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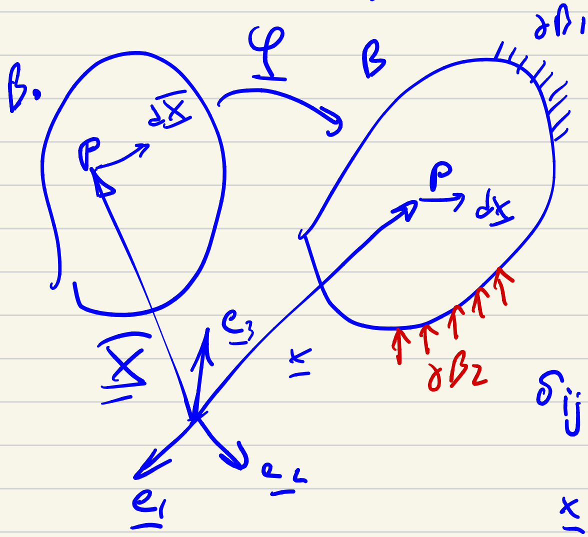
Linear Elasticity (constitutive Law)



$$F_{ij} = \frac{\partial \varphi_i}{\partial X_j}$$

$$dx_i = F_{ij}\, dX_j$$

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{x}_i \quad (\text{in } B)$$

$$\underline{x} = \underline{x}^* \text{ on } \partial B_1$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = t^* \text{ on } \partial B_2$$

So far we derived this

we still need something to relate measures of stress & kinematics:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{F}}) \quad \text{function of deformation gradient.}$$

we call this constitutive relation

can also be $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{F}}, \dot{\underline{\underline{F}}}, \underline{\rho})$

Everything discussed so far is very generic:

- applies to all material (fluids, solids, elastic, plastic, etc.)

- also applies to all deformation regimes (finite & infinitesimal)

- The only missing link to be able to solve real problems is the material behavior

  ie: how deformation & deformation rates relate to stress.

- Material behavior enters the mathematical formulation through <u>constitutive equations</u>.

- When stresses are <u>proportional to strains</u> we say that the material is <u>linear elastic</u>.

<u>Observations</u>:

- Most solids are linearly elastic for sufficiently small strains

- For large strains, solids typically deviate from linear elasticity to exhibit non-linear elastic behavior (ex: rubber), or to break (brittle material such as rock) or to plastically flow (ductile materials such as metals)

- Hence, small (infinitesimal) strains usually imply linear elasticity and the other way around
- That is why linear elasticity is typically formulated for infinitesimal deformations.

Hooke's Law

$$\underline{\underline{\sigma}} \leftrightarrow \underline{\underline{\varepsilon}}$$

$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) \rightarrow$ this means

$$\sigma_{11} = f_{11}(\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{33})$$
$$\sigma_{12} = f_{12}(\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{33})$$
$$\vdots$$
$$\sigma_{33} = f_{33}(\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{33})$$

(each component of $\underline{\underline{\sigma}}$ is a function of all components of $\underline{\underline{\varepsilon}}$)

$\downarrow$

9 functions

- The simplest relation would be a linear one:

$$\begin{cases} \sigma_{11} = C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} \\ \qquad + \dots + C_{1133}\varepsilon_{33} \\ \sigma_{12} = C_{1211}\varepsilon_{11} + C_{1212}\varepsilon_{12} + \\ \qquad \dots + C_{1233}\varepsilon_{33} \\ \vdots \\ \sigma_{33} = C_{3311}\varepsilon_{11} + \dots + C_{3333}\varepsilon_{33} \end{cases}$$

(81 coefficients!!)

Rank 4 tensor

$$\boxed{\sigma_{ij} = C_{ijkl}\varepsilon_{kl}}$$

In direct notation:

- we write: $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$ → Hooke's Law

where $\underline{\underline{\underline{\underline{C}}}}$ is a rank 4 tensor w/ components

C_{ijkl}. $\underline{\underline{\underline{\underline{C}}}}$ is called <u>the elasticity tensor</u>

or <u>tensor of elastic moduli</u>.

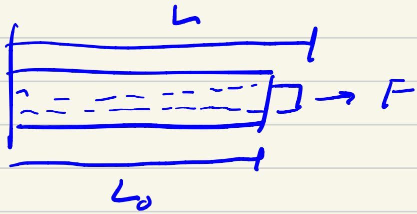
<u>Notes:</u>

- The component of $\underline{\underline{\sigma}}$ a linear combination of the components of $\underline{\underline{\varepsilon}}$, with the coefficients given by the components of $\underline{\underline{\underline{\underline{C}}}}$
- There might be a problem w/
  1) $\underline{\underline{\varepsilon}}$ is defined in the reference configuration whereas $\underline{\underline{\sigma}}$ is defined in the deformed configuration!
- In linear elasticity we assume that both config are indistinguishable, ie $\underline{x} \sim \underline{X}$

  $\underline{\underline{\sigma}}(x_1, x_2, x_3) = \underline{\underline{\sigma}}(X_1, X_2, X_3)$

  and the same $\underline{\underline{\varepsilon}}$

- under this assumption, the first P/K tensor is identical to the Cauchy stress tensor $\underline{\underline{P}} \approx \underline{\underline{\sigma}}$



Initial area A_0

Final area A

$\sigma_{11} = \frac{f}{A}$ $P_{11} = \frac{f}{A_0}$

for inf. def.

