NAME: ... KEY ....

CSCI E-89c Deep Reinforcement Learning

Part I of Assignment 11

Suppose each state $s \in \mathcal{S}$ of the Markov Decision Process can be represented by a vector of 2 real-valued features: $\mathbf{x}(s) = (x_1(s), x_2(s))^T$.

Given some policy π, suppose we model the state value function $v_\pi(s)$ with a *fully connected feedforward neural network* (please see the table below) which has two inputs ($x_1(s)$ and $x_2(s)$), one hidden layer that consists of two neurons (u_1 and u_2) with ReLU activation functions, and one output ($\hat{v}(s, \mathbf{w})$) with the ReLU activation function.

The explicit representation of this network is

input layer	hidden layer	output layer
x_1	$u_1 = f(w_{01}^{(1)} + w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2)$	$\hat{v} = f(w_0^{(2)} + w_1^{(2)} u_1 + w_2^{(2)} u_2)$
x_2	$u_2 = f(w_{02}^{(1)} + w_{12}^{(1)} x_1 + w_{22}^{(1)} x_2)$	

Here, $f(x)$ denotes the rectified linear unit (ReLU) defined as follows:

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Assume that the weights,

$$\mathbf{w} = \big(\underbrace{w_{01}^{(1)}, w_{11}^{(1)}, w_{21}^{(1)}, w_{02}^{(1)}, w_{12}^{(1)}, w_{22}^{(1)}}_{\text{hidden layer}}, \underbrace{w_0^{(2)}, w_1^{(2)}, w_2^{(2)}}_{\text{output layer}}\big)^T,$$

are currently estimated as follows:

hidden layer	output layer
$w_{01}^{(1)} = -1.2, w_{11}^{(1)} = 0.1, w_{21}^{(1)} = 0.5$	$w_0^{(2)} = 0.2, w_1^{(2)} = -0.8, w_2^{(2)} = 1.2$
$w_{02}^{(1)} = 0.9, w_{12}^{(1)} = 0.8, w_{22}^{(1)} = -0.3$	

Assume the agent minimizes the mean squared error loss function,

$$L \doteq \frac{1}{2} \left(\hat{v}(S_t, \mathbf{w}) - v_\pi(S_t)\right)^2,$$

using Stochastic Gradient Descent (SGD), i.e. the Neural Network is trained in mini-batches of size 1.

If for current state S_t, the features are $x_1(S_t) = 1.3$ and $x_2(S_t) = 0.7$; and the agent "observes" $v_\pi(S_t)$ (this, of course, means the agent uses MC return, 1-step TD return, etc. as a "measurement" of $v_\pi(S_t)$) to be 4.1, please find

(a) Error associated with the output layer:

$$\varepsilon^{(2)} \doteq \frac{\partial L}{\partial \hat{v}}.$$

(b) Errors associated with the hidden layer:

$$\varepsilon_h^{(1)} \doteq \frac{\partial L}{\partial u_h}, \quad h = 1, 2.$$

(c) Partial derivatives of the loss function with respect to weights in the output layer:

$$\frac{\partial L}{\partial w_h^{(2)}}, \quad h = 0, 1, 2.$$

(d) Partial derivatives of the loss function with respect to weights in the hidden layer:

$$\frac{\partial L}{\partial w_{jh}^{(1)}}, \quad j = 0, 1, 2 \text{ and } h = 1, 2.$$

(e) The next SGD update of the weights using $\alpha = 0.1$:

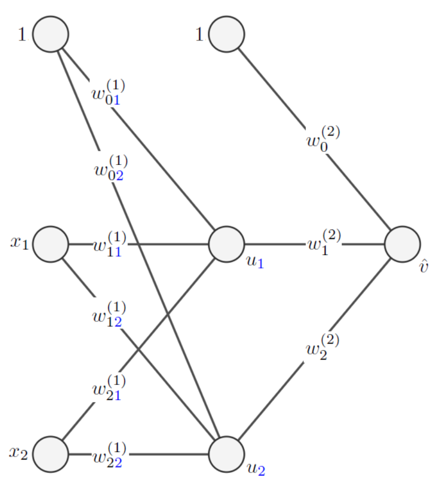
$$\mathbf{w} - \alpha \nabla L,$$

where $\nabla L \doteq \Big(\underbrace{\frac{\partial L}{\partial w_{01}^{(1)}}, \frac{\partial L}{\partial w_{11}^{(1)}}, \frac{\partial L}{\partial w_{21}^{(1)}}, \frac{\partial L}{\partial w_{02}^{(1)}}, \frac{\partial L}{\partial w_{12}^{(1)}}, \frac{\partial L}{\partial w_{22}^{(1)}}}_{\text{hidden layer}}, \underbrace{\frac{\partial L}{\partial w_0^{(2)}}, \frac{\partial L}{\partial w_1^{(2)}}, \frac{\partial L}{\partial w_2^{(2)}}}_{\text{output layer}} \Big)^T.$

