Fall 2020 Take Home Exam 1

Due by NLT November 02, 23:55 EDT

AE 6343: Aircraft Design I
Fall 2020

Karl Roush

AE6343 Aircraft Design I
Fall 2020 Take Home Exam 1
October 16 - October 23, 2020

Time: Take Home
Number of questions: 3
Total number of points: 100

Exam Rules

- Read the Honor Code Statement, print your name, sign it (electronic signature is ok), and date it.
- Students have until **Monday, November 2nd,** to complete it and submit all relevant information via Canvas (no later than 11:55PM EDT).
- Using word or Latex, write a report with your answers to these questions. Make sure that your report follows standard formatting guidelines, with elements such as figure and table captions, page numbers, justified margins, equations numbers, etc. Then, convert this document into a PDF for the Canvas submission.
- The questions in this exam have multiple parts, each one with some guiding questions. Make sure to, at least, address all of these questions in your answer.
- Please use the following naming convention for your file:

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- Late submissions will be allowed with a 10% penalty on the final grade for each day of delay. No exceptions.
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Full Name: KARL ROUSH

Signature: Date: 10/23/2020
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1 Table of Contents

2 List of Figures

3 Problem 1

Looking only the approach and turn segments for constraint analysis. Consider the energy balance of a system which leads to the "master equation".

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{n\beta W_{TO}}{qS}\right)^2 + K_2\frac{n\beta W_{TO}}{qS} + C_{D0} + \frac{R}{qS}\right] + \frac{1}{v}\frac{d}{dt}\left(h + \frac{v^2}{2g_0}\right)\right\} \quad \textbf{(1)}$$

This master equation assumes that:

- Aircraft is a point mass
- Installed thrust & aerodynamic drag are in the same direction as velocity
- Drag is described by a parabolic drag polar

3.1 Problem 1A

For the case of approach velocity, landing requirements are typically given as an approach speed, which is some safety factor above stall speed. Since lift is approximately equal to weight, the equation below relates approach speed to wing loading.

$$L = \beta W_{TO} = \frac{1}{2}\rho V^2 S C_{L,max} \quad \textbf{(2)}$$

$$V_{app} = k_{app}V_{stall} = k_{app}\sqrt{\frac{2\beta}{\rho C_{L,max}}\frac{W_{TO}}{S}} \quad \textbf{(3)}$$

If we rearrange the above equation, we can see that wing loading is a fixed value, which is represented by a vertical line in the constraint analysis plot.

$$\frac{W_{TO}}{S} = \frac{\rho {V_{app}}^2 C_{L,max}}{2{k_{app}}^2\beta} \quad \textbf{(4)}$$

For the case of minimum turn, it is given the fact that R=0 (clean configuration), allowing the master equation to be reduced to the following form:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 n^2\frac{\beta}{q}\frac{W_{TO}}{S} + K_2 n + \frac{C_{D0}}{\frac{\beta}{q}\frac{W_{TO}}{S}} + \frac{1}{v}\frac{d}{dt}\left(h + \frac{v^2}{2g_0}\right)\right\} \quad \textbf{(5)}$$

If constant speed and altitude are assumed, then T/W is the summation of a linear, a constant, and an inverse term with respect to W/S. From experience, K_2 (the constant term) is a rather low value so the trend for turn will be mostly comprised of the linear and inverse terms, as seen in the constraint plot.

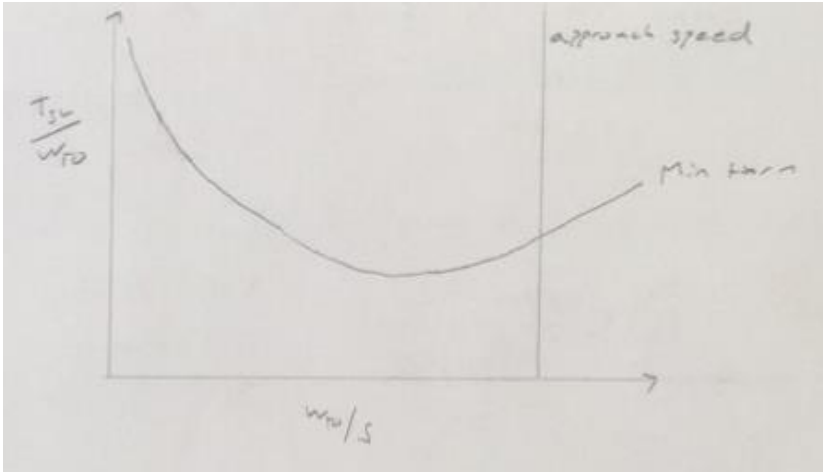

Figure 1: Problem 1A, notational constraint analysis plot

3.2 Problem 1B

This asks how the curves for both conditions will change if $g_0 = 2g_{Earth}$

Consider the approach velocity case first. Going back to level flight, lift equals weight. Increasing gravitational acceleration means the aircraft weights more and therefore needs more lift to sustain level flight. Lift is defined below and equated to weight.

$$L = \frac{1}{2}\rho V^2 S C_L = mg \tag{6}$$

In the equation above, the following is known when the conditions are changed.

- Rho (density) is assumed to be the same
- S (wing area) does not change
- C_L (lift coefficient) is a function of the wing and does not change
- M (mass) is fixed

Therefore, if g increases, velocity must increase to maintain this relationship. Specifically, doubling g requires the new velocity to be $V_{old}\sqrt{2}$. Applying this to the approach segment, it is now known that stall velocity has increased. Following the relationship in Equation (4), W/S is now a larger value.

