9/29

External forces can be of two kinds:

* Body forces
  ↳ from fields: gravity, electromagnetic, etc.

* Surface forces:
  ↳ short range interaction forces across ∂B resulting from interactions between B and its surrounding.

<u>Mathematically</u>:

Body Forces: $$\int_B \underline{b}\, dm = \int_B \underline{b} \rho\, dv$$

$\underline{b} \equiv$ body force per unit mass (ex: $\frac{N}{kg}$)

Surface Forces: $$\int_{\partial B} d\underline{f}^{surf} = \int_{\partial B} \underline{t}\, dA$$

goals

* <u>Note</u>: $$\check{\underline{t}} \equiv \lim_{\Delta A \to 0} \frac{\Delta f^{surf}}{\Delta A}$$

it is a fundamental assumption of continuum mechanics that this limit exists.

Thus, we have:

$$\underline{F}^{ext}(B) = \int_B \underline{b}\,\rho\, dV + \int_{\partial B} \underline{\breve{t}}\, dA$$

Substituting into (1) (from 9/22)

$$\boxed{\frac{D}{Dt}\int_B \rho\, \underline{\dot{x}}\, dV = \int_B \underline{b}\,\rho\, dV + \int_{\partial B} \underline{\breve{t}}\, dA} \quad (2)$$

over volume   over surface

Note:

$$\frac{d}{dt}\int_a^b f(x,t)\, dx = \int_a^b \frac{d}{dt} f(x,t)\, dx$$

but can't do this if a, b functions of time.

ie: $$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx$$

* **Note:** The integration domain in (2) changes w/ time.

↳ because the body is deforming.

* Thus, the time derivative cannot be just passed inside the integral.

To solve this issue, we make use of the conservation of mass principle:

$$\frac{D}{Dt}\int_{B} \rho \dot{\underline{x}}\, dv \overset{(*)}{=} \frac{D}{Dt}\int_{B} \frac{\rho_0}{J} \dot{\underline{x}}\, dv \overset{(**)}{=} \frac{D}{Dt}\int_{B_0} \frac{\rho_0}{J} \dot{x} J\, dV$$

(*) cons. of mass

(**) $dv = J\, dV$

Note: $J = 0$ means $dv = 0\, dV$ ((collapse the body)

↳ not possible, physically

$$= \frac{D}{Dt}\int_{B_0} \rho_0 \dot{\underline{x}}\, dV = \int_{B_0} \frac{D}{Dt}(\rho_0 \dot{\underline{x}})\, dV$$

$$= \int_{B_0} \rho_0 \ddot{\underline{x}}\, dV \underset{(*)}{=} \int_{B_0} J \rho \ddot{\underline{x}}\, dV \underset{(**)}{=} \int_{B} \rho \ddot{\underline{x}}\, dv$$

which allows us to express the spatial form of the global balance of linear momentum as:

$$\underbrace{\boxed{\int_B \rho \underline{\ddot{x}} \, dv = \int_B \rho \underline{b} \, dv + \int_{\partial B} \underline{\check{t}} \, dA}}_{\text{(global expression)}}$$

boundary of B.

Cauchy's stress principle.

- In order to obtain a local expression for the balance of linear momentum, it is necessary to write the previous expression for an arbitrary sub-body $E \subset B$
- This is a problem for the traction term since $\underline{\check{t}}$ is defined in terms of forces acting on B across its outer surfaces.
- This issue was addressed by Cauchy's stress principle.

Cauchy's stress principle.

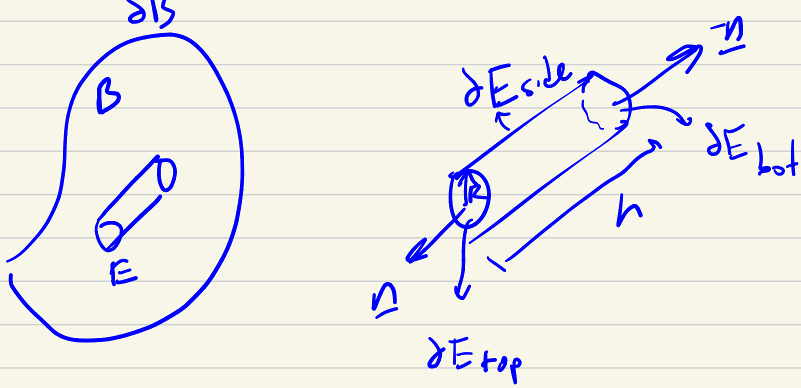
Material interactions across an internal surface in a body can be described as a distribution of tractions in the same way that the effect of external forces on physical surfaces of the body are described.

Thus, for an arbitrary subbody $E \subset B$ we can write the following:

$$\boxed{\int_E \rho \ddot{\underline{x}} \, dv = \int_E \rho \underline{b} \, dv + \int_{\partial E} \underline{t} \, dA}$$

(w/o the hat)

Let's consider an arbitrary "pillbox shaped" sub-body



$$\Rightarrow \int_E \rho\,(\ddot{\underline{x}} - \underline{b})\,dv = \int_{\partial E_{side}} \underline{t}\,dA + \int_{\partial E_{top}} \underline{t}\,dA + \int_{\partial E_{bot}} \underline{t}\,dA$$

Taking limit for $h \to 0$:

$$\int_{\partial E_{top}} \underline{t}\,dA + \int_{\partial E_{bot}} \underline{t}\,dA = \underline{0}$$

Finally, applying the mean value theorem and taking limit $R \to 0$: $\underline{t}(x)\Big|_{\partial E_{top}} = -\underline{t}(x)\Big|_{\partial E_{bot}}$