AE6114 - Fundamentals of Solid Mechanics

Fall 2020

Homework 4: Linear Elasticity - Basic concepts

Due at the indicated time on Canvas, on Thursday, October 22^{nd} 2020

Problem 1

Consider the following stress-strain relations for an isotropic linear elastic solid:

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \tag{1}$$

where λ and μ are Lame's moduli.

1. (a) Using equation 1 and without using the summation convention, write the explicit expressions for all six components of Cauchy's stress tensor $\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}$, and σ_{33}.
   (b) What are the units of λ and μ?

2. (a) Show that for the state of uniaxial stress (that is, when $\sigma_{11} \neq 0$ and all other components of $\boldsymbol{\sigma}$ are zero), we have:

   $$\sigma_{11} = E\epsilon_{11}, \quad \epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}$$

   where

   $$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

   are called the Young's modulus and the Poisson's ratio respectively.
   (b) What are the units of E and ν?
   (c) Sketch a circular cylindrical body that is homogeneously deformed in the uniaxial stress state with the tensile (i.e., positive) stress in the direction of the axis of the cylinder. Interpret $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$ and the Poisson's ratio in terms of the undeformed and deformed length and radius of the cylinder.

3. Consider the simple shear deformation given by

   $$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

   where γ is a small given constant.
   (a) Sketch this deformation in the $X_1 - X_2$ plane (using a unit square).
   (b) Mark the change of angle between directions $(1, 0, 0)^T$ and $(0, 1, 0)^T$ on your sketch.
   (c) Find the matrix form of the components of the infinitesimal strain tensor $\boldsymbol{\epsilon}$ and from equation 1, the Cauchy stress tensor $\boldsymbol{\sigma}$.

Problem 2

The strain energy density W of a material represents the energy stored due to elastic deformation. For a linear elastic material, it is given by:

$$W = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$$

Since it represents energy stored due to deformation, we require that W is a non-negative function for any elastic deformation, that is:

$$W \geq 0; \quad W = 0 \iff \epsilon_{ij} = 0$$

This imposes an extra constraint on the values of the elastic constants. We can study these constraints by looking at simple deformation states:

1. Consider the the simple shear problem solved in exercise 1. Find W for this case in terms of μ and γ. Show that $\mu > 0$.
2. Consider the hydrostatic state of stress $\sigma_{ij} = p\delta_{ij}$. Find W for this case in terms of the bulk modulus κ. Show that $\kappa > 0$ or $\lambda > -\frac{2}{3}\mu$.
3. How do these constraints translate to the elastic constants E and ν?

Problem 3

As we discussed in class, the Cauchy stress tensor $\underline{\underline{\sigma}}$ can be decomposed into its hydrostatic (spheric) and deviatoric components as follows:

$$\underline{\underline{\sigma}} = p\underline{\underline{I}} + \underline{\underline{\hat{\sigma}}}$$

where $p = \frac{1}{3}\text{tr}(\underline{\underline{\sigma}})$ and $\text{tr}(\underline{\underline{\hat{\sigma}}}) = 0$. Analogously, the infinitesimal strain tensor $\underline{\underline{\epsilon}}$ can be decomposed as follows:

$$\underline{\underline{\epsilon}} = \frac{1}{3}\epsilon_{\text{vol}}\underline{\underline{I}} + \underline{\underline{\hat{\epsilon}}}$$

with $\epsilon_{\text{vol}} = \text{tr}(\underline{\underline{\epsilon}})$ and $\text{tr}(\underline{\underline{\hat{\epsilon}}}) = 0$.

1. Show that, under isotropic linear elastic assumptions, the following relations hold:

$$p = \kappa\epsilon_{\text{vol}} \quad ; \quad \underline{\underline{\hat{\sigma}}} = 2\mu\underline{\underline{\hat{\epsilon}}}$$

2. Discuss the previous result.