AE6114 Final Exam

Problem 1

A strain gage rosette (see Figure 1) with 3 gages is placed at a point P on a body. One gage (B) is aligned in the x_2 direction with the other two (A and C) aligned $60°$ on either side of the x_2 direction. The body is then stressed and the strains from the 3 gages are measured to be ε_A, ε_B, and ε_C.

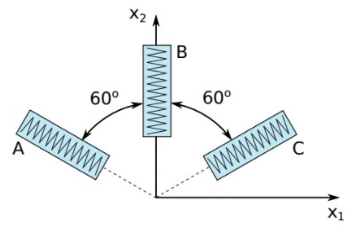


Figure 1: Strain gage rosette for problem 1.

1. Given that a strain gage gives the normal strain in the direction of its orientation, determine all components of the infinitesimal strain tensor, assuming that the components ε_{13}, ε_{23}, and ε_{33} are zero (i.e. plane strain).

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In class we found: $[\sigma]' = [Q][\sigma][Q]^T$

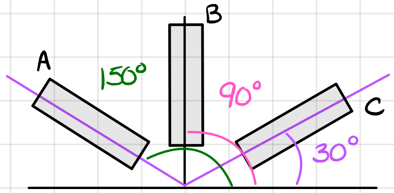
for $[\sigma]' = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix}$ and $[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

which gave us the equations:

$$\sigma_{rr} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta$$
$$\sigma_{r\theta} = -(\sigma_{11} + \sigma_{22})\cos\theta\sin\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)$$
$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\cos\theta\sin\theta$$

In the case of strain gages, each gage gives ε_{rr} at that particular orientation. We also know that the transformations for stress can be applied to transform strains. Using this information, we can say that:

$$\varepsilon_{rr} = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta$$



Gage A:

$\varepsilon_{rr} = \varepsilon_A$, $\theta = 90° + 60° = 150°$

$\varepsilon_A = \varepsilon_{11} \cos^2(150°) + \varepsilon_{22} \sin^2(150°) + 2\varepsilon_{12} \cos(150°) \sin(150°)$

$= \varepsilon_{11} \left(-\frac{\sqrt{3}}{2}\right)^2 + \varepsilon_{22} \left(\frac{1}{2}\right)^2 + 2\varepsilon_{12} \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$\varepsilon_A = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} - \frac{\sqrt{3}}{2}\varepsilon_{12}$ (1)

Gage B:

$\varepsilon_{rr} = \varepsilon_B$, $\theta = 90°$

$\varepsilon_B = \varepsilon_{11} \cos^2(90°) + \varepsilon_{22} \sin^2(90°) + 2\varepsilon_{12} \cos(90°) \sin(90°)$

$= \varepsilon_{11}(0) + \varepsilon_{22}(1) + 2\varepsilon_{12}(0)(1)$

$\varepsilon_B = \varepsilon_{22}$ (2)

Gage C:

$\varepsilon_{rr} = \varepsilon_C$, $\theta = 30°$

$\varepsilon_C = \varepsilon_{11} \cos^2(30°) + \varepsilon_{22} \sin^2(30°) + 2\varepsilon_{12} \cos(30°) \sin(30°)$

$= \varepsilon_{11} \left(\frac{\sqrt{3}}{2}\right)^2 + \varepsilon_{22} \left(\frac{1}{2}\right)^2 + 2\varepsilon_{12} \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$\varepsilon_C = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_{22} + \frac{\sqrt{3}}{2}\varepsilon_{12}$ (3)

Taking (3) − (1):

$\varepsilon_C - \varepsilon_A = \frac{\sqrt{3}}{2}\varepsilon_{12} + \frac{\sqrt{3}}{2}\varepsilon_{12} = \sqrt{3}\,\varepsilon_{12}$

$\Rightarrow \varepsilon_{12} = \frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A)$

plugging this and $\varepsilon_{22} = \varepsilon_B$ into (1):