Please notice that the "measurement" of the state-value $v_\pi(S_t)$ here is considered to be independent of $\mathbf{w}$ (please see, for example, the Semi-gradient 1-step Temporal-Difference (TD) prediction).

SOLUTION:

Graphical representation of this neural network:



The propagation forward results:

$$u_1 = f(\underbrace{w_{01}^{(1)} + w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2}_{z_1^{(1)}}) = f(\underbrace{-1.2 + 0.1 \cdot 1.3 + 0.5 \cdot 0.7}_{z_1^{(1)}}) = f(\underbrace{-0.72}_{z_1^{(1)}}) = 0$$

$$u_2 = f(\underbrace{w_{02}^{(1)} + w_{12}^{(1)} x_1 + w_{22}^{(1)} x_2}_{z_2^{(1)}}) = f(\underbrace{0.9 + 0.8 \cdot 1.3 - 0.3 \cdot 0.7}_{z_2^{(1)}}) = f(\underbrace{1.73}_{z_2^{(1)}}) = 1.73$$

then

$$\hat{v} = f(\underbrace{w_0^{(2)} + w_1^{(2)} u_1 + w_2^{(2)} u_2}_{z^{(2)}}) = f(\underbrace{0.2 - 0.8 \cdot 0 + 1.2 \cdot 1.73}_{z^{(2)}}) = f(\underbrace{2.276}_{z^{(2)}}) = 2.276.$$

(a) Error associated with the output layer:

$$\varepsilon^{(2)} \doteq \frac{\partial L}{\partial \hat{v}} = \frac{\partial}{\partial \hat{v}} \left[\frac{1}{2} \left(\hat{v} - v_\pi(S_t) \right)^2 \right] = \hat{v} - v_\pi(S_t) = 2.276 - 4.1 = -1.824.$$

(b) Errors associated with the hidden layer:

$$\varepsilon_h^{(1)} \doteq \frac{\partial L}{\partial u_h} = \frac{\partial L}{\partial \hat{v}} \frac{\partial \hat{v}}{\partial u_h} = \varepsilon^{(2)} f'(z^{(2)}) w_h^{(2)}, \quad h = 1, 2.$$

$$\varepsilon_1^{(1)} = \varepsilon^{(2)} f'(z^{(2)}) w_1^{(2)} = -1.824 \cdot f'(2.276) \cdot (-0.8) = -1.824 \cdot 1 \cdot (-0.8) = 1.4592$$
$$\varepsilon_2^{(1)} = \varepsilon^{(2)} f'(z^{(2)}) w_2^{(2)} = -1.824 \cdot f'(2.276) \cdot (1.2) = -1.824 \cdot 1 \cdot (1.2) = -2.1888$$

(c) Partial derivatives of the loss function with respect to weights in the output layer:

$$\frac{\partial L}{\partial w_h^{(2)}} = \frac{\partial L}{\partial \hat{v}} \frac{\partial \hat{v}}{\partial w_h^{(2)}} = \varepsilon^{(2)} \frac{\partial}{\partial w_h^{(2)}} \left[f(\underbrace{w_0^{(2)} + w_1^{(2)} u_1 + w_2^{(2)} u_2}_{z^{(2)}}) \right] = \varepsilon^{(2)} f'(z^{(2)}) u_h, \;\; h = 0, 1, 2.$$

$$\tfrac{\partial L}{\partial w_0^{(2)}} = \varepsilon^{(2)} f'(z^{(2)}) u_0 = -1.824 \cdot f'(2.276) \cdot 1 = -1.824 \cdot 1 \cdot 1 = -1.824$$
$$\tfrac{\partial L}{\partial w_1^{(2)}} = \varepsilon^{(2)} f'(z^{(2)}) u_1 = -1.824 \cdot f'(2.276) \cdot 0 = -1.824 \cdot 1 \cdot 0 = 0$$
$$\tfrac{\partial L}{\partial w_2^{(2)}} = \varepsilon^{(2)} f'(z^{(2)}) u_2 = -1.824 \cdot f'(2.276) \cdot 1 = -1.824 \cdot 1 \cdot 1.73 = -3.15552$$

