AE6114 Exam 1

Problem 1

1. For a second order tensor with components A_{ij}, show that the derivative of its inverse is given by the expression $$\frac{\partial (A^{-1})_{kl}}{\partial A_{ij}} = -(A^{-1})_{ki}(A^{-1})_{jl}$$

The identity is: $A^{-1}_{km} A_{mn} = \delta_{kn}$

Taking the derivative of both sides:

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} + A^{-1}_{km} \frac{\partial A_{mn}}{\partial A_{ij}} = 0$$

using $\frac{\partial A_{mn}}{\partial A_{ij}} = \delta_{mi}\delta_{nj}$:

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} + \underbrace{A^{-1}_{km}\delta_{mi}\delta_{nj}}_{\text{move to other side}} = 0$$

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} A_{mn} = -A^{-1}_{km}\delta_{mi}\delta_{nj} = -A^{-1}_{ki}\delta_{nj}$$

multiply both sides by A^{-1}_{nl}:

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} \underbrace{A_{mn}A^{-1}_{nl}}_{\delta_{ml}} = -A^{-1}_{ki}\delta_{nj}A^{-1}_{nl}$$

$$\frac{\partial A^{-1}_{km}}{\partial A_{ij}} \delta_{ml} = -A^{-1}_{ki}A^{-1}_{jl}$$

$$\frac{\partial (A^{-1})_{kl}}{\partial A_{ij}} = -(A^{-1})_{ki}(A^{-1})_{jl} \quad \checkmark$$

These solutions are intended to help everyone and anyone studying for Quals, so please feel free to share! No formal solutions were ever given, so these have been created by gathering the answers from students that were marked correct. As far as I'm aware, Exam 1 is the same for 2017, 2018, and 2019. If you find any errors, please let me know!

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2. Show that the change in length of an infinitesimal fiber in a given direction is frame indifferent, that is, that the operation $\underline{N}\cdot\underline{\underline{C}}\cdot\underline{N}$ gives the same result independently of the frame adopted to represent the Cauchy-Green strain tensor $\underline{\underline{C}}$ and the fiber direction $\underline{N}$.

$\underline{N}\,\underline{\underline{C}}\,\underline{N}$, $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$

For a given fiber direction $\underline{N}$: $\lambda(\underline{N}) = \sqrt{C_{ij}N_iN_j}$

In the current frame: $\underline{N}\,\underline{\underline{C}}\,\underline{N} = N_iC_{ij}N_j = N_iF_{ki}F_{kj}N_j$ (1)

$C_{ij} = F_{ki}F_{kj}$

In the adopted frame: $\underline{N}^*\underline{\underline{C}}^*\underline{N}^* = N_i^*C_{ij}^*N_j^*$

$N_i^* = l_{ik}N_k$
$N_j^* = l_{jp}N_p$

$C_{ij}^* = F_{li}^*F_{lj}^*$

$$= (\ell_{ik}N_k)(F^*_{\ell i}F^*_{\ell j})(\ell_{jp}N_p)$$

$F^*_{\ell i} = \ell_{c\ell}\,\ell_{id}\,F_{cd}$
$F^*_{\ell j} = \ell_{\ell e}\,\ell_{jg}F_{eg}$

$$\underline{N}^*\underline{\underline{C}}^*\underline{N}^* = (\ell_{ik}N_k)(\ell_{c\ell}\,\ell_{id}F_{cd})(\ell_{\ell e}\,\ell_{jg}F_{eg})(\ell_{jp}N_p)$$

$$= N_k F_{cd} F_{eg} N_p \underbrace{\ell_{ik}\,\ell_{id}}_{\delta_{kd}} \underbrace{\ell_{c\ell}\,\ell_{\ell e}}_{\delta_{ce}} \underbrace{\ell_{jg}\,\ell_{jp}}_{\delta_{gp}}$$

$$= N_k \delta_{kd} F_{cd} \delta_{ce} F_{eg} N_p \delta_{gp}$$

$$\underline{N}^*\underline{\underline{C}}^*\underline{N}^* = N_d F_{ed} F_{eg} N_g$$

d, e, and g are dummy indices

=> change: $d=i$
$g=j$
$e=K$

=> $\underline{N}^*\underline{\underline{C}}^*\underline{N}^* = N_i F_{Ki} F_{Kj} N_j$ which is the same as (1)

thus, the change in length of an infinitesimal fiber in a given direction is frame indifferent ✓

