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AE 6115 — Fundamentals of Aerospace Structural Analysis

Torsion

(Problems adapted from "Structural Analysis" by Bauchau and Craig)

Problem 1. Torsion of a bimetallic bar

A circular bar is constructed by bonding an aluminum shell around a solid steel cylinder. The radius of the steel cylinder is $R_S = 10$ mm, and the outer radius of the aluminum shell is $R_A = 20$ mm. The overall length of the bar is given by $L = 1$ m, and a torque $T = 1$ kN·m is applied at the ends. The shear moduli for the aluminum and steel are $G_A = 28$ GPa and $G_S = 76$ GPa, respectively.

1. Determine the torsional stiffness.
2. Plot the shear stress τ_α as a function of the radial coordinate r. (**Hint:** Similar to bending of multi-material beams, here both sections of the beam see the same twist rate κ_1)
3. Find the maximum shear stress in the steel and in the aluminum.
4. Determine the total twist angle of the bar.

Solution

1. Torsional stiffness here is found by adding the individual torsional stiffnesses of each material.

$$H_{11}^S = G_S \frac{\pi}{2}(R_S)^4 = 1193.8 \text{ Pa m}^4 \tag{1}$$

$$H_{11}^A = G_A \frac{\pi}{2}\left((R_A)^4 - (R_S)^4\right) = 6597.3 \text{ Pa m}^4 \tag{2}$$

$$\Rightarrow H_{11} = H_{11}^S + H_{11}^A = 7791.15 \text{ Pa m}^4 \tag{3}$$

2. Since we know shear stress is a function of $\kappa_1 = T/H_{11}$ which is the same for each material, then the circumferential shear stress of each material i is

$$\tau_i = G_i \kappa_1 r = G_i \frac{T}{H_{11}} r \tag{4}$$

where r is the radial distance from the center of twist (in this case, the center of the circular cross section). Using the above, the plot of the shear stress as a function of the radius is as shown in Figure 1.

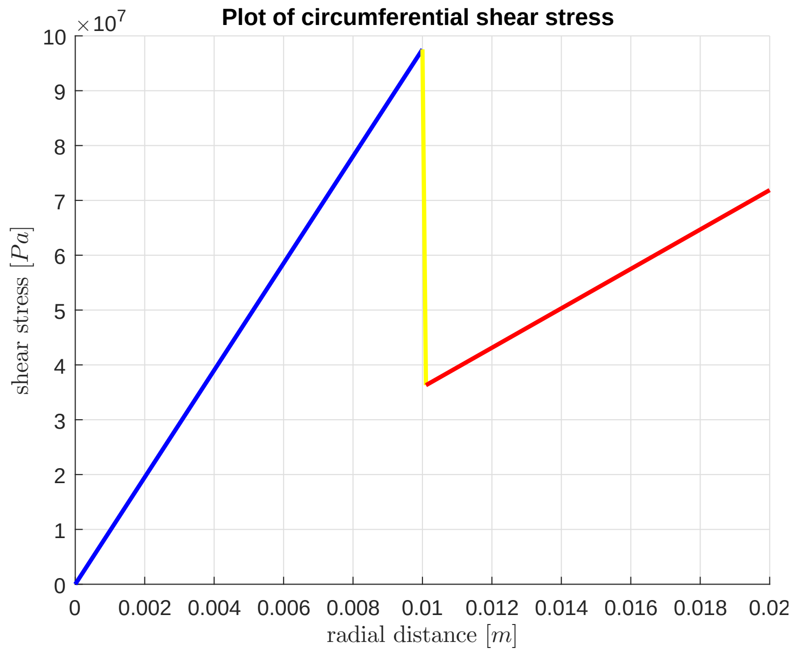


Figure 1: Shear stress as a function of radial coordinate.

3. The maximum shear stresses of each material is found at the outer most edge of the material away from the center of twist. Thus, the maximum shear stresses in each material are

$$\tau_A = G_A \frac{T}{H_{11}} R_A = 7.18764 \times 10^7 \text{ Pa} \tag{5}$$

$$\tau_S = G_S \frac{T}{H_{11}} R_S = 9.75466 \times 10^7 \text{ Pa} \tag{6}$$

$$\tag{7}$$

4. The total twist angle is calculated as follows:

$$\Phi_1(L) = \int_0^L \kappa_1 dx_1 = \kappa_1 L = \frac{TL}{H_{11}} = 0.128351 \text{ rad} \tag{8}$$

where

$$T = H_{11}\kappa_1 \tag{9}$$

Problem 2. Torsion of a circular bar with hollow segment

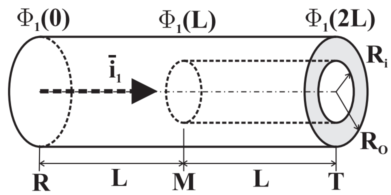


Figure 2: Circular bar with hollow segment.

