Fundamentals of Aircraft Performance

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Aircraft Performance

- For this class, performance metrics are classified by the <u>nature</u> of flight:
  - **Steady flight:** No acceleration, rectilinear flight-path
    - Maximum velocity
    - Minimum velocity
    - Climb performance – rate and angle of climb, time to climb
    - Service and absolute ceilings
    - Range and endurance
  - **Accelerated flight:** Non-zero acceleration, possibly curvilinear flight-path
    - Turning performance – turn rate, turn radius
    - Pull-up, pull-down, and push-over maneuvers
    - Accelerated rate of climb
    - Takeoff and landing performance

- Note that it is also possible to classify performance metrics based on the <u>phase</u> of flight to which they are most applicable

Equations of Motion

- Newton's 2nd Law relates forces acting on a mass to resulting acceleration:

$$\boxed{\sum \bar{\boldsymbol{F}} = \boldsymbol{m}\,\bar{\boldsymbol{a}}}$$

- This is a vector equation, and is equivalent to three scalar equations resolved along three orthogonal axes

- For instance, for the general case of an accelerating aircraft on a climbing, turning flight path, the resolution may be along:
  1. A first axis parallel to the flight path
  2. A second axis normal (perpendicular) to the flight path and in the vertical plane
  3. A third axis in the horizontal plane, perpendicular to the first two

$$\sum F_{||} = \boldsymbol{m}\,\boldsymbol{a}_{||}$$

$$\sum F_{\boldsymbol{V}\perp} = \boldsymbol{m}\,\boldsymbol{a}_{\boldsymbol{V}\perp}$$

$$\sum F_{\boldsymbol{H}} = \boldsymbol{m}\,\boldsymbol{a}_{\boldsymbol{H}}$$

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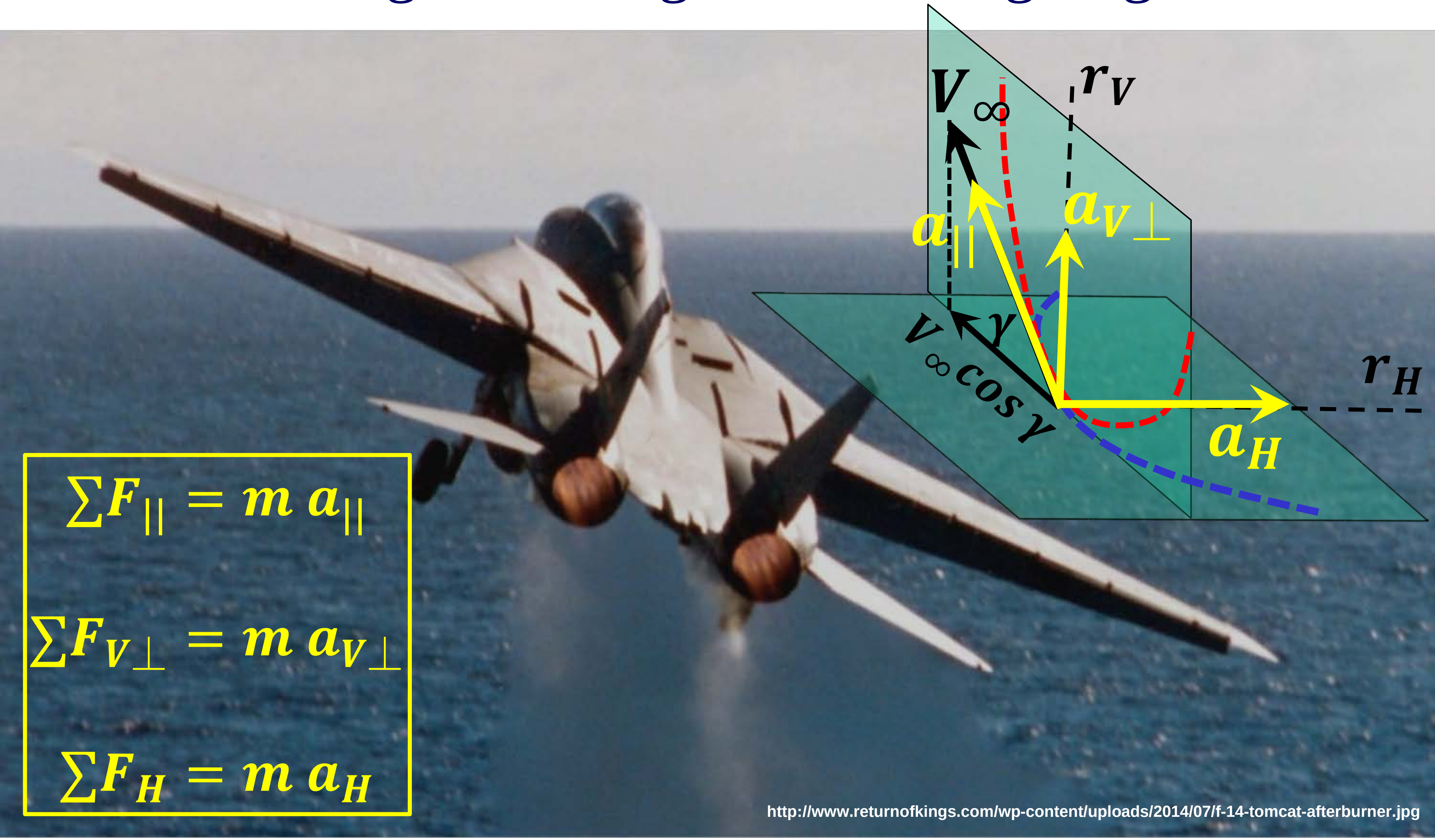
Accelerating, Climbing, and Turning Flight Path



$$\sum F_{||} = m\, a_{||}$$

$$\sum F_{V\perp} = m\, a_{V\perp}$$

$$\sum F_H = m\, a_H$$

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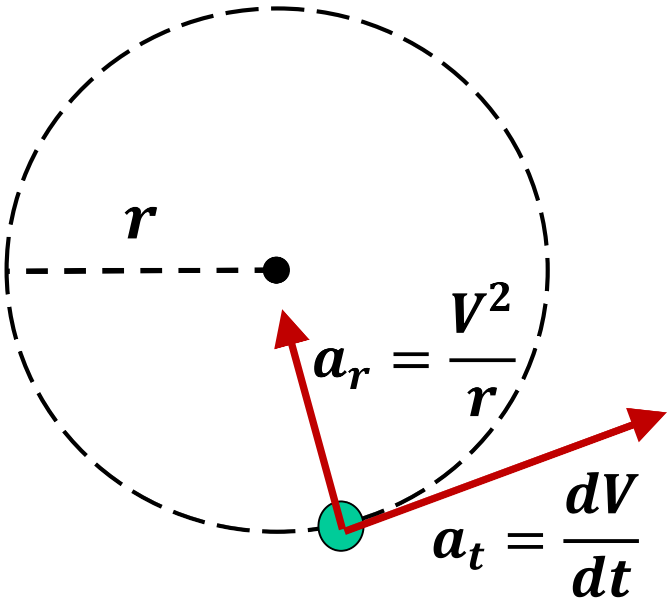
Accelerated Motion Along a Curved Path

- Consider the case of a particle moving along a planar circular path
  - The velocity vector is always tangential to the path, no radial component
- There are two perpendicular acceleration components for this motion:
  - Tangential acceleration, $a_t = \frac{dV}{dt}$
  - Radial (centripetal) acceleration, $a_r = \frac{V^2}{r}$
- If the motion is curved but not circular, these relationships still hold if $\boldsymbol{r}$ is used as the instantaneous radius of curvature



Let us apply these relationships to the accelerating, climbing, and turning flight path

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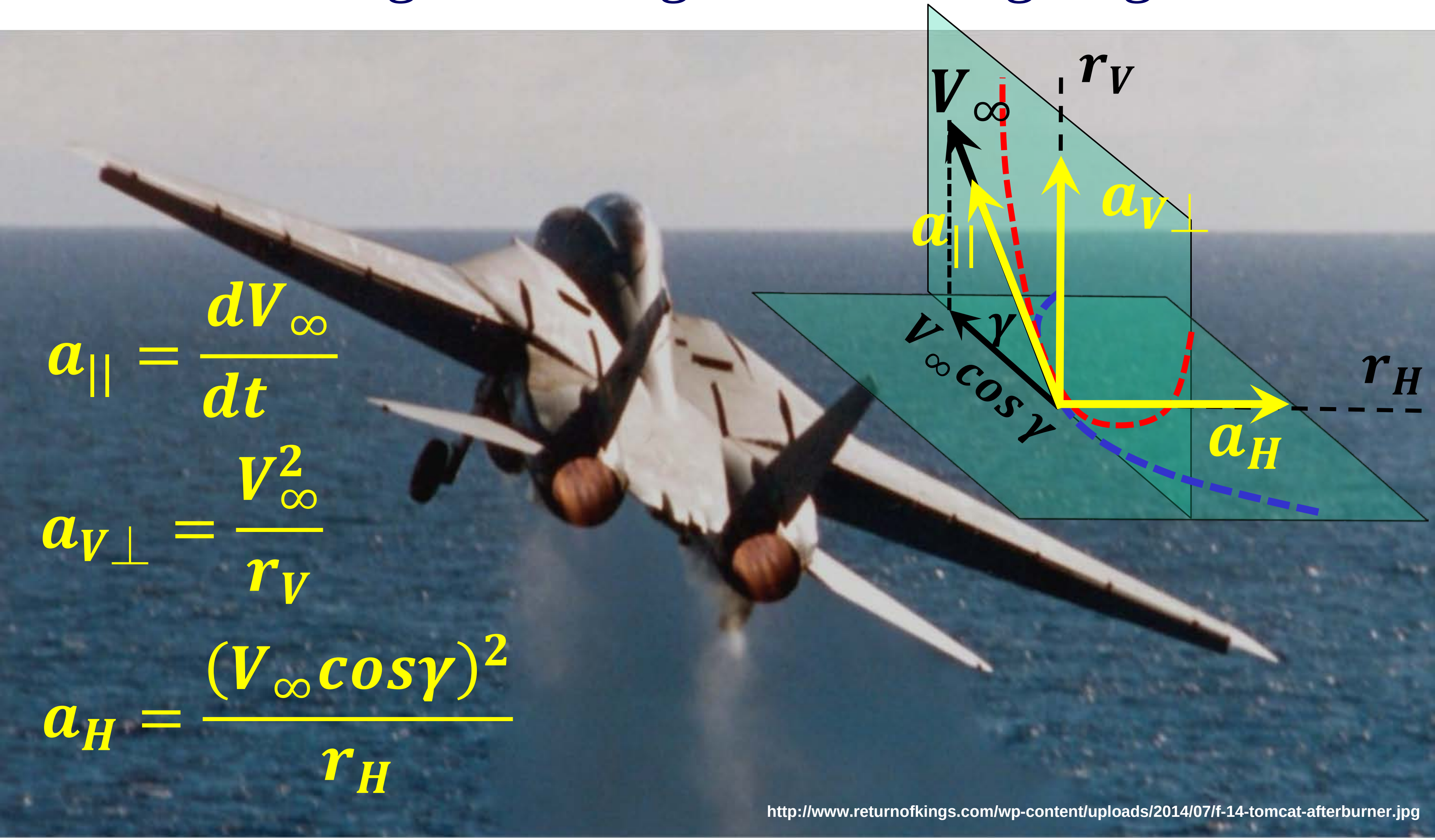
Accelerating, Climbing, and Turning Flight Path



$$a_{||} = \frac{dV_\infty}{dt}$$

$$a_{V\perp} = \frac{V_\infty^2}{r_V}$$

$$a_H = \frac{(V_\infty cos\gamma)^2}{r_H}$$

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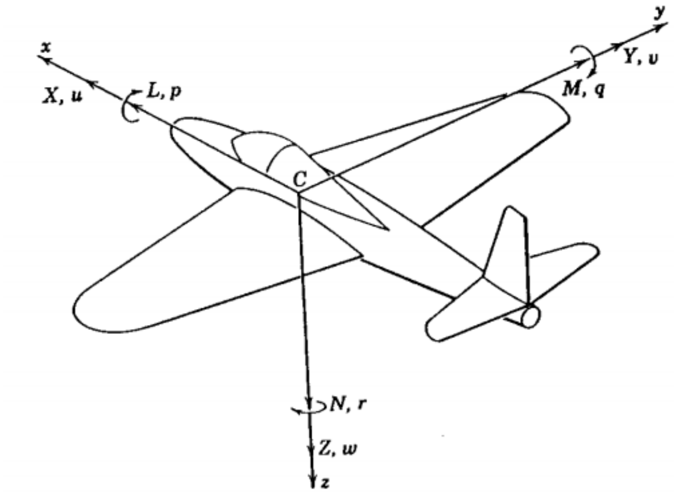
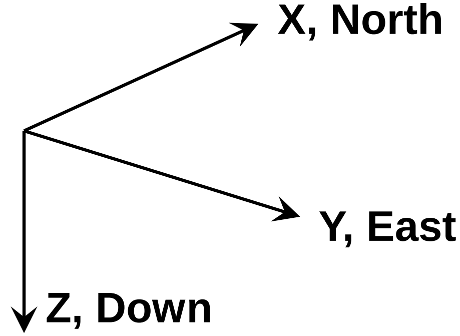
Euler Angles – An Airplane's "Attitude"

- The orientation of any reference frame relative to another can be given by <u>**three**</u> **Euler** angles:
  - When one reference frame is rotated by these angles about <u>*specified axes*</u> in a <u>*specified order*</u>, it is brought into coincidence with the other
- The following rotation <u>*sequence*</u> is typically used for airplane flight mechanics:
  - A rotation $\boldsymbol{\psi}$ about the Earth-fixed frame's **z-axis**. Called "azimuth angle", then…
  - A rotation $\boldsymbol{\theta}$ about the intermediate frame's **y-axis**. Called "elevation" or "pitch angle", …
  - And finally, a rotation $\boldsymbol{\phi}$ about the next intermediate frame's **x-axis**. Called "bank angle" or "roll angle"
- This sequence is called a **3-2-1 rotation sequence**. Can you see why?

