2\mu^* \underbrace{\left(\mathsf{D}_{ij} - \frac{\mathsf{D}_{kk}}{3}\delta_{ij} \right)}_{\mathsf{D}_{ij}^{dev}}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev}$$

Then

$$E_i = \int_S (-p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev}) \hat{\mathsf{n}}_j dS$$

and by applying the Gauss' theorem, we obtain:

$$E_i = \int_V (-p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev})_{,j} dV = \int_V (-p_{,j}\delta_{ij} + 2\mu^* \mathsf{D}_{ij,j}^{dev}) dV = \int_V (-p_{,i} + 2\mu^* \mathsf{D}_{ij,j}^{dev}) dV$$

where we have considered that $\mu^*_{,j} = 0_j$, i.e. μ^* is a homogenous scalar field (homogenous material). Then, the above equation in tensorial notation becomes:

$$\vec{\boldsymbol{E}} = \int_V [-\nabla_{\vec{x}} p + 2\mu^* (\nabla_{\vec{x}} \cdot \mathbf{D}^{dev})] dV \tag{11.22}$$

Problem 11.5

Let us consider a fluid at rest which has the mass density ρ^f. Prove **Archimedes' Principle**: *"Any body immersed in a fluid at rest experiences an upward buoyant force equal to the weight of the volume fluid displaced by the body".*

If mass density in the body is equal to ρ^s and the body force (per unit mass) is given by $\mathsf{b}_i = -g\delta_{i3}$, obtain the resultant force and acceleration acting on the body.

Solution:

In **Problem 11.4** we showed that $\vec{\boldsymbol{E}} = \int [-\nabla_{\vec{x}} p + 2\mu^* (\nabla_{\vec{x}} \cdot \mathbf{D}^{dev})] dV$. If the fluid is at rest $\mathbf{D}^{dev} = \mathbf{0}$ holds, and the thermodynamic pressure is equal to the hydrostatic pressure, i.e. $p = p_0$ whereby we have:

$$\vec{\boldsymbol{E}} = \int_V [-\nabla_{\vec{x}} p_0] dV \tag{11.23}$$

The weight of the fluid volume displaced by the body is given by:

$$\vec{\boldsymbol{W}}^f = \int_V \rho^f \vec{\mathbf{b}} dV \tag{11.24}$$

Then, by applying the equilibrium equations we have:

$$\begin{aligned}
&\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho^f \vec{\mathbf{b}} = \vec{\mathbf{0}} \\
&\Rightarrow \nabla_{\vec{x}} \cdot \boldsymbol{\sigma} = -\rho^f \vec{\mathbf{b}} \\
&\Rightarrow \nabla_{\vec{x}} \cdot (-p_0 \mathbf{1}) = -\rho^f \vec{\mathbf{b}} \\
&\Rightarrow \nabla_{\vec{x}} p_0 = \rho^f \vec{\mathbf{b}}
\end{aligned}
\qquad
\begin{aligned}
&\sigma_{ij,j} + \rho^f \mathsf{b}_i = 0_i \\
&\Rightarrow \sigma_{ij,j} = -\rho^f \mathsf{b}_i \\
&\Rightarrow (-p_0 \delta_{ij})_{,j} = -\rho^f \mathsf{b}_i \\
&\Rightarrow p_{0,i} = \rho^f \mathsf{b}_i
\end{aligned}
\tag{11.25}$$

Next, by considering both (11.23) and (11.24), we can conclude that:

$$\vec{W}^f = \int_V \rho^f \vec{\mathbf{b}} dV = \int_V \nabla_{\vec{x}} p_0 dV = -\vec{E} \tag{11.26}$$

which thereby proves Archimedes' principle.

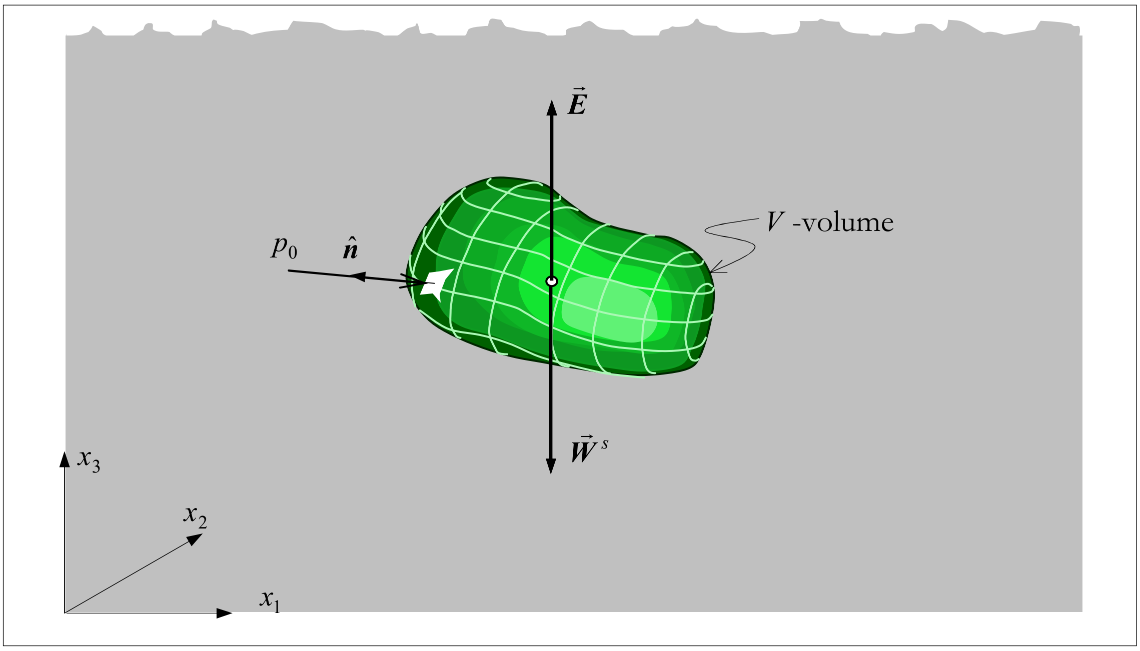

Figure 11.2

Now, the body weight, with mass density ρ_s, can be obtained as follows:

$$\vec{W}^s = \int_V \rho^s \vec{\mathbf{b}} dV$$

and the resultant force acting on the body is given by:

$$\vec{R} = \vec{E} + \vec{W}_s = -\int_V \rho^f \vec{\mathbf{b}} dV + \int_V \rho^s \vec{\mathbf{b}} dV = \int_V (\rho^s - \rho^f) \vec{\mathbf{b}} dV$$

whose components are:

$$R_i = \int_V (\rho^s - \rho^f) \mathsf{b}_i dV = \int_V -g(\rho^s - \rho^f)\delta_{i3} dV = \begin{bmatrix} 0 \\ 0 \\ \int_V g(\rho^f - \rho^s) dV \end{bmatrix}$$

thereby verifying that: if the body has a mass density lower than fluid mass density, e.g. if the body is a gas, the body rises, i.e. $\rho^f > \rho^s \Rightarrow \vec{\boldsymbol{R}} > \vec{\boldsymbol{0}}$, and if not the body falls. Moreover, if we consider that $\vec{\boldsymbol{R}} = m^s \vec{\boldsymbol{a}}$, where m^s is the total mass of the submerged body, we can obtain the acceleration of the body as:

$$a_3 = \frac{R_3}{m^s} = \frac{\int_V g(\rho^f - \rho^s) dV}{m^s} = \frac{\int_V g(\rho^f - \rho^s)\frac{\rho^s}{\rho^s} dV}{m^s} = \frac{\frac{g(\rho^f - \rho^s)}{\rho^s}\int_V \rho^s dV}{m^s} = \frac{g(\rho^f - \rho^s)}{\rho^s}$$

NOTE: It is interesting to note that if the medium (f) is such that $\rho^f = 0$ we have $a_3 = -g$, i.e. the acceleration is independent of the mass. Here we have clearly seen, as did Galileo, by means of a simple experiment, that a freely falling body was independent of the mass. For example, on the moon where we can consider that the mass density of air is equal to zero, two bodies with different masses in free fall, e.g. a feather and a hammer, will have the same acceleration and will reach the moon surface at the same time.

Problem 11.6

Prove that the Cauchy deviatoric stress tensor $\boldsymbol{\sigma}^{dev}$ is equal to $\boldsymbol{\tau}^{dev}$, where $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$.

Solution

If we consider that $\sigma_{kk} = -3p + \tau_{kk}$ we can obtain:

$$\sigma_{ij}^{dev} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} = -p\delta_{ij} + \tau_{ij} - \frac{(-3p + \tau_{kk})}{3}\delta_{ij} = \tau_{ij} - \frac{\tau_{kk}}{3}\delta_{ij} = \tau_{ij}^{dev}$$

Problem 11.7

Obtain the one-dimensional mass continuity equation for a non-viscous incompressible fluid flow through a pipeline. Then, consider the volume V between the two arbitrary cross sections A and B.

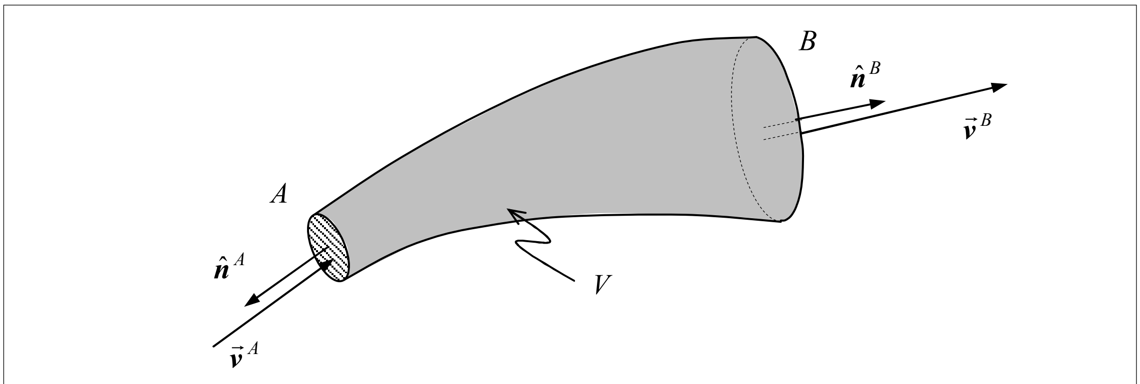

Figure 11.3

Solution:

In an incompressible medium, the mass density is independent of time $\frac{D\rho}{Dt} \equiv \dot{\rho} = 0$. Moreover, here we can consider the mass continuity equation $\dot{\rho} + \rho v_{k,k} = \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0$, where $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0$ or $v_{k,k} = 0$ holds. Then, by considering the entire volume we obtain:

$$\int_V \nabla_{\vec{x}} \cdot \vec{v}\, dV = 0 \qquad \Big| \qquad \int_V v_{k,k}\, dV = 0 \tag{11.27}$$

and by applying the divergence theorem (Gauss' theorem) we can obtain:

$$\int_S \vec{v} \cdot \hat{n}\, dS = 0 \qquad \Big| \qquad \int_S v_k \hat{n}_k\, dS = 0 \tag{11.28}$$

Thus:

$$\int_{S_A} \vec{v}_A \cdot \hat{\mathbf{n}}_A\, dS + \int_{S_B} \vec{v}_B \cdot \hat{\mathbf{n}}_B\, dS = 0$$

Next, the velocities at the cross sections S^A and S^B can be expressed as follows:

$$\vec{v}_A = -v_A \hat{\mathbf{n}}_A \quad ; \quad \vec{v}_B = v_B \hat{\mathbf{n}}_B$$

and by substituting the velocity into the integral we can obtain:

$$-v_A \int_{S_A} \hat{\mathbf{n}}_A \cdot \hat{\mathbf{n}}_A\, dS + v_B \int_{S_B} \hat{\mathbf{n}}_B \cdot \hat{\mathbf{n}}_B\, dS = 0$$

$$\boxed{v_A S_A = v_B S_B}$$

Problem 11.8

The velocity field of a gas in motion through a pipeline, whose prismatic axis is x_2, is defined by its components as follows:

$$v_1 = 0; \qquad v_2 = 0.02 x_2 + 0.05; \qquad v_3 = 0$$

At $x_2 = 0$ the mass density ρ is equal to $1.5 \frac{kg}{m^3}$. Find ρ at $x_2 = 5m$.

Solution:

Note that the velocity field is stationary, i.e. $\vec{v} = \vec{v}(\vec{x})$. From the mass continuity equation we can obtain:

$$\frac{\partial \rho}{\partial t} + \nabla_{\vec{x}} \cdot (\rho \vec{v}) = 0 \quad \Rightarrow \quad \nabla_{\vec{x}} \cdot (\rho \vec{v}) = 0$$

Then, we can conclude that $\rho \vec{v}$ is constant along x_2-direction, so:

$$(\rho v_2)\big|_{x_2=0} = (\rho v_2)\big|_{x_2=5}$$

$v_2(x_2 = 0) = 0.02 \times 0 + 0.05 = 0.05$ and $v_2(x_2 = 5) = 0.02 \times 5 + 0.05 = 0.15$, thus:

$$(\rho v_2)\big|_{x_2=0} = (\rho v_2)\big|_{x_2=5}$$

$$1.5 \times 0.05 = \rho 0.15 \quad \Rightarrow \quad \rho(x_2 = 5) = 0.5 \frac{kg}{m^3}$$

Alternative solution:

$$\nabla_{\vec{x}} \cdot (\rho \vec{v}) = 0 \quad \xrightarrow{indicial} \quad (\rho v_i)_{,i} = \rho_{,i} v_i + \rho v_{i,i} = 0$$

$$\frac{\partial \rho}{\partial x_i} v_i + \rho \frac{\partial v_i}{\partial x_i} = \left(\frac{\partial \rho}{\partial x_1} v_1 + \frac{\partial \rho}{\partial x_2} v_2 + \frac{\partial \rho}{\partial x_3} v_3 \right) + \rho \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) = 0$$

Thus:

$$\left(\frac{\partial\rho}{\partial x_2}(0.02x_2+0.05)\right)+\rho(0.02)=0 \quad \Rightarrow \quad \frac{\partial\rho}{\rho}=-\frac{0.02}{(0.02x_2+0.05)}\partial x_2$$

By integrating the above equation, we obtain:

$$\mathrm{Ln}\rho=\mathrm{Ln}(0.02x_2+0.05)+\mathrm{Ln}C \quad \Rightarrow \quad \mathrm{Ln}\rho=\mathrm{Ln}\left[\frac{C}{(0.02x_2+0.05)}\right]$$

$$\Rightarrow \rho=\frac{C}{(0.02x_2+0.05)}$$

The constant of integration can be obtained by applying the boundary condition, i.e. at $x_2=0 \Rightarrow \rho=1.5$, with that we obtain $C=0.075$:

$$\rho=\frac{0.075}{(0.02x_2+0.05)} \xrightarrow{x_2=5} \rho=\frac{0.075}{(0.02\times 5+0.05)}=0.5\frac{kg}{m^3}$$

Problem 11.9

The Cauchy stress tensor components at one point of a Newtonian fluid, in which the bulk viscosity coefficient is zero, are given by:

$$\sigma_{ij}=\begin{bmatrix}-6 & 2 & -1\\ 2 & -9 & 4\\ -1 & 4 & -3\end{bmatrix}Pa$$

Obtain the viscous stress tensor components.

Solution:

In the case in which the bulk viscosity coefficient is zero (Stokes' condition) we have $p=\overline{p}=p_0$, and, in addition, we can obtain:

$$\sigma_{ij}=-p\delta_{ij}+\tau_{ij} \quad ; \quad \kappa^*=\lambda^*+\frac{2}{3}\mu^*=0 \quad ; \quad \sigma_{ii}=-3p \quad ; \quad p=-\frac{\sigma_{ii}}{3}=-\frac{(-6-9-3)}{3}=6$$

Then:

$$\tau_{ij}=\sigma_{ij}+p\delta_{ij}=\begin{bmatrix}-6 & 2 & -1\\ 2 & -9 & 4\\ -1 & 4 & -3\end{bmatrix}+\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}=\begin{bmatrix}0 & 2 & -1\\ 2 & -3 & 4\\ -1 & 4 & 3\end{bmatrix}Pa$$

Problem 11.10

Determine the conditions needed for mean normal pressure $\overline{p}=-\frac{\sigma_{kk}}{3}=-\sigma_m$ to be equal to thermodynamic pressure p for a Newtonian fluid.

Solution:

It was deduced that:

$$\sigma_{ij}^{dev} = 2\mu^* \mathsf{D}_{ij}^{dev} \quad ; \quad \frac{1}{3}\sigma_{kk} = -p + k^* \mathsf{D}_{ii} \quad ; \quad \underbrace{\frac{\sigma_{kk}}{3}}_{-p} = -\overline{p} = -p + \kappa^* \mathsf{D}_{kk}$$

Thus, $\overline{p} = p$ holds when the following is satisfied:

$$\kappa^* = 0 \quad or \quad \begin{cases} \mathsf{D}_{ii} = 0 \\ \mathrm{Tr}(\mathbf{D}) = 0 \end{cases} \quad or \quad \lambda^* = -\frac{2}{3}\mu^*$$

Problem 11.11

A baratropic perfect fluid has as equation of state $\rho = \rho_0 + \dfrac{p}{k}$, where k is constant. Obtain the pressure field for a quasi-static regime (zero acceleration), under the action of the gravitational field $\mathsf{b}_i = [0 \quad 0 \quad -g]^T$.

Solution:

The constitutive equation in stress for a perfect fluid is given by:

$$\boldsymbol{\sigma} = -p\mathbf{1}$$

The equations of motion become:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{v}} \quad &\xrightarrow{indicial} \quad \sigma_{ij,j} + \rho\mathsf{b}_i = \underbrace{\rho\dot{v}_i}_{=0_i} = 0_i \\ &\qquad\qquad (-p\delta_{ij})_{,j} + \rho\mathsf{b}_i = 0_i \\ &\qquad\qquad -p_{,j}\delta_{ij} + \rho\mathsf{b}_i = 0_i \\ -\nabla_{\vec{x}} p + \rho\vec{\mathbf{b}} = \vec{\mathbf{0}} \quad &\xleftarrow{tensorial} \quad -p_{,i} + \rho\mathsf{b}_i = 0_i \end{aligned} \tag{11.29}$$

Considering the body force vector $\mathsf{b}_i = [0 \quad 0 \quad -g]$ we obtain:

$$-p_{,i} + \rho\mathsf{b}_i = 0_i \Rightarrow \begin{cases} (i=1) \Rightarrow -\dfrac{\partial p}{\partial x_1} + \rho\mathsf{b}_1 = 0 \Rightarrow \dfrac{\partial p}{\partial x_1} = 0 \Rightarrow p = p(\cancel{x_1}, x_2, x_3) \\ (i=2) \Rightarrow -\dfrac{\partial p}{\partial x_2} + \rho\mathsf{b}_2 = 0 \Rightarrow \dfrac{\partial p}{\partial x_2} = 0 \Rightarrow p = p(x_1, \cancel{x_2}, x_3) \\ (i=3) \Rightarrow -\dfrac{\partial p}{\partial x_3} + \rho\mathsf{b}_3 = 0 \Rightarrow \dfrac{dp(x_3)}{dx_3} + \rho g = 0 \end{cases} \tag{11.30}$$

With that we can conclude that the pressure field is only a function of the coordinate x_3, i.e. $p = p(x_3)$.

By the fact we are dealing with a barotropic fluid, this implies that the mass density is only a function of pressure $\rho = \rho(p)$. This relationship is precisely the equation of state of the problem statement:

$$\rho = \rho(p) \quad \Rightarrow \quad \rho = \rho_0 + \frac{p}{k}$$

Then:

$$\frac{dp(x_3)}{dx_3} + \rho g = 0 \quad \Rightarrow \quad \frac{dp(x_3)}{dx_3} + \left(\rho_0 + \frac{p}{k}\right) g = 0 \quad \Rightarrow \quad \frac{dp(x_3)}{dx_3} + \frac{g}{k} p = -\rho_0 g \tag{11.31}$$

The solution of the above differential equation is the sum of a particular solution and a homogeneous solution:

Homogeneous solution: $\dfrac{dp(x_3)}{dx_3} + \dfrac{g}{k} p = 0 \Rightarrow p = C \exp^{(\frac{-g}{k} x_3)}$

Particular solution: $p = -k\rho_0$

Thus:

$$p = C \exp^{(\frac{-g}{k} x_3)} - k\rho_0$$

Problem 11.12

A perfect gas is an ideal and incompressible fluid in which in the absence of heat sources the pressure is proportional to ρ^{γ} (barotropic motion), where γ is a constant and $\gamma > 1$. Show that when $r = 0$ (no internal heat source), the specific internal energy for a ideal gas is given by:

$$u = \frac{1}{(\gamma - 1)} \frac{p}{\rho} + constant$$

Solution:

For the proposed problem, the energy equation becomes:

$$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r = \boldsymbol{\sigma} : \mathbf{D} \quad \Rightarrow \quad \rho \dot{u} - \boldsymbol{\sigma} : \mathbf{D} = 0$$

For a perfect gas the stress tensor is a spherical tensor and is given by:

$$\boldsymbol{\sigma}(p) = -p\mathbf{1}$$

where p is the thermodynamic pressure. Then, the energy equation becomes:

$$\begin{aligned}
&\rho \dot{u} - \boldsymbol{\sigma} : \mathbf{D} = 0 \\
&\Rightarrow \rho \dot{u} + p\mathbf{1} : \mathbf{D} = 0 \\
&\Rightarrow \rho \dot{u} + p\mathrm{Tr}(\mathbf{D}) = 0 \\
&\Rightarrow \rho \dot{u} + p(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0
\end{aligned}$$

For a barotropic motion, the specific internal energy is a function of the mass density, $u = u(\rho)$, thus:

$$\rho \dot{u} + p(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0 \quad \Rightarrow \quad \rho \frac{\partial u}{\partial \rho} \dot{\rho} + p(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0$$

Considering the mass continuity equation $\dfrac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0 \Rightarrow \dot{\rho} = -\rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})$, the energy equation becomes:

$$\begin{aligned}
&\rho \frac{\partial u}{\partial \rho} \dot{\rho} + p(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0 \\
&\Rightarrow -\rho \frac{\partial u}{\partial \rho} \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) + p(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0 \\
&\Rightarrow \left(-\rho^2 \frac{\partial u}{\partial \rho} + p \right) \underbrace{\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}}_{\neq 0} = 0
\end{aligned}$$

with that the following holds:

$$-\rho^2 \frac{\partial u}{\partial \rho} + p = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial \rho} = \frac{p}{\rho^2}$$

As pressure is proportional to ρ^{γ}, we can state that $p = p(\rho) = k\rho^{\gamma}$, where k is a proportionality constant, then:

$$-\rho^2 \frac{\partial u}{\partial \rho} + p = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial \rho} = \frac{k\rho^{\gamma}}{\rho^2} = k\rho^{(\gamma-2)}$$

By integrating the above equation we obtain:

$$\begin{aligned} u &= \frac{k}{(\gamma-1)}\frac{\rho^{\gamma}}{\rho} + constant \\ &= \frac{1}{(\gamma-1)}\frac{p}{\rho} + constant \end{aligned}$$

Problem 11.13

A fluid moves with velocity $\vec{v}$ around a sphere of radius R, where the velocity components in spherical coordinates (r, θ, ϕ) are given by:

$$v_r = c\left(\frac{R^3}{2r^3} - \frac{3R}{2r} + 1\right)\cos(\theta) \quad ; \quad v_\theta = c\left(\frac{R^3}{4r^3} + \frac{3R}{4r} - 1\right)\sin(\theta) \quad ; \quad v_\phi = 0 \qquad (11.32)$$

where c is a positive constant.

Check whether we are dealing with an isochoric motion or not.

Note: Given a vector $\vec{\mathbf{u}}$, the divergence of this vector in spherical coordinates is:

$$\operatorname{div} \vec{\mathbf{u}} \equiv \nabla_{\vec{x}} \cdot \vec{\mathbf{u}} = \frac{\partial \mathrm{u}_r}{\partial r} + \frac{1}{r}\frac{\partial \mathrm{u}_\theta}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial \mathrm{u}_\phi}{\partial \phi} + \frac{\cot(\theta)}{r}\mathrm{u}_\theta + \frac{2}{r}\mathrm{u}_r$$

Solution:

To demonstrate that a motion is isochoric, we must show that $\nabla_{\vec{x}} \cdot \vec{v} = 0$.

From the velocity field we can obtain the following derivatives:

$$\frac{\partial v_r}{\partial r} = \frac{\partial}{\partial r}\left[c\left(\frac{R^3}{2r^3} - \frac{3R}{2r} + 1\right)\cos(\theta)\right] = c\left(\frac{-3R^3}{2r^4} + \frac{3R}{2r^2} + 1\right)\cos(\theta)$$

$$\frac{\partial v_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}\left[c\left(\frac{R^3}{4r^3} + \frac{3R}{4r} - 1\right)\sin(\theta)\right] = c\left(\frac{R^3}{4r^3} + \frac{3R}{4r} - 1\right)\cos(\theta)$$

With that it is possible to obtain the divergence of the velocity field:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \vec{v} &= \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin(\theta)}\underbrace{\frac{\partial v_\phi}{\partial \phi}}_{=0} + \frac{\cot(\theta)}{r}v_\theta + \frac{2}{r}v_r \\ &= c\left(\frac{-3R^3}{2r^4} + \frac{3R}{2r^2} + 1\right)\cos(\theta) + \frac{1}{r}c\left(\frac{R^3}{4r^3} + \frac{3R}{4r} - 1\right)\cos(\theta) + \\ &\quad + \frac{\cos(\theta)}{\sin(\theta)}\frac{1}{r}\left[c\left(\frac{R^3}{4r^3} + \frac{3R}{4r} - 1\right)\sin(\theta)\right] + \frac{2}{r}\left[c\left(\frac{R^3}{2r^3} - \frac{3R}{2r} + 1\right)\cos(\theta)\right] \end{aligned}$$

By simplifying the above equation we can conclude that:

$$\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0$$

So, we are dealing with an isochoric motion.

Problem 11.14

A barotropic fluid flows through a pipeline as shown in Figure 11.4, and said fluid has as equation of state:

$$p = \beta \,\mathrm{Ln}\left(\frac{\rho}{\rho_0}\right) \quad ; \quad (\beta \text{ and } \rho_0 \text{ are constants})$$

where p is pressure, and ρ is the mass density.

Calculate, in steady state, the output pressure $p_{(2)}$ in terms of variables presented in Figure 11.4.

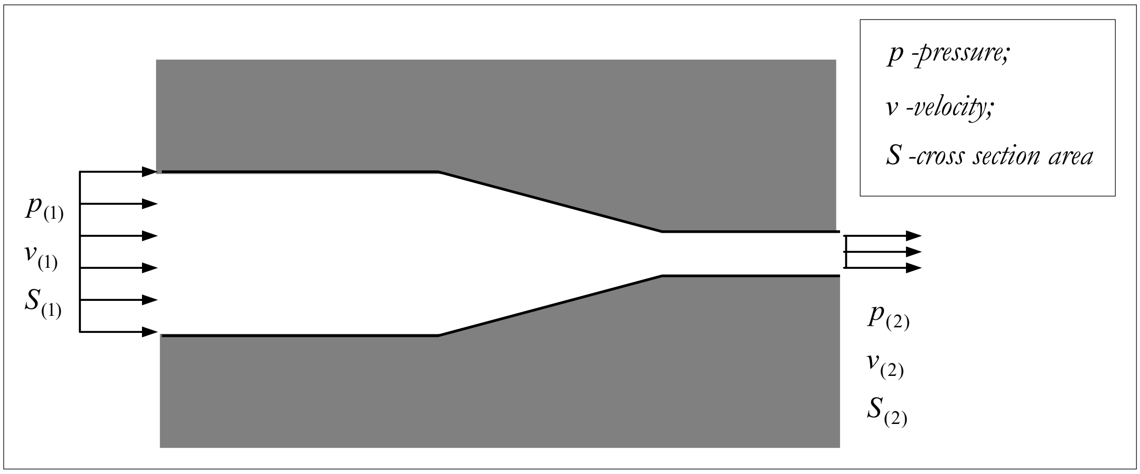

Figure 11.4

Solution:

According to the principle of conservation of mass we have:

$$\frac{D}{Dt}\int_V \rho dV = 0$$

and given a property $\Phi(\vec{\boldsymbol{x}},t)$, it fulfills that:

$$\begin{aligned}
\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t) dV &= \int_V \left(\frac{D\Phi(\vec{\boldsymbol{x}},t)}{Dt} dV + \Phi(\vec{\boldsymbol{x}},t)\frac{DdV}{Dt} \right) \\
&= \int_V \left(\frac{D\Phi(\vec{\boldsymbol{x}},t)}{Dt} dV + \Phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) dV \right) \\
&= \int_V \left(\frac{D\Phi(\vec{\boldsymbol{x}},t)}{Dt} + \Phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \right) dV \\
&= \int_V \left(\frac{\partial\Phi(\vec{\boldsymbol{x}},t)}{\partial t} + \nabla_{\vec{x}}\Phi(\vec{\boldsymbol{x}},t) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \right) dV
\end{aligned}$$

$$
\begin{aligned}
\frac{D}{Dt}\int_V \Phi(\vec{x},t)dV &= \int_V \left(\frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\Phi(\vec{x},t)\cdot\vec{v}(\vec{x},t) + \Phi(\vec{x},t)\nabla_{\vec{x}}\cdot\vec{v}(\vec{x},t) \right) dV \\
&= \int_V \left[\frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\cdot\left(\Phi(\vec{x},t)\vec{v}(\vec{x},t)\right) \right] dV \\
&= \int_V \frac{\partial \Phi(\vec{x},t)}{\partial t} dV + \int_V \nabla_{\vec{x}}\cdot\left[\Phi(\vec{x},t)\vec{v}(\vec{x},t)\right] dV \\
&= \int_V \frac{\partial \Phi(\vec{x},t)}{\partial t} dV + \int_S \left[\Phi(\vec{x},t)\vec{v}(\vec{x},t)\right]\cdot\hat{\mathbf{n}}dS = 0
\end{aligned}
\tag{11.33}
$$

by denoting $\Phi(\vec{x},t) = \rho(\vec{x},t)$ the above equation becomes:

$$
\frac{D}{Dt}\int_V \rho(\vec{x},t)dV = \int_V \frac{\partial \rho(\vec{x},t)}{\partial t} dV + \int_S \left[\rho(\vec{x},t)\vec{v}(\vec{x},t)\right]\cdot\hat{\mathbf{n}}dS = 0
$$

By applying the steady state condition, i.e. $\frac{\partial \rho(\vec{x},t)}{\partial t} = 0$, we obtain:

$$
\begin{aligned}
&\int_S \left[\rho(\vec{x})\vec{v}(\vec{x})\right]\cdot\hat{\mathbf{n}}dS = 0 \quad \Rightarrow \quad \int_{S_{(1)}} \left[\rho(\vec{x})\vec{v}(\vec{x})\right]\cdot\hat{\mathbf{n}}dS + \int_{S_{(2)}} \left[\rho(\vec{x})\vec{v}(\vec{x})\right]\cdot\hat{\mathbf{n}}dS = 0 \\
&\Rightarrow \int_{S_{(1)}} -\rho_{(1)}v_{(1)}dS + \int_{S_{(2)}} \rho_{(2)}v_{(2)}dS = 0 \quad \Rightarrow \quad -\rho_{(1)}v_{(1)}S_{(1)} + \rho_{(2)}v_{(2)}S_{(2)} = 0
\end{aligned}
$$

Thus:

$$
\boxed{\rho_{(1)}v_{(1)}S_{(1)} = \rho_{(2)}v_{(2)}S_{(2)}}
\tag{11.34}
$$

Remember that $\vec{\mathbf{q}} = \rho\,\vec{v}$ is the mass flux, and the SI unit is $[\vec{\mathbf{q}}] = \frac{kg}{m^2 s}$.

By means of the equation of state we can obtain an expression for mass density:

$$
p = \beta \operatorname{Ln}\left(\frac{\rho}{\rho_0}\right) \Rightarrow \frac{p}{\beta} = \operatorname{Ln}\left(\frac{\rho}{\rho_0}\right) \Rightarrow \exp^{\left(\frac{p}{\beta}\right)} = \left(\frac{\rho}{\rho_0}\right) \Rightarrow \rho(\boldsymbol{x}) = \rho_0 \exp^{\left(\frac{p}{\beta}\right)}
$$

Then:

$$
\begin{aligned}
&\rho_{(1)}v_{(1)}S_{(1)} = \rho_{(2)}v_{(2)}S_{(2)} \quad \Rightarrow \quad \rho_0 \exp^{\left(\frac{p_{(1)}}{\beta}\right)} v_{(1)}S_{(1)} = \rho_0 \exp^{\left(\frac{p_{(2)}}{\beta}\right)} v_{(2)}S_{(2)} \\
&\Rightarrow \exp^{\left(\frac{p_{(2)}-p_{(1)}}{\beta}\right)} = \frac{v_{(1)}S_{(1)}}{v_{(2)}S_{(2)}} \quad \Rightarrow \quad \frac{p_{(2)}-p_{(1)}}{\beta} = \operatorname{Ln}\left(\frac{v_{(1)}S_{(1)}}{v_{(2)}S_{(2)}}\right) \\
&\Rightarrow p_{(2)} - p_{(1)} = \beta \operatorname{Ln}\left(\frac{v_{(1)}S_{(1)}}{v_{(2)}S_{(2)}}\right) \quad \Rightarrow \quad p_{(2)} = p_{(1)} + \beta \operatorname{Ln}\left(\frac{v_{(1)}S_{(1)}}{v_{(2)}S_{(2)}}\right)
\end{aligned}
\tag{11.35}
$$

NOTE: The volumetric flow rate, (also known as volume flow rate), often represented by Q, is the *specific total flow*, i.e.:

$$
\boxed{Q = \int_S \frac{\vec{\mathbf{q}}\cdot d\vec{\boldsymbol{S}}}{\rho} = \int_S \frac{\rho\vec{v}\cdot d\vec{\boldsymbol{S}}}{\rho} = \int_S \vec{v}\cdot d\vec{\boldsymbol{S}}} \quad \textit{Volumetric flow rate} \qquad \left[\frac{m^3}{s}\right]
\tag{11.36}
$$

We check the SI unit $[Q]=\left[\int_S \frac{\vec{\mathbf{q}}\cdot d\vec{S}}{\rho}\right]=\frac{kg}{m^2 s}\frac{m^3}{kg}m^2=\frac{m^3}{s}$. In this example, we have obtained $\rho_{(1)}v_{(1)}S_{(1)}=\rho_{(2)}v_{(2)}S_{(2)}$, which can be rewritten as:

$$\rho_{(1)}v_{(1)}S_{(1)}=\rho_{(2)}v_{(2)}S_{(2)} \quad \Rightarrow \quad \rho_{(1)}Q_{(1)}=\rho_{(2)}Q_{(2)}$$

For the particular case of an incompressible medium we have $\rho_{(1)}=\rho_{(2)}$, then:

$$v_{(1)}S_{(1)}=v_{(2)}S_{(2)} \quad \Rightarrow \quad Q_{(1)}=Q_{(2)} \quad \text{(see **Problem 11.7**)}$$

Problem 11.15

Consider the water flow described in Figure 11.5 in which the jet is deflected by the curved vane. Obtain the total force applied to the curved vane.

Hypotheses (approximations): Consider that we are dealing with: a) the incompressible fluid; b) a steady flow; c) no body forces; d) the atmosphere pressure can be discarded.

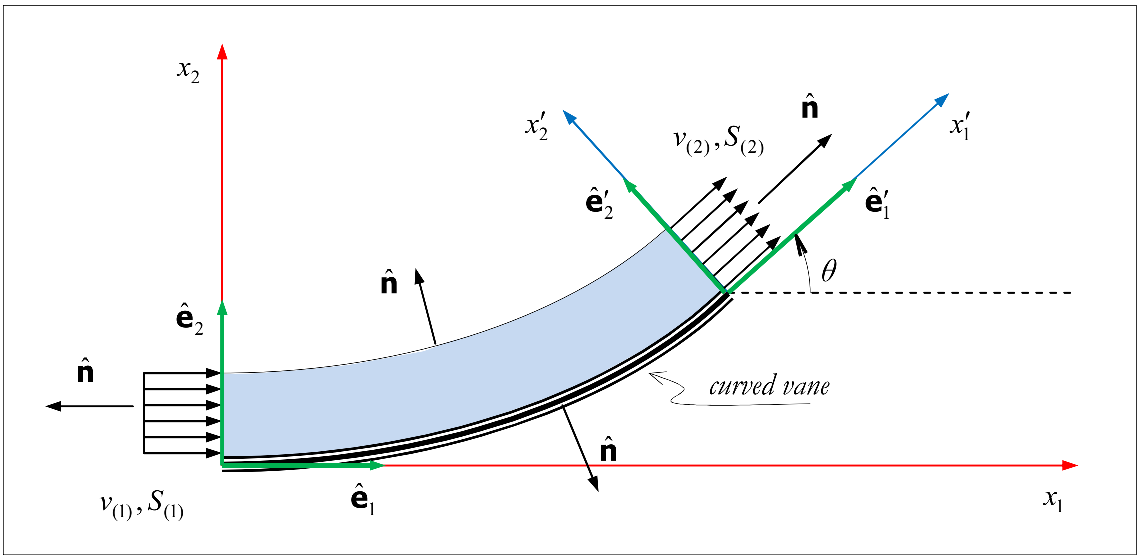

Figure 11.5: Jet deflected by the curved vane.

Solution:

By considering the homogeneous mass density field we can conclude

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\vec{x}}\cdot\frac{\partial\vec{x}}{\partial t}=\frac{\partial\rho}{\partial t}+\underbrace{\frac{\partial\rho}{\partial\vec{x}}}_{\substack{=\vec{\mathbf{0}}\\(\text{Homogeneous})}}\cdot\vec{v} \xrightarrow{\text{Homogeneous mass density field}} \frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}$$

Since we are dealing with steady flow the following is true $\frac{\partial\vec{v}}{\partial t}=\vec{\mathbf{0}}$, and in addition for an incompressible fluid the mass density it fulfills $\frac{D\rho}{Dt}=0$. Then, for this problem the equation $\frac{\partial(\rho\vec{v})}{\partial t}=\vec{\mathbf{0}}$ holds.

Recall from the textbook (Chapter 5 – Chaves (2013)) that the "Principle of Conservation of Linear Momentum" states

$$\vec{F} = \int_{S_\sigma} \vec{\mathbf{t}}^* dS + \underbrace{\int_V \rho \vec{\mathbf{b}} \, dV}_{=\vec{\mathbf{0}}} = \frac{D}{Dt} \int_V \rho \, \vec{v} \, dV \tag{11.37}$$

where $\vec{F}$ is the total force of the system. The above material time derivative can be expressed as follows, (see equation (11.33)):

$$\frac{D}{Dt} \int_V \rho \, \vec{v} dV = \underbrace{\int_V \frac{\partial(\rho \, \vec{v})}{\partial t} dV}_{=\vec{\mathbf{0}}} + \int_S (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS = \int_S (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS \tag{11.38}$$

Note that we are adopting the Eulerian formulation and the control volume and surface control can be appreciated in Figure 11.6.

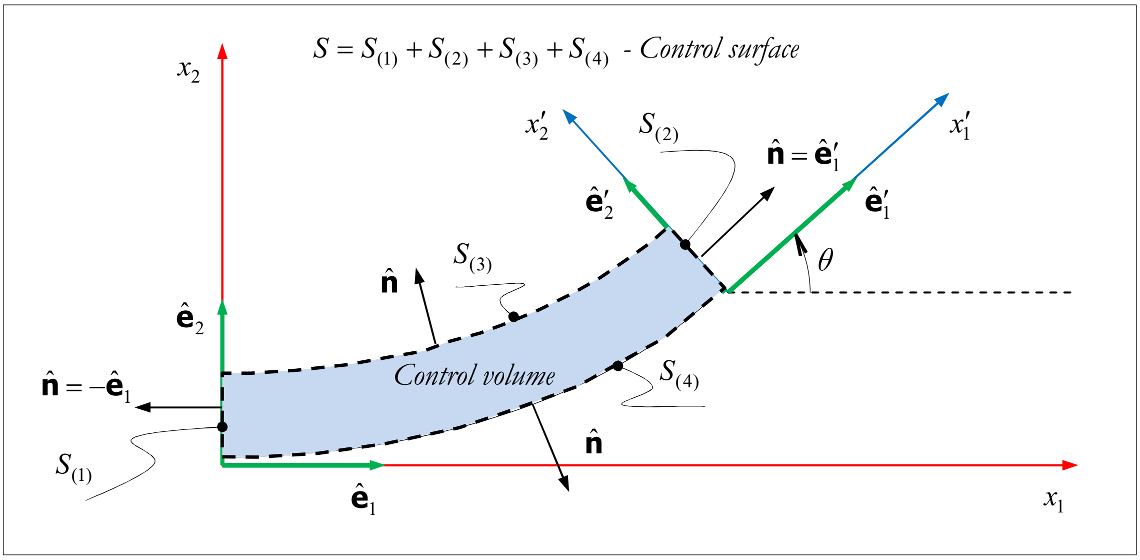

Figure 11.6: Control surface and control volume.

Then the equation (11.37) can be rewritten as follows

$$\begin{aligned} &\vec{F} = \int_{S_\sigma} \vec{\mathbf{t}}^* dS = \frac{D}{Dt} \int_V \rho \, \vec{v} \, dV = \int_S (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS \\ &\Rightarrow \vec{F} = \int_{S_{(1)}} (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS + \int_{S_{(2)}} (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS + \underbrace{\int_{S_{(3)}} (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS}_{=\vec{\mathbf{0}}} + \underbrace{\int_{S_{(4)}} (\rho \, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS}_{=\vec{\mathbf{0}}} \end{aligned} \tag{11.39}$$

Note that on the control surface $S_{(3)}$ and $S_{(4)}$ the velocity is always perpendicular to $\hat{\mathbf{n}}$, then $\vec{v} \cdot \hat{\mathbf{n}} = 0$. On surface $S_{(1)}$ we have $\vec{v} = v_{(1)} \hat{\mathbf{e}}_1$ and $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_1$ with that $\vec{v} \cdot \hat{\mathbf{n}} = (v_{(1)} \hat{\mathbf{e}}_1) \cdot (-\hat{\mathbf{e}}_1) = -v_{(1)}$, on the surface $S_{(2)}$ we have $\vec{v} = v_{(2)} \hat{\mathbf{e}}_1'$ and $\hat{\mathbf{n}} = \hat{\mathbf{e}}_1'$, with that $\vec{v} \cdot \hat{\mathbf{n}} = (v_{(2)} \hat{\mathbf{e}}_1') \cdot (\hat{\mathbf{e}}_1') = v_{(2)}$. Note also that the transformation law from the system $\vec{x}$ to the system $\vec{x}'$ is given by

$$\begin{Bmatrix} \hat{\mathbf{e}}_1' \\ \hat{\mathbf{e}}_2' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \end{Bmatrix} = \begin{Bmatrix} \cos\theta \, \hat{\mathbf{e}}_1 + \sin\theta \, \hat{\mathbf{e}}_2 \\ -\sin\theta \, \hat{\mathbf{e}}_1 + \cos\theta \, \hat{\mathbf{e}}_2 \end{Bmatrix}$$

Then, the velocity on the control surface $S_{(2)}$ can also be expressed as follows

$$\vec{v} = v_{(2)}\hat{\mathbf{e}}'_1 = v_{(2)}(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2)$$

By taking into account all the previous considerations the equation (11.39) becomes:

$$\begin{aligned}
&\vec{F} = \int_{S_{(1)}} (\rho\,\vec{v})\vec{v}\cdot\hat{\mathbf{n}}dS + \int_{S_{(2)}} (\rho\,\vec{v})\vec{v}\cdot\hat{\mathbf{n}}dS \\
&\Rightarrow \vec{F} = \int_{S_{(1)}} -(\rho\,v_{(1)}\hat{\mathbf{e}}_1)v_{(1)}dS + \int_{S_{(2)}} [\rho\,v_{(2)}(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2)]v_{(2)}dS \\
&\Rightarrow \vec{F} = -\rho\,v_{(1)}v_{(1)}S_{(1)}\hat{\mathbf{e}}_1 + \rho\,v_{(2)}v_{(2)}\cos\theta\,S_{(2)}\hat{\mathbf{e}}_1 + \rho\,v_{(2)}v_{(2)}\sin\theta\,S_{(2)}\hat{\mathbf{e}}_2 \\
&\Rightarrow \vec{F} = \rho\,[v_{(2)}^2\cos\theta\,S_{(2)} - v_{(1)}^2 S_{(1)}]\hat{\mathbf{e}}_1 + \rho\,v_{(2)}^2\sin\theta\,S_{(2)}\hat{\mathbf{e}}_2
\end{aligned} \tag{11.40}$$

For the particular case when $v_{(1)} = v_{(2)} = v_0$, $S_{(1)} = S_{(2)} = S_0$, the above equation becomes

$$\begin{aligned}
&\vec{F} = \rho\,[v_{(2)}^2\cos\theta\,S_{(2)} - v_{(1)}^2 S_{(1)}]\hat{\mathbf{e}}_1 + \rho\,v_{(2)}^2\sin\theta\,S_{(2)}\hat{\mathbf{e}}_2 \\
&\Rightarrow \vec{F} = \rho\,v_0^2 S_0[\cos\theta - 1]\hat{\mathbf{e}}_1 + \rho\,v_0^2 S_0\sin\theta\,\hat{\mathbf{e}}_2 \quad \Rightarrow \begin{cases} F_{x_1} = \rho\,v_0^2 S_0[\cos\theta - 1] \\ F_{x_2} = \rho\,v_0^2 S_0\sin\theta \end{cases}
\end{aligned} \tag{11.41}$$

And the reaction force in the curved vane is $\vec{R} = -\vec{F}$.

Problem 11.16

Consider the water flow described in Figure 11.7. Obtain the total force applied to the plate.

Hypotheses (approximations): Consider that we are dealing with: a) the incompressible fluid; b) a steady flow; c) no body forces; d) the atmosphere pressure can be discarded.

