h5-pca-cs

June 11, 2018

Topics: Sparsity (PCA and Compressive Sensing)

Assigned: Wednesday May 23

Due: Sunday June 10 by midnight

```
In [1]: # -*- coding: utf-8 -*-
        import numpy as np
        from math import *
        import matplotlib.pyplot as plt
        from matplotlib import cm
        from mpl_toolkits.mplot3d import Axes3D
        from scipy.stats import norm


        # Params
        N = 2000
        dims = 100
        ulength = 7
```

1 Code & Logs

1.1 Part I PCA

Generate a random vector u in d dimensions as follows: The components of u are i.i.d., with - P [u[i] = 0] = 2=3; P [u[i] = +1] = 1=6; P [u[i] = 1] = 1=6

```
In [2]: # Generate the IID
        def generateMultiDimGaussian(d, ulength):
            u = np.ndarray((ulength,d))
            angles = np.zeros((ulength,ulength))
            deviation = 10
            while np.amax(angles+90*np.eye(ulength)) > (90+deviation) or np.amin(angles+90*np.e
                for i in range(ulength):
                    for j in range(d):
                        r = np.random.rand()
                        if(r<4/6):
```

```
                    u[i,j] = 0
                elif(r<5/6):
                    u[i,j] = 1
                else:
                    u[i,j] = -1
        if(u.any(axis=1).all()):
            for i in range(ulength):
                for j in range(i):
                    angles[i,j] = np.arccos(  np.clip( np.dot(u[i,:]/np.linalg.norm(u[i
                    angles[j,i] = angles[i,j]
    #print(angles)
    return u

# Uj be i.i.d
Uj = generateMultiDimGaussian(dims, ulength)
print(Uj.shape)
```

```
(7, 100)
```

Generate d-dimensional data samples for a Gaussian mixture distribution with 3 equiprobable components 1. Zm : Standard Gaussian (N(0, 1)) distribution 2. N : noise vector" N N(0, 2Id) (default value 2 = 0:01) 3. Component 1: Generate X = u1 + Z1u2 + Z2u3 + N. 4. Component 2: Generate X = 2u4 + sqrt(2)Z1u5 + Z2u6 + N. 5. Component 3: Generate X = sqrt(2)u6 + Z1(u1 + u2) + (1/sqrt(2))Z2u5 + N

```
In [3]: """
        Generate the higher dimension dataset and sample equiprobable from components
        """
        def generateDataset(u, num_data = 50, d = 30):
            sigma_sq = 0.01
            #print('\nX(Nxd):',num_data,"x",d,',\tUj:',Uj[0].shape)
            dataset = np.ndarray((num_data,d))
            labels  = np.zeros((num_data,3))  # will be containing [0,1,0] one hot value
            # Assign the values based on the three component function
            for i in range(0, num_data):
                # Random numbers Zm {Z1, Z2} and N are drawn afresh
                Z1 = np.random.normal()
                Z2 = np.random.normal()
                noise = np.random.multivariate_normal(np.zeros(d), (sigma_sq)*np.eye(d))
                # choose which comnponent to pick from
                idx_comp = np.random.choice([0, 1, 2],1,p=[0.333, 0.333, 0.334])
                if(idx_comp == 0):   # Sample from component 1
                    dataset[i,:] = Uj[1,:] + Z1*Uj[2,:] + Z2*Uj[3,:] + noise
                elif(idx_comp == 1): # Sample from component 3
                    dataset[i,:] = 2*Uj[4,:] + np.sqrt(2)*Z1*Uj[5,:] + Z2*Uj[6,:] + noise
                elif(idx_comp == 2): # Sample from component 3
                    dataset[i,:] = np.sqrt(2)*Uj[6,:] + Z1*(Uj[1,:] + Uj[2,:]) + 1/np.sqrt(2) *
```

```
            # Assign a label
            labels[i,idx_comp]=1
        return dataset, labels

#X_data, Y_hot_labels = generateDataset(Uj, 50, 30)
#print("> X: Data Set:",X_data.shape,", Y: One hot :",Y_hot_labels.shape)
```

