Thin-Walled Beams: Bending & Shear

AE3140: Structural Analysis

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(Adapted from slides by M. Ruzzene from
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Contents

Stress resultants

Due to the thin wall assumption, integration over the beam's cross-sectional area reduces to an integration along curve $\mathcal{C}$, given that $\mathrm{d}\mathcal{A} = t\mathrm{d}s$. Axial stress resultant:

$$N_1(x_1) = \int_A \sigma_1 \, \mathrm{d}\mathcal{A} = \int_{\mathcal{C}} \sigma_1 \, t\mathrm{d}s = \int_{\mathcal{C}} n \, \mathrm{d}s, \tag{1}$$

Bending moments:

$$M_2(x_1) = \int_{\mathcal{C}} n \, x_3 \, \mathrm{d}s, \quad M_3(x_1) = -\int_{\mathcal{C}} n \, x_2 \, \mathrm{d}s. \tag{2}$$

Stress resultants

Shear forces acting along axes $\bar{\imath}_2$ and $\bar{\imath}_3$:

$$V_2(x_1) = \int_{\mathcal{C}} f \frac{\mathrm{d}x_2}{\mathrm{d}s}\ \mathrm{d}s$$

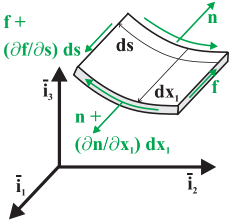
$$V_3(x_1) = \int_{\mathcal{C}} f \frac{\mathrm{d}x_3}{\mathrm{d}s}\ \mathrm{d}s$$

Torque computed about the origin, **O** requires a vector cross product

$$\underline{M}_O(x_1) = \int_{\mathcal{C}} \underline{r}_P \times f\ \mathrm{d}\underline{s},$$

where $\underline{r}_P = x_2\bar{\imath}_2 + x_3\bar{\imath}_3$ is the position vector of generic point **P**.

Local equilibrium equation



Equilibrium along $\bar{\imath}_1$:

$$\frac{\partial n}{\partial x_1} + \frac{\partial f}{\partial s} = 0. \tag{3}$$

Bending of thin-walled beams

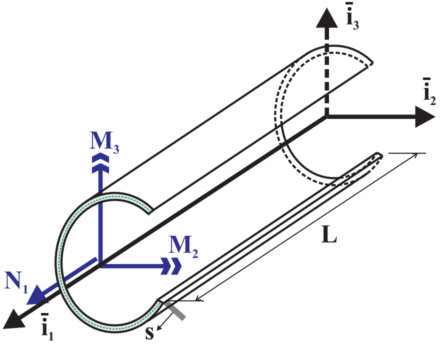


Figure: Thin-walled beam subjected to axial forces and bending moments.

Bending of thin-walled beams

Assumptions:

- Axes $\bar{\imath}_2$ and $\bar{\imath}_3$ are located at the centroid of the cross-section.
- The Euler-Bernoulli assumptions hold

The distribution of axial stresses can be expressed as

$$\sigma_1 = E\left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H} M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H} M_3\right], \qquad (4)$$

where $\Delta_H = H_{22}^c H_{33}^c - (H_{23}^c)^2$.

Bending of thin-walled beams

The axial flow distribution over the cross-section is therefore

$$n(x_1, s) = E(s)t(s)\left[\frac{N_1(x_1)}{S} - \frac{x_2(s)H_{23}^c - x_3(s)H_{33}^c}{\Delta_H}\ M_2(x_1)\right.$$
$$\left. - \frac{x_2(s)H_{22}^c - x_3(s)H_{23}^c}{\Delta_H}\ M_3(x_1)\right]. \tag{5}$$

NOTE: the thin-wall assumption may be used to simplify the evaluation of H_{22}, H_{33}, H_{23}.

In most cases, the bending moments are accompanied by transverse shear forces, which give rise to shear flow distributions over the cross-section. This distribution is evaluated through the local equilibrium equation, eq. (3), to find

$$\frac{\partial f}{\partial s} = -Et\left[\frac{1}{S}\frac{\mathrm{d}N_1}{\mathrm{d}x_1} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}\frac{\mathrm{d}M_2}{\mathrm{d}x_1} - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H}\frac{\mathrm{d}M_3}{\mathrm{d}x_1}\right]. \tag{6}$$

Enforcing shear force-feeding moment relations gives (assuming that the distributed axial loads, p_1, and moments, q_2 and q_3, are zero)

$$\frac{\partial f}{\partial s} = -E(s)t(s)\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H}V_2\right]. \tag{7}$$

Integration of this differential equation then yields the shear flow distribution arising from shear forces, V_2 and V_3, as

$$f(s) = c - \int_0^s Et \left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H} V_2 \right] \mathrm{d}s, \quad (8)$$

where:

- c is an integration constant corresponding to the value of the shear flow at $s = 0$.
- The procedure to determine this integration constant depends on whether the cross-section is *open* or *closed*.
- The bending stiffnesses and shear forces are functions of variable x_1 alone, and can be factored out of the integral

This gives:

$$f(s) = c + \frac{Q_3(s)H_{23}^c - Q_2(s)H_{33}^c}{\Delta_H}\ V_3 - \frac{Q_3(s)H_{22}^c - Q_2(s)H_{23}^c}{\Delta_H}\ V_2, \quad (9)$$

where

$$Q_2(s) = \int_0^s E(s)x_3(s)\ t(s)\mathrm{d}s; \quad Q_3(s) = \int_0^s E(s)x_2(s)\ t(s)\mathrm{d}s. \quad (10)$$

are the *stiffness static moments*, also called *stiffness first moments*.
NOTE: These integrals are the static moments for the portion of the cross-section from $s = 0$ to s, and thus, Q_2 and Q_3 are functions of s.

