Kinematics

(cont.)

	LAGRANGIAN	EULERIAN
Particle velocity field	$\underline{V}(\underline{X},t)=\frac{\partial \underline{\varphi}}{\partial t}(\underline{X},t)$	$\underline{v}(\underline{x},t)=\underline{V}(\underline{\varphi}^{-1}(\underline{x},t),t)$
Particle acceleration field	$\underline{A}(\underline{X},t)=\frac{\partial \underline{V}}{\partial t}(\underline{X},t)$	$\underline{a}(\underline{x},t)=\underline{A}(\underline{\varphi}^{-1}(\underline{x},t),t)$

(always capitalized)

Recall our previous example: $\underline{x}_1=\underline{X}_1(1+t^2)$;

$\underline{x}_2=\underline{X}_2$;

$\underline{x}_3=\underline{X}_3$

Lagrangian description

$$V_1(\underline{X},t)=\frac{\partial \varphi_1}{\partial t}=2t\underline{X}_1$$

→ points on $X_1=0$ face $v_1=0$

→ points more to the right moves faster

→ moves faster w/ time

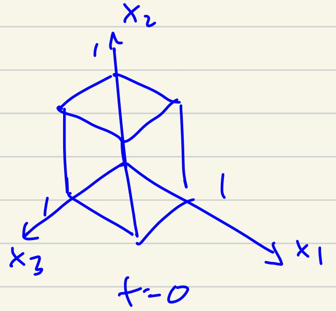
$$V_2(\underline{X},t)=\frac{\partial \varphi_2}{\partial t}=0$$

$$V_3(\underline{X},t)=\frac{\partial \varphi_3}{\partial t}=0$$

$$\underline{A}_1(\underline{X},t)=\frac{\partial V_1}{\partial t}=2\underline{X}_1$$

$$\underline{A}_2(\underline{X},t)=\frac{\partial V_2}{\partial t}=0$$

$$A_3(\underline{X},t)=\frac{\partial V_3}{\partial t}=0$$



(after we pick particle, relationship depends only on t for velocity)

For the ~~Eulerian description~~, we need to identify the inverse mapping $\underline{\varphi}^{-1}(\underline{x}, t)$

(takes from point in space, tell us which point it was in the reference.)

$$\underline{X}_1 = x_1/(1+t^2) \; ; \; \underline{X}_2 = x_2 \; ; \; \underline{X}_3 = x_3$$

Then we get:

Eulerian description:

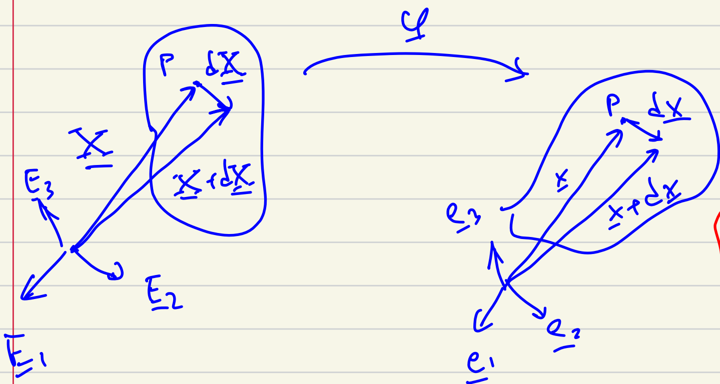
$$\begin{cases} v_1(\underline{x}, t) = V_1\left(\underset{\underline{X}_1}{\dfrac{x_1}{1+t^2}}, \underset{\underline{X}_2}{x_2}, \underset{\underline{X}_3}{x_3}, t\right) = \dfrac{2x_1 t}{1+t^2} \\ v_2 = v_3 = 0 \\ \\ a_1(\underline{x}, t) = A_1\left(\dfrac{x_1}{1+t^2}, x_2, x_3, t\right) = \dfrac{2x_1}{1+t^2} \\ a_2 = a_3 = 0 \end{cases}$$

__Note:__ In solid mechanics, the predominant description is Lagrangian.

Kinematics of local deformation.

Measures of local deformation play a prominent role in the formulation of material models when the principle of local action applies

Let us consider the mapping of a point within an infinitesimal neighborhood of the point labeled by $\underline{X}$



$$\underline{x} = \varphi(\underline{X})$$

$$\underline{x} + d\underline{x} = \varphi(\underline{X} + d\underline{X})$$

Neglect higher order terms

In indicial notation: $x_i + dx_i = \varphi_i(\underline{X} + d\underline{X}, t)$

$$x_i + dx_i = \varphi_i(\underline{X} + d\underline{X}, t) = \underbrace{\varphi_i(\underline{X}, t)}_{} + \frac{\partial \varphi_i}{\partial X_j}(\underline{X}, t)\, dX_j + \cancel{O(|d\underline{X}|^2)}$$

$$\cancel{x_i} + dx_i = \overset{x_i}{\cancel{x_i}} + \frac{\partial \varphi_i}{\partial X_j}(\underline{X}, t)\, dX_j \Rightarrow \boxed{dx_i = \underbrace{\frac{\partial \varphi_i}{\partial X_j}(\underline{X}, t)}_{\text{Rank 2}}\, \underbrace{dX_j}_{\text{Rank 1}}}$$

(matrix x vector)

That is, the deformation mapping of an infinitesimal material vector $d\underline{X}$ at $\underline{X}$ is completely determined by the deformation gradient

$$F : \ast \ \boxed{F_{ij}(\underline{X}, t) = \frac{\partial \varphi_i}{\partial X_j}(\underline{X}, t); \quad dx_i = F_{ij}(\underline{X}, t)\, dX_j}$$

So far, we can see F as a matrix w/ entries $\dfrac{\partial \varphi_i}{\partial X_j}$

But how does it behave under change of frames?

↪ Tensor or not?