An Energy Based Approach to Constraint Analysis – Part 1

Presented by
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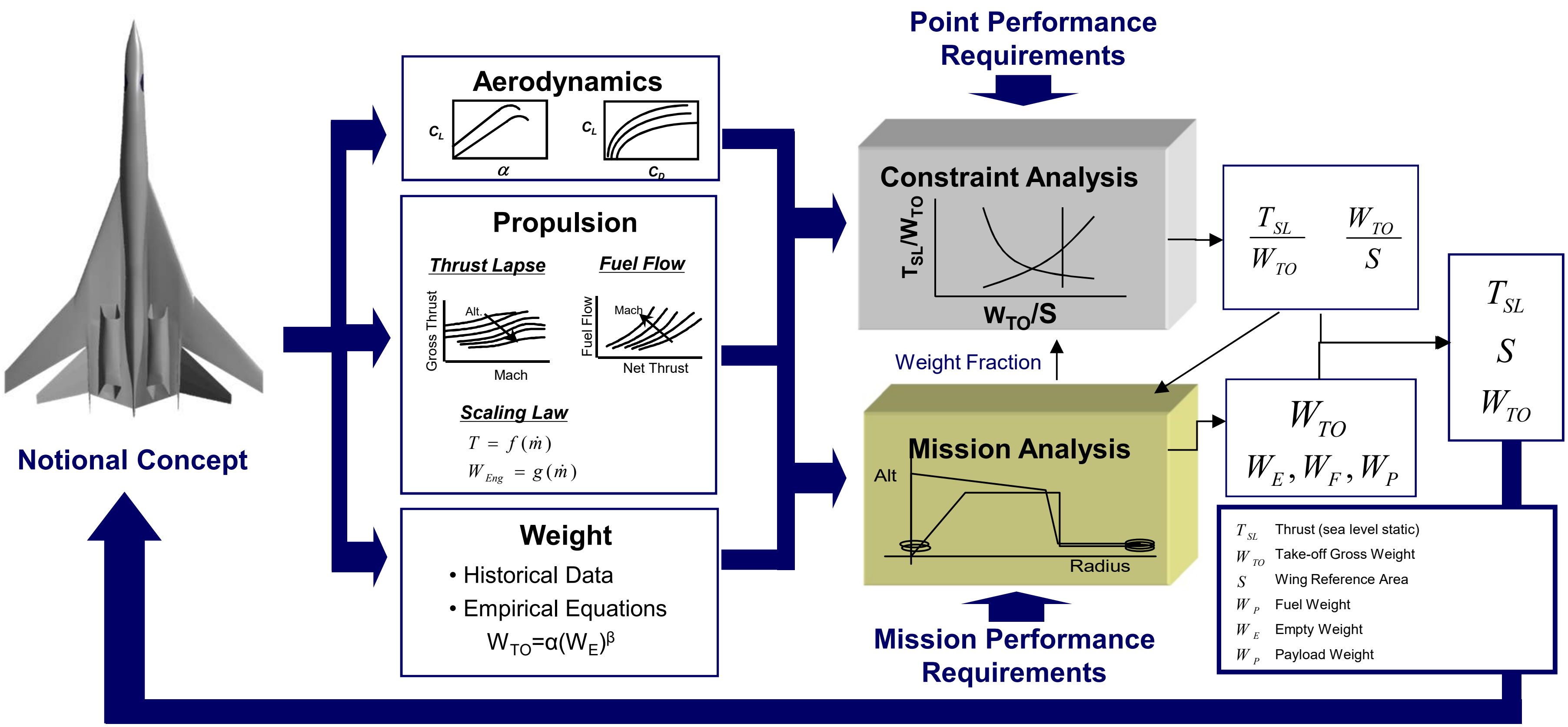
Disclaimer

The following presentation is based on the copyrighted work of Mattingly, Heiser and Daley. This presentation uses their formulation of an energy based approach to constraint analysis and mission sizing. The text associated with this material can be found in:

– Mattingly, J., D., Heiser, W., H., Daley, G., H., Aircraft Engine Design, AIAA, Washington, 1987.

Traditional Aircraft Sizing Process



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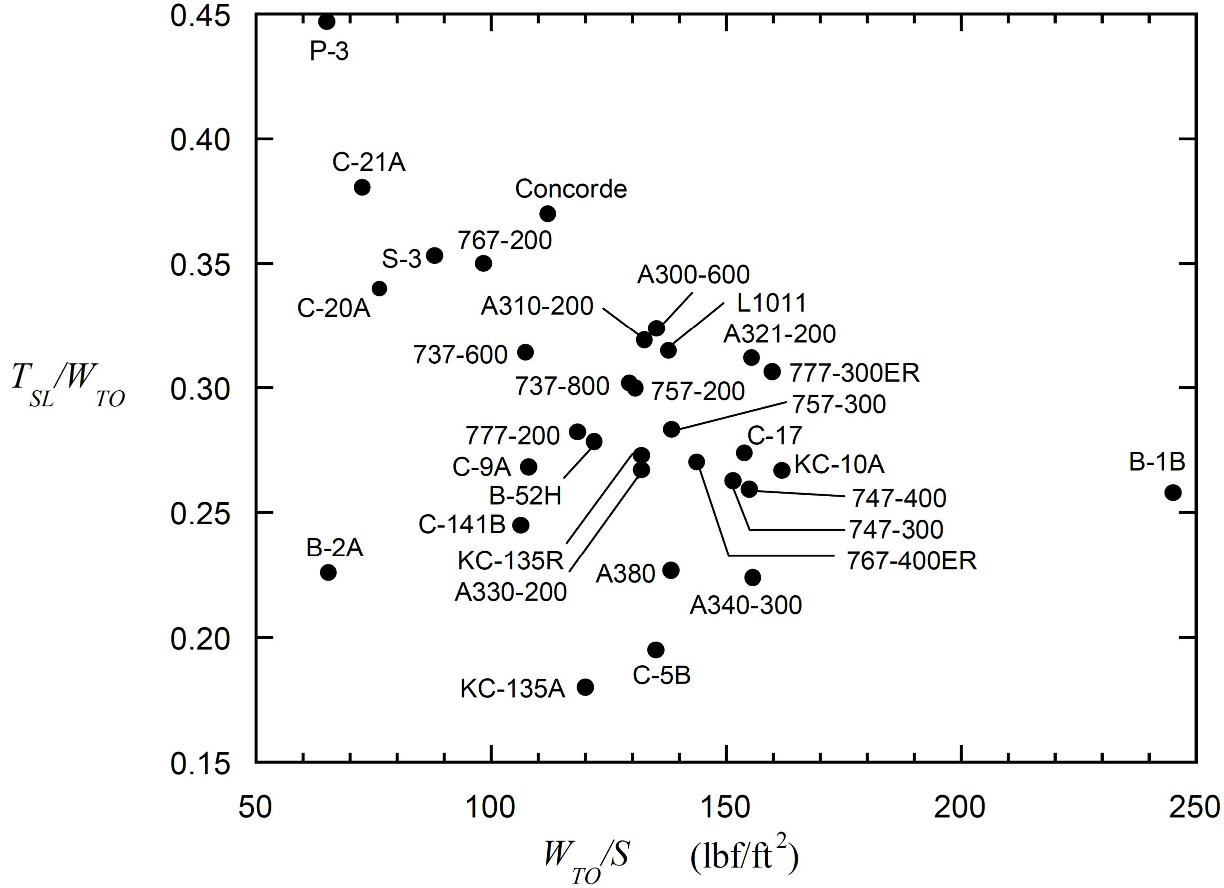
Importance of T/W and W/S

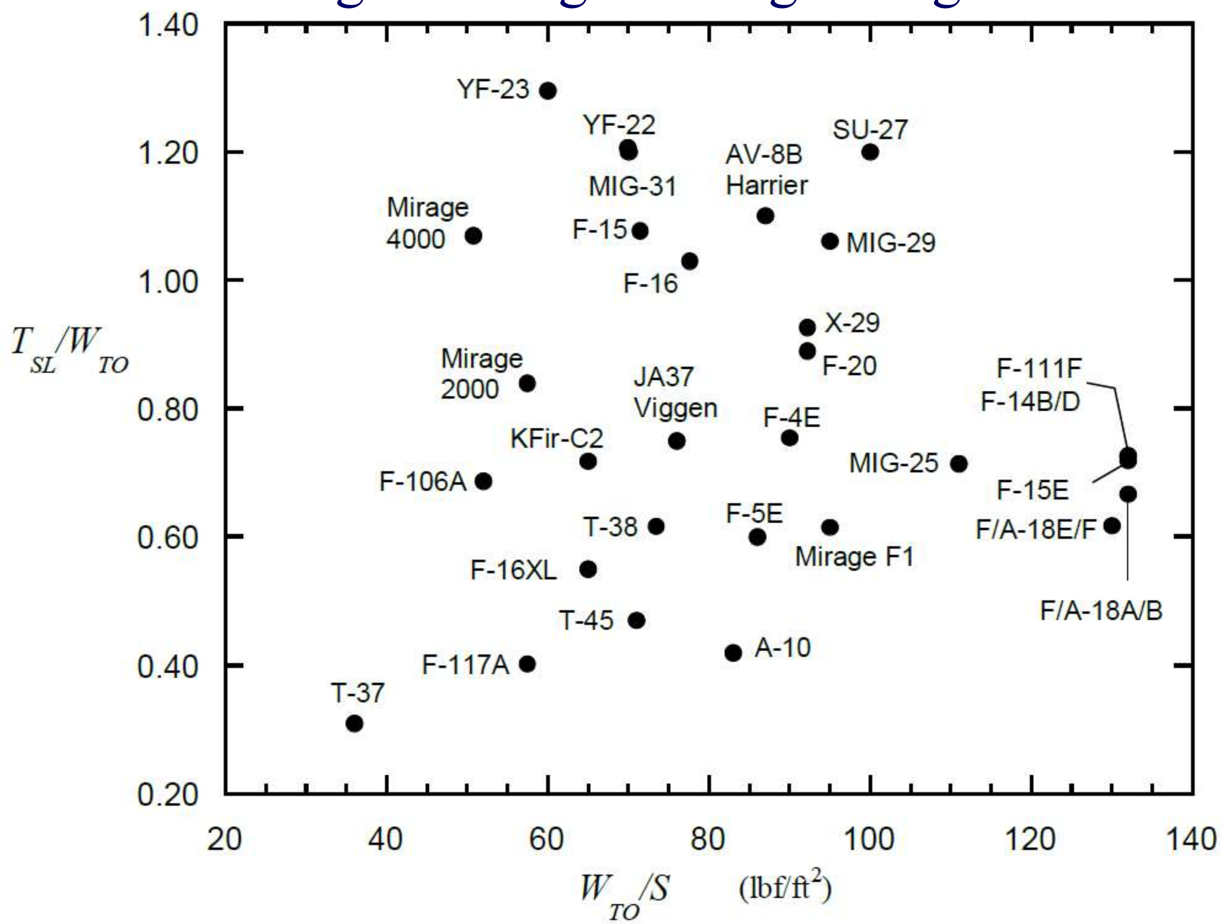
Thrust Loading vs. Wing Loading for cargo and passenger aircraft



Mattingly, J., D., Heiser, W., H., Pratt, D., T., *Aircraft Engine Design*, AIAA, Virginia, 2002.

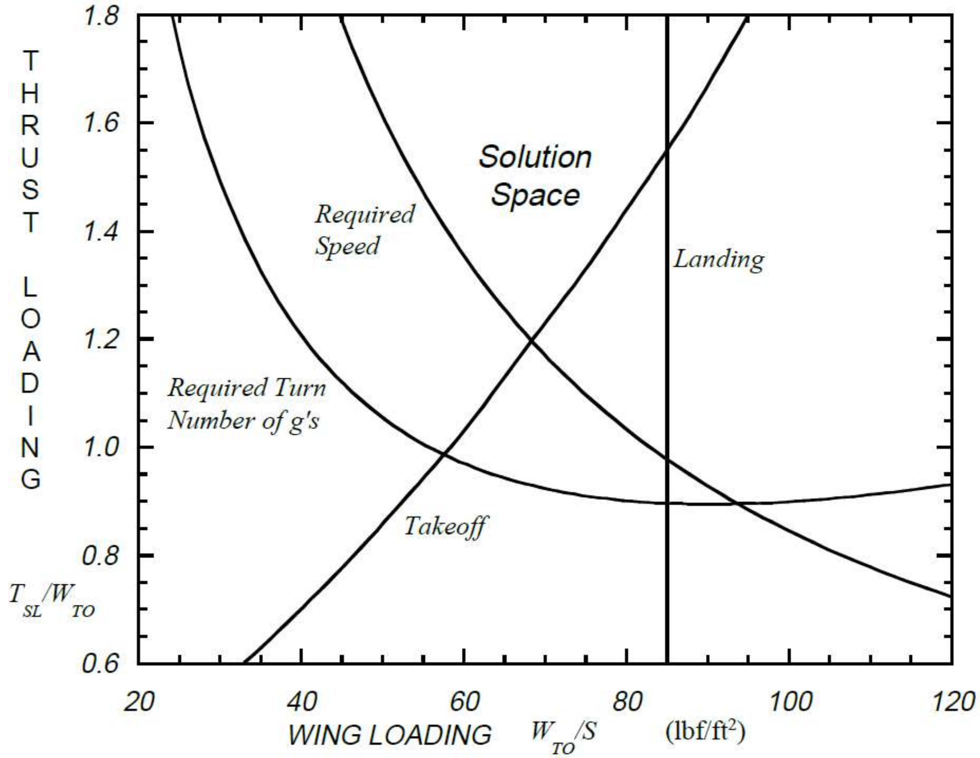
Importance of T/W and W/S

Thrust Loading vs. Wing Loading for fighter aircraft



Mattingly, J., D., Heiser, W., H., Pratt, D., T., *Aircraft Engine Design*, AIAA, Virginia, 2002.

Constraint Analysis Plot



Mattingly, J., D., Heiser, W., H., Pratt, D., T., *Aircraft Engine Design*, AIAA, Virginia, 2002.

- Constraint analysis can be visualized to identify the solution space
- Performance constraints and requirements are set as functions of Thrust Loading and Wing Loading
- Use two sizing parameters that contain key information about the top level characteristics of the system

Energy Balance of the System

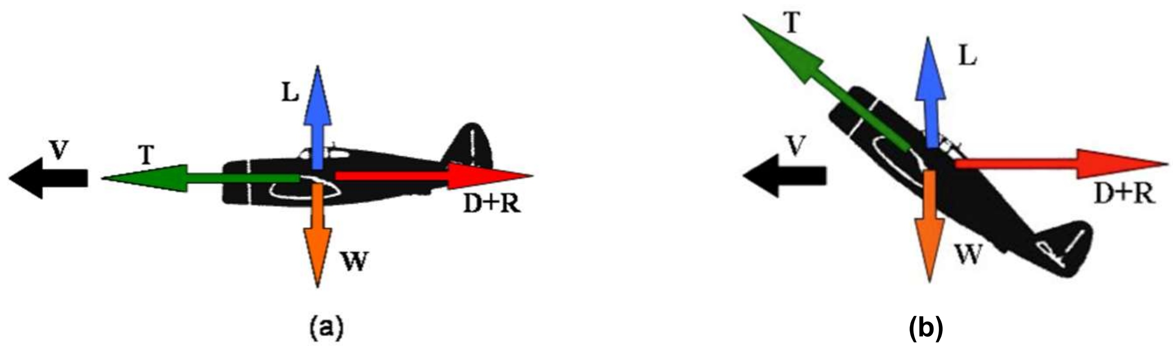
- An energy approach to the analysis of a dynamic system
- Offers a useful form of energy balance in terms of practical quantities

$$\{T-(D+R)\}V = W\frac{dh}{dt} + \frac{W}{g_o}\frac{d}{dt}\left(\frac{V^2}{2}\right) \quad (1)$$

Rate of Mechanical Energy Input — *Storage Rate Of Potential Energy* — *Storage Rate Of Kinetic Energy*

Assumptions

- Aircraft is represented as a moving point mass
  - What are the implications of this assumption?
  - What is being ignored by assuming the aircraft is a point?
- Installed thrust and aerodynamic drag act in the same direction as the velocity
  - Is this true?
  - Why can this assumption be made?