The cylindrical bar shown in fig. 2 consists of two segments, clamped at point **R** and **T**, $\Phi_1(0) = \Phi_1(2L) = 0$. The left segment of length L is a solid circular bar of radius R_O, while the right segment of length L is a hollow circular bar of inner radius R_i. A moment Q_1 is applied at point **M**.

1. Determine the twist angle at point **M**.
2. Determine the equivalent torsional stiffness, H, at point **M**, defined as $H = \dfrac{Q_1}{\Phi_1(L)}$.
3. Determine the torque carried in each segment.
4. Determine the maximum shear stress in each segment.

Solution

1. To find the twist angle, first the torsional stiffness of the bar needs to be calculated. For this, the bar is broken up into its two distinct sections, the section from R to M and from M to T, and their individual torsional stiffnesses found. The bar material is assumed to be the same material and thus have the same shear moduli.

$$H_{RM} = G\frac{\pi}{2}(R_O)^4 \tag{10}$$

$$H_{MT} = G\frac{\pi}{2}\left((R_O)^4 - (R_i)^4\right) \tag{11}$$

   Now, notice that by utilizing $\dfrac{d\Phi}{dx} = \kappa$, since the section is constant through RM and through MT (and so the twist rates throughout each of these two sections is constant), we may write $\Phi = \kappa x$, where x is the distance from side R along the beam's length. In addition, since the moment can be written as $Q = H\kappa = \dfrac{H}{x}\Phi = k\Phi$, then the bar segments RM and MT can be seen as torsional springs with the spring constants for each side being

$$k_{RM} = \frac{H_{RM}}{L} \tag{12}$$

$$k_{MT} = \frac{H_{MT}}{L} \tag{13}$$

Since the bar is hyperstatic, either the force method or the displacement method may be used to solve this problem. Here, we will use the displacement method and cut the beam at M. First, we attempt to find for each side the twist angles Φ_{RM} and Φ_{MT} located at M, and then solve for the torques in each side of M of the bar using the torsional spring stiffnesses found earlier. The torque equilibrium equation at M is $0 = -Q_{RM} + Q_1 + Q_{MT}$ and the compatibility equations for the twist angle are $\Phi_{RM} = -\Phi_{MT}$ and $\Phi_M = \Phi_{RM}$. Putting these together along with the constitutive equations yields that

$$Q_{RM} = k_{RM}\Phi_{RM} = \frac{G\pi R_O^4 \Phi_M}{2L} \tag{14}$$

$$Q_{MT} = k_{MT}\Phi_{MT} = -\frac{G\pi \left(R_O^4 - R_i^4\right)\Phi_M}{2L} \tag{15}$$

Utilizing the moment equilibrium equation gives

$$Q_1 - \frac{G\pi R_O^4 \Phi_M}{2L} - \frac{G\pi \left(R_O^4 - R_i^4\right)\Phi_M}{2L} = 0 \tag{16}$$

and solving for Φ_M gives

$$\Phi_M = \frac{2LQ_1}{G\pi \left(2R_O^4 - R_i^4\right)} \tag{17}$$

2. The torsional stiffness at M is

$$H_M = \frac{Q_1}{\Phi_M} = \frac{G\pi \left(2R_O^4 - R_i^4\right)}{2L} \tag{18}$$

3. As for the torque in each segment, Q_{RM} and Q_{MT}, they can be backed out from the equations for the torsional spring stiffness and the twist angle at M found earlier.

$$Q_{RM} = k_{RM}\Phi_M = \frac{G\pi R_O^4}{2L}\frac{2LQ_1}{G\pi \left(2R_O^4 - R_i^4\right)} = \frac{Q_1 R_O^4}{2R_O^4 - R_i^4} \tag{19}$$

$$Q_{MT} = -k_{MT}\Phi_M = \frac{Q_1 \left(R_O^4 - R_i^4\right)}{2R_O^4 - R_i^4} \tag{20}$$

