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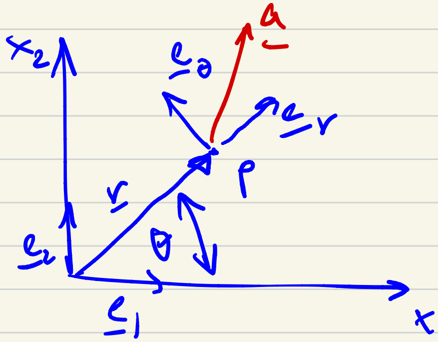
Plane problems in polar coordinates



- $(r, \theta) \equiv$ polar coordinates
- $\{\underline{e}_r, \underline{e}_\theta\} \equiv$ basis vectors in polar coords

Both coord systems are related through the following transformations:

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x_1^2 + x_2^2} \\ \theta = \arctan\left(\frac{x_2}{x_1}\right) \end{cases}$$

Which implies that the basis vectors relate as

$$\begin{cases} \underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2 & \rightarrow \ e_r \text{ at } \theta \\ \underline{e}_\theta = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2 & \rightarrow \ e_\theta \text{ at } \theta + 90 \end{cases}$$

That is, a general vector can be expressed in polar coordinates as:

$$\underline{a} = a_r \underline{e}_r + a_\theta \underline{e}_\theta = a_r(\cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2)$$
$$+ a_\theta(-\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2)$$
$$= \overbrace{(a_r\cos\theta - a_\theta\sin\theta)}^{a_1}\underline{e}_1 + \overbrace{(a_r\sin\theta + a_\theta\cos\theta)}^{a_2}\underline{e}_2$$
$$= a_1\underline{e}_1 + a_2\underline{e}_2$$

$$\Rightarrow \quad a_1 = a_r\cos\theta - a_\theta\sin\theta$$

$$a_2 = a_r\sin\theta + a_\theta\cos\theta$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \underbrace{\overbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}^{[Q]^T}}_{\text{rotation matrices}} \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a_r \\ a_\theta \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{[Q]} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Hence, we can get the stress components using the transformation $[Q]^T[\sigma][Q]$

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

From that we get

$$\sigma_{rr} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta$$

$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\cos\theta\sin\theta$$

$$\sigma_{r\theta} = -(\sigma_{11} - \sigma_{22})\cos\theta\sin\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)$$

Additionally, we can write:

$$\begin{cases} \text{position: } \underline{x} = r\underline{e}_r \\ \text{Displacement: } \underline{u} = u_r\underline{e}_r + u_\theta\underline{e}_\theta \\ \text{Body forces: } \underline{b} = b_r\underline{e}_r + b_\theta\underline{e}_\theta \end{cases}$$

- How about quantities derived from differential operators?

Just use direct notation, eg: $\underline{\nabla}\underline{u}$ instead of $\frac{\partial u_i}{\partial x_j}$ and apply the corresponding operator in the chosen coordinate system.

For example, for the inf. strain we have:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right);$$

$$\nabla \equiv \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

Thus, we can compute $\underline{\nabla}\,\underline{u}$ as follows:

$$\underline{\nabla}\,\underline{u} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{\partial}{\partial \theta}\right)\left(u_r \underline{e}_r + u_\theta \underline{e}_\theta\right)$$

$$= \underline{e}_r \frac{\partial}{\partial r}\left(u_r \underline{e}_r + u_\theta \underline{e}_\theta\right) + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left(u_r \underline{e}_r + u_\theta \underline{e}_\theta\right)$$

We can complete the calculation by applying differentiation rules while keeping in mind that basis vectors are no longer constant;

That is: $\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta$ and $\frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r$

In this way, we get:

$$\boxed{\begin{aligned} &\varepsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\ &\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\end{aligned}}$$

We can follow the same procedure to compute $\underline{\nabla} \cdot \underline{\underline{\sigma}}$ in the equation of equilibrium.