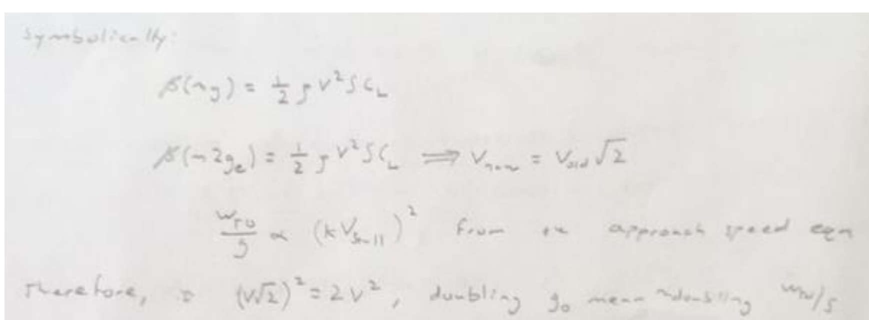

Figure 2: Problem 1B explanation from written work

Now consider the turning case. Load factor (n) is defined as the ratio of lift to weight. Since the aircraft is turning, apply the definition of centripetal acceleration in combination with the force balance ($L^2 = F_c^2 + W^2$). This yields the expression for load factor below.

$$n = \sqrt{\left(\frac{V^2}{g_0 R}\right)^2 + 1} \tag{7}$$

As g_0 increases, n decreases. This decreases the contribution of the linear component with n^2 in Equation (5), moving the line downwards. The contributions of n to the constant K_2 term are not as significant since K_2 is small.

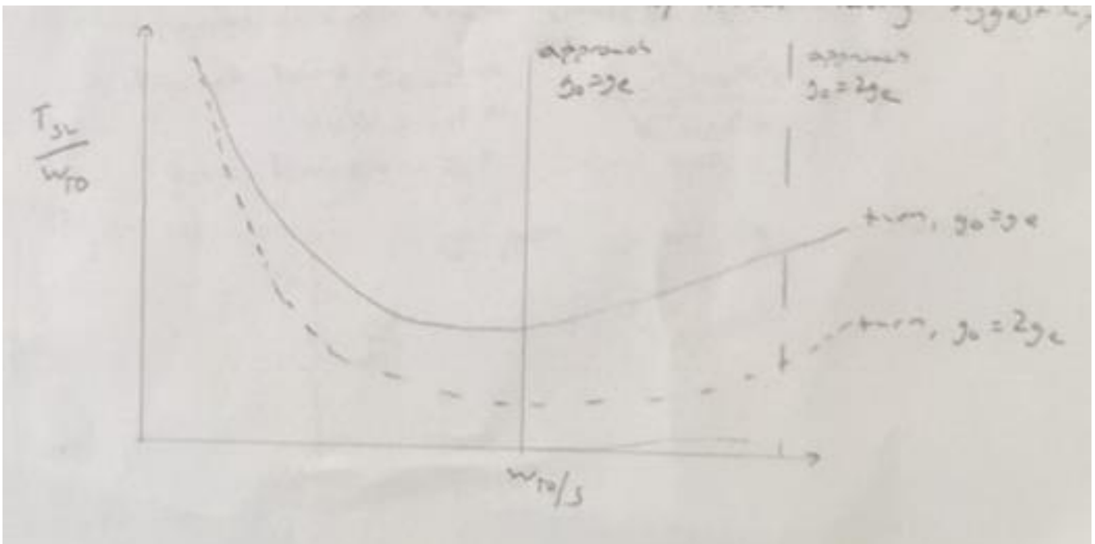

Figure 3: Problem 1B, constraint analysis plot increasing g0

3.3 Problem 1C

Adding external weapons to the aircraft means more drag, represented by $R \neq 0$. The general form of the energy balance of systems is as follows:

$$\{T - (D + R)\}V = W\frac{dh}{dt} + \frac{W}{g_0}\frac{d}{dt}\left(\frac{V^2}{2}\right) \quad \textbf{(8)}$$

Consider steady, level flight where thrust= drag + R. This means that when R is non-zero, there is more resisting force which in turn means that more thrust is required for the same conditions. Although the addition of $R \neq 0$ technically creates a 1/W term to the relations in Equation (1), we can generally say that more resisting force (D+R) requires more thrust for a given condition. This moves the curve up.

For the approach case, recall the equation defining W/S.

$$\frac{W_{TO}}{S} = \frac{\rho {V_{app}}^2 C_{L,max}}{2{k_{app}}^2\beta} \quad \textbf{(9)}$$

Technically, adding weapons would increase the total resistive forces on the aircraft, increasing the stall speed (and by extension, the approach speed). However, if approach speed is assumed to be fixed as a requirement, then adding external weapons would not change any of the variables in the above equation. Therefore the approach constraint curve would not move.

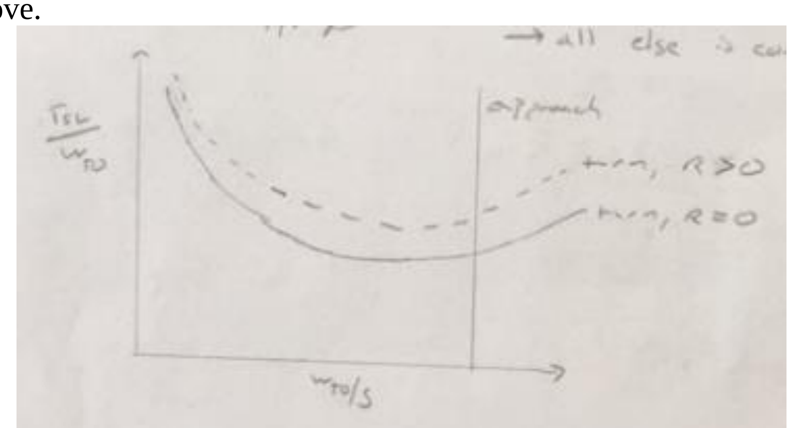

Figure 4: Problem 1C, constraint analysis plot for adding weapons

3.4 Problem 1D

Landing at a higher altitude means that the atmospheric density is lower compared to sea level. Referring to Equation (4), duplicated below, we can see there is a direct correlation between rho (density) and W/S.

$$\frac{W_{TO}}{S} = \frac{\rho {V_{app}}^2 C_{L,max}}{2{k_{app}}^2 \beta} \tag{10}$$

Assuming no other parameters change, a decrease in density (corresponding to an increase in altitude) will decrease the value for W/S.

