3D EULER-BERNOULLI BEAM THEORY

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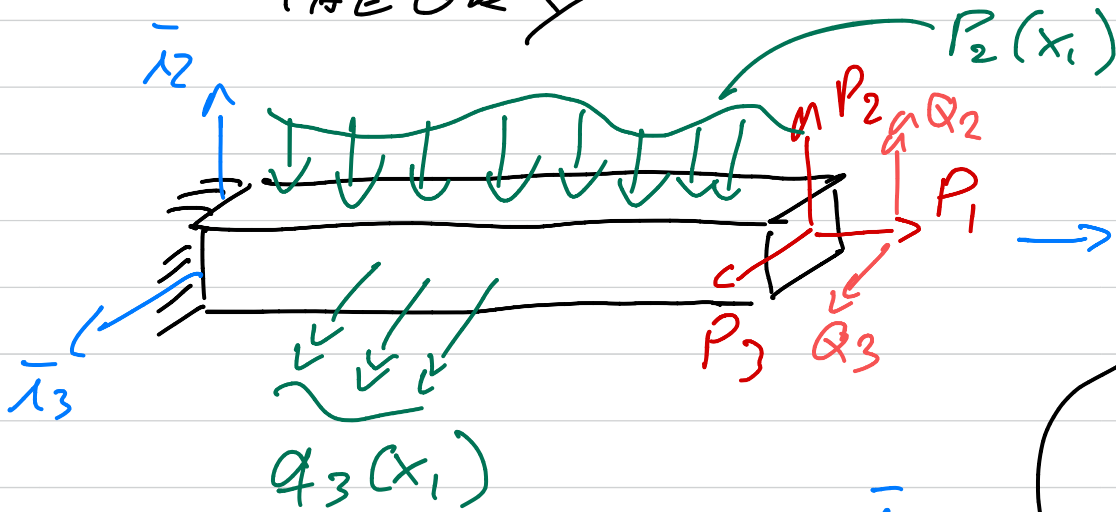
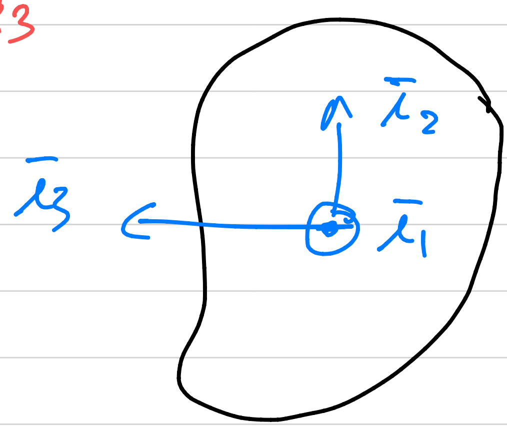




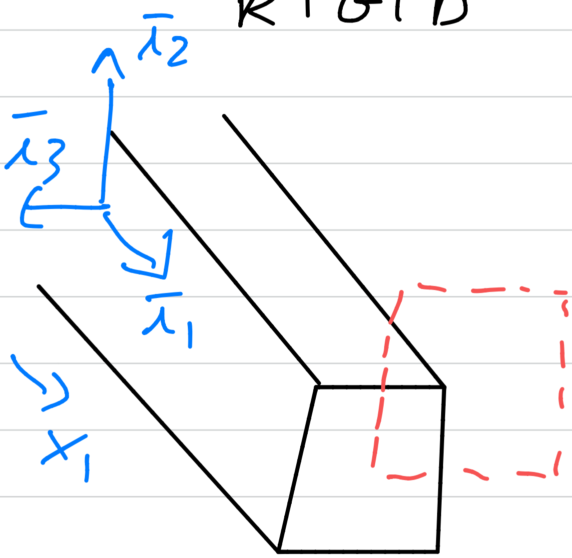
* Beams with loads applied in multiple directions
* Arbitrary cross-section shape
* Beam need not be homogeneous

EULER – BERNOULLI BEAM THEORY

* ONE PARTICULAR THEORY FOR DESCRIBING THE BEHAVIOR OF SLENDER BEAMS.