Problem 2

Let the material and spatial coordinates be measured in the same rectangular Cartesian coordinate system with basis vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$. Consider the following deformation:

$$x_1 = X_1$$
$$x_2 = X_2$$
$$x_3 = X_3 + \alpha(X_1 + X_2)$$

1. Find the matrices of the deformation gradient tensor $\underline{\underline{F}}$ and the Lagrangian strain tensor $\underline{\underline{E}}$. Does the deformation gradient change with position? Does this deformation change volume?

$$F_{ij} = \frac{\partial \phi_i}{\partial X_j}$$

$$F_{11} = \frac{\partial \phi_1}{\partial X_1} = 1 \qquad F_{12} = \frac{\partial \phi_1}{\partial X_2} = 0 \qquad F_{13} = \frac{\partial \phi_1}{\partial X_3} = 0$$

$$F_{21} = \frac{\partial \phi_2}{\partial X_1} = 0 \qquad F_{22} = \frac{\partial \phi_2}{\partial X_2} = 1 \qquad F_{23} = \frac{\partial \phi_2}{\partial X_3} = 0$$

$$F_{31} = \frac{\partial \phi_3}{\partial X_1} = \alpha \qquad F_{32} = \frac{\partial \phi_3}{\partial X_2} = \alpha \qquad F_{33} = \frac{\partial \phi_3}{\partial X_3} = 1$$

$$\boxed{\underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}}$$

$\underset{=}{C} = 2\underset{=}{E} + \underset{=}{I} \Rightarrow \underset{=}{E} = \frac{1}{2}(\underset{=}{C} - \underset{=}{I})$, $\underset{=}{C} = \underset{=}{F}^T \underset{=}{F}$

$$\underset{=}{F}^T = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{=}{F}^T \underset{=}{F} = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 & \alpha^2 & \alpha \\ \alpha^2 & 1+\alpha^2 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix} = \underset{=}{C}$$

$$\underset{=}{C} - \underset{=}{I} = \begin{bmatrix} \alpha^2 & \alpha^2 & \alpha \\ \alpha^2 & \alpha^2 & \alpha \\ \alpha & \alpha & 0 \end{bmatrix}$$

$\underset{=}{E} = \frac{1}{2}(\underset{=}{C} - \underset{=}{I})$:

$$\boxed{\underset{=}{E} = \begin{bmatrix} \frac{\alpha^2}{2} & \frac{\alpha^2}{2} & \frac{\alpha}{2} \\ \frac{\alpha^2}{2} & \frac{\alpha^2}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{\alpha}{2} & 0 \end{bmatrix}}$$

No, the deformation gradient does not change with position ($\underset{=}{F}$ is not a function of X_1, X_2, X_3).

$\det(\underset{=}{F}) = 1(1) - 0 + 0 = 1$

$dv = \det(\underset{=}{F})dV \Rightarrow$ No, the deformation does not change volume.

2. Without any further assumptions, find:

a) Stretch of the element that is in the direction $\underset{-}{e}_2$ in the reference configuration

$\lambda_{\underset{-}{N}}^2 = C_{jk} N_j N_k$ where $\underset{-}{N} = \underset{-}{e}_2$: $N = \langle 0 \ 1 \ 0 \rangle$

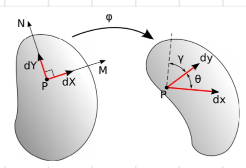
$$\lambda_{\underset{-}{e}_2}^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1+\alpha^2 & \alpha^2 & \alpha \\ \alpha^2 & 1+\alpha^2 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1+\alpha^2$$

← we also know that $\lambda(\underset{-}{e}_2) = \sqrt{C_{22}} = \sqrt{1+\alpha^2}$

$$\boxed{\lambda(\underset{-}{e}_2) = \sqrt{1+\alpha^2}}$$

b) Change of angle between the elements that are in the directions $\underset{-}{e}_2$ and $\underset{-}{e}_3$ in the reference configuration.

$\underset{-}{e}_2 \perp \underset{-}{e}_3$ in the reference configuration (so this is similar to Problem 2 from Assignment 2)



← $\cos\theta = \dfrac{\underset{-}{N} \underset{=}{C} \underset{-}{M}}{\lambda_N \lambda_M}$

when $\underset{-}{N} \perp \underset{-}{M}$, $\cos\theta = \sin\gamma$ where γ is the change in angle between the differential elements

$$\cos\Theta = \frac{C_{jk} N_j M_k}{\sqrt{C_{lm} N_l N_m}\sqrt{C_{pq} M_p M_q}} \quad , \quad \lambda_{\underline{N}} = \sqrt{C_{lm} N_l N_m} \, , \quad \lambda_{\underline{M}} = \sqrt{C_{pq} M_p M_q}$$

$$\sin\gamma = \cos\Theta(\underline{e}_2, \underline{e}_3) = \frac{C_{23}\, e_2 e_3}{\sqrt{C_{22} e_2 e_2}\sqrt{C_{33} e_3 e_3}} = \frac{C_{23}}{\sqrt{C_{22}}\sqrt{C_{33}}} = \frac{\alpha}{\sqrt{1+\alpha^2}\ \sqrt{1}}$$

$$\Rightarrow \boxed{\gamma = \sin^{-1}\left(\frac{\alpha}{\sqrt{1+\alpha^2}\ \sqrt{1}}\right)}$$

