PRINCIPLE OF VIRTUAL WORK

* Thus far we have relied on the requirement that the structure be in equilibrium (Newtonian mechanics) and that the deformation be compatible and satisfy boundary conditions.

  → Required us to work with vector forces and displacements.

* Work is a scalar quantity which depends on both forces and displacements.

* Formulations based on work have some unique advantages:

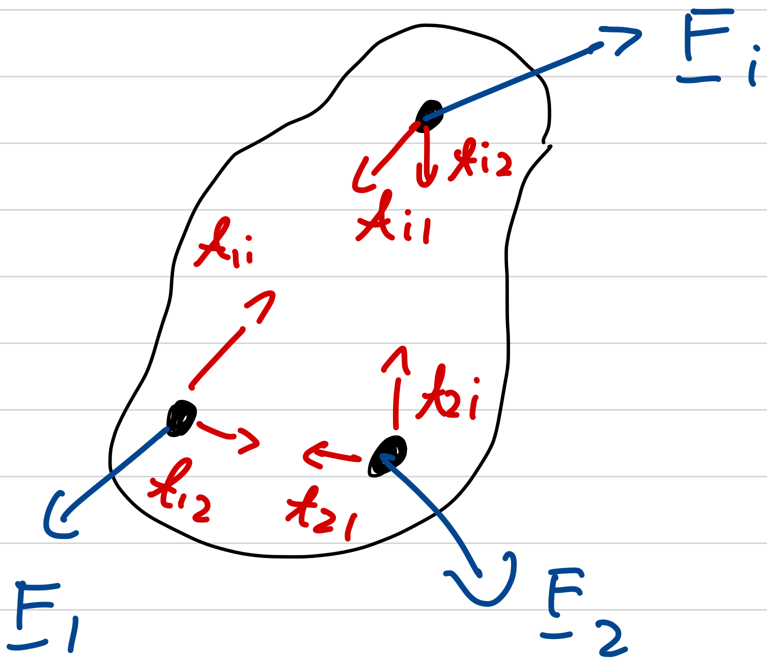
  → Deal with scalar quantities

  → Some forces (performing no work) can be eliminated from the formulation

  → Enables the systematic development of approximate solutions (FEA).

Equilibrium and Work

→ System of particles



For a single particle (i-th particle)

$$\underline{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{f}_{ij} = \underline{0}$$

Sum over all particles

$$\sum_{i=1}^{N} F_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{f}_{ij} = 0$$

Since $\underline{t}_{ij} + \underline{t}_{ji} = 0$

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}} \underline{t}_{ij} = 0$$

$$\rightarrow \sum_{i=1}^{N} \underline{F}_i = 0$$

→ EULER'S LAW

Mechanical Work

* Work done by a force is the scalar product of the force by the displacement of the point of application

$$W = \int_{\underline{r}_i}^{\underline{r}_f} dW = \int_{\underline{r}_i}^{\underline{r}_f} \underline{F} \cdot \underline{dr}$$

$$dW = \underbrace{\underline{F}}_{\text{Force}} \cdot \underline{dr}$$

← Incremental Work

Infinitesimal Desplacement

* Superposition holds

$\underline{F} = \underline{F}_1 + \underline{F}_2 \qquad dW = (\underline{F}_1 + \underline{F}_2) \cdot \underline{dr}$

$= \underline{F}_1 dr + \underline{F}_2 dr = dW_1 + dW_2$

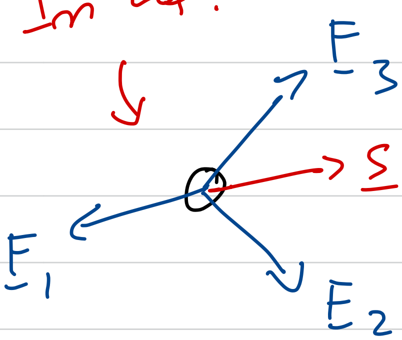
VIRTUAL WORK

* Introduce "arbitrary virtual displacement". Also known as "test" or "fictitious".