ASSUMPTIONS

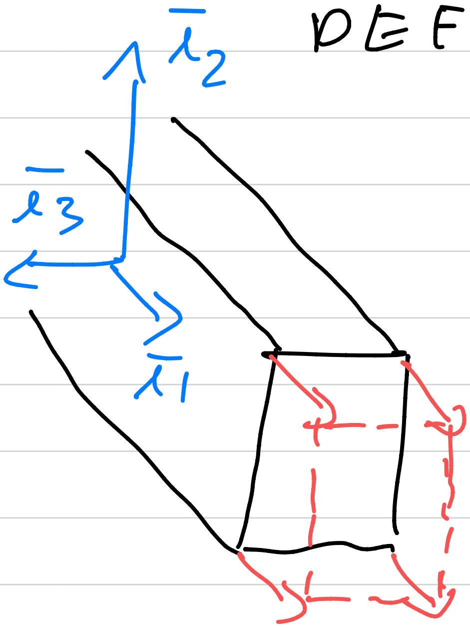
1) THE CROSS-SECTION IS RIGID IN ITS OWN PLANE

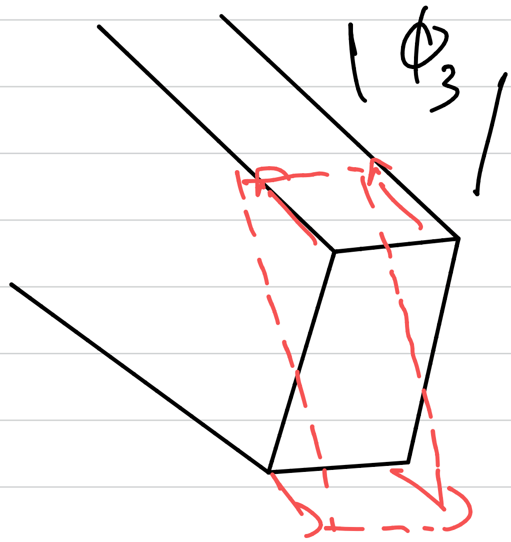


$$u_2 = u_2(x_1)$$

$$u_3 = u_3(x_1)$$

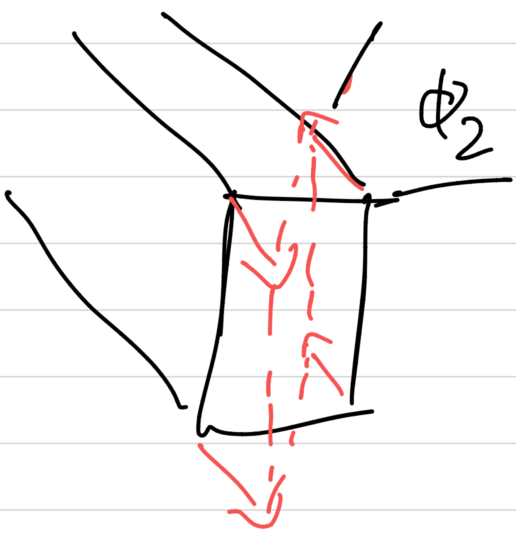
2) THE CROSS-SECTION REMAINS PLANE AFTER DEFORMATION.