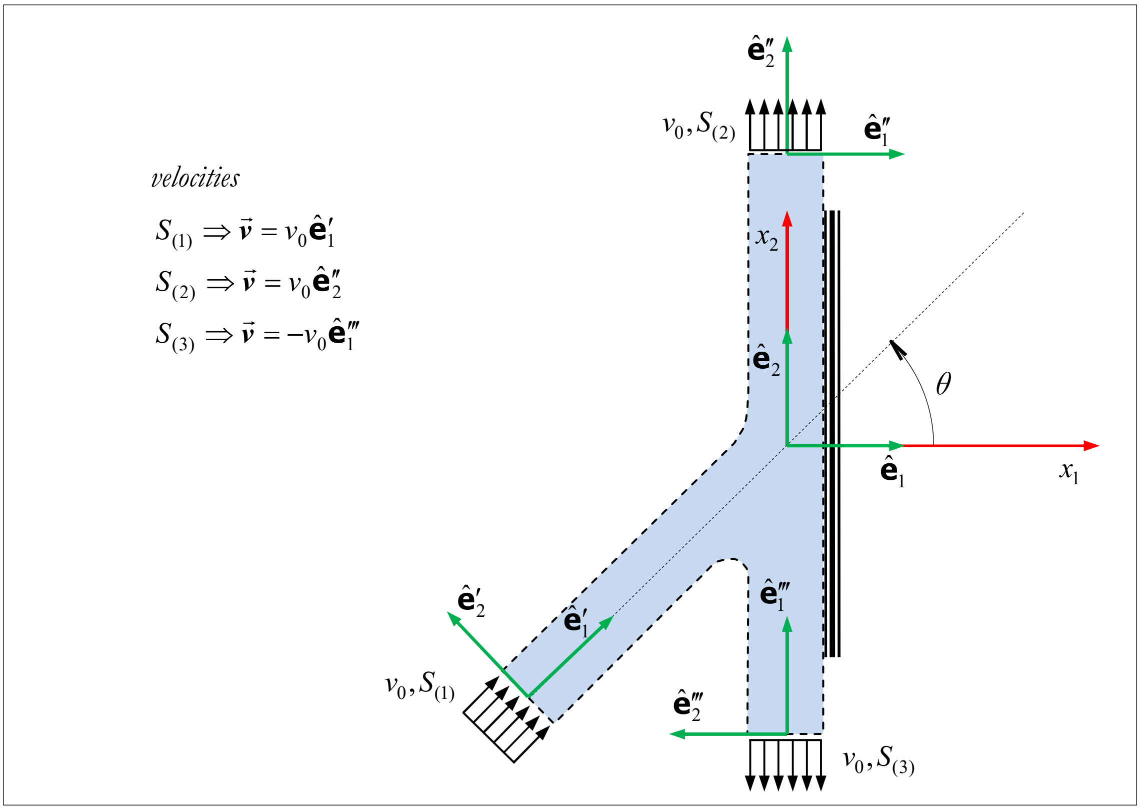

Figure 11.7

Solution:

Note that this problem is similar to the previous problem in which we have established that the total force, (see equation (11.39)), is given by

$$\vec{F} = \int_{S_\sigma} \vec{\mathbf{t}}^* dS = \frac{D}{Dt}\int_V \rho\,\vec{v}\; dV = \int_S (\rho\,\vec{v})\vec{v}\cdot\hat{\mathbf{n}}dS \tag{11.42}$$

According to Figure 11.8 we can obtain:

Transformation law from the system $\vec{x}$ to the system $\vec{x}'$, (angle θ):

$$\begin{Bmatrix}\hat{\mathbf{e}}'_1\\ \hat{\mathbf{e}}'_2\end{Bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\end{Bmatrix} = \begin{Bmatrix}\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2\\ -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2\end{Bmatrix}$$

Transformation law from the system $\vec{x}$ to the system $\vec{x}''$, (angle $\theta = 0^{\circ}$):

$$\begin{Bmatrix}\hat{\mathbf{e}}''_1\\ \hat{\mathbf{e}}''_2\end{Bmatrix} = \begin{bmatrix}\cos 0^{\circ} & \sin 0^{\circ}\\ -\sin 0^{\circ} & \cos 0^{\circ}\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\end{Bmatrix} = \begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\end{Bmatrix}$$

Transformation law from the system $\vec{x}$ to the system $\vec{x}'''$, (angle $\theta = 90^{\circ}$):

$$\begin{Bmatrix}\hat{\mathbf{e}}'''_1\\ \hat{\mathbf{e}}'''_2\end{Bmatrix} = \begin{bmatrix}\cos 90^{\circ} & \sin 90^{\circ}\\ -\sin 90^{\circ} & \cos 90^{\circ}\end{bmatrix}\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\end{Bmatrix} = \begin{Bmatrix}\hat{\mathbf{e}}_2\\ -\hat{\mathbf{e}}_1\end{Bmatrix}$$

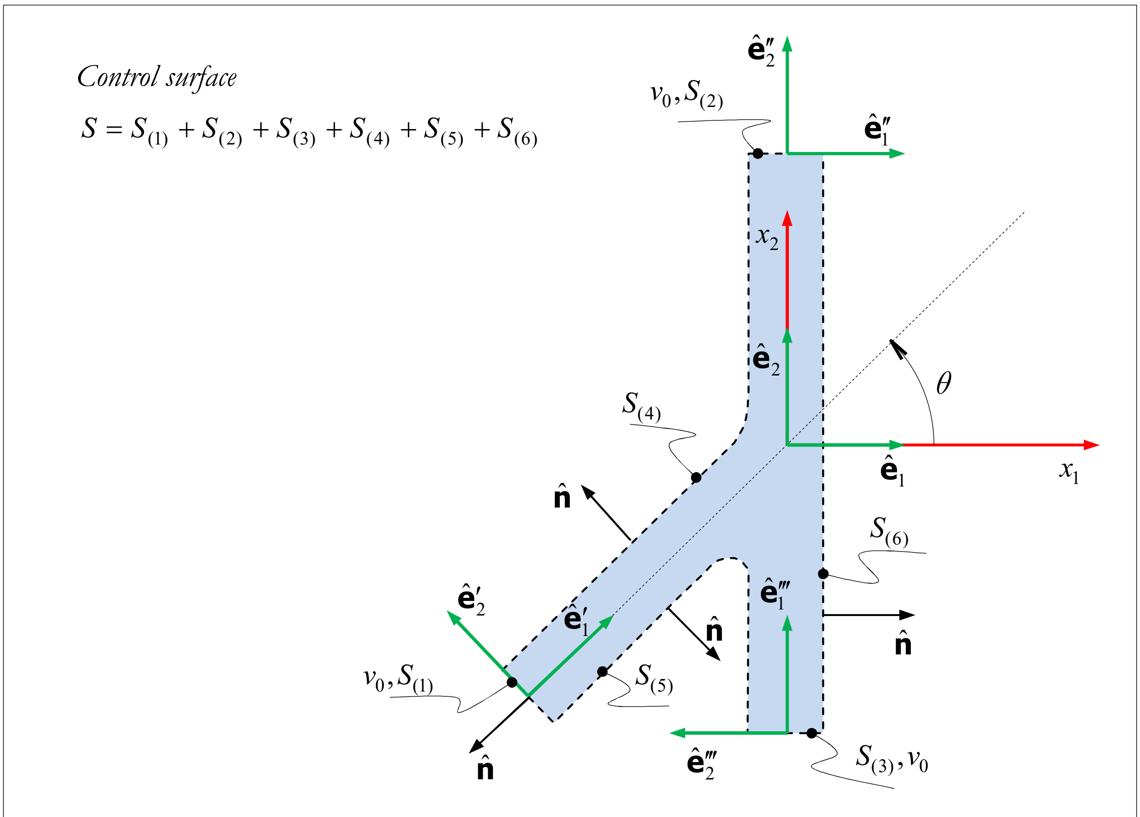

Figure 11.8

Then, on the control surface $S_{(1)}$:

$$S_{(1)} \Rightarrow \begin{cases}\vec{v} = v_0\hat{\mathbf{e}}'_1 = -v_0(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2)\\ \hat{\mathbf{n}} = -\hat{\mathbf{e}}'_1 = -(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2)\end{cases} \quad\Rightarrow\quad \vec{v}\cdot\hat{\mathbf{n}} = v_0$$

On the control surface $S_{(2)}$:

$$S_{(2)} \Rightarrow \begin{cases} \vec{v} = v_0 \hat{\mathbf{e}}_2'' = v_0 \, \hat{\mathbf{e}}_2 \\ \hat{\mathbf{n}} = \hat{\mathbf{e}}_2'' = \hat{\mathbf{e}}_2 \end{cases} \quad \Rightarrow \quad \vec{v} \cdot \hat{\mathbf{n}} = v_0$$

On the control surface $S_{(3)}$:

$$S_{(3)} \Rightarrow \begin{cases} \vec{v} = -v_0 \hat{\mathbf{e}}_1''' = -v_0 \, \hat{\mathbf{e}}_2 \\ \hat{\mathbf{n}} = -\hat{\mathbf{e}}_1''' = -\hat{\mathbf{e}}_2 \end{cases} \quad \Rightarrow \quad \vec{v} \cdot \hat{\mathbf{n}} = v_0$$

Note that for the control surfaces $S_{(4)}$, $S_{(5)}$ and $S_{(6)}$ the equation $\vec{v} \cdot \hat{\mathbf{n}} = 0$ holds, since the outward normal is always perpendicular to the velocity. Then, the equation in (11.42) becomes

$$\begin{aligned}
&\vec{F} = \int_S (\rho\, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS \\
&\Rightarrow \vec{F} = \int_{S_{(1)}} (\rho\, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS + \int_{S_{(2)}} (\rho\, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS + \int_{S_{(3)}} (\rho\, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS + \underbrace{\int_{S_{(4)}+S_{(5)}+S_{(6)}} (\rho\, \vec{v}) \vec{v} \cdot \hat{\mathbf{n}} dS}_{=\vec{\mathbf{0}}} \\
&\Rightarrow \vec{F} = \int_{S_{(1)}} \rho [-v_0 (\cos\theta\, \hat{\mathbf{e}}_1 + \sin\theta\, \hat{\mathbf{e}}_2)] v_0 dS + \int_{S_{(2)}} \rho\, (v_0 \, \hat{\mathbf{e}}_2) v_0 dS + \int_{S_{(3)}} \rho\, (-v_0 \, \hat{\mathbf{e}}_2) v_0 dS \\
&\Rightarrow \vec{F} = -\rho v_0^2 (\cos\theta\, \hat{\mathbf{e}}_1 + \sin\theta\, \hat{\mathbf{e}}_2) S_{(1)} + \rho v_0^2 S_{(2)} \hat{\mathbf{e}}_2 - \rho v_0^2 S_{(3)} \hat{\mathbf{e}}_2 \\
&\Rightarrow \vec{F} = -\rho v_0^2 S_{(1)} \cos\theta\, \hat{\mathbf{e}}_1 + \rho v_0^2 [-S_{(1)} \sin\theta + S_{(2)} - S_{(3)}\,] \hat{\mathbf{e}}_2
\end{aligned}$$

Problem 11.17

Starting from the Navier-Stokes-Duhem equations of motion, obtain the Bernoulli's equation:

$$\boxed{gh + \frac{p}{\rho} + \frac{v^2}{2} = constant} \quad \textit{Bernoulli's equation} \tag{11.43}$$

Hypothesis: incompressible and non-viscous fluid. Consider that the velocity field is steady and irrotational.

Solution:

Considering an incompressible medium $(\nabla_{\vec{x}} \cdot \vec{v}) = 0$, and a non-viscous fluid $(\lambda^* = \mu^* = 0)$, the Navier-Stokes-Duhem equations of motion become:

$$\begin{aligned}
&\rho \dot{v}_i = \rho \mathrm{b}_i - p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj} \\
&\rho \dot{\vec{v}} = \rho \vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*) \nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{v}) + \mu^* \nabla^2_{\vec{x}} \vec{v} \\
&\Rightarrow \rho \dot{\vec{v}} = \rho \vec{\mathbf{b}} - \nabla_{\vec{x}} p
\end{aligned} \tag{11.44}$$

Note that the $\rho \dot{\vec{v}} = \rho \vec{\mathbf{b}} - \nabla_{\vec{x}} p$ are the *Euler equations of motion*. The material time derivative of the velocity, (see equation (11.15)), becomes:

$$\dot{\vec{v}} = \frac{\partial \vec{v}}{\partial t} + \vec{\boldsymbol{\omega}} \wedge \vec{v} + \frac{1}{2} \nabla_{\vec{x}} (v^2) = \frac{1}{2} \nabla_{\vec{x}} (v^2)$$

where we have considered that the steady velocity field $\dfrac{\partial \vec{v}}{\partial t} = \vec{\mathbf{0}}$, and irrotational $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \mathrm{rot}\,\vec{v} = \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}$. With that the equation (11.44) can be rewritten as follows:

$$\frac{\rho}{2}\nabla_{\vec{x}}(v^2) = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p \quad \Rightarrow \quad \frac{1}{2}\nabla_{\vec{x}}(v^2) - \vec{\mathbf{b}} + \frac{1}{\rho}\nabla_{\vec{x}} p = \vec{\mathbf{0}} \tag{11.45}$$

Considering that the body force (conservative field) can be represented by $\vec{\mathbf{b}} = -\nabla_{\vec{x}}\varphi$, where φ is a potential, and also by considering that the mass density field is homogeneous the relationship $\nabla_{\vec{x}}\left(\dfrac{p}{\rho}\right) = \dfrac{1}{\rho}\nabla_{\vec{x}} p$ holds. Then, the equation in (11.45) becomes:

$$\nabla_{\vec{x}}\left(\varphi + \frac{p}{\rho} + \frac{v^2}{2}\right) = 0_i \quad \Rightarrow \quad \varphi + \frac{p}{\rho} + \frac{v^2}{2} = constant \tag{11.46}$$

Considering that the potential can be represented by $\varphi = gh$, where g is the acceleration of gravity and h is the piezometric height, we obtain the Bernoulli's equation:

$$gh + \frac{p}{\rho} + \frac{v^2}{2} = constant$$

We check the SI unit: $[gh] = \left[\dfrac{v^2}{2}\right] = \left[\dfrac{p}{\rho}\right] = \dfrac{N}{m^2}\dfrac{m^3}{kg} = \dfrac{Nm}{kg} = \dfrac{J}{kg} = \dfrac{m^2}{s^2}$, which is the unit of specific energy, i.e. unit of energy per unit mass.

Note that the Bernoulli's equation is the application of the conservation of energy, i.e. in the system there is no energy dissipation, (Figure 11.9).

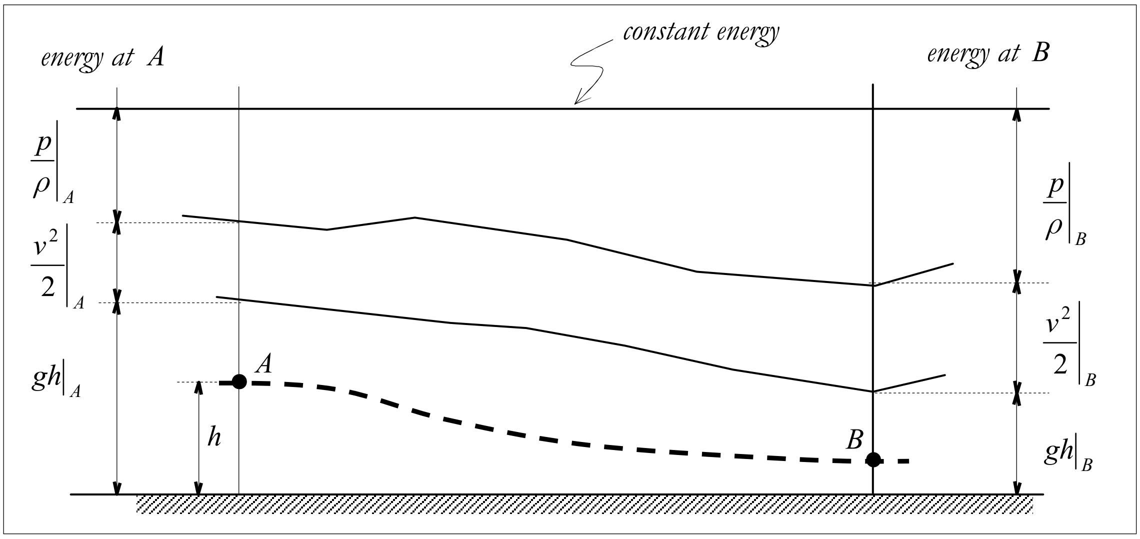

Figure 11.9

Problem 11.18

Let us consider a perfect and incompressible fluid in steady regime that is flowing through the channel as shown in Figure 11.10. Obtain the value of H.

Hypothesis: No energy dissipation is considered.

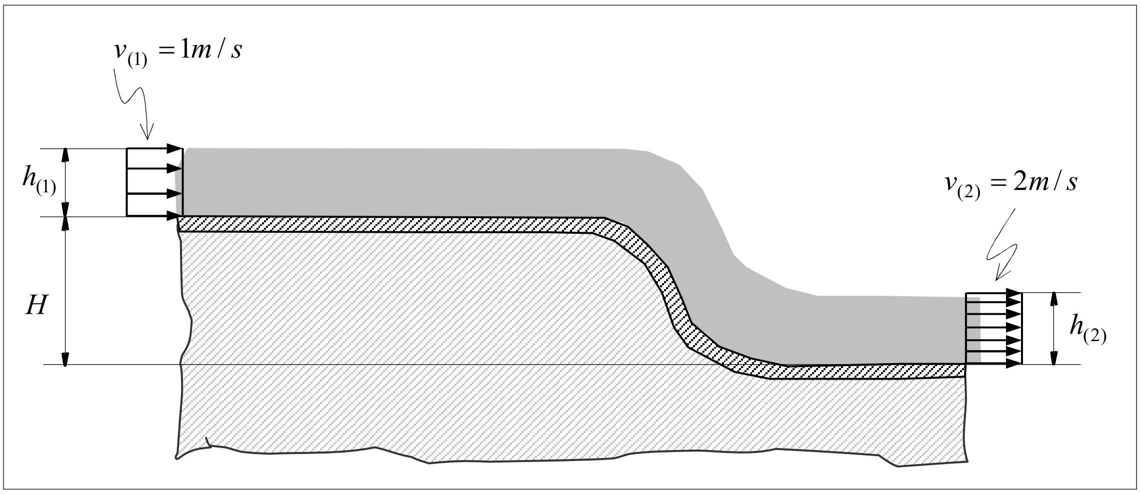

Figure 11.10

Solution:

The mass continuity equation:

$$v_{(1)} h_{(1)} = v_{(2)} h_{(2)} \quad \Rightarrow \quad h_{(2)} = \frac{v_{(1)}}{v_{(2)}} h_{(1)} = \frac{1}{2} h_{(1)}$$

The Bernoulli's equation:

$$\left.\begin{array}{l} (H + h_{(1)}) + 0 + \dfrac{v_{(1)}^2}{2g} \\ \\ h_{(2)} + 0 + \dfrac{v_{(2)}^2}{2g} \end{array}\right\} \quad \Rightarrow \quad H = h_{(2)} - h_{(1)} + \frac{v_{(2)}^2 - v_{(1)}^2}{2g} \quad \Rightarrow \quad H = \frac{-h_{(1)}}{2} + \frac{3}{2g}$$

Problem 11.19

A large diameter circular tank is filled with water. The water pours through a small orifice located at a height H below the water level of the reservoir, (see Figure 11.11). If the volumetric flow rate is Q, obtain the orifice diameter D.

Hypothesis: Consider that H does not vary with time (steady state). Consider that in the section BB', the flow pressure is equal to atmospheric pressure, (see Figure 11.11).

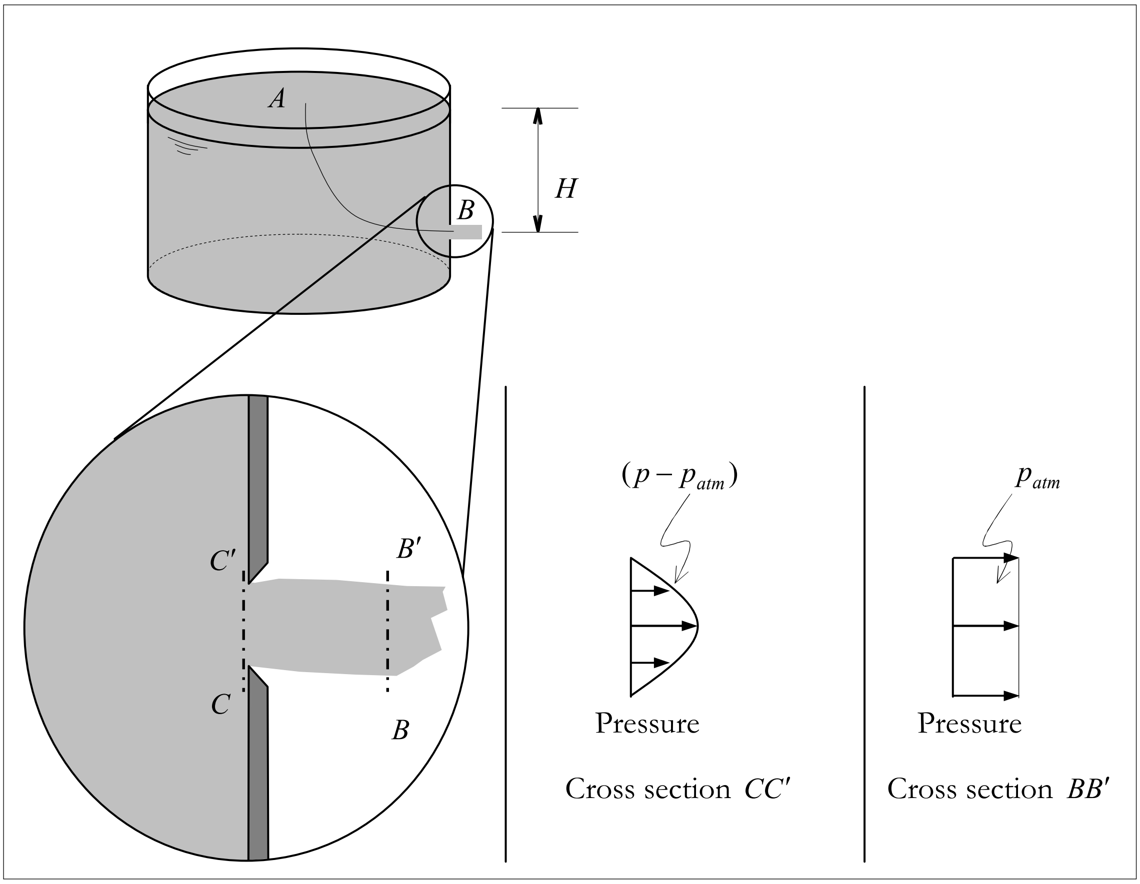

Figure 11.11

Solution:

The water can be considered as an incompressible perfect fluid. For this problem, we will consider the Bernoulli's equation:

$$z + \frac{p}{\rho g} + \frac{v^2}{2g} = const.$$

where it fulfills that:

$$\left.\begin{array}{ll} Point\ \ A \Rightarrow & H + \dfrac{p_{atm}}{\rho g} + 0 \\ Point\ \ B \Rightarrow & 0 + \dfrac{p_{atm}}{\rho g} + \dfrac{v_{(B)}^2}{2g} \end{array}\right\} \Rightarrow \quad v_{(B)} = \sqrt{2gH}$$

Considering that the volumetric flow rate is given by $Q = v_{(B)} S_{(B)}$, we can conclude that:

$$Q = v_{(B)} S_{(B)} = \sqrt{2gH}\,\frac{\pi D^2}{4} \quad \Rightarrow \quad D = \sqrt{\frac{4Q}{\pi\sqrt{2gH}}}$$

Problem 11.20

Consider a pipeline which has been introduced a pitot tube as shown in Figure 11.12. Obtain the velocity at the point 1 in terms of $h_{(1)}$ and $h_{(2)}$. Consider that there is no energy dissipation in the system.

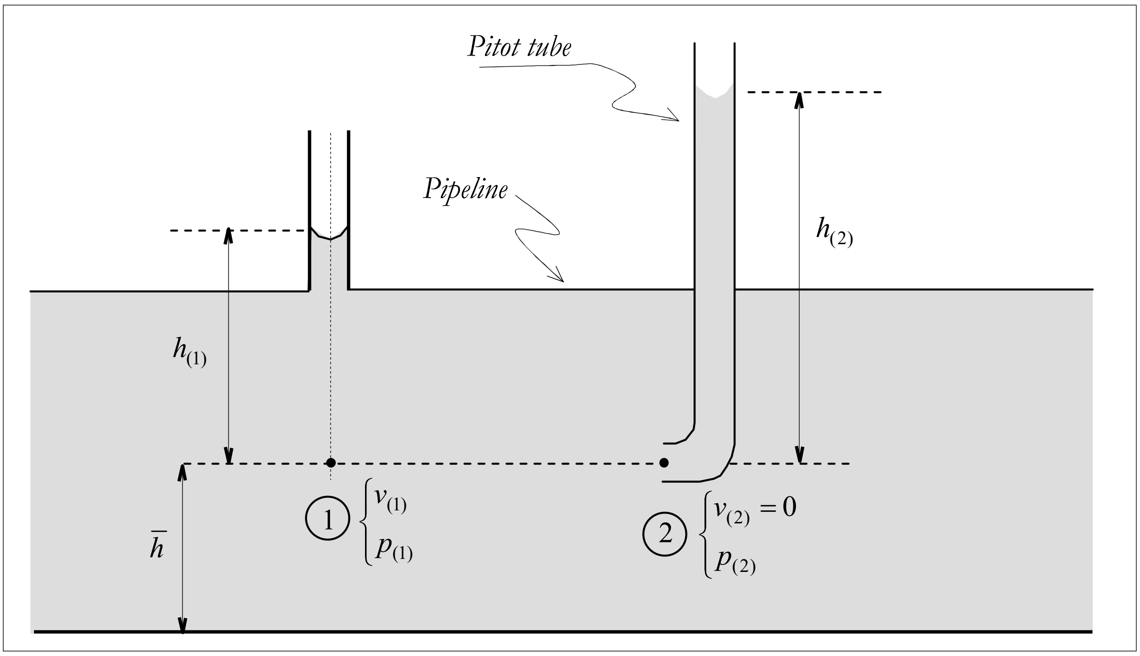

Figure 11.12: Pitot tube.

Solution:

By applying the Bernoulli's equation between the point 1 and 2, we obtain:

$$g\bar{h}+\frac{p_{(1)}}{\rho}+\frac{v_{(1)}^2}{2}=g\bar{h}+\frac{p_{(2)}}{\rho}+\frac{v_{(2)}^2}{2}$$

$$\Rightarrow \frac{p_{(1)}}{\rho}+\frac{v_{(1)}^2}{2}=\frac{p_{(2)}}{\rho}$$

$$\Rightarrow v_{(1)}=\sqrt{\frac{2(p_{(2)}-p_{(1)})}{\rho}}$$

The pressure values at the points 1 and 2 are given, respectively, by:

$$p_{(1)}=\rho g h_{(1)} \quad ; \quad p_{(2)}=\rho g h_{(2)}$$

with that the velocity $v_{(1)}$ is obtained as follows:

$$v_{(1)}=\sqrt{\frac{2(p_{(2)}-p_{(1)})}{\rho}}=\sqrt{\frac{2(\rho g h_{(2)}-\rho g h_{(1)})}{\rho}}=\sqrt{2g(h_{(2)}-h_{(1)})}$$

Problem 11.21

Consider an incompressible non-viscous fluid, which has a steady velocity field and irrotational. Consider also that the velocity field is independent of x_3-direction. Obtain the governing equations for the proposed problem in terms of the velocity potential ϕ and streamlines ψ.

Solution:

Velocity potential: In this example we can represent the velocity field by means of a potential ϕ, i.e. $\vec{\boldsymbol{v}} = \nabla_{\vec{x}}\phi$. With that we are considering that the velocity field is conservative, hence the curl of the velocity field is zero, i.e. $\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}} = \operatorname{rot} \vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$. Remember that a field whose curl is zero it does not necessarily imply that the field is conservative, but for a conservative field the curl is always equals zero.

Note that the velocity has the same direction as $\nabla_{\vec{x}}\phi$, i.e. it is normal to the isosurfaces $\phi = const$.

Streamline: Given a spatial velocity field at time t, we can define a streamline (ψ) to the curve in which the tangent at each point has the same direction as the velocity, (see Figure 11.13). In general, the streamline and trajectory do not coincide, but in steady state motion they do. Two streamlines cannot intersect.

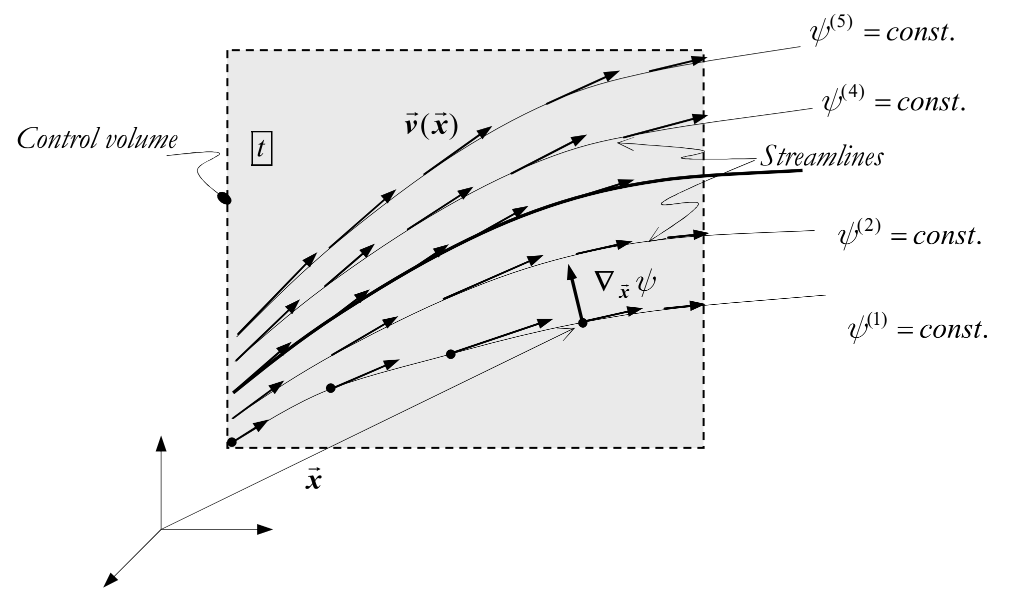

Figure 11.13

Based on the definition of differential $d\psi$ (the total derivative), and the definition of gradient $\nabla_{\vec{x}}\psi$ we obtain the relationship $d\psi = \nabla_{\vec{x}}\psi \cdot d\vec{\boldsymbol{x}}$.

Note that it holds that $(\nabla_{\vec{x}}\psi)\cdot(\nabla_{\vec{x}}\phi) = 0$, (Figure 11.14). The differential $d\vec{\boldsymbol{x}}$ in the streamline, at a point, has the same direction as the velocity at this point. Hence, the following is fulfilled:

$$d\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

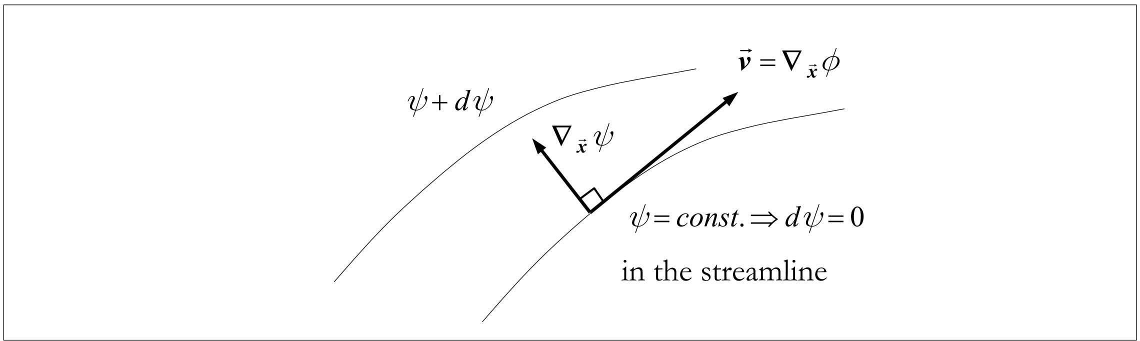

Figure 11.14

In the Cartesian system, $d\vec{x} \wedge \vec{v} = \vec{\mathbf{0}}$ is represented by:

$$d\vec{x} \wedge \vec{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ dx_1 & dx_2 & dx_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{\mathbf{0}}$$

$$= (v_3 dx_2 - v_2 dx_3)\hat{\mathbf{e}}_1 + (v_3 dx_1 - v_1 dx_3)\hat{\mathbf{e}}_2 + (v_2 dx_1 - v_1 dx_2)\hat{\mathbf{e}}_3 = \vec{\mathbf{0}}$$

Components:

$$(d\vec{x} \wedge \vec{v})_i = \begin{bmatrix} (v_3 dx_2 - v_2 dx_3) \\ (v_3 dx_1 - v_1 dx_3) \\ (v_2 dx_1 - v_1 dx_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For this example the velocity field is independent of x_3, i.e. the problem is stated on the plane $x_1 - x_2$ (2D-case). With that we can conclude that:

$$(d\vec{x} \wedge \vec{v})_i = \begin{bmatrix} 0 \\ 0 \\ (v_2 dx_1 - v_1 dx_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 dx_1 - v_1 dx_2 = 0 \tag{11.47}$$

Note that in a streamline it holds that $\psi = const. \Rightarrow d\psi = 0$ and also by applying the definition $d\psi = \nabla_{\vec{x}}\psi \cdot d\vec{x}$, we can obtain:

$$d\psi = \nabla_{\vec{x}}\psi \cdot d\vec{x} \xrightarrow{\text{indicial}} d\psi = \psi_{,i} dx_i = 0$$

$$\Rightarrow d\psi = \psi_{,1} dx_1 + \psi_{,2} dx_2 + \psi_{,3} dx_3 = 0$$

$$\Rightarrow d\psi = \frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 + \frac{\partial \psi}{\partial x_3} dx_3 = 0$$

For the 2D-case (two-dimensional case) we have:

$$\frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 = 0 \tag{11.48}$$

If we compare the equations (11.47) and (11.48) we can conclude that:

$$v_1 = -\frac{\partial \psi}{\partial x_2} \quad ; \quad v_2 = \frac{\partial \psi}{\partial x_1} \tag{11.49}$$

1) Starting from an incompressible fluid: $(\nabla_{\vec{x}} \cdot \vec{v}) = 0$ we can obtain:

$$v_{i,i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \quad \xrightarrow{2D} \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$$

Considering that $\vec{v} = \nabla_{\vec{x}} \phi$, we obtain:

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = 0 \quad \Rightarrow \quad \nabla^2_{\vec{x}} \phi = 0 \tag{11.50}$$

2) Based on the fact that the fluid is irrotational $\vec{\nabla}_{\vec{x}} \wedge \vec{v} \equiv \mathrm{rot}\, \vec{v} = \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}$:

$$\begin{aligned} \mathrm{rot}(\vec{v}) = \vec{\nabla}_{\vec{x}} \wedge \vec{v} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \epsilon_{ijk} \mathrm{v}_{k,j} \hat{\mathbf{e}}_i = \vec{\mathbf{0}} \\ &= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 = \vec{\mathbf{0}} \end{aligned} \tag{11.51}$$

Then:

$$\begin{bmatrix} \left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \right) \\ \left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \right) \\ \left(\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As we are dealing with a 2D-case, the above equations reduce to:

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0$$

Taking into account the equations in (11.49) we can conclude that:

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial x_1} + \frac{\partial^2 \psi}{\partial x_2} = 0 \quad \Rightarrow \quad \nabla^2_{\vec{x}} \psi = 0$$

With which the problem is stated by the equations:

$$\boxed{\nabla^2_{\vec{x}} \phi = 0 \quad ; \quad \nabla^2_{\vec{x}} \psi = 0} \tag{11.52}$$

Annex A

Numerical Integration over Time

A.1 Introduction

Before raising the case for multiple degrees of freedom we will analysis the numerical solution for the problem:

$$y'(x,t) = \frac{dy(x,t)}{dt} \tag{A.1}$$

which goal is to find the function $y(x,t)$.

The partial differential exact solution for the most engineering problems cannot be obtained due to the problem complexity. But, for a better understanding of the procedure numerical solution we will adopt the very simply differential equation, (see CHAPRA&CANALE (1988)):

$$y'(t) = \frac{dy(t)}{dt} = -2t^3 + 12t^2 - 20t + 8.5 \tag{A.2}$$

which is time dependent only. The exact solution can be obtained by integrate the above equation over time, i.e.:

$$y = -0.5t^4 + 4t^3 - 10t^2 + 8.5t + C \tag{A.3}$$

where C is the constant of integration. Note that there are infinite solutions, since C could assume infinite values, (see Figure A.1). The solution is unique if the initial condition is known. For this example (A.2) which is only time dependent the function value at $t=0$ is known and given by $y(t=0) \equiv y_0 = 1 \Rightarrow C = 1$. Then, the solution is unique:

$$y = -0.5t^4 + 4t^3 - 10t^2 + 8.5t + 1 \tag{A.4}$$

At $t=0$ we know the following parameters:

$$\begin{cases} y_0 = 1 \\ y_0' = 8.5 \end{cases} \tag{A.5}$$

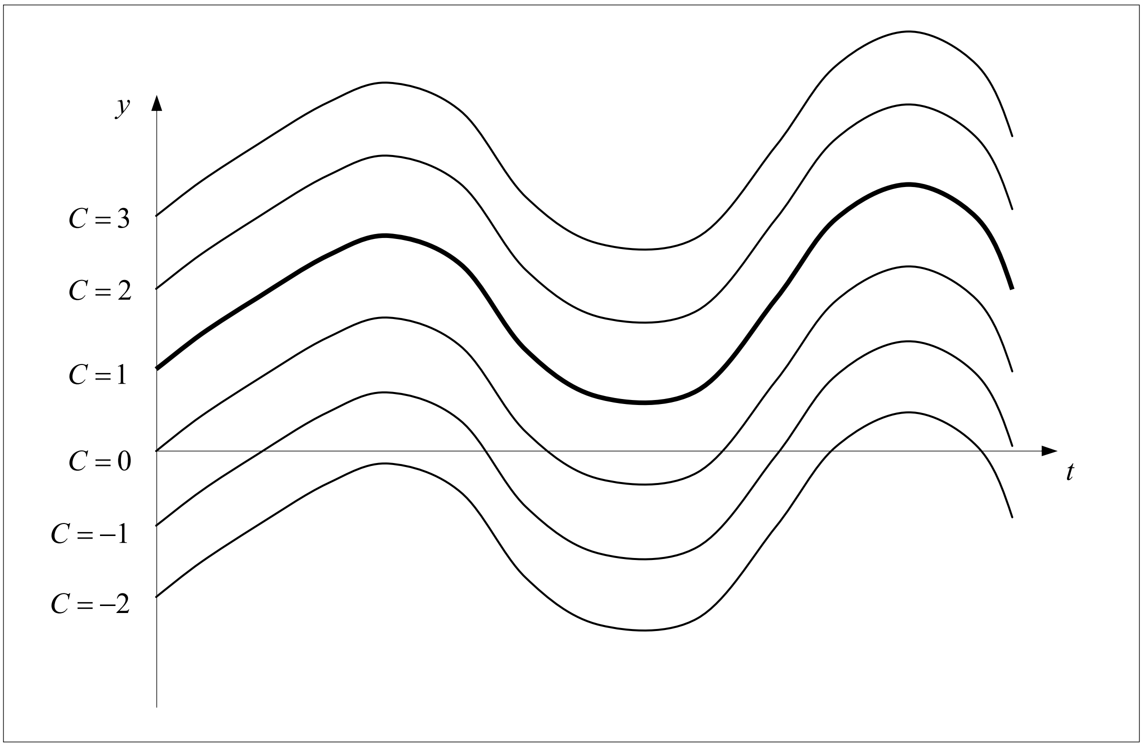

Figure A.1: Infinites solutions.

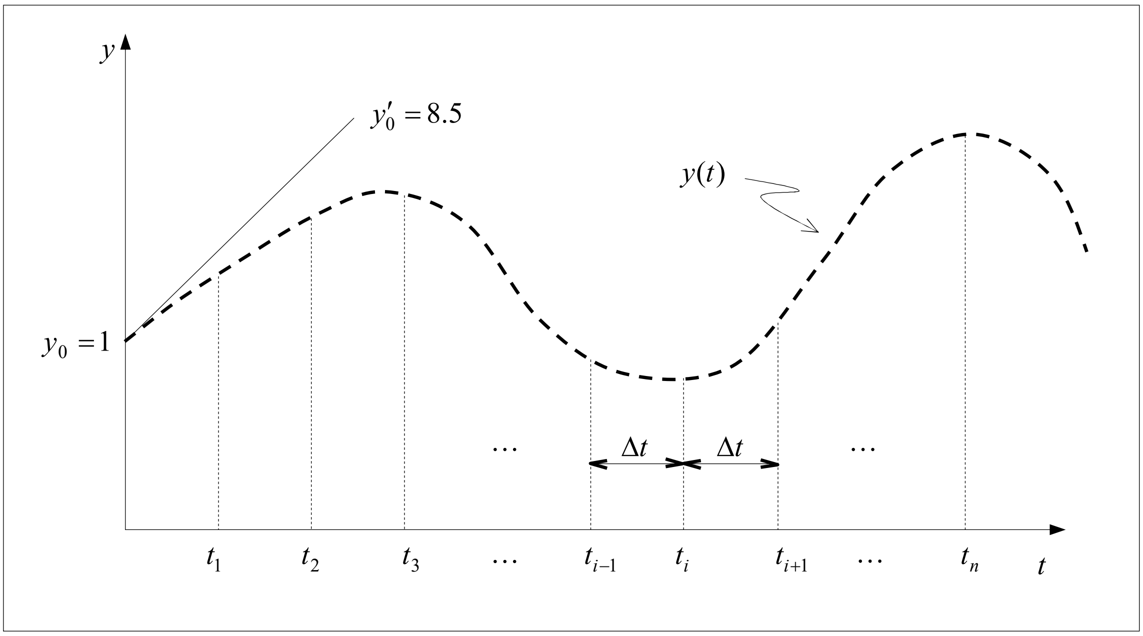

Figure A.2: Time discretization.

One of the most techniques used for numerical integration over time is the Finite Difference Method in which the "domain" is discretized by the finite value Δt (time increment). Next we will discuss some of these methods.

A.2 Euler's Method

Knowing the value of the curve slope at time t, i.e. y'_i, we can obtain the next approximated value for y_{i+1} by means of a lineal approach:

$$y'_i = \frac{y_{i+1} - y_i}{\Delta t} \quad \Rightarrow \quad y_{i+1} = y_i + y'_i \Delta t \tag{A.6}$$

The above approach is the same as forward finite difference.

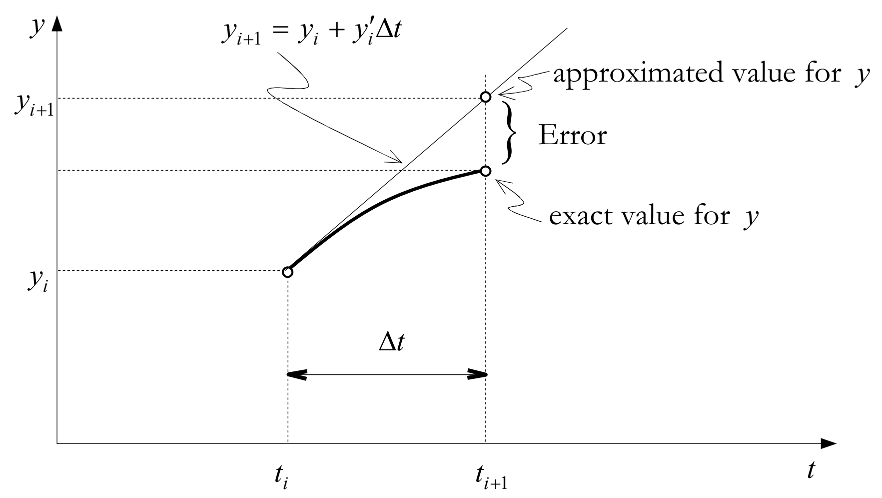

Figure A.3: Forward Finite Difference.

By means of Figure A.3 we can guess that when the time increment tends to zero we approach the exact value of the function. For problems with several unknowns working with very small time increment it can result in a high computational cost, so, to overcome this drawback some effective methods have been developed in order to guarantee result accuracy even when the time increment is big.

In previous example we have applied the forward finite difference using y'_i to obtain y_{i+1}. We can also use the following approach: we situate at y_{i+1} and we apply the backward finite difference, i.e.:

$$y'_{i+1} = \frac{y_{i+1} - y_i}{\Delta t} \quad \Rightarrow \quad y_{i+1} = y_i + y'_{i+1} \Delta t \tag{A.7}$$

This method is known as backward Euler's method (implicit method). With that we can summarize that:

$$y_{i+1} = y_i + y'_i \Delta t \quad \text{(Explicit method)} \tag{A.8}$$

$$y_{i+1} = y_i + y'_{i+1}\Delta t \text{ (Implicit method)} \tag{A.9}$$

Another approach we can adopt is by consider the curve slope as the average between y'_i and y'_{i+1}, i.e.:

$$y_{i+1} = y_i + \left(\frac{y'_i + y'_{i+1}}{2}\right)\Delta t \tag{A.10}$$

which method is more precise than forward/backward finite difference. This method is called Crank-Nicolson's method.

A.3 Alfa Method

We can generalize the above method in a single expression. To do this we consider the following approaches to the functions $y(t)$ and $y'(t)$, (see Figure A.4).

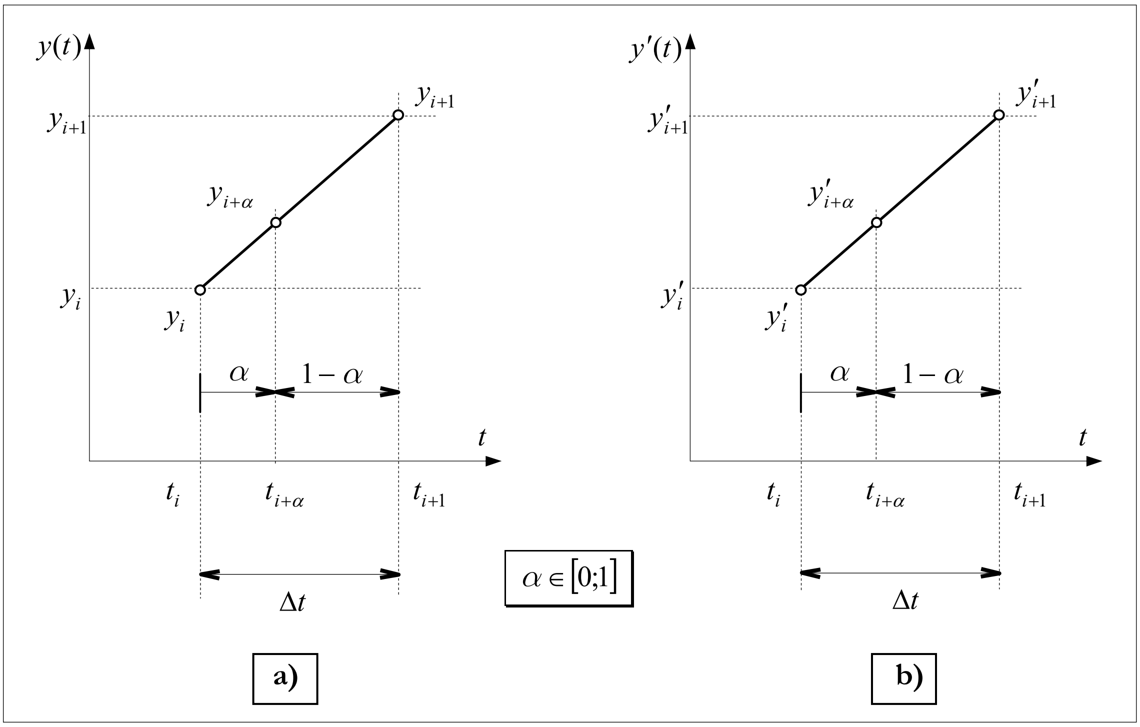

Figure A.4: Alfa method.

