Constitutive Behavior of Materials

AE3140: Structural Analysis

C.V. Di Leo
(Adapted from slides by M. Ruzzene from "Structural Analysis" by O.A. Bauchau and J.I. Craig)

School of Aerospace Engineering
Georgia Institute of Technology
Atlanta GA

Contents

Overview

The solution of elasticity problems requires three types of relationships:

- The equilibrium equations

Overview

The solution of elasticity problems requires three types of relationships:

- The equilibrium equations
- The strain-displacement relationships, and

Overview

The solution of elasticity problems requires three types of relationships:

- The equilibrium equations
- The strain-displacement relationships, and
- The stress and strain relations defined by a set of *constitutive laws*. These constitutive laws characterize the mechanical behavior of the material and consist of a set of mathematical idealization of their observed behavior.

Overview

The solution of elasticity problems requires three types of relationships:

- The equilibrium equations
- The strain-displacement relationships, and
- The stress and strain relations defined by a set of *constitutive laws*. These constitutive laws characterize the mechanical behavior of the material and consist of a set of mathematical idealization of their observed behavior.

Overview

The solution of elasticity problems requires three types of relationships:

- The equilibrium equations
- The strain-displacement relationships, and
- The stress and strain relations defined by a set of *constitutive laws*. These constitutive laws characterize the mechanical behavior of the material and consist of a set of mathematical idealization of their observed behavior.

Most practical constitutive models are based on empirical data, and various types of constitutive laws exist to represent the many types of experimentally observed material behaviors. If the deformation of the body remains very small, however, the stress-strain relationship can often be assumed to be linear. This widely use approximation in which stress is proportional to strain is used extensively to design structures.

Constitutive laws:

Homogeneous, isotropic, linear elastic materials

For specimens undergoing very small deformations, the stress-strain diagram often exhibit a linear behavior (*Hooke's Law*):

$$\sigma_1 = E\ \epsilon_1, \tag{1}$$

where the coefficient of proportionality is called *Young's modulus* or *modulus of elasticity*. ([Pa]).

Constitutive laws:

Homogeneous, isotropic, linear elastic materials

For specimens undergoing very small deformations, the stress-strain diagram often exhibit a linear behavior (*Hooke's Law*):

$$\sigma_1 = E\,\epsilon_1, \tag{1}$$

where the coefficient of proportionality is called *Young's modulus* or *modulus of elasticity*. ([Pa]).
The elongation of a bar in the direction of the applied stress is accompanied by a lateral contraction that is also proportional to the applied stress. The corresponding strains are

$$\epsilon_1 = \frac{1}{E}\,\sigma_1; \quad \epsilon_2 = -\frac{\nu}{E}\,\sigma_1; \quad \epsilon_3 = -\frac{\nu}{E}\,\sigma_1, \tag{2}$$

where ν is called *Poisson's ratio* and is a dimensionless constant.

If a stress component, σ_2, is applied to the same material, similar deformations will result

$$\epsilon_1 = -\frac{\nu}{E}\,\sigma_2; \quad \epsilon_2 = \frac{1}{E}\,\sigma_2; \quad \epsilon_3 = -\frac{\nu}{E}\,\sigma_2. \tag{3}$$

Note

The assumption of material isotropy implies identical values of Young's modulus and Poisson's ratio in eq. (2) and (3).

Similar relationships hold for an applied stress, σ_3.

When the three stress components are applied simultaneously, the resulting deformation is obtained through superposition. This results in the *generalized Hooke's law* for extensional strains

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]; \tag{4a}$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right]; \tag{4b}$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]. \tag{4c}$$

When the three stress components are applied simultaneously, the resulting deformation is obtained through superposition. This results in the *generalized Hooke's law* for extensional strains

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]; \tag{4a}$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right]; \tag{4b}$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]. \tag{4c}$$

Note

The extensional strains depend only on the direct stresses and not on the shear stresses. This is a key characteristic of isotropic materials and does not hold for all types of non-isotropic materials.