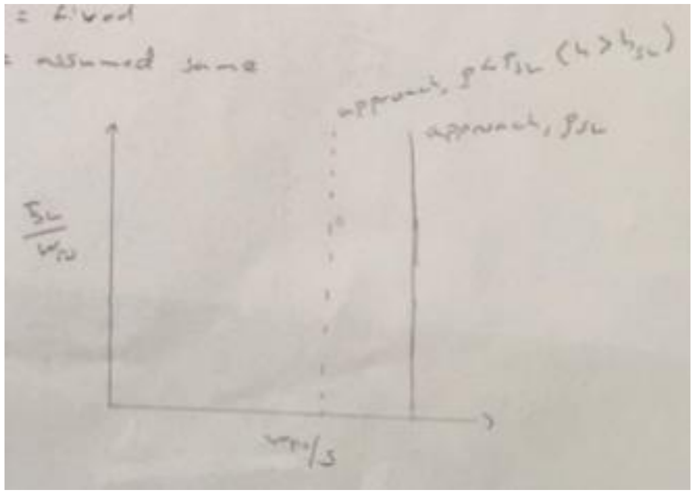

Figure 5: Problem 1D, constraint analysis plot for higher altitude landing

3.5 Problem 1E

To determine the effect of flying at a faster speed during a turn (all else staying the same), refer to Equation (7), duplicated below.

$$n = \sqrt{\left(\frac{V^2}{g_0 R}\right)^2 + 1} \tag{11}$$

Increasing the flight speed (V) will increase n. This increases the contribution of the linear component with n^2 in Equation (5), moving the line upwards. The contributions of n to the K_2 term are not as significant since K_2 is small.

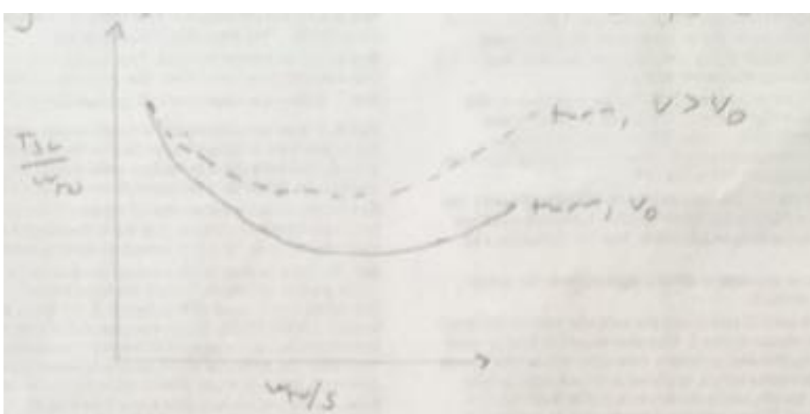

Figure 6: Problem 1E, constraint analysis plot for faster speed during turn

3.6 Problem 1F

The atmospheric density on mars is significantly less than on Earth. At the surface, the density of air is about 0.02 kg/m^3 which is about 1.6% of sea level atmosphere density on Earth (1.22 kg/m^3). Additionally, Mars has less gravity than Earth (~3.71m/s^2, or about 0.38g_e). Relatively speaking, the change in atmospheric density is larger, but the effects on the two flight conditions are summarized in the table below.

Table 1: Summary of Mars effects

	Decreasing g effect	Decreasing rho effect
Approach	W/S increase, move left (see 1b)	W/S decrease, move left (see 1d)
Min turn	Move curve down (see 1b)	Affects q

The equation for min turn (5) is duplicated below. The only terms impacted directly by air density are alpha (thrust lapse) and q (dynamic pressure).

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 n^2 \frac{\beta}{q}\frac{W_{TO}}{S} + K_2 n + \frac{C_{D0}}{\frac{\beta}{q}\frac{W_{TO}}{S}} + \frac{1}{v}\frac{d}{dt}\left(h + \frac{v^2}{2g_0}\right)\right\} \quad \textbf{(12)}$$

The effects of density on thrust lapse can be ignored since thrust lapse is driven by a density ratio at a given altitude (assume already defined with respect to Mars, assume same ratio as on Earth). This leaves dynamic pressure which is defined below.

$$q = \frac{1}{2}\rho V^2 \quad \textbf{(13)}$$

From this expression, dynamic pressure depends linearly on density. Since density decreases, dynamic pressure also decreases; we can examine the effect ions on Equation (12). The effect of the linear term (with K1 and 1/q) increases, while the effect of the inverse term (with q) decreases.

Since the change in density is so drastic, the approach curve should move left. The min turn curve should shift downwards.

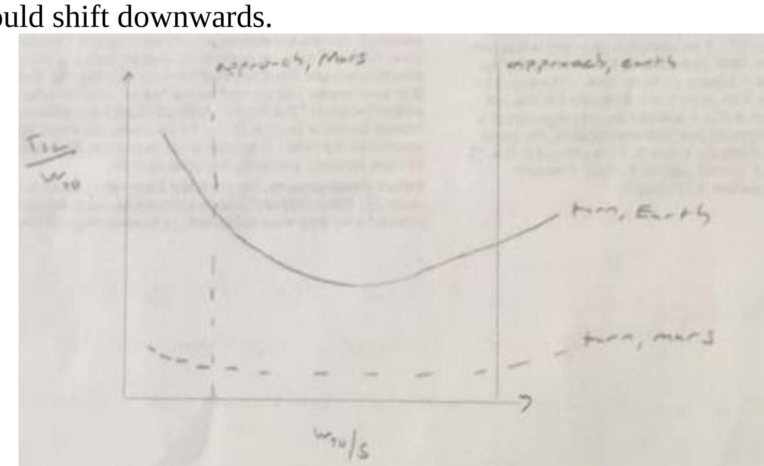

Figure 7: Problem 1F, constraint analysis plot for flying on Mars

4 Problem 2

4.1 Problem 2A

A vortex is defined as V_θ (rotational velocity) about a point. Specifically, it is a representation of the discontinuity in velocity between two points (like above and below an infinitely thin plate).

