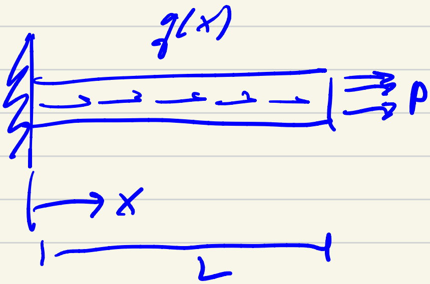
Energy Methods (cont)

11/19

$$\underset{\underline{u}}{\text{Min}} \quad \Pi(\underline{u}) = \min_{\underline{u}} \frac{1}{2}\int_{\Omega} \nabla\underline{u} : \underline{\underline{C}} : \nabla\underline{u}\, dV - \int_{\Omega} \underline{b}\cdot\underline{u}\, dV - \int_{\partial_2\Omega} \underline{t}^* \cdot \underline{u}\, dA$$

$\underline{u} = \underline{u}^*$ on $\partial_1\Omega$ $\qquad$ $\underline{u} = u^*$ on $\partial_1\Omega$ $\qquad \Longleftrightarrow$ BVP in elastostatics

Example: Uniaxial bar



The potential for this case reduces to:

$$\Pi = \frac{AE}{2}\int_0^L \left(\frac{\partial u_1}{\partial x}\right)^2 dx - A\int_0^L g(x) u_1\, dx - APu_1\Big|_{x=L}$$

$$\delta\Pi = 0 \Rightarrow \underbrace{E\frac{\partial^2 u_1}{\partial x^2} + g(x) = 0}$$ (recover strong form)

$$\underbrace{E\frac{\partial u_1}{\partial x} = P}$$ (recover traction BCs)

For example, for $g(x) = g$ and $p = 0$ we get:

$$E \frac{\partial^2 u_1}{\partial x^2} + g = 0 \Rightarrow \frac{1}{2}\frac{gx^2}{E} + C_1 x + C_2$$

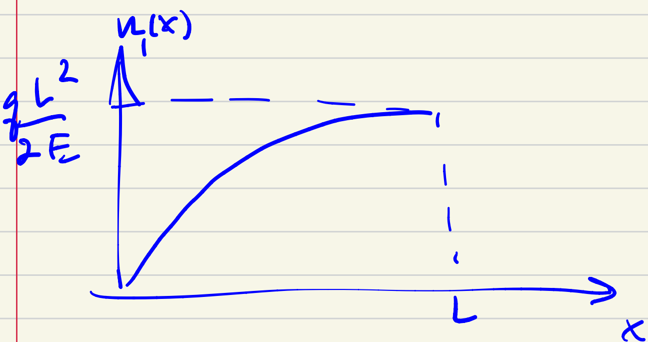
At $x = 0$, $u = 0 \Rightarrow C_2 = 0$

At $x = L$, $E \frac{\partial u}{\partial x} = 0 \Rightarrow -gL + C_1 E = 0$

$$\Rightarrow C_1 = \frac{gL}{E}$$

Thus we have:

$$u_1(x) = -\frac{1}{2}\frac{gx^2}{E} + \frac{gLx}{E}$$



Rayleigh - Ritz Method

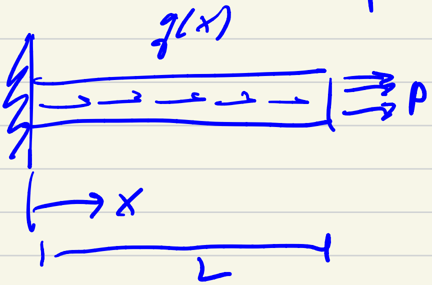
- Based on the principle of minimum potential energy

- Instead of solving for every compatible displacement field $\underline{u}$, we choose a set of functions $\underline{\tilde{u}}$, e.g. n^{th} grade polynomials and minimize the functional $\tilde{\Pi}$ wrt the parameters of those functions, eg, the coefficients of the polynomial.