Constitutive laws:

Homogeneous, isotropic, linear elastic materials

The relationship between the shear strains and the shear stresses is based on the study of a stress of pure shear stress (plane), i.e, assuming that the principal stresses are $\sigma_{p2} = -\sigma_{p1}$, $\sigma_{p3} = 0$. The corresponding extensional strain components then are

$$\epsilon_1 = \frac{1+\nu}{E}\sigma_{p1}; \quad \epsilon_2 = -\frac{1+\nu}{E}\sigma_{p1}; \quad \gamma_{12} = 0. \tag{5}$$

Constitutive laws:

Homogeneous, isotropic, linear elastic materials

The relationship between the shear strains and the shear stresses is based on the study of a stress of pure shear stress (plane), i.e, assuming that the principal stresses are $\sigma_{p2} = -\sigma_{p1}$, $\sigma_{p3} = 0$. The corresponding extensional strain components then are

$$\epsilon_1 = \frac{1+\nu}{E}\sigma_{p1}; \quad \epsilon_2 = -\frac{1+\nu}{E}\sigma_{p1}; \quad \gamma_{12} = 0. \tag{5}$$

The state of stress on faces oriented at a 45° angle with respect to the principal stress directions is instead:

$$\tau^*_{s12} = \sigma_{p2} = -\sigma_{p1}; \quad \sigma^*_{s1} = \sigma^*_{s2} = 0 \tag{6}$$

where the asterisk and subscript "s" are used to designate this special rotated axis system with maximum shear stresses.

The strains in this rotated axis system are readily obtained from the strain rotation equations, with $\theta_s = 45^\circ$ and are given by

$$\gamma^*_{s12} = -(\epsilon_1 - \epsilon_2) = -\frac{2(1+\nu)}{E}\,\sigma_{p1}; \quad \epsilon^*_{s1} = \epsilon^*_{s2} = 0. \tag{7}$$

The relationship between τ^*_{s12} and γ^*_{s12} is then obtained by comparing eq. (6) and eq. (7) to find:

$$\gamma^*_{s12} = -2(1+\nu)\sigma_{p1}/E = 2(1+\nu)\tau^*_{s12}/E$$

or

$$\tau^*_{s12} = E\gamma^*_{s12}/(2(1+\nu)) = G\,\gamma^*_{s12}$$

where

$$G = \frac{E}{2(1+\nu)} \tag{8}$$

is defined as the *shear modulus*.

The above reasoning can be repeated for a state of pure shear in the other two orthogonal planes leading to:

$$\gamma_{23} = \tau_{23}/G, \quad \gamma_{13} = \tau_{13}/G, \quad \gamma_{12} = \tau_{12}/G. \tag{9}$$

- Again, the shear modulus is assumed the same in all directions due to isotropy of the material.

The above reasoning can be repeated for a state of pure shear in the other two orthogonal planes leading to:

$$\gamma_{23} = \tau_{23}/G, \quad \gamma_{13} = \tau_{13}/G, \quad \gamma_{12} = \tau_{12}/G. \tag{9}$$

- Again, the shear modulus is assumed the same in all directions due to isotropy of the material.
- For more complex stress states involving axial stresses, because, in view of eq. (4), axial stresses create no shear strains. Similarly, the generalized Hooke's law, eq. (4), was established when only axial stresses are applied.

The above reasoning can be repeated for a state of pure shear in the other two orthogonal planes leading to:

$$\gamma_{23} = \tau_{23}/G, \quad \gamma_{13} = \tau_{13}/G, \quad \gamma_{12} = \tau_{12}/G. \tag{9}$$

- Again, the shear modulus is assumed the same in all directions due to isotropy of the material.
- For more complex stress states involving axial stresses, because, in view of eq. (4), axial stresses create no shear strains. Similarly, the generalized Hooke's law, eq. (4), was established when only axial stresses are applied.
- Similarly, in view of eq. (9), shear stresses create no axial strains.

Constitutive laws:

Generalized Hooke's Law

The constitutive laws can be expressed in a compact matrix form as

$$\underline{\epsilon} = \underline{\underline{C}}\,\underline{\sigma}, \tag{10}$$

where the strain array, $\underline{\epsilon}$ and the stress array, $\underline{\sigma}$, contain the six strain and six stress components, respectively, as

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \gamma_{12} & \gamma_{23} & \gamma_{31} \end{bmatrix}^T \tag{11}$$

and

$$\underline{\sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \tau_{12} & \tau_{23} & \tau_{31} \end{bmatrix}^T, \tag{12}$$

The material compliance matrix, $\underline{\underline{C}}$, is defined as

$$\underline{\underline{C}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}. \quad (13)$$

Constitutive laws:

Generalized Hooke's Law

The matrix equation, eq. (10) can be inverted to give:

$$\underline{\sigma} = \underline{\underline{S}}\, \underline{\epsilon}, \tag{14}$$

where the material stiffness matrix, $\underline{\underline{S}}$, is defined as

$$\underline{\underline{S}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}. \tag{15}$$

Constitutive laws:

Volumetric Strain

The volumetric strain is evaluated by combining the normal strains along the 3 reference directions:

$$e = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3), \tag{16}$$

In the special case of an applied hydrostatic pressure, $\sigma_1 = \sigma_2 = \sigma_3 = p$, a linear relationship is found between the applied pressure and the resulting volumetric strain

$$p = \kappa\ e, \tag{17}$$

where

$$\kappa = \frac{E}{3(1-2\nu)}, \tag{18}$$

is the *bulk modulus* of the material.

- When Poisson's ratio $\nu \approx 1/2$, the bulk modulus approaches infinity, i.e.:

$$\kappa \rightarrow \infty$$

- When Poisson's ratio $\nu \approx 1/2$, the bulk modulus approaches infinity, i.e.:

$$\kappa \to \infty$$

- Such a material is called an *incompressible material*. Many types of rubber are nearly incompressible materials, and metals undergoing plastic deformations are often assumed to be nearly incompressible.

Allowable stress

A central problem of structural analysis is to determine the configuration of a structure subjected to specific loads. The design is influenced by factors such as:

1. *The strength of the structure*: the local stress in the structure exceeds a specific value.
2. *The elastic deformations of the structure* under load: a structure can present undesirable levels of elastic deformations.
3. *The dynamics characteristics of the structure*: resonances, flutter.
4. *The stability characteristics of the structure*: parts of the structure become unstable, resulting in buckling.

This leads to the definition of an allowable stress, leading to the following strength criterion

$$\sigma \leq \sigma_{\text{allow}}. \tag{19}$$

Allowable stress:

Yielding under combined loading

A proper yield criterion must be used when *multiple stress components are acting simultaneously*.

The yield criteria to be presented are applicable to isotropic, homogeneous material subjected to a general three-dimensional state of stress. Since the material is isotropic, the direction of application of the stress is irrelevant.

Notes

- Empirical evidence indicates that hydrostatic stress does not cause yielding.

Allowable stress:

Yielding under combined loading

A proper yield criterion must be used when *multiple stress components are acting simultaneously*.
The yield criteria to be presented are applicable to isotropic, homogeneous material subjected to a general three-dimensional state of stress. Since the material is isotropic, the direction of application of the stress is irrelevant.

Notes

- Empirical evidence indicates that hydrostatic stress does not cause yielding.
- This implies that changes in the state of stress in which the principal stresses are increased equally will not result in yielding.

Allowable stress:

Yielding under combined loading

A proper yield criterion must be used when *multiple stress components are acting simultaneously*.
The yield criteria to be presented are applicable to isotropic, homogeneous material subjected to a general three-dimensional state of stress. Since the material is isotropic, the direction of application of the stress is irrelevant.

Notes

- Empirical evidence indicates that hydrostatic stress does not cause yielding.
- This implies that changes in the state of stress in which the principal stresses are increased equally will not result in yielding.
- Yielding is directly related to the maximum shear stress in the material which, in turn, is directly proportional to the differences between the principal stresses.

Allowable stress:

Tresca's criterion

Tresca's yield criterion is expressed in terms of the following three inequalities

$$|\sigma_{p1} - \sigma_{p2}| \leq \sigma_y, \quad |\sigma_{p2} - \sigma_{p3}| \leq \sigma_y, \quad |\sigma_{p3} - \sigma_{p1}| \leq \sigma_y, \tag{20}$$

where σ_y is the yield stress observed in a uniaxial test.
Note that a hydrostatic state of stress (*i.e.*, equal principal stresses) will not produce yielding.

Allowable stress:

Tresca's criterion

Consider a material under a plane state of stress. Let $\bar{\imath}_1^*$, $\bar{\imath}_2^*$ be the principal stress directions, for which $\sigma_1 = \sigma_{p1}$, $\sigma_2 = \sigma_{p2}$ and $\tau_{12} = 0$. The maximum shear stress is found on a face inclined at an angle $\theta = 45^o$:

$$\tau_{12\max} = |\sigma_{p1} - \sigma_{p2}|/2$$

Tresca's criterion can be expressed as:

$$\tau_{\max} \leq \sigma_y/2$$

the material reaches the yield condition when the maximum shear stress equals half the yield stress under a uniaxial stress state. This physical interpretation of Tresca's criterion helps explain why it is sometimes called the *"maximum shear stress criterion."*