1.1.1 1. SVD of the A(N Œ d) data matrix

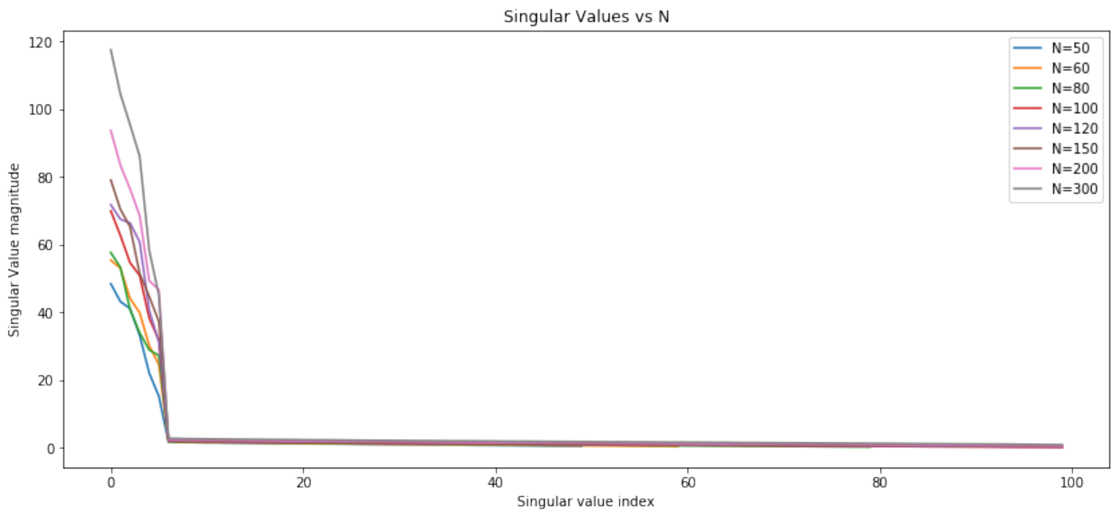
A = U SIGMA VT

1. V must diagonalize ATA and vi are eigenvectors of ATA.
2. SIGMA where SIGMAii are singular values of A.
3. U must diagonalize AAT and ui are eigenvectors of AAT

In [4]:
```
# input sample N=? from components
def getSingularValues(N = 50, d=30):
    X_data, Y_hot_labels = generateDataset(Uj, N, d)
    print("> X: Data Set:",X_data.shape,", Y: One hot :",Y_hot_labels.shape)
    #Xt = X_data.T
    U, S, V = np.linalg.svd(X_data, full_matrices=False)
    X_a = np.dot(np.dot(U, np.diag(S)), V)
    print("> SD-inp:", np.std(X_data), "\nSD-out:", np.std(X_a), "\nSD-Err:",np.std(X_d
    print("Singular values ",N,"Œ",d," : ",S[0:7])
    return S

plt.figure(figsize=(14,6))
plt.title('Singular Values vs N')
plt.xlabel('Singular value index')
plt.ylabel('Singular Value magnitude')

for Num in [50, 60, 80, 100, 120, 150, 200, 300]:
    S = getSingularValues(Num, dims)
    plt.plot(range(len(S)), S,label = str('N='+str(Num)))
plt.legend()
plt.show()
```

```
> X: Data Set: (50, 100) , Y: One hot : (50, 3)
> SD-inp: 1.2425699236269871
SD-out: 1.242569923626987
SD-Err: 3.048454173224238e-15
Singular values  50 Œ 100  :  [48.39048927 43.11330642 41.06337937 33.16084801 21.96589821 15.
  1.5994595 ]
> X: Data Set: (60, 100) , Y: One hot : (60, 3)
> SD-inp: 1.3498319267986385
SD-out: 1.3498319267986376
SD-Err: 3.4819962065686176e-15
```

```
Singular values  60 Œ 100  :  [55.31684119 53.0200592  44.0650705  39.84964441 30.09112566 24.5
  1.69841237]
> X: Data Set: (80, 100) , Y: One hot : (80, 3)
> SD-inp: 1.1498704791356968
SD-out: 1.1498704791356964
SD-Err: 1.806028648889445e-15
Singular values  80 Œ 100  :  [57.5893642  53.216893   40.83616548 33.89971405 28.80635373 27.
  1.76064902]
> X: Data Set: (100, 100) , Y: One hot : (100, 3)
> SD-inp: 1.2997668301927499
SD-out: 1.2997668301927496
SD-Err: 4.440394990401163e-15
Singular values  100 Œ 100  :  [69.90863454 62.67001101 54.6252193  50.84244765 38.22080618 31
  1.90438301]
> X: Data Set: (120, 100) , Y: One hot : (120, 3)
> SD-inp: 1.3060641823925117
SD-out: 1.3060641823925123
SD-Err: 1.6985712333628717e-15
Singular values  120 Œ 100  :  [71.74773111 67.46007854 66.34224228 60.79853815 40.5416491  31
  1.98174145]
> X: Data Set: (150, 100) , Y: One hot : (150, 3)
> SD-inp: 1.1982438899074066
SD-out: 1.198243889907406
SD-Err: 2.0649813402642227e-15
Singular values  150 Œ 100  :  [79.06779613 70.36741234 65.26769035 51.45642124 44.6072953  37
  2.11513307]
> X: Data Set: (200, 100) , Y: One hot : (200, 3)
> SD-inp: 1.2448438513004056
SD-out: 1.2448438513004045
SD-Err: 1.9863919347571614e-15
Singular values  200 Œ 100  :  [93.67871151 83.432265   76.45619056 68.60885691 49.28696661 46
  2.31861979]
> X: Data Set: (300, 100) , Y: One hot : (300, 3)
> SD-inp: 1.24972911012126
SD-out: 1.2497291101212598
SD-Err: 1.6247784093964919e-15
Singular values  300 Œ 100  :  [117.56893054 104.47921109  95.45651027  86.14543701  58.243263
  45.01554453   2.66531009]
```