Body-fixed Reference Frame





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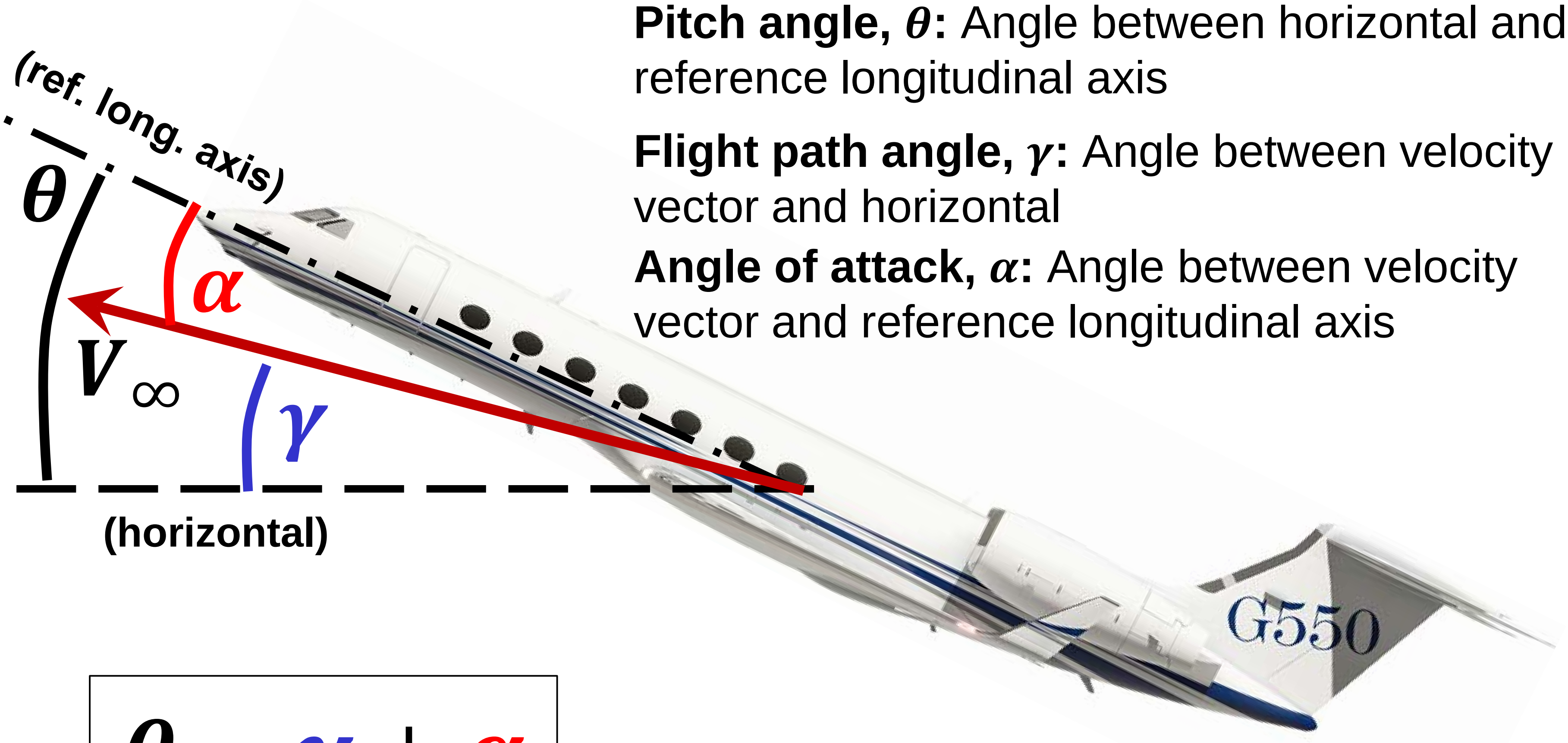
Pitch, Angle of Attack, and Flight Path Angle



Pitch angle, θ: Angle between horizontal and reference longitudinal axis

Flight path angle, γ: Angle between velocity vector and horizontal

Angle of attack, α: Angle between velocity vector and reference longitudinal axis

$$\boxed{\theta = \gamma + \alpha}$$

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Deriving the Equations of Motion

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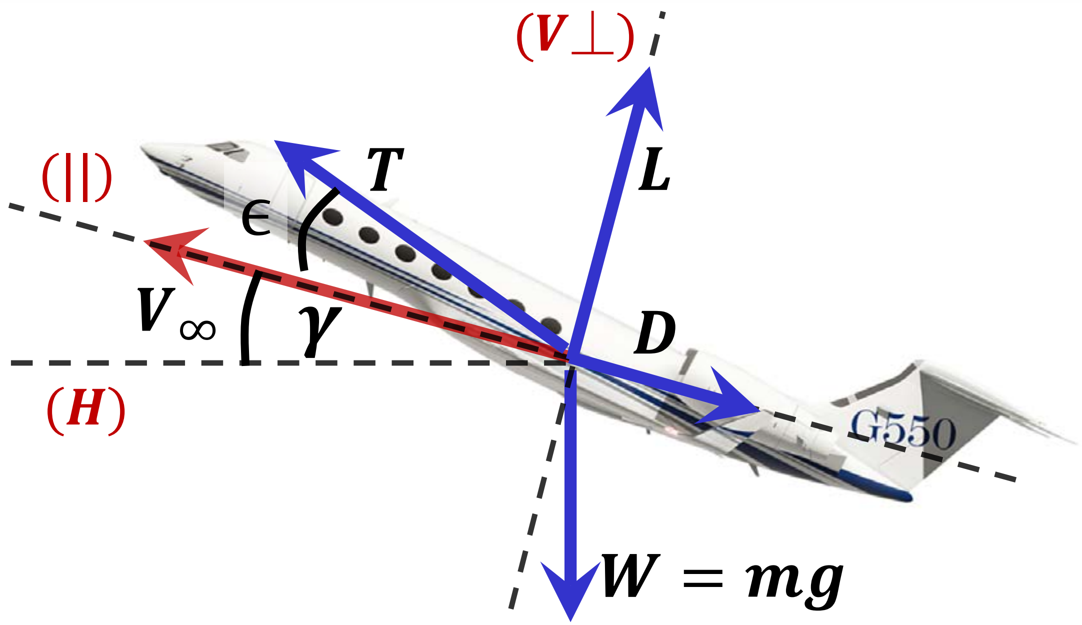

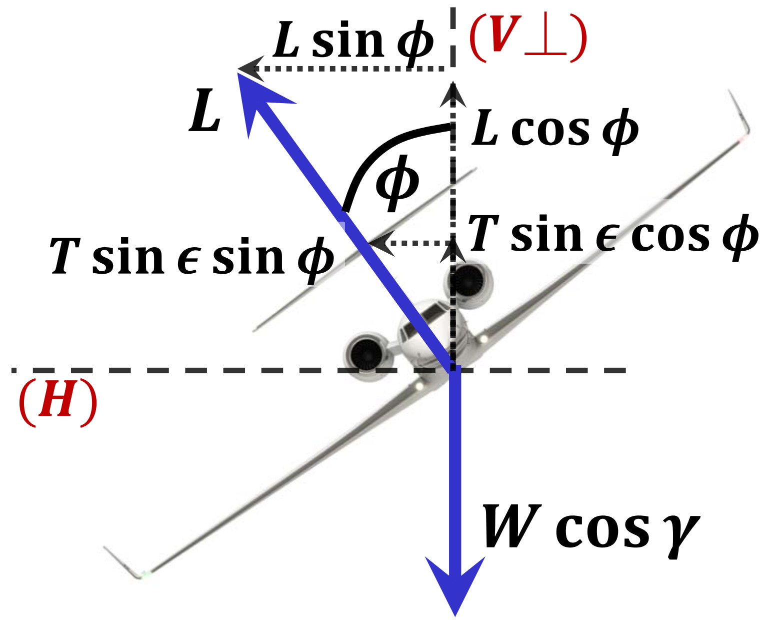

Along axis parallel to flight path ⇨ $m\dfrac{dV_\infty}{dt} = T\cos\epsilon - D - W\sin\gamma$

Along normal to flight path, in vertical plane ⇨ $m\dfrac{V_\infty^2}{r_V} = L\cos\phi + T\sin\epsilon\cos\phi - W\cos\gamma$

Along horizontal axis orthogonal to first two ⇨ $m\dfrac{(V_\infty\cos\gamma)^2}{r_H} = L\sin\phi + T\sin\epsilon\sin\phi$

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Steady, Level Flight

- Constant velocity, constant altitude, wings-level attitude
- Thrust assumed aligned with direction of flight

0; Const. V | 1; $\varepsilon = 0$ | 0; $\gamma = 0$

$$m\cancel{\frac{dV_\infty}{dt}} = T\cancel{\cos\varepsilon} - D - W\cancel{\sin\gamma} \quad \Longrightarrow \quad \boxed{\boldsymbol{T = D}}$$

0; $r_V \to \infty$ | 1; $\phi = 0$ | 0; $\varepsilon = 0$ | 1; $\gamma = 0$

$$m\cancel{\frac{V_\infty^2}{r_V}} = L\cancel{\cos\phi} + T\cancel{\sin\varepsilon}\cos\phi - W\cancel{\cos\gamma} \quad \Longrightarrow \quad \boxed{\boldsymbol{L = W}}$$

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Engine Specific Fuel Consumption (SFC)

- To reduce the effect of size (scale) and facilitate comparison of engine performance, *Specific Fuel Consumption (SFC)* is often used

- For **gas turbines**, the ***Thrust-Specific Fuel Consumption (TSFC)*** is defined as the fuel consumed in order to generate unit thrust

$$c_t\left[\frac{1}{h}\right] = \frac{Fuel\ flow\ \left[\frac{lb}{h}\right]}{Thrust\ [lb]}$$

$$\frac{dW_f}{dt} = -c_t\,T$$

- For **piston engines**, the ***Power-Specific Fuel Consumption (PSFC)*** is defined as the fuel consumed in order to generate unit shaft-power

$$c\left[{}^{lb}/_{hp.h}\right] = \frac{Fuel\ flow\ \left[{}^{lb}/_{h}\right]}{Power\ [hp]}$$

$$\frac{dW_f}{dt} = -cP$$

- Minimizing fuel consumption is naturally desirable from an efficiency perspective
- One way to do this is to design engines with lower c_t and c
- The other way is to minimize the required thrust $\boldsymbol{T_R}$ and required power $\boldsymbol{P_R}$

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Minimizing T_R for Steady, Level Flight

- Recall the simplified equations of motion for steady, level, unaccelerated flight:

$$T_R = D \qquad\qquad L = W$$

- Dividing the first equation by the second and re-arranging terms, the thrust required for steady, level flight is obtained as:

$$\boxed{T_R = \frac{W}{\left(\frac{L}{D}\right)}}$$

- For a given vehicle weight, the thrust required is minimum when the vehicle is flown at the maximum lift-to-drag ratio:

$$\boxed{T_{R_{min}} = \frac{W}{\left(\frac{L}{D}\right)_{max}}}$$

- How can the **lift-to-drag ratio** in steady, level, unaccelerated flight be **maximized**?

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Lift, Drag Coefficients and Drag Polar

- Lift and drag forces can be non-dimensionalized to give dimensionless lift and drag coefficients, C_L and C_D

$$C_L = \frac{L}{\left(\frac{1}{2}\rho_\infty V_\infty^2\right) S_{ref}} \qquad C_D = \frac{D}{\left(\frac{1}{2}\rho_\infty V_\infty^2\right) S_{ref}}$$

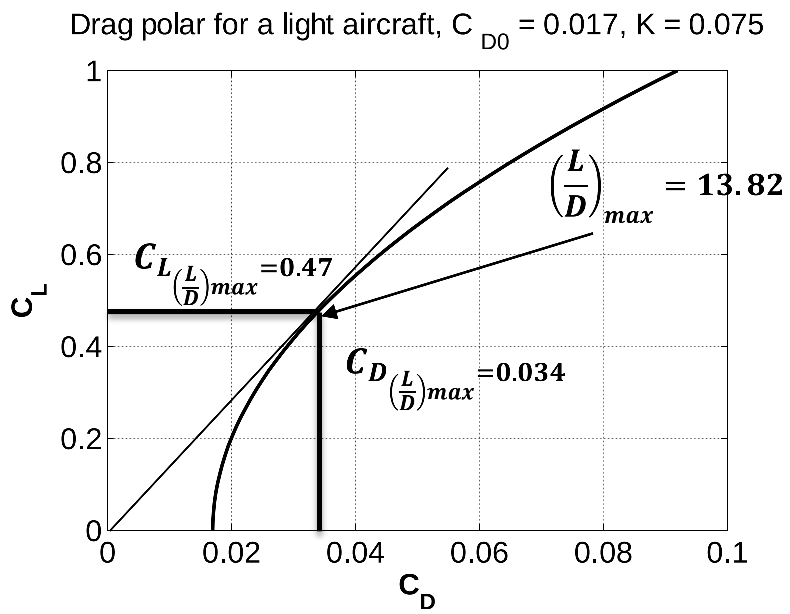
$\frac{1}{2}\rho_\infty V_\infty^2$: Freestream dynamic pressure ($\mathrm{N/m^2}$)

S_{ref}: Reference wing planform area ($\mathrm{m^2}$)

- The airplane's drag characteristics are commonly represented by the drag polar

$$C_D = C_{D_0} + K\, C_L^2$$

- Recall: $\min T_R \rightarrow \max\left(\frac{L}{D}\right) \rightarrow \max\left(\frac{C_L}{C_D}\right)$
- How can $\left(\frac{L}{D}\right)_{max}$ be obtained from the drag polar?



