Computer Graphics Assignment 5

Francesco Costa, Arnaud Fauconnet

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Exercise 1

Given $p_1 = (1,1)^T$ and $p = (1.5, 2.5)^T$ we define $\vec{u} = p - p_1 = (0.5, 1.5)^T$

Task 1

The transformation matrices in homogenous coordinates simply have an additional dimension with 1 in position $(3,3)$, so

$$R_{90} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Task 2

We start by representing the points and the vector in homogenous coordinates. This is done by adding a 1 for points and a 0 for directions.

$$p_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad p = \begin{pmatrix} 1.5 \\ 2.5 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{u} = \begin{pmatrix} 0.5 \\ 1.5 \\ 0 \end{pmatrix}$$

We now proceed by applying the rotation and transformation matrices to the points and the vector:

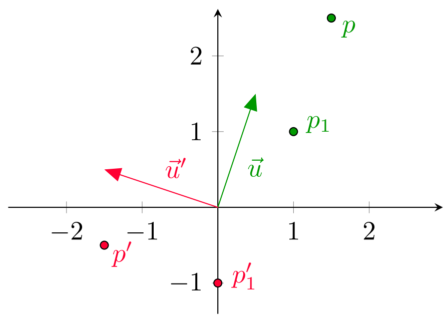
$$T\,R_{90}\,p_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$T\,R_{90}\,p = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.5 \\ 2.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2.5 \\ 1.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -0.5 \\ 1 \end{pmatrix}$$

$$T\,R_{90}\,\vec{u} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -1.5 \\ 0.5 \\ 0 \end{pmatrix}$$

so, since there was no scaling, we can simply remove the extra dimension

$$p_1' = (0, -1)^T, \quad p' = (-1.5, -0.5)^T \quad \text{and} \quad \vec{u}' = (-1.5, 0.5)^T$$



From the image we can clearly see that T had no effect on $\vec{u}$ since the direction has a 0 added in the homogenous coordinates, which negates the effect of the translation.

Task 3

A scaling matrix S is a matrix with the scaling factor in the diagonal, hence

$$S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and if we apply it to our points we get

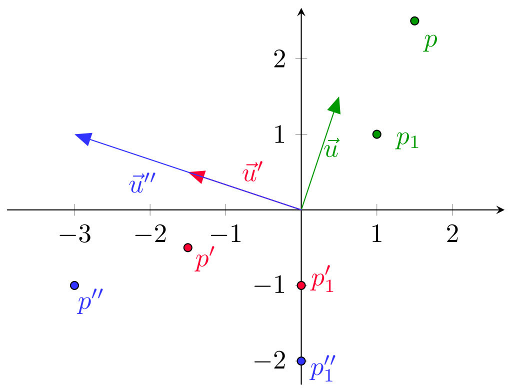
$$S\,p_1' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$S\,p' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1.5 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$S\,\vec{u}' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

and we can now convert them back to Cartesian coordinates:

$$p_1'' = (0, -2)^T, \quad p'' = (-3, -1)^T \quad \text{and} \quad \vec{u}'' = (-3, 1)^T$$



From the image we can now see that the vector has been scaled as expected.

Task 4

To compute the inverse matrices we simply think about the opposite effect we want to get, hence

$$R_{90}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^{-1} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by combining the matrices in a single transformation matrix we get

$$\begin{aligned} M = R_{90}^{-1}\, T^{-1}\, S^{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & -1 \\ 0 & 0.5 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We can now test our result by applying M to $p''_1, p'', \vec{u}''$

$$M\, p''_1 = \begin{pmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = p_1$$

$$M\, p'' = \begin{pmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 2.5 \\ 1 \end{pmatrix} = p$$

$$M\, \vec{u}'' = \begin{pmatrix} 0 & 0.5 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1.5 \\ 0 \end{pmatrix} = \vec{u}$$

Exercise 2

To know whether the point p lies in the triangle formed by p_1, p_2 and p_3 we must check the sign of the barycentric coordinates. If at least one of them is negative the point is outside our triangle. We first calculate the area of the triangle by using the cross product

$$\begin{aligned}\vec{n} &= (p_2 - p_1) \times (p_3 - p_1) = \\ &= \begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} -16 \\ 16 \\ -16 \end{pmatrix}\end{aligned}$$

and the area is

$$A = \frac{\|\vec{n}\|}{2} = \frac{16}{2}\sqrt{3} = 8\sqrt{3}$$

We can now calculate the areas of the sub triangles that originate from including p and two points of the triangle.

$$\begin{aligned}\vec{n}_1 &= (p_2 - p) \times (p_3 - p) = \\ &= \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \\ &= \begin{pmatrix} 8 \\ 8 \\ -8 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{n}_2 &= (p_3 - p) \times (p_1 - p) = \\ &= \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \\ &= \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\vec{n}_3 &= (p_1 - p) \times (p_2 - p) = \\ &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} = \\ &= \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}\end{aligned}$$

We can define the areas of the sub-triangles as

$$2w_i = \|\vec{n}_i\| \cdot \mathrm{sign}(\langle \vec{n}_i, \vec{n} \rangle)$$

so we simply look at the sign of $\langle \vec{n}_i, \vec{n} \rangle$ to determine if the barycentric coordinate is negative or not:

$$\langle \vec{n}_1, \vec{n} \rangle = \begin{pmatrix} 8 \\ 8 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} -16 \\ 16 \\ -16 \end{pmatrix} = 128$$

$$\langle \vec{n}_2, \vec{n} \rangle = \langle \vec{n}_3, \vec{n} \rangle = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -16 \\ 16 \\ -16 \end{pmatrix} = 64$$

and since all sub areas are positive, we can confidently say that p belongs to the triangle.

Exercise 3

Using barycentric coordinates we know that we can find a point inside a triangle by assigning different weights to them. Now, since the centroid of a triangle is the intersection of the medians, we want to give the same weight to each coordinate. Given that the sum of the barycentric coordinates $\lambda_1 + \lambda_2 + \lambda_3 = 1$ each λ will have the same weight of $\lambda_i = \frac{1}{3}$ for $i = 1, 2, 3$. If we want to consider a single median, we can divide it in two parts, namely the one between the vertex and the centroid (which we will call d) and the one between the centroid and the opposite edge (which is the barycentric coordinate λ). Since we know that the distance of the latter part is the barycentric coordinate, which is $\frac{1}{3}$, if we want to consider the whole segment we have to give the full weight to a single coordinate. We can now express d as the difference between 1 and λ, which results in $1 - \frac{1}{3} = \frac{2}{3}$, giving us the ratio 2:1 between d and λ.

Exercise 4

To construct the transformation matrix we will use the fact that to convert a point in homogenous coordinates back to Cartesian coordinates it must be scaled so that the coordinate of the extra dimension has value 1. From this we can define M as

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that when we apply the transformation M to a point we get that the extra coordinate will be equal to x, thus the resulting x component will forcibly be $\frac{x}{x} = 1$.

In the particular case of points starting on the y axis, it cannot be projected since the line starting from it and passing through the origin is parallel with the line $x = 1$. Mathematically, for any given point on the y axis its homogenous coordinates will be $(0, y, 1)^T$, $y \in \mathbb{R}$. The resulting vector would be $(0, y, 0)^T$ which cannot be rescaled so that the extra coordinate is equal to 1 (we can also see that it is not a point by definition of homogenous coordinates).