Econ 482

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FINAL EXAM

The goal of this exam is to estimate a 3-equation New-Keynesian DSGE model. I will derive all the equilibrium conditions of the model for you, find the non-stochastic steady state and log-linearize the model. Some of these derivations are not trivial (at least not to me), but you do not need to fully understand all of them. At the end of these derivations, we will end up with a log-linearized DSGE model, which is ready to be solved (using gensys) and estimated. This will be your job.

1. The Model

The model economy consists of four classes of agents: final-good producers, intermediate-good producers, households, and the Government. I will now illustrate the objective and constraints that define the behavior of each of these agents.

1.1. **Final-good producers.**

At every point in time t, perfectly competitive firms produce the final consumption good Y_t, using the intermediate goods $Y_t(i)$, $i \in [0,1]$, and the production technology

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\theta_t}} di \right]^{1+\theta_t},$$

where

$$log\ \theta_t \sim i.i.d.\ N\left(log\ \theta, \sigma_\theta^2\right)$$

Profit maximization and zero profit condition for the final goods producers imply the following relation between the price of the final good (P_t) and the prices of the intermediate goods ($P_t(i)$)

$$P_t = \left[\int_0^1 P_t(i)^{\frac{1}{\theta_t}} di \right]^{\theta_t},$$

and the following demand function for the intermediate good i:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\theta_t}{\theta_t}} Y_t.$$

1.2. **Intermediate-good producers.** A monopolist firm produces the intermediate good i using the following production function:

$$Y_t(i) = A_t L_t(i),$$

where $L_t(i)$ denotes labor input for the production of good i, and A_t is an exogenous stochastic process capturing the effects of technology. In particular, A_t is modeled as a unit root process, with a growth rate ($z_t \equiv \frac{A_t}{A_{t-1}}$) that follows the exogenous process

$$log\ z_t = (1-\rho_z)\gamma + \rho_z log\ z_{t-1} + \varepsilon_{z,t}$$

$$\varepsilon_{z,t} \sim i.i.d.\ N\left(0, \sigma_z^2\right).$$

At every point in time, a fraction ξ of firms cannot re-optimize their prices and, instead, set their prices following the indexation rule

$$P_t(i) = P_{t-1}(i)\pi_{t-1},$$

where π_t is the gross rate of inflation, defined as $\frac{P_t}{P_{t-1}}$. The remaining fraction $1-\xi$ of re-optimizing firms choose their price ($\tilde{P}_t(i)$) by maximizing the present value of future profits

$$E_t \sum_{s=0}^{\infty} \xi^s \beta^s \frac{\Lambda_{t+s}}{\Lambda_t} \left\{ \left[\tilde{P}_t(i) \left(\Pi_{j=0}^{s} \pi_{t-1+j} \right) \right] Y_{t+s}(i) - W_{t+s} L_{t+s}(i) \right\},$$

where $\beta^s \frac{\Lambda_{t+s}}{\Lambda_t}$ is the value in t of a dollar in $t+s$ (Λ_t denotes the Lagrange multiplier of the consumer maximization problem) and W_t denotes the nominal wage (we assume that all sectors draw their labor input from a unique labor market with nominal wage W_t).

1.3. **Households.** The firms are owned by a continuum of households, indexed by $j \in [0,1]$. Each household maximizes the utility function

$$E_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[log\ C_{t+s}(j) - \varphi \frac{L_{t+s}(j)^{1+\nu}}{1+\nu} \right],$$

where $C_t(j)$ is consumption and b_t is an intertemporal preference shock affecting both the marginal utility of consumption and the marginal disutility of labor. This shock follows the stochastic processes

$$\log b_t = \rho_b \log b_{t-1} + \varepsilon_{b,t}$$

$$\varepsilon_{b,t} \sim i.i.d.\ N\left(0, \sigma_b^2\right)$$

The household budget constraint is given by

$$P_{t+s}C_{t+s}(j) + B_{t+s}(j) \leq R_{t+s-1}B_{t+s-1}(j) + \Pi_{t+s} + W_{t+s}(j)L_{t+s}(j)$$

where $B_t(j)$ is holding of government bonds, R_t is the gross nominal interest rate, Π_t is the per-capita profit that households get from owning the firms.

1.4. **Monetary Policy.** Monetary policy sets short term nominal interest rates following a Taylor type rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi} \left(\frac{\frac{Y_t}{Y_{t-1}}}{e^\gamma}\right)^{\phi_Y}\right]^{1-\rho_R} e^{\varepsilon_{MP,t}},$$

where R is the steady state gross nominal interest rate, π is the steady state gross inflation rate and

$$\varepsilon_{MP,t} \sim i.i.d.\ N\left(0, \sigma_{MP}^2\right))$$

is a monetary policy shock. The term $\frac{Y_t/Y_{t-1}}{e^\gamma}$ represents current output growth relative to the steady state growth.