Energy Balance of the System

- The Energy Equation can be manipulated to obtain a dimensionless form

$$\left\{\frac{T-(D+R)}{W}\right\}=\frac{1}{V}\frac{d}{dt}\left\{h+\frac{V^2}{2g_o}\right\}=\frac{1}{V}\frac{dz_e}{dt} \qquad \text{(2a)}$$

Excess Power

- The left hand side, when multiplied by Velocity (V), becomes the equation for "weight specific excess power."
- What information is provided by this term?

$$P_s = \frac{d}{dt}\left\{h + \frac{V^2}{2g_o}\right\} = \frac{dz_e}{dt} \qquad (2b)$$

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Thrust and Weight Lapse

- Correct thrust using lapse rate, α
  - $T=\alpha T_{SL}$ (3)
- Correct weight using fuel/payload correction, β
  - $W=\beta W_{TO}$ (4)
- Why are these corrections made? How are they representative of the performance of the aircraft?

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\left(\frac{D+R}{\beta W_{TO}}\right) + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)\right\} \tag{5}$$

Definition of Lift and C_L

$$L = nW = qC_L S \tag{6}$$

Where n is the "load factor" (number of g's). n=1 for straight and level flight.

$$C_L = \frac{nW}{qS} = \frac{n\beta}{q}\left(\frac{W_{TO}}{S}\right) \tag{7}$$

Solving for C_L and substituting equation (4) to get the takeoff wing loading.

Drag, C_D and the Drag Polar

$$D = qC_D S \tag{8}$$

Equation (8) yields the relationship between drag coefficient and drag.

$$C_D = K_1 C_L^2 + K_2 C_L + C_{D_o} \tag{9}$$

The parabolic lift-drag polar equation is given above in equation (9).

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Combined Aerodynamic Drag Expression

Substitute Equation (9) into Equation (8) for C_D.

$$D = qSC_D \qquad D = qS\left(K_1 C_L^2 + K_2 C_L + C_{D_o}\right) \tag{9b}$$

Then substitute Equation (7) for C_L in the above Equation to yield :

$$D = qS\left(K_1\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} \right) \tag{10}$$

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The "Master Equation"

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\left(\frac{D+R}{\beta W_{TO}}\right) + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)\right\} \quad (5)$$

Substitute Drag in Energy Equation with Equation (10)

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \frac{R}{qS}\right] + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)\right\} \quad (11)$$

The above form of the Energy Equation is referred to by Mattingly as the "Master Equation". It can be manipulated to specific flight conditions.

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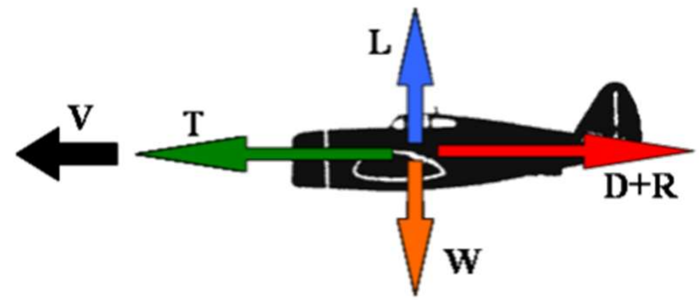
Using the Master Equation

- Case 1: Constant Altitude/Speed Cruise
- Case 2: Constant Speed Climb
- Case 3: Constant Altitude/Speed Turn
- Case 4: Horizontal Acceleration
- Case 5: Takeoff Ground Roll (lots of Thrust)
- Case 6: Takeoff Ground Roll (not so much Thrust)
  - A: Do NOT Clear the Obstacle During Transition
  - B: Clear the Obstacle During Transition
- Case 7: Braking Roll
- Case 8: Service Ceiling

Case 1: Constant Altitude/Speed Cruise (P_s=0)

Assumptions:	
dh/dt = 0	Constant Altitude
dV/dt =0	Constant Speed
n=1	Lift equals Weight
R=0	Clean configuration
h & V	Values are Given



$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \cancel{\frac{R}{qS}}\right] + \cancel{\frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)}\right\}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o}\right]\right\} \qquad (11b)$$

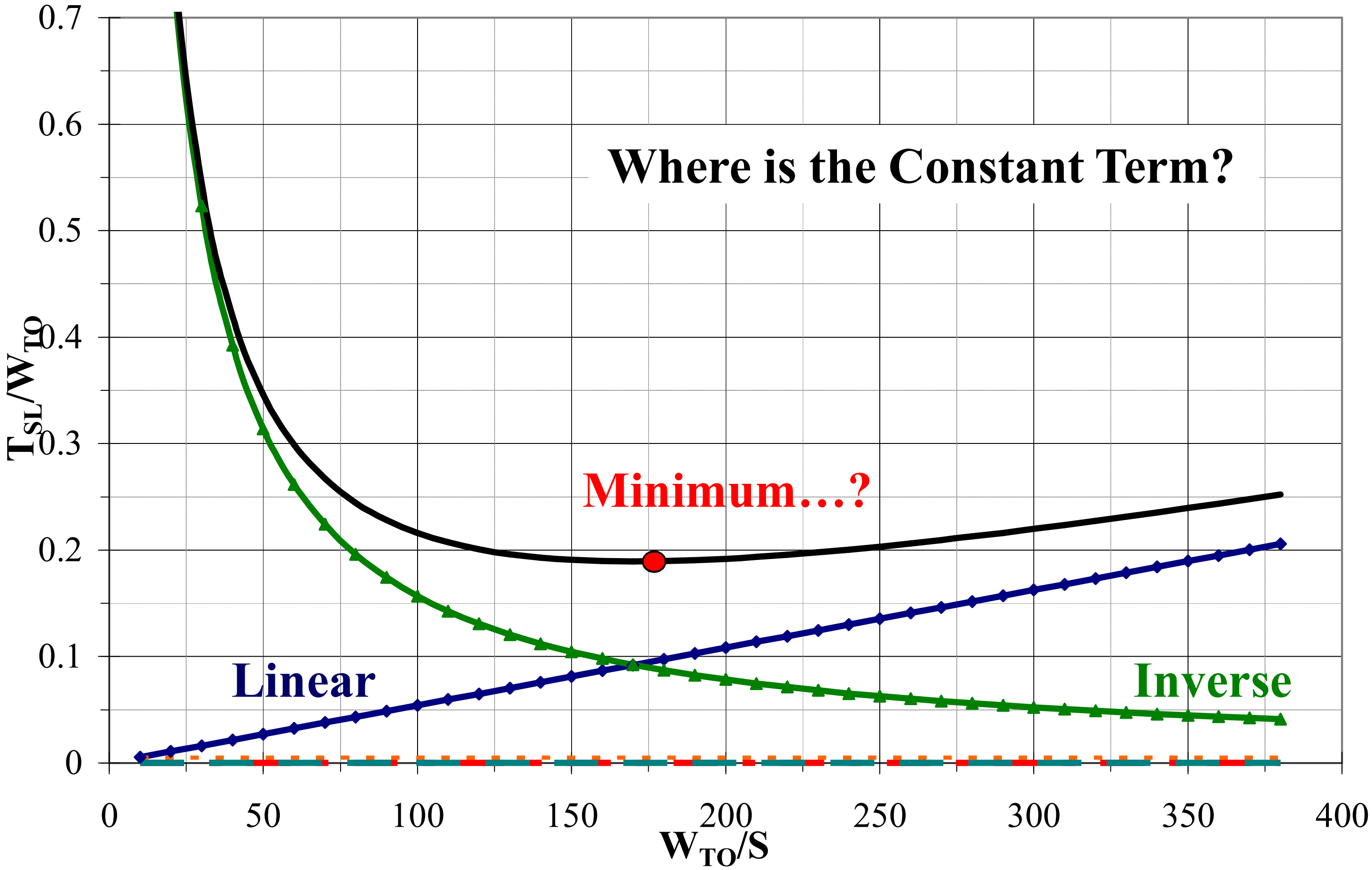
Case 1: Constant Altitude/Speed Cruise (P_s=0)

Equation 11b can be simplified by canceling out W_{TO} and qS terms

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} \right\} \quad (12)$$

Linear Term **Constant Term** **Inverse Term**

Plot for Case 1



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Case 1: Constant Altitude/Speed Cruise (P_s=0)

For very large and very small values of wing loading, the T/W ratio goes to infinity. To find the point at which the T_{SL}/W_{TO} ratio is minimum, take the partial derivative of Equation (12) with respect to W_{TO}/S and set it equal to zero. Then, solve for W_{TO}/S:

$$0 = \frac{d}{d\left(\frac{W_{TO}}{S}\right)}\left[\frac{\beta}{\alpha}\left\{K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}\right\}\right]$$

$$\frac{W_{TO}}{S} = \frac{q}{\beta}\sqrt{\frac{C_{D_o}}{K_1}}$$

Wing loading for minimum T_{SL}/W_{TO}

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Case 1: Constant Altitude/Speed Cruise (P_s=0)

Also, for very large *q* ($C_D \approx C_{D0}$), the linear term is reduced to a very low value and the inverse term increases significantly. The constant K_2 is inherently small so a hyperbolic relationship between T/W and W/S is observed.

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}\right\} \qquad (12)$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} \Rightarrow \left(\frac{T_{SL}}{W_{TO}}\right)\left(\frac{W_{TO}}{S}\right) = \frac{qC_{D_o}}{\alpha} \qquad (13)$$

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Using the Master Equation

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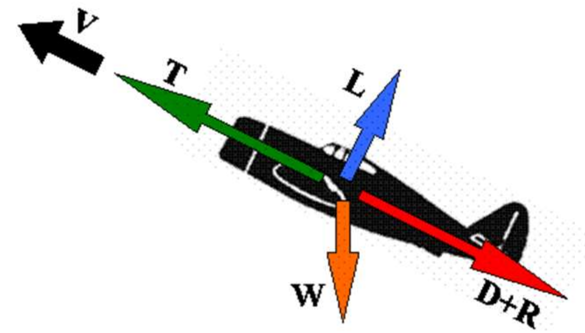
Case 2: Constant Speed Climb (P_s=dh/dt)

Assumptions:	
dV/dt =0	Constant Speed
n ≈ 1	Lift approximately equals Weight
R=0	Not on the ground
h, dh/dt & V	Values are Given



$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \cancel{\frac{R}{qS}}\right] + \frac{1}{V}\frac{d}{dt}\left(h + \cancel{\frac{V^2}{2g_o}}\right)\right\}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{V}\frac{dh}{dt}\right\} \qquad (14)$$

Case 2: Constant Speed Climb (P_s=dh/dt)

Note that the only difference between Case 1 and Case 2 is the final term, for climb rate. Assuming that the climb rate is constant the equation for Case 2 is the same as that for Case 1 except for the constant term added.

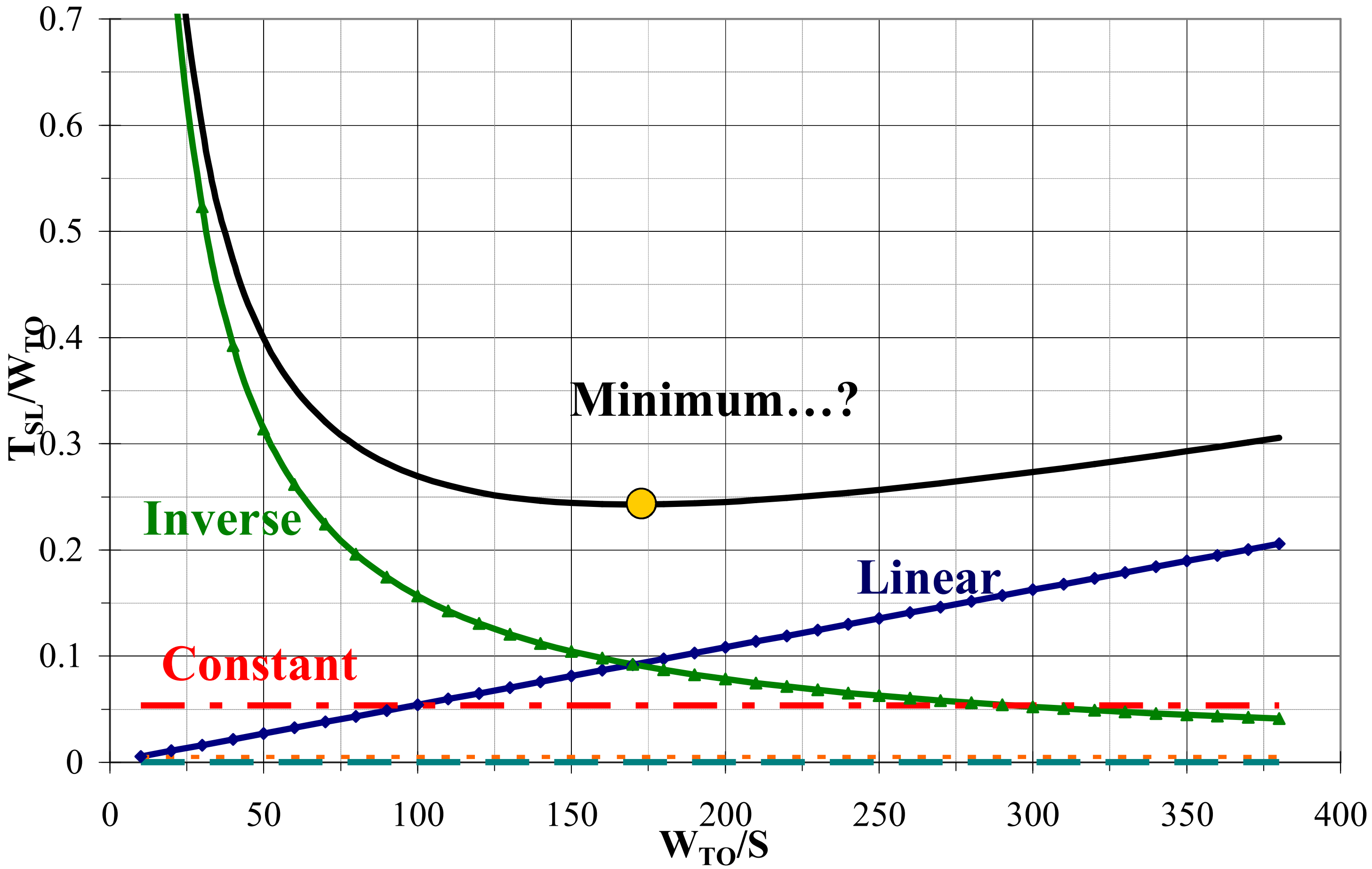
Case 1:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} \right\} \quad (12)$$

Case 2:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{V}\frac{dh}{dt} \right\} \quad (14)$$

Plot for Case 2



Case 2: Constant Speed Climb (P_s=dh/dt)

Since the rate of climb term is a constant and not a function of W/S, taking the derivative as was done in Case 1 will yield the same W/S value at which the minimum T/W occurs; however, the value of T/W at this point will be different based on the desired rate of climb.

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{V}\frac{dh}{dt} \right\} \quad (14)$$

$$\frac{W_{TO}}{S} = \frac{q}{\beta}\sqrt{\frac{C_{D_o}}{K_1}}$$

Wing loading for minimum T_{SL}/W_{TO}

Using the Master Equation

- Case 1: Constant Altitude/Speed Cruise
- Case 2: Constant Speed Climb
- **Case 3: Constant Altitude/Speed Turn**
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Case 3: Constant Altitude/Speed Turn (P_s=0)

Assumptions:	
dh/dt=0	Constant Altitude
dV/dt =0	Constant Speed
R=0	Not on the ground
h, n, & V	Values are Given

Question: Is there acceleration in this case? The dV/dt term is zero, but the load factor is NOT 1.