1.1.2 1.(a) d0 = 6 are the dominant singular values

we can see this based on the variation for N = [50, 60, 80, 100, 120, 150, 200, 300]

1.1.3 Now, project the data down to the dominant d0 components to obtain an N Œ d0 data matrix.

```
In [5]: X_data, Y_hot_labels = generateDataset(Uj, N, dims)
        print("> X: Data Set:",X_data.shape,", Y: One hot :",Y_hot_labels.shape)
        U, S, V = np.linalg.svd(X_data, full_matrices=False)
        X_a = np.dot(np.dot(U, np.diag(S)), V)
        print(">\nSD-inp:", np.std(X_data), "\nSD-out:", np.std(X_a), "\nSD-Err:",np.std(X_data
        print("Singular values ",N,"Œ",dims," : ",S[0:7])


        d0 = 6 # Dominant vectors d0 = 6
        print("Singular: ", np.diag(S[:d0]).shape)
        reconst_matrix = np.dot(U[:,:d0],np.dot(np.diag(S[:d0]),V[:d0,:]))
        print(">\nSD-inp:", np.std(X_data), "\nSD-out:", np.std(reconst_matrix), "\nSD-Err:",np


        # Using eigen vector V as the basis for projecting the data
        evecs = V[:, :d0]
        X_reduced_matrix = np.dot(X_data, evecs)
        print("Reduced Matrix[From =",N,"Œ",dims,"]:" ," To =",X_reduced_matrix.shape)
        #print(X_reduced_matrix)
```

```
> X: Data Set: (2000, 100) , Y: One hot : (2000, 3)
>
SD-inp: 1.2395057807174528
```

```
SD-out: 1.2395057807174528
SD-Err: 1.7785550318342964e-15
Singular values  2000 Œ 100  :  [282.76080712 266.71486558 242.47285716 229.01120717 161.325886
 133.6358575    5.38798619]
Singular:  (6, 6)
>
SD-inp: 1.2395057807174528
SD-out: 1.235722754594632
SD-Err: 0.09679696927444671
Reduced Matrix[From = 2000 Œ 100 ]:  To = (2000, 6)
```