$$u_1 = u_1(x_1)$$

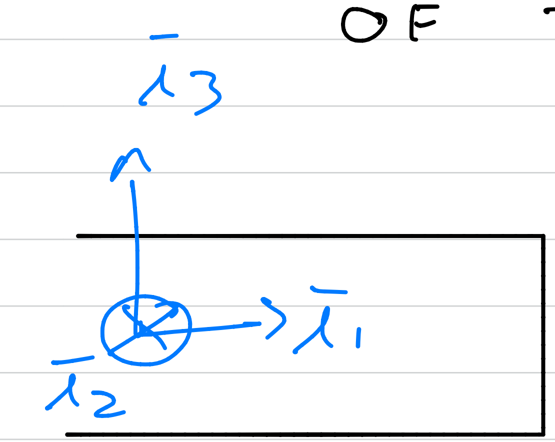
$$u_1 = -x_2\,\phi_3(x_1)$$

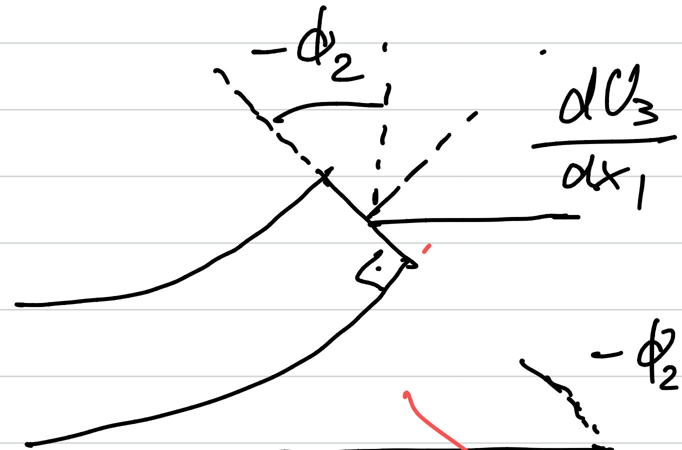


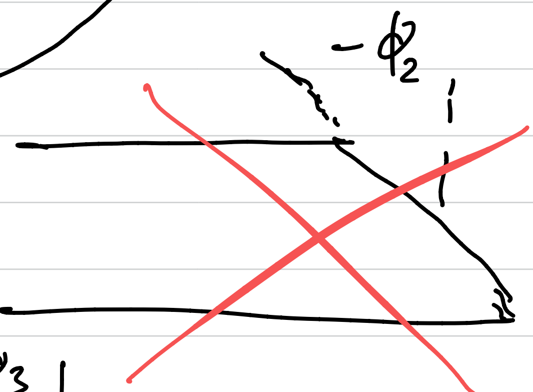
$$u_1 = x_3\,\phi_2(x_1)$$

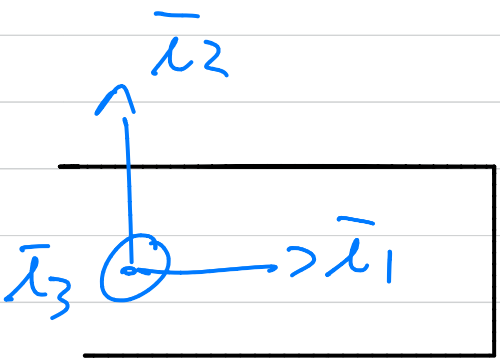
$$u_1 = \bar{u}_1(x_1) + x_3\,\phi_2(x_1) - x_2\,\phi_3(x_1)$$

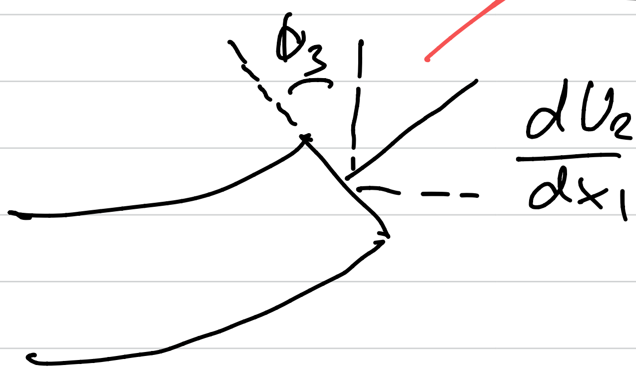
3) THE CROSS - SECTION REMAINS NORMAL TO THE DEFORMED AXIS OF THE BEAM











$$\phi_2 = -\frac{du_3}{dx_1}, \qquad \phi_3 = \frac{du_2}{dx_1}$$

$$\underline{u} \begin{cases} u_1 = \bar{u}_1(x_1) - x_3 \dfrac{du_3}{dx_1} - x_2 \dfrac{du_2}{dx_1} \\ u_2 = u_2(x_1) \\ u_3 = u_3(x_1) \end{cases}$$