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \cancel{\frac{R}{qS}}\right] + \cancel{\frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)}\right\}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 n^2 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 n + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}\right\} \qquad (15)$$

Case 3: Constant Altitude/Speed Turn (P_s=0)

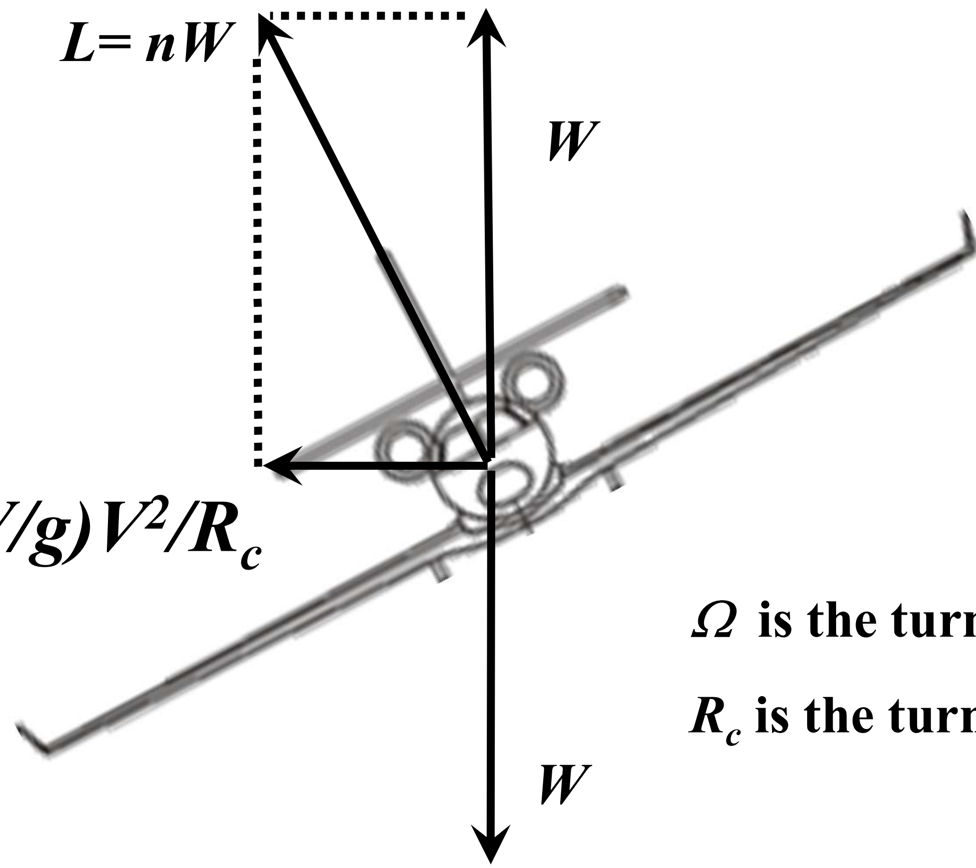
The behavior of Equation (15) is the same as that of Case 1 and Case 2. The point of minimum T/W, however, is now a function of the load factor, n.

$$\frac{W_{TO}}{S} = \frac{q}{n\beta}\sqrt{\frac{C_{D_o}}{K_1}}$$

Wing loading for minimum T_{SL}/W_{TO}

How can the load factor be determined?
How is its value relevant in constraint analysis?
Is it directly used to define constraints or can alternative performance metrics be used?
Can these alternative metrics help define load factor?

Turn Free Body Diagram



$$F_c = ma_c$$

$$= (W/g)\Omega V = (W/g)V^2/R_c$$

Ω is the turning rate

R_c is the turning radius

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Definitions of Load Factor

Using the free body diagram and the Pythagorean theorem to relate the forces the different relationships for n in the turn can be calculated.

$$n = \left\{ 1 + \left(\frac{\Omega V}{g_o} \right)^2 \right\}^{\frac{1}{2}} \quad (16)$$

$$n = \left\{ 1 + \frac{V^2}{g_0 R_C} \right\}^{\frac{1}{2}} \quad (17)$$

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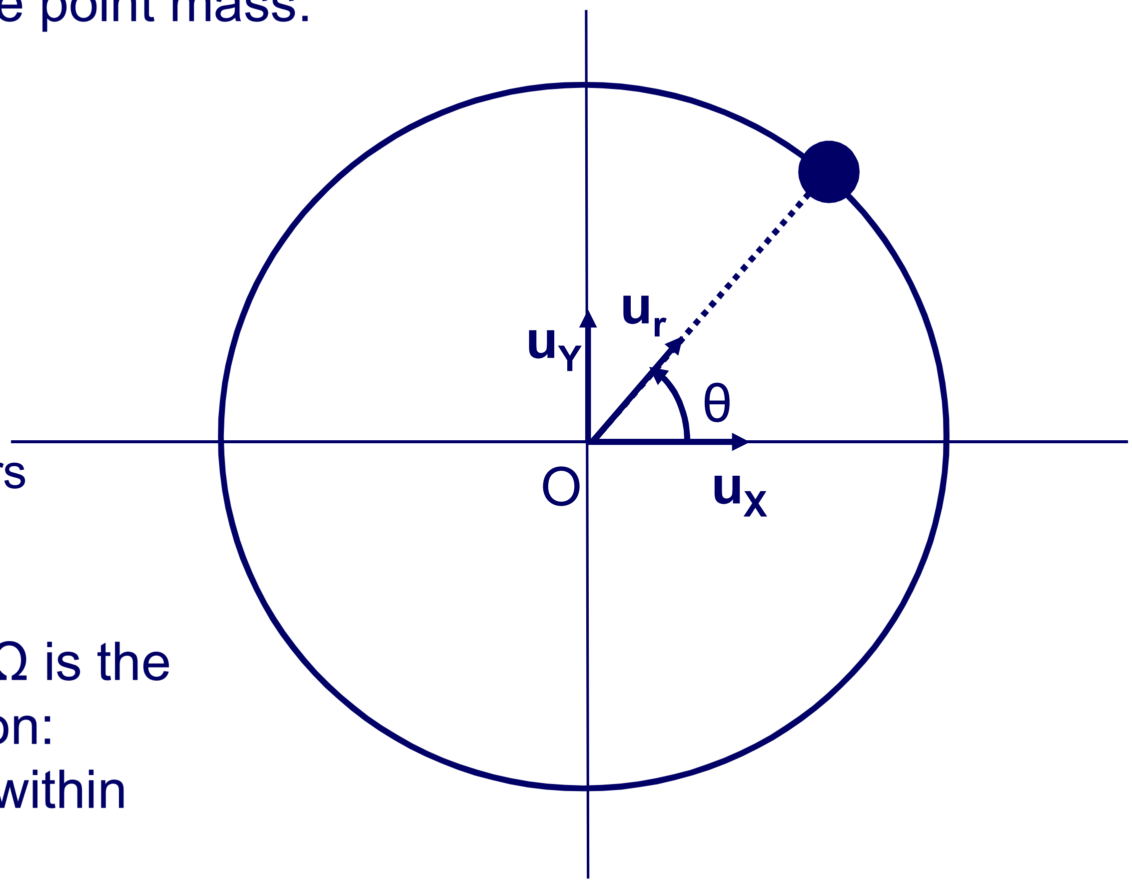
Background Information for Load Factor

r(t) defines the position of the point mass:

$$\vec{\mathrm{r}}(t) = R\,\vec{\mathrm{u}}_{\mathrm{r}}$$

with:

- R = radius of the circle
- $\mathbf{u}_r$, $\mathbf{u}_X$, and $\mathbf{u}_Y$ are unit vectors
- $\mathbf{u}_r = \cos\theta\,\mathbf{u}_X + \sin\theta\,\mathbf{u}_Y$
- and, $\theta = \theta(t) = \Omega t$, where Ω is the angular velocity (assumption: rotating at constant speed within the plane)



Definition of the Position and Velocity

Since Position, $\vec{\mathrm{r}}(t) = R\,\vec{\mathrm{u}}_{\mathrm{r}}$

and velocity,

$$\vec{\mathrm{V}} = \frac{d\vec{\mathrm{r}}(t)}{dt} = R\frac{d\vec{\mathrm{u}}_{\mathrm{r}}}{dt}$$

$$\frac{d\vec{\mathrm{u}}_{\mathrm{r}}}{dt} = \frac{d}{dt}(\cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}})$$

$$\frac{d\vec{\mathrm{u}}_{\mathrm{r}}}{dt} = \Omega(-\sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}})$$

$$\frac{d\vec{\mathrm{u}}_{\mathrm{r}}}{dt} = \Omega\,\vec{\mathrm{u}}_{\theta} \quad \text{where, } \vec{\mathrm{u}}_{\theta} = -\sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}}$$

velocity is also, $V = R\Omega\,\vec{\mathrm{u}}_{\theta} = R\,\Omega(-\sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}})$

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Definition of the Acceleration

Similarly, for acceleration:

$$\vec{a} = \frac{d\vec{\mathrm{V}}}{dt}$$

$$\vec{a} = \frac{d}{dt}(R\Omega(-\sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}}))$$

$$\vec{a} = -R\Omega^2(\cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}}) + R\frac{d\Omega}{dt}(-\sin\Omega t\,\vec{\mathrm{u}}_{\mathrm{X}} + \cos\Omega t\,\vec{\mathrm{u}}_{\mathrm{Y}})$$

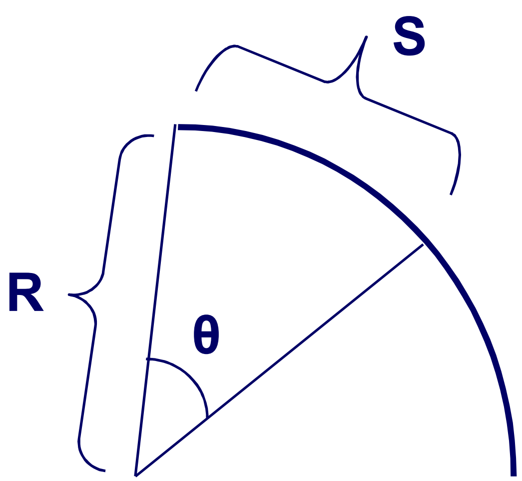
$$\vec{a} = -R\Omega^2\,\vec{\mathrm{u}}_{\mathrm{r}} + R\frac{d\Omega}{dt}\vec{\mathrm{u}}_{\theta}$$

Definition of the Centripetal Acceleration

$$\vec{a} = -R\Omega^2\,\vec{u}_r + R\frac{d\Omega}{dt}\vec{u}_\theta$$

The centripetal acceleration is the radial component of the acceleration:

$\vec{a}_c = -R\Omega^2\,\vec{u}_r$ pointing in the $-\vec{u}_r$ direction, where $a_c = R\Omega^2$



Since arc length, $S = R\theta$

then $$V = \frac{dS}{dt} = R\frac{d\theta}{dt} = R\Omega$$

Finally: $$a_c = R\Omega^2 = \frac{V^2}{R} = V\Omega$$

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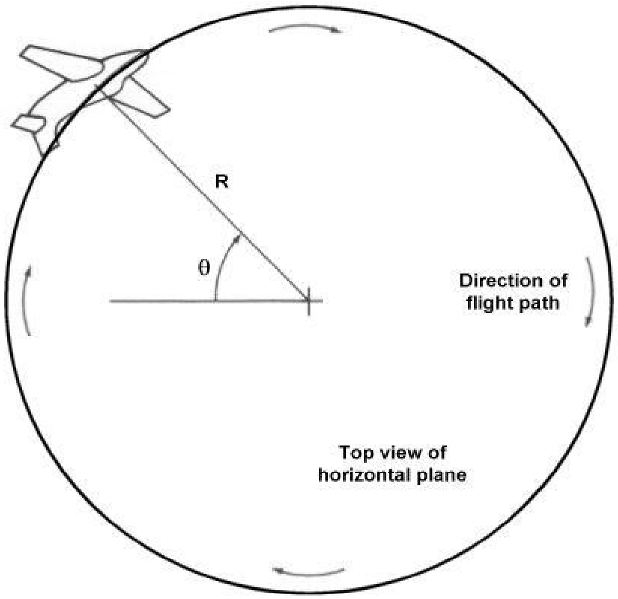
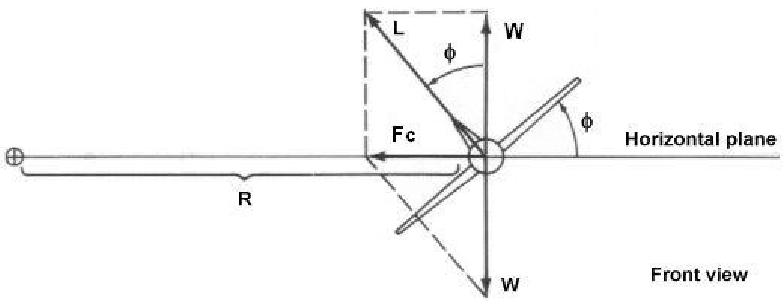
Definition of the Load Factor





R: Turn Radius

L: Lift

W: Weight

F_c = Centripetal force

Φ = bank angle

For a turn at constant altitude:

$$\boxed{W = L \cos \Phi}$$

Source: Aircraft Performance and Design, J.D. Anderson

Definition of the Load Factor

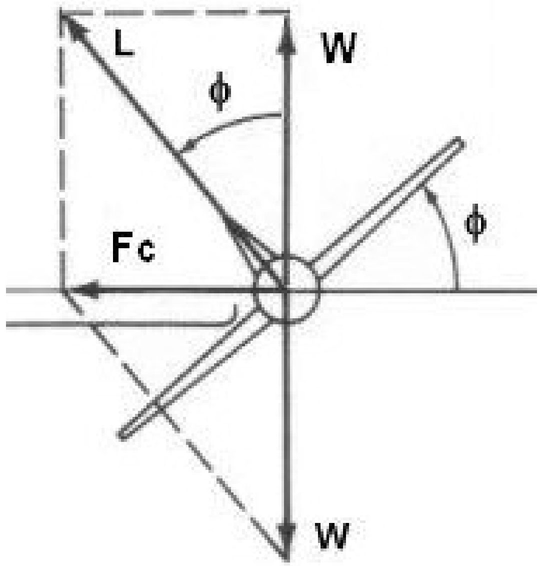
Load factor n: $$n = \frac{L}{W}$$

According to Pythagoras' Theorem: $$L^2 = F_c^2 + W^2$$

$$\frac{L^2}{W^2} = \frac{F_c^{\,2}}{W^2} + 1$$

$$n = \frac{L}{W} = \sqrt{\frac{F_c^{\,2}}{W^2} + 1}$$

$$n = \left\{ \left(\frac{m * a_c}{m * g_0} \right)^2 + 1 \right\}^{\frac{1}{2}} = \left\{ \left(\frac{a_c}{g_0} \right)^2 + 1 \right\}^{\frac{1}{2}}$$



Definition of the Load Factor

Recall that a_c is defined as: $$a_c = R\Omega^2 = \frac{V^2}{R} = V\Omega$$

Thus: $$n = \left\{ \left(\frac{a_c}{g_0} \right)^2 + 1 \right\}^{\frac{1}{2}} = \left\{ \left(\frac{\Omega V}{g_0} \right)^2 + 1 \right\}^{\frac{1}{2}}$$

Or: $$n = \left\{ \left(\frac{a_c}{g_0} \right)^2 + 1 \right\}^{\frac{1}{2}} = \left\{ \left(\frac{V^2}{g_0 R} \right)^2 + 1 \right\}^{\frac{1}{2}}$$