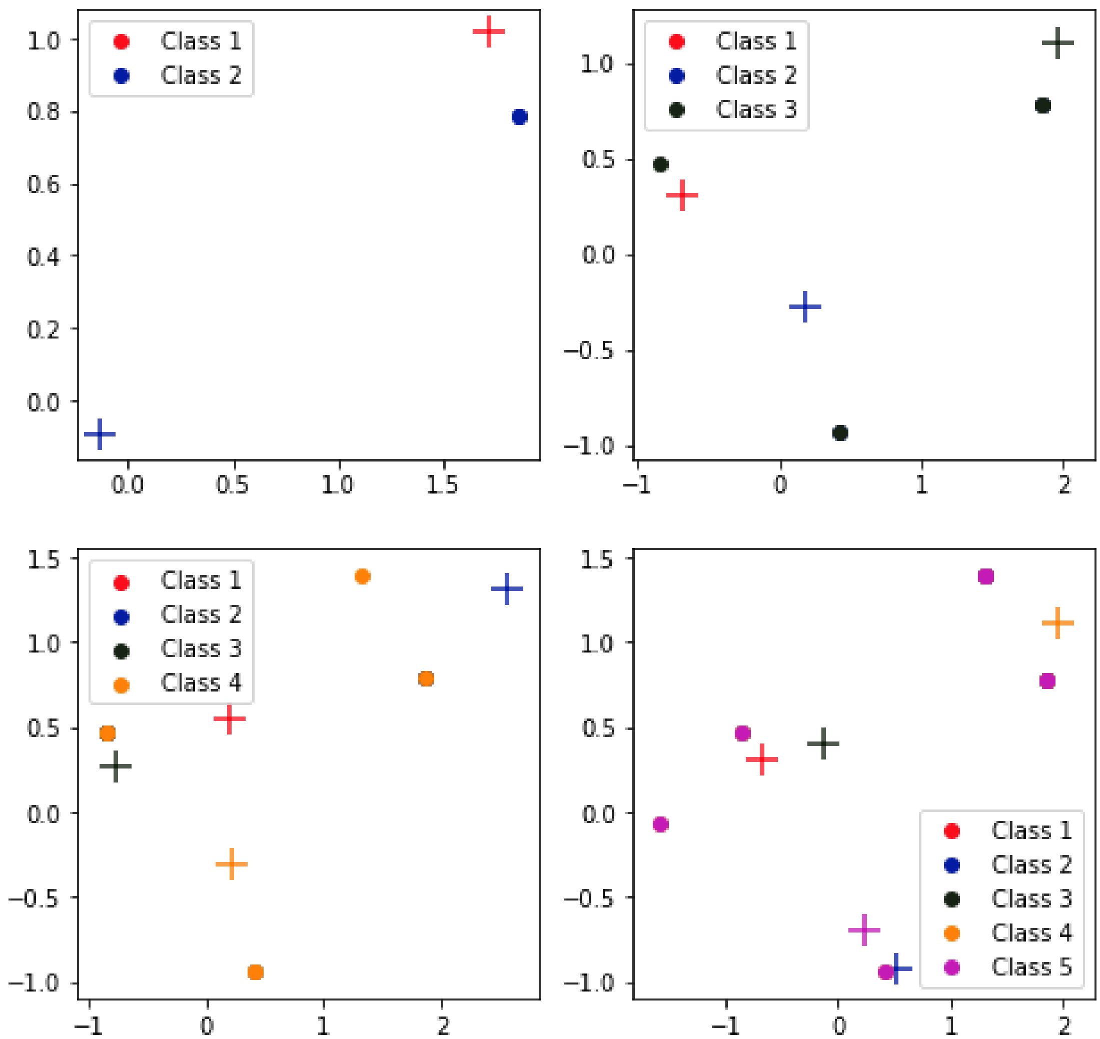
```
In [47]: def runKmeansCluster(data, labels, d):
             class_means_dict=dict()
             preds_dict=dict()
             fig = plt.figure(figsize=(8,8))
             subplot_id = 221
             print("Data Dims: ",N,"x",d)
             print("Dataset  : ", data.shape)
             # Kmeans trial
             for K in range(2,6):
                 pred = np.zeros((N, K))
                 class_means = np.ndarray((K, d))
                 initial_indices = np.random.choice(N,K)

                 for k in range(K):
                     class_means[k,:] = data[initial_indices[k],:]
                 old_class_means = np.zeros((K,d))

                 print("\n-------- K=",K,"----------")

                 while np.linalg.norm(old_class_means-class_means)/np.linalg.norm(class_means)
                     norm_mse = np.linalg.norm(old_class_means-class_means)
                     norm_mu = np.linalg.norm(class_means)
                     #print("> ",norm_mse,"/",norm_mu," \t=\t ",norm_mse/norm_mu)
                     old_class_means = np.array(class_means)
                     for i in range(N):
                         distance_to_means = np.zeros(K)
                         for k in range(K):
                             distance_to_means[k] = np.linalg.norm(class_means[k] - data[i])
                         #print("Distance: ",distance_to_means)
                         nearest_mean = np.argmin(distance_to_means)
                         #print("Nearest : ",nearest_mean)
                         # labels as the min dist
                         pred[i,:] = np.zeros(K)
                         pred[i, nearest_mean] = 1 # one hot encoding

                     # new mean
```

```
                for k in range(K):
                    class_means[k] = np.mean(data[np.where(pred[:,k]==1)], axis=0)
                #print("Class MU : ",class_means)
            ax = fig.add_subplot(subplot_id)
            subplot_id = subplot_id +1


            for k in range(K):
                colors = ('#fc0d1b','#041ca2','#162214','#fd8008','#c41bb6')
                ax.scatter(data[np.where(pred[:k] == 1),0], data[np.where(pred[:k] == 1),
                ax.scatter(class_means[k,0], class_means[k,1], s=200, marker='+', color =
            ax.legend()
            #plt.savefig('dim_%d_q1_%d_means.png' % (d,K), dpi=600)
            #plt.show

            # save means
            class_means_dict[K] = class_means
            preds_dict[K] = pred
            probabilities = np.ndarray((3,K))

            for predicted in range(K):
                print('______________________ \n')
                for true_label in range(3):
                    interssect = np.intersect1d(np.where(labels[:, true_label] == 1), np.
                    tots = np.where(labels[:, true_label] == 1)[0]
                    #print('K=',predicted,", idx=",true_label,"\nP -> ",interssect,"\nT -
                    probabilities[true_label, predicted] = len(interssect) / len(tots)
                    print('K=',predicted,", idx=",true_label,'  -> Prob=%.2f' % probabili
        fig.show
    print("Run on the reduced dimensions data with d0 = ", d0)
    runKmeansCluster(X_reduced_matrix, Y_hot_labels, d0)
```

```
Run on the reduced dimensions data with d0 =  6
Data Dims:  2000 x 6
Dataset  :  (2000, 6)

-------- K= 2 ----------
______________________

K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.80
K= 0 , idx= 2   -> Prob=0.00
______________________

K= 1 , idx= 0   -> Prob=1.00
K= 1 , idx= 1   -> Prob=0.20
K= 1 , idx= 2   -> Prob=1.00
```

```
-------- K= 3 ----------
----------------------
K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.30
K= 0 , idx= 2   -> Prob=0.49
----------------------
K= 1 , idx= 0   -> Prob=1.00
K= 1 , idx= 1   -> Prob=0.00
K= 1 , idx= 2   -> Prob=0.51
----------------------
K= 2 , idx= 0   -> Prob=0.00
K= 2 , idx= 1   -> Prob=0.70
K= 2 , idx= 2   -> Prob=0.00

-------- K= 4 ----------
----------------------
K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.51
K= 0 , idx= 2   -> Prob=0.00
----------------------
K= 1 , idx= 0   -> Prob=0.00
K= 1 , idx= 1   -> Prob=0.45
K= 1 , idx= 2   -> Prob=0.00
----------------------
K= 2 , idx= 0   -> Prob=0.00
K= 2 , idx= 1   -> Prob=0.04
K= 2 , idx= 2   -> Prob=0.56
----------------------
K= 3 , idx= 0   -> Prob=1.00
K= 3 , idx= 1   -> Prob=0.00
K= 3 , idx= 2   -> Prob=0.44

-------- K= 5 ----------
----------------------
K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.30
K= 0 , idx= 2   -> Prob=0.46
----------------------
K= 1 , idx= 0   -> Prob=0.50
```

```
K= 1 , idx= 1   -> Prob=0.00
K= 1 , idx= 2   -> Prob=0.00
----------------------

K= 2 , idx= 0   -> Prob=0.13
K= 2 , idx= 1   -> Prob=0.00
K= 2 , idx= 2   -> Prob=0.54
----------------------

K= 3 , idx= 0   -> Prob=0.00
K= 3 , idx= 1   -> Prob=0.70
K= 3 , idx= 2   -> Prob=0.00
----------------------

K= 4 , idx= 0   -> Prob=0.37
K= 4 , idx= 1   -> Prob=0.00
K= 4 , idx= 2   -> Prob=0.00
```



