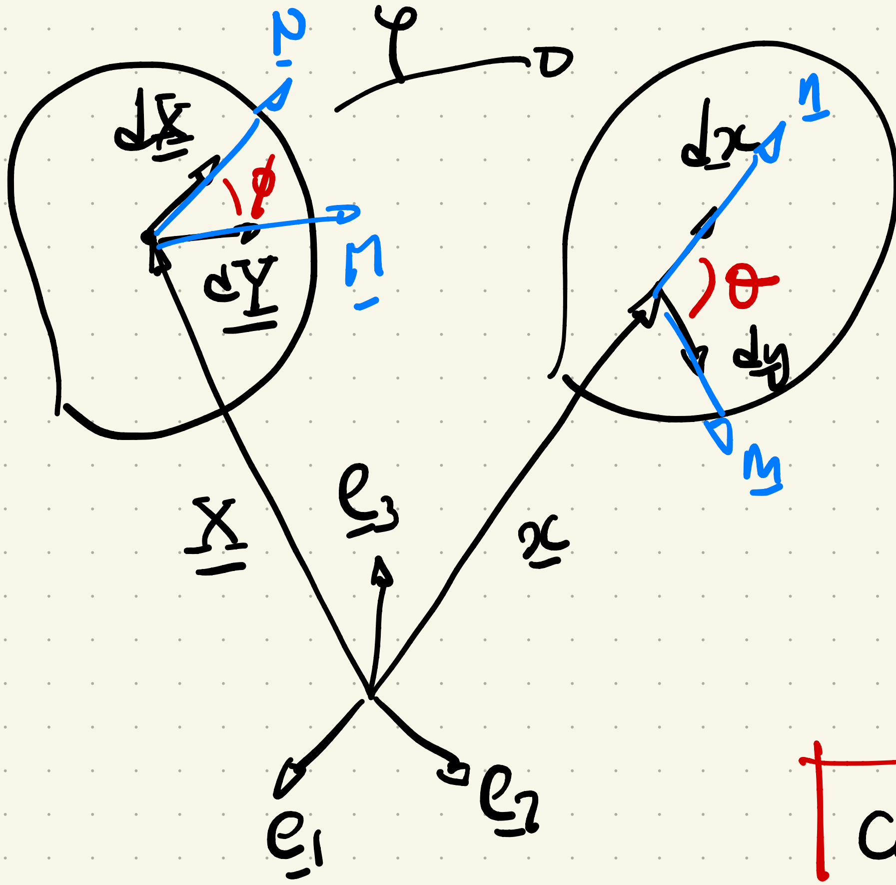
Changes in angles



Unit vectors:

$$\underline{N} = \frac{d\underline{X}}{\|d\underline{X}\|} \qquad \underline{n} = \frac{d\underline{x}}{\|d\underline{x}\|}$$

$$\underline{M} = \frac{d\underline{Y}}{\|d\underline{Y}\|} \qquad \underline{m} = \frac{d\underline{y}}{\|d\underline{y}\|}$$

$$\boxed{\cos\phi = \underline{N} \cdot \underline{M}}$$

Analogously: $\cos\theta = \underline{n} \cdot \underline{m} = n_i\, m_i = \dfrac{dx_i \; dy_i}{\|d\underline{x}\|\,\|d\underline{y}\|}$

$$\cos\theta = \frac{F_{ij}\, d\underline{X}_j \; F_{ik}\, d\underline{Y}_k}{\lambda_x \|d\underline{X}\| \, \lambda_y \|d\underline{Y}\|} = \frac{C_{jk} N_j M_k}{\lambda_x \lambda_y}$$

$$\lambda_x = \sqrt{C_{\ell m} N_\ell N_m} \qquad \lambda_y = \sqrt{C_{pq} M_p M_q}$$

