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Two dimensional problems

For some problems: $\underline{u}(x_1, x_2)$

$$\downarrow$$

$$\underline{\underline{\varepsilon}}(x_1, x_2)$$

$$\downarrow$$

$$\underline{\underline{\sigma}}(x_1, x_2)$$

This does <u>NOT</u> mean that u_3 doesn't exist (or $u_3 = 0$)

Using these assumptions we get:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \cancel{*} & \cancel{*} & \cancel{*} \\ & \cancel{*} & \cancel{*} \\ \text{Symm.} & & 0 \end{bmatrix}$$

$$\downarrow \text{ Hooke's law}$$

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \cancel{*} & \cancel{*} & \cancel{*} \\ \cancel{*} & \cancel{*} & \cancel{*} \\ \cancel{*} & \cancel{*} & \cancel{*} \end{bmatrix}$$

Using $[\underline{\underline{\sigma}}]$ we obtain equilibrium equations:

$$\left\{ (\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + \lambda\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1}\right) + \rho b_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \right\} \hookrightarrow (5a)$$

$$\left\{ \mu\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2}\right) + (\lambda+2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + \lambda\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \rho b_2 = \rho\frac{\partial^2 u_2}{\partial t^2} \right\} \hookrightarrow (5b)$$

$$\left\{ \mu\frac{\partial^2 u_3}{\partial x_1^2} + \mu\frac{\partial^2 u_3}{\partial x_2^2} + \rho b_3 = \rho\frac{\partial^2 u_3}{\partial t^2} \right\} \hookrightarrow (5c)$$

This naturally splits the problem into 2 subproblems:

(1) – In-plane problem w/ $u_1(x_1, x_2)$

$u_2(x_1, x_2)$

$u_3 \equiv 0$

(2) – Anti-plane problem w/ $u_1 \equiv 0$

$u_2 \equiv 0$

$u_3(x_1, x_2)$

And then we use principle of superposition for the general solution.

* Anti-plane problem (Anti-plane shear problem)

$u_1 \equiv u_2 \equiv 0$; looking for $u_3 = u_3(x_1, x_2)$

$$[\underset{=}{\varepsilon}] = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_1} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x_2} \\ \text{symm.} & & 0 \end{bmatrix}$$

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & 0 & \mu \dfrac{\partial u_3}{\partial x_1} \\ 0 & 0 & \mu \dfrac{\partial u_3}{\partial x_2} \\ \text{symm.} & & 0 \end{bmatrix}$$

Consequently, plugging $\underline{\underline{\sigma}}$ into cons. of linear mom. gives:

$$\boxed{\mu \frac{\partial^2 u_3}{\partial x_1^2} + \mu \frac{\partial^2 u_3}{\partial x_2^2} + \rho b_3 = \rho \frac{\partial^2 u_3}{\partial t^2}}$$

Recover (5C)

<u>Note:</u> if $b_3 = 0$, we get:

$$\boxed{\frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_2^2} = \left(\frac{\rho}{\mu}\right) \frac{\partial^2 u_3}{\partial t^2}}$$

The wave equation

wave speed (c)

$$c = \sqrt{\frac{\rho}{\mu}}$$

Also only material constant is μ (no λ)

For the in plane problem:

$$u_1(x_1, x_2), \quad u_2 = (x_1, x_2), \quad u_3 \equiv 0$$

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & 0 \\ & \frac{\partial u_2}{\partial x_2} & 0 \\ \text{symm.} & & 0 \end{bmatrix}$$

This problem is often <u>plane-strain</u>

Based on Hooke's law applied on the above $\underline{\underline{\varepsilon}}$, we get:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} * & * & 0 \\ & * & 0 \\ \text{sym} & & * \end{bmatrix}$$

non-zero in general

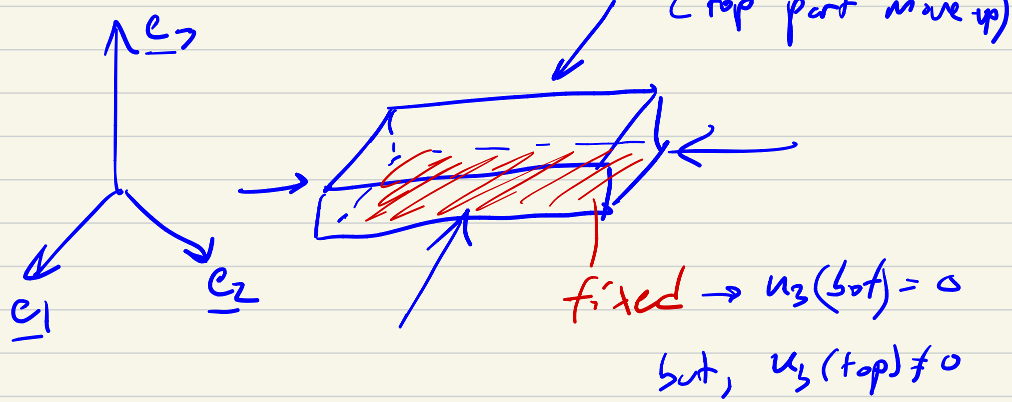
$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22})$$

prin. superposition.