* Virtual displacements can be chosen in an arbitrary manner without any restriction imposed on direction or magnitude

* Virtual displacements do not affect the forces acting on a particle.

In Eq.!



$$W = (\underbrace{\sum \underline{F}}_{=0}) \cdot \underline{s} = 0$$

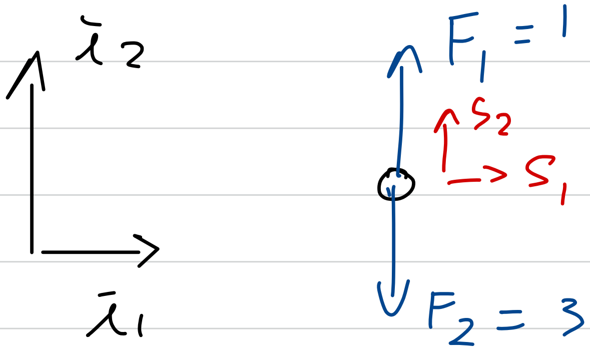
PVW:

* A particle is in static equilibrium if and only if the virtual work done is zero for all virtual displacements

* Practically we do not test all $\underline{s}$, but all independent virtual displacements

$$\underline{s} = s_1 \bar{\imath}_1 + s_2 \bar{\imath}_2 + s_3 \bar{\imath}_3$$

for a particle.

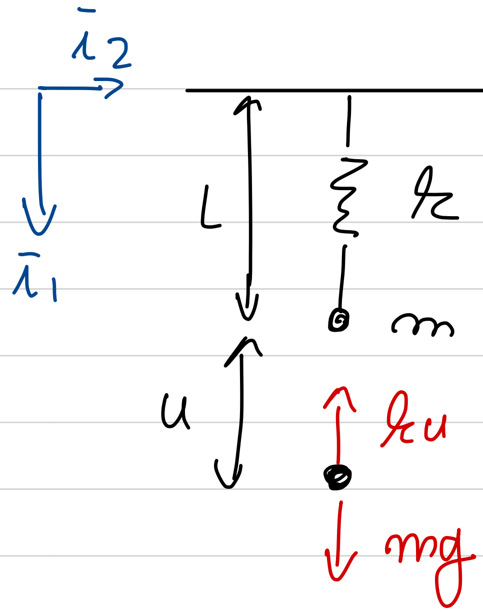


$$W = (1\,\bar{\imath}_2 - 3\,\bar{\imath}_2)\cdot(s_1\bar{\imath}_1 + s_2\bar{\imath}_2)$$

$$= -2\,s_2 \neq 0$$

$\rightarrow$ Not in Eq.

Note: If we had used $\underline{s} = s_1 \bar{\imath}_1$, we would not get the right answer since we are not checking all independent virtual displacements.



Find the displacement u at equilibrium

$$W = (mg\,\bar{\imath}_1 - ku\,\bar{\imath}_1)\cdot(\delta_1\,\bar{\imath}_1 + \delta_2\,\bar{\imath}_2)$$