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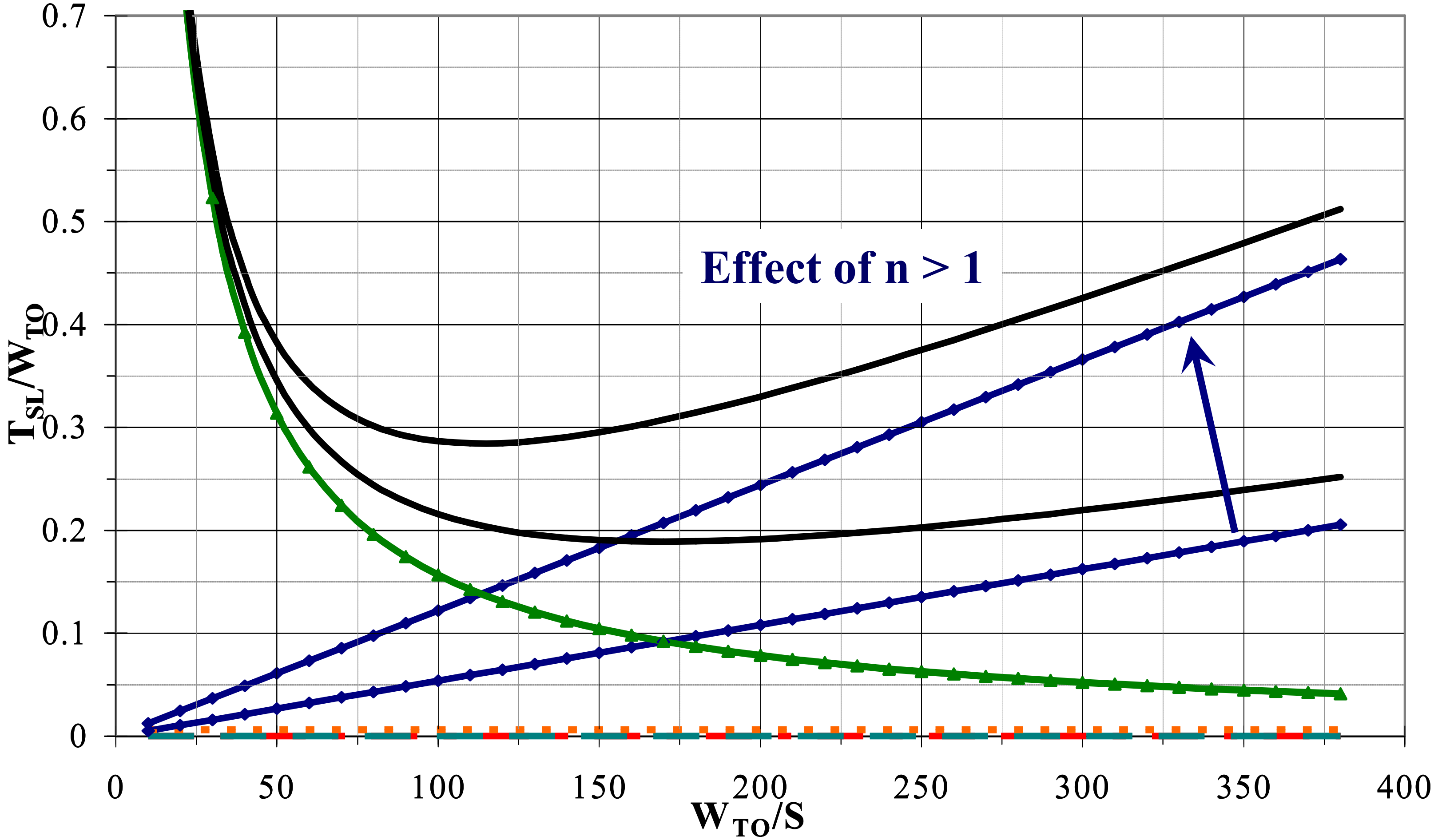
Plot for Case 3



Using the Master Equation

- Case 1: Constant Altitude/Speed Cruise
- Case 2: Constant Speed Climb
- Case 3: Constant Altitude/Speed Turn
- Case 4: Horizontal Acceleration
- Case 5: Takeoff Ground Roll (lots of Thrust)
- Case 6: Takeoff Ground Roll (not so much Thrust)
  - A: Do NOT Clear the Obstacle During Transition
  - B: Clear the Obstacle During Transition
- Case 7: Braking Roll
- Case 8: Service Ceiling

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Case 4: Horizontal Acceleration

Assumptions:	
n ≈ 1	Lift approximately equals Weight
R=0	Clean Configuration
dh/dt, h, t	Values are Given

$$P_s = {VdV}\Big/{g_0 dt}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \cancel{\frac{R}{qS}}\right] + \frac{1}{V}\frac{d}{dt}\left(\cancel{h} + \frac{V^2}{2g_o}\right)\right\}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D_0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{g_0}\frac{dV}{dt}\right\} \qquad (18)$$

Calculating Acceleration

The equation for horizontal acceleration conserves, like previous cases, a linear term, inverse term, and the constant K2. An additional constant for acceleration is present. How is this constant defined?

Acceleration can be rigorously integrated, or a simpler approach taken by assuming the following definition. Note that this expression uses terms that are likely to define an acceleration constraint, namely the initial and final velocity, and the allowable time.

$$\frac{1}{g_0}\frac{dV}{dt} = \frac{1}{g_0}\left(\frac{V_{Final} - V_{Initial}}{\Delta t_{Allowable}}\right)$$

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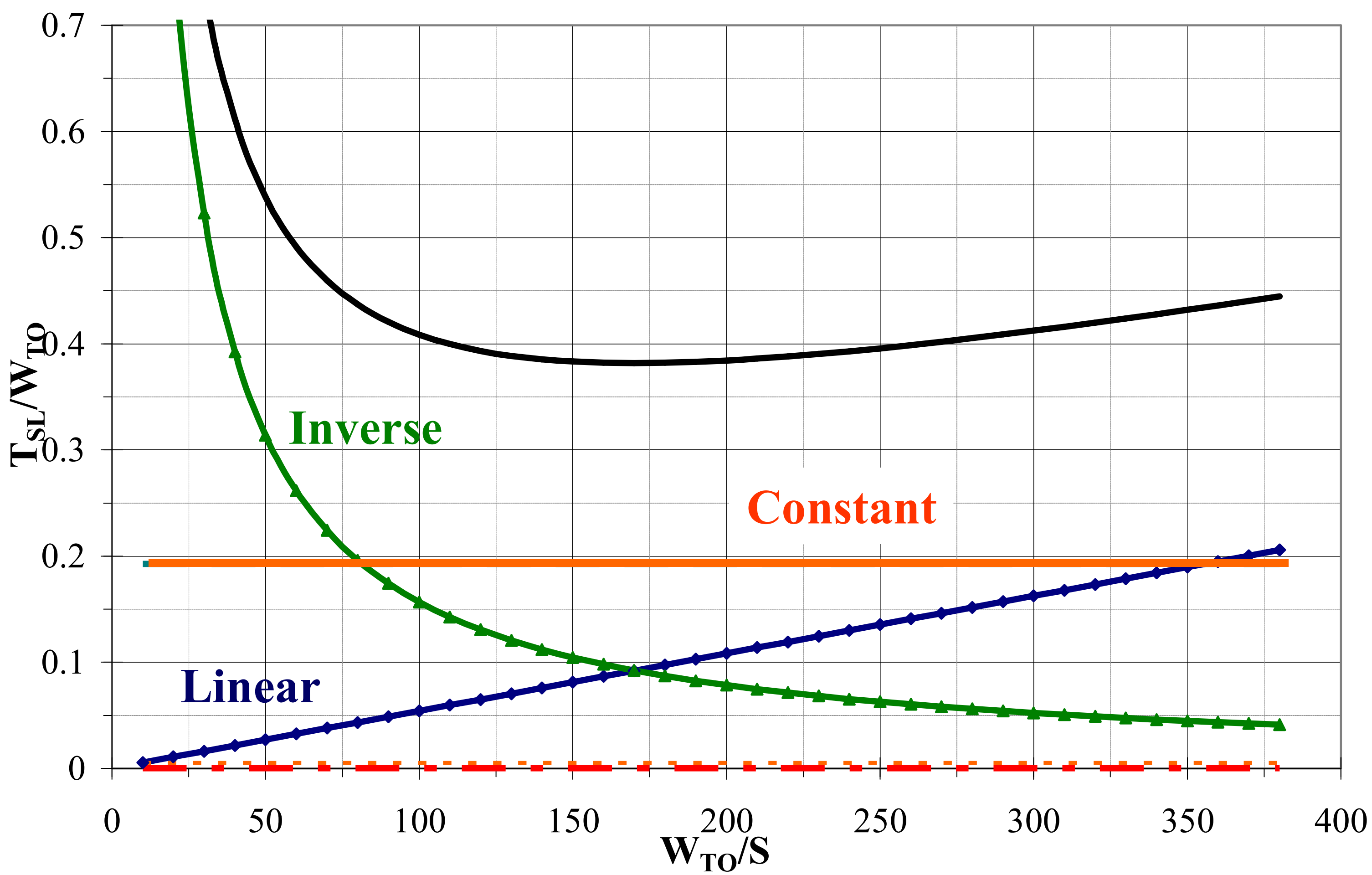
Plot for Case 4



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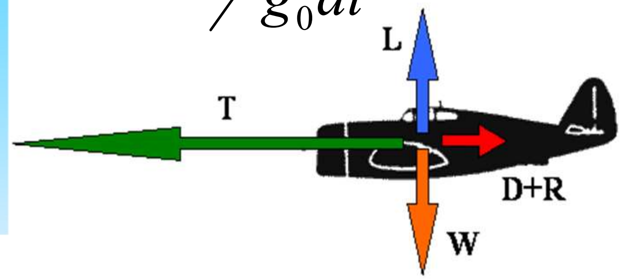
Case 5: Takeoff Ground Roll

Assumptions:

$dh/dt=0$	Constant Altitude
$R \neq 0$	Not clean configuration + Ground
s_G, □ρ, C_{Lmax}, k_{TO}	Given
$\boxed{TSL >> (D+R)}$	$\boxed{\text{Thrust is very large}}$

$$P_s = {VdV}\big/{g_0 dt}$$



$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\cancel{\left[K_1\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \frac{R}{qS}\right]} + \frac{1}{V}\frac{d}{dt}\left(\cancel{h} + \frac{V^2}{2g_o}\right)\right\}$$

By assuming that thrust at take off is much larger than the drag the drag-related terms can be eliminated.

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Case 5: Takeoff Ground Roll

After eliminating the drag related terms and rearranging the acceleration term, the energy equations is reduced to:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha g_o}\frac{dV}{dt} = \frac{\beta}{\alpha g_o}\frac{dV}{ds/V}$$

Note that the definition of velocity is used to eliminate the ***dt*** term

Solving for ***ds*** yields:

$$ds = \frac{\beta}{\alpha g_o}\left(\frac{W_{TO}}{T_{SL}}\right)V\,dV$$

Case 5: Takeoff Ground Roll

This expression can then be integrated over s=0, V=0 to takeoff ($s=s_G$, $V=V_{TO}$):

$$s_G = \frac{\beta}{\alpha}\left(\frac{W_{TO}}{T_{SL}}\right)\frac{V_{TO}^2}{2g_o}$$

What values for α and β should be used in this case?

Take off Velocity is defined through the stall velocity and a safety constant $k_{TO} > 1.0$ (usually 1.2 – 1.3)

$$V_{TO} = k_{TO} V_{STALL} \qquad (20)$$

Case 5: Takeoff Ground Roll

The take off velocity (and stall velocity) are used to incorporate aerodynamic information in the equation for Case 5. To achieve this it is possible to use the definition of lift and C_L for maximum conditions (which are representative of take off conditions):

$$qC_{L\max}S = \tfrac{1}{2}\rho V_{STALL}^2 C_{L\max}S = \beta W_{TO}$$

$$\frac{V_{TO}^2}{2} = k_{TO}^2\frac{V_{STALL}^2}{2} = \frac{\beta k_{TO}^2}{\rho C_{L\max}}\left(\frac{W_{TO}}{S}\right) \qquad (21)$$

Case 5: Takeoff Ground Roll

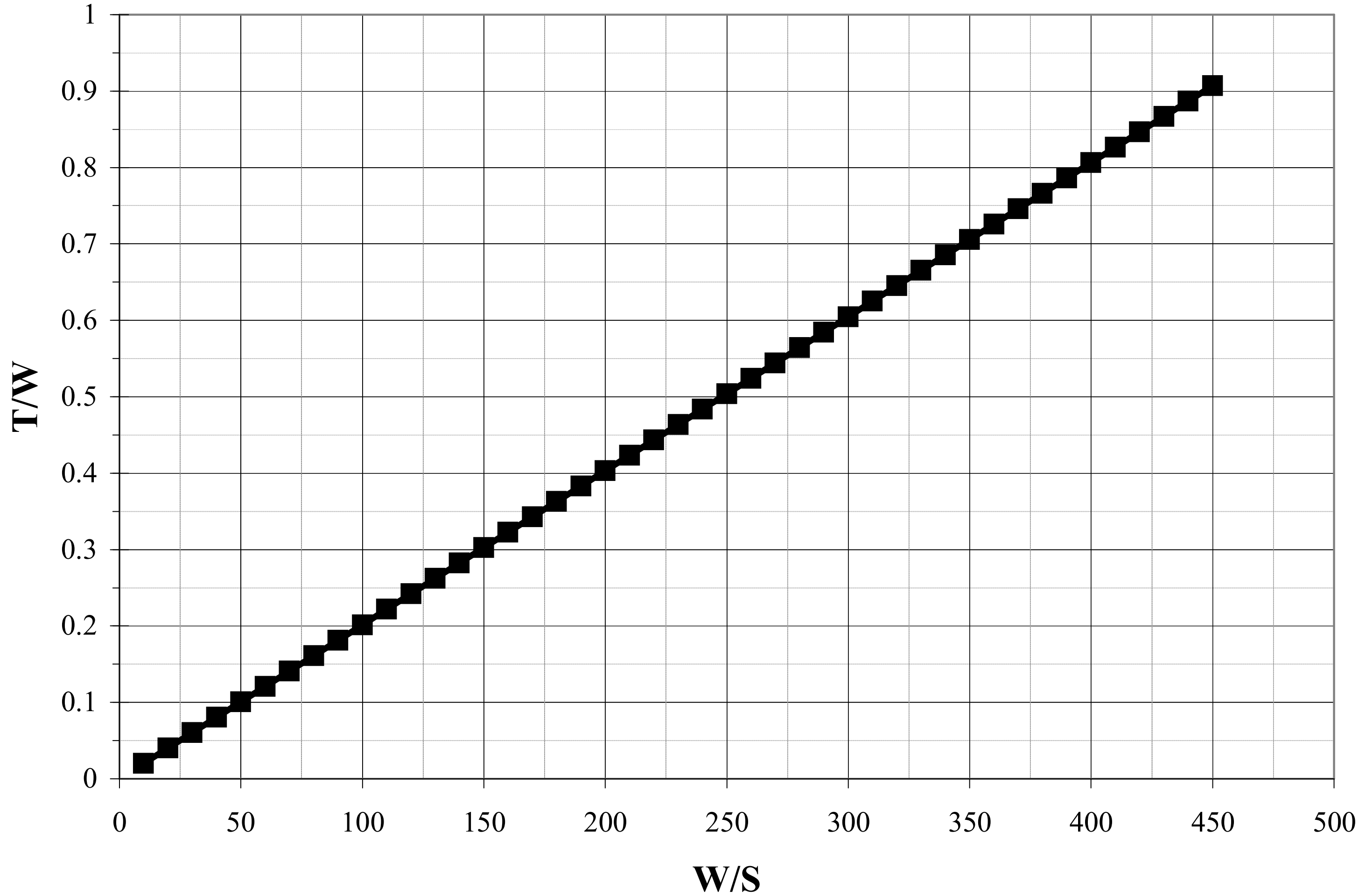
Replacing the expression for take off velocity (21) in the energy equation for Case 5 yields the final expression :

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta^2}{\alpha} \frac{k_{TO}^2}{s_G \rho g_o C_{L\max}} \left(\frac{W_{TO}}{S} \right) \qquad (22)$$

…in this limiting case: $\dfrac{T_{SL}}{W_{TO}}$ is proportional to: $\dfrac{W_{TO}}{S}$

and inversely proportional to: s_G

Case 5: Takeoff Ground Roll



Using the Master Equation

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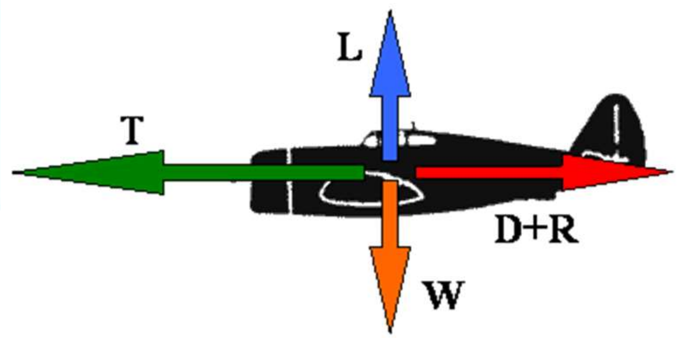
Case 6: Takeoff Ground Roll

$$P_s = \frac{VdV}{g_0 dt}$$

Assumptions:

$dh/dt=0$	Constant Altitude
$n \approx 1$	Lift approximately equals Weight
s_G, □ρ, C_{Lmax}, k_{TO}	Given
$D=qC_DS$	Equation (8)



The starting point for Case 6 is the Equation (5) for which the drag terms have not yet been decomposed into their components:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\left(\frac{D+R}{\beta W_{TO}}\right) + \frac{1}{V}\frac{d}{dt}\left(\cancel{h} + \frac{V^2}{2g_o}\right)\right\}$$

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Case 6: Takeoff Ground Roll

In this case for ground roll it is assumed that thrust is in the order of magnitude of the drag and resistance forces, which can no longer be ignored. The following is the expression for the resistance due to ground friction and non-clean configuration:

$$R = qC_{DR}S + \mu_{TO}\left(\beta W_{TO} - qC_L S\right)$$

The complete Drag term in Equation (5) can be re-written as follows:

$$\frac{D+R}{\beta W_{TO}} = \frac{\left(C_D + C_{DR} - \mu_{TO}C_L\right)qS + \mu_{TO}\beta W_{TO}}{\beta W_{TO}}$$

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Case 6: Takeoff Ground Roll

Rearranging. Eq. (5) becomes:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\xi_{TO}\frac{q}{\beta}\left(\frac{S}{W_{TO}}\right) + \mu_{TO} + \frac{1}{g_o}\frac{dV}{dt}\right\} \qquad (23)$$

Where:

$$\xi_{TO} = (C_D + C_{DR} - \mu_{TO} C_L) \qquad (24)$$

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Case 6: Takeoff Ground Roll

The term for ***dt*** is replaced with ***ds/V*** like in Case 5. Rearranging and integrating over s=0, V=0 to takeoff ($s=s_G$, $V=V_{TO}$). The final expression of the integral is as follows:

$$s_G = -\frac{\beta(W_{TO}/S)}{\rho g_o \xi_{TO}} \ln\left\{1 - \frac{\xi_{TO}}{\left[\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) - \mu_{TO}\right]\frac{C_{L\max}}{k_{TO}^2}}\right\}$$

Representative takeoff values for α and β must used.