1.5. **Market Clearing.** The resource constraint is given by

$$Y_t = C_t.$$

2. Nonlinear Equilibrium Conditions

2.1. **Households' problem.** Consider the households' maximization problem. Dropping the j index because of the complete market assumption, the Lagrangian of the maximization problem is

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \beta^s \left\{ \begin{array}{c} b_{t+s}\left[\log\left(C_{t+s}\right) - \varphi \frac{{L_{t+s}}^{1+\nu}}{1+\nu}\right] - \\ -\Lambda_{t+s}\left[P_{t+s}C_{t+s} + B_{t+s} - R_{t+s-1}B_{t+s-1} - Q_{t+s-1} - \Pi_{t+s} - W_{t+s}L_{t+s}\right] \end{array} \right\}$$

and the first order conditions

$$\begin{aligned} \partial C &: \Lambda_t P_t = \frac{b_t}{C_t} \\ \partial B &: \Lambda_t = \beta R_t E_t \Lambda_{t+1} \\ \partial L &: \varphi b_t L^\nu = W_t \Lambda_t \end{aligned}$$

2.2. **Firms' problem.** Since each producer is a monopolist and is bound to hire enough workers to satisfy demand at the posted price, this maximization is under the contraint of the demand function above. Therefore, the first order condition for the profits' maximization is then

(2.1)
$$0 = E_t \sum_{s=0}^{\infty} (\beta\xi)^s \Lambda_{t+s} \left(\prod_{j=1}^{s} \pi_{t-1+j} \right)^{-\frac{1}{\theta_t}} P_{t+s}^{\frac{1+\theta_t}{\theta_t}} Y_{t+s} \left\{ \tilde{P}_t - (1+\theta_t) \frac{W_{t+s}}{A_{t+s}} \left(\prod_{j=1}^{s} \pi_{t-1+j} \right)^{-1} \right\}$$

3. Normalized Nonlinear Equilibrium Conditions

Since the technology process A_t is assumed to have a unit root, consumption, real wages and output evolve along a stochastic growth path. Before computing the steady state and log-linearizing the model we must rewrite the model in terms of stationary variables. In order to do so, define

$$\begin{aligned} \lambda_t &\equiv P_t A_t \Lambda_t \\ c_t &\equiv \frac{C_t}{A_t} \\ y_t &\equiv \frac{Y_t}{A_t} \\ w_t &\equiv \frac{W_t}{P_t A_t}, \end{aligned}$$

and rewrite the equilibrium conditions of the model in terms of these new variables.

For the households' problem we get

$$\begin{aligned} \lambda_t &= \frac{b_t}{c_t} \\ \lambda_t &= \beta R_t E_t \left[\lambda_{t+1} \frac{1}{\pi_{t+1} z_{t+1}} \right] \\ \varphi b_t L_t^{\nu} &= w_t \lambda_t. \end{aligned}$$

For the producers' problem we get

$$y_t = L_t \qquad \text{(production fct)}$$

$$0 = E_t \sum_{s=0}^{\infty} (\beta\xi)^s \left(\frac{\prod_{j=1}^{s} \pi_{t-1+j}}{P_{t+s}/P_t} \right)^{-\frac{1}{\theta_t}} \lambda_{t+s} y_{t+s} \left\{ \tilde{p}_t - (1+\theta_t) w_{t+s} \left(\frac{\prod_{j=1}^{s} \pi_{t-1+j}}{P_{t+s}/P_t} \right)^{-1} \right\},$$

where $\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}$. For the monetary policy rule we obtain

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi} \right)^{\phi_\pi} \left(\frac{\frac{y_t}{y_{t-1}} z_t}{e^\gamma} \right)^{\phi_Y} \right]^{1-\rho_R} e^{\varepsilon_{MP,t}}.$$

Finally, the market clearing condition becomes

$$y_t = c_t.$$

4. Steady State

Once the model is rewritten in terms of these normalized variables, we can compute the non-stochastic steady state. Notice that in steady state

$$\begin{aligned} R &= \pi r \\ r &= \frac{e^\gamma}{\beta} \end{aligned}$$

where r is the gross real interest rate in steady state.