$$= \delta_1\,\underbrace{(mg - ku)}_{=0} = 0$$

$$\rightarrow \quad mg - ku = 0$$

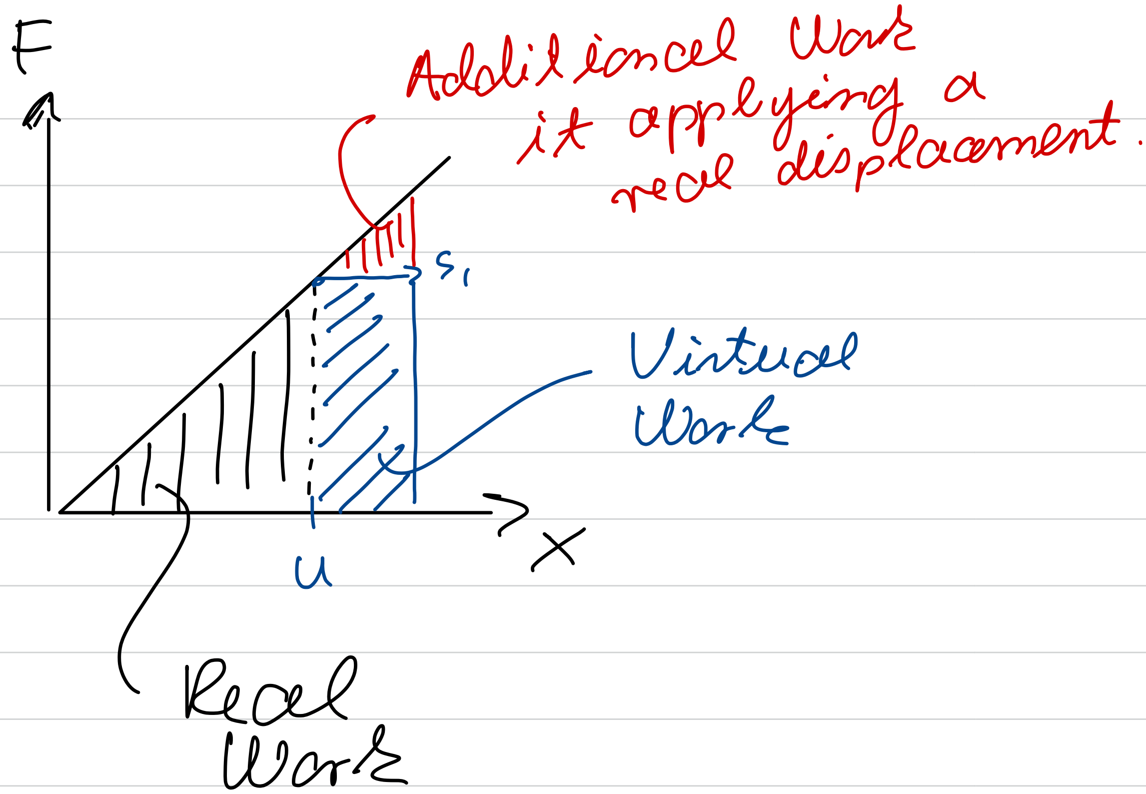
$$u = mg/k$$

* Note: The virtual displacement δ_1 about equilibrium does not change the force due to the spring!

Virtual $\quad W = \int_u^{u+\delta_1} F\,du = F\int_u^{u+\delta_1} du = (-ku)\,\delta_1$

Real $\quad W = \int_u^{u+d} F\,du = \int_u^{u+d} (-ku)\,du = \left[-\frac{ku^2}{2}\right]_u^{u+d}$

$$= (-ku)\,d - \frac{1}{2}kd^2$$



* We will use infinitesimal displacements as virtual displacements

$\delta W \rightarrow$ Notation for virtual work

$\underline{s} = \delta \underline{u}$

* Displacement - dependent forces automatically remain unaltered

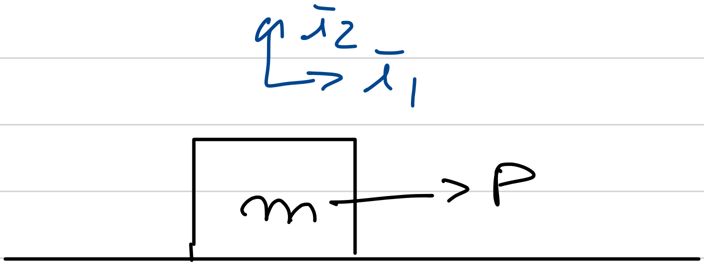
Real

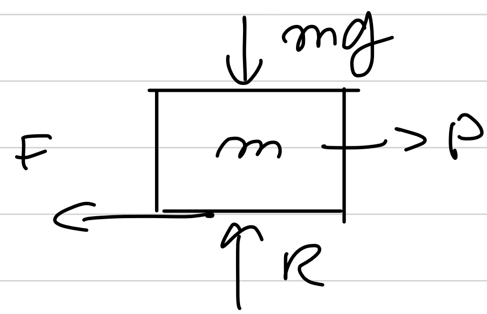
$$W = \int_{u}^{u+\delta u} F\,du = (-ku)\delta u - \frac{1}{2} k \underbrace{(\delta u)^2}_{\approx 0}$$

$$\approx (-ku)\delta u$$

Kinematically Admissible Virtual Displacements

* Virtual displacements that satisfy the kinematic constraints at the problem.