Vortices are used in aerodynamics to allow for mathematics to be applied to a singular point instead of the velocity difference between two points. Extending this, using vortices allows the airfoil to be modeled as a set of vortices (vortex sheet) on an infinitely thin plate.

Vortices represent the physical phenomena of the discontinuity in velocity (jump in tangential velocity across sheet) between two points. If considering a viscid fluid, vortices allow represent the impact of viscous effects on the fluid velocity, especially close to a surface.

The strength of these vortices is dependent on circulation (and therefore by extension, velocity) in an enclosed path. A single point is described as follows:

$$dV_\theta = \frac{-d\Gamma}{2\pi r}; \quad d\Gamma = \gamma d\xi \tag{14}$$

Therefore, the total strength can be calculated via integration.

$$\Gamma = \oint_0^c \gamma d\xi = \oint \vec{V} d\vec{s}; \quad \gamma = u_1 - u_2 \tag{15}$$

where u1 is parallel to chord and u2 is perpendicular to u1

4.2 Problem 2B

Circulation is the line integral of the velocity field along a closed path (see below).

$$\Gamma = \oint_0^c \gamma(s) ds = \oint \vec{V} d\vec{s} \tag{16}$$

Lift is related to circulation through the Kutta-Joukowski theorem below.

$$L' = -\rho V_\infty \Gamma \tag{17}$$

The KJ theorem relies on several concepts:

- Kutta condition= velocity of flow at top of airfoil and at bottom of airfoil are equal at the trailing edge (specifically a rear stagnation point)
- Non-penetration= flow cannot penetrate the airfoil (top and bottom are separate)
- There exists a pressure differential across the airfoil as shown by the Bernoulli equation
  - Say the flow velocity on one side is V and the other side has V+v
  - $0.5\rho V^2 + (P + \Delta P) = 0.5\rho(V + v)^2 + P$
  - $\Delta P = \rho V v$ if ignoring 0.5*rho*v2
  - $L' = c\Delta P = \rho V v c = -\rho V \Gamma$

4.3 Problem 2C

Thin airfoil theory converts the airfoil to an infinitely thin flat plate with a superimposed vortex sheet. In other words, this allows us to find L' from $\gamma(s)$, the strength of the vortex sheet per unit length along s.

This changes between uncambered (symmetric) and cambered airfoils since $\gamma(s)$ changes. In the case of a symmetric airfoil $\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}$, which eventually yields a lift curve slope of 2*pi*alpha (via Γ integration, then KJ theorem). This is not the case for a cambered airfoil which shifts the point of Cl=0 away from alpha=0.

5 Problem 3

5.1 Problem 3A

With a finite span airfoil (3D wing), there are now end effects which result in an induced drag. To extend thin airfoil theory to a finite wing, "Prandtl lifting line theory" is used. This theory models the effects of trailing vortices by using the superposition of a continuous distribution of circulation from bound vortices at the trailing edge.

Prandtl lifting line theory is limited in that it does not consider viscous, compressible, or unsteady flows. Additionally, it does not account for low aspect ratio wings (AR<4) or swept wings.

5.2 Problem 3B

The wing is represented in thin airfoil theory by the superposition of a continuous distribution of circulation from bound vortices at the trailing edge. In other words, there is a large number of vortices, each with a different length of the bound vortex, but all vortices are bounded along a single line (aka the lifting line). This sheet is parallel to the freestream velocity and the total strength along the sheet is zero since vortices are paired (same magnitude, but counter rotating).

There are several vortex systems at play here. At the wing tips, there are vortices which cause drag. Winglets on commercial aircraft are used to reduce their effect. Motion in trailing vortices also has downward component between two wingtips. This tends to push the free-stream air downwards, making the angle of attack effectively smaller (downwash). As discussed earlier, vortices are paired along the sheet- they have the same magnitude, but different directions. In other words, the flow behind the wing is not straight, rather it is at an angle. These vortices induce an overall downward velocity on the lifting line.

5.3 Problem 3C

Lift distribution across the wingspan is not constant; it is largest at the root and decreases to its minimum at the tips. The shape of this distribution is elliptical, but the most general form is given by

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{\infty} A_n \sin(n\theta) \qquad \textbf{(18)}$$

Where An are to be determined by the desired shape

The lift distribution is directly related to the induced drag, since it causes downwash reducing the effective angle of attack (aka induced drag). This is because the lift distribution directly impacts the freestream and disrupts the flow. More disruption means more induced drag ("losing velocity" that could be used for lift). The elliptical distribution has minimum lift at the edges of the wings, producing the least disruption. Specifically, the elliptical lift distribution has an Oswald efficiency (e) of 1. This makes it the distribution with the lowest induced drag, as show in the equation below.

$$C_{Di} = \frac{{C_L}^2}{\pi(AR)e} \qquad \textbf{(19)}$$

It is unclear if the question askes for a rigous proof of this formula, but it can be shown as follows (in LaTeX because Word sucks for proofs, using the process learned in AE3030).

