A Universal Turing Machine

A limitation of Turing Machines:

Turing Machines are $\underbrace{\text{“hardwired”}}_{\text{they execute only one program}}$

Real Computers are re-programmable

Solution: Universal Turing Machine

Attributes:

- Reprogrammable machine
- Simulates any other Turing Machine

Universal Turing Machine

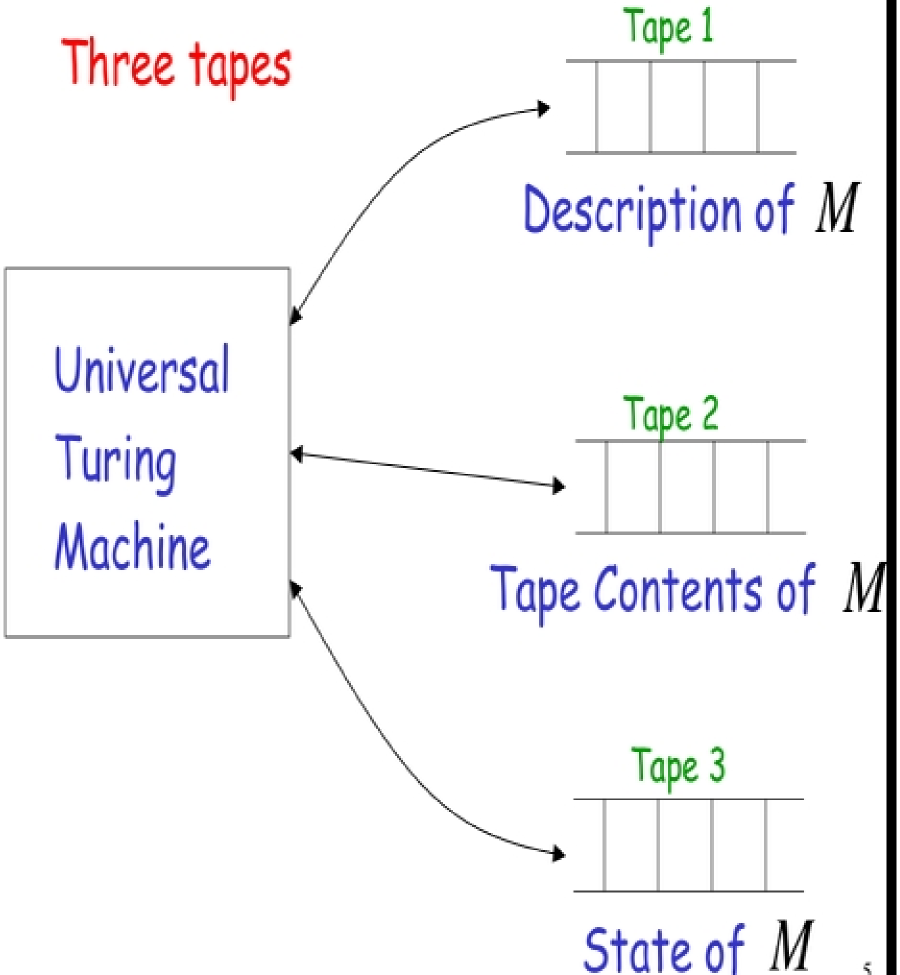
simulates any other Turing Machine M

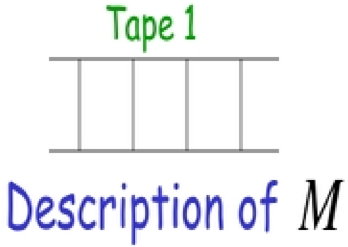
Input of Universal Turing Machine:

Description of transitions of M

Initial tape contents of M

Three tapes





We describe Turing machine M as a string of symbols:

We encode M as a string of symbols

Alphabet Encoding

Symbols:	a	b	c	d	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Encoding:	1	11	111	1111	

State Encoding

States:	q_1	q_2	q_3	q_4	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Encoding:	1	11	111	1111	

Head Move Encoding

Move:	L	R
	$\downarrow$	$\downarrow$
Encoding:	1	11

Transition Encoding

Transition: $\delta(q_1, a) = (q_2, b, L)$

Encoding: 10101101101

separator

Machine Encoding

Transitions:

$\delta(q_1, a) = (q_2, b, L)$ $\qquad$ $\delta(q_2, b) = (q_3, c, R)$

Encoding:

$10101101101 \; 00 \; 1101101110111011$

separator

Tape 1 contents of Universal Turing Machine:

encoding of the simulated machine M
as a binary string of 0's and 1's

A Turing Machine is described
with a binary string of 0's and 1's

Therefore:

The set of Turing machines forms a language:

each string of the language is
the binary encoding of a Turing Machine

Language of Turing Machines

L = { 010100101, (Turing Machine 1)

00100100101111, (Turing Machine 2)

111010011110010101, ...