$A \approx A_0 \rightarrow \sigma_{11} \approx P_{11}$

Symmetries of the elasticity tensor

- $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$ or $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$
- As a rank 4 tensor, $\underline{\underline{\underline{\underline{C}}}}$ has $3^4 = 81$ comps.
- However, to be consistent w/ the definitions of cont. mechanics, there are some constraints on $\underline{\underline{\underline{\underline{C}}}}$ that limit the number of independent components:

1) $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ are symmetric: $\Rightarrow$ Reduces # of independent comps to 36 *

$\sigma_{ij} = \sigma_{ji}$; $\varepsilon_{kl} = \varepsilon_{lk}$

$\Rightarrow C_{ijkl} = C_{jikl}, \; C_{ijkl} = C_{ijlk}$ (minor symmetry)

2) The strain energy density is quadratic

$$\Rightarrow E = \frac{1}{2} \underline{\underline{\varepsilon}} \cdot \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$$

or $$E = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

$$\Rightarrow C_{ijkl} = C_{klij}$$ *

(Major symmetry)

Minor + Major symmetries combined reduce the number of independent components of $\underline{\underline{\underline{\underline{C}}}}$ to

<u>21</u>

3) Material symmetry

- Most materials have some symmetry in their structure that translate in symmetries in its response to strains

- These symmetries may impose extra constraints in $\underline{\underline{\underline{\underline{C}}}}$.

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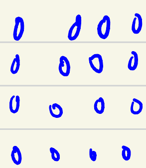
Rotation as symmetry

- Consider the following situation
  - Conduct experiment (1), in which a strain $\underline{\underline{\varepsilon}}$ is applied to a body and a stress $\underline{\underline{\sigma}}$ results
  - Now conduct experiment (2), in which the body is rotated w/ a rotation defined by $\underline{\underline{a}}$ and the same $\underline{\underline{\varepsilon}}$ is applied resulting in stress $\underline{\underline{\hat{\sigma}}}$ ← not rotated

- <u>Definition:</u> Rotation $\underline{\underline{a}}$ is called a symmetry of the elastic body if, for any $\underline{\underline{\varepsilon}}$, $\underline{\underline{\sigma}} = \underline{\underline{\hat{\sigma}}}$ (same stress)

- Practically speaking, we cannot determine whether or not the body was rotated

For example, a crystalline material



usually remain indifferent to several rotations $\underline{\underline{a}}^\alpha$.

Then for each

$$\underline{\underline{a}}^\alpha: \; C_{stpq} = C_{ijkl}\, a^\alpha_{is}\, a^\alpha_{jt}\, a^\alpha_{kp}\, a^\alpha_{lq}$$

- This set of equations provide further constraints on $\underline{\underline{C}}$ which reduce the number of independent components needed.

- The more symmetries a material has, the less independent component of $\underline{\underline{C}}$ are needed.

- For example, a material w/ one plane of symmetry, the number is reduced from 21 to 13.

- An orthotropic material (three mutually perpendicular symmetry plane) has 9 independent material constants.

- A cubic material can be characterized w/ only 3 elastic constants

Isotropic materials

- **Definition:** An elastic body is isotropic if every rotation $\underline{\underline{Q}}$ is a symmetry of that body.

- physically speaking: the body has no "special" direction

- Mathematically speaking,

$$C_{stpq} = C_{ijkl} Q_{is} Q_{jt} Q_{kp} Q_{lq} \quad \text{for all rotation } \underline{\underline{Q}}$$

- Such tensors ($\underline{\underline{\underline{\underline{C}}}}$) are called Isotropic

- It can be shown that their most general form:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu_1 \delta_{ij} \delta_{jl} + \mu_2 \delta_{il} \delta_{jk}$$

- However, the minor symmetry leads to $\mu_1 = \mu_2 = \mu$

- Hence, for an isotropic linear elastic material:

$$\boxed{C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}$$

2 independent constants

· we can derive the expression for the stress:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$= \left[\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\varepsilon_{kl}$$

$$= \lambda \delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl}\varepsilon_{kl} + \delta_{il}\delta_{jk}\varepsilon_{kl})$$

$$= \lambda \varepsilon_{kk}\delta_{ij} + \mu\underbrace{(\varepsilon_{ij} + \varepsilon_{ji})}_{2\varepsilon_{ij}}$$

$$\Rightarrow \boxed{\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}} \quad (*)$$

$$\boxed{\underline{\underline{\sigma}} = \lambda\, \mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{\varepsilon}}}$$

Remarks:

- Isotropic linear elastic material have only 2 independent elastic constants.
- λ & μ are called Lamé constants
- μ is also called the shear modulus
  ↳ if pure shear, $tr(\underline{\underline{\varepsilon}}) = 0$
- Lamé constants are usually combined and used in different forms. The most common combinations are:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{(Young's Modulus)}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad \text{(Poisson's Ratio)}$$

$$K = \frac{(3\lambda + 2\mu)}{3} \quad \text{(Bulk Modulus)}$$

- $(*)$ can be inverted in terms of E and ν

$$\varepsilon_{ij} = \frac{-\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij}$$

- Note that if we create a state of stress $\underline{\underline{\sigma}}$, strain $\underline{\underline{\varepsilon}}$ results.

  If we remove all loads so $\underline{\underline{\sigma}} = 0$, then $\underline{\underline{\varepsilon}}$ becomes 0 as well (no strain)

  This is a property of linear elastic materials.

- If the body is isotropic, its elastic properties can be described by just two elastic moduli eg: λ and μ

  If λ and μ are the same for all points of the body, then its elastic properties are homogeneous.

  Otherwise, we say the body has heterogeneous elastic properties.