STRAIN FIELD

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = \frac{d\bar{u}_1}{dx_1} - x_3 \frac{d^2 u_3}{dx_1^2} - x_2 \frac{d^2 u_2}{dx_1^2}$$

$$\epsilon_2 = \frac{\partial u_2}{\partial x_2} = 0 \, , \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3} = 0$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -\frac{du_2}{dx_1} + \frac{du_2}{dx_1} = 0$$

$$\gamma_{13} = 0 \, , \quad \gamma_{23} = 0$$

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) + x_3 k_2(x_1) - x_2 k_3(x_1)$$

$$\bar{\epsilon}_1(x_1) = \frac{d\bar{u}_1}{dx_1}$$ – SECTIONAL AXIAL STRAIN

$$k_2 = -\frac{d^2 u_3}{dx_1^2}$$

$$k_3 = \frac{d^2 u_2}{dx_1^2}$$

SECTIONAL CURVATURES ABOUT $\bar{\imath}_2$ AND $\bar{\imath}_3$

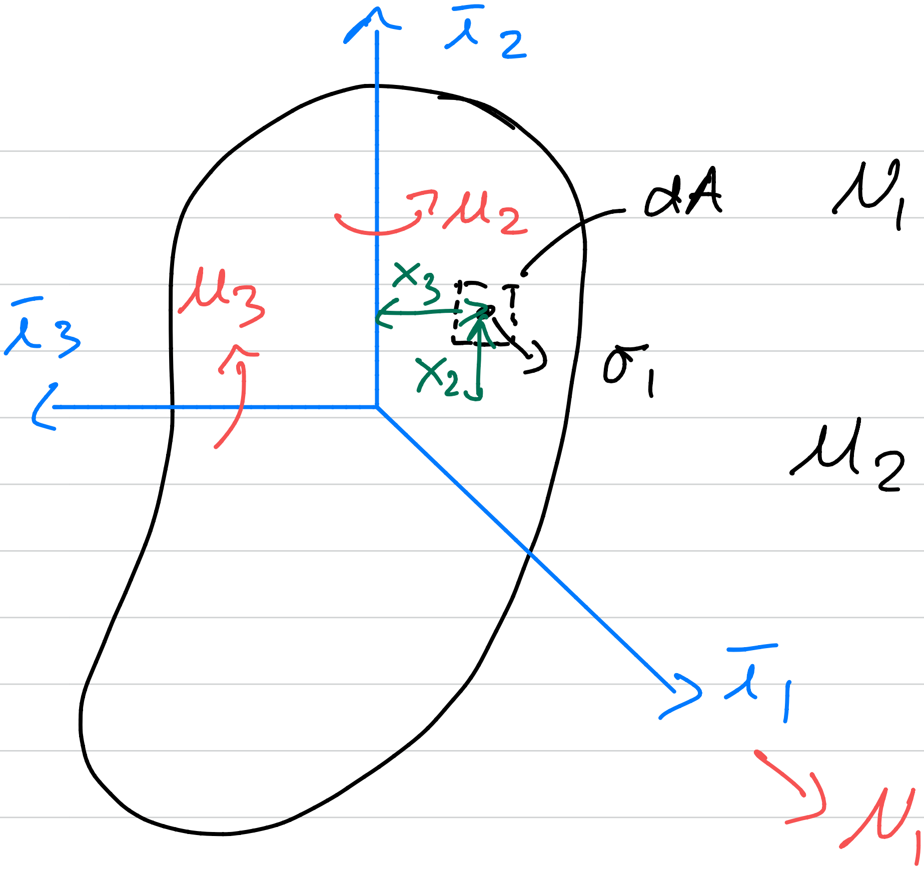
STRESS FIELD

* ASSUME LINEAR ELASTIC ISOTROPIC
* ASSUME THAT $\sigma_2 \ll \sigma_1$ AND $\sigma_3 \ll \sigma_1$ SUCH THAT