...... }

Countable Sets

Infinite sets are either:

Countable

or

Uncountable

Countable set:

Any finite set

or

Any *Countably infinite* set:

There is a one to one correspondence

between

elements of the set

and

Natural numbers

Example: The set of even integers is countable

Even integers: $0,\ 2,\ 4,\ 6,\ \ldots$

Correspondence: $\downarrow\ \downarrow\ \downarrow\ \downarrow$

Positive integers: $1,\ 2,\ 3,\ 4,\ \ldots$

$2n$ corresponds to $n+1$

Example: The set of rational numbers is countable

Rational numbers: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Naïve Proof

Rational numbers: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Correspondence: $\downarrow \quad \downarrow \quad \downarrow$

Positive integers: $1, 2, 3, \ldots$

Doesn't work:

we will never count
numbers with nominator 2: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$

Better Approach

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \ldots & \\
\frac{3}{1} & \frac{3}{2} & \ldots & & \\
\frac{4}{1} & \ldots & & &
\end{array}$$

$$
\begin{tikzcd}
\frac{1}{1} \arrow[r] & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\frac{1}{1} \arrow[r] & \frac{1}{2} \arrow[dl] & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{tikzcd}
$$

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}$$

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}$$

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \cdots & & \\
\frac{4}{1} & \cdots & & &
\end{array}
$$

$$
\begin{array}{lcccccc}
\text{Rational Numbers:} & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{1}, & \frac{1}{3}, & \frac{2}{2}, & \cdots \\
\text{Correspondence:} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\text{Positive Integers:} & 1, & 2, & 3, & 4, & 5, & \ldots
\end{array}
$$

We proved:

the set of rational numbers is countable by describing an enumeration procedure

Definition

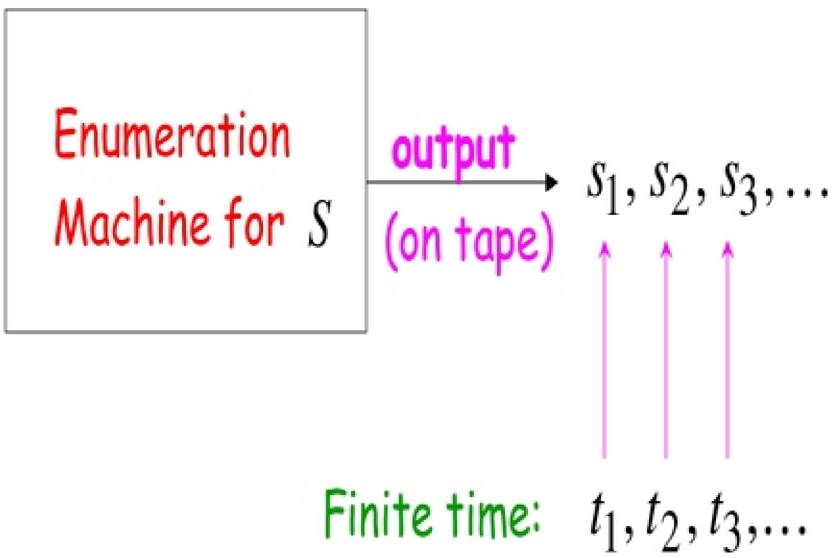
Let S be a set of strings

An **enumeration procedure** for S is a Turing Machine that generates all strings of S one by one

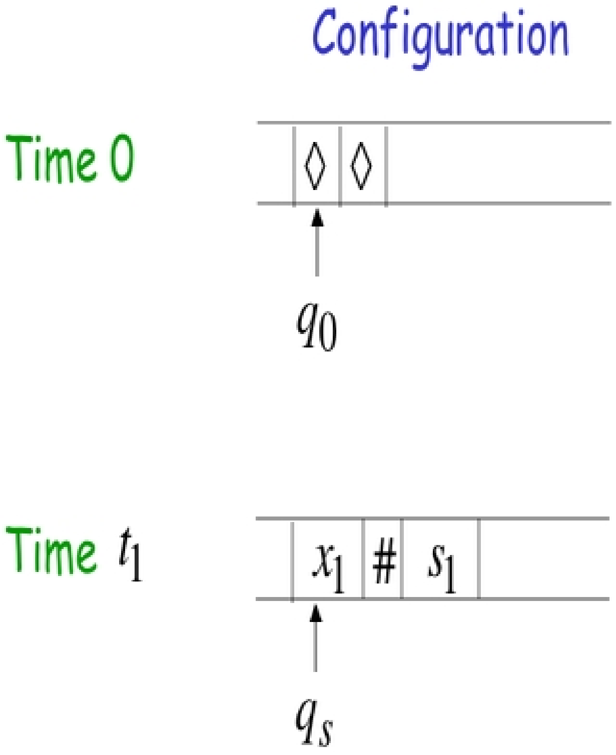
and

Each string is generated in finite time

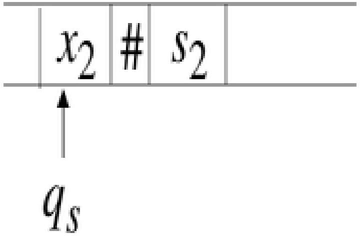
strings $s_1, s_2, s_3, \ldots \in S$



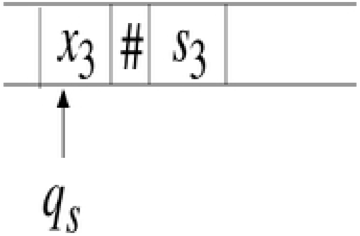
Enumeration Machine



Time t_2



Time t_3



Observation:

If for a set there is an enumeration procedure, then the set is countable

Example:

The set of all strings $\{a,b,c\}^+$ is countable

Proof:

We will describe an enumeration procedure

Naive procedure:

Produce the strings in lexicographic order:

$$a$$
$$aa$$
$$aaa$$
$$aaaa$$
$$\ldots\ldots$$

Doesn't work:

strings starting with b will never be produced

Better procedure: Proper Order

1. Produce all strings of length 1

2. Produce all strings of length 2

3. Produce all strings of length 3

4. Produce all strings of length 4

....

Produce strings in **Proper Order:**

$$\left.\begin{array}{l} a \\ b \\ c \end{array}\right\} \text{length 1}$$

$$\left.\begin{array}{l} aa \\ ab \\ ac \\ ba \\ bb \\ bc \\ ca \\ cb \\ cc \end{array}\right\} \text{length 2}$$

$$\left.\begin{array}{l} aaa \\ aab \\ aac \\ \ldots\ldots \end{array}\right\} \text{length 3}$$

Theorem: The set of all Turing Machines is countable

Proof: Any Turing Machine can be encoded with a binary string of 0's and 1's

Find an enumeration procedure for the set of Turing Machine strings

Enumeration Procedure:

Repeat

1. Generate the next binary string of 0's and 1's in proper order

2. Check if the string describes a Turing Machine

   if **YES:** print string on output tape

   if **NO:** ignore string

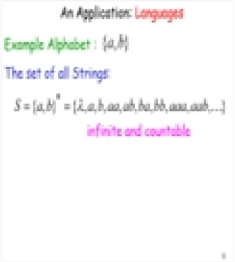
Uncountable Sets

Definition: A set is uncountable if it is not countable

Theorem:

Let S be an infinite countable set

The powerset 2^S of S is uncountable


An Application: Languages

Example Alphabet : $\{a,b\}$

The set of all Strings:

$S = \{a,b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

infinite and countable

Proof:

Since S is countable, we can write

$$S = \{s_1, s_2, s_3, \ldots\}$$

Elements of S

Elements of the powerset have the form:

$$\{s_1, s_3\}$$

$$\{s_5, s_7, s_9, s_{10}\}$$

......

We encode each element of the power set with a binary string of 0's and 1's

Powerset element	Encoding s_1	s_2	s_3	s_4	$\cdots$
$\{s_1\}$	1	0	0	0	$\cdots$
$\{s_2, s_3\}$	0	1	1	0	$\cdots$
$\{s_1, s_3, s_4\}$	1	0	1	1	$\cdots$

Let's assume (for contradiction)
that the powerset is countable.

Then: we can enumerate
the elements of the powerset

Powerset element	Encoding					
t_1	1	0	0	0	0	$\cdots$
t_2	1	1	0	0	0	$\cdots$
t_3	1	1	0	1	0	$\cdots$
t_4	1	1	0	0	1	$\cdots$
$\cdots$						

Take the powerset element whose bits are the complements in the diagonal

$$\begin{array}{lcccccc}
t_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
t_2 & 1 & 1 & 0 & 0 & 0 & \cdots \\
t_3 & 1 & 1 & 0 & 1 & 0 & \cdots \\
t_4 & 1 & 1 & 0 & 0 & 1 & \cdots
\end{array}$$

New element: $0011\ldots$

(birary complement of diagonal)

The new element must be some t_i of the powerset

However, that's impossible:

from definition of t_i

the i-th bit of t_i must be the complement of itself

Contradiction!!!

Since we have a contradiction:

The powerset 2^S of S is uncountable

An Application: Languages

Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S=\{a,b\}^{*}=\{\lambda,a,b,aa,ab,ba,bb,aaa,aab,\ldots\}$$

infinite and countable

Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S=\{a,b\}^{*}=\{\lambda,a,b,aa,ab,ba,bb,aaa,aab,\ldots\}$$

infinite and countable

A language is a subset of S :

$$L=\{aa,ab,aab\}$$

Example Alphabet : $\{a,b\}$

The set of all Strings:

$$S=\{a,b\}^{*}=\{\lambda,a,b,aa,ab,ba,bb,aaa,aab,\ldots\}$$

infinite and countable

The powerset of S contains all languages:

$$2^{S}=\{\{\lambda\},\{a\},\{a,b\}\{aa,ab,aab\},\ldots\}$$

$$L_1 \quad L_2 \quad L_3 \qquad L_4 \qquad \ldots$$

uncountable

Languages: uncountable

$$\begin{array}{cccccc} L_1 & L_2 & L_3 & \cdots & L_k & \cdots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ M_1 & M_2 & M_3 & & ? & \end{array}$$

Turing machines: countable

There are more languages
than Turing Machines

Conclusion:

There are some languages not accepted by Turing Machines

(These languages cannot be described by algorithms)

Languages not accepted by Turing Machines