1.2 Part II : Random Projections and Compressed Sensing

1.3 3.

(a.) Generate m x d matrix , IID drawn such as P [ij = +1] = 1/2; P [ij = 1] = 1/2

(b.) Compressive Projection, y = 1/sqrt(m) * (x) with sparse reconstruction of s based on y

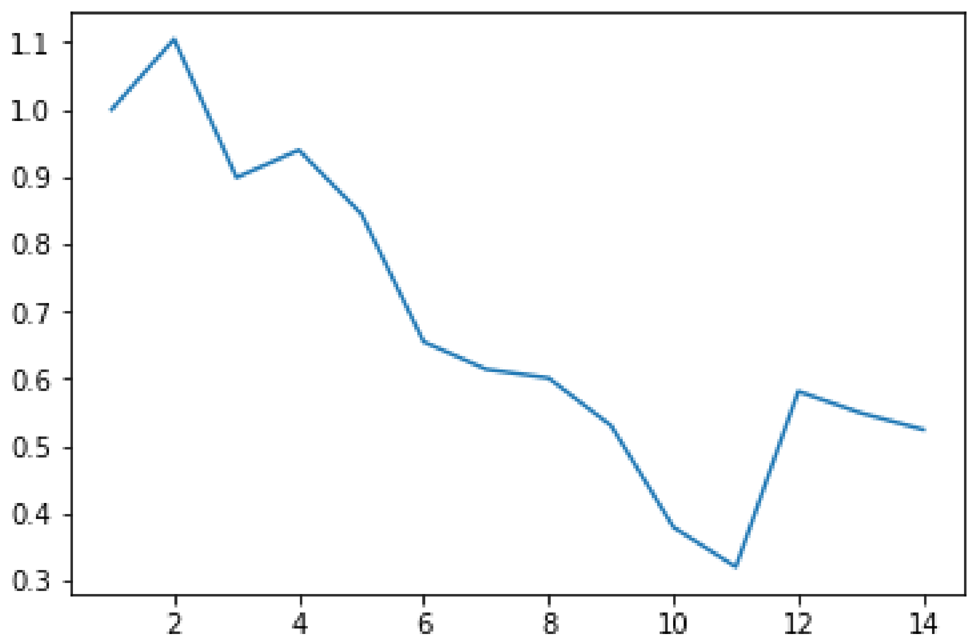
```
In [7]: def generatePhiMatrix(m, d):
            u = np.ndarray((m,d))
            for i in range(m):
                for j in range(d):
                    r = np.random.rand()
                    if(r<1/2):
                        u[i,j] = 1
                    else:
                        u[i,j] = -1
            #print("(mxd):",u.shape)
            return u

        def getCompressedProjection(m, d, log=False):
            # Keeping dimensions as dimensions d = 30 and number of data N = 50
            Xp_data, Zp_hot_labels = generateDataset(Uj, N, d)
            # Generate Phi
            phi = generatePhiMatrix(m,d)
            # Compressive Projection (m-dim projection of the d-dim matrix x)
            Y_xp =  (1/np.sqrt(m)) * np.matmul(Xp_data, phi.T)
            B = np.transpose(Uj)
            if(log == True):
                print("> N =",N,", M =",m,", Y_xp:", Y_xp.shape, ", Labels:",Zp_hot_labels.sha
            return Xp_data, Y_xp, Zp_hot_labels, phi, B

        M = 20
        X, Y_xp, labels, phi, B_evecs = getCompressedProjection(M, dims, True)
```

```
> N = 2000 , M = 20 , Y_xp: (2000, 20) , Labels: (2000, 3) , Phi : (20, 100) , B : (100, 7)
```