$$\sigma_3 \approx 0 \quad , \quad \sigma_2 \approx 0$$

$$\sigma_1(x_1, x_2, x_3) = E \cdot \epsilon_1(x_1, x_2, x_3)$$

$$\sigma_1 = E\left(\bar{\epsilon}_1(x_1) + x_3 K_2(x_1) - x_2 K_3(x_1)\right)$$



$$\mathcal{N}_1 = \int_A \sigma_1 \cdot dA$$

$$\mathcal{M}_2 = \int_A x_3 \, \sigma_1 \, dA$$

$$\mathcal{M}_3 = \int_A x_2 \, \sigma_1 \, dA$$

$$\mathcal{N}_1 = \int_A E \cdot \bar{\epsilon}_1 \, dA + \int_A E x_3 \kappa_2 \, dA - \int_A E x_2 \kappa_3 \, dA$$

$$\mathcal{N}_1 = \bar{\epsilon}_1 \underbrace{\int_A E \, dA}_{S} + \kappa_2 \underbrace{\int_A E x_3 \, dA}_{S_3} - \kappa_3 \underbrace{\int_A E x_2 \, dA}_{S_2}$$

S, S_2, S_3 – SECTIONAL STIFFNESS COEFFICIENTS

$$N_1 = S\,\bar{\epsilon}_1 + S_3\,\kappa_2 - S_2\,\kappa_3$$

$$M_2 = \bar{\epsilon}_1 \underbrace{\int_A E \cdot x_3\,dA}_{S_3} + \kappa_2 \underbrace{\int_A E\,x_3^2\,dA}_{H_{22}}$$

$$- \kappa_3 \underbrace{\int_A E\,x_2 x_3\,dA}_{H_{23}}$$

$$M_2 = S_3\,\bar{\epsilon}_1 + H_{22}\,\kappa_2 - H_{23}\,\kappa_3$$

$$M_3 = -S_2\,\bar{\epsilon}_1 - H_{23}\,\kappa_2 + H_{33}\,\kappa_3$$

H_{22}, H_{33} → BENDING STIFFNESSES

H_{23} → CROSS BENDING STIFFNESS

$$H_{22} = \int_A E\,x_3^2\,dA = E \underbrace{\int_A x_3^2\,dA}_{I} = \underline{EI}$$

$$H_{33} = \int_A E\,x_2^2\,dA$$

$$\begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} S & S_3 & -S_2 \\ S_3 & H_{22} & -H_{23} \\ -S_2 & -H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_1 \\ K_2 \\ K_3 \end{bmatrix}$$

↳ SECTIONAL STIFFNESS MATRIX

<u>SIMPLIFICATION</u>: CHOSE THE ORIGIN OF OUR AXIS SYSTEM AT THE CENTROID OF THE CROSS - SECTION

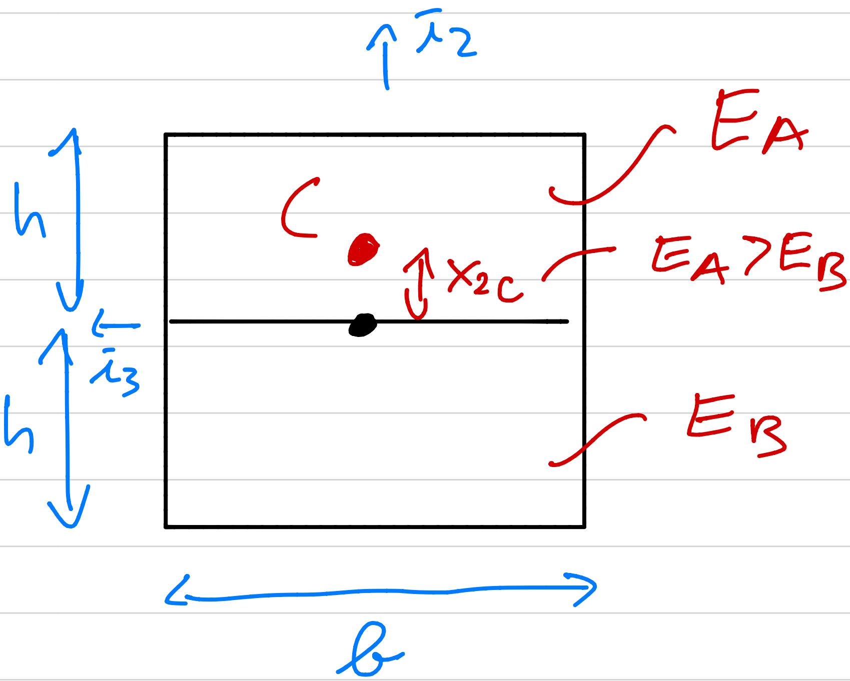
$$X_{2C} = \frac{S_2}{S} = \frac{\int E x_2 \, dA}{\int E \, dA}$$