By means of linearization for the functions $y(t)$ and $y'(t)$, we can obtain:

$$y'_{i+\alpha} = \frac{y_{i+1} - y_i}{\Delta t} \tag{A.11}$$

with that we can conclude that:

$$y_{i+1} = y_i + y'_{i+\alpha}\Delta t \tag{A.12}$$

Using similarity triangle, (see Figure A.4(b)), it is possible to express $y'_{i+\alpha}$ as follows:

$$\frac{y'_{i+1}-y'_i}{1}=\frac{y'_{i+\alpha}-y'_i}{\alpha} \quad \Rightarrow \quad y'_{i+\alpha}=y'_i+\alpha\left(y'_{i+1}-y'_i\right) \tag{A.13}$$

or:

$$y'_{i+\alpha}=\alpha\, y'_{i+1}+(1-\alpha)y'_i \tag{A.14}$$

The we summarize the Alfa method as follows:

$$\boxed{\begin{cases} y_{i+1}=y_i+y'_{i+\alpha}\Delta t \\ y'_{i+\alpha}=\alpha\, y'_{i+1}+(1-\alpha)y'_i \end{cases}} \tag{A.15}$$

in which depending on the α value we obtain::

- $\alpha=0$ (Explicit)

$$\begin{cases} y_{i+1}=y_i+y'_{i+\alpha}\Delta t \\ y'_{i+\alpha}=(1)y'_i \end{cases} \quad \Rightarrow \quad y_{i+1}=y_i+y'_i\Delta t \tag{A.16}$$

- $\alpha=1$ (Implicit)

$$\begin{cases} y_{i+1}=y_i+y'_{i+\alpha}\Delta t \\ y'_{i+\alpha}=y'_{i+1} \end{cases} \quad \Rightarrow \quad y_{i+1}=y_i+y'_{i+1}\Delta t \tag{A.17}$$

- $\alpha=\frac{1}{2}$ (Crank-Nicolson)

$$\begin{cases} y_{i+1}=y_i+y'_{i+\frac{1}{2}}\Delta t \\ y'_{i+\frac{1}{2}}=\frac{1}{2}y'_{i+1}+\left(1-\frac{1}{2}\right)y'_i=\frac{y'_{i+1}+y'_i}{2} \end{cases} \quad \Rightarrow \quad y_{i+1}=y_i+\left(\frac{y'_{i+1}+y'_i}{2}\right)\Delta t \tag{A.18}$$

The Crank-Nicolson's method is also called Heun's method. Geometrically we can interpret as indicated in Figure A.5. At t_i we make a prediction for $y^0_{i+1}=y_i+y'_i\Delta t$ and in turn we can obtain y'^0_{i+1}. Then we find the value for the new curve slope:

$$\bar{y}'_i=\frac{y'^0_{i+1}+y'_i}{2} \tag{A.19}$$

Then we obtain once again the new value for y_{i+1} by considering the slope $\bar{y}'_i$:

$$y_{i+1}=y_i+\left(\frac{y'^0_{i+1}+y'_i}{2}\right)\Delta t \tag{A.20}$$

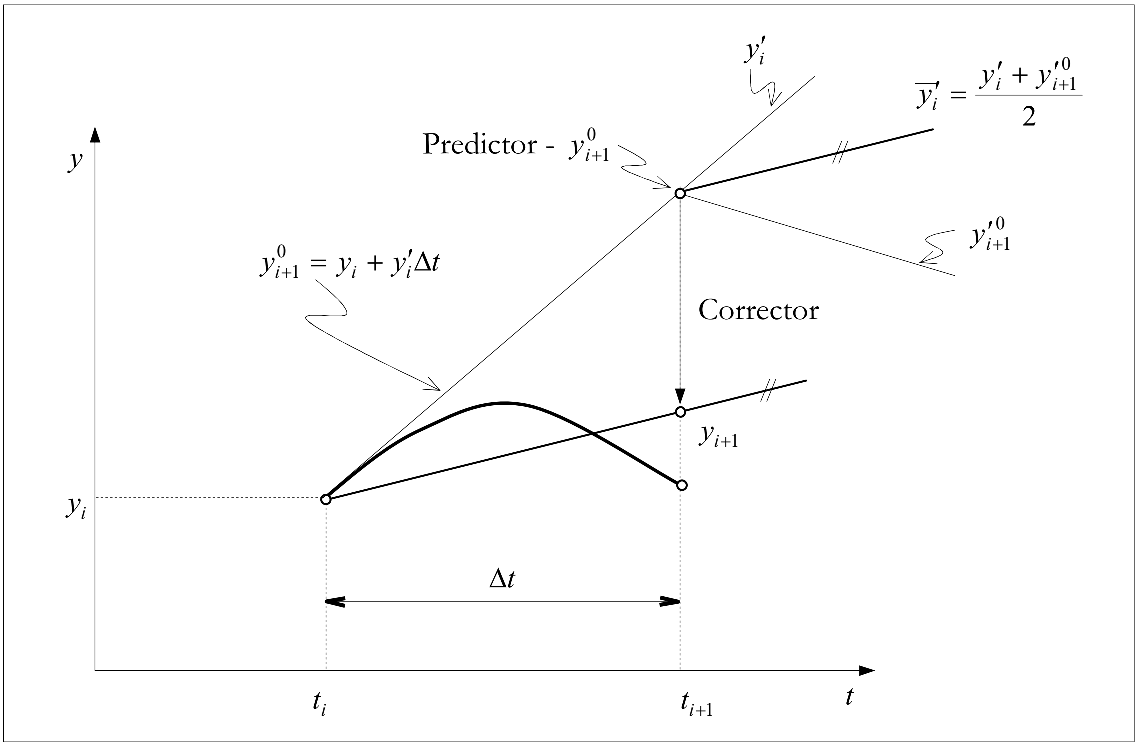

Figure A.5: Heun's method (predictor-corrector).

Returning to our example initially raised whose differential equations is:

$$y' = \frac{dy}{dt} = -2t^3 + 12t^2 - 20t + 8.5 \tag{A.21}$$

with the initial condition:

$$t = 0 \quad ; \quad y_0 = 1 \quad \Rightarrow \quad y'_0 = 8.5 \tag{A.22}$$

We will aplly the Euler's method and the Heun's method with time increment $\Delta t = 0.5s$. For the first time step $(t = 0.5)$ the exact value of the function can be obtained by means of (A.4), i.e.:

$$y(t = 0.5) = -0.5 \times 0.5^4 + 4 \times 0.5^3 - 10 \times 0.5^2 + 8.5 \times 0.5 + 1 = 3.21875 \tag{A.23}$$

The numerical procedure follows:

Predictor $i+1$

$y^0_1 = y_0 + y'_0 \Delta t = 1 + 8.5 \times 0.5 = 5.25$ (We stop here if we are using the Euler's method)

Corrector

$$y'(t = 0.5) = y'^0_1 = -2t^3 + 12t^2 - 20t + 8.5 = -2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5 = 1.25$$

$$\overline{y}'_i = \frac{y'^0_{i+1} + y'_i}{2} = \frac{1.25 + 8.5}{2} = 4.875$$

$$y_1 = y_0 + \overline{y}'_0 \Delta t = 1 + 4.875 \times 0.5 = 3.4375$$

A.4 Modified Euler's Method

In the modified Euler's method we use the Euler's method to predict the value function in the middle of the interval:

$$y_{i+\frac{1}{2}} = y_i + y'_i \frac{\Delta t}{2} \tag{A.24}$$

Next, we obtain the slope $y'_{i+\frac{1}{2}}$ at this point, which is used to obtain y_{i+1}:

$$y_{i+1} = y_i + y'_{i+\frac{1}{2}} \Delta t \tag{A.25}$$

Apply this methodology to the proposed example (A.21) we can obtain:

$$y_{i+\frac{1}{2}} = y_i + y'_i \frac{\Delta t}{2} = 1 + 8.5 \times \frac{0.5}{2} = 3.125$$

Slope calculation at the middle point:

$$y'(t = 0.25) = y'_{0+\frac{1}{2}} = -2t^3 + 12t^2 - 20t + 8.5 = -2(0.25)^3 + 12(0.25)^2 - 20(0.25) + 8.5 = 4.21875$$

$$y_1 = y_0 + y'_{\frac{1}{2}} \Delta t = 1 + 4.21875 \times 0.5 = 3.1093$$

In Figure A.6 we can appreciate this procedure to calculate the function, $y'(t) = \dfrac{dy(t)}{dt} = -2t^3 + 12t^2 - 20t + 8.5$ with $y'(0) = 1$, in which we have used the Euler's method, modified Euler's method and the Heun's method with time increment $\Delta t = 0.5$.

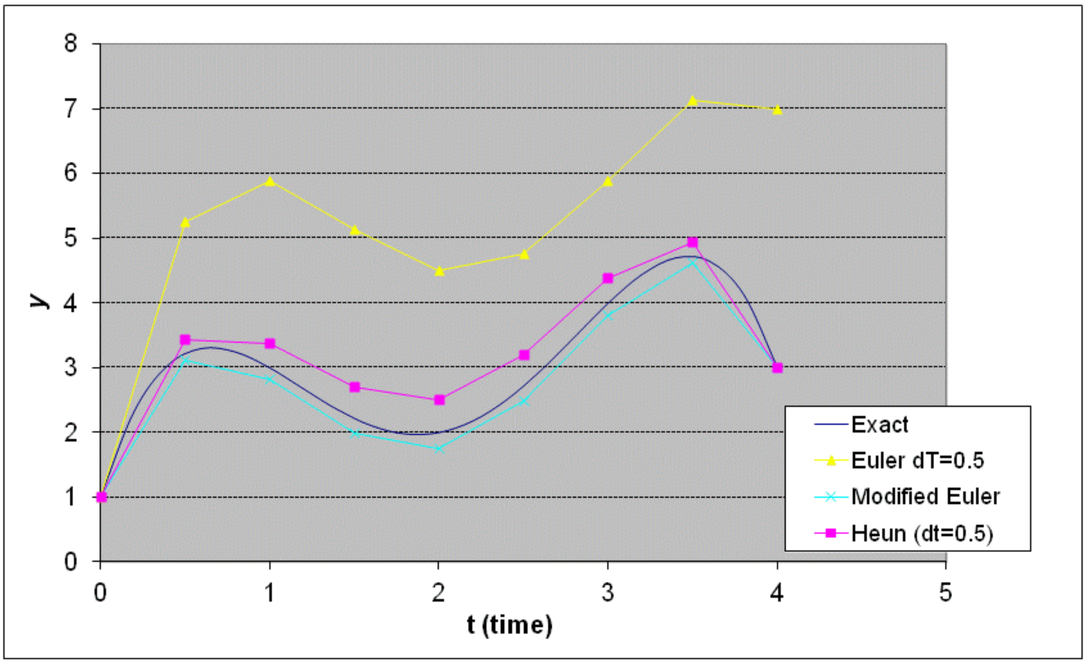

Figure A.6: Comparative responses between some methods.

As we can see the Heun's method (H) always overestimates the function value while the Modified Euler's method (ME) underestimates the function value. We can make the following approximation for y_{i+1}:

$$y_{i+1} = \frac{1}{3}\left(y_{i+1}^{H} + 2y_{i+1}^{ME}\right) \tag{A.26}$$

Using the equations (A.20) and (A.25) in the above equation we can obtain:

$$\begin{aligned} y_{i+1} &= \frac{1}{3}\left(y_{i+1}^{H} + 2y_{i+1}^{ME}\right) \\ &= \frac{1}{3}\left[y_i + \left(\frac{y_{i+1}'^{0} + y_i'}{2}\right)\Delta t + 2\left(y_i + y'_{i+\frac{1}{2}}\Delta t\right)\right] \\ &= \frac{1}{3}\left[y_i + \frac{y_{i+1}'^{0}}{2}\Delta t + \frac{y_i'}{2}\Delta t + 2y_i + 2y'_{i+\frac{1}{2}}\Delta t\right] \\ &= \frac{1}{3}\left[3y_i + \frac{y_{i+1}'^{0}}{2}\Delta t + \frac{y_i'}{2}\Delta t + 2y'_{i+\frac{1}{2}}\Delta t\right] \end{aligned} \tag{A.27}$$

thus:

$$y_{i+1} = y_i + \frac{\Delta t}{6}\left[y_i' + 4y'_{i+\frac{1}{2}} + y_{i+1}'^{0}\right] \tag{A.28}$$

whose equation is known as Runge-Kutta's integration method of third order (see CHAPRA&CANALE (1988)), which is a good approximation as we can see in Figure A.7.

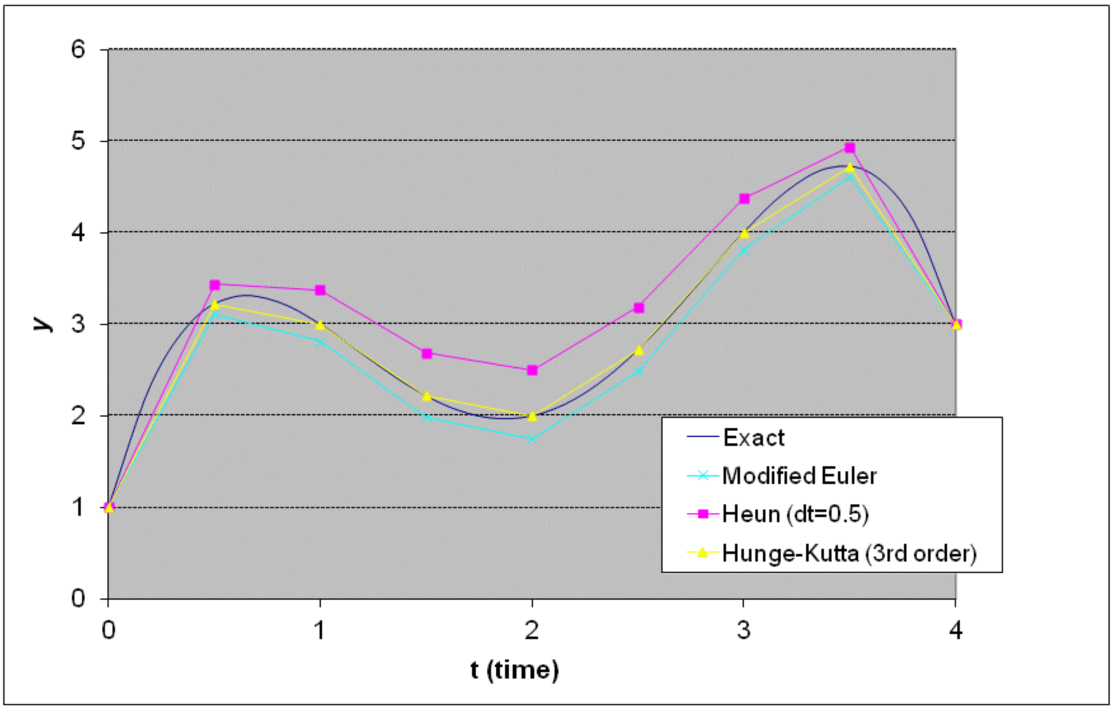

Figure A.7: Comparative responses between some methods (Hunge-Kutta).

A.5 Unsteady Case with Multiply Degree-of-Freedom

Let us consider the following set of equations:

$$\boldsymbol{D}\,\dot{\boldsymbol{T}} + \boldsymbol{K}\,\boldsymbol{T} = \boldsymbol{F} \tag{A.29}$$

For the thermal problem, $\boldsymbol{D}$ stands for capacitance matrix, $\boldsymbol{K}$ is the conductivity matrix, and $\boldsymbol{T}$ represents nodal temperature values.

By considering that

$$\dot{\boldsymbol{T}} = \frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t} \tag{A.30}$$

and by apply the Alfa method:

$$\begin{aligned} \boldsymbol{T}_\alpha &= \alpha \boldsymbol{T}_{t+1} + (1-\alpha)\boldsymbol{T}_t \\ \boldsymbol{F}_\alpha &= \alpha \boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t \end{aligned} \tag{A.31}$$

By substituting the equations (A.30) and (A.31) into the equation (A.29), we can obtain:

$$\boldsymbol{D}\left(\frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t}\right) + \boldsymbol{K}\,\boldsymbol{T}_\alpha = \boldsymbol{F}_\alpha \tag{A.32}$$

$$\boldsymbol{D}\left(\frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t}\right) + \boldsymbol{K}\left[\alpha \boldsymbol{T}_{\alpha+1} + (1-\alpha)\boldsymbol{T}_\alpha\right] = \alpha \boldsymbol{F}_{\alpha+1} + (1-\alpha)\boldsymbol{F}_\alpha \tag{A.33}$$

then:

$$\left[\frac{\boldsymbol{D}}{\Delta t} + \alpha \boldsymbol{K}\right]\boldsymbol{T}_{t+1} = \alpha\,\boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{\boldsymbol{D}}{\Delta t} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{T}_t \tag{A.34}$$

$$\boldsymbol{K}^{eff}\,\boldsymbol{T}_{t+1} = \boldsymbol{F}^{eff} \tag{A.35}$$

where

$$\boldsymbol{K}^{eff} = \frac{\boldsymbol{D}}{\Delta t} + \alpha \boldsymbol{K} \quad ; \quad \boldsymbol{F}^{eff} = \alpha\,\boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{\boldsymbol{D}}{\Delta t} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{T}_t \tag{A.36}$$

A.6 Dynamic Analysis with Numerical Integration

The more general approach to the solution of the dynamic response for structures is the direct numerical integration of the dynamic equilibrium:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{D}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \tag{A.37}$$

whose equations must fulfill for all time t, then it is also valid at time $t+\Delta t$:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.38}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{D}$ is the damping matrix, $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{F}$ is the nodal external force vector, and $\boldsymbol{U}$, $\dot{\boldsymbol{U}}$, $\ddot{\boldsymbol{U}}$ are displacement, velocity and acceleration, respectively. The absence of subscript time step in the matrices $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ indicates a linear problem, i.e. they do not depend on $\boldsymbol{U}$, $\dot{\boldsymbol{U}}$ and $\ddot{\boldsymbol{U}}$. In the case in which the structure presents a material non-linearity the matrix $\boldsymbol{K}$ depends on $\boldsymbol{U}$.

For a system without damping ($\boldsymbol{D}=\boldsymbol{0}$) the energy is conserved (system without energy dissipation), and the sum of the internal energy ($\frac{1}{2}\dot{\boldsymbol{U}}^{\boldsymbol{T}}\boldsymbol{M}\dot{\boldsymbol{U}}$) plus the strain energy ($\frac{1}{2}\boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{U}$) is constant at any time step:

$$2E = \dot{\boldsymbol{U}}_t^T\boldsymbol{M}\dot{\boldsymbol{U}}_t + \boldsymbol{U}_t^T\boldsymbol{K}\boldsymbol{U}_t = \dot{\boldsymbol{U}}_{t+\Delta t}^T\boldsymbol{M}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{U}_{t+\Delta t}^T\boldsymbol{K}\boldsymbol{U}_{t+\Delta t} \tag{A.39}$$

The numerical analysis of the dynamic system (A.37) can be inefficient where $\boldsymbol{D}$ is responsible for the damping (energy dissipation) in the structure. Therefore some methods were developed in order to introduce a numerical damping (artificial) which generally is controlled by a parameter. For example, we can replace the damping matrix $\boldsymbol{D}$ by a linear combination between $\boldsymbol{K}$ and $\boldsymbol{M}$ (Rayleigh damping):

$$\boldsymbol{D} = \alpha\boldsymbol{K} + \beta\boldsymbol{M} \tag{A.40}$$

Then, we will study some methods in which a numerical damping is introduced albeit the equation has no matrix $\boldsymbol{D}$.

Several numerical techniques have been developed for solving the set of equations (A.38). We can classify these techniques as *Explicit*, *Implicit*, or *Mixed*.

The Explicit methods do not require information at the time step $t+\Delta t$ to predict the response at time $t+\Delta t$, i.e.:

$$\boldsymbol{U}_{t+\Delta t} = f\left(\boldsymbol{U}_t, \dot{\boldsymbol{U}}_t, \ddot{\boldsymbol{U}}_t, \dot{\boldsymbol{U}}_{t-\Delta t}, \cdots\right) \tag{A.41}$$

These methods are *conditionally stable*, which implies that the time step size (Δt) must be less than a critical value (Δt_{cr}), otherwise the solution is not stable, i.e. the solution diverges.

The Implicit methods use the information at time step $t+\Delta t$ to predict the structural response at time $t+\Delta t$, i.e.:

$$\boldsymbol{U}_{t+\Delta t} = f\left(\boldsymbol{U}_t, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \cdots\right) \tag{A.42}$$

With these methods it is possible to use larger time steps than those used in the explicit method. The implicit method can be unconditionally or conditionally stable. In general, the implicit methods are unconditionally stable, and the only restriction for time step size is the solution accuracy.

A.6.1 Newmark's Family of Methods

Newmark in 1959 introduced a family of integration methods for solving dynamic structural problems. To illustrate these methods we will star from the following set of equations:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.43}$$

We can apply the Taylor series to approximate the functions $\boldsymbol{U}$ and $\dot{\boldsymbol{U}}$:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2!}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^3}{3!}\dddot{\boldsymbol{U}}_t + \cdots \tag{A.44}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2!}\dddot{\boldsymbol{U}}_t + \cdots \tag{A.45}$$

Newmark truncated the previous equations as follows:

$$\boldsymbol{U}_{t+\Delta t} \approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^3\dddot{\boldsymbol{U}}_t \tag{A.46}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} \approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t^2\dddot{\boldsymbol{U}}_t \tag{A.47}$$

Assuming that the acceleration varies linearly within the range $[t, t+\Delta t]$, we can apply the finite difference to approach $\dddot{\boldsymbol{U}}_t$, i.e.:

$$\dddot{\boldsymbol{U}}_t = \frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t} \tag{A.48}$$

Substituting the equation (A.48) into the equations (A.47) and (A.46) we can obtain:

$$\begin{aligned}\boldsymbol{U}_{t+\Delta t} &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^3\,\frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t}\\ &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} - \beta\Delta t^2\ddot{\boldsymbol{U}}_t\\ &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2}-\beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\end{aligned} \tag{A.49}$$

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &\approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t^2\,\frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t}\\ &\approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} - \gamma\Delta t\ \ddot{\boldsymbol{U}}_t\\ &\approx \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t}\end{aligned} \tag{A.50}$$

Then, we summarize the approaches used by Newmark for displacement, velocity and acceleration:

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2}-\beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\end{aligned}} \quad \textit{Newmark's method} \tag{A.51}$$

This method is unconditionally stable when:

$$2\beta \geq \gamma \geq \frac{1}{2} \tag{A.52}$$

Solving for $\ddot{\boldsymbol{U}}_{t+\Delta t}$ by using the displacement given by the equation (A.51) we can obtain:

$$\begin{aligned}
&\beta\Delta t^2 \ddot{\boldsymbol{U}}_{t+\Delta t} = \boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t - \Delta t \dot{\boldsymbol{U}}_t - \left(\frac{1}{2} - \beta\right)\Delta t^2 \ddot{\boldsymbol{U}}_t \\
&\Rightarrow \ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \frac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.53}$$

Substituting the equation (A.53) into the velocity equation (A.51) we can obtain:

$$\begin{aligned}
\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t \ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} \\
&= \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t \ddot{\boldsymbol{U}}_t + \gamma\Delta t \left[\frac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \frac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t\right] \\
&= \frac{\gamma}{\beta\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t \ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.54}$$

Thus

$$\begin{cases}
\ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dfrac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \dfrac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t \\
\dot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{\gamma}{\beta\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1 - \dfrac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1 - \dfrac{\gamma}{2\beta}\right)\Delta t \ddot{\boldsymbol{U}}_t
\end{cases} \tag{A.55}$$

By substituting the equations given by (A.55) into the equation (A.43) we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.56}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \left[\frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{\gamma}{\beta\Delta t}\boldsymbol{D} + \boldsymbol{K}\right] \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{\boldsymbol{M}}{\beta\Delta t^2} + \frac{\gamma}{\beta\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{\boldsymbol{M}}{\beta\Delta t} - \left(1 - \frac{\gamma}{\beta}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t - \left[\left(1 - \frac{1}{2\beta}\right)\boldsymbol{M} + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.57}$$

It is also possible to express the above equations as follows:

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \left[b_1\boldsymbol{M} + b_4\boldsymbol{D} + \boldsymbol{K}\right] \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \boldsymbol{M}\ \left[b_1\boldsymbol{U}_t - b_2\dot{\boldsymbol{U}}_t - b_3\ddot{\boldsymbol{U}}_t\right] + \boldsymbol{D}\left[b_4\boldsymbol{U}_t - b_5\dot{\boldsymbol{U}}_t - b_6\ddot{\boldsymbol{U}}_t\right]
\end{aligned} \tag{A.58}$$

where

$$\begin{aligned}
&b_1 = \frac{1}{\beta\Delta t^2} \quad ; \quad b_2 = -\frac{1}{\beta\Delta t} \quad ; \quad b_3 = 1 - \frac{1}{2\beta} \\
&b_4 = \gamma\Delta t b_1 \quad ; \quad b_5 = 1 + \gamma\Delta t b_2 \quad ; \quad b_6 = \Delta t\left[1 - \gamma + \gamma b_3\right]
\end{aligned} \tag{A.59}$$

The velocity and displacement fields can also be expressed in terms of the parameters (A.59):

$$\begin{cases}
\ddot{\boldsymbol{U}}_{t+\Delta t} = b_1\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + b_2\dot{\boldsymbol{U}}_t + b_3\ddot{\boldsymbol{U}}_t \\
\dot{\boldsymbol{U}}_{t+\Delta t} = b_4\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + b_5\dot{\boldsymbol{U}}_t + b_6\ddot{\boldsymbol{U}}_t
\end{cases} \tag{A.60}$$

Next we will apply the same methodology to solve the following system:

$$\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.61}$$

Given the vector $\boldsymbol{U}_{t+\Delta t}$ and its approach by using Taylor series (A.44), in which we truncate until second order term:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \alpha\Delta t^2\ddot{\boldsymbol{U}}_t \tag{A.62}$$

Considering the following approach for $\ddot{\boldsymbol{U}}_t$:

$$\ddot{\boldsymbol{U}}_t = \frac{\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t}{\Delta t} \tag{A.63}$$

Then, the vector $\boldsymbol{U}_{t+\Delta t}$ can be rewritten as follows:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \alpha\Delta t^2\frac{\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t}{\Delta t} \tag{A.64}$$

And the vector $\dot{\boldsymbol{U}}_{t+\Delta t}$ becomes:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\alpha\Delta t}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) + \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t \tag{A.65}$$

By substituting the equation (A.65) into the equation (A.61) we can obtain:

$$\begin{aligned}
&\boldsymbol{D}\left[\frac{1}{\alpha\Delta t}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) + \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t\right] + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\\
&\Rightarrow\left[\frac{1}{\alpha\Delta t}\boldsymbol{D} + \boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} + \boldsymbol{D}\left[\frac{1}{\alpha\Delta t}\boldsymbol{U}_t - \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t\right]\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\dot{\boldsymbol{U}}_t
\end{aligned} \tag{A.66}$$

And the vector $\dot{\boldsymbol{U}}_t$ can be expressed as follows:

$$\boldsymbol{D}\dot{\boldsymbol{U}}_t + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t \quad \Rightarrow \quad \dot{\boldsymbol{U}}_t = \boldsymbol{D}^{-1}(\boldsymbol{F}_t - \boldsymbol{K}\boldsymbol{U}_t) \tag{A.67}$$

Substituting (A.67) into (A.66) we can obtain:

$$\begin{aligned}
&\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\dot{\boldsymbol{U}}_t\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\boldsymbol{D}^{-1}(\boldsymbol{F}_t - \boldsymbol{K}\boldsymbol{U}_t)\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{F}_t - (1-\alpha)\boldsymbol{K}\boldsymbol{U}_t\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{1}{\Delta t}\boldsymbol{D} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{U}_t
\end{aligned} \tag{A.68}$$

or:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.69}$$

where

$$\boldsymbol{K}^{eff} = \frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K} \quad ; \quad \boldsymbol{F}^{eff} = \alpha\boldsymbol{F}_{t+\Delta t} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{1}{\Delta t}\boldsymbol{D} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{U}_t \tag{A.70}$$

We can verify that the equation (A.69) is the same equation obtained by means of Alfa method employed for unsteady temperature problem, (see equation (A.34)).

A.6.1.1 Newmark's Method Scheme

I. Initial Parameters

I.1. Construction of matrices $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Obtain the parameters:

$$b_1 = \frac{1}{\beta \Delta t^2} \quad ; \quad b_2 = -\frac{1}{\beta \Delta t} \quad ; \quad b_3 = 1 - \frac{1}{2\beta}$$

$$b_4 = \gamma \Delta t b_1 \quad ; \quad b_5 = 1 + \gamma \Delta t b_2 \quad ; \quad b_6 = \Delta t [1 - \gamma + \gamma b_3]$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = [b_1 \boldsymbol{M} + b_4 \boldsymbol{D} + \boldsymbol{K}]$$

I.4. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1} \left[\boldsymbol{F}_0 - \boldsymbol{D} \dot{\boldsymbol{U}}_0 - \boldsymbol{K} \boldsymbol{U}_0 \right]$$

I.5. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \boldsymbol{M} \left[b_1 \boldsymbol{U}_t - b_2 \dot{\boldsymbol{U}}_t - b_3 \ddot{\boldsymbol{U}}_t \right] + \boldsymbol{D} \left[b_4 \boldsymbol{U}_t - b_5 \dot{\boldsymbol{U}}_t - b_6 \ddot{\boldsymbol{U}}_t \right]$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff} \boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = b_1 \left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t \right) + b_2 \dot{\boldsymbol{U}}_t + b_3 \ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = b_4 \left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t \right) + b_5 \dot{\boldsymbol{U}}_t + b_6 \ddot{\boldsymbol{U}}_t \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots \right)$

Go to step II.1 with $t + \Delta t$.

A.6.2 Average Acceleration Method

The average acceleration method is identical to the trapezoidal rule, (see Figure A.8), in which we take the acceleration approach as follows:

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2} \quad ; \quad 0 \le \tau \le \Delta t \tag{A.71}$$

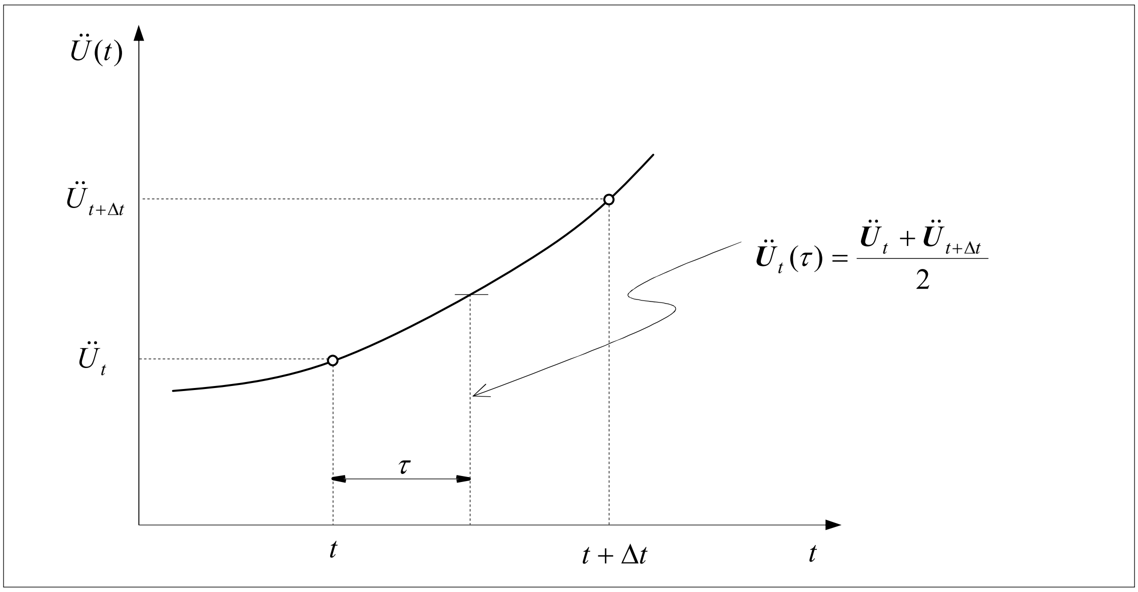

Figure A.8: Average acceleration.

Integrating the equation (A.71) we can obtain:

$$\dot{\boldsymbol{U}}_{t+\tau}(\tau) = \tau\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) + C_1 = \dot{\boldsymbol{U}}_t + \tau\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) \tag{A.72}$$

where the constant of integration C_1 was obtained by means of the initial condition $\dot{\boldsymbol{U}}_{t+\tau}(\tau = 0) = \dot{\boldsymbol{U}}_t \Rightarrow C_1 = \dot{\boldsymbol{U}}_t$. The displacement can be obtain by means of integration of the equation (A.72) over time, then:

$$\boldsymbol{U}_{t+\tau}(\tau) = \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) + C_2 = \boldsymbol{U}_t + \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) \tag{A.73}$$

Once again we use the initial condition to obtain the constant of integration $\boldsymbol{U}_{t+\tau}(\tau = 0) = \boldsymbol{U}_t \Rightarrow C_1 = \boldsymbol{U}_t$.

For $\tau = \Delta t$, the displacement and velocity vectors become:

$$\begin{aligned} \boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{4}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{4}\ddot{\boldsymbol{U}}_{t+\Delta t} \\ \dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_{t+\Delta t} \end{aligned} \tag{A.74}$$

The equations given by (A.74) are the same equations given by the Newmark's Method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, (see equations (A.49) and (A.50)).

By means of the equations (A.74) we can obtain the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$ in terms of $\boldsymbol{U}_{t+\Delta t}$ and $\boldsymbol{U}_t$, $\dot{\boldsymbol{U}}_t$, $\ddot{\boldsymbol{U}}_t$. By means of displacement vector given by (A.74) it is possible to obtain $\ddot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t \tag{A.75}$$

In turn we substitute the equation (A.75) into the velocity given by (A.74), thus:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left[\frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right] \\ &= \frac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t\end{aligned} \tag{A.76}$$

By substituting the velocity vector (A.76) and the acceleration vector (A.75) into the dynamic equilibrium equation at time $t+\Delta t$ we can obtain:

$$\begin{aligned}&\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \\ &\boldsymbol{M}\left\{\frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right\} + \boldsymbol{D}\left\{\frac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t\right\} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\end{aligned} \tag{A.77}$$

By restructuring the above equation we can obtain

$$\left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t \tag{A.78}$$

or

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}_{t+\Delta t} \tag{A.79}$$

where we have considered that:

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K} \\ \boldsymbol{F}^{eff}_{t+\Delta t} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.80}$$

A.6.2.1 Average Acceleration Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t+\Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \dot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t \\ \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{2}{\Delta t}\left[\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t\right] - \ddot{\boldsymbol{U}}_t \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t+\Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t+\Delta t$.

A.6.3 Linear Acceleration Method

For this method we consider a linear variation for the acceleration field within the range $[t, t+\Delta t]$, (see Figure A.9):

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \ddot{\boldsymbol{U}}_t + \frac{\tau}{\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.81}$$

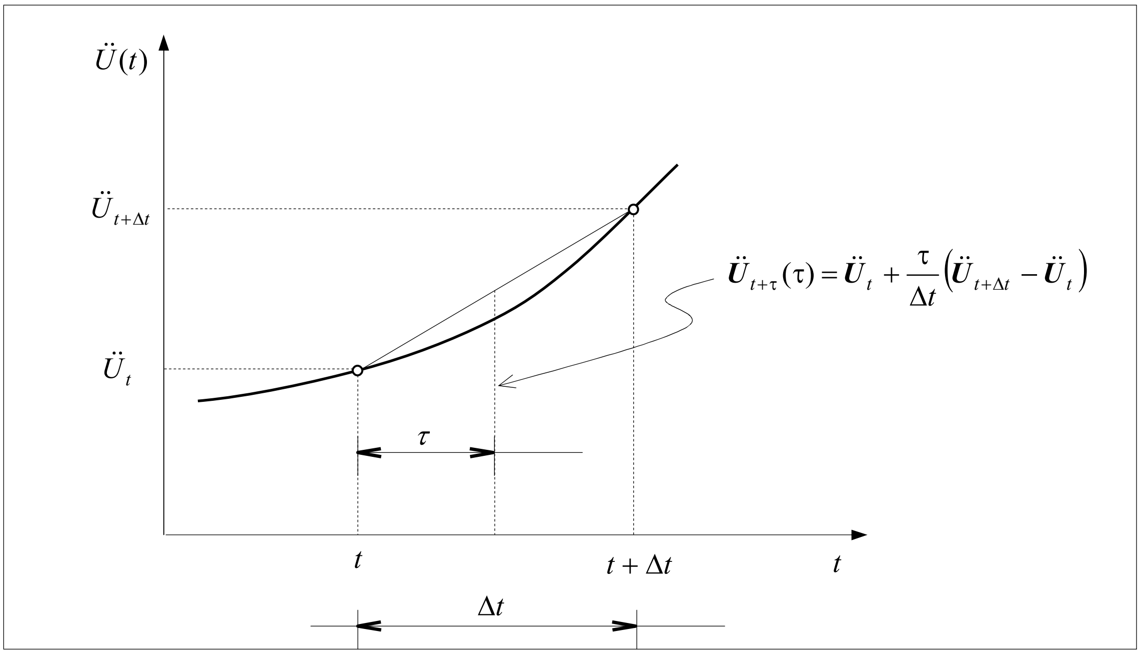

Figure A.9: Linear acceleration method.

The velocity vector can be obtained by integrate over time the equation (A.81), thus:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\tau}(\tau) &= \tau\ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) + C_1 \\ &= \dot{\boldsymbol{U}}_t + \tau\ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.82}$$

where we have apply the initial condition in order to obtain the constant of integration, i.e. at $\tau = 0$ we have that $\dot{\boldsymbol{U}}_{t+\tau}(\tau = 0) = \dot{\boldsymbol{U}}_t \Rightarrow C_1 = \dot{\boldsymbol{U}}_t$.

Then, integrate over time the equation (A.82) we can obtain the displacement vector:

$$\begin{aligned}\boldsymbol{U}_{t+\tau}(\tau) &= \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) + C_2 \qquad \therefore \quad \boldsymbol{U}_{t+\tau}(\tau = 0) = \boldsymbol{U}_t \Rightarrow C_2 = \boldsymbol{U}_t \\ &= \boldsymbol{U}_t + \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.83}$$

When $\tau = \Delta t$ we can obtain:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.84}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) = \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \tag{A.85}$$

By means of the equation (A.84) it is possible to solve for $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{6}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t \tag{A.86}$$

In turn, by substituting (A.86) into (A.85) we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{3}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - 2\dot{\boldsymbol{U}}_t - \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t \tag{A.87}$$

Then, by substituting the velocity vector (A.87) and the acceleration vector (A.86) into the dynamic equilibrium equation we can obtain:

$$\begin{aligned}
&\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\\
&\boldsymbol{M}\left\{\frac{6}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t\right\}\\
&\qquad + \boldsymbol{D}\left\{\frac{3}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - 2\dot{\boldsymbol{U}}_t - \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t\right\} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}
\end{aligned} \tag{A.88}$$

The above equation can be restructured as follows:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}_{t+\Delta t} \tag{A.89}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}\\
\boldsymbol{F}^{eff}_{t+\Delta t} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{6}{\Delta t}\boldsymbol{M} + 2\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.90}$$

The linear acceleration method is the same as Newmark's method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$. This fact can be verified by substituting $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ into the equations in (A.51), with which the displacement and velocity vectors become:

$$\begin{aligned}
\boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \frac{1}{6}\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \frac{1}{6}\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{3}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\ddot{\boldsymbol{U}}_{t+\Delta t}\\
\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \left(1 - \frac{1}{2}\right)\Delta t\ddot{\boldsymbol{U}}_t + \frac{1}{2}\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left[\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}\right]
\end{aligned} \tag{A.91}$$

which match the equations obtained by using the linear acceleration method, (see equations (A.84) and (A.85)).

A.6.3.1 Linear Acceleration Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{6}{\Delta t}\boldsymbol{M} + 2\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{6}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dfrac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \dfrac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

A.6.4 Central Finite Difference Method

This explicit method assumes that the velocity and acceleration vectors are obtained by using th central finite difference to approach the first and second derivatives:

$$\begin{aligned}\dot{\boldsymbol{U}}_t &= \frac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t}\\ \ddot{\boldsymbol{U}}_t &= \frac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2}\end{aligned} \tag{A.92}$$

By substituting the equations in (A.92) into the dynamic equilibrium equation at time t, $\boldsymbol{M}\ddot{\boldsymbol{U}}_t + \boldsymbol{D}\dot{\boldsymbol{U}}_t + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t$, we can obtain:

$$\begin{aligned}&\boldsymbol{M}\frac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2} + \boldsymbol{D}\frac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t} + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t\\ &\Rightarrow \left[\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}\\ &\Rightarrow \boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}\end{aligned} \tag{A.93}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}\\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}\end{aligned} \tag{A.94}$$

At time $t = 0$ we have to calculate $\boldsymbol{U}_{0-\Delta t}$, which value can be obtained by means of the equation (A.92):

$$\begin{aligned}\dot{\boldsymbol{U}}_0 &= \frac{\boldsymbol{U}_{0+\Delta t} - \boldsymbol{U}_{0-\Delta t}}{2\Delta t} \quad &\Rightarrow \quad \boldsymbol{U}_{0+\Delta t} &= 2\Delta t\dot{\boldsymbol{U}}_0 + \boldsymbol{U}_{0-\Delta t}\\ \ddot{\boldsymbol{U}}_0 &= \frac{\boldsymbol{U}_{0+\Delta t} - 2\boldsymbol{U}_0 + \boldsymbol{U}_{0-\Delta t}}{\Delta t^2} \quad &\Rightarrow \quad \boldsymbol{U}_{0+\Delta t} &= \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0-\Delta t}\end{aligned} \tag{A.95}$$

From the two above equations in (A.95) we can obtain:

$$\begin{aligned}&\Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0-\Delta t} = 2\Delta t\dot{\boldsymbol{U}}_0 + \boldsymbol{U}_{0-\Delta t}\\ &\Rightarrow 2\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - 2\Delta t\dot{\boldsymbol{U}}_0\\ &\Rightarrow \boldsymbol{U}_{0-\Delta t} = \boldsymbol{U}_0 - \Delta t\dot{\boldsymbol{U}}_0 + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_0\end{aligned} \tag{A.96}$$

A.6.4.1 Central Finite Difference Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}$$

I.4. Calculate $\boldsymbol{U}_{0-\Delta t}$:

$$\boldsymbol{U}_{0-\Delta t} = \boldsymbol{U}_0 - \Delta t\dot{\boldsymbol{U}}_0 + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_0$$

I.4. Update the variables:

$$\boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_{0-\Delta t} \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad F_t = F_0$$

II. For each time step t do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_t$, $\ddot{\boldsymbol{U}}_t$:

$$\begin{cases} \dot{\boldsymbol{U}}_t = \dfrac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t} \\ \ddot{\boldsymbol{U}}_t = \dfrac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2} \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_t \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_t \leftarrow \boldsymbol{F}\left(t+\Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

The central finite difference is a method with explicit integration and conditionally stable and requires time step Δt less than the critical value:

$$\Delta t \leq \Delta t_{cr} = \frac{T_{\min}}{\pi} = \frac{2}{\omega_{\max}} \tag{A.97}$$

where $T_{\min}$ stands for the smallest natural period, and $\omega_{\max}$ is the maximum frequency of the discrete system, which is greater eigenvalue of the characteristic determinant:

$$\det(\boldsymbol{K} + \omega^2 \boldsymbol{M}) \equiv \left|\boldsymbol{K} + \omega^2 \boldsymbol{M}\right| = \left|\boldsymbol{K} + \lambda \boldsymbol{M}\right| = 0 \tag{A.98}$$

A.6.5 Wilson-θ Method

The Wilson-θ method (1968) is an extension of the linear acceleration method (Newmark's method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$). The acceleration vector is approached within the range $0 \leq \tau \leq \theta \Delta t$ as showed in Figure A.10, where $\theta \geq 1$. When $\theta = 1$ we fall back into the linear acceleration method.

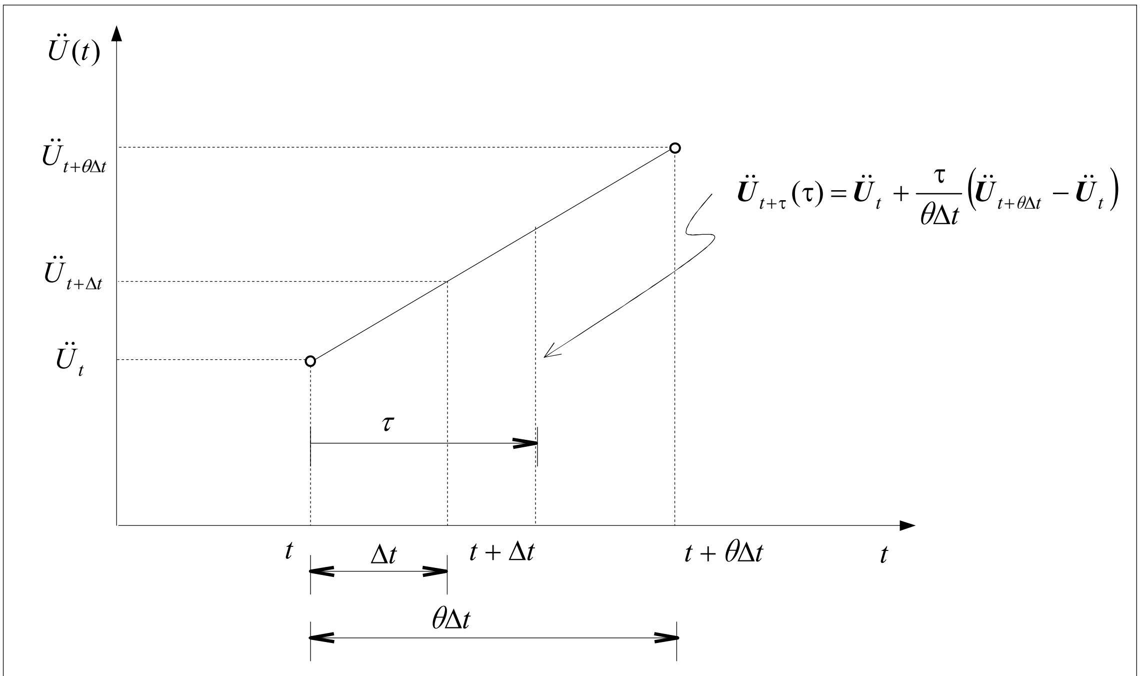

Figure A.10: Acceleration approach – Wilson-θ method.

By means of Figure A.10 the acceleration vector becomes:

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \ddot{\boldsymbol{U}}_t + \frac{\tau}{\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.99}$$

By integrate over time the above equation we can obtain the velocity vector:

$$\dot{\boldsymbol{U}}_{t+\tau}(\tau) = \dot{\boldsymbol{U}}_t + \tau \ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.100}$$

In turn, by integrate the above equation (A.100) we can obtain the displacement vector:

$$\boldsymbol{U}_{t+\tau}(\tau) = \boldsymbol{U}_t + \tau \dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.101}$$

By considering $\tau = \theta \Delta t$ the velocity vector becomes:

$$
\begin{aligned}
\dot{\boldsymbol{U}}_{t+\theta\Delta t} &= \dot{\boldsymbol{U}}_t + \theta\Delta t \ddot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^2}{2\theta\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\
&= \dot{\boldsymbol{U}}_t + \frac{\theta\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + \ddot{\boldsymbol{U}}_t\right)
\end{aligned}
\tag{A.102}
$$

and the displacement vector:

$$
\begin{aligned}
\boldsymbol{U}_{t+\theta\Delta} &= \boldsymbol{U}_t + \theta\Delta t \dot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^2}{2}\ddot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^3}{6\theta\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\
&= \boldsymbol{U}_t + \theta\Delta t \dot{\boldsymbol{U}}_t + \frac{\theta^2\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + 2\ddot{\boldsymbol{U}}_t\right)
\end{aligned}
\tag{A.103}
$$

Then, by solving for $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$:

$$
\ddot{\boldsymbol{U}}_{t+\theta\Delta t} = \frac{6}{\theta^2\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t
\tag{A.104}
$$

By substituting the equation (A.104) into the velocity equation (A.102) we can obtain:

$$
\dot{\boldsymbol{U}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - 2\dot{\boldsymbol{U}}_t - \frac{\theta\Delta t}{2}\ddot{\boldsymbol{U}}_t
\tag{A.105}
$$

Taking into account the dynamic equilibrium equation at time $t+\theta\Delta t$:

$$
\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\theta\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\theta\Delta t} = \hat{\boldsymbol{F}}_{t+\theta\Delta t}
\tag{A.106}
$$

where $\hat{\boldsymbol{F}}_{t+\theta\Delta t} = \theta\boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t$, and by substituting the values for $\dot{\boldsymbol{U}}_{t+\theta\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$ given respectively by the equations (A.105) and (A.104), we can obtain the following set of equations:

$$
\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\theta\Delta t} = \boldsymbol{F}^{eff}
\tag{A.107}
$$

where

$$
\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t} + \boldsymbol{K} \\
\boldsymbol{F}^{eff} &= \theta\boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t + \left(\frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t}\right)\boldsymbol{U}_t + \left(\frac{6\boldsymbol{M}}{\theta\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_t + \left(2\boldsymbol{M} + \frac{\theta\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_t
\end{aligned}
\tag{A.108}
$$

After the system (A.107) is solved, $\boldsymbol{U}_{t+\theta\Delta t}$ is determined and is possible to calculate $\boldsymbol{U}_{t+\Delta t}$, $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$. To do this, we consider the equation (A.99) when $\tau = \Delta t$, thus:

$$
\ddot{\boldsymbol{U}}_{t+\Delta t} = \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right)
\tag{A.109}
$$

By substituting the value of $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$ given by the equation (A.104), the above equation becomes:

$$
\begin{aligned}
\ddot{\boldsymbol{U}}_{t+\Delta t} &= \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\
&= \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left[\frac{6}{\theta^2\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right] \\
&= \frac{6}{\theta^3\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta^2\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \frac{3}{\theta}\right)\ddot{\boldsymbol{U}}_t
\end{aligned}
\tag{A.110}
$$

To obtain $\dot{\boldsymbol{U}}_{t+\Delta t}$ we use the equation in (A.100) by assuming $\tau = \Delta t$, thus:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t \ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.111}$$

Taking into account the equation (A.109), the relationship $\frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) = \ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t$ holds, and substituting into the above equation we can obtain:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \Delta t \ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.112}$$

In order to obtain the displacement $\boldsymbol{U}_{t+\Delta t}$ it is enough to enforce $\tau = \Delta t$ in the equation (A.101), thus:

$$\begin{aligned}\boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + 2\ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.113}$$

A.6.5.1 Wilson-θ Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Calculate $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{6\boldsymbol{M}}{\theta^2 \Delta t^2} + \frac{3\boldsymbol{D}}{\theta \Delta t} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \theta \boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t + \left(\frac{6\boldsymbol{M}}{\theta^2 \Delta t^2} + \frac{3\boldsymbol{D}}{\theta \Delta t}\right)\boldsymbol{U}_t + \left(\frac{6\boldsymbol{M}}{\theta \Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_t + \left(2\boldsymbol{M} + \frac{\theta \Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\theta\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\ddot{\boldsymbol{U}}_{t+\Delta t}$, $\dot{\boldsymbol{U}}_{t+\Delta t}$ y $\boldsymbol{U}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{6}{\theta^3 \Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \dfrac{6}{\theta^2 \Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \dfrac{3}{\theta}\right)\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \dfrac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \\ \boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \dfrac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + 2\ddot{\boldsymbol{U}}_t\right) \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\theta\Delta t} \leftarrow \boldsymbol{F}\left(t + \theta\Delta t, \boldsymbol{U}_{t+\theta\Delta t}, \dot{\boldsymbol{U}}_{t+\theta\Delta t}, \ddot{\boldsymbol{U}}_{t+\theta\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

A.6.6 Houbolt's Method

Houbolt's method (1950) is a mixed method, and approaches the displacement by a cubic function within the range between $t-2\Delta t$ and $t+\Delta t$, according to the function

$$U_{t+\tau}(\tau)=a\tau^3+b\tau^2+c\tau+d \tag{A.114}$$

where the coefficients a, b, c and d are given by:

$$\begin{aligned}
&a=\frac{1}{6\Delta t^3}\left[-U_{t-2\Delta t}+3U_{t-\Delta t}-3U_t+U_{t+\Delta t}\right] \quad ; \quad b=\frac{1}{2\Delta t^2}\left[U_{t-\Delta t}-2U_t+U_{t+\Delta t}\right]\\
&c=\frac{1}{6\Delta t}\left[U_{t-2\Delta t}-6U_{t-\Delta t}+3U_t+2U_{t+\Delta t}\right] \quad ; \quad d=U_t
\end{aligned} \tag{A.115}$$

The first and second derivatives in (A.114) are given by:

$$\dot{U}_{t+\tau}(\tau)=3a\tau^2+2b\tau+c \quad ; \quad \ddot{U}_{t+\tau}(\tau)=6a\tau+2b \tag{A.116}$$

Assuming $\tau=\Delta t$ we can obtain:

$$\dot{U}_{t+\Delta t}=3a\Delta t^2+2b\Delta t+c \quad ; \quad \ddot{U}_{t+\Delta t}=6a\Delta t+2b \tag{A.117}$$

By substituting the values for a, b and c given by (A.115) into the equations in (A.117) we can obtain:

$$\begin{aligned}
\dot{U}_{t+\Delta t}&=\frac{1}{6\Delta t}\left[11U_{t+\Delta t}-18U_t+9U_{t-\Delta t}-2U_{t-2\Delta t}\right]\\
\ddot{U}_{t+\Delta t}&=\frac{1}{\Delta t^2}\left[2U_{t+\Delta t}-5U_t+4U_{t-\Delta t}-U_{t-2\Delta t}\right]
\end{aligned} \tag{A.118}$$

NOTE: This method is unconditionally stable but provides artificial damping (numerical damping) which is very high for a low-frequency response.