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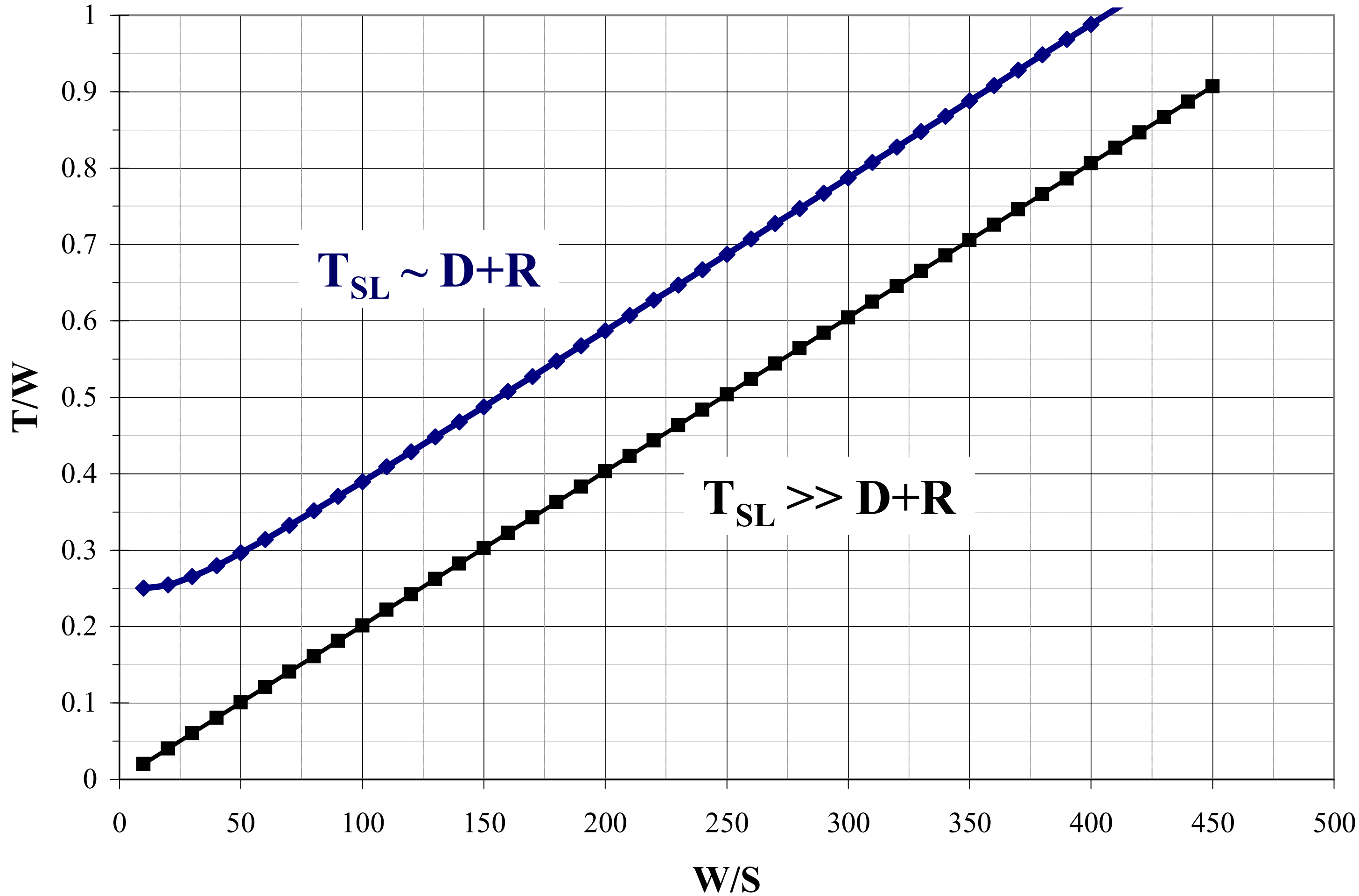
Plot for Case 6



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Case 6: Limiting Condition

It is assumed that T_{SL}/W_{TO} increases continuously with W_{TO}/S. In the limit as all terms in ξ_{TO} go to zero ln(1-ε) goes to – ε and the expression for s_G tends to:

$$s_G \longrightarrow \frac{\beta(W_{TO}/S)}{\rho g_o \xi_{TO}} \frac{\xi_{TO}}{\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right)\frac{C_{L\max}}{k_{TO}^2}}$$

Solving for T_{SL}/W_{TO} tends to the following equation, which is that found for the Case 5 where $T_{SL} >> D+R$

$$\frac{T_{SL}}{W_{TO}} \longrightarrow \frac{\beta^2}{\alpha} \frac{k_{TO}^2}{s_G \rho g_o C_{L\max}} \left(\frac{W_{TO}}{S}\right) \qquad (22)$$

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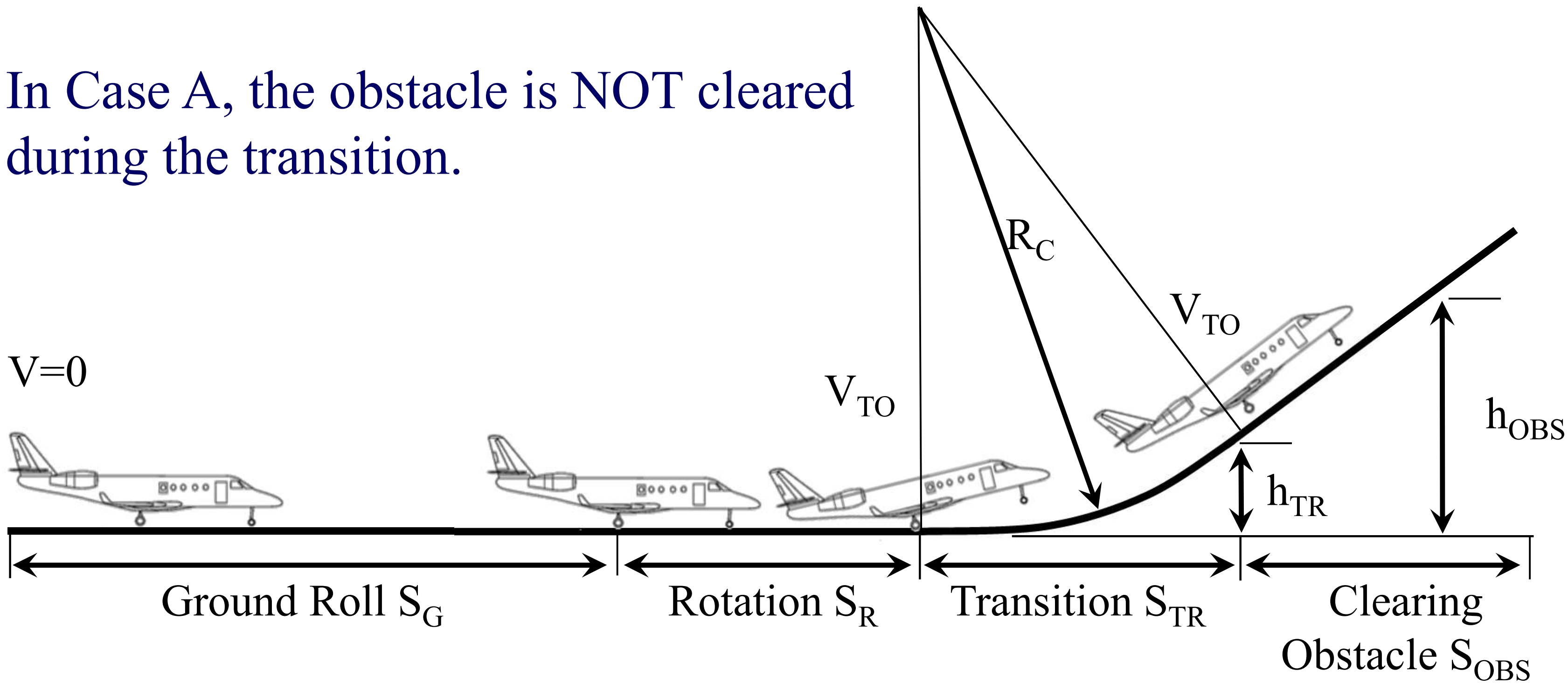
Calculating Takeoff Field Length (A)

In Case A, the obstacle is NOT cleared during the transition.



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Rotation Distance

Distance to rotate is calculated as the product of rotation time and velocity during rotation. This velocity is simply take off velocity:

$$s_R = t_R V_{TO}$$

t_R is roughly 3 seconds for modern aircraft – varies with rotational inertia and magnitude of control moments. The take off velocity is expressed, as shown previously, in terms of the definition of lift:

$$L_{TO} = \frac{1}{2}\rho V_{TO}{}^2 S C_{L-TO} = W \qquad V_{TO} = \sqrt{\left(\frac{W_{TO}}{S}\right)\frac{2\beta}{\rho C_{L\max}}}$$

$$s_R = t_R V_{TO} = t_R k_{TO} \sqrt{\{2\beta/(\rho C_{L\max})\}(W_{TO}/S)} \tag{26}$$

Transition Distance

Transition distance is defined as that necessary to bring the aircraft to its climb angle. This distance is the horizontal component of the arc defined by the trajectory of the aircraft during transition. Using geometry the transition distance is defined as:

$$s_{TR} = R_C \sin\theta_{CL}$$

The radius of the transition arch can be expressed in terms of take off velocity, which in turn can be used to incorporate the terms in the definition of lift as shown previously.

$$s_{TR} = \frac{V_{TO}^{\;2}\sin\theta_{CL}}{g_0\left(0.8k_{TO}^{\;2}-1\right)} = \frac{k_{TO}^{\;2}\sin\theta_{CL}}{g_0\left(0.8k_{TO}^{\;2}-1\right)}\frac{2\beta}{\rho C_{L\max}}\frac{W_{TO}}{S} \qquad (27)$$

Distance to clear obstacle

Outside the turn to climb, the flight path is a straight line at an angle to the ground defined by the vehicle climb angle θ_{CL}. The distance to clear the obstacle results from simple geometry:

$$s_{obs} = \frac{h_{obs} - h_{TR}}{\tan \theta_{CL}} \tag{28}$$

Climb angle is defined by climb rate and velocity (excess power)

$$\frac{1}{V}\frac{dh}{dt} = \sin \theta_{CL} = \frac{T - D}{W}$$

Altitude at the end of transition is found with the following expression:

$$h_{TR} = \frac{V_{TO}^{\ 2}(1 - \cos \theta_{CL})}{g_0(0.8k_{TO}^{\ 2} - 1)} = \frac{k_{TO}^{\ 2}(1 - \cos \theta_{CL})}{g_0(0.8k_{TO}^{\ 2} - 1)} \frac{2\beta}{\rho C_{L\max}} \frac{W_{TO}}{S} \tag{29}$$

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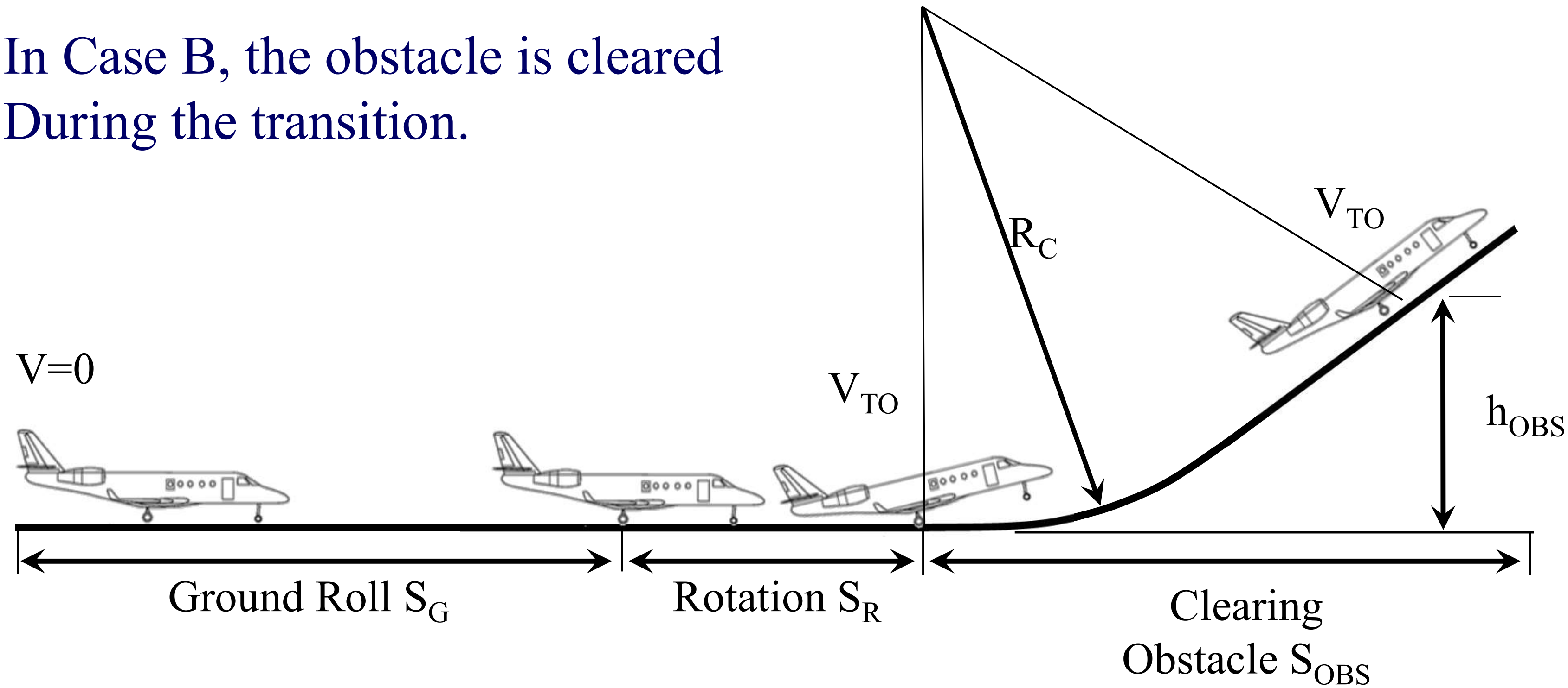
Calculating Take Off Field Length (B)

In Case B, the obstacle is cleared
During the transition.