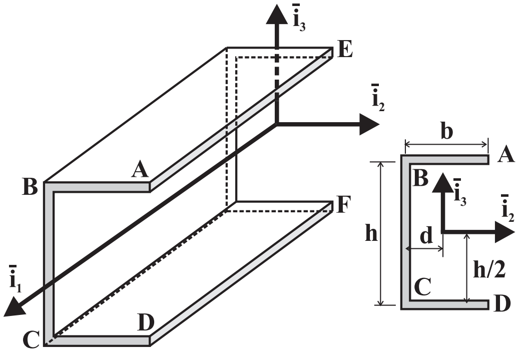


Figure: Cantilevered beam with a C-channel cross-section.

Shearing of open sections

- For open sections, the principle of reciprocity of shear stresses implies the vanishing of shear flow at the end points of curve $\mathcal{C}$.
- If the origin of the curvilinear coordinate, s, is chosen to be located at such a stress free edge, the integration constant, c, in eq. (9) must vanish.

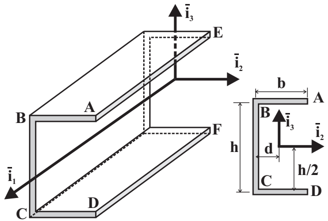
Shearing of open sections

Procedure:

1. Find centroid of the cross-section and select a set of centroidal axes, $\bar{\imath}_1$ and $\bar{\imath}_2$. Compute the sectional centroidal bending stiffnesses H_{22}^c, H_{33}^c and H_{23}^c (for principal centroidal axes of bending $H_{23}^c = 0$).
2. Select suitable curvilinear coordinates, s. Several curvilinear coordinates may be needed to describe the entire contour, $\mathcal{C}$.
3. Evaluate the first stiffness moments as functions of position, s, along contour, $\mathcal{C}$, of the cross-section.
4. The shear flow distribution, $f(s)$, then follows from eq. (9).

Example: Shear flow distribution in a C-channel

The considered section is subjected to a vertical shear force, V_3, at the specific span-wise location where the shear flow is to be computed.



The origin of the axes on the section is placed at the centroid, which is located at a distance $d = b/(2 + h/b)$ to the right of the web's mid-point. Because of symmetry about $\bar{\imath}_2$, these axes are principal axes of bending, *i.e.*, $H_{23}^c = 0$.

Given that loading is V_3 only, the shear flow distribution reduces to

$$f(s) = c - \frac{Q_2(s)}{H_{22}^c} V_3, \tag{11}$$

with

$$H_{22}^c = E\left[\frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2\right] = E\left(\frac{h^3}{12} + \frac{bh^2}{2}\right)t.$$

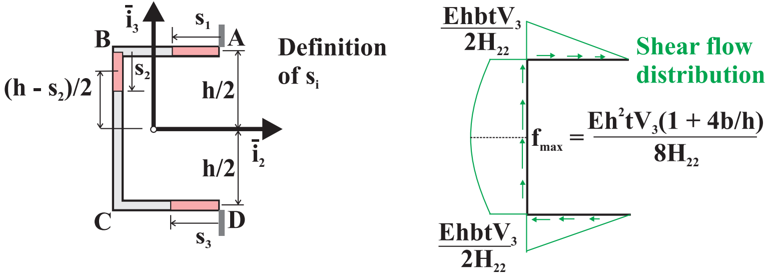


Figure: Distribution of shear flow over the C-channel cross-section.

Upper flange

For the section's upper flange, eq. (11) yields the shear flow distribution as

$$f(s_1) = c_1 - \frac{Q_2(s_1)}{H_{22}} V_3 = 0 - \frac{Ets_1 h/2}{H_{22}^c} V_3 = -\frac{Ehts_1}{2} \frac{V_3}{H_{22}^c}, \qquad (12)$$

where

$$Q_2(s) = Eth/2s_1$$

and the integration constant, c_1, vanishes because the shear flow must vanish at point **A**, where $s_1 = 0$.