$\varepsilon_A = \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_B - \frac{\sqrt{3}}{2}\left[\frac{1}{\sqrt{3}}(\varepsilon_C - \varepsilon_A)\right]$

$= \frac{3}{4}\varepsilon_{11} + \frac{1}{4}\varepsilon_B - \frac{1}{2}\varepsilon_C + \frac{1}{2}\varepsilon_A$

$\varepsilon_{11} = \left(\varepsilon_A - \frac{1}{4}\varepsilon_B + \frac{1}{2}\varepsilon_C - \frac{1}{2}\varepsilon_A\right)\frac{4}{3} = \left(\frac{1}{2}\varepsilon_A - \frac{1}{4}\varepsilon_B + \frac{1}{2}\varepsilon_C\right)\frac{4}{3}$

$\Rightarrow \varepsilon_{11} = \frac{2}{3}\varepsilon_A - \frac{1}{3}\varepsilon_B + \frac{2}{3}\varepsilon_C$

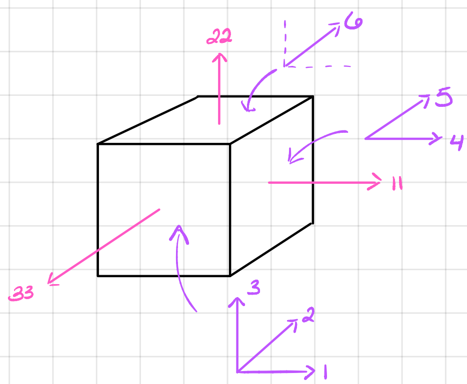
Thus, the components of the infinitesimal strain tensor are:

$$[\underline{\underline{\varepsilon}}] = \begin{bmatrix} \frac{1}{3}\left(2\varepsilon_A - \varepsilon_B + 2\varepsilon_C\right) & \frac{1}{\sqrt{3}}\left(\varepsilon_C - \varepsilon_A\right) & 0 \\ \frac{1}{\sqrt{3}}\left(\varepsilon_C - \varepsilon_A\right) & \varepsilon_B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

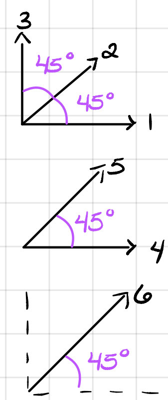
2. Assuming the body was not necessarily under plane strain, and you were allowed to attach a strain gage in any orientation, what is the smallest number of strain gages you would need to determine the entire strain tensor? Describe how you would place the strain gages.

We just used 3 gages to find ε_{11}, ε_{12}, and ε_{22}

If we want to find ε_{13}, ε_{23}, and ε_{33}, it would make sense that we would need 3 more gages: 6 equations $\longleftrightarrow$ 6 unknowns



Gage 1: in 11 direction
Gage 2: in 12 direction
Gage 3: in 22 direction
Gage 4: in 33 direction
Gage 5: in 23 direction
Gage 6: in 13 direction



Problem 2

An engineer conducts the following two tests to find the constitutive response of a thin sheet of material of thickness t.

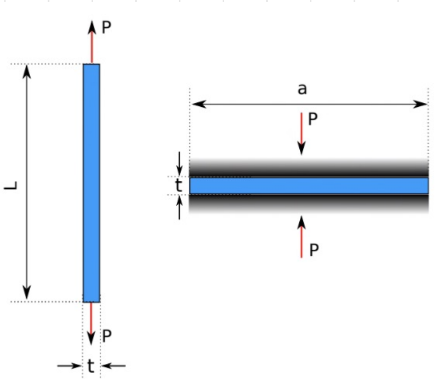


Figure 2: Schematics for problem 2.