Case of elliptical lift distribution

$$Let\ \Gamma(y) = \Gamma_0\sqrt{1-(\frac{y}{b/2})^2}$$

Lift

$$L = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y)dy \rightarrow \ = \rho V_\infty \Gamma_0 \int_{-b/2}^{b/2} (1-\frac{4y^2}{b^2})^{1/2} dy$$

Change of variables: let $y = \frac{b}{2}\cos\theta$

$$L = \rho V_\infty \Gamma_0 \int_\pi^0 \frac{-b}{2}\sin^2\theta d\theta \rightarrow \ = \rho V_\infty \Gamma_0 \frac{b\pi}{4}$$

$$\Gamma_0 = \frac{V_\infty S C_L}{b\pi}$$

Induced Drag

$$D_i = \int_{-b/2}^{b/2} L'(y)\alpha_i(y)dy =$$

Induced drag coefficient:

$$C_{Di} = \frac{D_i}{\frac{1}{2}\rho {V_\infty}^2 S} = \frac{2}{\rho {V_\infty}^2 S}\int_{-b/2}^{b/2} \rho V_\infty \Gamma(y)\alpha_i(y)dy$$

Induced angle of attack is independent of y for the elliptical case.

$$C_{Di} = \frac{{\Gamma_0}^2}{bSV_\infty}\frac{b\pi}{4} = \frac{S}{b^2}\frac{{C_L}^2}{\pi}$$

When substituting in Γ_0 $$C_{Di} = \frac{{C_L}^2}{\pi Æ}$$

Which is the same as the general case, except e=1, making it the minimum case.

5.4 Problem 3D

Starting from the fundamental equation of thin airfoil theory (from AE3030), we can find the induced velocity from the linear superposition of uniform flow with a vortex sheet γ(x) on the chord line. This assumes that the mean camber line, z(x), is a streamline. As a reminder, the fundamental "fundamental equation of thin airfoil theory" is as follows:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)d\zeta}{x-\zeta} = V_\infty\left(\alpha - \frac{d\zeta}{dx}\right) \quad \textbf{(20)}$$

Essentially, we want to solve for a $\gamma(x)$ function so that we can get velocity at any given point (and calculate circulation). Using the Kutta condition of $\gamma(x = c) = 0$, this is done as follows:

$$\zeta = \frac{c}{2}(1-\cos\theta); \quad d\,\zeta = \frac{c}{2}\sin\theta\, d\theta;$$

So that at leading edge $\zeta = 0, \theta = 0$

Trailing edge $\zeta = c, \theta = \pi$ **(21)**

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \quad \textbf{(22)}$$

The actual shape of the airfoil is given by dz/dx.

Generally, though if the circulation is already known, we can just apply the Biot-Savart law, described below.

$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{l}xd\vec{r}}{|r|^3} \quad \textbf{(23)}$$

This allows us to get the velocity directions from $d\vec{l}xd\vec{r}$ and the magnitudes from $d\vec{V}$ α $\frac{1}{|r|^2}$.

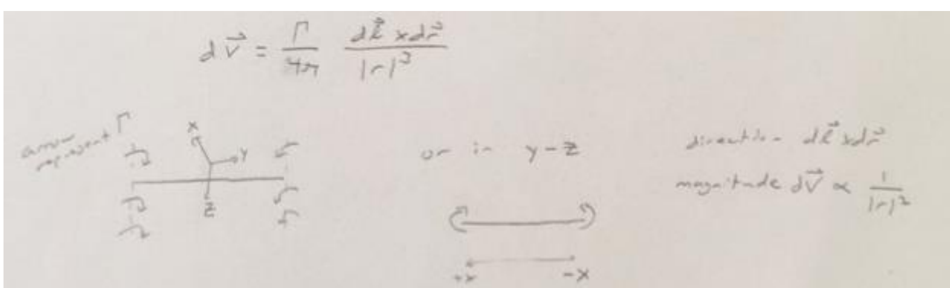

Figure 8: Graphical representation of Bio-Savart law to find induced velocity

6 Written Work

Combined into singular PDF starting on next page.
This work is simply the initial written version of the above content. Graphs are inserted in the context of how I thought about each problem.
The start of each problem is labeled as such and follows the page number format of "pg. problem- pg#".

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Full Name: KARL ROUSH

Signature: ____________ Date: 10/23/2020
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Fall 2020 AE6343 Aircraft Design I Take Home Exam 1 October 16 - October 23, 2020

1. Suppose that an aircraft is designed with only 2 constraints in mind, namely approach velocity and minimum turn. Assume for the turn that $R = 0$ (clean configuration). This question will ask you to evaluate and explain how the constraint curves change for different scenarios. Your answer should include a plot showing how the constraints change and any necessary explanations. Your plots should have at least 2, but no more than 4 curves. Also, in questions where the answer may vary depending on your assumptions, make sure to explicitly state your assumptions along with your answer for full credit.

   (a) (5 points) Provide a notional constraint analysis plot of this aircraft (with the original 2 constraints).

   (b) (6 points) Illustrate and explain how the constraint curves would change if the aircraft flies flies on a planet identical to Earth, except $g_0 = 2 * g_{earth}$ (Hint: Define your x- and y-axis appropriately)

   (c) (6 points) Add external weapons to the aircraft → $R \neq 0$ apply to turn as well?

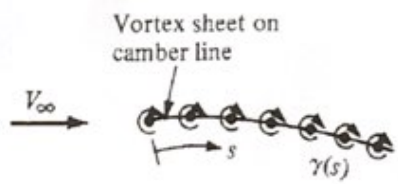
   (d) (6 points) Land at a higher altitude → ρ decrease

   (e) (6 points) Fly at a faster speed during the turn → V increase = n increase; Lower T/W

   (f) (6 points) Take this vehicle to Mars (significantly less dense atmosphere, but assume same composition) ⇒ $\rho\downarrow$, $g\downarrow$ consider g?

2. 2D aerodynamic theory:

   (a) (10 points) When we consider a thin airfoil, in 2D, we choose to represent it as a single line (the camber) with a distribution of vortices (see figure below). What is a vortex? Why are vortices used? What is the physical phenomena represented by these vortices? How is the strength of these vortices calculated?



   (b) (10 points) How are lift and circulation related? What theorems are used to describe it? Your answer should include not only equations, but also a description of the theory behind them.

   (c) (10 points) Thin airfoil theory: what are its main postulates/results? How do they change between uncambered and cambered airfoils?

3. Classic wing theory:

   (a) (5 points) The previous theory only considered a 2D section (the airfoil) of an "infinite" wing. Now we want to extend this theory to analyze a real 3D wing. What is the name of the simple aerodynamic theory used for this? What are the limitations of this theory?