By considering the dynamic equation at $t+\Delta t$, i.e. $\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t}+\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t}+\boldsymbol{K}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}_{t+\Delta t}$, and by substituting the values for $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$ given by (A.118), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}^{eff} \tag{A.119}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff}&=\frac{2}{\Delta t^2}\boldsymbol{M}+\frac{11}{6\Delta t}\boldsymbol{D}+\boldsymbol{K}\\
\boldsymbol{F}^{eff}&=\boldsymbol{F}_{t+\Delta t}+\left(\frac{5\boldsymbol{M}}{\Delta t^2}+\frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_t-\left(\frac{4\boldsymbol{M}}{\Delta t^2}+\frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{t-\Delta t}+\left(\frac{\boldsymbol{M}}{\Delta t^2}+\frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{t-2\Delta t}
\end{aligned} \tag{A.120}$$

Similarly to the central finite difference, the Houbolt's method needs a pretreatment at $t=0$ in order to obtain the values for $U_{0-\Delta t}$ and $U_{0-2\Delta t}$. Then, we express the values for $\dot{U}_{t+\tau}(\tau)$ and $\ddot{U}_{t+\tau}(\tau)$, (see equations in (A.116)), at time $\tau=0$, thus

$$\begin{aligned}
\dot{U}_{t+0}(\tau=0)&=c=\frac{1}{6\Delta t}\left[U_{t-2\Delta t}-6U_{t-\Delta t}+3U_t+2U_{t+\Delta t}\right]\\
\ddot{U}_{t+0}(\tau=0)&=2b=\frac{1}{\Delta t^2}\left[U_{t-\Delta t}-2U_t+U_{t+\Delta t}\right]
\end{aligned} \tag{A.121}$$

Thatt is, we apply in the third point of integration, (see Figure A.11).

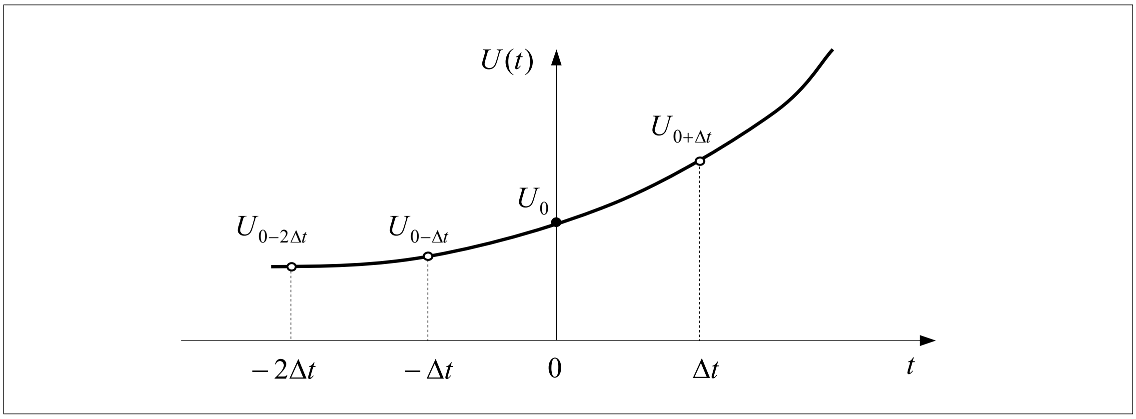

Figure A.11: Houbolt's method parameters at $t=0$.

Assuming the equations (A.121), at time $t=0$, we can obtain:

$$\dot{U}_0 = \frac{1}{6\Delta t}\left[U_{0-2\Delta t} - 6U_{0-\Delta t} + 3U_0 + 2U_{0+\Delta t}\right] \quad ; \quad \ddot{U}_0 = \frac{1}{\Delta t^2}\left[U_{0-\Delta t} - 2U_0 + U_{0+\Delta t}\right] \tag{A.122}$$

The value for $U_{0-\Delta t}$ can be obtained by means of the acceleration equation given by (A.122), i.e.:

$$U_{0-\Delta t} = \Delta t^2 \ddot{U}_0 + 2U_0 - U_{0+\Delta t} \tag{A.123}$$

Then, by substituting the value $U_{0-\Delta t}$ into the velocity equation ($\dot{U}_0$), given by (A.122), it is possible to obtain the value for $U_{0-2\Delta t}$, i.e.:

$$U_{0-2\Delta t} = 6\Delta t \dot{U}_0 + 6\Delta t^2 \ddot{U}_0 - 8U_{0+\Delta t} + 9U_0 \tag{A.124}$$

At time $t=0$ the set of equations (A.119) becomes:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{0+\Delta t} = \boldsymbol{F}^{eff} \tag{A.125}$$

where

$$\begin{aligned} \boldsymbol{K}^{eff} &= \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{D} + \boldsymbol{K} \\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_{0+\Delta t} + \left(\frac{5\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 - \left(\frac{4\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{0-\Delta t} + \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{0-2\Delta t} \end{aligned} \tag{A.126}$$

Note that $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ are also functions of the unknown $\boldsymbol{U}_{0+\Delta t}$. By substituting the values for $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ given by (A.123) and (A.124) into the set of equations (A.125) and after restructuring we can obtain:

$$\hat{\boldsymbol{K}}^{eff}\boldsymbol{U}_{0+\Delta t} = \hat{\boldsymbol{F}}^{eff} \tag{A.127}$$

$$\begin{aligned} \hat{\boldsymbol{K}}^{eff} &= \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K} \\ \hat{\boldsymbol{F}}^{eff} &= \boldsymbol{F}_{0+\Delta t} + \left(\frac{6\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 + \left(\frac{6\boldsymbol{M}}{\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_0 + \left(2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_0 \end{aligned} \tag{A.128}$$

The value for $\boldsymbol{U}_{0+\Delta t}$ is obtained after the system (A.127) is solved, with which the values for $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ can be obtained by means of the equations (A.123) and (A.124), respectively. Note that the equations in (A.128) are the same equations obtained by linear acceleration method, (see equations in (A.90)).

A.6.6.1 Houbolt's Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Calculate the matrices:

$$\hat{\boldsymbol{K}}^{eff} = \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

$$\hat{\boldsymbol{F}}^{eff} = \boldsymbol{F}_{0+\Delta t} + \left(\frac{6\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 + \left(\frac{6\boldsymbol{M}}{\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_0 + \left(2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_0$$

I.4. Solve the system:

$$\hat{\boldsymbol{K}}^{eff}\boldsymbol{U}_{0+\Delta t} = \hat{\boldsymbol{F}}^{eff}$$

I.5. Calculate the vectors $\boldsymbol{U}_{0-\Delta t}$, $\boldsymbol{U}_{0-2\Delta t}$ and $\dot{\boldsymbol{U}}_{0+\Delta t}$, $\ddot{\boldsymbol{U}}_{0+\Delta t}$:

$$\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0+\Delta t} \text{ y } \boldsymbol{U}_{0-2\Delta t} = 6\Delta t\dot{\boldsymbol{U}}_0 + 6\Delta t^2\ddot{\boldsymbol{U}}_0 - 8\boldsymbol{U}_{0+\Delta t} + 9\boldsymbol{U}_0$$

$$\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0+\Delta t} \quad ; \quad \boldsymbol{U}_{0-2\Delta t} = 6\Delta t\dot{\boldsymbol{U}}_0 + 6\Delta t^2\ddot{\boldsymbol{U}}_0 - 8\boldsymbol{U}_{0+\Delta t} + 9\boldsymbol{U}_0$$

$$\dot{\boldsymbol{U}}_{0+\Delta t} = \frac{1}{6\Delta t}\left[11\boldsymbol{U}_{0+\Delta t} - 18\boldsymbol{U}_0 + 9\boldsymbol{U}_{0-\Delta t} - 2\boldsymbol{U}_{0-2\Delta t}\right] \quad ; \quad \ddot{\boldsymbol{U}}_{0+\Delta t} = \frac{1}{\Delta t^2}\left[2\boldsymbol{U}_{0+\Delta t} - 5\boldsymbol{U}_0 + 4\boldsymbol{U}_{0-\Delta t} - \boldsymbol{U}_{0-2\Delta t}\right]$$

I.6. Calculate the effective matrix $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.7. Update the variables:

$$\boldsymbol{U}_{t-2\Delta t} \leftarrow \boldsymbol{U}_{0-\Delta t} \quad ; \quad \boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_0 \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{0+\Delta t}$$

II. For each time step $t+\Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left(\frac{5\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_t - \left(\frac{4\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{t-\Delta t} + \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{t-2\Delta t}$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{6\Delta t}\left[11\boldsymbol{U}_{t+\Delta t} - 18\boldsymbol{U}_t + 9\boldsymbol{U}_{t-\Delta t} - 2\boldsymbol{U}_{t-2\Delta t}\right] \quad ; \quad \ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\Delta t^2}\left[2\boldsymbol{U}_{t+\Delta t} - 5\boldsymbol{U}_t + 4\boldsymbol{U}_{t-\Delta t} - \boldsymbol{U}_{t-2\Delta t}\right]$$

II.4. Update tje variables:

$$\boldsymbol{U}_{t-2\Delta t} \leftarrow \boldsymbol{U}_{t-\Delta t} \quad ; \quad \boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_t \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t+\Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t+\Delta t$.

A.6.7 Hilber-Hughes-Taylor's Method (HHT)

The HHT method adopts the same displacement and velocity approximations used by Newmark's method. The difference between these two methods is how they treat the dynamic equilibrium equation:

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \beta_H\right)\Delta t^2 \ddot{\boldsymbol{U}}_t + \beta_H \Delta t^2 \ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma_H)\Delta t\ddot{\boldsymbol{U}}_t + \gamma_H \Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + (1+\alpha_H)\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} - \alpha_H \boldsymbol{D}\dot{\boldsymbol{U}}_t + (1+\alpha_H)\boldsymbol{K}\boldsymbol{U}_{t+\Delta t} - \alpha_H \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_{t+\Delta t}\end{aligned}} \quad \textit{Hilber-Hughes-Taylor's Method} \quad \text{(A.129)}$$

with $\alpha_H < 0$.

By means of the displacement $\boldsymbol{U}_{t+\Delta t}$, given by the equation (A.129), we can obtain $\ddot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta_H \Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta_H \Delta t}\dot{\boldsymbol{U}}_t - \left(\frac{1}{2\beta_H} - 1\right)\ddot{\boldsymbol{U}}_t \quad \text{(A.130)}$$

By substituting the equation (A.130) ino the velocity equation $\dot{\boldsymbol{U}}_{t+\Delta t}$, given by (A.129), we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{\gamma_H}{\beta_H \Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1 - \frac{\gamma_H}{\beta_H}\right)\dot{\boldsymbol{U}}_t + \left(1 - \frac{\gamma_H}{2\beta_H}\right)\Delta t\ddot{\boldsymbol{U}}_t \quad \text{(A.131)}$$

By substituting (A.130) and (A.131) into the dynamic equilibrium equation (A.129), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \quad \text{(A.132)}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{1}{\beta_H \Delta t^2}\boldsymbol{M} + \frac{(1+\alpha_H)\gamma_H}{\beta_H \Delta t}\boldsymbol{D} + (1+\alpha_H)\boldsymbol{K}\\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{\boldsymbol{M}}{\beta_H \Delta t^2} + \frac{(1+\alpha_H)\gamma_H}{\beta_H \Delta t}\boldsymbol{D} + \alpha_H \boldsymbol{K}\right]\boldsymbol{U}_t +\\ &+ \left[\frac{\boldsymbol{M}}{\beta_H \Delta t} - \left(1 - \frac{\gamma_H(1+\alpha_H)}{\beta_H}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[\left(\frac{1}{2\beta_H} - 1\right)\boldsymbol{M} - (1+\alpha_H)\left(1 - \frac{\gamma_H}{2\beta_H}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t\end{aligned} \quad \text{(A.133)}$$

For th particular case when $\alpha_H = 0$ we fall back into the Newmark's method. Besides, this method has second-order accuracy and is unconditionally stable when:

$$\frac{-1}{3} \le \alpha_H \le 0 \quad ; \quad \gamma_H = \frac{1}{2}(1-2\alpha_H) \quad ; \quad \beta_H = \frac{1}{4}(1-\alpha_H)^2 \quad \text{(A.134)}$$

The smaller α_H greater the numerical damping is. The numerical damping is small for a low frequency response and will be high for high frequency response.

A.6.8 Bossak's Method

The Bossak's method uses the same displacement and velocity approaches used by Newmark's method. And for the Bossak's method the dynamic equilibrium equation is treated as follows:

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+\Delta t}=\boldsymbol{U}_t+\Delta t\dot{\boldsymbol{U}}_t+\left(\frac{1}{2}-\beta_B\right)\Delta t^2\ddot{\boldsymbol{U}}_t+\beta_B\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\\&\dot{\boldsymbol{U}}_{t+\Delta t}=\dot{\boldsymbol{U}}_t+(1-\gamma_B)\Delta t\ddot{\boldsymbol{U}}_t+\gamma_B\Delta t\;\ddot{\boldsymbol{U}}_{t+\Delta t}\\&(1-\alpha_B)\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t}+\alpha_B\boldsymbol{M}\ddot{\boldsymbol{U}}_t+\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t}+\boldsymbol{K}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}_{t+\Delta t}\end{aligned}}\quad \textit{Bossak's method} \tag{A.135}$$

When $\alpha_B=0$ we fall back into the Newmark's method.

The stability condition is met when:

$$\alpha_B\le\frac{1}{2}\quad;\quad\beta_B\ge\frac{\gamma_B}{2}\ge\frac{1}{4}\quad;\quad\alpha_B+\beta_B\ge\frac{1}{2} \tag{A.136}$$

Using the displacement vector $\boldsymbol{U}_{t+\Delta t}$, given by (A.135), we can obtain the acceleration vector $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\ddot{\boldsymbol{U}}_{t+\Delta t}=\frac{1}{\beta_B\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t}-\boldsymbol{U}_t\right)-\frac{1}{\beta_B\Delta t}\dot{\boldsymbol{U}}_t-\left(\frac{1}{2\beta_B}-1\right)\ddot{\boldsymbol{U}}_t \tag{A.137}$$

By substituting the equation (A.137) into the velocity equation $\dot{\boldsymbol{U}}_{t+\Delta t}$, given by (A.135), we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t}=\frac{\gamma_B}{\beta_B\Delta t}\left(\boldsymbol{U}_{t+\Delta t}-\boldsymbol{U}_t\right)+\left(1-\frac{\gamma_B}{\beta_B}\right)\dot{\boldsymbol{U}}_t+\left(1-\frac{\gamma_B}{2\beta_B}\right)\Delta t\ddot{\boldsymbol{U}}_t \tag{A.138}$$

By substituting (A.137) and (A.138) into the dynamic equilibrium equation (A.135), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}^{eff} \tag{A.139}$$

where

$$\begin{aligned}&\boldsymbol{K}^{eff}=\frac{(1-\alpha_B)}{\beta_B\Delta t^2}\boldsymbol{M}+\frac{\gamma_B}{\beta_B\Delta t}\boldsymbol{D}+\boldsymbol{K}\\&\boldsymbol{F}^{eff}=\boldsymbol{F}_{t+\Delta t}+\left[\frac{(1-\alpha_B)}{\beta_B\Delta t^2}\boldsymbol{M}+\frac{\gamma_B}{\beta_B\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t+\\&\qquad+\left[\frac{(1-\alpha_B)}{\beta_B\Delta t}\boldsymbol{M}-\left(1-\frac{\gamma_B}{\beta_B}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t+\left[\left(\frac{1-\alpha_B}{2\beta_B}-1\right)\boldsymbol{M}-\left(1-\frac{\gamma_B}{2\beta_B}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.140}$$

A.6.9 Generalized α Method

The generalized α method was introduced by Chung&Hulbert (1993), and considers the following dynamic equilibrium equation:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t-\alpha_m}+\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t-\alpha_f}+\boldsymbol{K}\boldsymbol{U}_{t+\Delta t-\alpha_f}=\boldsymbol{F}_{t+\Delta t-\alpha_f}(\bar{t})\quad \textit{Generalized } \alpha \textit{ method} \tag{A.141}$$

where

$$\boldsymbol{U}_{t+\Delta t-\alpha_f} = (1-\alpha_f)\boldsymbol{U}_{t+\Delta t} + \alpha_f \boldsymbol{U}_t \tag{A.142}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t-\alpha_f} = (1-\alpha_f)\dot{\boldsymbol{U}}_{t+\Delta t} + \alpha_f \dot{\boldsymbol{U}}_t \tag{A.143}$$

$$\ddot{\boldsymbol{U}}_{t+\Delta t-\alpha_m} = (1-\alpha_m)\ddot{\boldsymbol{U}}_{t+\Delta t} + \alpha_m \ddot{\boldsymbol{U}}_t \tag{A.144}$$

$$\bar{t} = (1-\alpha_f)(t+\Delta t) + \alpha_f t \tag{A.145}$$

with

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2}-\beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} \tag{A.146}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ddot{\boldsymbol{U}}_{t+\Delta t} \tag{A.147}$$

Note that the displacement and velocity vectors are the same used for Bossak and HHT methods. Similarly for these methods we can express $\ddot{\boldsymbol{U}}_{t+\Delta t}$ and $\dot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t - \left(\frac{1}{2\beta}-1\right)\ddot{\boldsymbol{U}}_t \tag{A.148}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{\gamma}{\beta\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1-\frac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1-\frac{\gamma}{2\beta}\right)\Delta t\ddot{\boldsymbol{U}}_t \tag{A.149}$$

By substituting (A.142), (A.143), (A.144), and (A.148), (A.149) into the equation (A.141) we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.150}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{(1-\alpha_m)}{\beta\Delta t^2}\boldsymbol{M} + \frac{(1-\alpha_f)\gamma}{\beta\Delta t}\boldsymbol{D} + (1-\alpha_f)\boldsymbol{K} \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\alpha_f} + \left[\frac{(1-\alpha_m)}{\beta\Delta t^2}\boldsymbol{M} + \frac{(1-\alpha_f)\gamma}{\beta\Delta t}\boldsymbol{D} - \alpha_f\boldsymbol{K}\right]\boldsymbol{U}_t + \\
&\quad + \left[\frac{(1-\alpha_m)}{\beta\Delta t}\boldsymbol{M} - \left(1-\frac{\gamma(1-\alpha_f)}{\beta}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[\left(\frac{(1-\alpha_m)}{2\beta}-1\right)\boldsymbol{M} - (1-\alpha_f)\left(1-\frac{\gamma}{2\beta}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.151}$$

We can verify that when $\alpha_f = \alpha_m = 0$ we fall back into the equations obtained for Newmark's method, (see equations in (A.57)). When $\alpha_f = 0$ and $\alpha_m = \alpha_B$ we fall back into the Bossak's method, (see equations in (A.140)). When $\alpha_m = 0$ we fall back into the HHT method.

A.6.10 Park-Housner's Method

The Park-Housner's method is a semi-implicit method. The mass matrix ($\boldsymbol{M}$) is a diagonal matrix and the stiffness matrix $\boldsymbol{K}$ is split into triangular matrices, i.e. $\boldsymbol{K} = \boldsymbol{K}_L + \boldsymbol{K}_U$, in which $\boldsymbol{K}_L = \boldsymbol{K}_U^T$. The scheme for this method is presented next:

Construction of $\boldsymbol{K}$ and the diagonal mass matrix $\boldsymbol{M}$;

Construction of the matrices $\boldsymbol{K} = \boldsymbol{K}_L + \boldsymbol{K}_U$

Obtain the matrices:

$$\boldsymbol{L} = \boldsymbol{M}(\mathbf{1} + \alpha\beta\Delta t^2 \boldsymbol{M}^{-1}\boldsymbol{K}_L) \quad ; \quad \boldsymbol{Q} = \mathbf{1} + \alpha\beta\Delta t^2 \boldsymbol{M}^{-1}\boldsymbol{K}_U$$

$$\boldsymbol{g}_{t+\Delta t} = \alpha\beta\Delta t^2 \left[\beta \boldsymbol{f}_{t+\Delta t} + (1-\beta)\boldsymbol{f}_t\right] + \boldsymbol{M}(\boldsymbol{U}_t + \beta\Delta t \dot{\boldsymbol{U}}_t)$$

Solve the system:

$$\boldsymbol{L}\boldsymbol{y}_{t+\Delta t} = \boldsymbol{g}_{t+\Delta t} \quad ; \quad \boldsymbol{Q}\boldsymbol{U}^*_{t+\Delta t} = \boldsymbol{y}_{t+\Delta t}$$

Update the variables

$$\boldsymbol{U}_{t+\Delta t} \leftarrow \frac{1}{\beta}\left[\boldsymbol{U}^*_{t+\Delta t} - (1-\beta)\boldsymbol{U}_t\right]$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} \leftarrow \frac{}{\alpha\Delta t}\left[\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right] - \frac{(1-\alpha)}{\alpha}\dot{\boldsymbol{U}}_t$$

A.6.11 Trujillo's Method

Trujillo (1977) presented a semi-implicit method, and applied to solve a linear structural dynamic problem. The Trujillo's method separates the stiffness and damping matrices into two upper and lower matrices:

$$\boldsymbol{K} = \boldsymbol{K}_L + \boldsymbol{K}_U \quad ; \quad \boldsymbol{D} = \boldsymbol{D}_L + \boldsymbol{D}_U \tag{A.152}$$

The mass matrix ($\boldsymbol{M}$) is a diagonal matrix.

Trujillo's Method Scheme:

Backward substitution

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+\frac{1}{2}} = \boldsymbol{K}^{-1}_{(1)}\left\{\left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_L - \frac{\Delta t^2}{8}\boldsymbol{K}_U\right]\boldsymbol{U}_t + \frac{\Delta t}{2}\left[\boldsymbol{M} + \frac{\Delta t}{4}\left(\boldsymbol{D}_L - \boldsymbol{D}_U\right)\right]\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{16}\left[\boldsymbol{F}_{t+\Delta t} + \boldsymbol{F}_t\right]\right\}\\ &\textit{with}\\ &\boldsymbol{K}_{(1)} = \left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_L + \frac{\Delta t^2}{8}\boldsymbol{K}_L\right]\end{aligned}}$$

$$\boxed{\dot{\boldsymbol{U}}_{t+\frac{1}{2}} = \frac{4}{\Delta t}\left[\boldsymbol{U}_{t+\frac{1}{2}} - \boldsymbol{U}_t\right] - \dot{\boldsymbol{U}}_t}$$

Forward Substitution

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+1} = \boldsymbol{K}^{-1}_{(2)}\left\{\left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_U - \frac{\Delta t^2}{8}\boldsymbol{K}_L\right]\boldsymbol{U}_{t+\frac{1}{2}} + \frac{\Delta t}{2}\left[\boldsymbol{M} + \frac{\Delta t}{4}\left(\boldsymbol{D}_U - \boldsymbol{D}_L\right)\right]\dot{\boldsymbol{U}}_{t+\frac{1}{2}} + \frac{\Delta t^2}{16}\left[\boldsymbol{F}_{t+\Delta t} + \boldsymbol{F}_t\right]\right\}\\ &\textit{with}\\ &\boldsymbol{K}_{(2)} = \left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_U + \frac{\Delta t^2}{8}\boldsymbol{K}_U\right]\end{aligned}}$$

$$\boxed{\dot{\boldsymbol{U}}_{t+1} = \frac{4}{\Delta t}\left[\boldsymbol{U}_{t+1} - \boldsymbol{U}_{t+\frac{1}{2}}\right] - \dot{\boldsymbol{U}}_{t+\frac{1}{2}}}$$

A.7 Examples

Let us consider the mechanical model with on degree-of-freedom, (see Figure A.12). This mechanical model is made up by a mass body m which is connected by two devices, namely: spring (*Structural*), which is characterized by spring constant k; and by a dashpot with viscosity d (*Damping*), which is responsible for the system energy dissipation. The syste is conservative if $d = 0$.

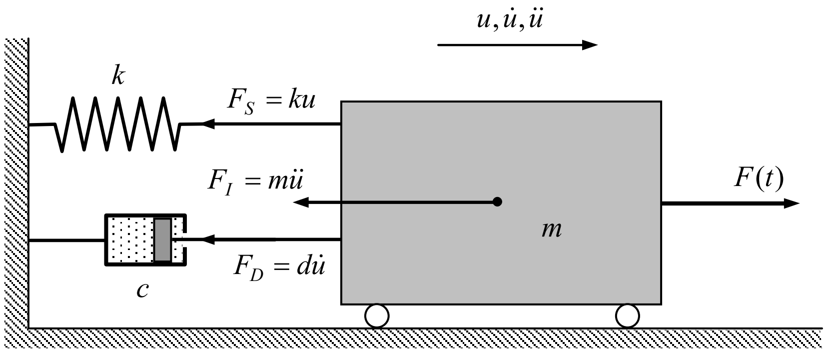

Figure A.12: Mechanical model.

The mechanical problem proposed here has three forces, namely: The inertial force $F_I = m\ddot{u}$, where $\ddot{u} \equiv a$ is the acceleration; F_D is the damping force; and F_S is the spring force associated with the structural stiffness.

The governing equation for this problem, (see Figure A.12), is obtained by force equilibrium:

$$F_I + F_D + F_S = F(t) \tag{A.153}$$

or:

$$\begin{aligned} m\ddot{u} + d\dot{u} + ku &= F(t) \\ m\frac{d^2u}{dt^2} + d\frac{du}{dt} + ku &= F(t) \end{aligned} \tag{A.154}$$

If the body is free of external forces $F(t) = 0$, the governing equation is called *free vibration*.

The equation in (A.154) can also be expressed as follows:

$$\begin{aligned} \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u &= \frac{F(t)}{m} \\ \ddot{u} + \frac{d}{m}\dot{u} + \omega^2 u &= \frac{F(t)}{m} \end{aligned} \tag{A.155}$$

where we have defined the parameter:

$$\omega = \sqrt{\frac{k}{m}} \qquad [\omega] \equiv rad/s \quad \textit{Natural circular frequency} \tag{A.156}$$

A.7.1 Oscillatory Motion

The mechanical model that represents an oscillatory motion is made up by a body mass m connected to the spring with constant k, (see Figure A.13).

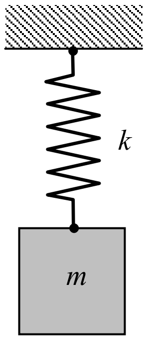

Figure A.13: Mechanical model for an oscillatory problem.

Oscillatory motion is a conservative system, since there is no energy dissipation. The governing equation is represented mathematically by:

$$m\ddot{u} + ku = F = 0 \tag{A.157}$$

with $F = 0$ (free vibration).

Consider as example, $m = 26$, $k = 21000$, and $F = 0$. As boundary and initial conditions we have:

$$u(t=0) \equiv u_0 = 2 \quad ; \quad \dot{u}(t=0) \equiv \dot{u}_0 = -3 \tag{A.158}$$

The exact solution is given by the following harmonic function:

$$u(t) = \frac{\dot{u}_0}{\omega}\sin(\omega t) + u_0 \cos(\omega t) \qquad where \qquad \omega = \sqrt{\frac{k}{m}} \tag{A.159}$$

We also define the following parameters for the model:

Natural frequency:

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \approx 2.52\,Hz \quad \textit{Natural frequency} \tag{A.160}$$

Natural period:

$$T = \frac{1}{f} \approx 0.22108\,sec \quad \textit{Natural period} \tag{A.161}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

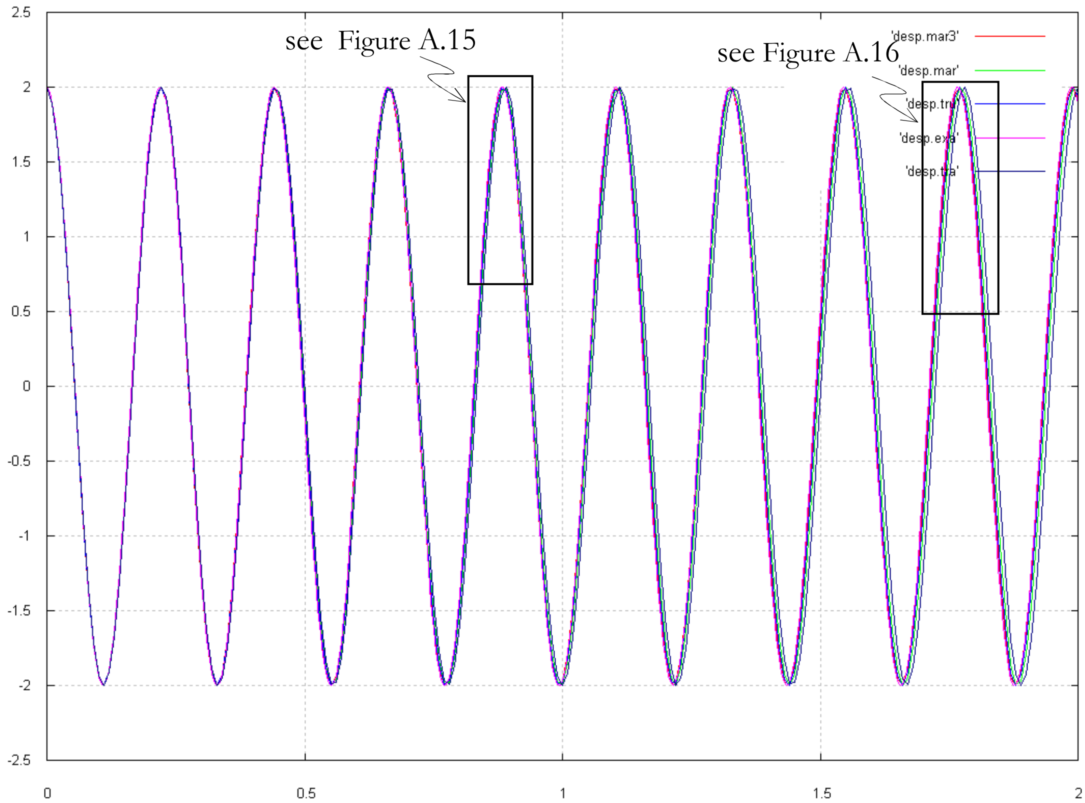

Figure A.14: Displacement vs. time curve ($\Delta t = 0.01$).

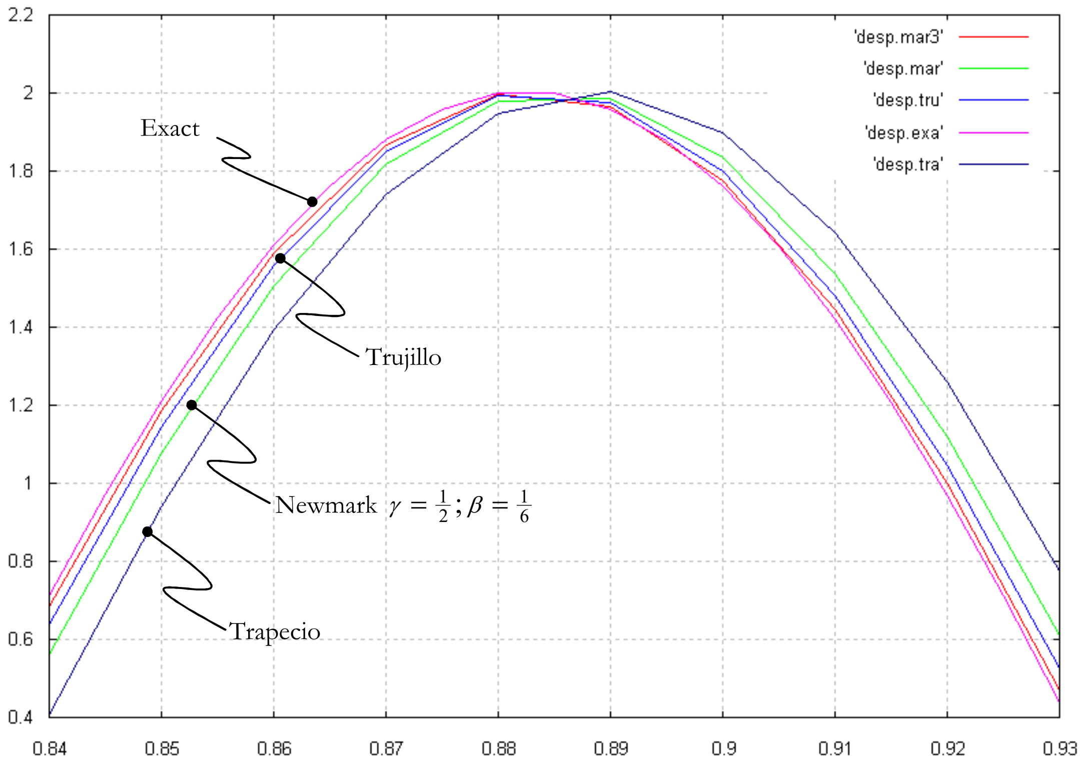

Figure A.15: Displacement vs. time curve[0.84:0.93], ($\Delta t = 0.01$).

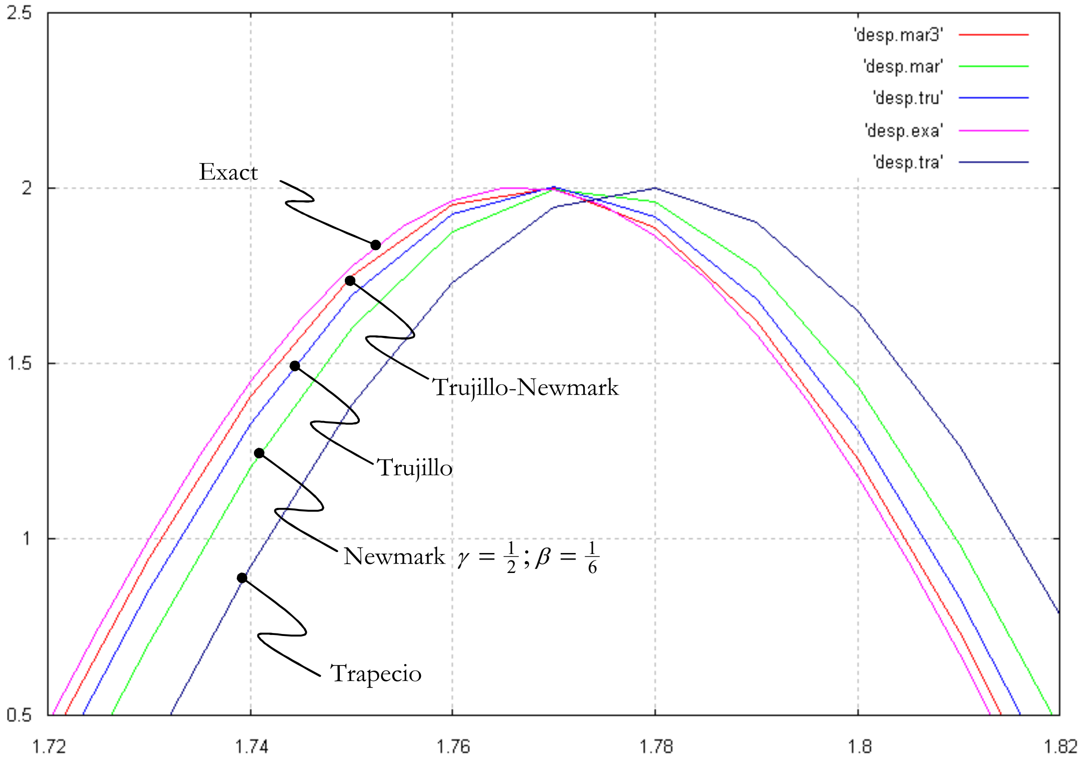

Figure A.16: Displacement vs. time curve [1.72:1.82], ($\Delta t = 0.01$).

A.7.2 Free Vibration with Damping

Let us consider a mechanical model, (see Figure A.17), which represents the free vibration problem ($F(t) = 0$) with damping ($d \neq 0$).

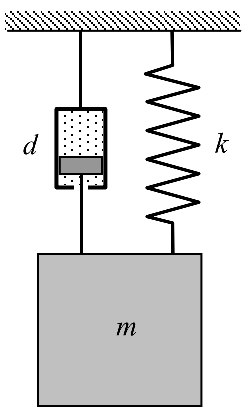

Figure A.17: Mechanical model for free vibration with damping.

As previously seen the governing equation is given by:

$$m\ddot{u} + d\dot{u} + ku = 0 \quad \Rightarrow \quad \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u = 0 \tag{A.162}$$

By assuming that $u(t) = C_1 \mathrm{e}^{st}$, the above equation becomes:

$$s^2+\frac{d}{m}s+\frac{k}{m}=0 \tag{A.163}$$

whose solutions are given by:

$$s_{1,2}=-\frac{d}{2m}\pm\sqrt{\left(\frac{d}{2m}\right)^2-\frac{k}{m}} \tag{A.164}$$

We could have three possibilities, namely: radicand equals zero (two identical solutions); radicand greater than zero (two different solutions); radicand less than zero (two complex solutions).

- *Radicand equals zero (Critical damping)*

In this case we have:

$$\left(\frac{d}{2m}\right)^2-\frac{k}{m}=0 \quad \Rightarrow \quad \frac{d}{2m}=\sqrt{\frac{k}{m}}=\omega \quad \Rightarrow \quad d=2m\omega \tag{A.165}$$

In this situation the damping coefficient is called *critical damping coefficient*:

$$\boxed{d_c=2m\omega} \quad \textit{critical damping coefficient} \tag{A.166}$$

In general, we define a parameter called damping factor (ζ) which is used to indicate whether the system is *underdamping* ($\zeta<1$, subcritical damping) or *overdamping* ($\zeta>1$, *supercritical damping*). This damping factor is defined by:

$$\boxed{\zeta=\frac{d}{d_c}=\frac{d}{2m\omega}} \quad \textit{Damping factor} \tag{A.167}$$

Note that the equation in (A.163) can be rewritten as follows:

$$s^2+2\omega\zeta s+\omega^2=0 \tag{A.168}$$

whose solutions are given by:

$$s_{1,2}=\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega \tag{A.169}$$

For the critical damping case, i.e $\zeta=1$, we have that $s_{1,2}=-\zeta\omega$.

The exact solution for the differential equation (A.162) is given by:

$$u(t)=e^{-\omega t}\left\{\left[\dot{u}_0+\omega u_0\right]t+u_0\right\} \quad \textit{with} \quad (\dot{u}_0\neq 0; u_0\neq 0) \tag{A.170}$$

- *Overcritical damping* $\zeta>1$

In this situation we have two different solutions (and real numbers) given by(A.169). And the solution for the differential equation (A.162) becomes:

$$u(t)=A\mathrm{e}^{-s_1t}+B\mathrm{e}^{-s_2t} \tag{A.171}$$

where

$$A=\frac{\dot{u}_0+\left(\zeta+\sqrt{\zeta^2-1}\right)\omega u_0}{2\omega\sqrt{\zeta^2-1}} \quad ; \quad B=\frac{-\dot{u}_0-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega u_0}{2\omega\sqrt{\zeta^2-1}} \tag{A.172}$$

- *Subcritical damping* $\zeta < 1$

In this case the solution for the equation (A.169) is given by:

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega \tag{A.173}$$

And the solution for the differential equation (A.162) becomes:

$$u(t) = \mathrm{e}^{-\zeta\omega t}\left[A\sin\left(\omega t\sqrt{1-\zeta^2}\right) + B\cos\left(\omega t\sqrt{1-\zeta^2}\right)\right] \tag{A.174}$$

or:

$$u(t) = \mathrm{e}^{-\zeta\omega t}\left[\frac{\dot{u}_0 + \zeta\omega u_0}{\omega_d}\sin(\omega_d t) + u_0\cos(\omega_d t)\right] \tag{A.175}$$

where $\omega_d = \omega\sqrt{1-\zeta^2}$

A.7.2.1 Free Vibration with Damping Example

As example, consider that $m = 0.0052$, $d = 0.1$, $k = 12$, and boundary and initial conditions:

$$u_0 = 1.5 \quad ; \quad \dot{u}_0 = 0 \tag{A.176}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.017$.

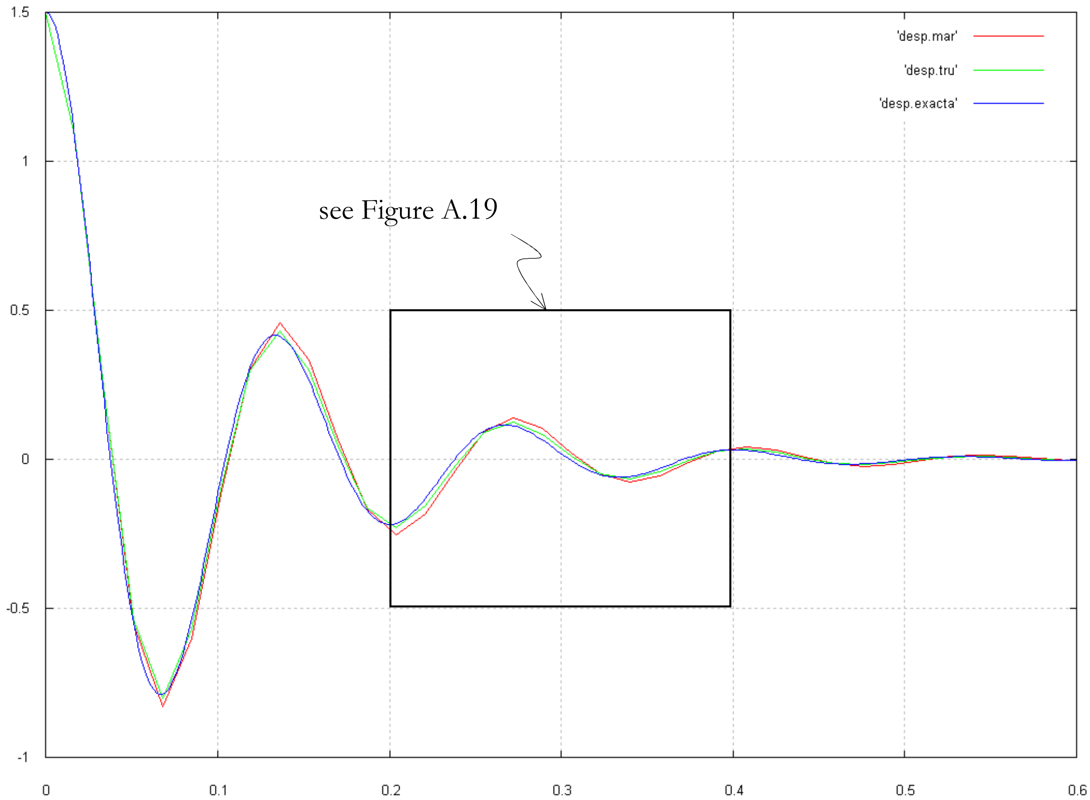

Figure A.18: Displacement vs. time curve, ($\Delta t = 0.017$).

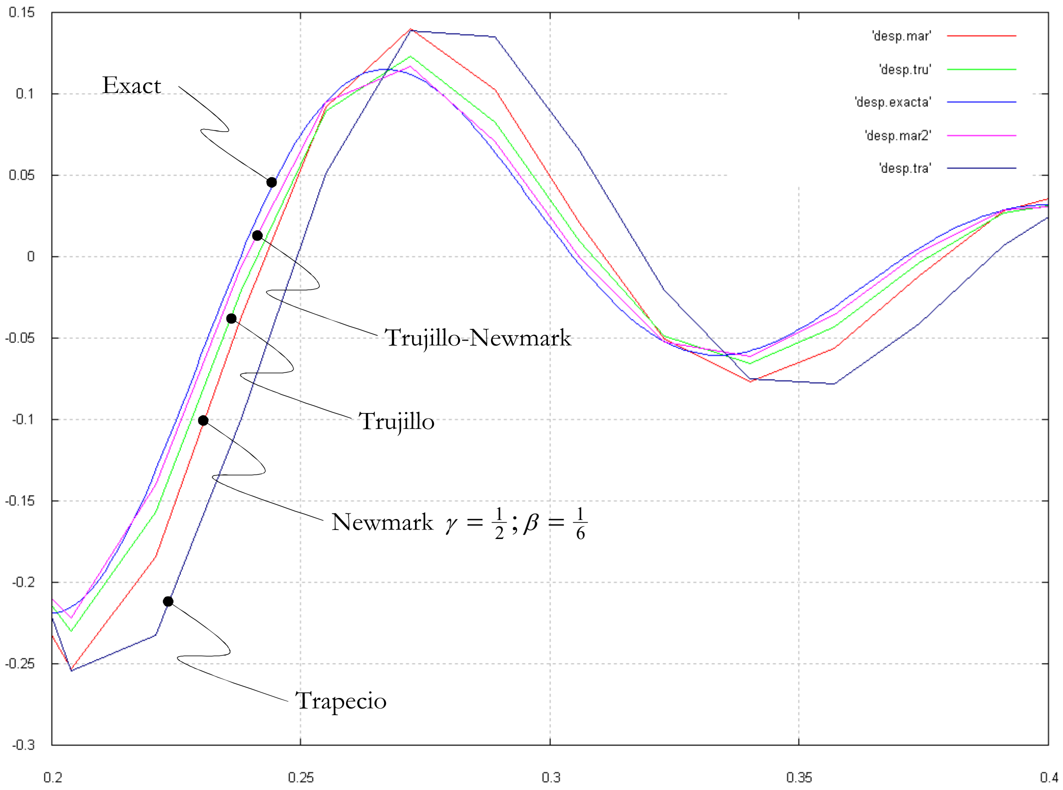

Figure A.19: Displacement vs. time curve [0.2:0.4], ($\Delta t = 0.017$).

A.7.3 Miscellaneous Examples

A.7.3.1 Pendulum

Let us consider the pendulum problem, (see Figure A.20).

