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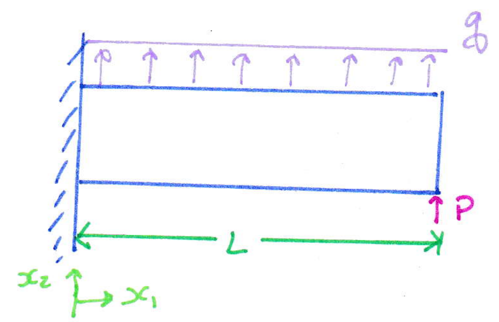
BONUS PROBLEM

Derive the strong formulation by using the principle of minimum potential energy.



let u_2 = vertical displacement of beam

Strain energy for beam in bending: $\frac{M^2 L}{2EI}$

so for this beam: $\int_0^L \frac{M^2}{2EI}\,dx_1$

then we have q: $A\int_0^L q u_2\,dx_1$ $\qquad M = EI\frac{d^2u_2}{dx_1^2}$

and finally P: $APu_2\big|_{x_1=L}$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - \int_0^L Aqu_2\,dx_1 - APu_2\big|_{x_1=L}$$

$$\delta\Pi = \delta\left[\int_0^L \frac{EI}{2}\left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1\right] - \delta\left[\int_0^L Aqu_2\,dx_1\right] - \delta\left[APu_2\big|_{x_1=L}\right]$$

$$= \frac{EI}{2}\int_0^L 2\left(\frac{d^2u_2}{dx_1^2}\right)\delta\left(\frac{d^2u_2}{dx_1^2}\right)dx_1 - A\int_0^L q\,\delta u_2\,dx_1 - AP\delta u_2\big|_{x_1=L}$$

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$$\delta\Pi = EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1 - Aq\int_0^L \delta u_2\, dx_1 - AP\delta u_2\Big|_{x_1=L}$$

$\delta u_2 = 0$ at $x_1 = 0$

$$EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1$$

integration by parts: $\int u\, dv = uv - \int v\, du$

$$u = \frac{\partial^2 u_2}{\partial x_1^2} \qquad du = \frac{\partial^3 u_2}{\partial x_1^3}dx_1$$

$$dv = \frac{\partial^2 \delta u_2}{\partial x_1^2} \qquad v = \frac{\partial \delta u_2}{\partial x_1}$$

$$EI\int_0^L \frac{\partial^2 u_2}{\partial x_1^2}\left(\frac{\partial^2 \delta u_2}{\partial x_1^2}\right)dx_1 = EI\left[\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_0^L - \int_0^L \frac{\partial \delta u_2}{\partial x_1}\frac{\partial^3 u_2}{\partial x_1^3}dx_1\right]$$

$$EI\int_0^L \frac{\partial \delta u_2}{\partial x_1}\frac{\partial^3 u_2}{\partial x_1^3}dx_1$$

$$u = \frac{\partial^3 u_2}{\partial x_1^3} \qquad du = \frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$dv = \frac{\partial \delta u_2}{\partial x_1} \qquad v = \delta u_2$$

$$EI\ \frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_{x_1=L} - \frac{\partial^3 u_2}{\partial x_1^3}\delta u_2\Big|_{x_1=L} + \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$\delta\Pi = EI\frac{\partial^2 u_2}{\partial x_1^2}\frac{\partial \delta u_2}{\partial x_1}\Big|_{x_1=L} - EI\frac{\partial^3 u_2}{\partial x_1^3}\delta u_2\Big|_{x_1=L} + EI\int_0^L \delta u_2\frac{\partial^4 u_2}{\partial x_1^4}dx_1$$

$$- Aq\int_0^L \delta u_2\, dx_1 - AP\delta u_2\Big|_{x_1=L}$$

Let's try taking the case $\delta u_2 \big|_{x_1=L} = 0$

$$0 = EI \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial \delta u_2}{\partial x_1} - EI \frac{\partial^3 u_2}{\partial x_1^3} \delta u_2 - AP \delta u_2$$

oh, wait, why am I using AP?

$$0 = EI \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial \delta u_2}{\partial x_1} - EI \frac{\partial^3 u_2}{\partial x_1^3} \delta u_2 - P \delta u_2$$

$x_1 = 0$:

$$0 = EI \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} dx_1 - Aq \int_0^L \delta u_2 \, dx_1$$

crap, I'm using Aq too...