4. Because each segment of the bar features the same length, maximum diameter, and the same total twist, their twist rates are the same $\kappa_{RM} = \kappa_{TM} = \kappa_1$. Therefore, their maximum stresses are the same and is

$$\tau_{max} = GR_O\frac{\Phi_M}{L} = \frac{2Q_1 R_O}{\pi \left(2R_O^4 - R_i^4\right)} \tag{21}$$

Problem 3. Torsional stiffness of a section with variable thickness

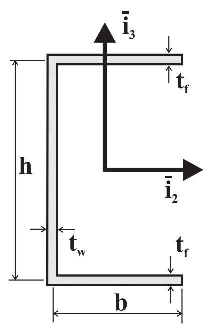


Figure 3: A thin-walled C-channel section

Figure 3 depicts the cross-section of a thin-walled beam with different thicknesses. For this problem, assume that $t_w = t$ and $t_f = 2t$.

1. Find the torsional stiffness of the section.

2. Find the magnitude and location of the maximum shear stress if the section is subjected to a torque Q.

3. Sketch the distribution of shear stress through the thickness of the wall for the two regions with different thicknesses.

Solution

1. Torsional stiffness is found by summing the torsional stiffnesses of each section. If a cross section is assumed to be thin-walled and one of the walls is a rectangle section of dimension $a \times c$ where $c >> a$, the torsional stiffness of that wall is found as approximately $H = G\dfrac{ca^3}{3}$. So, for our problem, the torsional stiffness is

$$H_{11} = \frac{1}{3}G(bt_f^3 + ht_w^3 + bt_f^3) = \frac{1}{3}G(16b + h)t^3 \tag{22}$$

2. The maximum shear stress of a thin walled section is

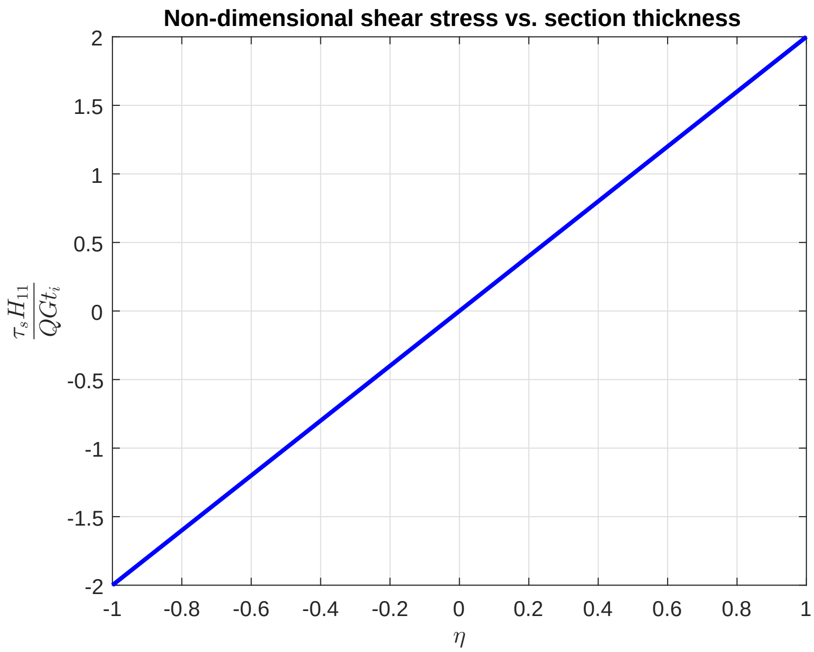
$$\tau_{max} = Gt_{max}\frac{Q}{H_{11}} = Gt_f\frac{Q}{H_{11}} = 2Gt\frac{Q}{H_{11}} = \frac{6Q}{(16b + h)\,t^2} \tag{23}$$

   and so since the thickest thickness t_{max} is the flange thickness t_f, the maximum shear stress occurs in the flanges.

3. Since $\tau_s = 2G\kappa_1 x_n$, where x_n is the perpendicular distance from the wall centerline and $-t_i \leq x_n \leq t_i$. Defining $\eta = x_n/t_i$ to be the nondimensional distance from the centerline so that $-1 \leq \eta \leq 1$, it follows that

$$\tau_s = 2G\kappa_1 t_i \eta = \frac{2Gt_iQ}{H_{11}}\eta \tag{24}$$

Letting either $t_i = t_w/2 = t/2$ or $t_i = t_f/2 = t$ makes no difference in the nondimensional plot of nondimensional stress $\frac{\tau_s H_{11}}{QGt_i}$ vs nondimensional position η



Problem 4. Torsional stiffness of semi-circular section

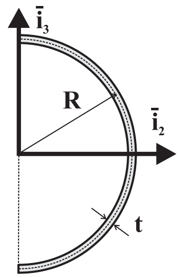


Figure 4: Semi-circular open cross-section.