Vertical web

For the vertical web:

$$Q_2(s_2) = Ets_2(h - s_2)/2$$

and the corresponding shear flow distribution is

$$f(s_2) = c_2 - \frac{h - s_2}{2} ts_2 \frac{EV_3}{H_{22}^c} = -\frac{1}{2}\left[bh + s_2(h - s_2)\right]\frac{tEV_3}{H_{22}^c}. \qquad (13)$$

The integration constant, c_2, is evaluated by enforcing the continuity of the shear flow:

$$f(s_2 = 0) = f(s_1 = b)$$

which gives:

$$c_2 = -hb/2 \ EV_3/H_{22}^c$$

Lower flange

For the lower flange, where $x_3 = -h/2$:

$$Q_2(s_3) = -E\ ts_3\ h/2$$

The shear flow then follows as

$$f(s_3) = c_3 + \frac{E\ ts_3\ h/2}{H_{22}^c}\ V_3 = \frac{hs_3}{2}\frac{tEV_3}{H_{22}^c}, \tag{14}$$

where the integration constant, c_3, vanishes because $f(s_3 = 0) = 0$.

Notes:

- The present solution also satisfies the shear flow continuity condition at point **C**, although this condition is not explicitly enforced.
- Indeed, the above results imply $f(s_2 = h) = -1/2\ bhtEV_3/H_{22}^c$ and $f(s_3 = b) = 1/2\ bhtEV_3/H_{22}^c$, *i.e.*, $f(s_2 = h) + f(s_3 = b) = 0$.
- The two shear flow add up to zero because curvilinear variables s_2 and s_3 both converge towards point **C**.
- The shear flows in the upper and lower flanges are linearly distributed along the flanges and vanish at the edges. The shear flow in the vertical web varies in a quadratic manner.
- The maximum shear flow is found at the mid-point of the vertical web.

Shear center for open sections

- In the previous derivations, the beam was assumed to be subjected to the transverse shear forces, V_2 and V_3 alone,
- No torque was applied, *i.e.*, $M_1 = 0$.
- Consequently, the net torque generated by the shear flow distribution should vanish.
- However, as stated, the problem is not precisely defined: whereas the magnitudes of the transverse shear forces are given, *their lines of action are not specified*.
- This precludes the computation of the torque generated by the applied shear forces, and hence, it is not possible verify the torque equilibrium of the cross-section.

Definition of shear center

The torque generated by the shear flow distribution associated with transverse shear forces must vanish when computed with respect to the shear center.

When transverse forces are applied at the shear center, shear occurs with no twist.

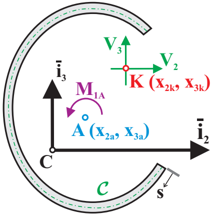


Figure: Thin-walled open cross-section subjected to shear forces.

Consider a beam with thin-walled, open cross-section

- the cross-section is subjected to horizontal and vertical shear forces of magnitudes V_2 and V_3, respectively, with lines of action passing through point **K**, with coordinates (x_{2k}, x_{3k}), which are, as yet, unknown.
- No external torque is applied with respect to point **K**, *i.e.*, $M_{1k} = 0$.

The torque about an arbitrary point **K** is:

$$M_{1k}(x_1) = \int_{\mathcal{C}} f r_k \, \mathrm{d}s, \tag{15}$$

where

$$r_k = (x_2 - x_{2k})\cos\alpha + (x_3 - x_{3k})\sin\alpha$$

$$r_k = r_o - x_{2k}\frac{\mathrm{d}x_3}{\mathrm{d}s} + x_{3k}\frac{\mathrm{d}x_2}{\mathrm{d}s}$$

Similarly, the perpendicular distance from an arbitrary point **A** to the line of action of the shear flow is:

$$r_a = r_o - x_{2a}\,\frac{\mathrm{d}x_3}{\mathrm{d}s} + x_{3a}\frac{\mathrm{d}x_2}{\mathrm{d}s}$$

Subtracting the last two equations gives:

$$r_k = r_a - (x_{2k} - x_{2a})\frac{\mathrm{d}x_3}{\mathrm{d}s} + (x_{3k} - x_{3a})\frac{\mathrm{d}x_2}{\mathrm{d}s}$$

Imposing that **K** is the *shear center*, gives:

$$M_{1k} = \int_{\mathcal{C}} f r_k \;\mathrm{d}s = 0. \tag{16}$$

Substituting r_k:

$$M_{1k} = \int_{\mathcal{C}} f r_a \;\mathrm{d}s - (x_{2k} - x_{2a})\left[\int_{\mathcal{C}} f\frac{\mathrm{d}x_3}{\mathrm{d}s}\;\mathrm{d}s\right] + (x_{3k} - x_{3a})\left[\int_{\mathcal{C}} f\frac{\mathrm{d}x_2}{\mathrm{d}s}\;\mathrm{d}s\right]$$

$$M_{1k} = \int_{\mathcal{C}} f r_a \;\mathrm{d}s - (x_{2k} - x_{2a})V_3 + (x_{3k} - x_{3a})V_2 = 0 \tag{17}$$

These equations locate the shear center **K**.

Notes:

- A beam bends without twisting *if and only* if the transverse loads are applied at the shear center.
- The beam both bends and twists if the transverse loads are not applied at the shear center.

Computation of the location of the shear center

Procedure:

1. Evaluate the shear flow distribution associated with a transverse shear force, V_2
2. Determine the location, x_{3k}, of the line of action of this shear force
3. Evaluate the shear flow distribution associated with a transverse shear force, V_3
4. Determine the location, x_{2k}, of the line of action of this shear force.