Test 1: a thin strip of material (thickness t, length L, and width w, with $L \gg w, t$) is subject to tension along its length

Test 2: the material is placed between two rigid square plates of side a ($a \gg t$) and is subject to uniform compression in the thickness direction while remaining laterally constrained

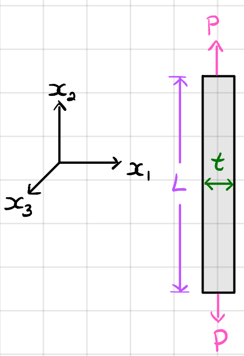
In each of the tests, the load and the displacement are measured. The engineer computes the elastic stress-strain response of the material from each of these tests and finds them to be different. Please help the engineer derive the stress-strain response of these two tests and investigate if any relation exists between them. State all your assumptions clearly. Assume the material is linearly elastic and isotropic, with Young's modulus E, and Poisson's ratio ν. How would you compute the elastic constants from the results of these two tests?

Assumptions

- material is linearly elastic and isotropic with E and ν
- frictionless plates
- no rigid body movement

Stress-Strain relations: Test 1

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad , \quad \varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$



← uniaxial stress: $\sigma_{22} \neq 0$ and all other components are zero

using $\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$:

$$\varepsilon_{11} = \varepsilon_{33} = -\frac{\nu}{E}\sigma_{22}$$

$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{22} + \frac{1+\nu}{E}\sigma_{22} = \frac{\sigma_{22}}{E}$$

$$\sigma_{22} = \frac{P}{wt}$$

$$\boxed{\text{Stress-strain response for Test 1:} \quad \varepsilon_{11} = \varepsilon_{33} = -\frac{\nu}{E}\sigma_{22} \ , \ \varepsilon_{22} = \frac{\sigma_{22}}{E}}$$

using $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$:

$$\left.\begin{aligned} &(1)\quad 0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11} \\ &(2)\quad \sigma_{22} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{22} \\ &(3)\quad 0 = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{33} \end{aligned}\right\}$$

adding these equations together:

$$\sigma_{22} = 3\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})$$

$$\Rightarrow \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \frac{\sigma_{22}}{3\lambda+2\mu}$$

plugging this back in to the equation for σ_{22}:

$$\sigma_{22} = \lambda\left(\frac{\sigma_{22}}{3\lambda+2\mu}\right) + 2\mu\varepsilon_{22}$$

$$\frac{\sigma_{22}(3\lambda+2\mu) - \lambda\sigma_{22}}{3\lambda+2\mu} = 2\mu\varepsilon_{22}$$

$$\left(\frac{2\lambda+2\mu}{3\lambda+2\mu}\right)\sigma_{22} = 2\mu\varepsilon_{22} \quad \Rightarrow \quad \sigma_{22} = \cancel{2}\mu\left(\frac{3\lambda+2\mu}{\cancel{2}\lambda+\cancel{2}\mu}\right)\varepsilon_{22}$$

$$\sigma_{22} = \underbrace{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}_{=E}\varepsilon_{22}$$

taking (1) − (3):

$$0 = \lambda\varepsilon_{kk} + 2\mu\varepsilon_{11}$$
$$-(0 = \lambda\varepsilon_{kk} + 2\mu\varepsilon_{33})$$

$$0 = 2\mu(\varepsilon_{11} - \varepsilon_{33}) \Rightarrow \varepsilon_{11} = \varepsilon_{33}$$

plugging in to (1) and solving for ε_{11}:

$$0 = \lambda\left(\varepsilon_{11} + \frac{\sigma_{22}}{E} + \varepsilon_{11}\right) + 2\mu\varepsilon_{11}$$

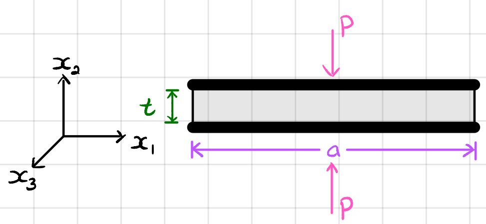
$$= \frac{\lambda\sigma_{22}}{E} + (2\lambda+2\mu)\varepsilon_{11} \Rightarrow \varepsilon_{11} = -\underbrace{\frac{\lambda}{2(\lambda+\mu)}}_{=\nu}\frac{\sigma_{22}}{E} = -\frac{\nu}{E}\sigma_{22}$$