Minimizing T_R for Steady, Level Flight

- An analytical expression for $(L/D)_{max}$ can be obtained by:
  1. Expressing the lift-to-drag ratio in terms of drag polar characteristics
  2. Differentiating with respect to C_L and equating to zero

$$\frac{d}{dC_L}\left[\frac{C_L}{C_D}\right] = \frac{d}{dC_L}\left[\frac{C_L}{C_{D_0} + K\,C_L^2}\right] = 0 \implies \frac{C_{D_0} + KC_L^2 - 2KC_L^2}{\left(C_{D_0} + KC_L^2\right)^2} = 0$$

- Simplifying the numerator gives the condition for max lift-to-drag: $\boxed{C_{D_0} = KC_L^2}$
  – Zero lift and induced drag have equal magnitudes

- The lift & drag coefficients at this flight condition are: $\boxed{C_L = \sqrt{\frac{C_{D_0}}{K}}, C_D = 2C_{D_0}}$

- The maximum lift-to-drag ratio itself, in terms of the drag polar characteristics, now follows as: $\boxed{\left(\frac{L}{D}\right)_{max} = \sqrt{\frac{1}{4KC_{D_0}}}}$

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Airspeed to Obtain Maximum Lift-to-Drag

- Recall the definition of lift coefficient and apply $L = W$ for steady, level flight:

$$C_L = \frac{L}{\left(\frac{1}{2}\rho_\infty V_\infty^2\right)S} = \frac{W}{\left(\frac{1}{2}\rho_\infty V_\infty^2\right)S}$$

- Rearranging the above into an expression for airspeed gives:

$$V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\frac{1}{C_L}}$$

- Recall that the lift coefficient for maximum lift-to-drag ratio is $C_L = \sqrt{\frac{C_{D_0}}{K}}$

- Substituting this yields the airspeed required to attain the maximum lift-to-drag ratio:

$$\boxed{V_{\infty_{\left(\frac{L}{D}\right)_{max}}} = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{K}{C_{D_0}}}}}$$

Minimizing P_R for Steady, Level Flight

- Recall from our discussion for T_R:

$$T_R = \frac{W}{\left(\frac{L}{D}\right)}$$

- Expressing Power using its definition $(P = FV)$, in terms of T and V_∞ gives,

$$P_R = T_R V_\infty = \frac{W}{L/D} V_\infty = \frac{W}{C_L/C_D}\sqrt{\frac{2W}{\rho_\infty S C_L}} = \sqrt{\frac{2W^3 C_D^2}{\rho_\infty S C_L^3}}$$

- Therefore, for a given V_∞, ρ_∞, W, and S,

$$\boxed{P_{R,min} \propto \frac{1}{\left(\frac{C_L^{\frac{3}{2}}}{C_D}\right)_{max}}}$$

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Minimizing P_R for Steady, Level Flight

- As was done for thrust, expressing $\left(\frac{C_L^{\frac{3}{2}}}{C_D}\right)$ in terms of the drag polar, differentiating, and simplifying gives the condition for $\left(\frac{C_L^{\frac{3}{2}}}{C_D}\right)_{max}$:

$$\boxed{3C_{D_0} = KC_L^2}$$

- The lift & drag coefficients at this flight condition are:

$$\boxed{C_L = \sqrt{\frac{3C_{D_0}}{K}}, C_D = 4C_{D_0}}$$

- To obtain velocity for $P_{R_{MIN}}$, recall, for steady, level flight: $V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\frac{1}{C_L}}$

- Substituting the corresponding C_L gives the airspeed at which P_R is minimized:

$$\boxed{V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{K}{3C_{D_0}}}}}$$

T_A vs T_R in-class Discussion

Thrust – Airspeed Relationship

- Recall: $C_L = \dfrac{W}{q_\infty S}$ $\quad C_D = \dfrac{D}{q_\infty S}$ $\quad q_\infty = \dfrac{1}{2}\rho_\infty V_\infty^2$ $\quad C_D = C_{D_0} + K\,C_L^2$
- For steady, level flight: $T_R = D = q_\infty S\,(C_{D_0} + K C_L^2)$
- Thrust required to achieve a desired airspeed:

$$\boxed{T_R = \left(\frac{1}{2}\rho_\infty S\,C_{D_0}\right)V_\infty^2 + \left\{\frac{2KS}{\rho_\infty}\left(\frac{W}{S}\right)^2\right\}\frac{1}{V_\infty^2}}$$

- Multiplying the above by V_∞^2 and solving the resulting bi-quadratic equation, the velocity attainable for a given thrust:

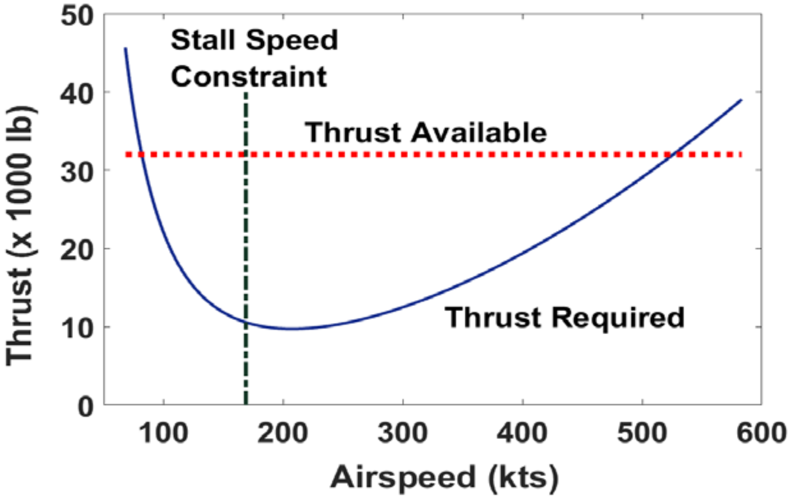
$$\boxed{V_\infty = \left[\frac{\left(\frac{T_R}{W}\right)\left(\frac{W}{S}\right) \pm \left(\frac{W}{S}\right)\sqrt{\left(\frac{T_R}{W}\right)^2 - 4C_{D_0}K}}{\rho_\infty C_{D_0}}\right]^{\frac{1}{2}}}$$

Maximum Velocity

- The extreme attainable velocities are obtained by setting the required thrust equal to the maximum available thrust: $T_R = T_{A,max}$

$$V_\infty = \left[\frac{\left(\frac{T_{A,max}}{W}\right)\left(\frac{W}{S}\right) \pm \left(\frac{W}{S}\right)\sqrt{\left(\frac{T_{A,max}}{W}\right)^2 - 4C_{D_0}K}}{\rho_\infty C_{D_0}}\right]^{\frac{1}{2}}$$

Sea-level, C_{D_0} = 0.025, K = 0.0396, W = 70,000 kg, S = 124 m^2, $C_{L_{max}}$ = 1.2



- The low-speed intersection (above solution using minus sign) typically falls below the stall speed and is not practically relevant

- The high-speed intersection (solution using plus sign) defines the maximum speed of the airplane in level flight

$$V_{max} = \left[\frac{\left(\frac{T_{A,max}}{W}\right)\left(\frac{W}{S}\right) + \left(\frac{W}{S}\right)\sqrt{\left(\frac{T_{A,max}}{W}\right)^2 - 4C_{D_0}K}}{\rho_\infty C_{D_0}}\right]^{\frac{1}{2}}$$

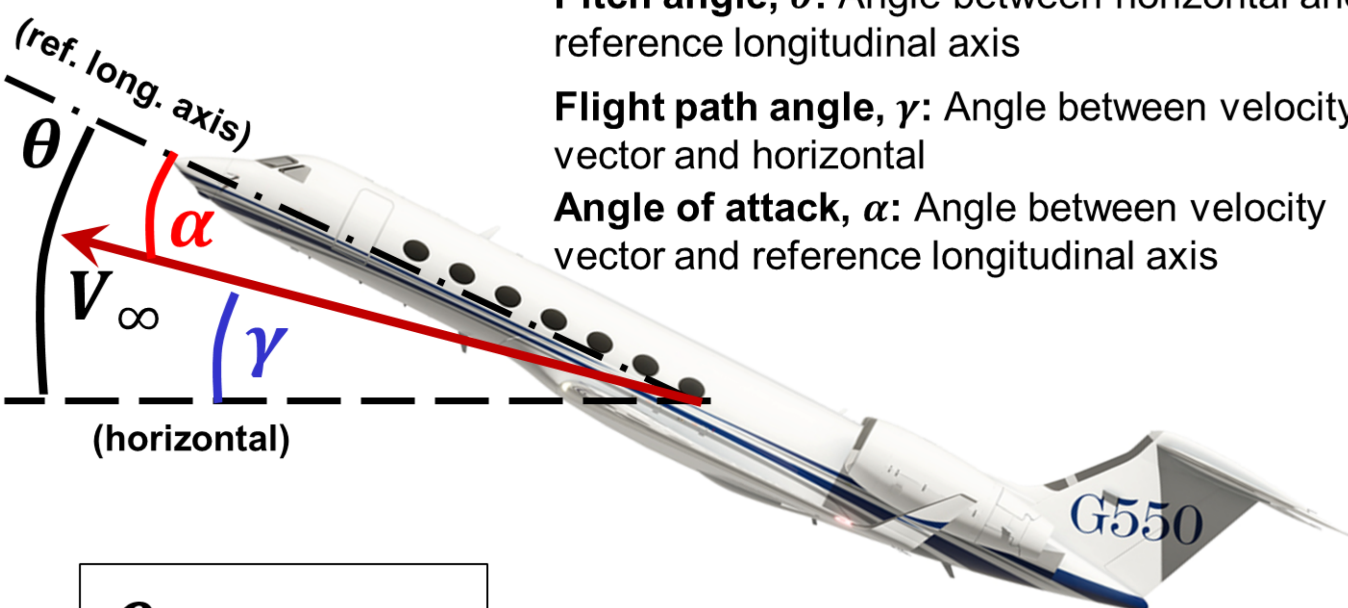
Climb Performance

- Climb performance is an essential part of the overall performance scenario
  - In previous sections we have dealt with *steady, level flight* of an airplane
  - Now we focus to an airplane in *steady, unaccelerated climbing* flight

Pitch angle, θ: Angle between horizontal and reference longitudinal axis

Flight path angle, γ: Angle between velocity vector and horizontal

Angle of attack, α: Angle between velocity vector and reference longitudinal axis



$$\boxed{\theta = \gamma + \alpha}$$

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Climb Performance - Equations of Motion

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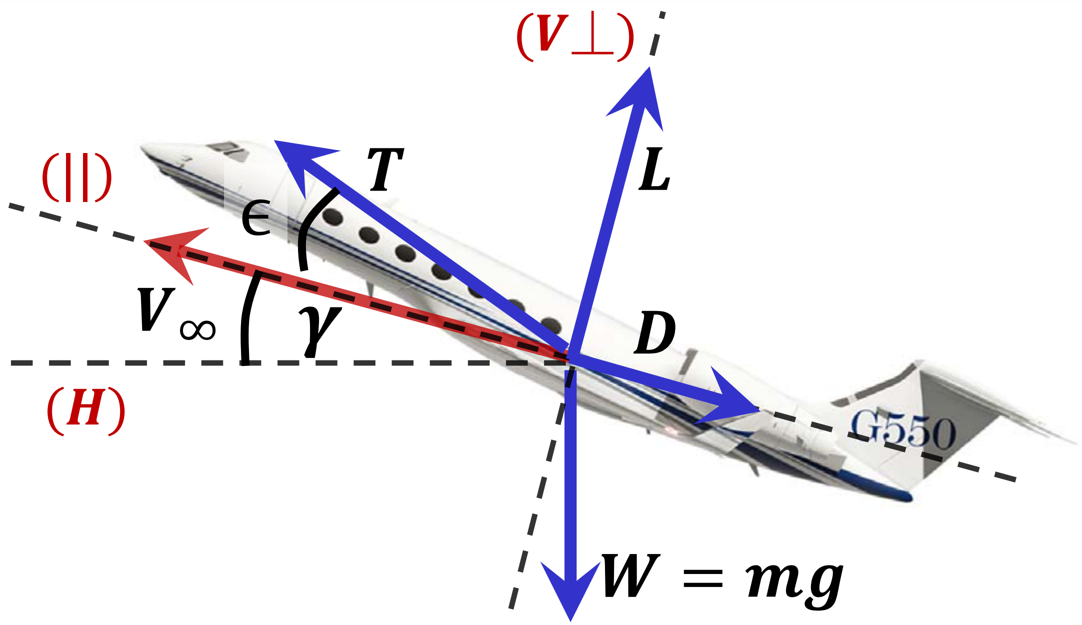

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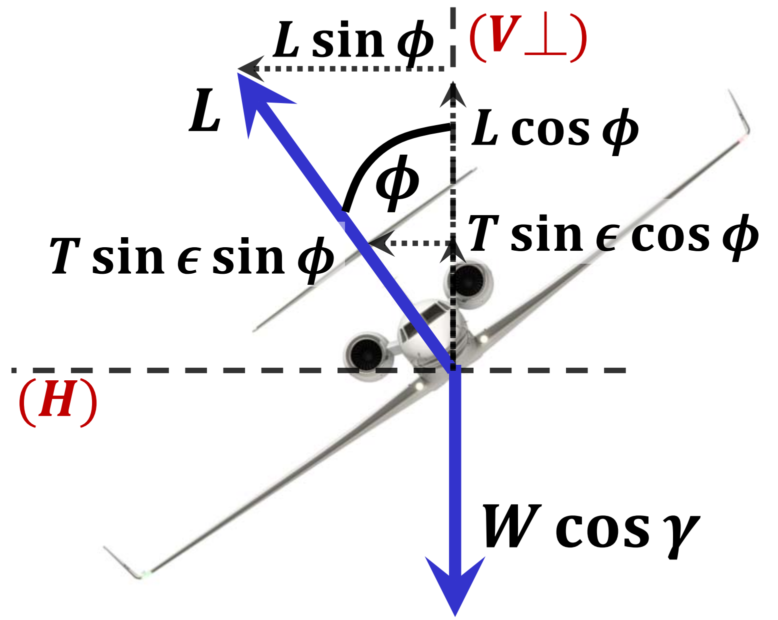

Along axis parallel to flight path ➡ $$m\frac{dV_\infty}{dt} = T\cos\epsilon - D - W\sin\gamma$$

Along normal to flight path, in vertical plane ➡ $$m\frac{V_\infty^2}{r_V} = L\cos\phi + T\sin\epsilon\cos\phi - W\cos\gamma$$

Along horizontal axis orthogonal to first two ➡ $$m\frac{(V_\infty\cos\gamma)^2}{r_H} = L\sin\phi + T\sin\epsilon\sin\phi$$

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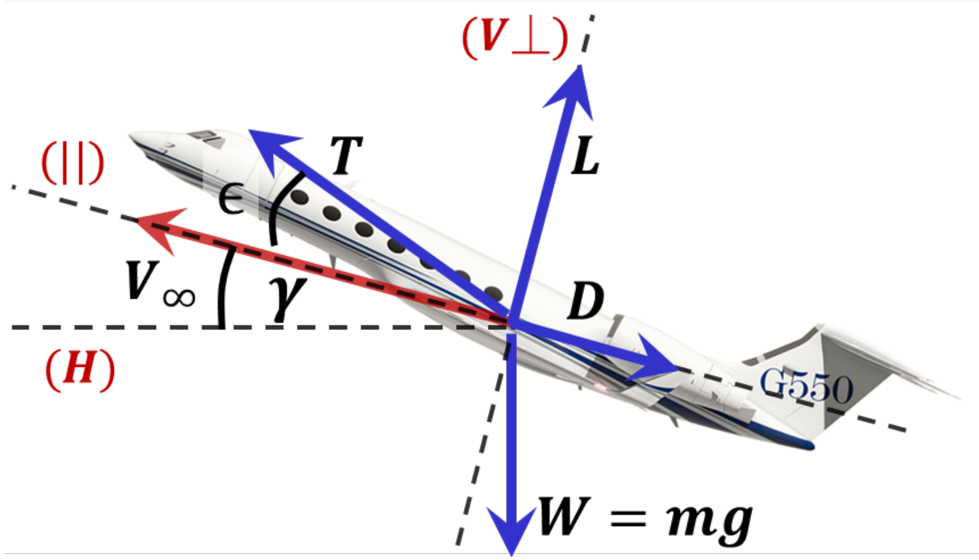
Climb Performance – Equations of Motion

- Constant velocity: $\frac{dV_\infty}{dt} = 0$
- Wings-level flight: $\phi = 0$
- Flight along a straight line, implying $r_H \to \infty, r_V \to \infty$
- This in turn means centripetal accelerations $\frac{V_\infty^2}{r_H}, \frac{(V_\infty \cos\gamma)^2}{r_V} = 0$



