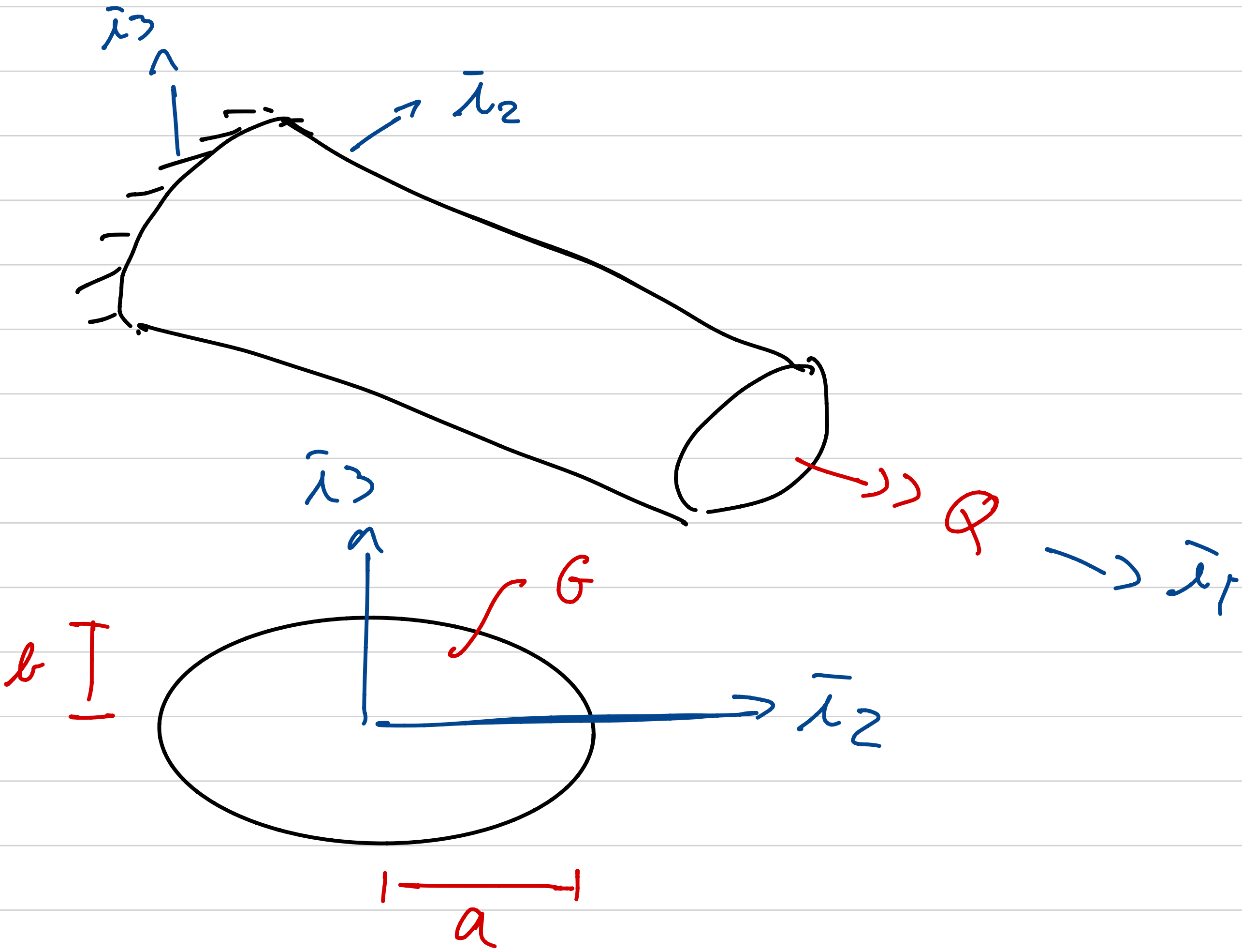
Torsion of an elliptical bar



a) Find the displacement field $\underline{u}$ (u_1, u_2, u_3)