Stress-strain response for Test 1:

$$\sigma_{22} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\varepsilon_{22} = E\varepsilon_{22}$$

$$\varepsilon_{11} = \varepsilon_{33} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\sigma_{22} = \frac{-\nu}{E}\sigma_{22}$$

Stress-strain relations: Test 2

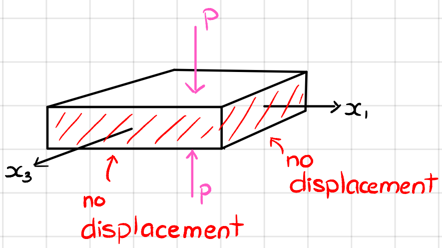


← laterally confined: $\varepsilon_{11} = \varepsilon_{33} = 0$ (σ_{22} and $\sigma_{33} \neq 0$)

using $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$:

$$\sigma_{22} = \lambda\varepsilon_{22} + 2\mu\varepsilon_{22}$$
$$\sigma_{11} = \sigma_{33} = \lambda\varepsilon_{22}$$

laterally confined:



Stress-strain response for Test 2:

$$\sigma_{22} = (\lambda+2\mu)\varepsilon_{22}, \quad \sigma_{11} = \sigma_{33} = \lambda\varepsilon_{22}$$

using $\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$:

$$(1)\quad 0 = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) + \frac{1+\nu}{E}\sigma_{11}$$

$$(2)\quad \varepsilon_{22} = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) + \frac{1+\nu}{E}\sigma_{22}$$

$$(3)\quad 0 = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) + \frac{1+\nu}{E}\sigma_{33}$$

(1)−(3):

$$0 = \frac{1+\nu}{E}(\sigma_{11}-\sigma_{33}) \Rightarrow \sigma_{11} = \sigma_{33}$$

plugging into (1) and solving for σ_{11} :

$$0 = -\frac{\nu}{E}(2\sigma_{11}+\sigma_{22}) + \frac{1+\nu}{E}\sigma_{11}$$

$$0 = -\frac{\nu}{E}\sigma_{22} + \frac{1-\nu}{E}\sigma_{11}$$

$$\nu\sigma_{22} = (1-\nu)\sigma_{11} \quad \Rightarrow \quad \sigma_{11} = \frac{\nu}{1-\nu}\sigma_{22}$$

plugging into (2):

$$\varepsilon_{22} = -\frac{\nu}{E}\left(\frac{\nu}{1-\nu}\sigma_{22} + \sigma_{22} + \frac{\nu}{1-\nu}\sigma_{22}\right) + \frac{1+\nu}{E}\sigma_{22}$$

$$= \left(-\frac{\nu^2}{E(1-\nu)} - \frac{\nu}{E} - \frac{\nu^2}{E(1-\nu)} + \frac{(1+\nu)}{E}\right)\sigma_{22}$$

$$= \left[\frac{-\nu^2 - \nu(1-\nu) - \nu^2 + (1+\nu)(1-\nu)}{E(1-\nu)}\right]\sigma_{22}$$

$$= \left[\frac{-\nu^2 - \nu + \nu^2 - \nu^2 + 1 - \nu^2}{E(1-\nu)}\right]\sigma_{22}$$

$$\varepsilon_{22} = \frac{-2\nu^2 - \nu + 1}{E(1-\nu)}\sigma_{22} = \underbrace{\frac{(1-2\nu)(1+\nu)}{E(1-\nu)}\sigma_{22}}$$

this is the same as what we found before

$$1-2\nu = 1 - \frac{\lambda}{\lambda+\mu} = \frac{\lambda+\mu-\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu}$$

$$1+\nu = 1 + \frac{\lambda}{2(\lambda+\mu)} = \frac{2\lambda+2\mu+\lambda}{2(\lambda+\mu)} = \frac{3\lambda+2\mu}{2(\lambda+\mu)}$$

$$1-\nu = 1 - \frac{\lambda}{2(\lambda+\mu)} = \frac{2\lambda+2\mu-\lambda}{2(\lambda+\mu)} = \frac{\lambda+2\mu}{2(\lambda+\mu)}$$