Note: If the cross-section features a plane of symmetry, the shear center must lie in that plane of symmetry

Example: Shear center for a C-channel

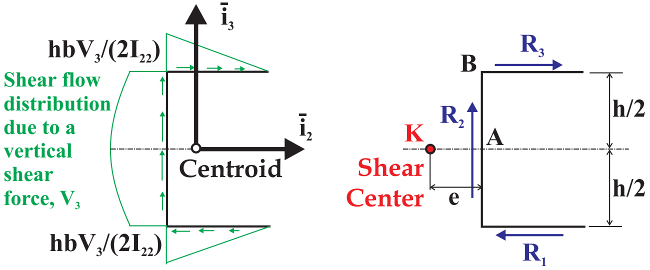


Figure: Shear center in C-channel section.

- Axis $\bar{\imath}_2$ is an axis of symmetry for the C-channel
- Hence, the shear center lies at a point along this axis.
- It is only necessary to evaluate the shear flow distribution generated by a vertical shear force, V_3, to determine the location of the shear center.
- The cross-section is made up of three straight line segments, the lower flange, the vertical web, and the upper flange.

The resultant force in each segment is easily evaluated by integrating the associated shear flow distribution to find (see fig. 5).

$$R_1 = \int_0^b f(s_1)\, \mathrm{d}s_1 = \frac{hb^2t}{4}\frac{EV_3}{H_{22}^c}, \quad R_2 = \int_{-h/2}^{h/2} f(s_2)\, \mathrm{d}s_2 = V_3,$$
$$R_3 = \int_0^b f(s_3)\, \mathrm{d}s_3 = \frac{hb^2t}{4}\frac{EV_3}{H_{22}^c} = R_1.$$

The equipollence condition gives:

$$\int_{\mathcal{C}} f r_k \,\mathrm{d}s = -R_1 \; h/2 + R_2 \; e - R_1 \; h/2 = 0$$

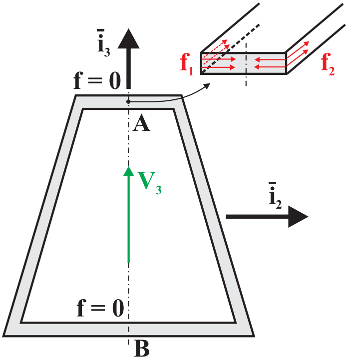
Solving for e leads to

$$e = \frac{hR_1}{R_2} = \frac{h^2 b^2 t}{4} \frac{E}{H_{22}^c} = \frac{3}{6 + h/b} \; b, \tag{18}$$

Shearing of closed sections

- In the case of closed sections, the governing equation for the shear flow distribution, eq. (9), still applies,
- No boundary condition is readily available to integrate this equation.
- One notable exception occurs when the section presents an axis of symmetry

Shearing of closed sections



Shearing of closed sections: symmetric sections

- If plane $(\bar{\imath}_1, \bar{\imath}_3)$ is a plane of symmetry of the section, and if a shear force, V_3, acts in this plane, the solution must be symmetric with respect to this plane.
- Thus, the shear flow distribution for the left half of the section must be the mirror image of that for the right half.
- Consider the free-body diagram of a small portion. Equilibrium implies that:

$$f_1 + f_2 = 0$$

whereas symmetry implies that:

$$f_1 = f_2$$

Hence:

$$f_1 = f_2 = 0$$

i.e., the shear flow must vanish at point **A**.

Shearing of closed sections: symmetric sections

- A similar reasoning will conclude that the shear flow also vanishes at point **B**, the other intersection of the section's wall with the plane of symmetry.
- The section's left and right halves can be analyzed separately, as if they are two independent open sections.
- If a horizontal shear force, V_2, is applied, the above symmetry argument is no longer applicable.

Shearing of closed sections: sections of arbitrary shape

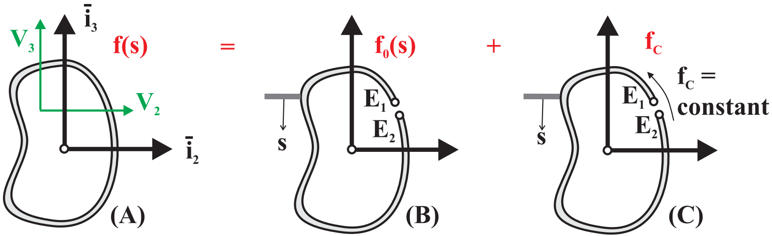


Figure: (A): a general closed section. (B): the auxiliary problem created by cutting the section open. (C): the constant closing shear flow.