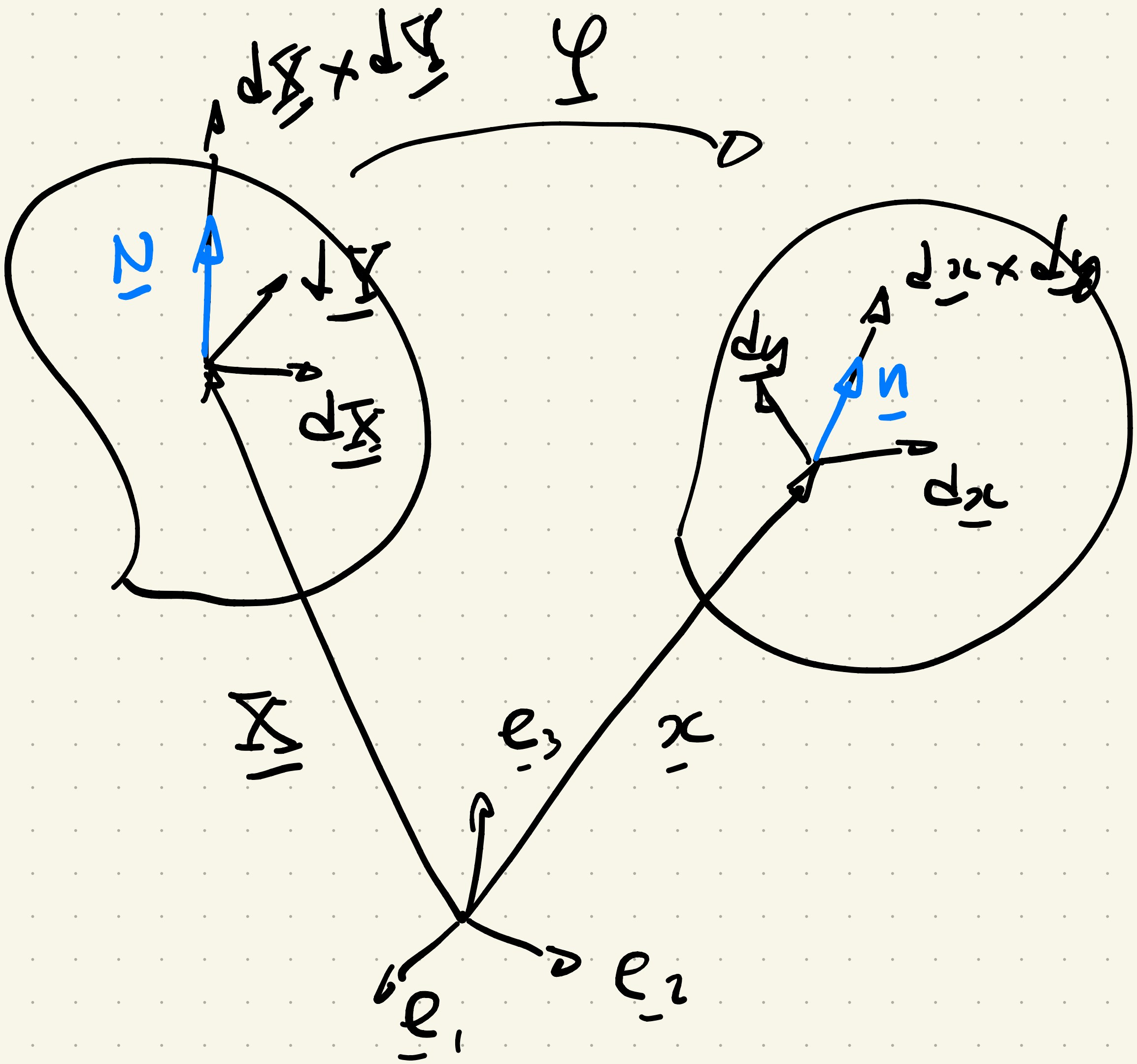
$$\boxed{\cos\theta = \frac{C_{jk} N_j M_k}{\sqrt{C_{\ell m} N_\ell N_m}\,\sqrt{C_{pq} M_p M_q}}}$$

We can specialize all previous relations for directions aligned with the basis vectors in the reference conf.

$$\lambda(\underline{e}_1) = \sqrt{C_{11}} \; ; \quad \lambda(\underline{e}_2) = \sqrt{C_{22}} \; ; \quad \lambda(\underline{e}_3) = \sqrt{C_{33}}$$

$$\cos\theta(\underline{e}_1, \underline{e}_2) = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} \quad \cos\theta(\underline{e}_1, e_3) = \frac{C_{13}}{\sqrt{C_{11}}\sqrt{C_{33}}} \quad \cos\theta(\underline{e}_2, e_3) = \frac{C_{23}}{\sqrt{C_{22}}\sqrt{C_{33}}}$$

Change in area



$$dA = \| d\underline{X} \times d\underline{Y} \|$$

$$\underline{N} = \frac{d\underline{X} \times d\underline{Y}}{\| d\underline{X} \times d\underline{Y} \|}$$

$$da = \| d\underline{x} \times d\underline{y} \|$$

$$\underline{n} = \frac{d\underline{x} \times d\underline{y}}{\| d\underline{x} \times d\underline{y} \|}$$

We define differential oriented areas as:

$$\left.\begin{aligned} d\underline{a} &= \underline{n}\, da = d\underline{x} \times d\underline{y} \Rightarrow da_i = \epsilon_{ijk}\, dx_j\, dy_k \\ d\underline{A} &= \underline{N}\, dA = d\underline{X} \times d\underline{Y} \Rightarrow dA_i = \epsilon_{ijk}\, dX_j\, dY_k \end{aligned}\right\} \text{by def. of cross product}$$

$$\Rightarrow da_i = n_i\, da = \epsilon_{ijk}\, F_{jp}\, dX_p\, F_{kq}\, dY_q$$

Multiplying by F_{io}: $F_{io}\, n_i\, da = \underbrace{\epsilon_{ijk}\, F_{io}\, F_{jp}\, F_{kq}}_{\text{by def. } \epsilon_{opq} \det(\underline{\underline{F}})}\, dX_p\, dY_q$