$$\frac{(1-2\nu)(1+\nu)}{(1-\nu)} = \left[\frac{\mu}{\lambda+\mu}\right]\left[\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right]\left[\frac{2(\lambda+\mu)}{\lambda+2\mu}\right] = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)(\lambda+2\mu)}$$

dividing by $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$:

$$\left[\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)(\lambda+2\mu)}\right]\left[\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\right] = \frac{1}{\lambda+2\mu} \quad \checkmark$$

plugging back in to (2):

$$\varepsilon_{22} = -\frac{\nu}{E}\left[(\lambda+2\mu)\varepsilon_{22} + \sigma_{11} + \sigma_{11}\right] + \frac{1+\nu}{E}(\lambda+2\mu)\varepsilon_{22}$$

$$\varepsilon_{22} = \frac{(\lambda+2\mu)}{E}\varepsilon_{22} - \frac{2\nu}{E}\sigma_{11}$$

$$\left[\frac{E-\lambda-2\mu}{E}\varepsilon_{22}\right]\left[\frac{-E}{2\nu}\right] = \sigma_{11}$$

$$\underbrace{\frac{\lambda+2\mu-E}{2\nu}}\varepsilon_{22} = \sigma_{11}$$

this is the same as what we found before

$$\lambda+2\mu-E = \lambda+2\mu - \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

$$= \frac{\lambda^2+\mu\lambda+2\mu\lambda+2\mu^2-3\mu\lambda-2\mu^2}{\lambda+\mu}$$

$$= \frac{\lambda^2}{\lambda+\mu}$$

$$\frac{\lambda+2\mu-E}{2\nu} = \left[\frac{\lambda^2}{\lambda+\mu}\right]\left[\frac{2(\lambda+\mu)}{2\lambda}\right] = \lambda \checkmark$$

Stress-strain response for Test 2:

$$\boxed{\varepsilon_{22} = \frac{(1-2\nu)(1+\nu)}{E(1-\nu)}\sigma_{22} = \frac{1}{\lambda+2\mu}\sigma_{22} \qquad \sigma_{11} = \sigma_{33} = \lambda\varepsilon_{22}}$$

To compute the elastic constants E and ν (2 unknowns), we need 2 equations:

From Test 1: $\sigma_{22} = E\varepsilon_{22}$

From Test 2: $\sigma_{22} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\varepsilon_{22}$

Problem 3

Consider the plate ($0<x_1<a$; $-b<x_2<b$) shown in Figure 3 and subject to the following loads:

- The surfaces $x_2 = \pm b$ are stress-free
- The surface $x_1 = a$ is subject to a force per unit length of $\tau(x_2)$ pointed in the x_2 direction, giving a total force equal to F, i.e.

$$F = \int_{-b}^{b} \tau(x_2)\,dx_2$$

- The surface $x_1 = 0$ is subject to suitable loads to keep the plate in equilibrium
- Zero body force

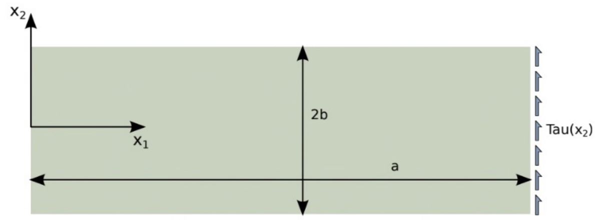


Figure 3: Schematics for problem 3.

Which of the following is a solution of the elasticity problem (for suitable constants A, B, C, and D)? Explain why and give the correct values of A, B, C, and D.