$$0 = EI \int_0^L \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} dx_1 - q \int_0^L \delta u_2 \, dx_1$$

$$0 = EI \, \delta u_2 \frac{\partial^4 u_2}{\partial x_1^4} - q \, \delta u_2$$

$$EI \frac{\partial^4 u_2}{\partial x_1^4} - q = 0$$

$$EI \frac{\partial^3 u_2}{\partial x_1^3} = P$$

now we want to solve for u_2

$$\frac{\partial^4 u_2}{\partial x_1^4} = \frac{q}{EI}$$

$$\int \frac{\partial^4 u_2}{\partial x_1^4} dx_1 = \int \frac{q}{EI} dx_1 \Rightarrow \frac{\partial^3 u_2}{\partial x_1^3} = \frac{q}{EI} x_1 + C$$

$$\int \frac{\partial^3 u_2}{\partial x_1^3}\,dx_1 = \int\left(\frac{q}{EI}x_1 + C\right)dx_1$$

$$\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{2}\frac{q}{EI}x_1^2 + C_1 x_1 + C_2$$

$$\int \frac{\partial^2 u_2}{\partial x_1^2} = \int\left(\frac{1}{2}\frac{q}{EI}x_1^2 + C_1 x_1 + C_2\right)dx_1$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{6}\frac{q}{EI}x_1^3 + \frac{1}{2}C_1 x_1^2 + C_2 x_1 + C_3$$

$$\int \frac{\partial u_2}{\partial x_1}\,dx_1 = \frac{1}{24}\frac{q}{EI}x_1^4 + \frac{1}{6}C_1 x_1^3 + \frac{1}{2}C_2 x_1^2 + C_3 x_1 + C_4 = u_2(x_1)$$

$$u_2(0) = 0 \Rightarrow C_4 = 0$$

at $x_1 = L$:

$$EI\frac{\partial^3 u_2}{\partial x_1^3} = P$$

$$\frac{\partial^3 u_2}{\partial x_1^3} = \frac{P}{EI} = \frac{q}{EI}x_1 + C_1$$

$$\frac{P}{EI} = \frac{qL}{EI} + C_1 \Rightarrow C_1 = \frac{P - qL}{EI}$$

$$\int \frac{\partial^3 u_2}{\partial x_1^3}\,dx_1 = \frac{P}{EI}x_1 = \frac{1}{2}\frac{q}{EI}x_1^2 + \frac{P - qL}{EI}x_1 + C_2$$

$$C_2 = \frac{PL}{EI} - \frac{1}{2}\frac{qL^2}{EI} - \frac{PL + qL^2}{EI} = -\frac{1}{2}\frac{qL^2}{EI} + \frac{qL^2}{EI} = \frac{qL^2}{2EI}$$

$$\int \frac{\partial^2 u_2}{\partial x_1^2}\,dx_1 = \frac{P}{2EI}x_1^2 = \frac{1}{6}\frac{q}{EI}x_1^3 + \frac{P - qL}{2EI}x_1^2 + \frac{qL^2}{2EI}x_1 + C_3$$

$$C_3 = \cancel{\frac{PL^2}{2EI}} - \frac{qL^3}{6EI} - \cancel{\frac{PL^2}{2EI}} + \cancel{\frac{qL^3}{2EI}} - \cancel{\frac{qL^3}{2EI}}$$

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$$u_z(x_1) = \frac{1}{24}\frac{q}{EI}x_1^4 + \frac{1}{6}\frac{P-qL}{EI}x_1^3 + \frac{1}{2}\frac{qL^2}{2EI}x_1^2 - \frac{qL^3}{6EI}x_1$$

$$u_z(0) = 0$$

$$u_z(L) = \frac{q}{24EI}L^4 + \frac{PL^3}{6EI} - \frac{qL^4}{6EI} + \frac{qL^4}{4EI} - \frac{qL^4}{6EI}$$

$$= \frac{qL^4}{24EI} - \frac{4qL^4}{24EI} + \frac{6qL^4}{24EI} - \frac{4qL^4}{24EI} + \frac{PL^3}{6EI}$$

$$= \frac{-qL^4}{24EI} + \frac{PL^3}{6EI}$$

I have no idea if this is correct but I've been working on this exam for over 24 hours straight now and I can't think any more

Thank you for a wonderful semester!