Georgia Institute of Technology
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AE 6115 — Fundamentals of Aerospace Structural Analysis — Spring 2020

Quiz No.1

Problem 1. Double Cantilevered Beam with Concentrated Moment (100 Points)

Consider the *multi-material* beam cantilevered on both ends as shown in Fig. 1. The beam is subjected to a *concentrated moment of magnitude* Q at its mid-span. The cross-section of the beam (right of Fig. 1) is composed of a multi-layered material with the left layer having a Young's modulus E_A, and the right layer having a Young's modulus E_B. Assume that $E_A > E_B$.

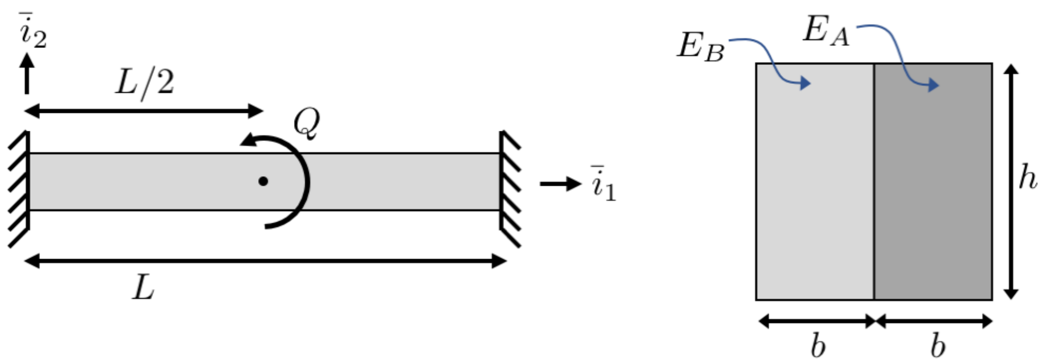


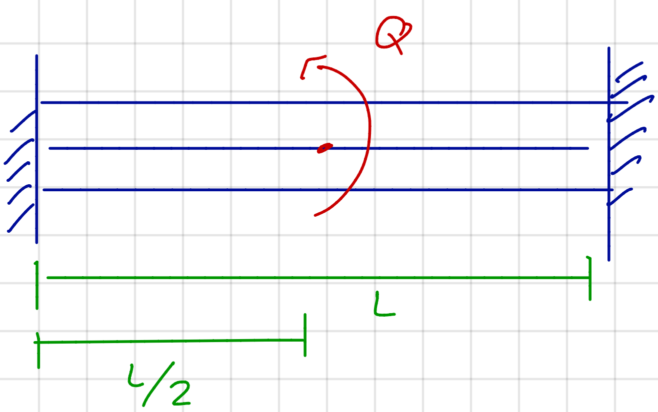
Figure 1: Multi-material beam cantilevered on both ends with a concentrated moment at its mid-span.

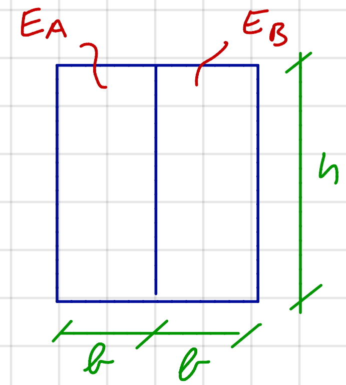
a) Solve for the displacement components $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$. You may leave your answer in terms of any non-zero stiffnesses.

b) Solve for the axial stress $\sigma_1(x_1, x_2, x_3)$.

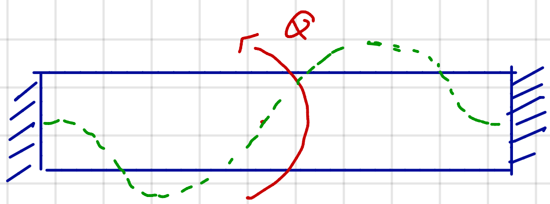
c) Sketch how you expect the beam to deform (i.e. sketch the u_2 transverse displacement). Sketch the internal moment distribution (i.e. sketch $M_3(x_1)$)

d) **Extra credit (attempt only after you've exhausted the other parts)**. Solve for the stiffnesses H^c_{22}, H^c_{33}, H^c_{23} and S as a function of the geometric properties $\{L, h, b\}$ and the material properties $\{E_A, E_B\}$