$$X_{3C} = \frac{S_3}{S} = \frac{\int E x_3 \, dA}{\int E \, dA}$$

IF THE COORDINATE SYSTEM IS AT THE CENTROID

$$x_{2c} = S_2/S = 0 \quad , \quad x_{3c} = S_3/S =$$

EXAMPLE :



$$x_{2c} = \frac{\int_A E x_2 \, dA}{\int_A E \, dA}$$

$$x_{2c} = \frac{\int_0^h E_A x_2 \, dx_2 \cdot \cancel{b} + \int_{-h}^0 E_B x_2 \, dx_2 \cdot \cancel{b}}{E_A \cdot h \cancel{b} + E_B \cdot h \cancel{b}}$$

$$x_{2c} = \frac{h}{2} \frac{(E_A - E_B)}{(E_A + E_B)}$$

$$\begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & H_{22}^c & -H_{23}^c \\ 0 & -H_{23}^c & H_{33}^c \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_1 \\ K_2 \\ K_3 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\epsilon}_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1/S & 0 & 0 \\ 0 & H_{33}^c/\Delta H & H_{23}^c/\Delta H \\ 0 & H_{23}^c/\Delta H & H_{22}^c/\Delta H \end{bmatrix} \begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix}$$

$$\Delta H = H_{22}^c H_{33}^c - H_{23}^c H_{23}^c$$

$$\sigma_1 = E(\bar{\epsilon}_1 + x_3 K_2 - x_2 K_3)$$

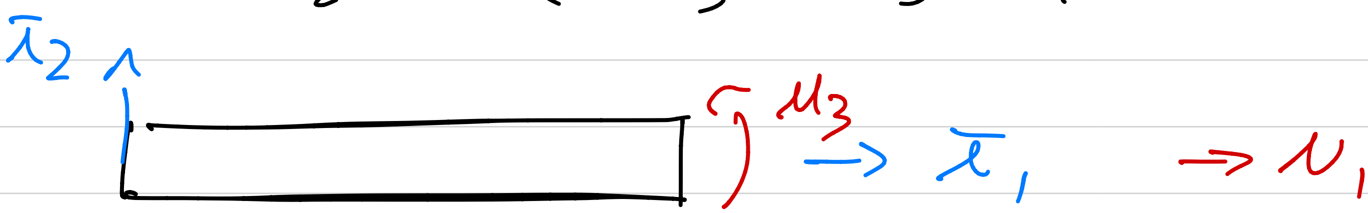
$$\sigma_1 = E\left[\frac{N_1}{S} + x_3 \frac{H_{33}^c M_2 + H_{23}^c M_3}{\Delta H} - x_2 \frac{H_{23}^c M_2 + H_{22}^c M_3}{\Delta H} \right]$$

Sanity Check

→ Symmetric Cross-Section ($H_{23}^c = 0$)

→ Homogeneous (E = constant)

→ $M_2 = 0$ ($M_3 \neq 0$, $N_1 \neq 0$)



$$\sigma_1 = E \frac{N_1}{S} - E x_2 \frac{H_{22}^c M_3}{\Delta H}$$

$$\Delta H = H_{22}^c H_{33}^c - \cancel{H_{23}^c H_{23}^c}$$

$$\sigma_1 = \frac{E N_1}{S} - \frac{E x_2 M_3}{H_{33}^c}$$

$$S = E \cdot A \qquad H_{33}^c = E I_{33}^c$$

$$\sigma_1 = \frac{N_1}{A} - x_2 \frac{M_3}{I_{33}^c} \longrightarrow \sigma = \frac{N}{A} - y \frac{M}{I}$$