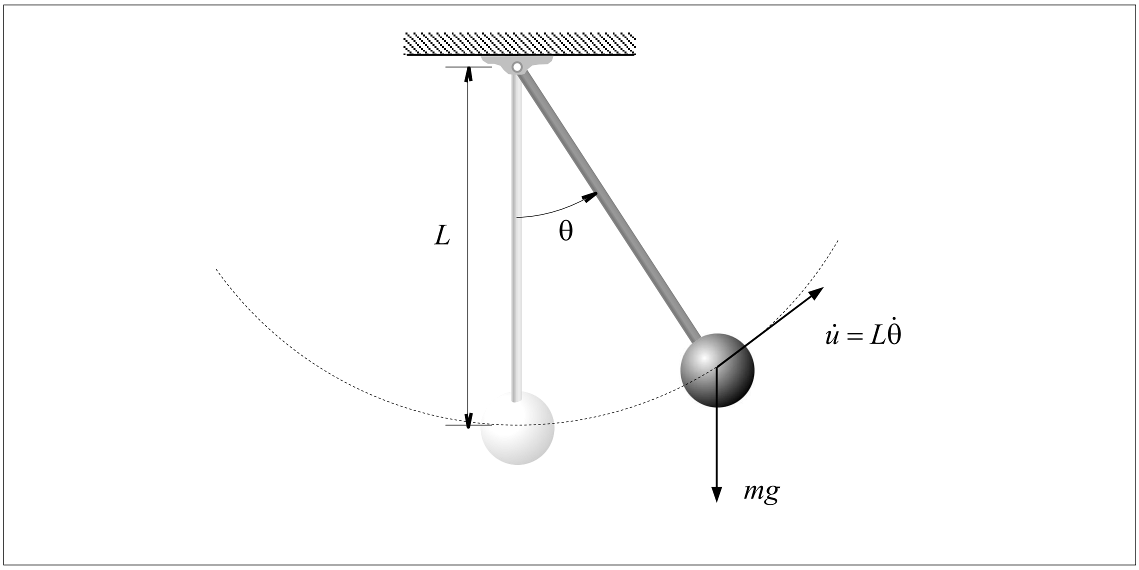

Figure A.20: Pendulum.

The governing equation for this problem is given by:

$$mL^2\ddot{\theta} + mgL\sin(\theta) + dL\dot{\theta} = 0 \tag{A.177}$$

The above equation can be rewritten as follows:

$$m\ddot{\theta} + \frac{d}{L}\dot{\theta} = -\frac{mg}{L}\sin(\theta) = F(t,\theta) \tag{A.178}$$

We consider the parameter values: $L = 10.0$, $m = 1.0$, $d = 2.0$, $g = 10$, and boundary and initial conditions:

$$\theta(t=0) = 0 \quad ; \quad \dot{\theta}(t=0) = 3 \tag{A.179}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.05$.

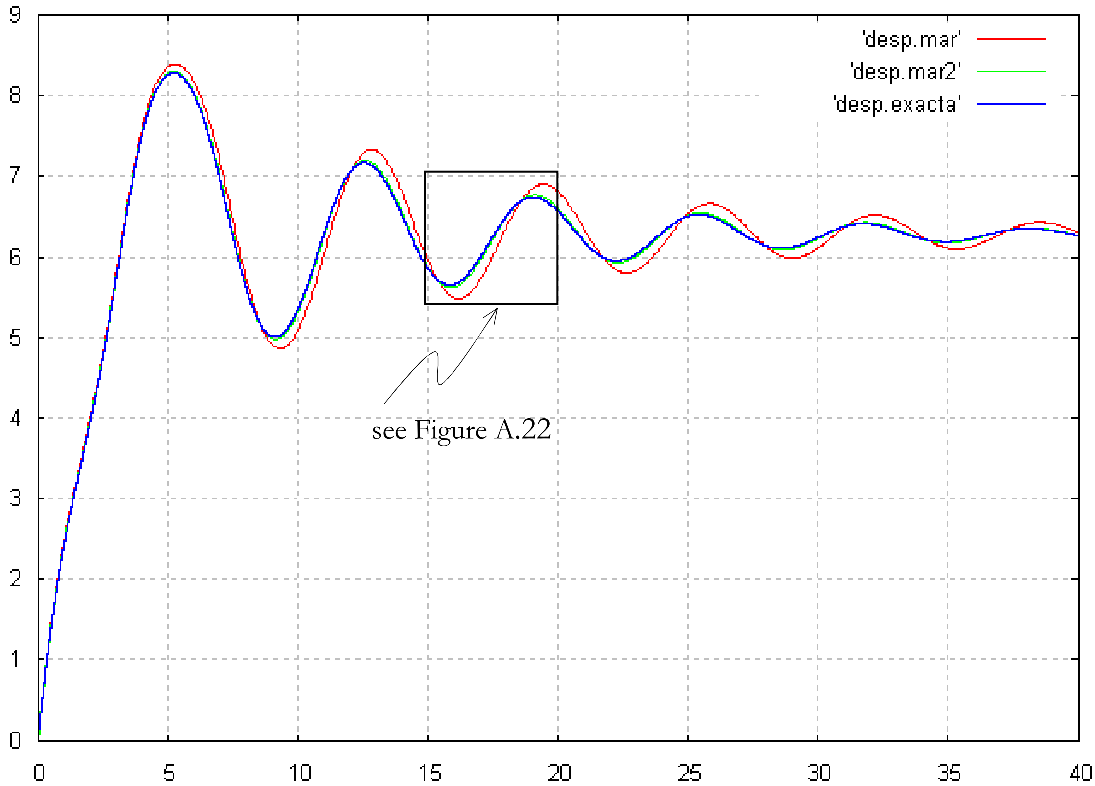

Figure A.21: Displacement vs. time curve, ($\Delta t = 0.05$).

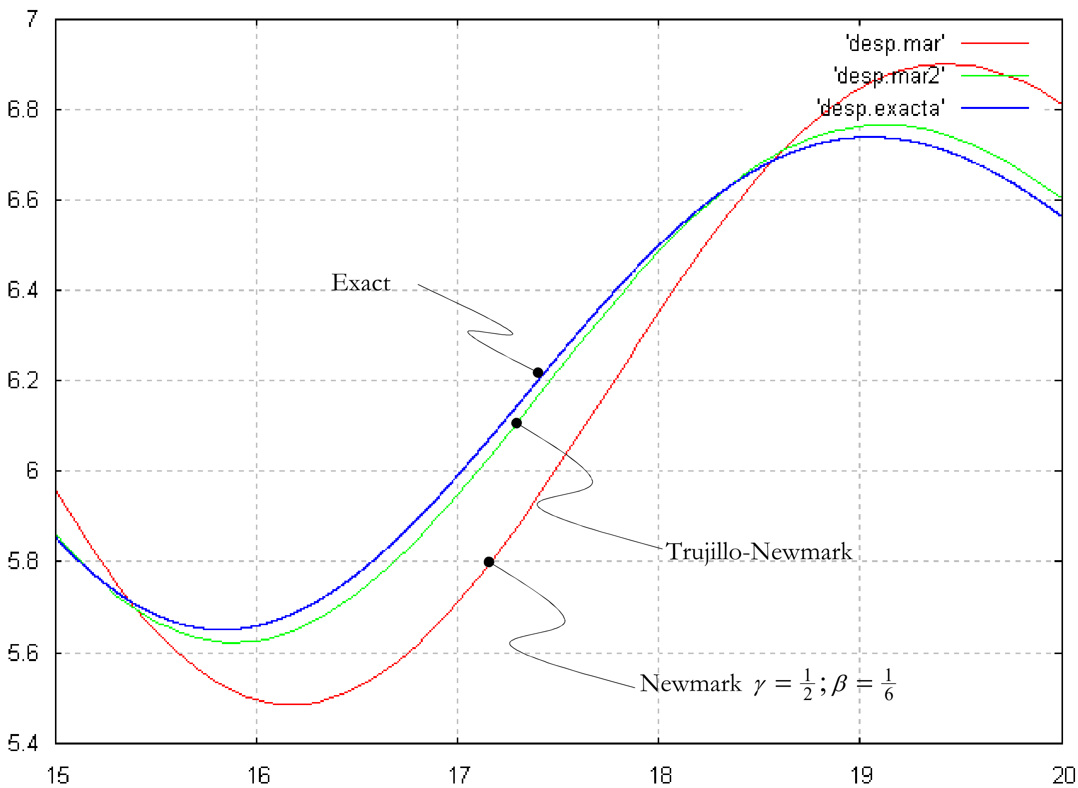

Figure A.22: Displacement vs. time curve [15:20], ($\Delta t = 0.05$).

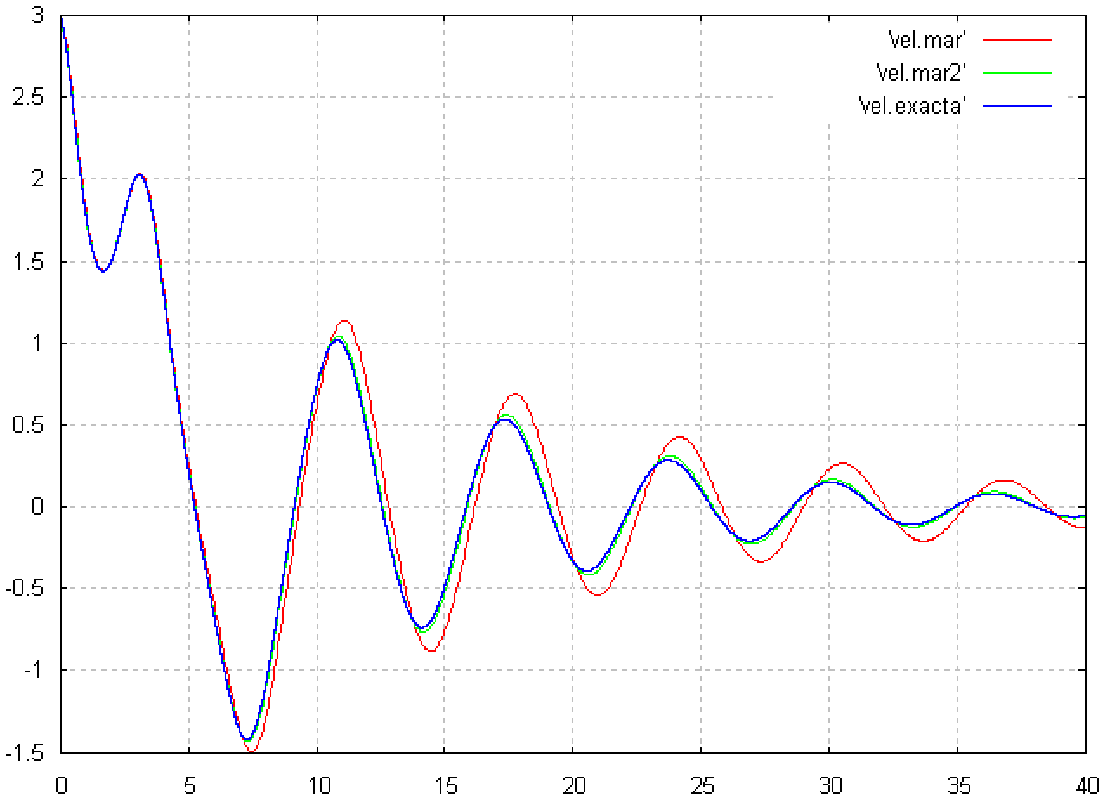

Figure A.23: Velocity vs. time curve, $(\Delta t = 0.05)$.

A.7.3.2 Dynamics of the van der Pol Equation

Let us consider the differential equation:

$$\ddot{\nu} + \alpha(\nu^2 - 1)\dot{\nu} + \omega^2 \nu = 0 \tag{A.180}$$

which is known as van der Pol oscillator. The above equation can be rewritten as follows:

$$\ddot{\nu} - \alpha\dot{\nu} + \omega^2 \nu = -\alpha\nu^2\dot{\nu} = F(t, \nu, \dot{\nu}) \tag{A.181}$$

By considering the dynamic equilibrium equation:

$$m\ddot{u} + d\dot{u} + ku = F \quad \Rightarrow \quad \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u = \frac{F}{m} \tag{A.182}$$

and by comparing the equations (A.181) and (A.182) we can conclude that: $\frac{d}{m} = -\alpha$, $\frac{k}{m} = \omega^2$, $\frac{F(t,\nu,\dot{\nu})}{m} = -\alpha\nu^2\dot{\nu}$. Consider as example the values: $m = 1$, $d = -\alpha = -8$, $k = \omega^2 = 0.25$, and boundary and initial conditions:

$$\nu(t=0) \equiv \nu_0 = 1 \quad ; \quad \dot{\nu}(t=0) \equiv \dot{\nu}_0 = 3 \tag{A.183}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

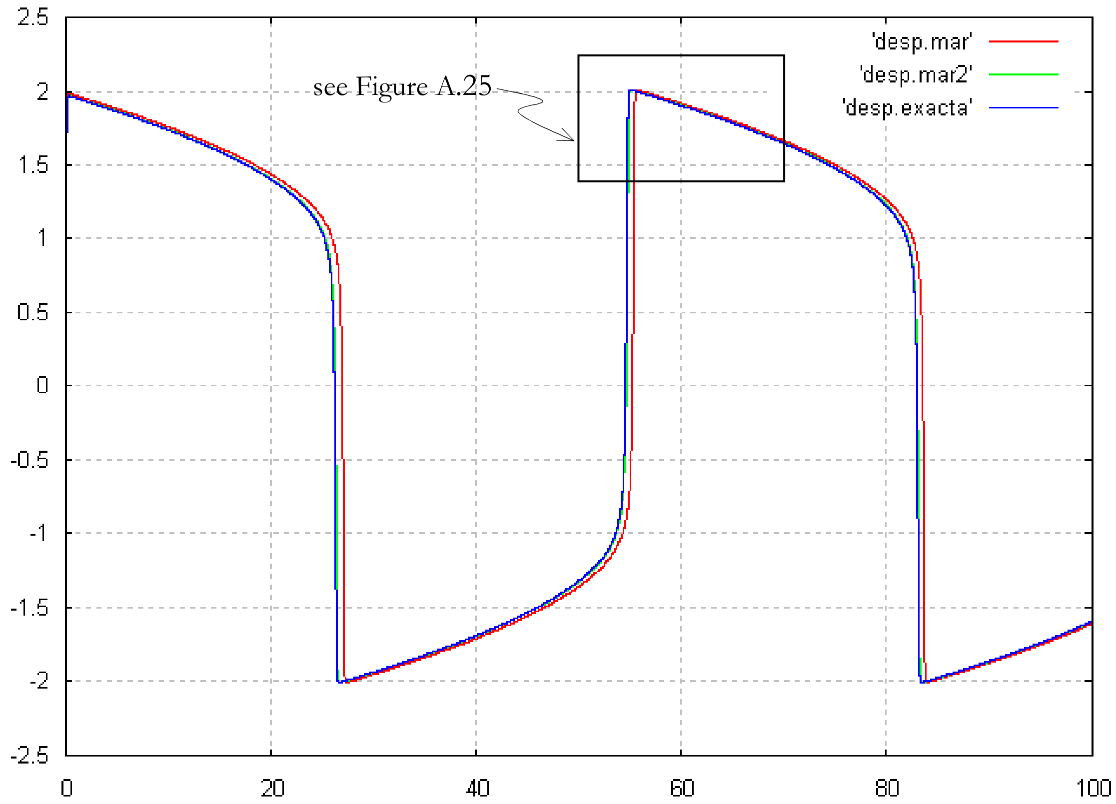

Figure A.24: "Displacement" vs. time curve, ($\Delta t = 0.01$).

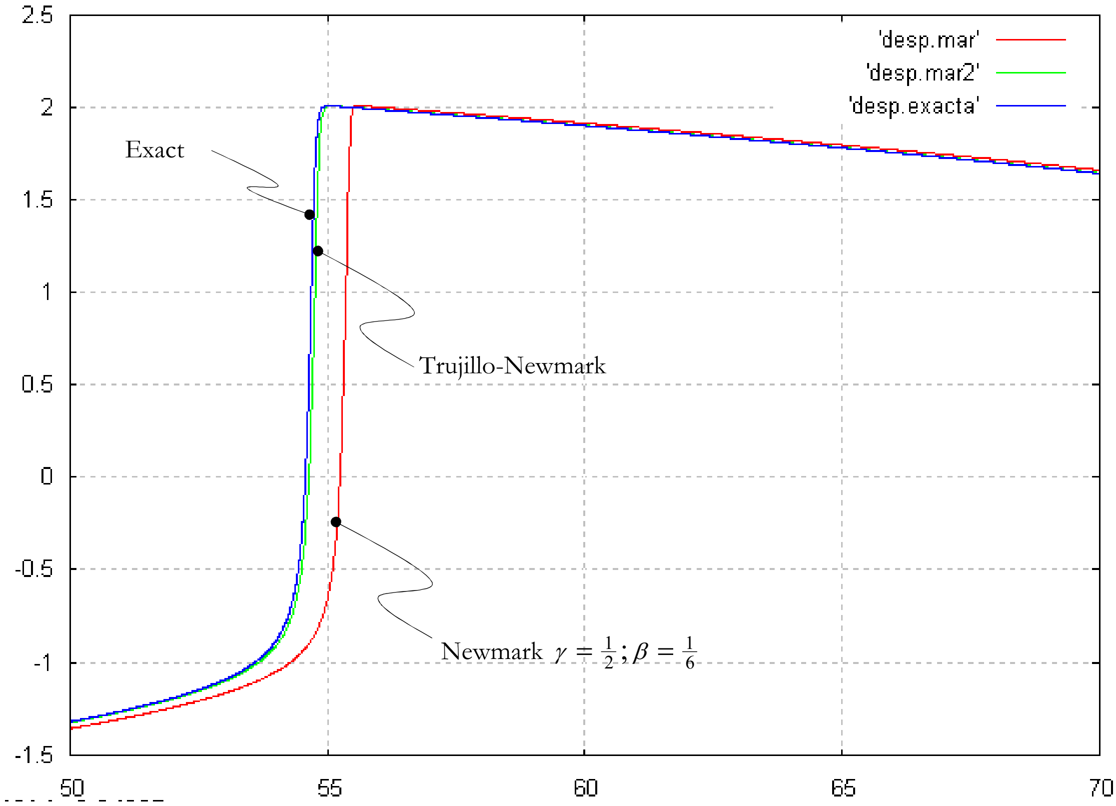

Figure A.25: "Displacement" vs. time curve [50:70], ($\Delta t = 0.01$).

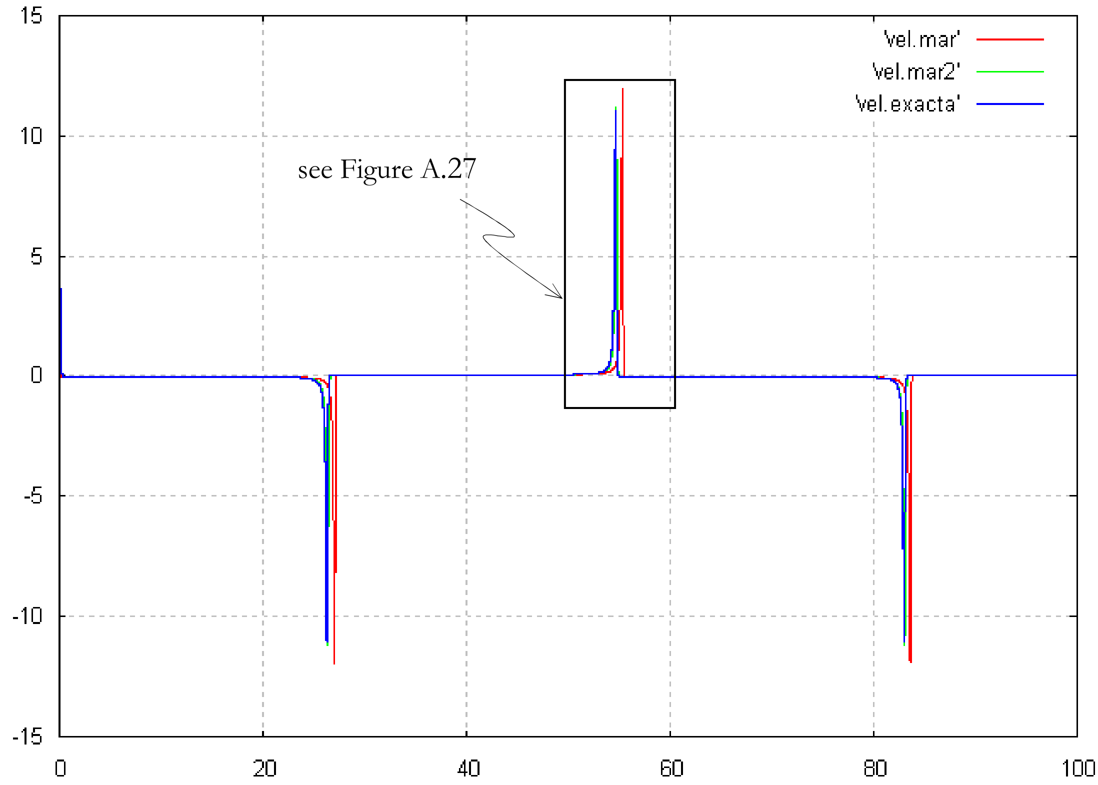

Figure A.26: “Velocity” vs. time curve, ($\Delta t = 0.01$).

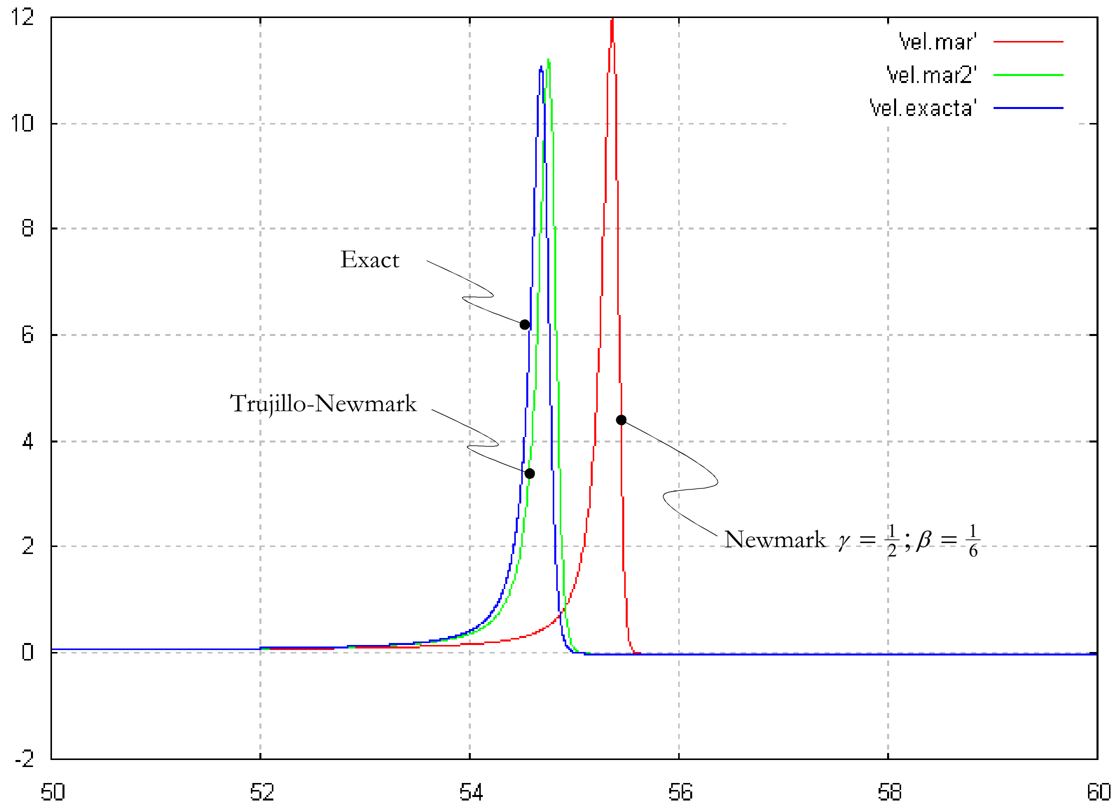

Figure A.27: “Velocity” vs. time [50:60], ($\Delta t = 0.01$).

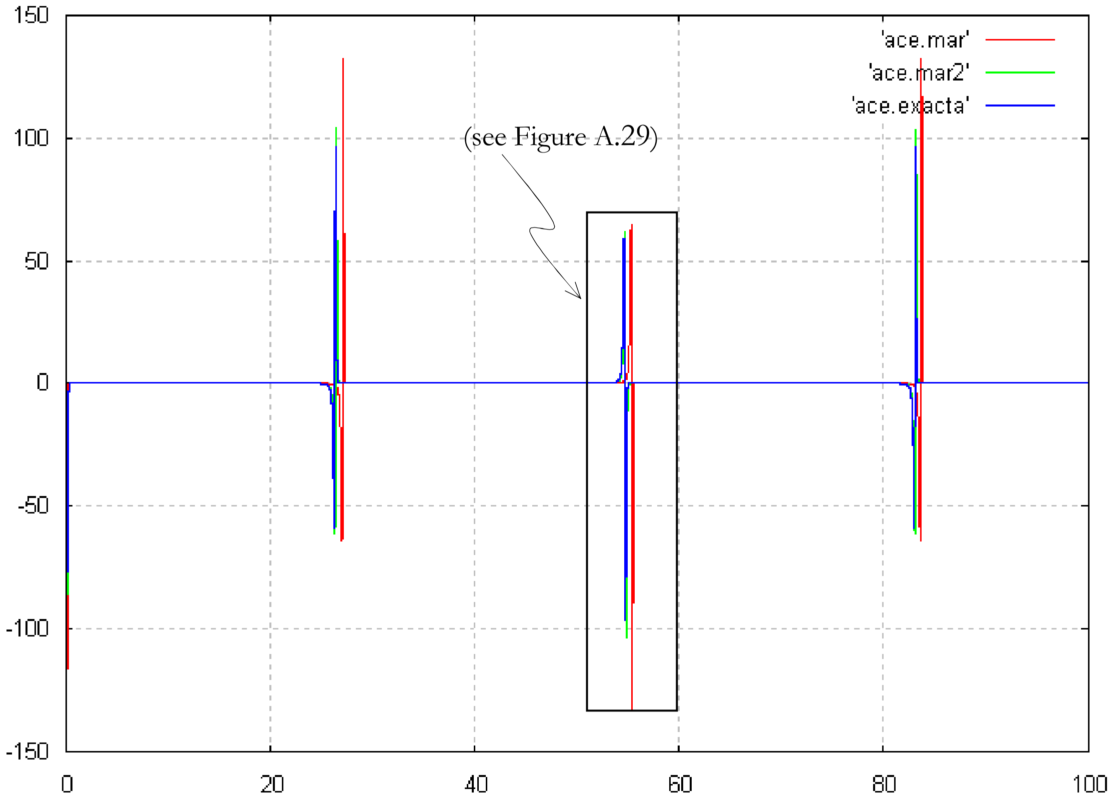

Figure A.28: "Acceleration" vs. time curve, ($\Delta t = 0.01$).

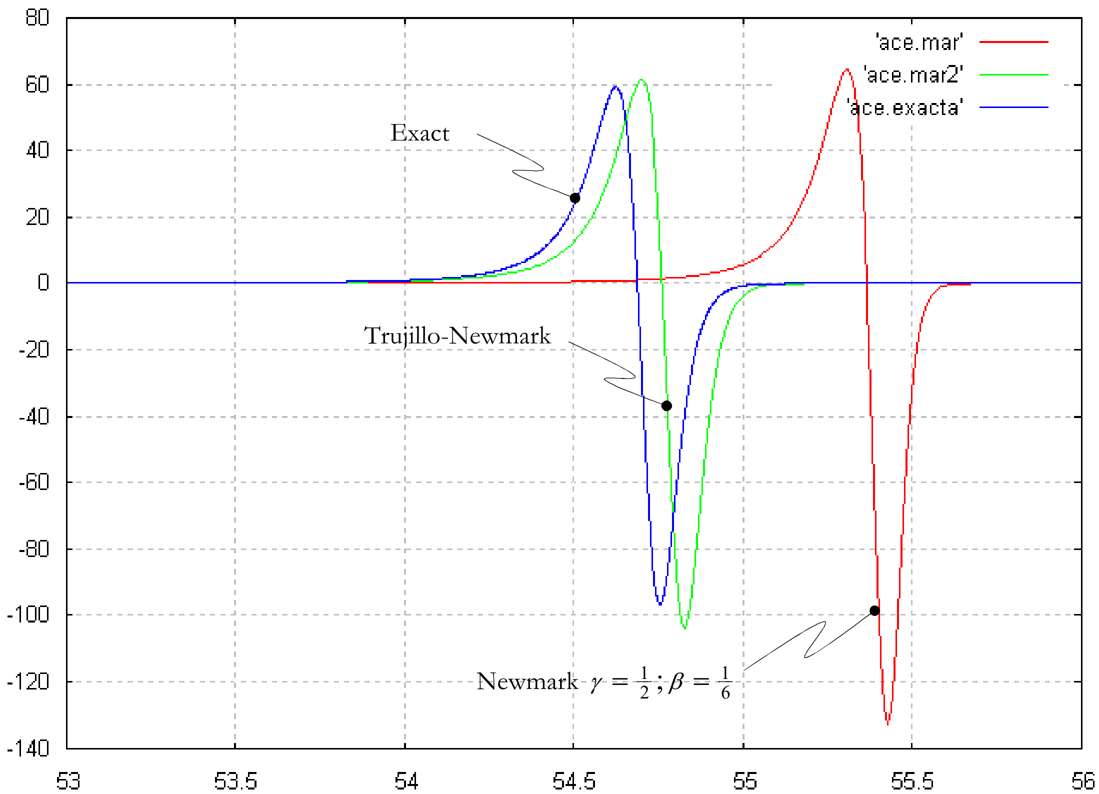

Figure A.29: "Acceleration" vs. time curve [53:56], ($\Delta t = 0.01$).

A continuación presentamos resultados utilizando integración directa con un incremento de tiempo igual a $\Delta t = 0.03$.

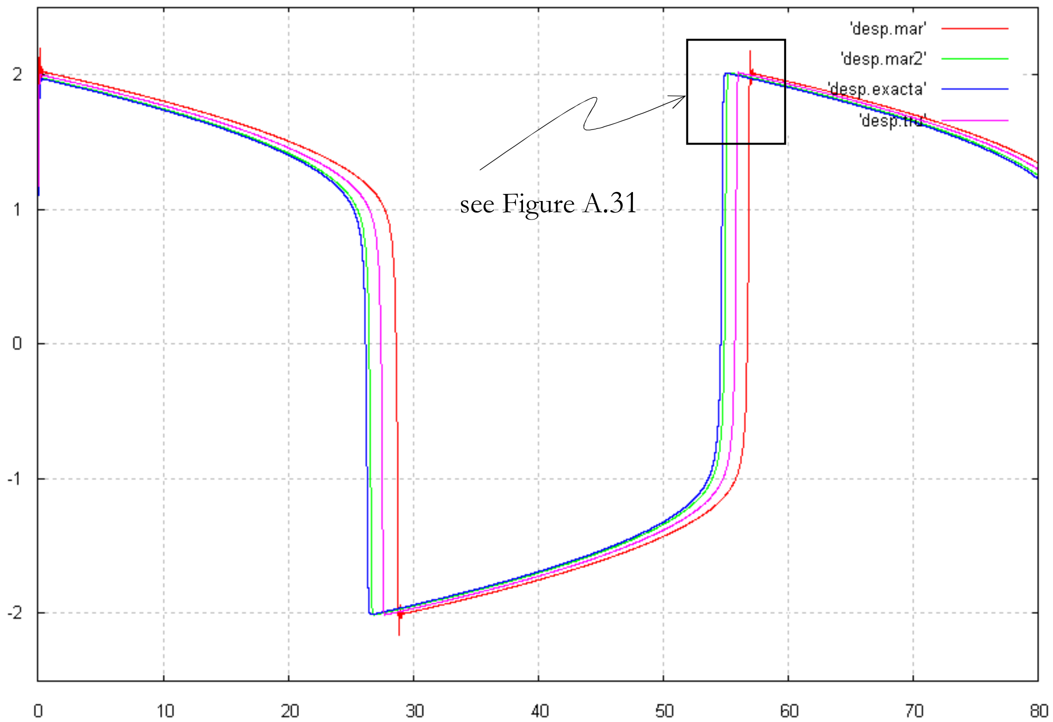

Figure A.30: "Displacement" vs. time curve, ($\Delta t = 0.03$).

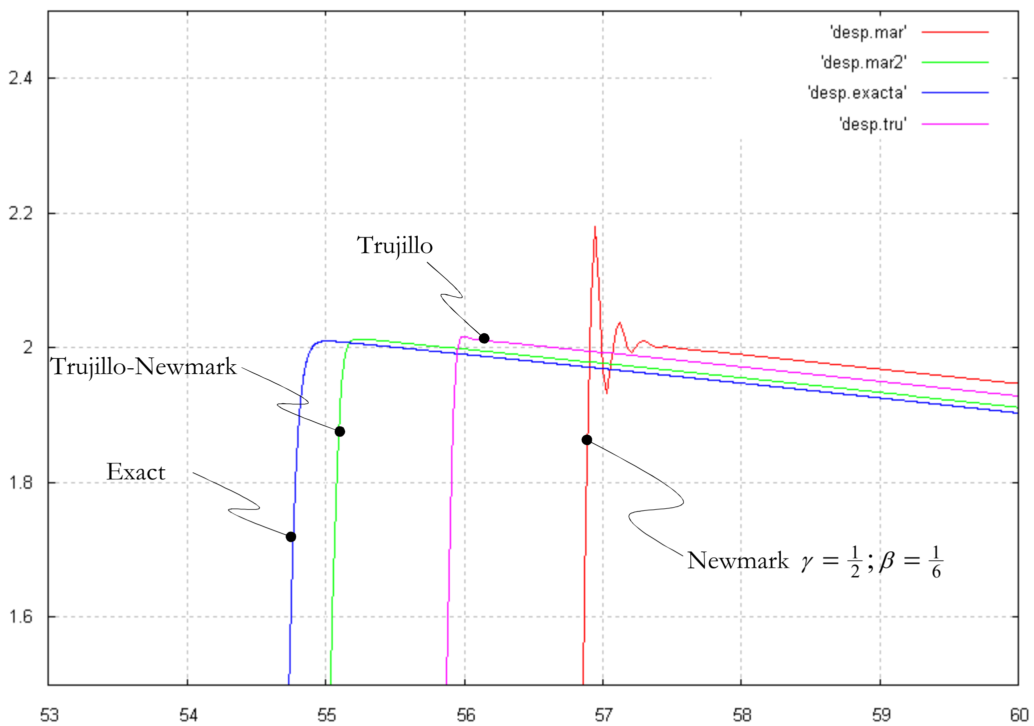

Figure A.31: "Displacement" vs. time curve [53:60], ($\Delta t = 0.03$).

By means of numerical integration we present the results using the time increment $\Delta t = 0.05$.

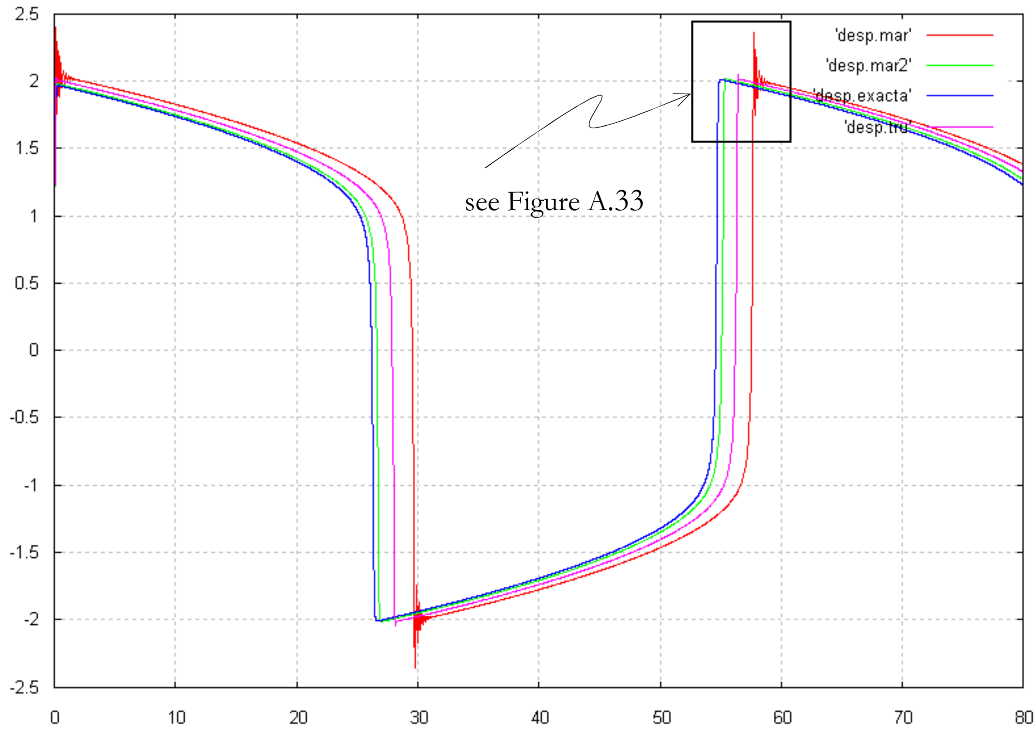

Figure A.32: "Displacement" vs. time curve, ($\Delta t = 0.05$).

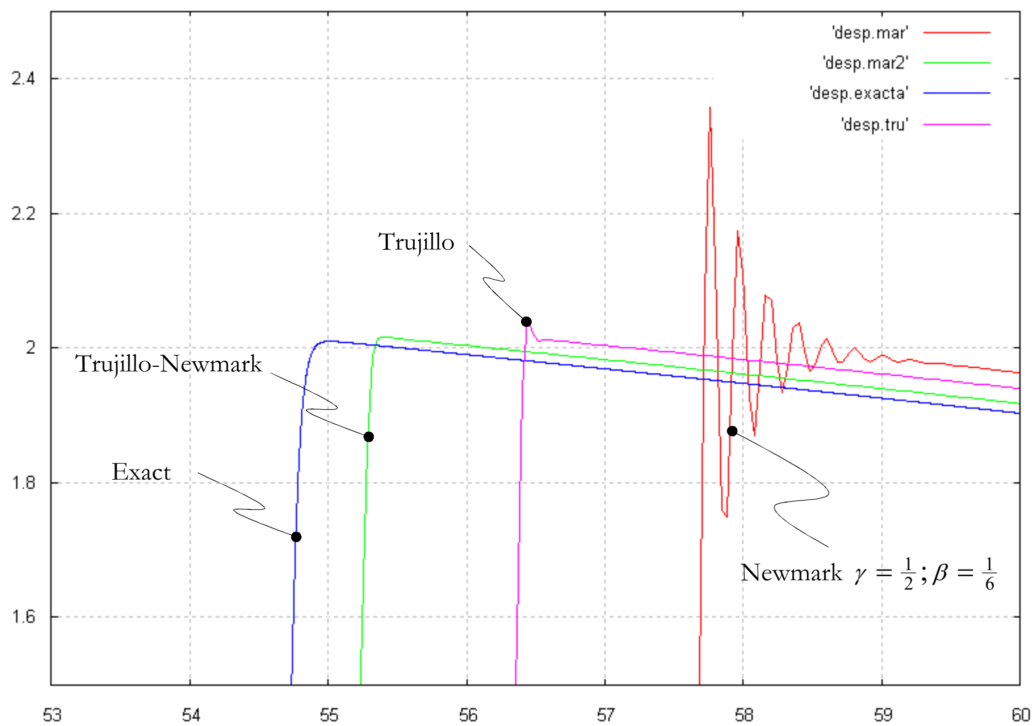

Figure A.33: "Displacement" vs. time curve [53:60], ($\Delta t = 0.05$).

In Figure A.34 we show the "displacement" vs. time curve for different values for α .

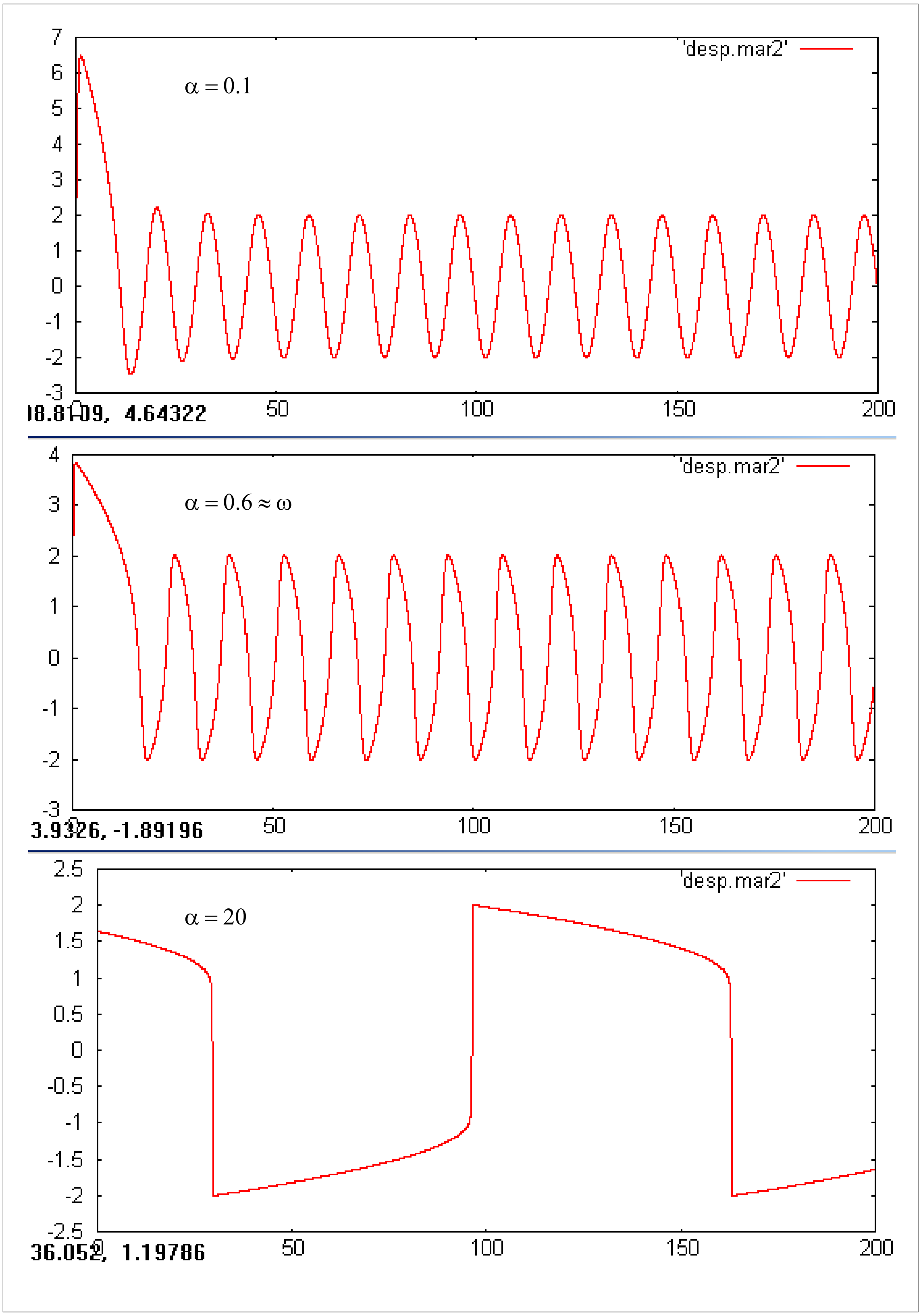

Figure A.34: "Displacement" vs. time curve.

A.7.3.3 Forced Harmonic Response without Damping

Consider the differential equation:

$$m\ddot{u} + ku = F_0 \sin(\Omega t) \tag{A.184}$$

where Ω is the excitation frequency. When $\Omega = \omega$, resonance phenomena appears.

As example consider that: $m = 4.5$, $k = 3500$, $F_0 = 100$, $\Omega = 18$, and boundary and initial conditions:

$$u(t=0) = 15 \quad ; \quad \dot{u}(t=0) = 150 \tag{A.185}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

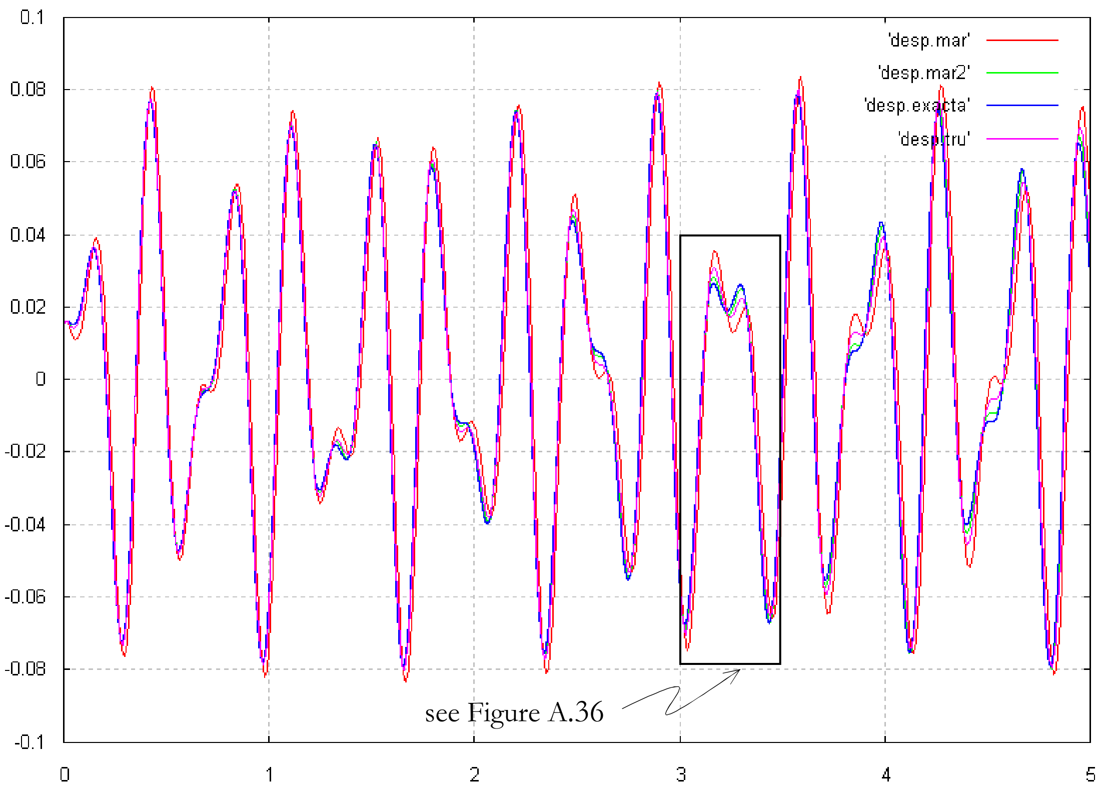

Figure A.35: Displacement vs. time curve, ($\Delta t = 0.01$).

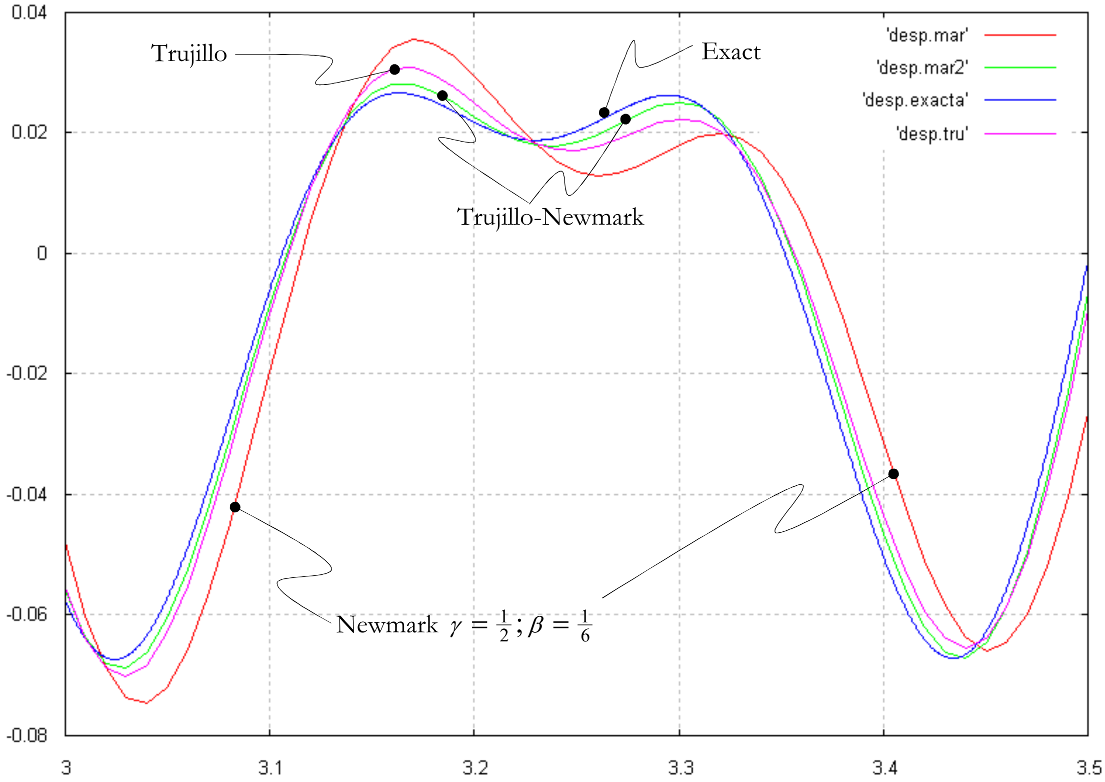

Figure A.36: Displacement vs. time curve [3:3.5], $(\Delta t = 0.01)$.