a) CONSIDER HOW THE BEAM WILL DEFORM

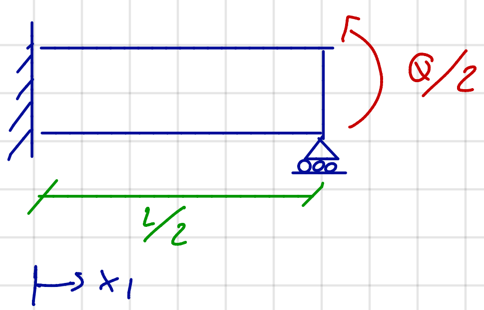


* By inspection $H_{23}^{C} = 0$

* WE CAN SEE THE SOLUTION IS ANTISYMMETRIC AND WE MAY SOLVE ONLY FOR HALF OF THE BEA

* FURTHER, WE CAN NOTE THAT DUE TO SYMMETRY THERE IS NO U_2 DISPLACMENT AT THE MID-SPAN

WE MAY THUS TREAT THE PROBLEM AS



B.C.'s

@ $x_1 = 0$

$u_2 = 0 \;,\; du_2/dx_1 = 0$

@ $x_1 = L$

$u_2 = 0, \quad H_{33}^{C} \dfrac{d^2 u_2}{dx_1^2} = \dfrac{Q}{2}$

Solving: $H_{33}^{C} \dfrac{d^4 u_2}{dx_1^4} = 0$

$$H_{33}^{C} \frac{d^3 u_2}{dx_1^3} = A$$

$$H_{33}^{C} \frac{d^2 u_2}{dx_1^2} = Ax_1 + B$$

Applying B.C. $H_{33}^c \frac{d^2 u_2}{dx_1^2} = \frac{Q}{2}$

$$\frac{Q}{2} = A\frac{L}{2} + B \qquad B = \frac{Q}{2} - A\frac{L}{2}$$

$$H_{33}^c \frac{d^2 u_2}{dx_1^2} = A\left(x_1 - \frac{L}{2}\right) + \frac{Q}{2}$$

$$H_{33}^c \frac{d u_2}{dx_1} = A\left(\frac{x_1^2}{2} - \frac{L}{2}x_1\right) + \frac{Q}{2}x_1 + \cancel{C}^{\,0}$$

Since $\frac{du_2}{dx_1} = 0$ @ $x_1 = 0$

$$H_{33}^c u_2 = A\left(\frac{x_1^3}{6} - \frac{L}{4}x_1^2\right) + \frac{Q}{4}x_1^2 + \cancel{D}^{\,0}$$

Since $u_2 = 0$ @ $x_1 = 0$

Now we use $u_2(x_1 = L/2) = 0$

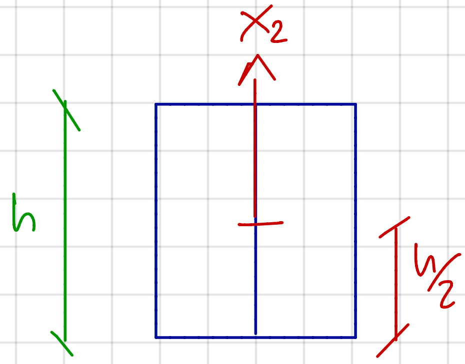
$$0 = A\left(\frac{1}{6}\left(\frac{L}{2}\right)^3 - \frac{L}{4}\left(\frac{L}{2}\right)^2\right) + \frac{Q}{4}\left(\frac{L}{2}\right)^2$$

$$0 = -\frac{AL^3}{24} + \frac{QL^2}{16} \rightarrow A = \frac{Q}{L}\frac{24}{16} = \frac{Q}{L}\frac{3}{2}$$

$$H_{33}^c u_2 = \frac{3}{2}\frac{Q}{L}\left(\frac{x_1^3}{6} - \frac{L}{4}x_1^2\right) + \frac{Q}{4}x_1^2$$

$$u_2 = \frac{3}{2}\frac{Q}{H_{33}^c L}\left(\frac{x_1^3}{6} - \frac{Lx_1^2}{12}\right)$$

$$\boxed{u_2 = \frac{Q}{H_{33}^c L}\left(\frac{x_1^3}{4} - \frac{Lx_1^2}{8}\right)}$$