Figure 4 depicts the thin-walled, semi-circular open cross-section of a beam. The wall thickness is t, and the material Young's and shear moduli are E and G, respectively.

1. Find the torsional stiffness of the section.
2. Find the distribution of shear stress due to an applied torque Q.
3. Indicate the location and magnitude of the maximum shear stress, τ_{max}.

Solution

1. Similar to the previous problem, torsional stiffness is calculated simply as

$$H_{11} = \frac{1}{3} G\pi R t^3 \tag{25}$$

2. The distribution of shear stress from Q, as measured from the centerline of the wall thickness, is

$$\tau_s = G\kappa_1 2t_w = 2G\frac{Q}{H_{11}}t_w = \frac{6Q}{\pi R t^3}t_w \tag{26}$$

for $-t/2 \leq t_w \leq t/2$.

3. τ_{max} is, according to the above equation for τ_s, therefore when $t_w = t/2$.

$$\tau_{max} = 2G\frac{Q}{H_{11}}\frac{t}{2} = \frac{3Q}{\pi R t^2} \tag{27}$$

The maximum shear stress occurs along the entire outer edge of the section. The dimensionless quantity of the maximum shear stress is

$$\frac{Rt^2\tau_{max}}{Q} = \frac{3}{\pi} \tag{28}$$

Problem 5. Torsion of a closed, semi-circular thin-walled section

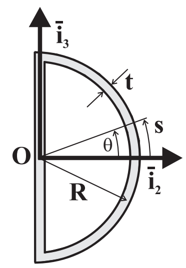


Figure 5: Thin-walled closed semi-circular section.

A beam has the closed, semi-circular thin-walled cross-section shown in fig. 5 and is subjected to a torque, Q_1.

1. Find the resulting shear flow distribution in the section.

2. Determine its torsional stiffness.

Solution

1. To find the shear flow of the section, first find the area enclosed by the section.

$$A = \frac{1}{2}\pi R^2 \tag{29}$$

   Then, the shear flow due to Q_1 is

$$f = \frac{Q_1}{2A} = \frac{Q_1}{\pi R^2} \tag{30}$$

2. Torsional stiffness is calculated as

$$H_{11} = \frac{Q_1}{\kappa_1} \tag{31}$$

   in which the twist rate is

$$\kappa_1 = \frac{1}{2A}\left(\frac{f(2R + \pi R)}{Gt}\right) = \frac{Q_1(2+\pi)}{\pi^2 R^3 G t} \tag{32}$$

   The torsional stiffness is then

$$H_{11} = \frac{G\pi^2 R^3 t}{2+\pi} \tag{33}$$

Problem 6. Torsion of a 2-cell rectangular cross-section

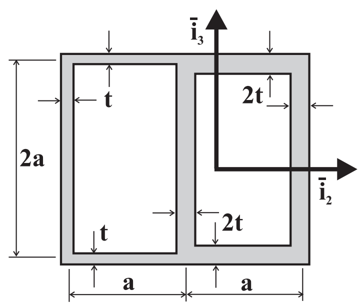


Figure 6: Thin-walled 2-cell rectangular section with variable wall thicknesses.

The cross-section of a thin-walled beam consists of two rectangular cells, as shown in fig. 6. The beam is subjected to a torque Q_1.

1. Determine the shear flow distribution in the cross-section.
2. Find the magnitude and location of the maximum shear stress in the section.
3. Determine its torsional stiffness.
4. Does the section's mid vertical web contribute significantly to the torsional stiffness? Explain.

Solution

Note: Much of this problem can be found by simply following the steps given in Example 8.17 in the course's textbook.

