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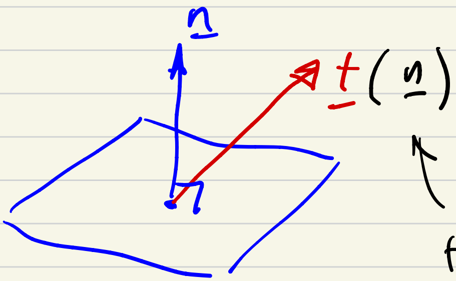
Some notes on the stress tensor



traction is a function of the normal

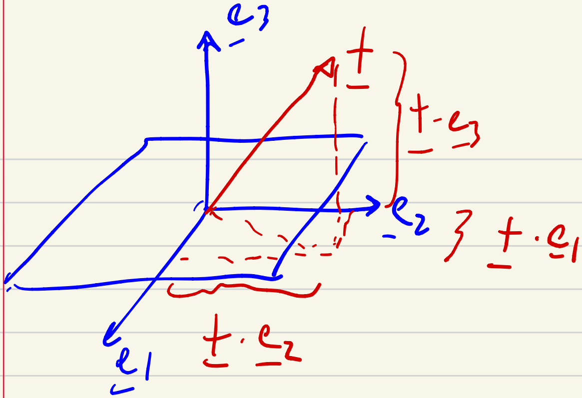
Also, $\underline{t}(-\underline{n}) = -\underline{t}(\underline{n})$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} \quad \begin{cases} t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{cases}$$

$\sigma_{ij} \rightarrow 9$ components,

<u>but</u> $\sigma_{ij} = \sigma_{ji} \rightarrow$ so 6 values

<u>NOTE</u>:
- σ_{ij} is the component of the traction on the face w/ normal $\underline{e}_j$ in the direction $\underline{e}_i$
- The diagonal components of σ_{ij} are the <u>normal</u> stresses, and the off diagonal are called <u>shear</u> stresses.



$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{e}_3$$

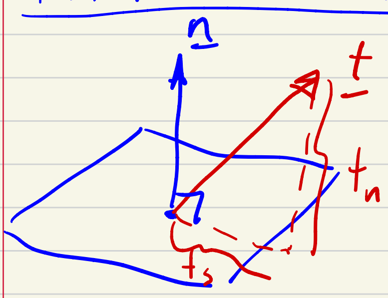
$$\{\underline{t}\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}$$

$\underline{t} \cdot \underline{e}_3 = \sigma_{33}$ → normal component

$\underline{t} \cdot \underline{e}_2 = \sigma_{23}$

$\underline{t} \cdot \underline{e}_1 = \sigma_{13}$

Normal & shear components of tractions



$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$

- In general, the component of $\underline{t}$ in the direction $\underline{d}$ is given by

$$t_d = \underline{t} \cdot \underline{d} = t_i d_i = \sigma_{ij} n_j d_i$$

In particular: $t_n = \underline{t} \cdot \underline{n} = \sigma_{ij} n_j n_i$

Using pythagoras: $t_s^2 + t_n^2 = |\underline{t}|^2$

$$t_s = \sqrt{\sigma_{ij} n_j \sigma_{ik} n_k - (\sigma_{ij} n_i n_j)^2}$$

Stress Decomposition.

- It is common to decompose the Cauchy stress tensor into its hydrostatic and deviatoric components:

$$\underline{\underline{\sigma}} = \underline{\underline{S}} - p\,\underline{\underline{I}} \,,$$

where $\left(p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}\sigma_{kk}\right)$ is the hydrostatic stress or pressure and

$$\underline{\underline{S}} = \underline{\underline{\sigma}} + p\,\underline{\underline{I}} = (\sigma_{ij} + p\,\delta_{ij})\,\underline{e}_i\underline{e}_j$$ is the deviatoric part of the Cauchy stress Tensor.