1.3.1 4. Lasso problem (using sklearn.linear_model.Lasso)

- (phi) = projection matrix also called A in examples
- x = latent data variables with gaussian noise
- y = observed results (projection matrix is applied on latent data variables and the result has been reduced in dimensions)

```
In [11]: from sklearn import linear_model

         def normalizedME(s, s_hat):
              return np.linalg.norm(s_hat - s)/np.linalg.norm(s)

         def findMinimumM():
             mses = []
             # Keeping dimension fixed and varying M for this
             rng = range(1,15)
             for m in rng:
                 s, Y_xp, labels, phi, B = getCompressedProjection(m, dims, False)
                 matrix = (1/np.sqrt(m)) * np.matmul(phi, B)
                 clf = linear_model.Lasso(alpha = 1.0) # Set lambda ( called alpha  here )
                 clf.fit(matrix, Y_xp.T) # Fit the reduced Y to the one hot
                 a_hat = clf.coef_ # Get a_hat
                 s_hat = np.matmul(a_hat, B.T) # this is the output
                 #print("Mat:",matrix.shape, ",\ts_hat:",s_hat.shape, ",\ts:",s.shape)
                 mses.append(normalizedME(s, s_hat))
             plt.plot(list(rng), mses)
             plt.show()
             return np.argmin(mses)+1

         minM = findMinimumM()
         print("The minimum M is ", minM)
```



