AE6114 Exam 2 Review

Problem 1

The components of the Cauchy stress tensor at a point in a solid are given, in the basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, by:

$$\underline{\underline{\sigma}} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

1. Calculate the stress vector (traction) $\underline{t}$ on a surface element with normal $[1, -1, 0]^T$

$$\underline{\underline{\sigma}} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$ ← this stress tensor is <u>not</u> in equilibrium!!

⇒ angular momentum is <u>not</u> conserved!

Normally, we would stop here, but if we pretend that it is in equilibrium...

$$\sigma_{ij} n_j = t_i$$

must be a unit vector!

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} - 1/\sqrt{2} \\ 4/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix} \Rightarrow \boxed{\underline{t} = \left[\frac{1}{\sqrt{2}}, \frac{4}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right]^T}$$

2. Give the normal and shear components of this stress vector (indicating both magnitude and direction of the components).

$$|\underline{t}_n| = \sigma_{ji} n_i n_j = t_i n_i$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 4/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{2} - 2 + 0 \Rightarrow \boxed{|\underline{t}_n| = \frac{-3}{2}}$$

$\underline{t}_n$ is in the $\underline{n}$ direction: $-\frac{3}{2}\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \Rightarrow \boxed{\underline{t}_n = \left[\frac{-3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, 0\right]^T}$

$$\underline{t}_s = \underline{t} - \underline{t}_n$$

$$\left[\frac{1}{\sqrt{2}}, \frac{4}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right] - \left[\frac{-3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, 0\right] = \left[\left(\frac{2}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\right), \left(\frac{8}{2\sqrt{2}} - \frac{3}{2\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}} - 0\right)\right] \Rightarrow \boxed{\underline{t}_s = \left[\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{-3}{\sqrt{2}}\right]^T}$$

$$|\underline{t}_s| = \sqrt{\left(\frac{5}{2\sqrt{2}}\right)^2 + \left(\frac{5}{2\sqrt{2}}\right)^2 + \left(\frac{-3}{\sqrt{2}}\right)^2}$$

$$\Rightarrow \boxed{|\underline{t}_s| = \frac{\sqrt{43}}{2}}$$

Another way to find $|\underline{t}_s|$:

$$|\underline{t}_s| = \sqrt{|\underline{t}|^2 - |\underline{t}_n|^2} \qquad |\underline{t}|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{4}{\sqrt{2}}\right)^2 + \left(\frac{-3}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{16}{2} + \frac{9}{2} = \frac{26}{2} = 13$$

$$= \sqrt{13 - \frac{9}{4}} = \frac{\sqrt{43}}{2}$$

3. Find the principal stresses and principal directions of the stress tensor.

$$\det[\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}] = 0$$

$$\det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 4 & -\lambda & 2 \\ 0 & 3 & 1-\lambda \end{bmatrix} = (2-\lambda)[-\lambda(1-\lambda) - 2(3)] - 1(4(1-\lambda) - 2(0)) + 0 = 0$$

$$= (2-\lambda)(-\lambda + \lambda^2 - 6) - (4 - 4\lambda) = 0$$

$$= -2\lambda + 2\lambda^2 - 12 + \lambda^2 - \lambda^3 + 6\lambda - 4 + 4\lambda = 0$$

$$= -\lambda^3 + 3\lambda^2 + 8\lambda - 16 = 0$$

$$\lambda^3 - 3\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)(\lambda^2 + \lambda - 4) = 0 \qquad \frac{-1 \pm \sqrt{1 - 4(1)(-4)}}{2(1)} = \frac{-1 \pm \sqrt{17}}{2} \simeq 1.56,\ -2.56$$

$$\lambda = 4,\ 1.56,\ -2.56$$

principal stresses:

$\sigma_1 = 4 \quad \sigma_2 = 1.56 \quad \sigma_3 = -2.56$

$\underline{\lambda = 4}$

$$\begin{bmatrix} 2-4 & 1 & 0 \\ 4 & -4 & 2 \\ 0 & 3 & 1-4 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 4 & -4 & 2 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -2v_1 + v_2 &= 0 \\ 4v_1 - 4v_2 + 2v_3 &= 0 \\ 3v_2 - 3v_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} v_1 &= \tfrac{1}{2} v_2 \\ v_3 &= v_2 \end{aligned}$$

let $v_2 = v_3 = 1 \Rightarrow v_1 = \frac{1}{2}$

make a unit vector: $\sqrt{1^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + 1 + \frac{1}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

$$\Rightarrow \underline{v}_1 = \left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]^T$$

Repeat this procedure for $\lambda = 1.56$ and $\lambda = -2.56$

Problem 2

A stress field is given by the following,

$$\sigma_{11} = \sigma_{22} = Cx_1x_2$$
$$\sigma_{12} = \sigma_{21} = C(a^2 - x_2^2)$$
$$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

1. Is the stress field in static equilibrium?

assuming the body is at rest:

$$\sigma_{ij,j} + \rho b_i = \cancel{\rho \ddot{x}_i} \rightarrow 0$$

$$\sigma_{ij,j} = -\rho b_i$$

assuming no body forces:

$$\sigma_{ij,j} = 0$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad \Rightarrow \quad Cx_2 - 2Cx_2 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0 \quad \Rightarrow \quad Cx_1 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \quad \Rightarrow \quad 0 = 0$$

$$\left.\begin{matrix} -Cx_2 = 0 \\ Cx_1 = 0 \end{matrix}\right\} \begin{matrix} C \neq 0 \\ x_1, x_2 \neq 0 \end{matrix} \quad \Rightarrow$$ no, the stress field is not in static equilibrium

2. If not, how would you modify the expressions for σ_{12} and σ_{21} to establish static equilibrium?

$$\frac{\partial \sigma_{12}}{\partial x_1} = -Cx_1$$ ← need this to be true

integrating both sides:

(1) $$\sigma_{12} = -\frac{1}{2}Cx_1^2 + A(x_2)$$

↑ Constant that is a function of x_2

$$\frac{\partial \sigma_{12}}{\partial x_2} = -Cx_2$$ ← need this to be true as well

↑ plug in (1) here:

$$-Cx_2 = \frac{\partial}{\partial x_2}\left[-\frac{1}{2}Cx_1 + A(x_2)\right] = \frac{\partial A(x_2)}{\partial x_2}$$

integrating both sides:

$$A(x_2) = -\frac{1}{2}Cx_2^2 + B$$

plug this back in to (1):

$$\sigma_{12} = -\frac{1}{2}Cx_1^2 - \frac{1}{2}Cx_2^2 + B$$

B can be anything - to keep it with the form of the original question, let $B = Ca^2$

then,

$$\boxed{\sigma_{12} = \sigma_{21} = C\left(a^2 - \frac{x_1^2}{2} - \frac{x_2^2}{2}\right)}$$