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HW #7

* Graded assignments were not given back (that I know of) so these are just my answers

↳ If you have corrected answers or find any mistakes, let me know! :)

Problem 2

Let us consider the same problem as before, and solve it using the Rayleigh-Ritz method. You don't need to re-derive the potential, just use the on derived before:

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 - A\int_0^L g(x_1)u_1\,dx_1 - APu_1\Big|_{x_1=L}$$

In particular, assume $g(x_1) = \text{constant} = g$ and proceed as follows:

1. First, consider an approximate solution $\hat{u} = ax_1 + b$, make sure it satisfies the displacement boundary conditions and minimize the corresponding functional. Plot this approximate solution and the exact one obtained for the previous problem. Comment on the results.

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 - A\int_0^L g(x_1)u_1\,dx_1 - APu_1\Big|_{x_1=L}$$

$$\hat{u} = ax_1 + b$$

Displacement BCs: $u_1(0)=0 \Rightarrow \hat{u}(0)=0 \Rightarrow b=0$

$$\frac{\partial \hat{u}}{\partial x_1} = a \quad , \quad \hat{u} = ax_1$$

plugging into Π:

$$\hat{\Pi} = \frac{EA}{2}\int_0^L a^2 dx_1 - A\int_0^L gax_1\,dx_1 - APaL$$

$$= \frac{EA}{2}a^2x_1\Big|_0^L - \frac{1}{2}Agax_1^2\Big|_0^L - APaL = \frac{1}{2}EAa^2L - \frac{1}{2}AgaL^2 - APaL$$

$$\frac{\partial \hat{\Pi}}{\partial a} = 0 = EAaL - \frac{1}{2}AgL^2 - APL$$

$$EAaL = \frac{1}{2}AgL^2 + APL$$

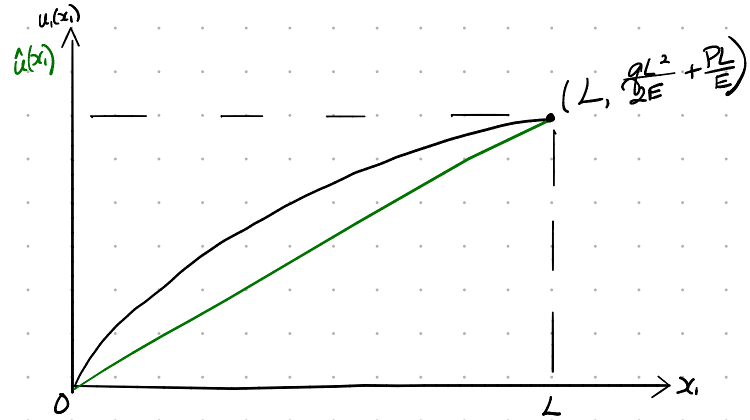
$$a = \frac{gL}{2E} + \frac{P}{E}$$

plugging back in to $\hat{u}$:

$$\hat{u}(x_1) = \left(\frac{qL}{2E} + \frac{P}{E}\right)x_1$$

$\hat{u}(0) = 0$ ✓

$\hat{u}(L) = \frac{qL^2}{2E} + \frac{PL}{E}$



Thus, with a linear approximate solution we linearly approximate the exact solution. If our solution was linear, our approximate solution would have given the exact solution.

2. Now consider an approximate solution of the form $\hat{u} = cx_1^2 + dx_1 + e$ and proceed as before. Plot this new solution versus the previous ones (exact and approximate) and comment about your results.

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 dx_1 - A\int_0^L q(x_1)u_1 dx_1 - APu_1\Big|_{x_1=L}$$

$$q(x_1) = q$$

$$\hat{u}_2 = cx_1^2 + dx_1 + e$$

Displacement BCs: $u_1(0) = 0$

$\Rightarrow \hat{u}_2(0) = 0 \Rightarrow e = 0$

$$\frac{\partial \hat{u}_2}{\partial x_1} = 2cx_1 + d, \quad \hat{u}_2 = cx_1^2 + dx_1$$

plugging into Π:

$$\hat{\Pi}_2 = \frac{EA}{2}\int_0^L (2cx_1 + d)^2 dx_1 - A\int_0^L q(cx_1^2 + dx_1)dx_1 - AP(cL^2 + dL)$$

$$= \frac{EA}{2}\int_0^L (4c^2x_1^2 + 4cdx_1 + d^2)dx_1 - A\int_0^L qcx_1^2 dx_1 - A\int_0^L qdx_1 dx_1 - APcL^2 - APdL$$

$$= \frac{EA}{2}\left[\frac{4}{3}c^2x_1^3 + 2cdx_1^2 + d^2x_1\right]_0^L - \frac{1}{3}Aqcx_1^3\Big|_0^L - \frac{1}{2}Aqdx_1^2\Big|_0^L - APcL^2 - APdL$$

$$\hat{\Pi}_2 = \frac{2}{3}EAc^2L^3 + EAcdL^2 + \frac{1}{2}EAd^2L - \frac{1}{3}AqcL^3 - \frac{1}{2}AqdL^2 - APcL^2 - APdL$$