Shearing of closed sections: sections of arbitrary shape

Solution procedure:

1. The beam is cut along its axis at an arbitrary point of the cross-section, thus defining an "auxiliary problem."
2. The shear flow distribution in this auxiliary problem is denoted $f_o(s)$, and is readily found using the procedure for open sections.
3. The shear flow $f_o(s)$ creates a shear strain, γ_s, and a corresponding infinitesimal axial displacement:

$$\mathrm{d}u_1 = \gamma_s \ \mathrm{d}s = \frac{\tau_s}{G} \ \mathrm{d}s = \frac{f_o(s)}{Gt} \ \mathrm{d}s, \tag{19}$$

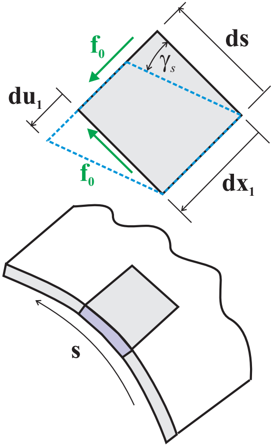


Figure: Axial displacement arising from the shear flow f_o.

Shearing of closed sections: sections of arbitrary shape

Solution procedure (*cont.d*):

1. The shear flow distribution over the entire section creates a finite relative axial displacement at the two edges of the cut, points $\mathbf{E_1}$ and $\mathbf{E_2}$
2. It can be evaluated by integrating around the section the infinitesimal axial displacement:
$$u_0 = \int_{\mathcal{C}} \frac{f_o(s)}{Gt} \,\mathrm{d}s.$$
3. A constant shear flow f_c (*closing shear flow*), is applied to eliminate the relative axial displacement, u_0, between the edges of the cut, thereby returning the section to its original, closed state.

Closing shear flow

The total shear flow is:

$$f(s) = f_o(s) + f_c$$

and the corresponding relative axial displacement vanishes

$$u_t = \int_{\mathcal{C}} \frac{f_o(s) + f_c}{Gt} \, \mathrm{d}s = 0. \tag{20}$$

This condition expresses the displacement compatibility for the closed section.
The closing shear flow therefore is

$$f_c = -\frac{\displaystyle\int_{\mathcal{C}} \frac{f_o(s)}{Gt} \, \mathrm{d}s}{\displaystyle\int_{\mathcal{C}} \frac{1}{Gt} \, \mathrm{d}s}. \tag{21}$$

Example: Shear flow distribution in a closed triangular section

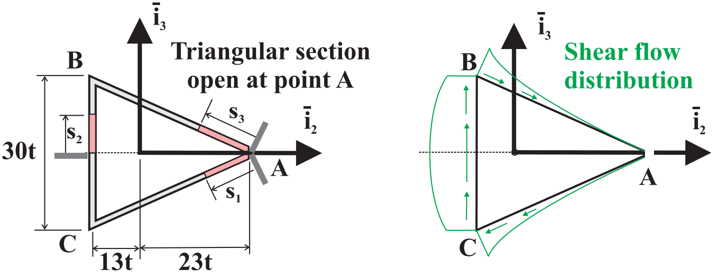


Figure: Thin-walled, open triangular section

Since $\bar{\imath}_2$ is axis of symmetry, then *i.e.*, $H_{23}^c = 0$.
The bending stiffness of the section is

$$H_{22}^c = 2E\left[\left(\frac{2}{3}39t^2\right)\left(\frac{15t}{2}\right)^2 + \left(\frac{30t^2}{6} + \frac{39t^2}{6}\right)(15t)^2\right] = 8100\ Et^4, \tag{22}$$

The curvilinear coordinates in the figure are considered for the subsequent analysis.

Open section analysis

Consider a cut at point **A**.
From open section analysis, one can find that

$$f_o(s_1) = c_1 - Q_2(s_1)V_3/H_{22}^c$$

where $Q_2(s_1) = -E\ s_1 t\ s_1/2 \sin\alpha$, with α denoting the angle between the upper flange and axis $\bar{\imath}_2$, with $\sin\alpha = 15/39$. Also $Q_2(s_2) = E\ s_2 t\ s_2/2$, so that the corresponding shear flow distribution is:

$$f_o(s_2) = c_2 - \frac{1}{2}s_2^2\frac{tEV_3}{H_{22}^c} = \frac{13}{360}\frac{V_3}{t} + \frac{1}{72}\left[1 - \left(\frac{s_2}{15t}\right)^2\right]\frac{V_3}{t}, \qquad (23)$$

where the integration constant, c_2, is evaluated using the joint equilibrium condition at point **C**: $f_o(s_1 = 39t) = f_o(s_2 = -15t)$.

Open section analysis

Due to symmetry, the shear flow distribution along the upper flange is identical to that along the lower flange, except for a change in sign due to the s_3:

$$f_o(s_3) = -\frac{13}{360}\left(\frac{s_3}{39t}\right)^2 \frac{V_3}{t}. \tag{24}$$

Note that the joint equilibrium condition at point **B** is satisfied by the present solution; indeed, $f_o(s_2 = 15t) + f_o(s_3 = 39t) = 0$.