- Thrust vector aligned with velocity, i.e., $\epsilon = 0$

- Simplifying the equations of motion with the above, the equations of motion for wings-level, unaccelerated climb are obtained as:

$$\boxed{\boldsymbol{T - D - W\sin\gamma = 0} \qquad \boldsymbol{L - W\cos\gamma = 0}}$$

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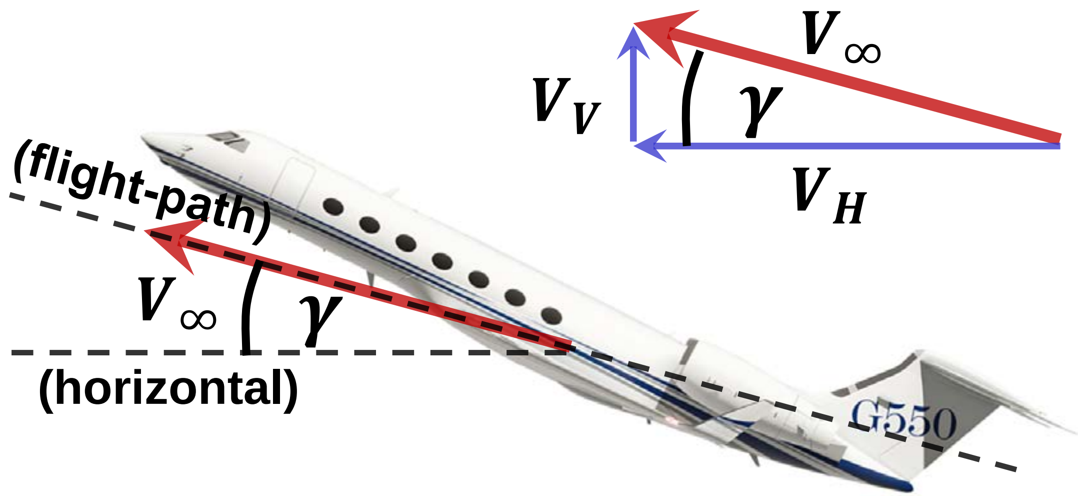
Climb Performance – Kinematic Parameters

- Kinematic parameters:
  - Flight velocity: V_∞
  - Horizontal velocity: V_H
  - Vertical velocity: V_V
  - Flight path angle: γ



- The following kinematic relationships relate the above parameters:
  - Rate of climb or vertical velocity, $V_V = V_\infty \sin\gamma$
  - Horizontal velocity, $V_H = V_\infty \cos\gamma$
  - Flight path angle, $\boldsymbol{\gamma} = \tan^{-1}\left(\frac{V_V}{V_H}\right)$

Climb Performance – Equations of Motion

- Constant velocity: $\frac{dV_\infty}{dt} = 0$
- Wings-level flight: $\phi = 0$
- Flight along a straight line, implying $r_H \to \infty, r_V \to \infty$
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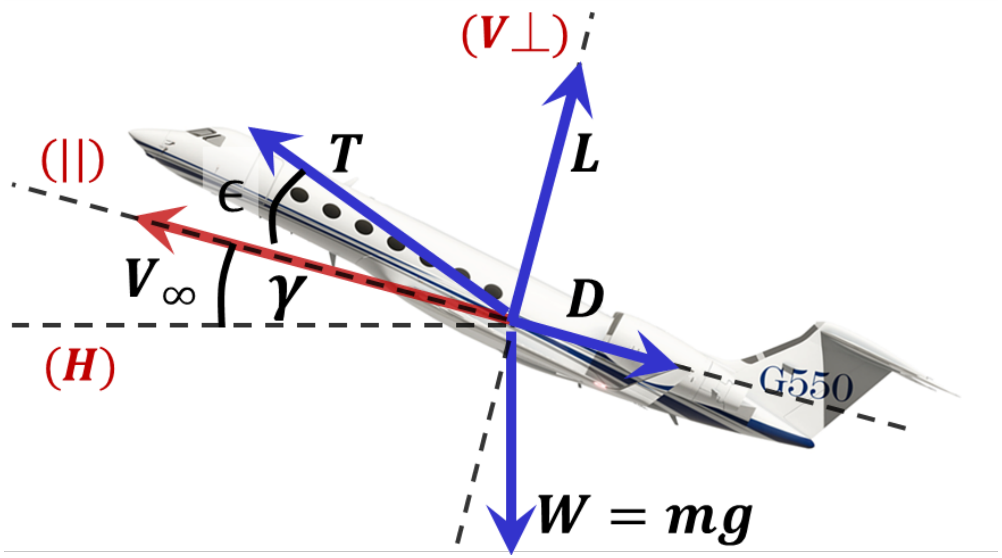

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Climb Performance: Excess & Specific Excess Power

- Recall the first equation of motion

$$T - D - W\sin\gamma = 0$$

- Rearranging and multiplying by V_∞/W

$$\frac{(TV_\infty - DV_\infty)}{W} = V_\infty \sin\gamma = Rate\ of\ Climb$$

- TV_∞ is the power available
- DV_∞ is the power required to overcome drag

$$TV_\infty - DV_\infty = \text{Excess Power} \ldots (8)$$

$$\frac{Excess\ Power}{W} = P_s = Specific\ Excess\ Power \ldots (9)$$

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Climb Performance: Derivation

- For steady climbing flight we can write

$$C_L = \frac{L}{q_\infty S} = \frac{W\cos(\gamma)}{q_\infty S}$$

- From the drag polar,

$$D = q_\infty S C_D = q_\infty S\left(C_{D,0} + KC_L^2\right)$$

$$D = q_\infty S\left(C_{D,0} + \frac{KW^2\cos^2\gamma}{q_\infty S}\right)$$

$$V_\infty \sin\gamma = V_\infty\left[\frac{T}{W} - \frac{1}{2}\rho_\infty V_\infty^2\left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K\cos^2\gamma}{\rho_\infty V_\infty^2}\right] - (10)$$

- Note again the importance of the key design parameters thrust-to-weight, T/W, and wing loading W/S. This time in regard to climb performance
- Eq. 10 is the key to the *exact* analytical solution of the climb performance of an airplane

Climb Performance

- The earlier equation is unwieldly to solve. V_∞ and γ appear on both sides of the equation. You can solve iteratively by either choosing a value for V_∞ or γ and then solving for the corresponding value of the other to get R/C = $V_\infty \sin\gamma$

- Fortunately, for preliminary performance analysis we can simplify the problem by assuming that for the drag expression only, $\cos\gamma \approx 1$

- This assumption leads to remarkably accurate results for climb angles as large as 50°[1]. The normal climb angles of conventional airplanes are usually less than 15°. Hence we will make this assumption for the rest of this section and use Eq. 11 later

$$\boxed{V_\infty \sin\gamma = V_\infty \left[\frac{T}{W} - \frac{1}{2}\rho_\infty V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K}{\rho_\infty V_\infty^2} \right] - (11)}$$

- Given a velocity, the climb angle can be calculated by rearranging Eq. 11 as:

$$\sin\gamma = \frac{T}{W} - \frac{1}{2}\rho_\infty V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K}{\rho_\infty V_\infty^2}$$

[1] Anderson: Aircraft Performance and Design

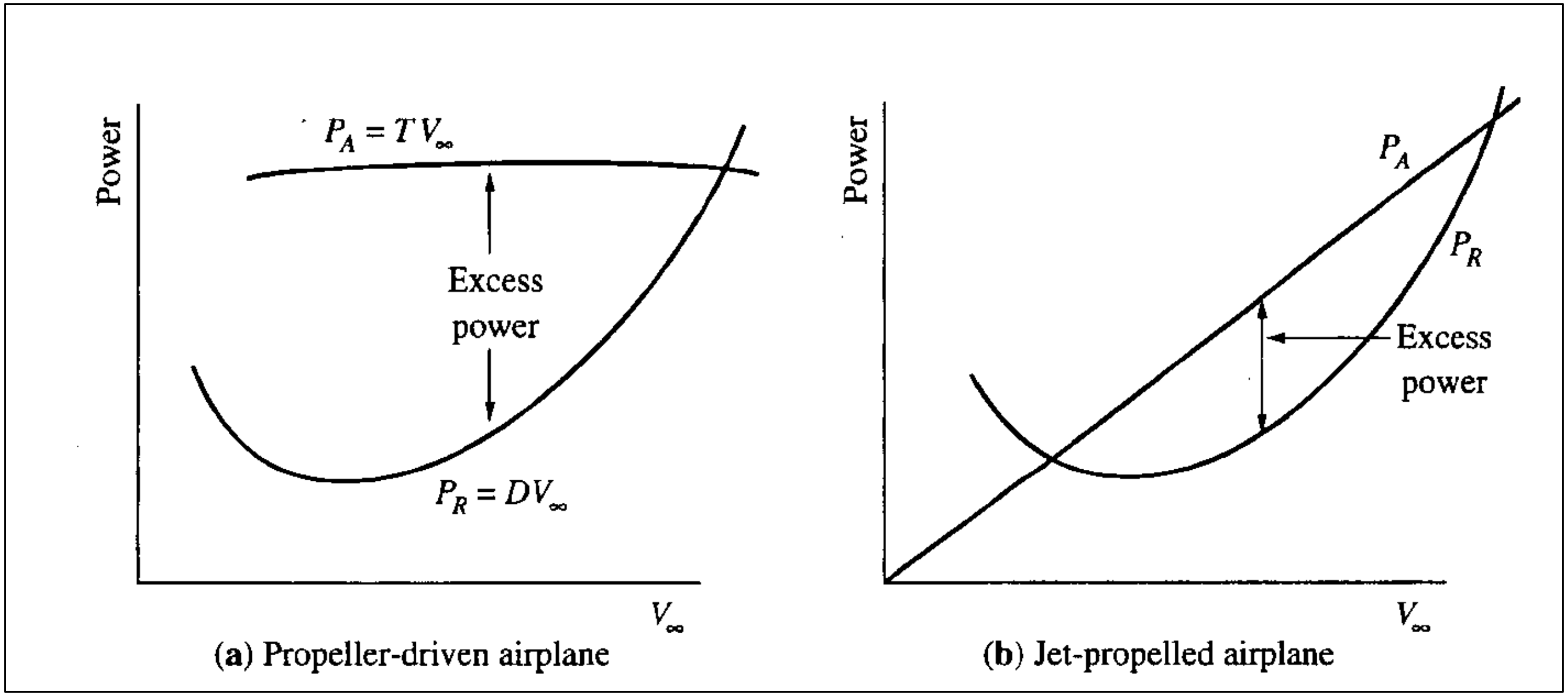
Rate of Climb – Analytical Approach

$$V_\infty \sin\gamma = R/C = V_\infty\left[\frac{T}{W} - \frac{1}{2}\rho_\infty V_\infty^2\left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K}{\rho_\infty V_\infty^2}\right] - (11)$$

- The effect of increasing altitude usually is to decrease R/C. All three terms in Eq. 11 are sensitive to altitude through ρ_∞
- The effect of altitude on T depends on the type of power plant used. For turbojets, turbofans, and unsupercharged piston engines with propellers, thrust at a given V_∞ decreases with altitude. The dominant term in Eq. 11 is T/W; hence when T/W decreases with increasing altitude, R/C also decreases
- For supercharged engine however, power output is reasonable constant; hence at a given V_∞ the thrust output of the propeller can be maintained reasonably constant by increasing the propeller pitch angle. Thus the altitude variation of R/C will depend on how the drag varies with altitude at the same V_∞
- Wing loading also affects R/C. Note that increasing W/S decreases the zero-lift drag and increases the drag due to lift. In the low-velocity range, drag due to lift is dominant; in the high-velocity range zero-lift drag is dominant

Rate of Climb - Graphical Approach

- Recall from earlier that rate of climb is excess power divided by weight. The excess power is shown in the following figure:



- Note that at any given V_∞, the excess power is just difference between the ordinates of the P_A and P_R curves

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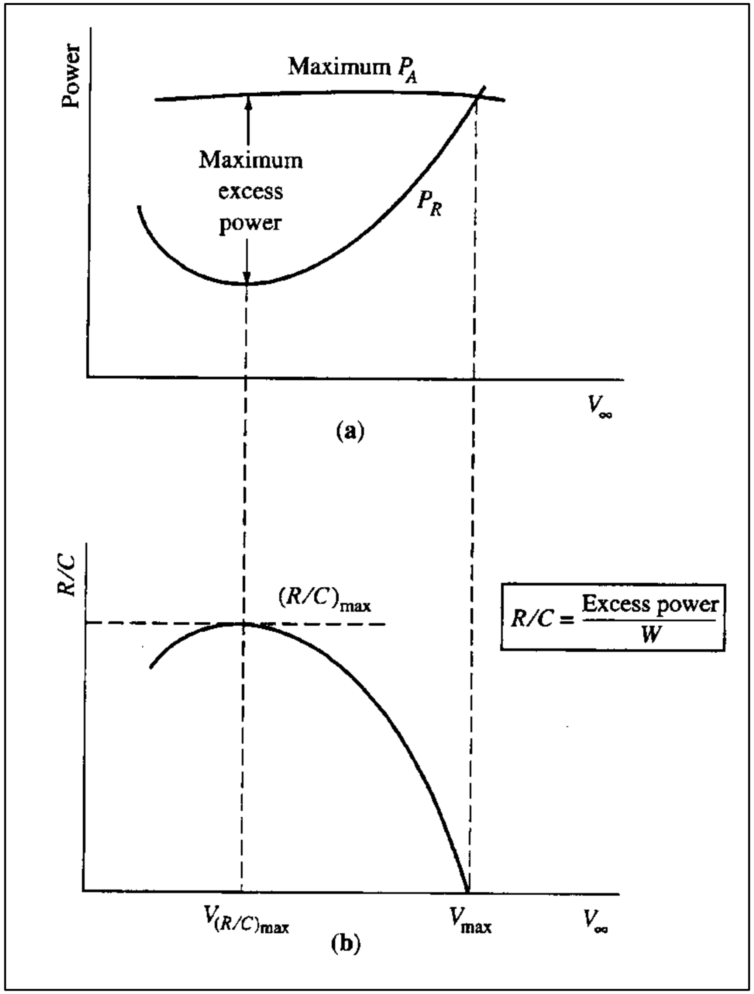
Rate of Climb - Graphical Approach



- This leads directly to a graphical construction of the variation of R/C with V_∞
    1. At any V_∞ measure the excess power
    2. Divide by W, obtaining the R/C at this velocity
    3. Carry out this procedure for a range of V_∞
- The corresponding locus of points is the graph of R/C versus velocity for the airplane
    - R/C curve shown here is at a particular altitude
    - At some velocity $P_A - P_R$ is maximum, this is velocity for R/C max
    - Point of intersection of P_A, P_R curves indicates the maximum speed attainable at that altitude. R/C = 0 at this point

