Higher moments of susie predictions

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Moment generating functions

$$M_X(t) = \mathbb{E}[\exp\{Xt\}]$$

MGF for Sum of independent random variables

For $Y = X_1 + \cdots + X_n$, where $X_1, \dots, X_n$ are independent. The moment generating function for X is the product of MGFs for the X_i.

$$M_Y(t) = \prod M_{X_i}(t)$$

Then derivates of the MGF for Y are given by "leave-one-out" sum

$$\frac{d}{dt} M_Y(t) = \sum_i M_{Y-X_i}(t) \frac{d}{dt} M_{X_i}(t)$$

$$\frac{d^m}{dt^m} M_Y(t) = \sum_i \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{d^j}{dt^j} M_{Y-X_i}(t) \frac{d^{m-j}}{dt^{m-j}} M_{X_i}(t)$$

So that

$$\mathbb{E}[Y^m] = \frac{d^m}{dt^m} M_X(0) = \sum_i \sum_{j=0}^{m-1} \binom{m-1}{j} \mathbb{E}\left[(Y - X_i)^j\right] \mathbb{E}\left[X_i^{m-j}\right]$$

Now, this is a pretty complicated sum!

We can also write

$$\mathbb{E}[(X_i + (Y - X_i))^m] = \mathbb{E}\left[\sum_{j=1}^{m} \binom{m}{j} X_i^j X_{-i}^{m-j}\right] = \sum_{j=1}^{m} \binom{m}{j} \mathbb{E}\left[X_i^j\right] \mathbb{E}\left[(Y - X_i)^{m-j}\right]$$

Which looks a bit simpler, but again the "leave-one-out" moment will need to be computed recursively.

$$\mathbb{E}\left[(\sum_i X_i)^m\right] = \mathbb{E}\left[\sum_{j_1,\ldots,j_n} \binom{m}{j_1,\ldots,j_n} \prod_{i=1}^n X_i^{j_i}\right] = \sum_{j_1,\ldots,j_n} \binom{m}{j_1,\ldots,j_n} \prod_{i=1}^n \mathbb{E}\left[X_i^{j_i}\right]$$

Which we can compute by (1) computing $m \times n$ moments for each X_i^j. (2) computing the multinomial sum over $\binom{n+m-1}{m-1}$ terms. That sum is quite large! And we might hope there is a way to simplify.

Simplification for mean 0, symmetric random variables

If X_i are mean 0 and symmetric, then $\mathbb{E}\left[X^{2k+1}\right] = 0 \forall k \geq 0$. Which can rather simplify the computation of the higher moments since may terms in the sum are 0.

"Rank one" updates

Suppose we've compute $\mathbb{E}\left[Y^k\right], k = 1 \ldots m$, but now we would like to (1) remove X_i and (2) add $\tilde{X}_i$ to Y. That is $\mathbb{E}\left[\tilde{Y}^k\right]$ where $\tilde{Y} = (Y - X_i + \tilde{X}_i)$.

Define $Y^- = Y - X$ and $Y^+ = Y - X$

It's straightforward to add to Y

$$\left(Y^+\right)^k = (Y + X)^k = \sum \binom{k}{i} X^i Y^{k-i}$$

Due to indpendence we can update the moments of Y easily

$$\mathbb{E}\left[\left(Y^+\right)^k\right] = \sum \binom{k}{i} \mathbb{E}[X]^i \mathbb{E}\left[Y^{k-i}\right]$$

We need to be careful when we remove from Y

$$\left(Y^-\right)^k = (Y - X)^k = \sum \binom{k}{i} (-1)^i X^i Y^{k-i}$$

Evaluating these expectations involve computing

$$\mathbb{E}\left[X^i Y^{k-i}\right]$$

MGF for Mixture distributions

Suppose $Z \sim \sum \pi_k X_k$ then

$$M_Z(t) = \sum \pi_k M_{X_k}(t)$$

SuSiE MGF

The linear predictions for the l-th single effect are a normal mixture

$$\psi_l = x^T \beta_l \sim \sum \pi_{li} N(x_i \mu_{b,il}, x_i^2 \sigma_{b,il}^2) = \pi_{li} N(\mu_{li}, \nu_{li}).$$

The linear prediction $\psi = \sum_l \psi_l$ is then also a normal mixture, now with L^p mixture components

$$\psi \sim \sum_{i_1,\ldots,i_L} \pi_{l,i_l} N(\sum_l \mu_{li_l}, \sum_l \nu_{li_l}).$$

So the question is this: can we compute (or accurately approximate) the higher moment of a normal mixture? Can this computation be simplified by this "sum of mixture" structure?

Normal moments

For $X \sim N(\mu, \nu)$,

$$M_X(t) = \exp\left\{\mu t + \frac{1}{2}\nu t^2\right\}$$

$$M_X^{(k)}(t) = M^{(k-1)}(t)(\mu + \nu t) + (k-1)M^{(k-2)}(t)\nu$$

So the moments of a normal distribution can be computed by a simple recursion. For $k \geq 2$

$$\mathbb{E}[X^m] = \mathbb{E}\left[X^{m-1}\right]\mu + (m-1)\mathbb{E}\left[X^{m-2}\right]\nu$$

Normal mixture mixture

Naively, the higher moments of a mixture distribution are just an average over the moments of each component

$$\mathbb{E}[Z^m] = \sum \pi_k \mathbb{E}[X_k^m]$$

$$\mathbb{E}[Z^m] = \sum_{j_1,\ldots j_L} \pi_{1j_1} \ldots \pi_{Lj_L} \mathbb{E}[X_k^m]$$