PROBLEM ... 1

(b) (10 points) How is the wing represented in thin-airfoil theory? (hint: need to talk about vortices). Describe the different types of vortex "systems" that are present here and their relationship with the flow structures in the chord and span-wise directions. Are these vortices equal in strength to each other? How do they affect the flowfield in the wake of the wing?

(c) (10 points) How does lift distribution changes across the wing span? What shape does it have, and why? How is this related with induced drag? What type of lift-distribution yields the lowest induced drag, and why?

(d) (10 points) Describe, using vortex theory, how induced velocity can be calculated for a real 3D wing of arbitrary shape. Make sure to accompany your description with proper illustrations.

PROBLEM 1

(1a) consider general energy balance of system via $\{T-(D+R)\}V = W\frac{dh}{dt} + \frac{W}{g_0}\frac{d}{dt}\left(\frac{V^2}{2}\right)$

this the reduces to the "master equation"

$$** \quad \frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{TO}}\left[K_1\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right)^2 + K_2\left(\frac{n\beta}{q}\frac{W_{TO}}{S}\right) + C_{D0} + \frac{R}{qS}\right] + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_0}\right)\right\}$$

This (**) assumes the following:

- aircraft is point mass
- Installed thrust & aerodynamic drag act in same direction as $\vec{V}$
- Drag is described by a parabolic drag polar ($C_D = K_1 C_L^2 + K_2 C_L + C_{D0}$)

Case 1: approach speed

Landing requirements are typically give as an approach speed $\vec{V}_{app}$

$\vec{V}_{app}$ is the stall speed with a margin of safety

$$\vec{V}_{app} = k_{app} V_{stall}$$

The stall speed can be found via the equation relating lift & weight

$$L = W \implies W\beta = L \longrightarrow W_{TO}\beta = \frac{1}{2}\rho V_{stall}^{2} S C_{L_{max}}$$

$$L = \tfrac{1}{2}\rho V^2 S C_L$$

Therefore we can re-arrange to get

$$\frac{W_{TO}}{S} = \frac{\rho (V_{app})^2 C_{L,max}}{2(k_{app})^2\beta}$$

note W/S does not depend on T/W so we have a vertical line on constraint plot (W/S = constant value)

Case 2: min turn

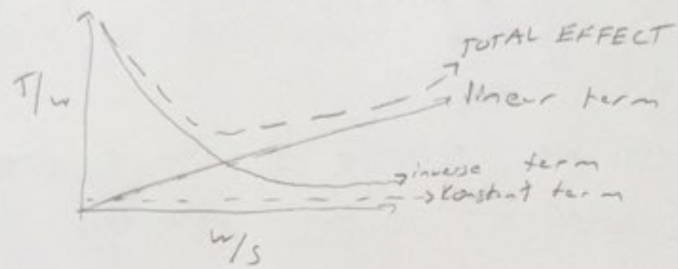
Given $R=0$, we can reduce (**)

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{ \underbrace{K_1 n^2 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}_{\text{linear term}} + \underbrace{K_2 n}_{\text{constant term}} + \underbrace{\frac{C_{D0}}{\frac{\beta}{q}\frac{W_{TO}}{S}}}_{\text{inverse term}} + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_0}\right) \right\}$$

↳ ignore if assume $dh/dt = \frac{dv}{dt} = \emptyset$

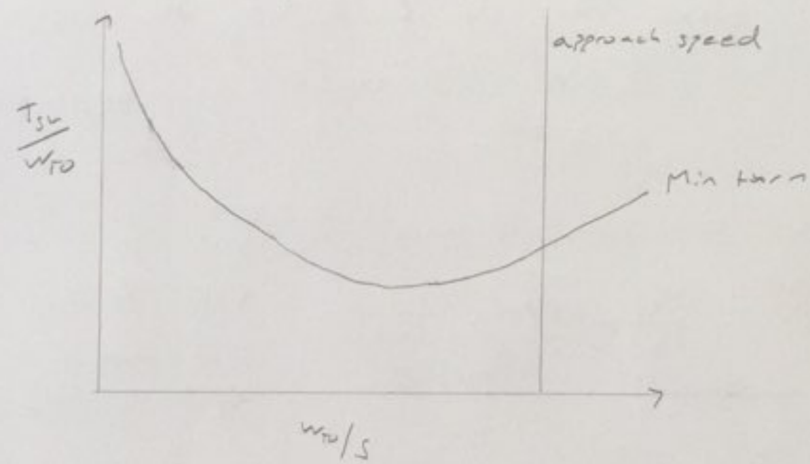
Therefore the constraint curve for turn is basically the summation of linear, constant, and inverse terms.

From experience, K_2 is small so linear & inverse dominate.



*sum together for turn curve

Plotting both turn & approach speed constraints on one plot:



(1b) Now consider $g_0 = 2g_{earth} \Rightarrow$ aircraft weighs more

consider level flight where $L = W$

$$\cancel{\beta} W = L = \tfrac{1}{2}\rho V^2 S C_L \quad \text{so all cannot change except } V$$

in other words if weight increases then aircraft must fly faster

symbolically:

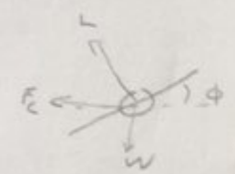
$$\cancel{\beta}(mg) = \tfrac{1}{2}\rho V^2 S C_L$$

$$\cancel{\beta}(m\,2g_e) = \tfrac{1}{2}\rho V^2 S C_L \Longrightarrow V_{new} = V_{old}\sqrt{2}$$

$$\frac{W_{TO}}{S} \propto (k V_{stall})^2 \quad \text{from the approach speed eqn}$$