b) Find shear stresses $\underline{\underline{\tau}}$

$$\frac{d}{dx_1}\Big[\overbrace{H_{11}\frac{d\phi_1}{dx_1}}^{M_1}\Big]=0$$

B.C. @ $x_1=0$ $\quad \phi_1=0$

@ $x_1=L$ $\quad M_1=Q=H_{11}\left.\frac{d\phi_1}{dx_1}\right|_L$

$$H_{11}\frac{d\phi_1}{dx_1}=C_1=Q$$

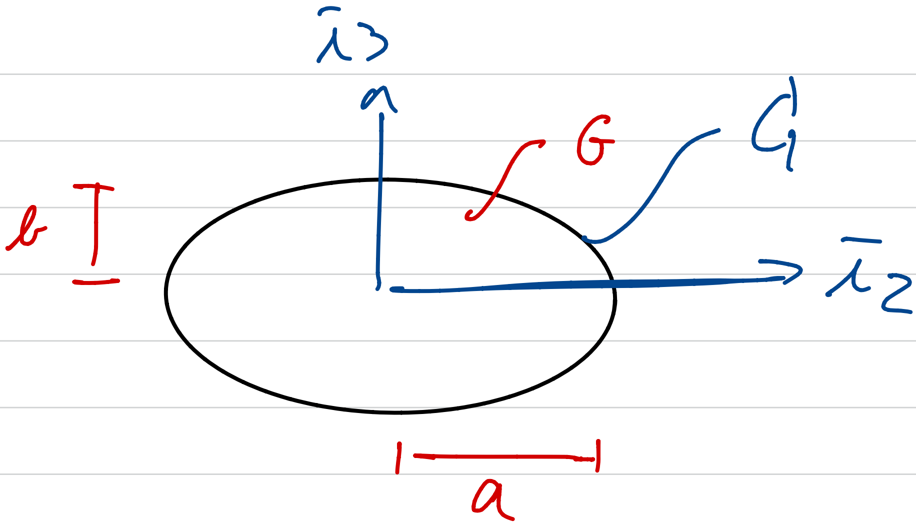
$$H_{11}\,\phi_1=Q\,x_1+\cancel{C_2}\rightarrow =0$$

$$\boxed{\phi_1=\frac{Q}{H_{11}}x_1}\qquad K_1=\frac{d\phi_1}{dx_1}=\frac{Q}{H_{11}}$$

$$u_2=-x_3\,\phi_1$$

$$u_3=x_2\,\phi_1$$

$$u_1=\psi\,K_1$$



Want Φ to be a constant on C_1'

Equation for C_1' is

$$\left(\frac{x_3}{b}\right)^2 + \left(\frac{x_2}{a}\right)^2 = 1$$

Chose

$$\boxed{\Phi = C_0\left[\left(\frac{x_3}{b}\right)^2 + \left(\frac{x_2}{a}\right)^2 - 1\right]}$$

→ Satisfies B.C. since $\Phi = 0$ on C_1'!

$$\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = -2GK_1$$

$$C_0 \frac{2}{a^2} + C_0 \frac{2}{b^2} = -2GK_1$$

$$C_0 \left(\frac{2b^2 + 2a^2}{a^2 b^2} \right) = -2GK_1$$

$$\rightarrow \quad C_0 = -GK_1 \frac{a^2 b^2}{a^2 + b^2}$$

$$\boxed{\Phi = -GK_1 \frac{a^2 b^2}{a^2 + b^2} \left[\left(\frac{x_3}{b}\right)^2 + \left(\frac{x_2}{a}\right)^2 - 1 \right]}$$

$$\tau_{12} = \frac{\partial \phi}{\partial x_3} = -GK_1 \frac{a^2 b^2}{a^2 + b^2} \frac{2x_3}{b^2}$$

$$= -2GK_1 \frac{a^2}{a^2 + b^2} x_3$$

$$\tau_{13} = -\frac{\partial \Phi}{\partial x_2} = 2GK_1 \frac{b^2}{a^2 + b^2} x_2$$

$$H_{11} = \frac{M_1}{\kappa_1} \qquad M_1 = 2\int_A \Phi \, dA$$

$$M_1 = -2G\kappa_1 \frac{a^2 b^2}{a^2+b^2} \int_A \left[\left(\frac{x_3}{b}\right)^2 + \left(\frac{x_2}{a}\right)^2 - 1 \right] dA$$

