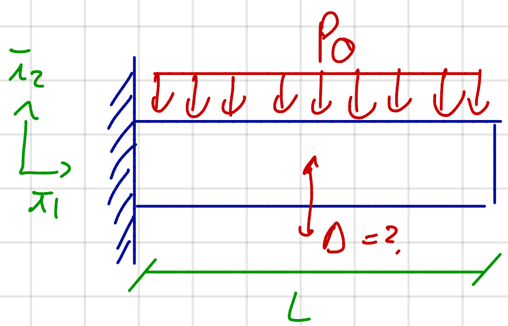
# Summary: Virtual Work & Energy Methods

## Unit load method (ULM)

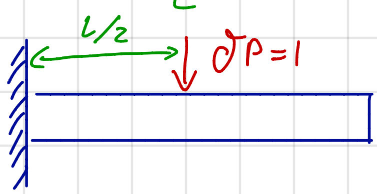
* Derived from the principle of complementary virtual work

* Yields the displacement (or rotation) in the direction of an applied virtual unit load (moment).




$H_{23}^c = 0$

Find $u_2$ @ $x_1 = L/2$




→ Unit Load System

1) Real internal moment $M_3(x_1)$

$$M_3(x_1) = -\frac{P_0}{2}(L - x_1)^2$$

2) Statically admissible $\hat{M}_3(x_1)$

$$\hat{M}_3(x_1) \begin{cases} -\delta P\,(L/2 - x_1) & x_1 < L/2 \\ 0 & x_1 > L/2 \end{cases}$$

$$\Delta = \int_0^L \frac{M_3 \hat{M}_3}{H_{33}^c}\, dx_1 = \int_0^{L/2} \frac{P_0}{2H_{33}^c}(L - x_1)^2\left(\frac{L}{2} - x_1\right) dx_1$$

$$\Delta = \int_0^L \left( \hat{N}_1 \bar{\epsilon}_1 + \hat{M}_2 \kappa_2 + \hat{M}_3 \kappa_3 \right) dx_1$$

$$\kappa_2 = \left( H_{33}^c M_2 + H_{23}^c M_3 \right) / \Delta H$$

$$\kappa_3 = \left( H_{23}^c M_2 + H_{22}^c M_3 \right) / \Delta H$$

$$\Delta H = H_{22}^c H_{33}^c - H_{23}^c H_{23}^c$$

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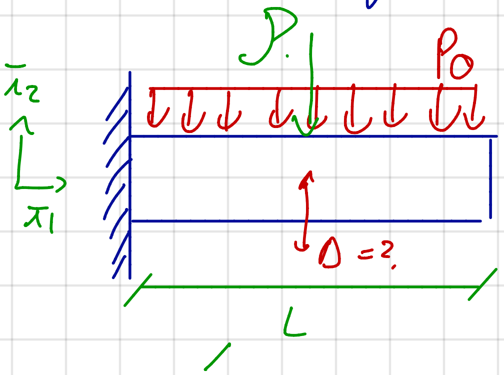
## Dummy Load Method

* Derived from the principle of minimum complementary energy and Castigliano's 2nd Theorem

* Yields the displacement (or rotation) under an applied concentrated dummy load (or moment)




$H_{23}^c = 0$

Find $u_2$ @ $x_1 = L/2$

$$\Delta = \lim_{\mathcal{P} \to 0} \frac{\partial A'}{\partial \mathcal{P}} = \lim_{\mathcal{P} \to 0} \frac{\partial}{\partial \mathcal{P}} \int_0^L \frac{1}{2} \frac{M_3^2}{H_{33}^c} dx_1$$

$$\Delta = \lim_{\mathcal{P} \to 0} \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial \mathcal{P}} dx_1$$

$$\mathcal{M}_3 = \begin{cases} -\dfrac{P_0}{2}(L-x_1)^2 - \mathcal{P}(L/2 - x_1) & x_1 < L/2 \\ -\dfrac{P_0}{2}(L-x_1)^2 & x_1 > L/2 \end{cases}$$

$$\frac{\partial \mathcal{M}_3}{\partial \mathcal{P}} = \begin{cases} -(L/2 - x_1) \\ 0 \end{cases}$$

$$\Delta = \lim_{\mathcal{P} \to 0} \int_0^{L/2} \frac{1}{H_{33}^c}\left(-\frac{P_0}{2}(L-x_1)^2 - \mathcal{P}(L/2 - x_1)\right)\left(-(L/2 - x_1)\right) dx_1$$