$$\Rightarrow F_{io}\, n_i\, da = \underbrace{\epsilon_{opq}\, dX_p\, dY_q}_{\text{by def. } N_o\, dA}\, \det(\underline{\underline{F}})$$

$$\Rightarrow F_{io}\, n_i\, da = \det(\underline{\underline{F}})\, N_o\, dA$$

Multiplying by $(F^{-1})_{op}$:

$$\underbrace{F_{io}(F^{-1})_{op}}_{\delta_{ip}}\, n_i\, da = (F^{-1})_{op} \det(\underset{=}{F}) N_o\, dA$$

$$\Rightarrow \boxed{n_p\, da = (F^{-1})_{op} \det(\underset{=}{F}) N_o\, dA}$$

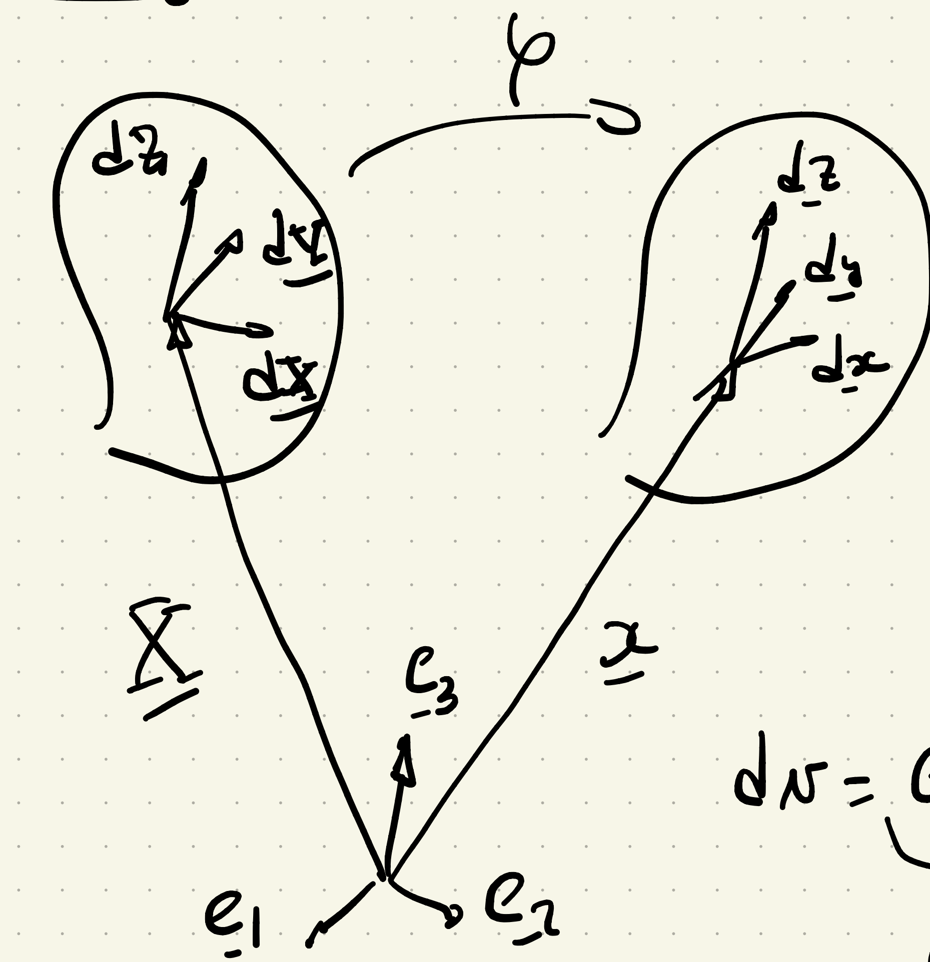
(Piola transformation)

In direct notation: $\underset{-}{n}\, da = \underset{=}{F}^{-T} \cdot \underset{-}{N}\, J\, dA$

where $J = \det(\underset{=}{F})$

(for stress)
↳ we look in the deformed config.

Change in volume



$$dV = \epsilon_{ijk}\, dX_i\, dY_j\, dZ_k$$

$$dv = \epsilon_{ijk}\, dx_i\, dy_j\, dz_k$$

$$dx_i = F_{im}\, dX_m$$

$$dy_j = F_{jn}\, dY_n$$

$$dz_k = F_{ko}\, dZ_o$$

$$dv = \underbrace{\epsilon_{ijk} F_{im} F_{jn} F_{ko}}_{\epsilon_{mno} \det(\underline{F})}\, dX_m\, dY_n\, dZ_o$$

$$\Rightarrow dv = \det(\underline{F})\, \underbrace{\epsilon_{mno}\, dX_m\, dY_n\, dZ_o}_{dV} \Rightarrow \boxed{dv = \det(\underline{F})\, dV}$$

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