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Takeoff Ground Roll (B)

The distance s_G and s_R are the same as in case A, but the obstacle is cleared during transition so:

$$s_{TO} = s_G + s_R + s_{obs}$$

where s_{obs} is the distance from the end of rotation to the point where the height h_{obs} is attained. From geometry (as with the previous case), the expression for s_{obs} is:

$$s_{obs} = R_C \sin\theta_{obs} = \frac{V_{TO}^{2} \sin\theta_{obs}}{g_0\left(0.8k_{TO}^{2} - 1\right)} \qquad (30)$$

where $$\theta_{obs} = \cos^{-1}\left(1 - \frac{h_{obs}}{R_C}\right)$$

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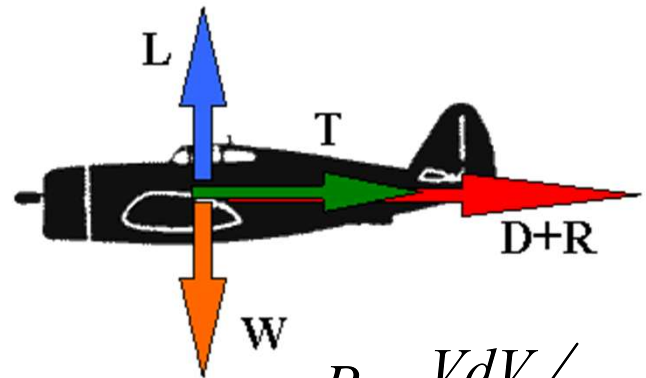
Case 7: Braking Roll

Assumptions:	
$dh/dt=0$	Constant Altitude
$n \approx 1$	Lift approximately equals Weight
s_B, □ρ, C_{Lmax}, k_{app}	Given
$D=qC_DS$	Equation (8)



$$P_s = VdV \big/ g_0 dt$$

The starting point for Case 6 is Equation (5) for which the drag terms have not yet been decomposed into their components:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\left(\frac{D+R}{\beta W_{TO}}\right) + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)\right\}$$

ξ_L is defined in Equation 2.32

Case 7: Braking Roll

In this case for braking roll it is necessary to assume that drag and resistance forces beyond those of the clean configuration are present. The following is the expression for the resistance due to ground friction/braking and aerodynamic breaking:

$$R = qC_{DR}S + \mu_B\left(\beta W_{TO} - qC_L S\right) \tag{31}$$

The complete Drag term in Equation (5) can be re-written as follows:

$$\frac{D+R}{\beta W_{TO}} = \frac{\left(C_D + C_{DR} + \mu_\beta C_L\right)qS + \mu_B \beta W_{TO}}{\beta W_{TO}}$$

Case 7: Braking Roll

Rearranging. Eq. (5) becomes:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\xi_L \frac{q}{\beta}\left(\frac{S}{W_{TO}}\right) + \mu_B + \frac{1}{g_o}\frac{dV}{dt}\right\}$$

Where:

$$\xi_L = (C_D + C_{DR} - \mu_B C_L) \qquad (32)$$

Case 7: Braking Roll

The term for ***dt*** is replaced with ***ds/V*** like in Case 5. Rearranging and integrating over s=0, V=V_{TD} to full stop conditions s=s_B, V=0. The final expression of the integral is as follows:

$$S_B = \frac{\beta}{\rho g\left(C_D + C_{DR} - \mu_B C_L\right)}\left(\frac{W_{TO}}{S}\right)\ln\left\{1 + \frac{\left(C_D + C_{DR} - \mu_B C_L\right)}{\left[\frac{(-\alpha)}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) + \mu_B\right]\frac{C_{L\max}}{k_{TD}^2}}\right\} \qquad (33)$$

Representative takeoff values for α and β must used. In this case β will be the value at the end of the mission. Notice that a negative sign is used to model thrust reversers. This value of α should be representative of the fraction of the thrust that is effective reversed, such as 0.65 – 0.85 for commercial airliners.

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Braking Roll Limiting Case

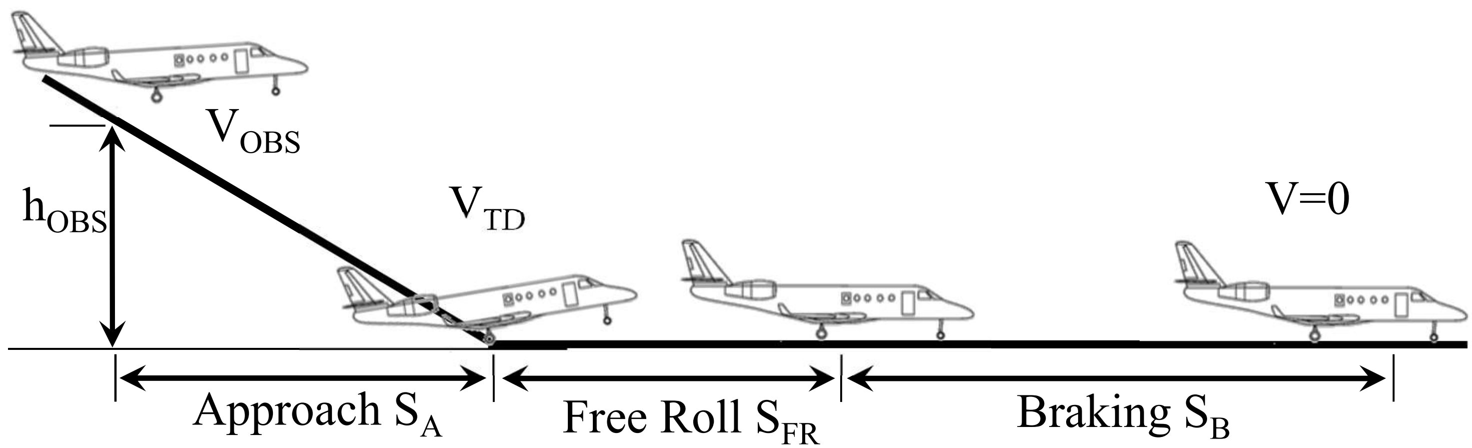
If the (-α) term is given a large enough value the expression for braking roll will tend to the following:

$$s_B \rightarrow \left(\frac{\beta (W_{TO} / S)}{\rho g_0 \xi_L} \right) \frac{\xi_L}{\dfrac{(-\alpha) T_{SL}}{\beta W_{TO}} \dfrac{C_{L\max}}{k_{TD}^2}}$$

In this case solving for T_{SL}/W_{TO} results in the following linear relationship:

$$\frac{T_{SL}}{W_{TO}} \rightarrow \left(\frac{\beta^2}{-\alpha} \right) \left(\frac{k_{TD}^2}{s_B \rho g_0 C_{L\max}} \right) \frac{W_{TO}}{S} \qquad (34)$$

Total Landing Distance



The total landing distance is calculated as the sum of the approach distance, the free roll distance and the braking roll distance. Notice that the braking roll distance is a function of k_{TD}, which is defined from the preceding segment.

Approach Distance

The approach distance is given by the following expression, which includes the landing clearance obstacle height h_{obs}, safety factor for approach k_{obs}, and the safety factor for touch down k_{TD}.

$$s_A = \frac{2\beta}{\rho g_0 (C_D + C_{DR})} \left(\frac{W_{TO}}{S} \right) \left(\frac{k_{obs}^2 - k_{TD}^2}{k_{obs}^2 + k_{TD}^2} \right) + \frac{C_{L\max}}{(C_D + C_{DR})} \frac{2h_{obs}}{(k_{obs}^2 + k_{TD}^2)} \quad (35)$$

The safety factor for approach k_{obs} defines the velocity when clearing the landing obstacle height V_{obs} in terms of the stall velocity V_{STALL}. V_{obs} is equal to the approach velocity V_{app} are used interchangeably, as k_{obs} and k_{app} are.

$$V_{obs} = k_{obs} V_{STALL} = V_{app} = k_{app} V_{STALL} \quad (36)$$

Remember that V_{STALL} can be expressed as: $V_{app} = k_{app} V_{stall} = k_{app} \sqrt{\frac{2\beta}{\rho C_{L\max}} \left(\frac{W_{TO}}{S} \right)}$

Free Roll Distance

The difference between approach and touchdown velocity is very small. However if a distinction is to be made the touchdown velocity can be expressed in terms of the stall velocity and k_{TD}. The definition of lift can be used to expand the expression.

$$V_{TD} = k_{TD} V_{stall} = k_{TD} \sqrt{\frac{2\beta}{\rho C_{L\max}} \left(\frac{W_{TO}}{S} \right)}$$

It may be assumed that the velocity at touchdown does not diminish during the free roll although some small reduction occurs due to friction. With this assumption the free roll distance is calculated as a product of the free roll time (3 s. approximately) and the touch down velocity.

$$s_{FR} = t_{FR} V_{TD} = t_{FR} k_{TD} \sqrt{\frac{2\beta}{\rho C_{L\max}} \left(\frac{W_{TO}}{S} \right)} \qquad (37)$$

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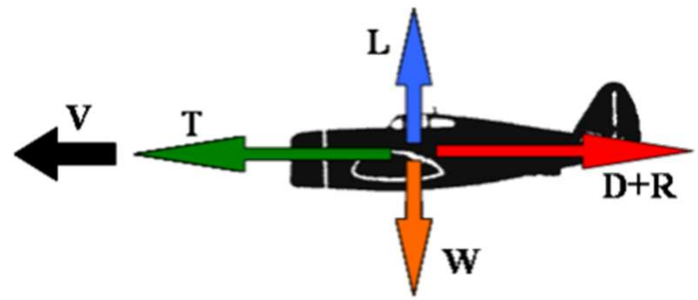
- Case 8: Service Ceiling

Case 8: Service Ceiling (P_s=dh/dt)

Assumptions:	
dV/dt = 0	No Acceleration
N=1	Straight and Level (L=W)
R=0	Clean configuration
dh/dt > 0	Value is given
h (i.e., σ) and CL	Given



Service ceiling is the altitude at which an aircraft's maximum climb rate has a specific value. This value represents a practical upper limit for steady flight, and is often set to 100 [ft/min] but may be set at other values if necessary.

Case 8: Service Ceiling

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ \frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{1\beta}{q}\frac{W_{TO}}{S}\right) + C_{D_o} + \cancel{\frac{R}{qS}} \right] + \frac{1}{V}\frac{d}{dt}\left(h + \cancel{\frac{V^2}{2g_o}} \right) \right\}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ K_1\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{V}\frac{dh}{dt} \right\} \qquad (38)$$

Notice that this form of the energy equation is not different from that used for the constant speed climb (Case 2).

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Case 8: Service Ceiling

In an RFP the operating conditions for service ceiling may be determined by lift coefficient or by velocity for a given altitude and climb rate. If V is known then calculate C_L as follows:

$$C_L = \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) = \frac{\beta}{\frac{1}{2}\rho V^2}\left(\frac{W_{TO}}{S}\right) \tag{39}$$

Otherwise if the velocity at the service ceiling is unknown but the CL is specified, the velocity can be found as follows:

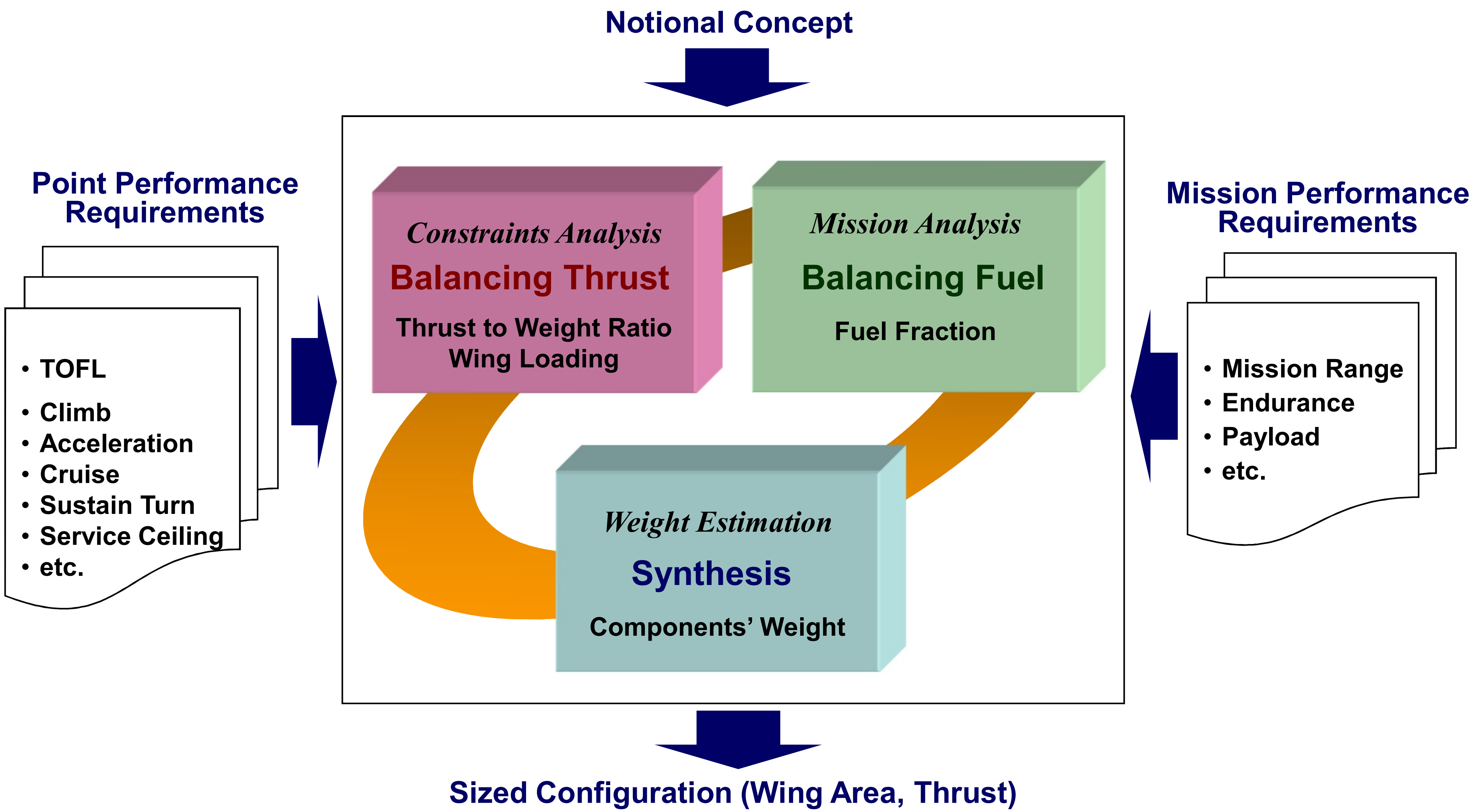
$$V = \sqrt{\frac{2\beta}{\sigma\rho_{SL}C_L}\left(\frac{W_{TO}}{S}\right)} \tag{40}$$

Case 8: Service Ceiling

When V and C_L are known the equation can be reduced to the following form:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 C_L + K_2 + \frac{C_{D_0}}{C_L} + \frac{1}{V}\frac{dh}{dt}\right\} \quad (41)$$