3. Find the displacement field and components of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$. For this specific case, under what conditions is $\underline{\underline{\varepsilon}}$ a good description of the strain? Show that the Lagrangian strain tensor $\underline{\underline{E}}$ reduces to $\underline{\underline{\varepsilon}}$ under those conditions.

$$\underline{u} = \underline{\Phi} - \underline{X}$$

$$\boxed{\begin{aligned} u_1 &= \Phi_1 - X_1 = 0 \\ u_2 &= \Phi_2 - X_2 = 0 \\ u_3 &= \Phi_3 - X_3 = \alpha(X_1 + X_2) \end{aligned}}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \underline{e}_i \underline{e}_j$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} = 0 \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial X_2} = 0 \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial X_3} = 0$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = \frac{1}{2}(0+0) = 0$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) = \frac{1}{2}(0+\alpha) = \frac{\alpha}{2}$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = \frac{1}{2}(0+\alpha) = \frac{\alpha}{2}$$

$$\boxed{\underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & \alpha/2 \\ 0 & 0 & \alpha/2 \\ \alpha/2 & \alpha/2 & 0 \end{bmatrix}}$$

$$\underline{\underline{E}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \underline{e}_i \underline{e}_j$$

From Part 1 we found:

$$\underline{\underline{E}} = \begin{bmatrix} \frac{\alpha^2}{2} & \frac{\alpha^2}{2} & \frac{\alpha}{2} \\ \frac{\alpha^2}{2} & \frac{\alpha^2}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{\alpha}{2} & 0 \end{bmatrix}$$

For this specific case, $\underline{\underline{\varepsilon}}$ is a good description of the strain when $\alpha \ll 1$. This would imply that $\alpha^2 \to 0$ (α^2 could be assumed $= 0$ because $\alpha \ll 1$)

$$\underline{\underline{E}} = \begin{bmatrix} \cancel{\frac{\alpha^2}{2}} \to 0 & \cancel{\frac{\alpha^2}{2}} \to 0 & \frac{\alpha}{2} \\ \cancel{\frac{\alpha^2}{2}} \to 0 & \cancel{\frac{\alpha^2}{2}} \to 0 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{\alpha}{2} & 0 \end{bmatrix} \Rightarrow \underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & \alpha/2 \\ 0 & 0 & \alpha/2 \\ \alpha/2 & \alpha/2 & 0 \end{bmatrix}$$

4. For the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ find:

a) The maximum value of extensional strain and its direction.

$$\det \begin{bmatrix} 0-\lambda & 0 & \alpha/2 \\ 0 & 0-\lambda & \alpha/2 \\ \alpha/2 & \alpha/2 & 0-\lambda \end{bmatrix} = -\lambda\left[-\lambda(-\lambda) - \left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right)\right] - 0 + \frac{\alpha}{2}\left[0 - (-\lambda)\left(\frac{\alpha}{2}\right)\right] = 0$$

$$= -\lambda\left(\lambda^2 - \frac{\alpha^2}{4}\right) + \frac{\alpha}{2}\left(\lambda\frac{\alpha}{2}\right) = 0$$

$$= -\lambda^3 + \lambda\frac{\alpha^2}{4} + \lambda\frac{\alpha^2}{4} = 0$$

$$= -\lambda^3 + \lambda\frac{\alpha^2}{2} = 0$$

$$\lambda\left(\lambda^2 - \frac{\alpha^2}{2}\right) = 0$$

$$\Rightarrow \lambda = 0, -\frac{\alpha}{\sqrt{2}}, \frac{\alpha}{\sqrt{2}}$$

$$\Rightarrow \varepsilon_1 = \frac{\alpha}{\sqrt{2}}$$

$\underline{\lambda = \frac{\alpha}{\sqrt{2}}}$

$$\begin{bmatrix} -\alpha/\sqrt{2} & 0 & \alpha/2 \\ 0 & -\alpha/\sqrt{2} & \alpha/2 \\ \alpha/2 & \alpha/2 & -\alpha/\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -\frac{\alpha}{\sqrt{2}}v_1 + \frac{\alpha}{2}v_3 &= 0 \\ -\frac{\alpha}{\sqrt{2}}v_2 + \frac{\alpha}{2}v_3 &= 0 \\ \frac{\alpha}{2}v_1 + \frac{\alpha}{2}v_2 - \frac{\alpha}{\sqrt{2}}v_3 &= 0 \end{aligned}\right\}$$