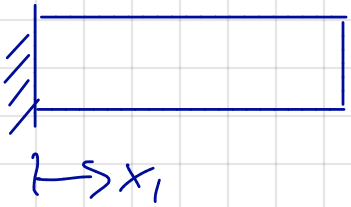
By inspection $\boxed{\bar{u}_3 = 0}$

$$\boxed{u_1 = -x_2\frac{du_2}{dx_1} = -x_2\frac{Q}{H_{33}^c L}\left(\frac{3}{4}x_1^2 - \frac{Lx_1}{4}\right)}$$

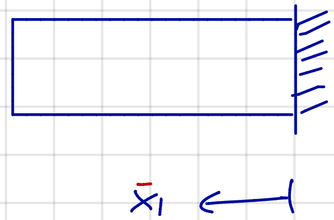
Appprach #2

* ONE MAY EQUALLY SOLVE THE PROBLEM W/OUT MAKING USE OF SYMMETRY.

LEFT HALF



RIGHT HALF



B. C.'s

@ $x_1 = 0$ $\quad u_2 = 0$ $\quad$ (1)

$\quad du_2/dx_1 = 0$ $\quad$ (2)

@ $\bar{x}_1 = 0$ $\quad u_2 = 0$ $\quad$ (3)

$\quad du_2/d\bar{x}_1 = 0$ $\quad$ (4)

@ $x_1 = L/2$ & $\bar{x}_1 = L/2$

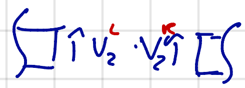
(5) $\quad u_2^L(x_1 = L/2) = u_2^R(\bar{x}_1 = L/2)$

(6) $\quad du_2^L/dx_1\,(x_1 = L/2) = -du_2^R/d\bar{x}_1\,(\bar{x}_1 = L/2)$

$$V_2^L + V_2^R = 0$$

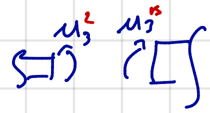
→ SIGNS ARE IMPORTANT HERE

(7) $\quad H_{33}^C\, d^3u_2^L/dx_1^3(x_1 = L/2) = H_{33}^C\, d^3u_2^R/d\bar{x}_1^3\,(\bar{x}_1 = L/2)$



$$\mathcal{M}_3^L - \mathcal{M}_3^R = +Q$$

→ THIS IS WHERE Q COMES INTO PLAY



(8) $\quad H_{33}^C\, d^2u_2^L/dx_1^2\,(x_1 = L/2) = H_{33}^C\, d^2u_2^R/d\bar{x}_1^2\,(\bar{x}_1 = L/2) + Q$

$$H_{33}^C \frac{d^4u_2}{dx_1^4} = 0 \qquad\qquad H_{33}^C \frac{d^4u_2}{d\bar{x}_1^4} = 0$$

$$H_{33}^C \frac{d^3u_2}{dx_1^3} = A_1 \qquad\qquad H_{33}^C \frac{d^3u_2}{d\bar{x}_1^3} = A_2$$

USING (7) YIELDS $A_1 = -A_2$

$$H_{33}^C \frac{d^3u_2}{dx_1^3} = A_1 \qquad\qquad H_{33}^C \frac{d^3u_2}{d\bar{x}_1^3} = -A_1$$

$$H_{33}^C \frac{d^3 u_2}{dx_1^3} = A_1$$

$$H_{33}^C \frac{d^2 u_2}{dx_1^2} = A_1 x_1 + B_1$$

$$H_{33}^C \frac{d^3 u_2}{d\bar{x}_1^3} = -A_1$$

$$H_{33}^C \frac{d^2 u_2}{d\bar{x}_1^2} = -A_1 \bar{x}_1 + B_2$$

Using (8)

$$A_1 \frac{L}{2} + B_1 = -A_1 \frac{L}{2} + B_2 + Q$$

$$B_1 = -A_1 L + B_2 + Q$$

$$H_{33}^C \frac{d^2 u_2}{dx_1^2} = A_1(x_1 - L) + B_2 + Q$$

$$H_{33}^C \frac{du_2}{dx_1} = A_1\left(\frac{x_1^2}{2} - Lx_1\right) + B_2 x_1 + Q x_1 + \cancel{C_1} \to 0$$

using (2)

$$H_{33}^C \frac{d^2 u_2}{d\bar{x}_1^2} = -A_1 \bar{x}_1 + B_2$$

$$H_{33}^C \frac{du_2}{d\bar{x}_1} = -A_1 \frac{\bar{x}_1^2}{2} + B_2 \bar{x}_1 + \cancel{C_2} \to 0$$