Therefore, $(V\sqrt{2})^2 = 2V^2$, doubling g_0 mean doubling W_{TO}/S

Now consider turning



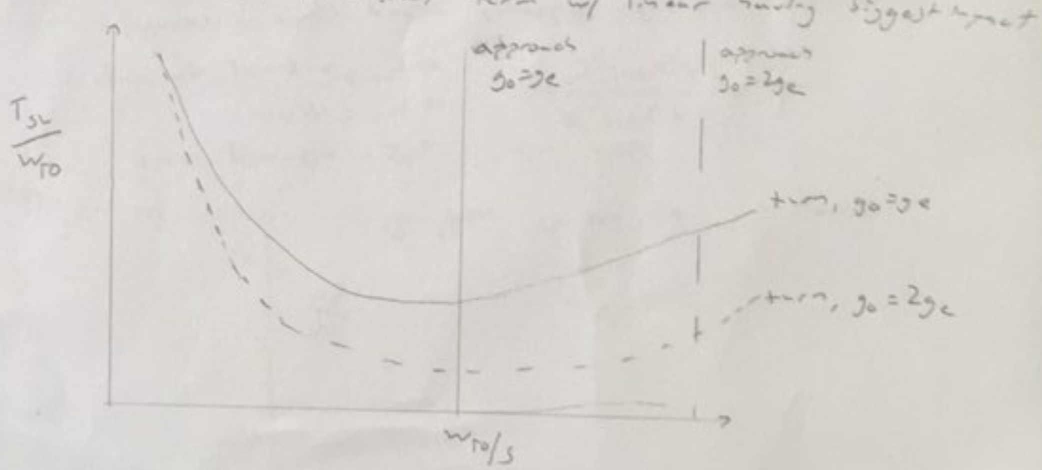
$$n = L/W \qquad L^2 = F_c^2 + W^2$$

$$\rightarrow n = \sqrt{\frac{F_c^2}{W^2} + 1} \qquad \text{since } a_c = V^2/R$$

$$n = \sqrt{\left(\frac{a_c}{g_0}\right)^2 + 1} \longrightarrow n = \sqrt{\left(\frac{V^2}{g_0 R}\right)^2 + 1}$$

therefore if g_0 doubles, n must decrease

recall n is in linear + constant term w/ linear having biggest impact



(1c) Consider $R \neq 0$

general energy balance $\{T-(D+R)\}V = W\frac{dh}{dt} + \frac{W}{g_0}\frac{d}{dt}\left(\frac{V^2}{2}\right)$

in steady level flight $T = D+R$

so having $R \neq 0$ means you need more thrust for same condition (also adds $\frac{1}{W_{TO}}$ term to **)

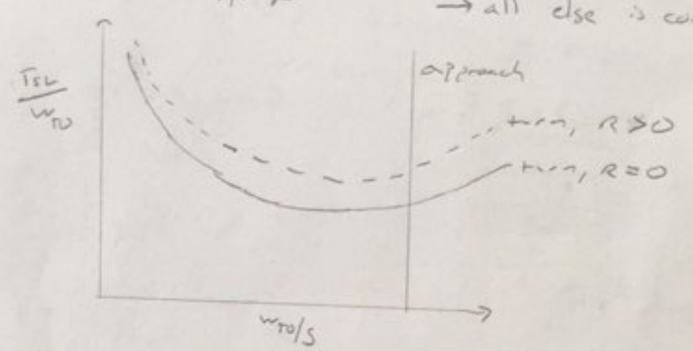
Needing more thrust is equivalent to moving the constraint curve up

for approach speed, $R \neq 0$ essentially means more resistance aka higher W/S
↳ this assumes a fixed W but we are in preliminary design

Recall equation

$$\frac{W_{TO}}{S} = \frac{\rho (V_{app})^2 C_{L_{max}}}{2(k_{app})^2 \beta}$$

→ V_{app} is fixed by requirements
→ $C_{L_{max}}$ is fixed by airfoil
→ all else is constant so $\frac{W_{TO}}{S}$ doesn't change



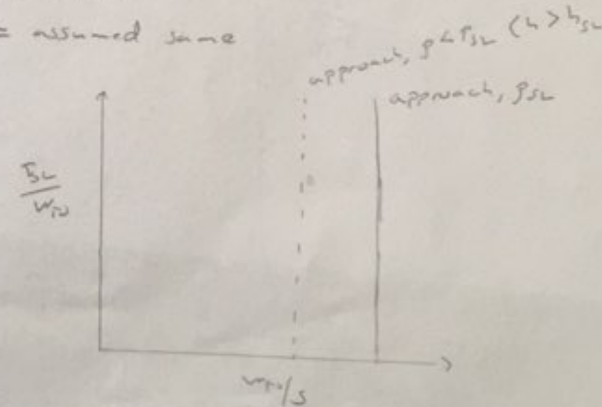
(1d) landing at a higher altitude means $\rho < \rho_{SL}$

Assume approach speed is fixed (V_{app} is same)

$$\frac{W_{TO}}{S} = \frac{\rho (V_{app})^2 C_{L_{max}}}{2(k_{app})^2 \beta}$$

→ $C_{L_{max}}$ = fixed by airfoil
→ K_{app} = fixed
→ β = assumed same

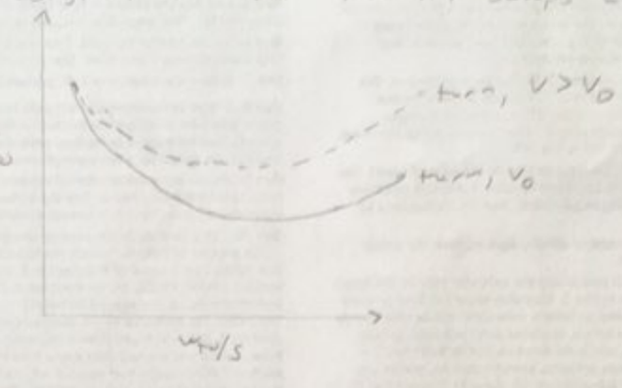
therefore $\rho \downarrow$ means $W_{TO}/S \downarrow$