1. $\sigma_{11} = Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)$

   $\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$

   $\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

2. $\sigma_{11} = Ax_2 + Bx_1x_2$

   $\sigma_{22} = D$

   $\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

3. $\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2$

   $\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

   $\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

We know that the plate is in equilibrium and that there are no body forces, so let's begin with checking the conservation of linear momentum (equilibrium equations). We also assume that $a, b \gg t$ such that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

Balance of linear momentum:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

<u>Case 1:</u>

$$\left.\begin{aligned} \sigma_{11} &= Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right) \\ \sigma_{22} &= B\sin\left(\frac{\pi x_2}{b}\right) \\ \sigma_{12} &= -\frac{1}{2}Bx_2^2 - C \end{aligned}\right\}$$

$$Bx_2 - \frac{\pi}{2a}D\sin\left(\frac{\pi x_1}{2a}\right) - Bx_2^2 = 0$$

$$\frac{\pi}{b}B\cos\left(\frac{\pi x_2}{b}\right) = 0$$

$$\Rightarrow \quad D\sin\left(\frac{\pi x_1}{2a}\right) = 0$$

$$B\cos\left(\frac{\pi x_2}{b}\right) = 0$$

$D\sin\left(\frac{\pi x_1}{2a}\right) = 0$ only if $D = 0$ or $\sin\left(\frac{\pi x_1}{2a}\right) = 0$

$\hookrightarrow \frac{\pi x_1}{2a} = n\pi, \; n = 0, 1, 2, \ldots$

x_1 varies from 0 to a, so $\sin\left(\frac{\pi x_1}{2a}\right) = 0$ only at $x_1 = 0$

$B\cos\left(\frac{\pi x_2}{b}\right) = 0$ only if $B = 0$ or $\cos\left(\frac{\pi x_2}{b}\right) = 0$

$\hookrightarrow \frac{\pi x_2}{b} = \frac{n\pi}{2}, \; n = 1, 3, 5, \ldots$

x_2 varies from $-b$ to b, so $\cos\left(\frac{\pi x_2}{b}\right) = 0$ only at $x_1 = \pm \frac{b}{2}$

Case 2:

$$\left.\begin{aligned} \sigma_{11} &= Ax_2 + Bx_1x_2 \\ \sigma_{22} &= D \\ \sigma_{12} &= -\tfrac{1}{2}Bx_2^2 - C \end{aligned}\right\} \qquad \begin{aligned} &Bx_2 - Bx_2 = 0 \\ &0 + 0 = 0 \end{aligned}$$

$$\Rightarrow \quad 0 = 0$$

Case 3:

$$\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2 = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1^2 - 2x_1a + a^2)$$

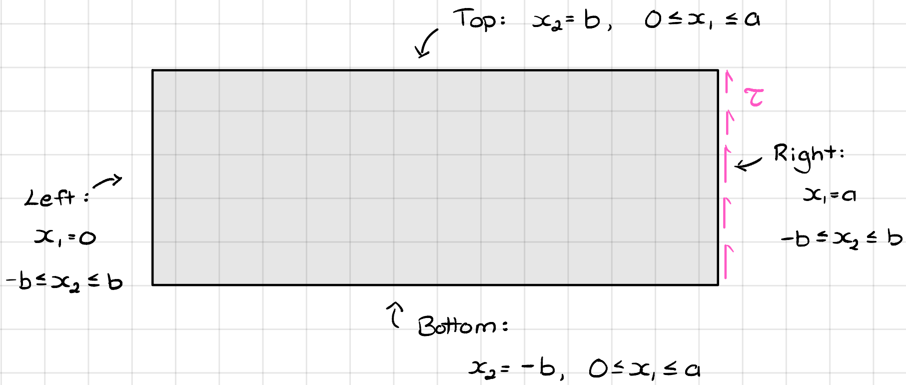
$$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$$

$$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C = -\frac{1}{2}Bx_2^2 - Bx_1x_2 + Bax_2 + C$$

$$\Rightarrow \quad \left.\begin{aligned} &Bx_2 + Bx_1 - Ba - Bx_2 - Bx_1 + Ba = 0 \\ &-Bx_2 + Bx_2 = 0 \end{aligned}\right\} \; 0 = 0$$

Right from the start, it looks like Case 1 is not a suitable solution.