Using (4)

Using (6):

$$A_1\left(\frac{1}{2}\left(\frac{L}{2}\right)^2 - \frac{L^2}{2}\right) + B_2 \frac{L}{2} + Q\frac{L}{2} = A_1 \frac{1}{2}\left(\frac{L}{2}\right)^2 - B_2 \frac{L}{2}$$

$$B_2 = \frac{A_1 L}{2} - \frac{Q}{2}$$

$$H_{33}^C \frac{du_2}{dx_1} = A_1\left(\frac{x_1^2}{2} - \frac{Lx_1}{2}\right) + \frac{Qx_1}{2}$$

$$H_{33}^C u_2 = A_1\left(\frac{x_1^3}{6} - \frac{Lx_1^2}{4}\right) + \frac{Qx_1^2}{4} + \cancel{D_1} \to 0$$

Using (1)

$$H_{33}^C \frac{du_2}{d\bar{x}_1} = A_1\left(-\frac{\bar{x}_1^2}{2} + \frac{L\bar{x}_1}{2}\right) - \frac{Q\bar{x}_1}{2}$$

$$H_{33}^C u_2 = A_1\left(-\frac{\bar{x}_1^3}{6} + \frac{L\bar{x}_1^2}{4}\right) - \frac{Q\bar{x}_1^2}{4} + \cancel{D_2} \to 0$$

Using (3)

Using (5):

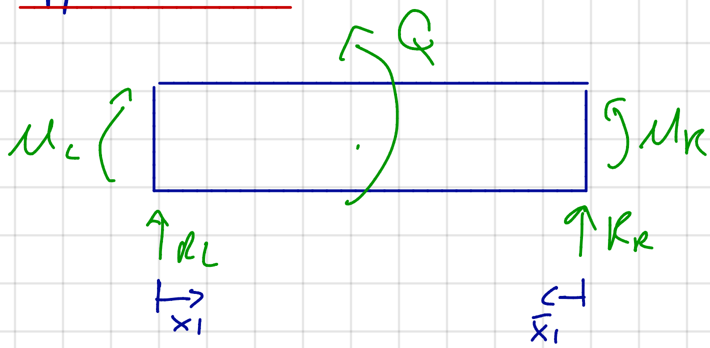
$$-\frac{A_1 L^3}{24} + \frac{QL^2}{16} = \frac{A_1 L^3}{24} - \frac{QL^2}{16} \longrightarrow A_1 = \frac{24}{16}\frac{Q}{L} = \frac{3}{2}\frac{Q}{L}$$

$$H_{33}^C = \frac{3Q}{2L}\left(\frac{x_1^3}{6} - \frac{Lx_1^2}{4}\right) + \frac{Qx_1^2}{4}$$

$$* \quad H_{33}^C = \frac{3}{2}\frac{Q}{L}\left(\frac{x_1^3}{6} - \frac{Lx_1^2}{12}\right)$$

$$* \quad H_{33}^C u_2 = -\frac{3}{2}\frac{Q}{L}\left(\frac{\bar{x}_1^3}{6} - \frac{L\bar{x}_1^2}{12}\right)$$

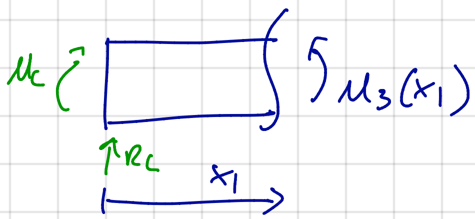
Approach #3



$\Sigma M(x_1=0): \quad M_L = Q + M_R + R_R L$

$\Sigma F_{x_2}: \quad R_L = -R_R$

Left:

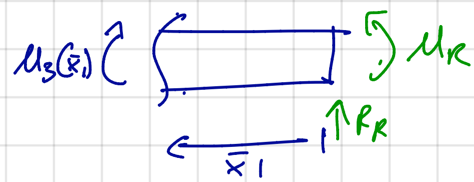


$$M_3(x_1) = M_L + R_L x_1$$

$$M_3(x_1) = Q + M_R + R_R L - R_R x_1$$

$$M_3(x_1) = Q + M_R + R_R(L - x_1)$$

Right:



$$M_3(\bar{x}_1) = M_R + R_R \bar{x}_1$$

$$H_{33}^C \frac{d^2 u_2}{dx_1^2} = \begin{cases} Q + M_R + R_R(L - x_1) & 0 < x_1 < L/2 \\ M_R + R_R \bar{x}_1 & 0 < \bar{x}_1 < L/2 \end{cases}$$