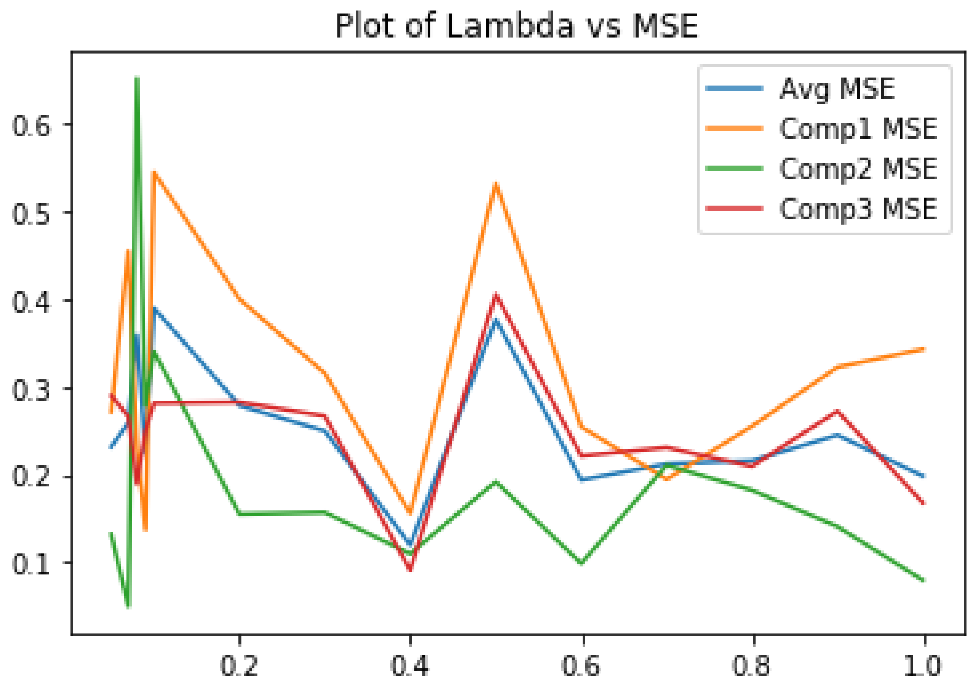

```
The minimum M is  11
```

1.3.2 5. Normalized MSE vs Lambda

In [37]:
```
def normalize(lst):
    s = sum(lst)
    return map(lambda x: float(x)/s, lst)

def mean(lst):
    return sum(lst)/float(len(lst))

def normalizedMSE(s, s_hat):
     return (np.linalg.norm(s_hat - s)/np.linalg.norm(s))**2

def computeMSELambda(N, m, d):
    mse_ = []
    mse_comp1_ = []
    mse_comp2_ = []
    mse_comp3_ = []
    s, Y_xp, labels, phi, B = getCompressedProjection(m, d, False)
    matrix = (1/np.sqrt(m)) * np.matmul(phi, B)
    clf = linear_model.Lasso(alpha = 1.0) # Set lambda ( called alpha  here )
    clf.fit(matrix, Y_xp.T) # Fit the reduced Y to the one hot
    a_hat = clf.coef_ # Get a_hat
    s_hat = np.matmul(a_hat, B.T) # this is the output
    for i in range(N):
        mse = normalizedMSE(s[i],s_hat[i])
        index = np.where(labels[i]==1)[0]
        if index == 0: #component 1
            mse_comp1_.append(mse)
        if index == 1:
            mse_comp2_.append(mse)
        if index == 2:
            mse_comp3_.append(mse)
        mse_.append(mse)
    return mean(mse_), mean(mse_comp1_),mean(mse_comp2_),mean(mse_comp3_)


mse_list=[]
mse_comp1_list = []
mse_comp2_list = []
mse_comp3_list = []

lambda_list = [0.05,0.07,0.08,0.09,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0]
for l in lambda_list:
    me, me1, me2, me3 = computeMSELambda(N, minM, dims)
    mse_list.append(me)
    mse_comp1_list.append(me1)
```

```
        mse_comp2_list.append(me2)
        mse_comp3_list.append(me3)
```

```
In [38]: plt.title('Plot of Lambda vs MSE')
         plt.plot(lambda_list, mse_list, label="Avg MSE")
         plt.plot(lambda_list, mse_comp1_list, label="Comp1 MSE")
         plt.plot(lambda_list, mse_comp2_list, label="Comp2 MSE")
         plt.plot(lambda_list, mse_comp3_list, label="Comp3 MSE")
         plt.legend()
         plt.show()
```



1.3.3 6. Compare the Euclidean distances squared vs the corresponding quantity in projected space

```
In [39]: m = minM
         J = 6
         B = np.transpose(Uj)
         projected_u = (1/np.sqrt(m)) * np.matmul(phi, B)
         projected_dist_matrix = np.zeros(shape=(J,J))
         for i in range(J):
             for j in range(J):
                 dist = np.linalg.norm(projected_u[:,i]-projected_u[:,j])**2
                 projected_dist_matrix[i][j] =  dist
         print("projected_dist_matrix\n",projected_dist_matrix)
```

```
        dist_matrix = np.zeros(shape=(J,J))
        for i in range(J):
            for j in range(J):
                dist = np.linalg.norm(Uj[:,i]-Uj[:,j])**2
                dist_matrix[i][j] =  dist
        print("dist_matrix\n",dist_matrix)
```

```
projected_dist_matrix
 [[  0.     37.45  56.36  73.45  23.64  46.82]
 [ 37.45   0.     66.91  66.55  34.91  87.55]
 [ 56.36  66.91   0.    131.27  59.64 114.45]
 [ 73.45  66.55 131.27   0.     70.18  67.18]
 [ 23.64  34.91  59.64  70.18   0.     43.18]
 [ 46.82  87.55 114.45  67.18  43.18   0.  ]]
dist_matrix
 [[ 0.  2.  2.  7.  3.  4.]
 [ 2.  0.  2.  3.  3.  2.]
 [ 2.  2.  0.  7.  3.  4.]
 [ 7.  3.  7.  0. 10.  7.]
 [ 3.  3.  3. 10.  0.  7.]
 [ 4.  2.  4.  7.  7.  0.]]
```

