AE6114 Exam 2

Problem 1

A cube made of two perfectly glued pieces of isotropic, linearly elastic material is subjected to a known uniform pressure P on its lateral faces as shown in Figure 1(a). The cube is clamped between fixed, frictionless plates so that the strain through its height is zero (Figure 1(b)). Assume that the stress state is homogeneous throughout the (it doesn't change with position) and the Lamé constants, λ and μ, are also known.

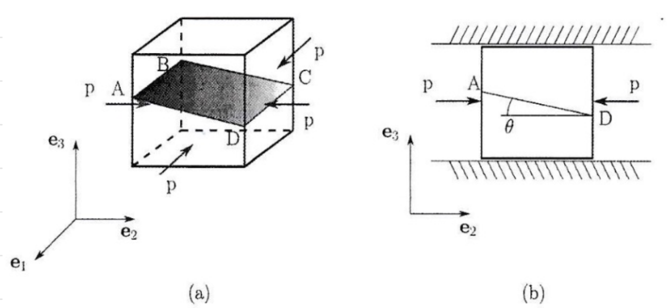
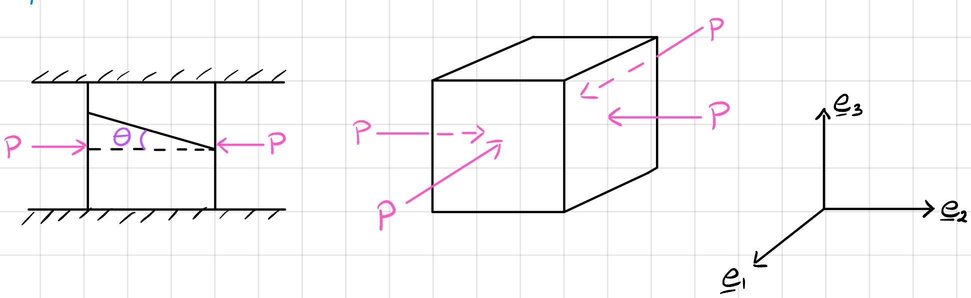


Figure 1: Schematics for Problem 1

1. Find the expression for the tractions on each face of the cube.



Tractions on face with normal e_1:

$$\underline{t}_{e_1} = -P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -P \\ 0 \\ 0 \end{bmatrix}$$

Tractions on face with normal e_2:

$$\underline{t}_{e_2} = -P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \\ 0 \end{bmatrix}$$

Tractions on face with normal e_3:

$$\underline{t}_{e_3} = -F \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -F \end{bmatrix}$$

← F is the reaction force from the plates

These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! These solutions were given in class by Dr. Rimoli. Exam 2 is normally the same every year, but in 2019 someone complained so this exam was changed to what you see here. If you find any errors, please let me know!

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2. With the tractions found in (1) and Hooke's Law, Find all the components of the stress tensor σ_{ij}.

e_1: $\quad \underset{\sim}{t} = \underset{\approx}{\sigma} \cdot e_1$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -P \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{11} = -P \\ \sigma_{12} = 0 \\ \sigma_{13} = 0 \end{matrix}$$

e_2: $\quad \underset{\sim}{t} = \underset{\approx}{\sigma} \cdot e_2$

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{22} = -P \\ \sigma_{23} = 0 \end{matrix}$$

*just because a strain component is zero does not mean the corresponding stress component is zero!