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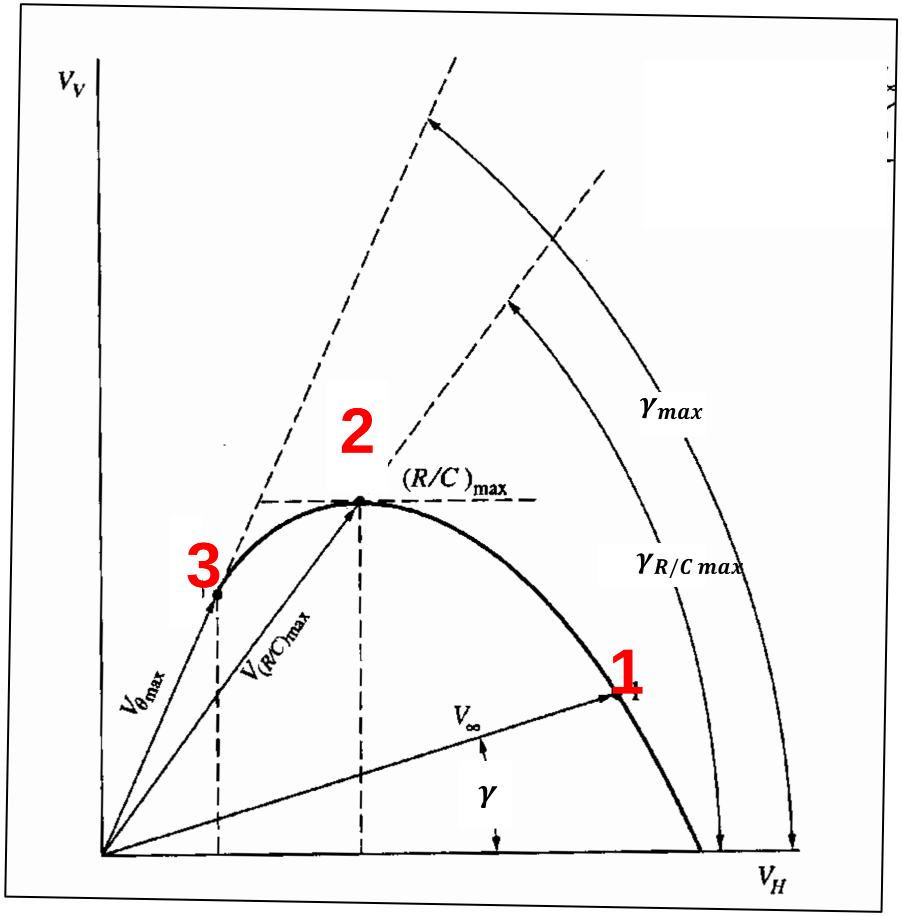
Graphical Approach– Hodograph Diagram

- Another useful graphical construction is the ***hodograph diagram***, which is a plot of the aircraft's vertical velocity V_V versus its horizontal velocity V_H



- Main difference between hodograph and previous curve – axes here are horizontal and vertical velocity
  1. Consider a line drawn from origin to any point (1). Length of this line is V_∞ and the angle it makes with horizontal is climb angle γ
  2. Now consider point (2) – this point corresponds to the maximum rate of climb
  3. A line drawn from origin and tangent to the curve intersects it at point (3). This point corresponds to the maximum climb angle

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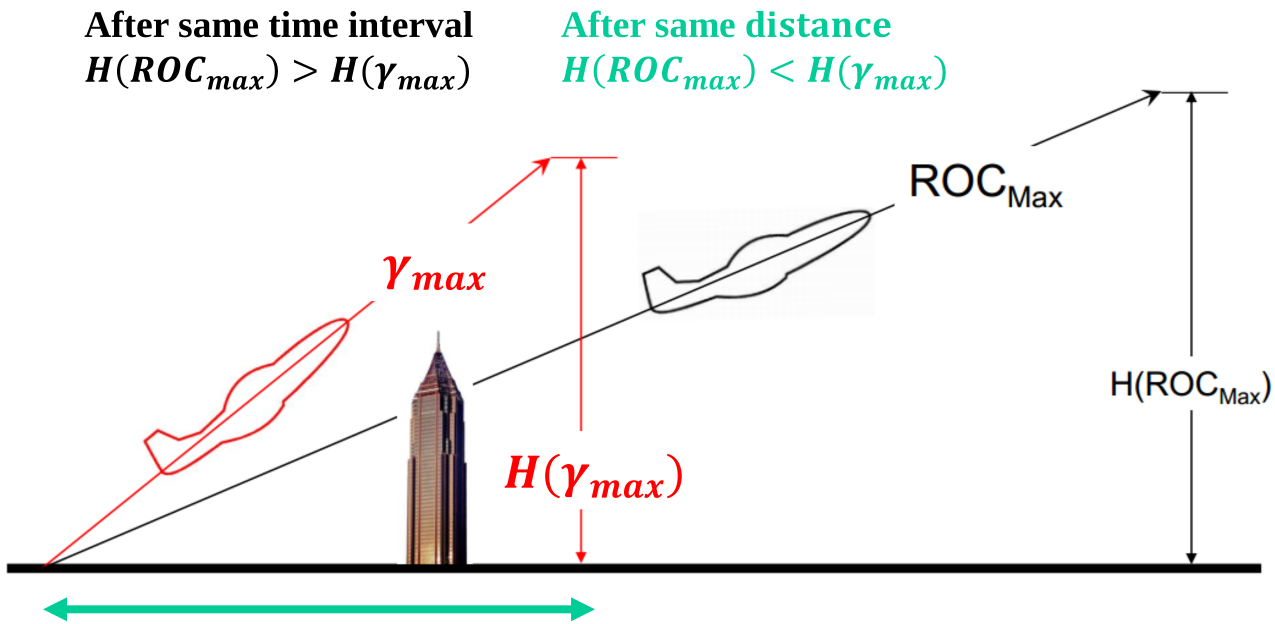
Climb Performance

- We will now go through the two important situations in aircraft steady climb performance identified earlier



Graphics source: Anderson, Aircraft Performance and Design

- We will examine the conditions for above situations for jet propelled and propeller airplanes

Climb Performance – Max climb angle

- Beginning with the simple expression obtained earlier:

$$\boldsymbol{sin(\gamma_{max}) = \frac{T}{W} - \frac{D}{W}}$$

$$\boldsymbol{W = \frac{L}{cos(\gamma)}}$$

$$\boldsymbol{sin(\gamma_{max}) = \frac{T}{W} - \frac{cos(\gamma)}{L/D}}$$

- Using the assumption that $\cos(\gamma) \approx 1$

$$\boldsymbol{sin(\gamma_{max}) = \frac{T}{W} - \frac{1}{L/D}}$$

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Climb Performance – Max climb angle

Jet-propelled airplanes

- For a **jet propelled** airplane thrust is essentially constant with velocity, therefore climb angle is maximum when L/D ratio is maximum -

$$\boldsymbol{sin(\gamma_{max}) = \frac{T}{W} - \frac{1}{(L/D)_{max}}}$$

- Maximum L/D is given by:

$$\boldsymbol{\left(\frac{L}{D}\right)_{max} = \frac{1}{\sqrt{4C_{D,0}K}}}$$

- Therefore, max climb angle for jet propelled aircraft is given by:

$$\boldsymbol{sin(\gamma_{max}) = \frac{T}{W} - \sqrt{4C_{D,0}K}}$$

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Climb Performance – Max climb angle

- Using this angle, the velocity for max climb angle and the rate of climb at this angle can be obtained

$$L = W\ cos\gamma = \frac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$W\ cos\gamma_{max} = \frac{1}{2}\rho_\infty V_{\gamma_{max}}^2 S\sqrt{\frac{C_{D,0}}{K}}$$

- The velocity corresponding to maximum climb angle for jet propelled airplane is

$$V_{\gamma_{max}} = \sqrt{\frac{2}{\rho_\infty}\left(\frac{K}{C_{D,0}}\right)^{1/2}\frac{W}{S}\cos\gamma_{max}}$$

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Climb Performance – Max climb angle

- Finally, the rate of climb that corresponds to maximum climb angle is given by

$$(R/C)_{\gamma_{max}} = V_{\gamma_{max}} \sin \gamma_{max}$$

$$V_{\gamma \max} = \sqrt{\frac{2}{\rho_\infty}\left(\frac{K}{C_{D,0}}\right)^{1/2}\frac{W}{S}\cos\gamma_{\max}} \qquad \sin\gamma_{\max} = \frac{T}{W} - \sqrt{4C_{D,0}K}$$

- γ_{max} does not depend on W/S, $V_{\gamma_{max}}$ varies directly as $(W/S)^{1/2}$
- Hence, everything else being equal, for flight at γ_{max}, the rate of climb is higher for higher wing loadings
- Since $(L/D)_{max}$ does not depend on altitude, γ_{max} decreases with altitude because T decreases with altitude
- However, note that $V_{\gamma_{max}}$ increases with altitude. These are competing effects in determining $(R/C)_{\gamma_{max}}$
- The altitude effect on γ_{max} usually dominates, and $(R/C)_{\gamma_{max}}$ usually decreases with increasing altitude

Climb Performance – Max climb angle

Propeller Airplanes:

- For a **propeller** airplane, the power available is approximately constant. Revisiting the basic equation

$$\sin\gamma = \frac{T}{W} - \frac{1}{L/D} \qquad T_A = \frac{\eta_{pr} P}{V_\infty}$$

- Using the drag polar mentioned earlier:

$$\sin\gamma = \frac{\eta_{pr} P}{V_\infty W} - \frac{1}{2}\rho_\infty V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K}{\rho_\infty V_\infty^2}$$

- Unlike a jet propelled airplane, γ_{max} cannot be easily obtained. BUT after a lot of excruciating math:

$$\boxed{V_{\gamma \max} \approx \frac{4(W/S)K}{\rho_\infty \eta_{pr} (P/W)}}$$

- Once $V_{\gamma_{max}}$ is obtained it can be used to get the actual climb angle - γ_{max}

Cautionary note: For a given airplane, it is possible for $V_{\gamma_{max}}$ to be less than the stalling velocity. For such a case, it is not possible for an airplane to the achieve the theoretical maximum climb angle

Climb Performance – Max rate of climb

- <u>Jet-propelled airplanes</u>:
  - For a jet-propelled airplane thrust T is relatively constant with V_∞. Max rate of climb can be obtained by differentiating the rate of climb equation (Eq.11) with velocity

$$\frac{d(R/C)}{dV_\infty} = \frac{T}{W} - \frac{3}{2}\rho_\infty V_\infty^2 \left(\frac{W}{S}\right)^{-1} C_{D,0} + \frac{W}{S}\frac{2K}{\rho_\infty V_\infty^2}$$

- Setting the right hand side equal to zero and simplifying, we get

$$V_\infty^2 - \frac{2(T/W)(W/S)}{3\rho_\infty C_{D,0}} - \frac{4K(W/S)^2}{3\rho_\infty^2 C_{D,0} V_\infty^2} = 0$$

- Multiplying by V_∞^2 and using $\left(\frac{L}{D}\right)_{max} = \frac{1}{\sqrt{4C_{D,0}K}}$ in the last term we get the following 4th order equation in V_∞

$$\boxed{\boldsymbol{V_\infty^4} - \frac{2\left(\frac{T}{W}\right)\left(\frac{W}{S}\right)}{3\rho_\infty C_{D,0}}\boldsymbol{V_\infty^2} - \frac{\left(\frac{W}{S}\right)^2}{3\rho_\infty^2 C_{D,0}^2\left(\frac{L}{D}\right)^2} = 0}$$

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Climb Performance – Max rate of climb

The 4th order equation can be solved analytically (not included here) to obtain:

$$V_{(R/C)_{\max}} = \left\{ \frac{(T/W)(W/S)}{3\rho_\infty C_{D,0}} \left[1 + \sqrt{1 + \frac{3}{(L/D)_{\max}^2 (T/W)^2}} \right] \right\}^{1/2}$$

Using the above expression for velocity at max rate of climb, the max rate of climb itself can be obtained using this velocity and an equation from earlier slides (detailed derivations in Anderson[1]):-

$$(R/C)_{\max} = \left[\frac{(W/S)Z}{3\rho_\infty C_{D,0}} \right]^{1/2} \left(\frac{T}{W} \right)^{3/2} \left[1 - \frac{Z}{6} - \frac{3}{2(T/W)^2 (L/D)_{\max}^2 Z} \right]$$

$$Z \equiv 1 + \sqrt{1 + \frac{3}{(L/D)_{\max}^2 (T/W)^2}}$$

1. Aircraft Performance and Design - Anderson

Climb Performance – Max rate of climb

- Implications

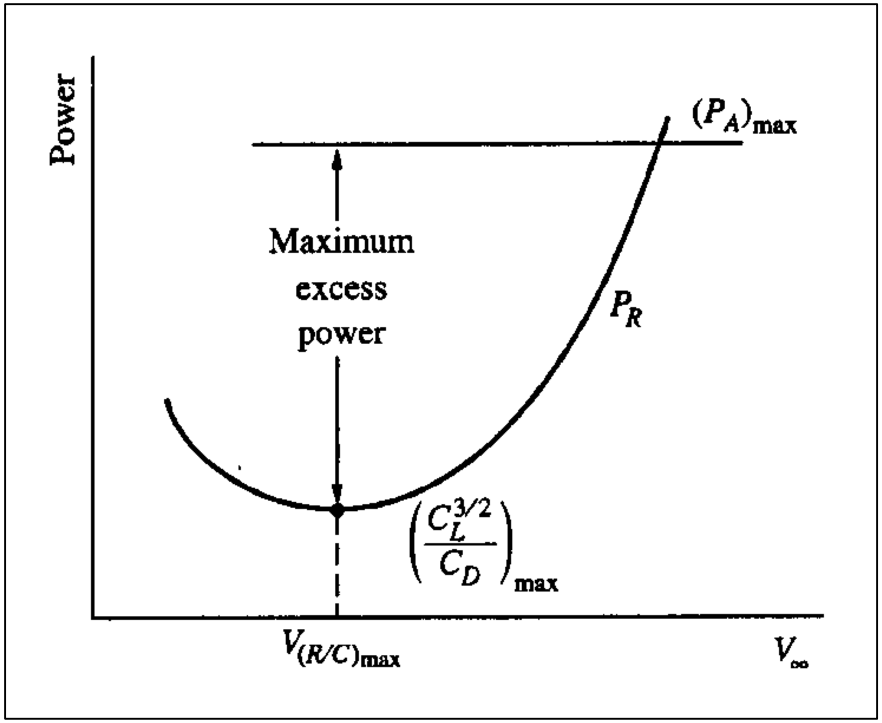
$$V_{(R/C)_{\max}} = \left\{ \frac{(T/W)(W/S)}{3\rho_\infty C_{D,0}} \left[1 + \sqrt{1 + \frac{3}{(L/D)^2_{\max} (T/W)^2}} \right] \right\}^{1/2}$$

$$(R/C)_{\max} = \left[\frac{(W/S)Z}{3\rho_\infty C_{D,0}} \right]^{1/2} \left(\frac{T}{W} \right)^{3/2} \left[1 - \frac{Z}{6} - \frac{3}{2(T/W)^2 (L/D)^2_{\max} Z} \right]$$

- Thrust-to-weight ratio plays a powerful role in determining $(R/C)_{max}$.
- Increasing wing loading increases both $V_{(R/C)\ max}$ and $(R/C)_{max}$. Both are proportional to $(W/S)^{1/2}$
- The effect of increasing altitude on $V_{(R/C)max}$ can be seen here. Assuming that T decreases with increasing altitude we see that $V_{(R/C)max}$ is increased with increasing altitude.
- However $(R/C)_{max}$, being dominated by the thrust-to-weight ratio, decreases with an increase in altitude.