Allowable stress:

Tresca's criterion

If σ_1, σ_2 and τ_{12} are the stress components in an arbitrary coordinate system, the principal stresses are readily found as

$$\sigma_{p1}, \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{12}^2}, \quad \sigma_{p3} = 0. \tag{21}$$

Tresca's criterion now implies the following conditions

$$2\sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{12}^2} \leq \sigma_y, \quad \left|\frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{12}^2}\right| \leq \sigma_y. \tag{22}$$

Allowable stress:

Von Mises' criterion

Von Mises' yield criterion is expressed by the following inequality

$$\sigma_{\mathrm{eq}} = \frac{1}{\sqrt{2}} \sqrt{[(\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p3} - \sigma_{p1})^2]} \leq \sigma_y, \qquad (23)$$

where the first equality defines the *equivalent stress*, σ_{eq}. Von Mises' criterion now states that *the yield condition is reached under the combined loading, when the equivalent stress*, σ_{eq}, *reaches the yield stress for a uniaxial stress state*, σ_y. Again, hydrostatic states of stress (*i.e.*, states with equal values of the principal stresses) will not produce yielding.

Consider a material under a plane state of stress. If σ_1, σ_2 and τ_{12} are the stress components in an arbitrary coordinate system, the equivalent stress reduces to

$$\sigma_{\mathrm{eq}} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 + 3\tau_{12}^2} \leq \sigma_y. \tag{24}$$

The state of pure shear is a special case of plane stress state where $\sigma_1 = \sigma_2 = 0$ and $\tau_{12} = \tau$.
Von Mises' criterion reduces to:

$$\tau \leq \sigma_y / \sqrt{3}$$

.

Notes

- According to von Mises' criterion, the shear stress level at which the material yields in a pure shear state is $1/\sqrt{3} \approx 0.577$ the level observed under uniaxial stress state.

The state of pure shear is a special case of plane stress state where $\sigma_1 = \sigma_2 = 0$ and $\tau_{12} = \tau$.
Von Mises' criterion reduces to:

$$\tau \leq \sigma_y/\sqrt{3}$$

.

Notes

- According to von Mises' criterion, the shear stress level at which the material yields in a pure shear state is $1/\sqrt{3} \approx 0.577$ the level observed under uniaxial stress state.
- Experimentation shows that this prediction is more accurate than the yield point predicted by Tresca's criterion. This is the reason why von Mises' criterion is more widely used than Tresca's.

Allowable stress:

Comparing Tresca's and von Mises' criteria

Consider a plane stress problem, Tresca's criterion reduces to three inequalities

$$|\frac{\sigma_{p1}}{\sigma_y}| \leq 1, \ |\frac{\sigma_{p2}}{\sigma_y}| \leq 1,$$
$$|\frac{\sigma_{p2}}{\sigma_y} - \frac{\sigma_{p1}}{\sigma_y}| \leq 1.$$

In a σ_{p1}, σ_{p2} plane, safe stress levels correspond to stress states falling within the irregular hexagon enclosed by the six dashed line segments.

For the same stress states, von Mises' criterion becomes the skewed ellipse defined by

$$\left(\frac{\sigma_{p1}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{p2}}{\sigma_y}\right)^2 - \left(\frac{\sigma_{p1}}{\sigma_y}\right)\left(\frac{\sigma_{p2}}{\sigma_y}\right) = 1.$$

Safe stress levels correspond to stress states falling within the ellipse which forms the yield envelope.

Notes

- At the six vertices of the hexagon, the yield conditions predicted by the two criteria are identical.

Notes

- At the six vertices of the hexagon, the yield conditions predicted by the two criteria are identical.
- For all other stress conditions, Tresca's criterion is slightly more conservative.

Notes

- At the six vertices of the hexagon, the yield conditions predicted by the two criteria are identical.
- For all other stress conditions, Tresca's criterion is slightly more conservative.
- In most experimental studies, yielding is observed to occur at points falling between these two criteria.

Notes

- At the six vertices of the hexagon, the yield conditions predicted by the two criteria are identical.
- For all other stress conditions, Tresca's criterion is slightly more conservative.
- In most experimental studies, yielding is observed to occur at points falling between these two criteria.
- As a purely practical matter, von Mises' criterion is often preferred because of its representation as a single analytical expression