AE6114 Assignment 4: Linear Elasticity - Basic Concepts

Problem 1

Consider the following stress-strain relations for an isotropic linear elastic solid:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \qquad (1)$$

where λ and μ are Lame's moduli.

1. a) Using Equation 1 and without using the summation convention, write the explicit expressions for all six components of Cauchy's stress tensor $\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}$, and σ_{33}.

$$\sigma_{11} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{11} + 2\mu\varepsilon_{11} = (\lambda+2\mu)\varepsilon_{11} + \lambda(\varepsilon_{22}+\varepsilon_{33})$$

$$\sigma_{12} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{12} + 2\mu\varepsilon_{12} = 2\mu\varepsilon_{12}$$

$$\sigma_{13} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{13} + 2\mu\varepsilon_{13} = 2\mu\varepsilon_{13}$$

$$\sigma_{22} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{22} + 2\mu\varepsilon_{22} = (\lambda+2\mu)\varepsilon_{22} + \lambda(\varepsilon_{11}+\varepsilon_{33})$$

$$\sigma_{23} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{23} + 2\mu\varepsilon_{23} = 2\mu\varepsilon_{23}$$

$$\sigma_{33} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\delta_{33} + 2\mu\varepsilon_{33} = (\lambda+2\mu)\varepsilon_{33} + \lambda(\varepsilon_{11}+\varepsilon_{22})$$

> These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering answers from students that were marked correct. If you find any errors, please let me know!
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b) What are the units of λ and μ?

$\varepsilon : [\varnothing] \Rightarrow \lambda$ and μ have the same units as $\underline{\sigma}$: $\left[\frac{F}{A}\right]$

↑Strain is unitless ↑ Force per area

2. a) Show that for the state of uniaxial stress (that is, when $\sigma_{11} \neq 0$ and all other components of $\underline{\sigma}$ are zero), we have:

$$\sigma_{11} = E\varepsilon_{11}, \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$, $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$

are called the Young's modulus and Poisson's ratio, respectively.

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\sigma_{11} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11} \qquad (1)$$
$$0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{22} \qquad (2)$$
$$0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{33} \qquad (3)$$

Subtracting (2) and (3):

$$
\begin{array}{rl}
 & 0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{22} \\
- & 0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{33} \\
\hline
\end{array}
$$

$$0 = 2\mu\varepsilon_{22} - 2\mu\varepsilon_{33}$$
$$0 = \varepsilon_{22} - \varepsilon_{33}$$
$$\Rightarrow \varepsilon_{22} = \varepsilon_{33}$$

replacing ε_{33} with ε_{22} in (2):

$$0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{22}) + 2\mu\varepsilon_{22}$$
$$0 = \lambda\varepsilon_{11} + 2\lambda\varepsilon_{22} + 2\mu\varepsilon_{22}$$

$$-\lambda\varepsilon_{11} = 2(\lambda+\mu)\varepsilon_{22}$$

$$\varepsilon_{22} = \frac{-\lambda}{2(\lambda+\mu)}\varepsilon_{11} = -\nu\varepsilon_{11}$$

thus, $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$ ✓

plugging $\varepsilon_{22} = \varepsilon_{33} = \frac{-\lambda}{2(\lambda+\mu)}\varepsilon_{11}$ into (1):

$$\sigma_{11} = \lambda\left(\varepsilon_{11} - \frac{\lambda}{2(\lambda+\mu)}\varepsilon_{11} - \frac{\lambda}{2(\lambda+\mu)}\varepsilon_{11}\right) + 2\mu\varepsilon_{11}$$

$$= \frac{2\lambda(\lambda+\mu)\varepsilon_{11} - \lambda^2\varepsilon_{11} - \lambda^2\varepsilon_{11} + 4\mu(\lambda+\mu)}{2(\lambda+\mu)}\varepsilon_{11}$$

$$= \frac{2\lambda^2 + 2\mu\lambda - 2\lambda^2 + 4\mu\lambda + 4\mu^2}{2(\lambda+\mu)}\varepsilon_{11}$$

$$= \frac{3\mu\lambda + 2\mu^2}{\lambda+\mu}\varepsilon_{11} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\varepsilon_{11} = E\varepsilon_{11}$$

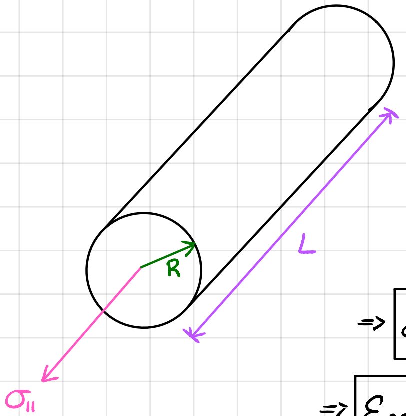
thus, $\sigma_{11} = E\varepsilon_{11}$ ✓

b) What are the units of E and ν?