Now consider that:

$$u(t=0)=0 \qquad ; \qquad \dot{u}(t=0)=0 \tag{A.186}$$

besides we consider that $\Omega = \omega$. With these conditions we can note that the system enter in resonance, (see Figure A.37).

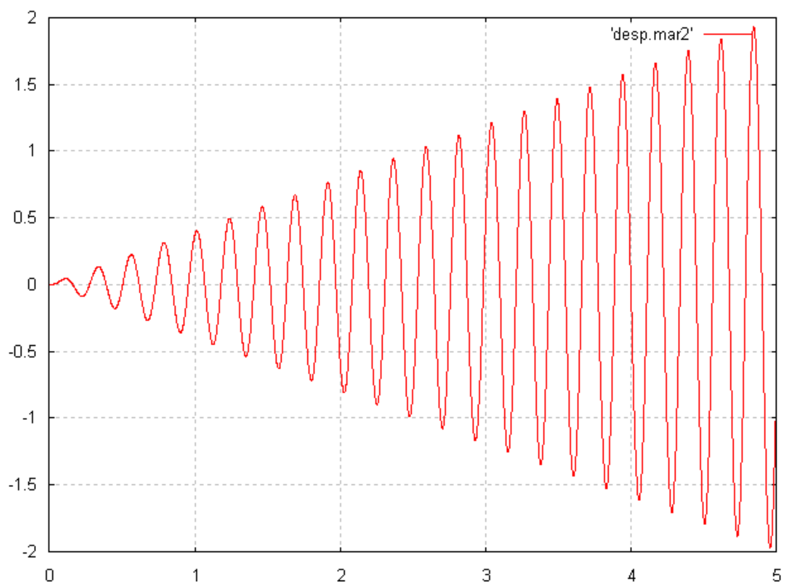

Figure A.37: Displacement vs. time curve.

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Annex B

Introduction to Finite Differences

B.1 Introduction

The finite difference method was the first technique that emerged to solve numerically partial differential equations related to practical engineering problems. Today this technique is now obsolete with respect to solving partial differential equations, for instance, to solve problems related to beams, plates, flux, etc. But the finite difference technique is widely spread when we are dealing with numerical integration over time (see Annex A).

B.2 The Finite Difference Method

Let us consider the function $y = y(x)$, and the derivative of y with respect to x is defined by:

$$y' \equiv \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \tag{B.1}$$

where y' indicates the slope of the function at the point x, (see Figure B.1).

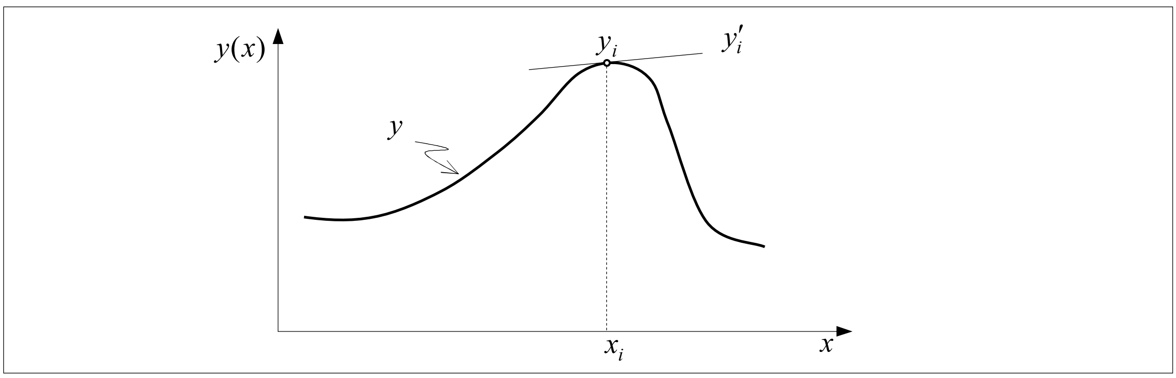

Figure B.1: Derivative of a function.

When the term Δx does not tend to zero but to a finite value, (see Figure B.2), the derivative at the point x_i can be defined in several ways. If we use the left neighbor point (y_{i-1}), *left finite difference*, the first derivative can be approached as follows:

$$y_i'^L = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_i - y_{i-1}}{\Delta x} \tag{B.2}$$

Otherwise, if we used information of the right neighbor point (y_{i+1}), *right finite difference*, the first derivative can be represented as follows:

$$y_i'^R = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_{i+1} - y_i}{\Delta x} \tag{B.3}$$

where we have adopted the nomenclature $y(x_{i-1}) = y_{i-1}$, $y(x_i) = y_i$, $y(x_{i+1}) = y_{i+1}$. As we can appreciate in Figure B.2, by using these techniques the first derivate is approximated to its exact value, i.e. there is an error associated with it. When $\Delta x \to 0$, the exact value for the first derivative is achieved.

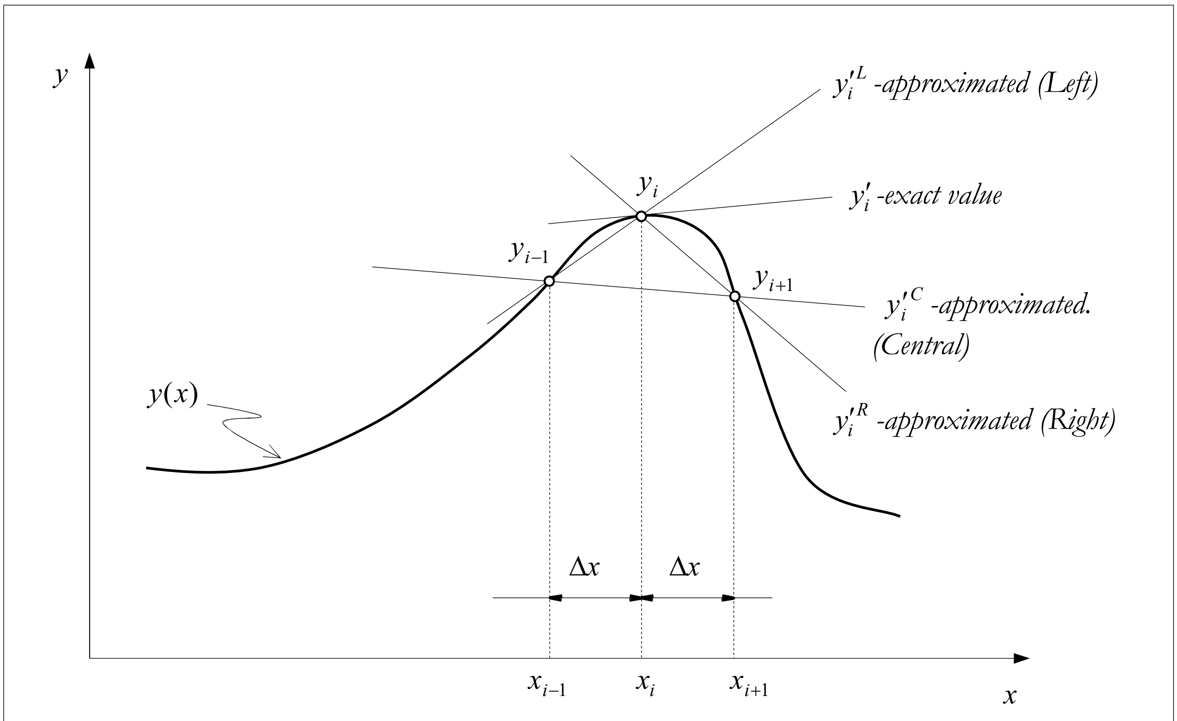

Figure B.2: Derivative of a function by means of finite differences.

We can still raise another possibility to approach the first derivative of the function at the point x_i by using simultaneously the left and the right points, *(central finite difference)*:

$$y_i'^C = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{B.4}$$

As we can appreciate in Figure B.2, the central finite difference approach is closer to the exact value. Note that the central finite difference, for the first derivative, is the average of left and right techniques:

$$\left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_i'^R + y_i'^L}{2} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{B.5}$$

Similarly we can obtain the derivatives for higher order, for example for the second derivative:

$$\frac{d^2y}{dx^2} = \lim_{\Delta x \to 0} \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{\frac{y(x+\Delta x)-y(x)}{\Delta x} - \frac{y(x)-y(x+\Delta x)}{\Delta x}}{\Delta x} \tag{B.6}$$

The left finite derivative:

$$\begin{aligned}\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i &= \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right)^L = \frac{\Delta}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x}\right) = \frac{1}{\Delta x}\left(\frac{\Delta y_i}{\Delta x} - \frac{\Delta y_{i-1}}{\Delta x}\right)\\ &= \frac{1}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x} - \frac{y_{i-1} - y_{i-2}}{\Delta x}\right)\\ &= \frac{y_i - 2y_{i-1} + y_{i-2}}{\Delta x^2}\end{aligned} \tag{B.7}$$

The right finite derivative:

$$\begin{aligned}\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i &= \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right)^R = \frac{\Delta}{\Delta x}\left(\frac{y_{i+1} - y_i}{\Delta x}\right) = \frac{1}{\Delta x}\left(\frac{\Delta y_{i+1}}{\Delta x} - \frac{\Delta y_i}{\Delta x}\right)\\ &= \frac{1}{\Delta x}\left(\frac{y_{i+2} - y_{i+1}}{\Delta x} - \frac{y_{i+1} - y_i}{\Delta x}\right) = \frac{y_{i+2} - 2y_{i+1} + y_i}{\Delta x^2}\end{aligned} \tag{B.8}$$

And by means of the central finite difference we can approach the second derivative as follows:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\frac{y_{i+1} - y_i}{\Delta x} - \frac{y_i - y_{i-1}}{\Delta x}}{\Delta x} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \tag{B.9}$$

B.2.1 Left Finite Differences

Next we will define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the left finite difference. As we have seen previously, for the first derivative we have $\Delta y = y_i - y_{i-1}$, (see equation (B.2)). If we want to obtain the operator for the second derivative we use the points located at the left side of the point x_i:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\Delta}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x}\right) = \frac{\Delta y_i - \Delta y_{i-1}}{\Delta x^2} \tag{B.10}$$

By applying once more the left derivative definition we get $\Delta y_i = y_i - y_{i-1}$ and $\Delta y_{i-1} = y_{i-1} - y_{i-2}$ and by substituting into the above equation we can obtain:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta y_i - \Delta y_{i-1}}{\Delta x^2} = \frac{(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})}{\Delta x^2} = \frac{(y_i - 2y_{i-1} + y_{i-2})}{\Delta x^2} \tag{B.11}$$

Then, we define the operator $\Delta^2 y = y_i - 2y_{i-1} + y_{i-2}$ for the left finite difference case. In Figure B.3 we constructed a figure in order to obtain automatically the operators for higher order operators.

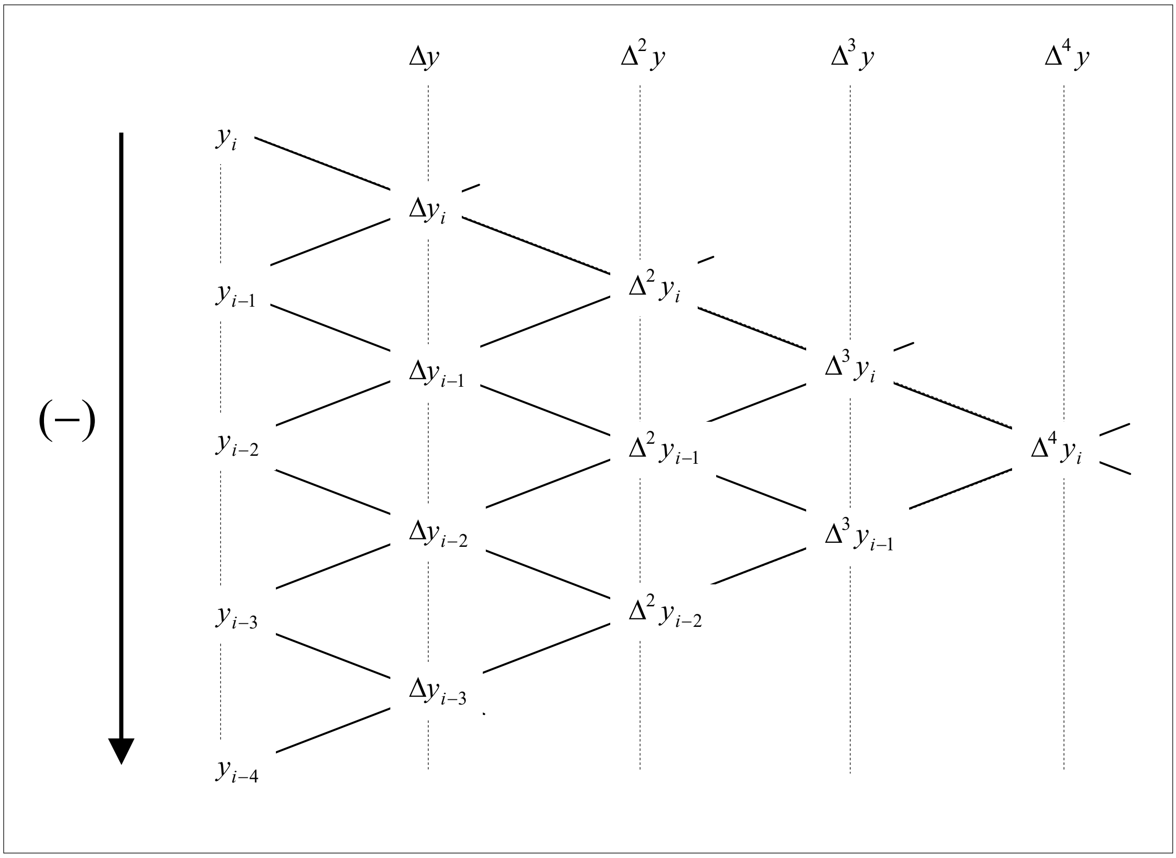

Figure B.3: Left finite differences.

For example, in order to obtain the operator $\Delta^4 y$ by means of Figure B.3 we localize the term $\Delta^4 y_i$ and we subtract the values as follows:

$$
\begin{aligned}
\Delta^4 y &= \Delta^3 y_i - \Delta^3 y_{i-1} = \left(\Delta^2 y_i - \Delta^2 y_{i-1}\right) - \left(\Delta^2 y_{i-1} - \Delta^2 y_{i-2}\right) = \Delta^2 y_i - 2\Delta^2 y_{i-1} + \Delta^2 y_{i-2} \\
&= \left(\Delta y_i - \Delta y_{i-1}\right) - 2\left(\Delta y_{i-1} - \Delta y_{i-2}\right) + \left(\Delta y_{i-2} - \Delta y_{i-3}\right) \\
&= \Delta y_i - 3\Delta y_{i-1} + 3\Delta y_{i-2} - \Delta y_{i-3} \\
&= \left(y_i - y_{i-1}\right) - 3\left(y_{i-1} - y_{i-2}\right) + 3\left(y_{i-2} - y_{i-3}\right) - \left(y_{i-3} - y_{i-4}\right) \\
&= y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}
\end{aligned}
\tag{B.12}
$$

With that we can define the fourth derivative by means of left finite difference as follows:

$$
\left(\frac{\Delta^4 y}{\Delta x^4}\right)_i = \frac{y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}}{\Delta x^4}
\tag{B.13}
$$

B.2.2 Right Finite Differences

Next we will define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the right finite difference. As we have seen previously, for the first derivative we have $\Delta y = y_{i+1} - y_i$, (see equation (B.3)). If we want to obtain the operator for the second derivative we use the points located at the right side of the point x_i:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\Delta}{\Delta x}\left(\frac{y_{i+1} - y_i}{\Delta x}\right) = \frac{\Delta y_{i+1} - \Delta y_i}{\Delta x^2} \tag{B.14}$$

By applying once more the right derivative definition we get $\Delta y_{i+1} = y_{i+2} - y_{i+1}$ and $\Delta y_i = y_{i+1} - y_i$ and by substituting into the above equation we can obtain:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta y_{i+1} - \Delta y_i}{\Delta x^2} = \frac{(y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)}{\Delta x^2} = \frac{(y_{i+2} - 2y_{i+1} + y_i)}{\Delta x^2} \tag{B.15}$$

Then, we define the operator $\Delta^2 y = y_{i+2} - 2y_{i+1} + y_i$ for the right finite difference case. Note that we only use the points on the right of the point x_i. In Figure B.4 we constructed a figure in order to obtain automatically the operators for higher order operators.

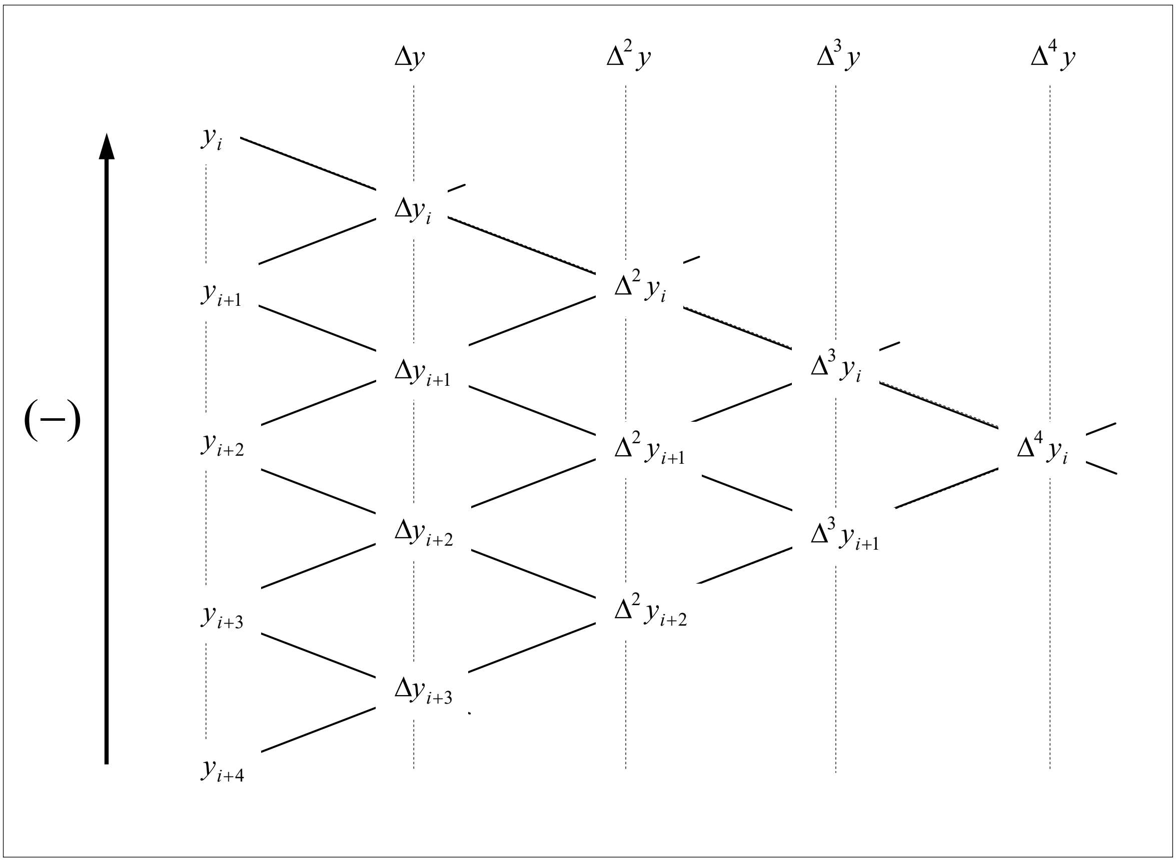

Figure B.4: Right finite differences.

For example, in order to obtain the operator $\Delta^3 y$ by means of Figure B.4 it is enough to do:

$$\begin{aligned}\Delta^3 y &= \Delta^2 y_{i+1} - \Delta^2 y_i = (\Delta y_{i+2} - \Delta y_{i+1}) - (\Delta y_{i+1} - \Delta y_i) = \Delta y_{i+2} - 2\Delta y_{i+1} + \Delta y_i \\ &= (y_{i+3} - y_{i+2}) - 2(y_{i+2} - y_{i+1}) + (y_{i+1} - y_i) \\ &= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i\end{aligned} \tag{B.16}$$

With that we can define the third derivative by means of right finite difference as follows:

$$\left(\frac{\Delta^3 y}{\Delta x^3}\right)_i = \frac{y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i}{\Delta x^3} \tag{B.17}$$

B.2.3 Central Finite Differences

The central finite difference uses simultaneously the points on the left and on the right. According to Figure B.4 we can define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the central finite difference.

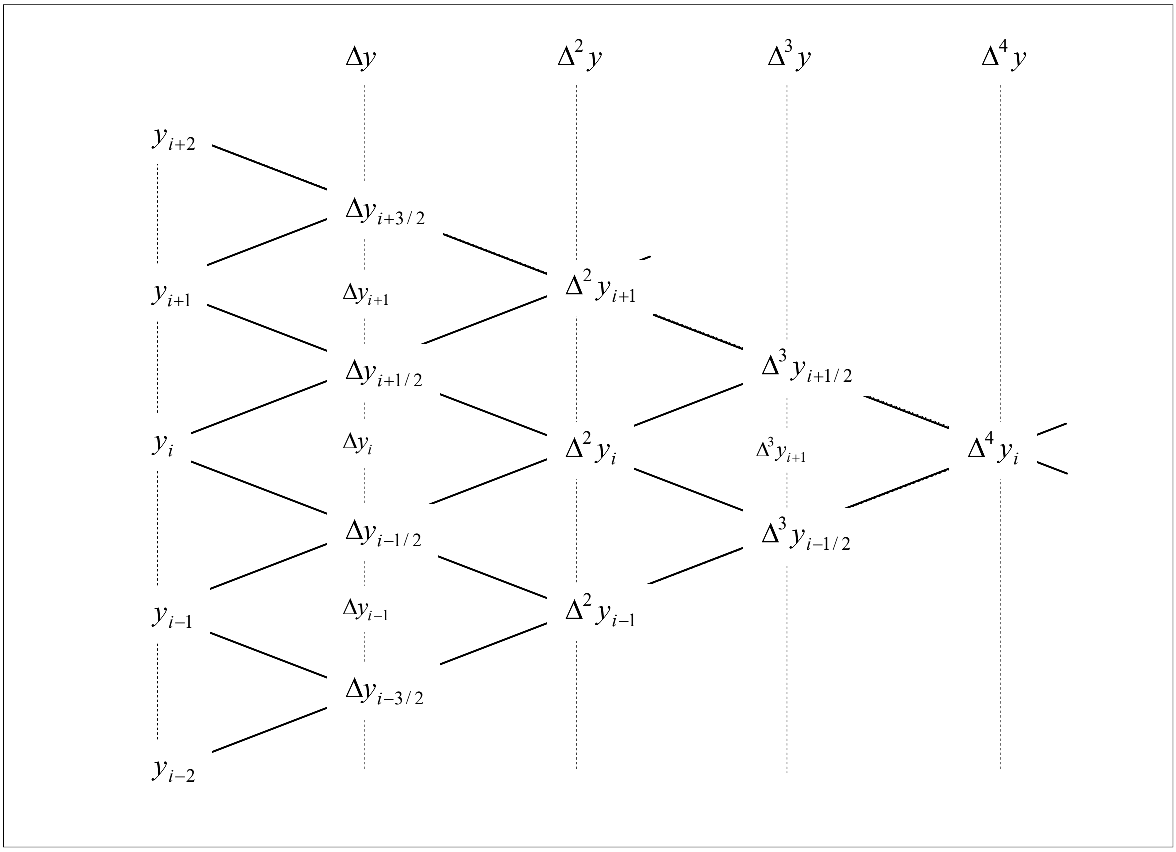

Figure B.5: Central finite differences.

In Figure B.5 the term $\Delta y_{i+3/2}$ characterizes finite difference taking at the point between x_{i+1} and x_{i+2}. For example, to obtain the first derivative, we localize the term Δy_i in Figure B.5, such term is between $\Delta y_{i+1/2}$ and $\Delta y_{i-1/2}$ and we take the average:

$$\begin{aligned} \Delta y_i &= \frac{\Delta y_{i+1/2} + \Delta y_{i-2}}{2} = \frac{(y_{i+1} - y_i) + (y_i - y_{i-1})}{2} = \frac{y_{i+1} - y_{i-1}}{2} \\ \Rightarrow \left(\frac{\Delta y}{\Delta x}\right)_i &= \frac{y_{i+1} - y_{i-1}}{2\Delta x} \end{aligned} \tag{B.18}$$

According to Figure B.5 we can represent the second derivative $\Delta^2 y_i = \Delta y_{i+1/2} - \Delta y_{i-2}$, then:

$$\begin{aligned} \Delta^2 y_i &= \Delta y_{i+1/2} - \Delta y_{i-2} = (y_{i+1} - y_i) - (y_i - y_{i-1}) = y_{i+1} - 2y_i + y_{i-1} \\ \Rightarrow \left(\frac{\Delta^2 y}{\Delta x^2}\right)_i &= \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \end{aligned} \tag{B.19}$$

In the same way the third derivative can be represented as follows:

$$
\begin{aligned}
\Delta^3 y_i &= \frac{\Delta^3 y_{i+1/2} + \Delta^3 y_{i-2}}{2} = \frac{(\Delta^2 y_{i+1} - \Delta^2 y_i) + (\Delta^2 y_i - \Delta^2 y_{i-1})}{2} \\
&= \frac{\Delta^2 y_{i+1} - \Delta^2 y_{i-1}}{2} = \frac{[\Delta y_{i+3/2} - \Delta y_{i+1/2}] - [\Delta y_{i-1/2} - \Delta y_{i-3/2}]}{2} \\
&= \frac{[(y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)] - [(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})]}{2} \\
&= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2}
\end{aligned} \tag{B.20}
$$

thus

$$
\left(\frac{\Delta^3 y}{\Delta x^3}\right)_i = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2\Delta x^3} \tag{B.21}
$$

Notice that when we are applying the central finite differences for derivatives of odd order it appears 2 in the denominator.

NOTE: For the finite differences of even order, e.g. $\Delta^2 y, \Delta^4 y, \Delta^6 y, \cdots$, the coefficients are the same as those for the binomial expression $(a-b)^n$, for instance

$$
(a-b)^2 = 1a^2 - 2ab + 1b^2 \tag{B.22}
$$

and the coefficients are $(1,-2,1)$. Another example

$$
(a-b)^4 = 1a^4 - 4a^3 b + 6a^2 b^2 - 4ab^3 + 1b^4 \tag{B.23}
$$

and the coefficients are $(1,-4,6,-4,1)$

B.3 Finite Differences to Partial Derivatives

Let us consider now the function $z = z(x,y)$. The partial derivatives can be approached by means of Central Finite Differences as follows:

$$
\begin{aligned}
&\left(\frac{\partial z}{\partial x}\right)_{i,j} \approx \frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x}; \qquad \frac{\partial^2 z}{\partial x^2} \approx \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \\
&\left(\frac{\partial z}{\partial y}\right)_{i,j} \approx \frac{z_{i,j+1} - z_{i,j-1}}{2\Delta y}; \qquad \frac{\partial^2 z}{\partial y^2} \approx \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2}
\end{aligned} \tag{B.24}
$$

with that we can also obtain:

$$
\begin{aligned}
&\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \approx \frac{\partial}{\partial y}\left[\frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x}\right] = \frac{1}{2\Delta x}\left[\frac{\partial}{\partial y}\left(z_{i+1,j}\right) - \frac{\partial}{\partial y}\left(z_{i-1,j}\right)\right] \\
&\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j} \approx \frac{1}{4\Delta x \Delta y}\left(z_{i+1,j+1} - z_{i+1,j-1} - z_{i-1,j+1} + z_{i-1,j-1}\right)
\end{aligned} \tag{B.25}
$$

If we consider the domain discretized into points, (see Figure B.6), we can represent the partial derivative $\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j}$ in operator form as indicated in Figure B.7, in which $\Delta x = h$, $\Delta y = k$.

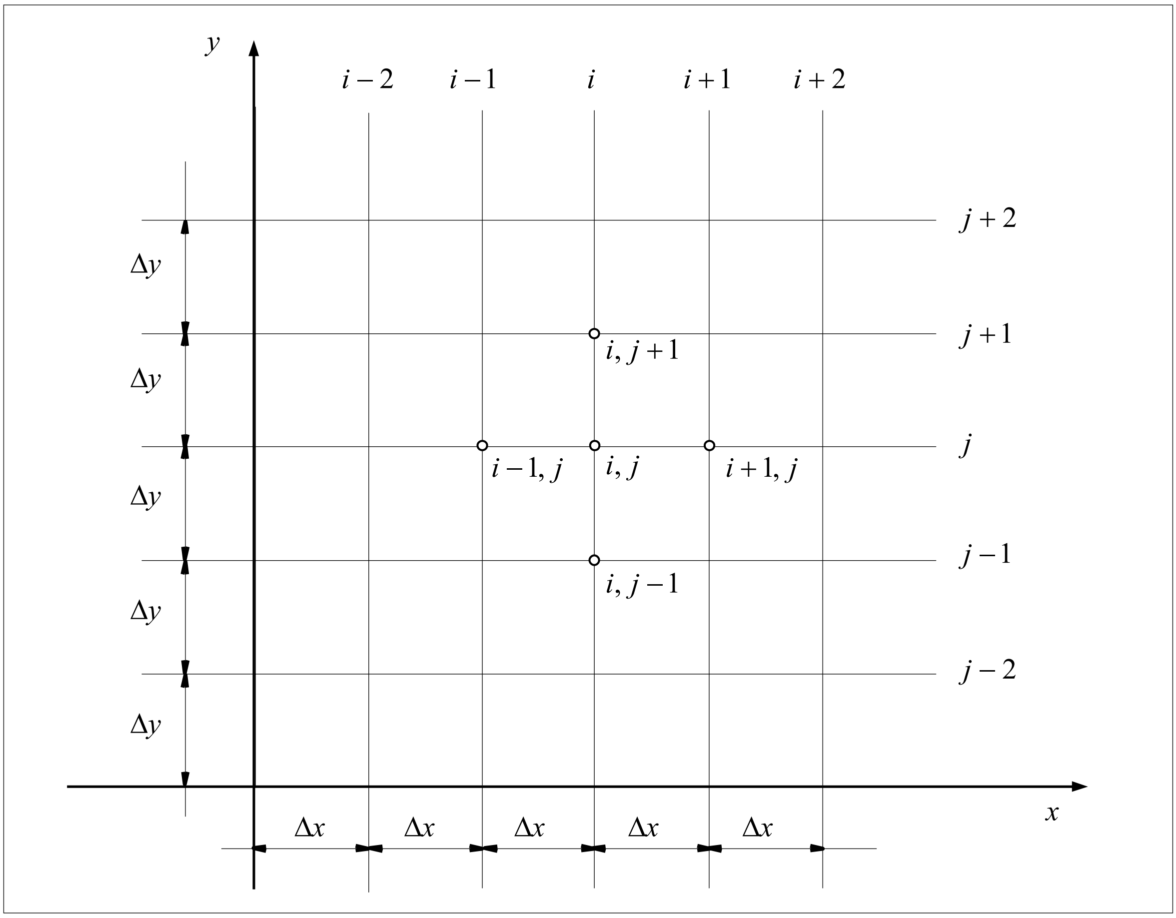

Figure B.6: Domain discretization.

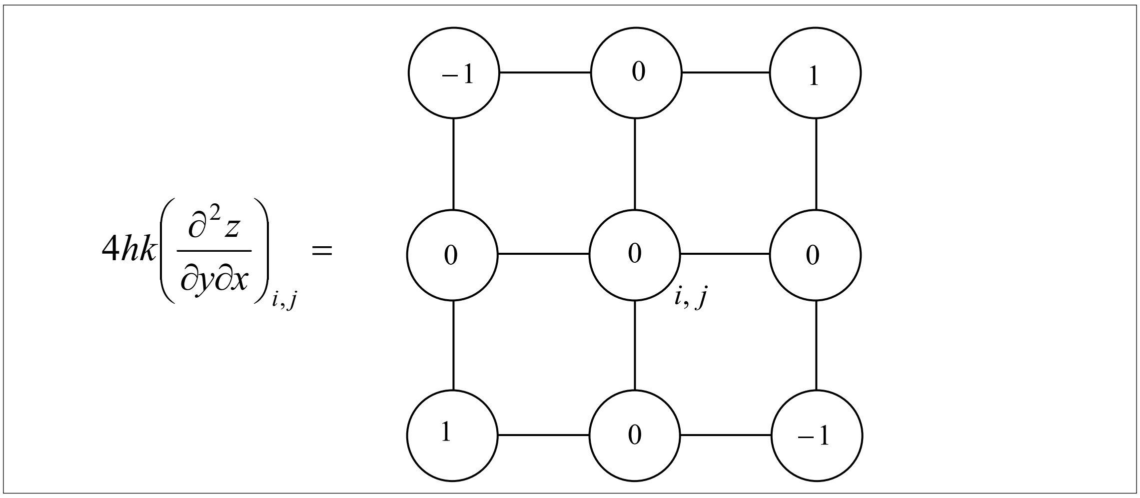

Figure B.7

In the same fashion we can represent

$$
\begin{aligned}
&\left(\frac{\partial^4 z}{\partial y^2 \partial x^2}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^2}{\partial y^2}\left[\frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2}\right] \\
&\Rightarrow \left(\frac{\partial^4 z}{\partial y^2 \partial x^2}\right) = \frac{1}{h^2 k^2}\left(\begin{aligned} &z_{i+1,j+1} - 2z_{i+1,j} + z_{i+1,j-1} - 2z_{i,j+1} \\ &\qquad + 4z_{i,j} - 2z_{i,j-1} + z_{i-1,j+1} - 2z_{i-1,j} + z_{i-1,j-1} \end{aligned}\right)
\end{aligned}
\tag{B.26}
$$

The above equation can be represented if form of operator as the one indicated in Figure B.8.

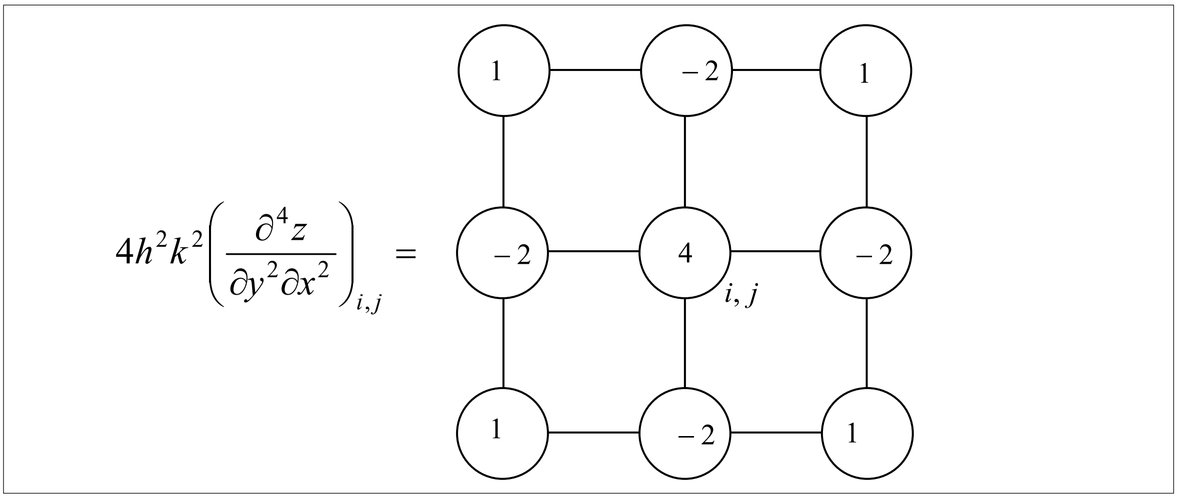

Figure B.8

As we have seen previously the following is true $\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$, then, the partial derivative can be written as follows:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j} = \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \tag{B.27}$$

In the same fashion we can obtain

$$\left(\frac{\partial^2 z}{\partial y^2}\right)_{i,j} = \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2} \tag{B.28}$$

with that, the Laplacian $\nabla^2 z$ becomes:

$$\nabla^2 z = \left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 z}{\partial y^2}\right)_{i,j} \approx \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2} + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \tag{B.29}$$

Example

Let us consider the following partial differential equation

$$\nabla^2 z = -\frac{p}{S} \tag{B.30}$$

where z represents the membrane deflection under the pressure p, in which the membrane deflection on the boundary is zero. The square cross section has side $b = 6h$ as indicated in Figure B.9. Obtain the membrane deflection z in the cross section.

Solution:

We can use the symmetry of the cross section and analyze only a quarter of the section. Besides, in this quarter there are points with the same displacement, with that we will need to analyze only the half of the quarter, (see Figure B.9).

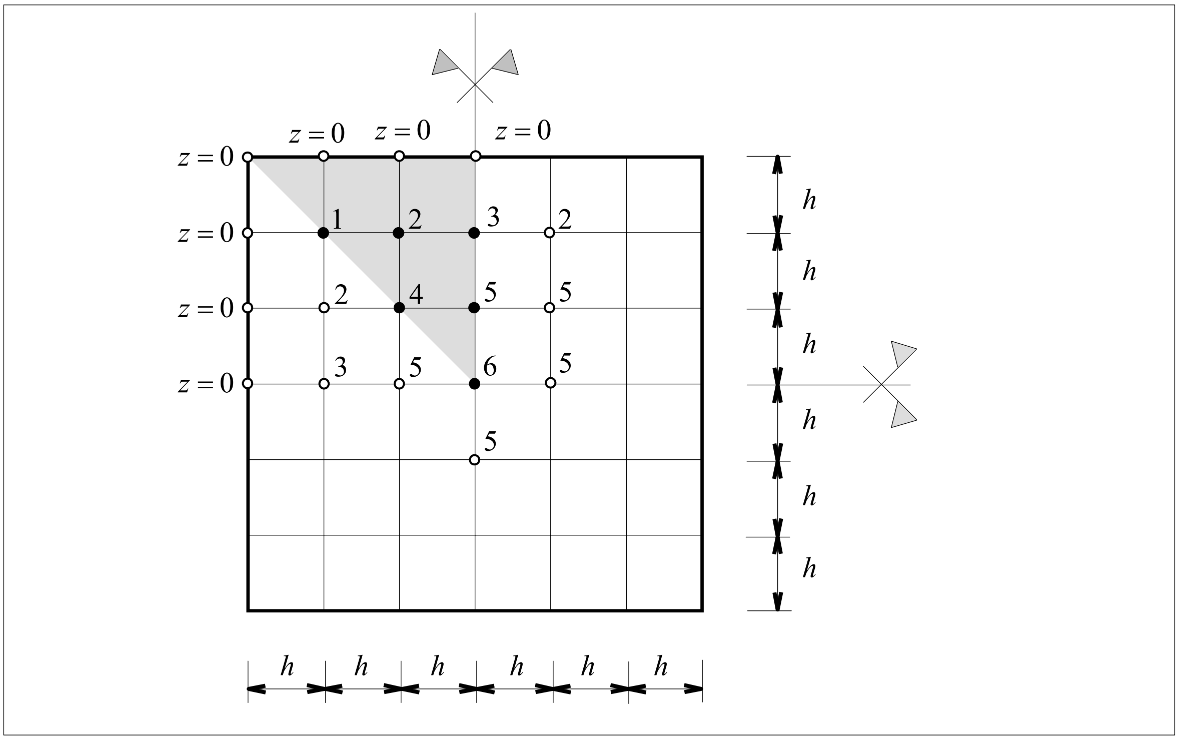

Figure B.9: Discretization of the domain.

As seen previously, the Laplacian can be approached by means of the finite difference:

$$\nabla^2 z \approx z_{i,j+1} + z_{i,j-1} + z_{i+1,j} + z_{i-1,j} - 4z_{i,j} = \frac{-h^2 p}{S} \tag{B.31}$$

where we have considered $\Delta x^2 = \Delta y^2 = h^2$. The operator can be appreciated in Figure B.10.

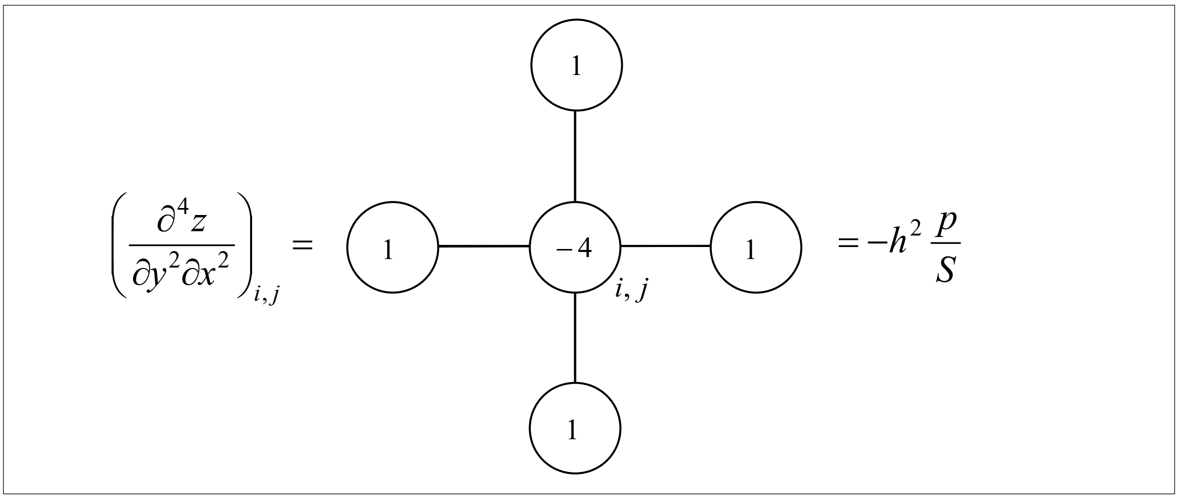

Figure B.10

By applying the operator described in Figure B.10 to the points of the mesh $(1,2,\cdots,6)$, (see Figure B.9), we can construct the following set of equations:

$$\begin{bmatrix} -4z_1 & +2z_2 & & & & \\ z_1 & -4z_2 & +z_3 & +z_4 & & \\ & +2z_2 & -4z_3 & & +z_5 & \\ & +2z_2 & & -4z_4 & +2z_5 & \\ & & z_3 & +2z_4 & -4z_5 & +z_6 \\ & & & & 4z_5 & -4z_6 \end{bmatrix} = \frac{-h^2 p}{S} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{B.32}$$

By restructuring the above set of equations we can obtain:

$$\begin{bmatrix} -4 & 2 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & -4 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 4 & -4 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix} = \frac{-h^2 p}{S} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{B.33}$$

By solving the above set of equations we can obtain:

$$\begin{aligned} z_1 &= 0{,}95192\frac{h^2 p}{S} \quad ; \quad z_2 = 1{,}4035\frac{h^2 p}{S} \quad ; \quad z_3 = 1{,}53846\frac{h^2 p}{S} \\ z_4 &= 2{,}1250\frac{h^2 p}{S} \quad ; \quad z_5 = 2{,}34615\frac{h^2 p}{S} \quad ; \quad z_6 = 2{,}59615\frac{h^2 p}{S} \end{aligned} \tag{B.34}$$

Annex C

Incremental-Iterative Strategy Solution

C.1 Introduction

For a better understanding of the behavior of structures and materials, it has increased the demand to formulate algorithms that are able to realistically simulate the complete behavior of the structure/material. Among one of the applications we can mention the simulation of a structure to its complete destruction. In this way it allows us to design more efficiently the structure in order to face a disaster (earthquakes, explosions, etc.). To achieve this objective we must take into account non-linearity behavior. Basically we can highlight two types of non-linearity:

- Material Non-Linearity;
- Geometric Non-Linearity.

The material non-linearity appears when the stress-strain relationship is non-linear. The geometric non-linearity occurs when deformed configuration (current state) has great influence in the outcome. When we are dealing with the Finite Element Formulation the strain field $\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}})$ is related to the nodal displacements $\{\boldsymbol{u}\}^{(e)}$ by means the matrix which contains the derivatives of the shape functions $[\boldsymbol{B}]$, which in small deformation regime is only a function of initial geometric parameters, $\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}}) = [\boldsymbol{B}]\{\boldsymbol{u}\}^{(e)}$. In the case of finite deformation regime the matrix $[\boldsymbol{B}]$ is also a function of the displacement field, *i.e.* $\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}}) = [\boldsymbol{B}(\boldsymbol{u})]\{\boldsymbol{u}\}^{(e)}$. And as consequence the stiffness matrix is also a function of the displacement field.

In general, the geometric non-linearity is a consequence of the large displacement that undergoes the structure. Then, the measure of strain adopted must be able to capture the real displacement of the structure. Several measures of strain have been established, e.g. Green-Lagrange strain tensor, Almansi strain tensor, Logarithmic strain tensor, etc.

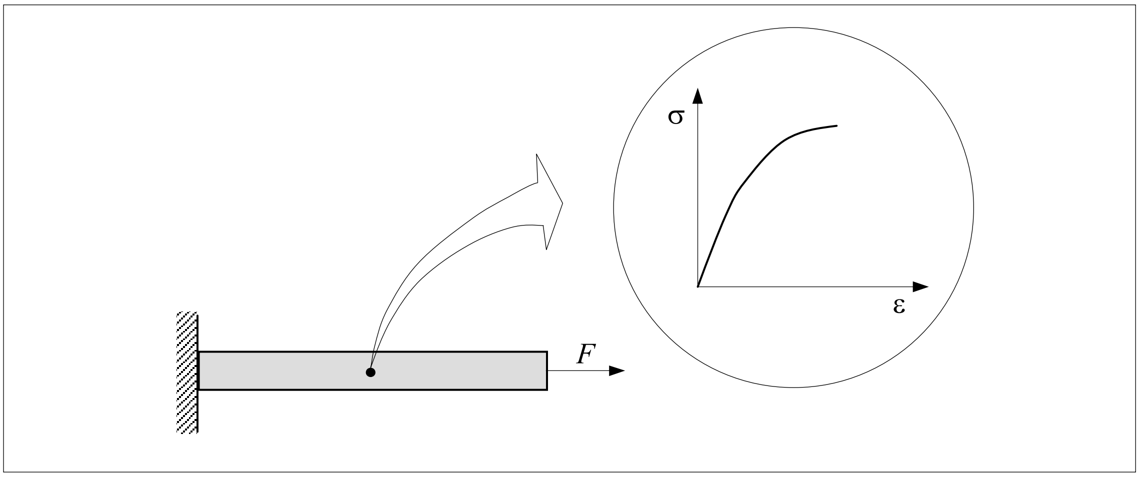

Figure C.1: Material non-linearity.