Climb Performance – Max rate of climb

Propeller-driven airplane



$$\left(\frac{R}{C}\right)_{max} = \frac{max\ excess\ power}{W}$$

- For a propeller-driven airplane with power available reasonably constant with velocity, condition for maximum rate of climb is shown here
- Note that the maximum excess power, hence $(R/C)_{max}$ occurs at the flight velocity for minimum power required
- Therefore, for propeller airplane it is easier to approach max rate of climb problem from this insight rather than through calculus as was done for jet

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Climb Performance – Max rate of climb

- Maximum excess power, hence $(R/C)_{max}$ occurs at the flight velocity for minimum power required, which occurs when the airplane is flying at $\left(\frac{C_L^{3/2.}}{C_D}\right)_{max}$

- The flight velocity for this condition was obtained earlier, therefore the *flight velocity for the maximum rate of climb* for propeller airplane is

$$V_{R/C_{max}} = \left(\frac{2}{\rho_\infty}\sqrt{\frac{K}{3C_{D,0}}}\left(\frac{W}{S}\right)\right)^{1/2}$$

- An expression for the maximum rate of climb can be found by inserting the above equation in Eq.11 and noting that $TV_\infty = P_A = \eta_{pr} P$ for a propeller-driven airplane

$$\left(\frac{R}{C}\right)_{max} = \frac{\eta_{pr}P}{W} - \left[\frac{2}{\rho_\infty}\sqrt{\frac{K}{3C_{D,0}}}\left(\frac{W}{S}\right)\right]^{\frac{1}{2}}\left(\frac{1.155}{\left(\frac{L}{D}\right)_{max}}\right)$$

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Climb Performance – Max rate of climb

Implications for propeller-driven airplane

$$\left(\frac{R}{C}\right)_{max} = \frac{\eta_{pr} P}{W} - \left[\frac{2}{\rho_\infty}\sqrt{\frac{K}{3C_{D,0}}}\left(\frac{W}{S}\right)\right]^{\frac{1}{2}}\left(\frac{1.155}{\left(\frac{L}{D}\right)_{max}}\right)$$

- Note that the dominant influence on $(R/C)_{max}$ is the power-to-weight ratio
- The effect of wing loading is secondary, but interesting. $V_{(R/C)max}$ increases with an increase in W/S. However, it is seen that $(R/C)_{max}$ decreases with an increase in W/S
- Note that this is in contrast to a jet-propelled airplane where an increase in W/S increases $(R/C)_{max}$
- Propeller-driven airplanes are penalized in terms of $(R/C)_{max}$ when they have a high wing loading
- The effect of increasing altitude is to increase $V_{(R/C)max}$ and decrease $(R/C)_{max}$. This is even true for supercharged reciprocating engines

Absolute, Service, Cruise, and Combat Ceilings

- **Absolute ceiling:** The altitude where the maximum rate of climb is zero: $(R/C)_{max} = 0$

- This is not a very practical quantity, since it can only be reached asymptotically. The following are more practical quantities:

- **Service ceiling:** The highest altitude where a specified maximum rate of climb is achievable.
  - For piston-engine aircraft: Typically 100 ft/min
  - For jet-engine aircraft: Typically 500 ft/min

- **Cruise ceiling:** Altitude where maximum rate of climb is 300 ft/min

- **Combat ceiling:** Altitude where maximum rate of climb is 500 ft/min

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Determination of Aircraft Ceilings

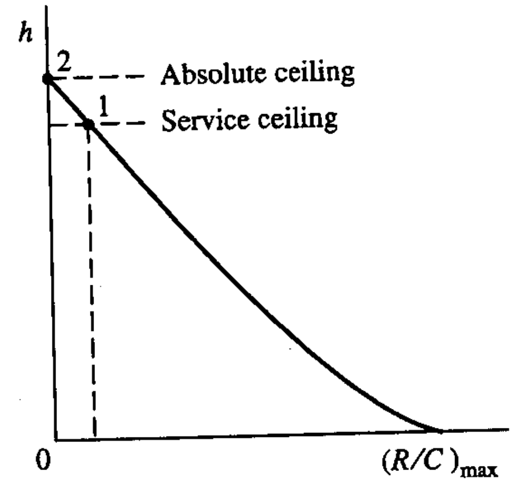
- For a range of altitudes, compute the maximum rate of climb $(R/C)_{max}$. Recall:

$$(R/C)_{max} = P_s = \frac{P_A - P_R}{W}$$

- Plot these results to obtain the variation of maximum climb performance with altitude

- Query the curve at $(R/C)_{max} = 0$ to obtain the absolute ceiling

- Query the curve at the relevant $(R/C)_{max} > 0$ to obtain service, cruise, and combat ceilings



Sketch of variation of maximum rate of climb with altitude, illustrating absolute and service ceilings.

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Time to Climb

- Consider an aircraft climbing at a rate of climb (R/C)
- By definition, $\frac{dh}{dt} = (R/C)$, which yields $dt = \frac{dh}{(R/C)}$
- The time Δt required to climb from initial altitude h_1 to final altitude h_2 is obtained by integrating the above

$$\boldsymbol{\Delta t = \int_{h_1}^{h_2} dt = \int_{h_1}^{h_2} \frac{dh}{(R/C)}}$$

- It is clear that the time to climb can be minimized by maximizing the rate of climb at each altitude

$$\boldsymbol{\Delta t_{\mathrm{min},\mathrm{h}_1 \rightarrow h_2} = \int_{h_1}^{h_2} \frac{dh}{(R/C)_{max}}}$$

- The above integral can be solved graphically or analytically

Minimum Time to Climb – Graphical Approach

- Recall that:

$$\Delta \mathbf{t}_{\mathbf{min,h_1} \to h_2} = \int_{h_1}^{h_2} \frac{dh}{(R/C)_{max}}$$

- Or, stated another way:

$$\Delta \mathbf{t}_{\mathbf{min,h_1} \to h_2} = \int_{h_1}^{h_2} (R/C)_{max}^{-1} \cdot dh$$

- Which is mathematically equal to the area under a graph of $(R/C)_{max}^{-1}$ versus altitude h

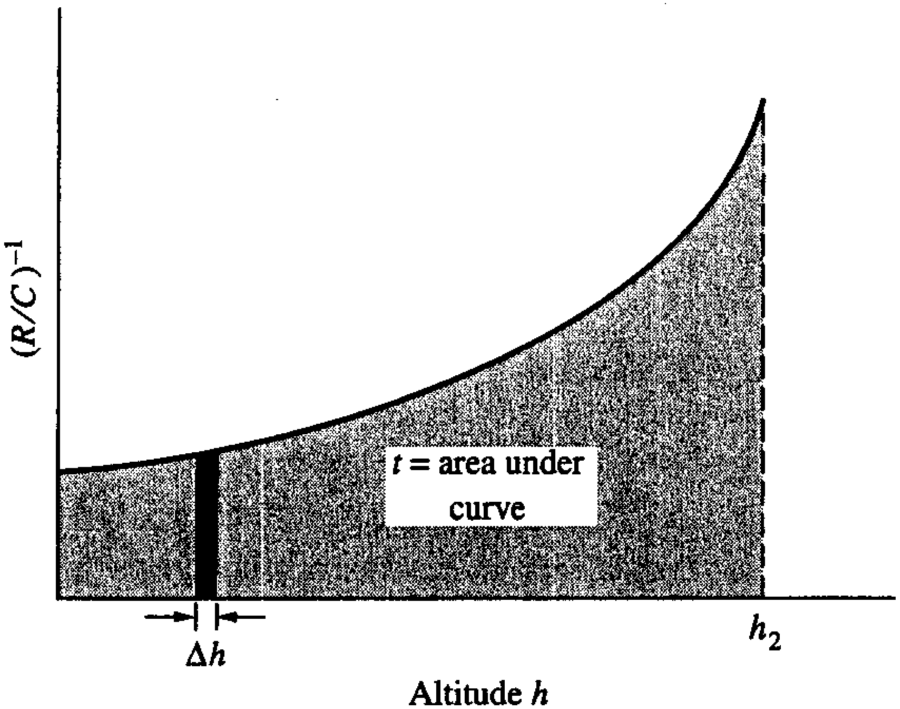

Figure 5.46 Graphical representation of the time to climb to altitude h_2.

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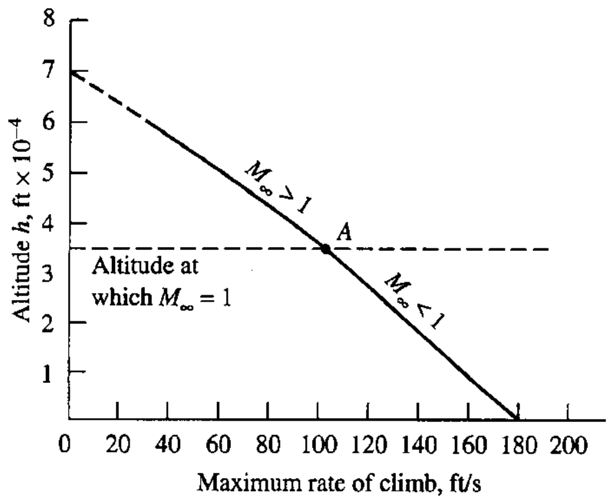
Minimum Time to Climb – Analytical Approach

- Both power required and power available vary nonlinearly with altitude and as such, there is no exact analytical solution for the integral

- However, it may be reasonable to use a linear approximation for variation of maximum rate of climb with altitude

$$(R/C)_{max} = a + bh$$

- In this case, the integral has an analytical solution which is given by



(a) Altitude variation of maximum rate of climb

Figure 5.45 Variations of maximum rate of climb and tl

$$\Delta t_{\min,h_1 \rightarrow h_2} = \int_{h_1}^{h_2} \frac{dh}{(R/C)_{max}} = \int_{h_1}^{h_2} \frac{dh}{a + bh} = ln\left[\frac{a + b\,h_2}{a + b\,h_1}\right]$$

Gliding (Unpowered) Flight

- During flight, if the power required is larger than the power available, the aircraft descends. In the extreme case of no power at all, the airplane will be in *gliding, or unpowered, flight*. This will be the case when
  - The engine(s) quit(s) during flight
    - *Foreign object debris (FOD)* – US Airways 1549 glided and successfully landed in the Hudson river (2009), - “The Miracle on the Hudson”
    - *Human Error* – Air Canada 143 ran out of fuel due to a fueling calculation error and glided to safety on a decommissioned runway (1983), - “The Gimli Glider”
  - Unpowered gliders and sailplanes
- Examples of gliding air vehicles:
  - A sailplane is an expensive, high-performance unpowered aircraft
  - A glider is a crude low-performance unpowered aircraft

Gliding flight is a special (opposite) case of our previous considerations dealing with climb

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In-class exercise - Paper airplane problem

- You are participating in a paper airplane competition, and have been asked to design a glider that performs the best in the following ways:
  a) Gliding the longest distance
  b) Staying aloft for the longest time period
  c) Accurately hitting a target

- In the capacity of an accomplished aerospace engineer, explain what parameters you would design the paper airplane for with sound quantitative reasoning for each of the three cases…

Gliding (Unpowered) Flight

- The force diagram for an unpowered aircraft in descending flight is shown in the following figure:

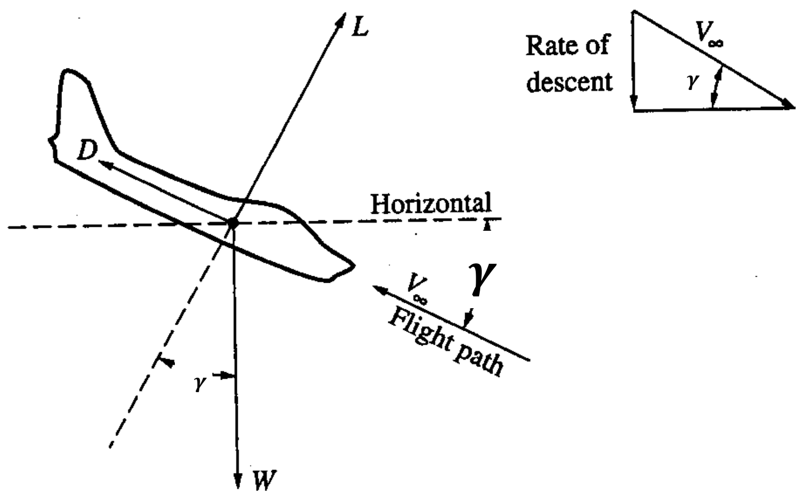

Figure 5.39 Force and velocity diagram for gliding flight.

- For steady, unaccelerated descent, where γ is the flight path angle:

$$L = W\cos\gamma$$

$$D = W\sin\gamma$$

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Gliding (Unpowered) Flight

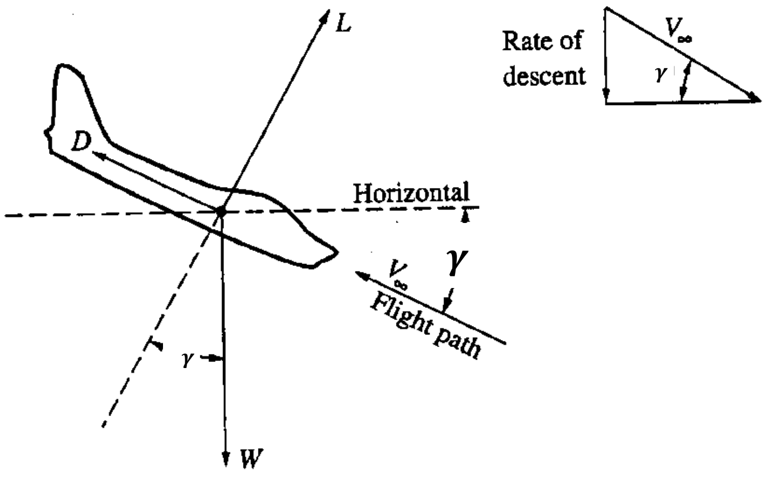

Figure 5.39 Force and velocity diagram for gliding flight.