Aircraft Sizing



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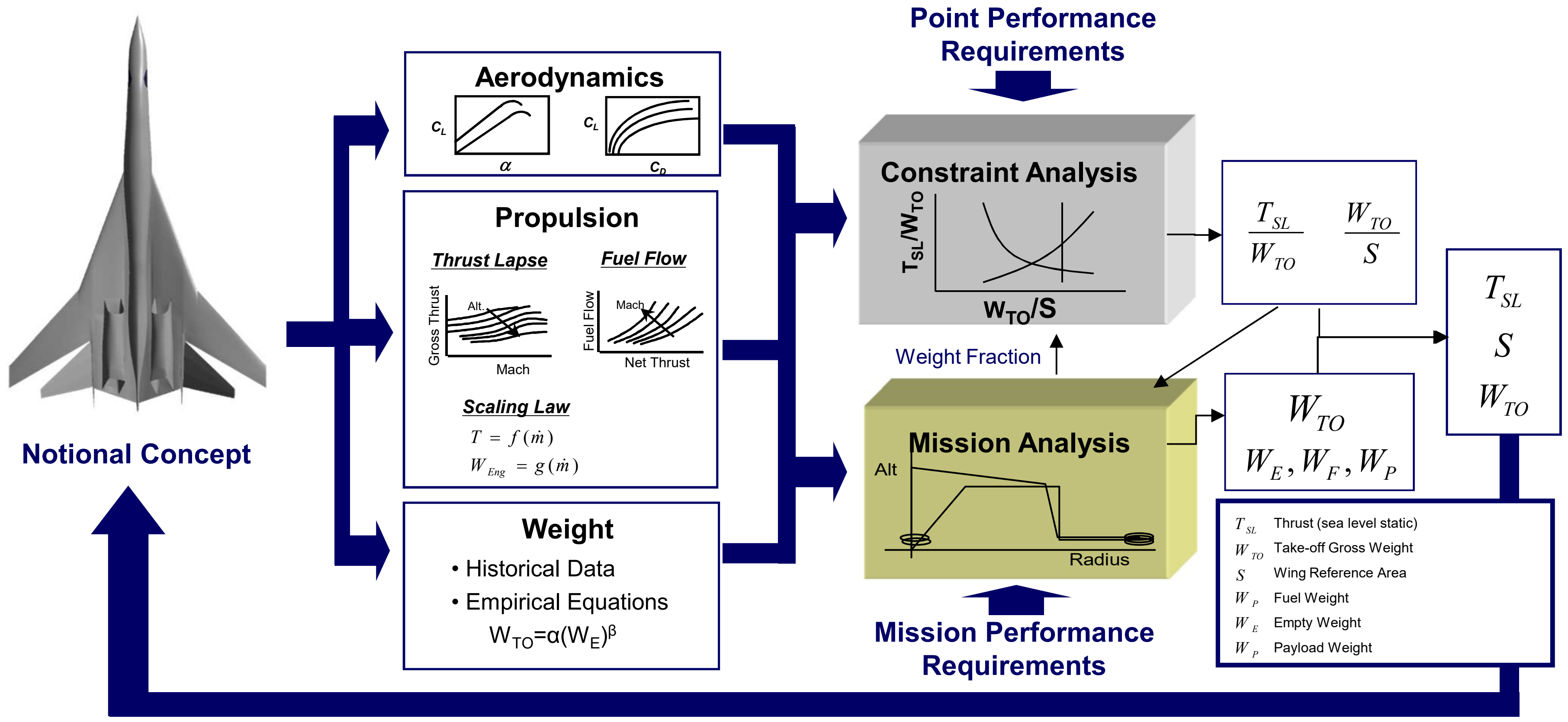
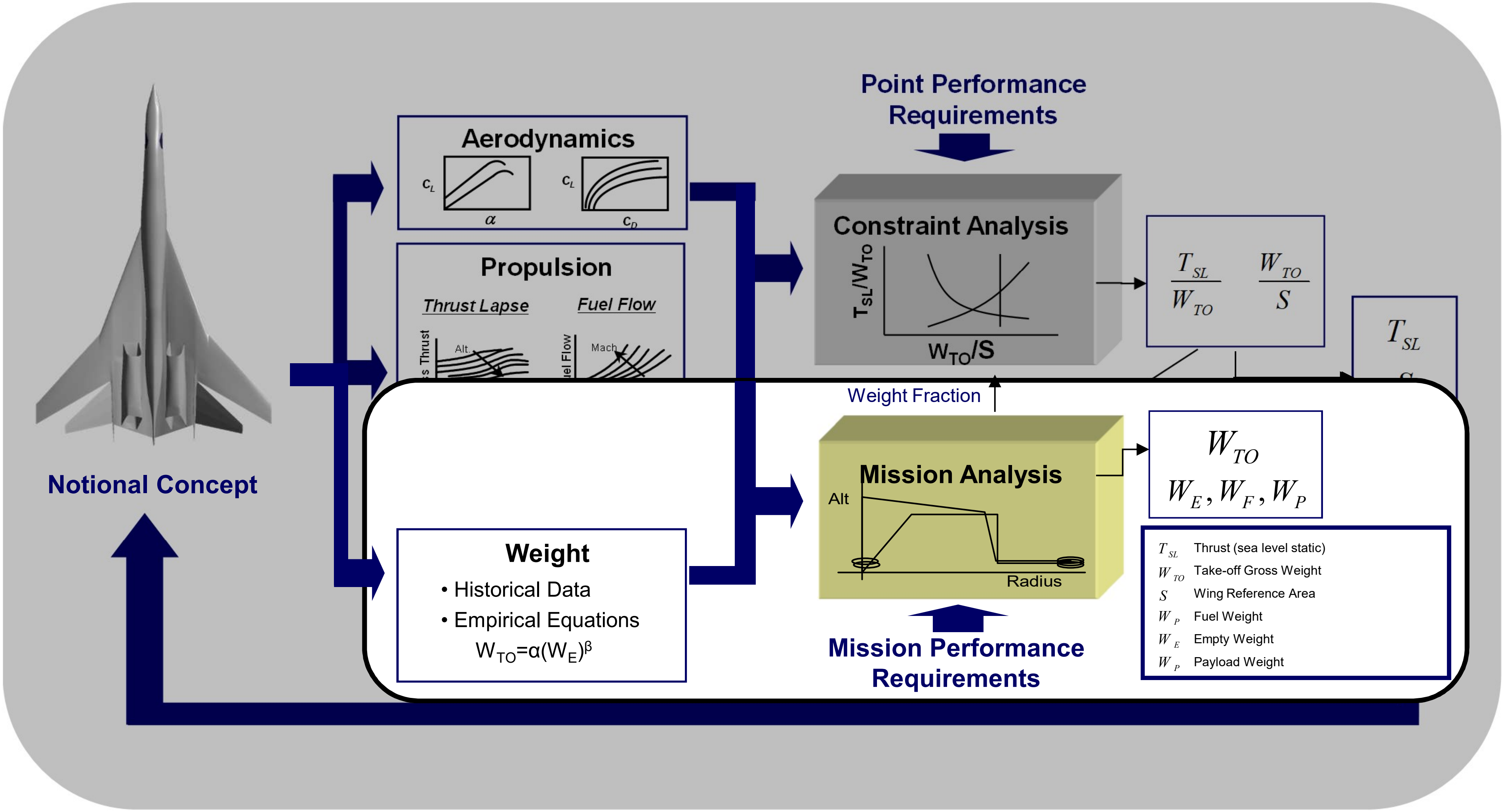
Traditional Aircraft Sizing Process



Traditional Aircraft Sizing Process



Traditional Mission Analysis

Weight Decomposition

$$W_{TO} = W_E + W_P + W_F$$

A/C weight change equals fuel weight change.

$$\frac{dW}{dt} = \frac{dW_F}{dt}$$

Fuel weight change (fuel flow) can be expressed as a function of thrust required.

$$\frac{dW}{dt} = \frac{dW_F}{dt} = -TSFC \times T$$

$TSFC$	Thrust Specific Fuel Consumption
T	Thrust
s	distance
V	Velocity
D	Basic configuration Drag
R	Additional drag to *D*
β	Weight fraction
h	Altitude
g_o	Gravity constant

A/C weight change (fuel flow) can be expressed as a function of thrust required.

$$\frac{dW}{W} = -TSFC\,\frac{T}{W}\,dt = -TSFC\,\frac{T}{W}\frac{ds}{V}$$

In addition, thrust-to-weight Ratio can be related to aerodynamic performance.

Where, $$\frac{T}{W} = \left(\frac{D+R}{\beta W_{TO}}\right) + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)$$

Breguet Range Equation:

$$\text{Range} = \int ds = \frac{V}{TSFC}\frac{L}{D}\ln\left(\frac{W_{final}}{W_{initial}}\right)$$

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Mission Analysis

- With initial values of thrust loading (T_{SL}/W_{TO}) and wing loading (W_{TO}/S) determined, now the focus shifts to the estimation of the gross takeoff weight W_{TO}, which is the sum of the component weights:

$$W_{TO} = W_C + W_P + W_E + W_F$$

- Where:
  - W_C is the crew weight,
  - W_P is the payload weight,
  - W_E is the empty weight, and
  - W_F is the fuel weight.

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W_{TO} Estimation

Using this weight breakdown for the crew, payload, fuel and empty weights components, the take-off gross weight can be expressed with fuel and empty weights as weight fractions:

$$W_{TO} = W_C + W_P + \frac{W_F}{W_{TO}} W_{TO} + \frac{W_E}{W_{TO}} W_{TO}$$

Solving for the take-off gross weight in terms of the crew and payload weights (constants), the fuel weight ratio (function of mission, aerodynamics and fuel consumption) and empty weight fraction (function of take-off gross weight from empirical data), provides:

$$W_{TO} = \frac{W_C + W_P}{1 - {W_F}/{W_{TO}} - {W_E}/{W_{TO}}}$$

Solving for the initial estimate of W_{TO} can be accomplished if estimates for the terms on the right hand side are found.

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Crew and Payload Weight

- It is assumed that the crew and payload weights as well as the mission are known (i.e. from the RFP)
- Payload weight may comprise of:
  - passengers
  - baggage
  - cargo
  - military loads (bombs, ammunition, etc)
- Rules of Thumb
  - For passenger aircraft: 175 lbs per person
    - 30 lbs luggage per person, short to medium flights
    - 40 lbs luggage per person, long flights

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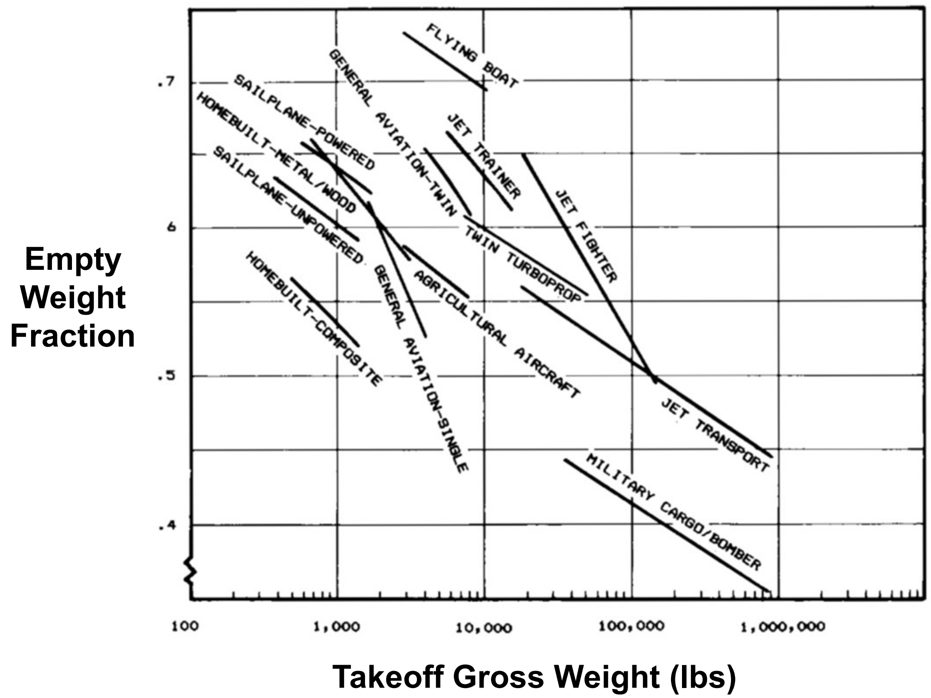
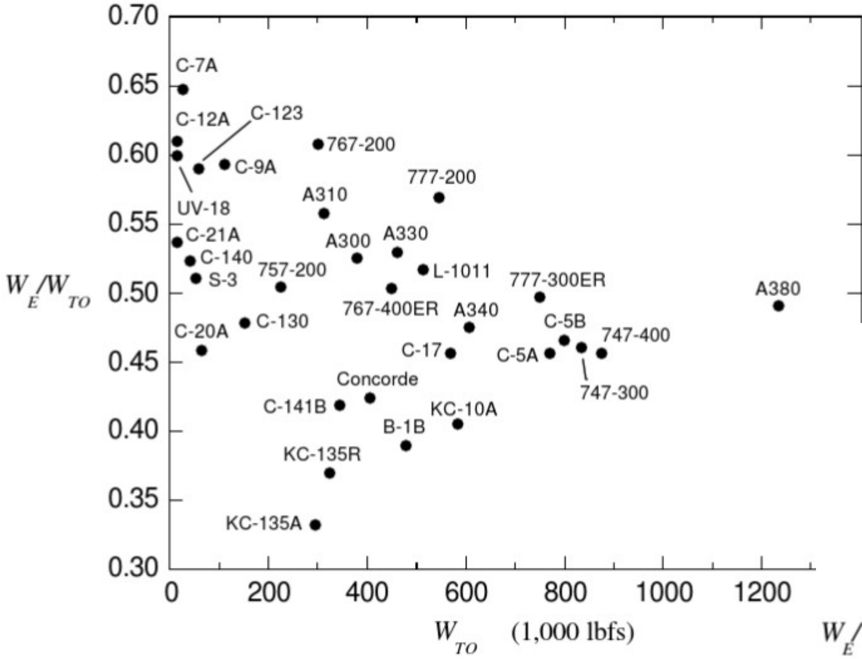
Empty Weight Fraction

- W_E consists of the basic aircraft structure and any permanently attached equipment
- There are strong empirical correlations between empty weight fraction (W_E/W_{TO}) and takeoff gross weight (W_{TO})
- Differences between aircraft type are due to different loading limits, structural design philosophies, etc.

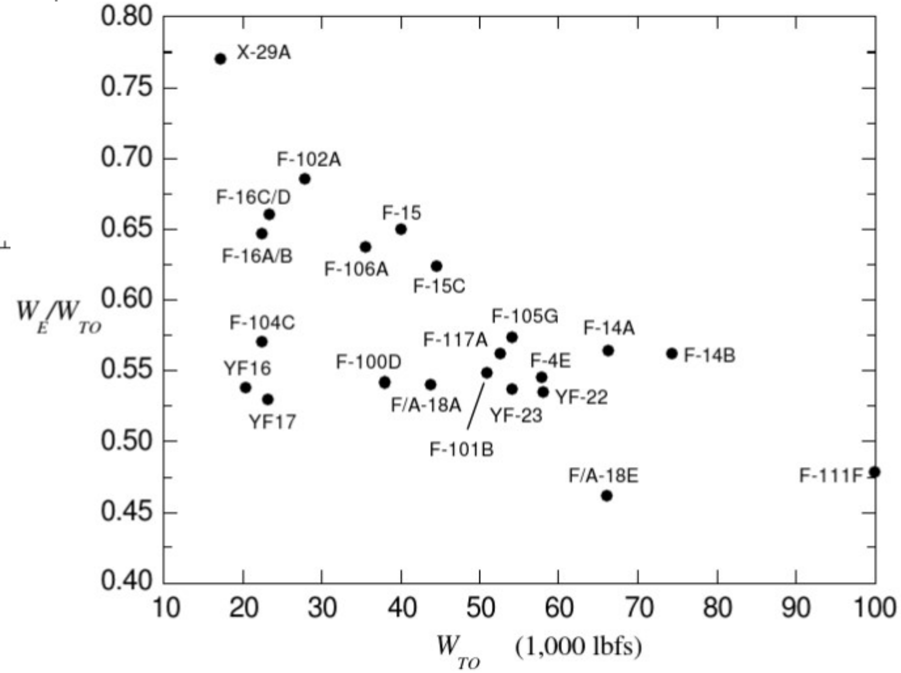


Raymer, D.P., *Aircraft Design: A Conceptual Approach*, AIAA, Washington, 1989.

Empty Weight Fraction



Weight Fractions of Cargo and Passenger Aircrafts



Weight Fractions of Fighter Aircraft

Mattingly, J., D., Heiser, W., H., Pratt, D., T., *Aircraft Engine Design*, AIAA, Virginia, 2002.

Empty Weight Fraction

- For initial sizing estimates, the empty weight fraction $\Gamma = W_E/W_{TO}$ may be approximated using an exponential regression of historical data.
- Based on the correlations of various authors, Γ can be approximated for the following classes of aircraft:
  - Cargo aircraft: $\Gamma = 1.26 W_{TO}^{-0.08}$
  - Passenger aircraft: $\Gamma = 1.02 W_{TO}^{-0.06}$
  - Fighter aircraft: $\Gamma = 2.34 W_{TO}^{-0.13}$
  - Twin turboprop aircraft: $\Gamma = 0.96 W_{TO}^{-0.05}$

Fuel Weight

- W_F represents the required fuel weight to fly the mission
- W_F depends on the rate of fuel consumption, which depends on the thrust specific fuel consumption (TSFC)
- TSFC depends on the engine cycle, flight conditions (altitude, Mach #, etc…)
- Fuel consumption analysis results in multiple benefits:
  - Little information needed for calculation
  - Reveals optimum solution (minimum fuel) for flying certain segments of the mission
  - Shows the fuel consumed during each mission segment as a fraction of the aircraft weight at the beginning of the mission segment, which makes W_F a calculable fraction of W_{TO}

Fuel Weight Fractions

To find W_F/W_{TO} use fuel fraction method:

Break down mission into a number of mission phases and calculate fuel used in each phase or segment based on simple calculations or experience.