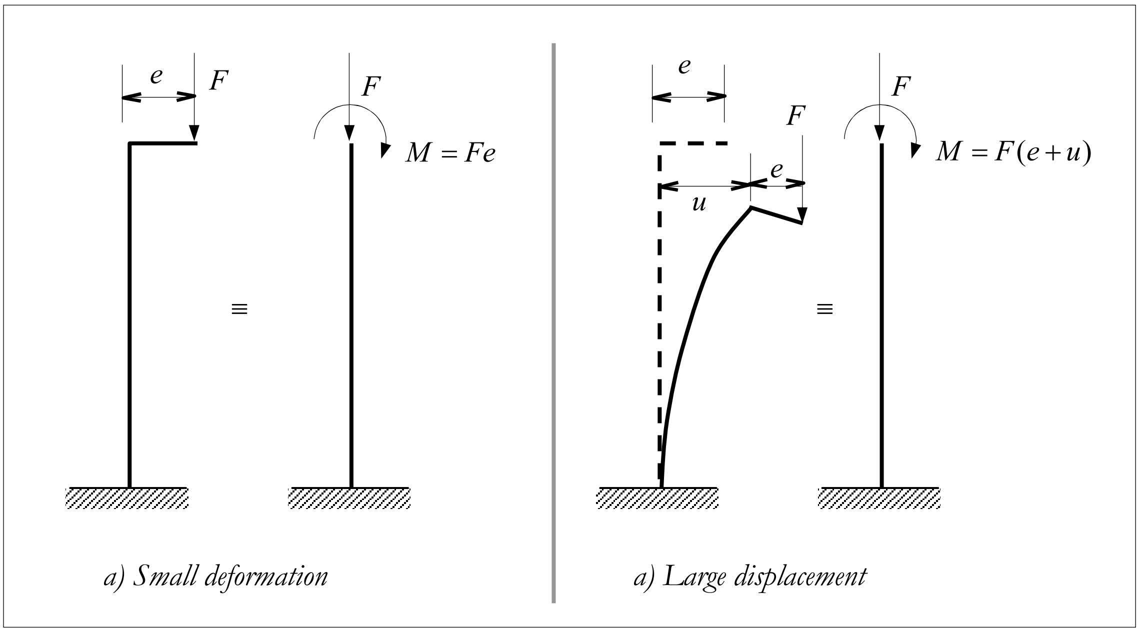

Figure C.2: Boundary conditions non-linearity.

C.1.1 Solution Strategies

To achieve the previous objectives, various solution techniques have been proposed. The choice of numerical algorithm depends on the given problem. For example, if the structure response is represented by a curve Force vs. Displacement, (see Figure C.3), the aim is to obtain the complete curve of the graph in question. As we will see later, we can use a strategy that is force increment (*force control*). But there may be a point, for instance, point A of the graph, where the force control procedure will get a no desired point as solution (point F) or even a divergence of the solution. Another strategy used is through incremental displacement (*displacement control*), which can also have undesirable solution if we are at point B and to a further increment in displacement we reach the point D of the graph missing all curve from B to D. Another strategy which can be employed is a combination of the two above methodologies, i.e. force control and displacement control simultaneously, which is known as *Arc-length control*.

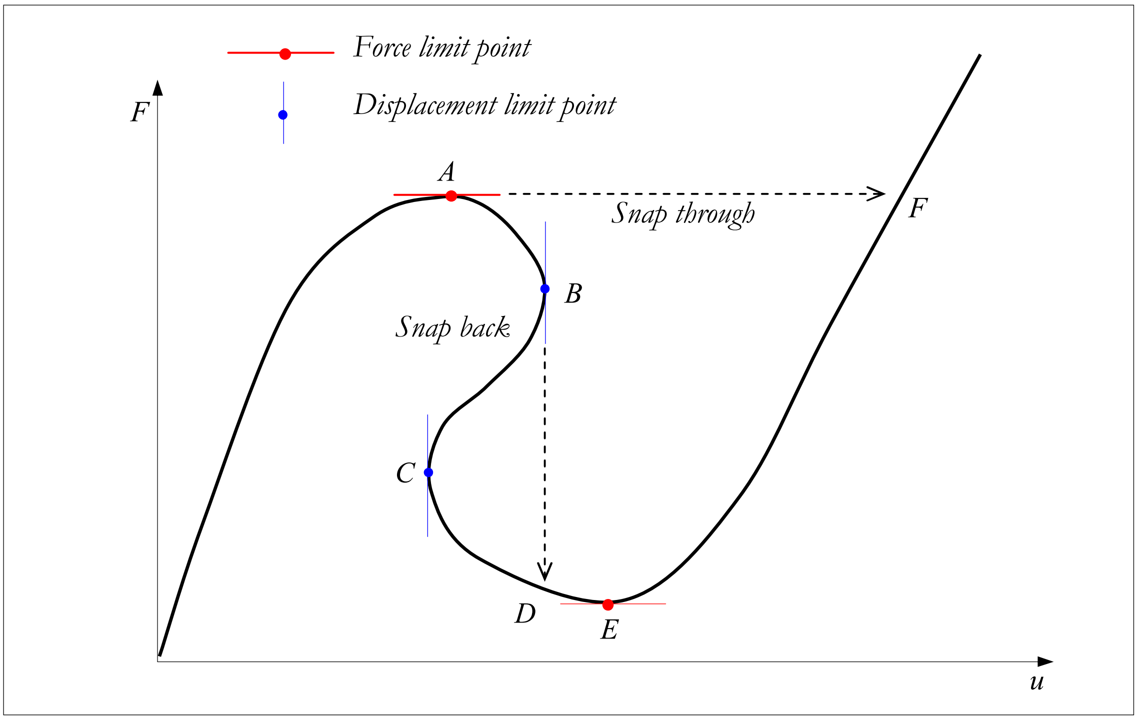

Figure C.3: Force-displacement curve.

Sometimes when we are using an incremental strategy it has an error associated with it or even the solution can diverge, (see Figure C.4). To overcome this drawback we must use an incremental-iterative scheme. Among the iterative methods we can mention the Newton-Raphson's method.

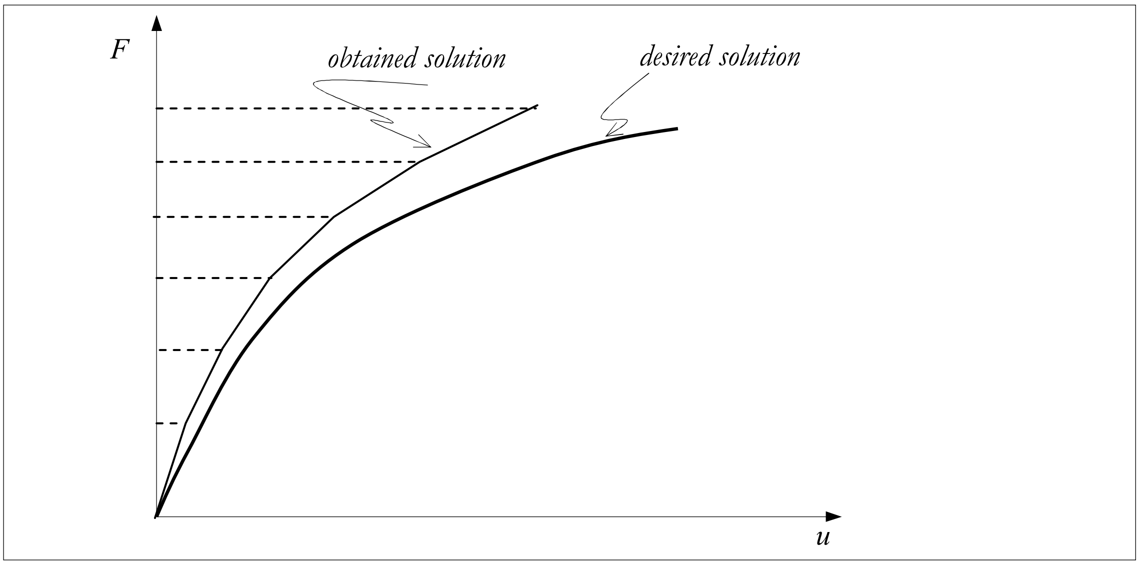

Figure C.4: Diverting of the solution

C.2 Total Potential Energy

The total potential energy (Π) of an elastic system is composed by two parts, namely:

- Internal potential energy (strain energy potential (U^{int}));
- External potential energy (U^{ext}):

$$\Pi(z) = U^{int}(z) - U^{ext}(z) \tag{C.1}$$

If the equation (C.1) represents the stationary condition for the total potential energy, it can be shown that the value of z is an extreme of $\Pi(z)$. This is the *Principle of stationary of the total potential energy*. Then, if we looking for the extremes (maximums and minimums) of the function, (see Figure C.5), the following must hold:

$$\frac{d\Pi(z)}{dz} = 0 \tag{C.2}$$

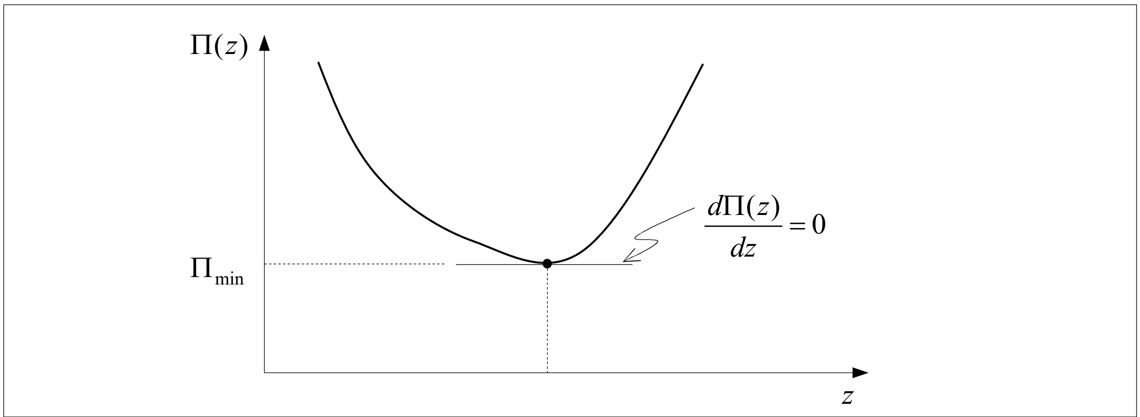

Figure C.5: Minimum of a function

Let us suppose a one-dimensional case in which $U^{ext}(u) = Wz$. With that the condition (C.2) becomes:

$$\begin{aligned} &\frac{d\Pi(z)}{dz} = \frac{dU^{int}(z)}{dz} - \frac{dU^{ext}(z)}{dz} = \frac{dU^{int}(z)}{dz} - W = B(z) - W = 0 \\ &\Rightarrow B(z) = W \end{aligned} \tag{C.3}$$

where we have considered $B(z) = \frac{dU^{int}(z)}{dz}$. We can rewrite the above equation as follows:

$$W = \left[\frac{B(z)}{z}\right] z = K^{Sec} z \tag{C.4}$$

where K^{Sec} is the secant of the curve $W \times z$ at the point z, (see Figure C.6).

Let us suppose that for a give increment Δz we have:

$$B(z + \Delta z) = W + \Delta W \tag{C.5}$$

In addition, by means of Taylor series the equation $B(z+\Delta z) = B(z) + \frac{\partial B(z)}{\partial z}\Delta z + \ldots$ holds, in which we have only considered linear terms. Then, the equation (C.5) can be rewritten as follows:

$$
\begin{aligned}
& B(z+\Delta z) = B(z) + \frac{\partial B(z)}{\partial z}\Delta z = W + \Delta W \\
& \Rightarrow \underbrace{B(z) - W}_{=0} + \frac{\partial B(z)}{\partial z}\Delta z = \Delta W \\
& \Rightarrow \frac{\partial B(z)}{\partial z}\Delta z = \Delta W \\
& \Rightarrow K^{T}\Delta z = \Delta W
\end{aligned}
\tag{C.6}
$$

where $K^{tan} = \dfrac{\partial B(z)}{\partial z} = \dfrac{\partial^2 U^{int}(z)}{\partial z^2}$ is the tangent of the curve $W \times z$ at the point z, (see Figure C.6).

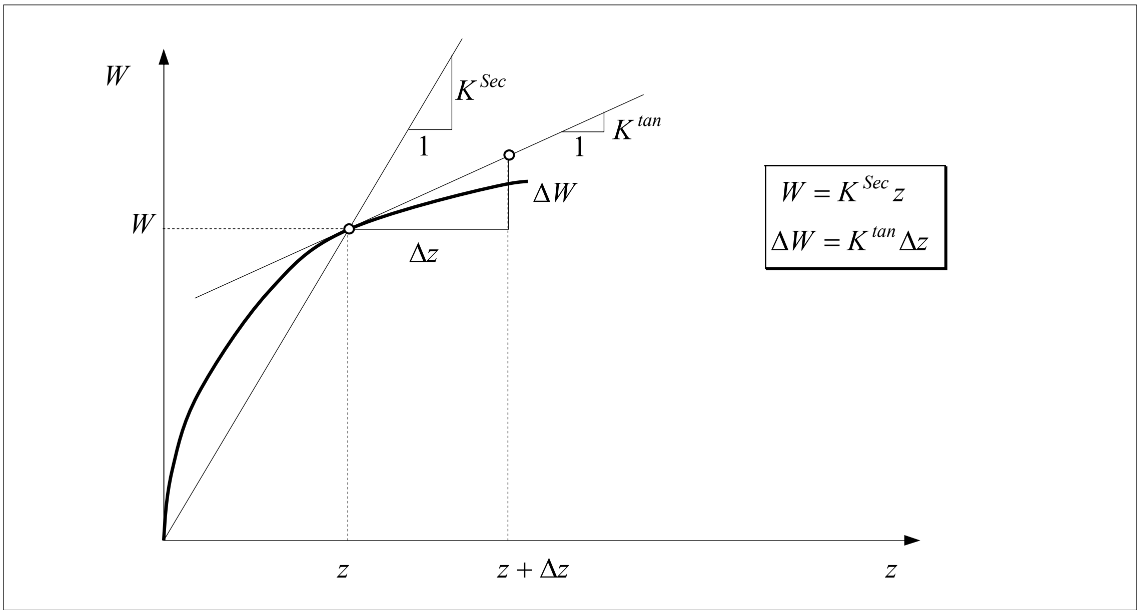

Figure C.6: Tangent *vs.* Secant.

C.2.1 Extension to *n* Dimensions

Suppose that the total potential energy is a function of n variables, i.e. $\Pi = \Pi(u_i)$ for $i = 1,2,3,\cdots,n$. Then, we can obtain that

$$
\begin{aligned}
& \frac{d\Pi(\boldsymbol{u})}{du_i} = \frac{dU^{int}(\boldsymbol{u})}{du_i} - \frac{dU^{ext}(\boldsymbol{u})}{du_i} = \frac{dU^{int}(\boldsymbol{u})}{du_i} - F_i = B_i(\boldsymbol{u}) - F_i = 0 \\
& \Rightarrow B_i(\boldsymbol{u}) = F_i
\end{aligned}
\tag{C.7}
$$

With that we can defined the Secant stiffness matrix $\boldsymbol{K}^{Sec}$ as follows:

$$
F_i = \left[\frac{B_i(\boldsymbol{u})}{u_j}\right] u_j = K_{ij}^{Sec} u_j
\tag{C.8}
$$

Note that we are using indicial notation. Similarly to obtain the equations in (C.6) we can define the Tangent stiffness matrix $\boldsymbol{K}^{tan}$ as follows:

$$
\frac{\partial B_i(\boldsymbol{u})}{\partial u_j}\Delta u_j = \Delta F_i \qquad \Rightarrow \qquad K_{ij}^{tan}\Delta u_j = \Delta F_i
\tag{C.9}
$$

where

$$K_{ij}^{tan} = \frac{\partial B_i(\boldsymbol{u})}{\partial u_j} = \frac{\partial^2 U^{int}(\boldsymbol{u})}{\partial u_i \partial u_j} \quad \textit{Tangent stiffness matrix} \tag{C.10}$$

C.3 Example with 1 Degree-of-Freedom

In this subsection we will adopt the example in the book Crisfield (1991), (see Figure C.7), where A stands for cross-section area of the bar and E represents the Young's modulus.

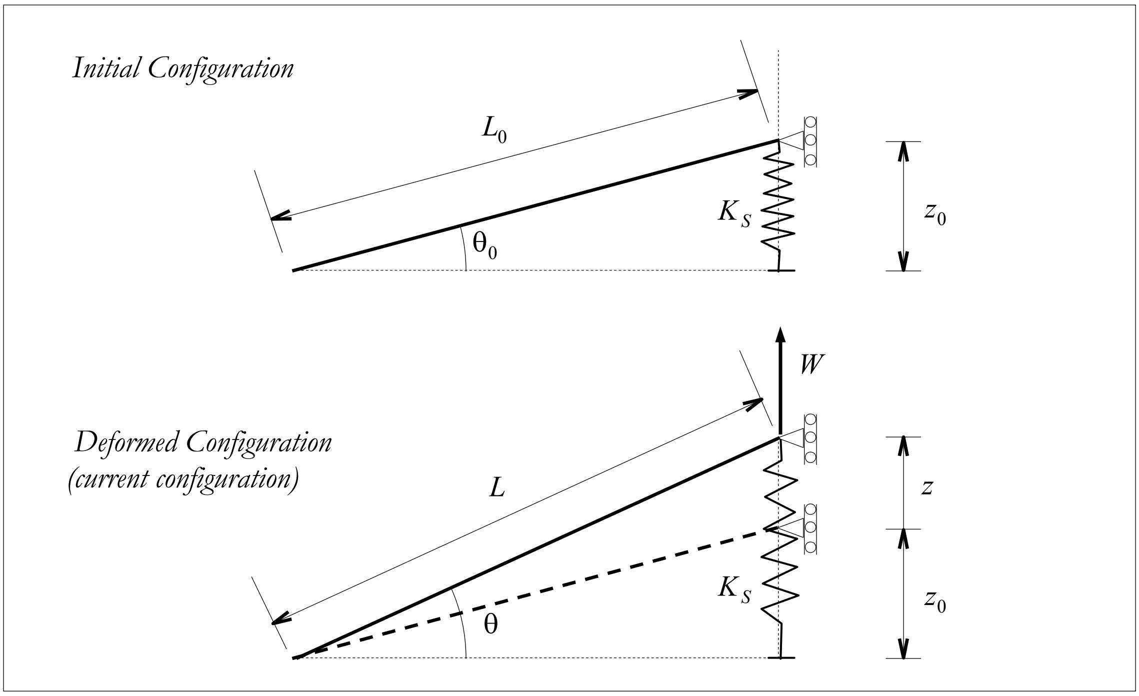

Figure C.7: Example with 1 degree-of-freedom, (Crisfield(1991)).

Making the vertical equilibrium at the node in which the force W is applied, (see Figure C.8), we can obtain:

$$W = N \sin\theta + K_S z = N \frac{(z_0 + z)}{L} + K_S z \cong N \frac{(z_0 + z)}{L_0} + K_S z \tag{C.11}$$

where N is the axial force of the bar and K_S is the spring coefficient. In Crisfield (1991) more detail about this formulation is provided.

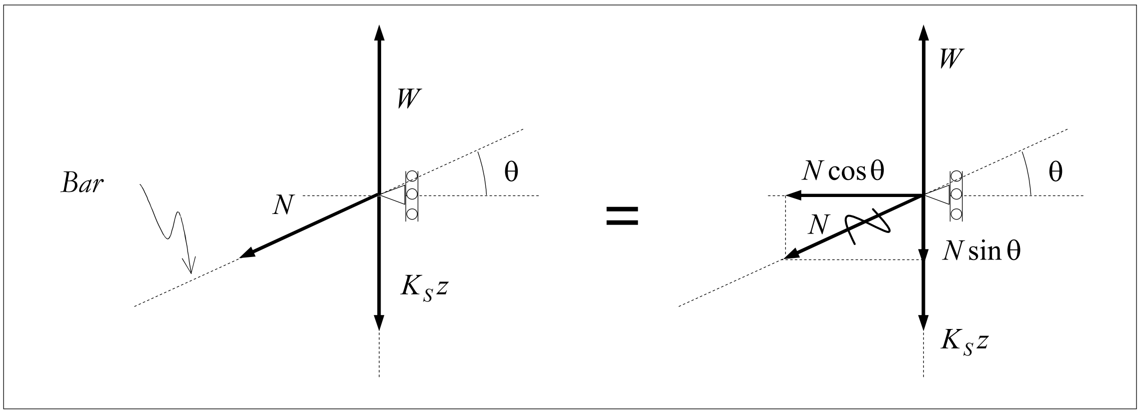

Figure C.8: Equilibrium at the node.

Additionally we can state that:

$$N = \sigma A = EA\varepsilon = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z}{L_0}\right) + \frac{1}{2}\left(\frac{z}{L_0}\right)^2\right] \tag{C.12}$$

and

$$W = W(z) = \frac{EA}{L_0^3}\left[z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right] + K_S z \tag{C.13}$$

For a given value of z we can calculate the tangent of the curve $W = W(z)$ as follows:

$$K^{tan} = \frac{dW}{dz} = \frac{d}{dz}\left[\frac{EA}{L_0^3}\left(z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right) + K_S z\right] = \frac{EA}{L_0^3}\left(z_0^2 + 3 z_0 z + \tfrac{3}{2} z^2\right) + K_S \tag{C.14}$$

We can also express K^{tan} in terms of axial force as follows:

$$\begin{aligned} K^{tan} &= \frac{dW}{dz} = \frac{(z_0 + z)}{L_0}\frac{dN}{dz} + \frac{N}{L_0} + K_S \\ &= \frac{EA}{L_0}\left(\frac{z_0 + z}{L_0}\right)^2 + \frac{N}{L_0} + K_S \\ &= \frac{EA}{L_0^3}(z_0 + z)^2 + \frac{N}{L_0} + K_S \end{aligned} \tag{C.15}$$

Next we will adopt values for $z < 0$ in order to draw the curve $W \times z$, (see Figure C.9), which represents the exact value of the function $W = W(z)$.

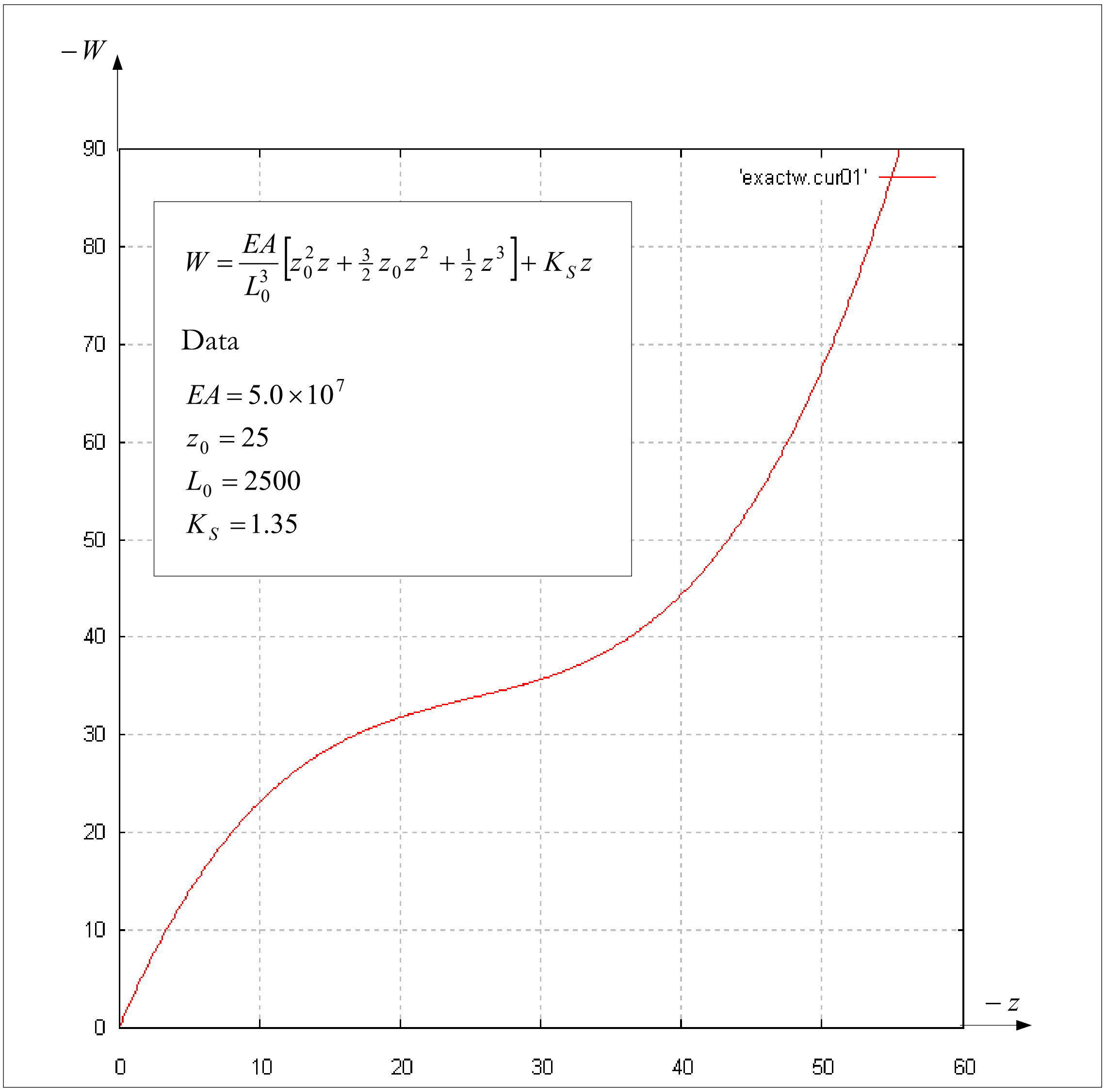

Figure C.9: Function $W(z)$.

Now let us assume that we do not know the function $W(z)$, but instead we know its first derivative $K^{tan} = \frac{dW}{dz}$. In order to obtain the function $W(z)$ we can adopt the Euler's method, (see Annex A). With this method we can control the force $W(z)$ throughout increment of ΔW or we can control the displacement z throughout increments of Δz, (see Figure C.10).

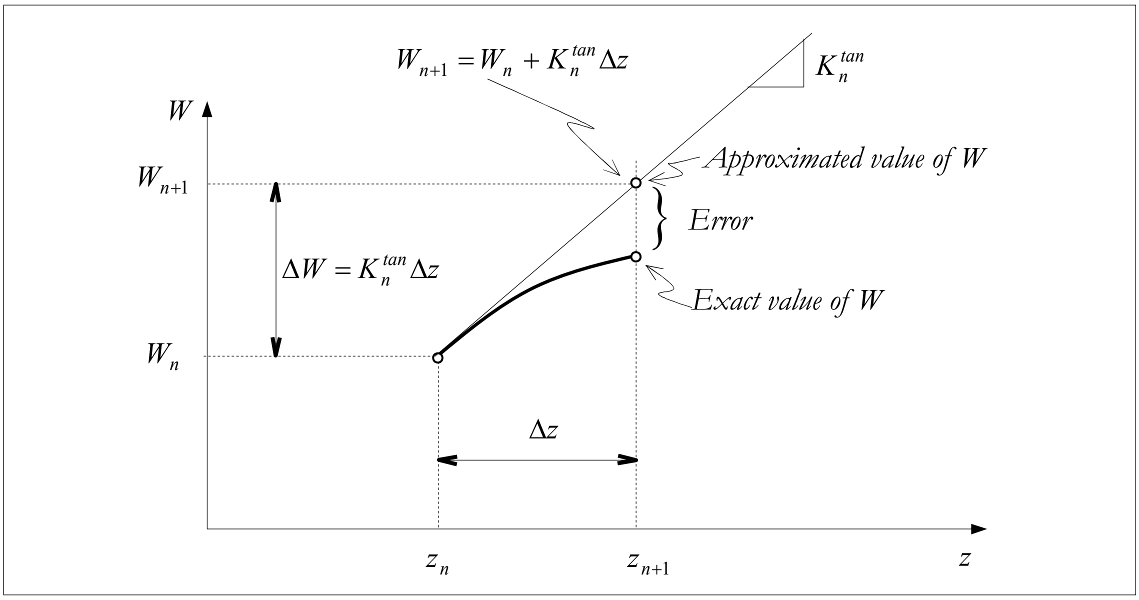

Figure C.10: Euler's method (Implicit method).

For increments of Δz we can apply:

$$K_n^{tan} = \frac{W_{n+1} - W_n}{\Delta z} \quad \Rightarrow \quad W_{n+1} = W_n + K_n^{tan} \Delta z \tag{C.16}$$

And for increments of ΔW we can apply:

$$K_n^{tan} = \frac{W_{n+1} - W_n}{\Delta z} = \frac{\Delta W}{\Delta z} \quad \Rightarrow \quad \Delta z = \left(K_n^{tan}\right)^{-1} \Delta W \tag{C.17}$$

We will apply these two methodologies in order to obtain numerically the curve $W \times z$. For the first case we will impose a displacement increment of $\Delta z = 5.0$ and for the second case an increment in force equal to $\Delta W = 7.0$, (see Figure C.11). If we want a more accurate solution we will need to adopt very small increments, but this procedure could cause a time-consuming from a computational point of view when we are dealing with several degrees-of-freedom. To overcome this drawback we can adopt the incremental-iterative scheme, in which even with big increments we can obtain good results.

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at label 20 for Displacement control and starting at label 30 for Load control).

When we are using force control the solution diverges when the force starts to decrease. To illustrate this behavior let us consider same example, but changing the value of K_S by $K_S = -0.35$, (see Figure C.9). In this situation we can verify that when we are adopting the force control the curve diverge, (see Figure C.12). For this example we have adopt a force increment equal to $\Delta W = 2$.

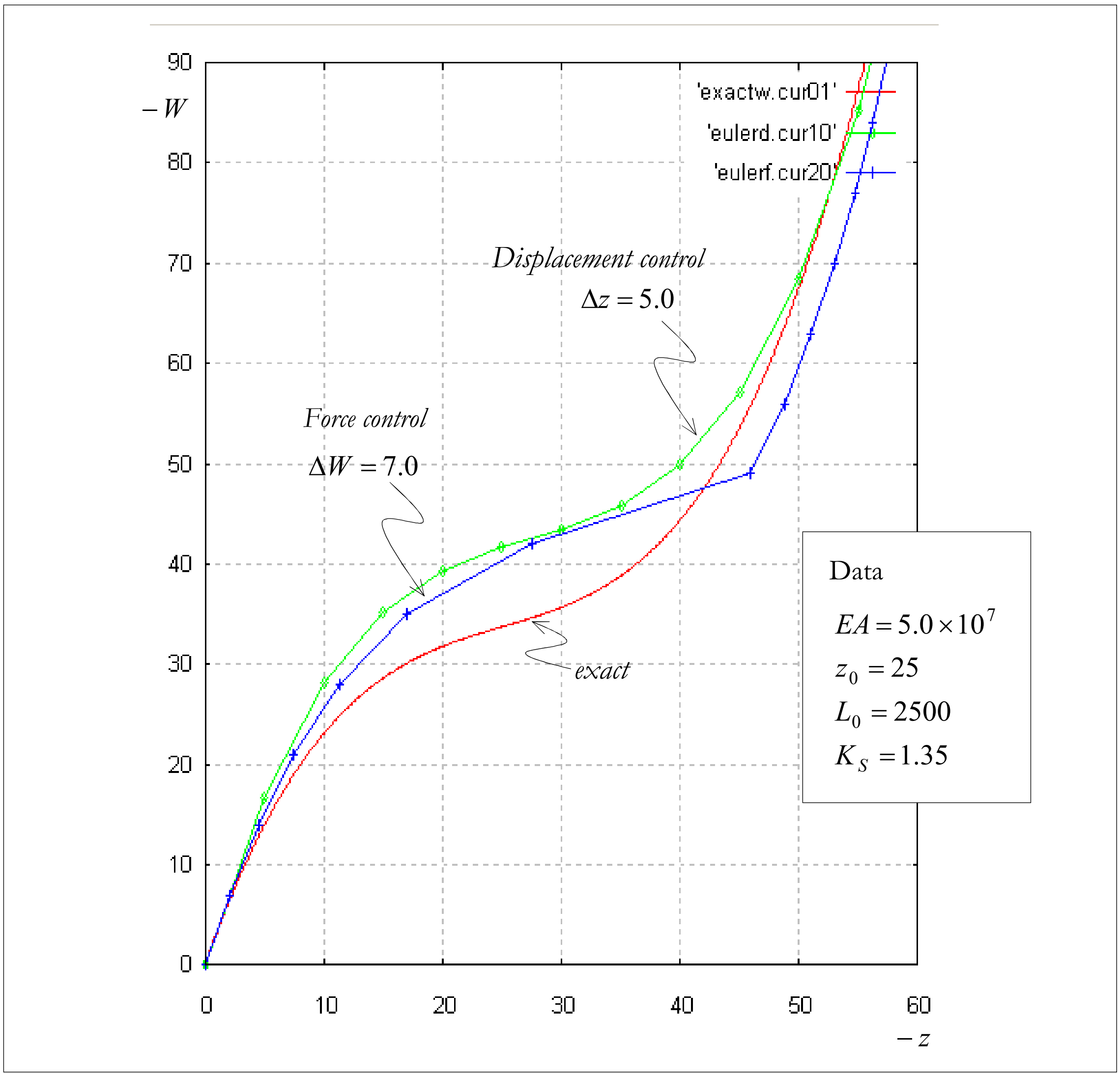

Figure C.11: Incremental solution.

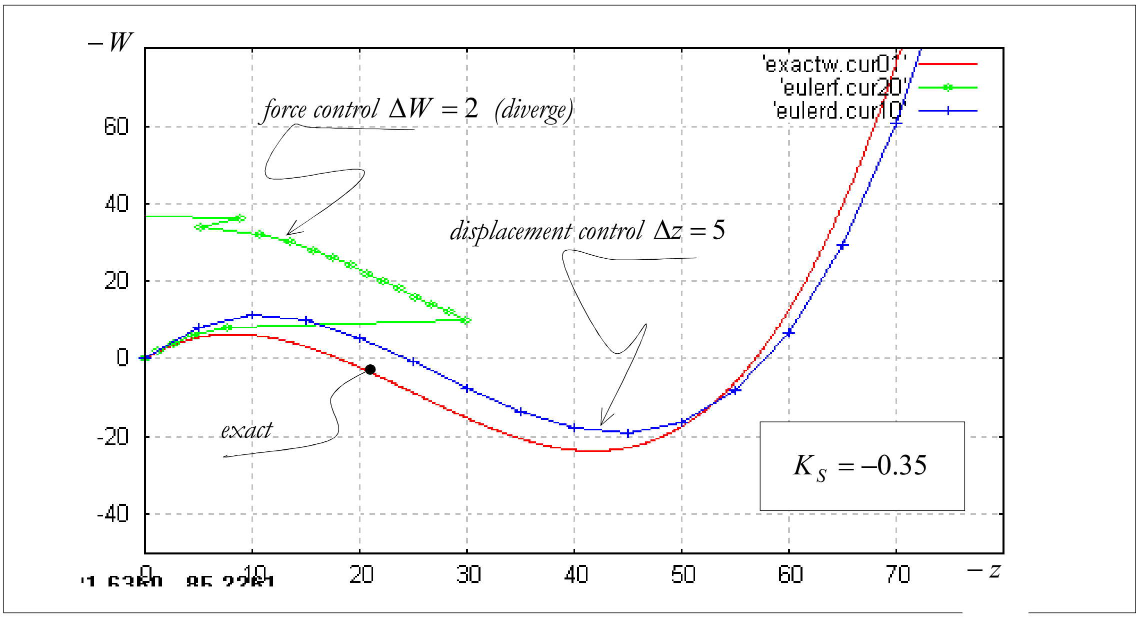

Figure C.12: Incremental solution.

C.3.1 Total Potential Energy

For the example given by Section C.3 the total potential energy is given by:

$$\Pi(z) = U^{int}(z) - U^{ext}(z) = \left(\frac{1}{2}\int_0^{L_0} \frac{N^2}{EA} d\overline{x} + \frac{1}{2} K_S z^2 \right) - (Wz) \tag{C.18}$$

which is the same as:

$$\Pi(z) = \frac{L_0}{2EA} N^2 + \frac{1}{2} K_S z^2 - Wz \tag{C.19}$$

Note that the axial force depends on z, i.e. $N = N(z)$.

By applying the Principle of stationary of the total potential energy we can obtain:

$$\begin{aligned} \frac{d\Pi(z)}{dz} &= \frac{d}{dz}\left(\frac{L_0}{2EA} N^2 + \frac{1}{2} K_S z^2 - Wz \right) = 0 \\ &\Rightarrow \underbrace{\frac{L_0}{EA} N \frac{dN}{dz} + K_S z}_{=\frac{dU^{int}}{dz}} - \underbrace{W}_{=\frac{dU^{ext}}{dz}} = 0 \end{aligned} \tag{C.20}$$

By considering the equation in (C.12) we can obtain:

$$\frac{dN}{dz} = \frac{EA}{L_0^2}\left[z_0 + z \right] \tag{C.21}$$

And by substituting the equation (C.21) into (C.20) we can obtain:

$$\begin{aligned} &\frac{L_0}{EA} N \frac{dN}{dz} + K_S z - W = 0 \quad \Rightarrow \quad \frac{L_0}{EA} N \frac{EA}{L_0^2}\left[z_0 + z \right] + K_S z - W = 0 \\ &\Rightarrow \frac{N}{L_0}\left[z_0 + z \right] + K_S z - W = 0 \\ &\Rightarrow W = \frac{N}{L_0}\left[z_0 + z \right] + K_S z \end{aligned} \tag{C.22}$$

As expected, the above equation is the same as the one given by the equation in (C.11). The tangent matrix can be obtained by means of:

$$\begin{aligned} \frac{d^2 U^{int}(z)}{dz^2} &= \frac{d}{dz}\left[\frac{N}{L_0}(z_0 + z) + K_S z \right] \\ &= \frac{1}{L_0}\frac{dN}{dz}(z_0 + z) + \frac{N}{L_0} + K_S \\ &= \frac{1}{L_0}\frac{EA}{L_0^2}\left[z_0 + z \right](z_0 + z) + \frac{N}{L_0} + K_S \end{aligned} \tag{C.23}$$

which results in:

$$\frac{d^2 U^{int}(z)}{dz^2} = K^{tan} = \frac{EA}{L_0^3}(z_0 + z)^2 + \frac{N}{L_0} + K_S \tag{C.24}$$

which matches the equation in (C.15).

Next, we will draw the curve $\Pi \times z$. To do so, let us adopt the following values for the load W, namely:

$$
\begin{aligned}
z_1 &= -2.5 & \Rightarrow \quad & W_1 = -7.65 \\
z_2 &= -4.0 & \Rightarrow \quad & W_2 = -11.5824 \\
z_3 &= -5.0 & \Rightarrow \quad & W_3 = -13.95
\end{aligned}
\tag{C.25}
$$

where we have used the data of Figure C.9. By means of the total potential energy given by equation (C.19) and with fixed values of W_1, W_2 and W_3 we can obtain, respectively, three curves Π_1, Π_2 and Π_3, (see Figure C.13). As we can verify, for each potential function the extreme (minimum) corresponds to the displacement associated with W_i.

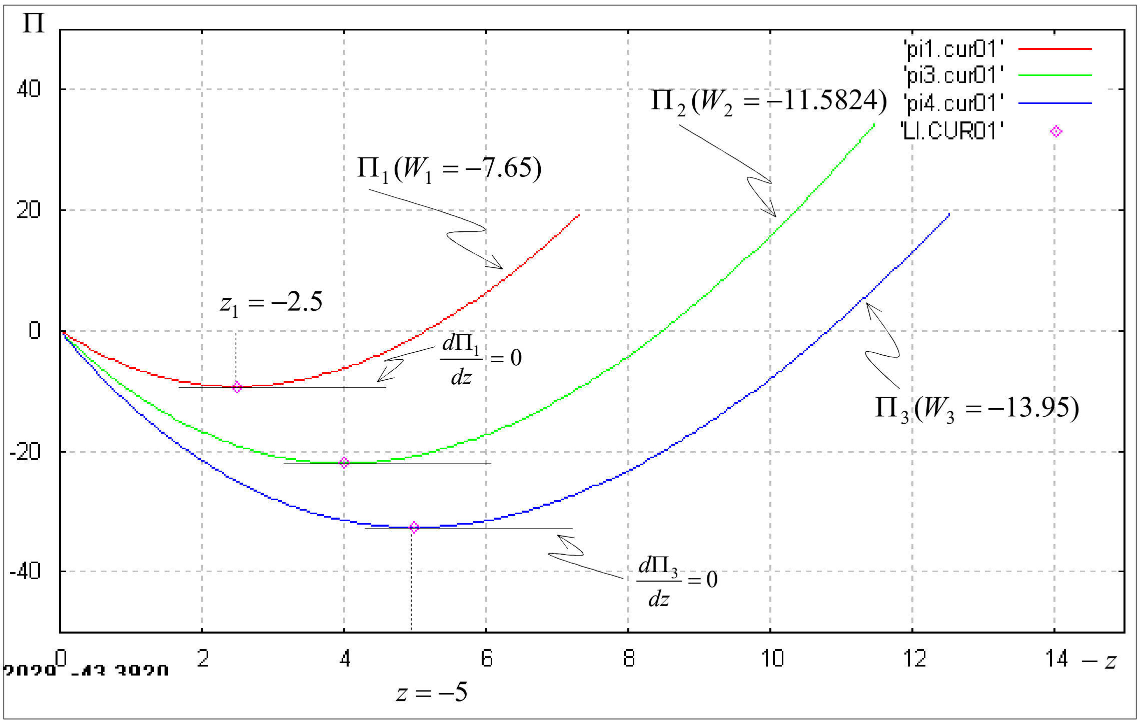

Figure C.13: Total potential ($K_S = 1.35$).

Let us now turn our attention to the example in Figure C.12, where we have considered that $K_S = -0.35$. And we draw the total potential when the load is equal to $W = -2$, (see Figure C.15). If we look at Figure C.14 we can verify that for the value $W = -2$ we have three solutions, namely $z_1 = -1.34047$, $z_2 = -16.24045$ and $z_3 = -57.41908$, (see Figure C.14), whose values are the roots of the cubic equation:

$$
\frac{EA}{L_0^3}\left[z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right] + K_S z - W = 0
$$

These roots correspond exactly to the extremes of Π, i.e. when $\frac{d\Pi}{dz} = 0$, (see Figure C.15). We also emphasize that at z_1 and at z_3 the tangent matrix is positive definite, while at z_2 the tangent matrix is negative definite, in this case indicating that the solution diverges for force increment, but not for an increment of displacement (displacement control).

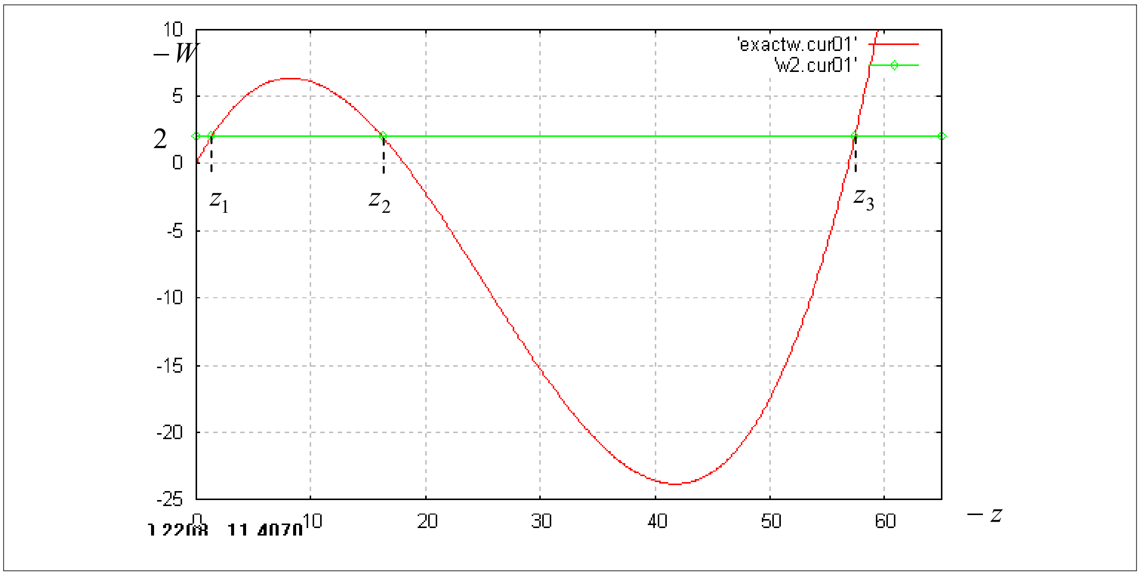

Figure C.14: Function W when $K_S = -0.35$.

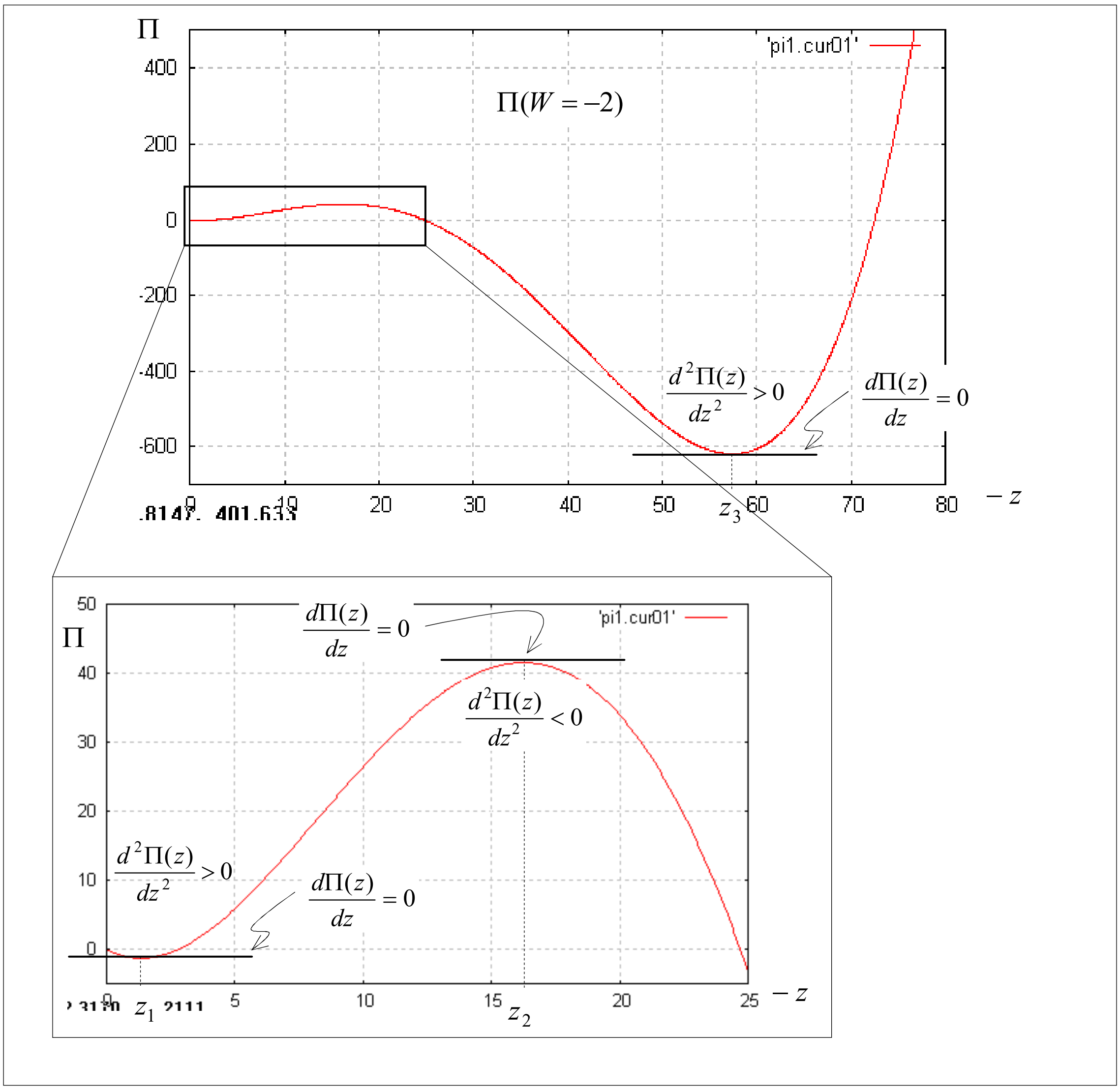

Figure C.15: Total potential $K_S = -0.35$.

C.4 Analyzing the Total Potential Energy

For the next analysis let us consider that we are at the equilibrium point and we add an increment of force ΔW and we search for the next equilibrium point due to this new increment. We know that the solution lies on an extreme point (minimum or maximum).

Let us consider the known displacement vector $\vec{z}_0$ in n dimensions, which is represented by its components $z_{0i} \quad (i=1,2,...,n)$. Let us suppose that $z_i = z_{0i} + \beta d_i$ where d_i represents a vector, supposedly known, that minimizes or maximizes the total potential energy. With that total potential energy is a function of β, i.e. $\Pi(z_i) = \overline{\Pi}(\beta)$.