B.C.s

(1) @ $x_1 = 0$ $\quad u_2 = 0, \; du_2/dx_1 = 0$

(2) @ $\bar{x}_1 = 0$ $\quad u_2 = 0, \; du_2/dx_1 = 0$

(3) @ $x_1 = \bar{x}_1 = L/2$ $\quad u_2|_L = u_2|_R$

$du_2/dx_1|_L = -\,du_2/dx_1|_R$

Negative due to x_1, $\bar{x}_1$ definitions.

$$H_{33}^C u_2 = (Q + M_R + R_R L)\frac{x_1^2}{2} - R_R \frac{x_1^3}{6} + \cancel{C_1}^{\,=0} x_1 + \cancel{C_2}^{\,=0}$$

Using (1)

$$H_{33}^C u_2 = M_R \frac{\bar{x}_1^2}{2} + R_R \frac{\bar{x}_1^3}{6}$$

Using (2) $+ \cancel{C_3}^{\,=0} x_1 + \cancel{C_4}^{\,=0}$

$$H^C_{33}\,u_2 = (Q + M_R + R_R L)\frac{x_1^2}{2} - R_R\frac{x_1^3}{6} \qquad\Big|\qquad H^C_{33}\,u_2 = M_R\frac{\bar{x}_1^2}{2} + R_R\frac{\bar{x}_1^3}{6}$$

Using (3.1)

$$(Q + M_R + R_R L)\frac{L^2}{8} - R_R\frac{L^3}{6\cdot 8} = M_R\frac{L^2}{8} + \frac{R_R L^3}{6\cdot 8}$$

$$Q\frac{L^2}{8} = R_R L^2\left(\frac{1}{6\cdot 8} + \frac{1}{6\cdot 8} - \frac{1}{8}\right) \longrightarrow \boxed{R_R = -\frac{3}{2}\frac{Q}{L}}$$

$$H^C_{33}\,u_2 = M_R\frac{x_1^2}{2} - \frac{3}{2}\frac{Q}{L}\left(\frac{-Lx_1^2}{3} + \frac{Lx_1^2}{2} - \frac{x_1^3}{6}\right) \qquad\Big|\qquad H^C_{33}\,u_2 = M_R\frac{\bar{x}_1^2}{2} - \frac{3}{2}\frac{Q}{L}\frac{\bar{x}_1^3}{6}$$

$$H^C_{33}\,u_2 = M_R x_1^2/2 - \frac{3}{2}\frac{Q}{L}\left(\frac{Lx_1^2}{6} - \frac{x_1^3}{6}\right)$$

Using (3.2)

$$du_2/dx_1\big|_L = -du_2/dx_1\big|_R \quad @\ x_1 = \bar{x}_1 = L/2$$

$$M_R\frac{L^2}{8} - \frac{3}{2}\frac{Q}{L}\left(\frac{L}{6}\frac{L^2}{4} - \frac{L^3}{6\cdot 8}\right) = -M_R\frac{L^2}{8} + \frac{3}{2}\frac{Q}{L}\frac{L^3}{6\cdot 8}$$

$$M_R\frac{L^2}{4} = \frac{3}{2}\frac{Q}{L}\left(\frac{L^3}{6\cdot 8} + \frac{L^3}{6\cdot 4} - \frac{L^3}{6\cdot 8}\right)$$

$$M_R = \frac{3}{2}\frac{Q}{L}\cdot\frac{L}{6}$$

Finally

$$\boxed{H^C_{33}\,u_2 = -\frac{3}{2}\frac{Q}{L}\left(\frac{Lx_1^3}{12} - \frac{x_1^3}{6}\right)}$$

For $x_1 < L/2$

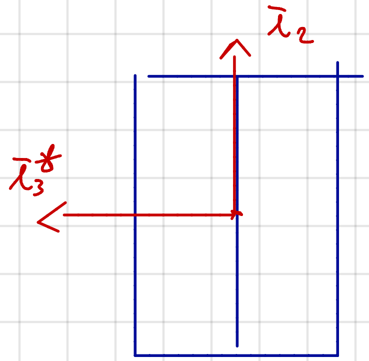
$$H^C_{33}\,u_2 = -\frac{3}{2}\frac{Q}{L}\left(\frac{\bar{x}_1^3}{6} - \frac{L\bar{x}_1^2}{12}\right)$$