Closing shear flow

The numerator in eq. (21) is :

$$\int_{\mathcal{C}} \frac{f_o}{Gt}\,\mathrm{d}s = \int_0^{39t} \frac{f_o(s_1)}{Gt}\mathrm{d}s_1 + \int_{-15t}^{15t} \frac{f_o(s_2)}{Gt}\mathrm{d}s_2 - \int_0^{39t} \frac{f_o(s_3)}{Gt}\mathrm{d}s_3 = \frac{23V_3}{10Gt},$$

Integrating in a clockwise direction around the section.
The denominator in eq. (21) is

$$\int_{\mathcal{C}} \frac{\mathrm{d}s}{Gt} = \frac{1}{Gt}(39t + 30t + 39t) = \frac{108}{G}.$$

Thus, the closing shear flow becomes

$$f_c = -\frac{23V_3/(10Gt)}{108/G} = -\frac{23}{1080}\frac{V_3}{t}. \tag{25}$$

Resulting shear flow $f(s) = f_o(s) + f_c$

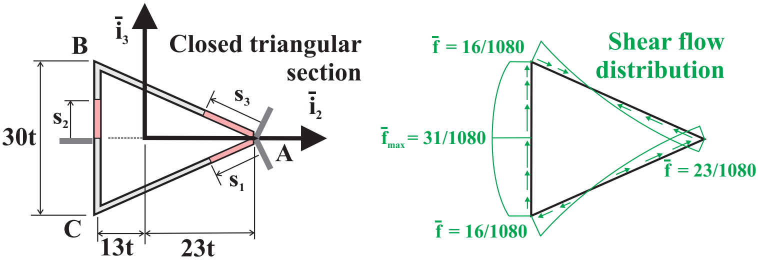


Figure: Non-dimensional shear flow $\bar{f}(s) = tf(s)/V_3$ distribution in a closed triangular section.

Shearing of multi-cellular sections

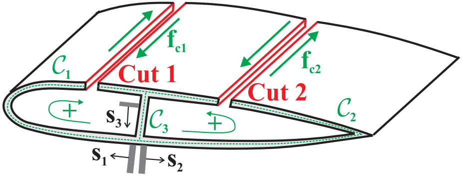


Figure: A thin-walled, multi-cellular section.

Shearing of multi-cellular sections

- The shear flow distribution must satisfy eq. (9), but no boundary condition is available to evaluate the integration constant.
- One cut per cell is required to eliminate all the closed paths of the section (fig. 10)
  Let $f_o(s_1)$, $f_o(s_2)$, and $f_o(s_3)$ be the shear flow distributions along curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively.
- Closing shear flows are applied at each of the cuts; f_{c1} and f_{c2} for the front and aft cells, respectively.
- The shear flow distributions are now $f_o(s_1) + f_{c1}$, $f_o(s_2) + f_{c2}$, and $f_o(s_3) + (f_{c1} + f_{c2})$, along curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively.

Shearing of multi-cellular sections

- There are two unknown closing shear flows
- These will be evaluated by enforcing the displacement compatibility condition for each of the two cells.
  - Front cell clockwise:

$$u_{t1} = \int_{\mathcal{C}_1} \frac{f_o(s_1) + f_{c1}}{Gt}\, \mathrm{d}s_1 + \int_{\mathcal{C}_3} \frac{f_o(s_3) + (f_{c1} + f_{c2})}{Gt}\, \mathrm{d}s_3 = 0. \quad (26)$$

  - Aft cell counterclockwise:

$$u_{t2} = \int_{\mathcal{C}_2} \frac{f_o(s_2) + f_{c2}}{Gt}\, \mathrm{d}s_2 + \int_{\mathcal{C}_3} \frac{f_o(s_3) + (f_{c1} + f_{c2})}{Gt}\, \mathrm{d}s_3 = 0. \quad (27)$$

Shearing of multi-cellular sections

- These compatibility conditions are recast as a set of two linear equations for the unknown closing shear flows,

$$\left[\int_{\mathcal{C}_1+\mathcal{C}_3} \frac{1}{Gt}\,\mathrm{d}s\right] f_{c1} + \left[\int_{\mathcal{C}_3} \frac{1}{Gt}\,\mathrm{d}s\right] f_{c2} = -\int_{\mathcal{C}_1+\mathcal{C}_3} \frac{f_o(s)}{Gt}\,\mathrm{d}s;$$
$$\left[\int_{\mathcal{C}_3} \frac{1}{Gt}\,\mathrm{d}s\right] f_{c1} + \left[\int_{\mathcal{C}_2+\mathcal{C}_3} \frac{1}{Gt}\,\mathrm{d}s\right] f_{c2} = -\int_{\mathcal{C}_2+\mathcal{C}_3} \frac{f_o(s)}{Gt}\,\mathrm{d}s.$$

  to be solved in terms of the two closing shear flows, f_{c1} and f_{c2}.
- The total shear flow in the multi-cellular section is found by $f_o(s_1) + f_{c1}, f_o(s_2) + f_{c2}$, and $f_o(s_3) + (f_{c1} + f_{c2})$, along curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.
- The procedure is readily extended to multi-cellular section possessing N closed cells, which needs the creation of N cuts

Example: Shear flow in thin-walled double-box section

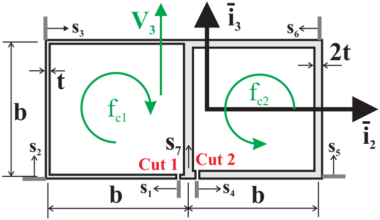


Figure: A thin-walled double-box section.