5. Log-linearized Equilibrium Conditions

We are now ready to log-linearize the model around the steady state.The log-linearized version of the model is described by the following equations:

$$
\begin{aligned}
\hat{\lambda}_t &= \hat{b}_t - \hat{c}_t \\
\hat{\lambda}_t &= \hat{R}_t + E_t\left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} - \hat{z}_{t+1}\right] \\
\hat{b}_t + \nu \hat{L}_t &= \hat{\lambda}_t + \hat{w}_t \\
\hat{y}_t &= \hat{L}_t \\
\hat{\pi}_t &= \frac{\beta}{1+\beta} E_t \hat{\pi}_{t+1} + \frac{1}{1+\beta}\hat{\pi}_{t-1} + \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta)\xi_p}\hat{w}_t + \hat{\theta}_t \\
\hat{R}_t &= \rho_R \hat{R}_{t-1} + (1-\rho_R)\left(\phi_\pi \hat{\pi}_t + \phi_y\left(\Delta \hat{y}_t + \hat{z}_t\right)\right) + \varepsilon_{MP,t} \\
\hat{y}_t &= \hat{c}_t \\
\hat{z}_t &= \rho_z \hat{z}_{t-1} + \varepsilon_{z,t} \\
\hat{b}_t &= \rho_b \hat{b}_{t-1} + \varepsilon_{b,t}
\end{aligned}
$$

where, for a generic variable x_t with steady state value x,

$$\hat{x}_t \equiv \log x_t - \log x.$$

6. Simplifying the System

All of these equilibrium conditions can be "compressed" using some simple substitutions. In the end, we will work with the following set of equilibrium conditions:

$$\hat{\pi}_t = \frac{\beta}{1+\beta} E_t \hat{\pi}_{t+1} + \frac{1}{1+\beta}\hat{\pi}_{t-1} + \frac{(1-\beta\xi_p)(1-\xi_p)(1+\nu)}{(1+\beta)\xi_p}\hat{y}_t + \hat{\theta}_t \tag{6.1}$$

$$\hat{y}_t = E_t \hat{y}_{t+1} - \left(\hat{R}_t - E_t\left(\hat{\pi}_{t+1}\right)\right) + (1-\rho_b)\hat{b}_t + \rho_z \hat{z}_t \tag{6.2}$$

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\left(\phi_\pi \hat{\pi}_t + \phi_y\left(\Delta \hat{y}_t + \hat{z}_t\right)\right) + \varepsilon_{MP,t} \tag{6.3}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{z,t}$$

$$\hat{b}_t = \rho_b \hat{b}_{t-1} + \varepsilon_{b,t}$$

In short, equation (6.1) is a log-linear New-Keynesian Phillips curve, which describes the pricing behavior of monopolistically competitive firms subject to a "Calvo" friction. Equation (6.2) is an aggregate demand equation, derived from the Euler equation of the maximization problem of the households. Equation (6.3) is a monetary policy rule, describing the systematic response of the short-term nominal interest rate to inflation and output growth. The remaining expressions describe the behavior of the exogenous disturbances that perturb this economy.

The model can be cast into "Gensys" canonical form

$$\Gamma_0(\chi)\,\zeta_t = \Gamma_1(\chi)\,\zeta_{t-1} + \Psi(\chi)\,\varepsilon_t + \Pi\eta_t,$$

where χ is the vector of model parameters, ζ_t is the vector of endogenous variables, ε_t is the vector of fundamental shocks, and η_t are 1-step ahead forecast errors. Specifically,

$$\begin{aligned}
\chi &\equiv [\gamma, \pi, r, \nu, \xi_p, \phi_\pi, \phi_y, \rho_R, \rho_z, \rho_b, \sigma_{MP}, \sigma_z, \sigma_\theta, \sigma_b] \\
\zeta_t &\equiv \left[\hat{\pi}_t, \hat{y}_t, \hat{R}_t, \hat{z}_t, \hat{b}_t, E_t\hat{\pi}_{t+1}, E_t\hat{y}_{t+1}, \hat{y}_{t-1}\right] \\
\varepsilon_t &\equiv [\varepsilon_{MP,t}, \varepsilon_{z,t}, \varepsilon_{\theta,t}, \varepsilon_{b,t}].
\end{aligned}$$

7. Observable variables

The dataset related to this exercise contains three time series for the U.S. economy:

- Quarterly growth rate of real GDP
- Annualized quarterly growth rate of the GDP deflator
- Federal funds rate (on an annual basis)

You are not supposed to transform the data in any way. The equations that relate these observable variables to the model's variables are the following:

$$100\Delta\log Y_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t + 100\gamma$$

$$400(\pi_t - 1) = 4\hat{\pi}_t + 400(\pi - 1)$$

$$400(R_t - 1) = 4\hat{R}_t + 400(R - 1),$$

where the left-hand side of these equations is what is observed in the data. More specifically, the measurement equation of the model has the form

$$x_t = H\zeta_t + C(\chi)$$

where

$$C = \begin{bmatrix} 100\gamma \\ 400(\pi - 1) \\ 400(R - 1) \end{bmatrix}$$

and

$$H = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. Priors

We will work with the following (independent) prior distributions for the coefficients of the model:

Coefficients	Density	mean	sd
γ	N	0.0045	0.00025
π	N	1.005	0.0015
r	N	1.005	0.0015
ν	C	2	0
ξ	B	0.75	0.15
ϕ_π	N	1.7	0.3
ϕ_y	N	0.3	0.15
ρ_R	B	0.5	0.25
ρ_z	B	0.5	0.25
ρ_b	B	0.5	0.25
σ_{MP}	$IG1$	0.5	2
σ_z	$IG1$	0.5	2
σ_θ	$IG1$	0.5	2
σ_b	$IG1$	0.5	2

,

where C stands for calibrated value, N, B and $IG1$ denote the normal, the beta and the inverse gamma-1 distributions respectively. See below, in the appendix, for more details. In addition, we place an a-priori zero probability that the model admits an indeterminate solution.

9. Questions

Estimate the model. You should approximately follow the procedure illustrated in class:

(1) The first thing you should do is write a code that takes as an input the value of the unknown parameter vector χ, and gives you as an output the value of the log-posterior density. Given that the posterior is the product of the prior and the likelihood, you should write a code that computes the value of the prior density, and another code that computes the value of the likelihood function. For a given value of the parameter vector, the latter should solve the model (i.e. forms the matrices of the "Gensys" canonical form and run "Gensys")[1] and use the "Gensys" solution and the observation equation to compute the likelihood with the Kalman Filter. I am providing a code with the basic steps of the Kalman Filter recursion.

(2) Perform a numerical maximization of the log-posterior using "csminwel.m" or a similar minimization routine.

(3) Use the results in (2) to choose "good" initial conditions and jump distribution for a Metropolis algorithm. In particular, choose a jump distribution of the form $N(0, k^2\hat{\Sigma})$, where $\hat{\Sigma}$ is the estimate of the inverse Hessian of the log-posterior at the peak (obtained in point (2)), and the constant k should be chosen to achieve an acceptance rate of about 25 percent.

(4) Take draws from the posterior distribution using the Metropolis algorithm designed in (3).

State and justify clearly all the choices you have to make (if any).

Present the following output of the estimation procedure:

(a) The value of the *log-likelihood* and the *log-posterior* at the posterior mode

(b) The constant k that you have chosen to calibrate the jump distribution

(c) The number of draws and the acceptance rate of the Metropolis algorithm

(d) The trace plots for a sample of 4 estimated parameters (ξ, ϕ_π, ϕ_y and ρ_R)

(e) A table with the prior mean, the posterior mode of the numerical maximization, the posterior mean, median, 5% and 95% quantiles from the MCMC draws for every parameter of the model

[1] The "Gensys" routine can be found on Chris Sims' webpage

(f) A plot of the impulse responses (medians and error bands, for 20 quarters) of *log-output*, the *inflation rate* and the *interest rate* to a monetary policy shock ($\varepsilon_{MP,t}$)

(g) Compute the marginal likelihood of the model in two alternative ways:

1. using the method based on a second order approximation of the posterior (to apply this method you should use the output of (a))
2. using Geweke's modified harmonic mean method (for 3 values of the truncation constant $\lambda = 0.3; 0.5; 0.9$)

Appendix A. Beta and IG1 distributions

A random variable $0 < X < 1$ has a BETA distribution with parameters $a > 0$ and $b > 0$ if its density function is given by

$$\begin{aligned} f_b(x|a,b) &= C_B^{-1}(a,b) x^{a-1}(1-x)^{b-1} \\ C_B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{aligned}$$

Therefore, to parameterize the Beta distribution you will need a and b and not mean and standard deviation as reported in table 1. However, there is a one-to-one mapping between mean and variance, and the parameters a and b of the Beta pdf. The table below reports the values of a and b corresponding to the means and variances of table 1:

Coefficients	Density	mean	sd	a	b
ξ	B	0.75	0.15	5.5	1.8333
ρ_R	B	0.5	0.25	1.5	1.5
ρ_z	B	0.5	0.25	1.5	1.5
ρ_b	B	0.5	0.25	1.5	1.5

A random variable $\sigma > 0$ has an INVERSE-GAMMA-1 distribution with degrees of freedom d and scale coefficient s if its density function is given by

$$\begin{aligned} f_{ig}(\sigma|d,s) &= C_g^{-1}(d,s) \cdot \sigma^{-(d+1)} \cdot e^{-\frac{1}{2}\frac{s}{\sigma^2}} \\ C_g(d,s) &= \frac{1}{2} \cdot \Gamma\left(\frac{d}{2}\right) \cdot \left(\frac{2}{s}\right)^{\frac{d}{2}}. \end{aligned}$$

The values of d and s that imply an inverse-gamma-1 density with mean 0.5 and standard deviation equal to 2 are $d = 2.0395$ and $s = 0.1679$.