$\varepsilon_{33} = 0 \not\Rightarrow \sigma_{33} = 0$

using Hooke's law (with $\varepsilon_{33} = 0$):

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\left.\begin{aligned} &(1) \quad \sigma_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{11} = -P \\ &(2) \quad \sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{22} = -P \\ &(3) \quad \sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}) \end{aligned}\right\} \text{3 equations, 3 unknowns}$$

(1) + (2):

$$\begin{aligned} -P &= \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{11} \\ +(-P &= \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{22}) \end{aligned}$$

$$-\cancel{2}P = \cancel{2}\lambda(\varepsilon_{11} + \varepsilon_{22}) + \cancel{2}\mu(\varepsilon_{11} + \varepsilon_{22})$$

$$-P = \lambda(\varepsilon_{11} + \varepsilon_{22}) + \mu(\varepsilon_{11} + \varepsilon_{22})$$

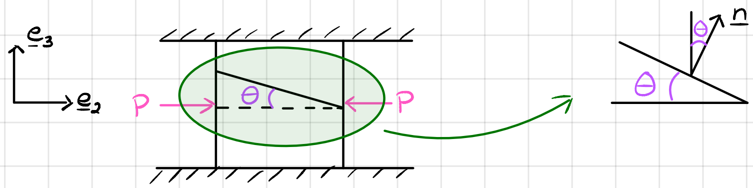
$$-P = (\lambda + \mu)(\varepsilon_{11} + \varepsilon_{22})$$

$\Rightarrow \varepsilon_{11} + \varepsilon_{22} = \dfrac{-P}{\lambda + \mu}$ plugging in to (3): $\sigma_{33} = \dfrac{-P\lambda}{\lambda + \mu}$

thus,

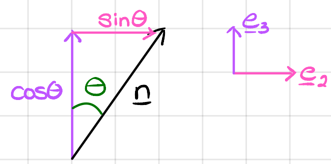
$$\boxed{\underset{\approx}{\sigma} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & \frac{-P\lambda}{\lambda + \mu} \end{bmatrix}}$$

3. Now that you know the full stress state inside the cube, determine the value of applied pressure P that would make the interface between the pieces fail, assuming that the glue has a shear strength τ_u (Hint: you can use the shear stress on the plane ABCD and compare it with the strength of the glue).



Find the shear component of traction:

$$\underline{n} = \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix}$$

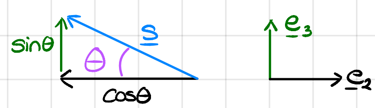


$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & \frac{-P\lambda}{\lambda+\mu} \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -P\sin\theta \\ \frac{-P\lambda\cos\theta}{\lambda+\mu} \end{bmatrix} = \underline{t}$$

we can use the Pythagorean theorem, but the dot product is faster:

$$|\underline{t}_s| = \underline{t} \cdot \underline{s}$$



$$\Rightarrow \quad \underline{s} = \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{bmatrix}$$

$$|\underline{t}_s| = \left[0 \quad -P\sin\theta \quad \frac{-P\lambda\cos\theta}{\lambda+\mu}\right] \cdot \left[0 \quad -\cos\theta \quad \sin\theta\right]$$

$$= 0 + (-P\sin\theta)(-\cos\theta) + \left(\frac{-P\lambda\cos\theta}{\lambda+\mu}\right)(\sin\theta)$$

$$|\underline{t}_s| = P\sin\theta\cos\theta - \frac{P\lambda}{\lambda+\mu}\sin\theta\cos\theta = P\left(1 - \frac{\lambda}{\lambda+\mu}\right)\sin\theta\cos\theta$$

Failure occurs when $t_s = \tau_u$

$$\Rightarrow \quad \boxed{P^{\mp}_{failure} = \frac{\tau_u}{\left(1 - \frac{\lambda}{\lambda+\mu}\right)\sin\theta\cos\theta}}$$

Problem 2

Let us consider the same cube as in the previous problem, but for a different loading scenario. Now, the side faces are free and the pressure P is applied to the top and bottom faces as shown in Figure 2.