(Details on the operator can be found in Bower's book (appendix))

The equations of equilibrium become:

$$\boxed{\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r = 0 \\ &\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \rho b_\theta = 0 \end{aligned}}$$

Finally, Hooke's law does not change, since we have

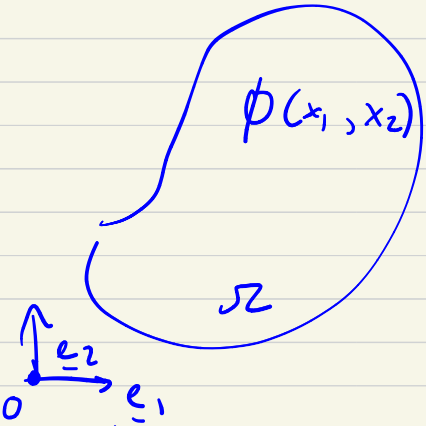
$$\underline{\underline{\sigma}} = \lambda^* tr(\underline{\underline{\varepsilon}}) + 2\mu^* \underline{\underline{\varepsilon}} \quad \text{remains the same}$$

$$\boxed{\begin{aligned} &\begin{cases} \sigma_{rr} = \lambda^*(\varepsilon_{rr} + \varepsilon_{\theta\theta}) + 2\mu^* \varepsilon_{rr} \\ \sigma_{\theta\theta} = \lambda^*(\varepsilon_{rr} + \varepsilon_{\theta\theta}) + 2\mu^* \varepsilon_{\theta\theta} \\ \sigma_{r\theta} = 2\mu^* \varepsilon_{r\theta} \end{cases} \\ \text{or} \; &\begin{cases} \varepsilon_{rr} = \frac{1}{E^*}(\sigma_{rr} - \nu^* \sigma_{\theta\theta}) \\ \varepsilon_{\theta\theta} = \frac{1}{E^*}(\sigma_{\theta\theta} - \nu^* \sigma_{rr}) \\ \varepsilon_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} \end{cases} \end{aligned}}$$

Airy stress for plane problems

Let us consider the plane equilibrium equations for the static case and w/ no body forces

$$(1)\quad \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0 \\[2ex] \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} = 0 \end{cases}$$



Let us consider a scalar function $\phi(x_1, x_2)$ s.t.

$$(2)\quad \left\{ \dfrac{\partial^2 \phi}{\partial x_2^2} = \sigma_{11} \;;\quad \dfrac{\partial^2 \phi}{\partial x_1^2} = \sigma_{22} \;;\quad -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} = \sigma_{12} \right.$$

We can see that such a representation for $\sigma_{\alpha\beta}$ $(\alpha, \beta = 1,2)$ automatically satisfies equations (1):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_2^2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)$$

$$= \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} - \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} = 0 \quad \checkmark$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = \frac{\partial}{\partial x_1}\left(-\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \phi}{\partial x_1^2}\right)$$

$$= -\frac{\partial^3 \phi}{\partial x_1^2 \partial x_2} + \frac{\partial^3 \phi}{\partial x_2 \partial x_1^2} = 0 \quad \checkmark$$

- That is, we can choose any scalar function $\phi(x_1, x_2)$ and compute the corresponding stresses given by (2) and equilibrium equations (1) will be automatically satisfied. But, would it make physical sense?

- To understand this question, let us think in terms of displacements:

$$\phi(x_1, x_2) \xrightarrow[\text{def}]{} \sigma_{\alpha\beta}(x_1, x_2) \xrightarrow[\text{Hooke's}]{} \varepsilon_{\alpha\beta}(x_1, x_2) \xrightarrow[?]{} u_i(x_1, x_2)$$

- That is, Hooke's law allows us to compute the strains given the stresses but that does not mean that we could integrate those strains to obtain displacements

- We usually refer to this question as

"Are strains compatible"

- It can be shown that in plane problems, there is just <u>one</u> compatibility condition to satisfy:

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0} \qquad (3)$$