- If the solution to the original problem lies within the set of chosen functions, then we can recover the exact solution.

- If this is not the case, then we get the "closest" possible solution in the sense of the energy norm, ie, Π as low as we can get for the chosen set of functions $\underline{\tilde{u}}$

Let's recall the previous example:



The potential for this case reduces to:

$$\Pi = \frac{AE}{2}\int_0^L \left(\frac{\partial u_1}{\partial x}\right)^2 dx - A\int_0^L q(x)\, u_1\, dx - A P u_1\Big|_{x=L}$$

Let's choose $\hat{u}_1$ to be a linear function: $\hat{u}_1(x) = ax + b$

Since it has to be compatible with the displacement BCs, we have:

$$\hat{u}_1(0) = 0 \Rightarrow b = 0 \Rightarrow \hat{u}_1(x) = ax$$

Assuming $q(x) = q$, $P = 0$ & plugging $\hat{u}_1$ into Π we get:

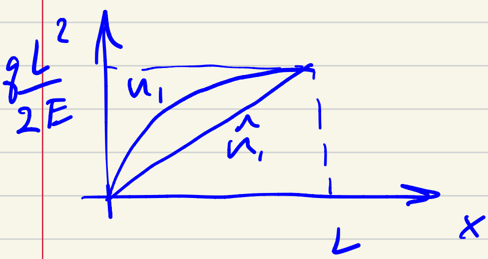
$$\hat{\Pi} = \frac{1}{2} AE a^2 L - \frac{1}{2} a q A L^2 \rightarrow$$ now func of a.

To minimize the potential we have:

$$\frac{\partial \hat{\Pi}}{\partial a} = 0$$

$$\Rightarrow \; AEaL - \frac{1}{2} g A L^2 = 0 \;\rightarrow\; a = \frac{1}{2}\frac{gL}{E}$$

$$\Rightarrow \; \underline{\hat{u}_1 = \frac{1}{2}\frac{gL}{E}x}$$



$\Rightarrow$ Note that we did not recover the exact solution.

The solution is as good as it can get for linear functions.

If instead, we choose $\hat{u}$ to be quadratic:

$\hat{u}_1 = cx^2 + dx + e$

Due to disp. BCs we need $e = 0 \Rightarrow \hat{u}_1 = cx^2 + dx$

substituting into the potential, we get:

$$\hat{\Pi} = \frac{1}{6}EL\left(4Ec^2L^2 + 6EcLd + 3Ed^2 - 2gL^2c - 3gLd\right)$$

Minimize wrt unknowns (c & d) simultaneously

$$\begin{cases} \dfrac{\partial \hat{\Pi}}{\partial c} = 0 \\ \dfrac{\partial \hat{\Pi}}{\partial d} = 0 \end{cases} \Rightarrow \begin{cases} 4EcL + 3Ed - gL = 0 \\ 2EcL + 2Ec - gL = 0 \end{cases}$$

$$\Rightarrow \quad c = -\frac{1}{2}\frac{g}{E} \quad ; \quad d = \frac{gL}{E}$$

$$\Rightarrow \quad \hat{u}_1(x) = -\frac{1}{2}\frac{gx^2}{E} + \frac{gLx}{E}$$

Notes:

- Since the exact solution is a quadratic polynomial and our family $\hat{u}$ includes all quadratic polynomials, we recovered the exact solution!

If we picked $u_1(x) = ax^3 + bx^2 + cx + d$

↳ get exact solution, but a & $d = 0$

If we pick $u_1(x) = ax^3 + bx^2 + c$

↳ don't get exact solution, need x term.

- Dec 1st release exam, not time.
- honor thing: 1 session
- Due at midnight Dec 3rd
  - Do-able in 3 hours.
  - Take as long as you want as long as its one session.
- Open notes, can use computer, submit intermediate steps from computer.