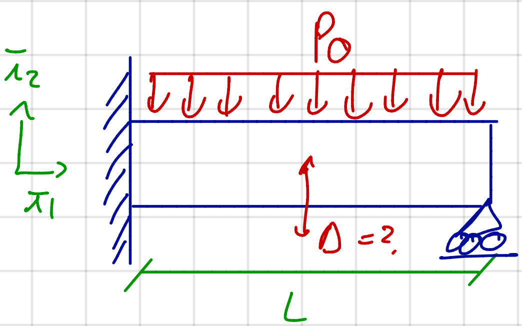
$$\Delta = \int_0^{L/2} \frac{P_0}{2H_{33}^c}(L-x_1)^2\left(\frac{L}{2} - x_1\right) dx_1$$

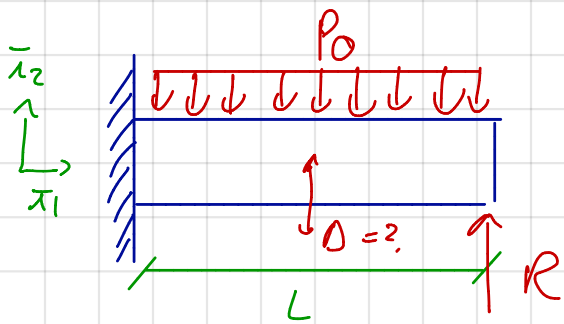
$$A' = \frac{1}{2}\int_0^L \left(\frac{\mathcal{N}_1^2}{S} + \frac{H_{33}^c}{\Delta H}\mathcal{M}_2^2 + \frac{H_{22}^c}{\Delta H}\mathcal{M}_3^2 + 2\frac{H_{23}^c}{\Delta H}\mathcal{M}_2\mathcal{M}_3\right) dx_1$$

# Castigliano's 2nd Theorem

* Yields the displacement under a point load. (Or rotation under a point moment).

* Useful for finding reaction forces that lead to kinematic constraints.




What is $R$ such that $u_2(x_1 = L) = 0$

$$\Delta_R = \frac{\partial A'}{\partial R} = 0 \quad \rightarrow \quad \text{Solve for } R$$

$$\Delta_R = \frac{\partial}{\partial R} \int_0^L \frac{1}{2} \frac{M_3^2}{H_{33}^c} dx_1 = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial R} dx_1$$

$$M_3(x_1) = -\frac{p_0}{2}(L - x_1)^2 + R(L - x_1)$$

$$\frac{\partial M_3}{\partial R} = (L - x_1)$$

$$\Delta_R = 0 = \int_0^L \frac{1}{H_{33}^c} \left( -\frac{p_0}{2}(L - x_1)^2 + R(L - x_1) \right) (L - x_1)\, dx_1$$

$$\frac{1}{H_{33}^c} \left[ -\frac{p_0}{2} \frac{L^4}{4} + \frac{R L^3}{3} \right] = 0 \qquad \underline{R = \frac{3}{8} p_0 L}$$

# Rayleigh Ritz

* Derived from the principle of minimum potential energy
* Yields an approximate solution by turning a continuous problem into one with a finite # of DOF.

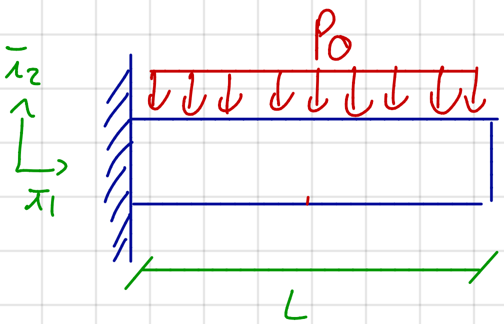
$$\hat{u} = \sum_{i=1}^{N} q_i \, h_i(x_1, x_2, x_3)$$

DOF

Shape Functions must satisfy kinematic B.C.s

Compute $q_i$ through $\frac{\partial \Pi}{\partial q_i} = 0$

$$\Pi = A + \Phi$$




$$\hat{u} = q \, x_1^2 = -\frac{P_0 L^2}{12 H_{33}^c} x_1^2$$

$$\Pi = \int_0^L \frac{1}{2} H_{33}^c \left(\frac{d^2 \hat{u}}{dx_1^2}\right)^2 dx_1 - \int_0^L (-P_0)\, \hat{u} \, dx_1$$

$$\Pi = \int_0^L \left(\frac{1}{2} H_{33}^c \, 4 q^2 + P_0 \, q \, x_1^2\right) dx_1$$

$$\Pi = 2 H_{33}^c q^2 L + P_0 \, q \frac{L^3}{3}$$

$$\frac{\partial \Pi}{\partial q} = 0 = 4 H_{33}^c q L + P_0 \frac{L^3}{3}$$

$$q = -\frac{P_0 L^2}{12 H_{33}^c}$$

$$A = \frac{1}{2}\int_0^L \left( S\,\bar{\epsilon}_1^{\,2} + H_{22}^c\,\kappa_2^{\,2} + H_{33}^c\,\kappa_3^{\,2} - 2H_{23}^c\,\kappa_2\kappa_3 \right) dx_1$$