Fuel Fraction: for each phase the ratio of end weight to starting weight is defined

- Phase 1 - Engine Start and Warm Up: $\mathbf{W_1/W_{TO}}$
  Use Existing Data for Comparable Aircraft
- Phase 2 - Taxi: $\mathbf{W_2/W_1}$
  Use Existing Data for Comparable Aircraft
- Phase 3 - Takeoff: $\mathbf{W_3/W_2}$
  Use Existing Data for Comparable Aircraft
- Phase 4 - Climb / accelerate to start of cruise: $\mathbf{W_4/W_3}$
  Use Existing Data for Comparable Aircraft, or
  Use Breguet's Endurance Equation, or
  Numerically integrate over a climb profile

Breguet's Endurance Eqn.

$$E = \frac{1}{c_t}\frac{L}{D}\ln\frac{W_{final}}{W_{initial}} \qquad \text{(jet)}$$

$$E = \frac{550\eta_{pr}}{cV}\frac{L}{D}\ln\frac{W_{final}}{W_{initial}} \qquad \text{(prop)}$$

Generic Fuel Burn ODE

$$\frac{dW}{dt} = \frac{dW_F}{dt} = -TSFC \times T$$

Fuel Weight Fractions

- Phase 5 – Cruise: $\mathbf{W_5/W_4}$
  - Use Breguet's Range Equation, or
  - Numerically integrate the generic fuel burn ODE
- Phase 6 - Loiter: $\mathbf{W_6/W_5}$
  - Use Breguet's Endurance Equation, or
  - Numerically integrate the generic fuel burn ODE
- Phase 7 - Descent: $\mathbf{W_7/W_6}$
  - Use Existing Data for Comparable Aircraft
- Phase 8 - Landing, Taxi, and Shutdown: $\mathbf{W_8/W_7}$
  - Use Existing Data for Comparable Aircraft

Breguet's Range Eqn.

$$R = \frac{V}{c_t}\frac{L}{D}\ln\frac{W_{final}}{W_{initial}} \qquad \text{(jet)}$$

$$R = \frac{550\eta_{pr}}{c}\frac{L}{D}\ln\frac{W_{final}}{W_{initial}} \qquad \text{(prop)}$$

Generic Fuel Burn ODE

$$\frac{dW}{dt} = \frac{dW_F}{dt} = -TSFC \times T$$

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Mission Fuel Weight Fraction

Now calculate the total mission fuel fraction using the weight fractions for each phase or segment:

Final to Initial weight ratio:

$$\frac{W_1}{W_{TO}} \cdot \frac{W_2}{W_1} \cdot \frac{W_3}{W_2} \cdots \frac{W_n}{W_{n-1}} = \frac{W_n}{W_{TO}}$$

Mission Fuel Weight Fraction:

$$\frac{W_F}{W_{TO}} = 1 - \frac{W_n}{W_{TO}}$$

Trapped fuel considerations (e.g. 6%):

$$\frac{W_F}{W_{TO}} = 1.06\left(1 - \frac{W_n}{W_{TO}}\right)$$

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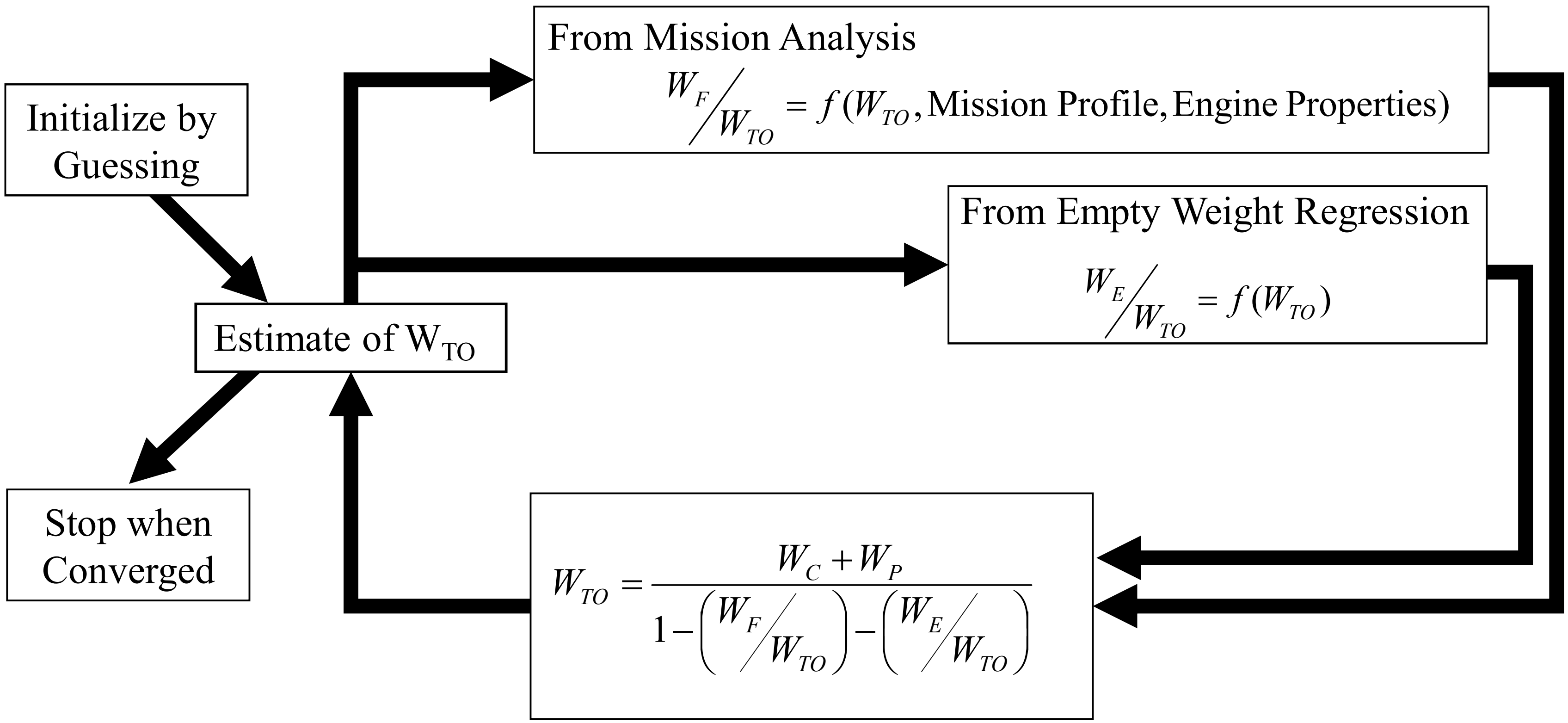
Weight Calculation

With these relationships and known values of payload and crew weight the takeoff gross weight can be calculated. With a good guess, the iteration should quickly converge on the minimum required gross takeoff weight.



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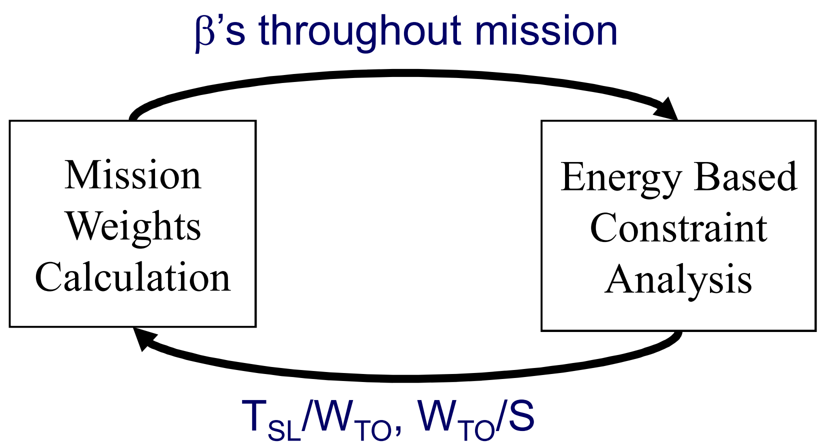
Weight Fractions (β)

- Instantaneous weight fraction, β, are also used for computing the constraints.
  - β is a fuel or payload correction for different points or segments along the mission
  - Initial values for β can be based on experience
  - Typical β values for different mission phases are shown on the mission profile figures on the following slides.
    - Typical β for Typical Fighter Aircraft
    - Typical β Cargo and Passenger Aircraft
  - The β value for each segment is essentially an intermediate weight fraction (i.e. $W_2 = \beta_2 W_{TO}$) thus:

$$\beta_n = \frac{W_1}{W_{TO}} \frac{W_2}{W_1} \frac{W_3}{W_2} \cdots \frac{W_n}{W_{n-1}}$$

Convergence of Weight Fractions (β)

- After the β values have been estimated or calculated, they must be reapplied to the constraint equations. With new βs, constraint analysis will yield new T_{SL}/W_{TO}, W_{TO}/S values
- With these new ratios, mission analysis is performed resulting in new βs.
- This process is repeated until a convergence on βs is reached.



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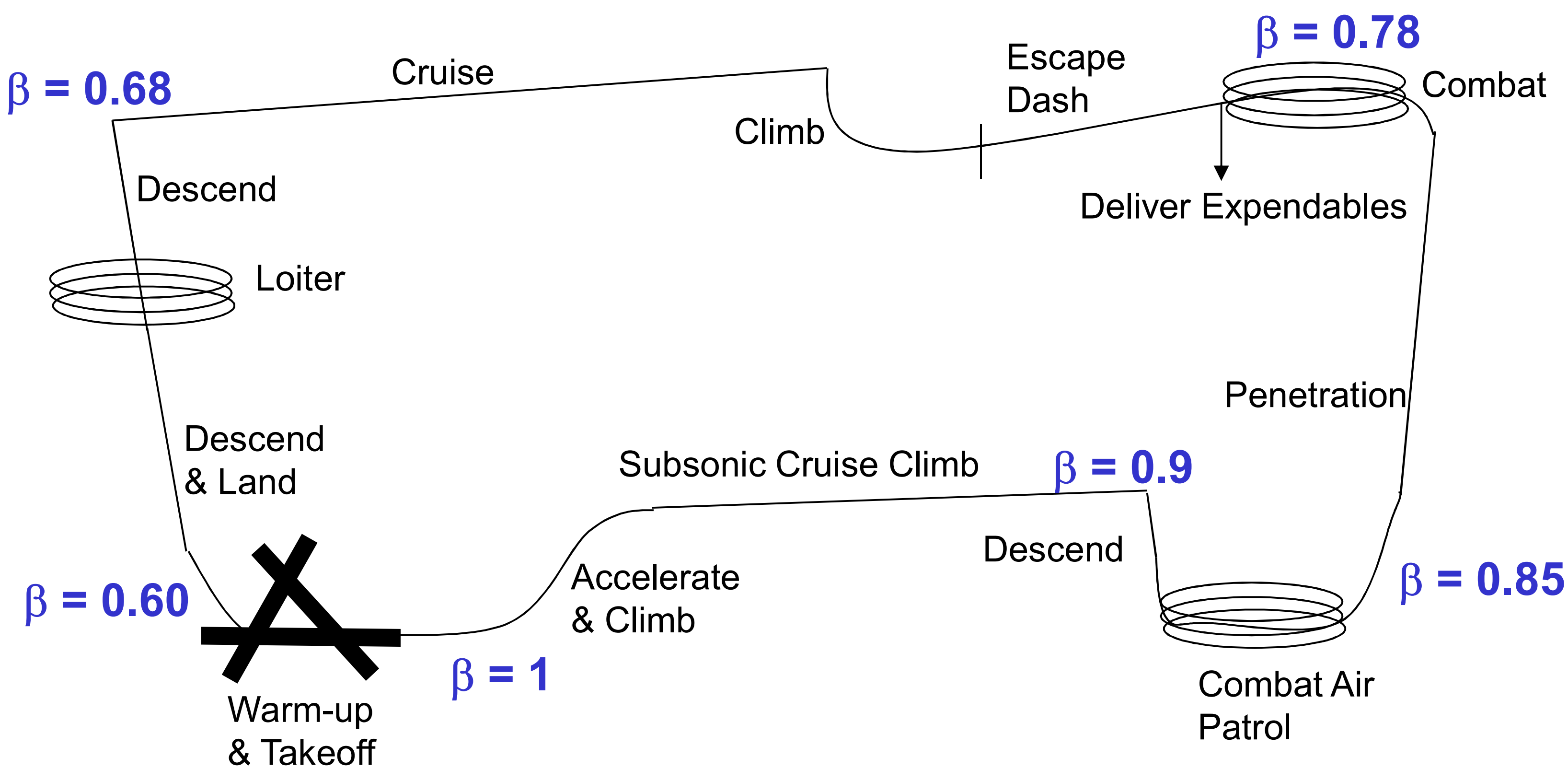
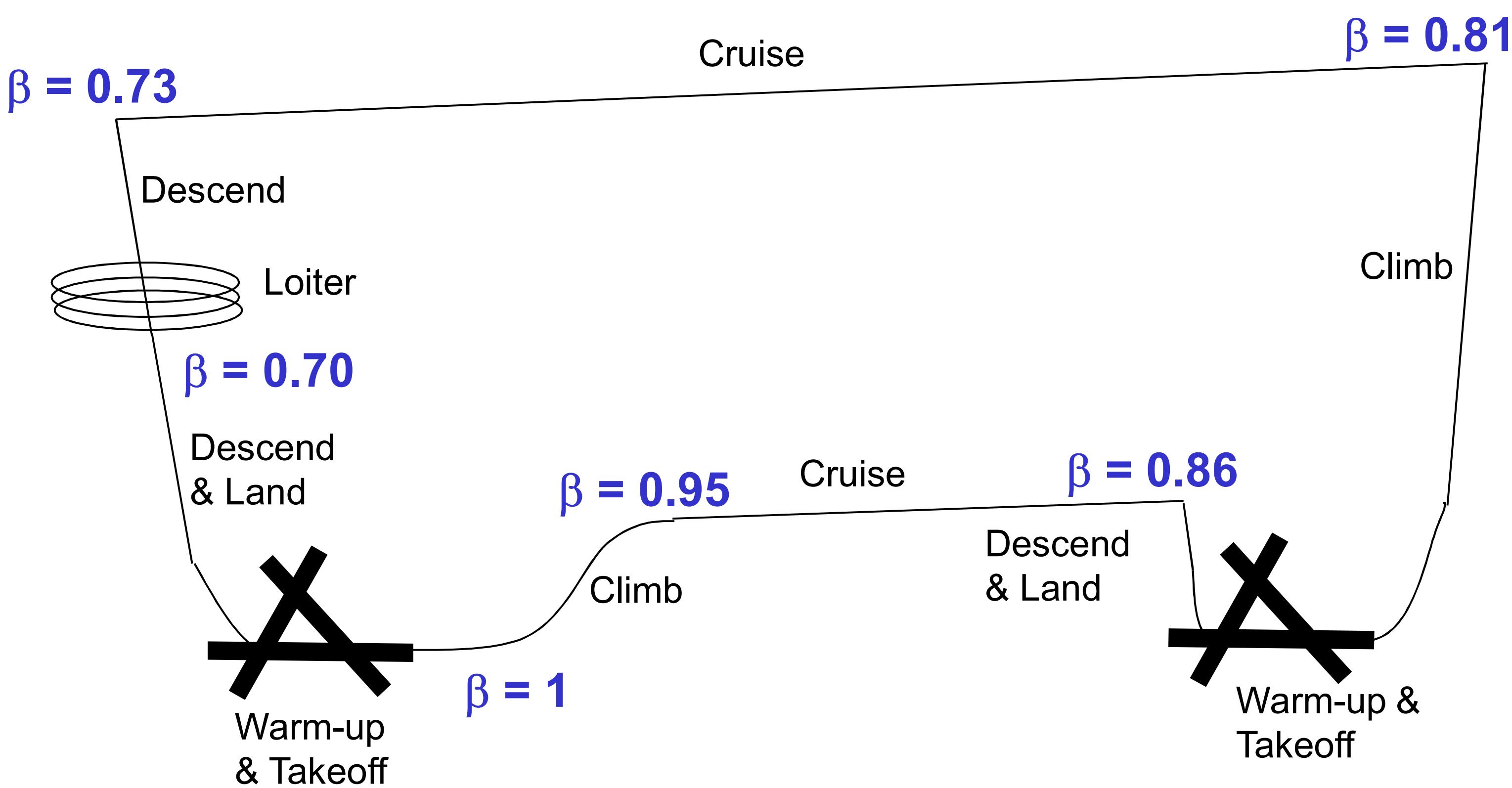
Weight Fraction (Fighter)

Instantaneous weight fraction, β, for a Typical Fighter Aircraft



Weight Fraction (Passenger)

Instantaneous weight fraction, β, for a Typical Cargo and Passenger Aircraft



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Top Level View of Sizing and Synthesis