- Due to symmetry, the centered horizontal axis $\bar{\imath}_2$ is a principal axis of bending, and hence, $H_{23}^c = 0$.
- The centroid of the section will be located in the right cell, as indicated in the figure.
- Using thin-wall assumptions, the bending stiffness is

$$H_{22}^c = E\left[2\left(\frac{2tb^3}{12}\right) + \frac{tb^3}{12} + 2(bt + b2t)\left(\frac{b}{2}\right)^2\right] = \frac{23}{12}tb^3E.$$

Open sections analysis

- Two cuts are considered on the two lower flanges where they connect to the center web.
- The locations of these cuts are arbitrary, but those selected here lead to simple definitions of the curvilinear coordinates, s_i, for each component of the section.
- The shear flow distribution in the open section, $f_o(s_i)$, becomes

$$f_o(s_1) = \frac{6V_3}{23b}\frac{s_1}{b}, \quad f_o(s_3) = \frac{6V_3}{23b}\left(1 - \frac{s_3}{b}\right), \quad f_o(s_4) = \frac{12V_3}{23b}\frac{s_4}{b},$$

$$f_o(s_2) = \frac{6V_3}{23b}\left[1 + \left(1 - \frac{s_2}{b}\right)\frac{s_2}{b}\right], \quad f_o(s_5) = \frac{12V_3}{23b}\left[1 + \left(1 - \frac{s_5}{b}\right)\frac{s_5}{b}\right],$$

$$f_o(s_6) = \frac{12V_3}{23b}\left(1 - \frac{s_6}{b}\right), \quad f_o(s_7) = -\frac{12V_3}{23b}\left(1 - \frac{s_7}{b}\right)\frac{s_7}{b}.$$

Closing shear flows

Next, two closing shear flows, denoted f_{c1} and f_{c2}, are added to the left and right cells, respectively.
Enforcing compatibility conditions gives for the left cell:

$$u_{t1} = \int_0^b \frac{f_o(s_1) + f_{c1}}{Gt} \, \mathrm{d}s_1 + \int_0^b \frac{f_o(s_2) + f_{c1}}{Gt} \, \mathrm{d}s_2 + \int_0^b \frac{f_o(s_3) + f_{c1}}{Gt} \, \mathrm{d}s_3 - \int_0^b \frac{f_o(s_7) - f_{c1} - f_{c2}}{G2t} \, \mathrm{d}s_7 = \frac{b}{Gt}\left(\frac{7f_{c1}}{2} + \frac{f_{c2}}{2} + \frac{12V_3}{23b}\right) = 0.$$

The right cell is described counterclockwise and the corresponding compatibility equation is

$$u_{t2} = \int_0^b \frac{f_o(s_4) + f_{c2}}{G2t} \, \mathrm{d}s_4 + \int_0^b \frac{f_o(s_5) + f_{c2}}{G2t} \, \mathrm{d}s_5 + \int_0^b \frac{f_o(s_6) + f_{c2}}{G2t} \, \mathrm{d}s_6 - \int_0^b \frac{f_o(s_7) - f_{c1} - f_{c2}}{G2t} \, \mathrm{d}s_7 = \frac{b}{Gt}\left(\frac{f_{c1}}{2} + 2f_{c2} + \frac{12V_3}{23b}\right) = 0$$

Evaluation of the integrals and solution of these two simultaneous algebraic equations yields $f_{c1} = -8V_3/(69b)$ and $f_{c2} = -16V_3/(69b)$. Finally, the total shear flow is

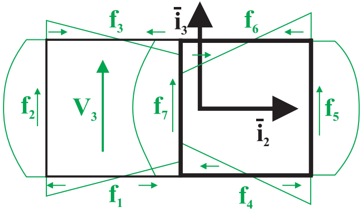


Figure: Shear flow in the thin-walled double-box section.

Notes:

- The net resultant of the shear flows in the flanges must vanish because no shear force is externally applied in the horizontal direction.
- The resultant of the shear flows in the webs must equal the externally applied vertical shear force, V_3.

Coupled bending-torsion problems

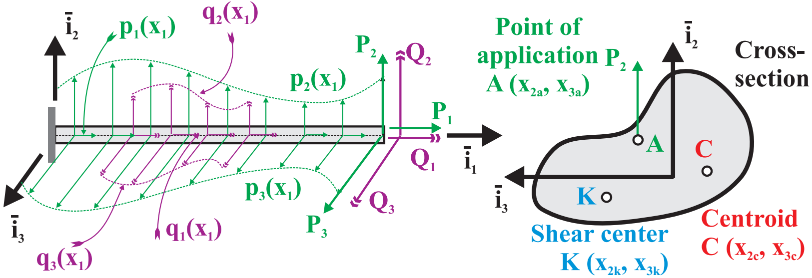


Figure: Beam under a complex loading condition.