E has the same units as $\underline{\sigma}$: $\left[\frac{F}{A}\right]$ ← force per area

ν has no units: $\left[\frac{L}{L}\right] = [\emptyset]$ ← dimensionless (same units as $\underline{\underline{\varepsilon}}$)

c) Sketch a circular cylindrical body that is homogeneously deformed in the uniaxial stress state with the tensile (i.e., positive) stress in the direction of the axis of the cylinder. Interpret ε_{11}, ε_{22}, ε_{33} and the Poisson's ratio in terms of the undeformed and deformed length and radius of the cylinder.



$$\varepsilon_{11} = \frac{\Delta L}{L} \qquad \varepsilon_{22} = \varepsilon_{33} = \frac{\Delta R}{R}$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$$

$L \equiv$ undeformed length $\qquad R \equiv$ undeformed radius

$L' \equiv$ deformed length $\qquad R'' \equiv$ deformed radius

$$\Rightarrow \boxed{\varepsilon_{11} = \frac{L'-L}{L}} \qquad \nu = \frac{R'-R}{R} \cdot \frac{L}{L'-L} = \frac{L(R'-R)}{R(L'-L)}$$

$$\Rightarrow \boxed{\varepsilon_{22} = \varepsilon_{33} = \frac{R'-R}{R}} \qquad \Rightarrow \boxed{\nu = \frac{L(R'-R)}{R(L'-L)}}$$

(ε_{22}, ε_{33}, and ν will be negative if diameter decreases as length increases)

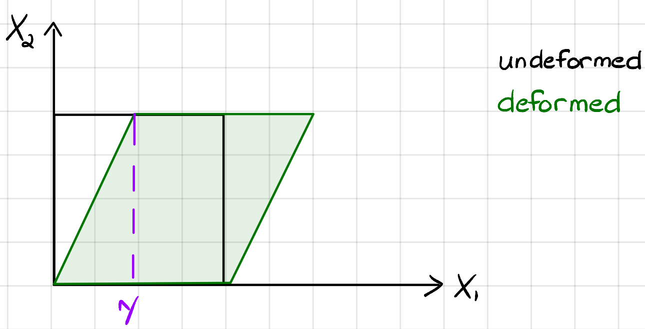
this problem assumes infinitesimal deformations

3. Consider the simple shear deformation given by

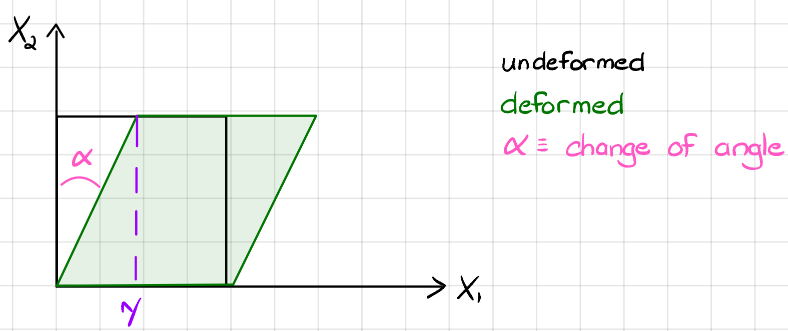
$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

where γ is a small given constant.

a) Sketch this deformation in the $X_1 - X_2$ plane (using a unit square).



b) Mark the change of angle between directions $(1,0,0)^T$ and $(0,1,0)^T$ on your sketch.



c) Find the matrix form of the components of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ and from Equation 1, the Cauchy stress tensor $\underline{\underline{\sigma}}$.

$$\underline{u} = \underline{\phi} - \underline{X} \quad \Rightarrow \quad u_1 = \gamma X_2 \quad u_2 = 0 \quad u_3 = 0$$

$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$

$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$

$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\gamma}{2}$

$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$

$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0$

$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$

$$\boxed{\underset{\approx}{\varepsilon} = \begin{bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$

$\sigma_{11} = 0$ $\quad$ $\sigma_{22} = 0$

$\sigma_{12} = 2\mu\left(\frac{\gamma}{2}\right) = \mu\gamma$ $\quad$ $\sigma_{23} = 0$