$$W = (P\bar{\imath}_1 - F\bar{\imath}_1 - mg\,\bar{\imath}_2 + R\,\bar{\imath}_2)\cdot(S_1\bar{\imath}_1 + S_2\bar{\imath}_2) = 0$$

$$(P-F)S_1 + (R-mg)S_2 = 0$$

$\rightarrow$ S_1 and S_2 are arbitrary

$\Rightarrow$ $P - F = 0$ $\qquad R - mg = 0$

Kinematically admissible

$$\underline{S} = S_1\bar{\imath}_1$$

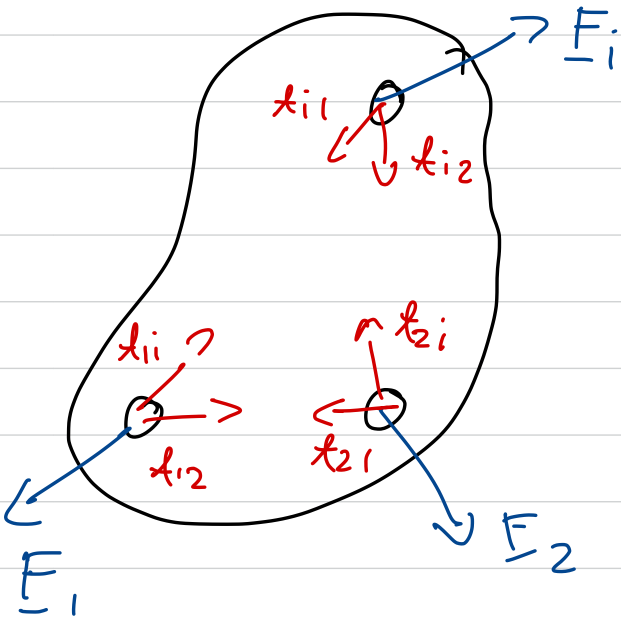
$$W = (P\bar{\imath}_1 - F\bar{\imath}_1)(S_1\bar{\imath}_1) = 0$$

$$\rightarrow P - F = 0$$

* Constraint forces (i.e. reaction forces) are no longer considered. However, also not computed!

* Virtual Work done by reaction forces (resulting in kinematic constraints) vanish for kinematically admissible fields.

System of Particles



$$\underline{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{t}_{ij} = \underline{0}$$

$$\delta W_i = \left(\underline{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{t}_{ij} \right) \cdot \delta \underline{u}_i$$

$$\delta W = \sum_{i=1}^{N} \left\{ \left(\underline{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{t}_{ij} \right) \cdot \delta \underline{u}_i \right\} = 0$$

External Applied Forces

Internal Forces.

$$\delta W = \delta W_E + \delta W_I = 0$$

$$\delta W_E = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{u}_i \quad \text{- External Virtual Work}$$

$$\delta W_I = \sum_{i=1}^{N} \left(\sum_{\substack{j=1 \\ i \neq j}}^{N} \underline{t}_{ij} \right) \cdot \delta \underline{u}_i \quad \text{- Internal Virtual Work.}$$

PVW: A system of particles is in static equilibrium if and only if the sum of internal and external virtual work vanishes for all virtual displacements.

Since δu is arbitrary

$$W = W_E + W_I = 0$$

* If the system is in equilibrium, the sum of the external and internal work vanish.

* The PVW above is general! It could be a rigid body, an elastic body, a fluid, etc ...