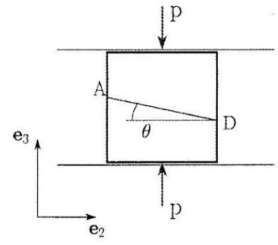
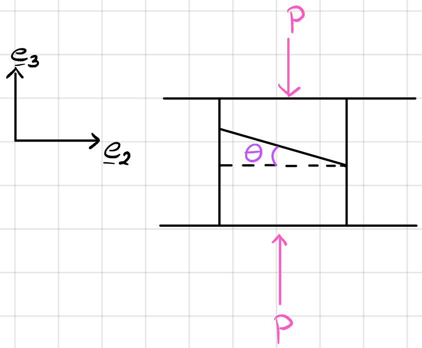


Figure 2: Schematics for Problem 2

1. Determine the value of applied pressure P that would make the interface between the pieces fail.



$$\underset{\sim}{t}_{e_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underset{\sim}{t}_{e_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underset{\sim}{t}_{e_3} = \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix}$$

$$\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{e}_1$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{11} = 0 \\ \sigma_{12} = 0 \\ \sigma_{13} = 0 \end{matrix}$$

$$\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{e}_2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{22} = 0 \\ \sigma_{23} = 0 \end{matrix}$$

$$\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{e}_3$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix} \Rightarrow \sigma_{33} = -P$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} \leftarrow \text{uniaxial stress}$$

Just as in Problem 1: $\underline{n} = \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix}$ $\underline{s} = \begin{bmatrix} 0 \\ -\cos\theta \\ \sin\theta \end{bmatrix}$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -P\cos\theta \end{bmatrix}$$

Side Note:

Problem 1: $\underline{t}$ depends on material constants

Problem 2: $\underline{t}$ is independent of material constants

$$|t_s| = \underline{t} \cdot \underline{s}$$

$$|t_s| = [0 \quad 0 \quad -P\cos\theta] \cdot [0 \quad -\cos\theta \quad \sin\theta] = -P\sin\theta\cos\theta$$

Failure occurs when $t_s = \tau_u$

$$\boxed{P^{II}_{failure} = \frac{\tau_u}{\sin\theta\cos\theta}}$$

2. Based on your calculations, at which angle θ should you glue the two pieces to get maximum strength. Justify your answer for both cases.

For both cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$ would give maximum strength.

$\underline{\theta = 0}$

Case 1: $|t_s| = P\left(1 - \frac{\lambda}{\lambda + \mu}\right)\sin(0)\cos(0) = 0$

Case 2: $|t_s| = P\sin(0)\cos(0) = 0$

} always $< \tau_u$

$\underline{\theta = \frac{\pi}{2}}$

Case 1: $|t_s| = P\left(1 - \frac{\lambda}{\lambda + \mu}\right)\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) = 0$

Case 2: $|t_s| = P\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) = 0$

} always $< \tau_u$

3. Considering that Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ must hold values between -1 and $\frac{1}{2}$, determine, for a given θ, which loading condition is more favorable. Justify your answer.

To compare both cases:

$$\frac{P^{II}_{failure}}{P^{I}_{failure}} = \frac{\tau_u}{\sin\theta\cos\theta} \cdot \frac{\left(1-\frac{\lambda}{\lambda+\mu}\right)\sin\theta\cos\theta}{\tau_u} = 1 - \frac{\lambda}{\lambda+\mu}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \implies \frac{\lambda}{\lambda+\mu} = 2\nu$$

$$\frac{P^{II}_{failure}}{P^{I}_{failure}} = 1-2\nu\,, \quad -1 \le \nu \le \frac{1}{2}$$

for $\nu = -1$:

$$\frac{P^{II}_{failure}}{P^{I}_{failure}} = 1-2(-1) = 3 \implies \text{Case II is better}$$

for $\nu = \frac{1}{2}$:

$$\frac{P^{II}_{failure}}{P^{I}_{failure}} = 1-2\left(\tfrac{1}{2}\right) = 0 \implies \text{Case I is better}$$

when material is incompressible

Case II and case I are the same for $1-2\nu = 1$ $(\nu = 0)$

Thus,

$-1 \le \nu < 0$: Case II is better

$\nu = 0$: cases are the same

$0 < \nu \le \frac{1}{2}$: Case I is better