- The equilibrium glide angle is obtained by dividing the equations for *L* and *D*:

$$\frac{\sin\gamma}{\cos\gamma} = \frac{D}{L} \quad \text{or,} \quad \boxed{Tan\gamma = \frac{1}{L/D}}$$

- Clearly, the glide angle is strictly a function of the lift-to-drag ratio; the higher the L/D, shallower the glide angle
- Note that the smallest equilibrium glide angle occurs at $(L/D)_{max}$

$$\boxed{Tan\gamma_{\min} = \frac{1}{(L/D)_{\max}}}$$

Gliding (Unpowered) Flight

- The rate of descent, a.k.a. the sink rate, is the downward vertical velocity of the airplane V_V. For unpowered flight, it is the analog of rate of climb for powered flight

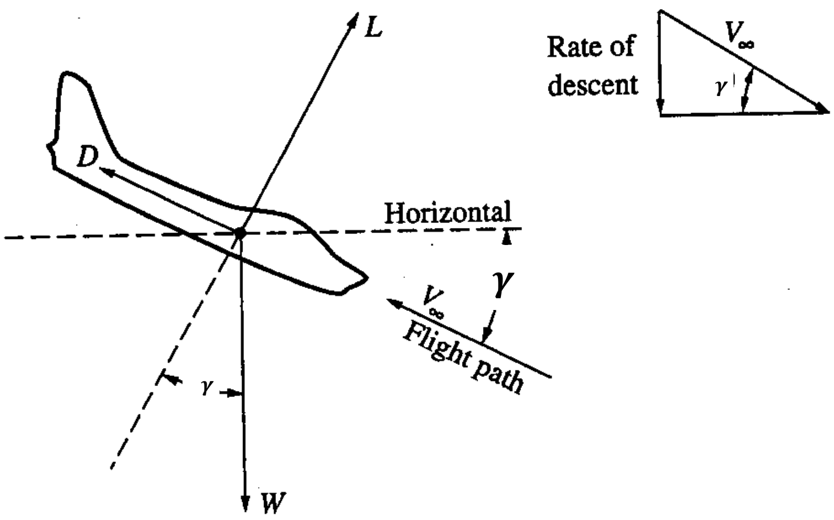

Figure 5.39 Force and velocity diagram for gliding flight.

- Rate of descent : $V_V = V_\infty \sin\gamma$
- The rate of descent, by convention, is a positive number in the downward direction
- Multiplying the equation for drag by V_∞ we get,

$$DV_\infty = WV_\infty \sin\gamma = WV_V$$

- Rearranging for the sink rate(V_V),

$$V_V = \frac{DV_\infty}{W}$$

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5.10.3 Gliding (Unpowered) Flight

- With the assumption that $\cos\gamma = 1, DV_\infty$ is simply the power required for steady, level flight
- Hence, the variation of V_V with velocity is the same as the power required curve, divided by weight. This variation is shown in the following figure:

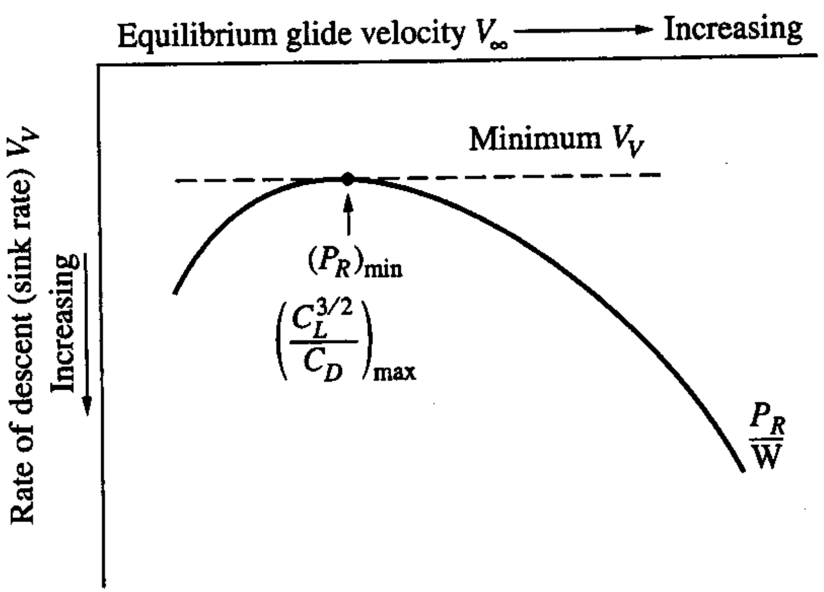

Figure 5.42 Rate of descent versus equilibrium guide velocity.

- The minimum sink rate occurs at V_∞ for minimum power required
- The conditions for minimum sink rate are the same as those for $(P_R)_{min}$, derived previously

$$\boxed{3C_{D_0} = KC_L^2}$$

$$\boxed{V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{K}{3C_{D_0}}}}}$$

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Gliding (Unpowered) Flight

- We will now develop an analytical expression for the sink rate V_V to demonstrate that indeed is equivalent to the condition for $(P_R)_{min}$, in steady, level flight,

$$V_V = V_\infty \sin\gamma = (\sin\gamma)\sqrt{\frac{2\cos\gamma}{\rho_\infty C_L}\frac{W}{S}}$$

- Dividing the equations for drag and lift in gliding flight, and rearranging gives,

$$\sin\gamma = \frac{D}{L}\cos\gamma = \frac{C_D}{C_L}\cos\gamma$$

- Substituting for $\sin\theta$ in the expression for V_V, and approximating $\cos\theta \approx 1$ gives,

$$V_V = \sqrt{\frac{2}{\rho_\infty (C_L^3/C_D^2)}\frac{W}{S}}$$

- $(V_V)_{min}$ occurs at $(C_L^{3/2}/C_D)_{max}$.
- The sink rate decreases with decreasing altitude.
- The sink rate increases as the square root of the wing loading.

Gliding (Unpowered) Flight

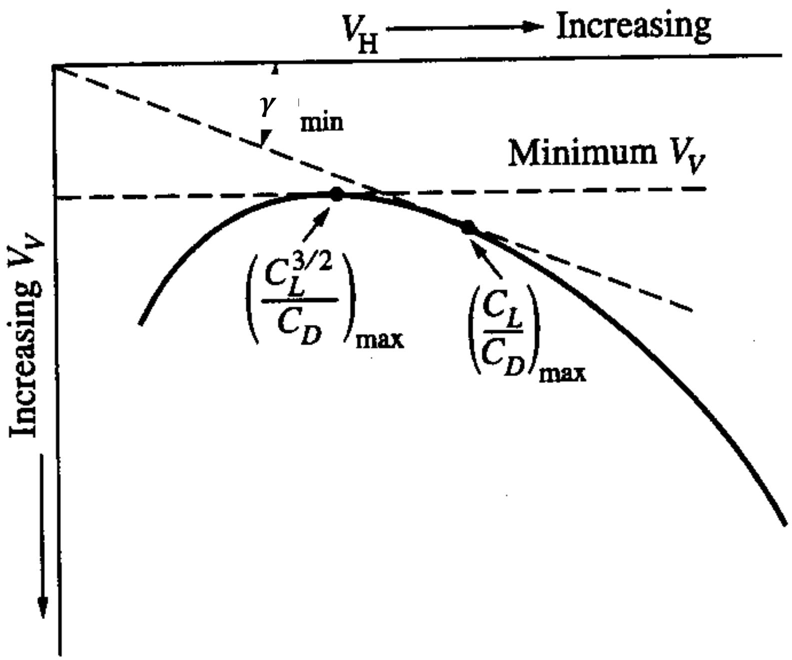

Figure 5.43 Hodograph for unpowered flight.

- A line from the origin tangent to the hodograph curve defines γ_{min}
- Note that the minimum sink rate does not correspond to the minimum glide angle
- V_∞ for the minimum sink rate is less than that for the minimum glide angle
- **Q.** Can you think of a situation where both the conditions(γ_{min} and $V_{V_{min}}$) might be useful?
  - **A.** Glider pilots switch between maximizing range (γ_{min}) when searching for thermals, and minimizing sink rate ($V_{V_{min}}$) when flying in a thermal

Range

By definition, *range* is the total distance (measured with respect to the ground) traversed by an airplane on one load of fuel. Range is denoted by R

- In order to calculate the range of a fuel burning airplane, the fuel depletion rate has to be related to the distance flown

- To that end, consider the following weight breakdown of a fuel burning airplane:

  1. W_0—gross weight of the airplane including everything; full fuel load, payload, crew, structure, etc.
  2. W_f—weight of fuel; this is an instantaneous value, and it changes as fuel is consumed during flight
  3. W_1—weight of the airplane when the fuel tanks are empty

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Range

- At any instant during the flight, the weight of the airplane W is,

$$W = W_1 + W_f$$

- Since W_f is decreasing during flight, *W* is also decreasing. The time rate of change of weight is,

$$\frac{dW}{dt} = \frac{dW_f}{dt} = \dot{W}_f$$

- Note that both $\frac{dW}{dt}$ and $\dot{W}_f$ are negative numbers because fuel is being consumed i.e. W and W_f are decreasing

- Engine performance manifests into $\dot{W}_f$ through the *Thrust*

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Recall: Engine Specific Fuel Consumption (SFC)

- For **gas turbines**, the ***Thrust-Specific Fuel Consumption (TSFC)*** is defined as the fuel consumed in order to generate unit thrust
- $c_t \left[\frac{1}{h}\right] = \frac{Fuel\ flow\ \left[\frac{lb}{h}\right]}{Thrust\ [lb]}$
- $\frac{dW_f}{dt} = -c_t\, T$

- For **piston engines**, the ***Power-Specific Fuel Consumption (PSFC)*** is defined as the fuel consumed in order to generate unit shaft-power
- $c \left[\frac{lb}{hp.h}\right] = \frac{Fuel\ flow\ \left[\frac{lb}{h}\right]}{Power\ [hp]}$
- $\frac{dW_f}{dt} = -cP$

- The time rate of weight change, can now be related to the to the thrust/power produced and the engine technology level(c_t, c) through the definition of TSFC and PSFC

- Note that c_t can be related to c using the definition of power, and propulsive efficiency that accounts for losses incurred between shaft power and propulsive power generation as:

$$\boxed{c_t = \frac{c\, V_\infty}{\eta_{pr}}}$$

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Range

- We are now ready to develop a general relation for the calculation of range
- Consider an airplane is steady, level flight. Let s denote horizontal distance covered over the ground. In the absence of wind the airplane's velocity is:

$$V_\infty = \frac{ds}{dt} \Longrightarrow ds = V_\infty dt$$

- Recall:

$$\frac{dW_f}{dt} = -c_t\, T \Longrightarrow dt = -\frac{dW_f}{c_t T}$$

- Substituting for dt in the expression for ds, and multiplying both the numerator and denominator by W we get,

$$ds = -\frac{V_\infty}{c_t}\frac{W}{T}\frac{dW}{W}$$

- Using $T = D$, and $L = W$ for steady, level cruise,

$$\boxed{ds = -\frac{V_\infty}{c_t}\frac{L}{D}\frac{dW}{W}}$$

The Breguet Range Equation

- The range between any 2 points in cruise can be calculated by integrating,

$$ds = -\int_{W_{ini}}^{W_{fin}} \frac{V_\infty}{c_t}\frac{L}{D}\frac{dW}{W} \quad \Longleftrightarrow \quad ds = \int_{W_{fin}}^{W_{ini}} \frac{V_\infty}{c_t}\frac{L}{D}\frac{dW}{W}$$

- W_{ini}, W_{fin} are initial and final weights respectively
- Restrictions are steady, level flight with no headwinds, or tailwinds
- The integrated form under the assumption that V_∞, c_t, and $\frac{L}{D}$ are constant is called the *"Breguet Range Equation"*,

$$R = \frac{V_\infty}{c_t}\frac{L}{D}\ ln\left(\frac{W_{ini}}{W_{fin}}\right)$$

- Range depends on V_∞, c_t, and $\frac{L}{D}$
- However, these parameters aren't independent of each other
- In practice, the differential form of the equation is numerically integrated to accurately calculate range
- Range is maximized at maximum $V_\infty \frac{L}{D}$
- This condition varies between piston and gas turbine powered airplanes

Range for Piston Engine Powered Airplanes

- The specific fuel consumption for propeller/reciprocating engine power plants is fundamentally expressed in terms of power
- Recall the relationship between c_t and c:

$$\boxed{c_t = \frac{c\,V_\infty}{\eta_{pr}}}$$

- Substituting for c_t in the integrated expression for range:

$$\boxed{R = \frac{\eta_{pr}}{c}\frac{L}{D}\,ln\left(\frac{W_{ini}}{W_{fin}}\right)}$$

- Range is maximized by:
  - Maximizing η_{pr}
  - Minimizing c
  - Operating at $\left(\frac{L}{D}\right)_{max}$
- Recall the conditions for $\left(\frac{L}{D}\right)_{max}$ from our discussion on steady, level flight:

$$\boxed{C_{D_0} = KC_L^2}$$

$$\boxed{C_L = \sqrt{\frac{C_{D_0}}{K}}, C_D = 2C_{D_0}}$$

$$\boxed{V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{K}{C_{D_0}}}}}$$

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Range for Gas Turbine Powered Airplanes

- Recall for steady, level flight: $$V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\frac{1}{C_L}}$$
- Substituting in the range equation, assuming constant $c_t, \rho_\infty, \frac{C_L^{1/2}}{C_D}$:

$$\boxed{ds = \frac{1}{c_t}\sqrt{\frac{2}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}\int_{W_{fin}}^{W_{ini}}\frac{dW}{W^{\frac{1}{2}}}} \Leftrightarrow \boxed{R = \frac{2}{c_t}\sqrt{\frac{2}{\rho_\infty S}}\frac{C_L^{1/2}}{C_D}\left(W_{ini}^{\frac{1}{2}} - W_{fin}^{\frac{1}{2}}\right)}$$

- Range is maximized by:
  - Operating at $\left(\frac{C_L^{0.5}}{C_D}\right)_{max}$
  - Minimizing c_t
  - Flying at high altitudes(lower ρ_∞)
- Conditions for $\left(\frac{C_L^{0.5}}{C_D}\right)_{max}$ can be obtained analytically as done for $\left(\frac{C_L}{C_D}\right)_{max}$

$$\boxed{C_{D_0} = 3KC_L^2}$$

$$\boxed{C_L = \sqrt{\frac{C_{D_0}}{3K}}, C_D = \frac{4}{3}C_{D_0}}$$

$$\boxed{V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{3K}{C_{D_0}}}}}$$

Range: Jet vs. Prop Comparison

- Range for gas turbine powered airplanes:

$$R = \frac{2}{c_t}\sqrt{\frac{2}{\rho_\infty S}}\,\frac{C_L^{1/2}}{C_D}\,(W_{ini}^{\frac{1}{2}} - W_{fin}^{\frac{1}{2}})$$