Taking a look at the boundary conditions:



<u>Top</u>: $\underline{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, traction free

$$\underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{12} = 0 \\ \sigma_{22} = 0 \end{matrix}$$

<u>Bottom</u>: $\underline{n} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, traction free

$$\underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{12} = 0 \\ \sigma_{22} = 0 \end{matrix}$$

<u>Right</u>: $\underline{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\underline{t} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{11} = 0 \\ \sigma_{12} = \tau \end{matrix}$$

<u>Left</u>: $\underline{n} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\underline{t} = \begin{bmatrix} 0 \\ -\tau \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sigma_{11} = 0 \\ \sigma_{12} = \tau \end{matrix}$$

<u>Case 1:</u>

$$\sigma_{11} = Ax_2 + Bx_1x_2 + D\cos\left(\frac{\pi x_1}{2a}\right)$$

$$\sigma_{22} = B\sin\left(\frac{\pi x_2}{b}\right)$$

$$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$$

<u>Top:</u> $0 \le x_1 \le a$, $x_2 = b$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - C \Rightarrow C = -\frac{1}{2}Bb^2$

$\sigma_{22} = 0 = B\sin(\pi) = B(0)$

<u>Bottom:</u> $0 \le x_1 \le a$, $x_2 = -b$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - C \Rightarrow C = -\frac{1}{2}Bb^2$

<u>Left:</u> $-b \le x_2 \le b$, $x_1 = 0$

$\sigma_{11} = 0 = Ax_2 + D\cos(0) \Rightarrow Ax_2 = -D$

$\sigma_{12} = \tau = -\frac{1}{2}Bx_2^2 - C$

<u>Right:</u> $-b \le x_2 \le b$, $x_1 = a$

$\sigma_{11} = 0 = Ax_2 + D\cos\left(\frac{\pi}{2}\right) \Rightarrow A = 0$

$\sigma_{12} = \tau = -\frac{1}{2}Bx_2^2 - C$

$$\int_{-b}^{b} \tau(x_2)\,dx_2 = F = \int_{-b}^{b} \left(-\frac{1}{2}Bx_2^2 - C\right)dx_2 = \left[-\frac{1}{6}Bx_2^3 - Cx_2\right]_{-b}^{b} = -\frac{1}{6}Bb^3 - Cb + \frac{1}{6}Bb^3 + Cb = F$$

$\Rightarrow F = 0$ which does not make any sense for the given problem.

<u>Case 1 is not a suitable solution</u> for this elasticity problem.

Case 2:

$\sigma_{11} = Ax_2 + Bx_1x_2$

$\sigma_{22} = D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - C$

Top: $0 \le x_1 \le a$, $x_2 = b$

$\sigma_{22} = 0 \Rightarrow D = 0$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - C \Rightarrow C = -\frac{1}{2}Bb^2$

Bottom: $0 \le x_1 \le a$, $x_2 = -b$

$\sigma_{22} = 0 \Rightarrow D = 0$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - C \Rightarrow C = -\frac{1}{2}Bb^2$

Left: $-b \le x_2 \le b$, $x_1 = 0$

$\sigma_{11} = 0 = Ax_2 \Rightarrow A = 0$

$\sigma_{12} = \tau(x_2) = -\frac{1}{2}Bx_2^2 - C = -\frac{1}{2}Bx_2^2 + \frac{1}{2}Bb^2$

$$\int_{-b}^{b} \tau(x_2)\,dx_2 = F \Rightarrow \int_{-b}^{b}\left(-\frac{1}{2}Bx_2^2 + \frac{1}{2}Bb^2\right)dx_2 = F$$

$$= \left[-\frac{1}{6}Bx_2^3 + \frac{1}{2}Bb^2x_2\right]_{-b}^{b} = F$$

$$-\frac{1}{6}Bb^3 + \frac{1}{2}Bb^3 - \left(\frac{1}{6}B(-b)^3 + \frac{1}{2}Bb^2(-b)\right) = -\frac{1}{6}Bb^3 + \frac{1}{2}Bb^3 + \frac{1}{6}Bb^3 + \frac{1}{2}Bb^3 = Bb^3 = F$$