$\sigma_{13} = 0$ $\quad$ $\sigma_{33} = 0$

$$\boxed{\underset{\approx}{\sigma} = \begin{bmatrix} 0 & \mu\gamma & 0 \\ \mu\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Problem 2

The strain energy density W of a material represents the energy stored due to elastic deformation. For a linear elastic material, it is given by:

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

Since it represents energy stored due to deformation, we require that W is a non-negative function for any elastic deformation, that is:

$$W \geq 0; \quad W = 0 \iff \varepsilon_{ij} = 0$$

This imposes an extra constraint on the values of the elastic constants. We can study these constraints by looking at simple deformation states:

1. Consider the simple shear problem solved in Exercise 1. Find W for this case in terms of μ and γ. Show that $\mu > 0$.

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\mu\gamma)\left(\frac{\gamma}{2}\right) + \frac{1}{2}(\mu\gamma)\left(\frac{\gamma}{2}\right) = \frac{\mu\gamma^2}{2}$$

$W > 0 \quad (\varepsilon_{ij} \neq 0) \Rightarrow \frac{\mu\gamma^2}{2} > 0 \quad \Rightarrow \mu > 0$ $\leftarrow$ because γ^2 is always > 0

2. Consider the hydrostatic state of stress $\sigma_{ij} = p\delta_{ij}$. Find W for this case in terms of the bulk modulus K. Show that $K > 0$ or $\lambda > -\frac{2}{3}\mu$.

$\sigma_{ij} = p\delta_{ij} \qquad p = \frac{1}{3}\sigma_{ii} = K\varepsilon_{vol} \qquad K = \frac{3\lambda + 2\mu}{3}$

When $\underset{\approx}{\varepsilon}$ is diagonal, $\underset{\approx}{\sigma}$ is diagonal and $W > 0 \quad (\varepsilon_{ij} \neq 0)$

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{3}{2}\kappa\varepsilon_{vol}\overset{=1}{\delta_{ij}} = \frac{3}{2}\kappa\varepsilon_{vol} > 0 \quad \Rightarrow \quad \kappa > 0$$

$$\kappa = \frac{3\lambda + 2\mu}{3} > 0 \quad \Rightarrow \quad 3\lambda + 2\mu > 0 \quad \Rightarrow \quad 3\lambda > -2\mu \quad \Rightarrow \quad \lambda > -\frac{2}{3}\mu$$

Another way:

$$\underline{\underline{\sigma}} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} = \kappa \begin{bmatrix} \varepsilon_{vol} & 0 & 0 \\ 0 & \varepsilon_{vol} & 0 \\ 0 & 0 & \varepsilon_{vol} \end{bmatrix} \qquad \underline{\underline{\varepsilon}} = \frac{1}{3}\begin{bmatrix} \varepsilon_{vol} & 0 & 0 \\ 0 & \varepsilon_{vol} & 0 \\ 0 & 0 & \varepsilon_{vol} \end{bmatrix}$$

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\frac{\kappa}{3}\varepsilon_{vol}^2 + \frac{\kappa}{3}\varepsilon_{vol}^2 + \frac{\kappa}{3}\varepsilon_{vol}^2\right) = \frac{1}{2}\kappa\varepsilon_{vol}^2$$

Since $W > 0$ and $\varepsilon_{vol}^2 > 0$ $(\varepsilon_{vol} \neq 0)$, κ must always be positive and > 0

3. How do these constraints translate to the elastic constants E and ν?

E can be written as $\boxed{E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}}$

$\lambda > -\frac{2}{3}\mu$, so $\boxed{\text{E must always be greater than 0}}$

$$\mu(3(-\tfrac{2}{3}\mu) + 2\mu) = 0$$

ν can be written as $\boxed{\nu = \frac{\lambda}{2(\lambda + \mu)}}$

$\lambda > -\frac{2}{3}\mu$, so $\boxed{\nu \text{ must be greater than } -1}$

as $\lambda \to \infty$:

$$\frac{\infty}{2\infty + 2\mu} \longrightarrow \frac{1}{2}$$ $\Leftarrow$ 2∞ is much greater than 2μ: $\frac{\cancel{\infty}}{2\cancel{\infty} + \cancel{2\mu}^{\,0}} = \frac{1}{2}$

$\hookrightarrow$ as $\lambda \to \infty$, $\nu \to \frac{1}{2}$ so $\nu < \frac{1}{2}$ is the upper limit of ν

$\boxed{\nu \text{ must be less than } \frac{1}{2}}$