(1e) fly at a higher speed during turn ($V \uparrow$)

recall $n=\sqrt{\left(\frac{V^2}{g_0 R}\right)^2+1}$ so if turn radius R = same, $V\uparrow$ yields $n\uparrow$

all else being constant in ~~**~~ turn $n\uparrow$ bumps curve up



(1f) ρ_{mars} @ $h=0$ is $0.02 \frac{kg}{m^3}$ aka 1.6% of ρ_e

	$g\downarrow$	$\rho\downarrow$
land	$W/S \downarrow$	$W/S \downarrow$
turn	decrease	check q

$g_{mars} \approx 3.721 \frac{kg}{m s^2}$ or $0.38 g_e$

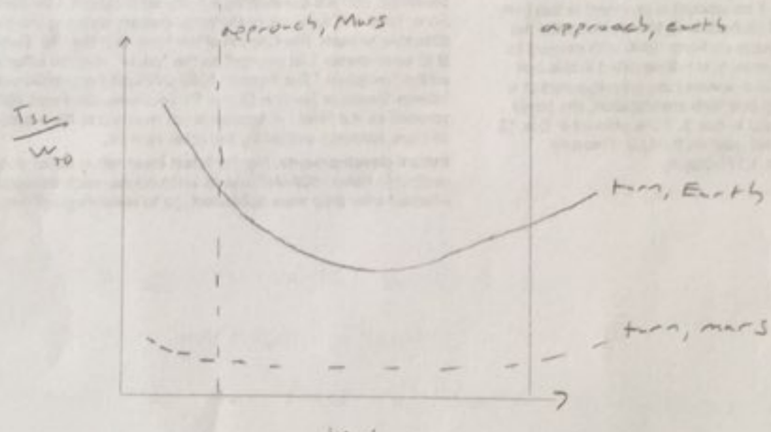
$\Longrightarrow$ effects dominated by density change

landing (approach): refer to 1d

turn: dynamic pressure is $q=\frac{1}{2}\rho V^2$ so $\rho\downarrow$, $q\downarrow$

therefore drastically reducing ρ, drastically reduces q

↳ move curve down



PROBLEMS 2

(2a) vortex = representation of discontinuity in velocity since velocity above & below are not equal ⇒ allows for mathematical analysis and to model airfoil as vortex sheet

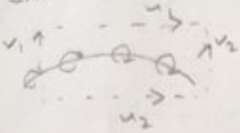
specifically a jump in tangential $\vec{V}$ for sheet

vortex = "V_θ about a point, ie rotational velocity"

calculate strength: $dV_\theta = \dfrac{-d\Gamma}{2\pi r}$; $d\Gamma = \gamma d?$

so integrate $\Gamma = \oint \gamma \, d? = \oint \vec{v} \, d\vec{s}$

ex.



$\Gamma = u_1 ds - v_2 dn - u_2 ds + v_1 dn$

$\gamma ds = (u_1 - u_2) ds + (v_1 - v_2) dn$ if $dn \to 0$ $\gamma = (u_1 - u_2)$

⇒ local sheet strength is equal to local jump in tangential $\vec{V}$

(2b) Circulation is line integral of the velocity field along a closed path

$$\Gamma = \oint_0^c \gamma(s) ds = \oint \vec{V} d\vec{s}$$

circulation is linked to lift via Kutta-Joukowski theorem

$L' = -\rho V_\infty \Gamma$ → equivalent to using Bernoulli w/ ΔP

* KJ is based on

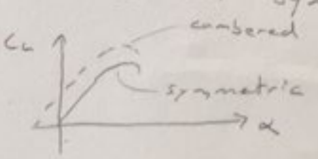
Kutta condition = V@ top of airfoil, V@ bottom of airfoil → equal at trailing edge

Non-penetration = flow does not go through airfoil

(2c) Thin airfoil theory converts airfoil to an infinitely thin plate w/ superimposed vortex sheet

↳ allows us to find L' from $\gamma(s)$ which is strength of vortex sheet per unit length

* $\gamma(s)$ is different between symmetric & cambered airfoils



this shifts lift curve

PROBLEM 3

(3a) finite span means now need to consider
↳ end effects ↳ trailing edge vortices

Prandtl lifting line theory models effects of trailing vortices by superposition of continuous distribution of circulation from bound vortices at TE

Does not apply to
- viscous flows
- compressible flows
- unsteady flows
- AR < 4
- swept wings

(3b) wing is represented by (")

in other words we have a large number of horseshoe vortices w/ different lengths of bound vortex but all bound to single line.

$\Rightarrow$ sheet is $\parallel$ to V_∞ & Σ sheet strength $= 0$

this is because sheet has paired vortices of equal strength but opposite direction

effects = down @ wing tips + trailing vortices

3c Lift is highest at root & least at wing tips

Lift causes downwash $\Rightarrow$ so angle of attack is effectively less $\Longleftrightarrow$ equivalent to induced drag

Lift also disrupts flow, "losing" velocity

$e=1$ for elliptical making it min $C_{D_i} = \frac{C_L^2}{\pi A\!\!R\, e}$

this can be proven by

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y}{b/2}\right)^2} \implies \text{type in LaTeX}$$

3d induced velocity is found via superposition of uniform flow and vortex sheet

This is from the fundamental equation of thin airfoil theory

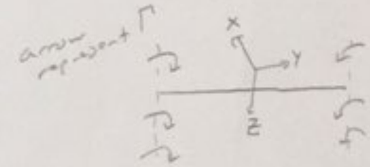
$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\, d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{d\xi}{dx}\right)$$

we want a function $\gamma(x)$ so we can use for any shape

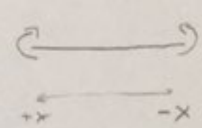
$\Rightarrow$ refer to LaTeX

generally though if circulation is known, can use Biot-Savart law

$$d\vec{V} = \frac{\Gamma}{4\pi}\, \frac{d\vec{l} \times d\vec{r}}{|r|^3}$$



or in y-z



direction $d\vec{l} \times d\vec{r}$

magnitude $d\vec{V} \propto \frac{1}{|r|^2}$