We apply the chain rule in order to obtain:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial z_i}\frac{\partial z_i}{\partial\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial z_i} d_i \tag{C.26}$$

or in tensorial notation:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}}\cdot\frac{\partial\vec{z}}{\partial\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}}\cdot\vec{\boldsymbol{d}} = \nabla_{\vec{z}}\overline{\Pi}(\beta)\cdot\vec{\boldsymbol{d}} \tag{C.27}$$

where $\vec{\boldsymbol{g}} = \dfrac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}} \equiv \nabla_{\vec{z}}\overline{\Pi}(\beta)$ represents the gradient of $\overline{\Pi}(\beta)$ in the space defined by $\vec{z}$. We rewrite the above equation as follows:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}} \xrightarrow{\textit{indicial}} \frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = g_i d_i \xrightarrow{\textit{matrix form}} \frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \{\boldsymbol{g}\}^T\{\boldsymbol{d}\} \tag{C.28}$$

where $\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}(\vec{z})$ is the residue vector.

Taking once again the derivative of the equation (C.26) we can obtain:

$$\begin{aligned}\frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} &= \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{\partial\overline{\Pi}(\beta)}{\partial z_i}d_i\right) = \left(\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\partial\overline{\Pi}(\beta)}{\partial z_i}d_i + \frac{\partial\overline{\Pi}(\beta)}{\partial z_i}\underbrace{\frac{\mathrm{d}d_i}{\mathrm{d}\beta}}_{=0_i}\right)\\ &= \frac{\partial}{\partial z_i}\left(\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta}\right)d_i = \frac{\partial}{\partial z_i}\left(\frac{\partial\overline{\Pi}(\beta)}{\partial z_j}d_j\right)d_i = d_i\frac{\partial^2\overline{\Pi}(\beta)}{\partial z_i\partial z_j}d_j = d_i S_{ij} d_j\end{aligned} \tag{C.29}$$

or in tensorial notation:

$$\frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = \vec{\boldsymbol{d}}\cdot\frac{\partial^2\overline{\Pi}(\beta)}{\partial\vec{z}\otimes\partial\vec{z}}\cdot\vec{\boldsymbol{d}} = \vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}} \xrightarrow{\textit{matrix form}} \frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = \{\boldsymbol{d}\}^T[\boldsymbol{S}]\{\boldsymbol{d}\} \tag{C.30}$$

where the matrix $\boldsymbol{S}$ is the Hessian of $\overline{\Pi}(\beta)$, and within the scope of structural analysis is called tangent stiffness matrix.

Next, we will express $\overline{\Pi}(z_{0i} + \beta d_i)$ my means of Taylor series:

$$\overline{\Pi}(z_{0i} + \beta d_i) = \overline{\Pi}(\vec{z}_0) + \frac{\partial\overline{\Pi}(\vec{z}_0)}{\partial\beta}\beta + \frac{1}{2!}\frac{\partial^2\overline{\Pi}(\vec{z}_0)}{\partial\beta^2}\beta^2 + ... \tag{C.31}$$

and taking into account $\dfrac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = g_i d_i$ and $\dfrac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = d_i S_{ij} d_j$, the above equation becomes

$$\begin{aligned}&\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})=\overline{\Pi}(\vec{z}_0)+g_i d_i\beta+0.5 d_i S_{ij} d_j\beta^2\\&\Rightarrow \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\Delta\overline{\Pi}=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j)\end{aligned}\tag{C.32}$$

$$\begin{aligned}&\xrightarrow{tensorial} \quad =\Delta\overline{\Pi}=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})\\&\xrightarrow{matrix\ form} \quad =\Delta\overline{\Pi}=\beta\left(\{\boldsymbol{g}\}^T\{\boldsymbol{d}\}+0.5\beta\{\boldsymbol{d}\}^T[\boldsymbol{S}]\{\boldsymbol{d}\}\right)\end{aligned}\tag{C.33}$$

We can deal with two scenarios:

Minimum

$$\begin{aligned}&\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})<\overline{\Pi}(\vec{z}_0) \quad\Rightarrow\quad \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j)<0\\&\Rightarrow \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})=\beta(\vec{\boldsymbol{g}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S})\cdot\vec{\boldsymbol{d}}<0\end{aligned}\tag{C.34}$$

In this situation, $\boldsymbol{S}$ is positive definite, i.e. all its eigenvalues are positive, and as consequence $d_i S_{ij} d_j>0$. with that we can conclude that $g_i d_i<0$ and is the predominate term for small values of $\beta>0$:

$$g_i d_i<0 \quad \xrightarrow{tensorial} \quad \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}<0\tag{C.35}$$

Maximum

$$\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})>\overline{\Pi}(\vec{z}_0) \quad\Rightarrow\quad \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j)>0\tag{C.36}$$

In this case, $\boldsymbol{S}$ is definite negative, i.e. all its eigenvalues are negative, and as consequence $d_i S_{ij} d_j<0$. With that we can conclude that $g_i d_i>0$ and it is the predominate term for small values of $\beta>0$:

$$g_i d_i>0 \quad \xrightarrow{tensorial} \quad \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}>0\tag{C.37}$$

For better illustration of the previous development, we will make an analogy where we have a mountain and depression in which they are represented by its level curves, (see Figure C.16). Recall that the gradient always points towards the growing sense of the function and is normal to the level curves.

If the parameters $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{d}}$ are fixed, the value of β that minimize or maximize the function $\Delta\overline{\Pi}(\beta)$, (see equation (C.34)), is given by:

$$\begin{aligned}&\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\Delta\overline{\Pi}=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})\\&\Rightarrow\frac{\mathrm{d}\Delta\overline{\Pi}}{\mathrm{d}\beta}=\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}=0\\&\Rightarrow\beta=\frac{-\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}}\end{aligned}\tag{C.38}$$

Given a known state $\vec{z}_0$, now the question is: Which is the value of $\vec{\boldsymbol{d}}$ in which we must adopt in order to guarantee that we are reaching an extreme?

If we are looking for a minimum, it is enough to adopt a matrix $\boldsymbol{A}$ which is positive definite, with that the equation $\vec{\boldsymbol{g}}\cdot\boldsymbol{A}\cdot\vec{\boldsymbol{g}}>0$ holds. Then, the vector $\vec{\boldsymbol{d}}$ could be defined such as:

$$\vec{\boldsymbol{d}}=-\boldsymbol{A}\cdot\vec{\boldsymbol{g}}\tag{C.39}$$

In this way we guarantee that:

$$\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}=-\vec{\boldsymbol{g}}\cdot\boldsymbol{A}\cdot\vec{\boldsymbol{g}}<0\tag{C.40}$$

The choice for the matrix $\boldsymbol{A}$ is what given origin for several iterative methods.

Taking into account that given a matrix which is positive definite its inverse is also definite positive, we adopt $\boldsymbol{A} = \boldsymbol{S}^{-1}$, with that $\vec{\boldsymbol{d}} = -\boldsymbol{A} \cdot \vec{\boldsymbol{g}} = -\boldsymbol{S}^{-1} \cdot \vec{\boldsymbol{g}} \Rightarrow \vec{\boldsymbol{g}} = -\boldsymbol{S} \cdot \vec{\boldsymbol{d}}$. Then, we can conclude that:

$$\beta = \frac{-\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}} \cdot \boldsymbol{S} \cdot \vec{\boldsymbol{d}}} = \frac{-(-\boldsymbol{S} \cdot \vec{\boldsymbol{d}}) \cdot \vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}} \cdot \boldsymbol{S} \cdot \vec{\boldsymbol{d}}} = \frac{\vec{\boldsymbol{d}} \cdot \boldsymbol{S} \cdot \vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}} \cdot \boldsymbol{S} \cdot \vec{\boldsymbol{d}}} = 1 \tag{C.41}$$

With these conditions we obtain the Newton-Raphson's method, in which the matrix $\boldsymbol{A}$ is the inverse of the Hessian matrix (tangent stiffness matrix) $\boldsymbol{A} = \boldsymbol{S}^{-1}$.

$$\beta^{(k)} = \frac{-\vec{\boldsymbol{g}}^{(k)} \cdot \vec{\boldsymbol{d}}^{(k)}}{\vec{\boldsymbol{d}}^{(k)} \cdot \boldsymbol{S}^{(k)} \cdot \vec{\boldsymbol{d}}^{(k)}} = \frac{-\vec{\boldsymbol{g}}^{(k)} \cdot \vec{\boldsymbol{d}}^{(k)}}{\vec{\boldsymbol{d}}^{(k)} \cdot (\vec{\boldsymbol{g}}^{(k+1)} - \vec{\boldsymbol{g}}^{(k)})} \tag{C.42}$$

Note that $\boldsymbol{S}^{(k)} \cdot \vec{\boldsymbol{d}}^{(k)} = \vec{\boldsymbol{g}}^{(k+1)} - \vec{\boldsymbol{g}}^{(k)}$.

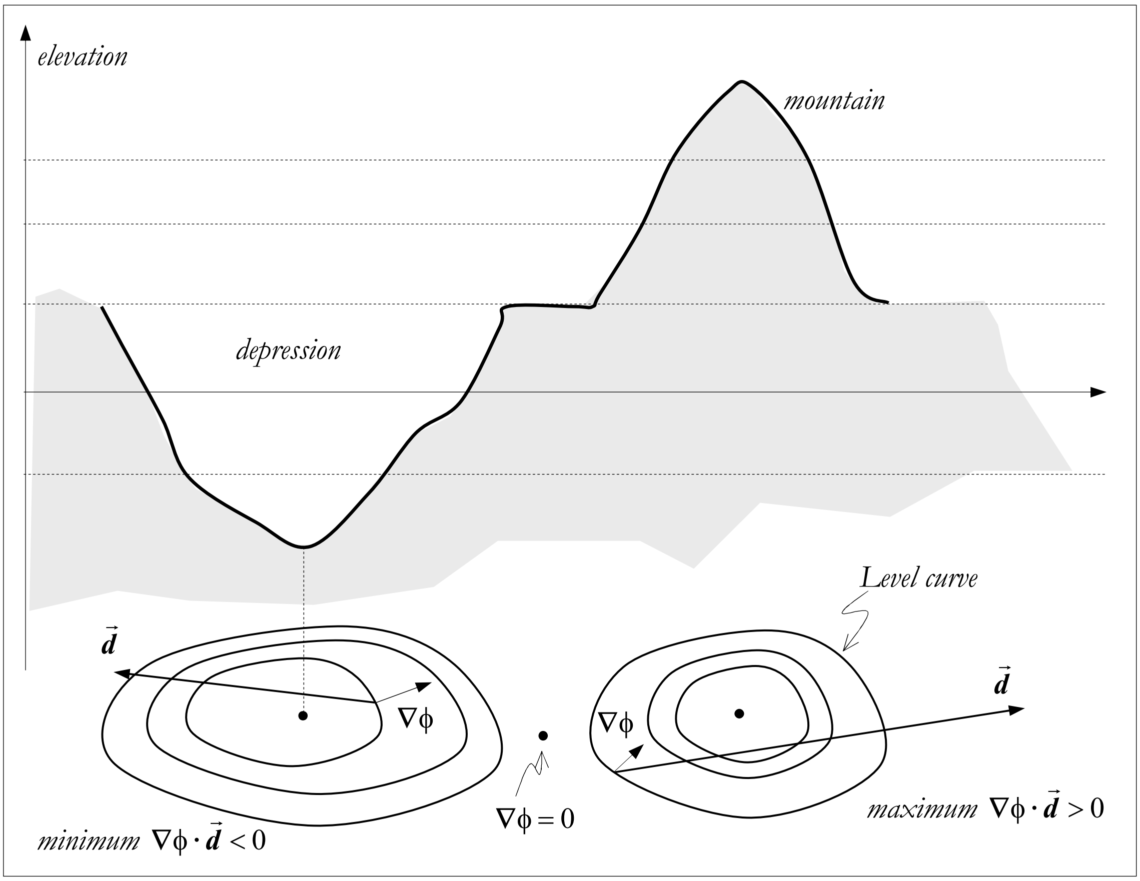

Figure C.16: Level curves.

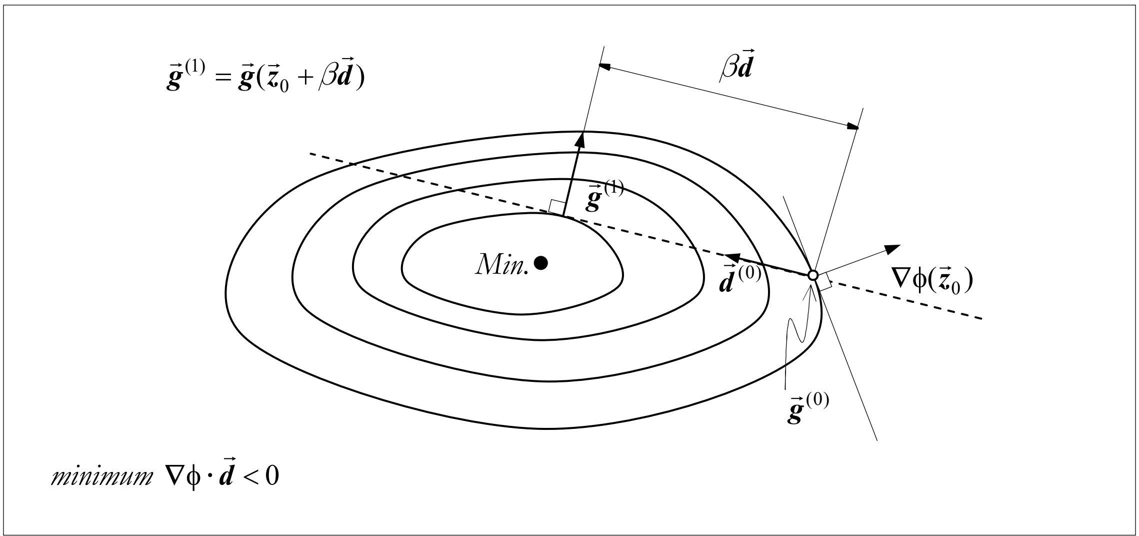

Figure C.17

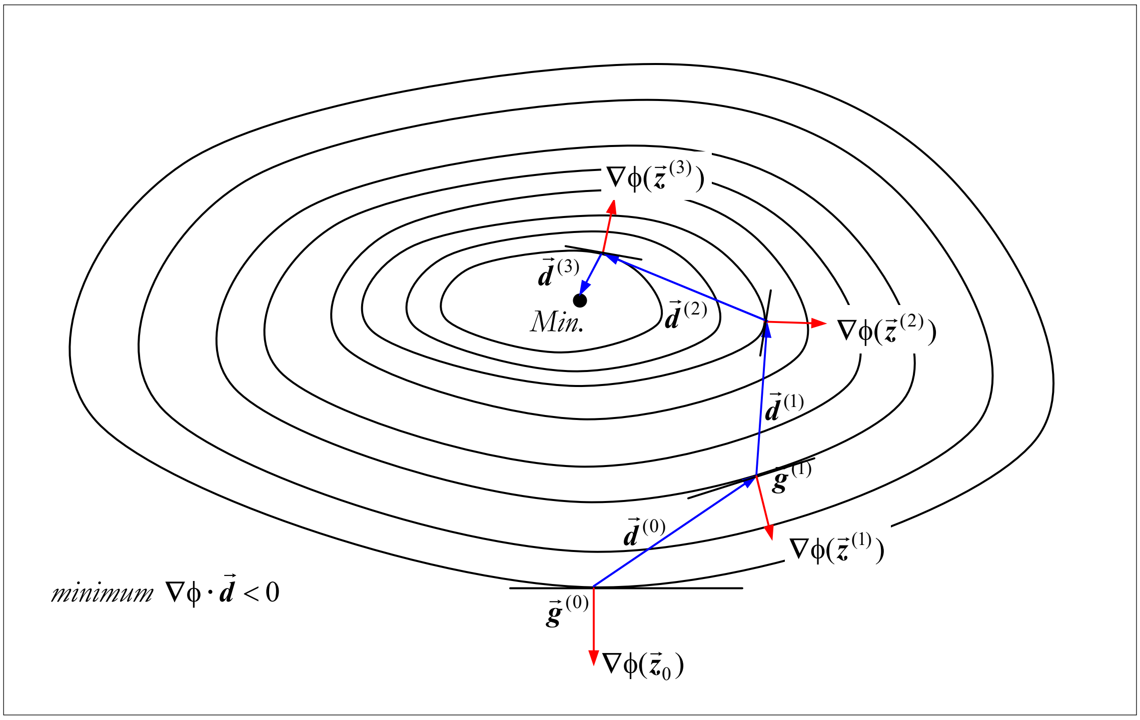

Figure C.18: Iterative process.

C.5 Classical Formulation of the Newton-Raphson's Method

The Newton-Raphson's is employed to obtain the roots of a function, i.e. $f(x)=0$, (Chapra&Canale(1988)). Newton in 1669 has obtained a version of the method and Raphson in 1690 has generalized the method.

The formulation of the Newton-Raphson's method can be obtained by means of Taylor series. For example, given the function $f(x)$ we can approach the value of the function $f(x+\Delta x)$ by Taylor series as follows:

$$\begin{aligned} f(x+\Delta x) &= f(x_0)+\frac{1}{1!}\frac{\partial f(x_0)}{\partial x}\Delta x+\frac{1}{2!}\frac{\partial^2 f(x_0)}{\partial x^2}\Delta x^2+\ldots \\ &\approx f(x_0)+f'(x_0)\Delta x \end{aligned} \tag{C.43}$$

where we have considered until linear terms. Let us adopt the following nomenclature: $x_{i+1}=x_i+\Delta x$, $\Delta x=x_{i+1}-x_i$ and for the application point $x_i=x_0$. With that the above equation can be rewritten as follows:

$$f(x_{i+1})=f(x_i)+f'(x_i)\Delta x=f(x_i)+f'(x_i)(x_{i+1}-x_i) \tag{C.44}$$

Here the index (i) does not indicate indicial notation.

As we are searching for the roots of the function $f(x_{i+1})=0$ we can obtain:

$$\begin{aligned} & f(x_{i+1})=f(x_i)+f'(x_i)(x_{i+1}-x_i)=0 \\ & \Rightarrow f(x_i)+x_{i+1}f'(x_i)-x_i f'(x_i)=0 \\ & \Rightarrow x_{i+1}f'(x_i)=x_i f'(x_i)-f(x_i) \\ & \Rightarrow x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \end{aligned} \tag{C.45}$$

Once x_{i+1} is obtained the values of the function $(f(x_{i+1}))$ and its derivative $(f'(x_{i+1}))$ at the point can be obtained. This procedure is repeated until $f(x_{i+n})\approx 0$ is reached. It can be shown that the error associated with it is proportionally to the square of the previous error, i.e. it has *quadratic convergence*. For more details the lector is referred to Chapra&Canale (1988).

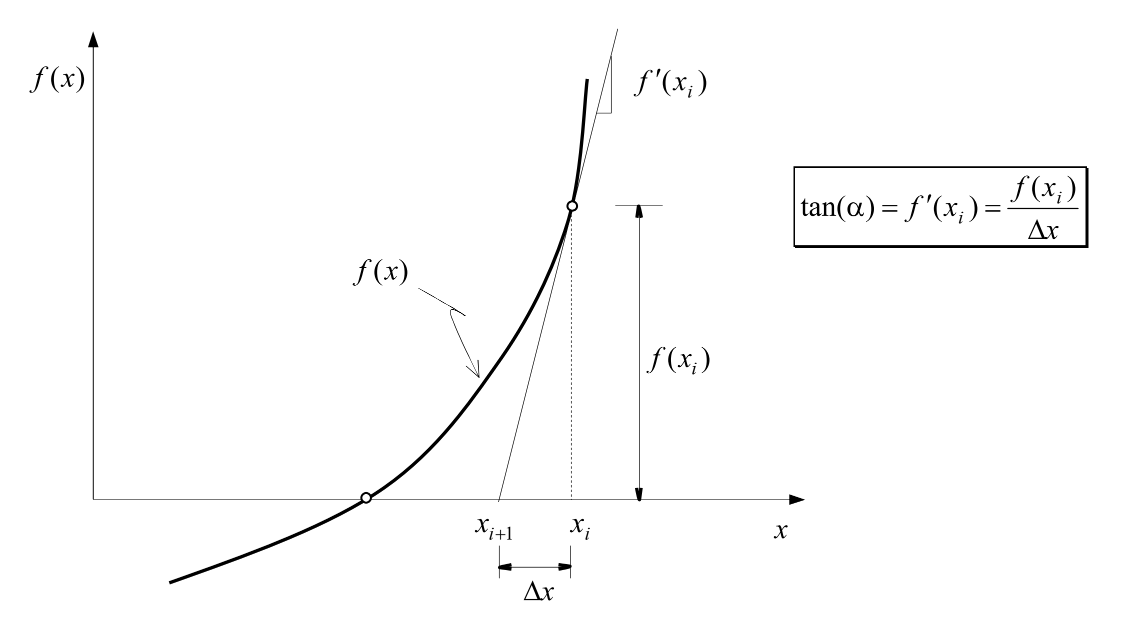

Figure C.19: Newton-Raphson's method.

In general, in structure analysis it is taken as a direct solution procedure to solve the set of equations. For a non-linear problem an incremental/iterative strategy is adopted. In this way we apply and increment of load (*load step*) and in each load step we apply iterative procedure in order to achieve the equilibrium point.

Let us consider that the function is the residue $\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}})$ and we know its value and its derivative, $\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}}$, at the application point $\vec{\boldsymbol{u}}_0$. Then, by means of Taylor series we can obtain:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) + \frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \cdot \Delta \vec{\boldsymbol{u}} \tag{C.46}$$

or in indicial notation:

$$\begin{aligned} g_i(\vec{\boldsymbol{u}}) &= g_i(\vec{\boldsymbol{u}}_0) + \frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j} \Delta u_j = g_i(\vec{\boldsymbol{u}}_0) + \frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j}(u_j - u_{0j}) \\ &= g_i(\vec{\boldsymbol{u}}_0) - K_{ij}^{tan}(u_j - u_{0j}) \end{aligned} \tag{C.47}$$

where $\frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j} = -K_{ij}^{tan}(\vec{\boldsymbol{u}}_0)$ is the Jacobian matrix which in structural analysis ambit represents the tangent stiffness matrix at the application point $\vec{\boldsymbol{u}}_0$. As we are looking for the value in which $\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{0}}$, we can state that:

$$\begin{aligned} & g_i(\vec{\boldsymbol{u}}) = g_i(\vec{\boldsymbol{u}}_0) - K_{ij}^{tan}(u_j - u_{0j}) = 0_i \quad \Rightarrow \quad -g_i(\vec{\boldsymbol{u}}_0) = -K_{ij}^{tan}(u_j - u_{0j}) \\ & \Rightarrow -(K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) = -(K_{ki}^{tan})^{-1} K_{ij}^{tan}(u_j - u_{0j}) \\ & \Rightarrow (K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) = \delta_{kj}(u_j - u_{0j}) = (u_k - u_{0k}) \\ & \Rightarrow u_k = u_{0k} + (K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) \end{aligned} \tag{C.48}$$

With that we can conclude:

$$u_i = u_{0i} + (K_{ij}^{tan})^{-1} g_j(\vec{\boldsymbol{u}}_0) = u_{0i} + \Delta u_i \tag{C.49}$$

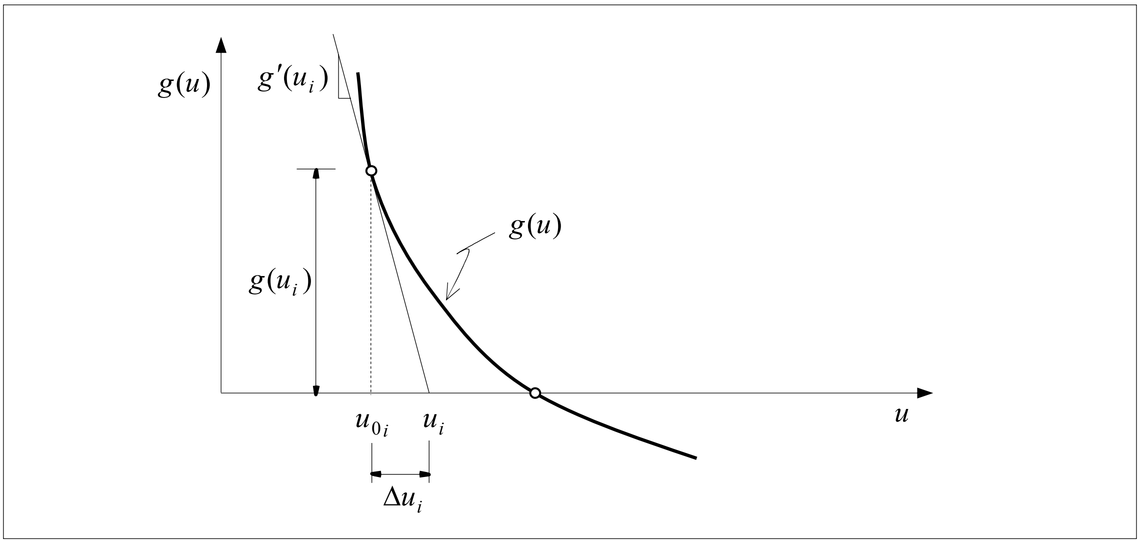

Figure C.20: Residue function.

We can generalized the equation in (C.46) such as:

$$\begin{aligned}
&\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) + \frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \cdot \Delta\vec{\boldsymbol{u}} \\
&\Rightarrow \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) - \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) = \left(\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}}\right) \cdot \Delta\vec{\boldsymbol{u}} \\
&\Rightarrow \Delta\vec{\boldsymbol{g}}^{(k)} = \left(\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}}\right)^{(k)} \cdot \Delta\vec{\boldsymbol{u}}^{(k)} \\
&\Rightarrow \Delta\vec{\boldsymbol{g}}^{(k)} = \boldsymbol{H}^{(k)} \cdot \Delta\vec{\boldsymbol{u}}^{(k)}
\end{aligned} \tag{C.50}$$

where (k) means iterations. When $\boldsymbol{H}^{(k)}$ represents the tangent matrix and changes for each iteration we fallback into the classical Newton-Raphson's method. When the matrix $\boldsymbol{H}^{(k)}$ does not change during the iterations we are dealing with the so-called Modified Newton-Raphson's method. We can also state that:

$$\begin{aligned}
&\Delta\vec{\boldsymbol{g}}^{(k)} = \boldsymbol{H}^{(k)} \cdot \Delta\vec{\boldsymbol{u}}^{(k)} \\
&\Rightarrow \Delta\vec{\boldsymbol{g}}^{(k)} = (\boldsymbol{H}^{(k-1)} + \boldsymbol{E}^{(k)}) \cdot \Delta\vec{\boldsymbol{u}}^{(k)}
\end{aligned} \tag{C.51}$$

where $\boldsymbol{E}^{(k)}$ is a correction matrix. In a generic way we can adopt:

$$\boldsymbol{E}^{(k)} = \alpha\vec{\boldsymbol{a}} \otimes \vec{\boldsymbol{a}} + \beta\vec{\boldsymbol{b}} \otimes \vec{\boldsymbol{b}} \tag{C.52}$$

where we can adopt $\vec{\boldsymbol{a}} = \vec{\boldsymbol{u}}^{(k)}$, $\vec{\boldsymbol{b}} = \boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}$. With that the above equation becomes:

$$\boldsymbol{E}^{(k)} = \alpha\vec{\boldsymbol{u}}^{(k)} \otimes \vec{\boldsymbol{u}}^{(k)} + \beta(\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \otimes (\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \tag{C.53}$$

The values for α and β can be determined in a such a way that the equation in (C.51) is fulfilled. For example:

$$\alpha\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{g}}^{(k)} = 1 \quad \Rightarrow \quad \alpha = \frac{1}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{g}}^{(k)}} = \frac{1}{\vec{\boldsymbol{u}}^{(k)} \cdot \vec{\boldsymbol{g}}^{(k)}} \tag{C.54}$$

$$\beta \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{g}}^{(k)} = -1 \quad \Rightarrow \quad \beta = \frac{-1}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{g}}^{(k)}} = \frac{-1}{(\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \cdot \vec{\boldsymbol{g}}^{(k)}} = \frac{-1}{\vec{\boldsymbol{g}}^{(k)} \cdot \boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}} \tag{C.55}$$

In this way we can obtain:

$$\boldsymbol{E}^{(k)} = \frac{\vec{\boldsymbol{u}}^{(k)} \otimes \vec{\boldsymbol{u}}^{(k)}}{\vec{\boldsymbol{u}}^{(k)} \cdot \vec{\boldsymbol{g}}^{(k)}} - \frac{(\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \otimes (\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)})}{\vec{\boldsymbol{g}}^{(k)} \cdot \boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}} \tag{C.56}$$

And in turn we define the matrix $\boldsymbol{H}^{(k)}$ as follows:

$$\begin{aligned} &\boldsymbol{H}^{(k)} = \boldsymbol{H}^{(k-1)} + \boldsymbol{E}^{(k)} \\ &\Rightarrow \boldsymbol{H}^{(k)} = \boldsymbol{H}^{(k-1)} + \frac{\vec{\boldsymbol{u}}^{(k)} \otimes \vec{\boldsymbol{u}}^{(k)}}{\vec{\boldsymbol{u}}^{(k)} \cdot \vec{\boldsymbol{g}}^{(k)}} - \frac{(\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \otimes (\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)})}{\vec{\boldsymbol{g}}^{(k)} \cdot \boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}} \end{aligned} \tag{C.57}$$

This method is called BFGS method (Broyden-Fletcher-Goldfarb-Shanno). Note that if we know the inverse of $\boldsymbol{H}^{(k-1)}$, the inverse of $\boldsymbol{H}^{(k)}$ can be easily obtained by means of **Problem 1.87**.

C.6 Newton-Raphson' Method

Let us return to our initial problem proposed in Section C.2 and let us assume that we know the values W_n and z_n. For the force increment ΔW_{n+1} we can state that $W_{n+1} = W_n + \Delta W_{n+1}$. For the next load step, represented by $n+1$, we have that for the first iteration (i):

PREDICTION $(i = 0)$

Tangent matrix calculation:

$$N_{n+1}^{i} = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z_{n+1}^{i}}{L_0}\right) + \frac{1}{2}\left(\frac{z_{n+1}^{i}}{L_0}\right)^2\right] \xrightarrow{if \quad i=1} N_{n+1}^{i} \leftarrow N_n = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z_n}{L_0}\right) + \frac{1}{2}\left(\frac{z_n}{L_0}\right)^2\right] \tag{C.58}$$

$$\begin{aligned} &(K^{tan})_{n+1}^{i} = \frac{EA}{L_0^3}(z_0 + z_{n+1}^{i})^2 + \frac{N_{n+1}^{i}}{L_0} + K_S \\ &if \quad i = 1 \Rightarrow (K^{tan})_{n+1}^{i} \leftarrow (K^{tan})_n = \frac{EA}{L_0^3}(z_0 + z_n)^2 + \frac{N_n}{L_0} + K_S \end{aligned} \tag{C.59}$$

Solve the system:

$$\Delta z_{n+1}^{i} = \left(K_{n+1}^{tan\,i}\right)^{-1} \mathcal{R}_{n+1}^{i} \xrightarrow{si \quad i=1} \Delta z_{n+1}^{i} = \left(K_n^{tan}\right)^{-1} \Delta W_{n+1} \tag{C.60}$$

where $\mathcal{R}_{n+1}^{i}$ is the residue. Once the above equation is solved, we can obtain:

$$z_{n+1}^{i} = z_n + \Delta z_{n+1}^{i} \tag{C.61}$$

And in turn we can calculate the internal forces for this iteration:

$$N^i_{n+1} = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z^i_{n+1}}{L_0}\right)+\frac{1}{2}\left(\frac{z^i_{n+1}}{L_0}\right)^2\right] \tag{C.62}$$

and

$$\overline{W}^i_{n+1} = N^i_{n+1}\frac{(z_0 + z^i_{n+1})}{L_0} + K_S z^i_{n+1} \tag{C.63}$$

and the residue can be obtained:

$$\mathcal{R}^{i=2}_{n+1} = \overline{W}^{i=1}_{n+1} - W_{n+1} \tag{C.64}$$

Next, we obtain a norm in order to check the convergence. For our one-dimensional example the residue is a scalar, and we check whether this norm is less than a tolerance. If so, we go to the next load step $(n \leftarrow n+1)$. Otherwise, we have to do a new iteration $(i \leftarrow i+1)$ until the convergence is achieved, (see Figure C.21).

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at *label 40).*

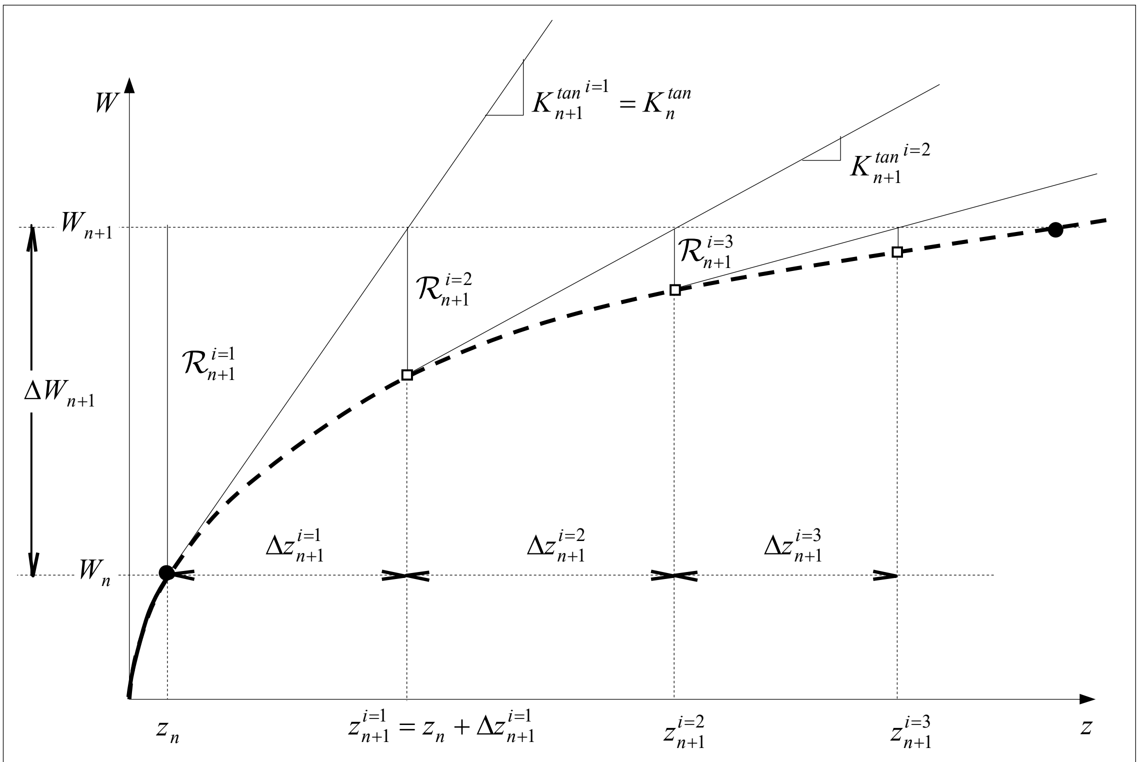

Figure C.21: Newton-Raphson iterative procedure.

For our example we will consider an load increment equal to $\Delta W = 30$. With that we can obtain the graph describe in Figure C.22. And in Figure C.23 the curve for several load steps.

As we can see in Figure C.22, for each iteration, we need to calculate the tangent matrix. This is a drawback of the Newton-Raphson's method, since for a system with a large number of variables it could be costly, from a computation point of view, to calculate the stiffness matrix for each iteration.

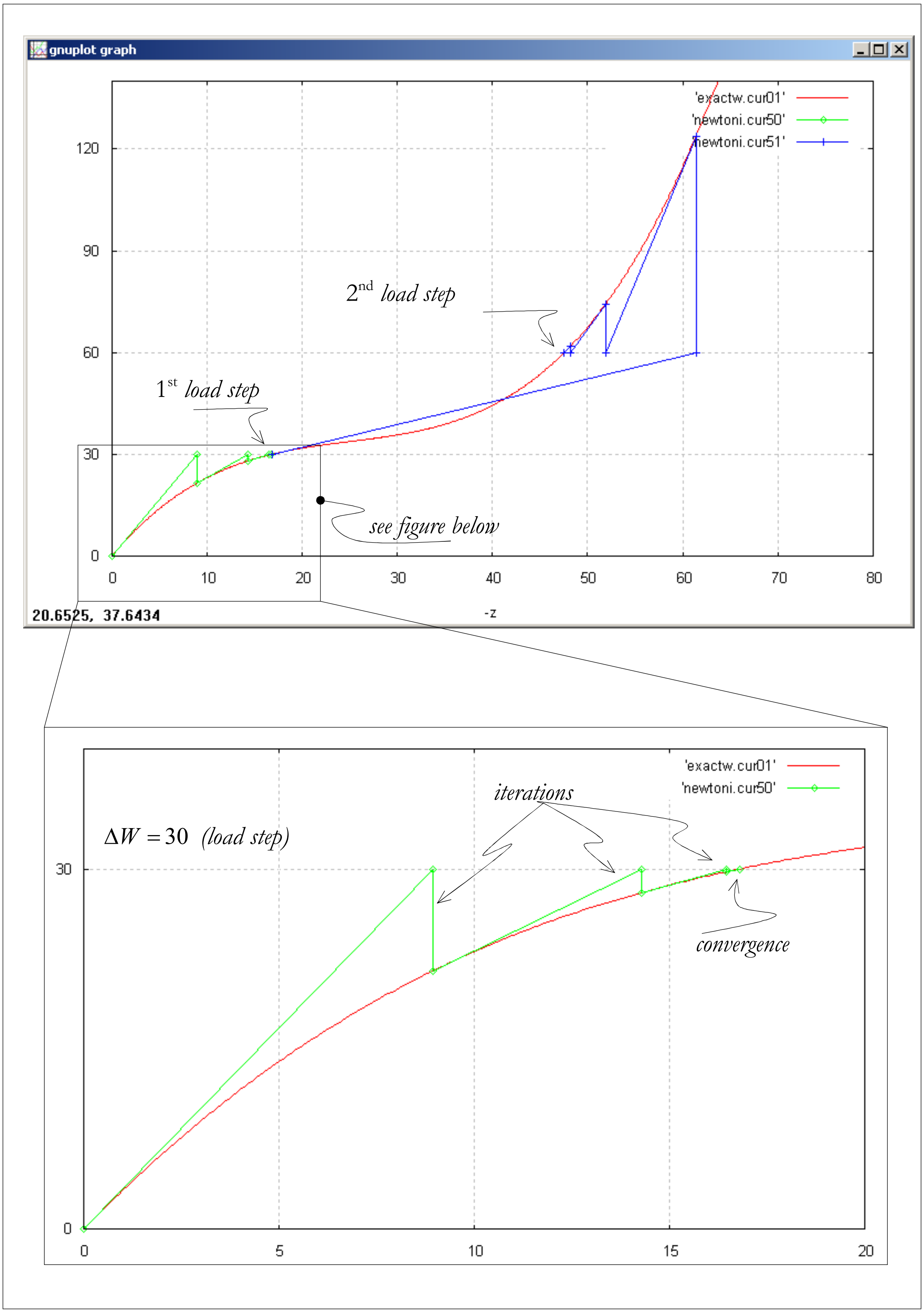

Figure C.22: Newton-Raphson iterative procedure.

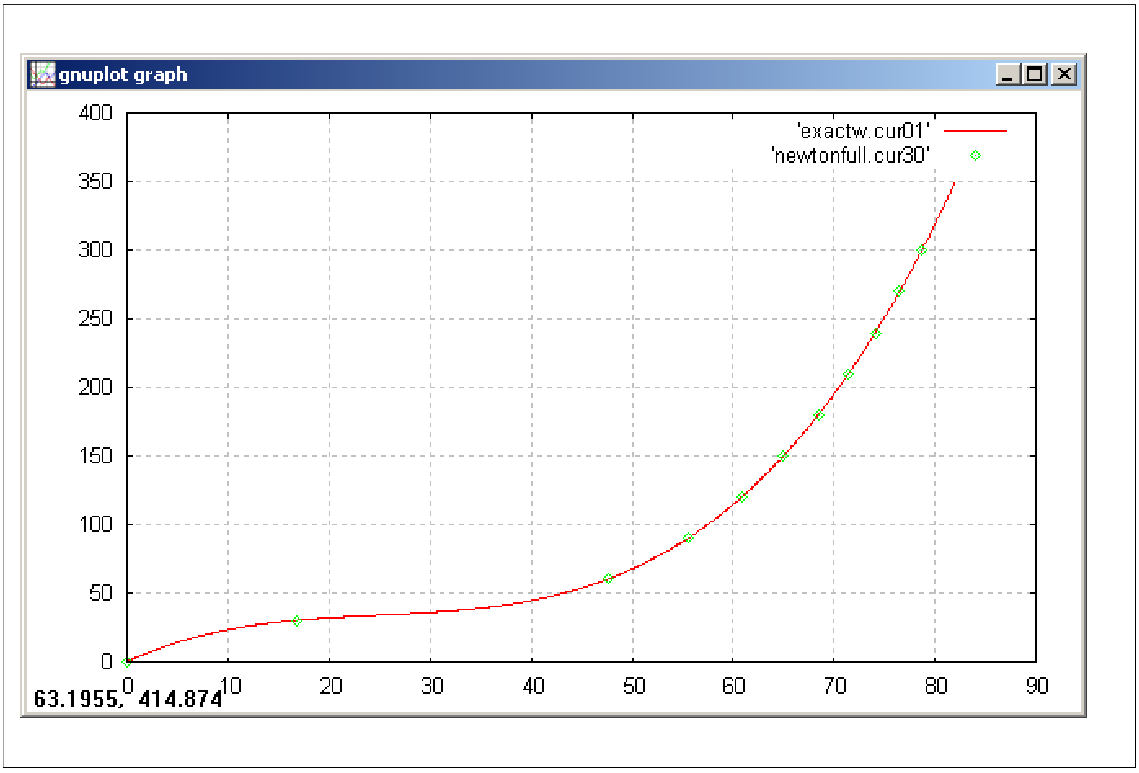

Figure C.23: Example of Newton-Raphson iterative procedure.

C.7 Modified Newton-Raphson Method

Several methods have been formulated based on the Newton-Raphson's method, e.g. the modified Newton-Raphson method, which basically consists in adopt the same tangent matrix for each iteration, (see Figure C.24). This method requires more iterations than the Full Newton-Raphson method and beside has no convergence if we are dealing with inflection point as the one describe in Figure C.22. Just to illustrate this method, we apply the load increment equal to $\Delta W = 30$ which is before the inflection point, (see Figure C.25). As we can see the modified Newton-Raphson method needs more iterations to achieve convergence.

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at *label 50).*

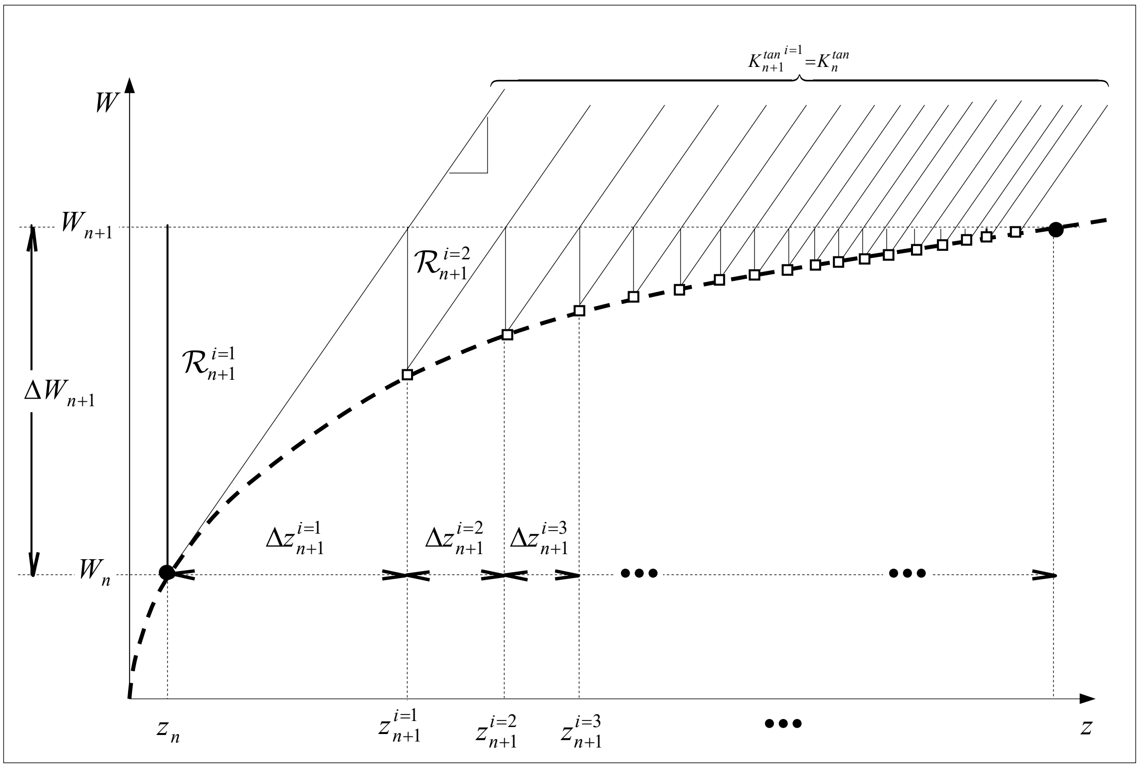

Figure C.24: Modified Newton-Raphson iterative procedure.

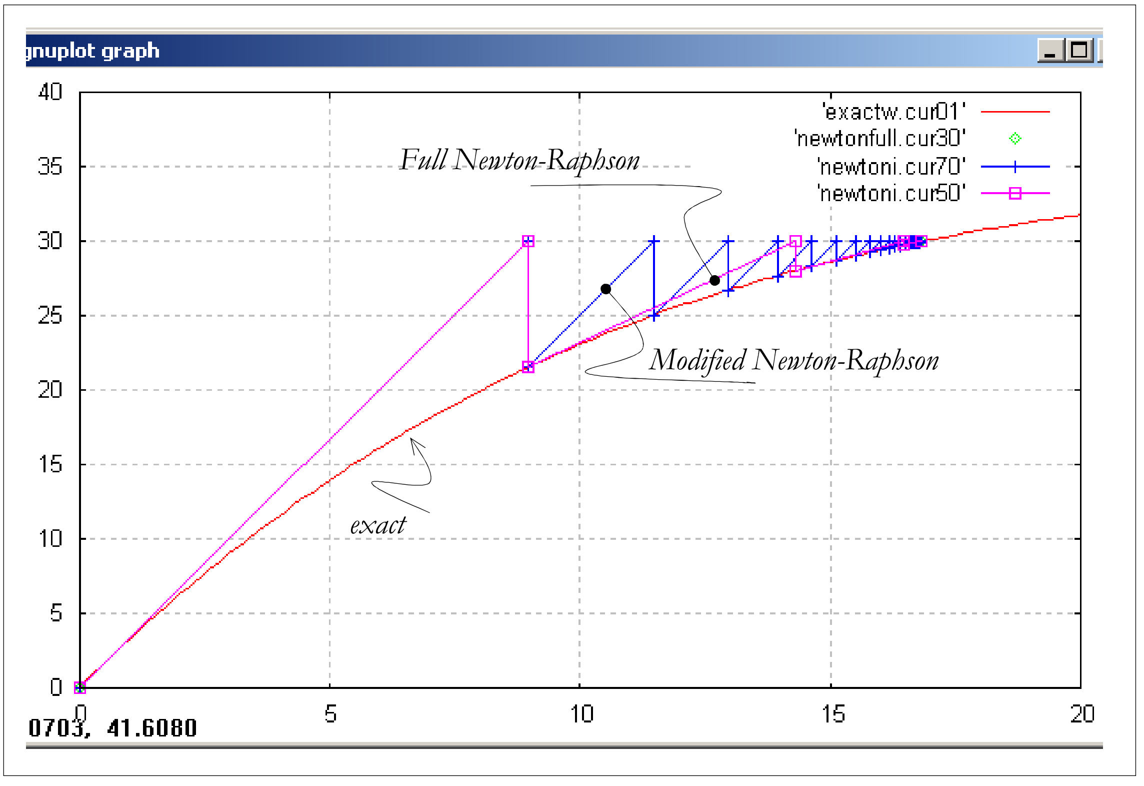

Figure C.25: Full Newton-Raphson *vs.* Modified Newton-Raphson.

Incremental-Iterative Solution References

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