- If all transverse loads are applied at the shear center, the bending and shearing analyses developed in this chapter are applicable.
- If a transverse load is not applied at the shear center, it can always be replaced by an equipollent system consisting of an equal transverse load applied at the shear center *plus a torque* equal to the moment of the transverse load about the shear center.
- In the figure, for example, the transverse load, $p_2(x_1)$, applied at point **A** is equivalent to a transverse load of equal magnitude, $p_2(x_1)$, applied at the shear center, point **K**, plus a distributed torque $-(x_{3a} - x_{3k})p_2(x_1)$.

Solution procedure.

1. Compute the location of the centroid, **C** (x_{2c}, x_{3c}), of the cross-section.
2. Compute the orientation of the principal axes of bending $\bar{\imath}_1^*$, $\bar{\imath}_2^*$, and $\bar{\imath}_3^*$, and the principal centroidal bending stiffnesses.
3. Compute the location of the shear center, **K** (x_{2k}, x_{3k}), of the cross-section.
4. Compute the torsional stiffness

Solution procedure (cont.d).

1. Solve the extensional problem, with appropriate boundary conditions.
2. Solve two decoupled bending problems in principal centroidal axes of bending planes with appropriate boundary conditions.
3. Solve the torsional problem governed by the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x_1^*}\left(H_{11}^* \frac{\mathrm{d}\Phi_1^*}{\mathrm{d}x_1^*}\right) = -\left[q_1^*(x_1^*) + (x_{2a}^* - x_{2k}^*)p_3^*(x_1^*) - (x_{3a}^* - x_{3k}^*)p_2^*(x_1^*)\right] \tag{28}$$

subjected to boundary conditions at the root, $\Phi_1^* = 0$, and at the tip

$$H_{11}^* \frac{\mathrm{d}\Phi_1^*}{\mathrm{d}x_1^*} = Q_1^* + (x_{2a}^* - x_{2k}^*)P_3^* - (x_{3a}^* - x_{3k}^*)P_2^*.$$

Example: Wing subjected to aerodynamic lift and moment

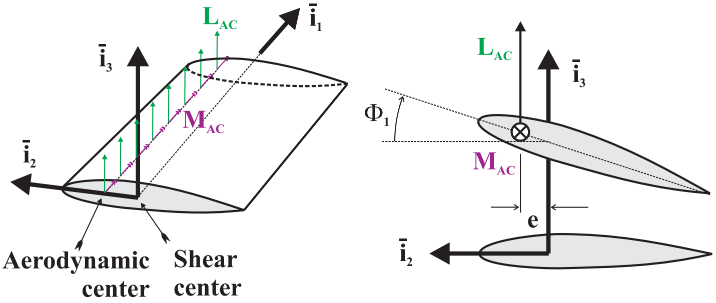


Figure: The wing bending torsion coupled problem.

- The principal axes of bending, $\bar{\imath}_2$ and $\bar{\imath}_3$, are selected with their origin at the shear center.
- Axis $\bar{\imath}_1$ is along the locus of the shear centers of all the cross-sections assumed to form a straight line called the *elastic axis*.
- The aerodynamic loading consists of a lift per unit span, L_{AC}, applied at the aerodynamic center and an aerodynamic moment per unit span, M_{AC}.

The differential equation for bending in plane $(\bar{\imath}_1, \bar{\imath}_3)$ is

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(H_{22}^c \frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}x_1^2}\right) = L_{AC}. \tag{29}$$

with boundary conditions (cantilevered wing of length L) are:

$$\bar{u}_3 = \mathrm{d}\bar{u}_3/\mathrm{d}x_1 = 0$$

at the root and

$$\mathrm{d}^2\bar{u}_3/\mathrm{d}x_1^2 = \mathrm{d}^3\bar{u}_3/\mathrm{d}x_1^3 = 0$$

at the unloaded tip.

The governing equation for torsion is

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(H_{11}\ \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}\right) = -\left(M_{AC} + eL_{AC}\right), \tag{30}$$

where e is the distance from the aerodynamic center to the shear center. The boundary condition at the wing's root is:

$$\Phi_1 = 0$$

and at its tip

$$\mathrm{d}\Phi_1/\mathrm{d}x_1 = 0$$

Notes

- For symmetric airfoils $M_{AC} = 0$, but the wing still twists because the lift is applied at the aerodynamic center, which does not coincide with the shear center for the case at hand.
- For typical transport aircraft, the aerodynamic and shear centers are located at 25% and 35% chord, respectively. Consequently, the lift generates a nose-up torque on the wing.

Notes

- For aircraft wing analysis, it is convenient to select the origin of the axes at the shear center, rather than at the centroid
- The main advantage of selecting the centroid as the origin is that the bending problems decouple from the axial problem.
- In this case, the beam is not subjected to any axial loads, therefore, the axial problem is of little interest.
- With the shear center as the origin of the axis system, the beam rotates about the origin of the axis system.
- The rotation, $\Phi_1(x_1)$, of the section is the geometric angle of attack of the airfoil.
- The lift, $L_{AC} = L_{AC}(\Phi_1)$, is a function of this angle of attack.
- The aerodynamic problem, which involves the computation of the lift as a function of the angle of attack, and the elastic problem, which involves the computation of the wing deflection and twist as a function of the applied loads, become coupled. (*aeroelasticity*.)