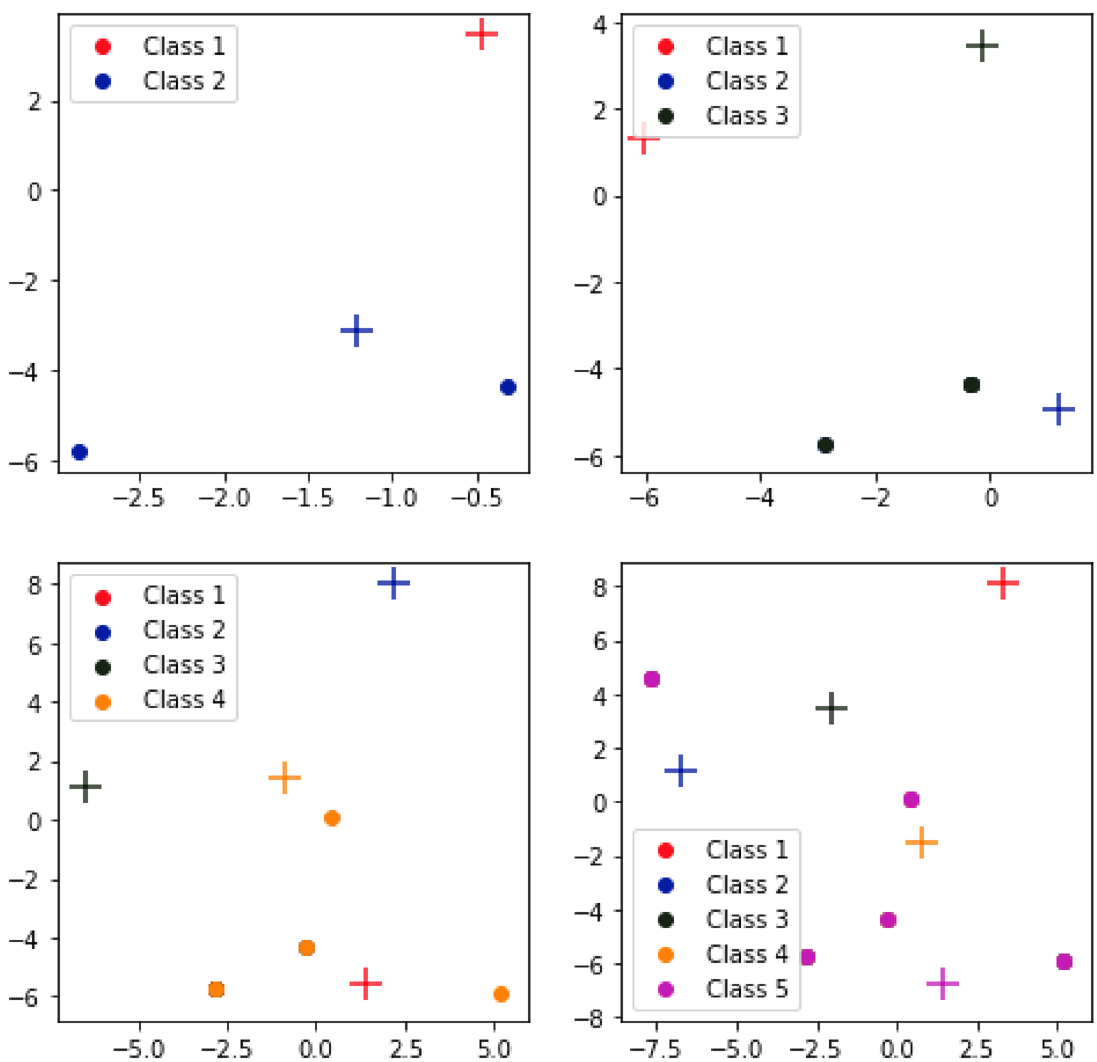
1.3.4 7. K-means algorithm post-projection

In [48]:
```
x, labels = generateDataset(Uj, N, dims)
phi = generatePhiMatrix(minM,dims)
Y = np.matmul(phi,np.transpose(x))/(np.sqrt(minM))
Z = np.transpose(Y)
runKmeansCluster(Z,labels, minM)
```

```
Data Dims:  2000 x 11
Dataset  :  (2000, 11)

-------- K= 2 ----------

----------------------

K= 0 , idx= 0   -> Prob=0.69
K= 0 , idx= 1   -> Prob=0.63
K= 0 , idx= 2   -> Prob=0.27
----------------------

K= 1 , idx= 0   -> Prob=0.31
K= 1 , idx= 1   -> Prob=0.37
K= 1 , idx= 2   -> Prob=0.73

-------- K= 3 ----------

----------------------
```

```
K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.43
K= 0 , idx= 2   -> Prob=0.08
----------------------

K= 1 , idx= 0   -> Prob=0.31
K= 1 , idx= 1   -> Prob=0.02
K= 1 , idx= 2   -> Prob=0.66
----------------------

K= 2 , idx= 0   -> Prob=0.69
K= 2 , idx= 1   -> Prob=0.55
K= 2 , idx= 2   -> Prob=0.26

-------- K= 4 ----------
----------------------

K= 0 , idx= 0   -> Prob=0.23
K= 0 , idx= 1   -> Prob=0.01
K= 0 , idx= 2   -> Prob=0.57
----------------------

K= 1 , idx= 0   -> Prob=0.03
K= 1 , idx= 1   -> Prob=0.30
K= 1 , idx= 2   -> Prob=0.01
----------------------

K= 2 , idx= 0   -> Prob=0.00
K= 2 , idx= 1   -> Prob=0.39
K= 2 , idx= 2   -> Prob=0.06
----------------------

K= 3 , idx= 0   -> Prob=0.74
K= 3 , idx= 1   -> Prob=0.30
K= 3 , idx= 2   -> Prob=0.36

-------- K= 5 ----------
----------------------

K= 0 , idx= 0   -> Prob=0.00
K= 0 , idx= 1   -> Prob=0.25
K= 0 , idx= 2   -> Prob=0.01
----------------------

K= 1 , idx= 0   -> Prob=0.00
K= 1 , idx= 1   -> Prob=0.37
K= 1 , idx= 2   -> Prob=0.05
```

```
----------------------

K= 2 , idx= 0   -> Prob=0.47
K= 2 , idx= 1   -> Prob=0.28
K= 2 , idx= 2   -> Prob=0.13
----------------------

K= 3 , idx= 0   -> Prob=0.43
K= 3 , idx= 1   -> Prob=0.10
K= 3 , idx= 2   -> Prob=0.41
----------------------

K= 4 , idx= 0   -> Prob=0.10
K= 4 , idx= 1   -> Prob=0.00
K= 4 , idx= 2   -> Prob=0.40
```



1.3.5 8. Geometric insights

1.3.6 END