$$v_1 = \left(-\frac{\alpha}{2}v_3\right)\left(-\frac{\sqrt{2}}{\alpha}\right) = \frac{\sqrt{2}}{2}v_3$$

$$v_2 = \left(-\frac{\alpha}{2}v_3\right)\left(-\frac{\sqrt{2}}{\alpha}\right) = \frac{\sqrt{2}}{2}v_3$$

$$\left.\begin{aligned} \frac{\alpha}{2}\left(\frac{\sqrt{2}}{2}v_3\right) + \frac{\alpha}{2}\left(\frac{\sqrt{2}}{2}v_3\right) - \frac{\alpha}{\sqrt{2}}v_3 &= 0 \\ \left(\frac{\alpha\sqrt{2}}{4} + \frac{\alpha\sqrt{2}}{4} - \frac{\alpha\sqrt{2}}{2}\right)v_3 &= 0 \end{aligned}\right\} \text{ let } v_3 = 1$$

$$\underline{n} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \right\rangle \Rightarrow |\underline{n}| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + 1^2} = \sqrt{\frac{2}{4} + \frac{2}{4} + 1} = \sqrt{2}$$

$$\Rightarrow \underline{n}_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right]^T$$

$$\boxed{\varepsilon_{max} = \frac{\alpha}{\sqrt{2}} \text{ in the direction } \underline{n} = \left[\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right]^T}$$

b) The maximum value of shear strain and its directions.

$$\varepsilon_{s_{max}} = \frac{1}{2}(\varepsilon_1 - \varepsilon_3) = \frac{1}{2}\left(\frac{\alpha}{\sqrt{2}} - \left(-\frac{\alpha}{\sqrt{2}}\right)\right) = \frac{\alpha}{\sqrt{2}}$$

← this was shown in Assignment 3 Problem 6 $\sigma_{s_{max}} = \frac{1}{2}(\sigma_1 - \sigma_3)$ and this corresponds to directions $\frac{1}{\sqrt{2}}(v_1 \pm v_3)$

need to find the eigenvector for $\lambda = -\frac{\alpha}{\sqrt{2}}$:

$$\begin{bmatrix} \frac{\alpha}{\sqrt{2}} & 0 & \frac{\alpha}{2} \\ 0 & \frac{\alpha}{\sqrt{2}} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{\alpha}{2} & \frac{\alpha}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} \frac{\alpha}{\sqrt{2}} w_1 + \frac{\alpha}{2} w_3 &= 0 \\ \frac{\alpha}{\sqrt{2}} w_2 + \frac{\alpha}{2} w_3 &= 0 \\ \frac{\alpha}{2} w_1 + \frac{\alpha}{2} w_2 + \frac{\alpha}{\sqrt{2}} w_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} w_1 &= -\frac{\sqrt{2}}{2} w_3 \\ w_2 &= -\frac{\sqrt{2}}{2} w_3 \end{aligned}$$

$$\left.\begin{aligned} \frac{\alpha}{2}\left(-\frac{\sqrt{2}}{2} w_3\right) + \frac{\alpha}{2}\left(-\frac{\sqrt{2}}{2} w_3\right) + \frac{\alpha}{\sqrt{2}} V_3 &= 0 \\ \left(-\frac{\alpha\sqrt{2}}{4} - \frac{\alpha\sqrt{2}}{4} + \frac{\alpha\sqrt{2}}{2}\right) V_3 &= 0 \end{aligned}\right\} \text{ let } V_3 = 1$$

$$\underline{n} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 \right\rangle \Rightarrow |\underline{n}| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + 1^2} = \sqrt{\frac{2}{4} + \frac{2}{4} + 1} = \sqrt{2}$$

$$\Rightarrow \underline{n}_3 = \left[-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right]^T$$

$$\frac{1}{\sqrt{2}}(\underline{n}_1 + \underline{n}_3) = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}[0, 0, \sqrt{2}] = [0, 0, 1]$$

$$\frac{1}{\sqrt{2}}(\underline{n}_1 - \underline{n}_3) = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}[1, 1, 0] = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]$$

$$\Rightarrow \boxed{\varepsilon_{s_{max}} = \frac{\alpha}{\sqrt{2}} \text{ in the directions } [0, 0, 1]^T \text{ and } \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T}$$