For $\bar{x}_1 < L/2$

$\bar{x}_1 = L - x_1$

PART B)

LETS DEFINE A COORDINATE SYSTEM WHICH WE NOTE IS NOT CENTROIDAL BUT WILL HELP DEFINE σ_1



$$\sigma_1 = -E(\bar{x}_3) \cdot x_2 K_3$$

WITH

$$E = \begin{cases} E_A & \bar{x}_3 > 0 \\ E_B & \bar{x}_3 < 0 \end{cases}$$

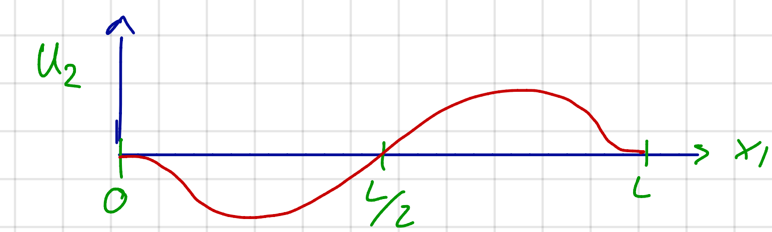
For $x_1 < L/2$

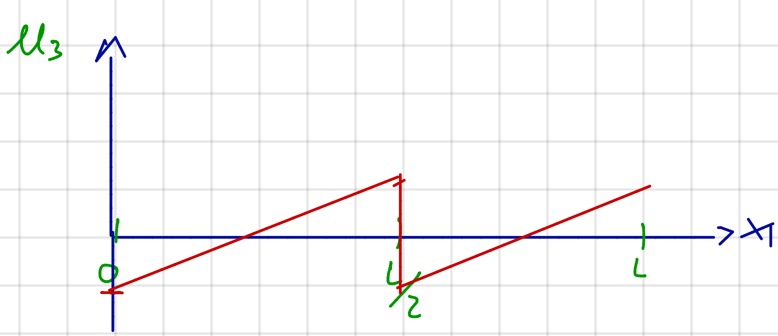
$$\sigma_1 = E(\bar{x}_3)\, x_2 \frac{Q}{H_{33}^c L}\left(\frac{3}{2} x_1 - \frac{L}{4}\right)$$

For $x_1 > L/2$

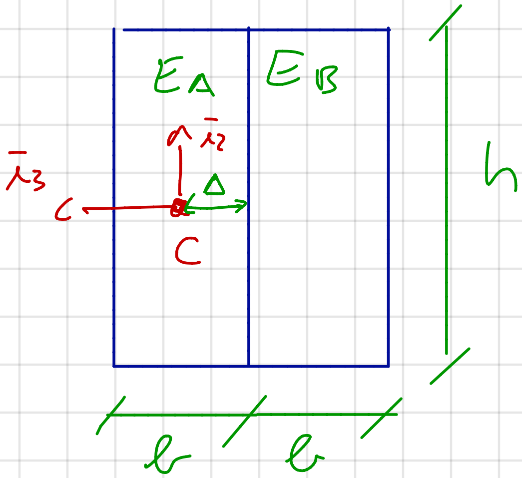
$$\sigma_1 = -E(\bar{x}_3)\, x_2 \frac{Q}{H_{33}^c L}\left(\frac{3}{2}(L - x_1) - \frac{L}{4}\right)$$

PART C)





PART D)



$$S = (E_A + E_B)(hb)$$

$$H_{23}^{C} = 0$$

$$H_{33}^{C} = \frac{bh^3}{12}(E_A + E_B)$$

$$H_{22}^{C} = E_A\left(\frac{hb^3}{12} + (bh)\left(\Delta - \frac{b}{2}\right)^2\right) + E_B\left(\frac{hb^3}{12} + (bh)\left(\Delta + \frac{b}{2}\right)^2\right)$$

$$\Delta = S_3 / S$$

$$\Delta = \frac{\left[\int_0^b E_A x_3 + \int_{-b}^0 E_B x_3\right] h}{(E_A + E_B)(hb)}$$

$$\Delta = \frac{b}{2}\frac{(E_A - E_B)}{(E_A + E_B)}$$