1. The box is made up of 2 rectangular cells, so let the "left" cell be labeled as Section 1 and the "right" cell be labeled as Section 2. For each closed section, we first find the area enclosed.

$$A_1 = (a)(2a) = 2a^2 \tag{34}$$
$$A_2 = (a)(2a) = 2a^2 \tag{35}$$

Now, we know that the total torque in the section is simply the sum of the torques experienced by each cell, and since the torque within a closed, thin-walled section i is $T_i = 2A_i f_i$, where f_i is the shear flow, we are left with the following equation

$$Q_1 = 2A_1 f_1 + 2A_2 f_2 = 4a^2(f_1 + f_2) \tag{36}$$

As for the twist rate, the twist rate for each section must be the same due to our initial assumptions of the compatibility conditions. Therefore, the twist rate κ_1 is for Section 1

$$\kappa_1 = \frac{1}{2A_1}\left(\frac{f_1(a+2a+a)}{Gt} + \frac{(f_1 - f2)(2a)}{G(2t)}\right) = \frac{1}{4a^2}\left(\frac{4af_1}{Gt} + \frac{a(f_1 - f2)}{Gt}\right) \tag{37}$$

and for Section 2

$$\kappa_1 = \frac{1}{2A_2}\left(\frac{f_2(a+2a+a)}{G(2t)} + \frac{(f_2 - f1)(2a)}{G(2t)}\right) = \frac{1}{4a^2}\left(\frac{2af_2}{Gt} + \frac{a(f_2 - f1)}{Gt}\right) \tag{38}$$

We would like to solve for the shear flow distribution in the cross-section, that is, we want to find the shear flow along all edges. The shear flow for the left box cell's walls not including the middle vertical web is f_1, for the right box cell's walls not including the middle vertical web is f_2, and for the middle vertical web is $f_1 - f_2$ if we consider downward the positive direction (or equivalently, in the same direction as f_1). To find these shear flows, notice that we have three equations, (36), (37), and (38); and we have three unknowns, κ_1, f_1, and f_2. Solving this system of equations gives us

$$f_1 = \frac{Q_1}{10a^2} \tag{39}$$

$$f_2 = \frac{3Q_1}{20a^2} \tag{40}$$

$$\kappa_1 = \frac{7Q_1}{80a^3Gt} \tag{41}$$

immediately giving us two of the three shear flows in the cross section we were wanting to find. The shear flow in the middle vertical web is then

$$f_1 - f_2 = \frac{Q_1}{20a^2} \tag{42}$$

2. Max shear stress can be found by computing the shear distribution per unit of cross section. The sections are all closed, and so we know the shear flows, or in other words, the shear per unit length along the box cell's walls, is constant. The shear per unit of cross section, or simply the shear, is then the shear flow divided by the wall thickness.

   For the Section 1 walls not including the vertical middle web, the shear flow is f_1 throughout and the wall thicknesses are all t, and so the shear is then

$$s_1 = \frac{f_1}{t} = \frac{0.1Q_1}{a^2t} \tag{43}$$

   For the Section 2 walls not including the vertical middle web, the shear flow is f_2 throughout and the wall thicknesses are all $2t$, and so the shear is then

$$s_2 = \frac{f_2}{2t} = \frac{0.075Q_1}{a^2t} \tag{44}$$

   As for the vertical middle web, the shear flow is $f_1 - f_2$ throughout and the wall thickness is $2t$, and so the shear is then

$$s_{1+2} = \frac{f_1 - f_2}{2t} = \frac{0.025Q_1}{a^2t} \tag{45}$$

   Therefore, the maximum shear stress is in the walls comprising the perimeter of Section 1 with thickness t, that is, $s_{max} = s_1$.

3. The torsional stiffness is given as Q_1/κ_1, in which κ_1 was found earlier.

$$H_{11} = \frac{Q_1}{\kappa_1} = \frac{80}{7}a^3Gt \tag{46}$$

4. Computing the torsional stiffness disregarding the middle vertical web requires finding a new twist rate. Since without the middle web there is a single closed section, then there is only one shear flow f' and area A'. The following then leads us to the new torsional stiffness.

$$A' = (2a)(2a) = 4a^2 \tag{47}$$

$$Q_1 = 2A'f' \quad \Rightarrow \quad f' = \frac{Q_1}{8a^2} \tag{48}$$

$$\kappa_1' = \frac{1}{2A'}\left(\frac{f'(a+2a+a)}{Gt} + \frac{f'(a+2a+a)}{G(2t)}\right) \tag{49}$$

$$H_{11}' = \frac{Q_1}{\kappa_1'} = \frac{32}{3}a^3Gt \tag{50}$$

Comparing this new torsional stiffness with the original, the increase of the torsional stiffness due to the web's contribution is

$$\frac{H_{11} - H_{11}'}{H_{11}'} = 0.00\bar{6} \tag{51}$$

or $\approx 6.667\%$ increase in stiffness.

So we may conclude that the middle vertical web does not contribute significantly to the torsional stiffness.