Aside: $\int_A x_2^2 \, dA = \pi a^3 b / 4$

$$M_1 = G \pi \frac{a^3 b^3}{a^2+b^2} \cdot \kappa_1$$

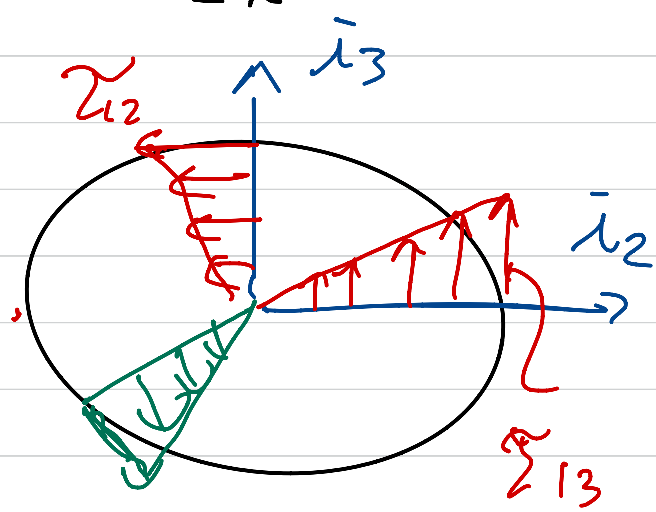
$$\boxed{H_{11} = G \pi \frac{a^3 b^3}{a^2+b^2}}$$

Sanity check

$a = b = R$

$$\checkmark H_{11} = \frac{G\pi R^6}{2R^2} = \frac{G\pi}{2} R^4$$

$$\begin{cases} \tau_{12} = -\dfrac{2M_1}{\pi a b^3} x_3 \\[2ex] \tau_{13} = \dfrac{2M_1}{\pi a^3 b} x_2 \end{cases}$$



Warping

$$\tau_{12} = GK_1\left(\frac{\partial\psi}{\partial x_2} - x_3\right) = \frac{\partial\Phi}{\partial x_3}$$

$$\tau_{13} = GK_1\left(\frac{\partial\psi}{\partial x_3} + x_2\right) = -\frac{\partial\Phi}{\partial x_2}$$

$$\frac{\partial\psi}{\partial x_2} = \frac{1}{GK_1}\frac{\partial\Phi}{\partial x_3} + x_3$$

$$= \frac{1}{\cancel{GK_1}}\left(-\cancel{GK_1}\frac{a^2\cancel{\ell^2}}{a^2+\ell^2}\,2\frac{x_3}{\cancel{\ell^2}}\right) + x_3$$

$$= -\frac{a^2}{a^2+\ell^2}2x_3 + x_3$$

$$\frac{\partial\psi}{\partial x_2} = -\frac{2a^2 + a^2 + \ell^2}{a^2+\ell^2}x_3 = -\frac{a^2-\ell^2}{a^2+\ell^2}x_3$$

$$\psi = -\frac{a^2-\ell^2}{a^2+\ell^2}x_3x_2 + f(x_3)$$

$$\frac{\partial\psi}{\partial x_3} = -\frac{a^2-\ell^2}{a^2+\ell^2}x_2$$

$$\psi = -\frac{a^2-\ell^2}{a^2+\ell^2}x_3x_2 + g(x_2)$$

Combining yields $t(x_3) = g(x_2) = 0$

$$\boxed{\varphi(x_2, x_3) = -\frac{a^2 - \ell^2}{a^2 + \ell^2} x_3 x_2}$$

Displacement Field

$$u_2 = -x_3 \frac{Q}{H_{11}} x_1$$

$$\rightarrow u_2 = -\frac{Q}{G\pi} \frac{a^2 + \ell^2}{a^3 \ell^3} x_3 x_1$$

$$u_3 = x_2 \frac{Q}{H_{11}} x_1$$

$$\rightarrow u_3 = \frac{Q}{G\pi} \frac{a^2 + \ell^2}{a^3 \ell^3} x_2 x_1$$

$$\rightarrow u_1 = -\frac{a^2 - \ell^2}{a^2 + \ell^2} x_3 x_2 \left(\frac{Q}{G\pi} \frac{a^2 + \ell^2}{a^3 \ell^3} \right)$$