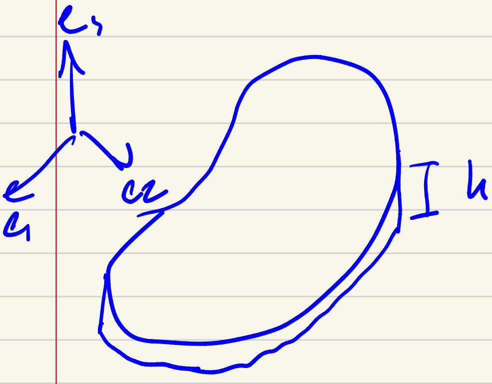
- Plugging the expressions for $\underline{\underline{\sigma}}$ into cons. lin. mom. yields (5a) & (5b).

- Hence, using such approach, we can solve the general 2D problem by decomposing it into the in-plane and anti-plane problems and adding the sol.

Plane Stress



There is still one more type of problem w/ 2D mathematical structure – plane stress problem.



- consider very thin cylinder
  (h is very small compared to other dimensions)
- In addition, assume that top & bottom faces are traction free

$\hookrightarrow \underline{\sigma}_3 \cdot \underline{e}_3 = \underline{0}$

$$\boxed{\sigma_{13} = \sigma_{23} = \sigma_{33} = 0}$$

- Since h is very small, we assume that the same holds true everywhere inside the domain

ie: $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ in Ω

Then, the stress tensor has the following form:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On the other hand, Hooke's Law can be expressed in terms of E and ν as:

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

For this problem, $\sigma_{kk} = \sigma_{11} + \sigma_{22}$

$$\Rightarrow [\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{1}{E}\sigma_{11} - \nu\sigma_{22} & \frac{1+\nu}{E}\sigma_{12} & 0 \\ & \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11} & 0 \\ \text{symm.} & & -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}) \end{bmatrix}$$

Note that $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$

But $\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$ which implies u_3 depends on x_3

- This problem is not 2D in the context of our previous definition.

(2D nature of this problem is approx.!)

- Equations of motion are given by (5a) & (5b)

- Hence, plane stress problem is very similar to plane strain, with the following important difference:

* Plane strain:

$$\sigma_{\alpha\beta} = \lambda \varepsilon_{kk} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} \; ; \; \alpha\beta = 1,2$$

For this problem: $\varepsilon_{33} = 0$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \cancel{\varepsilon_{33}}^{\,0}$$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}$$

$$= \varepsilon_{\gamma\gamma} \quad , \gamma = 1,2$$

$$\boxed{\sigma_{\alpha\beta} = \lambda \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta}}$$

* plane stress:

$$\sigma_{\alpha\beta} = \lambda \varepsilon_{kk} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta}$$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$= \varepsilon_{\gamma\gamma} + \varepsilon_{33}$$

$$\Rightarrow \sigma_{\alpha\beta} = \lambda(\varepsilon_{\gamma\gamma} + \varepsilon_{33}) \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} \hookrightarrow (*)$$

We know:

$$\varepsilon_{33} = \frac{-\nu}{E}(\sigma_{11} + \sigma_{22}) \rightarrow \boxed{A}$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} \; ; \; \varepsilon_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11}$$

so to get $\varepsilon_{\gamma\gamma} = \varepsilon_{11} + \varepsilon_{22}$

$$\varepsilon_{\gamma\gamma} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11}$$

$$= \frac{1}{E}(\sigma_{11} + \sigma_{22}) - \frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

$$\Rightarrow (\sigma_{11} + \sigma_{22}) = \left(\frac{E}{1-\nu}\right) \varepsilon_{\gamma\gamma} \rightarrow \boxed{B}$$

From $\boxed{A}$ & $\boxed{B}$

$$\Rightarrow \varepsilon_{33} = \left(\frac{-\nu}{1-\nu}\right) \varepsilon_{\gamma\gamma} = \left(\frac{\nu}{\nu - 1}\right) \varepsilon_{\gamma\gamma}$$

With some manipulation:

$$\varepsilon_{zz} = \left(\frac{-\lambda}{\lambda + 2\mu} \right) \varepsilon_{\gamma\gamma}$$

Plugging into ✱:

$$\sigma_{\alpha\beta} = \lambda \underbrace{\left(\varepsilon_{\gamma\gamma} - \frac{\lambda}{\lambda + 2\mu} \varepsilon_{\gamma\gamma} \right)}_{\left(\frac{2\mu}{\lambda + 2\mu} \right) \varepsilon_{\gamma\gamma}} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta}$$

$$\Rightarrow \boxed{\sigma_{\alpha\beta} = \left(\frac{2\mu\lambda}{\lambda + 2\mu} \right) \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta}}$$

Hence, if we would like to limit our consideration to plane values of $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ plane-strain & plane-stress are mathematically identical if we replace (λ) for plane-strain and $\left(\frac{2\lambda\mu}{\lambda + 2\mu} \right)$ for plane-stress

- In summary, both plane strain & plane-stress problems can be formulated as follows:

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho b_\alpha = 0 \quad \text{(static problems)}$$

$\alpha = 1, 2$

$\beta = 1, 2$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

$$\sigma_{\alpha\beta} = \lambda^* \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} \; ; \; \gamma = 1,2$$

$$\begin{cases} \lambda^* = \lambda & \text{for plane-strain} \\ \lambda^* = \dfrac{2\lambda\mu}{\lambda + 2\mu} & \text{for plane-stress.} \end{cases}$$

* plus appropriate B.Cs