- Max. range at $\left(\frac{C_L^{0.5}}{C_D}\right)_{max}$
- Conditions for maximum range:

$$C_{D_0} = 3KC_L^2$$

$$V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{3K}{C_{D_0}}}}$$

- Range for piston engine powered airplanes:

$$R = \frac{\eta_{pr}}{c}\frac{L}{D}\,ln\left(\frac{W_{ini}}{W_{fin}}\right)$$

- Max. range at $\left(\frac{C_L}{C_D}\right)_{max}$
- Conditions for maximum range:

$$C_{D_0} = KC_L^2$$

$$V_\infty = \sqrt{\frac{2}{\rho_\infty}\left(\frac{W}{S}\right)\sqrt{\frac{K}{C_{D_0}}}}$$

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Range: Cruise-Climb and Step-Climb

- Recall the Breguet range equation for jet and propeller airplanes: $R_{jet} = \frac{V_\infty}{c_t}\frac{L}{D}\, ln\left(\frac{W_{ini}}{W_{fin}}\right)$ $R_{prop} = \frac{\eta_{pr}}{c}\frac{L}{D}\, ln\left(\frac{W_{ini}}{W_{fin}}\right)$
- For both, maximizing range requires flying at a constant C_L
  - For jet: $C_L = \sqrt{C_{D_0}/3K}$, which maximizes $\left(\sqrt{C_L}/C_D\right)$
  - For prop: $C_L = \sqrt{C_{D_0}/K}$, which maximizes (C_L/C_D)
- Recall also, that for steady, level cruise flight: $C_L = \frac{W}{\frac{1}{2}\rho_\infty V_\infty^2\, S}$
- As the airplane consumes fuel and becomes lighter, maintaining the same C_L would require decreasing V_∞ or decreasing ρ_∞ (increasing altitude)
- If altitude is increased continuously to maintain C_L, a cruise-climb results
- In reality, a step-climb is preferred due to air traffic considerations
  - A stair-stepping profile: flat-cruise, climb, flat-cruise, climb, and so on

Carson's Speed for Propeller Aircraft – "The Least Wasteful Way of Wasting Fuel"

- Recall the following, related to range maximization for propeller aircraft

$$R_{prop} = \frac{\eta_{pr}}{c}\frac{L}{D}\, ln\left(\frac{W_{ini}}{W_{fin}}\right) \qquad C_L = \sqrt{C_{D_0}/K} \qquad V_{R,max} = \sqrt{\frac{2}{\rho_\infty}\sqrt{K/C_{D_0}}\frac{W}{S}}$$

- If the range-maximizing velocity is too low, it will take an inordinately long time to reach the destination
- Carson suggested an alternate figure of merit (FOM) considering the above
  - Conventional FOM: Minimize fuel consumed per unit distance
  - Carson's FOM: Minimize fuel consumed per unit velocity, i.e., minimize

$$\frac{|dW_f|}{V_\infty} = \frac{c\ P\ dt}{V_\infty} = \frac{c\ T\ V_\infty\ (ds/V_\infty)}{V_\infty} = \left(\frac{T}{V_\infty}\right) c\ ds$$

- This means minimizing $\frac{T}{V_\infty} = \frac{W}{V_\infty \frac{L}{D}}$, which, recall, occurs at maximum $\frac{\sqrt{C_L}}{C_D}$

- The corresponding speed has come to be called Carson's speed:

$$\boxed{V_{\infty,carson} = \sqrt{\frac{2}{\rho_\infty}\sqrt{3K/C_{D_0}}\,\frac{W}{S}}}$$

Endurance

- ***Endurance (E)*** is the time that an airplane can stay in the air on one load of fuel
- The flight conditions for maximum endurance are different than those for maximum range. They are also different for propeller and jet airplanes
- Recall that for steady, level flight, $T = \frac{W}{L/D}$, and also, by definition, $\frac{dW_f}{dt} = -c_t T$

$$dt = -\frac{dW_f}{c_t T} = -\frac{1}{c_t}\frac{L}{D}\frac{dW_f}{W} \Rightarrow E = \int dt = \int_{W_{fin}}^{W_{ini}} \frac{1}{c_t}\frac{L}{D}\frac{dW_f}{W}$$

- Assuming flight at constant specific fuel consumption and lift-to-drag ratio:

$$E = \frac{1}{c_t}\frac{L}{D}\int_{W_f}^{W_0}\frac{dW_f}{W}$$

- The solution of the above integral can now be branched out for the case of jet-powered and propeller-powered aircraft

Endurance – for Jet and Propeller Aircraft

- For jet aircraft, integrating the integral under the assumption of constant thrust-specific fuel consumption and lift-to-drag ratio yields:

$$\boxed{E_{jet} = \frac{1}{c_t}\frac{L}{D}\int_{W_{fin}}^{W_{ini}} \frac{dW_f}{W} = \frac{1}{c_t}\frac{L}{D} ln\left(\frac{W_o}{W_f}\right)}$$

Maximized by flying at maximum $\frac{C_L}{C_D}$

- For propeller-driven aircraft, recalling that $c_t = c\,\frac{V_\infty}{\eta_{pr}}$ and $V_\infty = \sqrt{\frac{2}{\rho_\infty}\frac{W}{S}\frac{1}{C_L}}$ and substituting these into the integral:

$$E = \int_{W_{fin}}^{W_{ini}} \frac{\eta_{pr}}{c}\sqrt{\frac{\rho_\infty S}{2}}\ \left(C_L^{\frac{3}{2}}/C_D\right)\frac{dW_f}{W^{\frac{3}{2}}}$$

- Integrating under the assumption of constant $C_L^{3/2}/C_D$:

$$\boxed{E_{prop} = \frac{\eta_{pr}}{c}\sqrt{2\rho_\infty S}\ \frac{C_L^{\frac{3}{2}}}{C_D}\left(\frac{1}{\sqrt{W_{fin}}} - \frac{1}{\sqrt{W_{ini}}}\right)}$$

Maximized by flying at maximum $\frac{C_L^{\frac{3}{2}}}{C_D}$

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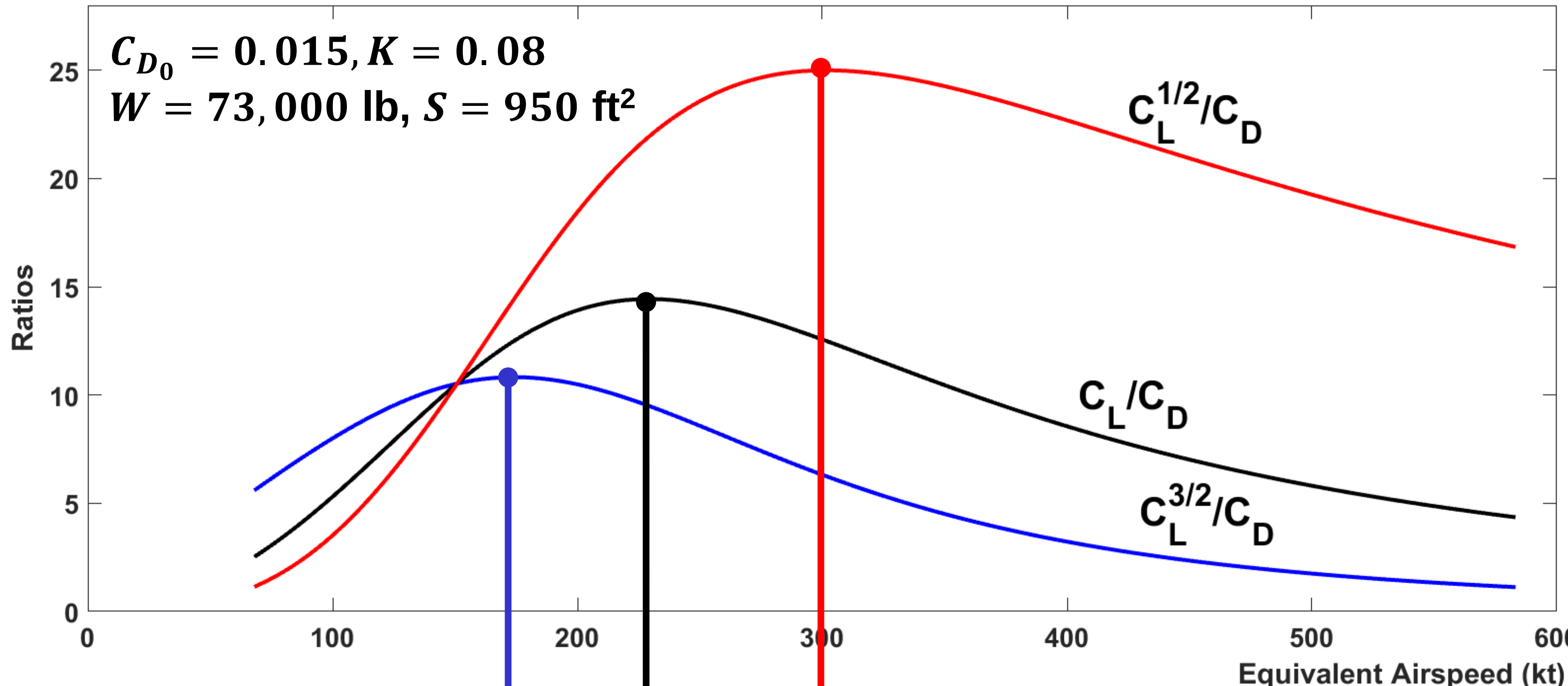
Range and Endurance – Concept Summary



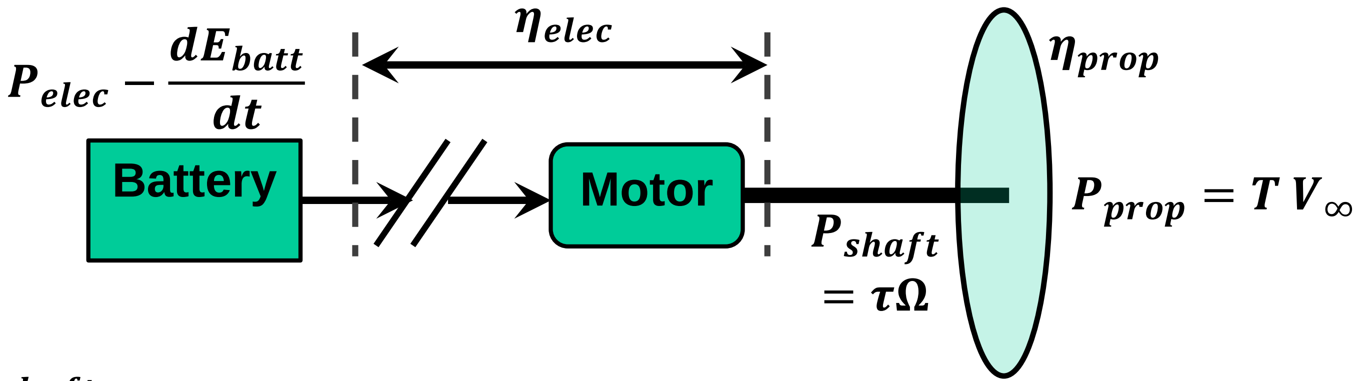
Max $C_L^{3/2}/C_D$, Min P_R
Best endurance for prop,
$3C_{D_0} = KC_L^2$

Max $C_L^{1/2}/C_D$, Min T_R/V_∞
Best range for jet
Carson's speed for prop, $C_{D_0} = 3KC_L^2$

Max C_L/C_D, Min T_R
Best endurance for jet
Best range for prop, $C_{D_0} = KC_L^2$

The Case of Electrically Powered Aircraft

- Electric aircraft are different in that
  - Propulsive power is supplied from a storage device such as a battery, whose energy content is progressively depleted
  - However, the depletion of battery energy does not result in any change in its weight, i.e., the aircraft weight does not change



$$\eta_{elec} = \frac{P_{shaft}}{P_{elec}}$$

$$\eta_{prop} = \frac{P_{prop}}{P_{shaft}}$$

$$\eta_o = \eta_{elec}\,\eta_{prop} = \frac{P_{prop}}{P_{elec}} = \frac{T\,V_\infty}{-\frac{dE}{dt}}$$

$$\boxed{-\frac{dE}{dt} = \frac{T\,V_\infty}{\eta_o}}$$

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Range for Electrically Powered Aircraft

- Recall from the power flow diagram that: $-\frac{dE_{batt}}{dt} = \frac{T\,V_\infty}{\eta_o}$
- Express the differential time as: $dt = \frac{ds}{V_\infty}$
- Recall that in steady, level, unaccelerated flight: $T = \frac{W}{L/D}$
- Combining the above relationships: $ds = -\frac{\eta_0}{W} \cdot \frac{L}{D}\; dE_{batt}$
- Integrating this yields an expression for range:

$$R = \frac{\eta_0}{W} \cdot \frac{L}{D}\; \Delta E_{batt} = \frac{\eta_0}{W}\,\frac{L}{D}\, K\, E_{batt,cap}$$

- Using only a fraction of battery capacity $\Delta E_{batt} = KE_{batt,cap}$, $K < 1$ ensures
  - The battery is not completely discharged (bad for battery health)
  - There is some margin for error (i.e., not run out of battery)

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Range for Electrically Powered Aircraft

- Battery energy capacity $E_{batt,cap}$ is related to battery weight W_{batt} through the battery energy density U_{batt} [*Wh/kg*], which represents battery technology state-of-the-art

$$W_{batt} = \frac{E_{batt,cap}}{U_{batt}}\ g$$

- A Breguet Range Equation for electric aircraft is then given by

$$\boxed{R = \eta_0 \left(\frac{L}{D}\right)\left(\frac{W_{batt}}{W}\right)\frac{\boxed{U_{batt}\ K}}{g}}$$

Increase energy (power) transmission efficiency

Increase aerodynamic efficiency

Reduce vehicle weight, or increase battery weight fraction

Improve battery technology (energy density), and discharge as much as you can (safely and successfully) get away with

Endurance for Electrically Powered Aircraft

- Recall from the power flow diagram that: $-\frac{dE_{batt}}{dt} = \frac{T\,V_\infty}{\eta_o}$
- Re-arranging the above expression: $dt = -\frac{\eta_0}{T\,V_\infty}\,dE_{batt}$
- Integrating both sides of the above equation: $T = \frac{\eta_0}{T\,V_\infty}\,\Delta E_{batt}$
- Applying balance of forces and the battery relationships

$$T = \frac{W}{L/D} \qquad \Delta E_{batt} = K E_{batt,cap} \qquad W_{batt} = \frac{E_{batt,cap}}{U_{batt}}\;g$$

- … the endurance T of an electric aircraft is given by

$$\boxed{T = \frac{\eta_0}{V_\infty}\left(\frac{L}{D}\right)\left(\frac{W_{batt}}{W}\right)\frac{U_{batt}\,K}{g}}$$