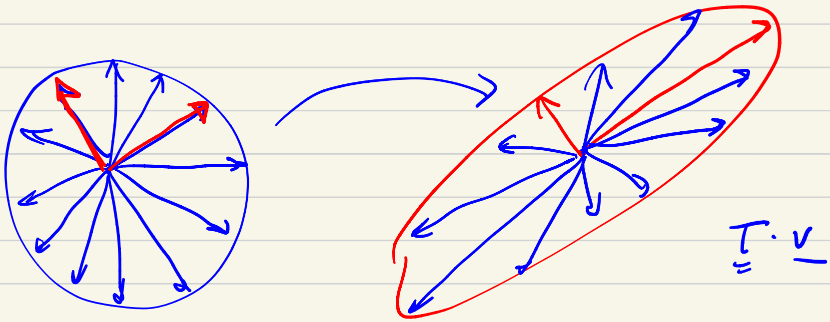
# Eigenvals/vecs

# Eigenvalues & Eigenvectors of rank 2 tensors.

Let $\underline{\underline{T}}$ be a rank 2 tensor, $v$ a vector, and $\lambda$ a scalar.

If $\underline{\underline{T}} \cdot \underline{v} = \lambda \underline{v}$ ; $\underline{v} \neq 0$, then:

- $\lambda$ is called an eigenvalue of $\underline{\underline{T}}$
- $\underline{v}$ is called an eigenvector of $\underline{\underline{T}}$ associated w/ $\lambda$.




(note, the red ones didn't change direction)

- Eigen {values / vectors} are also called proper, principal, characteristic {values / vectors}

- How do we find $\lambda$'s & $\underline{v}$'s ?

$$\underline{\underline{T}} \cdot \underline{v} = \lambda \underline{v} = \lambda \underline{\underline{I}} \cdot \underline{v} \Rightarrow \underline{\underline{T}} \cdot \underline{v} - \lambda \underline{\underline{I}} \cdot \underline{v} = 0$$

$$\Rightarrow \boxed{(\underline{\underline{T}} - \lambda \underline{\underline{I}}) \cdot \underline{v} = 0} \quad (1)$$

We can express (1) in matrix form, in terms of components in a basis $e^\alpha = \{ \underline{e}_1^\alpha, \underline{e}_2^\alpha, \underline{e}_3^\alpha \}$

$$\Rightarrow \left( [T]^\alpha - \lambda [I]^\alpha \right) \{v\}^\alpha = [0]^\alpha$$

which has a nontrivial solution for

$$\det \left( [T]^\alpha - \lambda [I]^\alpha \right) = 0$$

$$\det \begin{vmatrix} T_{11}^\alpha - \lambda & T_{12}^\alpha & T_{13}^\alpha \\ T_{21}^\alpha & T_{22}^\alpha - \lambda & T_{23}^\alpha \\ T_{31}^\alpha & T_{32}^\alpha & T_{33}^\alpha - \lambda \end{vmatrix} = 0$$

which can be written as

$$\boxed{\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0} \quad (2)$$

characteristic equation.

where

$I_1, I_2, I_3$

are real numbers

given by:

$$I_1 = T_{kk}^\alpha = \mathrm{Trace}(\underline{\underline{T}})$$

$$I_2 = \frac{1}{2}\left( T_{ii}^\alpha T_{jj}^\alpha - T_{ij}^\alpha T_{ji}^\alpha \right)$$

$$I_3 = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} T_{ip}^\alpha T_{jq}^\alpha T_{kr}^\alpha = \det(\underline{\underline{T}})$$

Equation (2) is called the characteristic equation

$I_1$, $I_2$, $I_3$ are called principal invariants of tensor $\underline{\underline{T}}$

The following questions arise:

- Does the solution of (2) hold for other basis?
- why do we call $I_1$, $I_2$, $I_3$ invariants?

To answer the first question, let's write the components of T in a different basis $e^{\beta} = \{ \underline{e}_1^{\beta}, \underline{e}_2^{\beta}, \underline{e}_3^{\beta} \}$. Then eq. (1) can be written as:

$$\left[ [T]^{\beta} - \lambda [I]^{\beta} \right] \{v\}^{\beta} = \{0\}^{\beta}$$

which has non-trivial solution for:

$$\det\left( [T]^{\beta} - \lambda [I]^{\beta} \right] = 0 \qquad (3)$$

If $[Q]$ is a rotation that relates the basis $e^\alpha$ & $e^\beta$ then we can write:

$$[T]^\beta = [Q]^T[T]^\alpha[Q] \quad \text{and}$$

$$[I]^\beta = [Q]^T[I]^\alpha[Q]$$

$$\Rightarrow \det\left([Q]^T[T]^\alpha[Q] - \lambda[Q]^T[I]^\alpha[Q]\right) = 0$$

$$\Rightarrow \det\left([Q]^T\left([T]^\alpha - \lambda[I]^\alpha\right)[Q]\right) = 0$$

$$\Rightarrow \underbrace{\det([Q]^T)}_{1} \det\left([T]^\alpha - \lambda[I]^\alpha\right) \underbrace{\det([Q])}_{1} = 0$$

$\longrightarrow$ because they are rotations! $\longleftarrow$

$$\Rightarrow \det\left([T]^\alpha - \lambda[I]^\alpha\right) = 0$$

└ which leads again to eq (2)

Cevahs question:

properties of rotation matrices.

det = 1, inverse = Transpose

That is, the values of $\lambda$ that satisfies eq (2) are independent of the basis chosen to represent $\underline{T}$

Since the invariants are the coefficients of the unique characteristics equation, they must be independent of the chosen basis as well

This can also be shown by writing

$$T^{\beta}_{ij} = a_{ip}\, a_{jq}\, T^{\alpha}_{pq}$$

and using orthogonality properties of $\underline{\underline{a}}$, for ex:

$$T^{\beta}_{ii} = \underbrace{a_{ip}\, a_{iq}}_{\delta_{pq}}\, T^{\alpha}_{pq} = \delta_{pq} T^{\alpha}_{pq} = T_{pp}{}^{\alpha} = I_1$$

The same procedure can be repeated for the other invariants.

# Properties of symmetric tensors

Let $\underline{\underline{T}}$ be a symmetric tensor, then:

1) All roots of its characteristic equation are real, ie: we can find three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$

2) Out of the associated eigenvectors, we can always form an orthonormal right-handed set of eigenvectors $\underline{e}^{\alpha} = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

3) In the eigenbasis $\underline{e}^{\alpha}$, the matrix representing the components of $\underline{\underline{T}}$ is diagonal, with the eigenvalues in the diagonal

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}^{\alpha} \Rightarrow \overbrace{\lambda_1, \lambda_2, \lambda_3}^{\text{Real}}$$

$$e_\beta = \{v_1, v_2, v_3\}$$

$$\Rightarrow \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(Try to interpret in terms of physics. ie Cauchy-green $\lambda$'s are stretch.)