(d) Partial derivatives of the loss function with respect to weights in the hidden layer:

$$\frac{\partial L}{\partial w_{jh}^{(1)}} = \frac{\partial L}{\partial u_h} \frac{\partial u_h}{\partial w_{jh}^{(1)}} = \varepsilon_h^{(1)} \frac{\partial}{\partial w_{jh}^{(1)}} \left[f(\underbrace{w_{0h}^{(1)} + w_{1h}^{(1)} x_1 + w_{2h}^{(1)} x_2}_{z_h{}^{(1)}}) \right] = \varepsilon_h^{(1)} f'(z_h{}^{(1)}) x_j,$$

for each $j = 0, 1, 2$ and $h = 1, 2$. Here, we define $x_0 \doteq 1$.medskip

$$\tfrac{\partial L}{\partial w_{01}^{(1)}} = \varepsilon_1^{(1)} f'(z_1{}^{(1)}) x_0 = 0$$
$$\tfrac{\partial L}{\partial w_{11}^{(1)}} = \varepsilon_1^{(1)} f'(z_1{}^{(1)}) x_1 = 0$$
$$\tfrac{\partial L}{\partial w_{21}^{(1)}} = \varepsilon_1^{(1)} f'(z_1{}^{(1)}) x_2 = 0$$

$$\tfrac{\partial L}{\partial w_{02}^{(1)}} = \varepsilon_2^{(1)} f'(z_2{}^{(1)}) x_0 = -2.1888$$
$$\tfrac{\partial L}{\partial w_{12}^{(1)}} = \varepsilon_2^{(1)} f'(z_2{}^{(1)}) x_1 = -2.84544$$
$$\tfrac{\partial L}{\partial w_{22}^{(1)}} = \varepsilon_2^{(1)} f'(z_2{}^{(1)}) x_2 = -1.53216$$

(e) The next SGD update of the weights using $\alpha = 0.1$:

$$\mathbf{w} - \alpha \nabla L,$$

where $\nabla L \doteq \Big(\underbrace{\frac{\partial L}{\partial w_{01}^{(1)}}, \frac{\partial L}{\partial w_{11}^{(1)}}, \frac{\partial L}{\partial w_{21}^{(1)}}, \frac{\partial L}{\partial w_{02}^{(1)}}, \frac{\partial L}{\partial w_{12}^{(1)}}, \frac{\partial L}{\partial w_{22}^{(1)}}}_{\text{hidden layer}}, \underbrace{\frac{\partial L}{\partial w_{0}^{(2)}}, \frac{\partial L}{\partial w_{1}^{(2)}}, \frac{\partial L}{\partial w_{2}^{(2)}}}_{\text{output layer}} \Big)^T$.

Please notice that the "measurement" of the state-value $v_\pi(S_t)$ here is considered to be independent of $\mathbf{w}$ (please see, for example, the Semi-gradient 1-step Temporal-Difference (TD) prediction).

$$\begin{aligned}
&\mathbf{w} - \alpha \nabla L \\
&= \big(\underbrace{w_{01}^{(1)}, w_{11}^{(1)}, w_{21}^{(1)}, w_{02}^{(1)}, w_{12}^{(1)}, w_{22}^{(1)}}_{\text{hidden layer}}, \underbrace{w_{0}^{(2)}, w_{1}^{(2)}, w_{2}^{(2)}}_{\text{output layer}} \big)^T \\
&- \alpha \Big(\underbrace{\frac{\partial L}{\partial w_{01}^{(1)}}, \frac{\partial L}{\partial w_{11}^{(1)}}, \frac{\partial L}{\partial w_{21}^{(1)}}, \frac{\partial L}{\partial w_{02}^{(1)}}, \frac{\partial L}{\partial w_{12}^{(1)}}, \frac{\partial L}{\partial w_{22}^{(1)}}}_{\text{hidden layer}}, \underbrace{\frac{\partial L}{\partial w_{0}^{(2)}}, \frac{\partial L}{\partial w_{1}^{(2)}}, \frac{\partial L}{\partial w_{2}^{(2)}}}_{\text{output layer}} \Big)^T \\
&= (-1.2,\ 0.1,\ 0.5,\ 1.11888,\ 1.084544,\ -0.146784,\ 0.3824,\ -0.8,\ 1.515552)^T
\end{aligned}$$