- A stress state with $\underline{\underline{S}} = \underline{\underline{0}}$ is called spherical. In this case all directions are principal directions

Material form of balance of linear momentum

Begin w/ spatial form:

$$\int_E \rho \ddot{x}_i \, dv = \int_E \rho b_i \, dv + \int_{\partial E} \sigma_{ij} n_j \, da$$

Using conservation of mass ($J\rho = \rho_0$) and $\frac{dv}{dV} = J$;

$J = \det(\underline{\underline{F}})$

$$\hookrightarrow \int_E \rho a_i \, dv = \int_{E_0} \rho_0 a_i \, dV \quad \text{and} \quad \int_E \rho b_i \, dv = \int_{E_0} \rho_0 b_i \, dV$$

Using the Piola transformation: $n_j \, da = J (F^{-1})_{kj} N_k \, dA$

$$\Rightarrow \int_{\partial E} \sigma_{ij} n_j \, da = \int_{\partial E_0} \sigma_{ij} J (F^{-1})_{kj} N_k \, dA$$

We define the first Piola-Kirchhoff stress tensor as:

$$\boxed{P_{ik} = J \sigma_{ij} (F^{-1})_{kj}} \qquad \boxed{\underline{\underline{P}} = J \underline{\underline{\sigma}} \cdot \underline{\underline{F}}^{-T}}$$

Material Form.

In this way, we get:

$$\boxed{\int_{E_0} \rho_0 a_i \, dV = \int_{E_0} \rho_0 b_i \, dV + \int_{\partial E_0} P_{ik} N_k \, dA}$$

(lost symmetry in $\underline{\underline{\sigma}}$, $\underline{\underline{P}}$ is not symmetric)

<u>Notes:</u>

- $\underline{\underline{P}}$ is just another representation (defined for convenience) of the Cauchy stress tensor and <u>does not</u> represent a new physical quantity.
- The Cauchy stress tensor can be expressed as

$$\underline{\underline{\sigma}} = \frac{1}{J} \underline{\underline{P}} \cdot \underline{\underline{F}}^T \quad \text{or} \quad \sigma_{ij} = \frac{1}{J} P_{ik} F_{jk}$$

<u>Nominal traction:</u> $\underline{t} = \frac{d\underline{f}}{da}$; $t_i = \frac{df_i}{da}$

with $da \equiv$ <u>deformed</u> differential of area

From Cauchy's relation: $t_i = \sigma_{ij} n_j$

$$\Rightarrow \sigma_{ij} n_j = \frac{dF_i}{da} \Rightarrow \sigma_{ij} n_j \, da = df_i$$

But $\sigma_{ij} = \frac{1}{J} P_{ik} F_{jk}$

$$\Rightarrow \frac{1}{J} P_{ik} F_{jk} n_j \, da = df_i$$

Using the transformation of area:

$$P_{ik} N_k \, dA = df_i$$

$$\Rightarrow \boxed{\frac{df_i}{dA} = P_{ik} N_k}$$ Nominal Traction

per unit referenced area.

Finally, we define the nominal traction as

$$\boxed{\begin{aligned} T_i &= \frac{df_i}{dA} \\ \underline{T} &= \frac{d\underline{f}}{dA} \end{aligned}}$$

In this way, we get the material form of Cauchy's relation

$$T_i = P_{ik} N_k \quad \text{or}$$

$$\underline{T} = \underline{\underline{P}} \cdot \underline{N}$$

Local expression of material form of cons. of linear momentum.

$$\int_{E_0} \rho_0 a_i \, dV = \int_{E_0} \rho_0 b_i \, dV + \int_{\partial E_0} P_{ik} N_k \, dA$$

Applying the divergence theorem:

$$\int_{E_0} \rho_0 a_i \, dV = \int_{E_0} \rho_0 b_i \, dV + \int_{E} P_{ik,k} \, dV$$

Collecting terms: $$\int_{E_0} \left(P_{ik,k} + \rho_0 b_i - \rho_0 a_i \right) dV = 0 \qquad \forall E_0 \subset B_0$$

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$$\Rightarrow \boxed{\begin{aligned} & P_{ik,k} + \rho_0 b_i = \rho_0 a_i \\ & \mathrm{Div}(\underline{\underline{P}}) + \rho_0 \underline{b} = \rho_0 \underline{a} \end{aligned}}$$

Pro: In ref config: easier to integrate

Cons: P is no longer symmetric.