Taking derivatives with respect to c and d and set equal to zero:

$$\frac{\partial \hat{\Pi}_2}{\partial c} = \frac{4}{3}E\cancel{A}cL^{\cancel{3}} + E\cancel{A}d\cancel{L^2} - \frac{1}{3}\cancel{A}qL^{\cancel{3}} - \cancel{A}P\cancel{L^2} = 0$$

$$4EcL + 3Ed - qL - 3P = 0$$

$$4EcL = qL + 3P - 3Ed$$

$$c = \frac{qL + 3P - 3Ed}{4EL}$$

$$\frac{\partial \hat{\Pi}_2}{\partial d} = E\cancel{A}cL^{\cancel{2}} + E\cancel{A}d\cancel{L} - \frac{1}{2}\cancel{A}qL^{\cancel{2}} - \cancel{A}P\cancel{L} = 0$$

$$EcL + Ed - \frac{1}{2}qL - P = 0$$

$$Ed = \frac{1}{2}qL + P - EcL$$

$$d = \frac{1}{2}\frac{qL}{E} + \frac{P}{E} - cL$$

plugging in for c:

$$d = \frac{1}{2}\frac{qL}{E} + \frac{P}{E} - \left(\frac{qL^2 + 3PL - 3EdL}{4EL}\right)$$

$$d = \frac{1}{2}\frac{qL}{E} + \frac{P}{E} - \frac{qL}{4E} - \frac{3P}{4E} + \frac{3d}{4}$$

$$\frac{d}{4} = \frac{1}{4}\frac{qL}{E} + \frac{1}{4}\frac{P}{E}$$

$$\Rightarrow d = \frac{qL}{E} + \frac{P}{E}$$

plugging in for d:

$$c = \frac{q}{4E} + \frac{3P}{4EL} - \frac{3}{4L}\left(\frac{qL}{E} + \frac{P}{E}\right) = \frac{q}{4E} + \frac{3P}{4EL} - \frac{3q}{4E} - \frac{3P}{4EL}$$

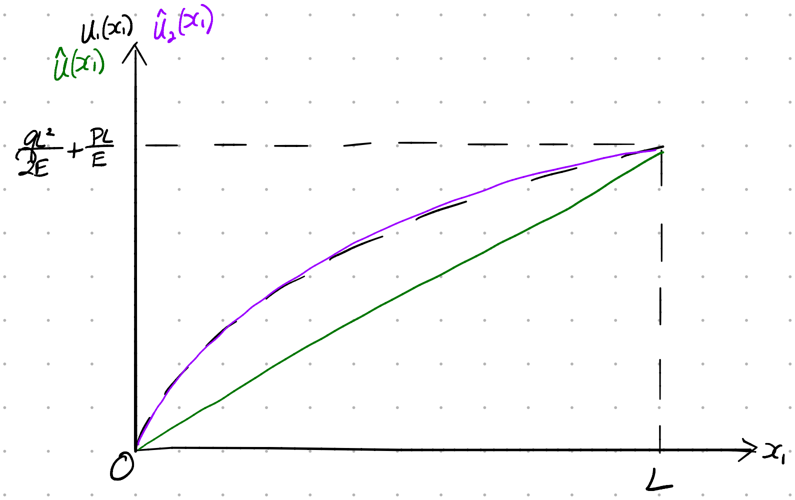
$$\Rightarrow c = -\frac{q}{2E}$$

plugging c and d back into $\hat{u}_2$:

$$\hat{u}_2 = -\frac{qx_1^2}{2E} + \left(\frac{qL+P}{E}\right)x_1$$ ← the exact solution!

$\hat{u}_2(0) = 0$ ✓

$\hat{u}_2(L) = -\frac{qL^2}{2E} + \frac{qL^2}{E} + \frac{PL}{E} = \frac{qL^2}{2E} + \frac{PL}{E}$



Thus, by having an approximate that was quadratic, we arrived at our exact solution because our exact solution is also quadratic. $\hat{u}$ included only linear functions, so we could not recover our exact solution, but $\hat{u}_2$ includes all quadratic polynomials, which, of course, included our exact solution.