$\Rightarrow B = \frac{F}{b^3}$

Right: $-b \le x_2 \le b$, $x_1 = a$

$\sigma_{11} = 0 = Ax_2 + Bax_2 \Rightarrow A = -Ba$

$\sigma_{12} = \tau(x_2) = -\frac{1}{2}Bx_2^2 - C = -\frac{1}{2}Bx_2^2 + \frac{1}{2}Bb^2 \Rightarrow B = \frac{F}{b^3} \Rightarrow C = -\frac{F}{2b}$

$A = 0$, $A = -\frac{Fa}{b^3}$, $B = \frac{F}{b^3}$, $C = -\frac{F}{2b}$, $D = 0$

Case 3:

$\sigma_{11} = Ax_2 + Bx_1x_2 + \frac{B}{2}(x_1 - a)^2$

$\sigma_{22} = \frac{1}{2}Bx_2^2 + D$

$\sigma_{12} = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

Top: $0 \le x_1 \le a$, $x_2 = b$

$\sigma_{22} = 0 = \frac{1}{2}Bx_2^2 + D \Rightarrow D = -\frac{1}{2}Bb^2$

$\sigma_{12} = 0 = -\frac{1}{2}Bx_2^2 - B(x_1 - a)x_2 + C$

$0 = -\frac{1}{2}Bb^2 - B(x_1 - a)b + C$

Bottom: $0 \le x_1 \le a$, $x_2 = -b$

$\sigma_{22} = 0 = \frac{1}{2}Bb^2 + D \Rightarrow D = -\frac{1}{2}Bb^2$

$\sigma_{12} = 0 = -\frac{1}{2}Bb^2 - B(x_1 - a)b + C$

Left: $-b \le x_2 \le b$, $x_1 = 0$

$\sigma_{11} = 0 = Ax_2 + \frac{1}{2}Ba^2$

$\sigma_{12} = \tau(x_2) = -\frac{1}{2}Bx_2^2 + Bax_2 + C$

$$\int_{-b}^{b} \tau(x_2)\,dx_2 = F = \int_{-b}^{b} \left(-\tfrac{1}{2}Bx_2^2 + Bax_2 + C\right)dx_2 = \left[-\tfrac{1}{6}Bx_2^3 + \tfrac{1}{2}Bax_2^2 + Cx_2\right]_{-b}^{b}$$

$$= -\tfrac{1}{6}Bb^3 + \tfrac{1}{2}Bab^2 + Cb - \left(\tfrac{1}{6}Bb^3 + \tfrac{1}{2}Bab^2 - Cb\right) = \tfrac{1}{3}Bb^3 + 2Cb = F$$

<u>Right:</u> $-b \le x_2 \le b,\ x_1 = a$

$$\sigma_{11} = 0 = Ax_2 + Bax_2$$

$$\sigma_{12} = \tau(x_2) = -\tfrac{1}{2}Bx_2^2 + C$$

$$\int_{-b}^{b}\left(-\tfrac{1}{2}Bx_2^2 + C\right)dx_2 = \left[-\tfrac{1}{6}Bx_2^3 + Cx_2\right]_{-b}^{b} = -\tfrac{1}{6}Bb^3 + Cb - \left(\tfrac{1}{6}Bb^3 - Cb\right) = -\tfrac{1}{3}Bb^3 + 2Cb = F$$

equating the results for F and σ_{12} gives

$$\left.\begin{aligned} F:&\quad -\tfrac{1}{3}Bb^3 + 2Cb = \tfrac{1}{3}Bb^3 + 2Cb \\ \sigma_{12}:&\quad -\tfrac{1}{2}Bb^2 + B(x_1 - a)b + C = -\tfrac{1}{2}Bb^2 - B(x_1 - a)b + C \end{aligned}\right\} \quad B = 0 \Rightarrow C = \frac{F}{2b} \Rightarrow \begin{aligned} A &= 0 \\ D &= 0 \end{aligned}$$